On the Asymptotic Expansion Method and its Applications

漸近展開法とその応用について

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博士 学位論文

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Contents

Introduction

This paper presents a new scheme for computation in the method so-called "an asymptotic expansion approach" and extensions of the method with various financial applications and numerical examples.

The 'asymptotic expansion method' was firstly introduced to a financial literature by [11] and [28] with an application to the evaluation of an average option that is a popular derivative in commodity markets. They derive the approximation formulas for the average option by the asymptotic expansion method based on log-normal approximations of a distribution of an average price when an underlying asset price follows a geometric Brownian motion. [45] applies a formula derived through the asymptotic expansion of certain statistical estimators for small diffusion processes to approximating average option prices. Thereafter, the asymptotic expansion have been applied to a broad class of problems in finance: See [29], [30], Kunitomo and Takahashi [12], [13], Matsuoka, Takahashi and Uchida [19], Takahashi and Yoshida [40], [41], Muroi [20], and Takahashi and Takehara [31], [32], [33].

It is notable that the method has flexible applicability to a broad class of diffusion-type stochastic settings in a unified way, and mathematical justification by Watanabe theory(Watanabe [43], Yoshida [44]) in Malliavin calculus.

There are also other various approaches for approximation of solutions to pricing PDEs, Greeks and heat kernels through certain asymptotic expansions: for instance, there are recent works such as Fouque, Papanicolaou and Sircar [6], [7], Hagan, Kumar, Lesniewski and Woodward [9], Henry-Labordere [14], [15], Siopacha and Teichmann [27], Ben Arous and Laurence [3] and Gatheral, Hsu, Laurence, Ouyang and Wang [8].

Recently, not only academic researchers but also many practitioners such as Antonov and Misirpashaev [1] or Andersen and Hutchings [2] have used the asymptotic expansion method based on Watanabe theory in or combined with their techniques for a variety of financial issues. e.g. pricing or hedging complex derivatives under high-dimensional underlying stochastic environments. These methods fully or partially rely on the framework developed by [11], [28], [29] in a financial literature.

In theory, this method provides us the expansion, which has a proper meaning in the limit of some ideal situations such as cases where these processes would be deterministic, of underlying stochastic processes (for the detail see [43], [44] or [13]). In practice, however, we are often interested in cases far from those situations, where the underlying processes are highly volatile as seen in recent financial markets especially after the crisis on 2008. Then from view points of accuracy or stability of the techniques in practical uses, it is desirable to investigate behaviors of its estimators especially with expansion up to high orders in such environments.

In Chapter 1, we introduce a new computational scheme for an asymptotic expansion method of an arbitrary order, based on the result in [34], [35], [37], and [38]. In the existing application of the asymptotic expansion based on Watanabe theory, they calculated certain conditional expectations which appear in their expansions and which play key roles in computation, by formulas up to the third order given explicitly in [28], [29] and [31]. In many applications, these formulas give sufficiently accurate approximation, but in some cases, for example in cases with long maturities or/and with highly volatile underlying variables, the approximations up to the third order may not provide satisfactory accuracies. Thus, formulas for higher-order computations are desirable. But to our knowledge, the asymptotic expansion formulas higher than the third order in a general setting have not been given yet. This paper provides a new scheme for computing unconditional expectations which is completely equivalent to direct calculation of the conditional expectations (Lemma 2). This enables us to derive the high-order approximation formulas in an automatic manner (Theorem 2 and Theorem 3).

In Chapter 2, we present various applications and practical techniques of the asymptotic expansion method including the high-order expansion introduced in Chapter 1 with numerical examples. In Section 2.1, we apply new computation algorithm in Chapter 1 to the concrete financial models, and confirms effectiveness of the high-order expansions by numerical examples in the (λ-)SABR model.

Section 2.2 applies the high-order expansion scheme to pricing average options. This section is based on the result in [26]. In particular, we describe the method using numerical examples under the λ-SABR and SABR models and show that the fourth asymptotic expansion scheme provides sufficiently accurate approximations.

In Section 2.3, we develop slightly different expansions from the usual asymptotic expansion, which have the benchmark distribution other than the normal distribution. Namely, asymptotic expansions around the log-normal distribution, the shifted log-normal distribution, and a expansion around the jump diffusion process are introduced with numerical examples. 2.4 presents an extension of a general computational scheme of an asymptotic expansion described in Chapter 1. This section is based on the result in [36]. In particular, through change of variable technique as well as the various ways of setting perturbation parameters in an expansion, we provide flexibility of setting the benchmark distribution around which the expansion is made. We also show some concrete examples with numerical experiment.

Chapter 1

A General Computation Scheme for the Asymptotic Expansion Method

In this chapter, we introduce a new scheme for computation in the asymptotic expansion method. This chapter is based on the paper [34], [35], [37], and [38].

1.1 An Asymptotic Expansion in a General Diffusion Setting

This section briefly describes an asymptotic expansion method in a general diffusion setting.

Let (W, P) be a r-dimensional Wiener space. We consider a d-dimensional diffusion process $X_t^{(\epsilon)} = (X_t^{(\epsilon),1}, \cdots, X_t^{(\epsilon),d})'$ which is the solution to the following stochastic differential equation:

$$\begin{aligned} dX_t^{(\epsilon),j} &= V_0^j(X_t^{(\epsilon)}, \epsilon)dt + \epsilon V^j(X_t^{(\epsilon)})dW_t \quad (j = 1, \cdots, d) \\ X_0^{(\epsilon)} &= x_0 \in \mathbf{R}^d \end{aligned} \tag{1.1}$$

where $W = (W^1, \cdots, W^r)'$ is a r-dimensional standard Wiener process, and $\epsilon \in (0, 1]$ is a known parameter.

Suppose that $V_0 = (V_0^1, \cdots, V_0^d)' : \mathbf{R}^d \times (0, 1] \mapsto \mathbf{R}^d$ and $V = (V^1, \cdots, V^d)$: $\mathbf{R}^d \mapsto \mathbf{R}^d \otimes \mathbf{R}^r$ satisfy some regularity conditions(for example, V_0 and V are smooth functions with bounded derivatives of all orders).

Next, let a function $g : \mathbf{R}^d \mapsto \mathbf{R}$ be smooth and all of its derivatives have polynomial growth. Then, a smooth Wiener functional $g(X_T^{(\epsilon)})$ has its asymptotic expansion:

$$g(X_T^{(\epsilon)}) \sim g_{0T} + \epsilon g_{1T} + \epsilon^2 g_{2T} + \cdots$$

in $\mathbf{D}^\infty$ as $\epsilon \downarrow 0$ where $g_{0T}, g_{1T}, g_{2T}, \cdots \in \mathbf{D}^\infty$. For any $k \in \mathbf{N}$, $q \in (1, \infty)$ and $s > 0$, this expansion means that

$$\frac{1}{\epsilon^k}\|g(X_T^{(\epsilon)}) - (g_{0T} + \epsilon g_{1T} + \cdots + \epsilon^{k-1} g_{k-1,T})\|_{q,s} = O(1) \ (as\ \epsilon \downarrow 0),$$

where $\|G\|_{q,s}$ represents the sum of L^q -norms of Malliavin derivatives of a Wiener functional G up to the s-th order. Further, a Banach space $\mathbf{D}_{q,s} = \mathbf{D}_{q,s}(\mathbf{R})$ can be regarded as the totality of random variables bounded with respect to (q,s)-norm $\|\cdot\|_{q,s}$, and $\mathbf{D}^\infty = \cap_{s>0} \cap_{1<q<\infty} \mathbf{D}_{q,s}$.

Coefficients $g_{nT} \in \mathbf{D}^\infty (n = 0, 1, \cdots)$ in the expansion can be obtained by Taylor's formula and represented based on multiple Wiener-Itô integrals. See chapter V of Ikeda and Watanabe [10] for the detail.

Let $A_{kt} = \frac{1}{k!}\frac{\partial^k X_t^{(\epsilon)}}{\partial \epsilon^k}|_{\epsilon=0}$ and A_{kt}^j, $j = 1, \cdots, d$ denote the j-th elements of A_{kt}. In particular, A_{1t} is represented by

$$A_{1t} = \int_0^t Y_t Y_u^{-1} \left(\partial_\epsilon V_0(X_u^{(0)}, 0)du + V(X_u^{(0)})dW_u \right) \tag{1.2}$$

where Y denotes the solution to the ordinary differential equation:

$$dY_t = \partial V_0(X_t^{(0)}, 0)Y_t dt; \ Y_0 = I_d.$$

Here, ∂V_0 denotes the $d \times d$ matrix whose (j,k)-element is $\partial_k V_0^j = \frac{\partial V_0^j(x,\epsilon)}{\partial x_k}$, V_0^j is the j-th element of V_0, and I_d denotes the $d \times d$ identity matrix.

For $k \geq 2$, A_{kt}^j, $j = 1, \cdots, d$ is recursively determined by the following equation:

$$\begin{aligned} A_{kt}^j &= \frac{1}{k!}\int_0^t \partial_\epsilon^k V_0^j(X^{(0)}, 0)du \\ &\quad + \sum_{l=1}^{k} \sum_{\vec{l}_\beta, \vec{d}_\beta}^{(l)} \frac{1}{(k-l)!}\frac{1}{\beta!}\int_0^t \left(\prod_{j=1}^{\beta} A_{l_j u}^{d_j} \right) \partial_{\vec{d}_\beta}^\beta \partial_\epsilon^{k-l} V_0^j(X_u^{(0)}, 0)du \\ &\quad + \sum_{\vec{l}_\beta, \vec{d}_\beta}^{(k-1)} \frac{1}{\beta!}\int_0^t \left(\prod_{j=1}^{\beta} A_{l_j u}^{d_j} \right) \partial_{\vec{d}_\beta}^\beta V^j(X_u^{(0)})dW_u \end{aligned} \tag{1.3}$$

where $\partial_\epsilon^l = \frac{\partial^l}{\partial \epsilon^l}$, $\partial_{\vec{d}_\beta}^\beta = \frac{\partial^\beta}{\partial x_{d_1} \cdots \partial x_{d_\beta}}$,

$$L_{n,\beta} = \left\{ \vec{l}_\beta = (l_1, \cdots, l_\beta); \sum_{j=1}^{\beta} l_j = n, l_j \geq 1, j = 1, \cdots, \beta \right\} \tag{1.4}$$

and

$$\sum_{\vec{l}_\beta, \vec{d}_\beta}^{(n)} = \sum_{\beta=1}^{n} \sum_{\vec{l}_\beta \in L_{n,\beta}} \sum_{\vec{d}_\beta \in \{1,\cdots,d\}^\beta}$$

for $n \geq 1$, and

$$\sum_{\vec{l}_\beta,\vec{d}_\beta}^{(0)} = \sum_{\beta=0} \sum_{\vec{l}_0=(\emptyset)} \sum_{\vec{d}_0=(\emptyset)} .$$

Then, g_{0T} and g_{1T} can be written as

$$\begin{aligned} g_{0T} &= g(X_T^{(0)}), \\ g_{1T} &= \sum_{j=1}^{d} \partial_j g(X_T^{(0)}) A_{1T}^j. \end{aligned}$$

For $n \geq 2$, g_{nT} is expressed as follows:

$$g_{nT} = \sum_{\vec{l}_\beta,\vec{d}_\beta}^{(n)} \frac{1}{\beta!} \partial_{\vec{d}_\beta}^{\beta} g(X_T^{(0)}) A_{l_1T}^{d_1} \cdots A_{l_\beta T}^{d_\beta}. \tag{1.5}$$

Here, we note that each $A_{lt}^j (j = 1, \cdots, d, l = 1, 2, \cdots, k, 0 \leq t \leq T)$ (and thus each g_{nT}s) has all finite moments due to a *grading* structure as follows: Consider the stochastic differential equation of the form

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t; \; S_0 = s_0 \in \mathbf{R}^d \tag{1.6}$$

where $\mu : \mathbf{R}^d \times \mathbf{R}^+ \to \mathbf{R}^d$ and $\sigma : \mathbf{R}^d \times \mathbf{R}^+ \to \mathbf{R}^d \otimes \mathbf{R}^r$.

Definition 1. *A grading of $\mathbf{R}^d$ is a decomposition $\mathbf{R}^d = \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_q}$ with $d = d_1 + \cdots + d_q$. The coordinates of a point in $\mathbf{R}^d$ are always arranged in an increasing order along the subspace $\mathbf{R}^{d_i}$, and we set $M_0 = 0$ and $M_l = d_1 + \cdots + d_l$ for $1 \leq l \leq q$. We say that the coefficients μ and σ are graded according to the grading $\mathbf{R}^d = \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_q}$ if $\mu^i(s,t)$ and $\sigma_j^i(s,t), j = 1, \cdots, r$ depend upon only through the coordinates $(s^k)_{1 \leq k \leq M_p}$ when $M_{p-1} \leq i \leq M_p$.*

Theorem 1. *We assume the coefficients μ and σ in (1.6) have a Lipschitz lower triangular structure, and are graded according to $\mathbf{R}^d = \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_q}$. Moreover for $F(s,t) = \mu(s,t)$ or $\sigma_j(s,t), j = 1, \cdots, r$, we assume F is differentiable in s in $\mathbf{R}^d$ and*

1. *$|F^i(0,t)| \leq Z_t$ for $i = 1, \cdots, d$*
2. *$|\frac{\partial}{\partial s^j} F^i(s,t)| \leq \hat{Z}_t(1 + |s|^\theta)$ for all i, j*
3. *$|\frac{\partial}{\partial s^j} F^i(s,t)| \leq \zeta$ if $M_{p-1} \leq i, j \leq M_p$ for some $p \leq q$*

where $\zeta, \theta \geq 0$ are constants, and Z, $\hat{Z}$ are predictable processes such that $\|Z\|_p$ and $\|\hat{Z}\|_p$ are finite for all $p \geq 1$ where $\|Z\|_p = \left\{ \int_0^T E[|Z_t|^p]dt \right\}^{1/p}$. Then (1.6) have a unique solution S, and for every $p \geq 1$ there are constants c_p and γ_p depending only upon $(\zeta, \theta, \{||\hat{Z}||_{p'}\}_{p' \geq 1})$, such that

$$|| \sup_{0 \leq t \leq T} S_t||_{L^p} \leq c_p(s_0 + ||Z||_{\gamma_p}).$$

For the detail of the definition and theorem above, see pp.45-47 in Bichteler, Gravereaux and Jacod [4].

Applying Theorem 1 to the system of stochastic differential equations consists of $A^i_{lt}(i = 1, \cdots, d, l = 1, \cdots, k, 0 \leq t \leq T)$ and any products of them, we obtain the following lemma.

Lemma 1. *Each coefficient of the expansion $A^i_{lt}(i = 1, \cdots, N, l = 1, \cdots, k, 0 \leq t \leq T)$ has all finite moments.*

(proof) Consider the system of stochastic differential equations which $A^1_1, \cdots, A^d_1$, $A^1_1A^1_1, \cdots, A^d_1A^d_1$, $A^1_2, \cdots, A^d_2, \cdots$ follow. Note that the system of equations is linear and the coefficients of the linear equations are represented by the derivatives at $\epsilon = 0$ of $\tilde{V}_0(X^{(\epsilon)}_u, \epsilon)$ and $\tilde{V}(X^{(\epsilon)}_u)$ which are bounded in $[0, T]$. Then it is easily shown that the coefficients of the equation have a grading structure and satisfy the conditions in Theorem 1. Hence the coefficients A^i_{kt} have all finite moments.□

Next, let normalize $g(X^{(\epsilon)}_T)$ to

$$G^{(\epsilon)} = \frac{g(X^{(\epsilon)}_T) - g_{0T}}{\epsilon}$$

for $\epsilon \in (0, 1]$. Then, we have

$$G^{(\epsilon)} \sim g_{1T} + \epsilon g_{2T} + \cdots$$

in $\mathbf{D}^\infty$. Moreover, let

$$\hat{V}(x, t) = (\partial g(x))'[Y_T Y_t^{-1} V(x)]$$

and make the following assumption:

$$\text{(Assumption 1)} \quad \Sigma_T = \int_0^T \hat{V}(X^{(0)}_t, t)\hat{V}(X^{(0)}_t, t)' dt > 0.$$

Note that g_{1T} follows a normal distribution with variance Σ_T; the density function of g_{1T} denoted by $f_{g_{1T}}(x)$ is given by

$$f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp\left(-\frac{(x - C)^2}{2\Sigma_T}\right)$$

where

$$C := \left(\partial g(X^{(0)}_T)\right)' \int_0^T Y_T Y_t^{-1} \partial_\epsilon V_0(X^{(0)}_t, 0) dt. \tag{1.7}$$

Hence, Assumption 1 means that the distribution of g_{1T} does not degenerate. In application, it is easy to check this condition in most cases. Hereafter, Let $\mathcal{S}$ be the real Schwartz space of rapidly decreasing $\mathbf{C}^\infty$-functions on $\mathbf{R}$ and $\mathcal{S}'$ be its dual space that is the space of the Schwartz tempered distributions. Next, take $\Phi \in \mathcal{S}'$. Then, by Watanabe theory(Watanabe [43], Yoshida [44]) a generalized Wiener functional $\Phi(G^{(\epsilon)})$ has an asymptotic expansion in $\mathbf{D}^{-\infty}$ as $\epsilon \downarrow 0$ where

$\mathbf{D}^{-\infty}$ denotes the set of generalized Wiener functionals. See chapter V of Ikeda and Watanabe [10] for the detail. Hence, the expectation of $\Phi(G^{(\epsilon)})$ is expanded around $\epsilon = 0$ as follows: For $N = 0, 1, 2, \cdots$,

$$\begin{aligned}
\mathbf{E}[\Phi(G^{(\epsilon)})] &= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \mathbf{E}\left[\Phi^{(\delta)}(g_{1T}) \left(\prod_{j=1}^{\delta} g_{(k_j+1)T} \right) \right] + o(\epsilon^N) \\
&= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \mathbf{E}\left[\Phi^{(\delta)}(g_{1T}) X^{\vec{k}_\delta} \right] + o(\epsilon^N) \\
&= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \int_{\mathbf{R}} \Phi^{(\delta)}(x) \mathbf{E}[X^{\vec{k}_\delta} | g_{1T} = x] f_{g_{1T}}(x) dx + o(\epsilon^N) \\
&= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \int_{\mathbf{R}} \Phi(x) (-1)^\delta \frac{d^\delta}{dx^\delta} \left\{ \mathbf{E}[X^{\vec{k}_\delta} | g_{1T} = x] f_{g_{1T}}(x) \right\} dx + o(\epsilon^N)
\end{aligned} \tag{1.8}$$

where $\Phi^{(\delta)}(g_{1T}) = \left. \frac{\partial^\delta \Phi(x)}{\partial x^\delta} \right|_{x = g_{1T}}$,

$$X^{\vec{k}_\delta} = \prod_{j=1}^{\delta} g_{(k_j+1)T}$$

for $\vec{k}_\delta \in L_{n,\delta}$, and

$$\sum_{\vec{k}_\delta}^{(n)} = \sum_{\delta=1}^{n} \sum_{\vec{k}_\delta \in L_{n,\delta}} .$$

In the preceding works on application of the asymptotic expansion, conditional expectations in (1.8) were directly computed with some formulas given in [29] or [31] (for example, see Appendix B of [31]). Recently, while the formulas had been given up to the third order by those papers, [34] developed a high-order computation scheme for the conditional expectations using the fact that each of these $\{A^j_{k,t}\}_{j,k}$, $\{g_{nT}\}_n$ and also $\{X^{\vec{k}_\delta}\}_{\vec{k}_\delta}$ can be decomposed into a finite sum of iterated multiple Wiener-Itô integrals by Itô's formula, and a certain property of iterated multiple Wiener-Itô integrals (see Nualart, Üstünel and Zakai [24] and Section 4 of [34]). On the other hand, as shown in the next section, this paper develops a new method computing unconditional expectations instead of the conditional ones.

1.2 A General Computation Scheme for a High-Order Asymptotic Expansion

In this section we propose the new computational scheme in the asymptotic expansion, which is an alternative to the direct calculation method for the conditional expectations given by [34].

To compute the conditional expectations in the right hand side of (1.8), we use the following lemma which can be derived from a property of Hermite polynomials and leads us to compute the unconditional expectations instead of the conditional ones.

Lemma 2. *Let (Ω, F, P) be a probability space. Suppose that $X \in L^2(\Omega, P)$ and Z is a random variable with Gaussian distribution with mean 0 and variance Σ. Then, the conditional expectation $E[X|Z = x]$ has the following expansion in $L^2(\mathbf{R}, \mu)$ where μ is the Gaussian measure on $\mathbf{R}$ with mean 0 and variance Σ:*

$$E[X|Z = x] = \sum_{n=0}^{\infty} \frac{a_n}{\Sigma^n} H_n(x; \Sigma) \tag{1.9}$$

where $H_n(x; \Sigma)$ is the Hermite polynomial of degree n which is defined as

$$H_n(x; \Sigma) = (-\Sigma)^n e^{x^2/2\Sigma} \frac{d^n}{dx^n} e^{-x^2/2\Sigma}$$

and coefficients a_n are given by

$$a_n = \frac{1}{n!} \frac{1}{i^n} \frac{\partial^n}{\partial \xi^n}\bigg|_{\xi=0} \left\{ e^{\frac{\xi^2}{2}\Sigma} \mathbf{E}[e^{i\xi Z} X] \right\}. \tag{1.10}$$

(proof) Since the system of Hermite polynomials $\{H_n(x; \Sigma)\}$ is an orthogonal basis of $L^2(\mathbf{R}, \mu)$, and $E[X|Z = x] \in L^2(\mathbf{R}, \mu)$, we have the following unique expansion of $E[X|Z = x]$ in $L^2(\mathbf{R}, \mu)$:

$$E[X|Z = x] = \sum_{n=0}^{\infty} \frac{a_n}{\Sigma^n} H_n(x; \Sigma).$$

Since we have another Taylor expansion

$$e^{i\xi x} = e^{-\frac{\xi^2}{2}\Sigma} \sum_{n=0}^{\infty} \frac{H_n(x; \Sigma)}{n!} (i\xi)^n,$$

then,

$$\begin{aligned} e^{\frac{\xi^2}{2}\Sigma} \mathbf{E}[e^{i\xi Z} X] &= e^{\frac{\xi^2}{2}\Sigma} \int_{\mathbf{R}} e^{i\xi x} \mathbf{E}[X|Z = x] \mu(dx) \\ &= \int_{\mathbf{R}} \sum_{m=0}^{\infty} \frac{H_m(x; \Sigma)}{m!} (i\xi)^m \sum_{n=0}^{\infty} a_n H_n(x; \Sigma) \mu(dx) \\ &= \sum_{n=0}^{\infty} a_n (i\Sigma)^n \xi^n. \end{aligned}$$

Comparing to the coefficients of the Taylor series of $e^{\frac{\xi^2}{2}\Sigma}\mathbf{E}[e^{i\xi Z}X]$ around 0 with respect to ξ, we see that a_n can be written as (1.10).□

Here, we define $\hat{g}_1 = \{\hat{g}_{1t}; t \in \mathbf{R}^+\}$ and $Z^{\langle\xi\rangle} = \{Z_t^{\langle\xi\rangle}; t \in \mathbf{R}^+\}$ as the stochastic processes

$$\hat{g}_{1t} = \int_0^t \hat{V}(X_u^{(0)}, u)dW_u$$

and

$$Z_t^{\langle\xi\rangle} = \exp\left(i\xi\hat{g}_{1t} + \frac{\xi^2}{2}\Sigma_t\right),$$

respectively.

Then, from Lemma 2, the conditional expectations appearing in the right hand side of the equation (1.8) is expressed as

$$\begin{aligned}\mathbf{E}[X^{\vec{k}_\delta}|g_{1T} = x] &= \mathbf{E}[X^{\vec{k}_\delta}|\hat{g}_{1T} = x - C] \\ &= \sum_{l=0}^{\infty} \frac{a_l^{\vec{k}_\delta}}{\Sigma_T^l} H_l(x - C, \Sigma_T) \end{aligned} \tag{1.11}$$

where

$$a_l^{\vec{k}_\delta} = \frac{1}{l!}\frac{1}{i^l}\frac{\partial^l}{\partial\xi^l}\mathbf{E}[X^{\vec{k}_\delta}Z_T^{\langle\xi\rangle}]\bigg|_{\xi=0}. \tag{1.12}$$

Here it is noted that with this expression we now need to compute unconditional expectations $\mathbf{E}[X^{\vec{k}_\delta}Z_T^{\langle\xi\rangle}]$ instead of the conditional expectations.

1.2.1 The Asymptotic Expansion of Density Function

In this subsection, we explain the new computational method through deriving a general formula for the expansion (1.8) with an arbitrary specification of its order N. In particular, we show that the coefficients in the expansion are obtained through a system of ordinary differential equations that is solved easily.

First, we define $\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(t;\xi)$ for $\vec{l}_\beta \in L_{n,\beta}$ and $\vec{d}_\beta \in \{1, \cdots, d\}^\beta$ $(n \geq \beta \geq 1)$ as

$$\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(t;\xi) = \mathbf{E}\left[\left(\prod_{j=1}^{\beta} A_{l_jt}^{d_j}\right)Z_t^{\langle\xi\rangle}\right], \tag{1.13}$$

and for $n = 0$ as

$$\eta_{(\emptyset)}^{(\emptyset)}(t;\xi) = \mathbf{E}\left[Z_t^{\langle\xi\rangle}\right]. \tag{1.14}$$

Then, unconditional expectations $\mathbf{E}[X^{\vec{k}_\delta} Z_T^{\langle\xi\rangle}]$ appearing in the definition of $a_l^{\vec{k}_\delta}$ (1.12) can be written in terms of η as follows:

$$
\begin{aligned}
\mathbf{E}[X^{\vec{k}_\delta} Z_T^{\langle\xi\rangle}] &= \mathbf{E}\left[\left(\prod_{j=1}^{\delta} g_{(k_j+1)T}\right) Z_T^{\langle\xi\rangle}\right] \\
&= \mathbf{E}\left[\left(\prod_{j=1}^{\delta} \sum_{\vec{l}^j_{\beta_j},\vec{d}^j_{\beta_j}}^{(k_j+1)} \frac{1}{\beta_j!}\partial^{\beta_j}_{\vec{d}^j_{\beta_j}} g(X_T^{(0)}) A^{d_1^j}_{l_1^j T}\cdots A^{d^j_{\beta_j}}_{l^j_{\beta_j}}\right) Z_T^{\langle\xi\rangle}\right] \\
&= \sum_{\vec{l}^1_{\beta_1},\vec{d}^1_{\beta_1}}^{(k_1+1)} \cdots \sum_{\vec{l}^\delta_{\beta_\delta},\vec{d}^\delta_{\beta_\delta}}^{(k_\delta+1)} \left(\prod_{j=1}^{\delta} \frac{1}{\beta_j!}\partial^{\beta_j}_{\vec{d}^j_{\beta_j}} g(X_T^{(0)})\right) \eta^{\vec{d}^1_{\beta_1}\otimes\cdots\otimes\vec{d}^\delta_{\beta_\delta}}_{\vec{l}^1_{\beta_1}\otimes\cdots\otimes\vec{l}^\delta_{\beta_\delta}}(T;\xi)
\end{aligned}
\tag{1.15}
$$

where

$$
\begin{aligned}
\vec{d}^i_{\beta_i}\otimes\vec{d}^j_{\beta_j} &:= (d_1^i,\cdots,d^i_{\beta_i},d^j_1,\cdots,d^j_{\beta_j}),\\
\vec{l}^i_{\beta_i}\otimes\vec{l}^j_{\beta_j} &:= (l_1^i,\cdots,l^i_{\beta_i},l^j_1,\cdots,l^j_{\beta_j}).
\end{aligned}
$$

So, we have to calculate $\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(T;\xi)$ to evaluate the asymptotic expansion (1.8).

In the following, we derive a system of ODEs satisfied by these $\{\eta^{\vec{d}_\beta}_{\vec{l}_\beta}\}$. Before showing a general result, we first derive the ODEs for few leading-low-order terms explicitly to give a better intuition of a key idea of our method. Consider the evaluation of $\eta^j_{(2)}(T;\xi) = E[A^j_{2T} Z_T^{\langle\xi\rangle}]$ which appears in the ϵ-order. Here, for simplicity, we assume that V_0 does not depend on ϵ, and write $V_0(x,\epsilon)$ as $V_0(x)$. First, applying Itô's formula to $A^j_{2t}Z_t^{\langle\xi\rangle}$, we have

$$
\begin{aligned}
d(A^j_{2t}Z_t^{\langle\xi\rangle}) &= A^j_{2t}dZ_t^{\langle\xi\rangle} + Z_t^{\langle\xi\rangle}dA^j_{2t} + d\langle A^j_2, Z^{\langle\xi\rangle}\rangle_t \\
&= \left\{(i\xi)\sum_{j'=1}^{d} A^{j'}_{1t}Z_t^{\langle\xi\rangle}\hat{V}(X_t^{(0)},t)\partial_{j'}V^j(X_t^{(0)})' + \sum_{j'=1}^{d} A^{j'}_{2t}Z_t^{\langle\xi\rangle}\partial_{j'}V_0^j(X_t^{(0)})\right.\\
&\quad\left. +\frac{1}{2}\sum_{j',k'=1}^{d} A^{j'}_{1t}A^{k'}_{1t}Z_t^{\langle\xi\rangle}\partial_{j'}\partial_{k'}V_0^j(X_t^{(0)})\right\}dt \\
&\quad + \left\{(i\xi)A^j_{2t}Z_t^{\langle\xi\rangle}\hat{V}(X_t^{(0)},t) + \sum_{j'=1}^{d} A^{j'}_{1t}Z_t^{\langle\xi\rangle}\partial_{j'}V^j(X_t^{(0)})\right\}dW_t.
\end{aligned}
$$

Since the last term is a martingale, taking expectation on both sides, we have the following ordinary differential equation for $\eta^j_{(2)}$:

$$\begin{aligned}\frac{d}{dt}\eta^j_{(2)}(t;\xi) &= (i\xi)\sum_{j'=1}^{d}\eta^{j'}_{(1)}(t;\xi)\hat{V}(X^{(0)}_t,t)\partial_{j'}V^j(X^{(0)}_t)' \\ &\quad +\sum_{j'=1}^{d}\eta^{j'}_{(2)}(t;\xi)\partial_{j'}V^j_0(X^{(0)}_t)+\frac{1}{2}\sum_{j',k'=1}^{d}\eta^{j',k'}_{(1,1)}(t;\xi)\partial_{j'}\partial_{k'}V^j_0(X^{(0)}_t).\end{aligned}$$

Here, $\eta^j_{(1)}(j=1,\cdots,d)$ appearing in the right hand side of the above ODE are evaluated in the similar manner:

$$\begin{aligned}d(A^j_{1t}Z^{\langle\xi\rangle}_t) &= A^j_{1t}dZ^{\langle\xi\rangle}_t+Z^{\langle\xi\rangle}_t dA^j_{1t}+d\langle A^j_1,Z^{\langle\xi\rangle}\rangle_t \\ &= \left\{(i\xi)Z^{\langle\xi\rangle}_t\hat{V}(X^{(0)}_t,t)V^j(X^{(0)}_t)'+\sum_{j'=1}^{d}A^{j'}_{1t}Z^{\langle\xi\rangle}_t\partial_{j'}V^j_0(X^{(0)}_t)\right\}dt \\ &\quad +\left\{(i\xi)A^j_{1t}Z^{\langle\xi\rangle}_t\hat{V}(X^{(0)}_t,t)+Z^{\langle\xi\rangle}_tV^j(X^{(0)}_t)\right\}dW_t,\end{aligned}$$

hence, we have

$$\frac{d}{dt}\eta^j_{(1)}(t;\xi)=(i\xi)\hat{V}(X^{(0)}_t,t)V^j(X^{(0)}_t)'+\sum_{j'=1}^{d}\eta^{j'}_{(1)}(t;\xi)\partial_{j'}V^j_0(X^{(0)}_t).$$

$\eta^{j,k}_{(1,1)}$ and other higher-order terms can be evaluated in the same way. The key observation is that each ODE does not involve any higher-order terms, and only lower- or the same order- terms appear in the right hand side of the ODE. So, one can easily solve (analytically or numerically) the system of ODEs and evaluate the expectations.

The following theorem provides a way to calculate general $\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(T;\xi)$ as a solution to the system of the ordinary differential equations:

Theorem 2. *For $\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(t;\xi)$ defined in (1.13), the following system of ordinary differential equations is satisfied:*

$$\begin{aligned}\frac{d}{dt}\left\{\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(t;\xi)\right\} &= \sum_{k=1}^{\beta}\frac{1}{l_k!}\left\{\eta^{\vec{d}_{\beta/k}}_{\vec{l}_{\beta/k}}(t;\xi)\right\}\left\{\partial^{l_k}_\epsilon V^{d_k}_0(X^{(0)}_t,0)\right\} \\ &+ \sum_{k=1}^{\beta}\sum_{l=1}^{l_k}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l)}\frac{1}{(l_k-l)!}\frac{1}{\gamma!}\left\{\eta^{(\vec{d}_{\beta/k})\otimes\vec{\tilde{d}}_\gamma}_{(\vec{l}_{\beta/k})\otimes\vec{m}_\gamma}(t;\xi)\right\}\left\{\partial^\gamma_{\vec{\tilde{d}}_\gamma}\partial^{l_k-l}_\epsilon V^{d_k}_0(X^{(0)}_t,0)\right\} \\ &+ \sum_{\substack{k,m=1\\k<m}}^{\beta}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)}\sum_{\vec{m}_\delta,\vec{\tilde{d}}_\delta}^{(l_m-1)}\frac{1}{\gamma!\delta!}\left\{\eta^{(\vec{d}_{\beta/k,m})\otimes\vec{\tilde{d}}_\gamma\otimes\vec{\tilde{d}}_\delta}_{(\vec{l}_{\beta/k,m})\otimes\vec{m}_\gamma\otimes\vec{m}_\delta}(t;\xi)\right\}\left\{\partial^\gamma_{\vec{\tilde{d}}_\gamma}V^{d_k}(X^{(0)}_t)\right\}\left\{\partial^\delta_{\vec{\tilde{d}}_\delta}V^{d_m}(X^{(0)}_t)\right\} \\ &+ (i\xi)\sum_{k=1}^{\beta}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)}\frac{1}{\gamma!}\left\{\eta^{(\vec{d}_{\beta/k})\otimes\vec{\tilde{d}}_\gamma}_{(\vec{l}_{\beta/k})\otimes\vec{m}_\gamma}(t;\xi)\right\}\left\{\partial^\gamma_{\vec{\tilde{d}}_\gamma}V^{d_k}(X^{(0)}_t)\right\}\hat{V}(X^{(0)}_t,t) \qquad (1.16)\end{aligned}$$

where

$$\begin{aligned}
\vec{l}_{\beta/k} &:= (l_1,\cdots,l_{k-1},l_{k+1},\cdots,l_\beta)\\
\vec{l}_{\beta/k,n} &:= (l_1,\cdots,l_{k-1},l_{k+1},\cdots,l_{n-1},l_{n+1},\cdots,l_\beta),\ 1\le k<n\le\beta\\
\vec{l}_\beta\otimes\vec{m}_\gamma &:= (l_1,\cdots,l_\beta,m_1,\cdots,m_\gamma)
\end{aligned}$$

for $\vec{l}_\beta=(l_1,\cdots,l_\beta)$ *and* $\vec{m}_\gamma=(m_1,\cdots,m_\gamma)$.

(Proof) First, Applying Itô's formula to $\left(\prod_{j=1}^{\beta}A_{l_jt}^{d_j}\right)$, we have

$$\begin{aligned}
d\left(\prod_{j=1}^{\beta}A_{l_jt}^{d_j}\right) &= \sum_{k=1}^{\beta}\left(\prod_{\substack{j=1\\j\neq k}}^{\beta}A_{l_jt}^{d_j}\right)dA_{l_kt}^{d_k}+\sum_{\substack{k,m=1\\k<m}}^{\beta}\left(\prod_{\substack{j=1\\j\neq k,m}}^{\beta}A_{l_jt}^{d_j}\right)d\langle A_{l_k}^{d_k},A_{l_m}^{d_m}\rangle_t\\
&= \sum_{k=1}^{\beta}\left(\prod_{\substack{j=1\\j\neq k}}^{\beta}A_{l_jt}^{d_j}\right)\frac{1}{l_k!}\partial_\epsilon^{l_k}V_0^{d_k}(X_t^{(0)},0)dt\\
&+ \sum_{k=1}^{\beta}\sum_{l=1}^{l_k}\left(\prod_{\substack{j=1\\j\neq k}}^{\beta}A_{l_jt}^{d_j}\right)\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l)}\frac{1}{(l_k-l)!}\frac{1}{\gamma!}\left(\prod_{j'=1}^{\gamma}A_{m_{j'}t}^{\tilde{d}_{j'}}\right)\partial_{\tilde{d}_\gamma}^{\gamma}\partial_\epsilon^{l_k-l}V_0^{d_k}(X_t^{(0)},0)dt\\
&+ \sum_{k=1}^{\beta}\left(\prod_{\substack{j=1\\j\neq k}}^{\beta}A_{l_jt}^{d_j}\right)\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)}\frac{1}{\gamma!}\left(\prod_{j'=1}^{\gamma}A_{m_{j'}t}^{\tilde{d}_{j'}}\right)\partial_{\tilde{d}_\gamma}^{\gamma}V^{d_k}(X_t^{(0)})dW_t\\
&+ \sum_{\substack{k,m=1\\k<m}}^{\beta}\left(\prod_{\substack{j=1\\j\neq k,m}}^{\beta}A_{l_jt}^{d_j}\right)\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)}\sum_{\vec{m}_\delta,\vec{\hat{d}}_\delta}^{(l_m-1)}\frac{1}{\gamma!\delta!}\\
&\times\left(\prod_{j'=1}^{\gamma}A_{m_{j'}t}^{\tilde{d}_{j'}}\right)\partial_{\tilde{d}_\gamma}^{\gamma}V^{d_k}(X_t^{(0)})\left(\prod_{j'=1}^{\delta}A_{m_{j'}t}^{\hat{d}_{j'}}\right)\partial_{\hat{d}_\delta}^{\delta}V^{d_m}(X_t^{(0)})dt.
\end{aligned}\tag{1.17}$$

Note also that

$$dZ_t^{\langle\xi\rangle}=(i\xi)\hat{V}(X_t^{(0)},t)Z_t^{\langle\xi\rangle}dW_t.\tag{1.18}$$

Then, applying Itô's formula again to $\left(\prod_{j=1}^{\beta}A_{l_jt}^{d_j}Z_t^{\langle\xi\rangle}\right)$ and taking expectations on both sides, we obtain the result. □

Remark 1. *Due to the hierarchical structure of the ODEs with respect to* $n=\sum_{j=1}^{\beta}l_j$ *and* $\eta_{(\emptyset)}^{(\emptyset)}(t;\xi)=\mathbf{E}[Z_t^{\langle\xi\rangle}]=1$, *one can easily solve these ODEs successively from lower-order terms to higher-order terms with initial conditions* $\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(0;\xi)=0$ *for* $(\vec{l}_\beta,\vec{d}_\beta)\neq(\emptyset,\emptyset)$.

Remark 2. *Further, due to the structure of the system of the differential equations, it is easily shown by induction that each* $\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(t;\xi)$ *is expressed as a polynomial of degree* $n=\sum_{j=1}^{\beta} l_j$ *with respect to* $(i\xi)$*. Then, we can also show that* $\mathbf{E}[X^{\vec{k}_\delta} Z_T^{\langle\xi\rangle}]$ *is a polynomial of degree* $(n+\delta)$ *with respect to* $(i\xi)$*, and thus* $a_l^{\vec{k}_\delta}=0(l>n+\delta)$ *for* $\vec{k}_\delta\in L_{n,\delta}$*. This ensures a convergence of the infinite sum in (1.11).*

Then, from Lemma 2 and (1.8), we have the following expression of $\mathbf{E}[\Phi(G^{(\epsilon)})]$:

$$\begin{aligned}\mathbf{E}[\Phi(G^{(\epsilon)})] &= \sum_{n=0}^{N}\epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!}\int_{\mathbf{R}} \Phi(x)(-1)^\delta \frac{d^\delta}{dx^\delta}\left\{\sum_{l=0}^{n+\delta}\frac{a_l^{\vec{k}_\delta}}{\Sigma_T^l} H_l(x-C,\Sigma_T) f_{g_{1T}}(x)\right\}dx + o(\epsilon^N)\\ &= \sum_{n=0}^{N}\epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!}\int_{\mathbf{R}} \Phi(x)\left\{\sum_{l=0}^{n+\delta}\frac{a_l^{\vec{k}_\delta}}{\Sigma_T^{l+\delta}} H_{l+\delta}(x-C,\Sigma_T) f_{g_{1T}}(x)\right\}dx + o(\epsilon^N)\end{aligned}$$

Here we used the relation

$$\frac{d^\delta}{dx^\delta}\left\{H_l(x-C,\Sigma_T)f_{g_{1T}}(x)\right\} = \frac{1}{\Sigma_T^\delta}H_{l+\delta}(x-C,\Sigma_T)f_{g_{1T}}(x).$$

In particular, let Φ be the delta function at $x\in\mathbf{R}$, δ_x, we obtain the asymptotic expansion of the density of $G^{(\epsilon)}$:

$$\begin{aligned}f_{G^{(\epsilon)}}(x) &= \mathbf{E}[\delta_x(G^{(\epsilon)})]\\ &= \sum_{n=0}^{N}\epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!}\sum_{l=0}^{n+\delta}\frac{a_l^{\vec{k}_\delta}}{\Sigma_T^{l+\delta}} H_{l+\delta}(x-C,\Sigma_T) f_{g_{1T}}(x) + o(\epsilon^N). \qquad (1.19)\end{aligned}$$

We summarize the discussion above as the following theorem:

Theorem 3. *The asymptotic expansion of the density function of* $G^{(\epsilon)}$ *up to* ϵ^N*-order is given by*

$$f_{G^{(\epsilon)}}(x) = f_{g_{1T}}(x) + \sum_{n=1}^{N}\epsilon^n\left(\sum_{m=0}^{3n} C_{nm} H_m(x-C,\Sigma_T)\right) f_{g_{1T}}(x) + o(\epsilon^N) \qquad (1.20)$$

where

$$\begin{aligned}C_{nm} &= \frac{1}{\Sigma_T^m}\sum_{\delta=1}^{m}\sum_{\vec{k}_\delta\in L_{n,\delta}}\sum_{\vec{l}^1_{\beta_1},\vec{d}^1_{\beta_1}}^{(k_1+1)}\cdots\sum_{\vec{l}^\delta_{\beta_\delta},\vec{d}^\delta_{\beta_\delta}}^{(k_\delta+1)}\frac{1}{\delta!(m-\delta)!}\\ &\quad \left(\prod_{j=1}^{\delta}\frac{1}{\beta_j!}\partial^{\beta_j}_{\vec{d}^j_{\beta_j}} g(X_T^{(0)})\right)\frac{1}{i^{m-\delta}}\frac{\partial^{m-\delta}}{\partial\xi^{m-\delta}}\left\{\eta^{\vec{d}^1_{\beta_1}\otimes\cdots\otimes\vec{d}^\delta_{\beta_\delta}}_{\vec{l}^1_{\beta_1}\otimes\cdots\otimes\vec{l}^\delta_{\beta_\delta}}(T;\xi)\right\}\Bigg|_{\xi=0} \qquad (1.21)\end{aligned}$$

and $\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(T;\xi)$ *are obtained as a solution to the system of ODEs given in Theorem 2.*

1.2.2 The Asymptotic Expansion of Option Prices

We apply the asymptotic expansion to option pricing. We consider a plain vanilla option on the underlying asset $g(X_T^{(\epsilon)})$ whose dynamics is given by (1.1).

For example, an asymptotic expansion up to $\epsilon^{(N+1)}$ of a call option price at time 0 with maturity T and strike price K where $K = X_T^{(0)} - \epsilon y$ for arbitrary $y \in \mathbf{R}$ is given by

$$\begin{aligned} \mathrm{Call}(K,T) &= P(0,T)\mathbf{E}[\max(g(X_T^{(\epsilon)}) - K, 0)] \\ &= \epsilon P(0,T)\int_{-y}^{\infty}(x+y)f_{G^{(\epsilon)},N}(x)dx + o(\epsilon^{(N+1)}). \end{aligned} \tag{1.22}$$

Here, $P(0,T)$ denotes the price at time 0 of a zero coupon bond with maturity T and $f_{G^{(\epsilon)},N}$ is the asymptotic expansion of the density of $G^{(\epsilon)}$ up to ϵ^N-order given by (1.20):

$$f_{G^{(\epsilon)},N}(x) = f_{g_{1T}}(x) + \sum_{n=1}^{N}\epsilon^n\left(\sum_{m=0}^{3n}C_{nm}H_m(x-C,\Sigma_T)\right)f_{g_{1T}}(x)$$

Integrals appearing in the right hand side of (1.22) can be calculated by following formulas related to the Hermite polynomials

$$\begin{aligned} \int_{-y}^{\infty}H_k(x;\Sigma)f_{g_{1T}}(x)dx &= \Sigma H_{k-1}(-y;\Sigma)f_{g_{1T}}(y)\ (k\geq 1), \\ \int_{-y}^{\infty}xH_k(x;\Sigma)f_{g_{1T}}(x)dx &= -\Sigma y H_{k-1}(-y;\Sigma)f_{g_{1T}}(y) \\ &\quad +\Sigma^2 H_{k-2}(-y;\Sigma)f_{g_{1T}}(y)\ (k\geq 2). \end{aligned}$$

1.2.3 Remarks on the Asymptotic Expansion for Multi-dimensional Density Functions

In this section, we extend Lemma 1 in [35], which easily leads to the asymptotic expansion of a multi-dimensional density function in the same manner as for the one dimensional case appearing in the previous section. That is, we obtain the following result as an extension of Lemma 1 in [35].

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Suppose that $X \in L^2(\Omega, P)$ and $\vec{Z}$ is a d-dimensional random variable with Gaussian distribution with mean $\vec{0}$ and variance-covariance matrix $\underline{\Sigma}$. Then, the conditional expectation $\mathbf{E}[X|\vec{Z} = \vec{x}]$ for $\vec{x} \in \mathbf{R}^d$ has the following expansion in $L^2(\mathbf{R}^d, \vec{\mu})$ where $\vec{\mu}$ is the Gaussian measure on $\mathbf{R}^d$ with mean $\vec{0}$ and variance $\underline{\Sigma}$:

$$\mathbf{E}[X|\vec{Z} = \vec{x}] = \sum_{|\vec{n}|=0}^{\infty} a_{\vec{n}!}H_{\vec{n}}(\vec{x} : \underline{\Sigma}), \tag{1.23}$$

where $\vec{n} = (n_1, n_2, \cdots, n_d)$, $|\vec{n}| = n_1 + n_2 + \cdots + n_d$, $\vec{n}! = n_1! n_2! \cdots n_d!$ and

$$a_{\vec{n}} = \frac{1}{\vec{n}} \frac{1}{i^{|\vec{n}|}} \frac{\partial^{\vec{n}}}{\partial \vec{\xi}}\bigg|_{\vec{\xi}=\vec{0}} \left\{ e^{\frac{1}{2}\vec{\xi}^\top \underline{\Sigma} \vec{\xi}} \mathbf{E}\left[e^{\vec{\xi}^\top \vec{Z}} X \right] \right\}. \tag{1.24}$$

Here, $H_{\vec{n}}(\vec{x} : \underline{\Sigma})$ stands for the d-dimensional multiple Hermite polynomial of degree $|\vec{n}|$ with $\vec{n} = (n_1, n_2, \cdots, n_d)$:

$$H_{\vec{n}}(\vec{x} : \underline{\Sigma}) \quad = \quad \frac{1}{n[\vec{x} : \underline{\Sigma}]} \left(-\frac{\partial}{\partial_{x_1}} \right) \left(-\frac{\partial}{\partial_{x_2}} \right) \cdots \left(-\frac{\partial}{\partial_{x_d}} \right) n[\vec{x} : \underline{\Sigma}]; \; \vec{x} = (x_1, x_2, \cdots, x_d) \tag{1.25}$$

where

$$n[\vec{x} : \underline{\Sigma}] \quad = \quad \frac{1}{(2\pi)^{d/2} |\underline{\Sigma}|^{1/2}} \exp\left\{ -\frac{1}{2} \vec{x}^\top \underline{\Sigma}^{-1} \vec{x} \right\}. \tag{1.26}$$

Indeed, since the system of Hermite polynomials:

$$\{ H_{\vec{n}}(\vec{x} : \underline{\Sigma}) : \vec{n} = (n_1, n_2, \cdots, n_d), n_i = 0, 1, 2 \cdots (i = 1, 2, \cdots, d) \}$$

is an orthogonal basis of $L^2(\mathbf{R}^d, \vec{\mu})$, and $E[X|\vec{Z} = \vec{x}] \in L^2(\mathbf{R}^d, \vec{\mu})$, we have the following unique expansion of $E[X|\vec{Z} = \vec{x}]$ in $L^2(\mathbf{R}^d, \vec{\mu})$:

$$\mathbf{E}[X|\vec{Z} = \vec{x}] = \sum_{|\vec{n}|=0}^{\infty} a_{\vec{n}} H_{\vec{n}}(\vec{x} : \underline{\Sigma}).$$

On the other hand, we know the relation:

$$\sum_{|\vec{j}|=0}^{\infty} \frac{(i\vec{\xi})^{\vec{j}}}{\vec{j}!} \tilde{H}_{\vec{j}}(\vec{x} : \underline{\Sigma}) = e^{i\vec{\xi}^\top \vec{x}} e^{\frac{1}{2}\vec{\xi}^\top \underline{\Sigma} \vec{\xi}}, \tag{1.27}$$

and hence,

$$e^{\vec{\xi}^\top \vec{x}} = e^{-\frac{1}{2}\vec{\xi}^\top \underline{\Sigma} \vec{\xi}} \sum_{|\vec{j}|=0}^{\infty} \frac{(i\vec{\xi})^{\vec{j}}}{\vec{j}!} \tilde{H}_{\vec{j}}(\vec{x} : \underline{\Sigma}),$$

where

$$\begin{aligned} \tilde{H}_{\vec{n}}(\vec{x} : \Sigma) \quad &= \quad \frac{1}{n[\vec{x} : \underline{\Sigma}]} \left(-\frac{\partial}{\partial_{y_1}} \right) \left(-\frac{\partial}{\partial_{y_2}} \right) \cdots \left(-\frac{\partial}{\partial_{y_d}} \right) n[\vec{x} : \underline{\Sigma}], \\ \vec{y} \quad &= \quad (y_1, y_2, \cdots, y_d) = \underline{\Sigma}^{-1} \vec{x}. \end{aligned} \tag{1.28}$$

Therefore,

$$\begin{aligned} e^{\frac{1}{2}\vec{\xi}^\top \underline{\Sigma} \vec{\xi}} \mathbf{E}\left[e^{\vec{\xi}^\top \vec{Z}} X \right] \quad &= \quad e^{\frac{1}{2}\vec{\xi}^\top \underline{\Sigma} \vec{\xi}} \mathbf{E}\left[e^{\vec{\xi}^\top \vec{Z}} \mathbf{E}\left[X | \vec{Z} = \vec{x} \right] \right] \\ &= \quad \int_{\mathbf{R}^d} \left\{ \sum_{|\vec{j}|=0}^{\infty} \tilde{H}_{\vec{n}}(\vec{x} : \underline{\Sigma}) (i\vec{\xi})^{\vec{j}} \right\} \left\{ \sum_{|\vec{n}|=0}^{\infty} a_{\vec{n}} H_{\vec{n}}(\vec{x} : \underline{\Sigma}) \right\} \mu(d\vec{x}) \quad (1.29) \\ &= \quad \sum_{|\vec{n}|=0}^{\infty} \vec{n}! a_{\vec{n}} i^{|\vec{n}|} \xi^{\vec{n}}; \; (\xi^{\vec{n}} = \xi_1^{n_1} \xi_2^{n_2} \cdots \xi_d^{n_d}), \quad (1.30) \end{aligned}$$

and making $\vec{n} = (n_1, \cdots, n_d)$-th order differentiation of both sides in the equation above with respect to $\vec{\xi} = (\xi_1, \cdots, \xi_d)$ at $\vec{\xi} = \vec{0}$, we obtain (1.24) and hence the result, (1.23) - (1.26).

Chapter 2

Applications and Extensions of the Asymptotic Expansion Method

2.1 High-Order Asymptotic Expansions of Stochastic Volatility Models

In this section, we test effectiveness of the asymptotic expansion method described in Chapter 1 through numerical examples. Also, we compare approximation accuracy of our method with that of another existing approximation method.

2.1.1 An Asymptotic Expansion of the λ-SABR Model

To test efficiency of the expansion, we first consider a European plain-vanilla call and put prices under the following λ-SABR model [14] (interest rate=0%) :

$$\begin{aligned} dS^{(\epsilon)}(t) &= \epsilon\sigma^{(\epsilon)}(t)(S^{(\epsilon)}(t))^{\beta}dW_t^1, \\ d\sigma^{(\epsilon)}(t) &= \lambda(\theta - \sigma^{(\epsilon)}(t))dt + \epsilon\nu_1\sigma^{(\epsilon)}(t)dW_t^1 + \epsilon\nu_2\sigma^{(\epsilon)}(t)dW_t^2, \\ S^{(\epsilon)}(0) &= S_0, \quad \sigma^{(\epsilon)}(0) = \sigma_0, \end{aligned}$$

where $\nu_1 = \rho\nu,\ \nu_2 = (\sqrt{1-\rho^2})\nu$ (an instantaneous correlation between $S^{(\epsilon)}$ and $\sigma^{(\epsilon)}$ is $\rho \in [-1,1]$). Note that when $\lambda = 0$ the model becomes the original SABR model [9]. Rigorously speaking, this model does not satisfy the regularity conditions since the coefficient function $V^1(\sigma, s) = \sigma s^{\beta}$ is unbounded and has non-smooth derivatives at $s = 0$. However, as seen in the following, our method is (formally) applicable to this model and gives better accuracies for approximate prices in higher-order expansions for various ranges of strikes and parameters.

To compute an option price on $S^{(\epsilon)}$, we need the density function of $S_T^{(\epsilon)}$ whose asymptotic expansion is given by (1.20) with setting $g(S,\sigma) = S$. The asymptotic expansion of the density function is obtained by solving the system of the ordinary differential equations given in Theorem 2. For example, the corresponding differential equations up to the second order are given by

$$\begin{aligned}
\frac{d}{dt}\eta_{(1)}^S(t;\xi) &= (i\xi)(S_t^{(0)})^{2\beta}(\sigma_t^{(0)})^2, \\
\frac{d}{dt}\eta_{(1)}^\sigma(t;\xi) &= (i\xi)\nu_1(S_t^{(0)})^{\beta}(\sigma_t^{(0)})^2 - \lambda\eta_{(1)}^\sigma(t;\xi), \\
\frac{d}{dt}\eta_{(2)}^S(t;\xi) &= (i\xi)\beta(S_t^{(0)})^{2\beta-1}(\sigma_t^{(0)})^2\eta_{(1)}^S(t;\xi) + (i\xi)(S_t^{(0)})^{2\beta}\sigma_t^{(0)}\eta_{(1)}^\sigma(t;\xi),
\end{aligned}$$

where $S_t^{(0)} = S_0$ and $\sigma_t^{(0)} = e^{-\lambda t}(\sigma_0 - \theta) + \theta$. Since these equations are linear and have the hierarchical structure, one can easily integrate them as

$$\begin{aligned}
\eta_{(1)}^S(t;\xi) &= (i\xi)\int_0^t (S_{t_1}^{(0)})^{2\beta}(\sigma_{t_1}^{(0)})^2 dt_1, \\
\eta_{(1)}^\sigma(t;\xi) &= (i\xi)\int_0^t e^{-\lambda(t-t_1)}\nu_1(S_{t_1}^{(0)})^{\beta}(\sigma_{t_1}^{(0)})^2 dt_1, \\
\eta_{(2)}^S(t;\xi) &= (i\xi)^2\int_0^t\int_0^{t_1} \beta(S_{t_1}^{(0)})^{2\beta-1}(\sigma_{t_1}^{(0)})^2(S_{t_2}^{(0)})^{2\beta}(\sigma_{t_2}^{(0)})^2 dt_2 dt_1 \\
&\quad +(i\xi)^2\int_0^t\int_0^{t_1} e^{-\lambda(t_1-t_2)}(S_{t_1}^{(0)})^{2\beta}\sigma_{t_1}^{(0)}\nu_1(S_{t_2}^{(0)})^{\beta}(\sigma_{t_2}^{(0)})^2 dt_2 dt_1.
\end{aligned}$$

Integrals appearing in the right hand side are analytically evaluated, which is omitted due to the limitation of the space (they are available upon request).

Then, from Theorem 3 the asymptotic expansion of the density function of $G^{(\epsilon)} = \frac{S_T^{(\epsilon)} - S_T^{(0)}}{\epsilon}$ can be expressed as

$$f_{G^{(\epsilon)}}(x) \sim f_{g_{1T}}(x) + \epsilon C_{13} H_3(x;\Sigma_T) f_{g_{1T}}(x) + \cdots \tag{2.1}$$

where

$$f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\Sigma_T}}\exp\left(-\frac{x^2}{2\Sigma_T}\right)$$

with

$$\Sigma_T = \int_0^T (S_t^{(0)})^{2\beta}(\sigma_t^{(0)})^2 dt$$

and

$$\begin{aligned}
C_{13} &= \frac{1}{\Sigma_T^3}\int_0^T\int_0^{t_1} \beta(S_{t_1}^{(0)})^{2\beta-1}(\sigma_{t_1}^{(0)})^2(S_{t_2}^{(0)})^{2\beta}(\sigma_{t_2}^{(0)})^2 dt_2 dt_1 \\
&\quad +\frac{1}{\Sigma_T^3}\int_0^T\int_0^{t_1} e^{-\lambda(t_1-t_2)}(S_{t_1}^{(0)})^{2\beta}\sigma_{t_1}^{(0)}\nu_1(S_{t_2}^{(0)})^{\beta}(\sigma_{t_2}^{(0)})^2 dt_2 dt_1.
\end{aligned}$$

Note also that C_{13} is calculated in closed form; the expression is omitted, which is available upon request. Moreover, by a similar calculation to that in Section 3.2, an approximate price of a call option on $S^{(\epsilon)}$ at time 0 with maturity T and strike $K = S_T^{(0)} - \epsilon y$ up to ϵ^2-order is given by

$$\begin{aligned} C(K,T) &= \epsilon P(0,T)\left(\Sigma_T f_{g_{1T}}(y) + yN\left(\frac{y}{\sqrt{\Sigma_T}}\right)\right) \\ &\quad -\epsilon^2 P(0,T) C_{13} \Sigma_T^2 y f_{g_{1T}}(y) + o(\epsilon^2) \end{aligned} \tag{2.2}$$

where $N(\cdot)$ is a cumulative distribution function of the standard normal distribution. Higher-order asymptotic expansions can be calculated in a similar manner.

2.1.2 Numerical Example: $\lambda = 0$ (SABR case)

First, we consider European plain-vanilla call and put prices under the original SABR case ($\lambda = 0$ in the λ-SABR model). We calculate approximated prices by the asymptotic expansion method up to the fifth order. Note that all the solutions to the differential equations are obtained in closed form. Thus, the computation is very fast(e.g. the computation time is within $10^{-5} \sim 10^{-6}$ second for the fifth-order expansion). We also calculate approximated prices by Hagan et al.[9] to compare accuracy of its approximation with ours. Benchmark values are computed by Monte Carlo simulations. In the simulations for the benchmark values, we use Euler-Maruyama scheme as a discretization scheme with 1024 time steps, and generate 10^8 paths in each simulation. ϵ is set to be one and other parameters used in the test are given in Table 2.1.

Table 2.1: Parameters used in the SABR ($\lambda = 0$) case

Parameter	$S(0)$	β	$\sigma(0)$	ν	ρ	T
i	100	0.5	3.0	0.3	-0.7	10

Results are in Table 2.3 and Figure 2.1. From the results, we can see that the higher-order asymptotic expansion almost always improves accuracy of the approximation by the lower ones. While sometimes the third-order approximation does not perform well, particularly in OTM options, the fifth-order one approximates the prices almost perfectly in these settings. This strongly supports importance of computing high-order terms, and hence of our method. We also see the fifth-order expansion has equal or smaller approximation errors than Hagan et al.[9]'s formula. Moreover, as seen in the next example, the asymptotic expansion method can be easily extend to the λ-SABR ($\lambda \neq 0$) case.

2.1.3 Numerical Example: $\lambda \neq 0$

Next, we consider the European option prices under the λ-SABR model with $\lambda \neq 0$. Parameters used in the test are given in Table 2.2 (and $\epsilon = 1$ as well as in the previous examples).

Table 2.2: Parameters used in the λ-SABR ($\lambda \neq 0$) case

Parameter	$S(0)$	β	$\sigma(0)$	λ	θ	ν	ρ	T
ii	100	0.5	3.0	0.1	3.0	0.3	-0.7	10
iii	100	1.0	0.3	0.1	0.3	0.3	-0.7	10

We calculate approximated prices by the asymptotic expansion method up to the fifth order. Note that all the solutions to the differential equations are obtained analytically. Further, for the case of $\beta = 1$ in the λ-SABR model (case iii), we can also apply the log-normal asymptotic expansion method given in Section 2.3. This gives the slightly different approximation formula from that with the normal asymptotic expansion method. Note also that the system of ODEs appearing in the log-normal expansion formula are solved analytically as in the normal asymptotic expansion case. We calculate approximated prices by the log-normal asymptotic expansion up to the forth order. We also calculate option prices by Hagan et al.[9]'s formula by setting $\lambda = 0$ in the model which can be thought as the SABR approximation to the λ-SABR model. Benchmark prices are computed by Monte Carlo simulations with Euler-Maruyama discretization scheme with 1024 time steps, and we generate 10^8 paths in each simulation.

Results for the normal asymptotic expansion are in Table 2.3 and Figure 2.2 and 2.3, and results for the log-normal expansion for case iii are in Table 2.4 and Figure 2.4. Note that the 0th-order log-normal expansion (indicated by 'LogNormal' in Table 4 and Figure 4) gives a simple log-normal approximation of the model.

From the results, in each case, as well as the examples in the original SABR model the higher-order expansion or log-normal expansion almost always improve accuracy of the approximation by the lower-order expansions. On the other hand, a naive application of Hagan et al.[9]'s formula to λ-SABR model($\lambda \neq 0$) seems to fail to capture the underlying distribution and the resulting option prices. This might be caused by the fact that it cannot be directly applied to the λ-SABR setting while our method is applicable to a general setting in the unified manner. Further, unlike Hagan et al.[9]'s one whose high-order expansions are difficult to calculate, our method easily provides us the approximation with an arbitrary-high order as we have already seen. These results support flexibility of ours in financial practice.

In addition, for SABR and λ-SABR models we compare computation times of our method with the ones of the method by Hagan et al. [9]. As the computation times of both methods are very fast ($10^{-5} \sim 10^{-6}$ second per option), we implement 10,000 times calculations of 20 options with different strike prices for comparison. Then, the computation times are of the same order for both methods: the ratios of the times based on our method relative to the ones by Hagan et al.[9] are approximately $0.3 \sim 1.6$ for the cases in which both methods achieve the similar accuracies.

Table 2.3: Monte Carlo prices and standard errors (S.E.) and approximated prices, relative errors, and average absolute relative errors by asymptotic expansions (A.E.) and Hagan et al.[9]'s formula (Hagan et al. or Hagan) in the SABR model (case i) and the λ-SABR model with $\beta = 0.5$ (case ii) and $\beta = 1$ (case iii).

Case	Strike	M.C.	Approximation Prices (Relative Errors)					
	(C/P)	Price (S.E.)	A.E. 1st	A.E. 2nd	A.E. 3rd	A.E. 4th	A.E. 5th	Haganet al.(SABR)
	50 Put	12.859 (0.002)	17.987 (39.88%)	19.634 (52.68%)	16.628 (29.31%)	14.679 (14.15%)	13.112 (1.97%)	15.031 (16.89%)
	80 Put	23.824 (0.003)	28.686 (20.41%)	29.426 (23.51%)	26.179 (9.88%)	25.443 (6.80%)	23.978 (0.64%)	25.891 (8.67%)
i	100 Call	32.971 (0.004)	37.847 (14.79%)	37.847 (14.79%)	34.554 (4.80%)	34.554 (4.80%)	33.108 (0.42%)	34.791 (5.52%)
	120 Call	23.887 (0.004)	28.686 (20.09%)	27.945 (16.99%)	24.698 (3.40%)	25.434 (6.48%)	23.968 (0.34%)	25.334 (6.06%)
	150 Call	13.619 (0.003)	17.987 (32.07%)	16.340 (19.98%)	13.334 (-2.09%)	15.284 (12.23%)	13.718 (0.73%)	14.597 (7.19%)
Average Error		-	25.45%	25.59%	9.90%	8.89%	0.82%	8.87%
	50 Put	13.058 (0.002)	17.987 (37.75%)	18.110 (38.69%)	15.423 (18.11%)	14.177 (8.57%)	13.370 (2.39%)	15.031 (15.11%)
	80 Put	24.670 (0.003)	28.686 (16.28%)	28.741 (16.51%)	25.990 (5.35%)	25.499 (3.36%)	24.838 (0.68%)	25.891 (4.95%)
ii	100 Call	34.325 (0.004)	37.847 (10.26%)	37.847 (10.26%)	35.087 (2.22%)	35.087 (2.22%)	34.452 (0.37%)	34.791 (1.36%)
	120 Call	25.654 (0.005)	28.686 (11.82%)	28.630 (11.60%)	25.879 (0.88%)	26.370 (2.79%)	25.709 (0.21%)	25.334 (-1.25%)
	150 Call	15.611 (0.004)	17.987 (15.22%)	17.863 (14.43%)	15.175 (-2.79%)	16.421 (5.19%)	15.614 (0.02%)	14.597 (-6.49%)
Average Error		-	18.27%	18.30%	5.87%	4.43%	0.73%	5.83%
	50 Put	9.429 (0.002)	17.985 (90.74%)	13.991 (48.39%)	8.982 (-4.74%)	8.721 (-7.51%)	8.989 (-4.66%)	8.867 (5.95%)
	80 Put	22.043 (0.003)	28.685 (30.13%)	26.890 (21.99%)	21.618 (-1.93%)	21.505 (-2.44%)	21.771 (-1.23%)	21.350 (-1.91%)
iii	100 Call	32.974 (0.004)	37.847 (14.78%)	37.847 (14.78%)	32.539 (-1.32%)	32.539 (-1.32%)	32.805 (-0.51%)	32.598 (-3.71%)
	120 Call	25.664 (0.006)	28.685 (11.77%)	30.480 (18.77%)	25.209 (-1.77%)	25.321 (-1.33%)	25.587 (-0.30%)	25.892 (-7.47%)
	150 Call	17.427 (0.005)	17.985 (3.20%)	21.979 (26.12%)	16.969 (-2.63%)	17.231 (-1.13%)	17.499 (0.41%)	18.723 (-14.20%)
Average Error		-	30.12%	26.01%	2.48%	2.75%	1.42%	6.65%

Table 2.4: Monte Carlo prices and standard errors (S.E.) and approximated prices, relative errors, and average absolute relative errors by log-normal (LogNormal) and log-normal asymptotic expansions (LN-A.E.) in the λ-SABR model with $\beta = 1$ (case iii).

Case	Strike	M.C.	Approximation Prices (Relative Errors)					
	(C/P)	Price (S.E.)	LogNormal	LN-A.E. 1st	LN-A.E. 2nd	LN-A.E. 3rd	LN-A.E. 4th	Hagan et al.(SABR)
	50 Put	9.429 (0.002)	8.532 (-9.51%)	9.679 (2.65%)	9.899 (4.99%)	9.206 (-2.37%)	9.450 (0.22%)	8.867 (5.95%)
	80 Put	22.043 (0.003)	23.661 (7.34%)	21.942 (-0.46%)	22.455 (1.87%)	21.851 (-0.87%)	22.080 (0.17%)	21.350 (-1.91%)
iii	100 Call	32.974 (0.004)	36.474 (10.62%)	32.555 (-1.27%)	33.350 (1.14%)	32.808 (-0.50%)	33.022 (0.15%)	32.598 (-3.71%)
	120 Call	25.664 (0.006)	30.804 (20.03%)	24.882 (-3.04%)	25.993 (1.29%)	25.520 (-0.56%)	25.718 (0.21%)	25.892 (-7.47%)
	150 Call	17.427 (0.005)	24.332 (39.62%)	16.004 (-8.17%)	17.681 (1.46%)	17.333 (-0.54%)	17.503 (0.43%)	18.723 (-14.20%)
Average Error		-	17.42%	3.12%	2.15%	0.97%	0.24%	6.65%

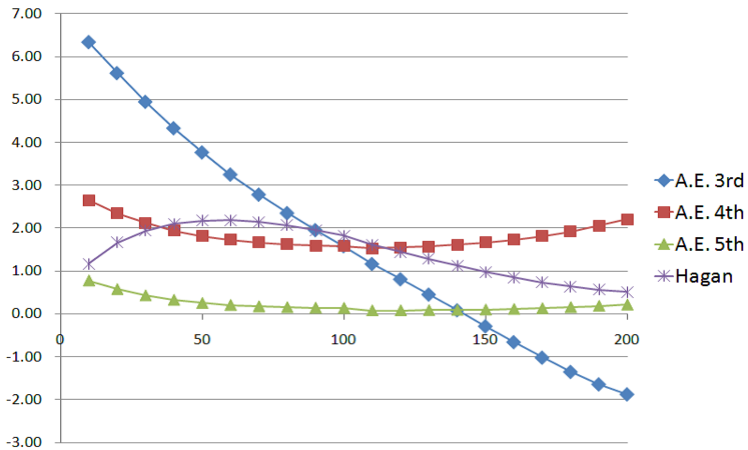


Figure 2.1: Approximation errors in price in case i

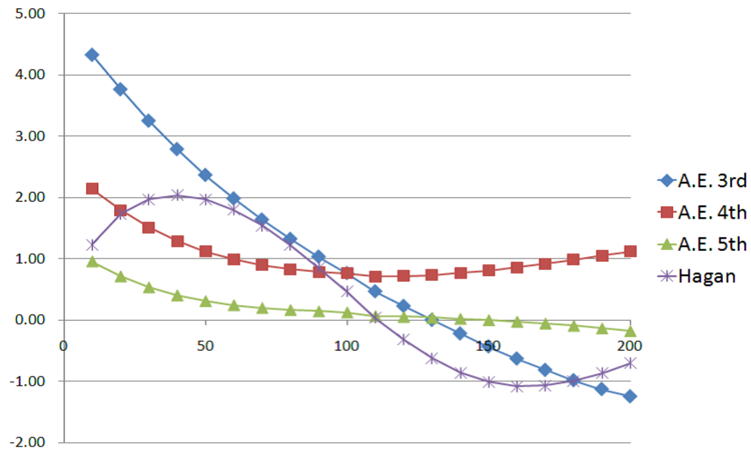


Figure 2.2: Approximation errors in price in case ii

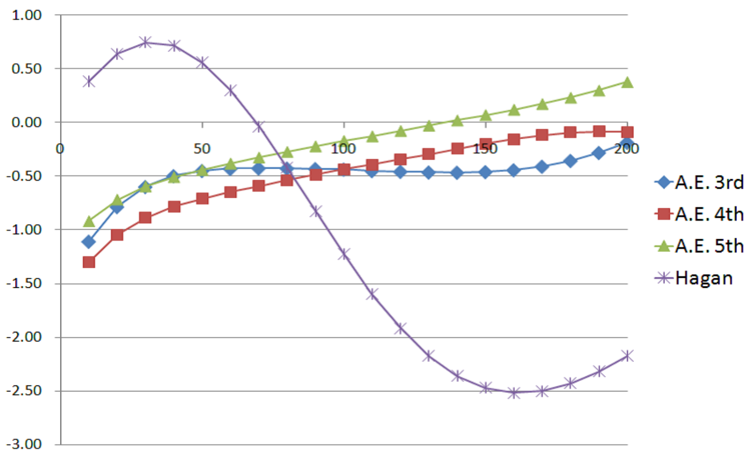


Figure 2.3: Approximation errors in price in case iii

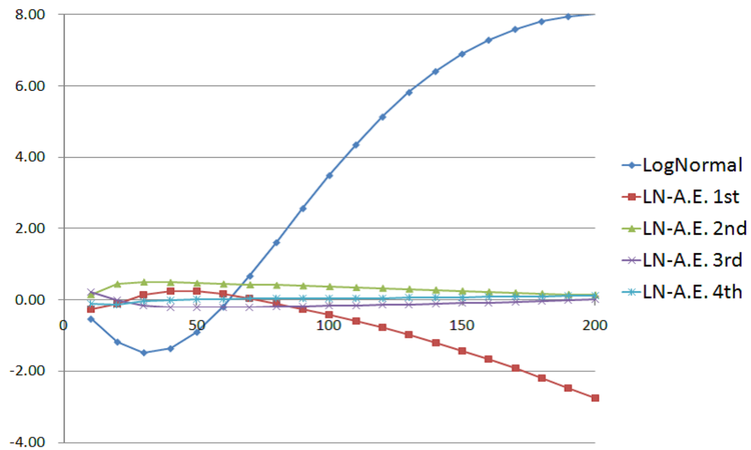


Figure 2.4: Approximation errors in price in case iii (log-normal asymptotic expansions)

2.2 Pricing Average Options under Stochastic Volatility

This section applies the high-order expansion scheme to pricing average options.

In particular, we describe the method using numerical examples under the λ-SABR and SABR models.

2.2.1 Average Options under λ-SABR and SABR Models

We consider the average European call and put options under the λ-SABR model ([14]) with interest rate=0% for simplicity. In particular, when $\lambda = 0$ the model becomes the SABR model. Further, we define

$$S_A^{(\epsilon)}(t) = \int_0^t S^{(\epsilon)}(u)du.$$

Then, the average European call option price with strike K and maturity T can be written as

$$C_A^{(\epsilon)}(K,T) = \mathbf{E}\left[\max\left\{\frac{1}{T}S_A^{(\epsilon)}(T) - K, 0\right\}\right].$$

Thus, if we consider the following three-dimensional diffusion process, we can easily see that it is a special case of (1.1) and the general method can be applied:

$$\begin{aligned}
dS_A^{(\epsilon)}(t) &= S^{(\epsilon)}(t)dt, \\
dS^{(\epsilon)}(t) &= \epsilon\sigma^{(\epsilon)}(t)(S^{(\epsilon)}(t))^\beta dW_t^1, \\
d\sigma^{(\epsilon)}(t) &= \lambda(\theta - \sigma^{(\epsilon)}(t))dt + \epsilon\nu_1\sigma^{(\epsilon)}(t)dW_t^1 + \epsilon\nu_2\sigma^{(\epsilon)}(t)dW_t^2
\end{aligned} \tag{2.3}$$

with $S_A^{(\epsilon)}(0) = 0$, $S^{(\epsilon)}(0) = S_0$ and $\sigma^{(\epsilon)}(0) = \sigma$.

The corresponding differential equations up to the second order are given by

$$\begin{aligned}
\frac{d}{dt}\eta_{1,1}^S(t;\xi) &= (i\xi)(S_t^{(0)})^\beta\sigma_t^{(0)}\hat{V}(t), \\
\frac{d}{dt}\eta_{1,1}^\sigma(t;\xi) &= (i\xi)\nu_1\sigma_t^{(0)}\hat{V}(t) - \lambda\eta_{1,1}^\sigma(t;\xi), \\
\frac{d}{dt}\eta_{2,1}^{S_A}(t;\xi) &= \eta_{2,1}^S(t;\xi), \\
\frac{d}{dt}\eta_{2,1}^S(t;\xi) &= 2(i\xi)\beta(S_t^{(0)})^{\beta-1}\sigma_t^{(0)}\hat{V}(t)\eta_{1,1}^S(t;\xi) + 2(i\xi)(S_t^{(0)})^\beta\hat{V}(t)\eta_{1,1}^\sigma(t;\xi),
\end{aligned}$$

where $S_t^{(0)} = S_0$, $\sigma_t^{(0)} = e^{-\lambda t}(\sigma - \theta) + \theta$ and

$$\hat{V}(t) = (T-t)(S_t^{(0)})^\beta\sigma_t^{(0)}.$$

Then, the asymptotic expansion of the density function of $\tilde{G}^{(\epsilon)} = \frac{S_{AT}{}^{(\epsilon)} - S_{AT}{}^{(0)}}{\epsilon}$ can be obtained as

$$f_{\tilde{G}^{(\epsilon)}}(x) \approx f_{g_{1T}}(x) + \epsilon\tilde{C}_{13}H_3(x;\tilde{\Sigma}_T)f_{g_{1T}}(x) + \cdots \tag{2.4}$$

where

$$f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\tilde{\Sigma}_T}} \exp\left(-\frac{x^2}{2\tilde{\Sigma}_T}\right)$$

with

$$\tilde{\Sigma}_T = \int_0^T \hat{V}(t)^2 dt$$

and

$$\begin{aligned}\tilde{C}_{13} &= \frac{1}{\tilde{\Sigma}_T^3}\int_0^T\int_0^{t_1}\int_0^{t_2} \beta(S_{t_2}^{(0)})^{\beta-1}\sigma_{t_2}^{(0)}\hat{V}(t_2)(S_{t_3}^{(0)})^{\beta}\sigma_{t_3}^{(0)}\hat{V}(t_3)dt_3dt_2dt_1 \\ &\quad +\frac{1}{\tilde{\Sigma}_T^3}\int_0^T\int_0^{t_1}\int_0^{t_2} e^{-\lambda(t_2-t_3)}(S_{t_2}^{(0)})^{\beta}\hat{V}(t_2)\nu_1\sigma_{t_3}^{(0)}\hat{V}(t_3)dt_3dt_2dt_1.\end{aligned}$$

As in the plain vanilla case described in the previous section, integrals appeared in the coefficients of the expansion can be analytically evaluated, but the expressions are lengthy and hence omitted. Moreover, by a similar calculation to the previous case, we have the following closed-form approximation formula for the average European call option up to ϵ^2:

$$\begin{aligned}C_A^{(\epsilon)}(K,T) &= \epsilon P(0,T)\left(\frac{\tilde{\Sigma}_T}{T}f_{g_{1T}}(y) + \frac{y}{T}N\left(\frac{y}{\sqrt{\tilde{\Sigma}_T}}\right)\right) \\ &\quad -\epsilon^2 P(0,T)\frac{\tilde{C}_{13}\tilde{\Sigma}_T^2 y}{T}f_{g_{1T}}(y) + o(\epsilon^2),\end{aligned} \tag{2.5}$$

where $y = \frac{TS_0 - TK}{\epsilon}$ and $P(0,T)$ denotes the price at time 0 of a zero coupon bond with maturity T .

2.2.2 Numerical Examples

This subsection provides some numerical examples of our asymptotic expansion method for pricing average options under the λ-SABR and SABR models to see the effectiveness of the higher order asymptotic expansions. Further, as a special case of the SABR model, we apply our method to the constant volatility case (Black-Scholes model) and compare approximation accuracies of our method with those of other approximation methods.

Constant Volatility Case

First, we apply our method to the constant volatility case (the Black-Scholes model) which is obtained by setting $\lambda = \nu_i = 0 (i = 1, 2)$ and $\beta = 1$ in (2.3). Then, the asymptotic expansion of the density function (2.4) can be simplified as

$$f_{\tilde{G}^{(\epsilon)}}^{BS}(x) \approx f_{g_{1T}}^{BS}(x) + \epsilon\tilde{C}_{13}^{BS}H_3(x;\tilde{\Sigma}_T^{BS})f_{g_{1T}}^{BS}(x) + \cdots,$$

where

$$f_{g_{1T}}^{BS}(x) = \frac{1}{\sqrt{2\pi\tilde{\Sigma}_T^{BS}}} \exp\left(-\frac{x^2}{2\tilde{\Sigma}_T^{BS}}\right)$$

with

$$\begin{aligned}
\tilde{\Sigma}_T^{BS} &= \int_0^T (T-t)^2\sigma^2 S_0^2 dt = \frac{1}{3}\sigma^2 S_0^2 T^3, \\
\tilde{C}_{13}^{BS} &= \frac{1}{(\tilde{\Sigma}_T^{BS})^3}\int_0^T\int_0^{t_1}\int_0^{t_2}(T-t_2)(T-t_3)\sigma^4 S_0^3 dt_3 dt_2 dt_1 = \frac{1}{5}\sigma^2 S_0 T^2. \qquad (2.6)
\end{aligned}$$

A closed-form approximation formula to the average European call option under the Black-Scholes model can be obtained by replacing $\tilde{\Sigma}_T$ and $\tilde{C}_{13}$ by $\tilde{\Sigma}_T^{BS}$ and $\tilde{C}_{13}^{BS}$ respectively in (2.5).

In the Black-Scholes case, unlike the stochastic volatility cases, there are several approximation methods for pricing an average option. Here we compare approximation accuracies of our asymptotic expansion method with those of these existing methods.

We consider the average European call option under the Black-Scholes model. We calculate approximated prices of average options by the asymptotic expansion method up to the fifth order and we also calculate approximated prices by the moment matching method given by Levy[16] and by the lower bound for average options given by Nielsen and Sandmann[21].

In the numerical examples, ϵ is set to be one and other parameters are given in Table 2.5.

Table 2.5: Parameters for the Black-Scholes Models

Case	$S(0)$	σ	T
i	100	0.3	1
ii	100	0.3	2
iii	100	0.5	2

Benchmark values are computed by Monte Carlo simulations. We use the second order scheme given by Ninomiya-Victoir[22] as a discretization scheme with 128 time steps for case i, and with 256 time steps for case ii and iii respectively. We adopt Mersenne-twister as a random number generating engine, and generate 5×10^7 paths with antithetic sampling in each simulation. We calculate the lower bound given by Nielsen and Sandmann with 1024 time steps.

Benchmark prices by Monte Carlo simulations and their standard errors are given in Table 2.6. Also, approximation errors of the moment matching method(Levy), the lower bound given by Nielsen and Sandmann(N-S) and of our asymptotic expansions are reported in Table 2.6.

From the results above, asymptotic expansions almost always improve the accuracy of the approximation as the order of expansion increases and the forth or fifth order asymptotic expansion

Table 2.6: Approximation errors for average call options under Black-Scholes model.

			Levy	N-S			A.E.(Difference)		
Case	Strike(C/P)	MC(s.e.)	(Diff.)	(Diff.)	1st	2nd	3rd	4th	5th
i	70 Call	30.081 (0.002)	0.026	-0.001	0.212	-0.065	-0.007	0.001	0.000
	90 Call	12.667 (0.001)	0.082	0.001	0.363	0.012	-0.002	-0.001	-0.001
	100 Call	6.896 (0.001)	0.031	0.003	0.014	0.014	-0.001	-0.001	-0.001
	110 Call	3.367 (0.001)	-0.031	0.002	-0.336	0.015	0.000	-0.001	-0.001
	130 Call	0.622 (0.000)	-0.054	-0.001	-0.329	-0.051	0.008	0.000	-0.001
ii	70 Call	30.555 (0.002)	0.126	-0.002	0.751	-0.080	-0.026	-0.002	0.001
	90 Call	14.993 (0.002)	0.169	0.003	0.582	0.043	-0.001	0.002	0.002
	100 Call	9.729 (0.002)	0.092	0.005	0.043	0.043	0.001	0.001	0.002
	110 Call	6.067 (0.001)	0.000	0.004	-0.491	0.048	0.004	0.001	0.002
	130 Call	2.168 (0.001)	-0.103	-0.001	-0.862	-0.031	0.022	-0.002	0.001
iii	70 Call	33.179 (0.004)	0.568	-0.016	2.319	0.081	-0.063	-0.006	0.002
	90 Call	20.639 (0.004)	0.536	-0.006	1.134	0.186	-0.014	0.000	0.003
	100 Call	16.095 (0.003)	0.415	-0.003	0.192	0.192	0.000	0.000	0.003
	110 Call	12.509 (0.003)	0.271	-0.003	-0.736	0.212	0.013	-0.001	0.002
	130 Call	7.542 (0.002)	0.008	-0.008	-2.045	0.193	0.050	-0.006	0.002

have smaller or equal approximation errors to those of other methods. Further, as seen in the next subsection, our method can be extended in the same framework to the stochastic volatility case where these other methods cannot be applied.

Stochastic Volatility Case

Next, we consider the stochastic volatility case such as λ-SABR/SABR model described in (2.3).

In the following numerical example, approximated prices by the asymptotic expansion method are calculated up to the fourth order for the λ-SABR model and up to the fifth order for the SABR model respectively. Note that all the solutions to differential equations are obtained analytically. Benchmark values are computed by Monte Carlo simulations. ϵ is set to be one and other parameters used in the test are given in Table 2.7 for the λ-SABR case (i, ii and iii) and the SABR case (iv, v and vi).

Table 2.7: Parameters for the λ-SABR models

Case	$S(0)$	β	$\sigma(0)$	λ	θ	ν	ρ	T
i	100	1.0	0.3	1.0	0.3	0.3	-0.5	1
ii	100	1.0	0.3	1.0	0.3	0.6	-0.5	1
iii	100	1.0	0.3	1.0	0.3	0.3	-0.5	2
iv	100	1.0	0.5	0	-	0.5	-0.5	1
v	100	0.5	3.0	0	-	0.3	-0.5	1
vi	100	1.0	0.5	0	-	0.5	-0.5	2

In Monte Carlo simulations for benchmark values, we use Euler-Maruyama scheme as a discretization scheme with extrapolation method with 256 and 512 time steps for case i, ii, iv, v and with 512 and 1024 time steps for case iii and vi respectively. In each simulation, we generate 5×10^7 paths with antithetic sampling.

Results are in Table 2.8 for the λ-SABR case and in Table 2.9 for the SABR case respectively. Since the solution to the system of ordinary differential equations is solved analytically, computing time for the asymptotic expansions is less than 10^{-3} seconds which is much shorter than that for the Monte Carlo simulations.

From the results above, in each case the higher order asymptotic expansion almost always improves the accuracy of approximation by the lower expansions. In particular, the higher order asymptotic expansions effectively approximate the prices in long-term cases or high-volatility of volatility (ν) cases in which the lower order asymptotic expansions can not approximate the prices well.

Finally, we remark that in the asymptotic expansion method the approximate density functions are expressed as a product of polynomials and the Gaussian density function: Because these polynomial-based approximation functions have wavy forms, higher order approximation sometimes provides worse approximation to the density at particular values (and to the option prices at particular strikes) than lower ones as seen in Table 12 and 13. However, on average absolute differences decrease as higher order correction terms are included.

2.3 A Log-Normal Asymptotic Expansion and its Family

In this section, we develop a slightly different expansion from the usual asymptotic expansion.

Table 2.8: Asymptotic expansions for average options under the λ-SABR model up to the fourth order

Case	Strike(C/P)	MC	A.E.(Difference) 1st	2nd	3rd	4th
i	50 Put	0.000 (0.000)	0.009	-0.010	0.003	-0.001
	80 Put	0.804 (0.000)	0.261	0.011	0.004	0.003
	100 Call	6.873 (0.001)	0.036	0.036	0.005	0.005
	120 Call	1.306 (0.000)	-0.240	0.010	0.004	0.005
	150 Call	0.046 (0.000)	-0.036	-0.017	-0.004	0.000
ii	50 Put	0.005 (0.000)	0.005	-0.001	0.007	-0.001
	80 Put	0.988 (0.000)	0.078	0.002	0.030	0.007
	100 Call	6.886 (0.001)	0.024	0.024	0.007	0.007
	120 Call	1.183 (0.000)	-0.117	-0.042	-0.014	0.009
	150 Call	0.035 (0.000)	-0.025	-0.020	-0.012	-0.004
iii	50 Put	0.024 (0.000)	0.162	-0.076	-0.001	0.001
	80 Put	2.251 (0.001)	0.609	0.060	0.004	0.003
	100 Call	9.685 (0.002)	0.088	0.088	0.001	0.001
	120 Call	3.348 (0.001)	-0.488	0.061	0.005	0.006
	150 Call	0.495 (0.000)	-0.309	-0.071	0.004	0.002

2.3.1 A Log-Normal Asymptotic Expansion for Stochastic Volatility Models

Suppose that an underlying one-dimensional asset process $S^{(\epsilon)}$ and d-dimensional stochastic process $X^{(\epsilon)}$ follow

$$\begin{aligned} dS_t^{(\epsilon)} &= g(X_t^{(\epsilon)})S_t^{(\epsilon)}\bar{\sigma}dW_t; \quad S_0^{(\epsilon)} = s_0, \\ dX_t^{(\epsilon)} &= V_0(X_t^{(\epsilon)}, \epsilon)dt + \epsilon V(X_t^{(\epsilon)})dW_t; \quad X_0^{(\epsilon)} = x_0 \in \mathbf{R}^d \end{aligned}$$

respectively, where g:$\mathbf{R}^d \to \mathbf{R}$ and $\bar{\sigma}$ is a constant vector in $\mathbf{R}^r$. First, let we define $\hat{X}^{(\epsilon)}$ as

$$\hat{X}_t^{(\epsilon)} = \log\left(\frac{S_t^{(\epsilon)}}{s_0}\right).$$

Then, we have

$$\hat{X}_t^{(\epsilon)} = -\frac{|\bar{\sigma}|^2}{2}\int_0^t g(X_u^{(\epsilon)})^2 du + \int_0^t g(X_u^{(\epsilon)})\bar{\sigma}dW_u,$$

Table 2.9: Asymptotic expansions for average options under the SABR model up to the fifth order

Case	Strike(C/P)	MC(s.e.)	A.E.(Difference) 1st	2nd	3rd	4th	5th
iv	50 Put	0.137 (0.000)	0.351	-0.034	0.027	-0.014	-0.012
	80 Put	3.496 (0.001)	0.679	0.136	0.038	0.014	0.002
	100 Call	11.359 (0.002)	0.158	0.158	0.020	0.020	0.007
	120 Call	4.623 (0.001)	-0.448	0.096	-0.001	0.023	0.011
	150 Call	0.964 (0.001)	-0.476	-0.091	-0.029	0.013	0.015
v	50 Put	0.008 (0.000)	0.002	0.002	0.003	0.001	0.000
	80 Put	1.054 (0.000)	0.012	0.012	0.013	0.005	0.004
	100 Call	6.897 (0.001)	0.013	0.013	0.007	0.007	0.006
	120 Call	1.070 (0.000)	-0.004	-0.004	-0.003	0.005	0.003
	150 Call	0.012 (0.000)	-0.002	-0.002	-0.002	0.000	-0.000
vi	50 Put	0.854 (0.000)	1.324	0.170	0.132	-0.020	-0.067
	80 Put	6.883 (0.001)	1.321	0.454	0.120	0.049	-0.020
	100 Call	15.824 (0.003)	0.463	0.463	0.073	0.073	0.002
	120 Call	8.713 (0.002)	-0.509	0.357	0.023	0.093	0.024
	150 Call	3.339 (0.001)	-1.162	-0.008	-0.046	0.106	0.060

and note that

$$\hat{X}_T^{(0)} \sim N(\hat{\mu}_T, \hat{\Sigma}_T),$$

where

$$\begin{aligned}\hat{\mu}_T &= -\frac{|\bar{\sigma}|^2}{2}\int_0^T g(X_u^{(0)})^2 du = -\frac{1}{2}\hat{\Sigma}_T,\\ \hat{\Sigma}_T &= |\bar{\sigma}|^2 \int_0^T g(X_u^{(0)})^2 du.\end{aligned}$$

Moreover, an asymptotic expansion of $\hat{X}_T^{(\epsilon)}$ up to ϵ^N-order is expressed as

$$\hat{X}_T^{(\epsilon)} = \hat{X}_T^{(0)} + \sum_{n=1}^{N} \epsilon^n \hat{A}_{nT} + o(\epsilon^N),$$

where $\hat{A}_{nt} = \frac{1}{n!}\frac{\partial^n \hat{X}_t^{(\epsilon)}}{\partial \epsilon^n}|_{\epsilon=0}$. Note that $S_T^{(\epsilon)}$ is now expanded around a log-normal distribution since $\hat{X}_T^{(0)}$ has the Gaussian distribution (hereafter we call this expansion 'the log-normal asymptotic expansion' of $S_T^{(\epsilon)}$ in contrast to calling the usual asymptotic expansion 'the normal asymptotic expansion').

Next, define $Z^{\langle\xi\rangle} = \{Z_t^{\langle\xi\rangle}; t \in \mathbf{R}^+\}$ as

$$Z_t^{\langle\xi\rangle} = \exp\left(i\xi \int_0^t g(X_u^{(0)})\bar{\sigma} dW_u\right).$$

Then, the result in the usual asymptotic expansion case is applied to deriving the density function of $\hat{X}_T^{(\epsilon)}$ with replacement of $G^{(\epsilon)}$ by $\hat{X}_T^{(\epsilon)}$.

Similar to the normal case, the log-normal asymptotic expansion of the price of the call option on $\hat{X}_T^{(\epsilon)}$ is given by

$$\mathrm{Call}(K,T) \quad = \quad P(0,T)\int_{\log \frac{K}{s_0}}^{\infty} (s_0 e^x - K) f_{\hat{X}_T^{(\epsilon)}}(x) dx.$$

2.3.2 An Asymptotic Expansion around the Shifted Log-Normal

In this subsection, we derive an approximation formula to the option price in the shifted log-normal model with stochastic volatility:

$$\begin{aligned}
&\frac{dS_t^{(\epsilon)}}{S_t^{(\epsilon)} + \alpha} = h(X_t^{(\epsilon)})dt + g(X_t^{(\epsilon)})dW_t \\
&dX_t^{(\epsilon),i} = V_0^i(X_t^{(\epsilon)}, \epsilon)dt + \epsilon V^i(X_t^{(\epsilon)})dW_t \quad (i = 1, \cdots, N) \\
&S_0^{(\epsilon)} = S_0 \in \mathbf{R}, \quad X_0^{(\epsilon)} = X_0 \in \mathbf{R}^N
\end{aligned} \tag{2.7}$$

First, we consider the change of variable:

$$F_t^{(\epsilon)} = S_t^{(\epsilon)} + \alpha.$$

Then,

$$dF_t^{(\epsilon)} = h(X_t^{(\epsilon)})F_t^{(\epsilon)}dt + g(X_t^{(\epsilon)})F_t^{(\epsilon)}dW_t; \quad F_0^{(\epsilon)} = S_0 + \alpha.$$

Thus, $F_t^{(\epsilon)}$ is a log-normal type diffusion with stochastic volatilities and the usual log-normal asymptotic expansion introduced in previous subsection can be applied.

Further, the price of the call option on $S^{(\epsilon)}$ with strike K and maturity T, $C_\alpha(K,T)$ can be written as

$$\begin{aligned}
C_\alpha(K,T) \quad &= \quad P(0,T)\mathbf{E}\left[(S_T^{(\epsilon)} - K)_+\right] \\
&= \quad P(0,T)\mathbf{E}\left[((S_T^{(\epsilon)} + \alpha) - (K+\alpha))_+\right] \\
&= \quad P(0,T)\mathbf{E}\left[(F_T^{(\epsilon)} - (K+\alpha))_+\right].
\end{aligned}$$

Thus, the approximation formula of the call option price is given by

$$C_\alpha(K,T) = P(0,T)\int_{\log\frac{K+\alpha}{S_0+\alpha}}^{\infty}((S_0+\alpha)e^x-(K+\alpha))f_{\hat{X}_T^{(\epsilon)}}(x)dx \tag{2.8}$$

where $f_{\hat{X}_T^{(\epsilon)}}$ is an asymptotic expansion of density of $\hat{X}_T^{(\epsilon)} := \frac{F_T^{(\epsilon)}}{S_0+\alpha}$ which is calculated in a similar way as in the previous subsection. Note that, since $S^{(0)}$ is a shifted log-normal process (displaced-diffusion), this gives an asymptotic expansion around a shifted log-normal process.

2.3.3 An Asymptotic Expansion around the Jump Diffusion

In this subsection, we apply the log-normal asymptotic expansion developed in previous subsections to the jump-diffusion stochastic volatility models where the underlying process $S^{(\epsilon)}$ is the solution to the following stochastic differential equation with jumps:

$$\begin{aligned}
\frac{dS_t^{(\epsilon)}}{S_{t-}^{(\epsilon)}} &= \left(h(X_t^{(\epsilon)})-\lambda m\right)dt + g(X_t^{(\epsilon)})dW_t + (e^{Y_t}-1)dN_t \qquad (2.9)\\
dX_t^{(\epsilon),i} &= V_0^i(X_t^{(\epsilon)},\epsilon)dt + \epsilon V^i(X_t^{(\epsilon)})dW_t \quad (i=1,\cdots,N)\\
S_0^{(\epsilon)} &= S_0 \in \mathbf{R}, \quad X_0^{(\epsilon)} = X_0 \in \mathbf{R}^N
\end{aligned}$$

where N_t is a counting Poisson process with intensity λ, $\{Y_t\}$ are i.i.d. Gaussian random variables with mean μ and variance δ^2, and define $m = \mathbf{E}[e^{Y_t}-1] = e^{\mu+\frac{1}{2}\delta^2}-1$. Note that W_t, N_t and $\{Y_t\}$ are mutually independent.

Define $\hat{X}_t^{(\epsilon)} := \log\frac{S_t^{(\epsilon)}}{S_0}$, then,

$$\hat{X}_t^{(\epsilon)} = \int_0^t \left(h(X_s^{(\epsilon)}) - \frac{1}{2}\|g(X_s^{(\epsilon)})\|^2 - \lambda m\right)ds + \int_0^t g(X_s^{(\epsilon)})dW_s + \sum_{i=1}^{N_t} Y_i, \tag{2.10}$$

and define a continuous part of $\hat{X}_t^{(\epsilon)}$ denoted by $\hat{X}_t^{(\epsilon),c}$ as

$$\hat{X}_t^{(\epsilon),c} = \int_0^t \left(h(X_s^{(\epsilon)}) - \frac{1}{2}\|g(X_s^{(\epsilon)})\|^2\right)ds + \int_0^t g(X_s^{(\epsilon)})dW_s. \tag{2.11}$$

Note that, we can apply the log-normal asymptotic expansion in the previous subsection to $\hat{X}_T^{(\epsilon),c}$, and obtain the approximation formula to the characteristic function of it:

$$\phi_{\hat{X}_T^{(\epsilon),c}}(\xi) = \left\{1+\epsilon\sum_{l=1}^{3}C_{1l}(i\xi)^l + \epsilon^2\sum_{l=1}^{6}C_{2l}(i\xi)^l + \cdots\right\}\exp\{i\xi\hat{\mu}_T - \frac{1}{2}\xi^2\hat{\Sigma}_T\} \tag{2.12}$$

where

$$\begin{aligned}
\hat{\mu}_T &= \int_0^T \left(h(X_t^{(0)}) - \frac{1}{2}\|g(X_t^{(0)})\|^2\right)dt,\\
\hat{\Sigma}_T &= \int_0^T \|g(X_t^{(0)})\|^2 dt,
\end{aligned}$$

and the coefficients $C_{1l}, C_{2l}, \cdots$ are obtained from the solutions to the differential equations.

Moreover, since N_t and $\{Y_i\}$ are independent of W, and $\{Y_i\}$ are i.i.d. normal random variables, the characteristic function of $\hat{X}_T^{(\epsilon)}$ can be written as

$$\begin{aligned}
\phi_{\hat{X}_T^{(\epsilon)}}(\xi) &= \mathbf{E}\left[\exp\{(i\xi)\hat{X}_T^{(\epsilon)}\}\right] \\
&= \mathbf{E}\left[\exp\left\{(i\xi)\left(\hat{X}_T^{(\epsilon),c} - \lambda m T + \sum_{i=1}^{N_T} Y_i\right)\right\}\right] \\
&= e^{-i\xi\lambda m T}\phi_{\hat{X}_T^{(\epsilon),c}}(\xi)\mathbf{E}\left[(\mathbf{E}[\exp\{(i\xi)Y_1\}])^{N_T}\right] \\
&= \sum_{n=0}^{\infty}\frac{e^{-\lambda T}(\lambda T)^n}{n!}e^{-i\xi\lambda m T}\phi_{\hat{X}_T^{(\epsilon),c}}(\xi)e^{i\xi n\mu - \frac{n}{2}\xi^2\delta^2} \\
&= \sum_{n=0}^{\infty}\frac{e^{-\lambda T}(\lambda T)^n}{n!}\left\{1 + \epsilon\sum_{l=1}^{3}C_{1l}(i\xi)^l + \epsilon^2\sum_{l=1}^{6}C_{2l}(i\xi)^l + \cdots\right\} \\
&\quad \times \exp\{i\xi\mu_n - \frac{1}{2}\xi^2\Sigma_n\}
\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}
\mu_n &= \hat{\mu}_T - \lambda m T + n\mu, \\
\Sigma_n &= \hat{\Sigma}_T + n\delta^2.
\end{aligned}$$

Then, applying the inverse Fourier transformation to $\phi_{\hat{X}_T^{(\epsilon)}}$, we have the following series expression of the asymptotic expansion of the density function of $\hat{X}_T^{(\epsilon)}$:

$$\begin{aligned}
f_{\hat{X}_T^{(\epsilon)}}(x) &= \sum_{n=0}^{\infty}\frac{e^{-\lambda T}(\lambda T)^n}{n!}\left\{1 + \epsilon\sum_{l=1}^{3}\frac{C_{1l}}{\Sigma_n^l}H_l(x-\mu_n;\Sigma_n) + \epsilon^2\sum_{l=1}^{6}\frac{C_{2l}}{\Sigma_n^l}H_l(x-\mu_n;\Sigma_n) + \cdots\right\} \\
&\quad \times n(x-\mu_n;\Sigma_n)
\end{aligned} \tag{2.14}$$

An approximation formula to a option price can be obtained from the approximation of the density above. Note that the first term of an approximation of the option price corresponds to the Merton's formula for the jump-diffusion process with a deterministic volatility function.

Remark 3. *The asymptotic expansion around jump-diffusion explained above can be applied to the shifted log-normal model with jumps where $S^{(\epsilon)}$ is given by the solution to the following stochastic differential equation:*

$$\frac{dS_t^{(\epsilon)}}{S_{t-}^{(\epsilon)} + \alpha} = \left(h(X_t^{(\epsilon)}) - \lambda m\right)dt + g(X_t^{(\epsilon)})dW_t + (e^{Y_t} - 1)dN_t. \tag{2.15}$$

In this case, as in the previous subsection, an asymptotic expansion around the jump-diffusion shifted log-normal model can be obtained by replacing $\hat{X}_t^{(\epsilon)} = \log \frac{S_t^{(\epsilon)}}{S_0}$ with $\hat{X}_t^{(\epsilon)} = \log \frac{S_t^{(\epsilon)}+\alpha}{S_0+\alpha}$ in the asymptotic expansion of jump-diffusion.

2.3.4 Numerical Examples

Shifted Log-Normal λ-SABR Model

Next, we consider the following shifted log-normal λ-SABR model:

$$\begin{aligned} dS(t) &= \sigma(t)(S(t)+\alpha)dW_t^1, \\ d\sigma(t) &= \lambda(\theta - \sigma(t))dt + \nu_1\sigma(t)dW_t^1 + \nu_2\sigma(t)dW_t^2, \end{aligned}$$

where α is a shift parameter and assume that $(S_0 + \alpha) > 0$.

Then, we can apply the shifted log-normal asymptotic expansion method introduced in the previous section. Parameter sets for the tests are given in Table 2.10, and the approximate prices and their errors are given in Table 2.11.

Table 2.10: Parameter sets for Shifted Log-Normal λ-SABR model

Parameter	$S(0)$	$\alpha/S(0)$	$\sigma(0)$	λ	θ	ν	ρ	T
iv	100	1.00	0.15	0.1	0.3	0.15	-0.7	10
v	100	1.00	0.15	0.1	0.3	0.15	0.0	10
vi	100	0.33	0.23	0.1	0.3	0.23	0.0	10

Table 2.11: Approximate prices and errors of shifted log-normal asymptotic expansion for shifted λ-SABR model

			Shifted Log Normal A.E.					Shifted Log Normal A.E. (Relative Difference)				
Case	Strike(C/P)	MC	J.D.	1st	2nd	3rd	4th	J.D.	1st	2nd	3rd	4th
i	Put 50	15.776	13.506	15.898	16.306	15.673	15.807	-14.39%	0.77%	3.36%	-0.65%	0.20%
	Put 70	22.464	21.656	22.480	22.888	22.380	22.509	-3.59%	0.07%	1.89%	-0.37%	0.20%
	Put 90	30.826	31.767	30.643	31.156	30.760	30.885	3.05%	-0.59%	1.07%	-0.21%	0.19%
	Call 100	35.667	37.495	35.363	35.952	35.610	35.733	5.12%	-0.85%	0.80%	-0.16%	0.18%
	Call 110	30.957	33.632	30.516	31.196	30.907	31.028	8.64%	-1.42%	0.77%	-0.16%	0.23%
	Call 130	22.864	27.009	22.100	23.006	22.830	22.949	18.13%	-3.34%	0.62%	-0.15%	0.37%
	Call 150	16.462	21.657	15.290	16.498	16.444	16.561	31.55%	-7.12%	0.22%	-0.11%	0.60%
ii	Put 50	15.698	13.506	14.531	16.243	15.810	15.713	-13.96%	-7.43%	3.48%	0.71%	0.10%
	Put 70	23.099	21.656	22.009	23.565	23.215	23.128	-6.25%	-4.72%	2.02%	0.50%	0.13%
	Put 90	32.397	31.767	31.286	32.798	32.524	32.437	-1.94%	-3.43%	1.24%	0.39%	0.13%
	Call 100	37.752	37.495	36.581	38.126	37.888	37.798	-0.68%	-3.10%	0.99%	0.36%	0.12%
	Call 110	33.561	33.632	32.297	33.910	33.706	33.611	0.21%	-3.77%	1.04%	0.43%	0.15%
	Call 130	26.445	27.009	24.905	26.749	26.615	26.503	2.13%	-5.82%	1.15%	0.64%	0.22%
	Call 150	20.828	21.657	18.928	21.092	21.029	20.894	3.98%	-9.12%	1.26%	0.96%	0.31%
iii	Put 50	12.604	11.056	11.777	13.224	12.733	12.594	-12.28%	-6.56%	4.92%	1.02%	-0.08%
	Put 70	20.655	19.882	19.890	21.211	20.799	20.659	-3.74%	-3.70%	2.69%	0.70%	0.02%
	Put 90	30.806	30.879	30.003	31.312	30.974	30.821	0.23%	-2.61%	1.64%	0.54%	0.05%
	Call 100	36.612	37.064	35.740	37.099	36.794	36.631	1.23%	-2.38%	1.33%	0.50%	0.05%
	Call 110	32.862	33.644	31.889	33.333	33.059	32.885	2.38%	-2.96%	1.43%	0.60%	0.07%
	Call 130	26.551	27.823	25.290	26.992	26.782	26.581	4.79%	-4.75%	1.66%	0.87%	0.11%
	Call 150	21.581	23.126	19.956	21.998	21.852	21.618	7.16%	-7.53%	1.93%	1.25%	0.17%

Stochastic Volatility Model with Jumps

We consider the European plain-vanilla call and put option prices under the following jump-diffusion stochastic volatility model:

$$\begin{aligned} dS^{(\epsilon)}(t) &= -\eta m S^{(\epsilon)}(t-)dt + \epsilon\sigma^{(\epsilon)}(t)S^{(\epsilon)}(t-)dW_t^1 + S^{(\epsilon)}(t-)(e^{Y_t}-1)dN_t, \\ d\sigma^{(\epsilon)}(t) &= \lambda(\theta - \sigma^{(\epsilon)}(t))dt + \epsilon\nu_1\sigma^{(\epsilon)}(t)dW_t^1 + \epsilon\nu_2\sigma^{(\epsilon)}(t)dW_t^2, \end{aligned}$$

where $W = (W^1, W^2)$ is a 2-dimensional standard Wiener process, N is compound Poisson process with intensity η, $\{Y_t\}$ are i.i.d. Gaussian random variables with mean μ, variance δ^2 and $m := \mathbf{E}[e^Y - 1] = e^{\mu+\frac{\delta^2}{2}} - 1$.

Note that, the log price of the underlying asset, denoted by $\hat{X}^{(\epsilon)}(t) = \log \frac{S^{(\epsilon)}(t)}{S_0}$ is expressed as

$$\hat{X}^{(\epsilon)}(t) = \int_0^t \left(-\frac{1}{2}\sigma^{(\epsilon)}(s)^2 - \eta m\right) ds + \int_0^t \sigma^{(\epsilon)}(s)dW_s + \sum_{i=1}^{N_t} Y_i.$$

The parameters for the test is given in Table 2.12.

Table 2.12: Parameter sets for the Jump-Diffusion Stochastic Volatility model

case	$S(0)$	$\sigma(0)$	λ	θ	ν	ρ	η	m	δ	T
i	100	0.3	0.1	0.3	0.3	-0.7	1.0	0.0	0.3	1
ii	100	0.3	0.1	0.3	0.3	-0.7	0.5	0.0	0.5	1
iii	100	0.3	0.1	0.3	0.3	-0.7	1.0	0.0	0.3	5

In Monte Carlo simulations for the benchmark values, we first simulate the continuous part of $\hat{X}$ discretized by Euler-Maruyama scheme with time steps 512 in case i and ii and 1024 in case iii, and combine $\hat{X}$ with the independently generated N_T and Y_ts to generate S_T.

Results for the numerical experiments are given in Table 2.13.

Table 2.13: Approximate prices and errors of asymptotic expansion for stochastic volatility model with jumps in Table 2.12.

			Log Normal A.E. for SVJ					Log Normal A.E. SVJ (Relative Difference)				
Case	Strike(C/P)	MC	J.D.	1st	2nd	3rd	4th	J.D.	1st	2nd	3rd	4th
i	Put 50	0.9679	0.7676	0.9015	0.9646	0.9723	0.9688	-20.69%	-6.85%	-0.34%	0.46%	0.10%
	Put 70	4.1730	3.8285	4.1233	4.1848	4.1759	4.1753	-8.26%	-1.19%	0.28%	0.07%	0.05%
	Put 90	11.1228	11.0060	11.0860	11.1331	11.1265	11.1263	-1.05%	-0.33%	0.09%	0.03%	0.03%
	Call 100	16.1989	16.2934	16.1552	16.2074	16.2023	16.2020	0.58%	-0.27%	0.05%	0.02%	0.02%
	Call 110	12.2939	12.5845	12.2371	12.2994	12.2957	12.2954	2.36%	-0.46%	0.04%	0.01%	0.01%
	Call 130	7.0322	7.5303	6.9347	7.0338	7.0330	7.0326	7.08%	-1.39%	0.02%	0.01%	0.01%
	Call 150	4.1295	4.5996	3.9864	4.1327	4.1305	4.1288	11.38%	-3.46%	0.08%	0.02%	-0.02%
ii	Put 50	9.0069	8.4574	8.8554	9.1729	9.0199	9.0019	-6.10%	-1.68%	1.84%	0.14%	-0.06%
	Put 70	17.8740	17.8102	17.6818	18.0495	17.8729	17.8751	-0.36%	-1.08%	0.98%	-0.01%	0.01%
	Put 90	29.0894	29.6041	28.7887	29.2578	29.0818	29.0898	1.77%	-1.03%	0.58%	-0.03%	0.00%
	Call 100	35.4199	36.2192	35.0590	35.5946	35.4209	35.4279	2.26%	-1.02%	0.49%	0.00%	0.02%
	Call 110	32.1576	33.2136	31.7252	32.3349	32.1626	32.1670	3.28%	-1.34%	0.55%	0.02%	0.03%
	Call 130	26.6669	28.1426	26.0708	26.8442	26.6697	26.6651	5.53%	-2.24%	0.66%	0.01%	-0.01%
	Call 150	22.2926	24.0834	21.5424	22.4902	22.3038	22.2870	8.03%	-3.37%	0.89%	0.05%	-0.03%
iii	Put 50	1.4370	1.2554	1.3653	1.4328	1.4417	1.4369	-12.64%	-4.99%	-0.29%	0.33%	-0.01%
	Put 70	4.8796	4.4952	4.8292	4.8905	4.8789	4.8790	-7.88%	-1.03%	0.22%	-0.01%	-0.01%
	Put 90	11.8184	11.6674	11.7849	11.8244	11.8176	11.8177	-1.28%	-0.28%	0.05%	-0.01%	-0.01%
	Call 100	16.8704	16.9734	16.8310	16.8749	16.8698	16.8698	0.61%	-0.23%	0.03%	0.00%	0.00%
	Call 110	12.9724	13.3125	12.9205	12.9737	12.9703	12.9704	2.62%	-0.40%	0.01%	-0.02%	-0.02%
	Call 130	7.8336	8.3998	7.7358	7.8294	7.8315	7.8320	7.23%	-1.25%	-0.05%	-0.03%	-0.02%
	Call 150	5.1017	5.5869	4.9458	5.0981	5.1012	5.1001	9.51%	-3.06%	-0.07%	-0.01%	-0.03%

2.4 An Asymptotic Expansion Method with Change of Variables

This section presents an extension of a general computational scheme of an asymptotic expansion described in Chapter 1. In particular, through change of variable technique as well as the various ways of setting perturbation parameters in an expansion, we provide flexibility of setting the benchmark distribution around which the expansion is made and an automatic way for computation up to an arbitrary order in the expansion. We also show some concrete examples with numerical experiment. This section is based on the paper [36].

2.4.1 A Framework

We consider a d-dimensional diffusion process $X_t = (X_t^1, \cdots, X_t^d)$ which is the solution to the following stochastic differential equation:

$$\begin{aligned} dX_t^j &= V_0^j(X_t)dt + V^j(X_t)dW_t \quad (j = 1, \cdots, d) \\ X_0 &= x_0 \in \mathbf{R}^d \end{aligned} \tag{2.16}$$

where $W = (W^1, \cdots, W^r)$ is an r-dimensional standard Wiener process; $V_0^j : \mathbf{R}^d \mapsto \mathbf{R}$ and $V^j : \mathbf{R}^d \mapsto \mathbf{R}^d$ are smooth functions with bounded derivatives of all orders.

Next, let $C : \mathbf{R}^d \mapsto \mathbf{R}^d$ be a $\mathbf{C}^2$-function which has the unique inverse function, C^{-1}, and define $\tilde{X}_t$ as $\tilde{X}_t = C(X_t)$. Then, the dynamics of $\tilde{X}$ is given by

$$\begin{aligned} d\tilde{X}_t^j &= \tilde{V}_0^j(\tilde{X}_t)dt + \tilde{V}^j(\tilde{X}_t)dW_t \quad (j = 1, \cdots, d), \\ \tilde{X}_0 &= \tilde{x}_0, \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} \tilde{V}_0^j(\tilde{x}) &:= \sum_{j'=1}^{d} \partial_{j'} C^j(\tilde{x}) V_0^{j'}(C^{-1}(\tilde{x})) + \frac{1}{2} \sum_{j',k'=1}^{d} \partial_{j'k'} C^j(\tilde{x}) V^{j'}(C^{-1}(\tilde{x})) V^{k'}(C^{-1}(\tilde{x}))', \\ \tilde{V}^j(\tilde{x}) &:= \sum_{j'=1}^{d} \partial_{j'} C^j(\tilde{x}) V^{j'}(C^{-1}(\tilde{x})), \end{aligned}$$

and $\tilde{x}_0 = C(x_0)$.

Next, we introduce a perturbation parameter $\epsilon \in (0, 1]$ as follows:

$$\begin{aligned} &\tilde{X}_t \mapsto \tilde{X}_t^{(\epsilon)} \\ &\tilde{V}_0^j(\tilde{x}, \epsilon) \mapsto \tilde{V}_0^{(\epsilon),j}(\tilde{x}, \epsilon) \\ &\tilde{V}^j(\tilde{x}) \mapsto \epsilon \tilde{V}^j(\tilde{x}), \end{aligned}$$

and hence, the dynamics of $\tilde{X}^{(\epsilon)}$ is expressed as

$$d\tilde{X}_t^{(\epsilon),j} = \tilde{V}_0^{(\epsilon),j}(\tilde{X}_t^{(\epsilon)},\epsilon)dt + \epsilon\tilde{V}^j(\tilde{X}_t^{(\epsilon)})dW_t \quad (j=1,\cdots,d). \tag{2.18}$$

Hereafter, let us apply the technique developed in [35] to the transformed SDE (2.18). Firstly, take a smooth function $g:\mathbf{R}^d \mapsto \mathbf{R}$ with all of the derivatives having polynomial growth orders. Then, a smooth Wiener functional $g(X_T^{(\epsilon)})$ has its asymptotic expansion:

$$g(\tilde{X}_T^{(\epsilon)}) \sim g_{0T} + \epsilon g_{1T} + \epsilon^2 g_{2T} + \cdots \tag{2.19}$$

in L^p for every $p>1$(or in $\mathbf{D}^\infty$) as $\epsilon \downarrow 0$.

Let $A_{kt} = \frac{1}{k!}\frac{\partial^k \tilde{X}_t^{(\epsilon)}}{\partial \epsilon^k}|_{\epsilon=0}$ and A_{kt}^j, $j=1,\cdots,d$ denote the j-th elements of A_{kt}. In particular, A_{1t} is represented by

$$A_{1t} = \int_0^t Y_t Y_u^{-1}\left(\partial_\epsilon \tilde{V}_0(\tilde{X}_u^{(0)},0)du, +\tilde{V}(\tilde{X}_u^{(0)})dW_u\right), \tag{2.20}$$

where $\tilde{V}_0 = (\tilde{V}_0^1,\cdots,\tilde{V}_0^d)$: $\mathbf{R}^d \times (0,1] \mapsto \mathbf{R}^d$, and $V = (\tilde{V}^1,\cdots,\tilde{V}^d)$: $\mathbf{R}^d \mapsto \mathbf{R}^d \otimes \mathbf{R}^r$; Y denotes the solution to the differential equation:

$$dY_t = \partial\tilde{V}_0(\tilde{X}_t^{(0)},0)Y_t dt;\ Y_0 = I_d.$$

For $k \geq 2$, A_{kt}^j, $j=1,\cdots,d$ is recursively determined by the following:

$$\begin{aligned} A_{kt}^j &= \frac{1}{k!}\int_0^t \partial_\epsilon^k \tilde{V}_0^j(\tilde{X}_u^{(0)},0)du \\ &\quad + \sum_{l=1}^k \sum_{\vec{l}_\beta,\vec{d}_\beta}^{(l)} \frac{1}{(k-l)!}\frac{1}{\beta!}\int_0^t \left(\prod_{j=1}^\beta A_{l_j u}^{d_j}\right)\partial_{\vec{d}_\beta}^\beta \partial_\epsilon^{k-l}\tilde{V}_0^j(\tilde{X}_u^{(0)},0)du \\ &\quad + \sum_{\vec{l}_\beta,\vec{d}_\beta}^{(k-1)} \frac{1}{\beta!}\int_0^t \left(\prod_{j=1}^\beta A_{l_j u}^{d_j}\right)\partial_{\vec{d}_\beta}^\beta \tilde{V}^j(\tilde{X}_u^{(0)})dW_u. \end{aligned} \tag{2.21}$$

Then, g_{0T} and g_{1T} can be written as

$$\begin{aligned} g_{0T} &= g(\tilde{X}_T^{(0)}), \\ g_{1T} &= \sum_{j=1}^d \partial_j g(\tilde{X}_T^{(0)})A_{1T}^j. \end{aligned}$$

For $n \geq 2$, $g_{nT} = \frac{1}{n!}\left.\frac{\partial^n g(\tilde{X}_T^{(\epsilon)})}{\partial\epsilon^n}\right|_{\epsilon=0}$ is expressed as follows:

$$g_{nT} = \sum_{\vec{l}_\beta,\vec{d}_\beta}^{(n)} \frac{1}{\beta!}\partial_{\vec{d}_\beta}^\beta g(\tilde{X}_T^{(0)})A_{l_1T}^{d_1}\cdots A_{l_\beta T}^{d_\beta}. \tag{2.22}$$

Next, normalize $g(\tilde{X}_T^{(\epsilon)})$ to

$$G^{(\epsilon)} = \frac{g(\tilde{X}_T^{(\epsilon)}) - g_{0T}}{\epsilon}$$

for $\epsilon \in (0,1]$. Then,

$$G^{(\epsilon)} \sim g_{1T} + \epsilon g_{2T} + \cdots$$

in L^p for every $p > 1$.

Moreover, let

$$\hat{V}(x,t) = (\partial g(x))' [Y_T Y_t^{-1} \tilde{V}(x)]$$

and make the following assumption:

$$\text{(Assumption 1')} \quad \Sigma_T = \int_0^T \hat{V}(\tilde{X}_t^{(0)}, t) \hat{V}(\tilde{X}_t^{(0)}, t)' dt > 0.$$

Note that g_{1T} follows a normal distribution with variance Σ_T; the density function of g_{1T} denoted by $f_{g_{1T}}(x)$ is given by

$$f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp\left(-\frac{(x-c)^2}{2\Sigma_T} \right) \tag{2.23}$$

where

$$c = (\partial g(\tilde{X}_T^{(0)}))' \int_0^T Y_T Y_t^{-1} \partial_\epsilon \tilde{V}_0(\tilde{X}_t^{(0)}, 0) dt.$$

Hence, (Assumption 1) means that the distribution of g_{1T} does not degenerate.

Let $\mathcal{S}$ be the real Schwartz space of rapidly decreasing C^∞-functions on $\mathbf{R}$ and $\mathcal{S}'$ be its dual space.

Next, take $\Phi \in \mathcal{S}'$. Then, the asymptotic expansion of a generalized Wiener functional $\Phi(G^{(\epsilon)})$ as $\epsilon \downarrow 0$ can be verified by Watanabe theory. In particular, if we take the delta function at $x \in \mathbf{R}$, δ_x as Φ, we obtain an asymptotic expansion of the density for $G^{(\epsilon)}$.

That is, the expectation of $\Phi(G^{(\epsilon)})$ is expanded as follows:

$$\begin{aligned}
\mathbf{E}[\Phi(G^{(\epsilon)})] &= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \mathbf{E}\left[\Phi^{(\delta)}(g_{1T}) \prod_{j=1}^{\delta} g_{(k_j+1)T} \right] + o(\epsilon^N) \\
&= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \int_{\mathbf{R}} \Phi^{(\delta)}(x) \mathbf{E}\left[\tilde{X}^{\vec{k}_\delta} | g_{1T} = x \right] f_{g_{1T}}(x) dx + o(\epsilon^N) \\
&= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \int_{\mathbf{R}} \Phi(x) (-1)^\delta \frac{d^\delta}{dx^\delta} \left\{ \mathbf{E}\left[\tilde{X}^{\vec{k}_\delta} | g_{1T} = x \right] f_{g_{1T}}(x) \right\} dx + o(\epsilon^N)
\end{aligned} \tag{2.24}$$

where $\Phi^{(\delta)}(g_{1T}) = \left. \frac{d^\delta \Phi(x)}{dx^\delta} \right|_{x=g_{1T}}$, $\sum_{\vec{k}_\delta}^{(n)} = \sum_{\delta=1}^{n} \sum_{\vec{k}_\delta \in L_{n,\delta}}$, and

$$\tilde{X}^{\vec{k}_\delta} := \prod_{j=1}^{\delta} g_{(k_j+1)T}. \tag{2.25}$$

To compute the asymptotic expansion (2.24), we need to evaluate the conditional expectations of the form

$$E\left[\tilde{X}^{\vec{k}_\delta}\middle| g_{1T} = x\right]$$

where $\tilde{X}^{\vec{k}_\delta}$ is represented by a product of multiple Wiener-Itô integrals.

The next theorem shows a general result for an asymptotic expansion of the density function for $G^{(\epsilon)}$. In particular, the coefficients in the expansion are obtained through the solution of a system of ordinary differential equations(ODEs). The key point is that each ordinary differential equation(ODE) does not involve any higher order terms, and only lower or the same order terms appear in the right hand side of the ODE. Hence, one can easily solve (analytically or numerically) the system of ODEs.

Theorem 4. *The asymptotic expansion of the density function of* $G^{(\epsilon)} = \frac{g(\tilde{X}_T^{(\epsilon)}) - g(\tilde{X}_T^{(0)})}{\epsilon}$ *up to* ϵ^N*-order is given by*

$$\begin{aligned} f_{G^{(\epsilon)}}(x) &= f_{g_{1T}}(x) \\ &\quad + \sum_{n=1}^{N} \epsilon^n \left(\sum_{m=0}^{3n} C_{nm} H_m(x - c, \Sigma_T) \right) f_{g_{1T}}(x) + o(\epsilon^N), \end{aligned} \tag{2.26}$$

where $H_n(x;\Sigma)$ *is the Hermite polynomial of degree* n *which is defined as*

$$H_n(x;\Sigma) = (-\Sigma)^n e^{x^2/2\Sigma} \frac{d^n}{dx^n} e^{-x^2/2\Sigma}, \tag{2.27}$$

and

$$\begin{aligned} C_{nm} &= \frac{1}{\Sigma_T^m} \sum_{\vec{k}_\delta}^{(m)} \sum_{\vec{l}^1_{\beta_1}, \vec{d}^1_{\beta_1}}^{(k_1+1)} \cdots \sum_{\vec{l}^\delta_{\beta_\delta}, \vec{d}^\delta_{\beta_\delta}}^{(k_\delta+1)} \frac{1}{\delta!(m-\delta)!} \\ &\quad \times \left(\prod_{j=1}^{\delta} \frac{1}{\beta_j!} \partial^{\beta_j}_{\vec{d}^j_{\beta_j}} g(\tilde{X}_T^{(0)}) \right) \frac{1}{i^{m-\delta}} \frac{\partial^{m-\delta}}{\partial \xi^{m-\delta}} \eta^{\vec{d}^1_{\beta_1} \otimes \cdots \otimes \vec{d}^\delta_{\beta_\delta}}_{\vec{l}^1_{\beta_1} \otimes \cdots \otimes \vec{l}^\delta_{\beta_\delta}}(T;\xi) \Bigg|_{\xi=0}, \quad \left(i = \sqrt{-1}\right). \end{aligned} \tag{2.28}$$

$\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(T;\xi)$ are obtained as a solution to the following system of ODEs:

$$\begin{aligned}
\frac{d}{dt}\left\{\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(t;\xi)\right\} &= \sum_{k=1}^{\beta}\frac{1}{l_k!}\eta^{\vec{d}_{\beta/k}}_{\vec{l}_{\beta/k}}(t;\xi)\partial_\epsilon^{l_k}\tilde{V}_0^{d_k}(\tilde{X}_t^{(0)},0) \\
&\quad +\sum_{k=1}^{\beta}\sum_{l=1}^{l_k}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l)}\frac{1}{(l_k-l)!}\frac{1}{\gamma!}\eta^{(\vec{d}_{\beta/k})\otimes\vec{\tilde{d}}_\gamma}_{(\vec{l}_{\beta/k})\otimes\vec{m}_\gamma}(t;\xi)\partial^\gamma_{\vec{\tilde{d}}_\gamma}\partial_\epsilon^{l_k-l}\tilde{V}_0^{d_k}(\tilde{X}_t^{(0)},0) \\
&+ \sum_{\substack{k,m=1\\k<m}}^{\beta}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)}\sum_{\vec{m}_\delta,\vec{\tilde{d}}_\delta}^{(l_m-1)}\frac{1}{\gamma!\delta!}\eta^{(\vec{d}_{\beta/k,m})\otimes\vec{\tilde{d}}_\gamma\otimes\vec{\tilde{d}}_\delta}_{(\vec{l}_{\beta/k,m})\otimes\vec{m}_\gamma\otimes\vec{m}_\delta}(t;\xi) \\
&\quad \times\partial^\gamma_{\vec{\tilde{d}}_\gamma}\tilde{V}^{d_k}(\tilde{X}_t^{(0)})\partial^\delta_{\vec{\tilde{d}}_\delta}\tilde{V}^{d_m}(\tilde{X}_t^{(0)}) \\
&+ (i\xi)\sum_{k=1}^{\beta}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)}\frac{1}{\gamma!}\eta^{(\vec{d}_{\beta/k})\otimes\vec{\tilde{d}}_\gamma}_{(\vec{l}_{\beta/k})\otimes\vec{m}_\gamma}(t;\xi)\partial^\gamma_{\vec{\tilde{d}}_\gamma}\tilde{V}^{d_k}(\tilde{X}_t^{(0)})\hat{V}(\tilde{X}_t^{(0)},t) \\
\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(0;\xi) &= 0 \text{ for } (\vec{l}_\beta,\vec{d}_\beta)\neq(\emptyset,\emptyset),\ \eta^{(\emptyset)}_{(\emptyset)}(t;\xi)=1.
\end{aligned} \tag{2.29}$$

Here, we use the following notations:

$$\begin{aligned}
\vec{l}_{\beta/k} &:= (l_1,\cdots,l_{k-1},l_{k+1},\cdots,l_\beta) \\
\vec{l}_{\beta/k,n} &:= (l_1,\cdots,l_{k-1},l_{k+1},\cdots,l_{n-1},l_{n+1},\cdots,l_\beta),\ 1\le k<n\le\beta \\
\vec{l}_\beta\otimes\vec{m}_\gamma &:= (l_1,\cdots,l_\beta,m_1,\cdots,m_\gamma)
\end{aligned}$$

for $\vec{l}_\beta=(l_1,\cdots,l_\beta)$ and $\vec{m}_\gamma=(m_1,\cdots,m_\gamma)$.

The proof is given in Sections 3 and 5 of [35].

Remark 4. *Due to the hierarchical structure of the ODEs with respect to $l=\sum_{j=1}^{\beta}l_j$ and $\eta^{(\emptyset)}_{(\emptyset)}(t;\xi)=1$, one can easily solve these ODEs successively from lower order terms to higher order terms with initial conditions $\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(0;\xi)=0$ for $(\vec{l}_\beta,\vec{d}_\beta)\neq(\emptyset,\emptyset)$. For instance, $\eta^j_{(1)}$, $\eta^{j,k}_{(1,1)}$ and $\eta^j_{(2)}$ are evaluated in the following order:*

$$\eta^j_{(1)}\to\ \eta^{j,k}_{(1,1)}\to\ \eta^j_{(2)}.$$

2.4.2 Applications to Option Pricing

Given the above theorem for an approximation of the density, we can easily derive approximation formulas for option prices under various models.

For instance, let us evaluate a plain-vanilla call option on the underlying asset whose price process is given by X^1 where X^1 denotes the first element of X. We first determine the *change* *of variable* function, C such that

$$C(x) = (C_1(x^1), C_{d-1}(x^2, \cdots, x^d)),$$

where x^j denotes the j-th element of $x \in \mathbf{R}^d$, and $C_1 : \mathbf{R} \mapsto \mathbf{R}$ and $C_{d-1} : \mathbf{R}^{d-1} \mapsto \mathbf{R}^{d-1}$ are some invertible functions. Then, we have $\tilde{X}_t = C(X_t)$ for all $t \in [0, T]$.

Next, we introduce a perturbation parameter $\epsilon \in [0, 1]$ to get $\tilde{X}_t^{(\epsilon)} = (\tilde{X}_t^{(\epsilon),1}, \cdots, \tilde{X}_t^{(\epsilon),d})$ for all $t \in [0, T]$ as in (2.18), and define $X_T^{(\epsilon),1} = C^{-1}\left(\tilde{X}_T^{(\epsilon),1}\right)$. (In particular, $X^1 = C^{-1}\left(\tilde{X}_T^{(1),1}\right)$.) Also, we set a smooth function $g : \mathbf{R}^d \mapsto \mathbf{R}$ (appearing in (2.19) of the previous subsection) as $g(x) = x^1$ for $x = (x^1, \cdots, x^d)$.

Let us consider an approximation of the call option price, $Call^{(\epsilon)}(K, T)$ with maturity T and strike price K, whose payoff is given by

$$\left(X_T^{(\epsilon),1} - K\right)_+ := \max\left\{X_T^{(\epsilon),1} - K, 0\right\}.$$

Then, we obtain an approximation of the call price as follows:

$$\begin{aligned} Call^{(\epsilon)}(K,T) &= P(0,T)\mathbf{E}\left[\left(C_1^{-1}\left(\tilde{X}_T^{(\epsilon),1}\right) - K\right)_+\right] = P(0,T)\mathbf{E}\left[\left(C_1^{-1}\left(\epsilon G^{(\epsilon)} + \tilde{X}_T^{(0),1}\right) - K\right)_+\right] \\ &\approx P(0,T)\int_{y^{(\epsilon)}}^{\infty}\left(C_1^{-1}\left(\epsilon x + \tilde{X}_T^{(0),1}\right) - K\right) f_{G^{(\epsilon)},N}(x)dx, \end{aligned}$$

where

$$G^{(\epsilon)} = \frac{\left(\tilde{X}_T^{(\epsilon),1} - \tilde{X}_T^{(0),1}\right)}{\epsilon}, \tag{2.30}$$

$$y^{(\epsilon)} = \frac{C_1(K) - \tilde{X}_T^{(0),1}}{\epsilon}. \tag{2.31}$$

Here, $P(0, T)$ stands for the price at time 0 of a zero coupon bond with maturity T, and $f_{G^{(\epsilon)},N}$ denotes the asymptotic expansion of density of $G^{(\epsilon)}$ up to ϵ^N-th order:

$$f_{G^{(\epsilon)},N}(x) = f_{g_{1T}}(x) + \sum_{n=1}^{N}\epsilon^n\left(\sum_{m=0}^{3n} C_{nm}H_m(x - c, \Sigma_T)\right) f_{g_{1T}}(x), \tag{2.32}$$

which comes from the first and second terms of (2.26) in Theorem 4.

Particularly, when $\epsilon = 1$, the payoff is given by

$$\left(X_T^1 - K\right)_+ = \left(X_T^{(1),1} - K\right)_+. \tag{2.33}$$

Then, an approximation of the call price, $Call(K, T) \equiv Call^{(1)}(K, T)$ with maturity T and strike price K is obtained by

$$\begin{aligned} Call(K,T) &= P(0,T)\mathbf{E}\left[\left(C_1^{-1}(\tilde{X}_T^1) - K\right)_+\right] = P(0,T)\mathbf{E}\left[\left(C_1^{-1}\left(G^{(1)} + \tilde{X}_T^{(0),1}\right) - K\right)_+\right] \\ &\approx P(0,T)\int_{y}^{\infty}\left(C_1^{-1}\left(x + \tilde{X}_T^{(0),1}\right) - K\right) f_{G^{(1)},N}(x)dx, \end{aligned} \tag{2.34}$$

where

$$G^{(1)} = \tilde{X}_T^{(1),1} - \tilde{X}_T^{(0),1}, \tag{2.35}$$

$$y = C_1(K) - \tilde{X}_T^{(0),1}, \tag{2.36}$$

and $f_{G^{(1)},N}$ is given by

$$f_{G^{(1)},N}(x) = f_{g_{1T}}(x) + \sum_{n=1}^{N}\left(\sum_{m=0}^{3n} C_{nm} H_m(x - C, \Sigma_T)\right) f_{g_{1T}}(x). \tag{2.37}$$

Various approximation formulas could be obtained through choice of change of variable function C or/and the way to setting the perturbation parameter ϵ in $\tilde{V}_0^j(\tilde{X}_t^{(\epsilon)}, \epsilon)$ of (2.18), for instance, $\tilde{V}_0^j(\tilde{X}_t^{(\epsilon)})$, $\epsilon\tilde{V}_0^j(\tilde{X}_t^{(\epsilon)})$, $\epsilon^2\tilde{V}_0^j(\tilde{X}_t^{(\epsilon)}), \cdots$. Then, the limiting distribution of the underlying asset price may become normal, log-normal, shifted log-normal, non-central chi-square, and so on. The next subsection will illustrate option pricing under a local-stochastic volatility model.

Option Pricing under Local-Stochastic Volatility Model

We assume the underlying process is the unique solution to the following SDE:

$$\begin{aligned}
&dS_t = \sigma(X_t)h(S_t)dW_t \\
&dX_t^j = V_0^j(X_t)dt + V^j(X_t)dW_t \quad (j = 2, \cdots, d) \\
&S_0 = s_0 \in \mathbf{R}, \quad X_0 = x_0 \in \mathbf{R}^{d-1},
\end{aligned} \tag{2.38}$$

where $\sigma : \mathbf{R}^{d-1} \to \mathbf{R}^r$, $h : \mathbf{R} \to \mathbf{R}$, and W is a r-dimensional Brownian motion. Then, we evaluate a call option with strike K and maturity T, whose underlying price process is given by S. Under the zero discount interest, for simplicity, the call price $Call(K,T)$ is obtained by

$$Call(K,T) = \mathbf{E}[(S_T - K)_+]. \tag{2.39}$$

First, for $x = (x^1, x^2, \cdots, x^d)$, let

$$C(x) = (C_1(x^1), x^2, \cdots, x^d),$$

where $C_1 : \mathbf{R} \to \mathbf{R}$ be an invertible $\mathbf{C}^2$-function. Then, $\tilde{S}_t = C_1(S_t)$, and the dynamics of $\tilde{S}$ is given by

$$d\tilde{S}_t = \frac{1}{2}||\sigma(X_t)||^2 h(C_1^{-1}(\tilde{S}_t))^2 C_1''(C_1^{-1}(\tilde{S}_t))dt + \sigma(X_t)C_1'(C_1^{-1}(\tilde{S}_t))dW_t, \quad \tilde{s}_0 = C_1(s_0). \tag{2.40}$$

Next, we introduce a perturbation parameter ϵ as follows:

$$\begin{aligned}
d\tilde{S}_t^{(\epsilon)} &= \frac{\eta(\epsilon)}{2}||\sigma(X_t^{(\epsilon)})||^2 h(C^{-1}(\tilde{S}_t^{(\epsilon)}))^2 C''(C^{-1}(\tilde{S}_t^{(\epsilon)}))dt + \epsilon\sigma(X_t^{(\epsilon)})C'(C^{-1}(\tilde{S}_t^{(\epsilon)}))dW_t \quad (2.41)\\
dX_t^{(\epsilon),j} &= V_0^j(X_t^{(\epsilon)}, \epsilon)dt + \epsilon V^j(X_t^{(\epsilon)})dW_t \quad (j = 1, \cdots, d),
\end{aligned}$$

where $\eta(\epsilon) = \epsilon^j$ and j is a nonnegative integer such as $j = 0, 1, 2, \cdots$. Note that

$$S_t = C_1^{-1}(\tilde{S}_t) = C_1^{-1}(\tilde{S}_t^{(1)}).$$

According to Theorem 4, we have already an asymptotic expansion of the density function of $G^{(\epsilon)} = \frac{\tilde{S}_T^{(\epsilon)} - \tilde{S}_T^{(0)}}{\epsilon}$ up to ϵ^N-order, denoted by $f_{G^{(\epsilon)},N}(x)$.

Therefore, an approximation formula of the call price is given as follows:

$$\begin{aligned} Call(K,T) &= \mathbf{E}[(S_T - K)_+] = \mathbf{E}\left[\left(C_1^{-1}\left(\tilde{S}_T^{(1)}\right) - K\right)_+\right] &(2.42)\\ &\approx \int_y^\infty \left(C_1^{-1}(x + \tilde{S}_T^{(0)}) - K\right) f_{G^{(1)},N}(x)dx, &(2.43) \end{aligned}$$

where $y = C_1(K) - \tilde{S}_T^{(0)}$.

A simple example is the following. Set the local volatility function to be linear:

$$\begin{aligned} &dS_t = \sigma(X_t)S_t dW_t \\ &dX_t^j = V_0^j(X_t)dt + V^j(X_t)dW_t \quad (j = 2, \cdots, d). \end{aligned} \qquad (2.44)$$

For $x = (x^1, x^2, \cdots, x^d)$, let

$$C(x) = (\log x^1, x^2, \cdots, x^d),$$

and set $\eta(\epsilon) = \epsilon^j$ where j is 0, 1 or 2. Then, we have $\tilde{S}_t^{(\epsilon)} = \log S_t^{(\epsilon)}$, where

$$\begin{aligned} d\tilde{S}_t^{(\epsilon)} &= -\frac{\epsilon^j}{2}\sigma(X_t^{(\epsilon)})^2 dt + \epsilon\sigma(X_t^{(\epsilon)})dW_t, &(2.45)\\ dX_t^{(\epsilon),j} &= V_0^j(X_t^{(\epsilon)}, \epsilon)dt + \epsilon V^j(X_t^{(\epsilon)})dW_t \quad (j = 1, \cdots, d). \end{aligned}$$

This case corresponds to some existing researches. (e.g. [32], [42], [33], [34], [35], [39])

2.4.3 Examples

This section will provide concrete examples with numerical examination.

Constant Elasticity of Variance(CEV) Model

The first example is on the well-known CEV model (Cox [5]) :

$$dS_t = \sigma(S_t^\beta S_0^{1-\beta})dW_t, \ \sigma \text{ and } S_0 \text{ are positive constants}, \ \beta \in [0, 1], \qquad (2.46)$$

where the term $S_0^{1-\beta}$ makes the level of σ is of the same order for different β. For $x > 0$, let us take the change of variable function to be $C(x) = \log(x/S_0)$, that is $x = C^{-1}(\tilde{x}) = S_0 \exp(\tilde{x})$. Hence, $\tilde{S}_t = \log \frac{S_t}{S_0}$ and we have

$$d\tilde{S}_t = -\frac{1}{2}\sigma^2 e^{2(\beta-1)\tilde{S}_t}dt + \sigma e^{(\beta-1)\tilde{S}_t}dW_t. \qquad (2.47)$$

Next, we introduce a perturbation $\epsilon \in [0,1]$, again as follows:

$$d\tilde{S}_t^{(\epsilon)} = -\frac{\eta(\epsilon)}{2}\sigma^2 e^{2(\beta-1)\tilde{S}_t^{(\epsilon)}}dt + \epsilon\sigma e^{(\beta-1)\tilde{S}_t^{(\epsilon)}}dW_t, \tag{2.48}$$

where $\eta(\epsilon) = \epsilon^j$ and j is a nonnegative integer.

Because

$$S_T = C^{-1}\left(\tilde{S}_T^{(1)}\right) = S_0 \exp\left(\tilde{S}_T^{(1}\right) = S_0 \exp\left(G^{(1)} + S_T^{(0)}\right),$$

an approximation formula of the call price with strike K and maturity T is given as follows:

$$\begin{aligned} Call(K,T) &= \mathbf{E}[(S_T - K)_+] = \mathbf{E}\left[\left(S_0 \exp\left(G^{(1)} + \tilde{S}_T^{(0)}\right) - K\right)_+\right] \\ &\approx \int_y^\infty \left(S_0 \exp\left(x + \tilde{S}_T^{(0)}\right) - K\right) f_{G^{(1)},N}(x)dx; \qquad (2.49) \\ y &= C(K) - \tilde{S}_T^{(0)} = \log\frac{K}{S_0} - \tilde{S}_T^{(0)}. \qquad (2.50) \end{aligned}$$

Note that $f_{g_{1T}}$, the first term in the asymptotic expansion of the density $f_{G^{(\epsilon)}}$ is a normal density and hence, the underlying asset price is expanded around a log-normal distribution. Thus, we could call this case a log-normal asymptotic expansion. We also remark that the case of $\eta(\epsilon) = \epsilon^0 = 1$ is harder to be evaluated than the other cases, which is essentially due to difficulty in computation of $\tilde{S}_t^{(0)}$ for $\eta(\epsilon) = 1$.

The λ-SABR Model

Let us consider a stochastic volatility model so called λ-SABR Model [14]:

$$\begin{aligned} dS_t &= \sigma_t(S_t^\beta S_0^{1-\beta})dW_t^1;\ S_0 > 0, \qquad (2.51) \\ d\sigma_t &= \lambda(\theta - \sigma_t)dt + \nu\sigma_t dW_t^2;\ \sigma_0 > 0 \end{aligned}$$

where $\beta \in [0,1]$, $\lambda \geq 0$, $\theta > 0$, $\nu > 0$, and $W = (W^1, W^2)$ is a two dimensional Wiener process with correlation $\rho \in [0,1]$.

Remark 5. *Previous works such as [42], [34] and [35] have considered an asymptotic expansion based on the following perturbed process, where the change of variable function, C is set by $C(x) = x$:*

$$\begin{aligned} dS_t^{(\epsilon)} &= \epsilon\sigma_t(S_t^{(\epsilon)})^\beta dW_t^1;\ S_0^{(\epsilon)} = S_0 > 0, \qquad (2.52) \\ d\sigma_t^{(\epsilon)} &= \lambda(\theta - \sigma_t^{(\epsilon)})dt + \epsilon\nu\sigma_t^{(\epsilon)}dW_t^2;\ \sigma_0^{(\epsilon)} = \sigma_0 > 0 \end{aligned}$$

From a viewpoint of mathematical justification of our asymptotic expansion, we may consider a smooth and bounded version of the local volatility function, x^β *in the above model as follows:*

$$\begin{aligned} dS_t^{(\epsilon)} &= \epsilon \sigma_t g_1(S_t^{(\epsilon)}) dW_t^1 \\ d\sigma_t^{(\epsilon)} &= \lambda(\theta - \sigma_t^{(\epsilon)})dt + \epsilon \nu \sigma_t^{(\epsilon)} dW_t^2 \end{aligned} \tag{2.53}$$

where for prefixed very small $K_3 > 0$ *and very large* $K_1 > K_2 (> K_3)$*,*

$$\begin{aligned} g_1(x) &= h_1(x) g_2(x) + h_2(x) K_1^\beta, \qquad (2.54) \\ g_2(x) &= h_3(x) x^\beta, \\ h_1(x) &= \frac{\psi(K_1 - x)}{\psi(x - K_2) + \psi(K_1 - x)}, \ 0 < K_2 < K_1, \\ h_2(x) &= \frac{\psi(x - K_2)}{\psi(x - K_2) + \psi(K_1 - x)}, \ 0 < K_2 < K_1, \\ h_3(x) &= \frac{\psi(x)}{\psi(x) + \psi(K_3 - x)}, \ 0 < K_3 < K_2 < K_1, \\ \psi(x) &= e^{-1/x} \ \textit{for } x > 0, \ \psi(x) = 0 \ \textit{for } x \leq 0. \qquad (2.55) \end{aligned}$$

Note that the local volatility function $g_1(x)$ *shows the following feature:*

$$\begin{aligned} g_1(x) &= 0, \ \textit{if } x \leq 0 \\ &= h_3(x) x^\beta, \ \textit{if } 0 < x \leq K_3 \\ &= x^\beta, \ \textit{if } K_3 < x \leq K_2 \\ &= h_1(x) x^\beta + h_2(x) K_1^\beta, \ \textit{if } K_2 < x \leq K_1 \\ &= K_1^\beta, \ \textit{if } x > K_1 (\textit{constant}). \end{aligned} \tag{2.56}$$

Hence, this model is be regarded as a smooth and bounded modification of the local volatility function:

$$\left(\min\{\max\{x, 0\}, K_1\}\right)^\beta. \tag{2.57}$$

Then, we are easily able to apply our asymptotic expansion to this modified λ*-SABR model up to an arbitrary order. In fact, because we can take* K_1 *and* K_2 *as arbitrarily large constants, and* K_3 *as arbitrarily positive small constant, we may use the same asymptotic expansion both for (2.52) and (2.53) as long as the deterministic process* $\{ S^{(\epsilon)}(t)\big|_{\epsilon=0} : 0 \leq t \leq T\}$*. is in the range between* K_2 *and* K_3*. If necessary, we could modify the volatility process as well.*

The similar modification and consideration could be applied to the asymptotic expansions appearing in the current paper.

Log-Normal Asymptotic Expansion Let us take a log-normal asymptotic expansion for the underlying asset price S, that is for $x_1 > 0$, set $C(x_1, x_2) = (\log(x_1/S_0), x_2)$ and $\tilde{S}_t = \log \frac{S_t}{S_0}$:

$$\begin{aligned} d\tilde{S}_t &= -\frac{1}{2}\sigma_t^2 e^{2(\beta-1)\tilde{S}_t}dt + \sigma_t e^{(\beta-1)\tilde{S}_t}dW_t^1;\ \tilde{S}_0 = 0 \\ d\sigma_t &= \lambda(\theta - \sigma_t)dt + \nu\sigma_t dW_t^2;\ \sigma_0 > 0. \end{aligned} \tag{2.58}$$

Next, we introduce a perturbation $\epsilon \in [0, 1]$, again as follows:

$$\begin{aligned} d\tilde{S}_t^{(\epsilon)} &= -\frac{\eta_1(\epsilon)}{2}\sigma^2 e^{2(\beta-1)\tilde{S}_t^{(\epsilon)}}dt + \epsilon\sigma e^{(\beta-1)\tilde{S}_t^{(\epsilon)}}dW_t;\ \tilde{S}_0 = 0, \\ d\sigma_t^{(\epsilon)} &= \eta_2(\epsilon)\lambda(\theta - \sigma_t^{(\epsilon)})dt + \epsilon\nu\sigma_t^{(\epsilon)}dW_t^2;\ \sigma_0^{(\epsilon)} = \sigma_0, \end{aligned} \tag{2.59}$$

where $\eta_i(\epsilon) = \epsilon^{j_i}$, $i = 1, 2$ and j_i is a nonnegative integer. For instance, typical cases are given as follows:

Case I

$$\begin{aligned} d\tilde{S}_t^{(\epsilon)} &= -\frac{1}{2}{\sigma_t^{(\epsilon)}}^2 e^{2(\beta-1)\tilde{S}_t^{(\epsilon)}}dt + \epsilon\sigma_t^{(\epsilon)} e^{(\beta-1)\tilde{S}_t^{(\epsilon)}}dW_t^1 \\ d\sigma_t^{(\epsilon)} &= \lambda(\theta - \sigma_t^{(\epsilon)})dt + \epsilon\nu\sigma_t^{(\epsilon)}dW_t^2 \end{aligned} \tag{2.60}$$

Case II (an extension of *the Log-Normal Asymptotic Expansion* in [42], [34])

$$\begin{aligned} d\tilde{S}_t^{(\epsilon)} &= -\frac{\epsilon}{2}{\sigma_t^{(\epsilon)}}^2 e^{2(\beta-1)\tilde{S}_t^{(\epsilon)}}dt + \epsilon\sigma_t^{(\epsilon)} e^{(\beta-1)\tilde{S}_t^{(\epsilon)}}dW_t^1 \\ d\sigma_t^{(\epsilon)} &= \lambda(\theta - \sigma_t^{(\epsilon)})dt + \epsilon\nu\sigma_t^{(\epsilon)}dW_t^2 \end{aligned} \tag{2.61}$$

Case III (an extension of [31] to the CEV-type local volatility)

$$\begin{aligned} d\tilde{S}_t^{(\epsilon)} &= -\frac{\epsilon^2}{2}{\sigma_t^{(\epsilon)}}^2 e^{2(\beta-1)\tilde{S}_t^{(\epsilon)}}dt + \epsilon\sigma_t^{(\epsilon)} e^{(\beta-1)\tilde{S}_t^{(\epsilon)}}dW_t^1 \\ d\sigma_t^{(\epsilon)} &= \lambda(\theta - \sigma_t^{(\epsilon)})dt + \epsilon\nu\sigma_t^{(\epsilon)}dW_t^2 \end{aligned} \tag{2.62}$$

An approximation formula of the call price with strike K and maturity T is given as follows:

$$\begin{aligned} Call(K,T) &= \mathbf{E}[(S_T - K)_+] = \mathbf{E}\left[\left(S_0\exp\left(G^{(1)} + \tilde{S}_T^{(0)}\right) - K\right)_+\right] \\ &\approx \int_y^\infty \left(S_0\exp\left(x + \tilde{S}_T^{(0)}\right) - K\right) f_{G^{(1)},N}(x)dx; \end{aligned} \tag{2.63}$$

$$y = C(K) - \tilde{S}_T^{(0)} = \log\frac{K}{S_0} - \tilde{S}_T^{(0)}. \tag{2.64}$$

Again, we note that Case I, that is $\eta(\epsilon) = \epsilon^0 = 1$ is harder to be evaluated than the other cases, which results from difficulty in computation of $\tilde{S}_t^{(0)}$ for $\eta(\epsilon) = 1$.

CEV Asymptotic Expansion Let us take change of variable function C as $C(x) = (C_1(x_1), x_2) for x = (x_1, x_2)$, where for $x > 0$ and $\beta \in [0, 1)$,

$$C_1(x) = \frac{1}{1-\beta}\frac{x^{1-\beta}}{S_0^{1-\beta}} \left(= \int^x \frac{dz}{z^\beta S_0^{1-\beta}} \right). \tag{2.65}$$

That is,

$$C_1^{-1}(\tilde{x}) = S_0(1-\beta)^{\frac{1}{(1-\beta)}} \tilde{x}^{\frac{1}{(1-\beta)}}. \tag{2.66}$$

Then, as $\tilde{S}_t = C_1(S_t)$, we have

$$\begin{aligned} d\tilde{S}_t &= -\frac{1}{2}\frac{\beta}{1-\beta}\sigma_t^2 \frac{1}{\tilde{S}_t} dt + \sigma_t dW_t^1;\ \tilde{S}_0 = \frac{1}{1-\beta} \\ d\sigma_t &= \lambda(\theta - \sigma_t)dt + \nu\sigma_t dW_t^2\ \sigma_0 > 0. \end{aligned} \tag{2.67}$$

Again, we obtain a perturbed process as follows:

$$\begin{aligned} d\tilde{S}_t^{(\epsilon)} &= -\frac{\eta_1(\epsilon)}{2}\frac{\beta}{1-\beta}(\sigma_t^{(\epsilon)})^2 \frac{1}{\tilde{S}_t^{(\epsilon)}} dt + \epsilon\sigma_t^{(\epsilon)} dW_t^1;\ \tilde{S}_0^{(\epsilon)} = \frac{1}{1-\beta} \\ d\sigma_t^{(\epsilon)} &= \eta_2(\epsilon)\lambda(\theta - \sigma_t^{(\epsilon)})dt + \epsilon\nu\sigma_t^{(\epsilon)} dW_t^2;\ \sigma_0^{(\epsilon)} = \sigma_0, \end{aligned} \tag{2.68}$$

where $\eta_i(\epsilon) = \epsilon^{j_i}$, $i = 1, 2$ and j_i is a nonnegative integer.

For illustrative purpose, let us set $\eta_1(\epsilon) = \eta_2(\epsilon) = \epsilon$. That is,

$$\begin{aligned} d\tilde{S}_t^{(\epsilon)} &= -\frac{\epsilon}{2}\frac{\beta}{1-\beta}(\sigma_t^{(\epsilon)})^2 \frac{1}{\tilde{S}_t^{(\epsilon)}} dt + \epsilon\sigma_t^{(\epsilon)} dW_t^1;\ \tilde{S}_0^{(\epsilon)} = \frac{1}{1-\beta}, \\ d\sigma_t^{(\epsilon)} &= \epsilon\lambda(\theta - \sigma_t^{(\epsilon)})dt + \epsilon\nu\sigma_t^{(\epsilon)} dW_t^2;\ \sigma_0^{(\epsilon)} = \sigma_0. \end{aligned} \tag{2.69}$$

In this case, as $\tilde{S}_t^{(0)} = \frac{1}{1-\beta}$ and $\sigma_t^{(0)} = \sigma_0$ for all $t \in [0, T]$, the first term in the asymptotic expansion, $g_{1t} = \frac{\partial}{\partial\epsilon}\big|_{\epsilon=0} \tilde{S}_t^{(\epsilon)}$ follows a Gaussian process:

$$dg_{1t} = \frac{-\beta\sigma_0^2}{2} dt + \sigma_0 dW^1;\ g_{10} = 0. \tag{2.70}$$

Then, by applying *Itô*'s formula to

$$\hat{g}_{1t} := C_1^{-1}(g_{1t}) = S_0(1-\beta)^{\frac{1}{(1-\beta)}} g_{1t}^{\frac{1}{(1-\beta)}}, \tag{2.71}$$

and using

$$g_{1t} = \frac{1}{1-\beta}\frac{\hat{g}_{1t}^{1-\beta}}{S_0^{1-\beta}}, \tag{2.72}$$

we formally obtain the dynamics of $\hat{g}_{1t}$ though it is well-defined only for $g_{1t} \geq 0$:

$$d\hat{g}_{1t} = \frac{\sigma_0^2}{2}\hat{g}_{1t}^\beta \left[-\beta S_0^{1-\beta} + S_0^{2(1-\beta)}\hat{g}_{1t}^{\beta-1} \right] dt + \sigma_0 S_0^{1-\beta}\hat{g}_{1t}^\beta dW_t^1;\ \hat{g}_{10} = 0. \tag{2.73}$$

Here, the diffusion coefficient of $\hat{g}_{1t} = C_1^{-1}(g_{1t})$ is given by $\sigma_0 S_0^{1-\beta}(\hat{g}_{1t})^{\beta}$. As we may think that S is expanded around $\hat{g}_1$, we call this case a CEV asymptotic expansion (though $\hat{g}_1$ is not exactly a CEV process).
In particular, when $\beta = 1/2$,

$$d\hat{g}_{1t} = \frac{\sigma_0^2}{2}\left[-\sqrt{S_0\hat{g}_{1t}}/2 + S_0\right]dt + \sigma_0\sqrt{S_0\hat{g}_{1t}}dW_t^1;\ \hat{g}_{10} = 0, \tag{2.74}$$

and because

$$\hat{g}_{1T} = \frac{S_0}{4}g_{1T}^2, \tag{2.75}$$

$\hat{g}_{1T}$ follows a non-central χ-square distribution, around which the original underlying asset price S_T is expanded.

Finally, for $\eta_i(\epsilon) = \epsilon^{j_i}$, $i = 1, 2$ and j_i is a nonnegative integer, an approximation formula of the call price with strike K and maturity T is obtained as follows:

$$\begin{aligned}
Call(K,T) &= \mathbf{E}[(S_T - K)_+] = \mathbf{E}\left[C_1^{-1}(\tilde{S}_T) - K\right] \\
&= \mathbf{E}\left[\left(\left\{S_0(1-\beta)^{\frac{1}{(1-\beta)}}(\tilde{S}_T)^{\frac{1}{(1-\beta)}}\right\} - K\right)_+\right] \\
&= \mathbf{E}\left[\left(\left\{S_0(1-\beta)^{\frac{1}{(1-\beta)}}(\tilde{S}_T^{(1)})^{\frac{1}{(1-\beta)}}\right\} - K\right)_+\right] \\
&= \mathbf{E}\left[\left(\left\{S_0(1-\beta)^{\frac{1}{(1-\beta)}}(G^{(1)} + \tilde{S}_T^{(0)})^{\frac{1}{(1-\beta)}}\right\} - K\right)_+\right] \\
&\approx \int_y^{\infty}\left(\left\{S_0(1-\beta)^{\frac{1}{(1-\beta)}}(x + \tilde{S}_T^{(0)})^{\frac{1}{(1-\beta)}}\right\} - K\right) f_{G^{(1)},N}(x)dx; \qquad (2.76) \\
y &= C_1(K) - \tilde{S}_T^{(0)} = \frac{1}{1-\beta}\left(\frac{K}{S_0}\right)^{1-\beta} - \tilde{S}_T^{(0)}. \qquad (2.77)
\end{aligned}$$

2.4.4 Numerical Examination

For numerical examination of approximation for European option prices, we take SABR [9] model (λ-SABR with $\lambda = 0$):

$$\begin{aligned}
dS_t &= \sigma_t(S_t^{\beta}S_0^{1-\beta})dW_t^1;\ S_0 > 0, \qquad (2.78) \\
d\sigma_t &= \nu\sigma_t dW_t^2;\ \sigma_0 > 0
\end{aligned}$$

In particular, we apply the following three different expansions for approximation. (Although we use the same notation $f_{G^{(1)},N}(x)$ for the density approximations in all expansions, each represents the density obtained by the corresponding expansion.)

1. Normal expansion

   This case corresponds to the original asymptotic expansion method. We apply the asymptotic expansion to the following perturbed stochastic differential equation:

$$
\begin{aligned}
dS_t^{(\epsilon)} &= \epsilon\sigma_t (S_t^{(\epsilon)})^\beta S_0^{1-\beta} dW_t^1;\ S_0^{(\epsilon)} = S_0 > 0, \\
d\sigma_t^{(\epsilon)} &= \epsilon\nu\sigma_t^{(\epsilon)} dW_t^2;\ \sigma_0^{(\epsilon)} = \sigma_0 > 0
\end{aligned} \tag{2.79}
$$

   Then, an approximation of a call option price with maturity T and strike price K is given by

$$
C(K,T) \approx \int_y^\infty (x-y) f_{G^{(1)},N}(x)dx, \tag{2.80}
$$

$$
y = K - S_T^{(0)} = K - S_0, \tag{2.81}
$$

   where $G^{(1)} = G^{(\epsilon)}\big|_{\epsilon=1}$,

$$
G^{(\epsilon)} = \frac{S_T^{(\epsilon)} - S_T^{(0)}}{\epsilon} = \frac{S_T^{(\epsilon)} - S_0}{\epsilon}, \tag{2.82}
$$

   and $f_{G^{(1)},N}$ denotes the asymptotic expansion of density of $G^{(\epsilon)}$ up to ϵ^N-th order evaluated at $\epsilon = 1$.

   Integrals may be calculated by the formulas:

$$
\int_y^\infty (x-y) H_k(x;\Sigma) f_{g_{1T}}(x)dx = \Sigma^2 H_{k-2}(-y;\Sigma) f_{g_{1T}}(y). \tag{2.83}
$$

2. Log-normal expansion

   We apply the expansion result in Section 2.4.3 with $\eta_1(\epsilon) = \epsilon$:

$$
\begin{aligned}
d\tilde{S}_t^{(\epsilon)} &= -\frac{\epsilon}{2} {\sigma_t^{(\epsilon)}}^2 e^{2(\beta-1)\tilde{S}_t^{(\epsilon)}} dt + \epsilon\sigma_t^{(\epsilon)} e^{(\beta-1)\tilde{S}_t^{(\epsilon)}} dW_t^1;\ \tilde{S}_0^{(\epsilon)} = 0, \\
d\sigma_t^{(\epsilon)} &= \epsilon\nu\sigma_t^{(\epsilon)} dW_t^2;\ \sigma_0^{(\epsilon)} = \sigma_0
\end{aligned} \tag{2.84}
$$

   In this case, an approximation of a call option price with maturity T and strike price K is given by

$$
Call(K,T) \approx \int_y^\infty (S_0 e^x - K) f_{G^{(1)},N}(x)dx; \tag{2.85}
$$

$$
y = C(K) - \tilde{S}_T^{(0)} = \log \frac{K}{S_0}. \tag{2.86}
$$

3. CEV expansion

We apply the result in Section 2.4.3 with $\eta_1(\epsilon) = \epsilon$, that is,

$$d\tilde{S}_t^{(\epsilon)} = -\frac{\epsilon}{2}\frac{\beta}{1-\beta}(\sigma_t^{(\epsilon)})^2\frac{1}{\tilde{S}_t^{(\epsilon)}}dt + \epsilon\sigma_t^{(\epsilon)}dW_t^1;\ \tilde{S}_0^{(\epsilon)} = \frac{1}{1-\beta}, \tag{2.87}$$
$$d\sigma_t^{(\epsilon)} = \epsilon\nu\sigma_t^{(\epsilon)}dW_t^2;\ \sigma_0^{(\epsilon)} = \sigma_0.$$

Hence, an approximation formula of the call price with strike K and maturity T is obtained as follows:

$$Call(K,T) \approx \int_y^\infty \left(\left\{S_0(1-\beta)^{\frac{1}{(1-\beta)}}\left(\frac{1}{(1-\beta)}+x\right)^{\frac{1}{(1-\beta)}}\right\} - K\right) f_{G^{(1)},N}(x)dx; \tag{2.88}$$
$$y = C_1(K) - \tilde{S}_T^{(0)} = \frac{1}{1-\beta}\left(\frac{K}{S_0}\right)^{1-\beta} - \frac{1}{(1-\beta)}. \tag{2.89}$$

In the numerical examples below, we set the parameters as follows:

- The option maturity T, the current underlying asset price S_0, the current volatility σ_0, the volatility on volatility ν:

$$S_0 = 100, T = 1, \sigma_0 = 0.30, \nu = 0.30.$$

- The instantaneous correlation ρ between the asset price S and its volatility σ: three different correlations;

$$\rho = 0.0, -0.5, -0.75.$$

- The CEV parameter β of the underlying asset price process S: nine different βs;

$$\beta = 0.0, 0.125, 0.25, 0.375, 0.50, 0.625, 0.75, 0.875, 1.0.$$

- Strike price K of the option: twenty different strikes;

$$K = 10, 20, \cdots, 100, 110, 120, \cdots, 200.$$

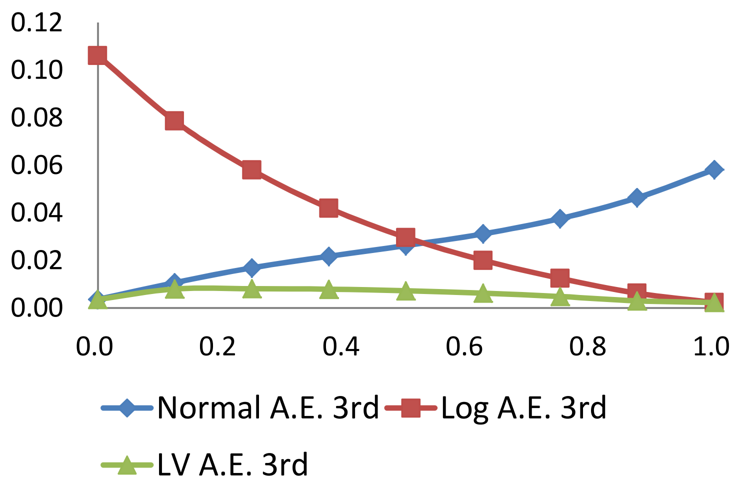
Benchmark prices are computed by Monte Carlo simulation with 10^8 trials, 1024 time steps and the antithetic variable method, where Euler-Maruyama scheme is used for the discretization of the stochastic differential equation (2.78). Then, the absolute error is given by |(approximation price)$-$(benchmark price)| for each case. We have computed each expansion up to the third order. That is, for each approximation we use ϵ^j, ($j = 1, 2, 3$)-order expansion for the density $f_{G^{(1)}}(x)$, that is $f^{cev}_{G^{(1)},j}(x)$:

$$f_{G^{(1)},j}(x) = f_{g_{1T}}(x) + \sum_{n=1}^{j}\left(\sum_{m=0}^{3n} C_{nm}H_m(x-C,\Sigma_T)\right) f_{g_{1T}}(x). \tag{2.90}$$

For each expansion, the higher order expansion provides the better approximation. Particularly, as for ϵ^3-order expansion Figure 1-3 below show the average values of the absolute errors for option prices with all the strikes K for each β, given the correlation value ρ.[1] In the figures, the horizontal axis is β while the vertical axis is the average absolute error; *Normal A.E. 3rd*, *Log A.E. 3rd* and *LV A.E. 3rd* represent *Normal expansion*, *Log-normal expansion* and *CEV expansion*, respectively. Because CEV expansion is not well-defined for $\beta = 1$, we use the same formula as the one of Log-normal expansion.

We find that *CEV expansion* provides the most stable approximations for all the cases. On the other hand, *Log-normal expansion* is not robust to the change in β in a sense that its approximation becomes worse as β deviates from 1. As for *Normal expansion*, although its approximation in zero correlation $\rho = 0.0$ becomes worse as β deviates from 0, it becomes stable for the higher (negative) correlations such as $\rho = -0.5, -0.75$. For completeness, Appendix provides the results of the first and second order expansions. Through investigation of the behavior of the the asymptotic expansions up to the third order, we observe that *CEV expansion* becomes more precise with the same level of absolute errors across the whole range of β along the higher order expansions. Thus, we expect a higher order *CEV expansion* will produce the better and more stable approximation than *normal* and *log-normal* expansions.

Figure 2.5: Correlation $\rho = 0.0$



[1]The details of the numerical analysis are given upon request.

Figure 2.6: Correlation: $\rho = -0.5$

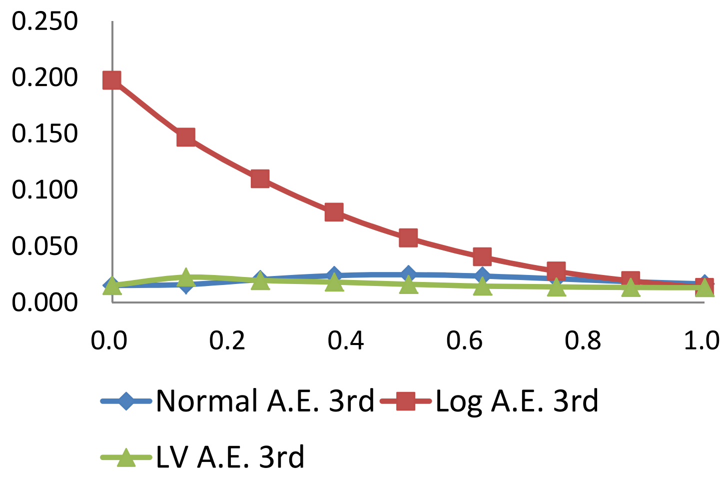
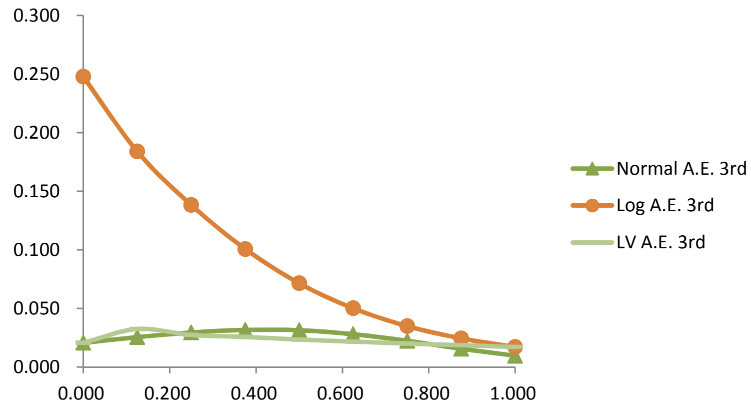


Figure 2.7: Correlation: $\rho = -0.75$



2.4.5 Conclusion

This note extends a general computational scheme proposed by our previous results [42], [34], and [35]. Particularly, we have constructed a scheme that enables us to set a distribution around which we would like to expand a target random variable, and to approximate the target variable up to any order based upon the distribution. As numerical examples, we have shown new *Log-normal* and *CEV* expansions up to the third order for approximations of option prices under SABR model, which demonstrate that the *CEV* expansion will be a candidate for a more precise and robust technique than other approximation schemes such as normal and log-normal expansions.

Bibliography

[1] Antonov, A. and Misirpashaev, T.[2009], "Projection on a Quadratic Model by Asymptotic Expansion with an Application to LMM Swaption," Working Paper.

[2] Andersen, L.B.G. and Hutchings, N.A.[2009], "Parameter Averaging of Quadratic SDEs With Stochastic Volatility," Working Paper.

[3] Ben Arous, G., Laurence, P.[2009], "Second order expansion for implied volatility in two factor local stochastic volatility models and applications to the dynamic λ-SABR model," Working Paper.

[4] Bichteler, K., Gravereaux, J.-B. and Jacod, J. [1987], Malliavin Calculus for Processes with Jumps, volume 2 of *Stochastic Monographs*, Gordon and Breach Science Publishers.

[5] Cox, J. [1975], "Notes on Option Pricing I: Constant Elasticity of Diffusions." Unpublished draft, Stanford University, 1975.

[6] Fouque, J.-P., Papanicolaou, G. and Sircar, K. R.[1999], "Financial Modeling in a Fast Mean-reverting Stochastic Volatility Environment, "*Asia-Pacific Financial Markets*, Vol. 6(1), pp.37-48.

[7] Fouque, J.-P., Papanicolaou, G. and Sircar, K. R.[2000], *Derivatives in financial Markets with Stochastic Volatility*, Cambridge University Press.

[8] Gatheral, J., Hsu, E.P., Laurence, P., Ouyang, C. and Wang, T-H.[2009], "Asymptotics of implied volatility in local volatility models," Working Paper.

[9] Hagan, P.S., Kumar, D., Lesniewski, A.S., and Woodward, D.E. [2002], "Managing smile risk," *Willmott Magazine*, 15, 84-108.

[10] Ikeda, N. and Watanabe, S. [1989], *Stochastic Differential Equations and Diffusion Processes*, Second Edition, North-Holland/Kodansha, Tokyo.

[11] Kunitomo, N. and Takahashi, A. [1992], "Pricing Average Options," *Japan Financial Review*, Vol. 14, 1-20. (in Japanese).

[12] Kunitomo, N. and Takahashi, A. [2001], "The Asymptotic Expansion Approach to the Valuation of Interest Rate Contingent Claims," *Mathematical Finance*, Vol. 11, 117-151.

[13] Kunitomo, N. and Takahashi, A. [2003a], "On Validity of the Asymptotic Expansion Approach in Contingent Claim Analysis," *Annals of Applied Probability* Vol. 13-3, 914-952.

[14] Labordere, P.H. [2005a], "A General Asymptotic Implied Volatility for Stochastic Volatility Models", cond-mat/0504317.

[15] Labordere, P.H. [2005b], "Solvable Local and Stochastic Volatility Models: Supersymmetric Methods in Option Pricing," Working Paper.

[16] Levy, Y. [1992], "The Valuation of Average Rate Currency Options," *Journal of International Money and Finance*, 11:474–491.

[17] Malliavin, P. [1997], "Stochastic Analysis," Springer.

[18] Malliavin, P. and Thalmaier, A. [2006], *Stochastic Calculus of Variations in Mathematical Finance*, Springer.

[19] Matsuoka, R. Takahashi, A. and Uchida, Y. [2004], "A New Computational Scheme for Computing Greeks by the Asymptotic Expansion Approach," *Asia-Pacific Financial Markets*, Vol.11, 393-430.

[20] Muroi, Y. [2005], "Pricing Contingent Claims with Credit Risk: Asymptotic Expansion Approach," *Finance and Stochastics*, Vol. 9(3), 415-427.

[21] Nielsen, J., and Sandmann, K.[2003], "Pricing Bounds on Asian Options," *Journal of Financial and Quantitative Analysis* 38(2), June 2003.

[22] Ninomiya, S. and Victoir, N. [2008], "Weak Approximation of Stochastic Differential Equations and Application to Derivative Pricing", Applied Mathematical Finance, Vol.15, No.2, 107-121.

[23] Nualart, D. [1995], "The Malliavin Calculus and Related Topics," Springer.

[24] Nualart, D., Üstünel A. S. and Zakai M. [1988], "On the moments of a multiple Wiener-Itô integral and the space induced by the polynomials of the integral," *Stochastics*, Vol. 25, 233-340.

[25] Rubinstein, M.[1983], “Displaced Diffusion Option Pricing,” *Journal of Finance*, Vol.38-1.

[26] Shiraya, K., Takahashi, A., Toda, M., *Pricing Barrier and Average Options under Stochastic Volatility Environment*, Journal of Computational Finance vol.15-2,winter 2011/12, 111-148.

[27] Siopacha, M. and Teichmann, J.[2007], “Weak and Strong Taylor Methods for Numerical Solutions of Stochastic Differential Equations,” Working paper.

[28] Takahashi, A. [1995], “Essays on the Valuation Problems of Contingent Claims,” Unpublished Ph.D. Dissertation, Haas School of Business, University of California, Berkeley.

[29] Takahashi, A. [1999], “An Asymptotic Expansion Approach to Pricing Contingent Claims,” *Asia-Pacific Financial Markets*, Vol. 6, 115-151.

[30] Takahashi, A. [2009], ”On an Asymptotic Expansion Approach to Numerical Problems in Finance,” *Selected Papers on Probability and Statistics*, pp. 199-217, 2009, American Mathematical Society.

[31] Takahashi, A. and Takehara, K.[2007], “An Asymptotic Expansion Approach to Currency Options with a Market Model of Interest Rates under Stochastic Volatility Processes of Spot Exchange Rates,” *Asia-Pacific Financial Markets*, Vol.14 , pp. 69-121.

[32] Takahashi, A. and Takehara, K.[2008a], “Fourier Transform Method with an Asymptotic Expansion Approach: an Applications to Currency Options,” *International Journal of Theoretical and Applied Finance*, Vol. 11(4), pp. 381-401.

[33] Takahashi, A., Takehara, K. [2010], *A Hybrid Asymptotic Expansion Scheme: an Application to Long-term Currency Options,’*, International Journal of Theoretical and Applied Finance, Vol.13-8, 1179-1221.

[34] Takahashi, A., Takehara, K., Toda. M.[2009], *Computation in an Asymptotic Expansion Method*, Working paper, CARF-F-149, the University of Tokyo.

[35] Takahashi, A., Takehara, K., Toda. M.[2012], “A General Computation Scheme for a High-Order Asymptotic Expansion Method,” *International Journal of Theoretical and Applied Finance*, 15(6), DOI: 10.1142/S0219024912500446.

[36] Takahashi, A. and Toda. M.[2013], “Note on an Extension of an Asymptotic Expansion Scheme,” forthcoming in *International Journal of Theoretical and Applied Finance.*

[37] Takehara, K., Takahashi, A. and Masashi Toda [2010], "New Unified Computation Algorithm in a High-Order Asymptotic Expansion Scheme," Recent Advances in Financial Engineering (the Proceedings of KIER-TMU International Workshop on Financial Engineering 2009), pp. 231-251.

[38] Takehara, K., Toda, M. and Takahashi, A. [2011], "Application of a High-Order Asymptotic Expansion Scheme to Long-Term Currency Options," *International Journal of Business and Finance Research*, vol. 5-3, pp. 87-100.

[39] Takahashi, A. and Yamada, T. [2009], "An Asymptotic Expansion with Push Down Malliavin Weights," Working paper, CARF-F-194, the University of Tokyo, http://www.carf.e.u-tokyo.ac.jp/workingpaper/

[40] Takahashi, A. and Yoshida, N. [2004], "An Asymptotic Expansion Scheme for Optimal Investment Problems," *Statistical Inference for Stochastic Processes*, Vol. 7-2, 153-188.

[41] Takahashi, A. and Yoshida, N. [2005], "Monte Carlo Simulation with Asymptotic Method," *The Journal of Japan Statistical Society*, Vol. 35-2, 171-203.

[42] Toda, M. [2009], "Notes on an Asymptotic Expansion - Applications and Extensions -," Master Thesis, Graduate School Economics, The University of Tokyo, 2009.

[43] Watanabe, S. [1987], "Analysis of Wiener Functionals (Malliavin Calculus) and its Applications to Heat Kernels," *The Annals of Probability*, Vol. 15, 1-39.

[44] Yoshida, N. [1992a], "Asymptotic Expansion for Small Diffusions via the Theory of Malliavin-Watanabe," *Probability Theory and Related Fields*, Vol. 92, 275-311.

[45] Yoshida, N. [1992b], "Asymptotic Expansions for Statistics Related to Small Diffusions," *The Journal of Japan Statistical Society*, Vol. 22, 139-159.