A few topics related to maximum principles
(最大値原理に関連する諸課題)

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Contents

Abstract

In this paper we present a few topics related to maximum principles (comparison principles) in partial differential equations. The equations we consider include Hamilton-Jacobi equations and curvature flow equations appearing in crystal growth phenomena. On the basis of the theory of viscosity solutions, we establish a unique solvability of the initial value problems via maximum principles and track behavior of the solution. We also apply the method for proving maximum principles to discrete analysis.

In Chapter 1 we study the initial-value problem for a Hamilton-Jacobi equation whose Hamiltonian is discontinuous with respect to state variables. Our motivation comes from a model describing the two-dimensional nucleation in crystal growth phenomena. A typical equation has a semicontinuous source term. We introduce a new notion of viscosity solutions and prove, among other results, that the initial-value problem admits a unique global-in-time uniformly continuous solution for any bounded uniformly continuous initial data. For Bellman equations we give a representation formula of the solution as a value function of the optimal control problem with a semicontinuous running cost function. The large time behavior of the unique solution is also studied. We prove that, when the source term has a compact support, the scaling limit of the solution to the initial-value problem is characterized as the unique self-similar solution of the limit problem with a jump discontinuity at the origin. In the case where the source term is periodic, it turns out that the solution is asymptotically constant. We also study equations for the profile function of the self-similar solution, and establish a comparison principle and an existence result of solutions to general stationary problems.

Chapter 2 is devoted to the asymptotic behavior of solutions to fully nonlinear second order parabolic equations including a generalized curvature flow equation which was introduced by Mullins in 1957 as a model of evaporation-condensation. We prove that, in the multi-dimensional half space, solutions of the problem with prescribed contact angle asymptotically converge to a self-similar solution of the associated problem under a suitable rescaling. Several properties of the profile function of the self-similar solution are also investigated. We show that the profile function has a corner and that its angle is determined by points at which the equation is degenerate. We also study the depth of the groove, which is represented by the value of the profile function at the boundary. Among other results it turns out that, as the contact angle tends to zero, the depth of the groove is well approximated by the linearized problem.

In Chapter 3 we establish an isoperimetric inequality constrained by n-

dimensional lattices. We prove that, among all sets which consist of lattice translations of a given rectangular parallelepiped, a cube is the optimal shape that minimizes the ratio involving its perimeter and volume as long as the cube is realizable by the lattice. For its proof, the solvability of finite difference Poisson-Neumann problems is verified. Our approach to the isoperimetric inequality is based on the technique used in a proof of the Aleksandrov-Bakelman-Pucci maximum principle, which was originally proposed by Cabré in 2000 to prove the classical isoperimetric inequality.

Chapter 4 is concerned with the level set method. In the classical level set method, a slope of a solution to level set equations can be close to zero as time develops even if the initial slope is large, and this prevents one from computing interfaces given as the level set of the solution. To overcome this issue we introduce an improved equation by adding an extra term to the original equation. Then, by applying a comparison principle to the signed distance function to the interface, we prove that, globally in time, the slope of a solution of the initial value problem is preserved near the zero level set.

Chapter 1 is combination of papers [1] and [4], while Chapter 2 is essentially based on [3]. Chapter 3 is essentially based on [2].

All Sections, formulas and theorems, etc., are cited only in the chapter where they appear.

References

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Acknowledgments

The author would like to first express his gratitude to his supervisor, Professor Yoshikazu Giga, for many helpful discussions and valuable suggestions. Without his constant support and encouragement, it would have been impossible to complete this paper. He also thanks Professor Etsuro Yokoyama for giving him several important references on crystal growth. Many discussions with Professor Yokoyama helped him to understand the mechanism of crystal growth.

The author is grateful to Dr. Qing Liu for many useful suggestions, especially on the optimal control theory discussed in Chapter 1. He would next send his gratitude to Professor Yoshihito Kohsaka, who gave him informative remarks on curvature flow equations studied in Chapter 2. He would also like to thank Professor Xavier Cabré for his explanation of a proof of the classical isoperimetric inequality by the ABP method in a school "New Trends in Nonlinear PDEs" held at Centro di Ricerca Matematica Ennio De Giorgi during 18–22 June 2012. The lectures in the school led him to the study of the discrete isoperimetric inequality in Chapter 3.

The work of the author is supported by Grant-in-aid for Scientific Research of JSPS Fellows No. 23-4365.

Chapter 1

Hamilton-Jacobi equations with discontinuous source terms

1 Introduction

We consider the initial-value problem for the Hamilton-Jacobi equation of the form

$$\text{(HJ)} \begin{cases} \partial_t u + H(x, \nabla u) = 0 & \text{in } \mathbf{R}^n \times (0,T) =: Q, & (1.1) \\ u|_{t=0} = u_0 & \text{in } \mathbf{R}^n, & (1.2) \end{cases}$$

when the Hamiltonian H is discontinuous in space variable $x \in \mathbf{R}^n$. Here ∇u denotes the spatial gradient, i.e., $\nabla u = \nabla_x u = (\partial_{x_1} u, \dots, \partial_{x_n} u)$. A typical example we consider is the case when

$$H(x,p) = -|p| - cI(x) \quad (c > 0) \tag{1.3}$$

with

$$I(x) := \begin{cases} 1 & (x = 0), \\ 0 & (x \neq 0), \end{cases} \tag{1.4}$$

where $|\cdot|$ stands for the standard Euclidean norm in $\mathbf{R}^n$. In other words, the source term can be discontinuous. Our main goal is to introduce a suitable definition of weak solution (by extending the theory of viscosity solutions) so that the initial-value problem admits a unique global-in-time solution for a general bounded Lipschitz continuous initial data $u_0 \in BLip(\mathbf{R}^n)$ or even just bounded uniformly continuous initial data $u_0 \in BUC(\mathbf{R}^n)$.

Our motivation comes from crystal growth phenomena. One of key mechanisms of crystal growth is the *two dimensional nucleation* ([6, 36]). This growth is started by external supply of crystal molecules for a flat face. Such a source of supply is called a *step source*. This is a macroscopic understanding of crystal growth. At a very initial stage of the two dimensional nucleation the step source catches crystal molecules so that a small disk-like island is formed at the step source on a flat face. Then the island grows to be a "wedding cake" consisting of several disks. The other mechanism is the spiral growth ([6]). As pointed out by [6] a pair of two spirals (with opposite orientations) whose centers are very close essentially forms a small island just like the two dimensional nucleation; see, e.g., [38] and [33]. According to [37], some high-temperature superconductor provides such a model and the authors proposed a macroscopic model including (1.1)–(1.3) approximating spiral growth on a crystal surface. Both situations can be modeled by Hamilton-Jacobi equations with discontinuous source terms if we interpret the phenomena in macroscopic point of view.

Let us consider the typical case that there is a step source only at the origin and crystals grow at the uniform velocity 1 horizontally. Assume that the step source supplies crystal molecules at a rate of c (> 0) and let $u(x,t)$ be the height of crystals at position $x \in \mathbf{R}^n$ and time $t \in (0,T)$. (See Figure 1.) Then,

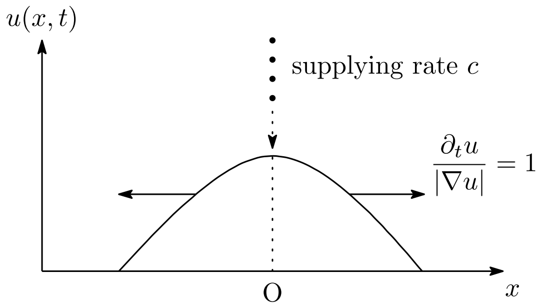

Figure 1: The step source at the origin.

the horizontal outward growth rate of crystals is given by $\partial_t u/|\nabla u|$. Since the horizontal growth speed is one, we have

$$\partial_t u = |\nabla u|$$

provided that $\nabla u \neq 0$. However, this equation does not include the effect of step sources. It is natural to postulate that the growth rate $\partial_t u$ at the origin should increase by c due to the step source. The resulting equation is formally of the form

$$\partial_t u - |\nabla u| = cI(x). \tag{1.5}$$

The corresponding Hamiltonian (1.3) is not continuous but lower semicontinuous. The equation (1.5) is a Hamilton-Jacobi equation with a discontinuous source term. The physical intuition suggests that

$$u^c(x,t) = c(t - |x|)_+ \tag{1.6}$$

is a solution for (1.5) when the initial-value equals zero (see Figure 2), where a_+ denotes the positive part of $a \in \mathbf{R}$, i.e., $a_+ = \max\{a,\ 0\}$. Such a "solution" is proposed in [30, 37] by variational principle. The function u^c is also obtained via approximation. More precisely, if we consider approximate Hamiltonians

$$H^\varepsilon(x,p) = -|p| - cI^\varepsilon(x) \tag{1.7}$$

with

$$I^\varepsilon(x) = \left(1 - \frac{|x|}{\varepsilon}\right)_+ \tag{1.8}$$

for $\varepsilon > 0$ and solve (HJ) with H^ε and $u_0 \equiv 0$, it turns out that the unique viscosity solution of the approximate problem uniformly converges to u^c as $\varepsilon \downarrow 0$. (See Example 3.2 for more details.) However, it is an important issue how to characterize u^c.

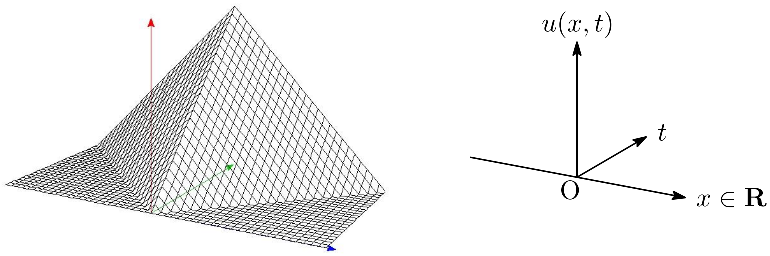

Figure 2: The intuitive solution of $\partial_t u - |\nabla u| = I(x)$, $u_0 \equiv 0$.

Unfortunately, we cannot expect the uniqueness of solutions for (1.5) even in Ishii's sense of viscosity solutions [26], where a discontinuous Hamiltonian is treated. Indeed, u^c is a solution but $u^\alpha(x,t) := \alpha(t - |x|)_+$ for $\alpha \in [0, c]$ is also a solution with the zero initial data. This is caused by an inadequate effect of the discontinuous term. More precisely, in the standard definition of supersolutions we use the upper semicontinuous envelope of H, that is $H^*(x,p) = -|p|$, but then the term $cI(x)$, which is a key term of our equation (1.5), disappears. Hence, in order to guarantee the uniqueness we must introduce some proper notion of supersolutions reflecting discontinuities and keep the notion of a subsolution in a standard way. Instead of using H^* we are tempted to define a supersolution (*D-supersolution*) by requiring

$$\tau + H(\hat{x}, p) \geqq 0 \quad \text{for all } (\hat{x}, \hat{t}) \in Q \text{ and } (p, \tau) \in D^- u(\hat{x}, \hat{t}), \tag{1.9}$$

where $D^- u$ denotes the subdifferential. However, u^α is a D-supersolution no matter how $\alpha \in (0, c]$ is taken. We are led to introduce a notion of *$\overline{D}$-supersolution* by replacing $D^- u$ in (1.9) with $\overline{D}^- u$, a kind of "closure" of $D^- u$. We shall establish a general comparison principle for $\overline{D}$-super- and subsolutions (Theorem 2.6). Applying this comparison principle we are able to establish a comparison principle for Lipschitz continuous $\overline{D}$-super- and subsolutions for (1.5) (Theorem 2.9). The reason we need Lipschitz continuity is that our general comparison principle needs continuity of H in x for large $|p|$ which excludes (1.5).

We next discuss the existence problem. Unfortunately, the intuitive solution u^c in (1.6) is not a $\overline{D}$-supersolution. We have to weaken the definition of supersolutions by regarding the infimum of a family of $\overline{D}$-supersolutions as a "supersolution". We call such a supersolution an *envelope supersolution*. By definition we have a comparison principle for envelope super- and subsolutions. In this way, we introduce a notion of an *envelope solution* (envelope super- and subsolution) and construct a global-in-time solution by approximating equations with continuous Hamiltonians (Proposition 3.7). It turns out that the envelope solution is a proper notion of the solution. Indeed, it is easy to see that u^c is a unique envelope solution of (1.5) with the zero initial data. Moreover, we show that our solution preserves the Lipschitz continuity and uniform continuity of the initial data if Hamiltonian is coercive. Thus the envelope solution is unique for Lipschitz continuous initial data. Moreover, by a suitable approximation argument one is able to conclude that the envelope solution we constructed is unique even for bounded uniformly continuous initial data (Theorem 3.19). The typical $H(x,p)$ we are concerned with is

$$H(x,p) = H_0(x,p) - r(x), \tag{1.10}$$

where H_0 is a continuous coercive Hamiltonian and r is a bounded lower semicontinuous function. No convexity (concavity) assumption on $p \mapsto H(x,p)$ is imposed, though our example (1.3) is concave in p. In this case, we prove that there exists a unique uniformly continuous envelope solution for all bounded uniformly continuous initial data (Theorem 3.20).

The name "an envelope solution" was also introduced in [2] and [1] in order to deal with boundary conditions. They considered equations with continuous Hamiltonians, and defined the notion of envelope supersolutions as the infimum of standard viscosity supersolutions. Except on the boundary their envelope supersolution is a standard viscosity supersolution since the infimum of supersolutions in a domain is known to be a supersolution. Different from their solutions, our envelope solutions for discontinuous Hamiltonians may not be a $\overline{D}$-supersolution. Moreover, it is not clear whether or not there is a way to characterize our envelope solutions by using a suitable class of test functions.

In the argument above we obtain the unique existence result for a Hamiltonian with the form (1.10) only when it is coercive. This is caused by a limitation of our comparison principle. In order to guarantee the continuity of $H(x,p)$ in x for large $|p|$, we define a *relaxed Hamiltonian* $\hat{H}$ by regularizing the discontinuity of H. It turns out in several interesting examples that our envelope solution of (1.1) is also an envelope solution of (1.1) with a relaxed Hamiltonian $\hat{H}$ which permits a general comparison principle without assuming the Lipschitz continuity of solutions. Then, by regarding our envelope solution of the original problem as that of the relaxed problem, we establish the uniqueness of the envelope solutions but only for more restrictive Hamiltonians (Proposition 4.5). Fortunately, this still applies to the problem with finitely many source terms. (See (1.11) with (1.12).) It turns out that the relaxed Hamiltonian corresponding to (1.3) is

$$\hat{H}(x,p) = -|p| - (cI(x) - |p|)_+,$$

which is continuous if $|p| \geqq c$. (This Hamiltonian is coercive but it is very instructive to calculate $\hat{H}$.) Also, u^c in (1.6) becomes an envelope solution of the relaxed problem.

If we consider the relaxed problem, there are more chances that an envelope solution is a $\overline{D}$-solution. We give its sufficient condition. Note that there still exists an envelope solution which is not a $\overline{D}$-solution.

Our theory applies to more physically interesting examples including

$$\partial_t u - |\nabla u| = r(x), \tag{1.11}$$

where $r : \mathbf{R}^n \to \mathbf{R}$ is bounded and upper semicontinuous, i.e., $r \in BUSC(\mathbf{R}^n)$. A typical example in our mind is

$$r(x) = \sum_{j=1}^{N} c_j I(x - a_j) \quad (c_j > 0,\ a_j \in \mathbf{R}^n,\ a_i \neq a_j (i \neq j)). \tag{1.12}$$

This is the case that the step source is distributed at several singletons. It turns out that the resulting unique envelope solution with zero initial data is

$$u(x,t) = \max_{j=1}^{N} c_j (t - |x - a_j|)_+, \tag{1.13}$$

which tells us that the envelope solution is the maximum of solutions for each step source. As an another example we have

$$r(x) = c\chi_S(x) \quad (c > 0 \text{ and } S \text{ is a nonempty closed subset of } \mathbf{R}^n). \tag{1.14}$$

Then (1.11) means that the step source is concentrated at a general set S. Here χ_S is the characteristic function of S, namely

$$\chi_S(x) := \begin{cases} 1 & (x \in S), \\ 0 & (x \notin S). \end{cases}$$

Our theory guarantees the unique existence of envelope solutions of (1.11) for general bounded uniformly continuous initial data. We are interested in establishing a representation formula of solutions based on the optimal control theory. However, the traditional method can be applied only for continuous equations. In this paper we adopt a discontinuous function appearing in our equation as a running cost function and prove that our envelope solution can be given via the value function of this discontinuous control problem under the some kind of controllability condition. Such an interpretation gives several explicit representation formulas of solutions. For example it guarantees that (1.13) is an envelope solution of (1.11) with (1.12) and $u_0 \equiv 0$.

Our theory applies to more general growth models including anisotropy. The typical form is

$$\partial_t u - |\nabla u| U\left(\frac{-\nabla u}{|\nabla u|}\right) \sigma(x) = r(x). \tag{1.15}$$

Here $U(\mathbf{n}) : S^{n-1} = \{x \in \mathbf{R}^n \mid |x| = 1\} \to \mathbf{R}$ is the growth rate in the direction $\mathbf{n} \in S^{n-1}$ and $-\nabla u/|\nabla u|$ means the outward unit normal vector to the level sets of u. The function $\sigma : \mathbf{R}^n \to \mathbf{R}$ is called the *surface supersaturation*.

Since $|p|U(-p/|p|) \to 0$ as $|p| \to 0$ provided that U is continuous, (1.15) has no singularity contrary to its seemingly singular appearance. The unique existence result for (1.15) is included in Theorem 3.20.

When the Hamiltonian is non-coercive ([37, 42]), the problem becomes more complicated. We cannot expect uniqueness results similar to the coercive cases. The difficulty may be seen from the following two examples. The first one is

$$H(x,p) = -cI(x) \quad (c > 0), \tag{1.16}$$

which means that there is no horizontal growth. Obviously $u(x,t) = ctI(x)$ seems to be the solution when the initial-value equals zero. However, the solution is not continuous and the uniqueness of solutions breaks down in our definition as will be mentioned in Example 3.16. The second one is

$$H^c(x,p) = -\frac{|p|}{1+|p|} - cI(x) \quad (c > 0). \tag{1.17}$$

This Hamiltonian arises in physical phenomena ([37]) where growth velocity is dependent on the gradient of the crystal surface. For $0 < c < 1$ we show in Theorem 4.8 that there exists a unique envelope solution for any bounded uniformly continuous initial data.

In [42] and [24] a step source is considered as a Dirichlet boundary condition. One may think that our envelope solution of (1.5) coincides with a solution of the Dirichlet boundary problem with $u(0,t) = ct + u_0(0)$ at the origin. This guess is correct provided that a slope of the initial data is less than or equal to c. However, if not, it turns out that the Dirichlet problem may give a different solution from our problem with (1.3). We also discuss a relation to the dynamic boundary condition $\partial_t u(0,t) = c$.

In the latter part of this chapter, we study the large time behavior of the unique envelope solution to the following problem:

$$(\mathrm{HJ1}) \begin{cases} \partial_t u(x,t) - H_1(\nabla u(x,t)) = r(x) & \text{in } \mathbf{R}^n \times (0,\infty), \\ (1.2). \end{cases} \tag{1.18}$$

Here H_1 is a continuous and coercive Hamiltonian and $r \in BUSC(\mathbf{R}^n)$ is non-negative. The large time behavior is discussed via the scaling method. Namely, we study the limit of the rescaled function of the solution u which is defined as

$$u_{(\lambda)}(x,t) := \frac{1}{\lambda} u(\lambda x, \lambda t)$$

for $\lambda > 0$. Our goal is to find a function v which is a solution to the associated problem of (HJ1) and characterizes the limit of $u_{(\lambda)}$ in the sense that

$$u_{(\lambda)}(x,t) \to v(x,t) \quad \text{as } \lambda \to \infty. \tag{1.19}$$

Once the convergence of the type (1.19) is obtained, putting $t = 1$ and $\lambda = t$, we get

$$\frac{1}{t} u(tx,t) \to v(x,1) \quad \text{as } t \to \infty, \tag{1.20}$$

which is the large time behavior of u. To study the scaling limit of u we consider the equation for $u_{(\lambda)}$. Differentiating the rescaled function, we see $u_{(\lambda)}$ satisfies the equation

$$\partial_t u_{(\lambda)} - H_1(\nabla u_{(\lambda)}) = r(\lambda x) \quad \text{in } \mathbf{R}^n \times (0, \infty) \tag{1.21}$$

and the initial data

$$u_{(\lambda)}(x, 0) = \frac{1}{\lambda} u_0(\lambda x).$$

Assuming that u_0 is bounded, the above initial data uniformly converges to zero. However, the source term in (1.21) may oscillate as $\lambda \to \infty$. In this paper we treat source terms r with special forms for which we are able to compute the scaling limit of solutions. The results include two types of source terms; source terms with compact support and periodic source terms.

We first suppose that the support of $r(x)$, which is

$$\operatorname{supp}(r) := \overline{\{x \in \mathbf{R}^n \mid r(x) \neq 0\}},$$

is compact in $\mathbf{R}^n$. In this case, since the graph of $r(\lambda x)$ concentrates into the origin and keeps its maximum value $c := \max_{\mathbf{R}^n} r$ as $\lambda \to \infty$, it is reasonable to expect that the scaling limit of the solution is characterized as the unique envelope solution of

$$\partial_t v - H_1(\nabla v) = cI(x) \quad \text{in } \mathbf{R}^n \times (0, \infty) \tag{1.22}$$

with the zero initial data. In fact, it will be shown in Theorem 6.5 that the convergence (1.19) holds locally uniformly in $\mathbf{R}^n \times [0, \infty)$ for the solution v. Under some restrictive assumptions we also give an example of the scaling limit of solutions when r has non-compact support (Example 6.7).

We are also interested in periodic source terms. It turns out that, if $r(x) = r(x + a)$ for all $a \in \mathbf{Z}^n$, then we have (1.19) in the sense of uniform convergence for $v(x, t) = ct$ with $c = \max_{\mathbf{R}^n} r$. This is a natural result since the source term is uniformly distributed in space as $\lambda \to \infty$. Our problem for periodic source terms is closely related to homogenization problems ([29, 18, 41]), and, when r is continuous, our result on the large time behavior is indeed consistent with the classical result in the homogenization theory (Remark 6.12).

The envelope solution (1.6) of (1.5) with the zero initial data is *self-similar* in the sense that u^c is scaling invariant, i.e., $u^c = (u^c)_{(\lambda)}$ for all $\lambda > 0$. More generally, as long as solutions are unique, the envelope solution v of (1.22) with the zero initial data is self-similar. Thus the result (1.19) means convergence to the self-similar solution of the limit problem. In other words, the solution u of the original problem is asymptotically self-similar although u itself is not necessarily self-similar. When a function $v(x, t)$ is self-similar, it is represented as $v(x, t) = tv(x/t, 1) = tV(x/t)$, where $V(x) := v(x, 1)$ is called a *profile function* of v. The profile function also appears in (1.20) as the limit of u, and so (1.20) asserts asymptotic convergence of u to the profile function. In Section 7 we will derive a stationary equation for the profile function and show that an envelope solution V of the stationary equation gives that of the time-dependent problem by letting $v(x, t) = tV(x/t)$. We also establish a comparison principle and an existence result of solutions to general stationary problems.

In this paper we mainly discuss the case when the given Hamiltonian $H(x,p)$ is lower semicontinuous with respect to x. We here recall some preceding studies about the viscosity solution theory for PDEs with discontinuous Hamiltonians. Shortly after the establishment of notions of viscosity solution, Ishii [26] studied discontinuous Hamiltonians with respect to the variables t and u. Discontinuities in the space variable x are investigated in many other works later.

For the stationary problem, the equation of eikonal type was studied by Newcomb and Su [32], Ostrov [34], Deckelnick and Elliott [13] and Soravia [40]. In [32] the authors considered the equation $H(\nabla u) = n(x)$. Here $H(p)$ is convex, coercive and positive except at $p = 0$ and n is assumed to be lower semicontinuous and positive. They introduced a suitable notion called Monge solutions, which, in the case of continuous Hamiltonians, are consistent with the usual viscosity solution. Briani and Davini [5] generalized the approach of Monge solutions for the equation $H(x, \nabla u) = 0$, where $H(x,p)$ is only assumed to be Borel measurable and quasi-convex in p. Although we did not check, we expect that our envelope solution should agree with the Monge solution when the latter is available. The work by Soravia [39] is related to our results concerning the optimal control theory. The author of [39] considered the equation

$$\lambda u(x) + \sup_{a\in A}\{-\langle f(x,a), \nabla u(x)\rangle - h(x,a)\} = g(x)$$

with a Borel measurable function g. Here $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product. The author established a general uniqueness result in the sense of lower semicontinuous solutions, which was introduced by Barron and Jensen [4]. However, the uniqueness result does not apply to our setting since the definition of solutions like lower semicontinuous solutions is not suitable for (1.4). The reason is that it is impossible to choose the intuitive solution u^c exclusively even if we impose an additional condition about test functions from the opposite side no matter which definition of solutions (standard, D- or $\overline{D}$-) we use.

For the time-dependent problem, Camilli and Siconolfi [9] considered the equation $\partial_t u + H(x, \nabla u) = 0$, where the Hamiltonian $H(x,p)$ is measurable in x and convex, coercive in p. The convexity is used to guarantee the Legendre transform and the equivalence of a.e. subsolution and viscosity subsolution. However, such measure theoretic approach does not give a suitable notion of solutions to our problems because the jump discontinuity such as $cI(x)$ is negligible with respect to the Lebesgue measure. Actually, it is reasonable to understand that the discontinuity depends on what kind of measure we consider as discussed at the end of this paper (Appendix C). See also [8] for an earlier work where a solution formula established for this type of equation under similar assumptions on H with positive 1-homogeneity on the function $p \mapsto H(x,p)$. For discontinuity of different types, there are a few works on the equations of the form

$$\partial_t u + f(x,t)h(x, \nabla u) = 0$$

with discontinuous f, which has important applications in front propagations. Deckelnick and Elliott [14] obtained the unique existence of continuous viscosity solutions for the one space dimensional case when $f(x,t) = a(x)$ and

$h(x,p) = \sqrt{1+p^2}$, where a is assumed to be bounded, of bounded variation and one-sided Lipschitz continuous. Afterwards Chen and Hu [10] studied a more general case when f depends on t but h depends only on p. They assumed that f is positive, bounded and measurable and h is non-negative and Lipschitz continuous. More recently, with the optimal control theory involved De Zan and Soravia [15] discussed the unique existence of solutions when h depends also on x while f is independent of t and piecewise Lipschitz continuous across Lipschitz hypersurfaces. Our results are therefore different from these above. The discontinuity of Hamiltonians we are concerned with is given as a source term instead of the jump of propagating speed, which is also studied recently in [23].

For the second order equation, Caffarelli, Crandall, Kocan and Święch [7] studied fully nonlinear and uniformly elliptic PDEs by utilizing L^p-viscosity solution theory. However, the situation is quite different from ours.

The large time behavior of solutions u to Hamilton-Jacobi equations is well studied in the sense that

$$u(x,t) \sim \lambda t + \phi(x) \quad \text{as } t \to \infty \tag{1.23}$$

when equations are continuous ([31, 20, 3, 35, 12]). However, the scaling limit of u in our sense gives a different limit from (1.23) both in the non-periodic case (Example 6.6) and in the periodic case (Example 6.13).

For the scaling limit of solutions to second order equations, the reader is referred to the book [22]. There, via the scaling method, self-similar solutions are analyzed for various equations including the heat equation, Navier-Stokes equation, porous medium equation and so on. The papers [16, 27] are concerned with the curvature flow equations for graphs. Under a suitable growth condition, the authors of these papers study the asymptotic behavior of solutions to the curvature flow equations over the whole space. Recently, the author of the present paper studied the asymptotic behavior of solutions to fully nonlinear parabolic equations including a generalized curvature flow equation which appears in a model of evaporation-condensation; see Chapter 2 in this paper. It was shown that the scaling limit of solutions to the problem is characterized as a self-similar solution to the associated problem, which is, in the evaporation-condensation model, the usual mean curvature flow equation for graphs.

This paper is organized as follows. In Section 2 we first define some notions of solutions. Then we establish two types of comparison principles for $\overline{D}$-solutions; a general version which excludes (1.5) and a Lipschitz version which includes (1.5) but needs Lipschitz continuity of solutions. Section 3 is devoted to existence problems of solutions. We prove that there exists a unique envelope solution of (HJ) when H is coercive. Section 4 deals with relaxed Hamiltonians. After introducing the relaxed Hamiltonians, we deduce a unique existence result of envelope solutions without the coercivity assumption. Also, we discuss the existence of $\overline{D}$-solutions. Section 5 is dedicated to showing some examples of envelope solutions. We also mention the relation between our envelope solutions and solutions of Dirichlet boundary problems. In Section 6 we study the large time behavior of envelope solutions to (HJ1). We prove that, if the support of the source term r is compact, then the solution converges to the self-similar solution of (1.22) under rescaling. Periodic source terms are also investigated. Section 7 is devoted to stationary problems. We first study equations for profile functions of the self-similar solutions. We next establish a comparison principle and an existence result of solutions to general stationary problems.

2 Proper definition of solutions and comparison principles

2.1 Definition of solutions

We first recall the notion of super- and subdifferentials to define a viscosity solution. For $u : Q \to \mathbf{R}$ and $(\hat{x}, \hat{t}) \in Q$ we set a *superdifferential* $D^+_Q u(\hat{x}, \hat{t})$ and an *extended superdifferential* $\overline{D}^+_Q u(\hat{x}, \hat{t})$ by

$$D^+_Q u(\hat{x}, \hat{t}) := \left\{ (p, \tau) \in \mathbf{R}^n \times \mathbf{R} \;\middle|\; \begin{array}{c} \exists \phi \in C^1(Q) \text{ such that } (p, \tau) = (\nabla\phi, \partial_t \phi)(\hat{x}, \hat{t}) \\ \text{and } \max_Q (u - \phi) = (u - \phi)(\hat{x}, \hat{t}) \end{array} \right\}, \tag{2.1}$$

$$\overline{D}^+_Q u(\hat{x}, \hat{t}) := \left\{ (p, \tau) \in \mathbf{R}^n \times \mathbf{R} \;\middle|\; \begin{array}{c} \exists \{(x_m, t_m)\}_{m \in \mathbf{N}} \subset Q,\ \exists \{(p_m, \tau_m)\}_{m \in \mathbf{N}} \subset \mathbf{R}^n \times \mathbf{R}, \\ \text{such that } (p_m, \tau_m) \in D^+_Q u(x_m, t_m),\ (x_m, t_m) \to (\hat{x}, \hat{t}), \\ (p_m, \tau_m) \to (p, \tau) \text{ and } u(x_m, t_m) \to u(\hat{x}, \hat{t}) \text{ as } m \to \infty \end{array} \right\}, \tag{2.2}$$

where $\mathbf{N} := \{1, 2, 3, \dots\}$. We denote a *subdifferential* $D^-_Q u(\hat{x}, \hat{t})$ and an *extended subdifferential* $\overline{D}^-_Q u(\hat{x}, \hat{t})$ by

$$D^-_Q u(\hat{x}, \hat{t}) := -D^+_Q(-u)(\hat{x}, \hat{t}) \quad \text{and} \quad \overline{D}^-_Q u(\hat{x}, \hat{t}) := -\overline{D}^+_Q(-u)(\hat{x}, \hat{t}).$$

We can also define D^-_Q and $\overline{D}^-_Q$ by replacing, respectively, max by min in (2.1) and D^+_Q by D^-_Q in (2.2). Index Q is often omitted. It is known that D^+ and D^- are closed convex subset of $\mathbf{R}^n \times \mathbf{R}$. (See [2, Lemma II.1.8.(a)].)

We call $\phi \in C^1(Q)$ a *corresponding test function* for $(p, \tau) \in D^+ u(\hat{x}, \hat{t})$, where ϕ appears in (2.1). One can take such ϕ as a separated form, i.e., $\phi(x, t) = \psi(x) + g(t)$ with $\psi \in C^1(\mathbf{R}^n)$ and $g \in C^1(0, T)$. (See [21, Proposition 2.2.3.(i)].) Moreover we call $\{(x_m, t_m), (p_m, \tau_m)\}_{m \in \mathbf{N}} \subset Q \times D^+ u(x_m, t_m)$ a *defining approximate sequence* for $(p, \tau) \in \overline{D}^+ u(\hat{x}, \hat{t})$, where (x_m, t_m) and (p_m, τ_m) are given in (2.2).

Definition 2.1. Let $H = H(x, p)$ be a real valued function defined in $\mathbf{R}^n \times \mathbf{R}^n$ and let u be a real valued function in Q.

(1) We call u a *(standard) viscosity supersolution* (resp. *subsolution*) of (1.1) if u is bounded from below (resp. from above) in Q and

$$\tau + H^*(\hat{x}, p) \geqq 0 \quad \text{for all } (\hat{x}, \hat{t}) \in Q \text{ and } (p, \tau) \in D^-_Q u_*(\hat{x}, \hat{t}). \tag{2.3}$$

$$(\text{resp. } \tau + H_*(\hat{x}, p) \leqq 0 \quad \text{for all } (\hat{x}, \hat{t}) \in Q \text{ and } (p, \tau) \in D^+_Q u^*(\hat{x}, \hat{t}).)$$

We denote by $SUP(H)$ and $SUB(H)$ respectively the set of all supersolutions and subsolutions of (1.1).

(2) If $u \in SUP(H)$ (resp. $u \in SUB(H)$) defined in $Q_0 := \mathbf{R}^n \times [0, T)$ is continuous on $\mathbf{R}^n \times \{0\}$ and satisfies the initial condition (1.2), it is called a viscosity supersolution (resp. subsolution) of (HJ) and then we write $u \in SUP(H, u_0)$ (resp. $u \in SUB(H, u_0)$).

(3) We say that u is a *viscosity solution* if it is both a viscosity supersolution and a viscosity subsolution. Define $SOL(H) := SUP(H) \cap SUB(H)$ and $SOL(H, u_0) := SUP(H, u_0) \cap SUB(H, u_0)$.

Remark 2.2. (1) For any subset $L \subset \mathbf{R}^N$ and $h : L \to \mathbf{R}$ we denote the *upper semicontinuous envelope* (resp. *lower semicontinuous envelope*) by h^* (resp. h_*) $: \overline{L} \to \mathbf{R} \cup \{\pm\infty\}$, which is as follows:

$$h^*(z) := \limsup_{y \to z} h(y) = \lim_{\delta \downarrow 0} \sup\{h(y) \mid y \in \overline{B}_\delta(z) \cap L\}$$

$$(\text{resp. } h_*(z) := \liminf_{y \to z} h(y) = \lim_{\delta \downarrow 0} \inf\{h(y) \mid y \in \overline{B}_\delta(z) \cap L\}) \quad (z \in \overline{L}),$$

where $\overline{B}_r(x)$ stands for the closed ball with center x and radius r. (We denote the open ball by $B_r(x)$.) The function h^* is characterized as the smallest upper semicontiuous function on $\overline{L}$ that is greater than h on L, while h_* is the greatest lower semicontiuous function on $\overline{L}$ that is smaller than h on L.

(2) We can replace D_Q^- in (2.3) by $\overline{D}_Q^-$ since H^* is applied in the definition. This can be easily shown by taking limits.

(3) It often assumes one side local boundedness of sub- and supersolutions instead of global boundedness in the literature. We impose the boundedness assumption to simplify the argument. Also, when we think of the initial value problem (HJ), we require solutions to be continuous at $t = 0$ for the sake of simplicity.

Example 2.3. Let us consider (HJ) with (1.3) and $u_0 \equiv 0$. It is easy to verify that functions $u^\alpha(x, t) := \alpha(t - |x|)_+$ $(0 \leqq \alpha \leqq c)$ are all viscosity solutions of this initial-value problem in the standard sense above. We must therefore strengthen the definition of solutions in order to get uniqueness. As mentioned in Section 1, we adopt a new definition in which H instead of H^* is used in (2.3). However, notice that in this case the definition by D_Q^- and that by $\overline{D}_Q^-$ are different.

Definition 2.4. Let $u : Q \to \mathbf{R}$. We call u a *D-viscosity supersolution* (resp. *$\overline{D}$-viscosity supersolution*) of (1.1) if u is bounded from below in Q and

$$\tau + H(\hat{x}, p) \geqq 0 \quad \text{for all } (\hat{x}, \hat{t}) \in Q \text{ and } (p, \tau) \in D_Q^- u_*(\hat{x}, \hat{t}).$$

$$(\text{resp. } \tau + H(\hat{x}, p) \geqq 0 \quad \text{for all } (\hat{x}, \hat{t}) \in Q \text{ and } (p, \tau) \in \overline{D}_Q^- u_*(\hat{x}, \hat{t}).)$$

We denote by D-$SUP(H)$ (resp. $\overline{D}$-$SUP(H)$) a set which consists of all D-supersolutions (resp. $\overline{D}$-supersolutions) of (1.1). Moreover we similarly define a corresponding viscosity subsolution, solution, solution of the initial-value problem, and set notations by marking D- or $\overline{D}$-.

About three notions of supersolutions defined so far we have the inclusion relation $SUP(H) \supset D\text{-}SUP(H) \supset \overline{D}\text{-}SUP(H)$ in general, and these sets are the same for upper semicontinuous H. Since we mainly think of a lower continuous H, as for subsolutions we have $SUB(H) = D\text{-}SUB(H) = \overline{D}\text{-}SUB(H)$ in many cases.

Example 2.5. We revisit the Example 2.3. How about solutions of our equation in the sense of D- or $\overline{D}$-solutions? Since there is no smooth function that touches u^α from below at $(0, \hat{t})$ $(\hat{t} > 0)$ when $\alpha \in (0, c]$, they all become D-solutions. (When $\alpha = 0$, the function $u \equiv 0$ is not a D-supersolution.) This suggests that D-solutions are still not unique. For this reason we adopt $\overline{D}$-solutions as a proper definition for the moment, and we will show comparison principles for such solutions in the next subsection.

2.2 Comparison principles

We will show comparison principles (CP for short), which are important to prove uniqueness of solutions. The following two assumptions are standard for usual CP.

- (H_p) There exists a modulus $\omega_1 \in \mathcal{M}$ such that $|H(x,p) - H(x,q)| \leqq \omega_1(|p-q|)$ for all $x, p, q \in \mathbf{R}^n$.
- (H_x) There exists a modulus $\omega_2 \in \mathcal{M}$ such that $|H(x,p) - H(y,p)| \leqq \omega_2((1 + |p|)|x-y|)$ for all $x, y, p \in \mathbf{R}^n$.

Here we denote by $\mathcal{M}$ the set of all moduli of continuity, namely

$$\mathcal{M} := \left\{ \omega : [0,\infty) \to [0,\infty) \;\middle|\; \begin{array}{c} \omega(0) = 0,\ \omega \text{ is continuous at } 0 \\ \text{and nondecreasing on } [0,\infty). \end{array} \right\}.$$

We still use (H_p) now. Since we should treat discontinuous Hamiltonians with respect to the space variable, we weaken (H_x) in the following manner.

- (H_{xN}) There exist a modulus $\omega_2 \in \mathcal{M}$ and a constant $N > 0$ such that $|H(x,p) - H(y,p)| \leqq \omega_2((1+|p|)|x-y|)$ for all $x, y \in \mathbf{R}^n$ and $p \in \mathbf{R}^n \setminus B_N(0)$.

This condition means that (H_x) holds if $|p|$ is large. Note that (1.3) does not satisfy (H_{xN}).

Before stating our CP, we check that H satisfying (H_p) and (H_{xN}) is locally bounded in $\mathbf{R}^n \times \mathbf{R}^n$. This fact will be used in the proof of CP. Since the local boundedness is clear in $\mathbf{R}^n \times (\mathbf{R}^n \setminus \overline{B}_N(0))$, we show that H is bounded in $\overline{B}_1(x) \times \overline{B}_1(p)$ for any $(x,p) \in \mathbf{R}^n \times \overline{B}_N(0)$. Take any $(y,q) \in \overline{B}_1(x) \times \overline{B}_1(p)$ and $p' \in \mathbf{R}^n$ such that $N \leqq |p'| \leqq N+1$. Then, we calculate

$$\begin{aligned}
&|H(x,p) - H(y,q)| \\
\leqq &|H(x,p) - H(x,p')| + |H(x,p') - H(y,p')| + |H(y,p') - H(y,q)| \\
\leqq &2\omega_1(|p-p'|+1) + \omega_2((1+|p'|)|x-y|) \\
\leqq &2\omega_1(2N+2) + \omega_2(N+2),
\end{aligned}$$

which yields our claim.

Theorem 2.6 (CP–general version)**.** *Assume that H satisfies* (H_p) *and* (H_{xN}). *Let u and $v : Q_0 \to \mathbf{R}$ be, respectively, bounded from above and bounded from below in Q_0. Assume that $u \in \overline{D}$-$SUB(H)$ and $v \in \overline{D}$-$SUP(H)$. If $u^*(\cdot, 0) \leqq v_*(\cdot, 0)$ in $\mathbf{R}^n$, then $u^* \leqq v_*$ in Q.*

Though our assumption for H is weaker than the classical one, our definition of solutions is stronger, and so we can keep balance.

Proof. 1. Suppose by contradiction that there would exist $(x_0, t_0) \in \mathbf{R}^n \times (0, T)$ such that $u^*(x_0, t_0) - v_*(x_0, t_0) =: A > 0$. We define a function $\mathcal{F} : (\mathbf{R}^n \times [0, T])^2 \to \mathbf{R} \cup \{-\infty\}$ by

$$\begin{aligned} \mathcal{F}(x, t, y, s) &:= u^*(x, t) - v_*(y, s) - \Psi(x, t, y, s) \\ \text{with } \Psi(x, t, y, s) &:= \frac{1}{2\varepsilon^2}\left(|x - y|^2 + |t - s|^2\right) + \beta f(x) + \frac{\alpha}{T - t}, \end{aligned}$$

where $\alpha \in (0, A(T - t_0))$, $\beta > 0$, $\varepsilon > 0$ and $f(x) = \sqrt{1 + |x - x_0|^2} - 1$. Note that $f \geqq 0$, $f \in C^1(\mathbf{R}^n)$ and $|\nabla f| \leqq 1$. Also, by the choice of α, we have

$$\mathcal{F}(x_0, t_0, x_0, t_0) = u^*(x_0, t_0) - v_*(x_0, t_0) - \frac{\alpha}{T - t_0} > 0.$$

Since u and $-v$ are bounded from above, $\mathcal{F}$ attains its maximum in $(\mathbf{R}^n \times [0, T])^2$ at some $(x^\varepsilon, t^\varepsilon, y^\varepsilon, s^\varepsilon) \in (\mathbf{R}^n \times [0, T))^2$. Then, we see

$$\mathcal{F}(x^\varepsilon, t^\varepsilon, y^\varepsilon, s^\varepsilon) \geqq \mathcal{F}(x_0, t_0, x_0, t_0) > 0. \tag{2.4}$$

2. Set $M := \sup_{Q_0} u^* + \sup_{Q_0}(-v_*)$ $(< \infty)$. Then we have $\beta f(x^\varepsilon) < M$ by (2.4), and hence $\{x^\varepsilon\}_{\varepsilon > 0}$ is bounded. Furthermore, since we also have $|x^\varepsilon - y^\varepsilon| \leqq \sqrt{2M}\varepsilon$ and $|t^\varepsilon - s^\varepsilon| \leqq \sqrt{2M}\varepsilon$ by (2.4), we may assume that there exists some $(\hat{x}, \hat{t}) \in \mathbf{R}^n \times [0, T]$ such that

$$\lim_{\varepsilon \downarrow 0}(x^\varepsilon, t^\varepsilon, y^\varepsilon, s^\varepsilon) = (\hat{x}, \hat{t}, \hat{x}, \hat{t}). \tag{2.5}$$

Here, we claim that $\hat{t} \in (0, T)$. By (2.4) we observe

$$0 < \limsup_{\varepsilon \downarrow 0} \mathcal{F}(x^\varepsilon, t^\varepsilon, y^\varepsilon, s^\varepsilon) \leqq \mathcal{F}(\hat{x}, \hat{t}, \hat{x}, \hat{t}).$$

However, since we have

$$\begin{aligned} \mathcal{F}(\hat{x}, 0, \hat{x}, 0) &= u^*(\hat{x}, 0) - v_*(\hat{x}, 0) - \beta f(\hat{x}) - \frac{\alpha}{T} < 0, \\ \mathcal{F}(\hat{x}, T, \hat{x}, T) &= -\infty, \end{aligned}$$

it follows that $\hat{t} \neq 0$ and $\hat{t} \neq T$.

3. We remark that

$$\mathcal{F}(\hat{x}, \hat{t}, \hat{x}, \hat{t}) \leqq \mathcal{F}(x^\varepsilon, t^\varepsilon, y^\varepsilon, s^\varepsilon) \tag{2.6}$$

because $(x^\varepsilon, t^\varepsilon, y^\varepsilon, s^\varepsilon)$ is the maximizer of $\mathcal{F}$. In view of (2.6) we calculate

$$
\begin{aligned}
&\limsup_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon^2}\left(|x^\varepsilon - y^\varepsilon|^2 + |t^\varepsilon - s^\varepsilon|^2\right)\\
\leqq &\limsup_{\varepsilon\downarrow 0}\left\{-\mathcal{F}(\hat{x},\hat{t},\hat{x},\hat{t}) + u^*(x^\varepsilon,t^\varepsilon) - v_*(y^\varepsilon,s^\varepsilon) - \beta f(x^\varepsilon) - \frac{\alpha}{T-t^\varepsilon}\right\}\\
\leqq &-\mathcal{F}(\hat{x},\hat{t},\hat{x},\hat{t}) + u^*(\hat{x},\hat{t}) - v_*(\hat{x},\hat{t}) - \beta f(\hat{x}) - \frac{\alpha}{T-\hat{t}} = 0.
\end{aligned}
$$

Hence

$$
\lim_{\varepsilon\downarrow 0}\frac{|x^\varepsilon - y^\varepsilon|}{\varepsilon} = 0, \quad \lim_{\varepsilon\downarrow 0}\frac{|t^\varepsilon - s^\varepsilon|}{\varepsilon} = 0. \tag{2.7}
$$

Also, by (2.6) and the upper semicontinuity of $\mathcal{F}$, we observe

$$
\mathcal{F}(\hat{x},\hat{t},\hat{x},\hat{t}) \leqq \liminf_{\varepsilon\downarrow 0}\mathcal{F}(x^\varepsilon,t^\varepsilon,y^\varepsilon,s^\varepsilon) \leqq \limsup_{\varepsilon\downarrow 0}\mathcal{F}(x^\varepsilon,t^\varepsilon,y^\varepsilon,s^\varepsilon) \leqq \mathcal{F}(\hat{x},\hat{t},\hat{x},\hat{t}),
$$

which means

$$
\lim_{\varepsilon\downarrow 0}\mathcal{F}(x^\varepsilon,t^\varepsilon,y^\varepsilon,s^\varepsilon) = \mathcal{F}(\hat{x},\hat{t},\hat{x},\hat{t}).
$$

This equality and (2.7) implies

$$
\lim_{\varepsilon\downarrow 0}\{u^*(x^\varepsilon,t^\varepsilon) - v_*(y^\varepsilon,s^\varepsilon)\} = u^*(\hat{x},\hat{t}) - v_*(\hat{x},\hat{t}).
$$

Now, we also observe

$$
\begin{aligned}
u^*(\hat{x},\hat{t}) &\geqq \limsup_{\varepsilon\downarrow 0} u^*(x^\varepsilon,t^\varepsilon) \geqq \liminf_{\varepsilon\downarrow 0} u^*(x^\varepsilon,t^\varepsilon)\\
&= \liminf_{\varepsilon\downarrow 0}\{(u^*(x^\varepsilon,t^\varepsilon) - v_*(y^\varepsilon,s^\varepsilon)) + v_*(y^\varepsilon,s^\varepsilon)\}\\
&\geqq (u^*(\hat{x},\hat{t}) - v_*(\hat{x},\hat{t})) + v_*(\hat{x},\hat{t}) = u^*(\hat{x},\hat{t}).
\end{aligned}
$$

Consequently it follows that

$$
\lim_{\varepsilon\downarrow 0} u^*(x^\varepsilon,t^\varepsilon) = u^*(\hat{x},\hat{t}), \quad \lim_{\varepsilon\downarrow 0} v_*(y^\varepsilon,s^\varepsilon) = v_*(\hat{x},\hat{t}). \tag{2.8}
$$

4. We next calculate the first derivatives of Ψ at $(x^\varepsilon, t^\varepsilon, y^\varepsilon, s^\varepsilon)$.

$$
\begin{aligned}
p_x^\varepsilon &:= \nabla_x\Psi(x^\varepsilon,t^\varepsilon,y^\varepsilon,s^\varepsilon) = \frac{1}{\varepsilon^2}(x^\varepsilon - y^\varepsilon) + \beta\nabla f(x^\varepsilon),\\
p_y^\varepsilon &:= \nabla_y\Psi(x^\varepsilon,t^\varepsilon,y^\varepsilon,s^\varepsilon) = -\frac{1}{\varepsilon^2}(x^\varepsilon - y^\varepsilon),\\
\tau^\varepsilon &:= \partial_t\Psi(x^\varepsilon,t^\varepsilon,y^\varepsilon,s^\varepsilon) = \frac{1}{\varepsilon^2}(t^\varepsilon - s^\varepsilon) + \frac{\alpha}{(T-t^\varepsilon)^2},\\
\sigma^\varepsilon &:= \partial_s\Psi(x^\varepsilon,t^\varepsilon,y^\varepsilon,s^\varepsilon) = -\frac{1}{\varepsilon^2}(t^\varepsilon - s^\varepsilon).
\end{aligned}
$$

By the definitions of $D^\pm$ we have

$$
\begin{cases}
(p_x^\varepsilon, \tau^\varepsilon) \in D^+u^*(x^\varepsilon,t^\varepsilon),\\
(-p_y^\varepsilon, -\sigma^\varepsilon) \in D^-v_*(y^\varepsilon,s^\varepsilon),
\end{cases} \tag{2.9}
$$

and therefore

$$\begin{cases} \tau^\varepsilon + H(x^\varepsilon, p_x^\varepsilon) \leqq 0, \\ -\sigma^\varepsilon + H(y^\varepsilon, -p_y^\varepsilon) \geqq 0 \end{cases} \tag{2.10}$$

since $u \in D\text{-}SUB(H)$ and $v \in D\text{-}SUP(H)$.

Here we discuss two different cases for subsequences of $\{p_y^\varepsilon\}_{\varepsilon>0}$:

There exists a sequence $\{\varepsilon(j)\}_{j\in\mathbf{N}}$ such that $\varepsilon(j) \downarrow 0$ $(j \to \infty)$ and

Case 1: $|p_y^{\varepsilon(j)}| \to \infty$.

Case 2: $p_y^{\varepsilon(j)} \to -\bar{p}$ for some $\bar{p} \in \mathbf{R}^n$.

We will reach to contradiction for both cases. From now on we simply write ε for $\varepsilon(j)$.

Case 1. In terms of (H_{xN}) it is enough to apply the classical method. Combining two inequalities in (2.10), we have

$$\frac{\alpha}{(T-t^\varepsilon)^2} \leqq \{H(y^\varepsilon, -p_y^\varepsilon) - H(x^\varepsilon, -p_y^\varepsilon)\} + \{H(x^\varepsilon, -p_y^\varepsilon) - H(x^\varepsilon, p_x^\varepsilon)\}.$$

Letting ε small and applying (H_{xN}) and (H_p), we compute

$$\begin{aligned} \frac{\alpha}{(T-t^\varepsilon)^2} &\leqq \omega_2((1+|p_y^\varepsilon|)|x^\varepsilon - y^\varepsilon|) + \omega_1(|p_x^\varepsilon + p_y^\varepsilon|) \\ &= \omega_2\left(|x^\varepsilon - y^\varepsilon| + \frac{1}{\varepsilon^2}|x^\varepsilon - y^\varepsilon|^2\right) + \omega_1(|\beta\nabla f(x^\varepsilon)|). \end{aligned}$$

Sending $\varepsilon \downarrow 0$ in the above and using $1/T^2 \leqq 1/(T-\hat{t})^2$, we obtain $\alpha/T^2 \leqq \omega_1(\beta|\nabla f(\hat{x})|) \leqq \omega_1(\beta)$. This is a contradiction for very small β.

Case 2. By (2.10) we see

$$\frac{\alpha}{(T-t^\varepsilon)^2} + H(x^\varepsilon, p_x^\varepsilon) \leqq \sigma^\varepsilon \leqq H(y^\varepsilon, -p_y^\varepsilon).$$

Thus we may assume that σ^ε converges to some $-\bar{\tau}$ as $\varepsilon \downarrow 0$ by the local boundedness of H. Now, since (2.5), (2.8) and

$$\lim_{\varepsilon\downarrow 0}(p_x^\varepsilon, \tau^\varepsilon, p_y^\varepsilon, \sigma^\varepsilon) = \left(\bar{p} + \beta\nabla f(\hat{x}), -\bar{p}, \bar{\tau} + \frac{\alpha}{(T-\hat{t})^2}, -\bar{\tau}\right)$$

hold, the definitions of $\overline{D}^{\pm}$ and (2.10) yield

$$\begin{cases} \left(\bar{p} + \beta\nabla f(\hat{x}), \bar{\tau} + \dfrac{\alpha}{(T-\hat{t})^2}\right) \in \overline{D}^+ u^*(\hat{x}, \hat{t}), \\ (\bar{p}, \bar{\tau}) \in \overline{D}^- v_*(\hat{y}, \hat{s}). \end{cases}$$

Therefore

$$\begin{cases} \bar{\tau} + \dfrac{\alpha}{(T-\hat{t})^2} + H(\hat{x}, \bar{p} + \beta\nabla f(\hat{x})) \leqq 0, \\ \bar{\tau} + H(\hat{x}, \bar{p}) \geqq 0 \end{cases}$$

since $u \in \overline{D}\text{-}SUB(H)$ and $v \in \overline{D}\text{-}SUP(H)$. Consequently

$$\frac{\alpha}{T^2} \leqq H(\hat{x}, \bar{p}) - H(\hat{x}, \bar{p} + \beta \nabla f(\hat{x})) \leqq \omega_1(\beta |\nabla f(\hat{x})|) \leqq \omega_1(\beta),$$

which is a contradiction for very small β. □

Remark 2.7. (1) In general, whenever CP holds, we have $\|u_1 - u_2\|_Q \leqq \|(u_1|_{t=0}) - (u_2|_{t=0})\|_{\mathbf{R}^n}$ for any two solutions u_1 and u_2 of (1.1), no matter which definition of solutions we use. This is continuous dependence of the solutions on the initial data. Here we write $\|f\|_U := \sup_U |f|$ for $f : U \to \mathbf{R}$.

(2) The term $\beta f(x)$ in the definition of Ψ is added in order that $\mathcal{F}$ attains the maximum in $(\mathbf{R}^n \times [0, T])^2$. If both u and v are periodic in $\mathbf{R}^n$, namely $u(x, t) = u(x + \sum_{i=1}^n e_i, t)$ and $v(x, t) = v(x + \sum_{i=1}^n e_i, t)$ for some linearly independent $e_1, \ldots, e_n \in \mathbf{R}^n$, the function $\mathcal{F}$ attains the maximum without $\beta f(x)$, and then we have $p_x^\varepsilon = -p_y^\varepsilon$. Therefore it is unnecessary to assume (H_p) in this periodic case.

Corollary 2.8 (uniqueness of $\overline{D}$-solutions)**.** *Assume that H satisfies* (H_p) *and* (H_{xN}). *Then there exists at most one $\overline{D}$-solution of* (HJ) *and it is continuous.*

Proof. Let $u, v \in \overline{D}\text{-}SOL(H, u_0)$. Applying Theorem 2.6 to a subsolution u and a supersolution v, we get $u^* \leqq v_*$ in Q. Next changing roles of u and v, we also see $v^* \leqq u_*$ in Q. Hence it follows that $u^* \leqq v_* \leqq v^* \leqq u_*$ in Q, which yields our claim. □

The assumption (H_{xN}) was used only in Case1 in the proof of Theorem 2.6 for the situation that elements in D^+u^* and D^-v_* are unbounded. For any Lipschitz continuous function w in Q, we have $|p| \leqq \mathrm{Lip}[w]$ and $\tau \leqq \mathrm{Lip}[w]$ whenever $(p, \tau) \in D^+w(\hat{x}, \hat{t})$ or $(p, \tau) \in D^-w(\hat{x}, \hat{t})$, where $\mathrm{Lip}[w]$ stands for the Lipschitz constant of w. Therefore it is unnecessary to assume (H_{xN}) in order to prove CP when one of solutions is Lipschitz continuous.

Theorem 2.9 (CP–Lipschitz version)**.** *Assume that H satisfies* (H_p). *Let u and $v : Q_0 \to \mathbf{R}$ be, respectively, bounded from above and bounded from below in Q_0. Assume that $u \in \overline{D}\text{-}SUB(H)$ and $v \in \overline{D}\text{-}SUP(H)$. Furthermore assume that either u or v is (space-time) Lipschitz continuous in Q. If $u^*(\cdot, 0) \leqq v_*(\cdot, 0)$ in $\mathbf{R}^n$, then $u^* \leqq v_*$ in Q.*

As mentioned in Remark 2.7 (2), the assumption (H_p) is unnecessary for the periodic case. It is not difficult to find that this version of CP applies to (1.3).

3 Existence results

3.1 Unique existence of envelope solutions

We adopted $\overline{D}$-solutions as a proper definition in Section 2.1 in order to guarantee the uniqueness of solutions of (HJ) with (1.3) and $u_0 \equiv 0$, but the existence turns out to be an issue for a discontinuous Hamiltonian. We give two examples to show the non-existence of $\overline{D}$-solutions.

Example 3.1. Let us consider (HJ) with (1.16) and $u_0 \equiv 0$. Then $u \equiv 0$ is a subsolution but is not a $\overline{D}$-supersolution. Also, one observes that $u^\varepsilon(x,t) = ctI^\varepsilon(x)$ with (1.8) is a $\overline{D}$-supersolution but is not a subsolution for each $\varepsilon > 0$. Therefore, if there would exist a $\overline{D}$-solution v, then $0 \leqq v_* \leqq v^* \leqq u^\varepsilon$ in Q by Theorem 2.9. Sending $\varepsilon \downarrow 0$, we see $0 \leqq v_* \leqq v^* \leqq ctI(x)$. Hence $v_* \equiv 0$ in Q, which contradicts the fact that 0 is not a $\overline{D}$-supersolution. (See Figure 3.)

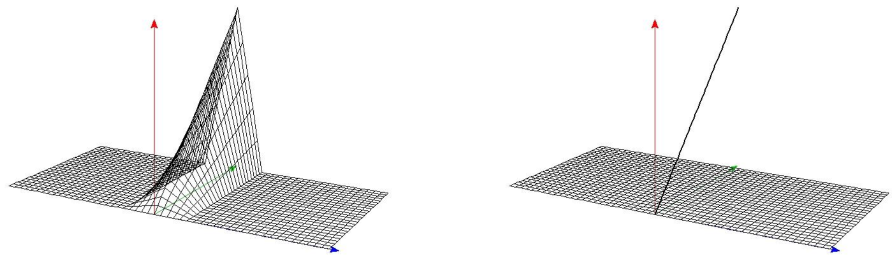

Figure 3: $\overline{D}$-supersolutions $tI^\varepsilon(x)$ of $\partial_t u = I(x)$, $u_0 \equiv 0$ (the left) and their limit $tI(x)$ (the right). The latter is an envelope solution.

Example 3.2. Let us consider (HJ) with (1.3) and $u_0 \equiv 0$. The intuitive solution $u(x,t) = c(t-|x|)_+$ is a subsolution but is not a $\overline{D}$-supersolution because $(p,c) \in \overline{D}^- u(0,1)$ ($|p| = c$) and $c - |p| < cI(0)$. Now we think of approximate problems

$$(\varepsilon.\mathrm{HJ}) \begin{cases} \partial_t u + H^\varepsilon(x, \nabla u) = 0 & \text{in } \ Q, \\ u|_{t=0} = u_0 & \text{in } \ \mathbf{R}^n, \end{cases}$$

where H^ε is given by (1.7). Since we can write $H^\varepsilon(x,p) = -\max_{a \in \overline{B}_1(0)} \langle a, p\rangle - cI^\varepsilon(x)$, the representation formula by the optimal control theory (see Section 5.1 for more details) implies that u^ε given by

$$u^\varepsilon(x,t) = \sup_{\alpha \in \mathcal{A}} \int_0^t cI^\varepsilon(X^\alpha(s))ds$$

is a unique viscosity solution of (ε.HJ). Here $\mathcal{A} = \{\alpha : [0,T] \to \overline{B}_1(0), \text{measurable}\}$ and $X^\alpha(s)$ is the solution of the state equation: $X'(s) = \alpha(s)$ in $(0,t)$, $X(0) = x$. In other words, $X^\alpha(s)$ describes a trajectory which leaves at time 0 from x and moves at velocity 1 or less. In this case for each $x \in \mathbf{R}^n$ the optimal control is the one that leads to a straight trajectory before it comes to the origin and stays there after that moment. A direct calculation yields

$$u^\varepsilon(x,t)/c = \begin{cases} \left(1 - \dfrac{|x|}{\varepsilon}\right) t + \dfrac{t^2}{2\varepsilon} & (t \leqq |x|), \\ t - \dfrac{|x|^2}{2\varepsilon} & (t \geqq |x|), \end{cases} \qquad \text{for} \quad |x| \leqq \varepsilon,$$

$$\text{and} \quad u^\varepsilon(x,t)/c = \begin{cases} 0 & (t \leqq |x| - \varepsilon), \\ \dfrac{(t - |x| + \varepsilon)^2}{2\varepsilon} & (|x| - \varepsilon \leqq t \leqq |x|), \\ t - |x| + \dfrac{\varepsilon}{2} & (t \geqq |x|), \end{cases} \qquad \text{for} \quad |x| \geqq \varepsilon.$$

The inequality $H \geqq H^\varepsilon$ implies that each u^ε is a $\overline{D}$-supersolution of the original (HJ). However, since $u^\varepsilon \downarrow u$ as $\varepsilon \downarrow 0$, it is shown by the similar argument in the previous example that there is no $\overline{D}$-solution of (HJ). (See Figure 4.)

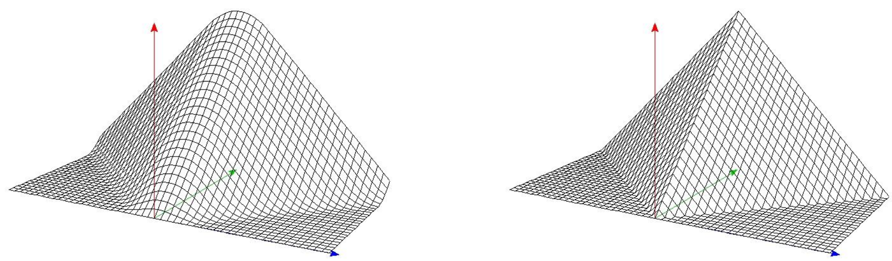

Figure 4: $\overline{D}$-supersolutions u^ε of $\partial_t u - |\nabla u| = I(x)$, $u_0 \equiv 0$ (the left) and their limit $(t - |x|)_+$ (the right). The latter is an envelope solution.

For (HJ) with (1.3) and $u_0 \equiv 0$, there are infinitely many D-solutions while there is no $\overline{D}$-solution. This suggests that we must define another proper notion of solutions.

Definition 3.3 (envelope solutions)**.** Let $\mathcal{S}$ be a nonempty subset of $\overline{D}$-$SUP(H)$. If $v := \inf_{w \in \mathcal{S}} w$ is bounded from below in Q, it is said to be an *envelope viscosity supersolution* of (1.1). Let $e.SUP(H)$ denote the set of all such solutions. If v is also an envelope viscosity subsolution (write $v \in e.SUB(H)$), i.e., $v = \sup_{w \in \mathcal{T}} w$ for some $\mathcal{T} \subset \overline{D}$-$SUB(H)$ and v is bounded from above in Q, we call it an *envelope viscosity solution*. Set $e.SOL(H) := e.SUB(H) \cap e.SUP(H)$. We also define $e.SUB(H, u_0)$, $e.SUP(H, u_0)$ and $e.SOL(H, u_0)$ as the sets of all (sub, super)solutions of (HJ) similarly as before.

Remark 3.4. (1) The function $ctI(x)$ is an envelope solution in Example 3.1 and $c(t - |x|)_+$ is an envelope solution in Example 3.2. Also, Example 3.1 suggests that our envelope solution is not always continuous.

(2) Since standard viscosity supersolutions have stability, that is, the infimum of them is still a supersolution (see for instance [11, Lemma 4.2]), the

class of solutions does not become large by taking infimum. As for $\overline{D}$-supersolutions, however, we observed that $\inf_{\varepsilon>0} u^\varepsilon \notin \overline{D}\text{-}SUP(H)$ in Examples 3.1 and 3.2. In other words, stability under infimum does not hold for $\overline{D}$-supersolutions in general. By contrast, our envelope supersolutions have such stability by the definition. We also learn by Example 3.2 that $\overline{D}$-supersolutions are not stable even under the uniform limit.

(3) We have $\overline{D}\text{-}SUP(H) \subset e.SUP(H) \subset SUP(H)$, but the inclusion relation $e.SUP(H) \subset D\text{-}SUP(H)$ does not hold in general. (See Figure 5.) The function $ctI(x)$ in Example 3.1 is its counter-example. If H is lower semicontinuous, then $e.SUB(H) = SUB(H)$.

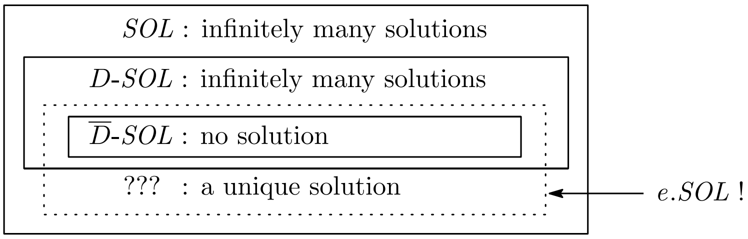

Figure 5: The notion of envelope solutions.

As was pointed out in Remark 3.4 (2) we do not have the stability under infimum for $\overline{D}$-supersolutions in general, but it is shown that the infimum of finitely many $\overline{D}$-supersolutions is still a $\overline{D}$-supersolution.

Proposition 3.5 (stability under infimum of finitely many solutions)**.** *Let $u_i \in \overline{D}\text{-}SUP(H)$ for all $i \in \{1, 2, \dots, M\}$. Then $u := \min_{i=1}^M u_i \in \overline{D}\text{-}SUP(H)$.*

Proof. We first remark that $u_* = \min_{i=1}^M (u_i)_*$. Fix $(\hat{x}, \hat{t}) \in Q$, $(p, \tau) \in \overline{D}^- u_*(\hat{x}, \hat{t})$ and take a defining sequence $(x_m, t_m) \in Q$, $(p_m, \tau_m) \in D^- u_*(x_m, t_m)$ $(m \in \mathbf{N})$. Then we have $\lim_{m\to\infty} u_*(x_m, t_m) = u_*(\hat{x}, \hat{t})$, and there exists a subsequence $\{m(k)\}_{k\in\mathbf{N}}$ of $\{m\}_{m\in\mathbf{N}}$ such that

$$u_*(x_{m(k)}, t_{m(k)}) = (u_i)_*(x_{m(k)}, t_{m(k)}) \quad (\forall k \in \mathbf{N})$$

for some $i \in \{1, 2, \dots, M\}$. Observe that

$$\begin{aligned} u_*(\hat{x}, \hat{t}) &= \lim_{m\to\infty} u_*(x_m, t_m) \\ &= \lim_{k\to\infty} u_*(x_{m(k)}, t_{m(k)}) \\ &= \lim_{k\to\infty} (u_i)_*(x_{m(k)}, t_{m(k)}) \geqq (u_i)_*(\hat{x}, \hat{t}). \end{aligned}$$

Therefore, it follows that $u_*(\hat{x}, \hat{t}) = (u_i)_*(\hat{x}, \hat{t})$. We thus have

$$(p_{m(k)}, \tau_{m(k)}) \in D^-(u_i)_*(x_{m(k)}, t_{m(k)}), \quad \lim_{k\to\infty} (u_i)_*(x_{m(k)}, t_{m(k)}) = (u_i)_*(\hat{x}, \hat{t}),$$

and hence $(p, \tau) \in \overline{D}^-(u_i)_*(\hat{x}, \hat{t})$. Since $u_i \in \overline{D}\text{-}SUP(H)$, we deduce that $\tau + H(\hat{x}, p) \geqq 0$. $\square$

We now present the uniqueness result for envelope solutions.

Proposition 3.6 (uniqueness of envelope solutions)**.** *Assume that H satisfies* (H_p) *and* (H_{xN})*. Then there exists at most one envelope solution of* (HJ)*. Moreover if H is lower semicontinuous, the unique envelope solution is upper semicontinuous.*

Proof. Let $u, v \in e.SOL(H, u_0)$. We first use the fact that $u \in e.SUB(H, u_0)$ and $v \in e.SUP(H, u_0)$. By the definition of envelope sub- and supersolutions there exists some $\mathcal{T} \subset \overline{D}\text{-}SUB(H, u_0)$ and $\mathcal{S} \subset \overline{D}\text{-}SUP(H, u_0)$ such that $u = \sup_{w\in\mathcal{T}} w$ and $v = \inf_{W\in\mathcal{S}} W$. Then applying Theorem 2.6 to $w \in \mathcal{T}$ and $W \in \mathcal{S}$, we get $w^* \leqq W_*$ in Q, which yields $u \leqq v$ in Q. Next changing roles of u and v, we also see $v \leqq u$ in Q, and hence our first claim is proved.

If H is lower semicontinuous, we apply Theorem 2.6 to u and $W \in \mathcal{S}$. Then we deduce that $u^* \leqq W_*$ in Q, hence that $u^* \leqq v$ in Q. Since we also have $v^* \leqq u$ in Q, it follows that $u^* \leqq v \leqq v^* \leqq u$ in Q, and so our second claim follows. □

We next consider the existence of envelope solutions. We will construct the solution as the infimum of u^ε, which are solutions for "good" Hamiltonians H^ε approximating H. Here "good" means that comparison and existence properties are ensured for solutions. We use the following assumption.

(H_ε) There exists a family $\{H^\varepsilon\}_{\varepsilon>0} \subset C(\mathbf{R}^n \times \mathbf{R}^n)$ such that $H^\varepsilon \uparrow H$ ($\varepsilon \downarrow 0$) pointwise, and for all $\varepsilon > 0$ and $u_0 \in BUC(\mathbf{R}^n)$ the following two statements hold.

(i) If $w^\varepsilon \in SUB(H^\varepsilon, u_0)$ and $v^\varepsilon \in SUP(H^\varepsilon, u_0)$, then $(w^\varepsilon)^* \leqq (v^\varepsilon)_*$ in Q.

(ii) There exists a bounded solution $u^\varepsilon \in SOL(H^\varepsilon, u_0)$.

If there is some $u^\varepsilon \in SOL(H^\varepsilon, u_0)$, it is automatically continuous and a unique solution by the comparison (i). Also, H satisfying (H_ε) is lower semicontinuous.

We here recall the *Perron's method* for constructing standard viscosity solutions. (See for instance [11, Theorem 4.1.].) Let $v \in SUB(H, u_0)$, $V \in SUP(H, u_0)$ and $v \leqq V$ in Q. Then u defined by

$$u := \sup\{w \in SUB(H, u_0) \mid v \leqq w \leqq V \text{ in } Q\}$$

is a viscosity solution of (HJ). Functions v and V are called respectively a *lower barrier* and an *upper barrier*. One can construct these barriers for all $u_0 \in BUC(\mathbf{R}^n)$ provided that $H(x, p)$ is bounded locally in p (see [21, Lemma 4.3.4.]), i.e.,

(H_m) $m(\rho) := \sup\{|H(x, p)| \mid (x, p) \in \mathbf{R}^n \times \overline{B}_\rho(0)\} < \infty$ for all $\rho \geqq 0$.

Proposition 3.7 (existence)**.** *Assume that H satisfies* (H_ε) *and* (H_m)*. Let $u^\varepsilon \in SOL(H^\varepsilon, u_0)$ in* (H_ε)*. Then $\overline{u} := \inf_{\varepsilon>0} u^\varepsilon$ is an envelope solution of* (HJ)*.*

We call $\overline{u}$ constructed in this way a *solution approximated from above*. By the definition $\overline{u}$ is upper semicontinuous.

Proof. We first show that u^ε is monotone in ε. Let $0<\varepsilon<\varepsilon'$. Then $u^\varepsilon \in SUB(H^\varepsilon,u_0)$, and also we see $u^{\varepsilon'} \in SUP(H^{\varepsilon'},u_0) \subset SUP(H^\varepsilon,u_0)$ since $H^{\varepsilon'} \leqq H^\varepsilon$. Therefore we conclude that $u^\varepsilon \leqq u^{\varepsilon'}$ by the comparison. This monotonicity implies that $\overline{u} = \limsup^*_{\varepsilon\downarrow 0} u^\varepsilon$ and that $\overline{u}$ is bounded from above. Now, we are able to take an upper semicontinuous lower barrier $v \in SUB(H,u_0)$ on account of the assumption (H_m). Since $v \in SUB(H^\varepsilon,u_0)$, we see by the comparison that $v \leqq u^\varepsilon$, and so $v \leqq \overline{u}$. We also find that $\overline{u}$ is bounded from below.

Since $u^\varepsilon \in SUB(H^\varepsilon)$, we see $\overline{u} \in SUB(\liminf_{*\varepsilon\downarrow 0} H^\varepsilon) = SUB(H)$ by the stability of viscosity subsolutions. Also, $\overline{u}$ is an envelope supersolution of (HJ) because $\overline{u} = \inf_{\varepsilon>0} u^\varepsilon$ and $u^\varepsilon \in SUP(H^\varepsilon) \subset \overline{D}\text{-}SUP(H)$. We finally show that $\overline{u}$ is continuous at the initial time. Take any $x \in \mathbf{R}^n$ and $(y,s) \in Q_0$. Then $v(y,s) - u_0(x) \leqq \overline{u}(y,s) - u_0(x) \leqq u^\varepsilon(y,s) - u_0(x)$ and both $v(y,s)$ and $u^\varepsilon(y,s)$ converge to $u_0(x)$ as $(y,s) \to (x,0)$. As a result we deduce that $\overline{u}(y,s) \to u_0(x)$. □

Remark 3.8. For any subset $L \subset \mathbf{R}^N$ and $h^\varepsilon : L \to \mathbf{R}$ $(\varepsilon > 0)$ we denote the *upper relaxed limit* (resp. *lower relaxed limit*) by $\overline{h} = \limsup^*_{\varepsilon\downarrow 0} h^\varepsilon$ (resp. $\underline{h} = \liminf_{*\varepsilon\downarrow 0} h^\varepsilon) : \overline{L} \to \mathbf{R} \cup \{\pm\infty\}$, which is defined as

$$\overline{h}(z) := \limsup_{(\varepsilon,y)\to(0,z)} h^\varepsilon(y) = \lim_{\delta\downarrow 0} \sup\{h^\varepsilon(y) \mid y \in \overline{B}_\delta(z) \cap L,\ 0<\varepsilon<\delta\}$$

$$(\text{resp. } \underline{h}(z) := \liminf_{(\varepsilon,y)\to(0,z)} h^\varepsilon(y) = \lim_{\delta\downarrow 0} \inf\{h^\varepsilon(y) \mid y \in \overline{B}_\delta(z) \cap L,\ 0<\varepsilon<\delta\})$$

for $z \in \overline{L}$. The following properties are easily seen by the definition: If $h^\varepsilon \equiv h$, then $\overline{h} = h^*$ and $\underline{h} = h_*$. If $h^\varepsilon \downarrow h$ (resp. $h^\varepsilon \uparrow h$) monotonously, then $\overline{h} = h^*$ (resp. $\underline{h} = h_*$). Also, $\overline{h} = \limsup^*_{\varepsilon\downarrow 0}(h^\varepsilon)^*$ and $\underline{h} = \liminf_{*\varepsilon\downarrow 0}(h^\varepsilon)_*$ in general.

We next present examples of H which satisfies the assumption (H_ε). In order to obtain the comparison and existence properties in (H_ε), it is sufficient that each H^ε satisfies $(\mathrm{H}_p), (\mathrm{H}_x)$ and (H_m).

Example 3.9. If H is lower semicontinuous and bounded in $\mathbf{R}^n \times \mathbf{R}^n$, then (H_ε) is fulfilled. In this case we take H^ε as the inf-convolution of H over $\mathbf{R}^n \times \mathbf{R}^n$. (See below about sup- and inf-convolution.) Each H^ε satisfies (H_p) and (H_x) since it is globally Lipschitz continuous, and (H_m) is clear from the boundedness of H^ε.

Example 3.10. Let H have the form of (1.10) with $r \in BUSC(\mathbf{R}^n)$. Assume that H_0 is uniformly continuous in $\mathbf{R}^n \times \mathbf{R}^n$ and satisfies (H_m). Then (H_ε) is fulfilled. The conditions $(\mathrm{H}_p), (\mathrm{H}_x), (\mathrm{H}_m)$ are all satisfied by $H^\varepsilon(x,p) = H_0(x,p) - r^\varepsilon(x)$, where r^ε is the sup-convolution of r.

Remark 3.11 (sup- and inf-convolution). For bounded $f : \mathbf{R}^N \to \mathbf{R}$ and $\varepsilon > 0$ we define the *sup-convolution* f^ε (resp. *inf-convolution* f_ε) of f by

$$f^\varepsilon(x) := \sup_{y\in\mathbf{R}^N} \left\{ f(y) - \frac{1}{2\varepsilon}|x-y|^2 \right\}.$$

$$\left(\text{resp. } f_\varepsilon(x) := \inf_{y\in\mathbf{R}^N} \left\{ f(y) + \frac{1}{2\varepsilon}|x-y|^2 \right\}.\right)$$

The following properties are easily found, and so we omit the verification.

- $-\|f\|_{\mathbf{R}^n} \leqq f \leqq f^\delta \leqq f^\varepsilon \leqq \|f\|_{\mathbf{R}^n}$ for $0 < \delta < \varepsilon$.
- f^ε is Lipschitz continuous in $\mathbf{R}^N$.
- If f is upper semicontinuous, then $f^\varepsilon(x) \downarrow f(x)$ $(\varepsilon \downarrow 0)$ for each $x \in \mathbf{R}^N$.
- If f is uniformly continuous, then f^ε converges to f uniformly in $\mathbf{R}^N$.

We mainly use these convolutions in order to approximate semicontinuous functions by Lipschitz continuous ones.

Combining Proposition 3.6 and Proposition 3.7, we obtain the unique existence result.

Theorem 3.12 (unique existence–general version)**.** *Assume that H satisfies* (H_p)*,* (H_{xN})*,* (H_ε) *and* (H_m)*. Then $\overline{u}$, a solution approximated from above, is a unique envelope solution of* (HJ)*.*

If we do not accept the assumption (H_{xN}), Lipschitz continuities of solutions are needed for CP.

Theorem 3.13 (unique existence–Lipschitz version)**.** *Assume that H satisfies* (H_p)*,* (H_ε) *and* (H_m)*. Let $u^\varepsilon \in SOL(H^\varepsilon, u_0)$ in* (H_ε) *and assume that u^ε $(\varepsilon > 0)$ and $\inf_{\varepsilon>0} u^\varepsilon$ are Lipschitz continuous in Q. Then $\overline{u}$, a solution approximated from above, is a unique envelope solution of* (HJ)*.*

Proof. We only need to show the uniqueness. Let $v \in e.SOL(H, u_0)$. An analogue of the proof of Proposition 3.6 works and yields the inequality $\overline{u} \leqq v_*$ in Q (but we use Theorem 2.9 here). Next, since $v \in SUB(H, u_0)$, $u^\varepsilon \in \overline{D}\text{-}SUP(H, u_0)$ and u^ε is Lipschitz continuous, Theorem 2.9 yields that $v^* \leqq u^\varepsilon$ in Q, and so $v^* \leqq \overline{u}$ in Q. Thus $\overline{u} = v$. □

Remark 3.14. Let $u \in e.SOL(H, u_0)$ and (H_p) hold. If there exists some $\mathcal{T} \subset \overline{D}\text{-}SUB(H, u_0) \cap BLip(Q)$ (resp. $\mathcal{S} \subset \overline{D}\text{-}SUP(H, u_0) \cap BLip(Q)$) such that $u = \sup_{v\in\mathcal{T}} v$ (resp. $u = \inf_{w\in\mathcal{S}} w$), then u is the minimal (resp. maximal) envelope solution. These facts are easily shown by using Theorem 2.9.

Remark 3.15. If the Lipschitz constants of u^ε are estimated uniformly in ε, then $\overline{u} = \inf_{\varepsilon>0} u^\varepsilon$ is also Lipschitz continuous (provided that u^ε are bounded uniformly in ε). In general, if u^ε have their modulus $\omega \in \mathcal{M}$ independent of ε, their infimum $\overline{u}$ also has the same ω as its modulus.

Example 3.16. In Example 3.2 the function $u(x,t) = c(t - |x|)_+$ is a unique envelope solution by Theorem 3.13 since u^ε and u are Lipschitz continuous. In Example 3.1, on the other hand, the envelope solutions are not unique in that $v^\alpha(x,t) = \alpha t I(x)$ $(\alpha \in (0, c])$ are all envelope solutions. Let us show this claim. It is easily seen that they are all subsolutions. Set $v^{\alpha,\varepsilon}(x,t) := \alpha t\{(1 - \sqrt{|x|/\varepsilon})_+\}^2$ for $\varepsilon > 0$. (See Figure 6.) Then one observes that $v^{\alpha,\varepsilon} \in \overline{D}\text{-}SUP(H, 0)$ since $\partial_t v^{\alpha,\varepsilon} \geqq 0$ and $\overline{D}^- v^{\alpha,\varepsilon}(0, \hat{t}) = \emptyset$. Hence the equality $v^\alpha = \inf_{\varepsilon>0} v^{\alpha,\varepsilon}$ implies our claim. Moreover $v^0 \equiv 0$ is also an envelope solution since $v^0 = \inf_{\alpha\in(0,c],\varepsilon>0} v^{\alpha,\varepsilon}$. The function v^c is the maximal envelope solution by

Remark 3.14 because $v^c(x,t) = \inf_{\varepsilon>0} ctI^\varepsilon(x)$ and $ctI^\varepsilon(x) \in \overline{D}\text{-}SUP(H,0) \cap BLip(Q)$, where $I^\varepsilon(x) := (1 - |x|/\varepsilon)_+$. Also, $v^0 \equiv 0$ is the minimal envelope solution.

For a general initial data $u_0 \in BUC(\mathbf{R}^n)$, it is also seen that $u_0(x) + \alpha tI(x)$ ($\alpha \in [0,c]$) are all envelope solutions. Uniqueness, therefore, always goes wrong for the initial-value problem with the equation $\partial_t u = cI(x)$. Such bad behavior can happen when a Hamiltonian is non-coercive. Indeed, we establish the uniqueness result for coercive Hamiltonians in the next subsection (Theorem 3.20).

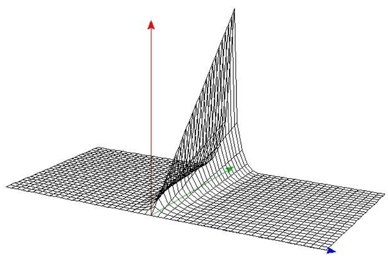

Figure 6: $\overline{D}$-supersolutions $v^{\alpha,\varepsilon}$ of $\partial_t u = I(x)$, $u_0 \equiv 0$.

3.2 Coercive Hamiltonians

In order to apply Theorem 3.13, we need to know what conditions guarantee the Lipschitz continuities of u^ε and $\inf_{\varepsilon>0} u^\varepsilon$. We therefore consider in this subsection whether the solutions preserve the continuity of initial data. For continuous Hamiltonians it is known that such preserving properties hold if they are *coercive*, namely

$$\lim_{|p|\to\infty} \inf_{x\in\mathbf{R}^n} H(x,p) = \infty \quad \text{or} \quad \lim_{|p|\to\infty} \sup_{x\in\mathbf{R}^n} H(x,p) = -\infty.$$

The coercivity of H is equivalent to (H_{R+}) or (H_{R-}) below.

(H_{R+}) $R_+(m) := \sup\{|p| \mid \exists x \in \mathbf{R}^n,\ H(x,p) \leqq m\} < \infty$ for all $m \geqq 0$.

(H_{R-}) $R_-(m) := \sup\{|p| \mid \exists x \in \mathbf{R}^n,\ H(x,p) \geqq -m\} < \infty$ for all $m \geqq 0$.

Here we use the convention that $\sup \emptyset = 0$. We first present Lipschitz continuity and BUC (bounded uniform continuity) preserving properties for continuous Hamiltonians. These results are more or less known. See for example [9], where they discussed for a.e. (sub)solutions. We give proofs based on the theory of viscosity solutions without using a.e. solutions in Appendix for completeness. By using these results we establish our preserving properties for discontinuous Hamiltonians via approximation by continuous ones.

For a function $u : Q \to \mathbf{R}$, we define

$$\operatorname{Lip}_t[u] := \sup_{x\in\mathbf{R}^n} \sup_{\substack{t,s\in(0,T)\\ t\neq s}} \frac{|u(x,t)-u(x,s)|}{|t-s|},$$

$$\operatorname{Lip}_x[u] := \sup_{t\in(0,T)} \sup_{\substack{x,y\in\mathbf{R}^n\\ x\neq y}} \frac{|u(x,t)-u(y,t)|}{|x-y|}.$$

Proposition 3.17 (Lipschitz continuity preserving property)**.** *Assume that H satisfies* (H_p)*,* (H_x)*,* (H_m) *and* (H_{R+})*. Let $u_0 \in BLip(\mathbf{R}^n)$ and $u \in SOL(H, u_0)$. Then $u \in BLip(Q)$ with the Lipschitz constant satisfying*

$$\operatorname{Lip}_t[u] \leqq m, \qquad \operatorname{Lip}_x[u] \leqq R_+(m),$$

where $m := m(\operatorname{Lip}[u_0])$ and $m(\cdot)$ is the function in (H_m)*.*

The assumption (H_{R+}) is able to be replaced by (H_{R-}). (The same is valid for Proposition 3.18.) For the proof see Appendix.

Proposition 3.18 (BUC preserving property)**.** *Assume that H satisfies* (H_p)*,* (H_x)*,* (H_m) *and* (H_{R+})*. Let $u_0 \in BUC(\mathbf{R}^n)$ and $u \in SOL(H, u_0)$. Furthermore let $\{u_0^\delta\}_{\delta>0} \subset BLip(\mathbf{R}^n)$ and assume that u_0^δ converges to u_0 uniformly in $\mathbf{R}^n$ as $\delta \downarrow 0$. Then $u \in BUC(Q)$ with modulus of continuity*

$$\omega(r) := \inf_{\delta>0} \left(2\|u_0 - u_0^\delta\|_{\mathbf{R}^n} + \sqrt{(m^\delta)^2 + (R_+(m^\delta))^2}\ r \right),$$

where $m^\delta := m(\operatorname{Lip}[u_0^\delta])$, $m(\cdot)$ is the function in (H_m) *and $R_+(\cdot)$ is the function in* (H_{R+})*.*

For a given $u_0 \in BUC(\mathbf{R}^n)$ one can always construct the family $\{u_0^\delta\}_{\delta>0}$ like the above by taking u_0^δ as the sup- or inf-convolution of u_0 for $\delta > 0$. For the proof of Proposition 3.18 see Appendix.

Since we should treat discontinuous H, we apply the above results to the solutions u^ε of the approximate equations and confirm that their infimum has a desired property. We use the fact in Remark 3.15 that if u^ε share a modulus independent of ε, then their infimum has the same modulus. In the case of non-coercive Hamiltonian, solutions cannot preserve even continuity of the initial data as we observed in Example 3.1, in which the envelope solution $u(x,t) = ctI(x)$ is not continuous in contrast to the initial data $u_0 \equiv 0$.

Theorem 3.19. *Assume that H satisfies* (H_ε)*,* (H_m) *and that each H^ε in* (H_ε) *satisfies* (H_m)*,* (H_{R+})*. Assume furthermore that*

$$\sup_{\varepsilon>0} m^\varepsilon(\rho) < \infty, \quad \sup_{\varepsilon>0} R_+^\varepsilon(m) < \infty$$

for all $\rho \geqq 0$ and $m \geqq 0$, where

$$m^\varepsilon(\rho) := \sup\{|H^\varepsilon(x,p)| \mid (x,p) \in \mathbf{R}^n \times \overline{B}_\rho(0)\}\ (< \infty),$$
$$R_+^\varepsilon(m) := \sup\{|p| \mid \exists x \in \mathbf{R}^n,\ H^\varepsilon(x,p) \leqq m\}\ (< \infty).$$

Let $u_0 \in BUC(\mathbf{R}^n)$. Then u, a solution approximated from above, has the following properties.

(1) $u \in BUC(Q)$.

(2) *If* $u_0 \in BLip(\mathbf{R}^n)$, *then* $u \in BLip(Q)$.

(3) *If* H *satisfies* (H_p), *then* u *is a unique envelope solution of* (HJ).

The condition (H_{R+}) is able to be replaced by (H_{R-}). In this case, if H itself satisfies (H_{R-}), then the assumption $\sup_{\varepsilon>0} R_-^\varepsilon(m) < \infty$ always holds since we have $R_-^\varepsilon(m) \leqq R_-(m)$ by $H^\varepsilon \leqq H$.

Proof. We first prove (2) and next show (1) by approximating the initial data. Take $u^\varepsilon \in SOL(H^\varepsilon, u_0)$ in (H_ε).

(2) Denote $l := \mathrm{Lip}[u_0]$. Now, Proposition 3.17 ensures that $u^\varepsilon \in BLip(Q)$ and

$$\mathrm{Lip}_t[u^\varepsilon] \leqq m^\varepsilon(l) \leqq \sup_{\varepsilon>0} m^\varepsilon(l), \quad \mathrm{Lip}_x[u^\varepsilon] \leqq R_+^\varepsilon(m^\varepsilon(l)) \leqq \sup_{\varepsilon>0} R_+^\varepsilon(m^\varepsilon(l)).$$

Since both Lipschitz constants are estimated independently of ε, we conclude $u = \inf_{\varepsilon>0} u^\varepsilon \in BLip(Q)$.

(1) Let $u_0^\delta = (u_0)^\delta$ be the sup-convolution of u_0 and denote $l^\delta := \mathrm{Lip}[u_0^\delta]$. Then, Proposition 3.18 ensures that $u^\varepsilon \in BUC(Q)$ and each u^ε has a modulus

$$\omega^\varepsilon(r) := \inf_{\delta>0} \left(2\|u_0 - u_0^\delta\|_{\mathbf{R}^n} + \sqrt{\{m^\varepsilon(l^\delta)\}^2 + \{R_+^\varepsilon(m^\varepsilon(l^\delta))\}^2}\ r \right).$$

Since $m^\varepsilon(l^\delta)$ and $R_+^\varepsilon(m^\varepsilon(l^\delta))$ are similarly estimated independently of ε, there exists a common modulus for u^ε. Thus we conclude $u \in BUC(Q)$.

(3) Since $u_0^\delta \in BLip(\mathbf{R}^n)$, there exists a Lipschitz continuous envelope solution $u^\delta \in e.SOL(H, u_0^\delta) \cap BLip(Q)$ for each $\delta > 0$ by (2) above. Moreover, there exist solutions of approximate equations $(u^\delta)^\varepsilon \in SOL(H^\varepsilon, u_0^\delta)$, which satisfy

$$u^\delta = \inf_{\varepsilon>0} (u^\delta)^\varepsilon \quad \text{and} \quad (u^\delta)^\varepsilon \in \overline{D}\text{-}SUP(H, u_0^\delta) \cap BLip(Q).$$

Then, by Theorem 2.9 we have $\|v - u^\delta\|_Q \leqq \|u_0 - u_0^\delta\|_{\mathbf{R}^n}$ for any envelope solution v of (HJ). Hence the uniqueness of u follows because $\lim_{\delta\downarrow 0} \|u_0 - u_0^\delta\| = 0$. □

We have given some examples of H satisfying (H_ε). In Example 3.9 H^ε are not coercive because of their boundedness. We therefore impose the coercivity assumption on H in Example 3.10 so as to apply Theorem 3.19.

Theorem 3.20. *Assume that* H *has the form of* (1.10) *with* $r \in BUSC(\mathbf{R}^n)$. *Assume that* H_0 *is coercive, uniformly continuous in* $\mathbf{R}^n \times \mathbf{R}^n$ *and satisfies* (H_m). *Let* $u_0 \in BUC(\mathbf{R}^n)$. *Then there exists a unique envelope solution* u *of* (HJ) *and it has the following properties.*

(1) $u \in BUC(Q)$.

(2) *If* $u_0 \in BLip(\mathbf{R}^n)$, *then* $u \in BLip(Q)$.

Proof. We assume (H_{R+}) because the proof in the case of (H_{R-}) is similar. Let $R_{0+}(\cdot)$ be the function in (H_{R+}) for H_0. It is clear that the above H fulfills (H_m). As observed in Example 3.10, we also learn that H satisfies (H_ε) by the approximation $H^\varepsilon(x,p) = H_0(x,p) - r^\varepsilon(x)$, where r^ε is the sup-convolution of r. Thus by Proposition 3.7 we obtain a solution approximated from above $u \in e.SOL(H, u_0)$. It remains to show the uniform boundedness of $m^\varepsilon(\rho)$ and $R^\varepsilon_+(m)$ in ε in order to apply Theorem 3.19. Since $H^\varepsilon \leqq |H_0| + \|r\|_{\mathbf{R}^n}$, we have $m^\varepsilon(\rho) \leqq \max_{\mathbf{R}^n \times \overline{B}_\rho(0)} |H_0| + \|r\| < \infty$, and hence $\sup_{\varepsilon>0} m^\varepsilon(\rho) < \infty$. Also, when $m \geqq H^\varepsilon(x,p)$, one observes that $H_0(x,p) \leqq m + r^\varepsilon \leqq m + \|r\|$ and so $|p| \leqq R_{0+}(m + \|r\|)$ by (H_{R+}). Therefore we obtain $R^\varepsilon_+(m) \leqq R_{0+}(m+\|r\|) < \infty$, which yields $\sup_{\varepsilon>0} R^\varepsilon_+(m) < \infty$. □

4 Relaxed Hamiltonians

In this section we establish a unique existence result without the coercivity assumption for H. Our existence result (Proposition 3.7) does not require the coercivity. The problem lies in the uniqueness part. In fact, we cannot expect the uniqueness in general as we observed in Example 3.16. However, we are able to show the uniqueness for more restrictive Hamiltonians without the coercivity. To apply our Lipschitz version of CP (Theorem 2.9) we need Lipschitz continuity of one of solutions, but the continuity preserving property does not hold in general without the coercivity. On the other hand, our general version of CP (Theorem 2.6) excludes Hamiltonians with discontinuous source terms. We solve this difficulty by considering a *relaxed problem.* If an envelope solution u of (HJ) can be regarded as an envelope solution of another problem (relaxed problem):

$$\text{(r.HJ)} \begin{cases} \partial_t u + \hat{H}(x, \nabla u) = 0 & \text{in} \quad Q, \\ (1.2). \end{cases} \tag{4.1}$$

with a *relaxed Hamiltonian* $\hat{H}$ satisfying (H_{xN}), then we conclude the uniqueness of u as envelope solutions of (HJ) by Theorem 2.6.

We define the relaxed Hamiltonians so that $\hat{H} \geqq H$. Then it is obvious that a supersolution of (1.1) is also a supersolution of (4.1). Therefore it is an important issue whether or not a subsolution of (1.1) is also a subsolution of (4.1). We will solve this problem after defining $\hat{H}$. In addition, as another topics about $\hat{H}$ we discuss existence of $\overline{D}$-solutions which are not guaranteed for original H.

4.1 Uniqueness revisited

In this section we treat special Hamiltonians with the following properties.

(H_r) (i) H is lower semicontinuous in $\mathbf{R}^n \times \mathbf{R}^n$ and is continuous in $(\mathbf{R}^n \setminus \Gamma) \times \mathbf{R}^n$ with some discrete set Γ, i.e.,

$$\text{for every } a \in \Gamma \text{ there exists a open set } V_a \text{ such that } \{a\} = \Gamma \cap V_a. \tag{4.2}$$

(ii) H^* is continuous in $\mathbf{R}^n \times \mathbf{R}^n$.

(iii) $H(a,p) \leqq \inf_{0 \leq \mu \leq 1} H^*(a, \mu p)$ for each $a \in \Gamma$ and $p \in \mathbf{R}^n$.

For such H, we define a *relaxed Hamiltonian* $\hat{H} : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ by

$$\hat{H}(x,p) := \begin{cases} H(x,p) & (x \notin \Gamma), \\ \min\{\inf_{0 \leq \mu \leq 1} H^*(x, \mu p),\ \sup_{0 \leq \mu \leq 1} H(x, \mu p)\} & (x \in \Gamma). \end{cases}$$

(See Figure 7.) The continuity of H^* implies that

$$H^*(x,p) = \lim_{\substack{(y,q) \to (x,p) \\ y \neq x}} H(y,q) \tag{4.3}$$

for all $(x,p) \in \mathbf{R}^n \times \mathbf{R}^n$. Also, since $H(x,p) \leqq \sup_{0 \leq \mu \leq 1} H(x, \mu p)$ and (H_r)(iii) holds, we have $H \leqq \hat{H}$ in $\mathbf{R}^n \times \mathbf{R}^n$. Besides, it is seen that $\hat{H} \leqq H^*$ in $\mathbf{R}^n \times \mathbf{R}^n$ and $\hat{H}$ is lower semicontinuous.

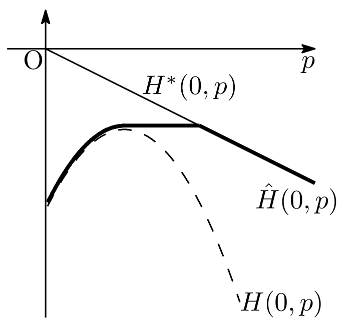

Figure 7: The definition of $\hat{H}(0,p)$ in the case $0 \in \Gamma$.

Example 4.1. Let H have the form of (1.10). Then the following (i)$'$–(iii)$'$ is one sufficient condition for (H_r).

(i)$'$ H_0 is continuous in $\mathbf{R}^n \times \mathbf{R}^n$. r is upper semicontinuous in $\mathbf{R}^n$ and is continuous in $\mathbf{R}^n \setminus \Gamma$ for some Γ which satisfies (4.2).

(ii)$'$ r_* is continuous in $\mathbf{R}^n$.

(iii)$'$ $H_0(a,p) - r(a) \leqq \inf_{0 \leq \mu \leq 1} H_0(a, \mu p) - r_*(a)$ for each $a \in \Gamma$ and $p \in \mathbf{R}^n$.

To show (iii)$'$ it is enough to prove that

$$\text{a function } \mu \mapsto H_0(a, \mu p) \text{ is nonincreasing on } \{\mu \geqq 0\} \text{ for each } a \in \Gamma \text{ and } p \in \mathbf{R}^n. \tag{4.4}$$

We here assume (4.4) and let $0 \in \Gamma$ (, i.e., r is discontinuous at 0). Then, since $H^*(0, \mu p) = H_0(0, \mu p) - r_*(0)$ and $H(0, \mu p) = H_0(0, \mu p) - r(0)$, we have

$$\hat{H}(0,p) = \min\{H_0(0,p) - r_*(0),\ H_0(0,0) - r(0)\}$$

for all $p \in \mathbf{R}^n$. By this equality we find that $\hat{H}(0,p)$ is a constant $H_0(0,0)-r(0) = H(0,0)$ on $P := \{p \in \mathbf{R}^n \mid H_0(0,p) \geqq H_0(0,0)-(r(0)-r_*(0))\}$. Furthermore, if P is bounded, namely $P \subset B_N(0)$ for some $N > 0$, then $\hat{H}(x,p) = H_0(x,p) - r_*(x)$ holds in $\mathbf{R}^n \times (\mathbf{R}^n \setminus B_N(0))$. Therefore assumptions (H_p) and (H_{xN}) required in Theorem 2.6 are fulfilled if H_0 and r_* are uniformly continuous.

Example 4.2. We see for (1.3) that $\hat{H}(x,p) = -|p| - (cI(x) - |p|)_+$. As for a unique envelope solution $u(x,t) = c(t-|x|)_+$ with $u_0 \equiv 0$, an easy computation shows that $u \in SUB(\hat{H}, 0)$, which implies $u \in e.SOL(\hat{H}, 0)$. Moreover one can also verify $u \in \overline{D}\text{-}SOL(\hat{H}, 0)$. This suggests that an envelope solution of (HJ) has a more chance to be a $\overline{D}$-solution of (r.HJ) than the original equation. The details will be discussed in the next subsection.

Example 4.3. For (1.16) we have $\hat{H} = H$. Hence the relaxation method does not give any new information to us.

The following is the key fact for relaxed Hamiltonians.

Lemma 4.4. *Assume that H satisfies* (H_r). *If $u \in SUB(H)$, then $u \in SUB(\hat{H})$.*

Proof. We simply write u for u^*. Take any $(\hat{x}, \hat{t}) \in Q$ and $(p, \tau) \in D^+u(\hat{x}, \hat{t})$. If $\hat{x} \notin \Gamma$ or $p = 0$, we deduce $\tau + \hat{H}(\hat{x}, p) \leqq 0$ since $\hat{H}(\hat{x}, p) = H(\hat{x}, p)$ and $u \in SUB(H)$. Therefore we need only consider the case that $\hat{x} \in \Gamma$ and $p \neq 0$. We may assume $\hat{x} = 0$ to simplify the notation. Our goal is now to show $\tau + \hat{H}(0,p) \leqq 0$, namely

$$\tau + \inf_{0 \leqq \mu \leqq 1} H^*(0, \mu p) \leqq 0 \qquad \text{or} \qquad \tau + \sup_{0 \leqq \mu \leqq 1} H(0, \mu p) \leqq 0.$$

Define

$$\Sigma := \{\mu \in [0,1] \mid (\mu p, \tau) \in D^+u(0, \hat{t})\}, \quad \mu_0 := \inf\{\mu \in [0,1] \mid [\mu, 1] \subset \Sigma\}.$$

Then $1 \in \Sigma$ and we also have $\mu_0 \in \Sigma$ since superdifferentials are closed. We discuss two different cases about μ_0.

Case 1: $\mu_0 = 0$. Since $(\mu p, \tau) \in D^+u(0, \hat{t})$ for each $\mu \in [0,1]$, it follows from $u \in SUB(H)$ that

$$\tau + H(0, \mu p) \leqq 0.$$

Thus we obtain

$$\tau + \sup_{0 \leqq \mu \leqq 1} H(0, \mu p) \leqq 0.$$

Case 2: $0 < \mu_0 \leqq 1$. Take a corresponding test function $\phi \in C^1(Q)$ for $(\mu_0 p, \tau) \in D^+u(0, \hat{t})$. We may assume $u - \phi$ attains its strict maximum at $(0, \hat{t})$. By the definition of μ_0 there exists a sequence $\{\mu_m\}_{m \in \mathbf{N}}$ such that $\mu_m \uparrow \mu_0$ and $\mu_m \notin \Sigma$. Define

$$\phi_m(x,t) = \phi(x,t) - (\mu_0 - \mu_m)\langle x, p \rangle$$

for each m. Since ϕ_m converges to ϕ locally uniformly, there exists some sequence $\{(x_m, t_m)\}_{m \in \mathbf{N}}$ such that $(x_m, t_m) \to (0, \hat{t})$ and $\max_{Q'}(u - \phi_m) = (u - \phi_m)(x_m, t_m)$. Here Q' is an arbitrary bounded open subset of Q containing $(0, \hat{t})$.

The facts that $\mu_m \notin \Sigma$ and $\nabla\phi_m(0,\hat{t}) = \mu_m p$ imply $(x_m, t_m) \neq (0, \hat{t})$. Moreover we find that $x_m \neq 0$ since $\phi_m(0,t) = \phi(0,t)$. Thus it follows from $u \in SUB(H)$ that

$$\partial_t\phi(x_m, t_m) + H(x_m, \nabla\phi(x_m, t_m) - (\mu_0 - \mu_m)p) \leqq 0$$

and by letting $m \to \infty$ we obtain

$$\tau + H^*(0, \mu_0 p) \leqq 0$$

on account of (4.3). As a result we have

$$\tau + \inf_{0 \leqq \mu \leqq 1} H^*(0, \mu p) \leqq 0,$$

which concludes the proof. □

We present a uniqueness result in a general form.

Proposition 4.5 (uniqueness by relaxation)**.** *Assume that H satisfies* (H_r) *and that $\hat{H}$ satisfies* (H_p), (H_{xN})*. Then there exists at most one envelope solution of* (HJ) *and it is upper semicontinuous.*

Proof. If $u_1, u_2 \in e.SOL(H, u_0)$, then $u_1, u_2 \in e.SOL(\hat{H}, u_0)$ by Lemma 4.4 and $H \leqq \hat{H}$. Since $\hat{H}$ satisfies (H_p), (H_{xN}) and is lower semicontinuous, we see $u_1 = u_2$ and they are upper semicontinuous in terms of Proposition 3.6. □

Here we give one sufficient condition to apply Proposition 4.5.

Proposition 4.6. *Assume that H has the form of* (1.10) *with* (1.12)*. Assume that H_0 is uniformly continuous in $\mathbf{R}^n \times \mathbf{R}^n$, satisfies* (4.4) *and*

$$R(\gamma) := \sup\{|p| \mid \exists x \in \mathbf{R}^n,\ H_0(x,p) \geqq -\gamma\} < \infty, \tag{4.5}$$

where $\gamma := \max_{j=1}^N (c_j - H_0(a_j, 0))$. Then $\hat{H}$ satisfies (H_p) *and* (H_{xN})*.*

Proof. According to Example 4.1 we have

$$\hat{H}(x,p) := \begin{cases} H_0(x,p) & (x \neq a_j), \\ \min\{H_0(a_j,p),\ H_0(a_j,0) - c_j\} & (x = a_j) \end{cases}$$

under the above assumptions. It suffices to check (H_{xN}). By (4.5) we see $H_0(a_j, p) < H_0(a_j, 0) - c_j$ for all $p \in \mathbf{R}^n \setminus \overline{B}_{R(\gamma)}(0)$ and $j \in \{1, 2, \dots, N\}$. Consequently

$$\hat{H}(x,p) = H_0(x,p) \quad \text{if } (x,p) \in \mathbf{R}^n \times (\mathbf{R}^n \setminus \overline{B}_{R(\gamma)}(0)),$$

and so (H_{xN}) is satisfied. □

Example 4.7. Let us consider the non-coercive Hamiltonian (1.17). Then (4.5) is fulfilled if and only if $0 < c < 1$, and therefore the uniqueness of envelope solutions follows. We will later see in Example 5.15 that there are infinitely many $\overline{D}$-solutions even with $u_0 \equiv 0$ when $c > 1$.

Theorem 4.8. *Assume that H has the form of* (1.17)*. Let $u_0 \in BUC(\mathbf{R}^n)$ and $0 < c < 1$. Then there exists a unique envelope solution u^c of* (HJ)*. Moreover $u^c \in BLip(Q)$ provided that* $\mathrm{Lip}[u_0] < (1-c)/c$.

Proof. Since $H = H^c$ satisfies (H_ε) and (H_m) for Proposition 3.7 (refer to Example 3.9 or 3.10 about (H_ε)), there exists $u^c \in e.SOL(H^c, u_0)$, a solution approximated from above, and its uniqueness follows from Example 4.7.

We next show the Lipschitz continuity preserving property of u^c. Define $H^{c,\varepsilon}(x,p) = -|p|/(1+|p|) - cI^\varepsilon(x)$ with (1.8). Observe that for fixed $\rho_0 > 0$

$$m^{c,\varepsilon}(\rho) := \sup\{|H^{c,\varepsilon}(x,p)| \mid (x,p) \in \mathbf{R}^n \times \overline{B}_\rho(0)\} \leqq c + \frac{\rho_0}{1+\rho_0} \quad (\forall \rho \in [0,\rho_0])$$

and

$$R_-^{c,\varepsilon}(m) := \sup\{|p| \mid \exists x \in \mathbf{R}^n,\ H^{c,\varepsilon}(x,p) \geqq -m\} \leqq \frac{m}{1-m} \quad (\forall m \in [0,1)).$$

Therefore we learn by Remark A.1 that solutions of (HJ) with $H^{c,\varepsilon}$ are Lipschitz continuous provided that $c + \rho_0/(1+\rho_0) < 1$, i.e., $\rho_0 < (1-c)/c$. Furthermore, we see by the estimate above that their Lipschitz constants are bounded uniformly in ε. Hence their infimum, which is u^c by the uniqueness, is also Lipschitz continuous if $\mathrm{Lip}[u_0] < (1-c)/c$. □

We think that the Lipschitz continuity preserving property may not hold if $\mathrm{Lip}[u_0] > (1-c)/c$.

4.2 Existence of $\overline{D}$-solutions

For (HJ) with (1.3) and $u_0 \equiv 0$, the unique envelope solution $u(x,t) = c(t-|x|)_+$ is not only an envelope solution of (r.HJ) but also a $\overline{D}$-solution of (r.HJ). In other words, we obtained a $\overline{D}$-solution by the relaxation method while our original problem (HJ) has no $\overline{D}$-solution. Unfortunately, for a general initial-value it is not always true that $u \in \overline{D}\text{-}SOL(\hat{H}, u_0)$ when $u \in e.SOL(H, u_0)$. Its counter-example is given by the lower left function in Figure 10 later. It is the envelope solution of (HJ) with $H(x,p) = -|p| - I(x)$, $u_0(x) = 2\min\{(|x|-1)_+,\ 1\}$ and is written as

$$u(x,t) = \max\left\{\max_{\overline{B}_t(x)} u_0,\ (t-|x|)_+\right\}. \tag{4.6}$$

For this u, we observe that $u \notin \overline{D}\text{-}SOL(\hat{H}, u_0)$ because $(0,0) \in \overline{D}^- u(0,2)$ and $0 + \hat{H}(0,0) = -1 < 0$. In this subsection we consider what conditions lead an envelope solution of (HJ) to a $\overline{D}$-solution of (r.HJ).

Recall that an envelope supersolution is not always a $\overline{D}$-supersolution because of a lack of stability. If it is guaranteed for (r.HJ), one can obtain a $\overline{D}$-solution. We shall explain the difficulty to show the stability in general. Let $u := \inf_{\varepsilon>0} u^\varepsilon$, $u^\varepsilon \in \overline{D}\text{-}SUP(\hat{H})$, $(p,\tau) \in \overline{D}^- u_*(\hat{x},\hat{t})$ and take a defining approximate sequence $(p_m,\tau_m) \in D^- u_*(x_m,t_m)$. Since $u^\varepsilon \in SUP(\hat{H})$ in particular, the stability for standard solutions ensures $\tau_m + (\hat{H})^*(x_m,p_m) \geqq 0$. Sending $m \to \infty$, we see

$\tau + (\hat{H})^*(\hat{x}, p) \geqq 0$ and $\tau + \hat{H}(\hat{x}, p) \geqq 0$ if $\hat{H}$ is continuous at $(\hat{x}, p)$. Hence the remaining problem is whether $\tau + \hat{H}(\hat{x}, p) \geqq 0$ holds for every $(p, \tau) \in \overline{D}^- u_*(\hat{x}, \hat{t})$ such that $(\hat{x}, p)$ is a discontinuous point of $\hat{H}$.

Let us come back to the example of (1.3). Since the set of discontinuous points of $\hat{H}(x, p) = -|p| - (cI(x) - |p|)_+$ is $\{(0, p) \mid |p| < c\}$, the problem is whether $\tau - c \geqq 0$ holds for all $(p, \tau) \in \overline{D}^- u_*(0, \hat{t})$ such that $|p| < c$. This can be regarded as a condition about growth rates of u in the t-direction near $\{0\} \times (0, T)$ and is satisfied for example if u has the form $u(x, t) = c(t - |x|)_+ + k$ for some $k \in \mathbf{R}$. According to Example 5.7 later, if the initial-value u_0 satisfies

$$u_0(x) < c|x| + u_0(0) \quad \text{for all } x \in \mathbf{R}^n \setminus \{0\}, \tag{4.7}$$

then

$$u(x, t) = c(t - |x|)_+ + u_0(0) \quad \text{near } \{0\} \times (0, T). \tag{4.8}$$

Thus we obtain a $\overline{D}$-solution of (r.HJ). Summarizing the above arguments we conclude

Theorem 4.9. *Let $H(x, p) = -|p| - cI(x)$ for $c > 0$. Let $u \in e.SOL(H, u_0)$. If* (4.7) *holds, then $u \in \overline{D}$-$SOL(\hat{H}, u_0)$.*

Remark 4.10. The assumption (4.7) is optimal since (4.6) is not a $\overline{D}$-solution of (r.HJ). (Note that $u_0(x) = |x|$ if $|x| = 0$ or 2 and $u_0(x) < |x|$ if $|x| \neq 0$ and 2 in the example.)

In Appendix B we discuss the existence of $\overline{D}$-solutions for more general equations.

5 Some examples of solutions

5.1 Representation by optimal control theory

Let us recall the representation formula of viscosity solutions by *optimal control theory*. (See for instance [19].) We consider the following *state equation*.

$$X'(s) = f(X(s), \alpha(s)) \quad \text{in } (0, t), \qquad X(0) = x. \tag{5.1}$$

Here the unknown is $X : [0, t] \to \mathbf{R}^n$ and

- $x \in \mathbf{R}^n$ is a given initial state and $t \in [0, T]$ is a terminal time.
- $A \subset \mathbf{R}^m$ is a compact control set and $\alpha \in \mathcal{A} := \{\alpha : [0, T] \to A, \text{ measurable}\}$ is a control.
- $f = f(x, a) : \mathbf{R}^n \times A \to \mathbf{R}^n$ is a given bounded and continuous function. Moreover $f(x, a)$ is Lipschitz continuous in x uniformly in a, that is $\mathrm{Lip}_x[f] < \infty$.

As for this ODE there exists for each $\alpha \in \mathcal{A}$ a unique Lipschitz continuous solution $X(s)$ which satisfies the first equation of (5.1) a.e. $s \in (0,t)$. Let us write $X(s) = X^\alpha(s) = X(s;\alpha,x,t)$ to denote the solution. Since $|X^\alpha(s_1) - X^\alpha(s_2)| = |\int_{s_2}^{s_1} f(X^\alpha(s),\alpha(s))ds| \leqq \|f\|_{\mathbf{R}^n\times A} \cdot |s_1 - s_2|$ for each $\alpha \in \mathcal{A}$, the solutions X^α are Lipschitz continuous uniformly in α with the Lipschitz constant smaller than $\|f\|$.

Next, for given $(x,t) \in \mathbf{R}^n \times [0,T]$ and $\alpha \in \mathcal{A}$ we define a corresponding *cost functional* $C_{x,t}[\alpha]$ by

$$C_{x,t}[\alpha] := \int_0^t r(X^\alpha(s),\alpha(s))ds + u_0(X^\alpha(t)),$$

where

- $r = r(x,a) : \mathbf{R}^n \times A \to \mathbf{R}$ is a given bounded and continuous function. Moreover $r(x,a)$ is Lipschitz continuous in x uniformly in a, that is $\mathrm{Lip}_x[r] < \infty$.
- $u_0 \in BUC(\mathbf{R}^n)$.

We call the above r a *running cost function* while u_0 serves as a *terminal cost function.* Then the *value function* $u : \mathbf{R}^n \times [0,T] \to \mathbf{R}$ is defined by

$$u(x,t) := \sup_{\alpha\in\mathcal{A}} C_{x,t}[\alpha]. \tag{5.2}$$

We are able to prove that u is a viscosity solution of a Hamilton-Jacobi-Bellman equation.

Theorem 5.1 (a PDE for the value function)**.** *Let u be defined as above. Then u is a unique viscosity solution of the initial-value problem*

$$\text{(HJB)} \begin{cases} \partial_t u - \max\limits_{a\in A}\{\langle f(x,a),\nabla u\rangle + r(x,a)\} = 0 & \text{in } Q, \quad (5.3) \\ u|_{t=0} = u_0 & \text{in } \mathbf{R}^n. \quad (5.4) \end{cases}$$

Remark 5.2. When the value function is defined as the infimum of costs, namely

$$u(x,t) := \inf_{\alpha\in\mathcal{A}} C_{x,t}[\alpha], \tag{5.5}$$

u becomes a solution of the same equation as above except that the max is replaced by min.

Our goal is to extend the classical theory above for discontinuous equations. Now we study Hamiltonians written by the form $H(x,p) = -\max_{a\in A}\langle f(x,a),p\rangle - r(x)$ with $r \in BUSC(\mathbf{R}^n)$. We hereafter assume that running costs are independent of the control variable a. Recall that as Example 3.2 and Proposition 3.7 we are able to construct an envelope solution by regularizing r from above to get r^ε (the sup-convolution method enables us to do that) and taking the infimum of solutions of the approximate problems. That means we take

$$u^\varepsilon(x,t) := \sup_{\alpha\in\mathcal{A}} C^\varepsilon_{x,t}[\alpha] \quad \text{with } C^\varepsilon_{x,t}[\alpha] = \int_0^t r^\varepsilon(X^\alpha(s))ds + u_0(X^\alpha(t)), \tag{5.6}$$

$$u(x,t) := \lim_{\varepsilon\downarrow 0} u^\varepsilon(x,t) = \inf_{\varepsilon>0} u^\varepsilon(x,t), \tag{5.7}$$

and prove that u is an envelope solution. On the other hand, since upper semi-continuous functions are integrable, it is possible to define a cost and value function for our original r which is not necessarily continuous, that is

$$v(x,t) := \sup_{\alpha \in \mathcal{A}} C_{x,t}[\alpha] \quad \text{with } C_{x,t}[\alpha] = \int_0^t r(X^\alpha(s))ds + u_0(X^\alpha(t)). \tag{5.8}$$

What is the relationship between this v of the discontinuous problem and u via the approximation (5.7)? Since problems including no ε-perturbation can be directly handled, if $u = v$, it characterizes the limit of u^ε. (See Figure 8.)

$$\begin{tikzcd}
r \arrow[r, "\text{approximation}"] \arrow[d, "\text{control}"'] & r^\varepsilon \arrow[d, "\text{control}"] \\
\underset{?}{v = u} & u^\varepsilon \arrow[l, "\varepsilon \downarrow 0"']
\end{tikzcd}$$

Figure 8: Control theory for a discontinuous running cost. The commutativity of this diagram is a problem.

In Example 3.2 we deduced that $u(x,t) = \lim_{\varepsilon \downarrow 0} u^\varepsilon(x,t) = c(t - |x|)_+$ by regarding the Hamiltonian as $H(x,p) = -\max_{a \in \overline{B}_1(0)} \langle a, p \rangle - cI(x)$. On the other hand, as for discontinuous case $v(x,t) = \sup_{\alpha \in \mathcal{A}} \int_0^t cI(X^\alpha(s))ds$, for each $x \in \mathbf{R}^n$ the optimal control is still the one that leads to a straight trajectory before it comes to the origin and stays there after that moment. Therefore we conclude that $v(x,t) = c(t-|x|)_+$, and so $u = v$. However, situations are different for another compact set A. For example if the control set A' is taken as S^{n-1}, the resulting Hamiltonian is the same as $H(x,p) = -\max_{a \in S^{n-1}} \langle a, p \rangle - cI(x)$. However, since X^α moves at a velocity of 1 all the time for each control α, it cannot stay at the origin. Hence we conclude that $v \equiv 0$.

We here give one sufficient condition for guaranteeing $u = v$.

Lemma 5.3 (controllability)**.** *Let $r \in BUSC(\mathbf{R}^n)$, $r^\varepsilon \in BLip(\mathbf{R}^n)$ $(\varepsilon > 0)$ and $r^\varepsilon \downarrow r$ in $\mathbf{R}^n$ pointwise, and define u^ε, u and v by* (5.6)–(5.8)*. Assume furthermore that*

(A1) *there exists a measurable function $\theta : \mathbf{R}^n \times \overline{B}_{\|f\|}(0) \to A$ such that $p = f(x, \theta(x,p))$ for all $(x,p) \in \mathbf{R}^n \times \overline{B}_{\|f\|}(0)$,*

where $\|f\| = \sup_{\mathbf{R}^n \times A} |f|$. Then $u = v$.

Proof. 1. We find that $v \leqq u$ because $C_{x,t}[\cdot] \leqq C^\varepsilon_{x,t}[\cdot]$ for all $\varepsilon > 0$. It remains to prove $u \leqq v$. Fix $(x,t) \in Q$. For each $\varepsilon > 0$ there is some $\alpha^\varepsilon \in \mathcal{A}$ such that $u^\varepsilon(x,t) - \varepsilon \leqq C^\varepsilon_{x,t}[\alpha^\varepsilon]$. Set $X^\varepsilon(s) := X(s; \alpha^\varepsilon, x, t)$, then one can easily check that the family $\{X^\varepsilon\}_{\varepsilon > 0} \subset C[0,t]$ is equicontinuous and uniformly bounded by using Lipschitz continuities of X^α. Consequently Ascoli-Arzelà theorem ensures that there exists a subseqence $\{X^{\varepsilon(j)}\}_{j \in \mathbf{N}}$ such that $X^{\varepsilon(j)}$ uniformly converges to some $\bar{X} \in C[0,t]$ as $j \to \infty$. The estimate $\mathrm{Lip}[X^\alpha] \leqq \|f\|$ $(\forall \alpha)$ implies

$\mathrm{Lip}[\bar{X}] \leqq \|f\|$, and so $\bar{X}$ is a.e. differentiable and $\bar{X}'(s) \in \overline{B}_{\|f\|}(0)$. Therefore by setting $\bar{\alpha}(s) := \theta(\bar{X}(s), \bar{X}'(s))$ we have $\bar{X}'(s) = f(\bar{X}(s), \bar{\alpha}(s))$ a.e. s, and also $\bar{X}(s) = X(s; \bar{\alpha}, x, t)$.

2. Fix $d > 0$. If $0 < \varepsilon(j) < d$, then we have

$$u^{\varepsilon(j)}(x,t) \leqq \varepsilon(j) + C^{\varepsilon(j)}_{x,t}[\alpha^{\varepsilon(j)}] \leqq \varepsilon(j) + C^{d}_{x,t}[\alpha^{\varepsilon(j)}]$$

and

$$\begin{aligned}\lim_{j\to\infty} C^{d}_{x,t}[\alpha^{\varepsilon(j)}] &= \lim_{j\to\infty} \left\{ \int_0^t r^d(X^{\varepsilon(j)}(s))ds + u_0(X^{\varepsilon(j)}(t)) \right\} \\ &= \int_0^t r^d(\bar{X}(s))ds + u_0(\bar{X}(t)).\end{aligned}$$

Hence it follows that

$$u(x,t) \leqq \int_0^t r^d(\bar{X}(s))ds + u_0(\bar{X}(t)).$$

Sending $d \downarrow 0$, we obtain by monotone convergence theorem

$$u(x,t) \leqq \int_0^t r(\bar{X}(s))ds + u_0(\bar{X}(t)) = C_{x,t}[\bar{\alpha}] \leqq v(x,t),$$

which completes the proof. □

Remark 5.4. To prove $u = v$ in the case that value functions are defined by (5.5), there is no need to assume (A1). It suffices to show $u \leqq v$ again. Take a minimizing sequence $\{\alpha_m\}$ of $v(x,t)$ (, i.e., $\lim_{m\to\infty} C_{x,t}[\alpha_m] = v(x,t)$), and then $u^\varepsilon(x,t) \leqq C^\varepsilon_{x,t}[\alpha_m]$ holds for all ε and m. Letting $\varepsilon \downarrow 0$, we see that $u(x,t) \leqq C_{x,t}[\alpha_m]$ by monotone convergence theorem. Finally send $m \to \infty$.

Let us calculate some examples of solutions by applying Lemma 5.3 or Remark 5.4.

At first we consider the case that $H(x,p) = -|p| - r(x) = -\max_{a\in\overline{B}_1(0)}\langle a, p\rangle - r(x)$ with $r \in BUSC(\mathbf{R}^n)$. Then (A1) is satisfied by taking $\theta(x,p) = p$, and so Lemma 5.3 guarantees that v defined by

$$v(x,t) := \sup_{\alpha\in\mathcal{A}} \left\{ \int_0^t r(X^\alpha(s))ds + u_0(X^\alpha(t)) \right\}$$

is the unique envelope solution.

Example 5.5. Let us consider the case of (1.14) and $u_0 \equiv 0$. In this case, for each $x \in \mathbf{R}^n$ the optimal control forces the state to move straight towards the nearest point in S and to stop moving after the arrival. Therefore we conclude that

$$v(x,t) = c(t - \mathrm{dist}(x,S))_+.$$

The solution in the case that $S = [-1,1]$ and $c = 1$ is given in Figure 9 (the left).

Example 5.6. Let us consider the case of (1.12) and $u_0 \equiv 0$. In this case, since we have obtained the optimal control for the case $r(x) = c_j I(x - a_j)$ for every j $(= 1, 2, \ldots, N)$, we only need to pick up the maximum of them. Hence we have

$$v(x,t) = \max_{j=1}^{N} c_j(t - |x - a_j|)_+.$$

The solution in the case that $a_1 = 1, v_1 = 1, a_2 = -1, v_2 = 1/3$ is given in Figure 9 (the right).

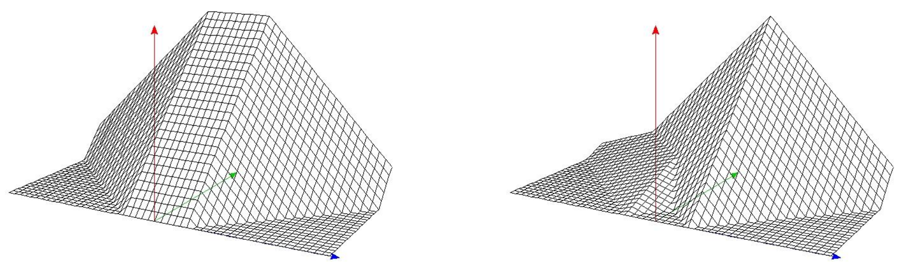

Figure 9: The envelope solution of $\partial_t u - |\nabla u| = \chi_{[-1,1]}(x)$, $u_0 \equiv 0$ (the left) and that of $\partial_t u - |\nabla u| = I(x+1)/3 + I(x-1)$, $u_0 \equiv 0$ (the right).

Example 5.7. Let us consider the case that $r(x) = cI(x)$ $(c > 0)$ with a general initial condition $u_0 \in BUC(\mathbf{R}^n)$. In this case, for each $x \in \mathbf{R}^n$ all of the controls can be categorized into two types. One type is to force the state to approach the origin. The other type results in trajectories without passing the origin. The optimal value for the former type is

$$\max_{s \in [0, t-|x|]} \left\{ cs + \max_{\overline{B}_{t-|x|-s}(0)} u_0 \right\} =: V(x,t)$$

provided that $t \geqq |x|$ while

$$\max_{\overline{B}_t(x)} u_0$$

is the maximal value for the latter type. Thus we conclude that

$$v(x,t) = \begin{cases} \max_{\overline{B}_t(x)} u_0 & (t \leqq |x|), \\ \max\left[\max_{\overline{B}_t(x)} u_0,\ V(x,t)\right] & (t \geqq |x|). \end{cases}$$

We will make this formula simpler by imposing some conditions on u_0. Assume that $u_0(0) = 0$ hereafter.

[1] The case that $u_0(x) \leqq c|x|$ in $\mathbf{R}^n$. Since $V(x,t) = c(t - |x|)$ $(s = t - |x|)$, we have

$$v(x,t) = \begin{cases} \max_{\overline{B}_t(x)} u_0 & (t \leqq |x|), \\ \max\left[\max_{\overline{B}_t(x)} u_0,\ c(t-|x|)\right] & (t \geqq |x|). \end{cases}$$

In particular, we see $u(0,t) = ct$ for all $t \in (0,T)$ because $\max_{\overline{B}_t(0)} u_0 \leqq ct$.

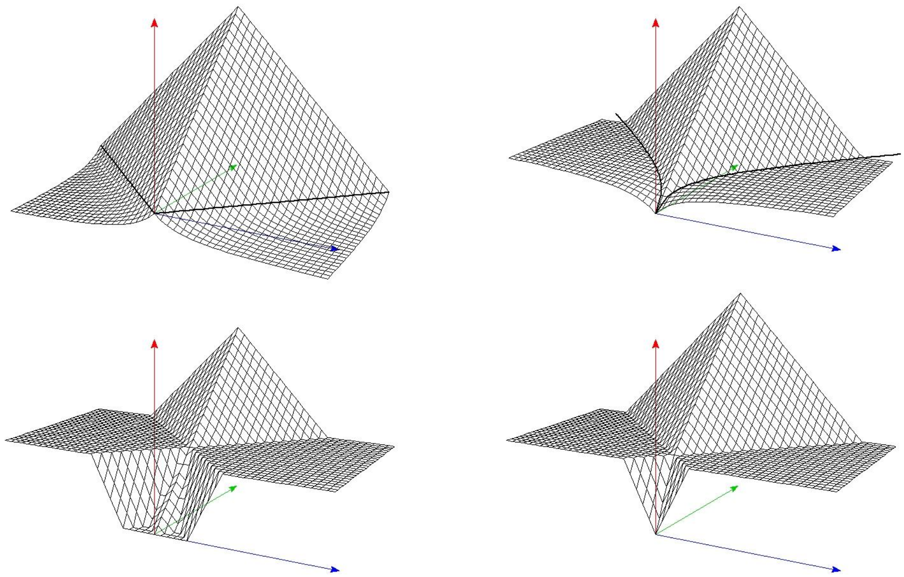

Figure 10: The envelope solutions of $\partial_t u - |\nabla u| = I(x)$ under several initial data u_0. The upper left is the case that $u_0(x) = -|x|/(1+|x|)$ and the solid curve is $(x,t,u) = (s,|s|,0)$. The upper right is the case that $u_0(x) = |x|/(1+|x|)$ and the solid curve is $(s, \sqrt{s^2+2|s|}, \sqrt{s^2+2|s|} - s)$. The lower left is the case that $u_0(x) = 2\min\{(|x|-1)_+,\ 1\}$. The lower right is the case that $u_0(x) = 2\min\{|x|,\ 1\}$. Each function is the unique envelope solution.

(a) If $u_0(x) \leqq 0$ in $\mathbf{R}^n$, then $v(x,t) = c(t-|x|)$ for $t \geqq |x|$. The solution for $c=1$ and $u_0(x) = -|x|/(1+|x|)$ is given in Figure 10 (the upper left).

(b) If $u_0(x) < c|x|$ in $\mathbf{R}^n \setminus \{0\}$, then for all $\hat{t} \in (0,T)$ we have $v(x,t) = c(t-|x|)$ in some open neighborhood of $(0,\hat{t}) \in Q$ because $\max_{\overline{B}_t(0)} u_0 < ct$. The solution for $c=1$ and $u_0(x) = |x|/(1+|x|)$ is given in Figure 10 (the upper right).

(c) If there is some $\hat{x} \neq 0$ such that $u_0(\hat{x}) = c|\hat{x}|$, it is unable to take the open neighborhood described in (b) at $(0,|\hat{x}|)$. The solution for $c=1$ and $u_0(x) = 2\min\{(|x|-1)_+,\ 1\}$ is given in Figure 10 (the lower left), where $\hat{x} = 2$.

[2] The case that $u_0(x) \not\leqq c|x|$ in $\mathbf{R}^n$. We assume that u_0 has the form $u_0(x) = b(|x|)$ and that $b(\rho_2) - b(\rho_1) > c(\rho_2 - \rho_1)$ $(0 \leqq \rho_1 < \rho_2 \leqq R)$, $b(\rho) = b(R)$ $(\rho \geqq R)$ for some $R > 0$. Note that we have $\max_{\overline{B}_t(x)} u_0 = b(|x|+t)$. Then we observe that $V(x,t) = b(t-|x|-s)$ $(s=0)$ for $0 \leqq t-|x| \leqq R$ and it is smaller than $b(|x|+t)$, and also $V(x,t) = c(t-|x|-R) + b(R)$ $(s = t-|x|-R)$ for $t-|x| \geqq R$ and it is bigger than $b(|x|+t) = b(R)$. Thus we conclude that

$$v(x,t) = \begin{cases} b(|x|+t) & (t \leqq |x|+R), \\ c(t-|x|-R) + b(R) & (t \geqq |x|+R). \end{cases}$$

It is seen that $v(x,t) \equiv b(R)$ if $-|x|+R \leqq t \leqq |x|+R$. In this case, there is no effect of the step source by time R on account of rapid growth of the initial data and v becomes flat at time R. The solution for $c=1$ and $b(\rho) = 2\min\{\rho,\ 1\}$ $(R=1)$ is given in Figure 10 (the lower right).

We next consider the case that $H(x,p) = |p| - r(x) = -\min_{a \in \overline{B}_1(0)} \langle a, p \rangle - r(x)$ with $r \in BUSC(\mathbf{R}^n)$, which describes the isotropic shrink at a velocity of 1. Then Remark 5.4 guarantees that v defined by

$$v(x,t) := \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^t r(X^\alpha(s))ds + u_0(X^\alpha(t)) \right\}$$

is the unique envelope solution.

Example 5.8. Let us consider the case of (1.12) with a general initial condition $u_0 \in BUC(\mathbf{R}^n)$. In this case, since for each $x \in \mathbf{R}^n$ the optimal control forces the corresponding state to go to the minimizer of u_0 on $\overline{B}_t(x)$ (and not to stay each a_j for a positive time), we have

$$v(x,t) = \min_{\overline{B}_t(x)} u_0.$$

This coincides with the solution of $\partial_t u + |\nabla u| = 0$, and hence we may think that there is no effect of the source term $r(x)$.

Example 5.9. Let us consider the case of (1.14) and $u_0 \equiv 0$. In this case, since for each $x \in \mathbf{R}^n$ the optimal control forces the corresponding state to leave S for the shortest time and stay in the outside of S after the exit, we have

$$v(x,t) = c \cdot \min\{t,\ \mathrm{dist}(x, S^c)\},$$

where S^c means a complementary set of S in $\mathbf{R}^n$. In particular, if S has no interior point, we see $v(x,t) = 0$, which reduces to a special case of Example 5.8. We also learn for a bounded S that $v(x,t) = c \cdot \mathrm{dist}(x, S^c)$ for every $t \geqq \sup_{x \in \mathbf{R}^n} \mathrm{dist}(x, S^c) \neq \infty$. The solution in the case that $S = [-1,1]$ and $c = 1$ is given in Figure 11.

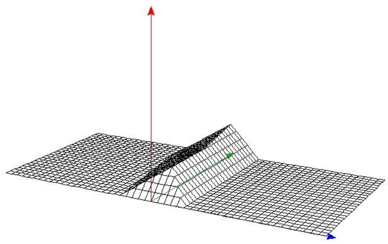

Figure 11: The envelope solution of $\partial_t u + |\nabla u| = \chi_{[-1,1]}(x)$, $u_0 \equiv 0$.

5.2 Solutions without coercivity assumption

In this subsection we focus on the equations of the form

$$\text{(HJ1c)} \begin{cases} \partial_t u - H_1(\nabla u) = cI(x) & \text{in } Q, \\ u|_{t=0} \equiv 0 & \text{in } \mathbf{R}^n, \end{cases} \tag{5.9}$$

i.e., $H(x,p) = -H_1(p) - cI(x)$ with $H_1 : \mathbf{R}^n \to \mathbf{R}$ and $c > 0$. We do not impose the coercivity assumption on H_1 here. Also, without loss of generality we may take

$$H_1(0) = 0; \tag{5.10}$$

if not, we replace $H_1(\nabla u)$ in (5.9) with $H_1(\nabla u) - H_1(0)$ and solve the new problem. For any solution $u(x,t)$ of the new one, $u(x,t) + H_1(0)t$ is a solution of the original (HJ1c).

The next proposition helps us to construct envelope supersolutions when the step source consists of a singleton.

Proposition 5.10 (construction of envelope supersolutions)**.** *Assume* (5.10). *Assume that $u : Q \to \mathbf{R}$ is bounded from below and satisfies the following three conditions.*

(i) $\tau + H(\hat{x}, p) \geqq 0$ *for all* $(\hat{x}, \hat{t}) \in (\mathbf{R}^n \setminus \{0\}) \times (0,T)$ *and* $(p, \tau) \in \overline{D}^- u_*(\hat{x}, \hat{t})$.

(ii) *u is continuous on* $\{0\} \times (0, T)$.

(iii) $u(0, t) = ct$ *in* $(0, T)$ *and* $u(x, t) \leqq ct$ *in* Q.

Then $u \in e.SUP(H)$.

Set $Q_* := (\mathbf{R}^n \setminus \{0\}) \times (0, T)$. We say that u is a *$\overline{D}$-viscosity supersolution in* Q_* (write $u \in \overline{D}\text{-}SUP(H)$ in Q_*) if u satisfies the condition (i) and is bounded from below.

Proof. For $\varepsilon > 0$ we define

$$u^\varepsilon(x, t) := \min\{u(x, t) + \varepsilon t,\ ct\}.$$

(See Figure 12.) Then, we deduce by Proposition 3.5 that $u^\varepsilon \in \overline{D}\text{-}SUP(H)$ since we have the following three facts.

- $u(x, t) + \varepsilon t \in \overline{D}\text{-}SUP(H)$ in Q_*.
- $ct \in \overline{D}\text{-}SUP(H)$.
- $u(x, t) + \varepsilon t > ct$ in some open neighborhood of $(0, \hat{t}) \in Q$, where $\hat{t} \in (0, T)$.

Also, it is clear that $u = \inf_{\varepsilon > 0} u^\varepsilon$. We thus conclude that $u \in e.SUP(H)$. $\square$

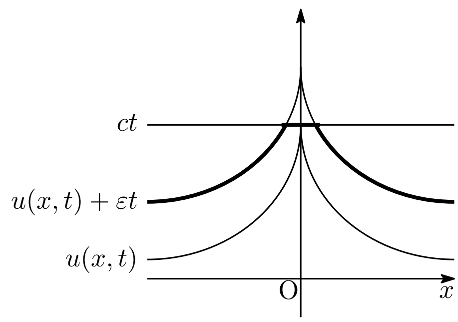

Figure 12: The definition of u^ε.

Let $a, b > 0$ and define a "cone-shaped" function $W_{a,b}(x, t) := (at - b|x|)_+$. Before describing the first existence result of (HJ1c), which claims that $W_{c,b}$ is an envelope solution for suitable b and H_1, we give simple observations about sub- and superdifferentials of the cone-shaped functions.

(CS) Let $(p_+, \tau_+) \in D^+ W_{a,b}(\hat{x}, \hat{t})$ and $(p_-, \tau_-) \in D^- W_{a,b}(\hat{x}, \hat{t})$ for $(\hat{x}, \hat{t}) \in Q$.

(1) If $a\hat{t} < b|\hat{x}|$, we have $p_\pm = 0$ and $\tau_\pm = 0$.

(2) If $a\hat{t} > b|\hat{x}|$ and $\hat{x} \neq 0$, we have $|p_\pm| = b$ and $\tau_\pm = a$.

(3) If $a\hat{t} = b|\hat{x}|$, we have $|p_-| \leqq b$, $0 \leqq \tau_- \leqq a$ and $b\tau_- = a|p_-|$. Also, $D^+ W_{a,b}(\hat{x}, \hat{t}) = \emptyset$.

(4) If $\hat{x} = 0$, we have $|p_+| \leqq b$ and $\tau_+ = a$. Also, $D^- W_{a,b}(\hat{x}, \hat{t}) = \emptyset$.

Proposition 5.11. *Assume* (5.10) *and let* $b > 0$. *Assume that* H_1 *satisfies the following.*

$$0 \leqq H_1(p) \leqq \frac{c}{b}|p| \quad \textit{for all } p \in \mathbf{R}^n \textit{ such that } |p| \leqq b. \tag{5.11}$$

$$H_1(p) = c \quad \textit{for all } p \in \mathbf{R}^n \textit{ such that } |p| = b. \tag{5.12}$$

Then $W_{c,b}(x,t) = (ct - b|x|)_+$ *is an envelope solution of* (HJ1c).

Proof. In view of (5.11), (5.12) and (CS) we see that $W_{c,b}$ is a $\overline{D}$-subsolution of (5.9) and is a $\overline{D}$-supersolution of (5.9) in Q_*. Since $W_{c,b}$ fulfills the assumptions in Proposition 5.10, we conclude that $W_{c,b}$ is an envelope solution of (HJ1c). □

Of course, if H_1 fulfills the regularity assumption (H_p) required in our comparison principles, it follows that the Lipschitz continuous function $W_{c,b}$ is a unique envelope solution.

Example 5.12. We consider the case that $H_1(p) = |p|^\alpha$ with $\alpha > 1$. Then, the conditions (5.11) and (5.12) are satisfied if we take $b = c^{1/\alpha}$. Hence we see that $W_{c,b}(x,t) = (ct - c^{1/\alpha}|x|)_+$ is an envelope solution of (HJ1c).

Example 5.13. We next consider the case that $H_1(p) = \sqrt{1 + |p|^2} - 1$. Then, the conditions (5.11) and (5.12) are satisfied if we take $b = \sqrt{c^2 + 2c}$. Hence we see that $W_{c,b}(x,t) = (ct - \sqrt{c^2 + 2c}|x|)_+$ is a unique envelope solution of (HJ1c). Equivalently, the function $u(x,t) = (ct - \sqrt{c^2 + 2c}|x|)_+ + t$ is a unique envelope solution of

$$\begin{cases} \partial_t u - \sqrt{1 + |\nabla u|^2} = cI(x) & \text{in } Q, \\ u|_{t=0} \equiv 0 & \text{in } \mathbf{R}^n. \end{cases}$$

Unfortunately, Proposition 5.11 does not include the case that H_1 is "spokewisely concave" from the origin, that is, the case when

$$H_1(p) = h(|p|), \quad \text{where } h : [0,\infty) \to \mathbf{R} \text{ is strictly concave.} \tag{5.13}$$

Here we say h is *strictly concave* if $h((1-\lambda)x + \lambda y) > (1-\lambda)h(x) + \lambda h(y)$ for every $\lambda \in (0,1)$ and $x, y \in [0,\infty)$ with $x \neq y$. For the purpose of finding envelope solutions of (HJ1c) in such cases we further assume that

$$h \in C^2(0,\infty) \cap C[0,\infty), \quad h \text{ is strictly increasing on } [0,\infty). \tag{5.14}$$

Then, it is easily seen by (CS) that

$$W_{a,h^{-1}(a)}(x,t) = (at - h^{-1}(a)|x|)_+$$

is a $\overline{D}$-subsolution of (HJ1c) for each $a \in (0,c)$ with $c \leqq \|h\|_{[0,\infty)}$. Besides, it turns out that these supremum

$$U_c(x,t) = \sup_{a \in (0,c)} (at - h^{-1}(a)|x|) \quad (c \leqq \|h\|)$$

becomes an envelope solution of (HJ1c).

Proposition 5.14. *Assume* (5.10), (5.13) *and* (5.14).

(1) *Assume* $c < \|h\|$. *Then* U_c *is a unique envelope solution of* (HJ1c).

(2) *Assume* $c = \|h\|$. *Then* $U_{\|h\|}$ *is a* $\overline{D}$*-solution of* (HJ1c) *and a unique envelope solution.*

(3) *Assume* $c > \|h\|$. *Then* $U_{\|h\|} + ktI(x)$ *are* $\overline{D}$*-solutions of* (HJ1c) *for all* $k \in [0, c - \|h\|]$.

We remark that the assumption (H_p) is satisfied because of the concavity of h; indeed, we now have $|H_1(p) - H_1(q)| \leqq h(|p-q|)$. Hence the uniqueness in (1) follows from the Lipschitz continuity of U_c.

Proof. At first, it is obvious that U_c is a standard subsolution due to the stability under supremum. In order to prove that U_c is an envelope supersolution we utilize Proposition 5.10. Notice that U_c is rewritten as

$$U_c(x,t) = \begin{cases} ct - b|x| & (|x| \leqq h'(b)t), \\ h\left(g\left(\dfrac{|x|}{t}\right)\right)t - g\left(\dfrac{|x|}{t}\right)|x| & (h'(b)t < |x| < h'(0)t), \\ 0 & (h'(0)t \leqq |x|) \end{cases} \tag{5.15}$$

by a direct calculation. Here $h'(0)$ means the right derivative at 0 and possibly equals $+\infty$. Also, we write $b = h^{-1}(c)$ and $g = (h')^{-1}$ for the inverse function of h'. If $c = \|h\|$, we read (5.15) as

$$U_c(x,t) = \begin{cases} ct & (x = 0), \\ h\left(g\left(\dfrac{|x|}{t}\right)\right)t - g\left(\dfrac{|x|}{t}\right)|x| & (0 < |x| < h'(0)t), \\ 0 & (h'(0)t \leqq |x|). \end{cases} \tag{5.16}$$

By the formula (5.15) we find $U_c \in C^1(Q_*)$ and its derivatives are as follows:

$$\partial_t U_c(x,t) = \begin{cases} c & (|x| \leqq h'(b)t), \\ h\left(g\left(\dfrac{|x|}{t}\right)\right) & (h'(b)t < |x| < h'(0)t), \\ 0 & (h'(0)t \leqq |x|). \end{cases}$$

$$\nabla U_c(x,t) = \begin{cases} -b\dfrac{x}{|x|} & (0 < |x| \leqq h'(b)t), \\ -g\left(\dfrac{|x|}{t}\right)\dfrac{x}{|x|} & (h'(b)t < |x| < h'(0)t), \\ 0 & (h'(0)t \leqq |x|). \end{cases}$$

Thus we deduce that

$$\partial_t U_c(x,t) - h(|\nabla U_c(x,t)|) = 0 \quad \text{for all } (x,t) \in Q_*,$$

and hence U_c satisfies the condition (i) in Proposition 5.10. Since the conditions (ii) and (iii) are clear, we conclude that U_c is an envelope supersolution. If

$c = \|h\|$ in particular, we see that U_c is also a $\overline{D}$-supersolution of (HJ1c) since $\overline{D}^- U_c(0,t) = \emptyset$.

(2) To show the uniqueness of U_c as an envelope solution we use the idea of Remark 3.14. It is seen that U_c is the minimal envelope solution because $U_c = \sup_{a\in(0,c)} W_{a,h^{-1}(a)}$ and $W_{a,h^{-1}(a)} \in SUB(H,0) \cap BLip(Q)$. Next, let $u^\varepsilon(x,t) := \min\{U_c(x,t)+\varepsilon t,\ ct\}$ for $\varepsilon > 0$. As we showed in the proof of Proposition 5.10, it turns out that $u^\varepsilon \in \overline{D}\text{-}SUP(H)$. We also have $u^\varepsilon \in BLip(Q)$ and $U_c = \inf_{\varepsilon>0} u^\varepsilon$, and therefore U_c is the maximal envelope solution.

(3) This claim follows from (2) and the fact that $(U_{\|h\|} + ktI(x))_* = U_{\|h\|}$. $\square$

Example 5.15. We consider the case that $H_1(p) = |p|/(1+|p|)$, which is a non-coercive Hamiltonian. Then, by substituting

$$b = \frac{c}{1-c}, \quad h(r) = \frac{r}{1+r}, \quad g(r) = \frac{1-\sqrt{r}}{\sqrt{r}}$$

into (5.15) we see that

$$U_c(x,t) = \begin{cases} ct - \dfrac{c}{1-c}|x| & (|x| \leqq (1-c)^2 t), \\ \{(\sqrt{t} - \sqrt{|x|})_+\}^2 & (|x| \geqq (1-c)^2 t) \end{cases}$$

is a unique envelope solution of (HJ1c) when $c \leqq 1 = \|h\|$, and in particular $U_1(x,t) = \{(\sqrt{t} - \sqrt{|x|})_+\}^2$ is a $\overline{D}$-solution of (HJ1c) with $c = 1$. In the case when $c > 1$, functions $U_1(x,t) + ktI(x)$ are all $\overline{D}$-solutions of (HJ1c) for $k \in [0, c-1]$. (See Figure 13.) Also, we see that the Lipschitz continuity preserving property breaks down for $c \geqq 1$ since U_1 is not Lipschitz continuous in Q.

Example 5.16. We next consider the case that $H_1(p) = |p|^\alpha$ with $0 < \alpha < 1$. Then, by substituting

$$b = c^{1/\alpha}, \quad h(r) = r^\alpha, \quad g(r) = \left(\frac{\alpha}{r}\right)^{1/(1-\alpha)}$$

into (5.15) we see that

$$U_c(x,t) = \begin{cases} ct - c^{1/\alpha}|x| & (|x| \leqq \alpha t / c^{(1-\alpha)/\alpha}), \\ \left(\dfrac{\alpha t}{|x|}\right)^{\alpha/(1-\alpha)} t - \left(\dfrac{\alpha t}{|x|}\right)^{1/(1-\alpha)} |x| & (|x| \geqq \alpha t / c^{(1-\alpha)/\alpha}) \end{cases}$$

is a unique envelope solution of (HJ1c) for any $c > 0$. This formula means that the present equation (HJ1c) has a some kind of infinite propagation property for a step source because $u > 0$ in Q. (See Figure 14.)

The formula (5.15) also applies to Hamiltonians with the hyperbolic tangent form ([42])

$$H(x,p) = -|p| \tanh \frac{1}{|p|} - cI(x),$$

but it is complicated to calculate the inverse function $g = (h')^{-1}$ in this case.

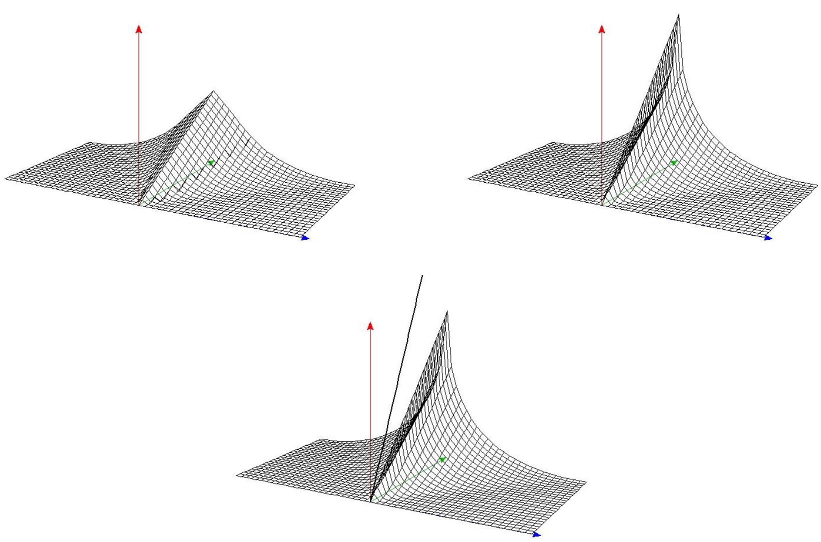

Figure 13: Solutions of $\partial_t u - |\nabla u|/(1+|\nabla u|) = cI(x)$, $u_0 \equiv 0$. The envelope solution for $c = 1/2$ (the upper left), the $\overline{D}$-solution for $c = 1$(the upper right) and one of the $\overline{D}$-solutions for $c = 2$ (the bottom).

5.3 Remark on relation to Dirichlet boundary problems

Let u be the unique envelope solution of the problem with a single step source:

$$\begin{cases} \partial_t u - |\nabla u| = cI(x) & \text{in } \mathbf{R}^n \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbf{R}^n. \end{cases}$$

We study in this subsection whether u is also a solution of the Dirichlet boundary problem:

$$\text{(Di)} \begin{cases} \partial_t u - |\nabla u| = 0 & \text{in } (\mathbf{R}^n \setminus \{0\}) \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbf{R}^n, \\ u(0,t) = ct & \text{in } (0,T). \end{cases}$$

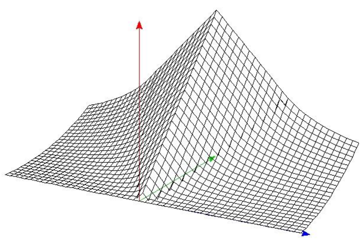

Figure 14: The envelope solution of $\partial_t u - \sqrt{|\nabla u|} = I(x)$, $u_0 \equiv 0$.

To simplify the argument we assume $u_0(0) = 0$. We first recall the following facts about u from the observation in Example 5.7.

(1) $u(0,t) \geqq ct$ for all $t \in (0,T)$.

(2) $u(0,t) = ct$ for all $t \in (0,T)$ provided that $u_0(x) \leqq c|x|$ in $\mathbf{R}^n$.

(3) If

$$u_0(x) = 2c \min\{|x|,\ 1\}, \tag{5.17}$$

whose slope is larger than c near the origin, then the unique envelope solution is

$$v(x,t) = \begin{cases} 2c\min\{|x|+t,\ 1\} & (t \leqq 1), \\ c(t-|x|-1)_+ + 2c & (t \geqq 1). \end{cases}$$

In particular, we have $v(0,t) = \min\{2ct,\ c(t+1)\} > ct$.

We see by (1) that u is always a supersolution of (Di). Also, by virtue of (2), if $u_0(x) \leqq c|x|$ in $\mathbf{R}^n$, then u is a viscosity solution of (Di) which indeed attains the boundary condition. What happens in the case that $u_0(x) \nleqq c|x|$ in $\mathbf{R}^n$? Unfortunately, we cannot expect that u is a subsolution on the boundary even in the weak sense, i.e., $u(0,t) \leqq ct$ or $\tau - |p| \leqq 0$ whenever $(p,\tau) \in D^+_{\mathbf{R}^n} u(0,t)$. In fact, when the initial data is given by (5.17), we have $v(0,2) > 2c$ and $\tau - |p| > 0$ for $(p,\tau) = (0,c) \in D^+ v(0,2)$. Instead of v, if we set

$$v^\tau(x,t) = \begin{cases} v(x,t) & (t \leqq 1), \\ 2c & (1 \leqq t \leqq 1+\tau), \\ v(x,t-\tau) & (t \geqq 1+\tau), \end{cases}$$

then each v^τ $(\tau \geqq 1)$ becomes a solution of (Di) with (5.17) in the weak sense. One can interpret the constant τ as a "waiting time". The Dirichlet problem (Di) forces its solution to stop the growth until it satisfies the Dirichlet boundary data at the origin.

As an another type of the boundary condition in the weak sense, one may think of the dynamic boundary condition ([17]); namely

$$\text{(Dy)} \begin{cases} \partial_t u - |\nabla u| = 0 & \text{in } (\mathbf{R}^n \setminus \{0\}) \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbf{R}^n, \\ \partial_t u(0,t) = c & \text{in } (0,T). \end{cases}$$

However, one cannot expect the uniqueness of solutions for (Dy) as well. Indeed, each v^τ $(\tau \geqq 0)$ is a solution of (Dy) with (5.17).

6 Large time behavior

6.1 Self-similar solution

We study the large time behavior of solutions to (HJ1). Hereafter we use a notation (HJ1; r, u_0) to represent the source term r and the initial data u_0 of

(HJ1). Our goal is to prove that a rescaled function of a solution to (HJ1) converges to a self-similar solution of the associated problem, which is (HJ1; $cI, 0$) if r has a compact support. In the case where r is periodic, we show that (HJ1; $c, 0$) with a constant source term gives the associated problem. Throughout our arguments we assume $u_0 \in BLip(\mathbf{R}^n)$ and

$$\begin{cases} \text{(i)} & H_1 \text{ is coercive and uniformly continuous in } \mathbf{R}^n. \\ \text{(ii)} & r \in BUSC(\mathbf{R}^n) \text{ and } r \geqq 0 \text{ in } \mathbf{R}^n. \end{cases} \tag{6.1}$$

These conditions guarantee that there exists a unique envelope solution of (HJ1) which is bounded in $\mathbf{R}^n \times [0, T)$ for all $T > 0$. Moreover, the unique solution is Lipschitz continuous in $\mathbf{R}^n \times [0, \infty)$ since Proposition 3.17 implies that the Lipschitz constant of the solution in $\mathbf{R}^n \times [0, T)$ does not depend on T.

Definition 6.1. Let $u : \mathbf{R}^n \times [0, \infty) \to \mathbf{R}$. For $\lambda > 0$ we define a *rescaled function* $u_{(\lambda)} : \mathbf{R}^n \times [0, \infty) \to \mathbf{R}$ of u by

$$u_{(\lambda)}(x, t) := \frac{1}{\lambda} u(\lambda x, \lambda t).$$

If $u = u_{(\lambda)}$ in $\mathbf{R}^n \times [0, \infty)$ for all $\lambda > 0$, we say u is *self-similar.*

When u is self-similar, letting $\lambda = 1/t$, we see $u(x, t) = u(\lambda x, \lambda t)/\lambda = tu(x/t, 1)$. Thus, setting $U(\xi) := u(\xi, 1)$, we get

$$u(x, t) = tU\left(\frac{x}{t}\right).$$

The function U is called a *profile function* of u. It is easy to see that the function (1.6), which is the unique envelope solution of (1.5) with the zero initial data, is self-similar in the sense of Definition 6.1 and that its profile function is

$$U(\xi) = c(1 - |\xi|)_+. \tag{6.2}$$

Self-similar solutions exist not only for the typical problem (1.5) but also for more general equations with a *positively 0-homogeneous* source term r, i.e., $r(\lambda x) = r(x)$ for all $\lambda > 0$ and $x \in \mathbf{R}^n$. The function $cI(x)$ is a trivial example of such positively 0-homogeneous functions.

Proposition 6.2. *Assume that H_1 and r satisfy* (6.1). *Assume that r is positively 0-homogeneous. Then the unique envelope solution of* (HJ1; $r, 0$) *is self-similar.*

The proof uses the uniqueness result. When u is an envelope solution of (HJ1; $r, 0$), noting that $\partial_t u_{(\lambda)}(x, t) = \partial_t u(\lambda x, \lambda t)$ and $\nabla u_{(\lambda)}(x, t) = \nabla u(\lambda x, \lambda t)$, we compute

$$\partial_t u_{(\lambda)}(x, t) - H_1(\nabla u_{(\lambda)}(x, t)) = \partial_t u(\lambda x, \lambda t) - H_1(\nabla u(\lambda x, \lambda t)) = r(\lambda x) = r(x)$$

as long as u is smooth. Thus by the uniqueness of solutions we obtain $u = u_{(\lambda)}$. In the general case where u is not smooth, we take smooth test functions in the definition of viscosity solutions and apply a similar calculation to them.

6.2 Source terms with compact support

Before we show a general result on the large time behavior, we present two simple examples of solutions which converge to the self-similar solution (1.6) under rescaling.

Example 6.3. We study (HJ1; $r, 0$) with $H_1(p) = |p|$ and r of the form (1.12). As we discussed in Example 5.6, the unique envelope solution is given by the formula 1.13. Now, let us compute the scaling limit of this solution. For $\lambda > 0$ we have

$$u_{(\lambda)}(x,t) = \frac{1}{\lambda}u(\lambda x, \lambda t) = \frac{1}{\lambda}\max_{j=1}^{N} c_j(\lambda t - |\lambda x - a_j|)_+ = \max_{j=1}^{N} c_j \left(t - \left|x - \frac{a_j}{\lambda}\right|\right)_+ .$$

Set $c := \max_{j=1}^{N} c_j = c_J$. Then

$$u_{(\lambda)}(x,t) \geqq c_J \left(t - \left|x - \frac{a_J}{\lambda}\right|\right)_+ \to c(t - |x|)_+$$

uniformly as $\lambda \to \infty$, while

$$u_{(\lambda)}(x,t) \leqq \max_{j=1}^{N} c \left(t - \left|x - \frac{a_j}{\lambda}\right|\right)_+ \to c(t - |x|)_+$$

since each functions $c(t - |x - (a_j/\lambda)|)_+$ uniformly converges to $c(t - |x|)_+$. We therefore conclude that

$$u_{(\lambda)}(x,t) \to c(t - |x|)_+ \tag{6.3}$$

uniformly as $\lambda \to \infty$.

Example 6.4. Let us consider (HJ1; $r, 0$) with $H_1(p) = |p|$ and r of the form (1.14). We further assume that S is bounded. The unique envelope solution which was computed in Example 5.5 is

$$u(x,t) = c(t - \operatorname{dist}(x, S))_+.$$

(This formula is valid even if S is unbounded.) For $\lambda > 0$ we observe

$$u_{(\lambda)}(x,t) = \frac{1}{\lambda}u(\lambda x, \lambda t) = \frac{c}{\lambda}(\lambda t - \operatorname{dist}(\lambda x, S))_+ = c\left(t - \frac{1}{\lambda}\operatorname{dist}(\lambda x, S)\right)_+ .$$

We now choose $R > 0$ large so that $S \subset \overline{B}_R(0)$ to see

$$\frac{1}{\lambda}\operatorname{dist}(\lambda x, S) \geqq \frac{1}{\lambda}\operatorname{dist}(\lambda x, \overline{B}_R(0)) = \frac{1}{\lambda}(|\lambda x| - R)_+ = \left(|x| - \frac{R}{\lambda}\right)_+ \to |x|$$

uniformly as $\lambda \to \infty$. Also, for a fixed $z \in S$ we compute

$$\frac{1}{\lambda}\operatorname{dist}(\lambda x, S) \leqq \frac{1}{\lambda}\operatorname{dist}(\lambda x, \{z\}) = \frac{1}{\lambda}|\lambda x - z| = \left|x - \frac{z}{\lambda}\right| \to |x|.$$

Hence the same conclusion (6.3) as the previous example holds.

We now state our result on the asymptotic behavior in the case the support of r is compact.

Theorem 6.5 (Large time behavior for source terms with compact support). *Assume that* H_1 *and* r *satisfy* (6.1). *Assume that* $\operatorname{supp}(r)$ *is a compact set in* $\mathbf{R}^n$. *Let* u *and* v *be, respectively, the unique envelope solution of* (HJ1; r, u_0) *and* (HJ1; $cI, 0$) *with* $c = \max_{\mathbf{R}^n} r$. *Then* $u_{(\lambda)}$ *converges to* v *locally uniformly in* $\mathbf{R}^n \times [0, \infty)$ *as* $\lambda \to \infty$.

This theorem especially implies that (1.20) holds locally uniformly in $\mathbf{R}^n$. This is an asymptotic convergence to a profile function of v.

Proof. 1. We fist show the relaxed limits $\overline{u} := \limsup^*_{\lambda\to\infty} u_{(\lambda)}$ and $\underline{u} := \liminf_{*\lambda\to\infty} u_{(\lambda)}$ fulfill the zero initial data. Set $M_0 := \sup_{\mathbf{R}^n} |u_0|$. Then it is easy to see that functions $H_1(0)t - M_0$ and $(c + H_1(0))t + M_0$ are, respectively, a subsolution and a $\overline{D}$-supersolution of (HJ1; r, u_0). From the comparison principle it follows that $H_1(0)t - M_0 \leqq u(x,t) \leqq (c + H_1(0))t + M_0$, and hence

$$H_1(0)t - \frac{M_0}{\lambda} \leqq u_{(\lambda)}(x,t) \leqq (c + H_1(0))t + \frac{M_0}{\lambda}.$$

We take $\limsup^*_{\lambda\to\infty}$ and $\liminf_{*\lambda\to\infty}$ in the above inequalities to obtain

$$H_1(0)t \leqq \underline{u}(x,t) \leqq \overline{u}(x,t) \leqq (c + H_1(0))t.$$

This ensures that these two relaxed limits are real-valued in $\mathbf{R}^n \times [0, \infty)$ and

$$\lim_{(y,t)\to(x,0)} \underline{u}(y,t) = \underline{u}(x,0) = 0, \qquad \lim_{(y,t)\to(x,0)} \overline{u}(y,t) = \overline{u}(x,0) = 0$$

for all $x \in \mathbf{R}^n$.

2. Take $z \in \mathbf{R}^n$ as a maximum point of r. We define $\tilde{v}(x,t) := v(x - z, t) - M_0$. Then $\tilde{v}$ solves (HJ1; $cI(\cdot - z), u_0(\cdot - z) - M_0$). Since $cI(\cdot - z) \leqq r$ and $u_0(\cdot - z) - M_0 \leqq u_0$, our comparison principle implies $\tilde{v} \leqq u$. We thus have

$$\frac{1}{\lambda} v(\lambda x - z, \lambda t) - \frac{M_0}{\lambda} \leqq u_{(\lambda)}(x,t) \tag{6.4}$$

for $\lambda > 0$. Now, by the self-similarity of v

$$\frac{1}{\lambda} v(\lambda x - z, \lambda t) = \frac{1}{\lambda} v\left(\lambda\left(x - \frac{z}{\lambda}\right), \lambda t\right) = v\left(x - \frac{z}{\lambda}, t\right).$$

Taking $\liminf_{*\lambda\to\infty}$ in (6.4), we obtain

$$v \leqq \liminf_{\lambda\to\infty}{}_* u_{(\lambda)}. \tag{6.5}$$

Here we have used the continuity of v.

3. Define $g(x) := c\chi_{\overline{B}_R(0)}(x)$, where we choose $R > 0$ large so that $\operatorname{supp}(r) \subset \overline{B}_R(0)$. We let w be the unique envelope solution of (HJ1; $g, 0$). Since $cI \leqq g$, by the comparison principle we have $u \leqq g + M_0$. This implies $u_{(\lambda)} \leqq w_{(\lambda)} + (M_0/\lambda)$ and

$$\limsup_{\lambda\to\infty}{}^* u_{(\lambda)} \leqq \limsup_{\lambda\to\infty}{}^* w_{(\lambda)}. \tag{6.6}$$

We write $g_\lambda(x) = g(\lambda x)$. Then $\limsup^*_{\lambda\to\infty} g_\lambda = cI$ by the definition of g. Now $w_{(\lambda)}$ is a standard viscosity subsolution of $(\mathrm{HJ1}; g_\lambda, 0)$ since w is a viscosity subsolution of $(\mathrm{HJ1}; g, 0)$ in the standard sense. Thus, the stability result for subsolutions under the relaxed limit ([11, Lemma 6.1, Remark 6.3]) ensures that $\limsup^*_{\lambda\to\infty} w_{(\lambda)}$ is a subsolution of $(\mathrm{HJ1}; \limsup^*_{\lambda\to\infty} g_\lambda, 0) = (\mathrm{HJ1}; cI, 0)$. This relaxed limit of $w_{(\lambda)}$ exists for the same reason as in Step 1. Since v is a solution of $(\mathrm{HJ1}; cI, 0)$, it follows from the comparison principle that

$$\limsup_{\lambda\to\infty}{}^* w_{(\lambda)} \leqq v. \tag{6.7}$$

Finally, combining (6.5), (6.6) and (6.7), we conclude

$$v = \limsup_{\lambda\to\infty}{}^* u_{(\lambda)} = \liminf_{\lambda\to\infty}{}_* u_{(\lambda)},$$

which implies the locally uniform convergence of $u_{(\lambda)}$ to v. □

Example 6.6. We consider the large time behavior in the sense (1.23). Let $H_1(p) = |p|$. A direct computation implies that, for the solution (1.6) of $(\mathrm{HJ1}; cI, 0)$, we have

$$u(x,t) - (ct - c|x|) \to 0 \quad \text{as } t \to \infty$$

locally uniformly in $\mathbf{R}^n$. Thus the limit $ct - c|x|$ is different from our scaling limit, which is (1.6) itself. More generally, the solution (1.13) of $(\mathrm{HJ1}; r, 0)$ with (1.12) satisfies

$$u(x,t) - (ct - c|x - a_J|) \to 0 \quad \text{as } t \to \infty,$$

where $c := \max_{j=1}^N c_j = c_J$, locally uniformly in $\mathbf{R}^n$ provided that $c_j < c_J$ $(j \neq J)$. Therefore the scaling limit in (6.3) gives another function.

In the next example we discuss the scaling limit of solutions when r has a non-compact support.

Example 6.7. We study the source term r which is non-zero near finitely many half-lines starting from the origin, but assume that r attains its maximum at the origin. We set half-lines l_i $(i = 1, \dots, M)$ as $l_i := \{\lambda z_i \in \mathbf{R}^n \mid \lambda > 0\}$, where $z_i \in \mathbf{R}^n$ $(i = 1, \dots, M)$ are different points satisfying $|z_i| = 1$, Then $l_i \cap l_j = \emptyset$ if $i \neq j$. We next take positive constants c_i $(i = 0, 1, \dots, M)$ such that $c_0 \geqq \max_{i=1}^M c_i$. Now, we define functions $\overline{r}, g \in BUSC(\mathbf{R}^n)$ as follows:

$$\overline{r}(x) := c_0 I(x) + \sum_{i=1}^M c_i \chi_{l_i}(x), \quad g(x) := \max\left\{ c_0 \chi_{\overline{B_\delta(0)}}(x),\ \max_{i=1}^M c_i \chi_{\overline{l_{i,\varepsilon}}}(x), \right\},$$

where $\delta, \varepsilon > 0$ and $l_{i,\varepsilon} := \bigcap_{x\in l_i} B_\varepsilon(x)$ is an ε-neighborhood of l_i. The function $\overline{r}$ plays a role as a source term of the limit problem, which is cI in Theorem 6.5. Typical examples of such $\overline{r}$ are

$$\overline{r}(x) = cI(x_1) \quad \text{and} \quad \overline{r}(x) = \sum_{i=1}^n c_i I(x_i),$$

where $I : \mathbf{R} \to \mathbf{R}$ is the function in (1.4). Since $\overline{r}$ is positively 0-homogeneous by definition, the unique envelope solution of (HJ1; $\overline{r}, 0$) is self-similar.

We now take the source term $r \in BUSC(\mathbf{R}^n)$ such that $\overline{r} \leqq r \leqq g$ in $\mathbf{R}^n$; clearly, this r has non-compact support. Then the same conclusion as Theorem 6.5 holds, i.e., if u is the unique envelope solution of (HJ1; r, u_0), the rescaled function $u_{(\lambda)}$ converges to the self-similar solution v of (HJ1; $\overline{r}, 0$) locally uniformly as $\lambda \to \infty$. The proof is similar to that of Theorem 6.5. Indeed, from the same argument as in Step 2, where we take $z = 0$ and replace "$cI(\cdot - z)$" by "$\overline{r}$", we deduce $v \leqq \liminf_{*\lambda\to\infty} u_{(\lambda)}$. Also, when we let w be the unique envelope solution of (HJ1; $g, 0$), a similar argument to Step 3 yields $\limsup^*_{\lambda\to\infty} u_{(\lambda)} \leqq \limsup^*_{\lambda\to\infty} w_{(\lambda)} \leqq v$.

6.3 Periodic source terms

We begin with a simple example where the source terms $cI(\cdot)$ are periodically distributed.

Example 6.8. We study (HJ1; $c\tilde{I}, 0$) with $H_1(p) = |p|$ and

$$\tilde{I}(x) := \max_{a\in\mathbf{Z}^n} I(x - a) = \begin{cases} 1 & (x \in \mathbf{Z}^n), \\ 0 & (x \notin \mathbf{Z}^n). \end{cases}$$

By Example 5.5 the unique envelope solution is given as

$$u(x,t) = c(t - \operatorname{dist}(x, \mathbf{Z}^n))_+ = \max_{a\in\mathbf{Z}^n} c(t - |x - a|)_+. \tag{6.8}$$

We shall show

$$u_{(\lambda)}(x,t) \to ct \quad \text{uniformly in } \mathbf{R}^n \times [0,\infty) \text{ as } \lambda \to \infty. \tag{6.9}$$

Obviously, we have $u(x,t) \leqq ct$ by (6.8), so that $u_{(\lambda)}(x,t) \leqq ct$. We next estimate $u_{(\lambda)}$ from below. Compute

$$u_{(\lambda)}(x,t) = \frac{1}{\lambda} u(\lambda x, \lambda t) = \frac{1}{\lambda} \max_{a\in\mathbf{Z}^n} c(\lambda t - |\lambda x - a|)_+ = \max_{a\in\mathbf{Z}^n} c\left(t - \left|x - \frac{a}{\lambda}\right|\right)_+.$$

Now, for fixed $(x,t) \in \mathbf{R}^n \times [0,\infty)$ and $\lambda > 0$, we choose $a \in \mathbf{Z}^n$ such that $|x - (a/\lambda)| \leqq \sqrt{n}/(2\lambda)$. We then see

$$u_{(\lambda)}(x,t) \geqq c\left(t - \frac{\sqrt{n}}{2\lambda}\right)_+ \to ct \quad \text{uniformly in } \mathbf{R}^n \times [0,\infty) \text{ as } \lambda \to \infty,$$

which gives (6.9).

It turns out in the next theorem that, for more general periodic source terms and general initial data, the scaling limits are still ct, where c is the maximum of r in $\mathbf{R}^n$. We remark that $\limsup^*_{\lambda\to\infty} r(\lambda x) = c$ by periodicity and that ct is the unique envelope solution of (HJ1; $c, 0$) provided that $H_1(0) = 0$.

Theorem 6.9 (Large time behavior for periodic source terms)**.** *Assume that H_1 and r satisfy* (6.1)*. Assume that $r(x) = r(x + a)$ for all $(x, a) \in \mathbf{R}^n \times \mathbf{Z}^n$, $H_1(0) = 0$ and $H_1 \geqq 0$ in $\mathbf{R}^n$. Let u and v be, respectively, the unique envelope solution of* (HJ1; r, u_0) *and* (HJ1; $c, 0$) *with $c = \max_{\mathbf{R}^n} r$. Then $v(x,t) = ct$ and $u_{(\lambda)}$ converges to v uniformly in $\mathbf{R}^n \times [0,\infty)$ as $\lambda \to \infty$.*

Proof. Take $z \in \mathbf{R}^n$ as a maximum point of r, so that $c\tilde{I}(x - z) \leqq r(x)$. We now claim

$$ct\tilde{I}(x - z) - M_0 \leqq u(x,t), \tag{6.10}$$

where $M_0 := \sup_{\mathbf{R}^n} |u_0|$. To prove (6.10) it is enough to show that $v_0(x,t) := ct\tilde{I}(x-z) - M_0$ is a standard subsolution of (HJ1; r, u_0). Let $(p, \tau) \in D^+ v_0(\hat{x}, \hat{t})$. If $\hat{x} - z \notin \mathbf{Z}^n$, then $(p, \tau) = (0, 0)$ since $v_0 \equiv -M_0$ near $(\hat{x}, \hat{t})$. Thus

$$\tau - H_1(p) = 0 \leqq r(\hat{x}).$$

We turn to the case $\hat{x} - z \in \mathbf{Z}^n$. Since $v_0(\hat{x}, t) = ct - M_0$, we then have $\tau = c$. Therefore

$$\tau - H_1(p) \leqq c = c\tilde{I}(\hat{x} - z) \leqq r(\hat{x}),$$

which proves (6.10). By the Lipschitz continuity of u we next see that (6.10) gives

$$\max_{a \in \mathbf{Z}^n} (ct - L|x - z - a|)_+ - M_0 \leqq u(x,t), \tag{6.11}$$

where L is the Lipschitz constant of u with respect to x, i.e.,

$$L = \sup_{t \geqq 0} \sup_{\substack{x,y \in \mathbf{R}^n \\ x \neq y}} \frac{|u(x,t) - u(y,t)|}{|x - y|}.$$

Rescaling the both functions in (6.11), we get

$$\max_{a \in \mathbf{Z}^n} \left(ct - L \left| x - \frac{z - a}{\lambda} \right| \right)_+ - \frac{M_0}{\lambda} \leqq u_{(\lambda)}(x,t)$$

for $\lambda > 0$. In the same manner as in Example 6.8 we see that the left hand side converges to ct uniformly in $\mathbf{R}^n \times [0, \infty)$ as $\lambda \to \infty$.

It remains to estimate $u_{(\lambda)}$ from above. Since $r \leqq c$ in $\mathbf{R}^n$, the comparison principle implies $u(x,t) \leqq ct + M_0$. Thus $u_{(\lambda)}(x,t) \leqq ct + (M_0/\lambda)$, and the right hand side converges to ct uniformly in $\mathbf{R}^n \times [0,\infty)$ as $\lambda \to \infty$. □

The following two examples show that the conclusion of Theorem 6.9 may not hold if we remove the non-negativity or coercivity of H_1.

Example 6.10. Let us study

$$\begin{cases} \partial_t u + |\nabla u| = c\tilde{I}(x) & \text{in } \mathbf{R}^n \times (0, \infty), \\ u(x, 0) = 0 & \text{in } \mathbf{R}^n. \end{cases} \tag{6.12}$$

Here $H_1(p) = -|p|$ is non-positive. According to Example 5.9, where we considered the value function defined as the imfimum of costs, the unique envelope solution of (6.12) is

$$u(x,t) = 0$$

since the interior of $\mathbf{Z}^n$ is empty. The rescaled function is $u_{(\lambda)} = 0$, but this does not converge to ct.

One can directly check that $u = 0$ is an envelope solution of (6.12). Indeed, it is easily seen that u is a subsolution. Next, since

$$u^\varepsilon(x,t) := \begin{cases} \max_{a\in\mathbf{Z}^n} c(t-|x-a|)_+ & (0 \leqq t \leqq \varepsilon), \\ \max_{a\in\mathbf{Z}^n} c(\varepsilon-|x-a|)_+ & (\varepsilon \leqq t) \end{cases}$$

is a $\overline{D}$-supersolution of (6.12) for every $\varepsilon > 0$, the infimum of u^ε, which equals zero, is an envelope supersolution.

Example 6.11. When a Hamiltonian is non-coercive, solutions may not have the scaling limit even in the sense of pointwise convergence. To see this let us consider the following problem:

$$\begin{cases} \partial_t u = c\tilde{I}(x) & \text{in } \mathbf{R}^n \times (0,\infty), \\ u(x,0) = 0 & \text{in } \mathbf{R}^n. \end{cases} \tag{6.13}$$

In the same manner as in Example 3.1 we see that $u(x,t) = ct\tilde{I}(x)$ is an envelope solution of (6.13). (This is not a unique envelope solution of the problem. In fact, $u^\alpha(x,t) = \alpha t\tilde{I}(x)$ solves the same problem for every $\alpha \in [0,c)$; see Example 3.16. However, among such solutions u is a natural one since it vertically grows at a speed c at each step source.) We now compute

$$u_{(\lambda)}(x,t) = \frac{1}{\lambda}u(\lambda x, \lambda t) = ct\tilde{I}(\lambda x),$$

and so the limit of $u_{(\lambda)}(x,t)$ as $\lambda \to \infty$ does not exist for every $(x,t) \in (\mathbf{R}^n \setminus \{0\}) \times (0,\infty)$. We also have

$$\limsup_{\lambda\to\infty}{}^* u_{(\lambda)}(x,t) = ct, \quad \liminf_{\lambda\to\infty}{}_* u_{(\lambda)}(x,t) = 0.$$

Remark 6.12. The result in Theorem 6.9 is consistent with the classical homogenization theory for continuous equations ([29]). To check this we set $H(x,p) = H_1(p) + r(x)$ so that H is coercive in the sense $\lim_{|p|\to\infty} \inf_{x\in\mathbf{R}^n} H(x,p) = \infty$, and solve (HJ) with this Hamiltonian. We let u be the unique envelope solution (an envelope subsolution and a standard supersolution) of (HJ), and set $u^\varepsilon := u_{(1/\varepsilon)}$. Then Theorem 6.9 implies that u^ε, which is a solution to

$$\partial_t u^\varepsilon - H_1(\nabla u^\varepsilon) = r\left(\frac{x}{\varepsilon}\right) \quad \text{in } \mathbf{R}^n \times (0,\infty),$$

uniformly converges to $-ct$ as $\varepsilon \to 0$, where $c = \max_{\mathbf{R}^n} r$. On the other hand, according to the homogenization theory, u^ε converges to the viscosity solution v of

$$\begin{cases} \partial_t v + \overline{H}(\nabla v) = 0 & \text{in } \mathbf{R}^n \times (0,\infty), \\ v(x,0) = 0 & \text{in } \mathbf{R}^n \end{cases} \tag{6.14}$$

when r is continuous. Here $\overline{H}$ is called an *effective Hamiltonian* chosen so that, for each $p \in \mathbf{R}^n$, the cell problem

$$H(x, p + \nabla w) = \overline{H}(p) \quad \text{in } \mathbf{R}^n \tag{6.15}$$

admits a periodic viscosity solution w. If $\overline{H}(0) = c$, then we see that the solution v of (6.14) is $v(x,t) = -ct$, and hence the conclusion is the same as Theorem 6.9.

Let us show $\overline{H}(0) = c$. (This fact is more or less known, but we give the proof for completeness.) We take $p = 0$ in (6.15). Then (6.15) is

$$H_1(\nabla w) = \overline{H}(0) - r(x) \quad \text{in } \mathbf{R}^n. \tag{6.16}$$

Suppose $\overline{H}(0) > c$. Since $0 \in D^- w(\hat{x})$ at a minimum point $\hat{x}$ of w, we would have

$$0 = H_1(0) \geqq \overline{H}(0) - r(\hat{x}) > c - r(\hat{x}) \geqq 0,$$

which is a contradiction. We next suppose $\overline{H}(0) < c$. Then, since $\overline{H}(0) - r(x) < 0$ in a small open neighborhood of a maximum point of r, we would reach a contradiction by substituting any element of $D^+ w$ in the neighborhood into (6.16); recall that H_1 is now assumed to be non-negative. As a result, we conclude $\overline{H}(0) = c$.

Example 6.13. We revisit Example 6.8 and consider the large time behavior in the sense (1.23). Since the unique envelope solution of (HJ1; $c\tilde{I}, 0$) is of the form (6.8), we see

$$u(x,t) - \left(ct - \max_{a \in \mathbf{Z}^n} |x - a|\right) \to 0 \quad \text{as } t \to \infty$$

uniformly in $\mathbf{R}^n$. Obviously, this limit is different from our scaling limit ct.

7 Stationary problem

We define a notion of solutions to stationary problems of the form

$$F(x, u(x), \nabla u(x)) = 0 \quad \text{in } \mathbf{R}^n \tag{7.1}$$

in the same way as the time-dependent problems. Namely, a function $u : \mathbf{R}^n \to \mathbf{R}$ such that $u_* > -\infty$ in $\mathbf{R}^n$ is called a $\overline{D}$-*supersolution* of (7.1) if

$$F(x, u_*(x), p) \geqq 0 \quad \text{for all } x \in \Omega \text{ and } p \in \overline{D}^- u_*(x). \tag{7.2}$$

Also, we call u an *envelope supersolution* of (7.1) if $u(x) = \inf_{w \in \mathcal{S}} w(x)$ $(x \in \mathbf{R}^n)$ for some family $\mathcal{S} \subset \{w \mid w \text{ is a } \overline{D}\text{-supersolution of (7.1)}\}$. A *(standard) viscosity supersolution* of (7.1) is defined by replacing "F" by "F^*" and "$\overline{D}^-$" by "D^-" in (7.2). Notions of *subsolutions* are defined in a similar way, i.e., we say u is a subsolution if $-u$ is a supersolution. For stationary problems we do not impose global boundedness on solutions.

7.1 Profile function

We first derive an equation for a profile function of a self-similar solution to (HJ1; $r, 0$). Let u be self-similar and suppose that its profile function U is smooth. Since $u(x,t) = tU(x/t)$, we compute

$$\partial_t u(x,t) = U(x/t) - \langle x/t, \nabla U(x/t) \rangle, \quad \nabla u(x,t) = \nabla U(x/t).$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product in $\mathbf{R}^n$. Thus, if u solves (HJ1; $r, 0$) with a 0-homogeneous r, substituting the above derivatives of u for (1.18) and letting $\xi := x/t$, we obtain

$$U(\xi) - \langle \xi, \nabla U(\xi) \rangle - H_1(\nabla U(\xi)) = r(\xi) \quad \text{in } \mathbf{R}^n. \tag{7.3}$$

The next proposition asserts that an envelope solution of the stationary problem (7.3) gives that of the time-dependent problem (HJ1; $r, 0$).

Proposition 7.1. *Assume $H_1 \in C(\mathbf{R}^n)$, $r \in BUSC(\mathbf{R}^n)$ and that r is positively 0-homogeneous. Let $U : \mathbf{R} \to \mathbf{R}$ be a bounded envelope solution of* (7.3). *Then $u : \mathbf{R}^n \times [0, \infty) \to \mathbf{R}$ defined as*

$$u(x,t) := \begin{cases} 0 & (t = 0), \\ tU(x/t) & (t > 0) \end{cases}$$

is an envelope solution of (HJ1; $r, 0$).

In contrast, it is not trivial whether the self-similar solution u gives the profile function U in the sense of envelope solutions.

Proof. 1. By the boundedness of U we easily see that u is continuous at $t = 0$, i.e., $u(x, 0) = \lim_{(y,t)\to(x,0)} u(y, t) = 0$ for all $x \in \mathbf{R}^n$.

2. Let us show that u is an envelope supersolution of (HJ1; $r, 0$). Since U is an envelope supersolution of (7.3), there is a family $\mathcal{T} \subset \{\overline{D}\text{-supersolutions of (7.3)}\}$ such that $U(\xi) = \inf_{W \in \mathcal{T}} W(\xi)$. Then $u(x,t) = tU(x/t) = \inf_{W\in\mathcal{T}} tW(x/t)$. We claim that $w(x,t) := tW(x/t)$ is a $\overline{D}$-supersolution of (1.18) for every $W \in \mathcal{T}$. To do this we first prove

$$\tau - H_1(p) \geqq r(\hat{x}) \tag{7.4}$$

for $(p, \tau) \in D^- w_*(\hat{x}, \hat{t})$. Take a test function ϕ for (p, τ), so that $w_*(x,t) - \phi(x,t) \geqq w_*(\hat{x},\hat{t}) - \phi(\hat{x},\hat{t})$ for all $(x,t) \in \mathbf{R}^n \times (0,\infty)$. Letting $\xi = x/t$, we see

$$tW_*(\xi) - \phi(\xi t, t) \geqq \hat{t}W_*(\hat{\xi}) - \phi(\hat{\xi}\hat{t}, \hat{t}) \quad \text{for all } (\xi, t) \in \mathbf{R}^n \times (0, \infty), \tag{7.5}$$

where $\hat{\xi} := \hat{x}/\hat{t}$. We now choose $t = \hat{t}$ in (7.5) to get

$$D^- W_*(\hat{\xi}) \ni \nabla_\xi (\phi(\xi\hat{t}, \hat{t})/\hat{t})\big|_{\xi=\hat{\xi}} = \nabla_x \phi(\hat{x}, \hat{t}) = p.$$

Since W is a $\overline{D}$-supersolution of (7.3), we have

$$W_*(\hat{\xi}) - \langle \hat{\xi}, p \rangle - H_1(p) \geqq r(\hat{\xi}) = r(\hat{x}). \tag{7.6}$$

Here we have used the 0-homogeneity of r. We next take $\xi = \hat{\xi}$ in (7.5). Since a map $t \mapsto tW_*(\hat{\xi}) - \phi(\hat{\xi}t, t)$ is smooth, we observe

$$\begin{aligned} 0 &= \frac{d}{dt}(tW_*(\hat{\xi}) - \phi(\hat{\xi}t, t))\bigg|_{t=\hat{t}} \\ &= W_*(\hat{\xi}) - \langle \hat{\xi}, \nabla_x \phi(\hat{x}, \hat{t}) \rangle - \partial_t \phi(\hat{x}, \hat{t}) \\ &= W_*(\hat{\xi}) - \langle \hat{\xi}, p \rangle - \tau. \end{aligned} \tag{7.7}$$

Substituting (7.7) into (7.6), we obtain (7.4).

3. We shall show that the inequality (7.4) holds for every $(p,\tau) \in \overline{D}^- w_*(\hat{x},\hat{t})$. Let us take an approximate sequence $\{((x_m,t_m),(p_m,\tau_m))\}_{m=1}^\infty$ for (p,τ) which appears in the definition of $\overline{D}^-$. Since $(p_m,\tau_m) \in D^- w_*(x_m,t_m)$ for each m, a similar argument to Step 2 yields $p_m \in D^- W_*(\xi_m)$ with $\xi_m = x_m/t_m$. Now, compute

$$\lim_{m\to\infty} W_*(\xi_m) = \lim_{m\to\infty} \frac{w_*(x_m,t_m)}{t_m} = \frac{w_*(\hat{x},\hat{t})}{\hat{t}} = W_*(\hat{\xi}),$$

where $\hat{\xi} = \hat{x}/\hat{t}$. We thus have $p \in \overline{D}^- W_*(\hat{\xi})$, which implies (7.6). In the same manner as in Step 2 we also see $0 = W_*(\xi_m) - \langle \xi_m, p_m\rangle - \tau_m$. Sending $m \to \infty$ gives (7.7), and hence (7.4) holds.

4. We now apply the argument in Step 2 again in order to show that u is a standard viscosity subsolution of (HJ1; $r,0$). Then, for $(p,\tau) \in D^+ u^*(\hat{x},\hat{t})$ we get $p \in D^+ U^*(\hat{\xi})$ and $U^*(\hat{\xi}) - \langle \hat{\xi}, p\rangle - \tau = 0$. Here $\hat{\xi} = \hat{x}/\hat{t}$. Using the fact that U is a subsolution of (7.3), we conclude $\tau - H_1(p) \leqq r(\hat{x})$. □

Example 7.2. The corresponding stationary problem to our typical equation (1.5) is

$$U(\xi) - \langle \xi, \nabla U\rangle - |\nabla U(\xi)| = cI(\xi) \quad \text{in } \mathbf{R}^n. \tag{7.8}$$

In Section 7.4 we will solve approximated problems of (7.8) with continuous source terms and show that (6.2), which is the profile function of the self-similar solution to (1.5), is indeed an envelope supersolution of (7.8). By contrast one can check that (6.2) is a subsolution of (7.8) in a direct way. Let us take $p \in D^+U(\xi)$, where U is given as (6.2). We first notice that $|\xi| \neq 1$ since $D^+U(\xi)$ is a empty set if $|\xi| = 1$. When $|\xi| > 1$, we have $p = 0$ and $U(\xi) = 0$. Thus $U(\xi) - \langle \xi, p\rangle - |p| = 0 - \langle \xi, 0\rangle - |0| = 0$. We next study the case $0 < |\xi| < 1$. Since $p = -c\xi/|\xi|$ and $U(\xi) = c(1-|\xi|)$, we compute

$$U(\xi) - \langle \xi, p\rangle - |p| = c(1-|\xi|) - \left\langle \xi, -c\frac{\xi}{|\xi|}\right\rangle - \left|-c\frac{\xi}{|\xi|}\right| = c(1-|\xi|) + c|\xi| - c = 0.$$

Finally, in the case $\xi = 0$ we observe $U(\xi) - \langle \xi, p\rangle - |p| = c - \langle 0, p\rangle - |p| \leqq c$.

7.2 Comparison principle

We study uniqueness of solutions to stationary problems with discontinuity. Let us consider a general equation of the form

$$\nu u(x) + H(x, \nabla u) = 0 \quad \text{in } \mathbf{R}^n, \tag{7.9}$$

where $\nu > 0$. We establish a comparison principle for $\overline{D}$-sub- and $\overline{D}$-supersolutions of (7.9) under a suitable growth condition for solutions.

Theorem 7.3 (Comparison principle)**.** *Assume that H satisfies* (H_p) *with $\omega_1(r) = L_1 r$ for some constant $L_1 > 0$. Let u and $v : \mathbf{R}^n \to \mathbf{R}$ be, respectively, a $\overline{D}$-subsolution and a $\overline{D}$-supersolution of* (7.9) *such that*

$$u(x) \leqq C_0(1+|x|) \quad \textit{and} \quad v(x) \geqq -C_0(1+|x|) \quad \textit{for all } x \in \mathbf{R}^n \tag{7.10}$$

for some $C_0 > 0$. Assume either (H_{xN}) *or Lipschitz continuity of u or v in $\mathbf{R}^n$. Then $u^* \leqq v_*$ in $\mathbf{R}^n$.*

Unfortunately, the assumption of Theorem 7.3 excludes Hamiltonians having the inner product $\langle x, p\rangle$ such as (7.3). The reader is referred to [28, Proposition 5.5] for the proof of a comparison principle in the whole space for standard viscosity solutions satisfying the same growth condition.

Proof. 1. Suppose by contradiction that there would exist some $x_0 \in \mathbf{R}^n$ such that $u^*(x_0) - v_*(x_0) =: A > 0$. Take a constant $\delta > 0$ small so that $0 < \delta \leqq A/4(1+|x_0|^2)$, and define

$$\theta := \sup_{x\in\mathbf{R}^n} \{u^*(x) - v_*(x) - 2\delta(1+|x|^2)\}.$$

Since u and v satisfy the growth condition (7.10), there exists a maximizer $z \in \mathbf{R}^n$ of the function in the right hand side; namely $\theta = u^*(z) - v_*(z) - 2\delta(1+|z|^2)$. Also, by the choice of δ, we have

$$\theta \geqq u^*(x_0) - v_*(x_0) - 2\delta(1+|x_0|^2) \geqq A - \frac{A}{2} = \frac{A}{2}. \tag{7.11}$$

We next define a function $\Psi : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ by

$$\Psi(x, y) := u^*(x) - v_*(y) - \phi(x, y),$$

where $\phi : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is a smooth function given as

$$\phi(x, y) := \frac{|x-y|^2}{2\varepsilon^2} + \delta(2 + |x|^2 + |y|^2).$$

Here $0 < \varepsilon < 1$. Then Ψ attains its maximum over $\mathbf{R}^n \times \mathbf{R}^n$ at some $(x^\varepsilon, y^\varepsilon) \in \mathbf{R}^n \times \mathbf{R}^n$ thanks to the growth condition (7.10). We observe

$$\Psi(x^\varepsilon, y^\varepsilon) \geqq \Psi(z, z) = u^*(z) - v_*(z) - 0 - \delta(2 + |z|^2 + |z|^2) = \theta. \tag{7.12}$$

From this inequality and (7.10) it follows that

$$\frac{|x^\varepsilon - y^\varepsilon|^2}{2\varepsilon^2} + \delta(2 + |x^\varepsilon|^2 + |y^\varepsilon|^2) \leqq C_0(2 + |x^\varepsilon| + |y^\varepsilon|).$$

This implies that $\{x^\varepsilon\}_\varepsilon$ and $\{y^\varepsilon\}_\varepsilon$ are bounded, so that we may assume

$$\lim_{\varepsilon\to 0}(x^\varepsilon, y^\varepsilon) = (\hat{x}, \hat{x}) \tag{7.13}$$

for some $\hat{x} \in \mathbf{R}^n$.

2. We shall show

$$\lim_{\varepsilon\to 0} \frac{|x^\varepsilon - y^\varepsilon|}{\varepsilon} = 0 \tag{7.14}$$

and

$$\lim_{\varepsilon\to 0} u^*(x^\varepsilon) = u^*(\hat{x}), \quad \lim_{\varepsilon\to 0} v_*(y^\varepsilon) = v_*(\hat{x}). \tag{7.15}$$

Since $(x^\varepsilon, y^\varepsilon)$ is a maximizer of Ψ, we have

$$\Psi(\hat{x}, \hat{x}) \leqq \Psi(x^\varepsilon, y^\varepsilon). \tag{7.16}$$

Using this inequality, we calculate

$$\begin{aligned}\limsup_{\varepsilon\to 0} \frac{|x^\varepsilon - y^\varepsilon|^2}{2\varepsilon^2} &\leqq -\Psi(\hat{x}, \hat{x}) + \limsup_{\varepsilon\to 0} \left\{u^*(x^\varepsilon) - v_*(y^\varepsilon) - \delta(2 + |x^\varepsilon|^2 + |y^\varepsilon|^2)\right\} \\ &\leqq -\Psi(\hat{x}, \hat{x}) + u^*(\hat{x}) - v_*(\hat{x}) - \delta(2 + |\hat{x}|^2 + |\hat{x}|^2) = 0,\end{aligned}$$

which implies (7.14). Next, by (7.16) and the upper semicontinuity of Ψ, we observe

$$\Psi(\hat{x}, \hat{x}) \leqq \liminf_{\varepsilon\to 0} \Psi(x^\varepsilon, y^\varepsilon) \leqq \limsup_{\varepsilon\to 0} \Psi(x^\varepsilon, y^\varepsilon) \leqq \Psi(\hat{x}, \hat{x}).$$

Thus

$$\lim_{\varepsilon\to 0} \Psi(x^\varepsilon, y^\varepsilon) = \Psi(\hat{x}, \hat{x}). \tag{7.17}$$

This equality and (7.14) implies

$$\lim_{\varepsilon\to 0} \{u^*(x^\varepsilon) - v_*(y^\varepsilon)\} = u^*(\hat{x}) - v_*(\hat{x}).$$

We now compute

$$\begin{aligned}u^*(\hat{x}) \geqq \limsup_{\varepsilon\to 0} u^*(x^\varepsilon) \geqq \liminf_{\varepsilon\to 0} u^*(x^\varepsilon) &= \liminf_{\varepsilon\to 0} \{(u^*(x^\varepsilon) - v_*(y^\varepsilon)) + v_*(y^\varepsilon)\} \\ &\geqq (u^*(\hat{x}) - v_*(\hat{x})) + v_*(\hat{x}) = u^*(\hat{x}).\end{aligned}$$

Consequently we obtain the both assertions in (7.15). Also, it is now easy to derive the inequality

$$\frac{A}{2} \leqq \Psi(\hat{x}, \hat{x}) \tag{7.18}$$

by (7.11), (7.12) and (7.17).

3. Set $p^\varepsilon := (x^\varepsilon - y^\varepsilon)/\varepsilon^2$. The first derivatives of ϕ at $(x^\varepsilon, y^\varepsilon)$ are given as follows:

$$\nabla_x \phi(x^\varepsilon, y^\varepsilon) = p^\varepsilon + 2\delta x^\varepsilon, \quad \nabla_y \phi(x^\varepsilon, y^\varepsilon) = -p^\varepsilon + 2\delta y^\varepsilon.$$

Since Ψ attains its maximum at $(x^\varepsilon, y^\varepsilon)$, we have

$$\begin{cases} p^\varepsilon + 2\delta x^\varepsilon \in D^+ u^*(x^\varepsilon), \\ p^\varepsilon - 2\delta y^\varepsilon \in D^- v_*(y^\varepsilon). \end{cases} \tag{7.19}$$

Now we divide the situation into two different cases.

Case 1: $\{p^\varepsilon\}_{0<\varepsilon<1}$ is unbounded, **Case 2:** $\{p^\varepsilon\}_{0<\varepsilon<1}$ is bounded.

We will reach to contradiction for both cases. Note that, if u or v is assumed to be Lipschitz continuous, then Case 1 does not occur because D^+u^* or D^-v_* is a bounded set by the Lipschitz continuity. On the other hand, Case 1 can happen when we assume (H_{xN}) instead of the Lipschitz continuity.

In Case 1 there exists a sequence $\{\varepsilon(j)\}_{j=1}^{\infty} \subset (0,1)$ such that $\lim_{j\to\infty} \varepsilon(j) = 0$ and $\lim_{j\to\infty} |p^{\varepsilon(j)}| = \infty$. To simplify the notation let us write ε for $\varepsilon(j)$. By (7.19) we see

$$\begin{cases} \nu u^*(x^\varepsilon) + H(x^\varepsilon, p^\varepsilon + 2\delta x^\varepsilon) \leqq 0, \\ \nu v_*(y^\varepsilon) + H(y^\varepsilon, p^\varepsilon - 2\delta y^\varepsilon) \geqq 0 \end{cases}$$

since u and v are a sub- and supersolution, respectively. Combining these two inequalities, we have

$$\begin{aligned} \nu(u^*(x^\varepsilon) - v_*(y^\varepsilon)) \leqq \{H(y^\varepsilon, p^\varepsilon - 2\delta y^\varepsilon) - H(y^\varepsilon, p^\varepsilon)\} + \{H(y^\varepsilon, p^\varepsilon) - H(x^\varepsilon, p^\varepsilon)\} \\ + \{H(x^\varepsilon, p^\varepsilon) - H(x^\varepsilon, p^\varepsilon + 2\delta x^\varepsilon)\}. \end{aligned}$$

We now let ε small so that $|p^\varepsilon| \geqq N$, where N is the constant in the assumption (H_{xN}). Applying (H_p) and (H_{xN}), we calculate

$$\begin{aligned} \nu(u^*(x^\varepsilon) - v_*(y^\varepsilon)) &\leqq 2\delta L_1 |y^\varepsilon| + \omega_2((1 + |p^\varepsilon|)|x^\varepsilon - y^\varepsilon|) + 2\delta L_1 |x^\varepsilon| \\ &= \omega_2\left(|x^\varepsilon - y^\varepsilon| + \frac{|x^\varepsilon - y^\varepsilon|^2}{\varepsilon^2}\right) + 2\delta L_1(|x^\varepsilon| + |y^\varepsilon|). \end{aligned}$$

Sending $\varepsilon \to 0$ yields

$$\nu(u^*(\hat{x}) - v_*(\hat{x})) \leqq 4\delta L_1 |\hat{x}|, \tag{7.20}$$

where we have used (7.14). Adding $-2\nu\delta(1 + |\hat{x}|^2)$ to the both sides, we compute

$$\begin{aligned} \nu\Psi(\hat{x}, \hat{x}) &\leqq 4\delta L_1 |\hat{x}| - 2\nu\delta(1 + |\hat{x}|^2) \\ &= 2\delta\left\{-\nu\left(|\hat{x}| - \frac{L_1}{\nu}\right)^2 + \frac{L_1^2}{\nu} - \nu\right\} \leqq 2\delta\left(\frac{L_1^2}{\nu} - \nu\right). \end{aligned}$$

Finally, we apply the inequality (7.18) to get

$$\frac{\nu A}{2} \leqq 2\delta\left(\frac{L_1^2}{\nu} - \nu\right).$$

This is a contradiction for small δ.

We next study Case 2. In this case we may assume $\lim_{\varepsilon\to 0} p^\varepsilon = \bar{p}$ for some $\bar{p} \in \mathbf{R}^n$. Since we have (7.13), (7.15) and

$$\lim_{\varepsilon\to 0}(p^\varepsilon + 2\delta x^\varepsilon, p^\varepsilon - 2\delta y^\varepsilon) = (\bar{p} + 2\delta\hat{x}, \bar{p} - 2\delta\hat{x}),$$

taking a limit in (7.19), we obtain

$$\begin{cases} \bar{p} + 2\delta\hat{x} \in \overline{D}^+ u^*(\hat{x}), \\ \bar{p} - 2\delta\hat{x} \in \overline{D}^- v_*(\hat{x}) \end{cases}$$

by the definitions of $\overline{D}^\pm$. Now u and v are solutions in the $\overline{D}$-sense, and so it follows that

$$\begin{cases} \nu u^*(\hat{x}) + H(\hat{x}, \bar{p} + 2\delta\hat{x}) \leqq 0, \\ \nu v_*(\hat{x}) + H(\hat{x}, \bar{p} - 2\delta\hat{x}) \geqq 0. \end{cases}$$

From these inequalities and (H_p) we are able to deduce (7.20) without (H_{xN}). The rest of the proof is similar to Case 1. $\square$

7.3 Existence result

We turn to the existence problem for (7.9). Similarly to the time-dependent case, we construct envelope solutions via approximated problems. More precisely, for each $\varepsilon > 0$ we first solve

$$\nu u^\varepsilon(x) + H^\varepsilon(x, \nabla u^\varepsilon) = 0 \quad \text{in } \mathbf{R}^n, \tag{7.21}$$

where H^ε is a continuous Hamiltonian approximating H. We next show that the infimum of u^ε gives an envelope solution of the original problem (7.9).

The following assumption requires an approximability of H:

(H'_ε) There exists $\{H^\varepsilon\}_{\varepsilon>0} \subset C(\mathbf{R}^n \times \mathbf{R}^n)$, a family of continuous Hamiltonians, such that $H^\varepsilon \uparrow H$ pointwise as $\varepsilon \to 0$ and the following conditions hold.

- (i) If u and $v : \mathbf{R}^n \to \mathbf{R}$ be, respectively, a standard viscosity sub- and supersolution of (7.21) satisfying (7.10) for some $C_0 > 0$, then $u^* \leqq v_*$ in $\mathbf{R}^n$.
- (ii) There exists a standard viscosity solution u^ε of (7.21) such that $-C^\varepsilon(1 + |x|) \leqq u^\varepsilon(x) \leqq C^\varepsilon(1 + |x|)$ in $\mathbf{R}^n$ for some $C^\varepsilon > 0$.

The condition (i) implies that the solution in (ii) is unique and continuous. We also remark that H satisfying (H'_ε) is always lower semicontinuous, so that we may discuss subsolutions of (7.9) in the standard viscosity sense.

Proposition 7.4 (Existence of envelope solutions). *Assume that H satisfies* (H'_ε). *Let u^ε be the unique solution of* (7.21), *and set $\overline{u} := \inf_{\varepsilon>0} u^\varepsilon$. Assume furthermore that there exists a subsolution w of* (7.9) *such that $-C'(1 + |x|) \leqq w(x)$ in $\mathbf{R}^n$ for some $C' > 0$. Then $\overline{u}$ is an envelope solution of* (7.9)*. Moreover, it hold that $-C'(1 + |x|) \leqq \overline{u}(x) \leqq C^\varepsilon(1 + |x|)$ in $\mathbf{R}^n$ for all $\varepsilon > 0$, where C^ε is the constant in* (H'_ε).

The proof of Proposition 7.4 uses a similar argument to that of Proposition 3.7.

Proof. Since $H^{\varepsilon'} \leqq H^\varepsilon \leqq H$ for $0 < \varepsilon < \varepsilon'$, the comparison principle (i) in (H'_ε) implies $w \leqq u^\varepsilon \leqq u^{\varepsilon'}$. Thus taking the infimum over $\{\varepsilon > 0\}$ yields $w \leqq \overline{u} \leqq u^{\varepsilon'}$. The growth condition on $\overline{u}$ follows from these inequalities.

We shall show that $\overline{u}$ is an envelope solution. Since u^ε is a standard subsolution of (7.21), the stability result for subsolutions under the relaxed limit ([11, Lemma 6.1, Remark 6.3]) implies that $\limsup^*_{\varepsilon\to 0} u^\varepsilon$, which is now equal to $\overline{u}$ by the monotonicity of u^ε, is a subsolution of (7.9). Here we have used $\liminf_{*\varepsilon\to 0} H^\varepsilon = H$. We also see that $\overline{u}$ is an envelope supersolution of (7.9) since u^ε is a $\overline{D}$-supersolution of (7.9) by $H^\varepsilon \leqq H$. $\square$

One of sufficient conditions for existence of the subsolution w in the statement of Proposition 7.4 is

$$m := \sup_{x\in\mathbf{R}^n} H(x, 0) < \infty.$$

In fact, if H satisfies this condition, then $w(x) = -m/\nu$ is a bounded subsolution of (7.9) since

$$\nu w(x) + H(x, \nabla w(x)) = -m + H(x, 0) \leqq 0.$$

7.4 Explicit solutions

Let us solve (7.8) via approximation. As we studied in Example 3.1 and 3.2, one of simple approximation to (7.8) is

$$u^\varepsilon(x) - \langle x, \nabla u^\varepsilon \rangle - |\nabla u^\varepsilon| = cI^\varepsilon(x) \quad \text{in } \mathbf{R}^n, \tag{7.22}$$

where $0 < \varepsilon < 1$ and I^ε is given by (1.8). Let us compute the exact solutions of the approximated problems (7.22). We fix $\varepsilon \in (0, 1)$ and simply write u for u^ε. Since (7.22) is written as a Bellman equation of the form

$$u(x) - \sup_{a \in \overline{B}_1(0)} \langle x + a, \nabla u \rangle = cI^\varepsilon(x) \quad \text{in } \mathbf{R}^n,$$

we are able to apply the classical optimal control theory. According to [2, Proposition III.2.8] the value function of the infinite horizon problem defined as

$$u(x) := \sup_{\alpha \in \mathcal{A}} \int_0^\infty e^{-s} cI^\varepsilon(X^\alpha(s))ds$$

is a standard viscosity solution of (7.22). Here $\alpha \in \mathcal{A} := \{\alpha : [0, \infty) \to \overline{B}_1(0), \text{measurable}\}$ is a control (the control set is $\overline{B}_1(0)$) and $X^\alpha : [0, \infty) \to \mathbf{R}^n$ solves the state equation

$$(X^\alpha)'(s) = X^\alpha(s) + \alpha(s) \quad \text{in } (0, \infty), \qquad X^\alpha(0) = x.$$

A discount factor λ, which usually appears in the exponential term $e^{-\lambda s}$, is now chosen as one. Studying the optimal strategy for each $x \in \mathbf{R}^n$, we shall simplify the representation formula of u.

We first notice that, if $|x| \geqq 1$, it is impossible for the state $X^\alpha(s)$ to reach $\overline{B}_\varepsilon(0) (= \operatorname{supp}(r))$ whatever control α is chosen because, for each $x' \in \mathbf{R}^n$ such that $|x'| \geqq 1$, the set $\{x' + a \mid a \in \overline{B}_1(0)\}$ is disjoint from the half space $\{y \in \mathbf{R}^n \mid \langle x', y \rangle < 0\}$. This implies that the state cannot get close to the origin. Thus

$$u(x) = 0 \quad (|x| \geqq 1). \tag{7.23}$$

We next study the case where $0 \leqq |x| < 1$. Then the optimal control forces to the state to move straight from x to the origin at a maximal speed and to stop at the origin after the arrival. In other words, the optimal strategy satisfies

$$|X^\alpha(s)| = \begin{cases} 1 - (1 - |x|)e^s & (0 \leqq s \leqq -\log(1 - |x|)), \\ 0 & (-\log(1 - |x|) \leqq s < \infty). \end{cases}$$

Therefore

$$\begin{aligned} \frac{u(x)}{c} &= \int_0^{-\log(1-|x|)} e^{-s} \left(1 - \frac{|X^\alpha(s)|}{\varepsilon}\right)_+ ds + \int_{-\log(1-|x|)}^\infty e^{-s} ds \\ &= \int_0^{-\log(1-|x|)} e^{-s} \left(1 - \frac{1 - (1 - |x|)e^s}{\varepsilon}\right)_+ ds + (1 - |x|). \end{aligned} \tag{7.24}$$

When $|x| \leqq \varepsilon$, the first term is

$$\int_0^{-\log(1-|x|)} e^{-s}\left(1-\frac{1-(1-|x|)e^s}{\varepsilon}\right) ds = -\left(\frac{1}{\varepsilon}-1\right)|x| - \frac{1-|x|}{\varepsilon}\log(1-|x|),$$

and consequently

$$\frac{u(x)}{c} = 1 - \frac{|x|}{\varepsilon} - \frac{1-|x|}{\varepsilon}\log(1-|x|) \quad (0 \leqq |x| \leqq \varepsilon). \tag{7.25}$$

We next compute the first term in (7.24) when $\varepsilon \leqq |x| < 1$. Then

$$\int_{-\log\{(1-|x|)/(1-\varepsilon)\}}^{-\log(1-|x|)} e^{-s}\left(1-\frac{1-(1-|x|)e^s}{\varepsilon}\right) ds = \left(\frac{\log(1-\varepsilon)}{\varepsilon}-1\right)(1-|x|),$$

which yields

$$\frac{u(x)}{c} = -\frac{\log(1-\varepsilon)}{\varepsilon}(1-|x|) \quad (\varepsilon \leqq |x| < 1). \tag{7.26}$$

Summarizing the above computations, we established the solution formula (7.23), (7.25) and (7.26). The solution u^ε of (7.22) is a $\overline{D}$-supersolution of (7.8) since $cI(x) \leqq cI^\varepsilon(x)$ for all $x \in \mathbf{R}^n$. Also, the infimum of u^ε is equal to (6.2), and therefore we conclude that (6.2) is an envelope supersolution of (7.8). (In this example, u^ε uniformly converges to (6.2) as $\varepsilon \to 0$.)

One can obtain functions of the form (7.25) and (7.26) via solving ordinary differential equations. To get the solutions we first simplify the equation (7.22). We consider (7.22) in the one-dimensional half space $(0, \infty)$ and suppose that the solution u is non-increasing in $(0, \infty)$ so that $|u'(x)| = -u'(x)$. Then the equation (7.22) simplifies to

$$u(x) + (1-x)u'(x) = cI^\varepsilon(x) \quad \text{in } (0, \infty). \tag{7.27}$$

Since this is a linear differential equation, we are able to employ the solution formula of the linear equation. For $0 < x < 1$ we have

$$u(x) = \left(u(0) + \int_0^x \frac{c}{1-s} I^\varepsilon(s) e^{\int_0^s dz/(1-z)} ds\right) e^{\int_0^x -ds/(1-s)}. \tag{7.28}$$

We consider the value of $u(0)$. If we suppose u is radially symmetric and differentiable at the origin, then $\nabla u(0) = 0$. Thus letting $x = 0$ in the original equation (7.22) implies $u(0) = c$. Substituting this for (7.28) and computing the integrals, we finally obtain (7.25) and (7.26), where "$|x|$" is replaced by "x", for $0 < x \leqq \varepsilon$ and $\varepsilon \leqq x < 1$, respectively.

In Example 3.2, considering a finite horizon optimal control problem, we computed the exact solution to the approximated time-dependent equation of the form $\partial_t v^\varepsilon - |\nabla v^\varepsilon| = cI^\varepsilon(x)$ under the zero initial data. However, the shape of the solution $v^\varepsilon(x,t)$ to this problem is different from that to the stationary problem (7.22). Indeed, for a fixed $t > 0$ the solution v^ε is a quadratic function of $|x|$ near the origin with the vertex $(0, ct)$, and the solution is smooth in $\mathbf{R}^n \times (0, T)$. Obviously, the solution of (7.22) does not enjoy these properties.

In Lemma 5.3 we proved that, under a suitable controllability condition, the value function of a finite horizon problem with a discontinuous running cost gives the unique envelope solution of Bellman equations. As its analogue let us study the value function

$$v(x) = \sup_{\alpha \in \mathcal{A}} \int_0^\infty e^{-s} cI(X^\alpha(s)) ds \tag{7.29}$$

with a discontinuous integrand. The optimal control for this problem is the same one as before, i.e., to maximize the cost the state first goes to the origin straight at a maximal speed, and, after the state arrives at the origin at a time $-\log(1-|x|)$, it stays there. Consequently, we see

$$\frac{v(x)}{c} = \int_{-\log(1-|x|)}^\infty e^{-s} ds = 1 - |x|.$$

In other words, (7.29) gives the envelope solution of (7.8). However, if we choose a control set as $\partial \overline{B}_1(0)$ instead of $\overline{B}_1(0)$, the value function with the discontinuous cost cI becomes zero because the state $X^\alpha(\cdot)$ is enable to stop at the origin whatever α is chosen. This implies that we need to take the control set suitably to represent the envelope solution.

A Preservation of Lipschitz and uniform continuity

Proof of Proposition 3.17. We first remark that $w(x,t) := u_0(x) + mt \in SUP(H, u_0)$. Take any $(\hat{x}, \hat{t}) \in Q$, $h \in (0, T - \hat{t})$ and define

$$\tilde{u}(x,t) := \begin{cases} w(x,t) & (t \in [0,h]), \\ u(x, t-h) + mh & (t \in (h, T)). \end{cases}$$

We claim $\tilde{u} \in SUP(H, u_0)$. Let $(p, \tau) \in D^- \tilde{u}(\hat{x}, \hat{t})$. Then it follows easily that $\tau + H(\hat{x}, p) \geqq 0$ when $\hat{t} \neq h$, and so we only consider the case $\hat{t} = h$. Since $u \in SUB(H, u_0)$ and $w \in SUP(H, u_0)$, we see by the comparison principle that $u \leqq w$ in Q. Take $(x,t) \in \mathbf{R}^n \times (h, T)$, and substitute $(x, t-h)$ into the inequality. Then we find $u(x, t-h) \leqq u_0(x) + m(t-h)$, namely $\tilde{u}(x,t) \leqq w(x,t)$. This implies the relation $D^- \tilde{u}(\hat{x}, h) \subset D^- w(\hat{x}, h)$, and hence our claim follows from $w \in SUP(H, u_0)$.

Applying the comparison principle to $u \in SUB(H, u_0)$ and $\tilde{u} \in SUP(H, u_0)$, we obtain $u \leqq \tilde{u}$ in Q. In particular, we have $u(\hat{x}, \hat{t} + h) \leqq \tilde{u}(\hat{x}, \hat{t} + h) = u(\hat{x}, \hat{t}) + mh$, that is

$$\frac{u(\hat{x}, \hat{t} + h) - u(\hat{x}, \hat{t})}{h} \leqq m.$$

By the similar argument we also deduce

$$\frac{u(\hat{x}, \hat{t} + h) - u(\hat{x}, \hat{t})}{h} \geqq -m$$

from the fact that $u_0(x) - mt \in SUB(H, u_0)$ and so on. Thus $\mathrm{Lip}_t[u] \leqq m$ is proved.

We next estimate $\mathrm{Lip}_x[u]$. Take any $(p,\tau) \in D^+u(\hat{x},\hat{t})$. Since the estimate $\mathrm{Lip}_t[u] \leqq m$ implies $|\tau| \leqq m$, we see from $u \in SUB(H)$ that $H(\hat{x},p) \leqq -\tau \leqq m$, hence that $|p| \leqq R_+(m)$ by (H_{R+}). This observation means

$$\sup_{\substack{(\hat{x},\hat{t})\in Q \\ (p,\tau)\in D^+u(\hat{x},\hat{t})}} |p| \leqq R_+(m),$$

and moreover Lemma A.2 (1) and (2) below ensure that

$$\sup_{\hat{t}\in(0,T)} \sup_{\substack{x,y\in\mathbf{R}^n \\ x\neq y}} \frac{|u(x,\hat{t})-u(y,\hat{t})|}{|x-y|} \leqq R_+(m).$$

We thus conclude that $\mathrm{Lip}_x[u] \leqq R_+(m)$. □

Remark A.1. From the proof it turns out that it is sufficient to assume that H satisfies (H_{R+}) or (H_{R-}) only for all $m \in I$, where I is the range of $m(\cdot)$ in (H_m) on $[0,\infty)$. For example $H(x,p) = -|p|/(1+|p|)$ is not coercive but the same conclusion in Proposition 3.17 still holds since we have $0 \leqq m(\rho) < 1$ $(\rho \geqq 0)$ and $R_-(m) < \infty$ $(0 \leqq m < 1)$.

Lemma A.2. (1) *Let $f : \mathbf{R}^N \to \mathbf{R}$ be bounded. Then we have*

$$\sup_{\substack{x,y\in\mathbf{R}^N \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|} = \sup_{\substack{\hat{x}\in\mathbf{R}^N \\ p\in D^+f(\hat{x})}} |p|.$$

(2) *Let $u : Q \to \mathbf{R}$ be continuous. Assume that $\mathrm{Lip}_t[u] < \infty$. Then we have*

$$\sup_{\substack{(\hat{x},\hat{t})\in Q \\ (p,\tau)\in D^+u(\hat{x},\hat{t})}} |p| = \sup_{\hat{t}\in(0,T)} \sup_{\substack{\hat{x}\in\mathbf{R}^n \\ p\in D^+(u|_{t=\hat{t}})(\hat{x})}} |p|.$$

Proof. (1) This is well-known even in multi-dimensional setting; see, e.g., [25, Proposition 5.8].

(2) Denote by L_u and R_u respectively the left hand side and the right hand side. Then we obtain $L_u \leqq R_u$ by the separation of variables of a test function. Let us show $L_u \geqq R_u$. Fix $(\hat{x},\hat{t}) \in Q$, $p \in D^+(u|_{t=\hat{t}})(\hat{x})$ and take a corresponding test function $\psi \in C^1(\mathbf{R}^n)$. We may assume that $u|_{t=\hat{t}} - \psi$ attains its strict maximum at $\hat{x}$. Define $C := \mathrm{Lip}_t[u]+1$, $g(t) := C|t-\hat{t}|$, $\phi := \psi + g$, $g^\varepsilon(t) := C\sqrt{|t-\hat{t}|^2+\varepsilon}$ and $\phi^\varepsilon := \psi + g^\varepsilon$. Then $u-\phi$ attains its strict maximum at $(\hat{x},\hat{t})$ and $u-\phi^\varepsilon$ converges to $u-\phi$ uniformly. Therefore, by the lemma on convergence of maximum points (see [21, Lemma 2.2.5]), there exists a sequence $\{(x^\varepsilon,t^\varepsilon)\}_{\varepsilon>0}$ such that $(x^\varepsilon,t^\varepsilon) \to (\hat{x},\hat{t})$ and $u-\phi^\varepsilon$ attains its local maximum at $(x^\varepsilon,t^\varepsilon)$ for each $\varepsilon > 0$. Then we have $(\nabla\psi(x^\varepsilon),(g^\varepsilon)'(t^\varepsilon)) \in D^+u^\varepsilon(x^\varepsilon,t^\varepsilon)$ and $\nabla\psi(x^\varepsilon) \to \nabla\psi(x) = p$, which yield $L_u \geqq R_u$. □

We next prove Proposition 3.18. In the following proof we use the fact that if uniformly continuous functions f^δ ($\delta > 0$) converges to f uniformly as $\delta \downarrow 0$, then f is also uniformly continuous. Let ω^δ be a modulus of f^δ. Then

$$\begin{aligned}|f(x) - f(y)| &\leqq |f(x) - f^\delta(x)| + |f^\delta(x) - f^\delta(y)| + |f^\delta(y) - f(y)| \\ &\leqq 2\|f - f^\delta\| + \omega^\delta(|x - y|),\end{aligned}$$

and hence our claim follows. We also find that f has

$$\omega(r) = \inf_{\delta>0} \Big(2\|f - f^\delta\| + \omega^\delta(r)\Big)$$

as its modulus and that there is no need to assume the existence of a common modulus of f^δ.

Proof of Proposition 3.18. By the assumption (H_m) there exists a solution $u^\delta \in SOL(H, u_0^\delta)$ for each $\delta > 0$, and Proposition 3.17 implies that $u^\delta \in BLip(Q)$ since $u_0^\delta \in BLip(\mathbf{R}^n)$. Now, by using the inequality

$$\|u - u^\delta\|_Q \leqq \|u_0 - u_0^\delta\|_{\mathbf{R}^n} \tag{A.1}$$

in Remark 2.7 (1) we find that u^δ converges to u uniformly in Q as $\delta \downarrow 0$. Besides, recalling the remark before this proof, we see $u \in BUC(Q)$ and

$$\omega_0(r) := \inf_{\delta>0} \Big(2\|u - u^\delta\|_Q + \mathrm{Lip}[u^\delta]r\Big)$$

is a modulus of u. Applying (A.1) and the estimate of $\mathrm{Lip}[u^\delta]$ in Proposition 3.17, we obtain the desired form of ω. $\square$

B Existence of $\overline{D}$-solutions to general equations

In this section we discuss a sufficient condition which guarantees that an envelope solution of the original problem is a $\overline{D}$-solution of the relaxed problem. As the original problem, we consider (HJ) with H of the form (1.10); that is

$$(\mathrm{HJ0}) \begin{cases} \partial_t u + H_0(x, \nabla u) = r(x) & \text{in } Q, \\ (1.2). \end{cases} \tag{B.1}$$

To study the relaxed problem we make the following assumptions on H_0 and r:

$$\begin{cases} \text{(i)} & H_0 \in C(\mathbf{R}^n \times \mathbf{R}^n) \text{ and the map } \mu \mapsto H_0(0, \mu p) \text{ is nonincreasing} \\ & \text{on } \{\mu \geqq 0\} \text{ for all } p \in \mathbf{R}^n. \\ \text{(ii)} & r \in C(\mathbf{R}^n \setminus \{0\}) \text{ and } r_* \in C(\mathbf{R}^n). \end{cases} \tag{B.2}$$

The condition (B.2) implies (i)$'$, (ii)$'$ and (4.4) in Example 4.1, and therefore (H_r) is satisfied. The corresponding relaxed Hamiltonian $\hat{H}$ is

$$\hat{H}(x, p) := \begin{cases} H(x, p) & (x \neq 0), \\ \min\{H^*(0, p),\ H(0, 0)\} & (x = 0). \end{cases}$$

Setting $P := \{p \in \mathbf{R}^n \mid H^*(0,p) \geqq H(0,0)\}$, we have

$$\hat{H}(0,p) = \begin{cases} H(0,0) & (p \in P), \\ H^*(0,p) & (p \notin P), \end{cases}$$

and so $\hat{H}$ is continuous in $\{(x,p) \mid x \neq 0 \text{ or } p \notin P\}$. We also notice that P is a closed set in $\mathbf{R}^n$ by the continuity of H^*. We have already shown in Section 4 that an envelope solution of (HJ0) is also an envelope solution of (r.HJ).

Our gold in the rest of this section is to prove that an envelope solution of (r.HJ) is a $\overline{D}$-solution of the same problem. To do this the following two assertions play crucial roles.

Lemma B.1. *Let $u : Q \to \mathbf{R}$ be continuous.*

(1) *Assume that u satisfies the following property:*

(U1) *For all $\hat{t} \in (0,T)$ there exist some $d > 0, \delta > 0$ and $C \geqq 0$ such that $u|_{x=x'} + Ct^2$ is convex in $(\hat{t}-\delta, \hat{t}+\delta)$ for each $x' \in B_d(0)$.*

Then

$$\tau \in D^-(u|_{x=0})(\hat{t}) \quad \text{for all } (p,\tau) \in \overline{D}^- u(0,\hat{t}) \text{ with } \hat{t} \in (0,T). \tag{B.3}$$

(2) *Assume that H_0 and r satisfies* (B.2). *Assume furthermore that u is a Lipschitz continuous $\overline{D}$-supersolution of* (4.1). *Then*

$$\tau + H(0,0) \geqq 0 \quad \text{for all } \tau \in D^-(u|_{x=0})(\hat{t}) \text{ with } \hat{t} \in (0,T). \tag{B.4}$$

A function $f : \mathbf{R}^n \to \mathbf{R}$ is said to be *semiconvex* if $f(x) + C|x|^2$ is convex for some $C > 0$. Thus the condition (U1) requires some kind of semiconvexity of u with respect to t. We remark that, when $f(x) + C|x|^2$ is convex, functions $f(x)+C|x-a|^2$ are also convex for all $a \in \mathbf{R}^n$. Indeed, we have $f(x)+C|x-a|^2 = f(x)+C|x|^2-2C\langle x,a\rangle+C|a|^2$, and the right hand side is the sum of two convex functions. This implies the convexity of $f(x) + C|x-a|^2$.

Proof. (1) Let $(p,\tau) \in \overline{D}^- u(0,\hat{t})$. Take an approximate sequence $(p_m,\tau_m) \in D^-u(x_m,t_m)$ for (p,τ) and their test functions $\phi_m \in C^1(\mathbf{R}^n \times (0,T))$. We may assume that ϕ_m is of the separated form, i.e, $\phi_m(x,t) = \psi_m(x) + g_m(t)$ for some $\psi_m \in C^1(\mathbf{R}^n)$ and $g_m \in C^1(0,T)$. Since $u|_{x=x_m} - g_m$ attains its minimum at t_m, we see that t_m is also a minimum point of $u|_{x=x_m} + C(t-t_m)^2 - g_m$, where C is the constant in (U1). As we remarked after the statement of Lemma B.1, the function $u|_{x=x_m} + C(t-t_m)^2$ is convex in $(\hat{t}-\delta, \hat{t}+\delta)$. Thus t_m is a minimum point of $u|_{x=x_m}+C(t-t_m)^2-g'(t_m)t$ over $(\hat{t}-\delta, \hat{t}+\delta)$. We now apply $g'(t_m) = \tau_m$ and send $m \to \infty$. Then, from the continuity of u at $(0,\hat{t})$, it follows that the function $u|_{x=0} + C(t-\hat{t})^2 - \tau t$ attains its minimum over $(\hat{t}-\delta, \hat{t}+\delta)$ at $\hat{t}$. This implies

$$D^-(u|_{x=0})(\hat{t}) \ni \frac{d}{dt}\{\tau t - C(t-\hat{t})^2\}\bigg|_{t=\hat{t}} = \tau,$$

which is the desired conclusion.

(2) Choose any $\tau \in D^-(u|_{x=0})(\hat{t})$, and take a test function $g \in C^1(0,T)$ for τ. We may assume that $u|_{x=0}-g$ attains its strict minimum at $\hat{t}$. For $\varepsilon > 0$ we define $\gamma := \mathrm{Lip}_x[u]+1$, $\psi(x) := -\gamma|x|$, $\phi(x,t) := \psi(x)+g(t)$, $\psi^\varepsilon(x) := -\gamma\sqrt{|x|^2+\varepsilon}$ and $\phi^\varepsilon(x,t) := \psi^\varepsilon(x)+g(t)$. Then $u-\phi$ attains its strict minimum at $(0,\hat{t})$, and $u-\phi^\varepsilon$ converges to $u-\phi$ uniformly as $\varepsilon \to 0$. Thus there exists a sequence $\{(x^\varepsilon,t^\varepsilon)\}_{\varepsilon>0}$ such that $(x^\varepsilon,t^\varepsilon) \to (0,\hat{t})$ as $\varepsilon \to 0$ and that $u-\phi^\varepsilon$ attains its local minimum at $(x^\varepsilon,t^\varepsilon)$ for each $\varepsilon > 0$ ([21, Lemma 2.2.5]). Now, since $|\nabla\psi^\varepsilon| \leqq \gamma$ for all $\varepsilon > 0$, taking a subsequence if necessarily, we may let $\nabla\psi^\varepsilon(x^\varepsilon)$ converge to some $\bar{p} \in \mathbf{R}^n$. Moreover $\lim_{\varepsilon\to 0} g'(t^\varepsilon) = g'(\hat{t}) = \tau$, and so we see $(\bar{p},\tau) \in \overline{D}^- u(0,\hat{t})$. Since u is a $\overline{D}$-supersolution of (4.1), we conclude $0 \leqq \tau + \hat{H}(0,\bar{p}) \leqq \tau + H(0,0)$. □

Remark B.2. (1) If a function u is of the form (4.8), then u satisfies (U1) with $C=0$. Thus, when a Hamiltonian is given as (1.3), we conclude that (4.7) is a sufficient condition for (U1) since (4.7) implies (4.8). For more general Hamiltonians, as far as the author knows, there is no convenient sufficient condition on the initial data which guarantees (U1) for envelope solutions u of (HJ0).

(2) The conclusion (B.3) in Lemma B.1 (1) may not hold if we remove (U1). To see this, let us consider the solution u given by (4.6). Concerning its subdifferentials, we have $0 \notin D^-(u|_{x=0})(2)$ and $(0,0) \in \overline{D}^- u(0,2)$; that is, (B.3) is violated. We also notice that u does not satisfy (U1) at $\hat{t}=2$.

Remark B.3. If we assume

$$\limsup_{(x,p)\to(0,\infty),\ x\neq 0} H(x,p) \leqq H(0,0) \tag{B.5}$$

in Lemma B.1 (2) instead of the Lipschitz continuity of u, the same conclusion follows. In this case we take ψ and ψ^ε in the proof as

$$\psi(x) = \begin{cases} 0 & (x=0) \\ -\infty & (x\neq 0) \end{cases}, \quad \psi^\varepsilon(x) = -\frac{|x|^2}{\varepsilon}.$$

Since $\limsup^*_{\varepsilon\to 0} \psi^\varepsilon = \psi$, it follows from [21, Lemma 2.2.5] that there exists a sequence $\{(x^\varepsilon,t^\varepsilon)\}_{\varepsilon>0}$ satisfying the same properties as in the proof of Lemma B.1 (2) by taking a subsequence if necessary. Set $p^\varepsilon := \nabla\psi^\varepsilon(x^\varepsilon) = -2x^\varepsilon/\varepsilon$ and assume that the sequence $\{p^\varepsilon\}_{\varepsilon>0}$ is not bounded; otherwise the same proof runs as before. By the unboundedness of p^ε we notice $x^\varepsilon \neq 0$ for very small ε. Since $(p^\varepsilon, g'(t^\varepsilon)) \in D^- u(x^\varepsilon,t^\varepsilon)$ and u is a supersolution of (4.1), we have $g'(t^\varepsilon) + H(x^\varepsilon,p^\varepsilon) \geqq 0$. Taking $\limsup^*_{\varepsilon\to 0}$ in the inequality, we obtain $\tau + H(0,0) \geqq 0$ by (B.5).

Proposition B.4 (Existence of $\overline{D}$-solutions)**.** *Assume that H_0 and r satisfies* (B.2)*. Let u be a continuous envelope solution of* (HJ0) *such that $u = \inf_{w\in\mathcal{S}} w$ for some $\mathcal{S}$ which consists of Lipschitz continuous $\overline{D}$-supersolutions of* (B.1)*. Assume furthermore that u satisfies* (U1)*. Then u is a $\overline{D}$-solution of* (r.HJ).

Proof. From Lemma 4.4 and the inequality $H \leqq \hat{H}$ it follows that u is an envelope solution of (r.HJ).

Let $(p,\tau) \in \overline{D}^- u(\hat{x},\hat{t})$. When $\hat{x} \neq 0$ or $p \notin P$, we apply the stability result under infimum for standard supersolutions ([11, Lemma 4.2]) to obtain

$$0 \leqq \tau + (\hat{H})^*(\hat{x},p) = \tau + \hat{H}(\hat{x},p)$$

since $\hat{H}$ is continuous at $(\hat{x},p)$. We next study the case where $\hat{x}=0$ and $p \in P$. By the definition of P our goal is to show

$$\tau + H(0,0) \geqq 0. \tag{B.6}$$

Since u satisfies (U1), Lemma B.1 (1) implies

$$\tau \in D^-(u|_{x=0})(\hat{t}). \tag{B.7}$$

We next apply Lemma B.1 (2) to each $w \in \mathcal{S}$. It then follows that $\partial_t(w|_{x=0}) + H(0,0) \geqq 0$ in $(0,T)$ in the standard viscosity sense. Thus, by the stability under infimum we see $\partial_t(u|_{x=0}) + H(0,0) \geqq 0$ in $(0,T)$ in the standard viscosity sense. Using (B.7), we finally deduce (B.6). □

C On effects of discontinuity and measures

If we see source terms with jump discontinuity such as $I(x)$ with respect to the Lebesgue measure, we cannot observe the discontinuity since, in the case of $I(x)$, it is equal to zero in the almost everywhere sense. However, if we use another measure, say the counting measure, we may not neglect such discontinuity at a singleton. It is thus reasonable to establish a notion of envelope solutions depending on a given measure. For this purpose we introduce an essential semicontinuous envelope as follows. Let μ be a Borel (regular) measure on $\mathbf{R}^n$. Assume that $0 < \mu(B) \leqq \infty$ for any open ball $B = B_r(x)$ with radius $r > 0$ centered at $x \in \mathbf{R}^n$. For a Borel measurable function $f : \mathbf{R}^n \to \mathbf{R}$ we define an *essential lower semicontinuous envelope of f with respect to μ* as

$$f_{\mu *}(x) := \lim_{\delta \to 0} \left(\operatorname*{ess\,inf}_{\overline{B}_\delta(x)} f \right),$$

where $\operatorname{ess\,inf}_{\overline{B}_\delta(x)} f := \sup\{\lambda \in \mathbf{R} \mid \mu(\{x \in \mathbf{R}^n \mid f(x) < \lambda\} \cap \overline{B}_\delta(x)) = 0\}$. In other words, we modify a set of discontinuity of f with μ-zero measures. Assuming that all balls have positive measure, we see that $f_{\mu *}$ is well-defined as a real-valued function provided that f is locally bounded. Also, as we will see below, $f_{\mu *}$ is a lower semicontinuous function.

We apply this essential envelope to the original Hamiltonian H, and solve (HJ) with $H_{\mu *}$ instead of H. We then obtain an envelope solution on which the measure μ has an effect. When a Hamiltonian is of the form (1.10), we have $H_{\mu *}(x,p) = H_0(x,p) - r^{\mu *}(x)$ with $r^{\mu *} := -(-r)_{\mu *}$. For instance, when the source term r is given as (1.12), we have $r^{\mu *} = 0$ if μ is the n-dimensional Lebesgue measure while $r^{\mu *} = r$ if μ is the counting measure. The former example implies that the inequality $r \leqq r^{\mu *}$ does not necessarily hold although we always have $r \leqq r^*$ for the usual upper semicontinuous envelope.

We prove that $f_{\mu *}$ is lower semicontinuous. Take $x \in \mathbf{R}^n$ and $\delta > 0$. We also choose $y \in B_\delta(x)$ and $d > 0$ small so that $\overline{B}_d(y) \subset \overline{B}_\delta(x)$. This inclusion relation implies $\operatorname{ess\,inf}_{\overline{B}_d(y)} f \geqq \operatorname{ess\,inf}_{\overline{B}_\delta(x)} f$. We send $d \to 0$ and then take $\liminf_{y \to x}$ in the inequality to obtain $\liminf_{y \to x} f_{\mu *}(y) \geqq \operatorname{ess\,inf}_{\overline{B}_\delta(x)} f$. Finally, sending $\delta \to 0$ gives $\liminf_{y \to x} f_{\mu *}(y) \geqq f_{\mu *}(x)$, which means that $f_{\mu *}$ is lower semicontinuous at x.

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Chapter 2

Asymptotically self-similar solutions to curvature flow equations with prescribed contact angle and their applications to groove profiles due to evaporation-condensation

1 Introduction

We are concerned with the asymptotic behavior of solutions to second order parabolic equations with the Neumann boundary condition of the form

$$\text{(NP)} \begin{cases} \partial_t u(x,t) = F(\nabla u(x,t), \nabla^2 u(x,t)) & \text{in } \Omega \times (0,\infty), & (1.1) \\ u(x,0) = u_0(x) & \text{on } \overline{\Omega}, & (1.2) \\ \partial_{x_1} u(x,t)|_{x_1=0} = \beta > 0 & \text{on } \partial\Omega \times (0,\infty), & (1.3) \end{cases}$$

which we also denote by (NP; F, u_0). Here $\Omega = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_1 > 0\}$ is the half space, ∇u and $\nabla^2 u$ denote, respectively, the gradient and Hessian matrix of u with respect to x, and the initial data u_0 is bounded and uniformly continuous, i.e., $u_0 \in BUC(\overline{\Omega})$. A given real-valued function F is continuous and degenerate elliptic. Our goal in this paper is to prove that (viscosity) solutions of (NP) asymptotically converge to a self-similar solution of the associated problem, and study properties of a profile function of the self-similar solution.

Our study is motivated by evaporation-condensation model which was first proposed by a material scientist Mullins in [43]. Consider the situation that there are two crystal grain regions (solid phases) on the plane which consist of the same matter and differ only in their relative crystalline orientation. Let the two region be $\{(x,y) \mid x \geqq 0,\ y \leqq u(x,t)\}$ and $\{(x,y) \mid x \leqq 0,\ y \leqq \tilde{u}(x,t)\}$ at time $t \geqq 0$, where we assume $u(0,t) = \tilde{u}(0,t)$ so that a triple junction appears at the point $(0, u(0,t))$; see Figure 1. Moreover, we assume the symmetry, i.e., $u(x,t) = \tilde{u}(-x,t)$ for $x > 0$. The rest part on the plane is filled by gas. The intersection between the two crystal regions, which is called a grain boundary, is assumed to be stable on the line $x = 0$. We suppose that due to evaporation and condensation crystal atoms move between solid phases and gas phase. This mechanism leads development of a surface groove at the grain boundary, which we call a thermal groove, as in Figure 1. In this setting we study evolution of

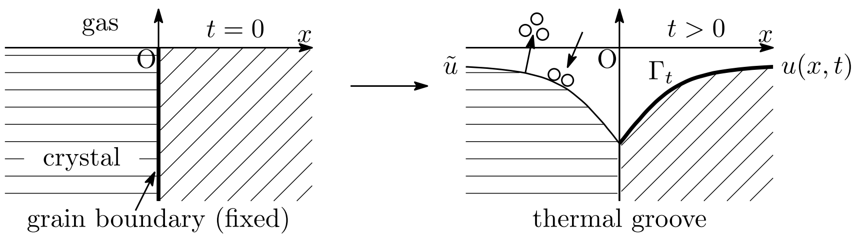

Figure 1: The thermal groove develops due to evaporation-condensation.

interfaces between crystal grains and gas. By symmetry we consider the interface only in the right region, which we represent as $\Gamma_t := \{(x, u(x,t)) \in \mathbf{R}^2 \mid x \geqq 0\}$. According to Mullins' theory in [43] the evolution equation for Γ_t is given as

$$V_n = C_0 \left(1 - e^{-C_1 k}\right) \quad \text{on } \Gamma_t, \tag{1.4}$$

where V_n is the upward normal velocity of Γ_t, k is the upward (mean) curvature, and C_0, C_1 are positive constants. Thus, taking $C_0 = C_1 = 1$ for simplicity, we obtain the following partial differential equation for u:

$$\frac{u_t}{\sqrt{1 + u_x^2}} = 1 - e^{-k} \tag{1.5}$$

in $\{x > 0\} \times \{t > 0\}$, where $(u_t, u_x, u_{xx}) = (\partial_t u, \partial_x u, \partial_{xx} u)$. Here we have invoked the formula $V_n = u_t / \sqrt{1 + u_x^2}$, and also the curvature k is represented by $k = u_{xx} / \sqrt{1 + u_x^2}^3$ ([22, Chapter 1.2, 1.4]). In this model a boundary condition on u at $x = 0$ is given as

$$u_x(0,t) \equiv \beta > 0, \tag{1.6}$$

which is the prescribed angle condition and results from equilibrium of tensions at the triple junction point $(0, u(0,t))$. Hence solving the Cauchy problem for (1.5) under the Neumann boundary condition (1.6) gives the surface profile due to evaporation-condensation. The problem (NP) is a generalized multidimensional case of this model.

In [43] Mullins approached the equation (1.5) via two approximations. He first applies the linear approximation of the exponential term, which is

$$1 - e^{-k} \approx k. \tag{1.7}$$

Then the original equation (1.5) simplifies to

$$v_t = \frac{v_{xx}}{1+v_x^2}, \tag{1.8}$$

which is the usual mean curvature flow equation for graphs. To solve (1.8) Mullins next applies the second approximation that

$$v_x \approx 0. \tag{1.9}$$

This condition comes from physical assumption that slopes on the surface are sufficiently small, which especially implies $\beta \ll 1$. Applying (1.9) to (1.8) finally yields

$$w_t = w_{xx}. \tag{1.10}$$

Since this is the simple heat equation, its classical solution w with the initial-boundary conditions $w(x,0) \equiv 0$ and $w_x(0,t) \equiv \beta > 0$ exists and has the explicit form; see Example 2.3. In this way Mullins concludes that the groove profile due to evaporation-condensation is given by the solution w. In particular, putting $x = 0$, Mullins computes the depth of the developing thermal groove at the origin, which is

$$-w(0,t) = 2\beta\sqrt{\frac{t}{\pi}} \approx 1.13\beta\sqrt{t}. \tag{1.11}$$

In this paper we aim at justifying these two approximations by Mullins. Namely, we rigorously discuss a relation among the three solutions u, v and w. The point in our study is that the solutions v of (1.8) and w of (1.10) are (forward) *self-similar*, i.e., they are of the form

$$v(x,t) = \sqrt{t}V\left(\frac{x}{\sqrt{t}}\right), \quad w(x,t) = \sqrt{t}W\left(\frac{x}{\sqrt{t}}\right).$$

The functions V and W are called *profile functions* of v and w, respectively. Then, as a justification for the first approximation, we prove

$$\frac{1}{\sqrt{t}}u(\sqrt{t}x, t) \to V(x) \quad \text{as } t \to \infty \tag{1.12}$$

in Theorem 3.4. This convergence result says that if we rescale the solution u of (1.5) in the above way, then it converges to the profile function V of the approximated equation. In other words, u itself is not necessarily self-similar, but it is asymptotically self-similar in the above sense.

We prove such an asymptotic result for more general problems of the form (NP) in Section 3. As a special structure of the equation (1.1) we direct our attention to homogeneity of F. Here we say F (or (1.1)) is *homogeneous* if F is positively homogeneous of degree 1 with respect to X, i.e., $F(p,X) = \lambda F(p, X/\lambda)$ for $\lambda > 0$. Evidently, the equation (1.8) is homogeneous. It also turns out that solutions of the homogeneous equations with the zero initial data are self-similar. Thus (1.8) can be generalized to homogeneous equations. In order to explain how we generalize (1.5) to the equation (1.1) with $G : \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}^n$, we shall give an idea of the proof of (1.12). Let u and v be, respectively, a solution of $(\mathrm{NP}; G, u_0)$ and $(\mathrm{NP}; F, 0)$, where F is homogeneous. We prove the result (1.12) by showing that rescaled functions of u converge to v; namely,

$$u_{(\lambda)}(x,t) := \frac{1}{\lambda} u(\lambda x, \lambda^2 t) \to v(x,t) \quad \text{as } \lambda \to \infty. \tag{1.13}$$

It is easy to see that this rescaled function $u_{(\lambda)}$ is a solution to the rescaled equation $(\mathrm{NP}; G_\lambda, (u_0)_{(\lambda)})$ with $G_\lambda(p, X) = \lambda G(p, X/\lambda)$ and $(u_0)_{(\lambda)}(x) = u_0(\lambda x)/\lambda$. Since $(u_0)_{(\lambda)} \to 0$ as $\lambda \to \infty$, we can conclude that if G_λ converges to F, then the limit of $u_{(\lambda)}$ solves $(\mathrm{NP}; F, 0)$. By uniqueness the limit should be v, and hence we obtain (1.13). Note that our convergence result (1.13) holds for a solution u of $(\mathrm{NP}; G, u_0)$ with an arbitrary initial data $u_0 \in BUC(\overline{\Omega})$.

In this way we are led to introduce a notion that G is *asymptotically homogeneous*, which roughly means that G approximates some homogeneous function in a suitable sense. To be more precise, we require that $G_\lambda(p, x) := \lambda G(p, X/\lambda)$ converge to some homogeneous F as $\lambda \to \infty$. By a simple calculation we see that (1.5) is asymptotically homogeneous with the limit (1.8). Accordingly the asymptotic homogeneity is a generalized notion containing (1.5), and the Mullins' first approximation is then generalized to

$$G \approx F.$$

To show the convergence of $u_{(\lambda)}$ to v rigorously we employ stability results of viscosity solutions. Due to comparison principle for (NP), we see that the upper and lower relaxed limit of $u_{(\lambda)}$, which are a sub- and supersolution respectively, should agree with v provided that the relaxed limits exist. Thus the remaining problem, which is our main difficulty, is to show the existence of the relaxed limits. This is achieved by constructing suitable barriers which are of order $O(\sqrt{t})$ as $t \to \infty$; see Lemma 3.5 and the proof of Theorem 3.4.

We turn to the second approximation by Mullins, to which we dedicate Section 5. Since the solution v of (1.8) and w of (1.10) are self-similar, we consider only their profile functions. Our main interest is to examine adequateness of Mullins' conclusion (1.11) concerning the depth of the thermal groove at the origin. For this purpose we compare the depths of two profile functions at the origin; one is the original depth $-V(0)(= -v(0,1))$ which comes from (1.8) and the other is the approximated depth $-W(0)(= -w(0,1))$ corresponding to (1.10). Recall that $-W(0)$ has the explicit form that $-W(0) = 2\beta/\sqrt{\pi}$ by (1.11). We prove among other results that, in Mullins' problem, $-W(0)$ is the third order approximation of $-V(0)$, i.e.,

$$-V(0) = -W(0) + O(\beta^3) \quad \text{as } \beta \to 0. \tag{1.14}$$

In this paper we discuss such comparison of the two depths for more general equations. From results for the general case we deduce (1.14). To discuss the general case let us consider (NP) with a homogeneous F. Since the problem $(\mathrm{NP}; F, 0)$ does not include the variables $x_2, \dots, x_n$, its self-similar solution depends only on x_1 and t. Thus, in what follows we let the spatial dimension n

be one so that the profile function V is defined on $\mathbf{R}$. Then it turns out that V satisfies the ordinary differential equation of the form

$$V(\xi) - \xi V'(\xi) = a(V'(\xi))V''(\xi) \quad \text{in } (0,\infty), \tag{1.15}$$

where a is given by $a(p) := -2F(p,-1)$. Note that $a(p) = 2/(1+p^2)$ in Mullins' case since $F(p,X) = X/(1+p^2)$ for (1.8). Let us recall the Mullins' second approximation which replaces the first derivative v_x by zero. As its analogue, for the general equation (1.15) we replace $a(V'(\xi))$ in the right hand side by $a(0)$, i.e., we apply

$$a(V'(\xi)) \approx a(0).$$

This is a generalized Mullins' second approximation. The resulting approximated equation is

$$W(\xi) - \xi W'(\xi) = a(0)W''(\xi) \quad \text{in } (0,\infty), \tag{1.16}$$

which represents the heat equation if we return (1.16) to the parabolic problem. Let V and W be, respectively, the unique viscosity solution of (1.15) and (1.16) with the boundary conditions that $V'(0) = \beta$ and $V(\infty) = 0$. A well-posedness of these equations in the viscosity sense is a consequence of that of parabolic equations (NP). We also remark that W has the explicit form. In this general setting we prove that the estimate

$$0 \leqq \frac{V(0) - W(0)}{\beta} \leqq C\left(a(0) - \min_{[0,\beta]} a\right) \tag{1.17}$$

holds for some positive constant C independent of β. This result implies $-V(0) = -W(0) + o(\beta)$ as $\beta \to 0$ for general equations and (1.14) for Mullins' case where $a(p) = 2/(1+p^2)$. The main tool for the proof of (1.17) is comparison principle. Namely, if we have a subsolution V_1 and a supersolution V_2, then we obtain an inequality $V_1 \leqq V_2$ and in particular $-V_1(0) \geqq -V_2(0)$. To this end we seek a suitable sub- or supersolution of the ordinary differential equation. We also deduce a couple of other estimates on the depth by the comparison method.

Our another interest is degenerate cases. We study (1.15) when $a(p)$ is allowed to be zero. Even in such degenerate cases the unique solution to (1.15) exists in the viscosity sense. As an instructive example, we now let $a(p) = 0$ for $p \in [q^-, q^+]$ and $a(p) > 0$ otherwise. Then a simple observation indicates

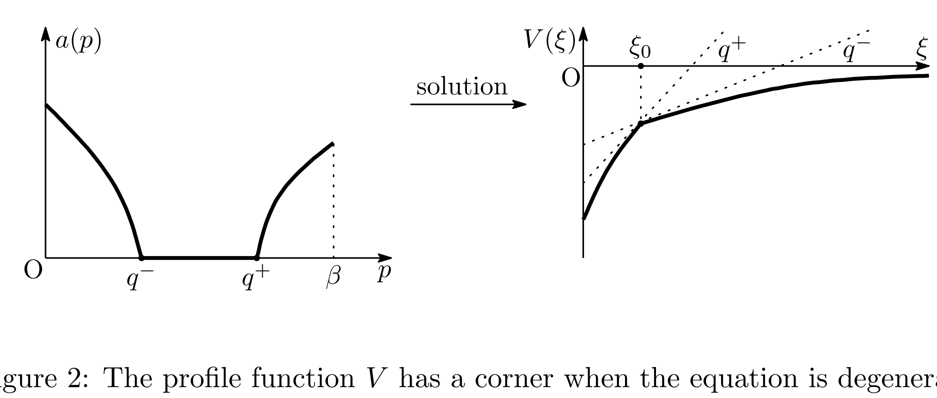

Figure 2: The profile function V has a corner when the equation is degenerate.

that the unique solution V has a corner whose angles are determined by q^- and q^+. Indeed, if we admit that V is negative and increasing (these properties are shown in Proposition 4.3), we notice by (1.15) that $0 > a(V'(\xi))V''(\xi)$. This implies $V'(\xi) \notin [q^-, q^+]$; in other words, the derivative of V jumps over the interval $[q^-, q^+]$. Rigorous statement and its proof on the corner of the viscosity solution V are given in Theorem 4.10, where we prove that there exists a unique $\xi_0 \in (0, \infty)$ such that the left and right derivatives of V at ξ_0 are, respectively, q^+ and q^-; see Figure 2.

Since the solution V of (1.15) is a profile function of the (forward) self-similar solution, it is natural to expect relation between V and the Wulff shape, which minimizes the total surface energy among all sets with the same volume. Although our interface Γ_t is now unbounded, we are able to relate the corner of the profile function V to that of the associated Wulff shape in the following way. For a given surface energy density $\gamma : S^{n-1} = \{x \in \mathbf{R}^n \mid |x| = 1\} \to (0, \infty)$ we define a Wulff shape associated with γ by

$$\mathrm{Wulff}(\gamma) = \bigcap_{|q|=1} \{x \in \mathbf{R}^n \mid \langle x, q\rangle \leqq \gamma(q)\},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbf{R}^n$. Let us consider the evolution equation of the form

$$V_n = M(\mathbf{n})k_\gamma \quad \text{on } \Gamma_t, \tag{1.18}$$

where $M : S^{n-1} \to (0, \infty)$ is the mobility, $\mathbf{n}$ is the oriented normal vector on Γ_t, and k_γ is the anisotropic curvature with respect to the surface energy density γ. See, e.g., [22, Chapter 1.3] for the definition of k_γ. We now let $n = 2$ and assume that Γ_t is represented by a graph, i.e., $\Gamma_t = \{(x, u(x,t)) \in \mathbf{R}^2\}$. Then, choosing $\mathbf{n}$ as the upward normal vector and using the formula

$$k_\gamma = (\tilde{\gamma}''(\arg \mathbf{n}) + \tilde{\gamma}(\arg \mathbf{n}))k,$$

where $\arg \mathbf{n}$ is the argument of $\mathbf{n}$ and $\tilde{\gamma}(\theta) := \gamma(\cos\theta, \sin\theta)$, we see that (1.18) is rewritten as

$$\frac{u_t}{\sqrt{1+u_x^2}} = M\left(\frac{(-u_x, 1)}{\sqrt{1+u_x^2}}\right)\left(\tilde{\gamma}''(\arg(-u_x,1)) + \tilde{\gamma}(\arg(-u_x,1))\right)\frac{u_{xx}}{\sqrt{1+u_x^2}^3}.$$

The profile function of the self-similar solution of this equation satisfies the ordinary differential equation (1.15) with a of the form

$$a(p) = 2M\left(\frac{(-p, 1)}{\sqrt{1+p^2}}\right)\left(\tilde{\gamma}''(\arg(-p,1)) + \tilde{\gamma}(\arg(-p,1))\right)\frac{1}{\sqrt{1+p^2}^3}.$$

Therefore we see that $a(p) = 0$ for all $p \in [q^-, q^+]$ if and only if $\tilde{\gamma}''(\theta) + \tilde{\gamma}(\theta) = 0$ for all $\theta \in [\arg(-q^-, 1), \arg(-q^+, 1)]$. The latter condition on γ leads the corner point of $\mathrm{Wulff}(\gamma)$ at which the slope of each tangent line is in $[q^-, q^+]$. This agrees with the corner of our profile function shown in Figure 2.

Let us explain why the equation (1.5) (or (1.4)) and the boundary condition (1.6) appear in Mullins' model. The exponential term in (1.4) comes from the Gibbs-Thompson formula in physics. This formula asserts that the vapor pressure p in equilibrium with the surface is given as

$$\log\left(\frac{p}{p_0}\right) = -C_1 k, \tag{1.19}$$

where p_0 is the atmospheric pressure and C_1 is a positive constant. Now, recall that the only mechanism operative in the transport of matter is evaporation-condensation. Thereby the normal velocity V_n is determined by the difference between the effect by condensation and that by evaporation. According to kinetic theory their effects are in proportion to pressures p_0 and p, respectively, and thus

$$V_n = C_2(p_0 - p) \tag{1.20}$$

with $C_2 > 0$. It is now clear that (1.19) and (1.20) lead the equation (1.4) by letting $C_0 = C_2 p_0$. The prescribed angle condition (1.6) is a consequence of equilibrium of tensions. More precisely, the resultant of the grain boundary tension $(0, -\gamma_b) \in \mathbf{R}^2$ and two surface tensions $(\pm\gamma_s \cos\theta, \gamma_s \sin\theta) \in \mathbf{R}^2$ is assumed to vanish at $(0, u(0,t))$, where $\gamma_b > 0$ and $\gamma_s > 0$ are, respectively, the boundary free energy and the surface free energy per unit area and θ is the slope angle of u at $x = 0$. Thus we have $2\gamma_s \sin\theta = \gamma_b$, which implies (1.6).

In [43] Mullins proposes another mechanism for the development of surface groove, which is surface diffusion. If we take the surface diffusion into account, the resulting equation describing the surface profile becomes a fourth order nonlinear parabolic equation. In this paper, however, we do not discuss such effect by surface diffusion so that only second order equations appear in our study. As a result, we are able to apply the viscosity solution theory ([16]) to study the problem. Mullins gives a criterion for judging which mechanism dominates the development of surface. According to [43] for magnesium under high pressure the profile is completely shaped by evaporation-condensation after a very short time while surface diffusion plays a dominant role for a very long time for gold under low pressure. See, e.g., [11, 32, 42, 58] for the studies of fourth order equations related to the surface diffusion.

We next state previous work related to our study. Many authors investigate asymptotic behaviors of solutions to curvature flow type equations. We first refer the reader to [26], where surfaces evolving by the mean curvature over a domain in $\mathbf{R}^n$ are studied under the zero Neumann boundary condition. It is shown that the solution converges to a constant function as $t \to \infty$. In [2] Altschuler and Wu study Cauchy problems for quasilinear equations of the form $u_t = (a(u_x))_x$ on $\{0 \leqq x \leqq d\} \times [0, \infty)$. They prove that solutions of the problem asymptotically converge to a solution which moves at a constant speed. The same authors obtain in [3] a similar convergence result for surfaces over a convex domain in $\mathbf{R}^2$, but they deal with only the curvature flow equation.

Asymptotic behaviors of graph solutions to free boundary problems are also studied in the literature. The paper [13] treats a quasilinear parabolic equation $u_t = (a(u_x))_x$ under a two point free boundary condition. (The same problem restricted to the equation (1.8) can be found in [15].) In [13] two half-lines are given radially from the origin and solutions are required to have intersections with them, which are the free boundary points, at prescribed contact angles. A global existence and uniqueness of solutions to the parabolic problem are established. A convergence result to a self-similar solution is deduced together with its convergence rate in the sense of the Hausdorff metric. The parabolic equation in [13] is not allowed to be degenerate, but our results concerning a well-posedness and the asymptotic behavior include degenerate cases. A similar setting to [13] is found in [39], where a one-point free boundary problem is considered. The paper [13] deals with expanding interfaces while the preserving case and the shrinking case for the same problem are discussed in [24].

For graphs defined on a whole space, their convergence results to a self-similar solution are obtained in [20, 31]. The paper [20] studies mean curvature evolutions written as graphs over $\mathbf{R}^n$. Under a suitable rescaling the convergence result is obtained for initial data satisfying a linear growth condition and further assumptions. Ishimura, the author of [31], considers the spatially one dimensional equation (1.8) in $\mathbf{R} \times (0, \infty)$ with prescribed opening angle conditions; that is, $v_x \to K_1$ as $x \to \infty$ and $v_x \to -K_2$ as $x \to -\infty$ for given constants $K_1, K_2 > 0$.

Curvature flow equations with constant driving force

$$\frac{v_t}{\sqrt{1+v_x^2}} = \frac{v_{xx}}{\sqrt{1+v_x^2}^3} + c \tag{1.21}$$

and asymptotic convergences to traveling fronts are studied in several works. In [17] the authors consider (1.21) for $(x,t) \in (0,\infty) \times (0,\infty)$ with the zero Neumann condition at $x = 0$ and the opening angle condition at $x = \infty$. It is shown that the solution v converges to a traveling wave solution as $t \to \infty$ when c is positive, while for a negative c convergence to a self-similar solution is proven in the sense that $t^{-1}|v(x,t) - tQ(x/t)| \to 0$ as $t \to \infty$, where $tQ(x/t)$ is a solution of (1.21). The explicit form of Q is also found in [17]. Note that the way of rescaling is different from ours. The papers [48, 45] studies asymptotics of solutions to (1.21) on $\mathbf{R} \times (0, \infty)$ when c is positive. Convergence results to a traveling V-shaped solution are obtained for spatially decaying and non-decaying initial perturbations in [48] and [45], respectively. For the explicit form of the V-shaped front, see [47]. The reader is also referred to [44] for convergence to a traveling line.

The paper [46] is related to the Mullins' second approximation (1.9) and asymptotic stability of constant solutions. There it is shown that

$$\sup_{x \in \mathbf{R}} |v(x,t) - w(x,t)| = O(1/\sqrt{t}) \quad \text{as } t \to \infty, \tag{1.22}$$

where v and w are, respectively, the solution of the Cauchy problem for (1.8) and (1.10) in $\mathbf{R} \times (0, \infty)$ with the same initial data. Moreover, using the results, the authors of [46] obtain a necessary and sufficient condition on initial data that ensures $u \to 0$ uniformly or pointwisely as $t \to \infty$. In our Neumann problem on the half space, however, a similar convergence result to (1.22) does not hold since

$$\sup_{x\in[0,\infty)} |v(x,t) - w(x,t)| = \sqrt{t} \sup_{\xi\in[0,\infty)} |V(\xi) - W(\xi)| \to \infty \quad \text{as } t \to \infty$$

for two different self-similar solutions v and w of the forms $v(x,t) = \sqrt{t}V(x/\sqrt{t})$ and $w(x,t) = \sqrt{t}W(x/\sqrt{t})$.

Asymptotic shapes of expanding interfaces represented by a level set function are obtained in [29]. There the evolution equation $V_n = -\mathrm{tr}(E(\mathbf{n})D\mathbf{n}) + \nu(\mathbf{n})$ on Γ_t is considered, and it is shown that $\Gamma_t/t \to \partial\mathrm{Wulff}(\nu)$ as $t \to \infty$ in the Hausdorff metric. We remark that the limit is not the Wulff shape of the surface energy density in this work. To prove this large time asymptotics the authors study the limit of rescaled viscosity solutions of second order parabolic equations, and consider the corresponding stationary equations which the limit function satisfies. The result says that if u is a viscosity subsolution (resp. supersolution) of $\partial_t u + F_1(\nabla u, \nabla^2 u) + F_2(\nabla u) = 0$, then the (relaxed) limit of $u(tx,t)$ as $t \to \infty$ is a viscosity subsolution (resp. supersolution) of $-\langle x, \nabla u\rangle + F_2(\nabla u) = 0$. Note that this limit equation is first order while the second order equation (1.15), which V in (1.12) should satisfy, appears in our study.

Motion by curvature with triple junctions such as the point $(0, u(0,t))$ in Mullins' model is studied in [14]. There a planar domain surrounded by other phase domains is considered, and at each junction point three intersection angles are assumed to satisfy the Herring condition which is determined by interfacial energies. The authors of [14] give conditions for existence of self-similar stationary, expanding or shrinking solutions to the problem. Plane curves having the triple junction are also treated in [57], where the authors study evolving three curves by curvature forming 120 degree angles at their common start point. The authors of [57] derive several properties of solutions to (1.15) with $a(p) = 1/(1+p^2)$ and prove the unique existence of self-similar expanding solutions. As a study of expanding self-similar solutions we finally refer the reader to [21] for evolution by a crystalline curvature flow.

A generalized Mullins' model is proposed in [56, 49]. The author of [56] considers the model including a strain energy. In [49] Ogasawara studies evaporation-condensation model under a temperature gradient and proves an existence of stationary solutions to the resulting parabolic equation of the form $u_t = F(u, u_x, u_{xx})$. See also [50] for flattening properties of solutions to the generalized problem. Such flattening properties are also studied in [33, 37, 34, 38] for equations of the type (1.8) and in [35, 36] for those of the type (1.5).

Interestingly, an exact representation of the solution to (1.8) with $v_x(0,t) \equiv \beta$ and $v(x,0) \equiv 0$ is obtained by Broadbridge in [10]. However, we do not employ the formula in the present paper since generalization of the problem is one of our aims and the formula is rather complicated to handle. In [5] the authors obtain upper and lower bounds on the solution to (1.15) of the Mullins' case by solving two auxiliary problems which are relatively easily solvable and employing the comparison principle. They conclude accurate estimates of the depth when β is large, but an estimate allowing β to be small such as (1.17) is not stated in [5]. See Remark 5.3 for comparison with our results concerning the depth. The paper [52] gives exact solutions to wider classes of nonlinear equations, but solutions to (1.8) constructed there do not satisfy the prescribed angle condition (1.6). In [53, 12] exact solutions of the separated form $\phi(x) + \psi(t)$ are investigated. We also refer [1] for solvablity of the equation (1.8) on $I \times (0, \infty)$, where I is a bounded interval. Under the zero Dirichlet or Neumann boundary condition, the authors of [1] establish the existence of weak, strong and classical solutions and asymptotic behaviors of the classical solutions. The paper [40] shows the existence of classical solutions to more general degenerate parabolic equations.

A well-posedness of the problem (NP) is established in the sense of viscosity solutions in Section 2. We thus interpret the boundary condition (1.3) in the viscosity sense, that is, we require solutions to satisfy either (1.1) or (1.3) on the boundary. As a result, we observe that the unique solution may not satisfy (1.3) in the classical sense when the equation is degenerate (Proposition 4.6 (1)). Such generalized boundary conditions, which naturally appear when we take the limit in the vanishing viscosity method, was first introduced by Lions in [41]; see also [51]. The well-posedness is obtained in [41] for first order equations with Neumann or oblique conditions involving applications to optimal control, differential games and ergodic problems. After their works, uniqueness and existence results for oblique boundary problems in the viscosity sense were established in [9] for first order cases and in [28, 27, 6] for second order cases. In [18, 19] the authors approach oblique problems on domains involving corners. All of these studies treat continuous equations while equations with singularity in ∇u like the mean curvature flow equation for level sets are discussed in [23, 54] under the zero Neumann boundary condition. As relatively general results for second order singular equations with nonlinear boundary conditions, we refer the reader to [7, 30]. Compared with [30], the paper [7] deals with more general equations and boundary conditions, but domains are more restrictive.

Unfortunately, all the above results treat a bounded domain with respect to the space variables. As far as the author know, [55] is the only paper which proves a well-posedness of the Neumann type problems on an unbounded domain. In [55] Sato established comparison and existence results for second order singular equations under the capillary boundary condition:

$$\partial_{x_1} u = k|\nabla u| \quad \text{with } -1 < k < 1,$$

which does not cover our boundary condition (1.3). Although it might be possible to extend the previous results for bounded domains to our problem (NP) by modifying their proofs suitably, we give in the present paper complete proofs of comparison and existence theorem for (NP) to make the paper self-contained. Neumann problems in half-space type domains are also treated in [4, 8], where the authors studies ergodic problems and homogenization.

This chapter is organized as follows. In Section 2 we establish comparison and existence results of viscosity solutions to (NP). Section 3 is devoted to the asymptotic profile. We prove (1.13), i.e., asymptotic self-similarity of the solution to the equation of the type (1.5). In Section 4 we consider the ordinary differential equation (1.15) and its solution. We show the solution has a corner if the equation is degenerate. Section 5 concerns the depth of the thermal groove at the origin. Several estimates for the depth including (1.17) are obtained.

2 A well-posedness of Neumann problems

2.1 Definition of solutions

Throughout this chapter we set $\Omega := \{(x_1, \dots, x_n) \in \mathbf{R}^n \mid x_1 > 0\}$. We first introduce a notion of viscosity solutions for (NP). The boundary condition (1.3) is interpreted in the (weak) viscosity sense. Our basic assumption on F is

(F0) $F : \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$ is continuous and degenerate elliptic.

Here $\mathbf{S}^n$ denotes the space of real $n \times n$ symmetric matrices with the usual ordering, i.e., $X \leqq Y$ if $\langle X\xi, \xi\rangle \leqq \langle Y\xi, \xi\rangle$ for all $\xi \in \mathbf{R}^n$. We say F is *degenerate elliptic* if $F(p, X) \leqq F(p, Y)$ for all $p \in \mathbf{R}^n$ and $X, Y \in \mathbf{S}^n$ with $X \leqq Y$.

Definition 2.1 (Viscosity solution)**.** We say $u : \overline{\Omega} \times [0, \infty) \to \mathbf{R}$ is a *viscosity subsolution* (resp. *supersolution*) of (NP) if u is bounded from above (resp. below) on $\overline{\Omega} \times [0, T)$ for every $T > 0$, $u^*(\cdot, 0) \leqq u_0$ (resp. $u_*(\cdot, 0) \geqq u_0$) on $\overline{\Omega}$ and

$$\begin{cases} \partial_t\phi(x,t) - F(\nabla\phi(x,t), \nabla^2\phi(x,t)) \leqq 0 \text{ (resp. } \geqq 0) & \text{if } x_1 > 0, \\ \partial_t\phi(x,t) - F(\nabla\phi(x,t), \nabla^2\phi(x,t)) \leqq 0 \text{ (resp. } \geqq 0) & \\ \text{or } \ \beta - \partial_{x_1}\phi(x,t) \leqq 0 \text{ (resp. } \geqq 0) & \text{if } x_1 = 0 \end{cases} \tag{2.1}$$

whenever $u^* - \phi$ (resp. $u_* - \phi$) attains its maximum (resp. minimum) at (x, t) for $\phi \in C^{2,1}(\overline{\Omega} \times [0, \infty))$. If u is both a viscosity sub- and supersolution, u is said to be a *viscosity solution*.

Here by a $C^{2,1}$ function we mean that derivatives $\partial_t\phi$, $\nabla\phi$ and $\nabla^2\phi$ are continuous. If $u^* < \infty$ (resp. $u_* > -\infty$) on $\overline{\Omega} \times [0, \infty)$ and u satisfies (2.1), u is said to be a viscosity subsolution (resp. supersolution) of (1.1) and (1.3). In the definition above, u^* and u_* stand for an *upper* and *lower semicontimuous envelope* of u respectively. Namely,

$$u^*(x,t) = \lim_{\delta\to 0} \sup\{u(y,s) \mid (y,s) \in \overline{\Omega} \times [0,\infty),\ |x-y| + |t-s| \leqq \delta\},$$
$$u_*(x,t) = \lim_{\delta\to 0} \inf\{u(y,s) \mid (y,s) \in \overline{\Omega} \times [0,\infty),\ |x-y| + |t-s| \leqq \delta\}.$$

As a boundary condition we consider not $\partial_{x_1}\phi(x,t) - \beta = 0$ but $\beta - \partial_{x_1}\phi(x,t) = 0$ so that consistency between a classical subsolution (resp. supersolution) and a viscosity subsolution (resp. supersolution) holds.

Proposition 2.2 (Consistency)**.** *Assume* (F0)*. Let $u \in C^{2,1}(\overline{\Omega} \times (0, \infty))$ and assume that*

$$\begin{cases} \partial_t u(x,t) \leqq F(\nabla u(x,t), \nabla^2 u(x,t)) & \text{if } \ x_1 > 0, \quad (2.2) \\ \beta - \partial_{x_1} u(x,t) \leqq 0 & \text{if } \ x_1 = 0. \quad (2.3) \end{cases}$$

Then u is a viscosity subsolution of (1.1) *and* (1.3)*.*

Proof. Take any $\phi \in C^{2,1}(\overline{\Omega} \times (0, \infty))$ such that $u - \phi$ attains its maximum at $(x, t) \in \overline{\Omega} \times (0, \infty)$. In the case where $x_1 > 0$, the inequality $\partial_t\phi(x,t) \leqq F(\nabla\phi(x,t), \nabla^2\phi(x,t))$ follows from (2.2) and the degenerate ellipticity of F. If $x_1 = 0$, we see at once that $\partial_{x_1}\phi(x,t) \geqq \partial_{x_1}u(x,t)$, and consequently $\beta - \partial_{x_1}\phi(x,t) \leqq 0$ by (2.3). □

It is known that for a general boundary condition $B(x, u(x), \nabla u(x)) = 0$ the consistency holds if a map $\lambda \mapsto B(x, r, p - \lambda\nu(x))$ is nonincreasing on $[0, \infty)$, where $\nu(x)$ is the unit outward normal vector at a boundary point x. We refer the reader to [16, Proposition 7.2] or [22, Proposition 2.3.3] for more details.

Example 2.3. We consider the heat equation

$$\partial_t u(x,t) = A\Delta u(x,t), \tag{2.4}$$

i.e., $F(p, X) = A \cdot \mathrm{tr}(X)$ with $A > 0$, where $\mathrm{tr}(X)$ denotes the trace of $X \in \mathbf{S}^n$. Then the unique solution of $(\mathrm{NP}; F, 0)$, which is also given by Mullins in [43], is

$$u(x,t) = h_{\beta,A}(x_1, t) := -2\beta\sqrt{At} \cdot \mathrm{ierfc}\left(\frac{x_1}{2\sqrt{At}}\right). \tag{2.5}$$

Here $\mathrm{ierfc}(x)$ is the integral error function

$$\mathrm{ierfc}(x) = \int_x^\infty \mathrm{erfc}(z)dz,$$

and $\mathrm{erfc}(x)$ is the error function

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-z^2}dz.$$

We now differentiate $h = h_{\beta,A}$ to obtain

$$\partial_t h(x_1,t) = -\beta\sqrt{\frac{A}{t}} \cdot \mathrm{ierfc}\left(\frac{x_1}{2\sqrt{At}}\right) - \frac{\beta x_1}{2t} \cdot \mathrm{erfc}\left(\frac{x_1}{2\sqrt{At}}\right),$$
$$\partial_{x_1} h(x_1,t) = \beta \cdot \mathrm{erfc}\left(\frac{x_1}{2\sqrt{At}}\right), \quad \partial_{x_1x_1} h(x_1,t) = \frac{-\beta}{\sqrt{\pi At}}e^{-x_1^2/(4At)}.$$

Employing the formula

$$\mathrm{ierfc}(\xi) + \xi \cdot \mathrm{erfc}(\xi) = \frac{1}{\sqrt{\pi}}e^{-\xi^2}$$

with $\xi = x_1/(2\sqrt{At})$, we observe that h indeed solves (2.4) in the classical sense. Thus h is also a viscosity solution of $(\mathrm{NP}; F, 0)$ by Proposition 2.2. By the formula (2.5) or the derivatives of h we notice that $h(\cdot, t)$ is negative, increasing and (strictly) concave on $[0, \infty)$. It turns out that these properties still hold for viscosity solutions of more general equations; see Proposition 4.3.

Example 2.4. We seek viscosity sub- and supersolutions of $(\mathrm{NP}; F, 0)$ which have the form of (2.5). Assume (F0) and

(F1) $F(p, X) = \lambda F(p, X/\lambda)$ for all $(p, X) \in \mathbf{R}^n \times \mathbf{S}^n$ and $\lambda > 0$.

We simply say F is *homogeneous* if F satisfies (F1). For $\gamma \geqq 0$ we set

$$m(\gamma) = \min_{0\leqq\theta\leqq 1}\{-F(\theta\gamma e_1, -I_{1,1})\}, \quad M(\gamma) = \max_{0\leqq\theta\leqq 1}\{-F(\theta\gamma e_1, -I_{1,1})\}, \tag{2.6}$$

where $e_1 = (1, 0, \dots, 0)$ and $I_{1,1}$ denotes the matrix with 1 in the $(1,1)$ entry and 0 elsewhere. We then notice that $m(\gamma) \geqq 0$ since F is degenerate elliptic by (F0) and satisfies $F(p, O) = 0$ for all $p \in \mathbf{R}^n$ by (F1), where O is the zero matrix. For the function $h = h_{\gamma,A}$ given as (2.5) we observe

$$F(\nabla h, \nabla^2 h) = F(\partial_{x_1} h \cdot e_1, \partial_{x_1 x_1} h \cdot I_{1,1}) = -\partial_{x_1 x_1} h \cdot F(\partial_{x_1} h \cdot e_1, -I_{1,1})$$
$$\begin{cases} \leqq m(\gamma) \cdot \partial_{x_1 x_1} h = (m(\gamma)/A) \cdot \partial_t h, \\ \geqq M(\gamma) \cdot \partial_{x_1 x_1} h = (M(\gamma)/A) \cdot \partial_t h. \end{cases}$$

Taking account of the boundary condition (1.3), we conclude that $h_{\gamma,A}$ is a viscosity subsolution of (NP; $F, 0$) if $\gamma \geqq \beta$ and $A \geqq M(\gamma)$ while $h_{\gamma,A}$ is a viscosity supersolution of (NP; $F, 0$) if $0 \leqq \gamma \leqq \beta$ and $0 < A \leqq m(\gamma)$.

2.2 Comparison principle

We show uniqueness of viscosity solutions to (NP) via comparison principle. Define $U_T := \overline{\Omega} \times \overline{\Omega} \times [0, T)$ for $T > 0$.

Theorem 2.5 (Comparison principle)**.** *Assume* (F0)*. Let u and v be, respectively, a viscosity subsolution and a viscosity supersolution of* (NP)*. Then*

$$K = K[u, v] := \lim_{\theta \to 0} \sup\{u^*(x, t) - v_*(y, t) \mid (x, y, t) \in U_T,\ |x - y| < \theta\} \leqq 0$$

for every $T > 0$. In particular, $u^* \leqq v_*$ on $\overline{\Omega} \times [0, \infty)$.

In the proof of Theorem 2.5 we use an auxiliary function $\mathcal{F} : \overline{U_T} \to \mathbf{R} \cup \{-\infty\}$ of the form

$$\mathcal{F}(x, y, t) = u^*(x, t) - v_*(y, t) - \Psi(x, y, t)$$

with

$$\Psi(x, y, t) = \frac{|x - y|^2}{2\varepsilon} + \beta(x_1 - y_1) + \delta\{\rho(x_1) + \rho(y_1)\} + \gamma(|x|^2 + |y|^2) + \frac{\alpha}{T - t}.$$

Here $\alpha, \gamma, \delta, \varepsilon \in (0, 1)$ are constants and ρ is given by $\rho(r) = (1 + r)^{-1}$. Note that $\rho'(0) = -1$. It then follows from an elementary calculation that for all $(x, y, t) \in U_T$

$$\beta - \partial_{x_1} \Psi(x, y, t) \geqq \delta \quad \text{if } x_1 = 0, \tag{2.7}$$
$$\beta + \partial_{y_1} \Psi(x, y, t) \leqq -\delta \quad \text{if } y_1 = 0 \tag{2.8}$$

and

$$\lim_{(\gamma,\delta) \to (0,0)} \nabla^2_{(x,y)} \Psi(x, y, t) = \frac{1}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{2.9}$$

where I is the identity matrix with dimension n.

Lemma 2.6. *Assume the same hypotheses of Theorem 2.5. Let $T > 0$ and suppose $K = K[u, v] > 0$. Then,*

(1) *$\mathcal{F}$ attains a maximum on $\overline{U_T}$ at some $(\hat{x}, \hat{y}, \hat{t})$ with $\hat{t} < T$.*

(2) *There exists a constant $\eta \in (0, 1]$ such that*

$$\max_{\overline{U_T}} \mathcal{F} > K' \tag{2.10}$$

for all $\alpha, \gamma, \delta \in (0, \eta)$, where $K' := K/7$.

(3) $\sup_{\gamma,\delta,\varepsilon \in (0,\eta)} |\hat{x} - \hat{y}| < \infty$ *and* $\lim_{\varepsilon \to 0} \sup_{\gamma,\delta \in (0,\eta)} |\hat{x} - \hat{y}| = 0$.

(4) $\lim_{(\gamma,\delta)\to(0,0)} (\gamma\hat{x}, \gamma\hat{y}) = (0, 0)$ *for all $\varepsilon \in (0, \eta)$.*

(5) *There exists a constant $\eta_0 \in (0, \eta)$ such that $\hat{t} > 0$ for all $\gamma, \delta, \varepsilon \in (0, \eta_0)$.*

Proof. (1) This follows from an upper semicontinuity of $\mathcal{F}$ and the facts that $\mathcal{F}(x, y, T) = -\infty$ and $\mathcal{F} \to -\infty$ as $|x| \to \infty$ or $|y| \to \infty$.

(2) By the definition of K there exists some $\theta_0 > 0$ such that for all $\theta \in (0, \theta_0]$

$$u^*(x_\theta, t_\theta) - v_*(y_\theta, t_\theta) > 6K' \tag{2.11}$$

holds for some $(x_\theta, y_\theta, t_\theta) \in U_T$ with $|x_\theta - y_\theta| < \theta$. Take

$$\theta = \min\left\{\theta_0,\ \sqrt{2K'\varepsilon},\ K'/\beta\right\}.$$

By this choice we have

$$\frac{|x_\theta - y_\theta|^2}{2\varepsilon} \leqq K', \quad \beta(x_{\theta 1} - y_{\theta 1}) \leqq K'. \tag{2.12}$$

We next choose $\eta \in (0, 1]$ as

$$\eta = \min\left\{1,\ K'/2,\ K'(|x_\theta|^2 + |y_\theta|^2 + 1)^{-1},\ K'(T - t_\theta)\right\},$$

and then for $\alpha, \gamma, \delta \in (0, \eta)$

$$\delta\{\rho(x_{\theta 1}) + \rho(y_{\theta 2})\} \leqq K', \quad \gamma(|x_\theta|^2 + |y_\theta|^2) \leqq K', \quad \frac{\alpha}{T - t_\theta} \leqq K'. \tag{2.13}$$

Thus (2.11)–(2.13) yield (2.10).

(3) Take $M > 0$ so that $u^* - v_* \leqq M$ on $\overline{U_T}$. By (2.10) we have

$$K' < u^*(\hat{x}, \hat{t}) - v_*(\hat{y}, \hat{t}) - \Psi(\hat{x}, \hat{y}, \hat{t}) \leqq M - \frac{|\hat{x} - \hat{y}|^2}{2\varepsilon} + \beta|\hat{x} - \hat{y}|.$$

Thus by an elementary calculation

$$|\hat{x} - \hat{y}| \leqq \varepsilon\beta + \sqrt{\varepsilon^2\beta^2 + 2\varepsilon M},$$

which implies our assertions.

(4) By (2.10) again we see

$$K' \leqq M + \beta|\hat{x} - \hat{y}| - \gamma(|\hat{x}|^2 + |\hat{y}|^2).$$

Therefore $\sup_{\gamma,\delta\in(0,\eta)} \gamma(|\hat{x}|^2 + |\hat{y}|^2) < \infty$, and so

$$\gamma(|\hat{x}| + |\hat{y}|) \leqq \sqrt{2\gamma}\sqrt{\gamma(|\hat{x}|^2 + |\hat{y}|^2)} \to 0 \quad \text{as } (\gamma,\delta) \to (0,0).$$

(5) Suppose by contradiction that there were some sequence $\{(\varepsilon_j, \delta_j, \gamma_j)\}_{j=1}^{\infty}$ which satisfies $\lim_{j\to\infty}(\varepsilon_j, \delta_j, \gamma_j) = (0,0,0)$ and $\hat{t} = \hat{t}(\varepsilon_j, \delta_j, \gamma_j) = 0$. Then

$$\mathcal{F}(\hat{x}, \hat{y}, \hat{t}) = u^*(\hat{x}, 0) - v_*(\hat{y}, 0) - \Psi(\hat{x}, \hat{y}, 0) \leqq u_0(\hat{x}) - u_0(\hat{y}) - \beta(\hat{x}_1 - \hat{y}_1),$$

and the right hand side converges to 0 as $j \to \infty$ by (3) and the uniform continuity of u_0. This is a contradiction to (2.10). □

Proof of Theorem 2.5. By virtue of (3) in Lemma 2.6 we may assume

$$\lim_{(\gamma,\delta)\to(0,0)} (\hat{x} - \hat{y}) = \bar{p}$$

for some $\bar{p} \in \mathbf{R}^n$ by taking a subsequence if necessary. We now apply the Crandall-Ishii lemma ([16, Theorem 8.3]) to $\mathcal{F}$. Since (2.7) and (2.8) hold, there exists $(X, Y) \in \mathbf{S}^n \times \mathbf{S}^n$ such that

$$\partial_t \Psi(\hat{x}, \hat{y}, \hat{t}) \leqq F(\nabla_x \Psi(\hat{x}, \hat{y}, \hat{t}), X) - F(-\nabla_y \Psi(\hat{x}, \hat{y}, \hat{t}), -Y) \tag{2.14}$$

and

$$-\left(\frac{1}{\varepsilon} + |A|\right) I \leqq \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leqq A + \varepsilon A^2. \tag{2.15}$$

Here $A := \nabla^2_{(x,y)} \Psi(\hat{x}, \hat{y}, \hat{t})$ and $|A| := \sup\{|\langle A\xi, \xi\rangle| \mid \xi \in \mathbf{R}^n,\ |\xi| = 1\}$. Note that

$$\begin{aligned}
\nabla_x \Psi(\hat{x}, \hat{y}, \hat{t}) &= \frac{\hat{x} - \hat{y}}{\varepsilon} + \{\beta + \delta\rho'(\hat{x}_1)\}e_1 + 2\gamma\hat{x}, \\
\nabla_y \Psi(\hat{x}, \hat{y}, \hat{t}) &= -\frac{\hat{x} - \hat{y}}{\varepsilon} - \{\beta - \delta\rho'(\hat{y}_1)\}e_1 + 2\gamma\hat{y}, \\
\partial_t \Psi(\hat{x}, \hat{y}, \hat{t}) &= \frac{\alpha}{(T - \hat{t})^2}.
\end{aligned}$$

In view of (2.9) and (2.15) we may assume that (X, Y) converges to some $(\bar{X}, \bar{Y}) \in \mathbf{S}^n \times \mathbf{S}^n$ as $(\gamma, \delta) \to (0,0)$. Then the limit $(\bar{X}, \bar{Y})$ satisfies

$$\begin{pmatrix} \bar{X} & O \\ O & \bar{Y} \end{pmatrix} \leqq \frac{3}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

and in particular $\bar{X} + \bar{Y} \leqq O$. Letting $(\gamma, \delta) \to (0,0)$ in (2.14), we have

$$\frac{\alpha}{T^2} \leqq F\left(\frac{\bar{p}}{\varepsilon} + \beta e_1, \bar{X}\right) - F\left(\frac{\bar{p}}{\varepsilon} + \beta e_1, -\bar{Y}\right).$$

This is a contradiction since F is degenerate elliptic. □

Corollary 2.7 (Uniqueness)**.** *Assume* (F0). *Then* (NP) *admits at most one viscosity solution, and the solution is continuous on* $\overline{\Omega} \times [0, \infty)$.

Proof. If u and v are two viscosity solutions of (NP), we have $u^* \leqq v_*$ and $v^* \leqq u_*$ on $\overline{\Omega} \times [0, \infty)$ by Theorem 2.5. These inequalities imply our assertions. $\square$

Corollary 2.8 (Contraction property)**.** *Assume* (F0). *Let* $u_{01}, u_{02} \in BUC(\overline{\Omega})$. *Let* u_1 *and* u_2 *be, respectively, a viscosity solution of* $(\mathrm{NP}; F, u_{01})$ *and that of* $(\mathrm{NP}; F, u_{02})$. *Then we have* $\sup_{\overline{\Omega} \times [0,\infty)} |u_1 - u_2| \leqq \sup_{\overline{\Omega}} |u_{01} - u_{02}|$.

Proof. Let $d = \sup_{\overline{\Omega}} |u_{01} - u_{02}|$. Then it is easily seen that $u_2 + d$ is a viscosity solution of $(\mathrm{NP}; F, u_{02} + d)$. Since $u_{01} \leqq u_{02} + d$ on $\overline{\Omega}$, Theorem 2.5 gives $u_1 \leqq u_2 + d$ on $\overline{\Omega} \times [0, \infty)$. In the same manner we obtain $u_2 \leqq u_1 + d$ on $\overline{\Omega} \times [0, \infty)$. $\square$

2.3 Existence result

We prove the existence of viscosity solutions by *Perron's method* ([16, Section 4]). An important step is to construct a *lower* and *upper barrier*, which are a viscosity sub- and supersolution of (NP) satisfying the given initial data. We first prepare stability results for viscosity solutions. For the proofs we refer the reader to [16, Lemma 4.2, Lemma 6.1] or [22, Lemma 2.4.1, Theorem 2.3.5].

Proposition 2.9 (Stability)**.** *Assume* (F0).

(1) *Let* $\mathcal{S}$ *be a nonempty subset of*

$$\{v \mid v \text{ is a viscosity subsolution of } (1.1) \text{ and } (1.3)\}.$$

Let $u(x,t) := \sup_{v \in \mathcal{S}} v(x,t)$. *If* $u^* < \infty$ *on* $\overline{\Omega} \times [0,\infty)$, *then* u *is a viscosity subsolution of* (1.1) *and* (1.3)

(2) *Assume that* F^ε *satisfies* (F0), *and let* u^ε *be a a viscosity subsolution of* (1.1) *with* $F = F^\varepsilon$ *and* (1.3) *for each* $\varepsilon > 0$. *If* $F \geqq \limsup^*_{\varepsilon \to 0} F^\varepsilon$ *on* $\mathbf{R}^n \times \mathbf{S}^n$ *and* $\overline{u} := \limsup^*_{\varepsilon \to 0} u^\varepsilon < \infty$ *on* $\overline{\Omega} \times [0, \infty)$, *then* $\overline{u}$ *is a viscosity subsolution of* (1.1) *and* (1.3).

To apply Perron's method we need only (1) while (2) plays an important role in Section 3, where we discuss a local uniform convergence of solutions. We recall a notion of relaxed limits appearing in (2). For a subset $L \subset \mathbf{R}^N$ and functions $h^\varepsilon : L \to \mathbf{R}$ with $\varepsilon > 0$ we define an *upper relaxed limit* $\overline{h} = \limsup^*_{\varepsilon \downarrow 0} h^\varepsilon$ (resp. *lower relaxed limit* $\underline{h} = \liminf_{*\varepsilon \downarrow 0} h^\varepsilon$) $: \overline{L} \to \mathbf{R} \cup \{\pm\infty\}$ as

$$\overline{h}(z) := \limsup_{(\varepsilon,y) \to (0,z)} h^\varepsilon(y) = \lim_{\delta \downarrow 0} \sup\{h^\varepsilon(y) \mid y \in L,\ |y - z| < \delta,\ 0 < \varepsilon < \delta\}$$

$$(\text{resp. } \underline{h}(z) := \liminf_{(\varepsilon,y) \to (0,z)} h^\varepsilon(y) = \lim_{\delta \downarrow 0} \inf\{h^\varepsilon(y) \mid y \in L,\ |y - z| < \delta,\ 0 < \varepsilon < \delta\}).$$

If $\overline{h} = \underline{h}$ in L, then h^ε converges to $h := \overline{h} = \underline{h}$ locally uniformly in L as $\varepsilon \to 0$.

Proposition 2.10 (Barriers)**.** *Assume* (F0). *Then* (NP) *has a viscosity subsolution* w^- *and a viscosity supersolution* w^+ *such that* $w^-(x,t) \leqq u_0(x) \leqq w^+(x,t)$ *for all* $(x,t) \in \overline{\Omega} \times [0,\infty)$ *and* $u_0(x) = w^\pm(x,0) = \lim_{(z,t) \to (x,0)} w^\pm(z,t)$ *for all* $x \in \overline{\Omega}$.

Proof. We give the proof only for a subsolution since a similar argument applies for a supersolution.

1. Let $\omega(r) = \sup_{|x-y|\leqq r} |u_0(x) - u_0(y)|$ and $f(r) = r - \arctan r$. Then for each $\varepsilon > 0$ there exists $C_0(\varepsilon) > 0$ such that $\omega(r) \leqq \varepsilon + C_0(\varepsilon) f(r)$ for all $r \geqq 0$. Set $C(\varepsilon) := \max\{4C_0(\varepsilon),\ 4C_0(\varepsilon)/\beta,\ 1\} \geqq 1$. Since $f(r+s) \leqq 4\{f(r)+f(s)\}$ for $r, s \geqq 0$, we see that

$$\omega(|x-y|) \leqq \varepsilon + \beta C(\varepsilon) f(|x_1 - y_1|) + C(\varepsilon) f(|x' - y'|) \tag{2.16}$$

for all $x = (x_1, x') \in \mathbf{R}^n$ and $y = (y_1, y') \in \mathbf{R}^n$. We also remark that $f \in C^2(\mathbf{R})$, $f(0) = f'(0) = f''(0) = 0$, $0 \leqq f' \leqq 1$ and $0 \leqq f'' \leqq 1/2$ in $\mathbf{R}$.

2. For $\varepsilon \in (0,1)$ and $y \in \Omega$ we define

$$v_{\varepsilon,y}(x,t) := u_0(y) - \varepsilon - \frac{\beta C(\varepsilon)}{f'(y_1)} f(|x_1 - y_1|) - C(\varepsilon) f(|x' - y'|) - Mt,$$

where $M = M(\varepsilon, y) > 0$ is a large constant. Then $v_{\varepsilon,y} \in C^{2,1}(\overline{\Omega} \times [0,\infty))$ and $v_{\varepsilon,y}(x,t) \leqq u_0(x)$ on $\overline{\Omega} \times [0,\infty)$ from (2.16). By the boundedness of f' and f'' we see that $|\nabla_x v_{\varepsilon,y}|$ and $|\nabla_x^2 v_{\varepsilon,y}|$ are also bounded on $\overline{\Omega} \times [0,\infty)$. We thus have

$$-M \leqq F(\nabla_x v_{\varepsilon,y}(x,t), \nabla_x^2 v_{\varepsilon,y}(x,t))$$

for sufficiently large M. We also compute

$$\partial_{x_1} v_{\varepsilon,y}(x,t)|_{x_1=0} = -\frac{\beta C(\varepsilon)}{f'(y_1)}\{-f'(y_1)\} \geqq \beta,$$

and therefore $v_{\varepsilon,y}$ is a viscosity subsolution of (1.1) and (1.3) by Proposition 2.2. Consequently Proposition 2.9 (2) ensures that the supremum of $v_{\varepsilon,y}$

$$w^-(x,t) = \sup\{v_{\varepsilon,y}(x,t) \mid \varepsilon \in (0,1),\ y \in \Omega\}$$

is also a viscosity subsolution of (1.1) and (1.3). By definition w^- is lower semicontinuous and satisfies $w^-(x,t) \leqq u_0(x)$ on $\overline{\Omega} \times [0,\infty)$. In particular w^- is bounded from above.

3. We next show $w^-(x,0) \geqq u_0(x)$ for all $x \in \overline{\Omega}$. We see $w^-(x,0) \geqq v_{\varepsilon,x}(x,0) = u_0(x) - \varepsilon$ if $x \in \Omega$, and so $w^-(x,0) \geqq u_0(x)$ holds. Let $x \in \partial\Omega$. Taking $y = (y_1, x')$, we then have

$$w^-(x,0) \geqq v_{\varepsilon,y}(x,0) \geqq u_0(x) - \varepsilon - \frac{\beta C(\varepsilon)}{f'(y_1)} f(y_1).$$

Letting $y_1 \to 0$ first and then $\varepsilon \to 0$, we obtain $w^-(x,0) \geqq u_0(x)$.

4. Since w^- is lower semicontinuous, for all $x \in \overline{\Omega}$

$$\begin{aligned} u_0(x) = w^-(x,0) &\leqq \liminf_{(z,t)\to(x,0)} w^-(z,t) \leqq \limsup_{(z,t)\to(x,0)} w^-(z,t) \\ &\leqq \limsup_{(z,t)\to(x,0)} u_0(z) = u_0(x). \end{aligned}$$

Hence $\lim_{(z,t)\to(x,0)} w^-(z,t) = u_0(x)$. We thus conclude that w^- satisfies the required properties. □

Remark 2.11. By the same way as in Step 3 we obtain a more general estimate that $w^-(x,t) \geqq u_0(x) - Mt$ for all $(x,t) \in \overline{\Omega} \times [0,\infty)$. This implies that w^- is bounded from below on $\overline{\Omega} \times [0,T)$ for every $T > 0$. Similarly, we are able to construct w^+ in Proposition 2.10 such that it is bounded from above on $\overline{\Omega} \times [0,T)$ for every $T > 0$.

Theorem 2.12 (Existence by Perron's method)**.** *Assume* (F0). *Then* (NP) *admits at least one viscosity solution.*

Proof. Let

$$\mathcal{S} = \left\{ v \;\middle|\; \begin{array}{c} v \text{ is a viscosity subsolution of (NP)} \\ \text{such that } w^- \leqq v \leqq w^+ \text{ on } \overline{\Omega} \times [0,\infty) \end{array} \right\},$$

where w^- and w^+ are functions in Proposition 2.10. Since $w^- \in \mathcal{S}$, the set $\mathcal{S}$ is nonempty. We demonstrate that $u(x) := \sup_{v \in \mathcal{S}} v(x)$ is a viscosity solution of (NP). By definition we have $w^- \leqq u \leqq w^+$ on $\overline{\Omega} \times [0,\infty)$. We then notice that $u^*(\cdot,0) = u_*(\cdot,0) = u_0$ on $\overline{\Omega}$ and that u is bounded on $\overline{\Omega} \times [0,T)$ for all $T > 0$ by Remark 2.11. Proposition 2.9 (1) ensures that u is a subsolution of (NP). We also see that u is a viscosity supersolution of (NP) since u is a maximal subsolution in the sense that $u(x_0,t_0) < v(x_0,t_0)$ for some $v \in \mathcal{S}$ and $(x_0,t_0) \in \overline{\Omega} \times (0,\infty)$ if u were not a supersolution. See [22, Lemma 2.4.2] for more details. $\square$

3 Asymptotic behavior

To study the asymptotic behavior self-similar solutions of (NP) play an important role in our study.

Definition 3.1. Let $u : \overline{\Omega} \times [0,\infty) \to \mathbf{R}$.

(1) We define a *rescaled function* $u_{(\lambda)}$ of u as $u_{(\lambda)}(x,t) := u(\lambda x, \lambda^2 t)/\lambda$ for $\lambda > 0$.

(2) We say u is *self-similar* if $u = u_{(\lambda)}$ for all $\lambda > 0$, or equivalently $u(x,t) = \sqrt{t}U(x/\sqrt{t})$ for some $U : [0,\infty) \to \mathbf{R}$. We call U a *profile function* of u.

Note that, if u is self-similar, the profile function U of u is represented by $U(x) = u(x,1)$. We next introduce a notion of asymptotic homogeneity. We consider $G : \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$ such that G is not necessarily homogeneous but it approximates some homogeneous F in a suitable sense. To state the rigorous meaning of the approximation we define

$$G_\lambda(p,X) := \lambda G\left(p, \frac{X}{\lambda}\right)$$

for $\lambda > 0$. We say G is *asymptotically homogeneous* if G satisfies the following:

(F2) G_λ converges to some $F : \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$ as $\lambda \to \infty$ locally uniformly in $\mathbf{R}^n \times \mathbf{S}^n$.

We call F in (F2) the limit of G. We also remark that the limit F satisfies (F0) and (F1) whenever G satisfies (F0) and (F2). Thus the limit F is always homogeneous. The function $G(p,X) = \sqrt{1+p^2}(1-e^{-k})$ with $k = X/\sqrt{1+p^2}^3$, which represents (1.5) in Mullins' case, is indeed asymptotically homogeneous with the limit $F(p,X) = X/(1+p^2)$ corresponding to (1.8). This follows from the fact that $f_\lambda(z) := \lambda(1-e^{-z/\lambda}) - z$ converges to 0 as $\lambda \to \infty$ locally uniformly in $\mathbf{R}$.

Remark 3.2. If u is a viscosity solution of (NP; G, u_0), then the rescaled function $u_{(\lambda)}$ is a viscosity solution of (NP; $G_\lambda, (u_0)_{(\lambda)}$), where

$$(u_0)_{(\lambda)}(x) = \frac{1}{\lambda} u_0(\lambda x).$$

Indeed, noting that

$$\begin{aligned}
&\partial_t u_{(\lambda)}(x,t) = \lambda \partial_t u(\lambda x, \lambda^2 t), \\
&\nabla u_{(\lambda)}(x,t) = \nabla u(\lambda x, \lambda^2 t), \quad \nabla^2 u_{(\lambda)}(x,t) = \lambda \nabla^2 u(\lambda x, \lambda^2 t),
\end{aligned}$$

we compute

$$\partial_t u_{(\lambda)}(x,t) = \lambda G(\nabla u(\lambda x, \lambda^2 t), \nabla^2 u(\lambda x, \lambda^2 t)) = \lambda G\left(\nabla u_{(\lambda)}(x,t), \frac{1}{\lambda}\nabla^2 u_{(\lambda)}(x,t)\right)$$

and

$$\partial_{x_1} u_{(\lambda)}(x,t) = \partial_{x_1} u(\lambda x, \lambda^2 t) = \beta$$

if u is a classical solution. In the case where u is not smooth, taking elements of semijets, we see that $u_{(\lambda)}$ solves (NP; $G_\lambda, (u_0)_{(\lambda)}$) in the viscosity sense. We also remark that if G is homogeneous, then $u_{(\lambda)}$ solves (NP; $G, (u_0)_{(\lambda)}$).

We prove that the unique solution of the homogeneous equation with the zero initial data is always self-similar. Several properties of the self-similar solution are also discussed.

Proposition 3.3 (Self-similar solution)**.** *Assume* (F0) *and* (F1). *Let u be the unique viscosity solution of* (NP; $F, 0$). *Then*

(1) *u is self-similar.*

(2) *$u \leqq 0$ on $\overline{\Omega} \times [0,\infty)$. If $F(0, -I_{1,1}) < 0$, then $u < 0$ on $\overline{\Omega} \times [0,\infty)$.*

(3) *$u(x,t) = u(x_1, 0, \dots, 0, t)$ for all $(x,t) \in \overline{\Omega} \times [0,\infty)$.*

(4) *$\lim_{x_1 \to \infty} u(x,t) = 0$ for all $t \geqq 0$.*

Proof. By Remark 3.2 we see that $u_{(\lambda)}$ is a viscosity solution of (NP; $F, 0$) for every $\lambda > 0$. Applying Theorem 2.5, we obtain $u = u_{(\lambda)}$. This implies (1). Combining Example 2.4 with Theorem 2.5, we observe that $h_{\beta,M(\beta)} \leqq u \leqq h_{\beta,m(\beta)}$ on $\overline{\Omega} \times [0,\infty)$, where h is the function in (2.5) and $m(\beta)$, $M(\beta)$ are defined as (2.6). Thus the first assertion in (2) and (4) hold. If $F(0, -I_{1,1}) < 0$, then we have $m(\gamma) < 0$ for sufficiently small $\gamma \in (0, \beta]$. Then a supersolution

$h_{\gamma,m(\gamma)}$ is negative on $\overline{\Omega} \times [0,\infty)$, so that u is also negative. We finally prove (3). For $a \in \mathbf{R}^{n-1}$ we set $w_a(x,t) := u(x_1, x' - a, t)$, where $x' = (x_2, \ldots, x_n)$. Then it is easy to see that w_a is also a viscosity solution of (NP; F, 0) since F and the initial-boundary conditions do not depend on x'. By the uniqueness we obtain $u = w_a$. In particular, for fixed $(x,t) \in \overline{\Omega} \times [0,\infty)$ we have $u(x,t) = w_{x'}(x,t) = u(x_1, 0, \ldots, 0, t)$. □

Our main result on asymptotic convergence is

Theorem 3.4 (Asymptotic behavior)**.** *Assume that G satisfies* (F0) *and* (F2) *with the limit F. Let u and v be, respectively, the unique viscosity solution of* (NP; G, u_0) *and that of* (NP; F, 0). *Then $u_{(\lambda)}$ converges to v as $\lambda \to \infty$ locally uniformly on $\overline{\Omega} \times [0,\infty)$.*

By Theorem 3.4 we see that $u_{(\sqrt{t})}(x,1)$ converges to $v(x,1)$ as $t \to \infty$ uniformly on every compact subset of $\overline{\Omega}$. This implies that (1.12) holds locally uniformly with respect to $x \in \overline{\Omega}$.

As is pointed out in Remark 3.2, the rescaled function $u_{(\lambda)}$ is a solution of (NP; $G_\lambda, (u_0)_{(\lambda)}$). Since the local uniform convergence of G_λ to F is assumed, in view of Proposition 2.9 (2) the relaxed limits $\overline{u}$ and $\underline{u}$ of $u_{(\lambda)}$ becomes a sub- and supersolution of (NP; F, 0), respectively, provided that the limits exist. To guarantee the existence of the relaxed limits we construct suitable barriers of (NP; G, u_0) whose rescaled families are locally uniformly bounded. Recalling Remark 2.11, we have rough estimates for u that $u_0(x) - Mt \leqq u(x,t) \leqq u_0(x) + Mt$. Then $u_0(\lambda x)/\lambda - M\lambda t \leqq u_{(\lambda)}(x,t) \leqq u_0(\lambda x)/\lambda + M\lambda t$, but this does not yields that $\overline{u}$ and $\underline{u}$ are real-valued. We construct the barriers so that they have the order $O(\sqrt{t})$ as $t \to \infty$.

Lemma 3.5. (1) *Assume that $g : [0,\infty) \to \mathbf{R}$ satisfies*

$$|g(t)| \leqq M(\sqrt{t} + 1) \quad \text{on } [0,\infty) \tag{3.1}$$

for some $M > 0$. Set $g_{(\lambda)}(t) := g(\lambda^2 t)/\lambda$, $\underline{g} := \liminf_{*\lambda\to\infty} g_{(\lambda)}$ and $\overline{g} := \limsup^*_{\lambda\to\infty} g_{(\lambda)}$. Then we have $-M\sqrt{t} \leqq \underline{g}(t) \leqq \overline{g}(t) \leqq M\sqrt{t}$ on $[0,\infty)$.

(2) *Assume that G satisfies* (F0) *and* (F2). *Then there exists $M_0 > 0$ such that*

$$\rho(t) := \sup_{|\theta|,|\sigma| \leqq 1} \left| G\left(\theta\beta e_1, \frac{\sigma I_{1,1}}{\sqrt{t}}\right)\right| \leqq \frac{M_0}{\sqrt{t}} \tag{3.2}$$

for all $t \geqq 1$. Moreover

$$g(t) := \begin{cases} 0 & (0 \leqq t \leqq 1), \\ \int_1^t \rho(s)ds & (t > 1) \end{cases} \tag{3.3}$$

satisfies (3.1) *with $M = 2M_0$.*

Obviously, the estimate (3.2) holds if G is homogeneous. For a general G, roughly speaking, (3.2) still holds since G is approximately homogeneous.

Proof. (1) Fix $t_0 \geqq 0$. Let $\delta > 0$ and take $t \geqq 0$, $\lambda > 0$ such that $|t - t_0| \leqq \delta$, $\lambda \geqq 1/\delta$. We then observe

$$|g_{(\lambda)}(t)| = \frac{1}{\lambda}|g(\lambda^2 t)| \leqq \frac{1}{\lambda}M(\sqrt{\lambda^2 t} + 1) = M\left(\sqrt{t} + \frac{1}{\lambda}\right) \leqq M(\sqrt{t_0 + \delta} + \delta).$$

Thus, sending $\delta \to 0$ gives $-M\sqrt{t_0} \leqq \underline{g}(t_0) \leqq \overline{g}(t_0) \leqq M\sqrt{t_0}$.

(2) The second assertion is obvious if (3.2) holds. For $t \geqq 1$ we observe

$$\sqrt{t}\rho(t) \leqq \sup_{|\theta|,|\sigma|\leqq 1}\left|\sqrt{t}G\left(\theta\beta e_1, \frac{\sigma I_{1,1}}{\sqrt{t}}\right) - F(\theta\beta e_1, \sigma I_{1,1})\right| + \sup_{|\theta|,|\sigma|\leqq 1}|F(\theta\beta e_1, \sigma I_{1,1})|.$$

The first term of the right hand side converges to 0 as $t \to \infty$ by assumption while the second term is a constant independent of t. Therefore (3.2) follows. $\square$

Proof of Theorem 3.4. Let w^- and w^+ be barriers constructed in the proof of Proposition 2.10. Then there exists $C > 0$ such that $-C \leqq w^- \leqq w^+ \leqq C$ on $\overline{\Omega} \times [0, 2]$ by Remark 2.11. Define $\Phi : \overline{\Omega} \times [0, \infty) \to \mathbf{R}$ as

$$\Phi(x, t) := -C + h(x_1, t) - g(t).$$

Here h and g are the functions given by (2.5) and (3.3), respectively. We choose $A = \beta^2/\pi$ in (2.5) so that $0 \leqq \partial_{x_1} h \leqq \beta$ and $-1/\sqrt{t} \leqq \partial_{x_1 x_1} h \leqq 0$ in $\overline{\Omega} \times (0, \infty)$. By the definition of g, we then find that Φ and $-\Phi$ are, respectively, a viscosity subsolution and a viscosity supersolution of

$$\partial_t u(x, t) = G(\nabla u(x, t), \nabla^2 u(x, t)) \tag{3.4}$$

in $\Omega \times (1, \infty)$ and (1.3). Indeed, the boundary condition is easy to check, and for $(x, t) \in \Omega \times (1, \infty)$ we compute

$$\partial_t \Phi(x, t) \leqq -g'(t) = -\sup_{|\theta|,|\sigma|\leqq 1}\left|G\left(\theta\beta e_1, \frac{\sigma I_{1,1}}{\sqrt{t}}\right)\right| \leqq G(\nabla\Phi(x, t), \nabla^2\Phi(x, t)).$$

Here we have used the facts that $\partial_t h \leqq 0$, $0 \leqq \partial_{x_1} h \leqq \beta$ and $-1/\sqrt{t} \leqq \partial_{x_1 x_1} h \leqq 0$. A similar argument yields that $-\Phi$ is a supersolution. Since $\Phi \leqq w^- \leqq w^+ \leqq -\Phi$ on $\overline{\Omega} \times [0, 2]$, we see that $\tilde{w}^- := \max\{w^-,\ \Phi\}$ and $\tilde{w}^+ := \min\{w^+,\ -\Phi\}$ are, respectively, a viscosity subsolution and a viscosity supersolution of (3.4) in $\Omega \times (0, \infty)$ and (1.3). Noting that $(\tilde{w}^-)^*(x, 0) = u_0(x) = (\tilde{w}^+)_*(x, 0)$ on $\overline{\Omega}$, we see by Theorem 2.5 that $(\tilde{w}^-)^* \leqq u \leqq (\tilde{w}^+)_*$ in $\overline{\Omega} \times [0, \infty)$. In particular, we have $\Phi_{(\lambda)} \leqq u_{(\lambda)} \leqq -\Phi_{(\lambda)}$. Taking $\liminf_{*\lambda\to\infty}$ and $\limsup^*_{\lambda\to\infty}$, we obtain

$$h(x_1, t) - \overline{g}(t) \leqq \underline{u}(x, t) \leqq \overline{u}(x, t) \leqq -h(x_1, t) + \overline{g}(t) \quad \text{in } \overline{\Omega} \times [0, \infty),$$

where $\underline{u} := \liminf_{*\lambda\to\infty} u_{(\lambda)}$ and $(\overline{u}, \overline{g}) := \limsup^*_{\lambda\to\infty}(u_{(\lambda)}, g_{(\lambda)})$. Therefore Lemma 3.5 implies that $\underline{u}$ and $\overline{u}$ are bounded on $\overline{\Omega} \times [0, T)$ for every $T > 0$ and that $\underline{u}(x, 0) = \overline{u}(x, 0) = 0$ on $\overline{\Omega}$.

Now, since $u_{(\lambda)}$ is a viscosity solution of $(\mathrm{NP}; G_\lambda, (u_0)_{(\lambda)})$ for every $\lambda > 0$, Proposition 2.9 (2) and (F2) imply that $\overline{u}$ and $\underline{u}$ are, respectively, a viscosity subsolution and a viscosity supersolution of $(\mathrm{NP}; F, 0)$. By Theorem 2.5 we have $\overline{u} \leqq \underline{u}$ in $\overline{\Omega} \times [0, \infty)$, and therefore $\overline{u} \equiv \underline{u} \equiv v$ since v is now the unique solution of $(\mathrm{NP}; F, 0)$. As a result, $u_{(\lambda)}$ converges to v locally uniformly in $\overline{\Omega} \times [0, \infty)$. $\square$

Remark 3.6. If G is homogeneous in Theorem 3.4, i.e., $G \equiv F$, then $u_{(\lambda)}$ converges to v uniformly on $\overline{\Omega} \times [0, \infty)$. Indeed, since $u_{(\lambda)}$ solves $(\mathrm{NP}; F, (u_0)_{(\lambda)})$, the contraction property (Corollary 2.8) ensures

$$\sup_{\overline{\Omega}\times[0,\infty)} |u_{(\lambda)} - v| \leqq \sup_{\overline{\Omega}} |(u_0)_{(\lambda)} - 0| = \frac{1}{\lambda} \sup_{\overline{\Omega}} |u_0|.$$

We thus obtain the uniform convergence of $u_{(\lambda)}$ together with its convergence rate.

Remark 3.7. We derive a sufficient condition for (F2). Let $G : \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$ and consider a linear approximation of G such as (1.7). Suppose that G is of the form $G(p, X) = H(p, f(p, X))$ with some continuous and homogeneous f. We expand H as $H(p, z) = z \cdot \partial_z H(p, 0) + z \cdot r(p, z)$, where we have assumed $H(p, 0) = 0$. Then

$$\lambda G\left(p, \frac{X}{\lambda}\right) = \lambda H\left(p, \frac{1}{\lambda} f(p, X)\right) = f(p, X) \cdot \partial_z H(p, 0) + f(p, X) \cdot r\left(p, \frac{1}{\lambda} f(p, X)\right).$$

Thus G satisfies (F2) with the limit $F(p, X) = f(p, X) \cdot \partial_z H(p, 0)$ if the reminder term $r(p, z/\lambda)$ converges to zero as $\lambda \to \infty$ locally uniformly with respect to (p, z). This setting includes Mullins' problem, which corresponds to the case where $H(p, X) = \sqrt{1+p^2}(1 - e^{-z})$ and $f(p, X) = X/\sqrt{1+p^2}^3$.

4 Profile functions

In this section we study the profile function of the unique self-similar solution to $(\mathrm{NP}; F, 0)$ with a homogeneous F. Our main interest is the configuration of its graph, especially the corner of the graph when $F(p, X)$ is allowed to be 0 even if $X \neq 0$.

We first derive the ordinary differential equation which the profile function should solve. Assume (F0) and (F1). Let v be a viscosity solution of $(\mathrm{NP}; F, 0)$. According to Proposition 3.3 (3), $v(x, t)$ is independent of $(x_2, \dots, x_n)$. Thus we hereafter assume $n = 1$ so that u and F are, respectively, defined on $\mathbf{R} \times [0, \infty)$ and $\mathbf{R} \times \mathbf{R}$. We let $V : [0, \infty) \to \mathbf{R}$ be the profile function of v, i.e.,

$$V(x) = v(x, 1). \tag{4.1}$$

Since v is self-similar, we have

$$v(x, t) = \sqrt{t}\, v\left(\frac{x}{\sqrt{t}}, 1\right) = \sqrt{t}\, V\left(\frac{x}{\sqrt{t}}\right). \tag{4.2}$$

We now differentiate v to find

$$\partial_t v(x, t) = \frac{1}{2\sqrt{t}} V\left(\frac{x}{\sqrt{t}}\right) - \frac{x}{2t} V'\left(\frac{x}{\sqrt{t}}\right),$$
$$\partial_x v(x, t) = V'\left(\frac{x}{\sqrt{t}}\right), \quad \partial_{xx} v(x, t) = \frac{1}{\sqrt{t}} V''\left(\frac{x}{\sqrt{t}}\right)$$

provided that v is smooth. Substituting these derivatives for (1.1), we have

$$\frac{1}{2\sqrt{t}}\left\{V\left(\frac{x}{\sqrt{t}}\right)-\frac{x}{\sqrt{t}}V'\left(\frac{x}{\sqrt{t}}\right)\right\}=F\left(V'\left(\frac{x}{\sqrt{t}}\right),\frac{1}{\sqrt{t}}V''\left(\frac{x}{\sqrt{t}}\right)\right).$$

Multiplying the both sides by $2\sqrt{t}$ and letting $\xi = x/\sqrt{t}$, we are led to

$$V(\xi)-\xi V'(\xi)=2F(V'(\xi),V''(\xi)).$$

Here we have used (F1). We consider this equation with the boundary condition at $\xi = 0$ and $\xi = \infty$:

$$\text{(FODE)}\begin{cases} V(\xi)-\xi V'(\xi)=2F(V'(\xi),V''(\xi)) \quad \text{in } (0,\infty), & (4.3)\\ V'(0)=\beta>0, & (4.4)\\ \lim_{\xi\to\infty} V(\xi)=0. & (4.5)\end{cases}$$

To impose (4.5) is natural in terms of Proposition 3.3 (4). Since F is now defined on $\mathbf{R}\times\mathbf{R}$ and satisfies (F1), we notice that F is written as

$$F(p,X)=\begin{cases}F(p,1)X & \text{if } X\geqq 0,\\ -F(p,-1)X & \text{if } X\leqq 0.\end{cases}$$

Thus the right hand side of (4.3) is linear with respect to $V''(\xi)$ as long as the sign of $V''(\xi)$ does not change. By (F0) we also find that $F(p,1)$ and $-F(p,-1)$ are nonnegative continuous functions of p.

We say a function $V : [0,\infty) \to \mathbf{R}$ is a *classical solution* of (FODE) if $V \in C^2(0,\infty)\cap C^1[0,\infty)$ and V satisfies (4.3)–(4.5). Here we define $V'(0)$ as the right derivative:

$$V'(0):=\lim_{\xi\downarrow 0}\frac{V(\xi)-V(0)}{\xi}.$$

A *viscosity subsolution* of (FODE) is a function $V : [0,\infty) \to \mathbf{R}$ such that V is bounded from above on $[0,\infty)$, V^* satisfies (4.5) and

$$\begin{cases}V^*(\xi)-\xi p\leqq 2F(p,X) & \text{if } \xi>0,\\ V^*(0)\leqq 2F(p,X) \ \text{ or } \ \beta-p\leqq 0 & \text{if } \xi=0\end{cases}$$

for all $(p,X)\in J^{2,+}V^*(\xi)$ with $\xi \geqq 0$. The definitions of a viscosity supersolution and a viscosity solution of (FODE) are similar so are omitted. The set of all second order superjets and subjets of V at ξ on $[0,\infty)$ are denoted by $J^{2,+}V(\xi)$ and $J^{2,-}V(\xi)$, respectively. Namely,

$$J^{2,+}V(\xi)=\{(\phi'(\xi),\phi''(\xi)) \mid \phi\in C^2[0,\infty) \text{ and } V-\phi \text{ attains its maximum at } \xi\},$$
$$J^{2,-}V(\xi)=\{(\phi'(\xi),\phi''(\xi)) \mid \phi\in C^2[0,\infty) \text{ and } V-\phi \text{ attains its minimum at } \xi\}.$$

Remark 4.1. Assume (F0) and (F1). Although (4.3) was derived under the assumption that v is smooth, the consistency between (NP; $F,0$) and (FODE) holds in the viscosity sense as well.

- (Consistency) If V is a viscosity subsolution of (FODE), then v given as (4.2) is a viscosity subsolution of (NP; $F, 0$). Conversely, if v is a viscosity subsolution of (NP; $F, 0$), then V given as (4.1) is a viscosity subsolution of (FODE). Similar statements hold for supersolutions.

Due to this consistency we have the comparison and existence of viscosity solutions to (FODE). These assertions follow from the results for the time-dependent case in Section 2.

- (Comparison principle) If U and V are, respectively, a viscosity subsolution and supersolution of (FODE), then $U^* \leqq V_*$ on $[0, \infty)$.
- (Existence) There exists a continuous viscosity solution of (FODE).

Example 4.2. Let us consider the linearized equation

$$\text{(LODE)} \begin{cases} V(\xi) - \xi V'(\xi) = BV''(\xi) & \text{in } (0, \infty), \\ (4.4), (4.5) \end{cases} \tag{4.6}$$

with $B > 0$. This equation corresponds to the case that $2F(p, 1) = -2F(p, -1) = B$ for all $p \in \mathbf{R}$ in (FODE). Choosing $A = B/2$ in (2.5), we see that the unique classical solution of (LODE) is

$$H_{\beta,B}(\xi) := -\beta\sqrt{2B} \cdot \operatorname{ierfc}\left(\frac{\xi}{\sqrt{2B}}\right). \tag{4.7}$$

Note that the derivatives of $H_{\beta,B}$ up to the second order are

$$H'_{\beta,B}(\xi) = \beta \cdot \operatorname{erfc}\left(\frac{\xi}{\sqrt{2B}}\right), \quad H''_{\beta,B}(\xi) = -\beta\sqrt{\frac{2}{\pi B}} \cdot e^{-\xi^2/2B}. \tag{4.8}$$

In the rest of this section we consider the problem of the form

$$\text{(ODE)} = (\text{ODE}; a, \beta) \begin{cases} V(\xi) - \xi V'(\xi) = a(V'(\xi))V''(\xi) & \text{in } (0, \infty), \\ (4.4), (4.5) \end{cases} \tag{4.9}$$

with nonnegative $a \in C(\mathbf{R})$. Although (ODE) is a special case of (FODE), it turns out that the both problems are equivalent; see Remark 4.8. Our basic assumption on a is

(A1) $a \in C(\mathbf{R})$, $a \geqq 0$ in $\mathbf{R}$ and $a(0) > 0$.

Recall that Mullins' equation (1.8) corresponds to (ODE) with $a(p) = 2/(1+p^2)$. We list fundamental properties of a viscosity solution to (ODE).

Proposition 4.3. *Assume* (A1). *Let V be the unique viscosity solution of* (ODE). *Let $(p, X) \in J^{2,-}V(\xi_0)$ with $\xi_0 > 0$. Then*

(1) $V < 0$ *on* $[0, \infty)$.

(2) V *is increasing on* $[0, \infty)$, *i.e,* $V(\xi_1) < V(\xi_2)$ *if* $0 \leqq \xi_1 < \xi_2$.

(3) $p > 0$ *and* $X < 0$.

(4) *V is strictly concave on $[0,\infty)$, i.e., $V((1-\lambda)\xi_1+\lambda\xi_2) > (1-\lambda)V(\xi_1) + \lambda V(\xi_2)$ for all $\lambda \in (0,1)$ and $\xi_1, \xi_2 \in [0,\infty)$ with $\xi_1 < \xi_2$.*

(5) $a(p) > 0$.

Proof. (1) This is a consequence of the second assertion of Proposition 3.3 (2).

(2) We suppose that $0 > V(\xi_1) \geqq V(\xi_2)$ with $\xi_1 < \xi_2$. In view of (4.5) we then have $\min_{[\xi_1,\infty)} V = V(\eta) < 0$ for some $\eta \in (\xi_1, \infty)$. Thus $(0,0) \in J^{2,-}V(\eta)$, so that $V(\eta) - \eta \cdot 0 \geqq a(0) \cdot 0 = 0$ since V is a supersolution. However, this is contract to (1).

(3) (5) The monotonicity of V yields that $p \geqq 0$. We then notice that $a(p)$ must be positive and that X must be negative since $0 > V(\xi) - \xi p \geqq a(p)X$. We show that $p > 0$ after the proof of (4).

(4) We suppose that $V((1-\lambda)\xi_1 + \lambda\xi_2) \leqq (1-\lambda)V(\xi_1) + \lambda V(\xi_2)$ for some $\lambda \in (0,1)$ and $\xi_1, \xi_2 \in [0,\infty)$ with $\xi_1 < \xi_2$. We now take the parabola $\phi \in C^2(\mathbf{R})$ which passes through three points $(\xi_1, V(\xi_1))$,$((1-\lambda)\xi_1+\lambda\xi_2, V((1-\lambda)\xi_1+\lambda\xi_2))$ and $(\xi_2, V(\xi_2))$. Then ϕ'' is a nonnegative constant c and $\min_{[\xi_1,\xi_2]}(V-\phi) = (V-\phi)(\eta)$ for some $\eta \in (\xi_1, \xi_2)$. Thus $(\phi'(\eta), c) \in J^{2,-}V(\eta)$, which contradicts (3) since $c \geqq 0$.

(3) Let $\xi_1, \xi_2 > 0$ with $\xi_1 < \xi_0 < \xi_2$. Since V is concave and increasing, we observe that

$$\frac{V(\xi_0) - V(\xi_1)}{\xi_0 - \xi_1} \geqq \frac{V(\xi_2) - V(\xi_0)}{\xi_2 - \xi_0} > 0.$$

We next take $\phi \in C^2(0,\infty)$ such that $\min_{(0,\infty)}(V-\phi) = (V-\phi)(\xi_0)$ and $(p, X) = (\phi'(\xi_0), \phi''(\xi_0))$. Then

$$\frac{\phi(\xi_0) - \phi(\xi_1)}{\xi_0 - \xi_1} \geqq \frac{V(\xi_0) - V(\xi_1)}{\xi_0 - \xi_1}.$$

Combining the two inequalities above and letting $\xi_1 \uparrow \xi_0$, we obtain $p > 0$. □

Remark 4.4. Since V is concave on $[0,\infty)$, we see by Aleksandrov's theorem ([16, Theorem A.2]) that V is twice differentiable almost everywhere on $[0,\infty)$. Namely, $J^{2,+}V(\xi) \cap J^{2,-}V(\xi)$ is nonempty a.e. $\xi \in [0,\infty)$. Accordingly, V solves (4.9) almost everywhere in the classical sense.

Remark 4.5. Although the viscosity solution V in Proposition 4.3 may not be differentiable, we are able to deduce several properties of its one-side derivatives mainly from the strict concavity of V. We define the right derivative V_r' of V and the left derivative V_l' of V as follows:

$$V_r'(\xi_0) := \lim_{\xi \downarrow \xi_0} \frac{V(\xi) - V(\xi_0)}{\xi - \xi_0} \quad \text{for } \xi_0 \in [0,\infty),$$
$$V_l'(\xi_0) := \lim_{\xi \uparrow \xi_0} \frac{V(\xi) - V(\xi_0)}{\xi - \xi_0} \quad \text{for } \xi_0 \in (0,\infty).$$

Under the same hypotheses of Proposition 4.3 these limits indeed exist and enjoy the following properties.

(a) (One-side continuity) $V_r'(\xi_0) = \lim_{\xi \downarrow \xi_0} V_r'(\xi)$ for all $\xi_0 \in [0,\infty)$ and $V_l'(\xi_0) = \lim_{\xi \uparrow \xi_0} V_r'(\xi)$ for all $\xi_0 \in (0,\infty)$.

(b) (Monotonicity) $\beta \geqq V_r'(\xi_1) > V_l'(\xi_2) \geqq V_r'(\xi_2) > V_l'(\xi_3) > 0$ if $0 \leqq \xi_1 < \xi_2 < \xi_3$.

(c) (Limit as $\xi \to \infty$) $\lim_{\xi\to\infty} V_r'(\xi) = \lim_{\xi\to\infty} V_l'(\xi) = 0$. (If the limit were positive, $V(\xi)$ would not converge to zero as $\xi \to \infty$.)

If V is a classical solution, it is obvious that the range of V' on $[0,\infty)$ is $(0,\beta]$. In Corollary 4.12 we will determine the range of V_r' and V_l' when V is not necessarily a classical solution.

We discuss the angle $V'(0)$ at the origin for a viscosity solution V of (ODE).

Proposition 4.6 (Angle at the origin)**.** *Assume* (A1)*. Let V be the unique viscosity solution of* (ODE)*.*

(1) *We have*

$$V'(0) = q^- := \begin{cases} \beta & \text{if } a(\beta) > 0, \\ \inf\{q \in (0,\beta] \mid a = 0 \text{ on } [q,\beta]\} & \text{if } a(\beta) = 0. \end{cases}$$

(2) *Let $\beta_1 > \beta$ and V_1 be the unique viscosity solution of* (ODE; a, β_1)*. If $a = 0$ on $[\beta, \beta_1]$, then $V = V_1$ on $[0,\infty)$.*

Proof. (1) 1. We first prove that $V'(0)$ exists and $0 < V'(0) \leqq \beta$. Since V is strictly concave, we see that $(V(\xi) - V(0))/\xi$ is increasing as $\xi \downarrow 0$. Thus $V'(0)$ exists and $V'(0) \in (0,\infty]$ by the monotonicity of V. Suppose $V'(0) > \beta$. Then $(p, 0) \in J^{2,-}V(0)$ for every $p \in (\beta, V'(0))$; however, $V(0) - 0 \cdot p < a(p) \cdot 0$ and $\beta - p < 0$. This is a contradiction.

2. We next show that $V'(0) \geqq q^-$. Suppose $0 < V'(0) < q^-$. Then, by the definition of q^- there exists some $p \in (V'(0), \beta)$ such that $a(p) > 0$. We let $\phi(\xi) = -c\xi^2 + p\xi + V(0)$ for $c > 0$. Since $\phi(0) = V(0)$ and $\phi'(0) = p$, it follows that $(p, -2c) \in J^{2,+}V(0)$. We thus have $V(0) - 0 \cdot p \leqq a(p) \cdot (-2c)$, which is a contradiction for large $c > 0$.

3. If $q^- = \beta$, the proof has already been completed. Let $q^- < \beta$ and suppose that $q^- < V'(0) \leqq \beta$. Since $V_r'(\xi) \to V'(0)$ as $\xi \downarrow 0$, we see that $q^- < V_r' \leqq \beta$ on $[0, \varepsilon]$ for some small $\varepsilon > 0$. We now take $(p, X) \in J^{2,+}V(\xi_0) \cap J^{2,-}V(\xi_0)$ with $\xi_0 \in (0, \varepsilon)$; see Remark 4.4 for the existence of such ξ_0. Then $p = V_r'(\xi_0)$. However, we reach a contradiction that $0 > V(\xi_0) - \xi_0 \cdot p \geqq a(p)X = 0$ since $q^- < p \leqq \beta$.

(2) If we prove that V is a viscosity solution of (ODE; a, β_1), the conclusion follows. We only need to consider the boundary condition. Evidently, V is a supersolution of (ODE; a, β_1) since $\beta_1 - p \geqq \beta - p \geqq 0$ whenever $(p, X) \in J^{2,-}V(0)$; see Remark 4.7. We next take $(p, X) \in J^{2,+}V(0)$ and let $p < \beta_1$; otherwise $\beta_1 - p \leqq 0$ holds. In (1) we have shown $V'(0) = \inf\{q \in (0,\beta] \mid a = 0 \text{ on } [q,\beta]\}$. Since $V'(0) \leqq p < \beta_1$, we now have $a(p) = 0$ and therefore $V(0) - 0 \cdot p \leqq 0 = a(p)X$. □

Remark 4.7. Since $0 < V'(0) \leqq \beta$ by (1) above, we always have $\beta - p \geqq 0$ if $(p, X) \in J^{2,-}V(0)$ for a viscosity solution V. Indeed, if $V - \phi$ has its minimum at the origin, then $\phi'(0) \leqq V'(0) \leqq \beta$.

Remark 4.8. Let $F : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ satisfy (F0), (F1) and $F(0,-1) < 0$. It is not difficult to see that, if we replace $a(p)$ by $-2F(p,-1)$, the assertions in Proposition 4.3 and 4.6 still hold for a viscosity solution of the general problem (FODE). We thus find that (FODE) and (ODE) are equivalent in the following sense.

(i) If V is a viscosity solution of (FODE) with F satisfying (F0), (F1) and $F(0,-1) < 0$, then V is also a viscosity solution of (ODE) with $a(p) = -2F(p,-1)$.

(ii) If V is a viscosity solution of (ODE) with a satisfying (A1), then V is also a viscosity solution of (FODE) with $F(p,X) = a(p)X/2$ if $X \leqq 0$, and $F(p,X) = b(p)X$ for some nonnegative $b \in C(\mathbf{R})$ if $X \geqq 0$.

Indeed, when V is concave, we have $2F(p,X) = a(p)X$ for $(p,X) \in J^{2,-}V(\xi)$ with $\xi > 0$. We next let $(p,X) \in J^{2,+}V(\xi)$ with $\xi \geqq 0$. If $X \leqq 0$, then $2F(p,X) = a(p)X$. If $X > 0$, we see $(p,0) \in J^{2,+}V(\xi)$ by concavity. Since $a(p) \cdot 0 \leqq a(p)X$ and $2F(p,0) \leqq 2F(p,X)$, we finally conclude (i) and (ii). (It is easy to check the boundary condition by virtue of Remark 4.7.) Also, similar assertions to (i) and (ii) hold for classical solutions.

We next establish a unique existence result of classical solutions to (ODE). Recalling the property (5) in Proposition 4.3, we see that there is no classical solution of (ODE) if $a(\beta_0) = 0$ for some $\beta_0 \in (0,\beta)$. We thus need the positivity of a for the existence. Conversely, it turns out that a viscosity solution of (ODE), for which we have already known the unique existence, is actually a classical solution of (ODE) if a is positive.

Proposition 4.9 (C^2-regularity of viscosity solutions)**.** *Assume* (A1)*. Let V be the unique viscosity solution of* (ODE)*. If $a > 0$ on $[0,\beta]$, then V is a classical solution of* (ODE)*.*

Proof. 1. By virtue of Proposition 4.6 (1) the boundary condition (4.4) is now fulfilled. Since V_r is right continuous, the condition $V \in C^1[0,\infty)$ is satisfied if we prove $V \in C^1(0,\infty)$. In the rest of the proof we show $V \in C^2(0,l)$ for every $l > 0$.

2. Let

$$\psi(r) := \begin{cases} V(0) & \text{if } r \leqq V(0), \\ r & \text{if } V(0) \leqq r \leqq \beta, \\ \beta & \text{if } \beta \leqq r \end{cases}$$

and

$$b(p) := \max\{a(p),\ m_\beta\} \quad \text{with } m_\beta = \min_{q \in [0,\beta]} a(q).$$

Then we observe that V also satisfies

$$\psi(W(\xi)) - \xi\psi(W'(\xi)) = b(W'(\xi))W''(\xi) \quad \text{in } (0,l) \tag{4.10}$$

in the viscosity sense because $V(0) \leqq V \leqq 0$ and $0 \leqq p \leqq \beta$ for every $p \in J^{2,+}V(\xi) \cup J^{2,-}V(\xi)$ with $\xi \in (0,l)$; recall Proposition 4.3 (1), (2) and Remark

4.5 (b). We now solve the ordinary differential equation (4.10) with the boundary condition

$$W(0) = V(0) \quad \text{and} \quad W(l) = V(l). \tag{4.11}$$

According to [25, Theorem XII.4.2] there exists $U \in C^2(0,l) \cap C[0,l]$ which satisfies (4.10) and (4.11) in the classical sense. The reason why we introduced (4.10) is to guarantee that

$$f(\xi, r, p) := \frac{\psi(r) - \xi\psi(p)}{b(p)}$$

is continuous and bounded on $[0,l] \times (-\infty, \infty) \times (-\infty, \infty)$, which is assumed in [25, Theorem XII.4.2].

3. We assert that $V(0) \leqq U \leqq 0$ on $[0,l]$. If $U(\eta) > 0$ at a maximum point $\eta \in (0,l)$ of U, noting that $U'(\eta) = 0$ and $U''(\eta) \leqq 0$, we would reach a contradiction that

$$\psi(U(\eta)) - \eta\psi(U'(\eta)) > 0 \geqq b(U'(\eta))U''(\eta).$$

Thus $U \leqq 0$. A similar argument yields $V(0) \leqq U$.

4. By Step 3 we find that U satisfies

$$W(\xi) - \xi\psi(W'(\xi)) = b(W'(\xi))W''(\xi) \quad \text{in } (0, \infty) \tag{4.12}$$

in the classical sense, and therefore in the viscosity sense. We now apply the comparison principle for a viscosity subsolution and a viscosity supersolution of (4.12). Such a comparison is ensured by [16, Theorem 3.3]; indeed, if we set $G(\xi, r, p, X) = r - \xi\psi(p) - b(p)X$, we have $G(\xi, r, p, X) - G(\xi, s, p, X) \geqq r - s$ for $r \geqq s$ and $G(\eta, r, \alpha(\xi - \eta), Y) - G(\xi, r, \alpha(\xi - \eta), X) \leqq \alpha|\xi - \eta|^2$ for $X \leqq Y$. We thus obtain $V = U$ on $[0,l]$, which implies $V \in C^2(0,l)$. □

Approximating a viscosity solution by classical solutions, we prove that its derivative takes the value p if $a(p) > 0$ and that the value of the derivative jumps over p if $a(p) = 0$. In other words, the solution has a corner when the equation is degenerate.

Theorem 4.10 (Corner of profile functions)**.** *Assume* (A1). *Let V be the unique viscosity solution of* (ODE). *Let $p \in (0, \beta)$.*

(1) *Assume that $a(p) > 0$. Then there exists a unique $\xi_p \in (0, \infty)$ such that $V \in C^2(I)$ and $V'(\xi_p) = p$ for some open interval I with $\xi_p \in I \subset (0, \infty)$.*

(2) *Assume that $a(p) = 0$. Let*

$$q^+ := \sup\{q \in [p, \beta] \mid a = 0 \text{ on } [p, q]\},$$
$$q^- := \inf\{q \in (0, p] \mid a = 0 \text{ on } [q, p]\}.$$

If $q^+ < \beta$, then there exists a unique $\xi_p \in (0, \infty)$ such that $V_l'(\xi_p) = q^+$ and $V_r'(\xi_p) = q^-$. If $q^+ = \beta$, then we have $V'(0) = q^-$.

Remark 4.11. If $q^- = q^+ = p$ in (2), then V is differentiable at ξ_p but not twice differentiable at ξ_p since $a(p) = 0$. See Proposition 4.3 (5).

Proof. The uniqueness assertions in (1) and (2) follow from the monotonicities of V_r' and V_l', which are ensured by Remark 4.5 (b). If $a > 0$ on $[0, \beta]$, the assertion in (1) is obvious since V' is bijection from $[0, \infty)$ to $(0, \beta]$; recall Remark 4.5 (b), (c) and Proposition 4.6 (1). Also, when $q^+ = \beta$ in (2), we have already proved $V'(0) = q^-$ in Proposition 4.6 (1).

(1) 1. Set $a_\delta(q) = \max\{a(q),\ \delta\}$ for $\delta \in (0, a(0)]$. Owing to the positivity of a_δ the unique solution V_δ of (ODE; a_δ, β) is smooth. Since a_δ converges to a uniformly, we see that V_δ converges to V as $\delta \to 0$ locally uniformly on $[0, \infty)$ by stability (Proposition 2.9 (2)).

2. Take $\varepsilon > 0$ small so that $[p-\varepsilon, p+\varepsilon] \subset (0, \beta)$ and $a > 0$ on $[p-\varepsilon, p+\varepsilon]$. Since V_δ is a classical solution of (ODE; a_δ, β) with a positive a_δ, there exist $\xi_\delta^-, \eta_\delta, \xi_\delta^+ \in (0, \infty)$ such that $\xi_\delta^- < \eta_\delta < \xi_\delta^+$ and $(V_\delta'(\xi_\delta^-), V_\delta'(\eta_\delta), V_\delta'(\xi_\delta^+)) = (p+\varepsilon, p, p-\varepsilon)$ for each $\delta > 0$. Then we observe

$$(p-\varepsilon)\xi_\delta^+ \leqq \int_0^{\xi_\delta^+} V_\delta'(\xi)d\xi = V_\delta(\xi_\delta^+) - V_\delta(0) \leqq -V_\delta(0).$$

Since $V_{a(0)}(0) \leqq V_\delta(0)$ by the comparison principle, we obtain $\xi_\delta^+ \leqq -V_{a(0)}(0)/(p-\varepsilon)$. Therefore we may assume that $(\xi_\delta^-, \eta_\delta, \xi_\delta^+) \to (\bar\xi^-, \bar\eta, \bar\xi^+)$ as $\delta \to 0$ by taking a subsequence if necessary.

3. We show that $-M \leqq V_\delta'' \leqq 0$ on $[\xi_\delta^-, \xi_\delta^+]$ for some $M > 0$ independent of δ. Take $c > 0$ such that $c \leqq a$ on $[p-\varepsilon, p+\varepsilon]$. Then, for $\xi \in [\xi_\delta^-, \xi_\delta^+]$ we have

$$V_\delta''(\xi) = \frac{V_\delta(\xi) - \xi V_\delta'(\xi)}{a(V_\delta'(\xi))} \geqq \frac{V_{a(0)}(0) - \xi_\delta^+(p+\varepsilon)}{c}.$$

Since $\{\xi_\delta^+\}_\delta$ is bounded by Step 2, we conclude that $V_\delta'' \geqq -M$ for some $M > 0$.

4. We next claim that $\bar\xi^- < \bar\eta < \bar\xi^+$. In fact, we compute

$$-\varepsilon = V_\delta'(\eta_\delta) - V_\delta'(\xi_\delta^-) = \int_{\xi_\delta^-}^{\eta_\delta} V_\delta''(\xi)d\xi \geqq -M(\eta_\delta^- - \xi_\delta^-),$$

which implies that $\bar\xi^- < \bar\eta$. The same argument yields that $\bar\eta < \bar\xi^+$.

5. Choose $\theta > 0$ small so that $J := [\bar\eta - \theta, \bar\eta + \theta] \subset (\bar\xi^-, \bar\xi^+)$. We then have $-M \leqq V_\delta'' \leqq 0$ on J for sufficiently small δ. Thus the Ascoli-Arzelà theorem ensures that V_δ' converges to some $U \in C(J)$ as $\delta \to 0$ uniformly on J by taking a subsequence. In particular, we have $U(\bar\eta) = \lim_{\delta\to 0} V_\delta'(\eta_\delta) = p$. Since V^δ converges to V pointwise, we learn that $V \in C^1(\bar\eta - \theta, \bar\eta + \theta)$ and $V' = U$. Consequently $V'(\bar\eta) = p$.

6. We are able to show the C^2-regularity of V in the same way as in the proof of Proposition 4.9. Let $I = (a, b) := (\bar\eta - \theta/2, \bar\eta + \theta/2)$. Since $V \in C^1(\bar\eta - \theta, \bar\eta + \theta)$, for every $\xi \in I$ and $(p, X) \in J^{2,+}V(\xi) \cup J^{2,-}V(\xi)$ we have $V'(a) \geqq p \geqq V'(b)$ and $a(p) \geqq m$ for some $m > 0$. Thus V solves

$$\psi(W(\xi)) - \xi\psi(W'(\xi)) = b(W'(\xi))W''(\xi) \quad \text{in } I \tag{4.13}$$

in the viscosity sense. Here $\psi(r)$ and $b(p)$ are suitable modification of functions r and $a(p)$ respectively; see the proof of Proposition 4.9. Then V must agree with a classical solution of (4.13) with the boundary condition $W(a) = V(a)$ and $W(b) = V(b)$. Hence $V \in C^2(I)$.

(2) 1. Let $q^+ < \beta$. By the definitions of q^- and q^+ there exist sequences $\{q_n^-\}_n$ and $\{q_n^+\}_n$ such that $0 < q_n^- < q^- \leqq q^+ < q_n^+ < \beta$, $a(q_n^-) > 0$, $a(q_n^+) > 0$, $q_n^- \uparrow q^-$ as $n \to \infty$ and $q_n^+ \downarrow q^+$ as $n \to \infty$. Then we see by (1) that $(V'(\xi_n^-), V'(\xi_n^+)) = (q_n^-, q_n^+)$ for some $\xi_n^-, \xi_n^+ \in (0,\infty)$ such that $0 < \xi_n^+ \leqq \xi_{n+1}^+ \leqq \xi_{n+1}^- \leqq \xi_n^-$. By this monotonicity we let $\lim_{n\to\infty}(\xi_n^-, \xi_n^+) = (\bar{\xi}^-, \bar{\xi}^+)$, and then we have

$$
\begin{aligned}
V_l'(\bar{\xi}^+) &= \lim_{\xi \uparrow \bar{\xi}^+} V_l'(\xi) = \lim_{n\to\infty} V_l'(\xi_n^+) = q^+, \\
V_r'(\bar{\xi}^-) &= \lim_{\xi \downarrow \bar{\xi}^-} V_r'(\xi) = \lim_{n\to\infty} V_r'(\xi_n^-) = q^-.
\end{aligned}
$$

2. It remains to prove that $\bar{\xi}^+ = \bar{\xi}^-$. Suppose that $\bar{\xi}^+ < \bar{\xi}^-$. We take $(p_0, X) \in J^{2,-}V(\eta_0)$ with $\eta_0 \in (\bar{\xi}^+, \bar{\xi}^-)$; recall Remark 4.4. We then have

$$
p_0 \leqq V_r'(\eta_0) \leqq V_r'(\xi_n^+) = q_n^+ \quad \text{and} \quad p_0 \geqq V_l'(\eta_0) \geqq V_l'(\xi_n^-) = q_n^-.
$$

Sending $n \to \infty$ yields that $q^- \leqq p_0 \leqq q^+$, and hence $a(p_0) = 0$. This is contrary to Proposition 4.3 (5). $\square$

We are now in a position to determine the range of V_r' and V_l'. Define $R(V_r') := \{V_r'(\xi) \mid \xi \geqq 0\}$, $R(V_l') := \{V_l'(\xi) \mid \xi > 0\}$ and

$$
\begin{aligned}
\overline{\{a > 0\}}^r &:= \left\{ p \in (0,\beta] \;\middle|\; \begin{array}{c} \text{there exists } \{q_n\}_{n=1}^\infty \subset (0,p] \text{ such that} \\ a(q_n) > 0 \text{ and } q_n \to p \text{ as } n \to \infty \end{array} \right\}, \\
\overline{\{a > 0\}}^l &:= \left\{ p \in (0,\beta) \;\middle|\; \begin{array}{c} \text{there exists } \{q_n\}_{n=1}^\infty \subset [p,\beta) \text{ such that} \\ a(q_n) > 0 \text{ and } q_n \to p \text{ as } n \to \infty \end{array} \right\}.
\end{aligned}
$$

Corollary 4.12 (Range of derivatives)**.** *Assume* (A1). *Let V be the unique viscosity solution of* (ODE). *Then we have $R(V_r') = \overline{\{a > 0\}}^r$ and $R(V_l') = \overline{\{a > 0\}}^l$.*

Proof. The inclusion $R(V_r') \supset \overline{\{a > 0\}}^r$ follows immediately from Theorem 4.10 (1) and (2). Let $p \in R(V_r')$, that is $p = V_r'(\xi)$ for some $\xi \geqq 0$. Evidently, we have $p \in \overline{\{a > 0\}}^r$ if $a(p) > 0$. We let $a(p) = 0$. When $\beta = q^+ := \sup\{q \in [p,\beta] \mid a = 0 \text{ on } [p,q]\}$, Proposition 4.6 implies $V'(0) = q^- := \inf\{q \in (0,p] \mid a = 0 \text{ on } [q,p]\}$. By definition $q^- \leqq p$. Since we also have $q^- \geqq V_r'$ on $[0,\infty)$ by monotonicity, it follows that $p = q^- \in \overline{\{a > 0\}}^r$. In the case where $\beta > q^+$, by Theorem 4.10 (2) we have $V_l'(\xi_p) = q^+$ and $V_r'(\xi_p) = q^-$ for some $\xi_p > 0$. Since $V_r'(\xi_p) = q^- \leqq p = V_r'(\xi)$, we see $\xi_p \geqq \xi$. If $\xi_p > \xi$, we would reach a contradiction that $V_r'(\xi) > V_l'(\xi_p) = q^+ \geqq p = V_r'(\xi)$. Thus $\xi = \xi_p$ and then $p = q^-$. This means $p \in \overline{\{a > 0\}}^r$. We have thus proved $R(V_r') = \overline{\{a > 0\}}^r$. A similar argument yields $R(V_l') = \overline{\{a > 0\}}^l$. $\square$

5 Depth of the thermal groove

We investigate the depth of the thermal groove. For a viscosity solution V of (ODE) we define

$$d(\beta) := -V(0). \tag{5.1}$$

This is the depth of the corresponding self-similar solution v in (4.2) at the origin when $t = 1$. Similarly, for the classical solution W of the linearized equation (LODE) with $B = a(0) > 0$ we define

$$L(\beta) := -W(0) = \beta\sqrt{\frac{2a(0)}{\pi}}, \tag{5.2}$$

where the second equality is due to (4.7) since $W = H_{\beta,a(0)}$.

Theorem 5.1 (Depth of the groove)**.** *Assume* (A1)*. Assume furthermore that* $a(p) \leqq a(0)$ *for all* $p > 0$*. Let* V *and* W *be, respectively, the unique viscosity solution of* (ODE) *and that of* (LODE) *with* $B = a(0)$*. Define* d *and* L *as in* (5.1) *and* (5.2)*. Then*

(1) $0 < d \leqq L$ *in* $(0, \infty)$.

(2) d *is nondecreasing in* $(0, \infty)$.

(3) $e_1(\beta) := \beta\sqrt{(2\min_{[0,\beta]} a)/\pi} \leqq d(\beta)$ *for all* $\beta > 0$.

(4)
$$0 \leqq \frac{L(\beta) - d(\beta)}{\beta} \leqq C\left(a(0) - \min_{[0,\beta]} a\right)$$
with $C = \sqrt{2/(\pi a(0))}$ *for all* $\beta > 0$*. In particular,* $\lim_{\beta\downarrow 0}(L(\beta) - d(\beta))/\beta = 0$.

(5) *If* a *is nonincreasing on* $[0, \infty)$*, then* $\lambda d(\beta) \leqq d(\lambda\beta)$ *for all* $\lambda \in [0, 1]$ *and* $\beta > 0$.

(6) $e_2(\beta) := \sqrt{\int_0^\beta a(p)p\,dp} \leqq d(\beta)$ *for all* $\beta > 0$.

(7) *If* $a(p) \geqq c/(1+p^2)$ *on* $[M, \infty)$ *for some* $c, M > 0$*, then* $\lim_{\beta\to\infty} d(\beta) = \infty$.

The estimate in (4) yields (1.14), which asserts that the depth of the linearized problem is the third order approximation in Mullins' case, i.e, $a(p) = 2/(1 + p^2)$. The main tool for the proof of (1)–(5) is the comparison principle while we calculate integrals to show (6) and (7).

Proof. (1) Fix $\beta > 0$. By Proposition 4.3 (1) the depth $d(\beta)$ is positive. We next observe that $W - \xi W' = a(0)W'' \leqq a(W')W''$ on $(0, \infty)$ since $W' \geqq 0$ and $W'' \leqq 0$. This inequality means that W is a subsolution of (ODE). We thus find by the comparison principle that $W \leqq V$ on $[0, \infty)$, and hence $d(\beta) \leqq L(\beta)$.

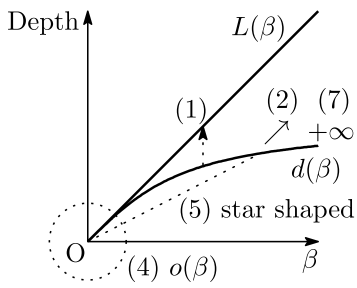

Figure 3: The assertions in Theorem 5.1 on the depth $d(\beta)$.

(2) Take $\beta_1, \beta_2 > 0$ with $\beta_1 < \beta_2$. Let V_1 and V_2 be, respectively, the unique viscosity solution of (ODE; a, β_1) and that of (ODE; a, β_2). It is then easily seen that V_1 is a supersolution of (ODE; a, β_2), and so $V_2 \leqq V_1$ on $[0, \infty)$ by the comparison principle. As a result we see that $d(\beta_1) \leqq d(\beta_2)$.

(3) Fix $\beta > 0$ and take $\beta_0 > 0$ such that $\min_{[0,\beta]} a = a(\beta_0)$. Clearly the claim holds if $a(\beta_0) = 0$. In the case where $a(\beta_0) > 0$ we consider the linearized equation (LODE) with $B = a(\beta_0)$. Then the unique classical solution is

$$U(\xi) := H_{\beta, a(\beta_0)}(\xi) = -\beta\sqrt{2a(\beta_0)} \cdot \mathrm{ierfc}\left(\frac{\xi}{\sqrt{2a(\beta_0)}}\right).$$

Since $0 \leqq U' \leqq \beta$ and $U'' \leqq 0$, we observe that $U - \xi U' = a(\beta_0)U'' \geqq a(U')U''$ on $(0, \infty)$. Thus U is a supersolution of (ODE; a, β). We now apply the comparison principle to obtain $V \leqq U$ on $[0, \infty)$. In particular, we have

$$d(\beta) \geqq -U(0) = \beta\sqrt{\frac{2a(\beta_0)}{\pi}} = e_1(\beta).$$

(4) It follows from (3) that

$$0 \leqq L(\beta) - d(\beta) \leqq \beta\sqrt{\frac{2a(0)}{\pi}} - \beta\sqrt{\frac{2\min_{[0,\beta]} a}{\pi}} \leqq C\beta\left(a(0) - \min_{[0,\beta]} a\right).$$

The second assertion in (4) is now obvious.

(5) Fix $\beta > 0$ and $\lambda \in (0, 1)$. Let V_λ be the unique viscosity solution of (ODE; $a, \lambda\beta$). Set $\tilde{V} = \lambda V$. We now claim that $\tilde{V}$ is a supersolution of (ODE; $a, \lambda\beta$). Let $(p, X) \in J^{2,-}\tilde{V}(\xi)$, i.e., $(p/\lambda, X/\lambda) \in J^{2,-}V(\xi)$. If $\xi = 0$, we derive $\beta - (p/\lambda) \geqq 0$ from Remark 4.7. This means $\lambda\beta - p \geqq 0$. If $\xi > 0$, noting that $p \geqq 0$, $X \leqq 0$ and $V(\xi) - \xi \cdot (p/\lambda) \geqq a(p/\lambda)X/\lambda$, we have $\tilde{V}(\xi) - \xi p \geqq a(p/\lambda)X \geqq a(p)X$ since a is monotone. We thus conclude that $\tilde{V}$ is a supersolution of (ODE; $a, \lambda\beta$). Hence $V_\lambda \leqq \tilde{V}$ on $[0, \infty)$, and so $d(\lambda\beta) \geqq \lambda d(\beta)$.

(6) 1. We first let $a > 0$ on $[0, \infty)$. Then V is a classical solution of (ODE) by Proposition 4.9. We multiply the both sides of (4.9) by $V'(\xi)$ and integrate

over $[0,\eta]$. We then have

$$I_1 := \int_0^\eta \{V(\xi) - \xi V'(\xi)\}V'(\xi)d\xi = \Big[\{V(\xi) - \xi V'(\xi)\}V(\xi)\Big]_0^\eta + \int_0^\eta \xi V''(\xi)V(\xi)d\xi$$
$$= \{V(\eta)\}^2 - \eta V'(\eta)V(\eta) - \{V(0)\}^2 + \int_0^\eta \xi V''(\xi)V(\xi)d\xi$$

from the left hand side while the right hand side becomes

$$I_2 := \int_0^\eta a(V'(\xi))V''(\xi)V'(\xi)d\xi = \int_\beta^{V'(\eta)} a(p)pdp = -\int_{V'(\eta)}^\beta a(p)pdp,$$

where we have used the change of variables that $p = V'(\xi)$. Since $V \leqq 0$, $V' \geqq 0$ and $V'' \leqq 0$, we see that $I_1 \geqq -\{V(0)\}^2 = -\{d(\beta)\}^2$. Thus

$$\{d(\beta)\}^2 \geqq -I_1 = -I_2 = \int_{V'(\eta)}^\beta a(p)pdp.$$

Letting $\eta \to \infty$ and recalling Remark 4.5 (c), we obtain the estimate in (6).

2. If a is not necessarily positive, we set $a_\delta(p) := \max\{a(p),\ \delta\}$ for $\delta > 0$. Then Step 1 yields $\int_0^\beta a_\delta(p)pdp \leqq \{V_\delta(0)\}^2$, where V_δ is the unique classical solution of (ODE; a_δ, β). Letting $\delta \to 0$ gives the desired conclusion since $V_\delta(0) \to V(0)$ by the stability; recall the argument in Step1 in the proof of Theorem 4.10 (1).

(7) For $\beta \geqq M$ we observe that

$$\{e_2(\beta)\}^2 = \int_0^\beta a(p)pdp \geqq \int_M^\beta \frac{cp}{1+p^2}dp = \frac{c}{2}\log\frac{1+\beta^2}{1+M^2}.$$

Thus (6) yields the claim. □

Remark 5.2. (1) We have actually derived several estimates not only at the origin but also on the whole $[0,\infty)$. In particular, by the proof of (1) and (3) we notice

$$0 \leqq V(\xi) - W(\xi)$$
$$\leqq \beta\left\{\sqrt{2a(0)}\cdot \operatorname{ierfc}\left(\frac{\xi}{\sqrt{2a(0)}}\right) - \sqrt{2a(\beta_0)}\cdot \operatorname{ierfc}\left(\frac{\xi}{\sqrt{2a(\beta_0)}}\right)\right\}$$

for all $\xi \in [0,\infty)$, where $\beta_0 > 0$ is chosen so that $a(\beta_0) = \min_{[0,\beta]} a$.

(2) By virtue of Proposition 4.6 (2) we see that $\lim_{\beta\to\infty} d(\beta) \neq \infty$ if $a = 0$ on $[M,\infty)$ for some $M > 0$. Namely, the depth does not necessarily go to infinity.

Remark 5.3. In [5] the authors gives upper and lower bounds on the solution V of (ODE) with $a(p) = 1/2(1+p^2)$. There two auxiliary (ODE) with $a_1(p) = 1/(1+p)^2$ and $a_2(p) = 1/2(1+p)^2$ are considered, and the exact solution V_1 of (ODE; a_1, β) and V_2 of (ODE; a_2, β) are given in the implicit forms. Since $a_1 \geqq a \geqq a_2$, employing the comparison theorem, the authors conclude $V_1 \leqq V \leqq V_2$, and in particular they derive the estimate at the origin of the form

$$\sqrt{2\log\left(\frac{\beta}{\sqrt{\pi}}\right)} \geqq d(\beta) \geqq \sqrt{\log\left(\frac{\beta}{2\sqrt{\pi}}\right)+\frac{1}{4}} - \frac{1}{2} =: l_1(\beta).$$

The both sides of the above inequality are of order $O(\sqrt{\log\beta})$ as $\beta \to \infty$. Our result (6) also gives a lower bound on $d(\beta)$, which is

$$d(\beta) \geqq \sqrt{\int_0^\beta \frac{p}{2(1+p^2)}dp} = \sqrt{\frac{1}{4}\log(1+\beta^2)} =: l_2(\beta).$$

The right hand side $l_2(\beta)$ is of order $O(\sqrt{\log\beta})$, the same order as in [5]; however, by a direct calculation we see $\lim_{\beta\to\infty}(l_1(\beta)-l_2(\beta)) = \infty$. Thus our estimate (6) in Theorem 5.1 is rough in this sense, but it is shown more simply and directly by integration and is enough to prove $d(\beta) \to \infty$ as $\beta \to \infty$ in the Mullins' example.

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Chapter 3

A discrete isoperimetric inequality on lattices

1 Introduction

The classical isoperimetric inequality asserts that for any bounded $E \subset \mathbf{R}^n$ we have

$$\frac{|\partial E|^n}{|E|^{n-1}} \geqq \frac{|\partial \mathbf{B}_1|^n}{|\mathbf{B}_1|^{n-1}}, \tag{1.1}$$

where $|E|$ and $|\partial E|$ denote, respectively, the volume of E and the perimeter of E, and $\mathbf{B}_r := \{x \in \mathbf{R}^n \mid |x| < r\}$ is a ball. This inequality says that among all sets a ball is the best shape to minimize the ratio given as the left-hand side of (1.1). Topics related to the classical isoperimetric problem or arguments on its generalization can be found in the book [5] and the survey paper [22]. See also the recent book [27] for connections with Sobolev inequalities and optimal transport.

In this paper we are concerned with the case where E is a collection of rectangular parallelepipeds with a common shape. To describe the situation more precisely we first define a weighted lattice. For each $i \in \{1, \ldots, n\}$ we fix a positive constant $h_i > 0$ as a step size in the direction of x_i. Then the resulting lattice is

$$h\mathbf{Z}^n := (h_1\mathbf{Z}) \times \cdots \times (h_n\mathbf{Z}) = \{(h_1x_1, \ldots, h_nx_n) \in \mathbf{R}^n \mid (x_1, \ldots, x_n) \in \mathbf{Z}^n\}.$$

Consider a subset $\Omega \subset h\mathbf{Z}^n$. We define $\overline{\Omega}$, the closure of Ω, as

$$\overline{\Omega} := \{x + \sigma h_i e_i \mid x \in \Omega,\ i \in \{1, \ldots, n\},\ \sigma \in \{-1, 0, 1\}\},$$

where $\{e_i\}_{i=1}^n \subset \mathbf{R}^n$ is the standard orthogonal basis of $\mathbf{R}^n$, e.g., $e_1 = (1, 0, \ldots, 0)$. Note that $\overline{\Omega}$ is not a closure in $\mathbf{R}^n$. We also set $\partial\Omega := \overline{\Omega} \setminus \Omega$, the boundary of Ω. Given a bounded $\Omega \subset h\mathbf{Z}^n$, we define the volume of Ω and the perimeter of Ω as, respectively,

$$\mathrm{Vol}(\Omega) := h^n \times (\#\Omega), \quad \mathrm{Per}(\Omega) := h^n \times \left(\sum_{i=1}^n \frac{\omega_i}{h_i}\right)$$

with

$$\omega_i = \omega_i[\Omega] = \sum_{x \in \Omega} \#(\{x \pm h_i e_i\} \cap \partial\Omega),$$

where $h^n := h_1 \times \cdots \times h_n$ and $\#A$ stands for the number of elements of a set A. The number ω_i counts the edges that are parallel to the x_i-direction and are connecting points of Ω with points of $\partial\Omega$. Our definitions of the volume and the perimeter are natural in that if we let

$$E = E[\Omega] := \bigcup_{(x_1,\dots,x_n)\in\Omega} \left[x_1 - \frac{h_1}{2}, x_1 + \frac{h_1}{2}\right] \times \cdots \times \left[x_n - \frac{h_n}{2}, x_n + \frac{h_n}{2}\right] \tag{1.2}$$

for a given $\Omega \subset h\mathbf{Z}^n$, we then have $\mathrm{Vol}(\Omega) = \mathcal{L}^n(E)$, the n-dimensional Lebesgue measure of E, and $\mathrm{Per}(\Omega) = \mathcal{H}^{n-1}(\partial E)$, the $(n-1)$-dimensional Hausdorff measure of ∂E (the boundary of E in $\mathbf{R}^n$). We say $\Omega \subset h\mathbf{Z}^n$ is *connected* if for all $x, y \in \Omega$ there exist $m \in \{1, 2, \dots\}$ and $z_1, \dots, z_m \in \Omega$ such that $z_1 \in \overline{\{x\}}$, $z_{k+1} \in \overline{\{z_k\}}$ $(k = 1, \dots, m-1)$ and $y \in \overline{\{z_m\}}$.

We denote by $\mathbf{Q}_r$ and $\bar{\mathbf{Q}}_r$, respectively, the open and closed cube in $\mathbf{R}^n$ centered at 0 with side-length $2r > 0$, i.e., $\mathbf{Q}_r := (-r, r)^n \subset \mathbf{R}^n$ and $\bar{\mathbf{Q}}_r := [-r, r]^n \subset \mathbf{R}^n$. Let $\bar{\mathbf{Q}}_r(a) := a + \bar{\mathbf{Q}}_r$ for $a \in \mathbf{R}^n$. The volume and perimeter of $\mathbf{Q}_r$ are, respectively, $|\mathbf{Q}_r| = (2r)^n$ and $|\partial\mathbf{Q}_r| = 2n(2r)^{n-1}$. We are now in a position to state our main result.

Theorem 1.1 (Discrete isoperimetric inequality)**.** *For any nonempty, bounded and connected $\Omega \subset h\mathbf{Z}^n$ we have*

$$\frac{\mathrm{Per}(\Omega)^n}{\mathrm{Vol}(\Omega)^{n-1}} \geq \frac{|\partial\mathbf{Q}_1|^n}{|\mathbf{Q}_1|^{n-1}}. \tag{1.3}$$

Moreover, the equality in (1.3) *holds if and only if $E[\Omega]$ is a cube, i.e., $E[\Omega] = \bar{\mathbf{Q}}_r(a)$ for some $r > 0$ and $a \in \mathbf{R}^n$.*

The isoperimetric constant for the cube is $|\partial\mathbf{Q}_1|^n/|\mathbf{Q}_1|^{n-1} = (2n)^n$. Although (1.3) can be regarded as a "continuous" isoperimetric inequality if we identify Ω with $E[\Omega]$ in (1.2), we call (1.3) a "discrete" isoperimetric inequality since our approach to Theorem 1.1 uses numerical techniques which study functions defined on the lattice $h\mathbf{Z}^n$. Note that our result is different from the classical one in that the minimizer of the left-hand side of (1.3) is a cube. This is a consequence of the constraint by square lattices; see Example 2.4. We also remark that the equality in (1.3) does not necessarily hold; consider the two dimensional case where $h_1 = 1$ and $h_2 = \sqrt{2}$.

Isoperimetric problems on discrete spaces are studied by many authors. The paper [1] gives a survey, and in the recent book [13, Chaper 8] isoperimetric problems are studied on graphs (networks). Various results including discrete Sobolev inequalities on finite graphs are also found in [9]. Isoperimetric problems concerning lattices are discussed in several previous works; however, their settings and problems are different from ours. The authors of [2, 15] study planar convex subsets and lattice points lying in them. In [4] isoperimetric inequalities for lattice-periodic sets are derived. The reader is also referred to its related work [14, 3, 24]. Properties of planar subsets with constraint by a triangular lattice are discussed in [11].

For the proof of our discrete isoperimetric inequality we employ the idea by Cabré. As an application of the technique used in a proof of the Aleksandrov-Bakelman-Pucci (ABP for short) maximum principle, Cabré pointed out in [8] (and the original paper [7] in Catalan) that the ABP method gives a simple proof of the classical isoperimetric inequality (1.1). The ABP maximum principle ([12, Theorem 9.1], [6, Theorem 3.2]) is a pointwise estimate for solutions of elliptic partial differential equations. In a typical case the principle asserts that if u is a (sub)solution of the equation $F(\nabla^2 u) = f(x)$ in $E \subset \mathbf{R}^n$, where F is a possibly nonlinear elliptic operator and $\nabla^2 u$ denotes the Hessian of u, then we have

$$\max_{\overline{E}} u \leqq \max_{\partial E} u + C\|f\|_{L^n(\Gamma)}.$$

Here $C > 0$, $\|f\|_{L^n(\Gamma)} = (\int_\Gamma |f(x)|^n dx)^{1/n}$ and Γ is an upper contact set of u which is defined as the set of points in E where the graph of u has a tangent plane that lies above u in E. Discrete versions of the ABP estimate are also established in a series of studies by Kuo and Trudinger; see [16, 21] for linear equations, [17] for nonlinear operators, [18, 20] for parabolic cases and [19, 20] for general meshes.

Unfortunately, the result in [8] does not cover subsets having corners such as (1.2) since domains E in [8] is assumed to be smooth in order to solve Neumann problems on E. To be more precise, the author of [8] takes a function u which solves the Poisson-Neumann problem

$$\begin{cases} -\Delta u = \dfrac{|\partial E|}{|E|} & \text{in } E, \\ \dfrac{\partial u}{\partial \nu} = -1 & \text{on } \partial E, \end{cases} \tag{1.4}$$

and proves (1.1) by studying the n-dimensional Lebesgue measure of $\nabla u(\Gamma)$, the image of the upper contact set of u under the gradient of u. Here ν is the outward unit normal vector to ∂E. In this paper we solve a finite difference version of (1.4) instead of the continuous equation. Considering such discrete equations and their discrete solutions enables us to deal with non-smooth domains.

Our proof is similar to that in [8] except that the minimizers are not balls but cubes and that a superdifferential of u, which is the set of all slopes of hyperplanes touching u from above, is used instead of the gradient of u ([16]). However, there are some extra difficulties in our case. One is a solvability of the discrete Poisson-Neumann problem. Such problems are discussed in the previous work [26, 23, 25, 28], but domains are restricted to rectangles [26, 23, 28] or their collections [25]. For the proof of our discrete isoperimetric inequality, fortunately, it is enough to require u to be a subsolution of the Poisson equation in (1.4) and to satisfy the Neumann condition in (1.4) with some direction ν. For this reason we are able to construct such solutions on general subsets of $h\mathbf{Z}^n$. Solutions of (1.4) are not unique in our discrete case as well as the continuous case since adding a constant gives another solution. Accordingly, the resulting coefficient matrix of a linear system which corresponds to the discrete (1.4) is not invertible. Thus an existence of solutions to the problem will be established by determining the kernel of the matrix. We will give a proof of this existence result separately from that of the isoperimetric inequality to increase readability. Another difficulty is to study a necessary and sufficient condition which leads to the equality in (1.3). This is not discussed in [8].

This chapter is organized as follows. In Section 2 we give a proof of the discrete isoperimetric inequality. Since we use a discrete solution of the Poisson-Neumann problem in the proof, we show the existence of such solutions in Section 3. In Appendix we present two results on maximum principles; one is an ABP maximum principle shown by a similar method to the isoperimetric inequality, and the other is a strong maximum principle which is used in Section 3.

2 A proof of the discrete isoperimetric inequality

Throughout this chapter we always assume

$$\Omega \subset h\mathbf{Z}^n \text{ is nonempty, bounded and connected.}$$

We first introduce a notion of *superdifferentials* and *upper contact sets*, and then study their properties. Let $u : \overline{\Omega} \to \mathbf{R}$. We denote by $\partial^+ u(z)$ a superdifferential of u on Ω at $z \in \Omega$, which is given as

$$\partial^+ u(z) := \{p \in \mathbf{R}^n \mid u(x) \leqq \langle p, x - z\rangle + u(z),\ \forall x \in \overline{\Omega}\},$$

where $\langle \cdot, \cdot \rangle$ stands for the Euclidean inner product in $\mathbf{R}^n$. It is easy to see that $\partial^+ u(z)$ is a closed set in $\mathbf{R}^n$. We next define $\Gamma[u]$, an upper contact set of u on Ω, as

$$\begin{aligned}\Gamma[u] :=& \{z \in \Omega \mid \partial^+ u(z) \neq \emptyset\} \\ =& \{z \in \Omega \mid \exists p \in \mathbf{R}^n \text{ such that } u(x) \leqq \langle p, x - z\rangle + u(z),\ \forall x \in \overline{\Omega}\}.\end{aligned}$$

For $x \in \Omega$ and $i \in \{1, \dots, n\}$ we define discrete differential operators as follows:

$$\begin{aligned}\delta_i^+ u(x) &:= \frac{u(x + h_i e_i) - u(x)}{h_i}, \qquad \delta_i^- u(x) := -\frac{u(x - h_i e_i) - u(x)}{h_i}, \\ \delta_i^2 u(x) &:= \frac{\delta_i^+ u(x) - \delta_i^- u(x)}{h_i} = \frac{u(x + h_i e_i) + u(x - h_i e_i) - 2u(x)}{h_i^2}, \\ \Delta' u(x) &:= \sum_{j=1}^n \delta_j^2 u(x) = \sum_{j=1}^n \frac{u(x + h_j e_j) + u(x - h_j e_j)}{h_j^2} - \left(2\sum_{j=1}^n \frac{1}{h_j^2}\right) u(x).\end{aligned}$$

Lemma 2.1. *Let $u : \overline{\Omega} \to \mathbf{R}$. For all $z \in \Gamma[u]$ we have $\delta_i^+ u(z) \leqq \delta_i^- u(z)$ for every $i \in \{1, \dots, n\}$ and*

$$\partial^+ u(z) \subset [\delta_1^+ u(z), \delta_1^- u(z)] \times \cdots \times [\delta_n^+ u(z), \delta_n^- u(z)]. \tag{2.1}$$

Proof. Let $p = (p_1, \ldots, p_n) \in \partial^+ u(z)$. From the definition of the superdifferential it follows that $u(x) \leqq \langle p, x - z\rangle + u(z)$ for all $x \in \overline{\Omega}$. In particular, taking $x = z \pm h_i e_i \in \overline{\Omega}$, we have

$$u(z \pm h_i e_i) \leqq \langle p, \pm h_i e_i\rangle + u(z);$$

that is,

$$\frac{u(z + h_i e_i) - u(z)}{h_i} \leqq p_i \leqq -\frac{u(z - h_i e_i) - u(z)}{h_i}.$$

This implies $\delta_i^+ u(z) \leqq \delta_i^- u(z)$ and (2.1). □

Remark 2.2. Since $\delta_i^+ u(z) \leqq \delta_i^- u(z)$ at $z \in \Gamma[u]$ by Lemma 2.1, we see that $\delta_i^2 u(z) \leqq 0$ for all $i \in \{1, \ldots, n\}$.

In the proof of the classical isoperimetric inequality proposed by Cabré [7, 8], solutions of the Poisson-Neumann problem (1.4) are studied, and actually the proof still works for a subsolution u of (1.4), i.e., $-\Delta u \leqq |\partial E|/|E|$ in E and $\partial u/\partial\nu = -1$ on ∂E. Similarly to this classical case, for the proof of Theorem 1.1 we consider the discrete version of (1.4) on Ω, which is

$$\text{(NP)} \begin{cases} -\Delta u \leqq \dfrac{\mathrm{Per}(\Omega)}{\mathrm{Vol}(\Omega)} & \text{in } \Omega, \qquad (2.2) \\ \dfrac{\partial u}{\partial \nu} = -1 & \text{on } \partial\Omega. \qquad (2.3) \end{cases}$$

The meaning of solutions of (NP) is given as follows. We say $u : \overline{\Omega} \to \mathbf{R}$ is a *discrete solution* of (NP) if

(a) $-\Delta' u(x) \leqq \mathrm{Per}(\Omega)/\mathrm{Vol}(\Omega)$ for all $x \in \Omega$;

(b) For all $x \in \partial\Omega$ there exist some $i \in \{1, \ldots, n\}$ and $\sigma \in \{-1, 1\}$ such that $x + \sigma h_i e_i \in \Omega$ and
$$\frac{u(x) - u(x + \sigma h_i e_i)}{h_i} = -1.$$

The condition (b) requires that the outward normal derivative of u be -1 for some direction $\nu = \pm e_i$. This boundary condition is also explained by saying that $\delta_i^+ u(x) = 1$ or $\delta_i^- u(x) = -1$ for all $x \in \partial\Omega$. We will prove the existence of discrete solutions of (NP) in the next section (Proposition 3.2).

Proof of Theorem 1.1. 1. Let $u : \overline{\Omega} \to \mathbf{R}$ be a discrete solution of (NP) and let $\Gamma[u]$ be the upper contact set of u on Ω. We claim

$$\mathbf{Q}_1 \subset \bigcup_{z \in \Gamma[u]} \partial^+ u(z). \tag{2.4}$$

Let $p \in \mathbf{Q}_1$. We take a maximum point $\hat{x} \in \overline{\Omega}$ of $u(x) - \langle p, x\rangle$ over $\overline{\Omega}$. To show (2.4) it is enough to prove that $\hat{x} \in \Omega$ since we then have $\hat{x} \in \Gamma[u]$ and $p \in \partial^+ u(\hat{x})$. Suppose by contradiction that $\hat{x} \in \partial\Omega$. Take any $i \in \{1, \ldots, n\}$

and $\sigma \in \{-1, 1\}$ such that $y := \hat{x} + \sigma h_i e_i \in \Omega$. Since $u(x) - \langle p, x\rangle$ attains its maximum at $\hat{x}$ and since p lies in the open cube Q_1, we compute

$$\frac{u(\hat{x}) - u(y)}{h_i} \geqq \frac{\langle p, \hat{x}\rangle - \langle p, y\rangle}{h_i} = \frac{\langle p, -\sigma h_i e_i\rangle}{h_i} \geqq -|p_i| > -1.$$

This implies that u does not satisfy the boundary condition (2.3) at $\hat{x} \in \partial\Omega$, a contradiction. Therefore (2.4) follows. We also remark that (2.4) guarantees $\Gamma[u]$ is nonempty.

2. By (2.4) we see

$$|\mathbf{Q}_1| = \mathcal{L}^n(\mathbf{Q}_1) \leqq \mathcal{L}^n\left(\bigcup_{z\in\Gamma[u]} \partial^+ u(z)\right) \leqq \sum_{z\in\Gamma[u]} \mathcal{L}^n(\partial^+ u(z)). \tag{2.5}$$

Also, for each $z \in \Gamma[u]$ Lemma 2.1 implies

$$\begin{aligned}\mathcal{L}^n(\partial^+ u(z)) &\leqq \mathcal{L}^n([\delta_1^+ u(z), \delta_1^- u(z)] \times \cdots \times [\delta_n^+ u(z), \delta_n^- u(z)]) \\ &= (\delta_1^- u(z) - \delta_1^+ u(z)) \times \cdots \times (\delta_n^- u(z) - \delta_n^+ u(z)) \\ &= h^n(-\delta_1^2 u(z)) \times \cdots \times (-\delta_n^2 u(z)). \end{aligned} \tag{2.6}$$

We next apply the arithmetic-geometric mean inequality to obtain

$$(-\delta_1^2 u(z)) \times \cdots \times (-\delta_n^2 u(z)) \leqq \left(\frac{-\delta_1^2 u(z) - \cdots - \delta_n^2 u(z)}{n}\right)^n = \left(\frac{-\Delta' u(z)}{n}\right)^n. \tag{2.7}$$

Consequently, combining (2.5)–(2.7) yields

$$|\mathbf{Q}_1| \leqq \sum_{z\in\Gamma[u]} h^n \left(\frac{-\Delta' u(z)}{n}\right)^n \leqq \sum_{z\in\Gamma[u]} h^n \frac{\mathrm{Per}(\Omega)^n}{n^n \mathrm{Vol}(\Omega)^n} \leqq \frac{\mathrm{Per}(\Omega)^n}{n^n \mathrm{Vol}(\Omega)^{n-1}}. \tag{2.8}$$

Since $n = |\partial\mathbf{Q}_1|/|\mathbf{Q}_1|$, it follows that

$$\frac{\mathrm{Per}(\Omega)^n}{\mathrm{Vol}(\Omega)^{n-1}} \geqq n^n|\mathbf{Q}_1| = \frac{|\partial\mathbf{Q}_1|^n}{|\mathbf{Q}_1|^n}|\mathbf{Q}_1| = \frac{|\partial\mathbf{Q}_1|^n}{|\mathbf{Q}_1|^{n-1}}.$$

3. We next assume that the equality in (1.3) holds. In view of Step 2, we then have $\Gamma[u] = \Omega$ by (2.8) and

$$\mathcal{L}^n(\mathbf{Q}_1) = \mathcal{L}^n\left(\bigcup_{x\in\Omega} \partial^+ u(x)\right), \tag{2.9}$$

$$\mathcal{L}^n(\partial^+ u(x)) = \mathcal{L}^n([\delta_1^+ u(x), \delta_1^- u(x)] \times \cdots \times [\delta_n^+ u(x), \delta_n^- u(x)]) \quad \text{for all } x \in \Omega, \tag{2.10}$$

$$\delta_1^2 u(x) = \cdots = \delta_n^2 u(x) =: \mu(x) \ (\leqq 0) \quad \text{for all } x \in \Omega \tag{2.11}$$

by (2.5), (2.6) and (2.7), respectively. Here we have derived (2.11) from the equality case of the arithmetic-geometric mean inequality. We claim

$$\partial^+ u(x) = [\delta_1^+ u(x), \delta_1^- u(x)] \times \cdots \times [\delta_n^+ u(x), \delta_n^- u(x)] \quad \text{for all } x \in \Omega. \tag{2.12}$$

One inclusion is known by (2.1). Also, the both sets in (2.12) are closed and have the same measure by (2.10). Thus they have to be the same set. For the same reason it follows from (2.4) and (2.9) that

$$\bar{\mathbf{Q}}_1 = \bigcup_{x\in\Omega} \partial^+ u(x). \tag{2.13}$$

4. Let $x, y \in \Omega$ be such that $y = x + h_i e_i$ for some $i \in \{1, \dots, n\}$. Then we show $\mu(x) = \mu(y)$ and

$$\partial^+ u(y) = \partial^+ u(x) + h_i \mu_0 e_i \tag{2.14}$$

with $\mu_0 := \mu(x)$, where $\mu(\cdot)$ is the function in (2.11). Without loss of generality we may assume $x = 0$, $y = h_1 e_1$ and $u(x) = 0$. We then notice that $u(y) = h_1 \delta_1^+ u(0)$. Fix $i \in \{2, \dots, n\}$ and set $p^\pm := \delta_1^+ u(0) e_1 + \delta_i^\pm u(0) e_i$. Because of (2.12) we see that $p^\pm$ belong to $\partial^+ u(0)$. Since $x = 0 \in \Gamma[u]$, we observe that $u(z) \leqq \langle p^\pm, z\rangle$ for all $z \in \overline{\Omega}$. In particular, letting $z = h_1 e_1 \pm h_i e_i$, we deduce $u(z) \leqq h_1 \delta_1^+ u(0) \pm h_i \delta_i^\pm u(0) = u(y) \pm h_i \delta_i^\pm u(0)$, i.e., $\delta_i^+ u(y) \leqq \delta_i^+ u(0)$ and $\delta_i^- u(0) \leqq \delta_i^- u(y)$. Changing the role of x and y we also have $\delta_i^+ u(y) \geqq \delta_i^+ u(0)$ and $\delta_i^- u(0) \geqq \delta_i^- u(y)$. Thus

$$\delta_i^+ u(y) = \delta_i^+ u(0) \quad \text{and} \quad \delta_i^- u(0) = \delta_i^- u(y) \tag{2.15}$$

for all $i \in \{2, \dots, n\}$. By (2.11) these equalities imply $\mu(x) = \mu(y)$, and then $\delta_1^\pm u(y)$ are computed as

$$\delta_1^- u(y) = \delta_1^+ u(x) = \delta_1^- u(x) + h_1\mu_0, \quad \delta_1^+ u(y) = \delta_1^- u(y) + h_1\mu_0 = \delta_1^+ u(x) + h_1 \mu_0.$$

Namely, we have $[\delta_1^- u(y), \delta_1^+ u(y)] = [\delta_1^- u(x), \delta_1^+ u(x)] + h_1\mu_0$, which together with (2.15) shows (2.14).

5. By translation we may let $0 \in \Omega$. Set $R := [-h_1/2, h_1/2] \times \cdots \times [-h_n/2, h_n/2]$ and in view of (2.11) and (2.12) there exists $z \in \mathbf{R}^n$ such that $\partial^+ u(0) = z + \mu R$ with $\mu := \mu(0)$. Since Ω is now connected, as a consequence of Step 4 we see $\mu(x) \equiv \mu$ and $\partial^+ u(x) = \partial^+ u(0) + \mu x = z + \mu x + \mu R$ for all $x \in \Omega$. Therefore (2.13) implies

$$\bar{\mathbf{Q}}_1 = \bigcup_{x\in\Omega} (z + \mu x + \mu R).$$

Finally, from translation and rescaling it follows that

$$\bar{\mathbf{Q}}_{1/|\mu|}(-z/\mu) = \bigcup_{x\in\Omega} (x + R) = E[\Omega],$$

which is the desired conclusion. □

Remark 2.3. If a nonempty and bounded subset $\Omega' \subset h\mathbf{Z}^n$ is not connected, then we have the strict inequality

$$\frac{\mathrm{Per}(\Omega')^n}{\mathrm{Vol}(\Omega')^{n-1}} > \frac{|\partial \mathbf{Q}_1|^n}{|\mathbf{Q}_1|^{n-1}}. \tag{2.16}$$

This is shown by docking one connected component with another one. To be more precise, translating two connected components Ω_1 and Ω_2, we are able to construct one connected set whose volume is equal to that of $\Omega_1 \cup \Omega_2$ and whose perimeter is strictly less than that of $\Omega_1 \cup \Omega_2$. Iterating this procedure, we finally obtain a connected set Ω such that $\mathrm{Vol}(\Omega') = \mathrm{Vol}(\Omega)$ and $\mathrm{Per}(\Omega') > \mathrm{Per}(\Omega)$. These relations and Theorem 1.1 imply (2.16).

Example 2.4. In the planar case ($n = 2$) it is easily seen that round-shaped subsets are not optimal. Let $h_1 = h_2 = 1$ for simplicity, and consider $\Omega \subset \mathbf{Z}^2$ which is nonempty, bounded and connected. We choose $R = \{a, a+1, \ldots, a+M-1\} \times \{b, b+1, \ldots, b+N-1\} \subset \mathbf{Z}^2$ as the minimal rectangle such that $\Omega \subset R$. Obviously, $\mathrm{Vol}(\Omega) < \mathrm{Vol}(R)$ if $\Omega \neq R$. We next consider their perimeters. Since Ω is connected, for each $x \in \{a, a+1, \ldots, a+M-1\}$ there exist $(x, y_-), (x, y_+) \in \Omega$ such that $(x, y_- - 1), (x, y_+ + 1) \notin \Omega$. This implies $\omega_1[\Omega] \geqq 2M = \omega_1[R]$. Similarly, we obtain $\omega_2[\Omega] \geqq 2N = \omega_2[R]$, and therefore $\mathrm{Per}(\Omega) \geqq \mathrm{Per}(R)$. We thus conclude that $\mathrm{Per}(\Omega)^2/\mathrm{Vol}(\Omega) > \mathrm{Per}(R)^2/\mathrm{Vol}(R)$, i.e., Ω is not optimal. Moreover, we see that, among all rectangles $R = \{a, a+1, \ldots, a+M-1\} \times \{b, b+1, \ldots, b+N-1\}$, a square is the best shape since

$$\frac{\mathrm{Per}(R)^2}{\mathrm{Vol}(R)} = \frac{\{2(M+N)\}^2}{MN} = 4\left(\frac{M}{N} + \frac{N}{M} + 2\right) \geqq 4(2+2) = 16$$

by the arithmetic-geometric mean inequality. Therefore, in the planar case Theorem 1.1 is easily shown. However, the above argument is not valid for $n \geqq 3$ since the inequalities $\omega_i[\Omega] \geqq \omega_i[R]$ do not necessarily hold.

On the contrary, if we define a volume and a perimeter of Ω as $\#\Omega$ and $\#(\partial\Omega)$, respectively, then a cube is not an optimal shape. This can be seen in the following simple example. Let $n = 2$, $h_1 = h_2 = 1$ again and consider planar subsets $\Omega_1 = \{(x, y) \in \mathbf{Z}^2 \mid |x| \leqq 1,\ |y| \leqq 1\}$ and $\Omega_2 = \{(x, y) \in \mathbf{Z}^2 \mid |x| + |y| \leqq 2\}$. We then have $\#\Omega_1 = 9$, $\#\Omega_2 = 13$ and $\#(\partial\Omega_1) = \#(\partial\Omega_2) = 12$. Thus the square Ω_1 is not a minimizer of the functional $(\#(\partial\Omega))^2/(\#\Omega)$. In the article [10] the author asserts that if Ω has a minimal $\#\partial\Omega$, then Ω is roughly diamond-shaped. The author of [10] also observes inequalities $(\#(\partial\Omega))^2/(\#\Omega) > 8$ for the two dimensional case and $(\#(\partial\Omega))^3/(\#\Omega)^2 > 36$ for the three dimensional case without detailed argument. We do not discuss such problems concerning the functional $(\#(\partial\Omega))^n/(\#\Omega)^{n-1}$ in the present paper.

3 An existence result for the Poisson-Neumann problem

We shall prove the solvability of (NP), the Poisson equation with the Neumann boundary condition which appeared in the proof of the discrete isoperimetric inequality. Before starting the proof, using a simple example, we explain how to construct the solutions.

Example 3.1. Consider $\Omega \subset h\mathbf{Z}^2$ which consists of three points P_1, P_2 and P_3 in the left lattice of Figure 1. We also denote by $S_1, \ldots, S_7$ all points on

$\partial\Omega$ as in the same figure. In order to determine values of u on $\overline{\Omega}$ we solve a system of linear equations of the matrix form $L\vec{a} = \vec{b}$ which corresponds to the finite difference equation (NP). However, if we require u to satisfy the Neumann condition (2.3) at S_1 toward the both adjacent points P_1 and P_3, the linear system may not be solvable since the number of the unknowns is less than that of equations; in the present example they are 10 and 11, respectively. Thus we are tempted to consider the Neumann condition toward either P_1 or P_3 since we are now allowed to relax (2.3) in this way by the meaning of solutions. Then the number of equations decreases to 10, but, unfortunately, it becomes difficult to study the linear system since the new matrix L is not symmetric. In addition, we do not know a priori how to choose the adjacent point toward which the Neumann condition is satisfied.

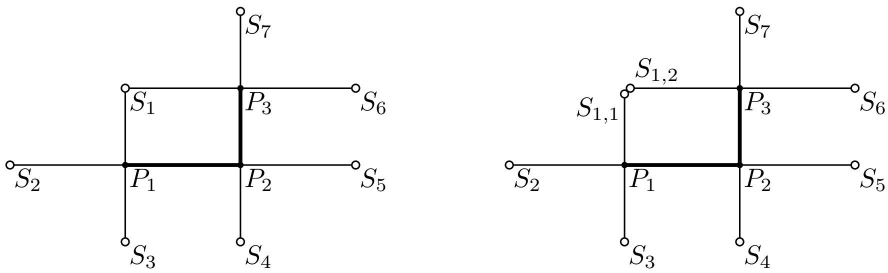

Figure 1: $\Omega = \{P_i\}_{i=1}^{3}$ and $\partial\Omega = \{S_i\}_{i=1}^{7}$. We solve a system of linear equations for the right lattice, and then define $u(S_1) := \max\{u(S_{1,1}),\ u(S_{1,2})\}$.

To avoid these situations we regard S_1 as two different points $S_{1,1}$ and $S_{1,2}$ which are connected to P_1 and P_3, respectively, and consider a modified system with new unknowns $u(S_{1,1})$ and $u(S_{1,2})$ instead of $u(S_1)$; see the right lattice in Figure 1. Then the number of the unknowns in our example becomes 11. Thanks to this increase of the unknowns, it turns out that the modified linear system admits at least one solution $(u(P_1), u(P_2), u(P_3), u(S_{1,1}), u(S_{1,2}), u(S_2), \dots, u(S_7))$. (In the notation of the proof below we write $u(S_{1,1}) = \beta(1,1)$ and $u(S_{1,2}) = \beta(1,2)$.) In the process of proving the solvability we find that the right-hand side of (2.2) should be $\mathrm{Per}(\Omega)/\mathrm{Vol}(\Omega)$. Also, for its proof we employ the strong maximum principle for the discrete Laplace equation.

The remaining problem is how to define $u(S_1)$. We define $u(S_1)$ as the maximum of $u(S_{1,1})$ and $u(S_{1,2})$, so that, if $u(S_{1,1}) \geqq u(S_{1,2})$, we have $-\Delta' u(P_3) \leqq \mathrm{Per}(\Omega)/\mathrm{Vol}(\Omega)$ since $u(S_1) \geqq u(S_{1,2})$ and $\{u(S_1) - u(P_1)\}/h_2 = -1$ since $u(S_1) = u(S_{1,1})$. In this way we obtain a solution of (NP).

Proposition 3.2. *The problem* (NP) *admits at least one discrete solution.*

Proof. 1. We first introduce notations. Let $\Omega = \{P_1, \dots, P_M\}$ and $\partial\Omega = \{S_1, \dots, S_{N_0}\}$, where $M := \#\Omega$ and $N_0 := \#(\partial\Omega)$. For each $i \in \{1, \dots, M\}$ we define subsets $\mathcal{M}(i) \subset \{1, \dots, M\}$ and $\mathcal{N}(i) \subset \{1, \dots, N_0\}$ so that $\overline{\{P_i\}} \setminus \{P_i\} = \{P_j\}_{j \in \mathcal{M}(i)} \cup \{S_j\}_{j \in \mathcal{N}(i)}$. We also set $s_i := \#(\overline{\{S_i\}} \cap \Omega)$ for $i \in \{1, \dots, N_0\}$, which stands for the number of points of Ω adjacent to S_i, and $N := \sum_{j=1}^{N_0} s_j$.

Next, for $i \in \{1, \ldots, N_0\}$ we define a map $n_i : \{1, \ldots, s_i\} \to \{1, \ldots, M\}$ such that $n_i(1) < n_i(2) < \cdots$ and $\overline{\{S_i\}} \cap \Omega = \{P_{n_i(j)}\}_{j=1}^{s_i}$. We denote by n_i^{-1} the inverse map of n_i; that is, P_j is the $n_i^{-1}(j)$-th point of $(P_{n_i(1)}, \ldots, P_{n_i(s_i)})$ if $P_j \in \overline{\{S_i\}} \cap \Omega$.

For $x, y \in \overline{\Omega}$ such that $y = x + \sigma h_i e_i$ with $\sigma = \pm 1$ and $i \in \{1, \ldots, n\}$ we set $h(x, y) := h_i$. Obviously, we then have $h(x, y) = h(y, x)$. We denote by $E(i, j)$ the $(M + N) \times (M + N)$ matrix with 1 in the (i, j) entry and 0 elsewhere. Given a vector

$$\vec{a} = {}^t(\alpha(1), \ldots, \alpha(M), \beta(1, 1), \ldots, \beta(1, s_1), \ldots, \beta(N_0, 1), \ldots, \beta(N_0, s_{N_0})) \in \mathbf{R}^{M+N}, \tag{3.1}$$

where ${}^t\vec{v}$ means the transpose of a vector $\vec{v}$, we define $u = u[\vec{a}] : \overline{\Omega} \to \mathbf{R}$ as

$$u(x) := \begin{cases} \alpha(i) & (x = P_i \in \Omega,\ i \in \{1, \ldots, M\}), \\ \max\{\beta(i, j) \mid 1 \leqq j \leqq s_i\} & (x = S_i \in \partial\Omega,\ i \in \{1, \ldots, N_0\}). \end{cases}$$

2. We consider the following system of linear equations

$$L\vec{a} = \vec{b}, \tag{3.2}$$

where $\vec{a} \in \mathbf{R}^{M+N}$ is the unknown vector and $\vec{b} = (b_k)_{k=1}^{M+N} \in \mathbf{R}^{M+N}$ is given as

$$b_k = \begin{cases} \dfrac{\mathrm{Per}(\Omega)}{\mathrm{Vol}(\Omega)} & (k = 1, \ldots, M), \\ \dfrac{-1}{h(S_j, P_{n_j(i)})} & (k = M + \displaystyle\sum_{l=0}^{j-1} s_l + i \text{ with } j \in \{1, \ldots, N_0\},\ i \in \{1, \ldots, s_j\}). \end{cases}$$

Here $s_0 = 0$. Also, the $(M + N) \times (M + N)$ matrix L is defined by

$$\begin{aligned} L :=& \\ &\begin{pmatrix} \theta I_M & 0 \\ 0 & 0 \end{pmatrix} - \sum_{i=1}^{M} \left\{ \sum_{j \in \mathcal{M}(i)} \frac{E(i, j)}{h(P_i, P_j)^2} + \sum_{j \in \mathcal{N}(i)} \frac{E\left(i, M + \sum_{l=0}^{j-1} s_l + n_j^{-1}(i)\right)}{h(P_i, S_j)^2} \right\} \\ &+ \sum_{j=1}^{N_0} \sum_{i=1}^{s_j} \frac{E\left(M + \sum_{l=0}^{j-1} s_l + i, M + \sum_{l=0}^{j-1} s_l + i\right) - E\left(M + \sum_{l=0}^{j-1} s_l + i, n_j(i)\right)}{h(S_j, P_{n_j(i)})^2}, \end{aligned}$$

where I_M is the identity matrix of dimension M and $\theta := 2\sum_{i=1}^{n}(1/h_i^2)$. (See Example 3.4, where we will give a small sized matrix L along the example of Figure 1.) By definition L is symmetric. To check the symmetricity we first take $i \in \{1, \ldots, M\}$ and $j \in \mathcal{M}(i)$. Then the (i, j) entry of L is $-1/h(P_i, P_j)^2$. Since $j \in \mathcal{M}(i)$, we see $P_j \in \overline{\{P_i\}}$. Thus $P_i \in \overline{\{P_j\}}$ and this implies $i \in \mathcal{M}(j)$. As a result, it follows that the (j, i) entry of L is $-1/h(P_j, P_i)^2$. We next let $i \in \{1, \ldots, M\}$ and $j \in \mathcal{N}(i)$, so that the $(i, M + \sum_{l=0}^{j-1} s_l + n_j^{-1}(i))$ entry of L is $-1/h(P_i, S_j)^2$. In this case we have $S_j \in \overline{\{P_i\}}$, and so $P_i \in \overline{\{S_j\}}$. Then from the definition of n_j it follows that $n_j(t) = i$ for some $t \in \{1, \ldots, s_j\}$, i.e., $t = n_j^{-1}(i)$.

Since $(M+\sum_{l=0}^{j-1} s_l + n_j^{-1}(i), i) = (M+\sum_{l=0}^{j-1} s_l + t, n_j(t))$, we conclude that the $(M+\sum_{l=0}^{j-1} s_l + n_j^{-1}(i), i)$ entry of L is $-1/h(S_j, P_{n_j(t)})^2 = -1/h(S_j, P_i)^2$. Hence the symmetricity of L is proved.

3. We claim that if $\vec{a} \in \mathbf{R}^{M+N}$ is a solution of (3.2), then $u = u[\vec{a}]$ is a discrete solution of (NP). Let $x \in \Omega$, i.e., $x = P_i$ for some i. Without loss of generality we may assume $x = P_1$. Since $\vec{a}$ satisfies (3.2), comparing the first coordinates of the both sides in (3.2), we observe

$$\begin{aligned}\frac{\mathrm{Per}(\Omega)}{\mathrm{Vol}(\Omega)} &= \theta\alpha(1) - \sum_{j \in \mathcal{M}(1)} \frac{\alpha(j)}{h(P_1, P_j)^2} - \sum_{j \in \mathcal{N}(1)} \frac{\beta(j, n_j^{-1}(1))}{h(P_1, S_j)^2} \\ &\geqq \theta u(P_1) - \sum_{j \in \mathcal{M}(1)} \frac{u(P_j)}{h(P_1, P_j)^2} - \sum_{j \in \mathcal{N}(1)} \frac{u(S_j)}{h(P_1, S_j)^2} \\ &= -\Delta' u(P_1).\end{aligned}$$

We next let $x \in \partial\Omega$. Again we may assume $x = S_1$. We also let $\beta(1, j_0) = \max\{\beta(1, j) \mid 1 \leqq j \leqq s_1\}$. Then the $(M + j_0)$-th coordinates in (3.2) implies

$$\frac{\beta(1, j_0) - \alpha(n_1(j_0))}{h(S_1, P_{n_1(j_0)})^2} = \frac{-1}{h(S_1, P_{n_1(j_0)})},$$

that is,

$$\frac{u(S_1) - u(P_{n_1(j_0)})}{h(S_1, P_{n_1(j_0)})} = -1.$$

Consequently, we see that u is a discrete solution of (NP) in our sense.

4. We shall show that (3.2) is solvable. For this purpose, we first assert that $\mathrm{Ker}L = \mathbf{R}\vec{\xi}$, where $\mathrm{Ker}L$ is the kernel of L and

$$\vec{\xi} = {}^t(1, \ldots, 1) \in \mathbf{R}^{M+N}.$$

By the definition of L we see that the sum of each row of L is zero. This implies $\mathrm{Ker}L \supset \mathbf{R}\vec{\xi}$. We next let $\vec{a} \in \mathrm{Ker}L$, i.e., $L\vec{a} = 0$. We represent each component of $\vec{a}$ as in (3.1). Now, by the same argument as in Step 3 we see that $u = u[\vec{a}]$ is a discrete solution of

$$(\mathrm{NP0}) \begin{cases} -\Delta u \leqq 0 & \text{in } \Omega, & (3.3) \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, & (3.4) \end{cases}$$

where the notion of a discrete solution of (NP0) is the same as that of (NP). We take a maximum point $z \in \overline{\Omega}$ of u over $\overline{\Omega}$. If $z \in \partial\Omega$, there exists some $y \in \overline{\{z\}} \cap \Omega$ such that $u(y) = u(z)$ since u satisfies the Neumann boundary condition (3.4) at z. Thus u attains its maximum at some point in Ω. Since Ω is now bounded and connected, the strong maximum principle for the Laplace equation (Corollary A.5) ensures that u must be some constant $c \in \mathbf{R}$ on $\overline{\Omega}$. From this it follows that $\alpha(1) = \cdots = \alpha(M) = c$. Also, since $L\vec{a} = 0$, we have $\beta(i, j) = \alpha(n_i(j))$ for all $i \in \{1, \ldots, N_0\}$ and $j \in \{1, \ldots, s_i\}$. As a result, we see $\vec{a} = c\vec{\xi} \in \mathbf{R}\vec{\xi}$. We thus conclude that $\mathrm{Ker}L = \mathbf{R}\vec{\xi}$.

5. Since L is symmetric and $\mathrm{Ker}L = \mathbf{R}\vec{\xi}$, we see that $(\mathrm{Im}L)^{\perp} = \mathbf{R}\vec{\xi}$, where $(\mathrm{Im}L)^{\perp}$ stands for the orthogonal complement of $\mathrm{Im}L$, the image of L. Thus, for $\vec{b}' \in \mathbf{R}^{M+N}$ it follows that $\vec{b}' \in \mathrm{Im}L$ if and only if $\langle \vec{\xi}, \vec{b}' \rangle = 0$. Noting that $-1/h_i$ appears ω_i times in a sequence $\{b_k\}_{k=M+1}^{M+N}$ for each $i \in \{1, \dots, n\}$, we compute

$$\langle \vec{\xi}, \vec{b} \rangle = \frac{\mathrm{Per}(\Omega)}{\mathrm{Vol}(\Omega)} \times M + \sum_{i=1}^{n} \left(\frac{-1}{h_i} \times \omega_i \right) = \frac{\mathrm{Per}(\Omega)}{h^n} - \sum_{i=1}^{n} \frac{\omega_i}{h_i} = 0.$$

Consequently $\vec{b} \in \mathrm{Im}L$, and therefore the problem (3.2) has at least one solution $\vec{a} \in \mathbf{R}^{M+N}$. Hence by Step 3 the corresponding $u = u[\vec{a}]$ solves (NP). $\square$

Remark 3.3. We have actually proved that u, which we constructed as a subsolution, is a solution of (2.2) in $\Omega \setminus \overline{\partial\Omega}$. Namely, we have $-\Delta' u(x) = \mathrm{Per}(\Omega)/\mathrm{Vol}(\Omega)$ for all $x \in \Omega \setminus \overline{\partial\Omega}$. This is clear from the construction of u.

Example 3.4. We revisit Example 3.1 and consider Ω given in Figure 1. Let us solve the system (3.2). For simplicity we assume $h_1 = h_2 =: h > 0$. In the notation used in the proof of Proposition 3.2, the unknown vector $\vec{a}$ is given as

$${}^t\vec{a} = \left(\begin{array}{ccc||cc|cccc} \alpha(1) & \alpha(2) & \alpha(3) & \beta(1,1) & \beta(1,2) & \beta(2,1) & \beta(3,1) & \dots & \beta(7,1) \end{array} \right).$$

Here $\alpha(i)$ $(i = 1, 2, 3)$ represents the value of $u(P_i)$. Also, $\beta(1, j)$ $(j = 1, 2)$ and $\beta(k, 1)$ $(k = 2, 3, \dots, 7)$ represent the values of $u(S_{1,j})$ and $u(S_k)$, respectively. Since $\mathrm{Vol}(\Omega) = 3h^2$ and $\mathrm{Per}(\Omega) = 8h$ in this example, we see

$${}^t\vec{b} = \frac{1}{h} \left(\begin{array}{ccc||cc|cccccc} \frac{8}{3} & \frac{8}{3} & \frac{8}{3} & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{array} \right),$$

and the coefficient matrix L is

$$L = \frac{1}{h^2} \left(\begin{array}{ccc||cc|cccccc} 4 & -1 & & -1 & & -1 & -1 & & & & \\ -1 & 4 & -1 & & & & & -1 & -1 & & \\ & -1 & 4 & & -1 & & & & & -1 & -1 \\ \hline\hline -1 & & & 1 & & & & & & & \\ & & -1 & & 1 & & & & & & \\ \hline -1 & & & & & 1 & & & & & \\ -1 & & & & & & 1 & & & & \\ & -1 & & & & & & 1 & & & \\ & -1 & & & & & & & 1 & & \\ & & -1 & & & & & & & 1 & \\ & & -1 & & & & & & & & 1 \end{array} \right).$$

The rest entries in L are zeros. A direct computation shows that $\vec{a_0}$ given as

$${}^t\vec{a_0} = \frac{h}{3} \left(\begin{array}{ccc||cc|cccccc} 3 & 4 & 3 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{array} \right)$$

is a particular solution of (3.2). Since the kernel of L is known, we conclude that the general solution of (3.2) is $\vec{a} = \vec{a_0} + c\ {}^t(1, \dots, 1)$ with $c \in \mathbf{R}$.

A Maximum principles

A.1 An ABP maximum principle

In Appendix we consider the second order fully nonlinear elliptic equations of the form

$$F(\partial_{x_1}^2 u, \dots, \partial_{x_n}^2 u) = f(x) \quad \text{in } \Omega, \tag{A.1}$$

where $F : \mathbf{R}^n \to \mathbf{R}$ and $f : \Omega \to \mathbf{R}$ are given function such that $F(0, \dots, 0) = 0$. Let $\vec{\delta}^2 u(x) := (\delta_1^2 u(x), \dots, \delta_n^2 u(x))$. We say $u : \overline{\Omega} \to \mathbf{R}$ is a *discrete subsolution* of (A.1) if $F(\vec{\delta}^2 u(x)) \leqq f(x)$ for all $x \in \Omega$. As an ellipticity condition on F for our ABP estimate, we use the following:

(F1) $-\lambda \sum \vec{X} \leqq F(\vec{X})$ for all $\vec{X} \in \mathbf{R}^n$ with $\vec{X} \leqq 0$.

Here $\lambda > 0$. Also, $\sum \vec{X} := \sum_{i=1}^n X_i$ for $\vec{X} = (X_1, \dots, X_n) \in \mathbf{R}^n$ and the inequality $\vec{X} \leqq 0$ means that $X_i \leqq 0$ for every $i \in \{1, \dots, n\}$. For $K \subset h\mathbf{Z}^n$ and $g : K \to \mathbf{R}$ the n-norm of g over K is given as $\|g\|_{\ell^n(K)} := \left(\sum_{x \in K} h^n |g(x)|^n\right)^{1/n}$. We also set $\operatorname{diam}(\Omega) := \max_{x \in \Omega, y \in \partial\Omega} |x - y|$ and $|\mathbf{B}_r| := \mathcal{L}^n(\mathbf{B}_r)$.

Theorem A.1 (ABP maximum principle)**.** *Assume* (F1)*. Let $u : \overline{\Omega} \to \mathbf{R}$ be a discrete subsolution of* (A.1)*. Then the estimate*

$$\max_{\overline{\Omega}} u \leqq \max_{\partial\Omega} u + C_A \operatorname{diam}(\Omega) \|f\|_{\ell^n(\Gamma[u])} \tag{A.2}$$

holds, where $C_A = C_A(\lambda, n)$ is given as $C_A = (\lambda n |\mathbf{B}_1|^{1/n})^{-1}$.

A crucial estimate to prove Theorem A.1 is

Proposition A.2. *For all $u : \overline{\Omega} \to \mathbf{R}$ we have*

$$\max_{\overline{\Omega}} u \leqq \max_{\partial\Omega} u + \frac{\operatorname{diam}(\Omega)}{n |\mathbf{B}_1|^{1/n}} \| - \Delta' u\|_{\ell^n(\Gamma[u])}. \tag{A.3}$$

Proof. 1. We first prove $\mathbf{B}_d \subset \bigcup_{z \in \Gamma[u]} \partial^+ u(z)$, where d is a constant given as $d = (\max_{\overline{\Omega}} u - \max_{\partial\Omega} u)/\operatorname{diam}(\Omega)$. If $d = 0$, the assertion is obvious. We assume $d > 0$, i.e., $u(\hat{x}) = \max_{\overline{\Omega}} u > \max_{\partial\Omega} u$ for some $\hat{x} \in \Omega$. Let $p \in \mathbf{B}_d$ and set $\phi(x) := \langle p, x - \hat{x} \rangle$. We take a maximum point z of $u - \phi$ over $\overline{\Omega}$. Then we have $z \in \Omega$. Indeed, for all $x \in \partial\Omega$ we observe

$$u(x) - \phi(x) \leqq \max_{\partial\Omega} u + |p| \cdot |x - \hat{x}| < \max_{\partial\Omega} u + d \cdot \operatorname{diam}(\Omega) = \max_{\overline{\Omega}} u = u(\hat{x}) - \phi(\hat{x}).$$

Thus $z \in \Omega$, and so we conclude that $z \in \Gamma[u]$ and $p \in \partial^+ u(z)$.

2. By Step 1 the estimate (2.5) with $\mathbf{B}_d$ instead of $\mathbf{Q}_1$ holds. Thus the same argument as in the proof of Theorem 1.1 yields

$$|\mathbf{B}_d| \leqq \sum_{z \in \Gamma[u]} h^n \left(\frac{-\Delta' u(z)}{n} \right)^n = \frac{1}{n^n} \| - \Delta' u\|^n_{\ell^n(\Gamma[u])}.$$

Applying $|\mathbf{B}_d| = d^n |\mathbf{B}_1|$ to the above inequality, we obtain (A.3) by the choice of d. $\square$

Proof of Theorem A.1. By Remark 2.2 we have $\vec{\delta}^2 u(z) \leqq 0$ for $z \in \Gamma[u]$, and therefore the condition (F1) yields $-\lambda\Delta' u(z) = -\lambda \sum \vec{\delta}^2 u(z) \leqq F(\vec{\delta}^2 u(z))$. Since u is a discrete subsolution of (A.1), we also have $F(\vec{\delta}^2 u(z)) \leqq f(z)$. Applying these two inequalities to (A.3), we obtain (A.2). □

A.2 A strong maximum principle

Although the strong maximum principle for the Laplace equation is enough for the proof of Proposition 3.2, we consider a wider class of equations in this subsection. We study homogeneous equations of the form

$$F(\partial_{x_1}^2 u, \ldots, \partial_{x_n}^2 u) = 0 \quad \text{in } \Omega. \tag{A.4}$$

From the ABP maximum principle (A.2) we learn that all discrete subsolutions u of (A.4) satisfy

$$\max_{\overline{\Omega}} u \leqq \max_{\Omega} u$$

if (F1) holds. This is the so-called weak maximum principle. Our aim in this subsection is to prove that a certain weaker condition on F actually leads to the strong maximum principle and conversely the weaker condition is necessary for it. Here the rigorous meaning of the strong maximum principle is

(SMP) If $u : \overline{\Omega} \to \mathbf{R}$ is a discrete subsolution of (A.4) such that $\max_{\overline{\Omega}} u = \max_{\Omega} u$, then u must be constant on $\overline{\Omega}$.

Following the classical theory of partial differential equations, we consider bounded and connected subsets $\Omega \subset h\mathbf{Z}^n$ for (SMP). It turns out that the strong maximum principle holds if and only if F satisfies the following weak ellipticity condition (F2). It is easily seen that (F1) implies (F2).

(F2) If $\vec{X} \in \mathbf{R}^n$, $\vec{X} \leqq 0$ and $F(\vec{X}) \leqq 0$, then $\vec{X}$ must be zero, i.e., $\vec{X} \equiv 0$.

Theorem A.3 (Strong maximum principle)**.** *The two conditions* (SMP) *and* (F2) *are equivalent.*

To show this theorem we first study discrete quadratic functions. They will be used when we prove that (SMP) implies (F2).

Example A.4. Let $(A_1, \ldots, A_n) \in \mathbf{R}^n$. We define a quadratic function $q : h\mathbf{Z}^n \to \mathbf{R}$ as

$$q(x) := \sum_{j=1}^{n} (h_j x_j)^2 A_j \quad \text{for } x = (h_1 x_1, \ldots, h_n x_n) \in h\mathbf{Z}^n.$$

Then $\delta_i^2 q$ is a constant for each $i \in \{1, \ldots, n\}$. Indeed, we observe

$$\begin{aligned}
\delta_i^2 q(x) &= \frac{q(x + h_i e_i) + q(x - h_i e_i) - 2q(x)}{h_i^2} \\
&= \frac{h_i^2 (x_i + 1)^2 A_i + h_i^2 (x_i - 1)^2 A_i - 2h_i^2 x_i^2 A_i}{h_i^2} = 2A_i
\end{aligned}$$

for all $x = (h_1 x_1, \ldots, h_n x_n) \in h\mathbf{Z}^n$.

Proof of Theorem A.3. 1. We first assume (F2). Let $u : \overline{\Omega} \to \mathbf{R}$ is a discrete subsolution of (A.4) such that $u(\hat{x}) = \max_{\overline{\Omega}} u$ for some $\hat{x} \in \Omega$. This maximality implies that for each $i \in \{1, \dots, n\}$

$$\delta_i^2 u(\hat{x}) = \frac{u(\hat{x} + h_i e_i) + u(\hat{x} - h_i e_i) - 2u(\hat{x})}{h_i^2} \leqq \frac{u(\hat{x}) + u(\hat{x}) - 2u(\hat{x})}{h_i^2} = 0.$$

Thus $\vec{\delta}^2 u(\hat{x}) \leqq 0$. Since u is a discrete subsolution, we also have $F(\vec{\delta}^2 u(\hat{x})) \leqq 0$. It now follows from (F2) that $\vec{\delta}^2 u(\hat{x}) \equiv 0$, and hence we see that $u(\hat{x}) = u(\hat{x} \pm h_i e_i)$ for all i. We next apply the above argument with the new central point $\hat{x} \pm h_i e_i$ if the point is in Ω. Iterating this procedure, we finally conclude that $u \equiv u(\hat{x})$ on $\overline{\Omega}$ since Ω is now connected.

2. We next assume (SMP). Take any $\vec{X} = (X_1, \dots, X_n) \in \mathbf{R}^n$ such that $\vec{X} \leqq 0$ and $F(\vec{X}) \leqq 0$. We may assume $0 \in \Omega$. Now, we take the quadratic function q in Example A.4 with $A_i = X_i/2 \leqq 0$. By the calculation in Example A.4 we then have $\delta_i^2 q(x) = X_i$ for all i, i.e., $\vec{\delta}^2 q(x) = \vec{X}$. Thus $F(\vec{\delta}^2 q(x)) = F(\vec{X}) \leqq 0$, which means that q is a discrete subsolution of (A.4). Next, we deduce from the nonpositivity of each A_i that q attains its maximum over $\overline{\Omega}$ at $0 \in \Omega$. Therefore (SMP) ensures that $q \equiv q(0) = 0$ on $\overline{\Omega}$, which implies that $A_i = 0$ for all $i \in \{1, \dots, n\}$. Consequently, we find $\vec{X} \equiv 0$. $\square$

A simple example of F satisfying (F2) is $F(\vec{X}) = -\sum \vec{X}$, and then (A.4) represents the Laplace equation for u. We therefore have

Corollary A.5. *Let $u : \overline{\Omega} \to \mathbf{R}$. If $-\Delta' u(x) \leqq 0$ for all $x \in \Omega$ and $\max_{\overline{\Omega}} u = \max_{\Omega} u$, then u is constant on $\overline{\Omega}$.*

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Chapter 4

An improved level set method for Hamilton-Jacobi equations

1 Introduction

In the classical level set method, a motion of an interface $\{\Gamma(t)\}_t$ in $\mathbf{R}^n$ is studied by representing the interface $\Gamma(t)$ as the zero level set of an auxiliary function $u(x,t)$, that is,

$$\Gamma(t) = \{x \in \mathbf{R}^n \mid u(x,t) = 0\},$$

and solving the associated initial value problem of a partial differential equation for u. In this paper we are concerned with the case where the associated problem is given as the Hamilton-Jacobi equation of the form

$$\partial_t u(x,t) + H(x, \nabla u(x,t)) = 0 \quad \text{in } \mathbf{R}^n \times (0,T) \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0(x) \quad \text{in } \mathbf{R}^n. \tag{1.2}$$

Here H is a continuous Hamiltonian and $\nabla u = (\partial_{x_i} u)_{i=1}^n$ denotes the gradient of u with respect to x. In practice, it might be difficult to compute the zero level set of u because the spatial gradient of u can be close to zero near $\Gamma(t)$ as time develops even if the initial gradient is large. To overcome this issue, in this paper we propose an improved equation of the form

$$\partial_t u(x,t) + H(x, \nabla u(x,t)) = u(x,t) G(x, \nabla u(x,t)) \tag{1.3}$$

with a continuous G. Our goal is to demonstrate that a solution u of (1.3) with a suitably defined G gives the same zero level set as (1.1), and that, globally in time, the slope of u is preserved near the zero level set.

We illustrate our approach on a typical example of (1.1), the transport equation of the form

$$\partial_t u(x,t) + \langle X(x), \nabla u(x,t) \rangle = 0, \tag{1.4}$$

where $X : \mathbf{R}^n \to \mathbf{R}^n$ is a vector field and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbf{R}^n$. In this case, as we will see in Section 2, a formal argument implies that the slope of a solution u of

$$\partial_t u(x,t) + \langle X(x), \nabla u(x,t)\rangle = u(x,t)\frac{\langle X'(x)\nabla u(x,t), \nabla u(x,t)\rangle}{|\nabla u(x,t)|^2} \tag{1.5}$$

is preserved along the flow determined by the vector field X. Here $X'(x)$ stands for the Jacobian matrix of $X(x) = (X_i(x))_{i=1}^n$, i.e., $X'(x) = (\partial_{x_j} X_i(x))_{ij}$. Our general theory for (1.3) will be constructed so that, in the case of transport equations, G agrees with the function appearing on the right-hand side of (1.5) except on a small neighborhood of the singular point at $|\nabla u(x,t)| = 0$.

We employ the theory of viscosity solutions to solve the improved equation (1.3) since it is a nonlinear equation. However, viscosity solutions are not smooth in general, and so it is an issue how to show that the "slope" of the viscosity solution of (1.3) is preserved near the zero level set. In this paper we establish the preservation of the slope of the solution by comparing it with the signed distance function $d(x,t)$ to the zero level set, which is defined as

$$d(x,t) = \begin{cases} \operatorname{dist}(x,\Gamma(t)) & \text{if } u(x,t) > 0, \\ 0 & \text{if } u(x,t) = 0, \\ -\operatorname{dist}(x,\Gamma(t)) & \text{if } u(x,t) < 0. \end{cases}$$

Here $\operatorname{dist}(x,\Gamma(t)) = \inf\{|x-y| \mid y \in \Gamma(t)\}$. The distance function is known to be a solution of the eikonal equation

$$|\nabla d(x,t)| = 1,$$

both in the almost everywhere sense and in the viscosity sense. It is thus reasonable to use the signed distance function in order to guarantee that the slope of u remains one. It turns out that, if the initial data u_0 in (1.2) is equal to the signed distance function near the initial interface and if d is smooth near $\Gamma(t)$, then, for every $\varepsilon > 0$, the estimates

$$\begin{cases} e^{-\varepsilon t} d(x,t) \leqq u(x,t) \leqq e^{\varepsilon t} d(x,t) & \text{if } d(x,t) \geqq 0, \\ e^{\varepsilon t} d(x,t) \leqq u(x,t) \leqq e^{-\varepsilon t} d(x,t) & \text{if } d(x,t) \leqq 0 \end{cases} \tag{1.6}$$

hold near $\Gamma(t)$. In this sense the slope of u is preserved near the zero level set.

This chapter is organized as follows. In Section 2 we formally explain how to derive the improved equation (1.5) in the case of transport equations. Section 3 contains a brief summary of the theory of viscosity solutions and the level set method. In Section 4 we prove that the zero level set of a solution of (1.3) agrees with that of a solution of (1.1), and finally Section 5 establishes the estimates (1.6) near the zero level set of a solution u of (1.3) with a suitably defined G.

2 Transport equations

Let us show how the improved equation (1.5) is formally derived in the case of transport equations. Assume that a smooth function $u(x,t)$ solves

$$\partial_t u(x,t) + \langle X(x), \nabla u(x,t)\rangle = u(x,t) G(x, \nabla u(x,t))$$

with (1.2). Let $\xi_x : (-\infty, \infty) \to \mathbf{R}^n$ be a solution of the ordinary differential equation

$$\begin{cases} \xi_x'(t) = X(\xi_x(t)) & \text{in } (-\infty, \infty), \\ \xi_x(0) = x. \end{cases}$$

This is the flow determined by X. Then the solution w of the original problem, (1.4) and (1.2), is given as $w(x,t) = u_0(\xi_x(-t))$. We now compute

$$\begin{aligned} \frac{d}{dt} u(\xi_x(t), t) &= \langle \nabla u(\xi_x(t), t), \xi_x'(t) \rangle + \partial_t u(\xi_x(t), t) \\ &= \langle \nabla u(\xi_x(t), t), X(\xi_x(t)) \rangle + \partial_t u(\xi_x(t), t) \\ &= u(\xi_x(t), t) G(x, \nabla u(\xi_x(t), t)). \end{aligned}$$

Thus we notice that

$$u(\xi_x(t), t) = 0 \quad \text{if } u_0(x) = 0, \quad u(\xi_x(t), t) > 0 \quad \text{if } u_0(x) > 0 \tag{2.1}$$

for all $t \in (-\infty, \infty)$. This implies that the zero level set of u agrees with that of the solution w of (1.4) and (1.2).

We now define $\phi(x,t) := |\nabla u(x,t)|^2$ and assume that

$$\partial_t \phi(x,t) + \langle X(x), \nabla \phi(x,t) \rangle = 0 \quad \text{on } \{u(x,t) = 0\}. \tag{2.2}$$

(We denote the set $\{(x,t) \in \mathbf{R}^n \times (0,T) \mid u(x,t) = 0\}$ briefly by $\{u(x,t) = 0\}$ unless confusion can occur.) If (2.2) holds, it then follows that ϕ is constant along each flow. Indeed, for $x \in \mathbf{R}^n$ such that $u_0(x) = 0$, we have

$$\frac{d}{dt} \phi(\xi_x(t), t) = \langle \nabla \phi(\xi_x(t), t), X(\xi_x(t)) \rangle + \partial_t \phi(\xi_x(t), t) = 0$$

since $u(\xi_x(t), t) = 0$ by (2.1). Thus we see that $|\nabla u|$ is constant along the flow.

Let us study a condition on G which leads to (2.2). We have

$$\begin{aligned} \partial_t \phi(x,t) &= 2 \langle \nabla u(x,t), \partial_t (\nabla u(x,t)) \rangle, \\ \nabla \phi(x,t) &= 2 (\nabla^2 u(x,t)) \nabla u(x,t). \end{aligned}$$

Here $\nabla^2 u = (\partial^2_{x_i x_j} u)_{ij}$ denotes the Hessian matrix with respect to x. The second derivative $\partial_t(\nabla u)$ is computed as

$$\begin{aligned} \partial_t(\nabla u) &= \nabla(\partial_t u) \\ &= \nabla(u G(x, \nabla u) - \langle X(x), \nabla u \rangle) \\ &= G(x, \nabla u) \nabla u + u \nabla(G(x, \nabla u)) - {}^t(X'(x)) \nabla u - (\nabla^2 u) X(x), \end{aligned}$$

where ${}^t(X'(x))$ stands for the transposed matrix of $X'(x)$. On $\{u(x,t) = 0\}$ we have $u \nabla(G(x, \nabla u)) = 0$, and therefore

$$\begin{aligned} &\frac{1}{2}(\partial_t \phi + \langle X(x), \nabla \phi \rangle) \\ =& \langle \nabla u, G(x, \nabla u) \nabla u - {}^t(X'(x)) \nabla u - (\nabla^2 u) X(x) \rangle + \langle X(x), (\nabla^2 u) \nabla u \rangle \\ =& G(x, \nabla u) |\nabla u|^2 - \langle X'(x) \nabla u, \nabla u \rangle. \end{aligned}$$

In the last equality we have used the relations $\langle {}^t(X'(x))\nabla u, \nabla u\rangle = \langle X'(x)\nabla u, \nabla u\rangle$ and $\langle X(x), (\nabla^2 u)\nabla u\rangle = \langle \nabla u, (\nabla^2 u)X(x)\rangle$. We thus conclude that, if

$$G(x, \nabla u) = \frac{\langle X'(x)\nabla u, \nabla u\rangle}{|\nabla u|^2},$$

then (2.2) holds, and hence the slope of u is preserved along the flow. This choice of G yields (1.5).

3 Viscosity solutions

In this section we first recall a notion of viscosity solutions and then describe basic results of the level set method. For the theory of viscosity solutions we refer the reader to [1] and [2, Section 2, 3], while analytic foundations of the level set method are presented in [2, Section 4].

We consider a general first order equation of the form

$$\partial_t u(x,t) + F(x, u(x,t), \nabla u(x,t)) = 0, \tag{3.1}$$

where $F : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ is a continuous function. Let us introduce notions of a sub- and superdifferential. For a function $u : \mathbf{R}^n \times (0,T) \to \mathbf{R}$ we define a *superdifferential* $D^+u(z,s)$ of u at $(z,s) \in \mathbf{R}^n \times (0,T)$ by

$$D^+u(z,s) := \left\{ (p,\tau) \in \mathbf{R}^n \times \mathbf{R} \;\middle|\; \begin{array}{c} \exists \phi \in C^1(\mathbf{R}^n \times (0,T)) \text{ such that} \\ (p,\tau) = (\nabla\phi, \partial_t\phi)(z,s) \text{ and} \\ \max_{\mathbf{R}^n \times (0,T)}(u-\phi) = (u-\phi)(z,s) \end{array} \right\}. \tag{3.2}$$

A *subdifferential* $D^-u(z,s)$ is defined by replacing "max" by "min" in (3.2). We call ϕ appearing in (3.2) a *corresponding test function* for $(p,\tau) \in D^+u(z,s)$.

Definition 3.1 (Viscosity solution)**.** We say an upper semicontinuous (resp. lower semicontinuous) function $u : \mathbf{R}^n \times (0,T) \to \mathbf{R}$ is a *viscosity subsolution* (resp. *viscosity supersolution*) of (3.1) if

$$\tau + F(z, u(z,s), p) \leqq 0 \text{ (resp. } \geqq 0)$$

for all $(z,s) \in \mathbf{R}^n \times (0,T)$ and $(p,\tau) \in D^+u(z,s)$ (resp. $(p,\tau) \in D^-u(z,s)$). If u is both a viscosity sub- and supersolution, then it is called a *viscosity solution*.

A class of viscosity subsolutions and supersolutions are known to be closed under the operation of supremum and infimum respectively.

Proposition 3.2 (Stability)**.** *Let $\mathcal{S}$ be a nonempty subset of viscosity subsolutions (resp. supersolutions) of* (3.1). *Set $u(x,t) := \sup_{v \in \mathcal{S}} v(x,t)$. If u is upper semicontinuous (resp. lower semicontinuous) in $\mathbf{R}^n \times (0,T)$, then u is a viscosity subsolution (resp. supersolution) of* (3.1).

We next present a comparison principle, which guarantees uniqueness of viscosity solutions of the initial value problem. We make the following two assumptions on F.

(F1) There exists some nondecreasing function $\omega \in C([0,\infty))$ satisfying $\omega(0) = 0$ such that

$$|F(x,r,p) - F(x,r,q)| \leqq \omega(|p-q|),$$
$$|F(x,r,p) - F(y,r,p)| \leqq \omega((1+|p|)|x-y|)$$

for all $x, y, p, q \in \mathbf{R}^n$ and $r \in \mathbf{R}$.

(F2) There exists some constant $\gamma \in \mathbf{R}$ such that $F(x,r,p)+\gamma r$ is nondecreasing in $r \in \mathbf{R}$.

Theorem 3.3 (Comparison principle)**.** *Assume* (F1) *and* (F2)*. Let $u, v : \mathbf{R}^n \times [0,T) \to \mathbf{R}$ and assume that u and $-v$ are upper semicontinuous and bounded from above on $\mathbf{R}^n \times [0,T)$. Assume that u and v are, respectively, a viscosity sub- and supersolution of* (3.1)*. If $u(x,0) \leqq v(x,0)$ for all $x \in \mathbf{R}^n$, then $u(x,t) \leqq v(x,t)$ for all $(x,t) \in \mathbf{R}^n \times (0,T)$.*

When F is written as $F(x,r,p) = H(x,p) - rG(x,p)$, which represents the equation (1.3), the following conditions imply (F1) and (F2).

(CP) (i) H and G satisfy (F1).

(ii) G is bounded in $\mathbf{R}^n \times \mathbf{R}^n$, i.e., $\|G\| := \sup_{\mathbf{R}^n \times \mathbf{R}^n} |G| < \infty$.

Indeed, (F1) is obvious while (F2) is fulfilled with $\gamma = \|G\|$.

Existence of viscosity solutions is shown by Perron's method, but we omit it in this paper; see [1, Section 4] or [2, Section 2.4].

We turn to the level set method for (1.1). To carry out the level set method, the geometricity (H1) is a basic assumption on H.

(H1) $H(x, \lambda p) = \lambda H(x,p)$ for all $(x,p) \in \mathbf{R}^n \times \mathbf{R}^n$ and $\lambda > 0$.

Note that (H1) implies that $H(x,0) = 0$ for all $x \in \mathbf{R}^n$. One of important properties of geometric equations is invariance under change of dependent variables. This invariance property as well as the comparison principle play a crucial role for the proof of uniqueness of evolutions.

Theorem 3.4 (Invariance)**.** *Assume* (H1)*. Let $\theta : \mathbf{R} \to \mathbf{R}$ be a nondecreasing and upper semicontinuous (resp. lower semicontinuous) function. If u is a viscosity subsolution (resp. supersolution) of* (1.1)*, then $\theta \circ u$ is a viscosity subsolution (resp. supersolution) of* (1.1).

The next theorem guarantees that, for a given initial surface, the evolution is independent of a choice of the initial auxiliary function, which is u_0 in (1.2). Let us denote by $BUC(\mathbf{R}^n)$ the set of all bounded and uniformly continuous functions in $\mathbf{R}^n$.

Theorem 3.5 (Uniqueness of evolutions)**.** *Assume that H satisfies* (F1) *and* (H1)*. Let $u_{01}, u_{02} \in BUC(\mathbf{R}^n)$ and assume that $\{u_{01}(x) = 0\} = \{u_{02}(x) = 0\}$, $\{u_{01}(x) > 0\} = \{u_{02}(x) > 0\}$ and that $\{u_{01}(x) > 0\}$ is compact. Let w_1 and w_2 be a viscosity solution of* (1.1) *with the initial data u_{01} and u_{02} respectively. Then we have $\{w_1(x,t) = 0\} = \{w_2(x,t) = 0\}$ and $\{w_1(x,t) > 0\} = \{w_2(x,t) > 0\}$.*

We conclude this section by giving simple sub- and supersolutions of (1.3). In the proof we use the fact that a classical subsolution (resp. supersolution) of (3.1) is always a viscosity subsolution (resp. supersolution) of (3.1).

Lemma 3.6. *Assume* (ii) *in* (CP) *and* (H1). *Let* $c > 0$, $M \geqq \|G\|$ *and define* $w^+(x,t) = ce^{Mt}$, $w^-(x,t) = ce^{-Mt}$. *Then* w^+ *(resp.* w^-*) is a supersolution (resp. subsolution) of* (1.3), *and* $-w^+$ *(resp.* $-w^-$*) is a subsolution (resp. supersolution) of* (1.3).

Proof. Since $H(x,0) = 0$ by (H1), we compute

$$\partial_t w^+ + H(x, \nabla w^+) = Mce^{Mt} + H(x,0) = Mw^+ + 0 \geqq w^+ G(x, \nabla w^+),$$

which implies that w^+ is a supersolution. The rest assertions also follows from similar calculations. □

4 Preservation of the zero level set

We demonstrate that a solution of the improved problem (1.3) gives the same zero level set of a solution of the original problem (1.1).

Proposition 4.1 (Preservation of the zero level set)**.** *Assume* (CP) *and* (H1). *Let* w *and* u *be, respectively, a viscosity solution of* (1.1) *and* (1.3) *with the same initial data. Then we have* $\{w(x,t) = 0\} = \{u(x,t) = 0\}$ *and* $\{w(x,t) > 0\} = \{u(x,t) > 0\}$.

Since we have already known uniqueness of evolutions for (1.1), Proposition 4.1 yields uniqueness of evolutions for (1.3). Namely, the same conclusion in Theorem 3.5 holds if we replace "(1.1)" by "(1.3)" and add the assumption (CP) in the statement of Theorem 3.5.

Proof. 1. We define

$$v^+(x,t) = \begin{cases} e^{\|G\|t} u(x,t) & \text{if } u(x,t) \geqq 0, \\ e^{-\|G\|t} u(x,t) & \text{if } u(x,t) < 0 \end{cases}$$

and

$$v^-(x,t) = \begin{cases} e^{-\|G\|t} u(x,t) & \text{if } u(x,t) \geqq 0, \\ e^{\|G\|t} u(x,t) & \text{if } u(x,t) < 0 \end{cases}$$

for $(x,t) \in \mathbf{R}^n \times [0,T)$. We claim that v^+ and v^- are, respectively, a viscosity supersolution and subsolution of (1.1).

2. We shall show that v^+ is a supersolution. If u is smooth and $u(x,t) > 0$, we compute

$$\begin{aligned} \partial_t v^+ + H(x, \nabla v^+) &= \|G\| e^{\|G\|t} u + e^{\|G\|t} \partial_t u + H(x, e^{\|G\|t} \nabla u) \\ &= \|G\| e^{\|G\|t} u + e^{\|G\|t} \{\partial_t u + H(x, \nabla u)\} \\ &= \|G\| e^{\|G\|t} u + e^{\|G\|t} u G(x, \nabla u) \geqq 0. \end{aligned}$$

In the general case where u is not necessarily smooth, taking an element of the subdifferential of u, we see that v^+ is a viscosity supersolution of (1.1). Since similar arguments apply to the case when $u(x,t) < 0$, it follows that v^+ is a supersolution in $\{u(x,t) > 0\} \cup \{u(x,t) < 0\}$.

3. It remains to prove that v^+ is a supersolution on $\{u(x,t) = 0\}$. For this purpose, we first claim that v_0^+ defined by

$$v_0^+(x,t) = \max\{v^+(x,t),\ 0\} = \begin{cases} e^{\|G\|t}u(x,t) & \text{if } u(x,t) \geqq 0, \\ 0 & \text{if } u(x,t) < 0 \end{cases}$$

is a supersolution of (1.1). Fix $\varepsilon > 0$ and define $\theta^\varepsilon : \mathbf{R} \to \mathbf{R}$ by $\theta^\varepsilon(r) := \max\{r,\ \varepsilon\}$. Since v^+ is a supersolution of (1.1) in $\{u(x,t) > 0\}$, Theorem 3.4 implies that $\theta^\varepsilon \circ v^+ = \max\{v^+,\ \varepsilon\}$ is also a supersolution of (1.1) in $\{u(x,t) > 0\}$. Now, the constant ε is a solution of (1.1) and $\theta^\varepsilon \circ v^+ = \varepsilon$ in $\{0 < u(x,t) < \varepsilon e^{-\|G\|t}\}$. Thus we see that $\theta^\varepsilon \circ v^+$ is a supersolution of (1.1) on the whole of $\mathbf{R}^n \times (0,T)$. Finally, by Proposition 3.2, taking the infimum over $\{\varepsilon > 0\}$ implies that $v_0^+ = \inf_{\varepsilon>0}(\theta^\varepsilon \circ v^+)$ is a supersolution of (1.1).

4. Let $(z,s) \in \mathbf{R}^n \times (0,T)$ be a point such that $u(z,s) = 0$, and take $(p,\tau) \in D^- v^+(z,s)$. Since $v^+(z,s) = v_0^+(z,s) = 0$ and $v^+ \leqq v_0^+$ in $\mathbf{R}^n \times (0,T)$, it is easily seen that $(p,\tau) \in D^- v_0^+(z,s)$. In the previous step we proved that v_0^+ is a supersolution, and thus we have $\tau + H(z,p) \geqq 0$. Summarizing the above argument, we conclude that v^+ is a supersolution of (1.1). Also, in the same manner we are able to prove that v^- is a subsolution of (1.1).

5. Since $v^\pm(x,0) = u_0(x)$ for all $x \in \mathbf{R}^n$, the comparison principle (Theorem 3.3) yields $v^-(x,t) \leqq w(x,t) \leqq v^+(x,t)$ for all $(x,t) \in \mathbf{R}^n \times (0,T)$. In particular, we have $\{v^-(x,t) > 0\} \subset \{w(x,t) > 0\} \subset \{v^+(x,t) > 0\}$. Since $\{v^\pm(x,t) > 0\} = \{u(x,t) > 0\}$ by the definition of $v^\pm$, we conclude that $\{w(x,t) > 0\} = \{u(x,t) > 0\}$. Similarly, we obtain $\{w(x,t) < 0\} = \{u(x,t) < 0\}$, and hence $\{w(x,t) = 0\} = \{u(x,t) = 0\}$. □

5 Comparison with the signed distance function

We study a bounded evolution $\{(\Gamma(t), D(t))\}_{0 \leqq t < T}$ in $\mathbf{R}^n$, Namely, we assume

(I) $\Gamma(t) \cup D(t)$ is a bounded set in $\mathbf{R}^n$ for every $t \in [0,T)$;

(II) there exists a continuous viscosity solution $w : \mathbf{R}^n \times [0,T) \to \mathbf{R}$ of (1.1) such that

$$\Gamma(t) = \{x \in \mathbf{R}^n \mid w(x,t) = 0\},$$
$$D(t) = \{x \in \mathbf{R}^n \mid w(x,t) > 0\}$$

for all $t \in [0,T)$.

For this evolution we define the signed distance function $d : \mathbf{R}^n \times [0,T) \to \mathbf{R}$ by

$$d(x,t) = \begin{cases} \operatorname{dist}(x, \Gamma(t)) & \text{if } x \in D(t), \\ 0 & \text{if } x \in \Gamma(t), \\ -\operatorname{dist}(x, \Gamma(t)) & \text{if } x \in \mathbf{R}^n \setminus (D(t) \cup \Gamma(t)). \end{cases}$$

We intend to prove that a viscosity solution u of the improved problem (1.3) satisfies the estimate (1.6) which involves the signed distance function d to the interface. This shows that the slope of u is preserved near the zero level set. For this purpose, we first derive the equation for the signed distance function d. In this paper we assume that d is smooth near the zero level set, and so the derivatives of d are interpreted in the classical sense. It is future work to extend the theory presented in this paper to the case of non-smooth signed distance functions. Our assumption concerning smoothness is

(SM) There exist a constant $\delta > 0$ and a function $w_0 : \mathbf{R}^n \times (0,T) \to \mathbf{R}$ such that

- (i) $d, w_0 \in C^1$ on $\{|d(x,t)| < \delta\}$ as a function of (x,t);
- (ii) $\Gamma(t) = \{w_0(x,t) = 0\}$ and $D(t) = \{w_0(x,t) > 0\}$ for all $t \in (0,T)$;
- (iii) for all $(x,t) \in \mathbf{R}^n \times (0,T)$ such that $d(x,t) = 0$, w_0 solves (1.1) and $|\nabla w_0(x,t)| \neq 0$;
- (iv) for all $(x,t) \in \mathbf{R}^n \times (0,T)$ such that $|d(x,t)| < \delta$,

$$\bar{x} := x - d(x,t)\nabla d(x,t) \in \Gamma(t)$$

and

$$\partial_t d(x,t) = \frac{\partial_t w_0(\bar{x},t)}{|\nabla w_0(\bar{x},t)|}, \quad \nabla d(x,t) = \frac{\nabla w_0(\bar{x},t)}{|\nabla w_0(\bar{x},t)|}. \tag{5.1}$$

It is known that, under a suitable smoothness assumption on the interface, the time derivative and the spatial gradient of d are, respectively, the normal velocity and the normal vector to the interface. (See, e.g., [3].) Instead of assuming that the interface possesses sufficient smoothness, we assume that the formulas (5.1) hold for d.

Lemma 5.1. *Assume* (SM) *and* (H1). *Then*

$$\partial_t d(x,t) + H(x - d(x,t)\nabla d(x,t), \nabla d(x,t)) = 0 \tag{5.2}$$

for all $(x,t) \in \mathbf{R}^n \times (0,T)$ *such that* $|d(x,t)| < \delta$.

Proof. Set $\bar{x} := x - d(x,t)\nabla d(x,t)$. Using the formulas in (5.1), we compute

$$\begin{aligned}
&\partial_t d(x,t) + H(x - d(x,t)\nabla d(x,t), \nabla d(x,t)) \\
=&\frac{\partial_t w_0(\bar{x},t)}{|\nabla w_0(\bar{x},t)|} + H\left(\bar{x}, \frac{\nabla w_0(\bar{x},t)}{|\nabla w_0(\bar{x},t)|}\right) \\
=&\frac{1}{|\nabla w_0(\bar{x},t)|}\{\partial_t w_0(\bar{x},t) + H(\bar{x}, \nabla w_0(\bar{x},t))\}.
\end{aligned}$$

Here we have used (H1) in the last equality. Since w_0 solves (1.1) at $(\bar{x},t)$, the last quantity is zero. □

The equation (5.2) is equivalent to

$$\partial_t d + H(x, \nabla d) = H(x, \nabla d) - H(x - d\nabla d, \nabla d). \tag{5.3}$$

If H is smooth and $|d|$ is sufficiently small, the right-hand side of (5.3) is approximated by $d\langle \nabla_x H(x, \nabla d), \nabla d\rangle$. It is thus reasonable to study the equation of the form

$$\partial_t d + H(x, \nabla d) = d\langle \nabla_x H(x, \nabla d), \nabla d\rangle$$

as an improved problem for (1.1). In this paper we define G as a suitably modified function of $\langle \nabla_x H(x,p), p\rangle$.

We shall give our assumptions on H and G precisely. Concerning differentiablity of H with respect to x, we require the following condition (H2). We denote by S^{n-1} the unit sphere in $\mathbf{R}^n$, i.e., $S^{n-1} = \{x \in \mathbf{R}^n \mid |x| = 1\}$.

(H2) For any compact set $K \subset \mathbf{R}^n$

$$\lim_{\mathbf{R} \ni h \to 0} \sup_{(x,p) \in K \times S^{n-1}} \frac{|H(x,p) - H(x - hp, p) - h\langle \nabla_x H(x,p), p\rangle|}{|h|} = 0. \tag{5.4}$$

For later use we state (5.4) in an equivalent way as follows:

$$\begin{gathered} \text{For all } r > 0 \text{ there exists some } a(r) > 0 \text{ such that,} \\ \text{if } |h| < a(r) \text{ and } (x,p) \in K \times S^{n-1}, \text{ then} \\ |H(x,p) - H(x - hp, p) - h\langle \nabla_x H(x,p), p\rangle| \leqq r|h|. \end{gathered} \tag{5.5}$$

The function $a : (0, \infty) \to (0, \infty)$ in (5.5) may depend on a choice of the compact set K. We next state how to define a function $G : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ appearing in the improved problem (1.3).

(G1) There exists some $\sigma \in (0,1)$ such that $G(x, \lambda p) = \langle \nabla_x H(x,p), p\rangle$ for all $(x,p) \in \mathbf{R}^n \times S^{n-1}$ and $\lambda \geqq \sigma$.

Example 5.2. We consider the transport equation (1.4), in which the Hamiltonian H is given by $H(x,p) = \langle X(x), p\rangle$. The gradient of H with respect to x is $\nabla_x H(x,p) = X'(x)p$. We next let $G : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be a continuous function such that

$$G(x,p) = \frac{\langle X'(x)p, p\rangle}{|p|^2} \quad \text{if } |p| \geqq \sigma$$

for some $\sigma \in (0,1)$. This is the function appearing in the right-hand side of (1.5). Now, for $(x,p) \in \mathbf{R}^n \times S^{n-1}$ and $\lambda \geqq \sigma$, we compute

$$G(x, \lambda p) = \frac{\langle X'(x)\lambda p, \lambda p\rangle}{|\lambda p|^2} = \frac{\langle X'(x)p, p\rangle}{|p|^2} = \langle X'(x)p, p\rangle = \langle \nabla_x H(x,p), p\rangle.$$

Thus we see that G satisfies (G1).

Example 5.3. We study the equation

$$\partial_t u(x,t) + b(x)|\nabla u(x,t)| = 0,$$

where $b \in C^1(\mathbf{R}^n)$. The corresponding Hamiltonian $H(x,p) = b(x)|p|$ satisfies (H1), and its spatial gradient is $\nabla_x H(x,p) = |p| \cdot \nabla b(x)$. Let $\sigma \in (0,1)$ and define

$$G(x,p) := \frac{\langle \nabla b(x), p\rangle}{|p|} \quad \text{if } |p| \geqq \sigma.$$

Then G satisfies (G1) since, for all $(x,p) \in \mathbf{R}^n \times S^{n-1}$ and $\lambda \geqq \sigma$, we have

$$G(x,\lambda p) = \frac{\langle \nabla b(x), \lambda p\rangle}{|\lambda p|} = \langle |p| \cdot \nabla b(x), p\rangle = \langle \nabla_x H(x,p), p\rangle.$$

Therefore the improved equation is given as

$$\partial_t u(x,t) + b(x)|\nabla u(x,t)| = u(x,t)\frac{\langle \nabla b(x), \nabla u(x,t)\rangle}{|\nabla u(x,t)|}$$

when $|\nabla u(x,t)| \geqq \sigma$.

Our next assertion is that functions $e^{\varepsilon t} d(x,t)$ and $e^{-\varepsilon t} d(x,t)$, which appear in our objective estimates (1.6), are a subsoltion and a supersolution of (1.3) near the zero level set respectively.

Proposition 5.4 (Sub- and supersolutions near the zero level set)**.** *Assume* (SM), (H1), (H2) *and* (G1). *Let* $\varepsilon \in (0, -(\log\sigma)/T]$ *and define*

$$d^+(x,t) := e^{\varepsilon t} d(x,t), \quad d^-(x,t) := e^{-\varepsilon t} d(x,t). \tag{5.6}$$

Let $K \subset \mathbf{R}^n$ *be a compact set such that* $\{d(x,t) < \delta\} \subset K$*, and let* a *be the function in* (5.5). *Let* $r > 0$ *be a constant satisfying* $a(r) \leqq \delta$*. Then, for all* $(x,t) \in \mathbf{R}^n \times (0,T)$ *such that* $|d(x,t)| < a(r)$*, we have*

$$|\partial_t d^+(x,t) + H(x, \nabla d^+(x,t)) - d^+(x,t)\{G(x,\nabla d^+(x,t)) + \varepsilon\}| \leqq r|d^+(x,t)|, \tag{5.7}$$

$$|\partial_t d^-(x,t) + H(x, \nabla d^-(x,t)) - d^-(x,t)\{G(x,\nabla d^-(x,t)) - \varepsilon\}| \leqq r|d^-(x,t)|. \tag{5.8}$$

In particular, if $r \leqq \varepsilon$*, then* d^+ *is a supersolution (resp. subsolution) of* (1.3) *on* $\{0 \leqq d(x,t) < a(r)\}$ *(resp. on* $\{-a(r) < d(x,t) \leqq 0\}$*), and* d^- *is a subsolution (resp. supersolution) of* (1.3) *on* $\{0 \leqq d(x,t) < a(r)\}$ *(resp. on* $\{-a(r) < d(x,t) \leqq 0\}$*).*

Proof. By the choice of ε we have $e^{-\varepsilon t} > e^{-\varepsilon T} \geqq \sigma$ for every $t \in (0,T)$. Thus the assumption (G1) implies

$$G(x, \nabla d^\pm) = G(x, e^{\pm\varepsilon t}\nabla d) = \langle \nabla_x H(x,\nabla d), \nabla d\rangle. \tag{5.9}$$

Let us fix $(x,t) \in \mathbf{R}^n \times (0,T)$ such that $|d(x,t)| < a(r)$. Since $a(r) \leqq \delta$, choosing $h = d(x,t)$ and $p = \nabla d(x,t)$ in (5.5), we have

$$|H(x,\nabla d) - H(x - d\nabla d, \nabla d) - d\langle \nabla_x H(x,\nabla d), \nabla d\rangle| \leqq r|d|.$$

Now, we apply (5.9) to the left-hand side and multiply the both sides by $e^{\pm\varepsilon t}$ to get

$$|e^{\pm\varepsilon t}\{H(x,p) - H(x - d\nabla d, \nabla d)\} - d^{\pm}G(x, \nabla d^{\pm})| \leqq r|d^{\pm}|. \tag{5.10}$$

Using (H1) and (5.3), we compute

$$\begin{aligned}
\partial_t d^{\pm} + H(x, \nabla d^{\pm}) &= \pm\varepsilon d^{\pm} + e^{\pm\varepsilon t}\partial_t d + H(x, e^{\pm\varepsilon t}\nabla d) \\
&= \pm\varepsilon d^{\pm} + e^{\pm\varepsilon t}\{\partial_t d + H(x, \nabla d)\} \\
&= \pm\varepsilon d^{\pm} + e^{\pm\varepsilon t}\{H(x, \nabla d) - H(x - d\nabla d, \nabla d)\}. \qquad (5.11)
\end{aligned}$$

Finally, combining (5.10) and (5.11), we arrive at both (5.7) and (5.8). The assertions in the case $r \leqq \varepsilon$ are clear from (5.7) and (5.8). □

In order to derive the estimates (1.6) by the comparison principle, we need to extend a local subsolution and supersolution $d^{\pm}$ in Proposition 5.4 so that they are a subsolution and supersolution on the whole of $\mathbf{R}^n \times (0,T)$. To do this, we first study superdifferentials of a function which is written as the minimum of two functions.

Lemma 5.5. *Let $f_1, f_2 : \mathbf{R}^n \to \mathbf{R}$ and define $g(x) := \min\{f_1(x),\ f_2(x)\}$. Let $z \in \mathbf{R}^n$ be a point such that $f_1(z) = f_2(z)$. Assume that $f_1, f_2 \in C^1$ near z. If $p \in D^+g(z)$, then $p = \lambda\nabla f_1(z) + (1-\lambda)\nabla f_2(z)$ for some $\lambda \in [0,1]$.*

Proof. 1. We first give the proof in the case $f_2 \equiv 0$. If $p = 0$, the assertion is obvious because we have $p = \lambda\nabla f_1(z) + (1-\lambda)\nabla f_2(z)$ with $\lambda = 0$. Assume $p \neq 0$. Take a corresponding test function $\phi \in C^1(\mathbf{R}^n)$ for $p \in D^+g(z)$ such that $\phi(z) = 0$. Since $\nabla\phi(z) = p \neq 0$, we may assume $\partial_{x_n}\phi(z) > 0$ without loss of generality.

2. We claim

$$\partial_{x_n} f_1(z) \geqq \partial_{x_n}\phi(z)\ (> 0). \tag{5.12}$$

Let us write $z = (z', z_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$. Since $\partial_{x_n}\phi(z) > 0$, we see that $\phi(z', z_n - h) < 0$ for sufficiently small $h > 0$. Thus $0 \geqq (g - \phi)(z', z_n - h) > g(z', z_n - h)$, which implies $g(z', z_n - h) = f_1(z', z_n - h)$. Now, we compute

$$\begin{aligned}
\partial_{x_n} f_1(z) &= \lim_{h\downarrow 0} \frac{f_1(z', z_n - h) - f_1(z)}{-h} \\
&= \lim_{h\downarrow 0} \frac{g(z', z_n - h) - g(z)}{-h} \\
&\geqq \lim_{h\downarrow 0} \frac{\phi(z', z_n - h) - \phi(z)}{-h} \\
&= \partial_{x_n}\phi(z).
\end{aligned}$$

We thus obtain (5.12).

3. By (5.12) the zero level sets $\{f_1(x) = 0\}$ and $\{\phi(x) = 0\}$ are written as the graphs of implicit functions on some open neighborhood U of z'. We represent $\{f_1(x) = 0\}$ as $x_n = h(x')$ and $\{\phi(x) = 0\}$ as $x_n = \psi(x')$, where $x' = (x_1, \ldots, x_{n-1}) \in U$. We then have $h \geqq \psi$ on U. Indeed, if $h(x') < \psi(x')$ for

some $x' \in U$, we would have $f_1(x', h(x')) = 0$ and $\phi(x', h(x')) < \phi(x', \psi(x')) = 0$. This is a contradiction to the fact that $f_1 - \phi$ takes its maximum value 0 at z. Since $h \geqq \psi$ on U and $h(z') = \psi(z')$, we have

$$\nabla_{x'} h(z') = \nabla_{x'} \psi(z'). \tag{5.13}$$

The derivatives of implicit functions are given as

$$\nabla_{x'} h(z') = -\frac{\nabla_{x'} f_1(z)}{\partial_{x_n} f_1(z)}, \quad \nabla_{x'} \psi(z') = -\frac{\nabla_{x'} \phi(z)}{\partial_{x_n} \phi(z)}. \tag{5.14}$$

Substituting (5.14) for (5.13) and setting $\lambda := \partial_{x_n} \phi(z)/\partial_{x_n} f_1(z)$, we see

$$\nabla_{x'} \phi(z) = \frac{\partial_{x_n} \phi(z)}{\partial_{x_n} f_1(z)} \nabla_{x'} f_1(z) = \lambda \nabla_{x'} f_1(z).$$

By the definition of λ and (5.12), we also have $\partial_{x_n} \phi(z) = \lambda \partial_{x_n} f_1(z)$ and $0 < \lambda \leqq 1$. Thus the proof is complete when $f_2 \equiv 0$.

4. For a general f_2 we study the function $(g - f_2)(x) = \min\{(f_1 - f_2)(x),\ 0\}$. Since $p \in D^+ g(z)$, we have $p - \nabla f_2(z) \in D^+(g - f_2)(z)$. Thus the result in the case $f_2 \equiv 0$ implies that there exists some $\lambda \in [0, 1]$ such that $p - \nabla f_2(z) = \lambda \nabla (f_1 - f_2)(z)$, i.e., $p = \lambda \nabla f_1(z) + (1 - \lambda) \nabla f_2(z)$. This is precisely the assertion of the lemma. $\square$

Using Lemma 5.5, we construct a global subsolution and supersolution of (1.3) which equal to d^+ or d^- near the zero level set.

Proposition 5.6 (Extension of sub- and supersolutions)**.** *Assume* (SM)*,* (H1)*,* (H2) *and* (G1)*. Assume the same hypotheses of Proposition 5.4 concerning ε and a. Let $c, L, M \in \mathbf{R}$ be positive constants such that*

$$0 < c \leqq \min\{a(\varepsilon),\ \delta\}, \quad L \geqq c, \quad M \geqq \frac{2\|G\|}{1 - \sigma e^{\varepsilon T}}, \quad M > \varepsilon.$$

Define $d^{\pm}(x, t)$ as in (5.6) *and*

$$V(x, t) := \frac{3L}{c} e^{\|G\| t} d^+(x, t).$$

We further define

$$u^+(x,t) = \begin{cases} \min\left\{\max\{d^+(x,t),\ V(x,t) - L\},\ L e^{\|G\| t}\right\} & \text{if } d(x,t) \geqq 0, \\ \max\left\{d^-(x,t),\ -c e^{-Mt}\right\} & \text{if } d(x,t) < 0 \end{cases} \tag{5.15}$$

and

$$u^-(x,t) = \begin{cases} \min\left\{d^-(x,t),\ c e^{-Mt}\right\} & \text{if } d(x,t) \geqq 0, \\ \max\left\{\min\{d^+(x,t),\ V(x,t) + L\},\ -L e^{\|G\| t}\right\} & \text{if } d(x,t) < 0. \end{cases} \tag{5.16}$$

Then u^+ and u^- are, respectively, a viscosity supersolution and a viscosity subsolution of (1.3) *in $\mathbf{R}^n \times (0, T)$.*

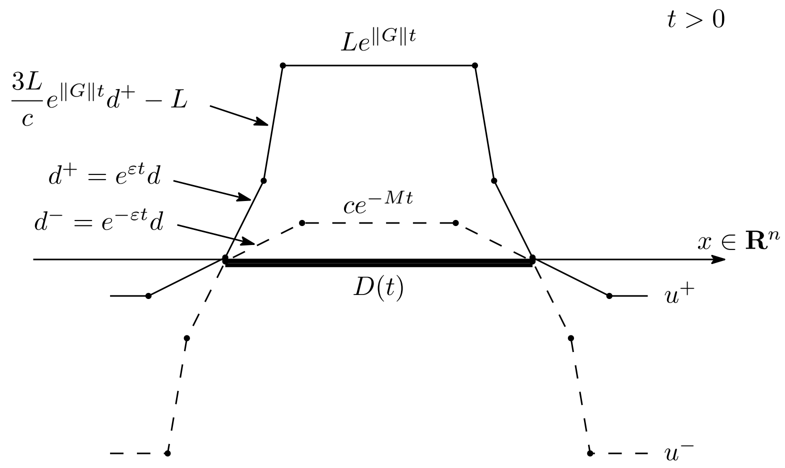

Figure 1: Definitions of u^+ and u^-.

Proof. 1. We only prove that u^+ and u^- are, respectively, a supersolution and a subsolution of (1.3) on $\{d(x,t) \geqq 0\}$, because the same arguments work in $\{d(x,t) < 0\}$. On $\{d(x,t) \geqq 0\}$ we see that u^+ and u^- are represented as follows:

$$u^+(x,t) = \begin{cases} d^+(x,t) & \text{if } 0 \leqq d^+(x,t) \leqq cL/(3Le^{\|G\|t} - c), \\ V(x,t) - L & \text{if } cL/(3Le^{\|G\|t} - c) \leqq d^+(x,t) \leqq c(1 + e^{-\|G\|t})/3, \\ Le^{\|G\|t} & \text{if } c(1 + e^{-\|G\|t})/3 \leqq d^+(x,t) \end{cases}$$

and

$$u^-(x,t) = \begin{cases} d^-(x,t) & \text{if } 0 \leqq d^-(x,t) \leqq ce^{-Mt}, \\ ce^{-Mt} & \text{if } ce^{-Mt} \leqq d^-(x,t). \end{cases}$$

(See also Figure 1.) When $0 \leqq d^+(x,t) \leqq c(1 + e^{-\|G\|t})/3$, we have

$$0 \leqq d(x,t) \leqq \frac{c(1 + e^{-\|G\|t})e^{-\varepsilon t}}{3} \leqq \frac{2}{3}c \leqq \frac{2}{3}\min\{a(\varepsilon),\ \delta\},$$

which implies that d^+ is a supersolution of (1.3) on $\{0 \leqq d^+(x,t) \leqq c(1 + e^{-\|G\|t})/3\}$. Also, if $0 \leqq d^-(x,t) \leqq ce^{-Mt}$, then

$$0 \leqq d(x,t) \leqq ce^{(\varepsilon - M)t} < c \leqq \min\{a(\varepsilon),\ \delta\}$$

since $M > \varepsilon$. Thus d^- is a subsolution of (1.3) on $\{0 \leqq d^-(x,t) \leqq ce^{-Mt}\}$.

2. We prove that u^- is a viscosity subsolution of (1.3) in $\{0 < d^-(x,t)\}$. As we stated in Step 1, $u(x,t) = d^-(x,t)$ is a subsolution in $\{0 < d^-(x,t) < ce^{-Mt}\}$. Also, since we have

$$M \geqq \frac{2\|G\|}{1 - \sigma e^{\varepsilon T}} > 2\|G\| \geqq \|G\|,$$

Lemma 3.6 guarantees that $u(x,t) = ce^{-Mt}$ is a subsolution in $\{ce^{-Mt} < d^-(x,t)\}$. What is left is to show that u^- is a subsolution on $\{d^-(x,t) = ce^{-Mt}\}$.

Let $(z,s) \in \mathbf{R}^n \times (0,T)$ be a point such that $d^-(z,s) = ce^{-Ms} =: \alpha$, and take any $(p,\tau) \in D^+u^-(z,s)$. Our goal is to show that

$$I := \tau + H(z,p) - \alpha G(z,p) \leqq 0.$$

We apply Lemma 5.5 to u^-. It then follows that

$$\begin{aligned} p &= \lambda \nabla d^-(z,s) + (1-\lambda)\nabla(ce^{-Mt})(z,s) \\ &= \lambda \nabla d^-(z,s), \\ \tau &= \lambda \partial_t d^-(z,s) + (1-\lambda)\partial_t(ce^{-Mt})(z,s) \\ &= \lambda \partial_t d^-(z,s) - M(1-\lambda)\alpha \end{aligned}$$

for some $\lambda \in [0,1]$, and thus

$$\begin{aligned} I &= \lambda \partial_t d^- - M(1-\lambda)\alpha + H(z, \lambda\nabla d^-) - \alpha G(z, \lambda \nabla d^-) \\ &= \lambda\{\partial_t d^- + H(z,\nabla d^-)\} - \alpha G(z,\lambda\nabla d^-) - M(1-\lambda)\alpha \\ &\leqq \lambda \alpha G(z, \nabla d^-) - \alpha G(z, \lambda \nabla d^-) - M(1-\lambda)\alpha. \end{aligned}$$

We now divide the situation into two different cases.

Case 1: $|\lambda \nabla d^-(z,s)| \geqq \sigma$. In this case, we have $G(z, \lambda\nabla d^-(z,s)) = G(z, \nabla d^-(z,s))$ by (G1). Thus

$$\begin{aligned} I/\alpha &= \lambda G(z,\nabla d^-) - G(z,\nabla d^-) - M(1-\lambda) \\ &= (1-\lambda)\{-G(z,\nabla d^-) - M\}. \end{aligned}$$

Recalling $M \geqq \|G\|$, we see that $I \leqq 0$.

Case 2: $|\lambda\nabla d^-(z,s)| < \sigma$. We first remark that

$$\lambda < \sigma/|\nabla d^-(z,s)| = \sigma e^{\varepsilon s} < \sigma e^{\varepsilon T}.$$

Using this estimate, we observe

$$I/\alpha \leqq \lambda\|G\| + \|G\| - M(1-\sigma e^{\varepsilon T}) \leqq 2\|G\| - M(1-\sigma e^{\varepsilon T}).$$

The right-hand side is nonpositive by the choice of M, and therefore $I \leqq 0$. As a result, we conclude that u^- is a subsolution in $\{0 < d^-(x,t)\}$.

3. We assert that u^- is a subsolution on $\{d^-(x,t) = 0\}$. Let us define $\theta_1(r) = 0$ if $r \geqq 0$, $\theta_1(r) = -Le^{\|G\|T}$ if $r < 0$ and

$$w_1(x,t) = (\theta_1 \circ w)(x,t) = \begin{cases} 0 & \text{if } d^-(x,t) \geqq 0, \\ -Le^{\|G\|T} & \text{if } d^-(x,t) < 0, \end{cases}$$

where w is the viscosity solution of (1.1) which appears in (II) at the beginning of this section. Since θ_1 is nondecreasing and upper semicontinuous, Theorem 3.4 implies that w_1 is a viscosity subsolution of (1.1). We now fix a point $(z,s) \in \{d^-(x,t) = 0\}$ and take $(p,\tau) \in D^+u^-(z,s)$. Then, since $u^-(z,s) = w_1(z,s) = 0$ and $w_1 \leqq u^-$ in $\mathbf{R}^n \times (0,T)$, we see that $(p,\tau) \in D^+w_1(z,s)$. Therefore

$$\tau + H(z,p) \leqq 0 = u^-(z,s)G(z,p),$$

which shows our assertion.

4. We next prove that u^+ is a viscosity supersolution of (1.3) in $\{0 < d^+(x,t)\}$. To do this, we first claim that

$$V_0(x,t) = Ke^{\|G\|t}d^+(x,t) - \eta$$

is a supersolution of (1.3) on $\{V_0(x,t) \geqq 0\} \cap \{d(x,t) < c\}$ for all $K \geqq 1$ and $\eta > 0$. Fix $(x,t) \in \mathbf{R}^n \times (0,T)$ such that $V_0(x,t) \geqq 0$ and $d(x,t) < c$. We then have $|\nabla V_0| = Ke^{\|G\|t}|\nabla d^+| \geqq |\nabla d^+| \geqq 1$, which yields $G(x, \nabla V_0) = G(x, \nabla d^+)$ by (G1). Since d^+ is a supersolution, we calculate

$$\begin{aligned}
\partial_t V_0 + H(x, \nabla V_0) &= K\{\|G\|e^{\|G\|t}d^+ + e^{\|G\|t}\partial_t d^+\} + H(x, Ke^{\|G\|t}\nabla d^+) \\
&= \|G\|(V_0 + \eta) + Ke^{\|G\|t}\{\partial_t d^+ + H(x, \nabla d^+)\} \\
&\geqq \|G\|(V_0 + \eta) + Ke^{\|G\|t}d^+ G(x, \nabla d^+) \\
&= (V_0 + \eta)\{\|G\| + G(x, \nabla V_0)\}.
\end{aligned}$$

Noting that $\|G\| + G \geqq 0$ and $V_0(x,t) \geqq 0$, we conclude

$$\partial_t V_0 + H(x, \nabla V_0) \geqq V_0\{\|G\| + G(x, \nabla V_0)\} \geqq V_0 G(x, \nabla V_0),$$

which shows our claim. Hereafter we choose $K = 3L/c$ and $\eta = L$, so that $V_0(x,t) = V(x,t) - L$.

5. We shall show that u^+ is a viscosity supersolution of (1.3) in $\{0 < d^+(x,t)\}$. Since we have already shown that d^+, V_0 and $Le^{\|G\|t}$ are supersolutions, we only need to study u^+ on $\{d^+(x,t) = V_0(x,t)\}$ and $\{V_0(x,t) = Le^{\|G\|t}\}$.

On $\{V_0(x,t) = Le^{\|G\|t}\}$ it is easily seen that subdifferentials $D^- u^+$ are empty, and so u^+ is a supersolution. (Applying Proposition 3.2 also shows that u^+ is a supersolution on $\{V_0(x,t) = Le^{\|G\|t}\}$ because u^+ is written as the infimum of two supersolutions V_0 and $Le^{\|G\|t}$ near $\{V_0(x,t) = Le^{\|G\|t}\}$.)

We next let $(z,s) \in \mathbf{R}^n \times (0,T)$ be a point such that $d^+(z,s) = V_0(z,s) =: \beta$, and take any $(p,\tau) \in D^- u^+(z,s)$. Our goal is to show that

$$J := \tau + H(z,p) - \beta G(z,p) \geqq 0.$$

Similarly to the case of subsolutions in Step 2, Lemma 5.5 implies that there exists some $\lambda \in [0,1]$ such that

$$\begin{aligned}
p &= \lambda \nabla d^+(z,s) + (1-\lambda)\nabla V_0(z,s) \\
&= \lambda \nabla d^+(z,s) + (1-\lambda)Ke^{\|G\|s}\nabla d^+(z,s), \\
\tau &= \lambda \partial_t d^+(z,s) + (1-\lambda)\partial_t V_0(z,s) \\
&= \lambda \partial_t d^+(z,s) + (1-\lambda)\{\|G\|(\beta + \eta) + Ke^{\|G\|s}\partial_t d^+(z,s)\}.
\end{aligned}$$

Set $\lambda' := \lambda + (1-\lambda)Ke^{\|G\|s}$. Then

$$\begin{aligned}
p &= \lambda' \nabla d^+(z,s), \\
\tau &= \lambda' \partial_t d^+(z,s) + (1-\lambda)\|G\|(\beta + \eta).
\end{aligned}$$

We also see $\lambda' \geqq \lambda + (1-\lambda) = 1$, which gives $G(z.\lambda'\nabla d^+) = G(z.\nabla d^+)$. We thus have

$$\begin{aligned} J &= \lambda'\partial_t d^+ + (1-\lambda)\|G\|(\beta+\eta) + H(z,\lambda'\nabla d^+) - \beta G(z,\lambda'\nabla d^+) \\ &= \lambda'\{\partial_t d^+ + H(z,\nabla d^+)\} + (1-\lambda)\|G\|(\beta+\eta) - \beta G(z,\nabla d^+) \\ &\geqq \lambda'\beta G(z,\nabla d^+) + (1-\lambda)\|G\|(\beta+\eta) - \beta G(z,\nabla d^+) \\ &= -(1-\lambda')\beta G(z,\nabla d^+) + (1-\lambda)\|G\|(\beta+\eta). \end{aligned}$$

The definition of λ' and β implies

$$-(1-\lambda')\beta = (1-\lambda)(Ke^{\|G\|s}\beta - \beta) = (1-\lambda)\eta,$$

and so we see

$$\begin{aligned} J &\geqq (1-\lambda)\eta G(z,\nabla d^+) + (1-\lambda)\|G\|(\beta+\eta) \\ &= (1-\lambda)[\eta\{G(z,\nabla d^+) + \|G\|\} + \beta\|G\|] \\ &\geqq 0. \end{aligned}$$

6. Finally, in a similar way to Step 3, we see that u^+ is a viscosity supersolution of (1.3) on $\{d^+(x,t)=0\}$ by studying the composite function $w_2 := \theta_2 \circ w$ with θ_2 defined by $\theta_2(r) = Le^{\|G\|T}$ if $r>0$, $\theta_2(r)=0$ if $r \leqq 0$. □

Remark 5.7. Define

$$\rho_0 := \min\left\{ \frac{cLe^{-\varepsilon T}}{3Le^{\|G\|T} - c},\ ce^{(\varepsilon - M)T} \right\}. \tag{5.17}$$

Then, by the definition of u^+ and u^- we see

$$(u^+(x,t), u^-(x,t)) = \begin{cases} (e^{\varepsilon t}d(x,t), e^{-\varepsilon t}d(x,t)) & \text{if } 0 \leqq d(x,t) \leqq \rho_0, \\ (e^{-\varepsilon t}d(x,t), e^{\varepsilon t}d(x,t)) & \text{if } -\rho_0 \leqq d(x,t) \leqq 0. \end{cases} \tag{5.18}$$

We are now in a position to state our main theorem. As the initial data u_0, we take a bounded and uniformly continuous function in $\mathbf{R}^n$ which agrees with the signed distance function near $\Gamma(0)$. Namely, we assume that there exists some $m > 0$ such that

$$\begin{cases} u_0(x) = d_0(x) & \text{if } |d_0(x)| \leqq m, \\ u_0(x) \geqq m & \text{if } d_0(x) > m, \\ u_0(x) \leqq m & \text{if } d_0(x) < -m, \end{cases} \tag{5.19}$$

where we set $d_0(x) := d(x,0)$.

Theorem 5.8 (Comparison with the signed distance function near the zero level set)**.** *Assume that the initial data $u_0 \in BUC(\mathbf{R}^n)$ satisfies* (5.19) *for some $m>0$. Assume* (SM), (CP), (H1), (H2) *and* (G1). *Let u be a viscosity solution of* (1.3) *with* (1.2). *Then for every $\varepsilon > 0$ there exists a positive constant $\rho(\varepsilon) > 0$ such that*

$$\begin{aligned} &e^{-\varepsilon t}d(x,t) \leqq u(x,t) \leqq e^{\varepsilon t}d(x,t) \quad \text{if } 0 \leqq d(x,t) \leqq \rho(\varepsilon), \\ &e^{\varepsilon t}d(x,t) \leqq u(x,t) \leqq e^{-\varepsilon t}d(x,t) \quad \text{if } -\rho(\varepsilon) \leqq d(x,t) \leqq 0. \end{aligned}$$

Proof. Fix $\varepsilon > 0$. Since it suffices to show the theorem for small ε, we may assume that $\varepsilon \leqq -(\log \sigma)/T$, where σ appears in (G1). We also choose a compact set $K \subset \mathbf{R}^n$ so that $\{|d(x,t)| < \delta\} \subset K$, and then take the function a in (5.5). Here δ is the constant in (SM). We define

$$\begin{aligned} c &:= \min\{\delta,\ a(\varepsilon),\ m\}, \\ L &:= \|u_0\| = \sup_{x\in\mathbf{R}^n} |u_0(x)|, \\ M &:= \max\left\{ \frac{2\|G\|}{1-\sigma e^{\varepsilon T}},\ \frac{3\varepsilon}{2} \right\}, \end{aligned}$$

and let $u^\pm : \mathbf{R}^n \times [0,T) \to \mathbf{R}$ be the functions in (5.15) and (5.16) with these constants. We then have

$$u^-(x,0) \leqq u_0(x) \leqq u^+(x,0) \quad \text{for all } x \in \mathbf{R}^n. \tag{5.20}$$

We shall prove (5.20) on $\{d_0(x) \geqq 0\}$; similar arguments work in $\{d_0(x) < 0\}$. Let $x \in \mathbf{R}^n$ be a point such that $d_0(x) \geqq 0$. We notice that

$$\begin{aligned} u^+(x,0) &= \min\left\{ \max\left\{ d_0(x),\ \frac{3\|u_0\|}{c} d_0(x) - \|u_0\| \right\},\ \|u_0\| \right\}, \\ u^-(x,0) &= \min\{d_0(x),\ c\} \end{aligned}$$

by definitions. When $0 \leqq d_0(x) \leqq m$, noting that $u_0(x) = d_0(x)$ by (5.19), we compute

$$\begin{aligned} u^+(x,0) &\geqq \min\{d_0(x),\ \|u_0\|\} = \min\{u_0(x),\ \|u_0\|\} = u_0(x), \\ u^-(x,0) &\leqq \min\{d_0(x),\ m\} = d_0(x) = u_0(x). \end{aligned}$$

In the case where $d_0(x) > m$, we estimate

$$\begin{aligned} u^+(x,0) &\geqq \min\left\{ \frac{3\|u_0\|}{c} d_0(x) - \|u_0\|,\ \|u_0\| \right\} \\ &\geqq \min\{2\|u_0\|,\ \|u_0\|\} \\ &\geqq u_0(x) \end{aligned}$$

and

$$u^-(x,0) \leqq \min\{d_0(x),\ m\} = m \leqq u_0(x).$$

Therefore (5.20) is proved.

By (5.20) the comparison principle (Theorem 3.3) implies

$$u^-(x,t) \leqq u(x,t) \leqq u^+(x,t) \quad \text{for all } (x,t) \in \mathbf{R}^n \times (0,T). \tag{5.21}$$

We set $\rho(\varepsilon) := \rho_0$, where ρ_0 is the constant in (5.17) with c, L and M given as above. Finally, combing (5.18) and (5.21) gives the conclusion of the theorem. $\square$

Remark 5.9. In view of the proof we notice that the constant $\rho(\varepsilon)$ also depends on T, $\|G\|$, σ, a, m and $\|u_0\|$.

Corollary 5.10. *Assume the same hypotheses of Theorem 5.8. Let $(z,s) \in \mathbf{R}^n \times (0,T)$ be a point such that $d(z,s)=0$.*

(1) *We have*

$$\lim_{\substack{(x,t)\to(z,s)\\ d(x,t)\neq 0}} \frac{u(x,t)}{d(x,t)} = 1.$$

(2) *The solution u is differentiable at (z,s), and the derivatives are given as*

$$\partial_t u(z,s) = \partial_t d(z,s), \quad \nabla u(z,s) = \nabla d(z,s).$$

Proof. (1) For a fixed $\varepsilon > 0$ we have

$$\frac{u(x,t)}{d(x,t)} \leqq \frac{e^{\varepsilon t} d(x,t)}{d(x,t)} = e^{\varepsilon t}$$

for all $(x,t) \in \mathbf{R}^n \times (0,T)$ such that $0 < |d(x,t)| \leqq \rho_0(\varepsilon)$. Thus

$$\limsup_{\substack{(x,t)\to(z,s)\\ d(x,t)\neq 0}} \frac{u(x,t)}{d(x,t)} \leqq e^{\varepsilon s}.$$

Since ε is arbitrary, we see that

$$\limsup_{\substack{(x,t)\to(z,s)\\ d(x,t)\neq 0}} \frac{u(x,t)}{d(x,t)} \leqq 1.$$

Similarly, we obtain

$$\liminf_{\substack{(x,t)\to(z,s)\\ d(x,t)\neq 0}} \frac{u(x,t)}{d(x,t)} \geqq 1.$$

and hence the assertion follows.

(2) We only give the proof for the time derivative since a similar argument applies to the spatial derivative. Fix $\varepsilon > 0$ and let $h \in \mathbf{R}$. If $|h|$ is sufficiently small, we have

$$\frac{u(z,s+h) - u(z,s)}{h} \leqq \frac{e^{\varepsilon(s+h)} d(z,s+h) - e^{\varepsilon(s+h)} d(z,s)}{h},$$

which yields

$$\limsup_{h\to 0} \frac{u(z,s+h) - u(z,s)}{h} \leqq e^{\varepsilon s} \partial_t d(z,s).$$

The rest of the proof runs as in (1). □

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