博士論文

論文題目

Examples of factors which have no Cartan subalgebras

（カルタン部分環を持たない 因子環の例）

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Examples of factors which have no Cartan subalgebras

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Abstract

We consider some conditions similar to Ozawa's condition (AO), and prove that if a non-injective factor satisfies such a condition and has the W*CBAP, then it has no Cartan subalgebras. As a corollary, we prove that II$_1$ factors of universal orthogonal and unitary discrete quantum groups have no Cartan subalgebras. We also prove that continuous cores of type III$_1$ factors with such a condition are semisolid as a II$_\infty$ factor.

1 Introduction

In the von Neumann algebra theory, the Cartan subalgebras give us many important information and fascinating examples. In fact, Cartan subalgebras always come from some orbit equivalence classes in the following sense: for a given separable factor M and its Cartan subalgebra $A \subset M$, there exists the unique orbit equivalence class $\mathcal{R}$ (and the cocycle σ) on a standard space X such that $(L^\infty(X) \subset L(\mathcal{R}, \sigma)) \simeq (A \subset M)$ [11]. This correspondence sometimes enables us to make use of the ergodic theory to analyze such class of factors. This is one of the main reasons why Cartan subalgebras have been studied for a long time.

For example, Sorin Popa gave first examples of II$_1$ factors whose fundamental groups are trivial [26]. In the proof, he identified fundamental groups of these factors as that of obit equivalence classes, by some deformation/intertwining arguments between two Cartan subalgebras. Hence he essentially investigated their Cartan subalgebras. This is the first result of the rigidity theory of II$_1$ factors.

From this pioneering work, there has been many remarkable works: realization of many outer automorphism groups and fundamental groups; new examples of prime factors; uniqueness and non-existence of Cartan subalgebras; W*-superrigidity and so on. In this paper, we concentrate our attention on a negative type result, that is, non-existence of Cartan subalgebras.

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Here we recall the definition of Cartan subalgebras. Let M be a von Neumann algebra and A an abelian subalgebra of M. We say A is a *Cartan subalgebra* of M if it satisfies the following conditions:

- there exists a faithful normal conditional expectation from M onto A;

- A is maximal abelian in M, that is, $A' \cap M = A$;

- the normalizer group $\mathcal{N}_M(A)$ generates M, that is, $\mathcal{N}_M(A)'' = M$.

Here the normalizer group is defined as $\mathcal{N}_M(A) := \{u \in \mathcal{U}(M) \mid uAu^* = A\}$. We note that Cartan subalgebras of M are diffuse (i.e. which have no minimal projections) if so is M.

Historically, examples of von Neumann algebras which have no Cartan subalgebras were first discovered by Voiculescu [35]. He showed that the free group factors $L(\mathbb{F}_n)$ ($n \geq 2$) have no Cartan subalgebras and his method relied on the free probability theory. Shlyakhtenko also proved that free Araki–Woods factors of type III$_\lambda$ have no Cartan subalgebras [29].

In the rigidity theory, Ozawa and Popa gave first examples [23]. They proved that the free group factors are such examples, and they actually proved that these factors are strongly solid. Here we recall that a finite von Neumann algebra M is *strongly solid* if for any diffuse injective subalgebra $A \subset M$, the normalizer $\mathcal{N}_M(A)$ generates an injective von Neumann algebra, that is, $\mathcal{N}_M(A)''$ is injective. It is easy to see that if a finite von Neumann algebra M is strongly solid, then any non-injective diffuse von Neumann subalgebra of M has no Cartan subalgebras. Hence their result is stronger than that of Voiculescu.

After the work of Ozawa and Popa, there has been many non-existence results, and in the present paper we follow [27], in which Popa and Vaes proved remarkable uniqueness and non-existence results of Cartan subalgebras. In the same paper they gave a new proof of the fact that factors of weakly amenable and bi-exact groups are strongly solid (this was first proved by Chifan, Sinclair, and Udrea [8, Corollary 0.2] with an equivalent notion of bi-exactness called *array*, see [7] and [8, Proposition 2.1]). We refer to this new proof. In fact, we will prove the same statement for more general von Neumann algebras which are not necessarily group von Neumann algebras.

For this purpose, we need notions of general von Neumann algebras which correspond weak amenability and bi-exactness. It is known that weak amenability has such a notion called the W*CBAP (see Subsection 2.3), but bi-exactness does not. Ozawa's condition (AO) is a candidate but this is not enough for us. We will investigate it in Section 3. See [28], [13], and [17] for other notions similar to condition (AO).

After this consideration, we prove the following main theorems.

Theorem A (Theorem 4.2.1). *Let M be a II$_1$ factor with separable predual. If M satisfies condition* (AO)$^+$ *(see Definition 3.1.1) and has the W*CBAP, then M is strongly solid.*

Theorem B (Theorem 4.3.1). *Let M be a non-injective type* III *factor with separable predual and ϕ a faithful normal state on M. If (M, ϕ) satisfies condition* $(\mathrm{AOC})^+$ *(see Definition* 3.2.1*) and has the* W^*CBAP*, then M has no ϕ-Cartan subalgebras.*

Here ϕ-*Cartan subalgebra* means a Cartan subalgebra which has a ϕ-preserving faithful normal conditional expectation E, that is, $\phi = \phi \circ E$.

To prove Theorem A, we need only slight modifications of the proof of (a special case of) [27, Theorem 3.1]. Theorem B can be proved by seeing its continuous core, and this idea comes from [15] and [16]. Since condition $(\mathrm{AOC})^+$ is similar to condition (AO) with respect to the continuous core, we naturally deduce the following primeness result. In the theorem below, Tr means the canonical semifinite trace on the continuous core.

Theorem C. *Let M be a von Neumann algebra with separable predual and ϕ a faithful normal state on M. Let $\mathcal{M}$ be its continuous core with respect to ϕ and p a projection in $L\mathbb{R}$ such that $\mathrm{Tr}(p) < \infty$. If (M, ϕ) satisfies condition* $(\mathrm{AOC})^+$*, then $p\mathcal{M}p$ is semisolid. In particular, $\mathcal{M}$ is a semisolid type* II_∞ *factor if M is a type* III_1 *factor.*

Here we recall that a finite von Neumann algebra M is *semisolid* (respectively, *solid*) if for any type II (respectively, diffuse) subalgebra $N \subset M$, the relative commutant $N' \cap M$ is injective. For a semifinite von Neumann algebra M, semisolidity (and solidity) is defined as that of pMp for all finite projections $p \in M$. We also recall that M is *prime* if for any tensor decomposition $M = M_1 \otimes M_2$, one of M_i $(i = 1, 2)$ is of type I. It is not difficult to see that semisolidity implies primeness for non-injective semifinite factors. Hence the conclusion of the theorem above implies primness.

The aim of our generalization is, of course, to find new examples. Factors of universal discrete quantum groups $A_o(F)$ and $A_u(F)$ (see Subsection 2.4) are our main targets. On the one hand, it is known that they satisfy condition (AO) [33][32], and we will observe that they in fact satisfy a little stronger conditions. See Section 3 for the details. On the other hand, weak amenability of them were shown very recently [12] but only for the case that they are monoidally equivalent to $A_o(1_n)$ or $A_u(1_n)$. Thus combined with the main theorems, we have the following corollary.

Corollary. *Let $\mathbb{G}$ be a universal discrete quantum group $A_o(F)$ or $A_u(F)$ for $F \in \mathrm{GL}(n, \mathbb{C})$ $(n \geq 3)$. Denote the Haar state by h.*

(1) *If $F = 1_n$, then $L^\infty(\mathbb{G})$ is strongly solid. In particular, $L^\infty(\mathbb{G})$ has no Cartan subalgebras.*

(2) *If $L^\infty(\mathbb{G})$ is non-injective and has the W^*CBAP, then it has no h-Cartan subalgebras.*

(3) If $L^\infty(\mathbb{G})$ is a non-injective type III_1 factor, then the continuous core $L^\infty(\mathbb{G}) \rtimes_{\sigma^h} \mathbb{R}$ is a semisolid, in particular prime, II_∞ factor.

We will observe in Subsection 5.2 that the continuous core of $L^\infty(A_o(F))$ is semisolid but never solid for some concrete matrix F.

Theorem B works for the case that F is not an identity matrix, but we do not know whether $L^\infty(\mathbb{G})$ has the W^*CBAP or not for a general matrix F. If one obtains this property, Theorem B is applicable for every non-injective von Neumann algebras of $A_o(F)$ and $A_u(F)$, and hence one has non-existence results for them. We leave this problem as follows.

Problem. *When do von Neumann algebras of universal discrete quantum groups $A_o(F)$ and $A_u(F)$ have the W^*CBAP?*

2 Preliminaries

2.1 Tomita–Takesaki theory

For Tomita–Takesaki theory, we refer the reader to [30].

Let M be a von Neumann algebra and ϕ a faithful normal state on M. We first consider the following anti-linear map:

$$S \colon M\Omega \to M\Omega \subset L^2(M,\phi); \ a\Omega \mapsto a^*\Omega,$$

where Ω is the canonical cyclic separating vector associated with ϕ. This map is closable on $L^2(M,\phi)$ and write as $S = J\Delta^{1/2}$ the polar decomposition of S. We call Δ the *modular operater* and J the *modular conjugation*. The following fundamental relations are important:

$$JMJ = M', \quad \Delta^{it}M\Delta^{-it} = M \quad (t \in \mathbb{R}).$$

In the paper, we frequently identify JMJ as the opposite algebra M^{op} with the obvious correspondence. The GNS-representation on the Hilbert space $L^2(M,\phi)$ (with a faithful normal state ϕ) is called a *standard representation* (see [30, Definition IX.1.14] for the intrinsic definition).

From the relation above, $\sigma_t^\phi(a) := \Delta^{it}a\Delta^{-it}$ $(a \in M, t \in \mathbb{R})$ defines a one parameter automorphism group on M, which is called the *modular automorphism group* on M associated with ϕ. The *continuous core* of M is defined as the crossed product von Neumann algebra $\widetilde{M} := M \rtimes_{\sigma^\phi} \mathbb{R}$ and it does not depend on the choice of ϕ. We can then construct a semifinite weight $\tilde{\phi}$ on the core called the *dual weight of* ϕ [30, Definition X.1.16], which of course depends on ϕ. The dual weights are always faithful and its modular action becomes inner (more precisely $\sigma_t^{\tilde{\phi}} = 1 \otimes \lambda_t$) so that $\widetilde{M}$ is always semifinite. A canonical semifinite trace on $\widetilde{M}$ is given by $\mathrm{Tr} := \tilde{\phi}(h\cdot)$, where h is the self-adjoint map satisfying $h^{it} = 1 \otimes \lambda_t$ $(t \in \mathbb{R})$. We say a

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type III factor M is of type III$_1$ if the continuous core is a factor of type II$_\infty$ (see [30, Definition XII.1.3] for definitions of type III$_\lambda$).

The associated representations

$$\pi\colon M \to \mathbb{B}(L^2(M) \otimes L^2(\mathbb{R})); \ x \mapsto \int_{\mathbb{R}} \sigma^\phi_{-t}(x) \otimes e_t \cdot dt,$$

$$u\colon \ \mathbb{R} \to \mathcal{U}(L^2(M) \otimes L^2(\mathbb{R})); \ t \mapsto 1 \otimes \lambda_t,$$

where $(\int_{\mathbb{R}} \sigma^\phi_{-t}(x) \otimes e_t \cdot dt\xi)(s) := \sigma^\phi_{-s}(x)\xi(s)$ and $((1 \otimes \lambda_t)\xi)(s) := \xi(-t+s)$ for any $\xi \in L^2(M) \otimes L^2(\mathbb{R})$, give a standard representation of $M \rtimes_{\sigma^\phi} \mathbb{R}$ on $L^2(M) \otimes L^2(\mathbb{R})$ with respect to the dual weight $\tilde{\phi}$. Since the standard representation is unique, this space is also standard with respect to Tr. More precisely, we have the following canonical unitary U which preserves standard representation structures:

$$U\colon L^2(M) \otimes L^2(\mathbb{R}) \to L^2(M) \otimes L^2(\mathbb{R}); \ \Lambda_{\tilde{\phi}}(x) \mapsto \Lambda_{\mathrm{Tr}}(xh^{-1/2}),$$

where Λ are canonical images of GNS representations and $x \in M$ satisfies $\tilde{\phi}(x^*x) < \infty$ and $xh^{-1/2}$ is bounded with closure in M. We note that

$$U(\mathbb{K}(L^2(M)) \otimes \mathbb{B}(L^2(\mathbb{R})))U^* = \mathbb{K}(L^2(M)) \otimes \mathbb{B}(L^2(\mathbb{R})).$$

A conditional expectation from $\widetilde{M}$ onto $L\mathbb{R}$ is defined by $E_{L\mathbb{R}}(x\lambda_t) := \phi(x)\lambda_t$ ($x \in M, t \in \mathbb{R}$). Then $E_{L\mathbb{R}}$ is $\tilde{\phi}$-preserving and Tr-preserving.

The modular conjugation $\tilde{J}$ on $\widetilde{M}$ is given by

$$(\tilde{J}\xi)(t) := \Delta^{-it}J\xi(-t) \quad (t \in \mathbb{R}, \xi \in L^2(M) \otimes L^2(\mathbb{R}))$$

[30, Lemma X.1.13], and one can easily verify that

$$\tilde{J}\pi(x)\tilde{J} = JxJ \otimes 1 \quad (x \in M), \quad \tilde{J}(1 \otimes \lambda_t)\tilde{J} = \Delta^{it} \otimes \rho_t \quad (t \in \mathbb{R}),$$

where ρ_t is the right translation defined by $(\rho_t\eta)(s) := \eta(s+t)$ ($\eta \in L^2(\mathbb{R})$). Hence we have

$$\begin{aligned}
(M \rtimes_{\sigma^\phi} \mathbb{R})' = \tilde{J}(M \rtimes_{\sigma^\phi} \mathbb{R})\tilde{J} &= W^*\{JxJ \otimes 1 \ (x \in M), \ \Delta^{it} \otimes \rho_t \ (t \in \mathbb{R})\} \\
&= W^*\{M' \otimes 1, \ \Delta^{it} \otimes \rho_t \ (t \in \mathbb{R})\}.
\end{aligned}$$

Next, we investigate how Cartan subalgebras of M behave in the continuous core of M. Let M be a general von Neumann algebra, $A \subset M$ a Cartan subalgebra of M, and let E_A be an associated conditional expectation. Take a faithful normal state ϕ on A and extend it on M via E_A (still denote it by ϕ). Then by the proof of Takesaki's conditional expectation theorem [30, Theorem IX.4.2], the restriction of σ^ϕ_t on A coincides with the modular automorphism group on A associated with ϕ. This implies $\sigma^\phi_t(A) = A$ so that we have a natural inclusion $A \rtimes_{\sigma^\phi} \mathbb{R} \subset M \rtimes_{\sigma^\phi} \mathbb{R}$. Since A is abelian (and so ϕ is tracial), $\sigma^\phi_t = \mathrm{id}_A$ on A and hence we have $A \rtimes_{\sigma^\phi} \mathbb{R} = A \otimes L\mathbb{R}$. Then it is known that for any Tr-finite projection $p \in L\mathbb{R}$, the reduced subalgebra $A \otimes pL\mathbb{R}p$ is a Cartan subalgebra of a finite von Neumann algebra $p(M \rtimes_{\sigma^\phi} \mathbb{R})p$ (e.g. [15, Propositions 2.6 and 2.7] and [10, Lemma 2.2]).

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2.2 Popa's intertwining techniques

As explained in Introduction, Sorin Popa introduced a useful tool which gives a good sufficient condition for unitary conjugacy of Cartan subalgebras. Here we recall only the precise statement which we need later. See [4, Theorem F.12] for another proof.

Theorem 2.2.1 ([26][25])**.** *Let M be a finite von Neumann algebra with separable predual, τ a faithful normal trace on M, and let $A, B \subset M$ be (possibly non-unital) von Neumann subalgebras. We denote by E_B the unique τ-preserving conditional expectation from $1_B M 1_B$ onto B. Then the following conditions are equivalent.*

(1) *There exists no sequences $(w_n)_n$ of unitaries in A such that $\lim_n \|E_B(b^* w_n a)\|_{2,\tau} = 0$ for any $a, b \in 1_A M 1_B$.*

(2) *There exists a non-zero A-B-submodule H of $1_A L^2(M) 1_B$ such that $\dim_{(B,\tau)} H < \infty$.*

(3) *There exist non-zero projections $e \in A$ and $f \in B$, a unital normal $*$-homomorphism $\theta \colon eAe \to fBf$, and a partial isometry $v \in M$ such that*

- $vv^* \leq e$ *and* $v^*v \leq f$,
- $v\theta(x) = xv$ *for any* $x \in eAe$.

We write as $A \preceq_M B$ if one of these conditions holds.

2.3 Weak amenability and W*CBAP

Weak amenability is an approximation property for discrete groups (more generally, locally compact groups) weaker than amenability, and the W*CBAP is a corresponding notion for von Neumann algebras.

To introduce these notions, we first recall the definition of a Herz–Shur multiplier. Let Γ be a discrete group and ϕ a map from Γ to $\mathbb{C}$. Consider a linear map

$$m_\phi \colon \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma]; \ \sum_{s \in \Gamma} a_s \cdot s \mapsto \sum_{s \in \Gamma} \phi(s) a_s \cdot s.$$

Then regarding $\mathbb{C}[\Gamma] \subset C_\lambda^*(\Gamma)$, we define the Herz–Shur norm of ϕ as $\|\phi\|_{\text{c.b.}} := \|m_\phi\|_{\text{c.b.}}$ (possibly infinite). We say ϕ (or m_ϕ) is a *Herz–Shur multiplier* if $\|\phi\|_{\text{c.b.}}$ is finite.

Then Recall that a discrete group Γ is *weakly amenable* if there exists a net $(\phi_i)_i$ of finitely supported Herz–Shur multipliers satisfying $\limsup_i \|\phi_i\|_{\text{c.b.}} < \infty$ and $\phi_i(g) \to 1$ as $i \to \infty$ for any $g \in \Gamma$. We also recall that a von Neumann algebra M has the *weak* completely approximation property* (or W^*CBAP,

in short) if there exists a net $(\psi_i)_i$ of normal c.b. maps on M with finite rank such that $\limsup_i \|\psi_i\|_{\text{c.b.}} < \infty$ and ψ_i converges to id_M in the point σ-weak topology.

Then optimal constants

$$\Lambda_{\text{c.b.}}(\Gamma) := \inf\{ \limsup_i \|\phi_i\|_{\text{c.b.}} \mid (\phi_i) \text{ satisfies the above condition}\}$$

$$\Lambda_{\text{c.b.}}(M) := \inf\{ \limsup_i \|\psi_i\|_{\text{c.b.}} \mid (\psi_i) \text{ satisfies the above condition}\}$$

are invariants of Γ and M respectively, both of which are called the *Cowling–Haagerup constant*. It is known that $\Lambda_{\text{c.b.}}(\Gamma) = \Lambda_{\text{c.b.}}(L\Gamma)$ (see for example [4, Section 12.3]). Freslon recently proved that $\Lambda_{\text{c.b.}}(L^\infty(\mathbb{G})) = 1$, where $\mathbb{G}$ is monoidally equivalent to $A_o(1_n)$ or $A_u(1_n)$ [12]. However the general case is still open.

We will use these properties in two ways: one is Theorem 4.1.1 to have weakly compact actions; the other is as follows with local reflexivity. Recall a C^*-algebra A is *locally reflexive* if for any finite dimensional subspace $E \subset A^{**}$, there exists a net $(\mu_j)_j$ of c.c. maps from E to A such that $(\mu_j)_j$ converges to id_E in the point σ-weak topology.

Lemma 2.3.1. *Let M be a von Neumann algebra and $A \subset M$ a σ-weakly dense C^*-subalgebra. Let $(\phi_i)_i$ be a net of normal c.b. maps on M with finite rank such that $\limsup_i \|\phi_i\|_{\text{c.b.}} =: k < \infty$ and ϕ_i converges id_M in the point σ-weak topology. Assume A is locally reflexive. Then we can find a net $(\psi_j)_j$ of normal c.b. maps from M into A with finite rank satisfying the same conditions as $(\phi_i)_i$.*

Proof. Let $z \in A^{**}$ be the central projection satisfying $M \simeq zA^{**}$. Put $E_i := \phi_i(M)$ and regard as a subset of A^{**} via $E_i \subset M \simeq zA^{**}$. Then, by local reflexivity of A, we can find a net $(\mu_j^i)_j$ of c.c. maps from E_i into A such that μ_j^i converges id_{E_i} in the point σ-weak topology. Now, putting $\tilde{\mu}_j^i(a) := z\mu_j^i(a)$, we have a net $(\tilde{\mu}_j^i \circ \phi_i)_{i,j}$ of c.b. maps from M into zA and this makes our desired net by using the identification $(A \subset M) \simeq (zA \subset zA^{**})$. $\qquad\square$

2.4 Universal discrete quantum groups

In the paper, we use the quantum group theory only for Propositions 3.1.2 and 3.2.4. We accept all the the basics of compact and discrete quantum groups and we refer the reader to [36] and [18] for the details. Our notations are very similar to those of [33].

Let $C(\mathbb{G})$ be a compact quantum group. We denote by Φ the comultiplication, by h the Haar state, and by $L^2(\mathbb{G})$ the GNS-representation of h. Then the Hilbert space $L^2(\mathbb{G})$ can be decomposed as follows:

$$L^2(\mathbb{G}) = \sum_{x \in \text{Irred}(\mathbb{G})} \oplus(H_x \otimes H_{\bar{x}}),$$

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where $\mathrm{Irred}(\mathbb{G})$ is the set of equivalent classes of all irreducible unitary representations of $\mathbb{G}$ and $\bar{x}$ is the contragredient of x. Let t_x be the unique unit vector (up to multiplication by $\mathbb{T}$) in $H_x \otimes H_{\bar{x}}$ such that $(U^x \boxtimes U^{\bar{x}})$-invariant, where U^x is the unitary element corresponding to x. Identify t_x as an antilinear map from $H_{\bar{x}}$ to H_x with the Hilbert–Schmidt correspondence. Then we have two representations

$$\rho\colon C(\mathbb{G}) \to \mathbb{B}(L^2(\mathbb{G})); \ \rho(\omega_{\eta,\xi} \otimes \iota(U^x))\Omega = \xi \otimes t^{\bar{x}}\eta \in H_x \otimes H_{\bar{x}},$$
$$\lambda\colon C(\mathbb{G}) \to \mathbb{B}(L^2(\mathbb{G})); \ \lambda(\omega_{\eta,\xi} \otimes \iota(U^x))\Omega = t^{\bar{x}}\eta \otimes \xi \in H_{\bar{x}} \otimes H_x,$$

for all $x \in \mathrm{Irred}(\mathbb{G})$ and $\xi, \eta \in H_x$. Here Ω is the canonical cyclic vector. We note that these representations are unitarily equivalent to the GNS-representation for the Haar state h. Define the dual discrete quantum group as

$$c_0(\hat{\mathbb{G}}) := \bigoplus_{x \in \mathrm{Irred}(\mathbb{G})} \mathbb{B}(H_x),$$
$$\ell^\infty(\hat{\mathbb{G}}) := \prod_{x \in \mathrm{Irred}(\mathbb{G})} \mathbb{B}(H_x),$$

and define two representations of them on the same Hilbert space $L^2(\mathbb{G})$ by

$$\hat{\lambda}\colon \ell^\infty(\hat{\mathbb{G}}) \to \prod_{x \in \mathrm{Irred}(\mathbb{G})} \mathbb{B}(H_x) \otimes \mathbb{C} \subset \mathbb{B}(L^2(\mathbb{G})),$$
$$\hat{\rho}\colon \ell^\infty(\hat{\mathbb{G}}) \to \prod_{\bar{x} \in \mathrm{Irred}(\mathbb{G})} \mathbb{C} \otimes \mathbb{B}(H_x) \subset \mathbb{B}(L^2(\mathbb{G})).$$

All dual objects are written with hat (e.g. $\hat{\Phi}$, $\hat{h}$). We have a natural unitary

$$\mathbb{V} = \bigoplus_{x \in \mathrm{Irred}\mathbb{G}} U^x.$$

From now on, we assume that the Haar state h is faithful on $C(\mathbb{G})$ and recall modular objects of them. We use similar notations to [31] which has a good survey of the modular theory on compact quantum groups. Let $A(\mathbb{G})$ be the dense Hopf $*$-algebra of $C(\mathbb{G})$, κ the antipode, and let ϵ be the counit of $C(\mathbb{G})$. Let $\{f_z\}_z$ $(z \in \mathbb{C})$ be the Woronowicz characters on $C(\mathbb{G})$, that is, a family of homomorphisms from $A(\mathbb{G})$ to $\mathbb{C}$ satisfying conditions in [36, Theorem 1.4]. Put $F_x := (\iota \otimes f_1)(U^x)$ for $x \in \mathrm{Irred}(\mathbb{G})$. Then we have the following useful relations to the modular group associated with the Haar state h:

- $(\iota \otimes \sigma_t^h)(U^x) = (F_x^{it} \otimes 1)U^x(F_x^{it} \otimes 1)$ $\quad(t \in \mathbb{R}, x \in \mathrm{Irred}(\mathbb{G}))$,

- $\Delta^{it} = \sum_{x \in \mathrm{Irred}(\mathbb{G})} \oplus(F_x^{it} \otimes F_{\bar{x}}^{-it})$ $\quad$ on $L^2(\mathbb{G}) = \sum_{x \in \mathrm{Irred}(\mathbb{G})} \oplus(H_x \otimes H_{\bar{x}})$.

We denote the scaling automorphism group by τ_t and the unitary antipode by R. Define a conjugate unitary $\hat{J}$ on $L^2(\mathbb{G})$ by $\hat{J}x\hat{1} := R(x^*)\hat{1}$ for $x \in C(\mathbb{G})$ and put $U := J\hat{J} = \hat{J}J$. Then we can identify all compact quantum group C^*-algebras as these opposite algebras, for example $\rho(C(\mathbb{G}))^{\mathrm{op}} \simeq J\rho(C(\mathbb{G}))J = U\rho(C(\mathbb{G}))U \simeq \rho(C(\mathbb{G}))$.

Next we recall universal discrete quantum groups introduced in [34] which are our main objects. Let F be an element in $\mathrm{GL}(n, \mathbb{C})$ ($n \geq 2$). Then the C^*-algebra $C(A_u(F))$ is defined as the universal unital C^*-algebra generated by all the entries of a unitary n by n matrix $u = (u_{i,j})_{i,j}$ satisfying

- $F\bar{u}F^{-1}$ is unitary,

where $\bar{u} = (u_{i,j}^*)_{i,j}$. Next assume that $F\bar{F} = \pm 1$. Then the C^*-algebra $C(A_o(F))$ is defined as the universal unital C^*-algebra generated by all the entries of a unitary n by n matrix $u = (u_{i,j})_{i,j}$ satisfying

- $F\bar{u}F^{-1} = u$.

Following [33] and [32], we treat only the case $n \geq 3$.

Put $\mathbb{G} := A_o(F)$ or $A_u(F)$. Then $C(\mathbb{G})_{\mathrm{red}}$ is defined as an image of $C(\mathbb{G})$ in $\mathbb{B}(L^2(\mathbb{G}))$ via the GNS-representation and it is still a compact quantum group. Write $L^\infty(\mathbb{G}) := C(\mathbb{G})_{\mathrm{red}}''$. Since previous two representations λ and ρ are unitary equivalent, we naturally have

$$C(\mathbb{G})_{\mathrm{red}} \simeq \lambda(C(\mathbb{G})) \simeq \rho(C(\mathbb{G})), \quad L^\infty(\mathbb{G}) \simeq \lambda(C(\mathbb{G}))'' \simeq \rho(C(\mathbb{G}))''.$$

We regard $\rho(C(\mathbb{G})) \subset \rho(C(\mathbb{G}))'' \subset \mathbb{B}(L^2(\mathbb{G}))$ as our main objects and, in the next section, we will prove that they satisfy some conditions similar to condition (AO). We note that factoriality and these types were studied in [2] and [33] (but not solved completely).

All the irreducible representations of $A_o(F)$ and $A_u(F)$ were completely classified in the following sense [3][2]: $\mathrm{Irred}(A_o(F))$ is identified with $\mathbb{N}$ in such a way that

$$x \otimes y \simeq |x - y| \oplus (|x - y| + 2) \oplus \cdots \oplus (x + y) \quad (x, y \in \mathbb{N});$$

$\mathrm{Irred}(A_u(F))$ is identified with $\mathbb{N} * \mathbb{N}$ in such a way that

$$x \otimes y \simeq \bigoplus_{z \in \mathbb{N}*\mathbb{N}, x = x_0 z, y = \bar{z}y_0} x_0 y_0 \quad (x, y \in \mathbb{N} * \mathbb{N}).$$

From now on, for simplicity, we treat only $A_o(F)$ and all the cases of $A_u(F)$ in this paper can be treated in the same way as that of $A_o(F)$. See [32, Section 5].

Let z be any irreducible representation contained in $x \otimes y$ as a subrepresentation (write as $z \in x \otimes y$). Let $p_z^{x \otimes y}$ be the unique projection in $\mathbb{B}(H_x \otimes H_y)$ satisfying $(U^x \boxtimes U^y)(p_z^{x \otimes y} \otimes 1) \simeq U^z$. Then take an intertwiner

$V(x \otimes y, z)$ between $(U^x \boxtimes U^y)(p_z^{x \otimes y} \otimes 1)$ and U^z and it is unique up to multiplication by $\mathbb{T}$. Define a u.c.p. map $\psi_{x+y,x} \colon \mathbb{B}(H_x) \to \mathbb{B}(H_{x+y})$ by

$$\psi_{x+y,x}(A) := V(x \otimes y, x+y)^*(A \otimes 1)V(x \otimes y, x+y),$$

and notice that this map does not depend on the choice of $V(x \otimes y, x+y)$. This independence sometimes allows us to use special properties of some special $V(x \otimes y, x+y)$. For example, they have the following useful inequality [33, (8.6)]:

$$d_{\mathbb{T}}((V(a \otimes b, a+b)^* \otimes 1)(1 \otimes V(b \otimes c, z)), V((a+b) \otimes c, a+z)V(a \otimes z, a+z)^*) \le Cq^{(z+b-c)/2},$$

where $d_{\mathbb{T}}(V, W) := \inf\{\|V - \lambda W\| \mid \lambda \in \mathbb{T}\}$ and $0 < q < 1$ with $\mathrm{Tr}(F^*F) = q + q^{-1}$.

Finally we recall a nuclear C^*-subalgebra $\mathcal{B}$ of $\ell^\infty(\hat{\mathbb{G}})$ which plays a significant role for us. We first put

$$\mathcal{B}_0 := \{a \in \ell^\infty(\hat{\mathbb{G}}) \mid \text{there exists } x \text{ such that } ap_y = \psi_{y,x}(ap_x) \text{ for all } y \ge x\}.$$

Let $\pi \colon \mathbb{B}(L^2(\mathbb{G})) \to \mathbb{B}(L^2(\mathbb{G}))/\mathbb{K}(L^2(\mathbb{G}))$ be the quotient map. In [33], Vaes and Vergnioux proved that

- the norm closure $\mathcal{B}$ of $\mathcal{B}_0$ is a C^*-algebra containing $c_0(\hat{\mathbb{G}})$ so that the C^*-algebra $\mathcal{B}_\infty := \mathcal{B}/c_0(\hat{\mathbb{G}})$ is defined;

- $\mathcal{B}$ and $\mathcal{B}_\infty$ are nuclear;

- $\hat{\Phi}$ induces a left action of $\hat{\mathbb{G}}$ on $\mathcal{B}$ and $\mathcal{B}_\infty$;

- this left action on $\mathcal{B}_\infty$ is amenable so that $\mathcal{B}_\infty \rtimes_r \hat{\mathbb{G}}$ is nuclear and $\mathcal{B}_\infty \rtimes_r \hat{\mathbb{G}} = \mathcal{B}_\infty \rtimes_{\mathrm{full}} \hat{\mathbb{G}}$;

- $\hat{\Phi}$ induces the trivial right action of $\hat{\mathbb{G}}$ on $\mathcal{B}_\infty$ so that $\hat{\lambda}(\mathcal{B}_\infty)$ commutes with $\pi \circ \lambda(C(\mathbb{G}))$, where we identify $\hat{\lambda}$ as a map from $\ell^\infty(\mathbb{G})/c_0(\mathbb{G})$ to $\mathbb{B}(L^2(\mathbb{G}))/\mathbb{K}(L^2(\mathbb{G}))$.

Since $\hat{\lambda} \colon \mathcal{B}_\infty \to \mathbb{B}(L^2(\mathbb{G}))/\mathbb{K}(L^2(\mathbb{G}))$ and $(\iota \otimes \pi \circ \rho)(\mathbb{V}) \in M(c_0(\hat{\mathbb{G}}) \otimes \mathbb{B}/\mathbb{K})$ are a covariant representation for the left action $\hat{\Phi}$, we have the following $*$-homomorphism

$$\pi_l(= \hat{\lambda} \rtimes \pi \circ \rho) \colon \mathcal{B}_\infty \rtimes_r \hat{\mathbb{G}} = \mathcal{B}_\infty \rtimes_{\mathrm{full}} \hat{\mathbb{G}} \longrightarrow \mathbb{B}(L^2(\mathbb{G}))/\mathbb{K}(L^2(\mathbb{G}))$$

by universality. Putting $\pi_r := \mathrm{Ad}U \circ \pi_l$, where $U = J\hat{J}$, we have the following algebraic $*$-homomorphism

$$\pi_l \times \pi_r \colon (\mathcal{B}_\infty \rtimes_r \hat{\mathbb{G}}) \odot (\mathcal{B}_\infty \rtimes_r \hat{\mathbb{G}}) \longrightarrow \mathbb{B}(L^2(\mathbb{G}))/\mathbb{K}(L^2(\mathbb{G}))$$
$$a \otimes b \longmapsto \pi_l(a)\pi_r(b),$$

since $\hat{\lambda}(\mathcal{B}_\infty)$ commutes with $\pi \circ \lambda(C(\mathbb{G}))$. Here $\odot$ means the algebraic tensor product. By nuclearity of $\mathcal{B}_\infty \rtimes_r \hat{\mathbb{G}}$, this map is min-bounded and the restriction of the map on $(\mathbb{C} \rtimes_r \hat{\mathbb{G}}) \otimes (\mathbb{C} \rtimes_r \hat{\mathbb{G}}) \simeq C(\mathbb{G})_{\mathrm{red}} \otimes C(\mathbb{G})_{\mathrm{red}}$ gives the min-boundedness of the multiplication map on $C(\mathbb{G})_{\mathrm{red}}$ after taking the quotient with $\mathbb{K}(L^2(\mathbb{G}))$. This is the proof of the fact that $L^\infty(\mathbb{G})$ satisfies condition (AO) given in [33].

We should mention that the multiplication map from $C(\mathbb{G})_{\mathrm{red}} \otimes C(\mathbb{G})_{\mathrm{red}}$ to $\mathbb{B}(L^2(\mathbb{G}))/\mathbb{K}(L^2(\mathbb{G}))$ is nuclear, since so is $(\mathcal{B}_\infty \rtimes_r \hat{\mathbb{G}}) \otimes (\mathcal{B}_\infty \rtimes_r \hat{\mathbb{G}})$ (and hence is $\pi_l \times \pi_r$). We will use this observation in the next section.

3 Conditions Similar to Ozawa's Condition (AO)

In this section, we introduce some similar conditions to condition (AO). We will prove that von Neumann algebras of $A_o(F)$ and $A_u(F)$ satisfy these conditions.

3.1 Condition $(\mathrm{AO})^+$

Let us first recall Ozawa's condition (AO). We say a von Neumann algebra $M \subset \mathbb{B}(H)$ satisfies *condition* (AO) if there exist σ-weakly dense unital C^*-subalgebras $A \subset M$ and $B \subset M'$ such that

(i) A is locally reflexive;

(ii) the multiplication map $\nu \colon A \odot B \to \mathbb{B}(H)/\mathbb{K}(H); \ a \otimes b \mapsto ab + \mathbb{K}(H)$ is min-bounded.

In [21], Ozawa proved his celebrated theorem: if a finite von Neumann algebra satisfies condition (AO), then it is solid. As we mentioned, solidity (or semisolidity) implies primeness for non-injective II_1 factors.

The most important examples of von Neumann algebras with condition (AO) comes from bi-exact groups [4, Definition 15.1.2]. In fact, Ozawa proved that they have the following characterization [4, Lemma 15.1.4]: a countable discrete group Γ is bi-exact if and only if Γ is exact and satisfies the following condition

- there exists a u.c.p. map $\theta \colon C_\lambda^*(\Gamma) \otimes C_\rho^*(\Gamma) \to \mathbb{B}(\ell^2(\Gamma))$ such that $\theta(a \otimes b) - ab \in \mathbb{K}(\ell^2(\Gamma))$ for any $a \in C_\lambda^*(\Gamma)$ and $b \in C_\rho^*(\Gamma)$.

It is now obvious that the group von Neumann algebras of bi-exact groups satisfy condition (AO). Thus he proved that factors of bi-exact non-amenable i.c.c. groups are solid, in particular, prime.

Here is another significant view point. To see solidity, we do *not* need the existence of a u.c.p. map θ above. We need only the property that the multiplication map ν is min-bounded after taking the quotient with

$\mathbb{K}(\ell^2(\Gamma))$. This is why condition (AO) is weaker than bi-exactness for group von Neumann algebras.

On the other hand, in [27], Popa and Vaes proved that the group von Neumann algebras of bi-exact and weakly amenable groups are strongly solid. In the proof, they used such a u.c.p. map θ as an essential tool.

Motivated these observation, we define the first condition similar to condition (AO) as follows.

Definition 3.1.1. Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra with standard representation and denote by J the modular conjugation. We say $M \subset \mathbb{B}(H)$ satisfies *condition* $(\mathrm{AO})^+$ if there exists a unital σ-weakly dense C^*-subalgebra A such that

(i) A is locally reflexive;

(ii) there exists a u.c.p. map $\theta\colon A \otimes JAJ \to \mathbb{B}(H)$ such that $\theta(a \otimes JbJ) - aJbJ \in \mathbb{K}(H)$ for any $a, b \in A$.

The difference of conditions (AO) and $(\mathrm{AO})^+$ is of course the existence of a u.c.p. map θ. So it may be useful to consider how we get such a θ for von Neumann algebras satisfying condition (AO). For this purpose, we translate the second condition as follows.

(ii') The multiplication map ν is min-bounded and it has a u.c.p. lift, that is, there exists a u.c.p. map $\theta\colon A \otimes JAJ \to \mathbb{B}(H)$ such that $\nu = \pi \circ \theta$, where $\pi\colon \mathbb{B}(H) \to \mathbb{B}(H)/\mathbb{K}(H)$ is the quotient map.

With this trivial translation, we can apply lifting theorems in some concrete cases. For example, if A is separable C^*-algebra and the multiplication map ν is nuclear, then ν has a u.c.p. lift by the lifting theorem due to Choi and Effros [5]. This method has been used by Ozawa (see the proof of [4, Proposition 15.2.3]).

Now combined with the observation in Subsection 2.4, we can easily deduce that our main targets satisfy condition $(\mathrm{AO})^+$.

Proposition 3.1.2. *Von Neumann algebras* $L^\infty(A_o(F))$ *and* $L^\infty(A_u(F))$ *for* $F \in \mathrm{GL}(n, \mathbb{C})$ $(n \geq 3)$ *satisfy Condition* $(\mathrm{AO})^+$.

3.2 A similar condition for continuous cores

To see strong solidity in the rigidity theory, finiteness assumption is essential since all the known proofs require the theory of amenable trace, which works only for finite von Neumann algebras. However our main targets $L^\infty(A_o(F))$ and $L^\infty(A_u(F))$ are hardly finite. So it is natural for us to see the continuous cores of such factors which are always semifinite.

In this subsection, we investigate some conditions for continuous cores of general von Neumann algebras. The following condition is a natural analogue of condition $(\mathrm{AO})^+$ for continuous cores.

Definition 3.2.1. Let M be a von Neumann algebra, ϕ a faithful normal state on M, and let $\tilde{J}$ be the modular conjugation for $M \rtimes_{\sigma^\phi} \mathbb{R} \subset \mathbb{B}(L^2(M,\phi) \otimes L^2(\mathbb{R}))$. We say the pair (M,ϕ) satisfies *condition* $(\mathrm{AO})^+$ *with respect to its continuous core* (say *condition* $(\mathrm{AOC})^+$, in short) if there exists a σ-weakly dense unital C^*-subalgebra $A \subset M$ such that

(i) σ^ϕ defines a norm continuous action on A (so that we can define $A \rtimes_r \mathbb{R}$);

(ii) $A \rtimes_r \mathbb{R}$ is locally reflexive;

(iii) there exists a u.c.p. map

$$\theta \colon (A \rtimes_r \mathbb{R}) \odot \tilde{J}(A \rtimes_r \mathbb{R})\tilde{J} \longrightarrow \mathbb{B}(L^2(M,\phi) \otimes L^2(\mathbb{R}))$$

such that $\theta(a \otimes \tilde{J}b\tilde{J}) - a\tilde{J}b\tilde{J} \in \mathbb{K}(L^2(M,\phi)) \otimes \mathbb{B}(L^2(\mathbb{R}))$ for any $a, b \in A \rtimes_r \mathbb{R}$.

Remark 3.2.2. In the definition, we regard $L^2(M,\phi) \otimes L^2(\mathbb{R}) = L^2(M \rtimes_{\sigma^\phi} \mathbb{R}, \tilde{\phi})$. Since $\mathbb{K}(L^2(M,\phi)) \otimes \mathbb{B}(L^2(\mathbb{R}))$ is preserved by the canonical unitary U (see Subsection 2.1), we indeed have the same condition on $L^2(M \rtimes_{\sigma^\phi} \mathbb{R}, \mathrm{Tr})$.

Our goal in the subsection is to show that $A_o(F)$ and $A_u(F)$ with the Haar states satisfy this condition. For this, we investigate a sufficient condition for condition $(\mathrm{AOC})^+$.

Let M be a von Neumann algebra and ϕ a faithful normal state on M. Write $H := L^2(M,\phi)$ and $\mathcal{K} := \mathbb{K}(H) \otimes \mathbb{B}(L^2(\mathbb{R}))$ and let J be the modular conjugation on H. Consider the multiplier algebra $\mathcal{L} := M(\mathcal{K})$ of $\mathcal{K}$ and denote $\mathcal{C} := \mathcal{L}/\mathcal{K}$.

Assume first that there exists a σ-weakly dense unital C^*-subalgebra $A \subset M$ such that

(a) σ^ϕ defines a norm continuous action on A (so that we can define $A \rtimes_r \mathbb{R}$).

Let π be a $*$-homomorphism from $\mathbb{B}(H)$ into $\mathbb{B}(H \otimes L^2(\mathbb{R}))$ given by $(\pi(x)\xi)(t) := \Delta_\phi^{-it} x \Delta_\phi^{it} \xi(t)$ for $x \in \mathbb{B}(H)$ and $t \in \mathbb{R}$. Consider the C^*-algebra D generated by following elements

- $\pi(a)$, $JbJ \otimes 1$ $(a, b \in A)$;

- $1 \otimes \lambda_t$, $\Delta_\phi^{it} \otimes \rho_t$ $(t \in \mathbb{R})$;

- $\int_\mathbb{R} f(s)(1 \otimes \lambda_s) \cdot ds$, $\int_\mathbb{R} f(s)(\Delta_\phi^{is} \otimes \rho_s) \cdot ds$ $(f \in L^1(\mathbb{R}))$.

Then we assume that

(b) D is contained in $\mathcal{L}$.

In particular, we have natural maps from A and $A^{\mathrm{op}}(= JAJ)$ to $\mathcal{C}$. We denote these maps by π_l and π_r respectively.

Next we assume that

(c) there exist separable nuclear C^*-algebras C_l and C_r containing A and A^{op} respectively (so that A is exact);

(d) there exist $*$-homomorphisms from C_l and C_r to $\mathcal{C}$ such that they are extensions of π_l and π_r, respectively. We still denote them by π_l and π_r.

Then we want to define the following $*$-homomorphism

$$\nu \colon C_l \odot C_r \longrightarrow \mathcal{C}; \ a \otimes b \mapsto \pi_l(a)\pi_r(b).$$

However we do not know whether ranges of C_l and C_r commute, and hence we further assume that

(e) ν is a well-defined $*$-homomorphism, that is, $[\pi_l(a), \pi_r(b)] = 0$ $(a \in C_l,\ b \in C_r)$.

We can extend ν on $C_l \otimes C_r$ by the nuclearity. Restricting this map, we have a natural multiplication $*$-homomorphism

$$
\begin{aligned}
\nu \colon A \otimes A^{\mathrm{op}} \longrightarrow \mathcal{C}\ ;\ a \otimes 1 \ &\longmapsto [\pi(a)], \\
1 \otimes a^{\mathrm{op}} &\longmapsto [a^{\mathrm{op}} \otimes 1].
\end{aligned}
$$

Next consider norm continuous $(\mathbb{R} \times \mathbb{R})$-actions on $A \otimes A^{\mathrm{op}}$ and $\nu(A \otimes A^{\mathrm{op}})$ given by

$$
\begin{aligned}
\mathbb{R} \times \mathbb{R} &\longrightarrow \mathrm{Aut}(A \otimes A^{\mathrm{op}}) \quad ;\ s \otimes t \longmapsto \quad \sigma_s^\phi \otimes \tilde{\sigma}_t^\phi, \\
\mathbb{R} \times \mathbb{R} &\longrightarrow \mathrm{Aut}(\nu(A \otimes A^{\mathrm{op}}))\ ;\ s \otimes t \longmapsto \mathrm{Ad}([1 \otimes \lambda_s][\Delta_\phi^{it} \otimes \rho_t]).
\end{aligned}
$$

Here $\tilde{\sigma}_t^\phi(a^{\mathrm{op}}) = \tilde{\sigma}_t^\phi(Ja^*J) := J\sigma_t^\phi(a^*)J = \sigma_t^\phi(a)^{\mathrm{op}}$. It is easily verified that ν is $(\mathbb{R} \times \mathbb{R})$-equivariant and hence we have the following $*$-homomorphism:

$$\tilde{\nu} \colon A \rtimes_r \mathbb{R} \otimes \tilde{J}(A \rtimes_r \mathbb{R})\tilde{J} \simeq (A \otimes A^{\mathrm{op}}) \rtimes_r (\mathbb{R} \times \mathbb{R}) \to (\nu(A \otimes A^{\mathrm{op}})) \rtimes_r (\mathbb{R} \times \mathbb{R}) \to \mathcal{C}.$$

Here the continuity of the final map comes from the amenability of $\mathbb{R} \times \mathbb{R}$. The resulting map says that the multiplication map on $A \rtimes_r \mathbb{R} \odot (A \rtimes_r \mathbb{R})^{\mathrm{op}}$ to $\mathcal{L} \subset \mathbb{B}(H \otimes L^2(\mathbb{R}))$ is min-bounded after taking the quotient with $\mathcal{K}$. Now the C^*-algebra $A \rtimes_r \mathbb{R}$ is exact (and hence locally reflexive) since so is A, and it is σ-weakly dense in $M \rtimes_{\sigma^\phi} \mathbb{R}$. At this point, $M \rtimes_{\sigma^\phi} \mathbb{R}$ satisfies a similar condition to condition (AO).

Finally we assume that

(f) $\mathrm{Ad}([1 \otimes \lambda_s][\Delta_\phi^{it} \otimes \rho_t])\nu(C_l \otimes C_r) = \nu(C_l \otimes C_r)$ for any $(s, t) \in \mathbb{R} \times \mathbb{R}$ and this defines a norm continuous action on $\nu(C_l \otimes C_r)$.

In this case there exists a ∗-homomorphism from $\nu(C_l \otimes C_r) \rtimes_r (\mathbb{R} \times \mathbb{R})$ into $\mathcal{C}$ and hence the image of the map is nuclear. Since ran$\tilde{\nu}$ is contained in this image, $\tilde{\nu}$ is a nuclear map into $\mathcal{C}$. Thus the lifting theorem of Choi and Effros is again applicable so that $\tilde{\nu}$ has a u.c.p. lift. Summary we have the following lemma.

Lemma 3.2.3. *Let M be a von Neumann algebra, ϕ a faithful normal state on M, and let $A \subset M$ be a σ-weakly dense unital C^*-subalgebra. If they satisfy all the conditions from* (a) *to* (f), *then* (M, ϕ) *satisfies condition* $(\text{AOC})^+$.

Now we turn to show our main objects satisfy these conditions.

Proposition 3.2.4. *Von Neumann algebras $L^\infty(A_o(F))$ and $L^\infty(A_u(F))$ for $F \in \mathrm{GL}(n, \mathbb{C})$ $(n \geq 3)$ with the Haar state h satisfy condition* $(\text{AOC})^+$.

Proof. We keep the notations in Subsection 2.4. Put $A := C^*_{\text{red}}(\mathbb{G}) = \rho(C(\mathbb{G})) \subset \mathbb{B}(L^2(\mathbb{G}))$ and $C_l = C_r = \mathcal{B}_\infty \rtimes_r \mathbb{G}$. We will verify all the conditions from (a) to (f) above. Note that the condition (a) is a well-know property.

For this, recall the following formula: for any irreducible decomposition $x \otimes y \simeq \sum_{z \in x \otimes y} \oplus z$, we have

$$F_x \otimes F_y \simeq \sum_{z \in x \otimes y} \oplus F_z \quad \text{on } H_x \otimes H_y \simeq \sum_{z \in x \otimes y} \oplus H_z.$$

Indeed this follows from a direct calculation of $(\iota \otimes \iota \otimes f_1)(U^x_{13} U^y_{23})$. By the formula, we have the following relation:

$$\Delta^{it} \hat{\lambda}(\psi_{x+y,x}(B)) \Delta^{-it} = \hat{\lambda}(\psi_{x+y,x}(F^{it}_x B F^{-it}_x)) \quad (B \in \mathbb{B}(H_x)).$$

In this sense, the modular group Δ^{it} commutes with all $\psi_{x+y,y}$.

To see the condition (b), observe first that for any $c \in \mathbb{B}(H \otimes L^2(\mathbb{R}))$, c is contained in $\mathcal{K}$ if and only if

$$\| \sum_{x=0}^{y} (p_x \otimes 1) c \sum_{x=0}^{z} (p_x \otimes 1) - c \| \to 0 \quad (y, z \to \infty).$$

Let a be an element in $\mathcal{K}$ and b a generator of D. We will show $ba \in \mathcal{K}$. The cases $b = 1 \otimes \lambda_t$, $\Delta^{it}_\phi \otimes \rho_t$, and $d \otimes 1$ $(d \in UAU^*)$ are trivial. The cases $b = \int_{\mathbb{R}} f(s)(1 \otimes \lambda_s) \cdot ds$ and $\int_{\mathbb{R}} f(s)(\Delta^{is}_\phi \otimes \rho_s) \cdot ds$ $(f \in L^1(\mathbb{R}))$ are easy since they commute with all p_x. For the final case $b = \pi(d)$ $(d \in A)$, we may assume $d = (\omega_{\xi,\eta} \otimes \rho)(\mathbb{V})$ for $\xi, \eta \in H_z$ and $z \in \mathrm{Irred}(\mathbb{G})$. Let $(\xi^z_k)^{n_z}_{k=1}$ be a

fixed orthonormal basis of H_z. Then we have

$$
\begin{aligned}
p_x \sigma_t(d) &= p_x \sigma_t((\omega_{\xi,\eta} \otimes \rho)(\mathbb{V})) \\
&= (\omega_{\xi,\eta} \otimes \iota)((1 \otimes p_x)\mathscr{V}_t) \qquad (\mathscr{V}_t := (\iota \otimes \sigma_t \circ \rho)(\mathbb{V})) \\
&= (\omega_{\xi,\eta} \otimes \iota)(\mathscr{V}_t \mathscr{V}_t^*(1 \otimes p_x)\mathscr{V}_t) \\
&= \sum_{k=1}^{n_z} (\omega_{\xi,\xi_k^z} \otimes \iota)(\mathscr{V}_t)(\omega_{\xi_k^z,\eta} \otimes \iota)((1 \otimes \Delta^{it})\hat{\Psi}(p_x)(1 \otimes \Delta^{-it}))
\end{aligned}
$$

for all $x \in \mathrm{Irred}(\mathbb{G})$, where σ_t is the modular group for the Haar state and $\hat{\Psi}(p_x) := (\iota \otimes \rho)(\mathbb{V})^*(1 \otimes p_x)(\iota \otimes \rho)(\mathbb{V})$. Since $\sum_{x=0}^{y} p_x$ converges to 1 in the strong topology as $y \to \infty$ and $\hat{\Psi}$ is normal, each $(\omega_{\xi_k^z,\eta} \otimes \iota)((1 \otimes \Delta^{it})\hat{\Psi}(\sum_{x=0}^{y} p_x)(1 \otimes \Delta^{-it}))$ converges to $\omega_{\xi_k^z,\eta}(1)1$ in the strong topology. Hence for any compact operator $T \in \mathbb{K}(H)$, the equation

$$
\sum_{x=0}^{y} p_x \sigma_t(d)T = \sum_{k=1}^{n_k} (\omega_{\xi,\xi_k^z} \otimes \iota)(\mathscr{V}_t)(\omega_{\xi_k^z,\eta} \otimes \iota)((1 \otimes \Delta^{it})\hat{\Psi}(\sum_{x=0}^{y} p_x)(1 \otimes \Delta^{-it}))T
$$

implies that $\sum_{x=0}^{y} p_x \sigma_t(d)T$ converges to $\sigma_t(d)T$ in the *norm* topology as $y \to \infty$. We choose T as an element of the set of all linear combinations of the form $\xi_k^a \otimes \xi_l^b$ for $a, b \in \mathrm{Irred}\mathbb{G}$ and k, l. Then it is easy to verify that this convergence is uniform with respect to $t \in \mathbb{R}$, that is,

$$
\sup_{t \in \mathbb{R}} \|(1 - \sum_{x=0}^{y} p_x)\sigma_t(d)T\| \to 0 \qquad (y \to \infty).
$$

Now, for any $S \in \mathbb{B}(L^2(\mathbb{R}))$, we have

$$
\begin{aligned}
&\|\sum_{x=0}^{y}(p_x \otimes 1)\pi(d)(T \otimes S) - \pi(d)(T \otimes S)\| \\
&= \|\int_{\mathbb{R}} (\sum_{x=0}^{y} p_x - 1)\sigma_{-t}(d) \otimes e_t \cdot dt(T \otimes S)\| \\
&\leq \sup_{t \in \mathbb{R}} \|(\sum_{x=0}^{y} p_x - 1)\sigma_t(d)T\|\|S\| \to 0 \quad (y \to \infty).
\end{aligned}
$$

Hence, for any $a \in \mathcal{K} = \mathbb{K}(H) \otimes \mathbb{B}(L^2(\mathbb{R}))$, we have

$$
\|\sum_{x=0}^{y}(p_x \otimes 1)\pi(d)a - \pi(d)a\| \to 0 \quad (y \to \infty).
$$

Since $\pi(d)a \sum_{x=0}^{y}(p_x \otimes 1)$ converges to $\pi(d)a$ in the norm topology, $\pi(d)a$ is contained in $\mathcal{K}$, and we get the condition (b).

Next we define two maps π_l and π_r. We begin with following maps

$$\mathcal{B} \longrightarrow \mathcal{L} \;;\; a \longmapsto \pi(\hat{\lambda}(a)),$$
$$\mathcal{B} \longrightarrow \mathcal{L} \;;\; a \longmapsto U\hat{\lambda}(a)U^* \otimes 1.$$

It is not difficult to see that ranges of these maps are really contained in $\mathcal{L}$. Since images of $c_0(\hat{\mathbb{G}})$ by these maps are contained in $\mathcal{K}$, we have induced maps from $\mathcal{B}_\infty$ to $\mathcal{C}$. Simple calculations show that these maps make two covariant representations of $\mathcal{B}_\infty$ and the natural left action of $\hat{\mathbb{G}}$. Since this action is amenable we have following desired maps:

$$\pi_l \colon C_l = \mathcal{B}_\infty \rtimes_r \hat{\mathbb{G}} \longrightarrow \mathcal{C},$$
$$\pi_r \colon C_r = \mathcal{B}_\infty \rtimes_r \hat{\mathbb{G}} \longrightarrow \mathcal{C}.$$

Finally we prove the condition (e) (and then the condition (f) is easily verified). For this, it suffices to see that $\pi \circ \hat{\lambda}(\mathcal{B})$ (respectively, $\hat{\rho}(\mathcal{B}) \otimes 1$) commutes with $\lambda(C(\mathbb{G})) \otimes 1$ (respectively, $\pi \circ \rho(C(\mathbb{G}))$) after taking the quotient with $\mathcal{K}$. Here we treat only the case of $\pi \circ \hat{\lambda}(\mathcal{B})$ and $\lambda(C(\mathbb{G})) \otimes 1$, and the other case follows from the same manner.

Let z be an element of $\mathrm{Irred}(\mathbb{G})$ and write as $U^z = \sum_{i,j} u_{i,j}^z \otimes e_{i,j}$, where $(e_{ij})_{ij}$ is a fixed matrix unit in $\mathbb{B}(H_z)$. Our goal is to show

$$[\pi \circ \hat{\lambda}(b), \lambda(u_{i,j}^z{}^*) \otimes 1] \in \mathbb{K}(L^2(\mathbb{G})) \otimes \mathbb{B}(L^2(\mathbb{R}))$$

for any z, i, j and any $b \in \mathcal{B}$, where $[\cdot, \cdot]$ is the commutator. Since this term coincides with

$$\int_{\mathbb{R}} [\Delta^{-it}\hat{\lambda}(b)\Delta^{it}, \lambda(u_{i,j}^z{}^*)] \otimes e_t \cdot dt,$$

running over all i and j, our goal is equivalent to

$$\int_{\mathbb{R}} \sum_{i,j} [\Delta^{-it}\hat{\lambda}(b)\Delta^{it}, \lambda(u_{i,j}^z{}^*)] \otimes e_{i,j} \otimes e_t \cdot dt \in \mathbb{K}(L^2(\mathbb{G})) \otimes \mathbb{B}(H_z) \otimes \mathbb{B}(L^2(\mathbb{R})),$$

and using $(\lambda \otimes \hat{\lambda})(\mathbb{V}_{21}^*)(1 \otimes p_z) = \sum_{i,j} u_{i,j}^z{}^* \otimes e_{i,j} \otimes \mathrm{id}_{H_{\bar{z}}}$ (write $W := (\lambda \otimes \hat{\lambda})(\mathbb{V}_{21})$), we further translate it as

$$\int_{\mathbb{R}} [\Delta^{-it}\hat{\lambda}(b)\Delta^{it} \otimes p_z, W^*(1 \otimes p_z)] \otimes e_t \cdot dt \in \mathbb{K}(L^2(\mathbb{G})) \otimes \mathbb{B}(H_z \otimes H_{\bar{z}}) \otimes \mathbb{B}(L^2(\mathbb{R})).$$

For simplicity, we denote it by $\int_{\mathbb{R}} \mathrm{T}_t \otimes e_t \cdot dt$. If $b \in \mathcal{B}$ is finitely supported, that is, contained in a finite direct sum of $\mathbb{B}(H_x)$ ($x \in \mathrm{Irred}(\mathbb{G})$), then this final condition holds since this term is contained in $(\oplus_{\mathrm{fin}}\mathbb{B}(H_x \otimes H_{\bar{x}})) \otimes \mathbb{B}(H_z \otimes H_{\bar{z}}) \otimes \mathbb{B}(L^2(\mathbb{R}))$. Hence we may assume that $b = \psi_{\infty,x}(A) \in \mathcal{B}$ for some $A \in \mathbb{B}(H_x)$, where $\psi_{\infty,x}(A)$ is defined by $\psi_{\infty,x}(A)p_y := \psi_{y,x}(A)$ if $y \geq x$ and 0 otherwise.

Now by the proof of [33, Proposition 3.8], for any $y \in \mathrm{Irred}(\mathbb{G})$ with $y \geq z$ we have

$$
\begin{aligned}
&\|\mathrm{T}_t(p_{x+y} \otimes p_z)\| \\
=\ & \|[\Delta^{-it}\hat{\lambda}(\psi_{\infty,x}(A))\Delta^{it} \otimes 1, W^*](p_{x+y} \otimes p_z)\| \\
=\ & \|[\hat{\lambda}(\psi_{\infty,x}(F_x^{-it}AF_x^{it})) \otimes 1, W^*](p_{x+y} \otimes p_z)\| \\
=\ & \|\{W(\hat{\lambda}(\psi_{\infty,x}(F_x^{-it}AF_x^{it})) \otimes 1)W^* - \hat{\lambda}(\psi_{\infty,x}(F_x^{-it}AF_x^{it})) \otimes 1\}(p_{x+y} \otimes p_z)\| \\
=\ & \|(\hat{\lambda} \otimes \hat{\lambda})\{\hat{\Phi}(\psi_{\infty,x}(F_x^{-it}AF_x^{it}) \otimes 1) - \psi_{\infty,x}(F_x^{-it}AF_x^{it}) \otimes 1\}(p_{x+y} \otimes p_z)\| \\
\leq\ & C(z)\|F_x^{-it}AF_x^{it}\|q^y = C(z)\|A\|q^y,
\end{aligned}
$$

where $C(z)$ and $0 < q < 1$ are constants ($C(z)$ depends on z). Since this estimate does not depend on $t \in \mathbb{R}$, we have the following norm convergent sequence

$$
\int_{\mathbb{R}} \sum_{k=0}^{y} (\mathrm{T}_t(p_{x+k} \otimes p_z)) \otimes e_t \cdot dt \to \int_{\mathbb{R}} \sum_{k=0}^{\infty} (\mathrm{T}_t(p_{x+k} \otimes p_z)) \otimes e_t \cdot dt \quad (y \to \infty).
$$

Now each element in this sequence is contained in $\mathbb{K}(L^2(\mathbb{G})) \otimes \mathbb{B}(H_z \otimes H_{\bar{z}}) \otimes \mathbb{B}(L^2(\mathbb{R}))$ and the limit element coincides with $\int_{\mathbb{R}} \mathrm{T}_t \otimes e_t \cdot dt$. Hence we can end the proof. $\square$

4 Absence of Cartan subalgebras

In this section, we prove Theorem A and B in the almost same way as [27, Theorem 3.1]. Since many proofs are same, we often omit them.

4.1 Preparation with the W^*CBAP and condition $(\mathrm{AO})^+$

Since we have similar arguments for proofs of both theorems, we first assume that M is arbitrary tracial von Neumann algebra with separable predual, and we will give other assumptions in each lemma.

Let M be a finite von Neumann algebra with separable predeal, τ a faithful normal tracial state, and let $A \subset M$ be a von Neumann subalgebra. Write $P := \mathcal{N}_M(A)''$ and set $N := M \otimes J_P P J_P \subset \mathbb{B}(L^2(M) \otimes L^2(P))$, where J_P is the modular conjugation for P and $\tau \mid_P$. Here we naturally have $L^2(N, \tau \otimes \tau) \simeq L^2(M) \otimes L^2(P)$ with $J_N = J_M \otimes J_P$. As usual, we identify JMJ and JPJ as opposite algebras M^{op} and P^{op} via natural identifications and write $\bar{a} := (a^{\mathrm{op}})^* = JaJ$ for $a \in P$ or M. Put $D := M \odot M^{\mathrm{op}} \odot P^{\mathrm{op}} \odot P$ and define two $*$-homomorphisms

$$
\Psi \colon D \longrightarrow \mathbb{B}(L^2(M) \otimes L^2(M) \otimes L^2(P)); \ a \otimes b^{\mathrm{op}} \otimes x^{\mathrm{op}} \otimes y \longmapsto a \otimes b^{\mathrm{op}} \otimes x^{\mathrm{op}}y,
$$
$$
\Theta \colon D \longrightarrow \mathbb{B}(L^2(M) \otimes L^2(P)) \qquad\quad ; \ a \otimes b^{\mathrm{op}} \otimes x^{\mathrm{op}} \otimes y \longmapsto ab^{\mathrm{op}} \otimes x^{\mathrm{op}}y.
$$

The following theorem is due to Popa and Ozawa.

Theorem 4.1.1 ([23][20])**.** *If M has the W^*CBAP and A is injective, then the conjugate action of $\mathcal{N}_M(A)$ on A is weakly compact, that is, there exists a net $(\xi_i)_i$ of unit vectors in $L^2(A) \otimes L^2(A) \subset L^2(M) \otimes L^2(P)$ satisfying the following conditions;*

(i) *$\langle (x \otimes 1)\xi_i, \xi_i \rangle \to \tau(x)$ for any $x \in M$;*

(ii) *$\|(a \otimes \bar{a})\xi_i - \xi_i\| \to 0$ for any $a \in \mathcal{U}(A)$;*

(iii) *$\|\xi_i - (u \otimes \bar{u})J_N(u \otimes \bar{u})J_N)\xi_i\| \to 0$ for any $u \in \mathcal{N}_M(A)$.*

By the proof, this net $(\xi_i)_i$ is contained in the positive cone of $L^2(A) \otimes L^2(A)$ and hence satisfies $(J_M \otimes J_M)\xi_i = \xi_i$ and $\langle (x^{\mathrm{op}} \otimes 1)\xi_i, \xi_i \rangle \to \tau(x)$ for any $x \in M$. We fix the above net $(\xi_i)_i$ and put $\Omega_1(x) := \mathrm{Lim}_i \langle x\xi_i, \xi_i \rangle$ for $x \in \mathbb{B}(L^2(M) \otimes L^2(P))$, where Lim is taken by a fixed free ultra filter. Then conditions (i) and (iii) in Theorem 4.1.1 is translated as follows:

- $\Omega_1(x \otimes 1) = \tau(x)$ for any $x \in M$,

- $\Omega_1(\Theta(u \otimes \bar{u} \otimes \bar{u} \otimes u)) = 1$ for any $u \in \mathcal{N}_M(A)$.

The condition (ii) is used only in the following lemma.

In the original paper of Popa and Vaes, the proof following lemma is very technical (whose origin is [8, Lemma 6.2]). In the present paper, we give a very simple proof which works only in a special situation. We are indebted to Eric Ricard for kindly demonstrating to the author this simple proof.

Lemma 4.1.2. *Let $(\xi_i)_i$ be a net of unit vectors in $L^2(A) \otimes L^2(A)$ satisfying conditions* (i) *and* (ii) *in Theorem 4.1.1. Assume A has a decomposition $A = B \bar{\otimes} C$ with B diffuse injective (possibly C is trivial). Then, there exists an increasing net $(p_j)_j$ of range finite projections in $\mathbb{B}(L^2(B))$ such that*

- *$p_j \to 1_B$ σ-weakly;*

- *$\limsup_i \|(p_j \otimes 1_C \otimes 1_A)\xi_i\| = 0$ for any p_j.*

Proof. Since B is diffuse injective and has separable predual, we can find a sequence (u_n) of unitaries in B satisfying

- $(u_n)_n$ converges to 0 in the σ-weak topology;

- $(\frac{1}{n} \sum_{k=1}^n u_k p u_k^*)_n$ converges to 0 in the *norm* topology for any $p \in \mathbb{K}(L^2(B))$.

Since the condition (ii) implies $\Omega_1((u_n \otimes 1_C) \otimes 1_A) = 1$, we have, for any $p \in \mathbb{K}(L^2(B))$

$$
\begin{aligned}
\Omega_1(p \otimes 1_C \otimes 1_A) &= \Omega_1(u_k p u_k^* \otimes 1_C \otimes 1_A) \\
&= \Omega_1(\frac{1}{n} \sum_{k=1}^n u_k p u_k^* \otimes 1_C \otimes 1_A) \to 0.
\end{aligned}
$$

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Hence we can find an increasing net p_j of projections in $\mathbb{K}(L^2(B))$ s.t. $\Omega_1(p_j \otimes 1_C \otimes 1_A) = 0$. Now it is easy to find our desired net. $\qquad\square$

Next, we assume that M satisfies condition $(\mathrm{AO})^+$. Put $D_0 := M_0 \odot M_0^{\mathrm{op}} \odot P^{\mathrm{op}} \odot P \subset D$, where M_0 is a σ-weakly dense C^*-subalgebra of M as in Definition 3.1.1. We use the u.c.p. lift θ only in the following simple lemma.

Lemma 4.1.3. *Let p_j be range finite projections in $\mathbb{B}(L^2(M))$ with $p_j \to 1$ σ-weakly and assume M satisfies condition $(\mathrm{AO})^+$. Then we have $\limsup_j \|\Theta(S)(p_j^\perp \otimes 1)\| \leq \|\Psi(S)\|$ for any $S \in D_0$, where $p_j^\perp := 1 - p_j$.*

Proof. Let θ be a u.c.p. lift of the multiplication map on $M_0 \otimes M_0^{\mathrm{op}}$. Then for $S := a \otimes b^{\mathrm{op}} \otimes x^{\mathrm{op}} \otimes y \in D_0$, we have

$$(\theta \otimes \mathrm{id}) \circ \Psi(S) - \Theta(S) = (\theta(a \otimes b^{\mathrm{op}}) - ab^{\mathrm{op}}) \otimes x^{\mathrm{op}}y \in \mathbb{K}(L^2(M)) \otimes \mathbb{B}(L^2(P)).$$

Since $p_j^\perp$ converges to 0 in the strong topology and $\theta(a \otimes b^{\mathrm{op}}) - ab^{\mathrm{op}} \in \mathbb{K}(L^2(M))$, the net $(\theta(a \otimes b^{\mathrm{op}}) - ab^{\mathrm{op}})p_j^\perp$ converges to 0 in the norm topology. Thus we have

$$
\begin{aligned}
\limsup_j \|\Theta(S)(p_j^\perp \otimes 1)\| &\leq \limsup_j \|((\theta \otimes \mathrm{id}) \circ \Psi(S) - \Theta(S))(p_j^\perp \otimes 1)\| + \|\Psi(S)\| \\
&= \|\Psi(S)\|.
\end{aligned}
$$

This holds for any $S \in D_0$ by the completely same manner. $\qquad\square$

4.2 Proof of Theorem A

We prove the following theorem which is a slight generalization of Theorem A.

Theorem 4.2.1. *Let M be a semifinite von Neumann algebra with separable predual and p a finite projection in M. If M satisfies condition $(\mathrm{AO})^+$ and has the W^*CBAP, then pMp is strongly solid.*

Proof. We first assume $p = 1$. Let $(\xi_i)_i$, Ω_1, D, D_0, Ψ, and Θ be as in previous subsections and take $(p_j)_j$ be an increasing net of finite projections in $\mathbb{B}(L^2(A))$ as in Lemma 4.1.2 in the case where C is trivial.

By the definition of Ω_1 and Lemma 4.1.2, we have $\Omega_1(p_j \otimes 1) = 0$ and this implies $\Omega_1(S(p_j^\perp \otimes 1)) = \Omega_1(S)$ for any $S \in \mathbb{B}(L^2(M) \otimes L^2(P))$. Hence for any $S \in D_0$, we have

$$|\Omega_1(\Theta(S))| = \limsup_j |\Omega_1(\Theta(S)(p_j^\perp \otimes 1))| \leq \limsup_j \|\Theta(S)(p_j^\perp \otimes 1)\| \leq \|\Psi(S)\|,$$

where we used Lemma 4.1.3. We can extend this inequality on D. Indeed, since M has the W^*CBAP and by Lemma 2.3.1, we can find a net $(\phi_j)_j$ of

normal c.b. map from M into M_0 with finite rank satisfying $\limsup_j \|\phi_j\|_{\text{c.b.}} =:$ $k < \infty$ and ϕ_j converges id_M in the point σ-weak topology. Then by the basic inequality $|\Omega_1(S(x \otimes 1))| \leq \|S\|\|x\|_2$ $(x \in M)$, we have

$$\lim_j \Omega_1(\Theta(\phi_j(x) \otimes \phi_j(y)^{\text{op}} \otimes a^{\text{op}} \otimes b)) = \Omega_1(\Theta(x \otimes y^{\text{op}} \otimes a^{\text{op}} \otimes b))$$

for $x, y \in M$ and $a, b \in P$. Then for any $S \in D$ we have

$$
\begin{aligned}
|\Omega_1(\Theta(S))| &= \limsup_i |\Omega_1(\Theta \circ (\phi_i \otimes \phi_i^{\text{op}} \otimes \text{id} \otimes \text{id})(\text{S}))| \\
&\leq \limsup_i \|\Psi \circ (\phi_i \otimes \phi_i^{\text{op}} \otimes \text{id} \otimes \text{id})(\text{S})\| \\
&= \limsup_i \|(\phi_i \otimes \phi_i^{\text{op}} \otimes \text{id}) \circ \Psi(\text{S})\| \\
&\leq k^2 \|\Psi(S)\| \quad (S \in D).
\end{aligned}
$$

Thus we can extend the inequality. We note that this extension result holds directly by local reflexivity of M_0, but here we follow the manner of Popa and Vaes.

Now we can define a state Ω_2 on $C^*(\Psi(D))$ by $\Omega_2(\Psi(S)) = \Omega_1(\Theta(S))$. Conditions of Ω_1 are easily translated as follows:

- $\Omega_2(x \otimes 1) = \tau(x)$ for $x \in M$,

- $\Omega_2(\Psi(u \otimes \bar{u} \otimes \bar{u} \otimes u)) = 1$ for $u \in \mathcal{N}_M(A)$.

Extend Ω_2 on $\mathbb{B}(L^2(M) \otimes L^2(M) \otimes L^2(P))$ by the Hahn–Banach Theorem and still denote it by Ω_2. Now the restriction of Ω_2 on $\mathbb{B}(L^2(M)) \otimes 1 \otimes 1$ gives an amenable trace of P, that is, Ω_2 is $(P \otimes 1)$-central on $\mathbb{B}(L^2(M)) \otimes 1 \otimes 1$. To see this, observe that $\Psi(u \otimes \bar{u} \otimes \bar{u} \otimes u)$ $(u \in \mathcal{N}_M(A))$ is contained in the multiplitative domain of Ω_2 (e.g. [4, Proposition 1.5.7]), that is, $\Omega_2(S\Psi(u \otimes \bar{u} \otimes \bar{u} \otimes u)) = \Omega_2(\Psi(u \otimes \bar{u} \otimes \bar{u} \otimes u)S)$ for any $S \in \mathbb{B}(L^2(M) \otimes L^2(M) \otimes L^2(P))$. Then it is easily checked that Ω_2 is $(P \otimes 1)$-central and this implies injectivity of P.

Now we prove the general case but it is routine. The pair $pMp \subset \mathbb{B}(L^2(pMp))$ satisfies the same conditions as the pair for $p = 1$ except for Lemma 4.1.3. This lemma follows from the same manner if one notice that $L^2(pMp)$ is isomorphic as a standard representation to $pJpJL^2(M)$. $\qquad\square$

4.3 Proof of Theorem B

We next prove Theorem B with a very similar argument. We actually prove the following generalized theorem.

Theorem 4.3.1. *Let M be a von Neumann algebra with separable predual and ϕ a faithful normal state on M. Let $N \subset M$ be a diffuse non-injective von Neumann subalgebra with a faithful normal expectation E_N which preserves ϕ. If (M, ϕ) satisfies condition $(\text{AOC})^+$ and has the W^*CBAP, then N has no ϕ-Cartan subalgebras.*

Proof. Let B be a ϕ-Cartan subalgebra of N and E_B a ϕ-preserving faithful normal conditional expectation from N onto B. Taking crossed products by $\mathbb{R}$ with the modular action of ϕ, we have $B \otimes L\mathbb{R} \subset N \rtimes \mathbb{R} := \mathcal{N} \subset M \rtimes \mathbb{R} := \mathcal{M}$ and denote by Tr the canonical trace. Then we can find a non zero projection p in $L\mathbb{R} \subset \mathcal{M}$ with $\mathrm{Tr}(p) < \infty$ such that $p\mathcal{N}p$ is still non-injective. We will apply the same argument to the inclusion $B \otimes pL\mathbb{R}p \subset p\mathcal{M}p$.

Let $(\xi_i)_i$ be a net as in Theorem 4.1.1 ($\mathcal{M}$ has the W*CBAP in this setting, see [1, 4.10] for example). We apply Lemma 4.1.2 with $C = pL\mathbb{R}p$ so that we have an increasing net $(p_j)_j$ of range finite projections in $\mathbb{B}(L^2(B))$ with some conditions. Then by the existence of a u.c.p. lift in condition $(\mathrm{AOC})^+$, we have the conclusion of Lemma 4.1.3, that is, $\limsup_j \|\Theta(S)(p_j^\perp \otimes p \otimes 1_{p\mathcal{M}p})\| \leq \|\Psi(S)\|$ for any $S \in D_0$, where we regard $A \rtimes_r \mathbb{R}$ as M_0. Then the same manner as in the previous subsection says that $\mathcal{N}_{p\mathcal{M}p}(B \otimes pL\mathbb{R}p)''$ is injective. Recall $B \otimes pL\mathbb{R}p \subset p\mathcal{N}p$ is a Cartan subalgebra (Subsection 2.1) and hence $p\mathcal{N}p = \mathcal{N}_{p\mathcal{N}p}(B \otimes pL\mathbb{R}p)'' \subset \mathcal{N}_{p\mathcal{M}p}(B \otimes pL\mathbb{R}p)''$. Since $\mathcal{N}_{p\mathcal{M}p}(B \otimes pL\mathbb{R}p)''$ is finite, there exists a conditional expectation from $\mathcal{N}_{p\mathcal{M}p}(B \otimes pL\mathbb{R}p)''$ onto $p\mathcal{N}p$, and this implies $p\mathcal{N}p$ is injective. Thus contradiction. $\square$

5 Semisolidity of continuous cores

In the section, we prove Theorem C. Our proof is a variant of the proof of [19, Theorem 4.6] and is very similar to that of [17, Theorem 5.3.3].

5.1 Proof of Theorem C

For simplicity, we write the core of M as $\mathcal{M} := M \rtimes_{\sigma^\phi} \mathbb{R}$ and identify $L^2(\mathcal{M}) = L^2(M) \otimes L^2(\mathbb{R})$. Let N be a type II$_1$ subalgebra of $p\mathcal{M}p$. Since N contains a copy of the AFD II$_1$ factor, we may assume that N itself is the AFD II$_1$ factor. Then N has finite dimensional unital C^*-subalgebras N_n ($n \in \mathbb{N}$) with $N_n \subset N_{n+1}$ such that $N = (\bigcup_n N_n)''$. Define a conditional expectation from $\mathbb{B}(L^2(\mathcal{M}))$ onto $N' \cap p\mathbb{B}(L^2(\mathcal{M}))p = N'p$ by

$$\Psi_N(x) := \lim_{n \to \omega} \int_{\mathcal{U}(N_n)} uxu^* du,$$

where du is the normalized Haar measure on $\mathcal{U}(N_n)$, ω is a fixed free ultra filter, and the limit is taken by the σ-weak topology. Then Ψ_N satisfies a properness condition

$$\Psi_N(x) \in \overline{\mathrm{co}}\{uxu^* \mid u \in \mathcal{U}(N)\}$$

for any $x \in \mathbb{B}(H)$, where the closure is taken by the σ-weak topology. We first prove that $\mathbb{K}(L^2(M)) \otimes \mathbb{B}(L^2(\mathbb{R})) \subset \ker \Psi_N$.

For this, it suffice to show that $\Psi_N(q \otimes 1) = 0$ for any $q \in \mathbb{K}(L^2(M))$. Since the image of $JMJ \ni JaJ \mapsto J\hat{a} \in L^2(M)$ is dense, we may assume $q = J\hat{a} \otimes J\hat{b}$ (with the Hilbert–Schmidt correspondence). Then since we have

$$
\begin{aligned}
\Psi_N((J\hat{a} \otimes J\hat{b}) \otimes 1) &= \Psi_N((JaJ\hat{1} \otimes JbJ\hat{1}) \otimes 1) \\
&= \Psi_N(JaJ(\hat{1} \otimes \hat{1})Jb^*J \otimes 1) \\
&= (JaJ \otimes 1)\Psi_N((\hat{1} \otimes \hat{1}) \otimes 1)(Jb^*J \otimes 1),
\end{aligned}
$$

we may further assume $a = b = 1$. Write $e := \hat{1} \otimes \hat{1}$.

Let q be a projection in $L\mathbb{R}$ with $\hat{\phi}(q) < \infty$ and $p \le q$. Put $\tilde{q} := \tilde{J}q\tilde{J} \in \mathcal{M}' \subset N'$, $\widetilde{N} := N\tilde{q}$ and $\Psi_{\widetilde{N}}(x) := \tilde{q}\Psi_N(x)\tilde{q}$ for $x \in \mathbb{B}(L^2(\mathcal{M}))$. We actually prove $\Psi_{\widetilde{N}}(e \otimes 1) = 0$, and this means $\Psi_N(e \otimes 1) = 0$ since q runs over sufficiently large projections.

Since e commutes with Δ_ϕ^{it} and since Ψ_N is proper, $\Psi_N(e \otimes 1)$ commutes with $\Delta_\phi^{it} \otimes \rho_t$ and so it is contained in $pN' \cap \rho(L\mathbb{R})'$, where $\rho(\lambda_t) := \Delta_\phi^{it} \otimes \rho_t$. Hence $\Psi_{\widetilde{N}}(e \otimes 1)$ is contained in $\tilde{q}(pN' \cap \rho(L\mathbb{R})')\tilde{q} \subset (N\tilde{q})' \cap \rho(qL\mathbb{R}q)'$.

Let r be any spectral projection of $\Psi_{\widetilde{N}}(e \otimes 1)$. Then $rL^2(\mathcal{M})q (= rqL^2(\mathcal{M})q \simeq rL^2(q\mathcal{M}q)$ has a natural N-$qL\mathbb{R}q$-submodule structure of $L^2(q\mathcal{M}q)$ since r is also contained in $(N\tilde{q})' \cap \rho(qL\mathbb{R}q)'$. By the comment below Theorem 2.2.1, we have $N \not\preceq_{q\mathcal{M}q} qL\mathbb{R}q$, and hence $\dim_{qL\mathbb{R}q} rL^2(\mathcal{M})q$ is zero or infinite.

Let W be the unitary on $L^2(\mathcal{M}) = L^2(M) \otimes L^2(\mathbb{R})$ given by $(W\xi)(t) := \Delta_\phi^{it}\xi(t)$. Then easy calculations show that

$$
W(1 \otimes \lambda_t)W^* = \Delta^{it} \otimes \lambda_t, \quad W(\Delta^{it} \otimes \rho_t)W^* = 1 \otimes \rho_t
$$

and hence we have

$$
\begin{aligned}
\rho(qL\mathbb{R}q)' \cap \mathbb{B}(L^2(\mathcal{M})q) &= \tilde{q}\rho(L\mathbb{R})'\tilde{q} \\
&= \tilde{q}\{W^*(1 \otimes R\mathbb{R})W\}'\tilde{q} \\
&= W^*(1 \otimes \bar{q}R\mathbb{R})'W \quad (1 \otimes \bar{q} := W\tilde{q}W^*) \\
&= W^*(\mathbb{B}(L^2(M)) \otimes \bar{q}L\mathbb{R})W \\
&\subset W^*\mathbb{B}(L^2(M) \otimes \bar{q}L^2(\mathbb{R}))W,
\end{aligned}
$$

where $R\mathbb{R}$ is the von Neumann algebra generated by ρ_t ($t \in \mathbb{R}$). We note that $\bar{q} = J_{L\mathbb{R}}qJ_{L\mathbb{R}}$, $\bar{q}L^2(\mathbb{R}) = L^2(\mathbb{R})q$, and the natural right action of $qL\mathbb{R}q$ induced from the above isomorphism is of the form $1 \otimes \bar{q}\rho_t$ ($t \in \mathbb{R}$). Hence $\dim_{qL\mathbb{R}q} rL^2(\mathcal{M})q$ coincides with that of $WrW^*(L^2(M) \otimes L^2(\mathbb{R})q)$ with respect to the trace $\mathrm{Tr}_{L^2(M)} \otimes \mathrm{Tr}_{qL\mathbb{R}q}$, where $\mathrm{Tr}_{qL\mathbb{R}q}$ is defined by $\mathrm{Tr}_{L\mathbb{R}}(\cdot)/\mathrm{Tr}_{L\mathbb{R}}(q)$. Now the value of this is smaller than $\mathrm{Tr}_{L^2(M)} \otimes \mathrm{Tr}_{qL\mathbb{R}q}(WrW^*)$

and this is finite since

$$
\begin{aligned}
(\mathrm{Tr}_{L^2(M)} \otimes \mathrm{Tr}_{qL\mathbb{R}q})(WrW^*) &\leq C(\mathrm{Tr}_{L^2(M)} \otimes \mathrm{Tr}_{qL\mathbb{R}q})(W\Psi_{\widetilde{N}}(e \otimes 1)W^*) \\
&\leq C(\mathrm{Tr}_{L^2(M)} \otimes \mathrm{Tr}_{qL\mathbb{R}q})(W\tilde{q}(e \otimes 1)\tilde{q}W^*) \\
&= C(\mathrm{Tr}_{L^2(M)} \otimes \mathrm{Tr}_{qL\mathbb{R}q})((1 \otimes \bar{q})W(e \otimes 1)W^*(1 \otimes \bar{q})) \\
&= C(\mathrm{Tr}_{L^2(M)} \otimes \mathrm{Tr}_{qL\mathbb{R}q})((1 \otimes \bar{q})(e \otimes 1)(1 \otimes \bar{q})) \\
&= C\mathrm{Tr}_{L^2(M)}(e)\mathrm{Tr}_{qL\mathbb{R}q}(\bar{q}) < \infty,
\end{aligned}
$$

where C is a positive constant and we used properness of Ψ_N. Hence we have $r = 0$ and $\Psi_N(e \otimes 1) = 0$. Thus we proved the claim.

Composing with a u.c.p. map θ as in the definition of condition $(\mathrm{AOC})^+$, we have a well define map

$$
\Phi := \Psi_N \circ \theta \colon A \rtimes_r \mathbb{R} \otimes (A \rtimes_r \mathbb{R})^{\mathrm{op}} \longrightarrow \mathbb{B}(L^2(\mathcal{M})) \longrightarrow N' \cap \mathbb{B}(L^2(\mathcal{M})),
$$

and observe that $\Phi(xy^{\mathrm{op}}) = \Psi_N(x)y^{\mathrm{op}}$. Since $A \rtimes_r \mathbb{R}$ is locally reflexive, Φ can be extended on $(A \rtimes_r \mathbb{R})^{**} \otimes (A \rtimes_r \mathbb{R})^{\mathrm{op}}$. Since Ψ_N is proper, it is a trace preserving map on $p\mathcal{M}p$ and hence is normal on $\mathcal{M}$. Then the restriction of Φ on $z(A \rtimes_r \mathbb{R})^{**} \otimes (A \rtimes_r \mathbb{R})^{\mathrm{op}}$ coincides with the extension of Φ on $\mathcal{M} \otimes (A \rtimes_r \mathbb{R})^{\mathrm{op}}$, by using the identification $\mathcal{M} \simeq z(A \rtimes_r \mathbb{R})^{**}$. Thus Φ is min-bounded on $\mathcal{M} \otimes (A \rtimes_r \mathbb{R})^{\mathrm{op}}$.

By Arveson's extension theorem, we can extend Φ on $\mathbb{B}(L^2(\mathcal{M})) \otimes (A \rtimes_r \mathbb{R})^{\mathrm{op}}$. Then the restriction of this map on $\mathbb{B}(L^2(\mathcal{M})) \otimes \mathbb{C}$, say $\widetilde{\Psi}_N$, gives an extension of Ψ_N and the image of this map is contained in $((A \rtimes_r \mathbb{R})^{\mathrm{op}})' = \mathcal{M}$ (e.g. [4, Proposition 3.6.5]). Finally $\Psi_N \circ \widetilde{\Psi}_N$ is a conditional expectation from $\mathbb{B}(L^2(\mathcal{M}))$ onto $N' \cap p\mathcal{M}p$.

Remark 5.1.1. By the same manner, we can prove that for any (possibly non-unital) subalgebra $N \subset p\mathcal{M}p$, the relative commutant $N' \cap 1_N\mathcal{M}1_N$ is injective. Then we can deduce primeness of $p\mathcal{M}p$ for any non-injective von Neumann algebra M (possibly not type III$_1$) with condition $(\mathrm{AOC})^+$ and any large projection p (e.g. the proof of [13, Proposition 2.7]). For example, $p(L^\infty(\mathbb{G}) \rtimes_{\sigma^h} \mathbb{R})p$ is prime (or injective) for any projection $p \in L\mathbb{R}$ with $\mathrm{Tr}(p) < \infty$, where $\mathbb{G} = A_o(F)$ or $A_u(F)$.

5.2 A remark on solidity and centralizer algebras

In the previous subsection, we proved semisolidity of continuous cores of some type III$_1$ factors. This property itself has nothing to say about original type III$_1$ factors at a first glance, but it has an interesting application once we get a stronger property, namely, solidity of the continuous cores. Indeed Houdayer gave the following observation [14, Subsection 3.3].

Let M be a type III$_1$ factor and assume that the continuous core $M \rtimes \mathbb{R}$ is solid as a II$_\infty$ factor. Let ϕ be a faithful normal state on M and M_ϕ be the centralizer of ϕ (see for example [30, Definition VIII.2.1]). Then

by Takesaki's conditional expectation theorem [30, Theorem IX.4.2], there exists the unique ϕ-preserving conditional expectation E from M onto M_ϕ. Hence by the observation in Subsection 2.1, we have $M_\phi \otimes L\mathbb{R} = M_\phi \rtimes_{\sigma^\phi} \mathbb{R} \subset M \rtimes_{\sigma^\phi} \mathbb{R}$. Then since $L\mathbb{R}$ is diffuse, it is easy to see that solidity of the continuous core forces all the centralizers (with respect to states) to be injective.

To apply this observation to our main objects, we next recall Connes' discrete decomposition of full type III factors [6, Section 4] (See also [9, Section 2]).

Theorem 5.2.1. *Let M be a full type* III *factor with separable predual and ϕ a faithful normal state on M. Assume that ϕ is almost periodic. Then there exists a decomposition $M \simeq (M_\phi \otimes \mathbb{B}(H)) \rtimes \Gamma$ with a separable Hilbert space H and an abelian discrete group Γ. In particular M_ϕ is a non-injective subalgebra of M.*

We turn to see our main objects. Let $\mathbb{G}$ be a universal quantum group $A_o(F)$ for $F \in \mathrm{GL}(n, \mathbb{C})$ ($n \geq 3$) with $F\bar{F} = \pm 1$. Assume that $\|F\|^2 \leq \mathrm{Tr}(FF^*)/\sqrt{5}$ and recall from [33, Theorem 7.1] that $L^\infty(\mathbb{G})$ then satisfies following conditions:

- $L^\infty(\mathbb{G})$ is a full factor and the Haar state h is almost periodic;

- for the subgroup Γ of $\mathbb{R}_+^*$ generated by eigenvalues of $Q \otimes Q^{-1}$, where $Q^{-1} := FF^*$, $L^\infty(\mathbb{G})$ is of type II$_1$ if $FF^* = 1$; of type III$_\lambda$ ($0 < \lambda < 1$) if $\Gamma = \lambda^{\mathbb{Z}}$; of type III$_1$ in the other cases.

In the case where $FF^* \neq 1$, we have non-injectivity of $L^\infty(\mathbb{G})_h$ by the theorem above. Hence Houdayer's observation says that the continuous core of $L^\infty(\mathbb{G})$ is not solid. We summary this result as follows.

Corollary 5.2.2. *Let $\mathbb{G}$ be a universal discrete group $A_o(F)$ for $F \in \mathrm{GL}(n, \mathbb{C})$ ($n \geq 3$) with $F\bar{F} = \pm 1$. Assume that $\|F\|^2 \leq \mathrm{Tr}(FF^*)/\sqrt{5}$ and $FF^* = 1$. Denote the continuous core of $L^\infty(\mathbb{G})$ by $\mathcal{M}$ and the canonical trace on $\mathcal{M}$ by Tr. Then for any Tr-finite projection p in $L\mathbb{R}$ with $p\mathcal{M}p$ non-injective, $p\mathcal{M}p$ is semisolid but never solid.*

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