博 士 論 文

論文題目　Supersymmetric gauge theories on various four-dimensional spaces and two-dimensional conformal field theories

(様々な四次元空間上の超対称ゲージ理論と二次元共形場理論)

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Contents

1 Introduction and Review

M-theory is believed to be the fundamental theory from which many known field theories emerge as certain limits. M5-branes are the most mysterious objects in the theory but once we assume the existence of 6d (2,0) superconformal theory, which describes them in the low energy limit, its compactification gives us predictions about many dualities and correspondences between lower dimensional field theories. Studying these correspondences will help us understand the 6d (2,0) theory.

Apart from M-theory, supersymmetric gauge theories are interesting in its own right. Although they have rich dynamics, we can obtain many exact results about them such as Nekrasov formulae for instanton partition functions, which are very nontrivial functions. These exact results enable us to study nontrivial properties of the theories such as S-duality. Therefore, among the correspondences obtained from M5-branes, I have been especially interested in correspondences between 4d $\mathcal{N} = 2$ supersymmetic gauge theories and 2d conformal field theories (CFTs). We call it 4d-2d correspondence.

4d-2d correspondence was inspired by the following idea. Let us put the 6d theory on $M \times C$, with M a four-manifold and C a Riemann surface. If we compactify it on C or M, we obtain the 4d theory or the 2d theory respectively. If we assume that certain quantities in the 6d theory are independent of the relative size of C and M, the two theories should share the same quantities. This correspondence is useful because it gives us predictions and confirmations of exact results of the 4d supersymmetric gauge theories from the dual 2d theories.

This thesis is based on my two papers [1] and [2]. Both of them are related to this topic. Section 2 in this thesis is based on the first one [1] and Section 3 is based on the second one [2]. Before going to these sections, we need to review the past researches that are relevant for my papers.

The Riemann surfaces related to 4d $\mathcal{N} = 2$ theories first appeared in [3, 4]. The authors found that the exact mass of BPS states in 4d $\mathcal{N} = 2$ gauge theories can be obtained from certain period integral over one-cycles on the corresponding Riemann surfaces.

Later this calculation was interpreted in terms of M-theory. The low energy theory for M5-branes wrapped on the Riemann surface is reduced to the 4d $\mathcal{N} = 2$ gauge theory and the period integral over the cycles corresponds to the energy of M2-branes that are reduced to the BPS particles in the 4d theory.

By using this Riemann surface and the information of the integrand in the period integral, the authors of [5] found the first example of 4d-2d correspondence. They put a certain CFT called Liouville theory on the Riemann surface corresponding to 4d $\mathcal{N} = 2$ $SU(2)$ superconformal gauge theories such as the theories with 4 fundamental hypermuletiplets (denoted by $N_f = 4$) and with 1 adjoint hypermultiplet (denoted by $\mathcal{N} = 2^*$). Then they considered the correlation functions of the Vertex operators in the CFT that reflect the information of the singularities of the integrand in the period integral. When we calculate the correlation functions in CFT, there are certain parts called conformal blocks that depend only on the central charge and the conformal dimensions of the operators. I will explain these quantities in Section 1.4. They found that the instanton partition functions of the $SU(2)$ superconformal gauge theories on $\mathbb{R}^4$ match the corresponding conformal blocks in Liouville theory up to so called $U(1)$ factor. We will explain the $U(1)$ factor in Section 1.4.2. It is expected to be the contribution from $U(1)$ part in the gauge theories. On the 2d side, there is a parameter b in the Liouville theory as I will explain in Section 1.4. In order for the match to hold, we need to identify this parameter with the regularization parameters for the instanton partition functions. Once we do so, the parameter b can take any value with keeping the correspondence hold. The first row in the table (1.1) indicates this correspondence.

$4d$	$2d$
instanton partition functions on $\mathbb{R}^4$	conformal blocks
full partition functions on S^4	correlation functions for $b = 1$
the vevs of the loop operators on S^4	Verlinde operators for $b = 1$

(1.1)

In addition, they found that the full partition functions on S^4 match the correlation functions for $b = 1$. Since S^4 is compact, there are no free parameters for regularization unlike the case of $\mathbb{R}^4$. That is why we have to fix $b = 1$ on the 2d side in order for the correspondence to hold. The second row in the table (1.1) indicates this correspondence.

While we have restricted to $SU(2)$ gauge theories on the 4d side in the above story, there is an extension [6] where $SU(N)$ superconformal gauge theories correspond to A_{N-1} Toda theory on the Riemann surfaces. In case of $SU(N)$, the gauge theories with $2N$ fundamental hypermultiplets (denoted by $N_f = 2N$) and with 1 adjoint hypermultiplet (still denoted by $\mathcal{N} = 2^*$) are considered as examples of $\mathcal{N} = 2$ superconformal theories.

A more relevant example of 4d-2d correspondence for my paper [1] is the correspondence with loop operators. Wilson loop and 't Hooft loop operators are well known non-local operators in the 4d gauge theories. There are also loop operators on the 2d CFTs and they are called Verlinde operators. The correspondence between the Wilson and 't Hooft loop operators in $SU(2)$ superconformal gauge theories on S^4 and the Verlinde operators for $b = 1$ is proposed in [7], [8]. This proposal was based on the exact results of the Wilson loop operators on S^4 obtained in [9] and the calculation of the Verlinde operators in Liouville theory. Later the exact results of the 't Hooft loop operators on S^4 was obtained in [10] and the authors checked that the Wilson and the 't Hooft loop operators in $SU(N)$ superconformal gague theoris on S^4 match the Verlinde operators for $b = 1$ in the Toda theories obtained in [11]. The third row in the table (1.1) indicates this correspondence.

In our paper [1], which Section 2 is based on, we proposed a new version of 4d-2d correspondence. We put 4d $\mathcal{N} = 2$ $SU(N)$ superconformal gauge theories on $S^1 \times \mathbb{R}^3$ and considered the indices in presence of the loop operators. Indices count the number of the BPS states weighted by their quantum numbers such as angular momenta, so they are useful quantities for obtaining the information about the BPS states. We calculated the indices for 4d $\mathcal{N} = 2$ superconformal gauge theories in the presence of infinitely massive particles with electric or magnetic charges, which are Wilson loop or 't Hooft loop operators along S^1. In doing so, we reinterpreted the indices as the partition functions where all fields have twisted periodicity along S^1 related to the above quantum numbers and performed localization calculation to obtain the exact results. Our results will be a step for studying the BPS states that reflect interesting non-perturbative dynamics.

We found that these indices correspond to the Verlinde operators with general b in the Toda theories. In this correspondence, the fugacity for the angular momentum in the above indicies is identified with the parameter b in the Toda theories. This is the first example that corresponds to the Verlinde operators with general b. This correspondence raises an interesting question about how we can explain it in terms of M5-branes and we will obtain a hint of interesting properties of M5-branes during answering the question.

Having explained Section 2 based on [1], we will turn to Section 3 based on [2]. To prepare for Section 3, let us return to the first example of 4d-2d correspondence indicated in the first row in the table (1.1). After the correspondence in case of 4d $\mathcal{N} = 2$ $SU(2)$ superconformal gauge theories on the 4d side was found in [5], the correspondence between 4d $\mathcal{N} = 2$ $SU(2)$ non-conformal gauge theoreis and Liouville theory was found in [12]. The author of [12] considered 4d $\mathcal{N} = 2$ $SU(2)$ gauge theories with $N_f = 0, 1, 2, 3$ where N_f is the number of fundamental hypermultiplets. Especially the cases with $N_f = 0, 1$ is relevant for Section 3 in this thesis. In these cases, the author of [12] found that the instanton partition functions match the inner products of certain states in Liouville theory side. These states are eigenstates of the Virasoro generators L_1 and L_2 and called Whittaker states. These correspondences are simpler than the cases with superconformal gauge theories in a sense that the $U(1)$ factor does not appear in the former. Therefore when we would like to find a new correspondence with different spacetimes on the 4d side, it is easier to begin with the case with pure gauge theories, i.e. theories without hypermultiplets, or theories with small numbers of fundamental hypermultiplets.

From now on, we consider 4d $\mathcal{N} = 2$ gauge theories on $\mathbb{R}^4/\mathbb{Z}_2$. This geometry was considered in the context of 4d-2d correspondence in [13] for the first time. The authors proposed that certain instanton partition functions of 4d $\mathcal{N} = 2$ $SU(2)$ pure gauge theory on $\mathbb{R}^4/\mathbb{Z}_2$ corresponds to the inner products of Whittaker states in super Liouville theory. In super CFTs, there are Neveu-Schwarz and Ramond sectors where fields are periodic and anti-periodic along the cycle corresponding to the space direction. In [13], the authors considered only the instanton partition functions corresponding to the Neveu-Schwarz sector. The same is true in [14]. In [15, 16], this correspondence was extended to the case with $SU(2)$ superconformal gauge theories on $\mathbb{R}^4/\mathbb{Z}_2$ but the authors still considered the instanton partition functions corresponding to the Neveu-Schwarz sector.

I found the instanton partition functions corresponding to the Ramond sector in [2] for the first time. Let us see the gauge theory side also has sectors. The asymptotic region of $\mathbb{R}^4/\mathbb{Z}_2$ is $S^3/\mathbb{Z}_2$ and it has a noncontractible cycle. Since $\pi_1(S^3/\mathbb{Z}_2) = \mathbb{Z}_2$, if we go round the cycle twice, it is contractible. Therefore the holonomy of the gauge field along the cycle $U = \exp(i \oint A)$ should satisfy $U^2 = 1$ if we consider flat connections. In case of $U(2)$ gauge theories, the value of the holonomy can divided into four inequivalent classes $U = \mathrm{diag}(1, 1)$, $\mathrm{diag}(-1, -1)$, $\mathrm{diag}(1, -1)$ and $\mathrm{diag}(-1, 1)$. In [13, 14, 15, 16], the authors considered only the instanton partition functions in the sectors with the holonomy $U = (1, 1)$ and $(-1, -1)$ and that is why their counterparts on the 2d side are all in the Neveu-Schwarz sector. Instead I considered the sector with the holonomy $U = (1, -1)$ and $(-1, 1)$ and found that the instanton partition functions match the inner products of Whittaker states in the Ramond sector. I checked it in case of $U(2)$ gauge theories with $N_f = 0, 1$.

In the remaining of this section, we will review the important examples of 4d-2d correspondences that were found before ones in Section 2 and 3 and relevant for them. In Section 1.1, we explain how Riemann surfaces are introduced for the analysis of the 4d theories in [3, 4]. From Section 1.2 we will explain 4d-2d correspondence written in the table (1.1). In Section 1.2 we review how to obtain instanton partition functions on $\mathbb{R}^4$. In Section 1.3 we write the results of full partition function and loop operators on S^4. Having review the 4d side, we turn to 2d side from Section 1.4. In Section 1.4 we consider correlators in 2d CFT and define conformal blocks. Then we will see the relation between these 2d quantities and the 4d quantities mentioned in Section 1.2 and Section 1.3. In Section 1.5 we will define Verlinde loop operators on the 2d side. We see that it corresponds to the loop operators on S^4 mentioned in Section 1.3 when $b = 1$. It is relevant for Section 2. In Section 1.6, we prepare for Section 3 and review the correspondence with non-conformal gauge theories on $\mathbb{R}^4$. We explain the countarpart of the instanton partition functions in Liouville theory.

Section 2 is based on my first paper [1]. We consider 4d $\mathcal{N} = 2$ $SU(N)$ gauge theories on $S^1 \times \mathbb{R}^3$. The exact results of the loop operators along S^1 are obtained and we compare them with the Verlinde operators for general b in Toda theories.

Section 3 is based on my second paper [2]. We calculate the instanton partition functions of 4d $\mathcal{N} = 2$ $U(2)$ gauge theories on $\mathbb{R}^4/\mathbb{Z}_2$ and compared them with inner products of Whittaker states in the Ramond sector.

In Section 4, we discuss future works related to Section 2 and Section 3.

1.1 Seiberg-Witten curve

The discovery of the 4d-2d correspondence came from the fact that for a certain class of 4d $\mathcal{N} = 2$ gauge theories, there are certain complex-one dimensional curves called Seiberg-Witten curves [3, 4]. It describes the exact mass of BPS states in the 4d $\mathcal{N} = 2$ gauge theory. It is a double cover of a Riemann surface and when we consider 4d-2d correspondence later, we will put certain 2d theories on this Riemann surface.

Before explaining the correspondence in Section 1.4, we need to know which Riemann surface we should introduce for a given 4d $\mathcal{N} = 2$ gauge theory and the information about the singularities of the Seiberg-Witten curve on the Riemann surface. We will explain about them in this subsection. In Section 1.1.1 and Section 1.1.2, we will give the Seiberg-Witten curve for 4d $SU(2)$ gauge theories with $N_f = 0$ and $N_f = 1$. They are relevent for Section

1.6 and Section 3. Next we will give the Seiberg-Witten curve for 4d $SU(2)$ gauge theory with $N_f = 4$. It is relevent for Section 1.4 and Section 2. This subsection is mostly based on [17].

Before giving the expressions of Seiberg-Witten curves, we explain how we can obtain mass of BPS states from the Seiberg-Witten curves. In 4d $\mathcal{N} = 2$ $SU(2)$ gauge theories, mass of BPS states with the electric chage n_e, the magnetic charge n_m and the flavor charges f_i are given as follows

$$M = |n_e\, a + n_m a_D + \sum_i f_i \mu_i| \,, \tag{1.2}$$

where a represents the vev of the complex scalar field ϕ in the vector multiplet so that $\langle\phi\rangle = \mathrm{diag}(a, -a)$. a_D is defined to be $\partial\mathcal{F}/\partial a$ where $\mathcal{F}$ is the prepotential for the low energy effective theory. μ_i are the mass parameters in Lagrangian. The running coupling constant is obtained as $\tau(a) = \partial a_D(a)/\partial a$. For a given theory, we can construct the Seiberg-Witten curve and a certain 1-form λ called Seiberg-Witten differential. The Seiberg-Witten curve has two cycles denoted by A and B and the variables a and a_D are obtained as follows

$$a = \frac{1}{2\pi i}\oint_A \lambda \,, \qquad a_D = \frac{1}{2\pi i}\oint_B \lambda \,.$$

1.1.1 $N_f = 0$

We describe the Seiberg-Witten curve in complex two dimensional space parameterized by the two complex coordinates x, z. The Seiberg-Witten curve for $\mathcal{N} = 2$ $SU(2)$ pure gauge theory is given as follows

$$z + \frac{1}{z} = \frac{x^2}{\Lambda^2} - \frac{u}{\Lambda^2} \tag{1.3}$$

where u is a moduli of the vacuum defined as $u := \langle \mathrm{Tr}\phi^2\rangle$ and Λ is the dynamical scale. Let us denote the roots of $x^2(z)$ by $z_\pm$, such that $u + \Lambda^2(z + \frac{1}{z}) = \frac{\Lambda^2}{z}(z - z_+)(z - z_-)$. The branch points of $x(z)$ are $z = 0, z_-, z_+, \infty$. Therefore there are two cycles on the Riemann surface parameterized by z as described in Figure 1.[1] If we identify the Seiberg-Witten differential as $\lambda = x dz/z$, a and a_D are given as follows

$$a = \frac{1}{2\pi i}\oint x\frac{dz}{z} \,, \qquad a_D = \frac{2}{2\pi i}\int_{z_+}^{z_-} x\frac{dz}{z} \,. \tag{1.4}$$

Let us check that they give the known expressions in the classical limit. When $|u| \gg \Lambda^2$, $z_+ \sim -u/\Lambda^2$ and $z_- \sim -\Lambda^2/u$, so indeed we can obtain the following result

$$a \sim \sqrt{u} \,, \qquad a_D = -\frac{8a}{2\pi i}\log\frac{a}{\Lambda} \tag{1.5}$$

[1] Figure 1, Figure 2 and Figure 3 were drawn by the author of [17] and I thank him for permitting me to use these figures in this thesis.

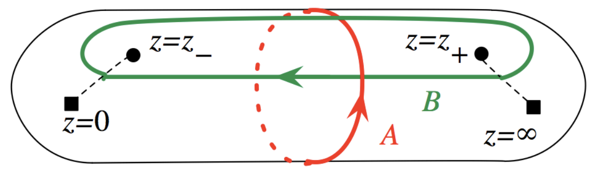

Figure 1: The Riemann surface corresponding to $SU(2)$ $N_f = 0$ theory

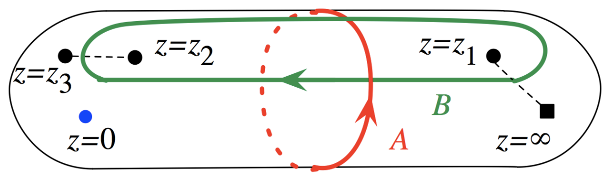

Figure 2: The Riemann surface corresponding to $SU(2)$ $N_f = 1$ theory

as expected in [3]. Let us turn to the singularities in the moduli space. These singularities are caused by the collision of the branch points z_+ and z_- on the Seiberg-Witten curve. We can identify the singularities as $u = \pm 2\Lambda^2$. We can check that (1.4) gives the correct monodromy of $a(u)$ and $a_D(u)$ around them. These observations confirm that the curve (1.3) is correct. We will use (1.3) in Section 1.6.

1.1.2 $N_f = 1$

The Seiberg-Witten curve for $\mathcal{N} = 2$ $SU(2)$ with the number of fundamental hypermultiplet $N_f = 1$ is given as follows

$$\frac{2\Lambda(x-\mu)}{z} + \Lambda^2 z = x^2 - u \tag{1.6}$$

where μ is the mass. The branch points of $x(z)$ are $z = z_1, z_2, z_3, \infty$, where z_1, z_2 and z_3 make the square root in the expression

$$x = \Lambda/z \pm \sqrt{(\Lambda/z)^2 - (2\Lambda\mu/z - \Lambda^2 z - u)} \tag{1.7}$$

zero. The variables a and a_D are given by

$$a = \frac{1}{2\pi i}\oint_A \lambda\,, \qquad a_D = \frac{1}{2\pi i}\oint_B \lambda\,, \tag{1.8}$$

where $\lambda := x dz/z$ and the cycles A and B are described in Figure 2. Note also that x has a residue $\pm\mu$ at $z = 0$. Therefore when a closed cycle L on the Seiberg-Witten curve winds the A cycle n_e times, the B cycle n_m times and the pole $z = 0$ f times, the integral of λ is then

$$\frac{1}{2\pi i}\oint \lambda = n_e a + n_m a_D + f\mu\,,$$

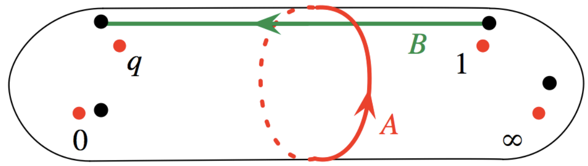

Figure 3: The Riemann surface corresponding to $SU(2)$ $N_f = 4$ theory

just as in the BPS mass formula.

In the classical limit where $|u| \gg |\Lambda|$, (1.9) gives the correct result

$$a \sim \sqrt{u}\,, \qquad a_D \sim \frac{2}{2\pi i}\int_{u/\Lambda^2}^{\Lambda/\sqrt{u}} a dz/z = -\frac{6}{2\pi i} a \log\frac{a}{\Lambda}\,, \tag{1.9}$$

where we used $z_1 \sim u/\Lambda^2$ and $z_2, z_3 \sim \Lambda/\sqrt{u}$ in this limit. In addition, the curve gives the correct monodromy of $a(u), a_D(u)$ around the singular points in the vacuum moduli space. We can confirm that this is the correct curve by considering them. We will use (1.6) in Section 1.6.

1.1.3 $N_f = 4$

The Seiberg-Witten curve for $\mathcal{N} = 2$ $SU(2)$ gauge theory with $N_f = 4$ is given as follows

$$(x - \widetilde{\mu}_1)(x - \widetilde{\mu}_2) + c\,\frac{(x - \widetilde{\mu}_3)(x - \widetilde{\mu}_4)}{z} = x^2 - u\,, \tag{1.10}$$

where $\widetilde{\mu}_{1,2,3,4}$ are related to the masses $\mu_{1,2,3,4}$ and c is a certain constant related to the coupling constant. Let us write the above expression simply as $Ax^2 - Bx - C = 0$ where A, B, C are independent of x. We denote the solutions of $A := 1 - z - c/z = 0$ by $z = z_\pm$. If we introduce the variable $\lambda = \widetilde{x} dz/z$ where $\widetilde{x} := x - B/(2A)$,

$$\lambda^2 = \frac{B^2 + 4AC}{4A^2}\frac{dz^2}{z^2} \tag{1.11}$$

has four poles $z = 0, z_-, z_+, \infty$. If we write the above expression in terms of $z' := z/z_+$,

$$\lambda^2 = \frac{D dz'^2}{(z' - 1)^2 (z' - q)^2 z'^2}\,, \tag{1.12}$$

where $q := z_-/z_+$. The poles are at $z' = 0, 1, q, \infty$ as indicated by the red points in Figure 3. D is a quartic polynomial of z' so it means that λ has four branch points. They are indicated by the black points on the Riemann surface parameterized by z in Figure 3. Next we will find that λ has the following residues at the singularities

$$\begin{array}{ll} \pm\frac{\mu_1 - \mu_2}{2} & \text{at } z = \infty \\ \pm\frac{\mu_1 + \mu_2}{2} & \text{at } z = 1 \\ \pm\frac{\mu_3 + \mu_4}{2} & \text{at } z = q \\ \pm\frac{\mu_3 - \mu_4}{2} & \text{at } z = 0\,, \end{array} \tag{1.13}$$

where z means the new z'.

We can find that q is related to the coupling constant in UV as $q = e^{2\pi i \tau_{UV}}$ as follows. From now on, let us write z' as z simply. When u is large,

$$a = \frac{1}{2\pi i}\oint_A \lambda \sim \sqrt{u}\,, \qquad a_D = \frac{1}{2\pi i}\oint_B \lambda = \frac{2}{2\pi i}\int_{z=1}^{z=q} a\frac{dz}{z} = \frac{2a}{2\pi i}\log q\,. \tag{1.14}$$

Combining it with the relation $a_D = 2\tau_{UV} a + \cdots$, we get $q = e^{2\pi i \tau_{UV}}$. We wil use the information of the singularities (1.13) in Section 1.4.

1.1.4 $\mathcal{N} = 2^*$

The Seiberg-Witten curve for $\mathcal{N} = 2^*$ $SU(2)$ gauge theory is

$$\lambda^2 - \phi(z) = 0 \tag{1.15}$$

where z is now a coordinate of the torus, which we take to be the complex plane with the identification $z \sim z + 1 \sim z + \tau$. As the origin of the coordinate is arbitrary, we put the puncture at the origin. The $\phi(z)$ is given by the condition that it has a double pole with a given strength at $z = 0$. This uniquely fixes the form of $\phi(z)$ to be

$$\phi(z) = (\mu^2 \wp(z;\tau) + u)dz^2\,, \tag{1.16}$$

where $\wp$ is the Weierstrass function, and u is the Coulomb branch vev $u = \langle \mathrm{tr}\phi^2 \rangle / 2$.

1.2 Instanton

The first example of 4d-2d correspondence was discovered in [5]. As indicated in the first row in the table (1.1), the authors considered the instanton partition functions on the 4d side. Let us review how to obtain them.

The moduli space of k instantons in $U(N)$ gauge theory can be described by two $k \times k$ matrices B_1 and B_2, a $k \times N$ matrix I and an $N \times k$ matrix J. It is the space of the solutions of the following equation

$$[B_1, B_2] + IJ = 0\,, \tag{1.17}$$

where we consider the solutions related by the transformation

$$(B_1, B_2, I, J) \sim (gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1}) \qquad g \in GL(k,\mathbb{C})\,, \tag{1.18}$$

as equivalent elements. We parameterize g by ϕ so that $g = e^{i\phi}$.

The instanton partition function can be written as certain integral over the moduli space. To calculate this, we use the localization theorem. M is a manifold with $2n$ dimension. A group action G acts on this manifold and causes the vector field $\xi = \xi^i X_i$ where $\{\xi^i\}$ is a set of the parameters of the group action G and X_i is a basis of the vector fields. We define equivariant derivative $d_\xi := d - i_\xi$ where i_ξ is the interior product with ξ. Note that $d_\xi^2 = -\mathcal{L}_\xi$ where $\mathcal{L}$ is a Lie derivative. Then we consider a closed form $\widetilde{\omega}$ over M

$$d_\xi \widetilde{\omega} = 0 \quad \text{where } \widetilde{\omega} = \sum_{k=0}^{2n} \omega_k \tag{1.19}$$

where ω_k is a certain k-form. If M is compact, the following theorem holds.

$$\int_M \widetilde{\omega} = (-2\pi)^n \sum_{p \in F} \frac{\omega_0(p)}{\sqrt{\det \mathcal{L}_\xi(p)}} \tag{1.20}$$

where F is the set of the fixed points of M under the action G. The determinant on the right-hand side is over the tangent space T_pM of M at a fixed point p. Even though it looks like that the right-hand side depends on the group action parameters ξ_i, it is independent in case of compact M. In our problem, we want to consider M as the instanton moduli spaces and the left-hand side of (1.20) as the instanton partition functions. Since this M is non-compact, the left-hand side is divergent. So let us define regularized volumes of instanton moduli spaces by the right-hand side, which depneds on ξ in this case. We take the action $U(1)^2 \times U(1)^N$ parameterized by $(\epsilon_1, \epsilon_2, a_\alpha)$ as the group action G. From now on, we sometimes call this action the equivariant action. The infinitesimal transformation of B_1, B_2, I, J under this action is as follows

$$\begin{aligned}
&\delta_{U(1)^2 \times U(1)^N} B_1 = -i\epsilon_1 B_1 \\
&\delta_{U(1)^2 \times U(1)^N} B_2 = -i\epsilon_2 B_2 \\
&\delta_{U(1)^2 \times U(1)^N} I = -iIa \\
&\delta_{U(1)^2 \times U(1)^N} J = i(\epsilon_1 + \epsilon_2) J + iaJ \,,
\end{aligned}$$

where $a = \text{diag}(a_1 \cdots a_N)$. At each fixed point, they should be invariant up to the transformation (1.18). Its infinitesimal one is as follows

$$\begin{aligned}
&\delta_{U(k)} B_1 = i[\phi, B_1] \\
&\delta_{U(k)} B_2 = i[\phi, B_2] \\
&\delta_{U(k)} I = i\phi I \\
&\delta_{U(k)} J = -iJ\phi \,.
\end{aligned}$$

Each fixed point corresponds to a solution to $\left(\delta_{U(1)^2 \times U(1)^N} + \delta_{U(k)}\right)(B_{1,2}, I, J) = 0$, i.e.,

$$\begin{aligned}
&-\epsilon_1 B_1 + [\phi, B_1] = 0 \\
&-\epsilon_2 B_2 + [\phi, B_2] = 0 \\
&-Ia + \phi I = 0 \\
&-(\epsilon_1 + \epsilon_2) J + aJ - J\phi = 0
\end{aligned} \tag{1.21}$$

with a certain ϕ. We label each fixed point by a set of the eigenvalues of the ϕ.

In order to obtain the eigenvectors of ϕ, let us decompose I into N vectors I_α with k components, where $\alpha = 1, \cdots, N$. From the third equation in (1.21), we can read that I_α is one of the eigenvalues with an eigenvalue a_α.

$$\phi I_\alpha = a_\alpha I_\alpha \tag{1.22}$$

where we do not sum over α. The other eigenvectors can be found as follows. If $B_1 I_\alpha$ is a nonzero vector, we can find that it is an eigenvector with an eigenvalue $a_\alpha + \epsilon_1$ by using the first equation of (1.21)

$$\phi(B_1 I_\alpha) = (B_1\phi + \epsilon_1 B_1) I_\alpha = (a_\alpha + \epsilon_1)(B_1 I_\alpha). \tag{1.23}$$

Similarly, $B_2 I_\alpha$ is an eigenvector with an eigenvalue $a + \epsilon_2$ if it is nonzero vector. More generally $B_1^p B_2^q I_\alpha$ is an eigenvector with an eigenvalue $a_\alpha + p\epsilon_1 + q\epsilon_2$ where p and q are non negative. From the last equation in (1.21), we get $IJ = 0$ and the equation (1.17) becomes $[B_1, B_2] = 0$. Therefore if we change the ordering of B_1 and B_2, the eigenstate $B_1^p B_2^q I_\alpha$ does not change. It was proven that all the eigenvectors can be written in this form. There are exactly k eigenvectors so $B_1^p B_2^q I_\alpha$ must be zero vector with enough large p and q. The set of the two integers (p, q) where $B_1^p B_2^q I_\alpha$ are nonzero vectors are described by the Young tableau for each α. Let $Y_\alpha = \{\lambda_{\alpha,1}, \lambda_{\alpha,2}, \cdots\}$ $(1 \leq \alpha \leq N)$ be a Young tableau where $\lambda_{\alpha,i}$ is the height of the i-the column. We set $\lambda_{\alpha,i} = 0$ when i is larger than the width of the tableau Y_α. This Young tableau corresponds to a solution where $B_1^{i-1} B_2^{j-1} I_\alpha$ is nonzero vector for $j \leq \lambda_{\alpha,i}$ and zero vector otherwise. Since each box in the tableau corresponds to each eigenvector, the N-tuple of Young tableaux $\vec{Y} = Y_1, \cdots, Y_N$ contains k boxes. Each fixed point is labeled by a N-tuple of Young tableaux.

Next we turn to the contribution to the instanton partition functions from each fixed point labeled by the N-tuple of the Young tableaux $\vec{Y}$. The det $\mathcal{L}$ in (1.20) is the product of the weights $\prod_i \omega_i$ where $e^{i\omega_i}$ are eigenvalues of the action on T_pM induced by $G = U(1)^2 \times U(1)^N$.Instead of the product, we consider the trace $\chi := \sum_i e^{i\omega_i}$ here. This trace can be easily calculated if we consider the ADHM complex in Figure 4. We define the spaces $V := \mathbb{C}^k$ and $W := \mathbb{C}^N$. For exmple, ϕ is a $k \times k$ matrix, so it is an element of the space $(V \otimes V^*)$. Similarly I is a $k \times N$ matrix, so it is an element of the space $(V \otimes W^*)$. The maps σ_p and τ_p in Figure 4 are as follows

$$\sigma_p(\phi) = \begin{pmatrix} \delta B_1 \\ \delta B_1 \\ \delta I \\ \delta J \end{pmatrix}, \quad \tau_p \begin{pmatrix} \delta B_1 \\ \delta B_1 \\ \delta I \\ \delta J \end{pmatrix} = [B_1, \delta B_2] + [\delta B_1, B_2] + I\delta J + (\delta I)J\,. \tag{1.24}$$

An element (B_1, B_2, I, J) of the instanton moduli space is a solution to the equation (1.17)

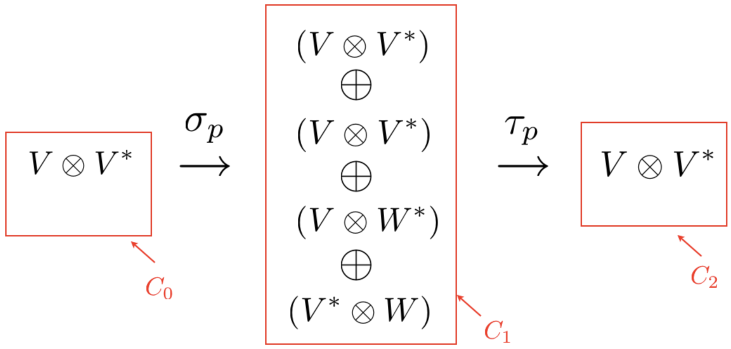

Figure 4: The ADHM complex

and identified by the transformation (1.18). Therefore tangent vectors of this space can be written as $\mathrm{Ker}\,\tau_p/\mathrm{Im}\,\sigma_p$. So the trace over this tangent space can be obtained by $\chi(C_1) - \chi(C_0) - \chi(C_2)$, where $\chi(X)$ is a trace of $U(1)^2 \times U(1)^N \times U(k)$ over the space X. The spaces C_0, C_1 and C_2 are denoted in Figure 4.

First let us give expressions of $\chi(V)$ and $\chi(W)$. If we define $T_1 := e^{-i\epsilon_1}, T_2 := e^{-i\epsilon_2}$ and $T_{a_\alpha} := e^{ia_\alpha}$,

$$\chi(W_{\vec{a}}) = \sum_{\alpha=1}^{N} T_{a_\alpha}\,. \tag{1.25}$$

At the fixed point labeled by $\vec{Y}$, the eigenvalues of the matrix ϕ are

$$\phi_s = a_{\alpha(s)} + (i_s - 1)\epsilon_1 + (j_s - 1)\epsilon_2\,, \tag{1.26}$$

so

$$\chi(V_{\vec{a},\vec{Y}}) = \sum_{\alpha}\sum_{s\in Y_\alpha} T_{a_\alpha} T_1^{-i_s+1} T_2^{-j_s+1}\,.$$

From now let us consider general case with quiver gauge theory $U(M) \times U(N)$ instead of $U(N)$ gauge theory in preparation for obtaining the contribution from fundamental hypermultiplets. We consider the contribution to the instanton partition functions from a bifundamental hypermultiplet with respect to this quiver gauge theory. When we count the trace $\chi(W), \chi(V)$, we use $(a_1, \cdots, a_N)$ as the parameters of the action onto W and Young tableau $Y_1, \cdots, Y_N$. When we count the trace $\chi(W^*), \chi(V^*)$, we use $(b_1, \cdots, b_M)$ as the parameters of the action onto W and Young tableau $X_1, \cdots, X_M$. $\chi(C_1) - \chi(C_0) - \chi(C_2)$

are as follows.

$$\chi_{\vec{a},\vec{b}} = \chi(V_{\vec{a},\vec{Y}})\,\chi(V^*_{\vec{b},\vec{X}})(T_1+T_2-T_1T_2-1)+\chi(V_{\vec{a},\vec{Y}})\,\chi(W^*_{\vec{b}})+T_1T_2\,\chi(W_{\vec{a}})\chi(V^*_{\vec{b},\vec{X}})$$
$$= \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N'} e^{i(a_\alpha-b_\beta)}\chi_{(Y_\alpha,X_\beta)}$$

where

$$\chi_{(Y_\alpha,X_\beta)} := -(1-T_1)(1-T_2)\chi(V_{Y_\alpha})\chi(V^*_{X_\beta})+\chi(V_{Y_\alpha})+T_1T_2\,\chi(V^*_{X_\beta})$$
$$\chi(V_{Y_\alpha}) := \sum_{s\in Y_\alpha} T_1^{-i_s+1}T_2^{-j_s+1} \tag{1.27}$$

After calculation, we get

$$\chi_{(Y_\alpha,X_\beta)} = \sum_{s\in Y_\alpha} T_1^{-L_{X_\beta}(s)}T_2^{A_{Y_\alpha}(s)+1}+\sum_{t\in X_\beta} T_1^{L_{Y_\alpha}(t)+1}T_2^{-A_{X_\beta}(t)} \tag{1.28}$$

For a box s in the i-the column and the j-th row, we defined its arm-length $A_{Y_\alpha}(s)$ and leg-length $L_{Y_\alpha}(s)$ with respect to the tableau Y_m by

$$A_{Y_\alpha}(s)=\lambda_{\alpha,i}-j\,,\qquad L_{Y_\alpha}(s)=\lambda'_{\alpha,j}-i\,. \tag{1.29}$$

We denoted the height of each column of the transpose of Y_α by $Y_\alpha^T=\{\lambda'_{\alpha,1},\lambda'_{\alpha,2},\cdots\}$. Using them, we define a function E by

$$E(a,Y_m,Y_n,s)=a-\epsilon_1 L_{Y_n}(s)+\epsilon_2(A_{Y_m}(s)+1)\,. \tag{1.30}$$

Changing the trace $\sum_i e^{i\omega_i}$ into the products of the weights $\prod_i \omega_i$, we get the contribuion z_{bifund} from the bifundamental hypermultiplet with mass μ to the summand in (1.20) at the fixed point labeled by $(\vec{Y},\vec{X})$. It is given as

$$z_{\text{bifund}}(\vec{a},\vec{Y};\vec{b},\vec{X};\mu)$$
$$=\prod_{\alpha,\beta}\prod_{s\in Y_\alpha}(E(a_\alpha-b_\beta,Y_\alpha,X_\beta,s)-\mu)\prod_{t\in X_\beta}(\epsilon_1+\epsilon_2-E(b_\beta-a_\alpha,X_\beta,Y_\alpha,t)-\mu)\,. \tag{1.31}$$

When we got the above expression, we added the flavor symmetry action to the equivariant action G. This action is parameterized by the mass μ.

The contribution from a vector multiplet and an adjoing hypermultiplet in $U(N)$ gauge theory is obtained as follows

$$z_{\text{vect}}(\vec{a},\vec{Y})=z_{\text{bifund}}(\vec{a},\vec{Y};\vec{a},\vec{Y};0)^{-1}$$
$$z_{\text{adj}}(\vec{a},\vec{Y},\mu)=z_{\text{bifund}}(\vec{a},\vec{Y};\vec{a},\vec{Y};\mu) \tag{1.32}$$

The contribution from a fundamental hypermultiplet with mass m is written as follows

$$z_{\text{fund},i}(\vec{a},\vec{Y},\mu_i)=\prod_{n=1}^{N}\prod_{s\in Y_n}\left(\phi(a_n,s)-\mu_i+\epsilon_1+\epsilon_2\right), \tag{1.33}$$

where

$$\phi(a,s)=a+\epsilon_1(i-1)+\epsilon_2(j-1) \tag{1.34}$$

for the box s in the i-th column and the j-th row. This mass is transformed into $Q-m$ by Weyl symmetry. An anti-fundamental hypermultiplet with mass m gives the following contribution

$$z_{\text{antifund},i}(\vec{a},\vec{Y},\mu)=z_{\text{fund},i}(\vec{a},\vec{Y},\epsilon_1+\epsilon_2-\mu) \tag{1.35}$$

The instanton partition function for $U(N)$ gauge theory with N_f fundamental matters and N_a adjoint matters is written as follows

$$\begin{aligned} Z_{\text{inst}}(a,\{\mu\},\epsilon_1,\epsilon_2,q)&=\sum_k q^k Z_{k\text{-inst}}(a,\{\mu\},\epsilon_1,\epsilon_2), \\ Z_{k\text{-inst}}(a,\{\mu\},\epsilon_1,\epsilon_2)&=\sum_{\vec{Y}:|\vec{Y}|=k}\frac{\prod_{i=1}^{N_f}z_{\text{fund},i}(\vec{a},\vec{Y},\mu_i)\prod_{j=1}^{N_a}z_{\text{adj},i}(\vec{a},\vec{Y},\mu_j)}{z_{\text{vector}}(\vec{a},\vec{Y})}, \end{aligned} \tag{1.36}$$

where q is the 1-instanton factor. In case of asymptotic free gauge theories, $q:=\Lambda^{2N-N_f}$ and Λ denotes the QCD scale. In case of conformal gauge theories such as $N_f=2N$ or $\mathcal{N}=2^*$, $q:=e^{2\pi i\tau}$ where τ is the complexified coupling constant.

1.3 Full partition functions and Loop Operators

In addition to the correspondence where the instantons partition functions are considered on the 4d side, the authors in [5] found the correspondence between the full partition functions on S^4 and certain 2d quantities. Moreover, the authors of [7],[8] and [11] found that the vevs of the loop operators on S^4 have counterparts on the 2d side. In this subsection let us see the results of 4d theories that appear in the above correspondences.

The full partition functions of 4d $\mathcal{N}=2$ gauge theories with the gauge group G on S^4 with the radius r were calculated in [9]. The results are as follows

$$Z^{\text{full}}_{S^4}(a,\{m\},q)=\int da\, Z_{\text{north}}(a,\{m\},q)\cdot Z_{\text{south}}(a,\{m\},\bar{q}) \tag{1.37}$$

where $a\in\mathfrak{t}$ takes values in the Cartan subalgebra of G and

$$Z_{\text{north}}(a,\{m\},q)=Z_{\text{cl}}(a,q)\,Z_{\text{1-loop,pole}}(a,\{m\})Z_{\text{inst}}\left(a,\{\frac{1}{r}+m\},\frac{1}{r},\frac{1}{r},q\right) \tag{1.38}$$

$$Z_{\text{south}}(a,\{m\},\bar{q})=Z_{\text{cl}}(a,\bar{q})\,Z_{\text{1-loop,pole}}(a,\{m\})Z_{\text{inst}}\left(a,\{-\frac{1}{r}+m\},-\frac{1}{r},-\frac{1}{r},\bar{q}\right). \tag{1.39}$$

$Z_{\rm cl}$ and $Z_{\text{1-loop,pole}}$ are given by

$$
\begin{aligned}
&Z_{\rm cl}\,(a,q) = \exp\left[\pi i\, r^2 \tau\, {\rm Tr}\, a^2\right] \,, \\
&Z_{\text{1-loop,pole}}(a,\{m\}) \propto \frac{\prod_{\alpha}(\alpha\cdot a)^{1/2}\prod_{n\geq 1}(\frac{n}{r}+\alpha\cdot a)^n\,,}{\prod_i\prod_{\omega_i\in R_i}\prod_{n\geq 1}(\frac{n}{r}+w_i\cdot a-m_i)^{n/2}(\frac{n}{r}-w_i\cdot a+m_i)^{n/2}}\,.
\end{aligned}
\tag{1.40}
$$

α are the roots of the Lie algebra $\mathfrak{g}$. m_i and w_i are the mass and the weights of the representation R_i of G for the i-th hypermultiplet.

Next we insert the half-BPS Wilson loop operator. The half-BPS Wilson loop operators W_R in the representation R along the great S^1 on S^4 is defined as

$$
W_R := {\rm Tr}_R \exp\left(\oint_{S^1}\left[A_\mu \frac{dx^\mu}{ds} + i\left|\frac{dx^\mu}{ds}\right|\Phi_0\right]ds\right)\,, \tag{1.41}
$$

where Φ_0 is defined so that the complex scalar field ϕ in the vector multiplet is written as $\phi = \Phi_0 + i\Phi_9$. The vev was also calculated in [9] as

$$
\langle W_R\rangle = \int da\, Z_{\rm north}(a,\{m\},q)\, Z_{\rm south}(a,\{m\},\overline{q})\, {\rm Tr}_R\, e^{2\pi r i a}\,. \tag{1.42}
$$

Next let us turn to the half-BPS 't Hooft loop operators along the great S^1 on S^4. The 't Hooft loop operators are labeled by coweights B. The coweights B can be seen as weights in the Langland dual group G^L. For $G = U(N)$, G^L is also $U(N)$. The vev of the 't Hooft loop operator labeled by B is defined as the sum of the sectors associated with weights v of the representation of G^L whose highest weight is B. The sector with w is defined by the path integral around the following singular configuration whose behavior near the loop is described as

$$
\begin{aligned}
F &= \frac{v}{4}\epsilon_{ijk}\frac{x^i}{|\vec{x}|^3}dx^k\wedge dx^j\,, \\
\Phi_9 &= \frac{v}{2|\vec{x}|}\,,
\end{aligned}
\tag{1.43}
$$

where $i,j,k = 1,2,3$ and $\vec{x} = (x_1,x_2,x_3)$. We used the coordinate of $\mathbb{R}^3$ above since the transverse directions to the loop are locally the same as $\mathbb{R}^3$. We consider B as the magnetic charge of the 't Hooft loop operator. Let us focus on the cases where all the weights are related to B by the action of Weyl group. In such cases, the vev of the 't Hooft loop operator with the magnetic charge B can be written as

$$
\begin{aligned}
\langle T_B\rangle = \sum_{v\in{\rm Rep}(B)}\int da\, Z_{\rm north}(a^{(N)},\{m\},q)\, Z_{\rm south}(a^{(S)},\{m\},\overline{q})& \\
\times Z_{\text{1-loop,equator}}(a,\{m\},v)\,,&
\end{aligned}
\tag{1.44}
$$

where $a^{(N)}$ and $a^{(S)}$ in Z_{north} and Z_{south} are $a^{(N)} := a - v/(2r)$ and $a^{(S)} := a + v/(2r)$ and $Z_{\text{1-loop,equator}}$ is the contribution from the equator where the loop operator is inserted.

For example, $Z_{\text{1-loop,equator}}$ of the 't Hooft loop operator with $B = (1, 0^{N-1})$ in $\mathcal{N} = 2^*$ $U(N)$ theory is given as

$$\begin{aligned} &Z_{\text{1-loop,equator}}(a, m, v) \\ &= \left(\prod_{i<j} \frac{\sinh\left[\pi(a_i - a_j) - \pi m + \pi \frac{v_i - v_j}{2}\right] \sinh\left[\pi(a_i - a_j) + \pi m + \pi \frac{v_i - v_j}{2}\right]}{\sinh^2\left[\pi(a_i - a_j) + \pi \frac{v_i - v_j}{2}\right]} \right)^{|v_i - v_j|/2} . \end{aligned} \tag{1.45}$$

v runs over $(1, 0^{N-1}), (0, 1, 0^{N-2}), \ldots, (0^{N-1}, 1)$. m denotes the mass of the hypermultiplet.

In case with general B, there is a phenomenon where dynamical monopoles surround the 't Hooft loop operator in the sectors associated with the weights that are not related to B by the action of Weyl group. As we will explain in Section 2.4, these sectors give another contribution to the vev in addition to the right-hand side in (1.44).

1.4 Conformal Blocks and Correlation functions in 2d CFT

Having reviewed the 4d side, we will turn to the 2d side from this subsection. First we review Liouville theory in Section 1.4.1. When the gauge group is $SU(2)$ on 4d side, we should put Liouville theory on the Riemann surfaces corresponding to the gauge theories.

Then if we consider certain correlation functions in Liouville theory related to the singularities of Seiberg-Witten curve, it will give the same quantities as the 4d theory side. In Section 1.4.2 we will define certain quantities called conformal blocks that appear during the calculation of the correlation functions. Then we will see the correspondence between these quantities and the instanton partition functions on $\mathbb{R}^4$ explained in Section 1.2. Next we will consider the correlation functions and see the correspondense between them and the full partition functions on S^4 mentioned in Section 1.4.2. In Section 1.4.4 we will extend the above correspondence and obtain the correspondence where the gauge group is $SU(N)$ with general N. In this case, we should put A_{N-1} Toda theory on the 2d side. We will review this theory and see the extended correspondence between this and $SU(N)$ gauge theories.

1.4.1 Liouville theory

We consider Liouville theory on the 2d side to obtain quantities corresponding to 4d $SU(2)$ gauge theories.

The action of Liouville theory is as follows

$$S = \int d^2\sigma \sqrt{g} \Big[\frac{1}{4\pi} g^{xy} \partial_x \phi \cdot \partial_y \phi + \mu e^{2b\phi} + \frac{Q}{4\pi} R\phi \Big] . \tag{1.46}$$

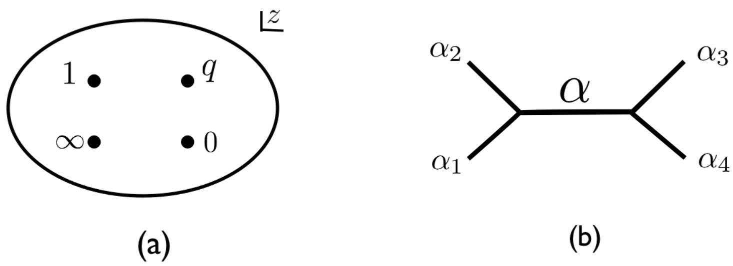

Figure 5: The sphere with four punctures and the conformal block related to it

where $Q = b + 1/b$. The central charge of this conformal field theory is

$$c = 1 + 6Q^2 \,. \tag{1.47}$$

This theory has the conformal symmmetry and the algebra is as follows

$$[L_n, L_m] \ = (n-m)L_{n+m} + \frac{c}{8}(n^3 - n)\delta_{n+m} \,. \tag{1.48}$$

$$\tag{1.49}$$

The primary fields can be written as $V_\alpha(z) := e^{2\alpha\phi(z)}$ and its conformal dimension is

$$\Delta_\alpha = \alpha(Q - \alpha) \,. \tag{1.50}$$

α is called its momentum.

1.4.2 Sphere with four punctures

Let us consider the quantity in Liouville theory correponding to 4d $U(2)$ gauge theories with $N_f = 4$. Based on the singularities and the residues in (1.13), we put the following vertex operators on the points on a sphere

$$\begin{array}{ll} V_{\alpha_1}(z) & \text{at } z = \infty \\ V_{\alpha_2}(z) & \text{at } z = 1 \\ V_{\alpha_3}(z) & \text{at } z = q \\ V_{\alpha_4}(z) & \text{at } z = 0 \,, \end{array} \tag{1.51}$$

where

$$\alpha_1 = \frac{\mu_1 - \mu_2}{2} + \frac{Q}{2} \,, \ \alpha_2 = \frac{\mu_1 + \mu_2}{2} \,, \ \alpha_3 = \frac{\mu_3 + \mu_4}{2} \,, \ \alpha_4 = \frac{\mu_3 - \mu_4}{2} + \frac{Q}{2} \,. \tag{1.52}$$

These points are indicated in (a) in Figure 5. Then we consider the following correlation functions on a sphere

$$\langle V_{\alpha_1}(\infty)V_{\alpha_2}(1)V_{\alpha_3}(q)V_{\alpha_4}(0)\rangle_{\text{sphere}}\,. \tag{1.53}$$

We calculate it by inserting the identity operators made from the complete sets.

$$\begin{aligned}(1.53) &= \langle V_{\alpha_1}(\infty)V_{\alpha_2}(1)\Big(\int_{\alpha\in\frac{Q}{2}+i\mathbb{R}} d\alpha \sum_{I,J} |\alpha, I\rangle\,(K_\alpha)^{-1}_{IJ}\,\langle\alpha, J|\Big)V_{\alpha_3}(q)V_{\alpha_4}(0)\rangle \\ &= \int d\alpha\,\langle V_{\alpha_1}(\infty)V_{\alpha_2}(1)\,|\alpha\rangle\langle\alpha|\,V_{\alpha_3}(q)V_{\alpha_4}(0)\rangle \\ &\qquad\times\sum_{I,J}\frac{\langle V_{\alpha_1}(\infty)V_{\alpha_2}(1)\,|\alpha, I\rangle}{\langle V_{\alpha_1}(\infty)V_{\alpha_2}(1)\,|\alpha\rangle}\,(K_\alpha)^{-1}_{I,J}\,\frac{\langle\alpha, J|\,V_{\alpha_3}(q)V_{\alpha_4}(0)\rangle}{\langle\alpha|\,V_{\alpha_3}(q)V_{\alpha_4}(0)\rangle}\end{aligned} \tag{1.54}$$

where $\{|\alpha, I\rangle\}_{I=1,2,3,\cdots} = \{|\alpha\rangle, L_{-1}|\alpha\rangle, \overline{L}_{-1}|\alpha\rangle, \cdots\}$ is the set of basis in the Verma module constructed from the primary state $|\alpha\rangle$. $L_0|\alpha\rangle = \Delta_\alpha|\alpha\rangle$. $(K_\alpha)^{-1}$ is the inverse of the matrix with the components $(K_\alpha)_{IJ} = \langle\alpha, I|\alpha, J\rangle$. We decompose it into chiral half parts. $\{|\alpha, I'\rangle^{(\text{c.h.})}\}_{I'=1,2,3,\cdots} = \{|\alpha\rangle, L_{-1}|\alpha\rangle, L_{-2}|\alpha\rangle, \cdots\}$, which does not include $\overline{L}_{-n}$. The superscript (c.h.) stands for chiral half.

$$\begin{aligned}(1.54) &= \int d\alpha\,\langle V_{\alpha_1}(\infty)V_{\alpha_2}(1)\,|\alpha\rangle\langle\alpha|\,V_{\alpha_3}(q)V_{\alpha_4}(0)\rangle \\ &\quad\times\Big|\sum_{I',J'}\frac{\langle V_{\alpha_1}(\infty)V_{\alpha_2}(1)\,|\alpha, I'\rangle^{(\text{c.h.})}}{\langle V_{\alpha_1}(\infty)V_{\alpha_2}(1)\,|\alpha\rangle}\,(K_\alpha^{(\text{c.h.})})^{-1}_{I',J'}\,\frac{{}^{(\text{c.h.})}\langle\alpha, J'|\,V_{\alpha_3}(q)V_{\alpha_4}(0)\rangle}{\langle\alpha|\,V_{\alpha_3}(q)V_{\alpha_4}(0)\rangle}\Big|^2\,.\end{aligned} \tag{1.55}$$

The general three-point function The three-point function is given by the DOZZ formula [18, 19]

$$\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)\rangle = |z_{12}|^{2(\Delta_3-\Delta_1-\Delta_2)}|z_{23}|^{2(\Delta_1-\Delta_2-\Delta_3)}|z_{31}|^{2(\Delta_2-\Delta_3-\Delta_1)}C(\alpha_1,\alpha_2,\alpha_3) \tag{1.56}$$

where Δ_i is the dimension of the operators $V_{\alpha_i} = e^{2\alpha\phi}$ given by

$$\Delta_i = \alpha_i(Q-\alpha_i),$$

and

$$\begin{aligned}&C(\alpha_1,\alpha_2,\alpha_3) \\ &\propto \frac{\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_1-\alpha_2+\alpha_3)\Upsilon(-\alpha_1+\alpha_2+\alpha_3)}\end{aligned} \tag{1.57}$$

where

$$\Upsilon(x) = \frac{1}{\Gamma_2(x|b,b^{-1})\Gamma_2(Q-x|b,b^{-1})}\,. \tag{1.58}$$

One can think of Γ_2 as the regularized infinite product

$$\Gamma_2(x|\epsilon_1,\epsilon_2) \propto \prod_{m,n\geq 0} (x+m\epsilon_1+n\epsilon_2)^{-1} \,. \tag{1.59}$$

The inside of $|\cdot|$ in the last line of (1.55) is the conformal block for the correlation function (1.53). We denote it by $\mathcal{F}_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}}(\alpha,q)$ (,where we sometimes omit q) and it is often described as (b) in Figure 5. The states $|\alpha,I'\rangle$ are called internal states and $V_{\alpha_1},\cdots,V_{\alpha_4}$ are called external fields. Collecting above, we rewrite (1.55) as

$$(1.54) = \int d\alpha\, |q|^{2(\Delta_\alpha-\Delta_{\alpha_3}-\Delta_{\alpha_4})}\, C(Q-\alpha_1,\alpha_2,\alpha)C(Q-\alpha,\alpha_3,\alpha_4)\left|\mathcal{F}_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}}(\alpha,q)\right|^2 \tag{1.60}$$

Since $\langle A|V_\alpha(z)V_\beta(0)\rangle \sim z^{\Delta_A-\Delta_\alpha-\Delta_\beta}$, the conformal block can be written as expansion of q whose power is the level of the internal states.

$$\mathcal{F}_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}}(\alpha,q) = \sum_{n=0}^{\infty} q^n\, \mathcal{F}^{(n)}_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}}(\alpha)\,, \tag{1.61}$$

$$\mathcal{F}^{(n)}_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}}(\alpha) := \sum_{i',j'} \frac{\langle V_{\alpha_1}(\infty)V_{\alpha_2}(1)\,|\alpha,i'\rangle_n}{\langle V_{\alpha_1}(\infty)V_{\alpha_2}(1)\,|\alpha\rangle}\,(K_{\alpha,n})^{-1}_{i',j'}\,\frac{{}_n\langle\alpha,j'|\,V_{\alpha_3}(1)V_{\alpha_4}(0)\rangle}{\langle\alpha|\,V_{\alpha_3}(1)V_{\alpha_4}(0)\rangle}\,, \tag{1.62}$$

where $\{\,|\alpha,i'\rangle_n\,\}_{i'=1,2,\cdots} = \{\,L_{-n_1}\cdots|\alpha\rangle \in \{|\alpha,I'\rangle\}_{I'}\,\Big|\, n_1+\cdots=n\}$ is the set of the basis with level n in the chiral half part and we omit the superscript (c.h.) in the above expression.

We can calculate the conformal blocks by using the following Ward identity

$$\langle L_{-n}V_1|V_2(1)V_3(0)\,\rangle = \langle V_1|V_2(1)(L_nV_3)(0)\rangle + \sum_{\ell=0}^{\infty}\begin{pmatrix} n+1\\ \ell\end{pmatrix}\langle V_1|L_{\ell-1}V_2(1)V_3)(0)\rangle\,. \tag{1.63}$$

Let us compare the gauge theory side with the corresponding conformal block. On the gauge theory side, we considered $U(2)$ gauge theory with $N_f = 4$. More precisely, we introduce two antifundamental hypermultiplets with the mass μ_1,μ_2 and two fundamental hypermultiplets with the mass μ_3,μ_4. We can obtain the instanton partition functions of the theory on $\mathbb{R}^4$ by using the formulae (1.36). Since the spacetime is noncompact, we have to fix the boundary condition and that is why the instanton partition function $Z_{\rm inst}(\vec{a},\{\mu\},\epsilon_1,\epsilon_2,)$ depends on the scalar vev. Even though we are considering $U(2)$ gauge theory, we take the scalar vev $\vec{a}=(a,-a)$, which is included in $SU(2)$. We set the parameter of the equvariant action $\epsilon_{1,2}$ as $\epsilon_1=b,\epsilon_2=1/b$ where b is the parameter in the Lagrangian of Liouville theory. The instanton partition function coincides with the corresponding conformal block up to the factor $Z^{C_{0,4}}_{U(1)} := (1-q)^{2\alpha_2(Q-\alpha_3)}$ as follows

$$Z_{\rm inst}(\vec{a},\{\mu_f\}_f,b,1/b,q) = Z^{C_{0,4}}_{U(1)}\,\mathcal{F}_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}}(\alpha,q)\,, \tag{1.64}$$

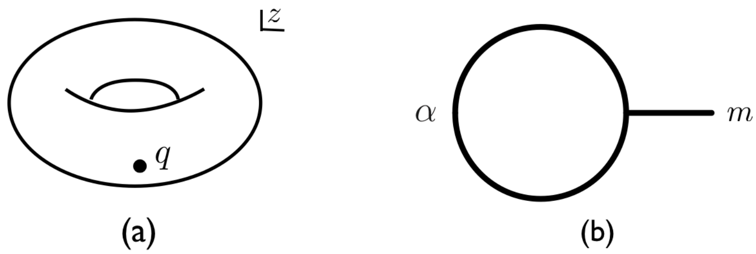

Figure 6: The torus with one puncture and the conformal block related to it

if we identify $\alpha = Q/2 + a$. The factor $Z^{C_{0,4}}_{U(1)}$ is called $U(1)$ factor and is believed to be contribution from the $U(1)$ part in the gauge theory. $C_{g,n}$ denotes a genus g Riemann surface with n punctures.

Moreover there is a relation between the full partition function of this gauge theory on S^4 with radius $r = 1$ and the correlation function (1.53) with $b = 1$. To calculate the correlation function (1.60), we need to take complex conjugate of $\mathcal{F}$. When we do so, we use the fact that all the Liouville momenta α take values in $Q/2 + i\mathbb{R}$. Noting that $Z_{\rm inst}(\vec{a}, \{\mu\}, \epsilon_1, \epsilon_2, q) = Z_{\rm inst}(-\vec{a}, \{-\mu\}, -\epsilon_1, -\epsilon_2, q)$, we find that $\overline{\mathcal{F}_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}}(\alpha, q)}$ coincides with $Z_{\rm inst}$ in the right-hand side of (1.39) up to the $U(1)$ factor.

Let us turn to the remaining part. If we compare $Z_{\text{1-loop,pole}}$ in $U(2)$ $N_f = 4$ theory with the three point functions in (1.60), we find

$$Z_{\text{1-loop,pole}}(a, \{m_f\}_f)^2 \propto C(Q - \alpha_1, \alpha_2, \alpha)C(Q - \alpha, \alpha_3, \alpha_4)\,.$$

Note that the notation of the mass μ_f in (1.52) and the notation of the mass m_f in Section 1.3 are different as $\mu_f = \frac{1}{r} + m_f$. When we obtained the above relation, we used the formula

$$\Gamma_2(x + \epsilon_1|\epsilon_1, \epsilon_2)\Gamma_2(x + \epsilon_2|\epsilon_1, \epsilon_2) = x\,\Gamma_2(x|\epsilon_1, \epsilon_2)\Gamma_2(x + \epsilon_1 + \epsilon_2|\epsilon_1, \epsilon_2) \qquad (1.65)$$

and ignored the factors independent of α. Collecting above, we get

$$Z^{\rm ful}_{S^4}(\{\mu_f\}_f, q) \propto \langle V_{\alpha_1}(\infty)V_{\alpha_2}(q)V_{\alpha_3}(1)V_{\alpha_4}(0)\rangle_{\rm sphere}\big|_{b=1}\,. \qquad (1.66)$$

1.4.3 Torus with one puncture

Here we consider the Liouville theory quantity correponding to $\mathcal{N} = 2^*$ $SU(2)$ gauge theory. The mass of the adjoint hypermultiplet is μ. On the Liouville theory side, we put the vertex operator $V_{\rm m}$ at the point $z = q$ on a torus indicated in (a) in Figure 6. The correlation function $\langle V_m(q)\rangle$ on the torus can be calculated as follows

$$\langle V_{\rm m}(q)\rangle_{\rm torus} = \int d\alpha\,\langle\alpha|V_{\rm m}(q)|\alpha\rangle\Big|\sum_{I',J'}\frac{\langle\alpha, J'|V_{\rm m}(q)|\alpha, I'\rangle}{\langle\alpha|V_{\rm m}(q)|\alpha\rangle}\,(K_\alpha)^{-1}_{I',J'}\Big|^2\,. \qquad (1.67)$$

The inside of the $|\cdot|$ is the conformal block for the one-point correlation function and it is denoted by $\mathcal{F}_{\rm m}(\alpha, q)$ or $\mathcal{F}_{\rm m}(\alpha)$. It is often described as (b) in Figure 6. The author of [5] found the following relation similar to the previous subsubsection

$$Z_{\rm inst}(\vec{a}, \mu, b, 1/b, q) = Z_{U(1)}^{C_{1,1}}\, \mathcal{F}_{\rm m}(\alpha, q)\,, \tag{1.68}$$

where we identify $\alpha = a + Q/2$ and $\rm m = \mu$ and

$$Z_{U(1)}^{C_{1,1}} := \Big[\prod_{i=1}^{\infty}(1-q^i)\Big]^{-1+2m(Q-m)}\,. \tag{1.69}$$

$C_{1,1}$ denotes a torus with one puncture. Again there is a relation between the full partition function on S^4 with radius $r = 1$ and the correlation function with $b = 1$.

$$Z_{S^4}^{\rm full}(m, q) \propto \langle V_{\rm m}(q)\rangle_{\rm torus}\big|_{b=1}\,. \tag{1.70}$$

Since the notation of mass in Section 1.2 and Section 1.3 are different, we have to substitute $\mu - 1/r$ into m in (1.40) when we calculate $Z_{S^4}^{\rm full}(m, q)$.

1.4.4 A_{N-1} Toda theory

In this subsubsection, we review an extension of the previous subsubsections. We consider A_{N-1} Toda theory, which corresponds to 4d $\mathcal{N} = 2$ $SU(N)$ gauge theories. The A_{N-1} Toda field theories are defined by the action

$$S = \int d^2\sigma\sqrt{g}\left[\frac{1}{8\pi}g^{ad}\langle\partial_a\phi, \partial_d\phi\rangle + \mu\sum_{i=1}^{N-1} e^{b\langle e_i,\phi\rangle} + R\frac{\langle Q\rho, \phi\rangle}{4\pi}\right], \tag{1.71}$$

where g_{ad} $(a, d = 1, 2)$ is the metric on the two-dimensional worldsheet, and R is its associated curvature. $e_1, \ldots, e_{N-1}$ are the simple roots of the A_{N-1} Lie algebra, i.e. the vectors with N components

$$(e_i)_j = \begin{cases} 1 & j = i \\ -1 & j = i+1 \\ 0 & \text{otherwise}\,. \end{cases} \tag{1.72}$$

The brackets $\langle\cdot,\cdot\rangle$ denotes the scalar product on the root space, and the $(N-1)$-dimensional vector of fields ϕ can be expanded as $\phi = \sum_i \phi_i e_i$. ρ is the Weyl vector (half the sum of all positive roots)

$$\rho = (\frac{N-1}{2}, \frac{N-3}{2}, \cdots, -\frac{N-1}{2})\,, \tag{1.73}$$

and $Q := b+1/b$. From the above action it is easy to see that the Liouville theory is identical to the A_1 Toda field theory. Toda theories can be defined for any simple Lie algebra by taking the e_i to be the simple roots of the corresponding Lie algebra. The central charge is

$$c = N - 1 + 12\langle Q\rho, Q\rho\rangle = (N-1)(1 + N(N+1)(b + \frac{1}{b})^2)\,. \tag{1.74}$$

This theory has $\mathcal{W}_N$ symmetry. Let us focus on the case where $N = 3$. The $\mathcal{W}_3$ algebra can be written as

$$\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}\,,\\
&[L_n, W^{(3)}_m] = (2n - m)W^{(3)}_{n+m}\,,\\
&[W^{(3)}_n, W^{(3)}_m] = \frac{c}{3\cdot 5!}(n^2-1)(n^2-4)n\,\delta_{n,-m} + \frac{16}{22+5c}(n-m)\Lambda_{n+m}\\
&\qquad + (n-m)\left(\frac{1}{15}(n+m+2)(n+m+3) - \frac{1}{6}(n+2)(m+2)\right)L_{n+m}\,,
\end{aligned} \tag{1.75}$$

where c is the central charge and

$$\begin{aligned}
\Lambda_n &= \sum_{k=-\infty}^{\infty} : L_k L_{n-k} : + \frac{1}{5}x_n L_n\,,\\
x_{2l} &= (1+l)(1-l)\,, \quad x_{2l+1} = (2+l)(1-l)\,.
\end{aligned} \tag{1.76}$$

In case of $\mathcal{W}_N$ algebra wigh general N, there are holomorphic symmetry currents $L, \mathcal{W}^{(k)}$ where $k = 3, \cdots, N$. Primary fields can be defined in analogy with the Virasoro case. A $\mathcal{W}$-primary field satisfies

$$W^{(k)}_0 V = w^{(k)} V\,, \qquad W^{(k)}_n V = 0 \quad \text{when } n > 0\,. \tag{1.77}$$

In the A_{N-1} Toda theories the primary fields are written as

$$V_\alpha = e^{\langle\alpha,\phi\rangle}\,, \tag{1.78}$$

and α are called their momentum. In the particular example of the A_2 theory the primary fields satisfy

$$L_0 V_\alpha = \Delta_\alpha V_\alpha\,, \qquad W^{(3)}_0 V_\alpha = w(\alpha)V_\alpha\,, \qquad L_n V_\alpha = W^{(3)}_n V_\alpha = 0 \quad \text{when } n > 0\,, \tag{1.79}$$

where

$$\Delta_\alpha = \frac{\langle 2Q\rho - \alpha, \alpha\rangle}{2}\,, \tag{1.80}$$

is the conformal dimension and

$$w(\alpha) = i\sqrt{\frac{48}{22+5c}}\,\langle\alpha - Q, h_1\rangle\langle\alpha - Q, h_2\rangle\langle\alpha - Q, h_3\rangle\,, \tag{1.81}$$

is the quantum number of the $\mathcal{W}^{(3)}$ current. Here the h_i are the weights of the fundamental representation of the A_2 Lie algebra. In case of A_{N-1} with general N, $h_1, \ldots, h_N$ are N-component vectors

$$(h_i)_j = \begin{cases} 1 - \frac{1}{N} & j = i \\ -\frac{1}{N} & j \neq i \,. \end{cases} \tag{1.82}$$

The author of [6] found the relations between $\mathcal{N} = 2$ $SU(N)$ superconformal gauge theory and A_{N-1} Toda theory , which is similar to the relations in Section 1.4.2 and Section 1.4.3. First let us look at the case of $SU(N)$ gauge theory with $N_f = 2N$. The mass of the antifundamental and fundamental hypermultiplets are denoted by $\mu_1, \cdots, \mu_N$ and $\mu_{N+1}, \cdots, \mu_{2N}$ respectively. On the A_{N-1} Toda theory, we consider the four-point correlation functions $\langle V_{\mathrm{m}_1}(\infty) V_{\widehat{\mathrm{m}}_2}(1) V_{\widehat{\mathrm{m}}_3}(q) V_{\mathrm{m}_4}(0)\rangle$ on a sphere, where

$$\begin{aligned} &\widehat{\mathrm{m}}_2 = \left(\frac{Q}{2} - \widehat{m}_2\right) N\omega_1 \,, && \widehat{\mathrm{m}}_3 = \left(\frac{Q}{2} + \widehat{m}_3\right) N\omega_{N-1} \,, \\ &\mathrm{m}_1 = Q\rho - \widetilde{m}_1 \,, && \mathrm{m}_4 = Q\rho + \widetilde{m}_4 \,, \\ &\mu_f = \begin{cases} Q/2 + \widehat{m}_2 + \widetilde{m}_1 \cdot h_f & \text{for } \ f = 1, \ldots, N \,, \\ Q/2 + \widehat{m}_3 + \widetilde{m}_4 \cdot h_{f-N} & \text{for } \ f = N+1, \ldots, 2N. \end{cases} \end{aligned} \tag{1.83}$$

$\omega_1, \ldots, \omega_{N-1}$ are N-component vectors where

$$(\omega_i)_j = \begin{cases} 1 - \frac{i}{N} & (1 \leq j \leq i) \\ -\frac{i}{N} & (i+1 \leq j \leq N) \,. \end{cases} \tag{1.84}$$

Momenta proportional to ω_1 or ω_{N-1} such as $\widehat{\mathrm{m}}_2$ and $\widehat{\mathrm{m}}_3$ are called semi-degenerate momentum. $a, \widetilde{m}_1$ and $\widetilde{m}_4$ are also traceless N-component vectors. $\widehat{m}_2$ and $\widehat{m}_3$ are 1-component numbers. The above correlation function can be written as follows

$$\begin{aligned} &\langle V_{\mathrm{m}_1}(\infty) V_{\widehat{\mathrm{m}}_2}(1) V_{\widehat{\mathrm{m}}_3}(q) V_{\mathrm{m}_4}(0)\rangle_{\text{sphere}} \\ &= \int d\alpha \, \langle V_{\mathrm{m}_1}(\infty) V_{\widehat{\mathrm{m}}_2}(1) V_\alpha\rangle \langle V_{2Q\rho-\alpha}(\infty) V_{\widehat{\mathrm{m}}_3}(q) V_{\mathrm{m}_4}(0)\rangle |\mathcal{F}_{\{\mathrm{m}_1,\widehat{\mathrm{m}}_2,\widehat{\mathrm{m}}_3,\mathrm{m}_4\}}(\alpha, q)|^2 \,, \\ &= \int d\alpha \, C(2Q\rho - \mathrm{m}_1, \widehat{\mathrm{m}}_2, \alpha)\, C(2Q\rho - \alpha, \widehat{\mathrm{m}}_3, \mathrm{m}_4)\, |\mathcal{F}_{\{\mathrm{m}_1,\widehat{\mathrm{m}}_2,\widehat{\mathrm{m}}_3,\mathrm{m}_4\}}(\alpha, q)|^2 \,, \end{aligned} \tag{1.85}$$

as in the Liouville theory case (1.55). $C(\alpha_1, \alpha_2, \alpha_3)$ is the coefficient that appears in the expression of the three-point function of Toda theory

$$\langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3)\rangle = |z_{12}|^{2(\Delta_3-\Delta_1-\Delta_2)} |z_{23}|^{2(\Delta_1-\Delta_2-\Delta_3)} |z_{31}|^{2(\Delta_2-\Delta_3-\Delta_1)} C(\alpha_1, \alpha_2, \alpha_3) \,. \tag{1.86}$$

In case with two generic momenta α_1, α_2 and one semi-degenerate momentum $\alpha_3 = \kappa\omega_{N-1}$, the three-point function was obtained in [20] as

$$C(\alpha_1, \alpha_2, \alpha_3 = \kappa\omega_{N-1}) \propto C^{(1)}(\alpha_1, \alpha_2, \kappa) := \frac{\prod_{i<j} \Upsilon(\langle Q\rho - \alpha_1, h_{ij}\rangle) \Upsilon(\langle Q\rho - \alpha_2, h_{ij}\rangle)}{\prod_{i,j=1}^{N} \Upsilon(\kappa/N + \langle \alpha_1 - Q\rho, h_i\rangle + \langle \alpha_2 - Q\rho, h_j\rangle)} \,. \tag{1.87}$$

When $\alpha_1,\ \alpha_2$ are generic momentum and $\alpha_3 = \kappa\omega_1$, the three-point function was presented in [21]:

$$C(\alpha_1, \alpha_2, \alpha_3 = \kappa\omega_1) \propto C^{(2)}(\alpha_1, \alpha_2, \kappa) := \frac{\prod_{i<j} \Upsilon(\langle Q\rho - \alpha_1, h_{ij}\rangle)\Upsilon(\langle Q\rho - \alpha_2, h_{ij}\rangle)}{\prod_{i,j=1}^{N} \Upsilon(\kappa/N - \langle \alpha_1 - Q\rho, h_i\rangle - \langle \alpha_2 - Q\rho, h_j\rangle)}\,. \tag{1.88}$$

$\mathcal{F}_{\{\mathrm{m}_1,\widehat{\mathrm{m}}_2,\widehat{\mathrm{m}}_3,\mathrm{m}_4\}}(\alpha, q)$ in (1.91) is a conformal block, which is a generalization of (1.61) to Toda theory. The author of [6] found the relation between $SU(3)$ gauge theory and A_2 Toda theory as follows

$$\begin{aligned} &Z_{\mathrm{inst}}(\vec{a}, \{\mu_f\}_f, \epsilon_1 = b, \epsilon_2 = 1/b, -q) = Z_{U(1)}^{C_{0,4},\,A_2}\, \mathcal{F}_{\{\mathrm{m}_1,\widehat{\mathrm{m}}_2,\widehat{\mathrm{m}}_3,\mathrm{m}_4\}}(\alpha, q)\,, \\ &Z_{S^4}^{\mathrm{full}}(\{m_f\}_f, q) \propto \langle V_{\mathrm{m}_1}(\infty)V_{\widehat{\mathrm{m}}_2}(1)V_{\widehat{\mathrm{m}}_3}(q)V_{\mathrm{m}_4}(0)\rangle\big|_{b=1}\,, \end{aligned} \tag{1.89}$$

where $\vec{a}$ and α are related as $\alpha = Q\rho + \vec{a}$. $Z_{U(1)}^{C_{0,4},\,A_2}$ is a generalization of $Z_{U(1)}^{C_{0,4}}$ in (1.64) to $SU(3)$ gauge theory and given as

$$Z_{U(1)}^{C_{0,4},\,A_2} := (1-q)^{3(\frac{Q}{2}+\widehat{m}_2)(\frac{Q}{2}-\widehat{m}_3)}\,. \tag{1.90}$$

Next let us look at the case of $\mathcal{N} = 2^*$ $SU(N)$ gauge theory where the mass of the hypermultiplet is μ. Correspondingly we consider the one-point function $\langle V_{\widehat{\mathrm{m}}}(q)\rangle$ on a torus in A_{N-1} Toda theory,

$$\langle V_{\widehat{\mathrm{m}}}(q)\rangle_{\mathrm{torus}} = \int d\alpha\, C(2Q\rho - \alpha, \widehat{\mathrm{m}}, \alpha)\, |\mathcal{F}_{\widehat{m}}(\alpha, q)|^2\,, \tag{1.91}$$

where the momentum $\widehat{m}$ is related to the gauge theory mass μ as

$$\widehat{\mathrm{m}} = N\left(\frac{Q}{2} + m\right)\omega_{N-1}\,, \qquad \mu = \frac{Q}{2} + m\,. \tag{1.92}$$

The conformal block $\mathcal{F}_{\widehat{\mathrm{m}}}(\alpha, q)$ was defined in the same way as (1.67). Then the following relations hold

$$\begin{aligned} &Z_{\mathrm{inst}}(\vec{a}, \mu, \epsilon_1 = b, \epsilon_2 = 1/b, q) = Z_{U(1)}^{C_{1,1},\,A_2}\, \mathcal{F}_{\widehat{\mathrm{m}}}(\alpha, q) \\ &Z_{S^4}^{\mathrm{full}}(m, q) \propto \langle V_{\widehat{\mathrm{m}}}(q)\rangle_{\mathrm{torus}}\big|_{b=1}\,, \end{aligned} \tag{1.93}$$

where a and α are related as $\alpha = Q\rho + \vec{a}$ again. $Z_{U(1)}^{C_{1,1},\,A_{N-1}}$ is a generalization of $Z_{U(1)}^{C_{1,1}}$ in (1.68) to $SU(N)$ theory and independent of $\vec{a}$.

$$Z_{U(1)}^{C_{1,1},\,A_2} := \Big[\prod_{i=1}^{\infty}(1-q^i)\Big]^{-1+3(\frac{Q}{2}+m)(\frac{Q}{2}-m)}\,. \tag{1.94}$$

1.5 Verlinde Operators

There are certain loop operators on the 2d side corresponding to Wilson and 't Hooft loop operators on the 4d side. These operators are called Verlinde operators. In this subsection we will review these operators in A_{N-1} Toda theory and see the correspondence between them and the loop operators in 4d gauge theories mentioned in Section 1.3. This correspondence is relevant for Section 2. On the 4d side, we consider $\mathcal{N} = 2^* \; SU(N)$ gauge theory and

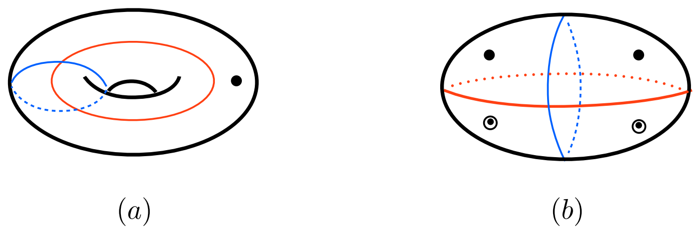

Figure 7: The closed curves on the Riemann surfaces

$\mathcal{N} = 2 \; SU(N)$ gauge theory with $N_f = 2N$. The corresponding Riemann surfaces are described in (a) and (b) in Figure 7. In each Riemann surface, the blue curve and red curve correspond to electric charge and magnetic charge on the gauge theory side. Therefore the Verlinde operators along the blue curve and the red curve correspond to Wilson and 't Hooft loop operators on the 4d side.

In this subsection, we focus on the Verlinde operators Λ_γ along the red curve γ. Before giving the definition of the Verlinde operators in 1.5.1, 1.5.2, let us turn to necessary formulae about conformal blocks. The conformal blocks that will appear there can be made from gluing four-point conformal blocks, we will focus on four-point conformal blocks here.

Let us consider four-point conformal blocks for the correlation functions $\langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle$ here. There are three types with different channels as described in Figure 8. The s in $\mathcal{F}^{(s)}$

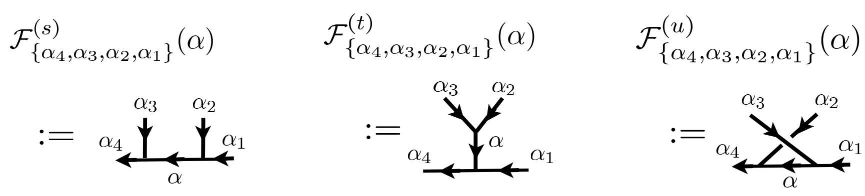

Figure 8: The s, t and u-channels

stands for s-channel. The same is true for t, u. They are related as follows

$$\mathcal{F}^{(s)}_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}}(\alpha) = \int d\alpha' F_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}}(\alpha,\alpha') \mathcal{F}^{(t)}_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}}(\alpha)$$
$$\mathcal{F}^{(s)}_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}}(\alpha) = \int d\alpha' B^{\epsilon}_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}}(\alpha,\alpha') \mathcal{F}^{(u)}_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}}(\alpha)\,, \tag{1.95}$$

where F and B are the fusion and braiding matrices respectively and $\epsilon = \pm 1$ determines the direction of the braiding. In order to describe the s-channel conformal block on the left-hand side in this figure, we have to consider continuously infinite kinds of momentum on the internal line in the t-channel or u-channel conformal block on the right-hand side in case with general external fields.

Unlike the general case (1.95), if we choose a completely degenerate field V_μ with the momentum $\mu = -bh_1 = -b\omega_1$ as one of the external fields in (a) in Figure 9, the momentum on the internal line can take only finite kinds of values. The reason is that the OPE of this field and a general field are as follows

$$[V_\mu] \cdot [V_\alpha] = \sum_{l=1}^{N} [V_{\alpha - bh_l}] \ . \tag{1.96}$$

Therefore the right-hand side in Figure 9 is not an integral but a sum over k, which labels

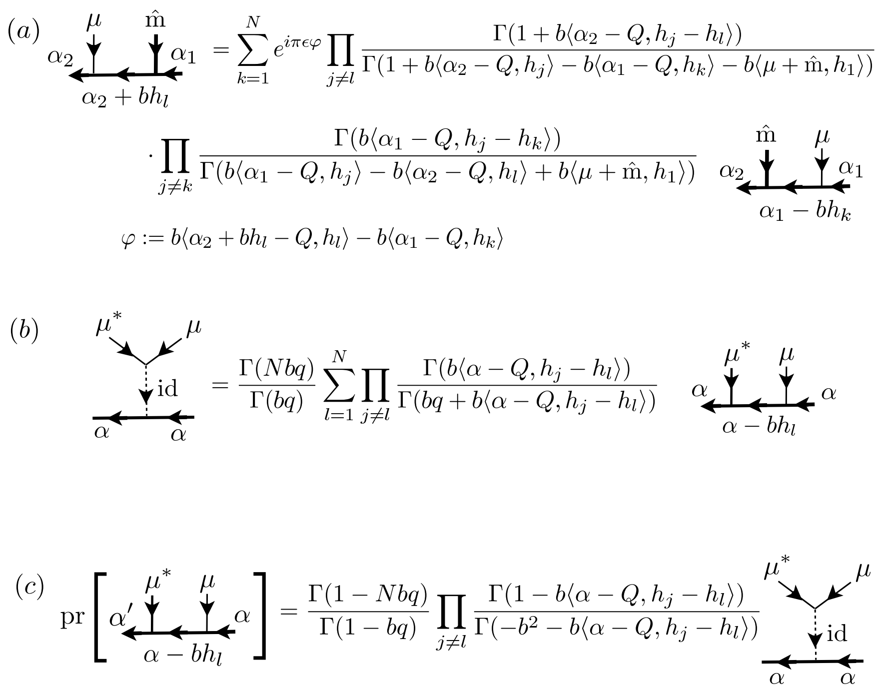

Figure 9: The fusion and braiding matrices

the internal momentum. We will use the formula in this figure.

Next let us consider more special cases where we take OPE between V_μ and V_{μ^*} where $\mu^* = -b\omega_{N-1} = bh_N$.

$$[V_\mu] \cdot [V_{\mu^*}] = [V_0] + \left[V_{-b(h_1 - h_N)}\right] \ . \tag{1.97}$$

This OPE appears in (b) and (c) in Figure 9. Zero momenta are denoted by id in the figures.

In (c) in Figure 9, we did projection. In order to describe the s-channel conformal block inside pr[$\cdot$] on the left-hand side, we have to consider the t-channel conformal blocks with the internal momentum 0 and $-b(h_1 - h_N)$ on the right-hand side. However we only picked up the contribution from the t-channel conformal block with the momentum 0. This is the definition of pr[$\cdot$].

1.5.1 Torus with one puncture

In this section, we compute the Verlinde operator Λ_γ on a torus with one puncture. This Verlinde operator acts on the conformal block (b) in Figure 6 as follows. First we insert the two external fields V_μ and V_{μ^*} where $\mu = -b\omega_1 = -bh_1$ and $\mu^* = -b\omega_N = bh_N$ as indicated as $\mathcal{F}^{(1)}_{\widehat{\mathrm{m}}}(\alpha)$ in Figure 10. Then we move V_μ along the red curve γ in Figure 7 as we go from $\mathcal{F}^{(1)}_{\widehat{\mathrm{m}}}(\alpha)$ to $\mathcal{F}^{(4)}_{\widehat{\mathrm{m}}}(\alpha'_k) \cdot \delta_{\alpha'_k,\alpha'_l}$ in Figure 10. We describe the conformal block $\mathcal{F}^{(1)}_{\widehat{\mathrm{m}}}(\alpha)$ in terms of the conformal block $\mathcal{F}^{(2)}_{\widehat{\mathrm{m}}}(\alpha'_l, \alpha)$. Repeating this procedure, we describe the conformal block $\mathcal{F}^{(1)}_{\widehat{\mathrm{m}}}(\alpha)$ in terms of $\mathcal{F}^{(1)}_{\widehat{\mathrm{m}}}(\alpha'_k) \cdot \delta_{\alpha'_k,\alpha'_l}$. The expression describing this is the conformal block acted on by the Verlinde operator.

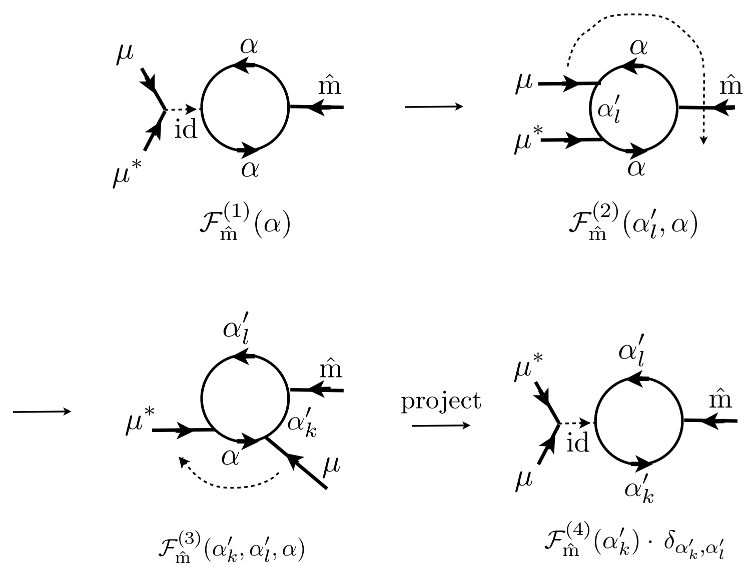

Figure 10: The Verlinde operator corresponding to γ on the torus

Let us denote the value of the first conformal block in Figure 10 by $\mathcal{F}^{(1)}_{\widehat{\mathrm{m}}}(\alpha)$. It is the same as that of the conformal block without V_μ and V_{μ^*} since we put the identity state on the dotted internal line. In the next step, we rewrite the value $\mathcal{F}^{(1)}_{\widehat{\mathrm{m}}}(\alpha)$ of the first conformal block in terms of that of the second one, which is denoted by $\mathcal{F}^{(2)}_{\widehat{\mathrm{m}}}(\alpha'_l, \alpha)$. If we consider the OPE (1.96), we understand that the internal momentum in the second one can take value of $\alpha'_l \equiv \alpha - bh_l$. By using the formula (b) in Figure 9, we can rewrite as follows

$$\mathcal{F}^{(1)}_{\widehat{\mathrm{m}}}(\alpha) = \frac{\Gamma(NbQ)}{\Gamma(bQ)} \sum_{l=1}^{N} \left[\left(\prod_{j\neq l} \frac{\Gamma(b\langle \alpha - Q\rho, h_j - h_l\rangle)}{\Gamma(bq + b\langle \alpha - Q\rho, h_j - h_l\rangle)} \right) \mathcal{F}^{(2)}_{\widehat{\mathrm{m}}}(\alpha'_l, \alpha) \right] \tag{1.98}$$

Next we describe the conformal block $\mathcal{F}^{(2)}_{\widehat{\mathrm{m}}}(\alpha'_l, \alpha)$ in terms of the conformal block $\mathcal{F}^{(3)}_{\widehat{\mathrm{m}}}(\alpha'_k, \alpha'_l, \alpha)$ in Figure 10. To do this, we use the formula (a) in Figure 9 with $\alpha_1 = \alpha$ and $\alpha_2 = \alpha_l{}'$. Then we get the following expression

$$\begin{aligned}\mathcal{F}^{(1)}_{\widehat{\mathrm{m}}}(\alpha) = \frac{\Gamma(NbQ)}{\Gamma(bQ)} \sum_{l,k=1}^{N} e^{i\pi\epsilon b\langle \alpha - Q\rho, h_l - h_k\rangle} \left(\prod_{j\neq l} \frac{\Gamma(b\langle \alpha - Q\rho, h_j - h_l\rangle)}{\Gamma(bq + b\langle \alpha - Q\rho, h_j - h_k\rangle - b\langle \widehat{\mathrm{m}}, h_1\rangle)} \right) \cdot \\ \cdot \left(\prod_{j\neq k} \frac{\Gamma(b\langle \alpha - Q\rho, h_j - h_k\rangle)}{\Gamma(b\langle \alpha - Q\rho, h_j - h_l\rangle + b\langle \widehat{\mathrm{m}}, h_1\rangle)} \right) \cdot \mathcal{F}^{(3)}_{\widehat{\mathrm{m}}}(\alpha'_k, \alpha'_l, \alpha) \,. \end{aligned} \tag{1.99}$$

Next we project the conformal block $\mathcal{F}^{(3)}_{\widehat{\mathrm{m}}}(\alpha'_k, \alpha'_l, \alpha)$. This projection is defined in (c) in Figure 9. We replace $\mathcal{F}^{(3)}_{\widehat{\mathrm{m}}}(\alpha'_k, \alpha'_l, \alpha)$ by $\mathrm{pr}[\mathcal{F}^{(3)}_{\widehat{\mathrm{m}}}(\alpha'_k, \alpha'_l, \alpha)]$ in the above expression and describe it in terms of $\mathcal{F}^{(4)}_{\widehat{\mathrm{m}}}(\alpha'_k)$ in Figure 10. by using the formula (c) in Figure 9. Thus we get the following expression.

$$\begin{aligned}\frac{\sin \pi bQ}{\sin \pi NbQ} \sum_{k=1}^{N} \prod_{j\neq k} \frac{\Gamma(b\langle \alpha - Q\rho, h_j - h_k\rangle)}{\Gamma(bq + b\langle \alpha - Q\rho, h_j - h_k\rangle - b\langle \widehat{\mathrm{m}}, h_1\rangle)} \frac{\Gamma(bq + b\langle \alpha - Q\rho, h_j - h_k\rangle)}{\Gamma(b\langle \alpha - Q\rho, h_j - h_k\rangle + b\langle \widehat{\mathrm{m}}, h_1\rangle)} \cdot \\ \cdot \mathcal{F}^{(4)}_{\widehat{\mathrm{m}}}(\alpha'_k) \,.\end{aligned}$$

Therefore the Verlinde operator acts on the conformal block as follows

$$\begin{aligned}\Lambda_\gamma :\ & \mathcal{F}_{\widehat{\mathrm{m}}}(\alpha) \to \sum_{k=1}^{N} H_{k\,;\,\widehat{\mathrm{m}}}(\alpha)\, \mathcal{F}_{\widehat{\mathrm{m}}}(\alpha - bh_k) \,, \\ H_{k\,;\,\widehat{\mathrm{m}}}(\alpha) &= \frac{\sin \pi bQ}{\sin \pi NbQ} \sum_{k=1}^{N} \prod_{j\neq k} \frac{\Gamma(b\langle \alpha - Q\rho, h_j - h_k\rangle)}{\Gamma(bQ + b\langle \alpha - Q\rho, h_j - h_k\rangle - b\langle \widehat{\mathrm{m}}, h_1\rangle)} \\ &\qquad \times \frac{\Gamma(bQ + b\langle \alpha - Q\rho, h_j - h_k\rangle)}{\Gamma(b\langle \alpha - Q\rho, h_j - h_k\rangle + b\langle \widehat{\mathrm{m}}, h_1\rangle)}\end{aligned} \tag{1.100}$$

In case of $N = 2$, where A_{N-1} Toda theory becomes Liouville theory, this operator was calculated in [7, 8]. The operator acts on the conformal block (b) in Figure 6 as follows

$$\begin{aligned}\Lambda_\gamma :\ & \mathcal{F}_m(\alpha) \to \sum_{\pm} H_{\pm\,;\,m}(\alpha)\, \mathcal{F}_m(\alpha \pm \frac{b}{2}) \,, \\ H_{+\,;\,m}(\alpha) &= \frac{\Gamma(2b\alpha - bQ)\Gamma(2b\alpha)}{\Gamma(2b\alpha - bQ + bm)\Gamma(2b\alpha - bm)} \,, \\ H_{-\,;\,m}(\alpha) &= \frac{\Gamma(-2b\alpha + bQ)\Gamma(-2b\alpha + 2bQ)}{\Gamma(-2b\alpha + bQ + bm)\Gamma(-2b\alpha - bm + 2bQ)} \,.\end{aligned} \tag{1.101}$$

The vev of the Verlinde operator is defined as

$$\int d\alpha\, C(2Q\rho-\alpha,\widehat{\mathrm{m}},\alpha)\,\overline{\mathcal{F}_{\widehat{m}}(\alpha,q)}\sum_{k=1}^{N} H_{k\,;\,\widehat{\mathrm{m}}}(\alpha)\,\mathcal{F}_{\widehat{\mathrm{m}}}(\alpha-bh_k,q) \tag{1.102}$$

$$=\int_{a\in i\mathbb{R}} da\, C(Q\rho-a,\widehat{\mathrm{m}},Q\rho+a)\,\mathcal{F}_{\overline{\widehat{m}}}(Q\rho-a,\overline{q})\sum_{k=1}^{N} H_{k\,;\,\widehat{\mathrm{m}}}(Q\rho+a)\,\mathcal{F}_{\widehat{\mathrm{m}}}(Q\rho+a-bh_k,q)\,.$$

We will denote $C(Q\rho-a,\widehat{\mathrm{m}},Q\rho+a)$ by $C(Q\rho+a)$. To compare this expression with (1.44), we shift the dummy variable a by $bh_k/2$

$$\begin{aligned}
&\int da\, C(Q\rho+a+bh_k/2)\,\mathcal{F}_{\overline{\widehat{m}}}(Q\rho-a-bh_k/2,\overline{q})\\
&\qquad\times\sum_{k=1}^{N} H_{k\,;\,\widehat{\mathrm{m}}}(Q\rho+a+bh_k/2)\,\mathcal{F}_{\widehat{\mathrm{m}}}(Q\rho+a-bh_k/2,q)\\
&=\int da\,\sum_{k=1}^{N} C(Q\rho+a+bh_k/2)^{\frac{1}{2}}\,\mathcal{F}_{\overline{\widehat{m}}}(Q\rho-a-bh_k/2,\overline{q})\\
&\qquad\times H_{k\,;\,\widehat{\mathrm{m}}}(Q\rho+a+bh_k/2)\,\frac{C(Q\rho+a+bh_k/2)^{\frac{1}{2}}}{C(Q\rho+a-bh_k/2)^{\frac{1}{2}}}\\
&\qquad\times C(Q\rho+a-bh_k/2)^{\frac{1}{2}}\,\mathcal{F}_{\widehat{\mathrm{m}}}(Q\rho+a-bh_k/2,q)\,.
\end{aligned} \tag{1.103}$$

If we compare (1.103) with $b=1$ and the vev of the 't Hooft loop operator (1.44) with $B=(1,0^{N-1})$ in $\mathcal{N}=2^*$ $SU(N)$ gauge theory where we substitute (1.40) and (1.45), we find that the first line in (1.103) coincides with Z_{south}, the third line in (1.103) coincides with Z_{north} and the second line in (1.103) coincides with $Z_{\text{1-loop,equator}}$.

1.5.2 Sphere with four punctures

The Verlinde operator along the red curve γ in Figure 7 (b) is defined in the same way as the previous subsubsection. It acts on the conformal block on the sphere with four punctures as follows

$$\Lambda_\gamma:\ \mathcal{F}_{\{\mathrm{m}_4^*,\widehat{\mathrm{m}}_3^*,\widehat{\mathrm{m}}_2,\mathrm{m}_1\}}(\alpha)\to\sum_{l,k=1}^{N} H_{(l,k)\,;\,\{\mathrm{m}_4^*,\widehat{\mathrm{m}}_3^*,\widehat{\mathrm{m}}_2,\mathrm{m}_1\}}(\alpha)\,\mathcal{F}_{\{\mathrm{m}_4^*,\widehat{\mathrm{m}}_3^*,\widehat{\mathrm{m}}_2,\mathrm{m}_1\}}(\alpha-bh_l+bh_k)\,,$$

where for $l\neq k$,

$$\begin{aligned}
H_{(l,k)\,;\,\{\mathrm{m}_4^*,\widehat{\mathrm{m}}_3^*,\widehat{\mathrm{m}}_2,\mathrm{m}_1\}} &= 4\pi^2\frac{\sin\pi bQ}{\sin\pi NbQ}e^{-i\pi b(N-2)\langle\widehat{\mathrm{m}}_2-\widehat{\mathrm{m}}_3,h_1\rangle}\\
&\cdot\prod_{j\neq l}\Gamma(b\langle\alpha-Q\rho,h_{jl}\rangle)\Gamma(bQ+b\langle\alpha-Q\rho,h_{jl}\rangle)\\
&\cdot\prod_{j\neq k}\Gamma(b^2\delta_{jl}+b\langle\alpha-Q\rho,h_{kj}\rangle)\Gamma(bQ+b^2\delta_{lj}+b\langle\alpha-Q\rho,h_{kj}\rangle)
\end{aligned}$$

$$
\begin{aligned}
&\cdot \Big[\prod_{f=1}^{N} \Gamma(b(-\langle \alpha - Q\rho, h_l\rangle + \langle \widehat{\mathrm{m}}_2, h_1\rangle + \langle \mathrm{m}_1 - Q\rho, h_f\rangle)) \\
&\qquad \cdot \Gamma(b(\langle \alpha - Q\rho, h_k\rangle + Q - \langle \widehat{\mathrm{m}}_2, h_1\rangle - \langle \mathrm{m}_1 - Q\rho, h_f\rangle)) \\
&\quad \cdot \prod_{f'=1}^{N} \Gamma(b(-\langle \alpha - Q\rho, h_l\rangle + \langle \widehat{\mathrm{m}}_3, h_1\rangle + \langle \mathrm{m}_4 - Q\rho, h_{f'}\rangle)) \\
&\qquad \cdot \Gamma(b(\langle \alpha - Q\rho, h_k\rangle + Q - \langle \widehat{\mathrm{m}}_3, h_1\rangle - \langle \mathrm{m}_4 - Q\rho, h_{f'}\rangle)) \Big]^{-1},
\end{aligned}
\tag{1.104}
$$

and the remaining part is as follows

$$
\begin{aligned}
&\sum_{l=1}^{N} H_{(l,l)\,;\,\{\mathrm{m}_4^*, \widehat{\mathrm{m}}_3^*, \widehat{\mathrm{m}}_2, \mathrm{m}_1\}} = \frac{\sin(\pi(N-2)bQ)}{\sin \pi N b Q} e^{-2b\pi i[\widehat{\mathrm{m}}_3 - \widehat{\mathrm{m}}_2]} \\
&+ \frac{\sin \pi b Q}{\sin \pi N b Q} e^{-(N-2)b\pi i[\widehat{\mathrm{m}}_2 - \widehat{\mathrm{m}}_3]} \Bigg(2\cos \pi \Big[b\langle \widehat{\mathrm{m}}_2 + \widehat{\mathrm{m}}_3, h_1\rangle \Big] \\
&\qquad + 4 \sum_{l} \prod_{f=1}^{N} \sin \pi [b(\langle Q\rho - \alpha, h_l\rangle + \langle \widehat{\mathrm{m}}_2, h_1\rangle + \langle \mathrm{m}_1 - Q\rho, h_f\rangle)] \\
&\qquad\qquad \cdot \frac{\prod_{f'=1}^{N} \sin \pi [b(\langle Q\rho - \alpha, h_l\rangle + \langle \widehat{\mathrm{m}}_3, h_1\rangle + \langle \mathrm{m}_4 - Q\rho, h_{f'}\rangle)]}{\prod_{j\neq l} \sin \pi (b\langle \alpha - Q\rho, h_{jl}\rangle) \sin \pi (b\langle \alpha, h_{jl}\rangle)} \Bigg).
\end{aligned}
\tag{1.105}
$$

1.6 AGT correspondence for $N_f = 0, 1$

This subsection is based on [12] and relevent for Section 3. In this subsection we define Whittaker states in Liouville theory and they correspond to the instanton partition functions of the theory with $N_f = 0, 1$ on $\mathbb{R}^4$. In Section 3, we will do the same thing in super Liouville theory.

1.6.1 $SU(2)$ $N_f = 0$

This theory is associated to a sphere with two punctures, such that the Seiberg-Witten differential squared ϕ_2 has a pole of degree 3 at each puncture. If the punctures are set at $z = 0, \infty$ we can take

$$
\phi_2 = \frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z}. \tag{1.106}
$$

Here Λ is fixed, and coincides with the scale of the $SU(2)$ theory, while u parameterizes the Coulomb branch. This ϕ_2 is related to λ in Section 1.1.1 as $\phi_2 dz^2 = \lambda^2 = (x^2/z^2)dz^2$.

ϕ_2 has been identified in [5] with the semiclassical limit of the energy momentum tensor $T(z)$. We would like to consider a two dimensional "conformal block" with two special punctures on the sphere, i.e. the inner product

$$\langle \Delta, \Lambda^2 | \Delta, \Lambda^2 \rangle. \tag{1.107}$$

The state $|\Delta, \Lambda^2\rangle$ should live in the Verma module of a highest weight state of conformal dimension $\Delta = \alpha(Q - \alpha) = \frac{Q^2}{4} - a^2$. We hope to identify $\pm a$ with the eigenvalues of the vector multiplet scalar in the instanton partition function. To reproduce the singularity of $\phi_2(z)$ at $z = 0$ we are led to the requirements

$$L_1 |\Delta, \Lambda^2\rangle = \Lambda^2 |\Delta, \Lambda^2\rangle \qquad\qquad L_2 |\Delta, \Lambda^2\rangle = 0. \tag{1.108}$$

Comparing $T(z) = \sum_n L_n z^{-n-2}$ with $\phi_2 = \Lambda^2 z^{-3} + \cdots$, we got the above coefficient on the right-hand side. Notice that the Virasoro commutation relations are sufficient to imply then $L_n |\Delta, \Lambda^2\rangle = 0$ for all $n > 2$.

We aim to define $|\Delta, \Lambda^2\rangle$ as a (possibly formal) power series in Λ^2, i.e,

$$|\Delta, \Lambda^2\rangle = v_0 + \Lambda^2 v_1 + \Lambda^4 v_2 + \cdots \tag{1.109}$$

Here v_0 is the highest weight vector $|\Delta\rangle$ and v_n is a level n descendant such that $L_1 v_n = v_{n-1}$ and $L_2 v_n = 0$. It is not fully clear to us why such vectors should exist. The author of [6] found that these equations can be recursively, and uniquely solved to as high a level n as we cared to check. (level 8)

The inner product

$$\langle \Delta, \Lambda^2 | \Delta, \Lambda^2 \rangle = \sum \Lambda^{4n} |v_n|^2 \tag{1.110}$$

coincides order by order (again, it was only checked up to level 8) with the instanton partition function for $SU(2)$ $N_f = 0$, with the simple identification of instanton factor as $q = \Lambda^4$

For reference, we report here the first few v_n

$$\begin{aligned}
v_0 &= |\Delta\rangle \\
v_1 &= \frac{1}{2\Delta} L_{-1} |\Delta\rangle \\
v_2 &= \frac{1}{4\Delta\,(2c\Delta + c + 16\Delta^2 - 10\Delta)} ((c + 8\Delta) L_{-1}^2 - 12\Delta L_{-2}) |\Delta\rangle \\
v_3 &= \frac{1}{24\Delta(2 + c - 7\Delta + c\Delta + 3\Delta^2)(c + 2c\Delta + 2\Delta(-5 + 8\Delta))} \\
&\quad (12\Delta(-3 - c + 7\Delta) L_{-3} - 12(c + 3c\Delta + \Delta(-7 + 9\Delta)) L_{-2} L_{-1} + \\
&\quad + (c^2 + c(8 + 11\Delta) + 2\Delta(-13 + 12\Delta)) L_{-1}^3) |\Delta\rangle \,,
\end{aligned} \tag{1.111}$$

where $c = 1 + 6Q^2$.

1.6.2 $SU(2)$ $N_f = 1$

This theory is associated to a sphere with two punctures, such that the quadratic differential ϕ_2 has a pole of degree 3 at a puncture (say at $z = 0$) and a pole of degree 4 at the other puncture (say at $z = \infty$). We can make some convenient choices

$$\phi_2 = \frac{\Lambda^2}{2z^3} + \frac{2u}{z^2} - \frac{2\Lambda m}{z} - \Lambda^2. \tag{1.112}$$

Here Λ is fixed, and coincides with the scale of the $SU(2)$ theory. m is also fixed, and coincides with the mass of the single flavor hypermultiplet. u parameterizes the Coulomb branch. The minus signs and the $\frac{1}{2}$ factor are introduced to simplify later expressions.

We will consider the following inner product:

$$\langle \Delta, \Lambda, m | \Delta, \Lambda^2/2 \rangle. \tag{1.113}$$

Both states should live in the Verma module of the highest weight state of conformal dimension $\Delta = \frac{Q^2}{4} - a^2$. $|\Delta, \Lambda^2/2\rangle$ is the same state as in the previous section, with a trivial redefinition of Λ^2 . We require $|\Delta, \Lambda, m\rangle$ to satisfy

$$L_2|\Delta, \Lambda, m\rangle = -\Lambda^2|\Delta, \Lambda, m\rangle \qquad L_1|\Delta, \Lambda, m\rangle = -2m\Lambda|\Delta, \Lambda, m\rangle. \tag{1.114}$$

Notice that the Virasoro commutation relations are again sufficient to imply $L_n|\Delta, \Lambda, m\rangle = 0$ for all $n > 2$.

We aim to define $|\Delta, \Lambda, m\rangle$ as a (possibly formal) power series in Λ, i.e,

$$|\Delta, \Lambda, m\rangle = w_0 + \Lambda w_1 + \Lambda^2 w_2 + \cdots \tag{1.115}$$

Here w_0 is the highest weight vector $|\Delta\rangle$ and w_n is a level n descendant such that $L_1 w_n = -2m w_{n-1}$ and $L_2 w_n = -w_{n-2}$. Again, it is not fully clear to us why such vectors should exist, but their existence and uniqueness were checked for the first few n.

The inner product

$$\langle \Delta, \Lambda, m | \Delta, \Lambda^2/2 \rangle = \sum \Lambda^{3n} 2^{-n} \langle w_n | v_n \rangle \tag{1.116}$$

coincides order by order (again, we only checked the first few levels) with the instanton partition function for $SU(2)$ $N_f = 1$, with $q = \Lambda^3$ and with mass parameter m.

As a reference, we report here the first few w_n

$$\begin{aligned} w_0 &= |\Delta\rangle \\ w_1 &= -\frac{m}{\Delta} L_{-1}|\Delta\rangle \\ w_2 &= \frac{1}{\Delta(c + 2c\Delta + 2\Delta(-5 + 8\Delta))}((cm^2 + \Delta(3 + 8m^2))L_{-1}^2 - 2\Delta(1 + 2\Delta + 6m^2)L_{-2})|\Delta\rangle. \end{aligned} \tag{1.117}$$

2 Loop operators on $S^1 \times \mathbb{R}^3$

This section is based on our paper [1].

One of the exactly calculable quantities in supersymmetric gauge theories on S^1 is an index. It counts the number of the BPS states weighted by their quantum numbers such as angular momenta, so it is a useful quantity for obtaining information about the BPS states. We calculated the index for 4d $\mathcal{N} = 2$ superconformal gauge theories in the presence of infinitely massive particles with electric or magnetic charges, which are Wilson loop or 't Hooft loop operators. In doing so, we reinterpreted the index as the partition functions where all fields have twisted periodicity along S^1 related to the above quantum numbers and performed localization calculation to obtain the exact results.

In case with the 't Hooft loop operators, there is a nontrivial dynamics. Dynamical monopoles in the theories arise and screen the Dirac monopoles. In Section 2.4 we obtain this contribution to the index by using the correspondence between these dynamical monopoles and instantons in Section 2.3.3.

Our result is one of 4d-2d correspondences. We found that the vevs of the loop operators are related to Verlinde operators in Toda theory reviewed in Section 1.5. We check it in Section 2.7. This correspondence raises an interesting question about how we can explain it in terms of M5-branes and we will obtain a hint of interesting properties of M5-branes during answering the question.

We confirmed our result by considering the relation between the vevs of the 't Hooft loop operators with minimal charge and with higher charges. The moduli space of the multiple monopoles has singularity and its resolution requires separating them into minimal-charge monopoles. It suggests that the vevs of the higher-charge 't Hooft loop operators should be related to a product of the vevs of the minimal ones. Indeed we observed that the vevs of the higher-charge 't Hooft loop operators are realized by so-called Moyal product of the vevs of the minimal ones. This observation is written in Section 2.5. It is good that we understood how to treat the singularities of the moduli spaces. We also explain how the noncommutativity arises in Section 2.6.

2.1 $\mathcal{N} = 2$ gauge theories on $S^1 \times \mathbb{R}^3$ and loop operators

In this thesis we study four-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry on $S^1 \times \mathbb{R}^3$ in the Coulomb branch. For notational convenience, we will use the notation appropriate for $\mathcal{N} = 2^*$ theory, which can be thought of as a dimensional reduction of the ten-dimensional super Yang-Mills, though we will state general results applicable to other field contents [9, 10]. The ten-dimensional gauge field A_M $(M = 1, \ldots, 9, 0)$ gives rise to the four-dimensional gauge field A_μ $(\mu = 1, \ldots, 4)$, hypermultiplet scalars $A_i \equiv \Phi_i$ $(i = 5, \ldots, 8)$, and vector multiplet scalars $A_A \equiv \Phi_A$ $(A = 0, 9)$. The ten-dimensional chiral spinor Ψ also decomposes into the gaugino $\psi \equiv \frac{1-\Gamma_{5678}}{2}\Psi$ and hypermultiplet fermion $\chi \equiv \frac{1+\Gamma_{5678}}{2}\Psi$. Our spinor and gamma matrix conventions are summarized in Appendix A.1. Real fields are hermitian matrices, and the gauge covariant derivative is $D_\mu = \partial_\mu + iA_\mu$. In terms of the coordinates $x^\mu = (x^i, \tau)$ ($\mu = 1, \ldots, 4$, $i = 1, 2, 3$), the metric is simply $ds^2 = d\tau^2 + dx^i dx^i$. We denote the radius of the Euclidean time circle by R.

The theory is defined by the physical action

$$S = S_{\rm vec} + S_{\rm hyp}\,, \tag{2.1}$$

where the two terms describing the vector multiplets and hypermultiplets are given by

$$\begin{aligned} S_{\rm vec} &= \frac{1}{g^2}\int_{S^1\times\mathbb{R}^3} d^4x\,\mathrm{Tr}\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\Phi_A D^\mu\Phi_A - [\Phi_0,\Phi_9]^2 - \psi\Gamma^\mu D_\mu\psi - i\psi\Gamma^A[\Phi_A,\psi]\right) \\ &\quad + \frac{i\vartheta}{8\pi^2}\int_{S^1\times\mathbb{R}^3}\mathrm{Tr}\,(F\wedge F)\,, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} S_{\rm hyp} &= \frac{1}{g^2}\int_{S^1\times\mathbb{R}^3} d^4x\,\mathrm{Tr}\Big(D_\mu\Phi_i D^\mu\Phi_i - \frac{1}{2}[\Phi_i,\Phi_j]^2 - ([\Phi_A,\Phi_i] - iM_{A\,ij}\Phi_j)^2 - \chi\Gamma^\mu D_\mu\chi \\ &\qquad - i\chi\Gamma^A\left([\Phi_A,\chi] - \frac{i}{4}M_{A\,ij}\Gamma^{ij}\chi\right) - i\chi\Gamma^i[\Phi_i,\chi]\Big)\,. \end{aligned} \tag{2.3}$$

Here Tr denotes an invariant metric on the Lie algebra of the gauge group G, ϑ is the theta angle, and $i, j = 5, 6, 7, 8$ denote the hypermultiplet scalar directions. The two real anti-symmetric matrices $M_{ij} \equiv M_{0\,ij}$ and $M_{9\,ij}$ are proportional to a single pure-imaginary anti-symmetric matrix F_{ij},[2] which is normalized as $F_{ij}F_{ji} = 4$ and is taken to be anti-self-dual in the 5678 directions so that only the hypermultiplet fermions get massive. The flavor generator F is represented as F_{ij} on the scalars and as $\frac{1}{4}F_{ij}\Gamma^{ij}$ on spinors. The real mass parameters $M \equiv M_0$ and M_9 are defined by $M_{A\,ij} = iM_A F_{ij}$ ($A = 0, 9$). The massless limit is $\mathcal{N} = 4$ super Yang-Mills. General $\mathcal{N} = 2$ theories have several mass parameters M_{Af} with $A = 0, 9$ and $f = 1, \ldots, N_f$. These can be thought of as the vevs of the scalars in the vector multiplets that weakly gauge the flavor symmetries. Only $M_f \equiv M_{A=0,f}$, which are the analog of Φ_0, will enter the loop operator vevs.

Our aim is to compute the expectation value of half-BPS loop operators along S^1, placed at a point on the 3-axis of $\mathbb{R}^3$. The most basic loop operator is the Wilson loop operator defined as

$$W_R = \mathrm{Tr}_R P\exp\oint_{S^1}\left(-iA_\tau + \Phi_0\right)d\tau\,. \tag{2.4}$$

[2] The flavor symmetry generator F_{ij} ($i, j = 5, \ldots, 8$) should not be confused with the field strength $F_{MN} = -i[D_M, D_N]$ ($M, N = 1, \ldots, 9, 0$).

This is labeled by the representation R of the gauge group, or equivalently its highest weight. The supersymmetric 't Hooft loop operator with charge B is defined by integrating over the fluctuations of the fields around the configuration

$$\begin{aligned} A \equiv A_\mu dx^\mu &= \left(ig^2\vartheta \frac{B}{16\pi^2}\frac{1}{r} + A_\tau^{(\infty)} \right) d\tau + \frac{B}{2}\cos\theta d\varphi \\ \Phi_0 &= -g^2\vartheta\frac{B}{16\pi^2}\frac{1}{r} + \Phi_0^{(\infty)} , \qquad \Phi_9 = \frac{B}{2r} + \Phi_9^{(\infty)} \end{aligned} \quad \text{in the background.} \tag{2.5}$$

We recall that $\tau \equiv x^4$ and that ϑ is the gauge theory theta angle. We have also introduced polar coordinates $(r \equiv |\vec{x}|, \theta, \varphi)$ for $\mathbb{R}^3$. Our choice of scalars in (2.4) and (2.5) ensures that the Wilson and 't Hooft loop operators preserve the same sets of supercharges. The action of the $U(1)$ R-symmetry rotates $\Phi_0 + i\Phi_9$ and changes the set of preserved supercharges. Note that we define the electric Wilson line $A_\tau^{(\infty)}$ in the local trivialization such that the $d\varphi$ term is given by $(B/2)\cos\theta d\varphi$ rather than the more familiar $-(B/2)(\pm 1 - \cos\theta)d\varphi$. Our choice guarantees that when $\lambda \neq 0$, the holonomy at the spatial infinity with $\theta = \pi/2$ is $\exp(-2\pi i R A_\tau^{(\infty)})$. This will play a role in Section 2.6.

More general loop operators are dyonic and carry both electric and magnetic charges. Such operators are defined by a path integral for a 't Hooft loop operator with charge B, with the insertion of a Wilson loop operator for the stabilizer of B in G. The dyonic charges are elements of the sum of coweight and weight lattices of G

$$\Lambda_{cw} \oplus \Lambda_w \,, \tag{2.6}$$

and the charges related by a simultaneous action of the Weyl group to the two lattices are equivalent [22]. Due to Dirac quantization, the magnetic charge must be a coweight which has integer inner products with all the weights in the matter representation.[3]

Having defined the loop operators whose vevs we wish to compute, let us explain the parameters of the theory those vevs will depend on. We are studying the theory in the Coulomb branch, so the real scalars in the vector multiplet have the expectation values

$$\langle \Phi_A \rangle =: \Phi_A^{(\infty)} \in \mathfrak{t} \qquad A = 0\,, 9\,, \tag{2.7}$$

which are the asymptotic values at $|\vec{x}| = \infty$. Since we compactify the theory on S^1, we also have the electric and magnetic Wilson lines. The electric Wilson line is the asymptotic value of the τ-component of the gauge field

$$A_\tau^{(\infty)} \in \mathfrak{t}\,. \tag{2.8}$$

[3] In the theories whose gauge group is a product of $SU(2)$'s, the electric and magnetic charges with these constraints and equivalence relations match the homotopy classes of non-self-intersecting curves on the corresponding Riemann surface [23].

Due to potential terms in the action (2.2), $\Phi_A^{(\infty)}$ and $A_\tau^{(\infty)}$ can be simultaneously diagonalized, i.e., they can take values in the Cartan subalgebra $\mathfrak{t}$.

We also need to consider the magnetic Wilson line. In the IR theory this is the vev of the scalar dual to the gauge field in three dimensions. In the UV theory we define it as follows. At a generic point of the Coulomb branch, the scalar vevs $\Phi_A^{(\infty)}$ classically breaks the gauge group G to the maximal torus T. The path integral includes infinitely many sectors classified by the magnetic charges at infinity. The general boundary condition is such that asymptotically as $|\vec{x}| \to \infty$, we allow $\Phi_A(\vec{x})$ to take any values that are gauge equivalent to $\Phi_A^{(\infty)}$, i.e., there is a map $g : S^2 \to G$ such that

$$\Phi_A(\vec{x}) \to g(\vec{n}) \cdot \Phi_A^{(\infty)} \cdot g^{-1}(\vec{n}) \quad \text{as} \quad |\vec{x}| \to \infty \tag{2.9}$$

with $\vec{n} \equiv \vec{x}/|\vec{x}| \in S^2$. Then the scalars $\Phi_A(\vec{x})|_{|\vec{x}|=\infty}$ themselves define a map from S^2 to the orbit $\{g(\Phi_0^{(\infty)}, \Phi_9^{(\infty)})g^{-1}|g \in G\}$, which is diffeomorphic to G/T because the stabilizer of a generic element of $\mathfrak{t} \times \mathfrak{t}$ is T. We can demand that $g = 1$ at the north pole of S^2, so that Φ_A at $|\vec{x}| = \infty$ define a homotopy class in $\pi_2(G/T)$ with a base point at the north pole. If G is simply connected, the maximal torus can be identified with the quotient of the Cartan subalgebra by the coroot lattice[4] $T \simeq \mathfrak{t}/\Lambda_{cr}$, so $\pi_2(G/T) \simeq \pi_1(T) = \Lambda_{cr}$. In fact G/T depends only on the Lie algebra of G, so $\pi_2(G/T) = \Lambda_{cr}$ for any G. The infinitely many topological sectors are therefore classified by Λ_{cr}. Physically this makes sense because Λ_{cr} is the lattice of magnetic charges carried by Polyakov-'t Hooft monopoles. This lattice is more coarse than the coweight lattice Λ_{cw} in which the magnetic charge B of the 't Hooft loop operator takes values, $\Lambda_{cr} \subset \Lambda_{cw}$. With generic matter representations, the lattice of 't Hooft charges B allowed by Dirac quantization would be smaller than Λ_{cw}.

Let us now insert a 't Hooft loop operator with magnetic charge $B \in \Lambda_{cw}$ at the origin. The insertion of the 't Hooft loop operator changes the topology of the vector bundles in which the fields take values, and in particular the structure of the boundary conditions at spatial infinity. One can classify the allowed configurations by the asymptotic magnetic charges taking values in the shifted lattice $\Lambda_{cr} + B \subset \Lambda_{cw}$. We define the magnetic Wilson line $\Theta \in \mathfrak{t}^*$ as the chemical potential for the magnetic charges. The expectation value of the 't Hooft loop operator is given by the sum

$$\langle T_B \rangle = \sum_{v \in \Lambda_{cr}+B} e^{iv\cdot\Theta} \int_v \mathcal{D}A\mathcal{D}\Psi \, e^{-S} \,, \tag{2.10}$$

where the path integral in each summand is performed with the boundary condition specified by v. In the three-dimensional Abelian gauge theory that arises via dimensional reduction, Θ is identified with the expectation values of scalars dual to the photons [25], and the UV and IR definitions of Θ are consistent.

[4]See [24] for a review of lattices in the Cartan subalgebra $\mathfrak{t}$ and its dual $\mathfrak{t}^*$.

Along the circle S^1 we can impose various twisted boundary conditions on the fields. It is convenient to exhibit them by representing the loop operator vev as a supersymmetric index, taking S^1 as a time direction. The loop operator L modifies the Hilbert space of the theory, rather than acts on the original Hilbert space as a linear transformation. We define our observable, the expectation value of the loop operator L, to be a trace in the modified Hilbert space $\mathcal{H}_L$

$$\langle L \rangle = \mathrm{Tr}_{\mathcal{H}_L}(-1)^F e^{-2\pi RH} e^{2\pi i\lambda(J_3+I_3)} e^{2\pi i\mu_f F_f}\,, \tag{2.11}$$

where J_3 and I_3 are the generators of the Lorentz $SU(2)$ and the R-symmetry $SU(2)$. Here J_3 generates a rotation along the 3-axis: $iJ_3 = x^1\partial_2 - x^2\partial_1$ when acting on a scalar. As we will see below, the combination $J_3 + I_3$ commutes with the supercharge we use for localization. We have also included the twist by the flavor symmetries with generators F_f and chemical potentials μ_f, $f = 1, \ldots, N_f$. The definition (2.11) of the loop operator vev coincides with the one used in [26]. The system may be realized in terms of a path integral over the fields with appropriate twisted boundary conditions along S^1. In this thesis we adopt the equivalent formulation where everywhere in the action (2.1) on $\mathbb{R}^4$ the time derivative is shifted as

$$\partial_\tau \to \partial_\tau - \frac{i}{R}\lambda(J_3+I_3) - \frac{i}{R}\sum_{f=1}^{N_f}\mu_f F_f \tag{2.12}$$

and the fields are periodic in τ. The electric and magnetic Wilson lines can also be regarded as the chemical potentials for the corresponding charges.

As we will see all the parameters except λ will enter the loop operator vevs in specific complex combinations. These are the moduli

$$\mathbf{a} := R\,(A_\tau^{(\infty)} + i\Phi_0^{(\infty)}) \in \mathfrak{t}_{\mathbb{C}}\,, \quad \mathbf{b} := \frac{\Theta}{2\pi} - \frac{4\pi i R}{g^2}\Phi_9^{(\infty)} + \frac{\vartheta}{2\pi}\mathbf{a} \in \mathfrak{t}^*_{\mathbb{C}}\,. \tag{2.13}$$

and the complexified mass parameters

$$m_f \equiv -\mu_f + iRM_f \in \mathbb{C} \qquad f = 1, \ldots, N_f\,. \tag{2.14}$$

We use the Lie algebra metric Tr in the action to regard $\Phi_9^{(\infty)}$ and $\mathbf{a}$ as elements of $\mathfrak{t}^*_{\mathbb{C}}$.

2.2 Localization for gauge theories on $S^1 \times \mathbb{R}^3$

We apply the localization technique introduced for calculations in gauge theory on S^4 [9]. In this formalism, one adds a new term $tQ \cdot V$ to the action, so that the path integral takes the form

$$\int \mathcal{D}A\mathcal{D}\Psi\, e^{-S-tQ\cdot V}\,. \tag{2.15}$$

Here A and Ψ include all the bosons and fermions, respectively. We will also need to add ghost fields after gauge-fixing. For observables that are invariant under the supercharge Q of choice, the path integral is independent of the parameter t. The localization action is chosen to be $V = (\Psi, \overline{Q \cdot \Psi}) = (\psi, \overline{Q \cdot \psi}) + (\chi, \overline{Q \cdot \chi})$, where ψ and χ denote the fermions in the vector multiplet and the hypermultiplet respectively. Since the bosonic part of $Q \cdot V$ is a positive definite term $||Q \cdot \Psi||^2$, the path integral is dominated by the solutions of $Q \cdot \Psi = 0$ in the limit $t \to +\infty$ and can be calculated exactly by summing the fluctuation determinants at all the saddle points.

2.2.1 Symmetries

For localization we need to close off-shell the relevant subalgebra of the whole superalgebra. For this we introduce seven auxiliary fields K_j as in [9]. The supersymmetry transformations in $\mathcal{N} = 2^*$ theory are given by

$$
\begin{aligned}
Q \cdot A_M &= \epsilon \Gamma_M \Psi \,, &(2.16)\\
Q \cdot \Psi &= \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon + i K^i \nu_i \,, &(2.17)\\
Q \cdot K_j &= i \nu_j \Gamma^M D_M \Psi \,. &(2.18)
\end{aligned}
$$

The gamma matrices and the constant spinors ν_i ($i = 1, \ldots, 7$) are defined in Appendix A.1. The gauge fields in F_{MN} and D_M include mass matrices $M_{Aij} = i M_A F_{ij}$ through the Scherk-Schwarz mechanism [9]. The spinor ϵ must be chosen so that the loop operators are invariant under the supersymmetry transformation Q. We will use the same spinor as used in [10]

$$
\epsilon = \frac{1}{\sqrt{2}} (1, 0^7, 1, 0^7) \,, \tag{2.19}
$$

where the power indicates the number of repeated entries. It satisfies[5]

$$
\Gamma_{5678} \epsilon = -\epsilon \,, \quad \Gamma_{04} \epsilon = -i \epsilon \,, \quad \Gamma_{1239} \epsilon = \epsilon \,, \quad (2\Gamma_{12} + \Gamma_{56} + \Gamma_{78}) \epsilon = 0 \,. \tag{2.20}
$$

The last condition implies that the supercharge commutes with the combination $J_3 + I_3$ of spatial and R-symmetry rotations. This explains why this particular combination entered the definition (2.11) of the vev.

We will need later the square of the supersymmetry transformation given by the spinor ϵ in (2.19). Using the vector

$$
v^M \equiv \epsilon \Gamma^M \epsilon = (i, 0^3, 1, 0^5) \qquad M = 0, 1, \ldots, 9 \,, \tag{2.21}
$$

[5]The third condition implies that Q corresponds to the fermionic symmetry for the Donaldson-Witten twist [27] in the 1239-directions. Thus $\langle L \rangle$ is a limit of the five-dimensional Nekrasov partition function [28] for a theory on $S^1 \times \mathbb{R}^4$ with a loop operator insertion, where one of the equivariant parameter for the rotation in the 39 plane is set to zero and a direction in $\mathbb{R}^4$ is compactified on an infinitely small circle.

we find that Q^2 generates time translation, minus the complexified gauge transformation G_Λ with gauge parameter $\Lambda = A_\tau + i\Phi_0$, and the flavor symmetry transformation iMF:

$$\begin{aligned} Q^2 \cdot A_M &= -F_{\tau M} - [i\Phi_0, D_M] - i\delta^i_M M_{ij}\Phi_j \,, \\ Q^2 \cdot \Psi &= -\partial_\tau \Psi - i[A_\tau + i\Phi_0, \Psi] - \frac{i}{4} M_{ij}\Gamma^{ij}\Psi \,, \\ Q^2 \cdot K_i &= -\partial_\tau K^i - i[A_\tau + i\Phi_0, K_i] \,. \end{aligned} \tag{2.22}$$

See Appendix C and (2.27) of [9].

2.2.2 Localization equations

Let us study the localization equations $Q \cdot \Psi = 0$, whose solutions the path integral localizes to. We decompose Ψ as

$$\Psi = \sum_{M=1}^{9} \Psi_M \widetilde{\Gamma}^M \overline{\epsilon} + i \sum_{j=1}^{7} \Upsilon_j \nu^j \,. \tag{2.23}$$

Noting that

$$\Psi_M = \epsilon \Gamma_M \Psi \,, \qquad i\Upsilon_j = \overline{\nu}_j \Psi \,. \tag{2.24}$$

we obtain

$$0 \;=\; Q \cdot \Psi_M = \frac{1}{2} F_{PQ}\, \epsilon\Gamma_M \Gamma^{PQ}\epsilon \qquad M = 1, \ldots, 9 \,, \tag{2.25}$$

$$0 \;=\; iQ \cdot \Upsilon_j = \frac{1}{2} F_{MN}\, \overline{\nu}_j \Gamma^{MN}\epsilon + iK_j \qquad j = 1, \ldots, 7 \,. \tag{2.26}$$

The equations (2.25) reduce to[6]

$$0 = Q \cdot \Psi_M = -v^N F_{NM} \,. \tag{2.27}$$

According to (2.22), these are equivalent to Q^2-invariance, i.e., invariance under a combination of τ-translation, gauge transformations, and flavor transformations. Due to the replacement of the τ-derivative in (2.12), for generic λ the bosonic fields must also be invariant under the combination $J_3 + I_3$ of spatial and R-symmetry rotations. Among the various components of (2.26), the most important equations are[7]

$$0 = iQ \cdot \Upsilon_j = D_j \Phi_9 - \frac{1}{2} \sum_{k,l=1}^{3} \epsilon_{jkl} F_{kl} + iK_j \qquad j, k, l = 1, 2, 3 \,. \tag{2.28}$$

[6] To show this we used the identities $\Gamma_M \widetilde{\Gamma}_{[P}\Gamma_{Q]} = \Gamma_{[M}\widetilde{\Gamma}_P \Gamma_{Q]} + 2\delta_{M[P}\Gamma_{Q]}$ and $\epsilon\Gamma_{[M}\widetilde{\Gamma}_P\Gamma_{Q]}\epsilon = 0$.

[7] We used the following facts: $\overline{\nu}_j \Gamma^{kl}\epsilon = -\epsilon_{jkl}$ for $j,k,l \in \{1,2,3\}$, $\overline{\nu}_j\Gamma^{kl}\epsilon = 0$ for $j,k \in \{1,2,3\}$ and $l \in \{5,6,7,8\}$, $\overline{\nu}_j \Gamma^{k9}\epsilon = \delta_{jk}$ for $j,k \in \{1,2,3\}$, and $\overline{\nu}_j\Gamma^{9l}\epsilon = 0$ for $j \in \{1,2,3\}$ and $l \in \{5,6,7,8\}$. We also went ahead and set the hypermultiplets to zero. This is justified below by Q^2-invariance.

The imaginary part sets K_j to zero. The real part is precisely the Bogomolny equations

$$*_3 F = D\Phi_9 \tag{2.29}$$

that describe monopoles on $\mathbb{R}^3$! Thus we conclude that the path integral localizes to the fixed points on the monopole moduli space with respect to spatial rotations and gauge transformations.

Four other components of (2.26) read

$$0 = iQ \cdot \Upsilon_j = \sum_{k=1}^{3}\sum_{l=5}^{8}(\overline{\nu}_j \Gamma^{kl}\epsilon) D_k \Phi_l + \sum_{l=5}^{8}(\overline{\nu}_j \Gamma^{9l}\epsilon) i[\Phi_9, \Phi_l] + iK_j \qquad j = 4,5,6,7\,. \tag{2.30}$$

Again the imaginary part requires K_j to vanish. The real part of (2.30) is in fact the "realification" of the Dirac-Higgs equation

$$\sum_{i=1}^{3} \sigma^i D_i q + [\Phi_9, q] = 0\,, \tag{2.31}$$

where the two-component "spinor" q is a linear combination of Φ_i with $i = 5,6,7,8$. See Appendix A.2 for a related discussion. As in topological twist, the hypermultiplet scalars behave as a spinor under the combination J_3+I_3. Though generically (2.31) itself admits non-zero solutions, the Q^2-invariance, in particular the invariance under flavor transformations, requires q to vanish.

Thus localization on $S^1 \times \mathbb{R}^3$ leaves no bosonic zero-mode to be integrated over, and the final answer for the vev will be expressed as a finite sum. This is in contrast with the results for S^4 [9, 10] where the path integral reduced to a finite dimensional matrix integral.

2.2.3 On-shell action

Let us work out the classical contribution $e^{-S_{\rm cl}}$, given by the on-shell action evaluated in the background (2.5). The on-shell action for the hypermultiplet simply vanishes, therefore we focus on the action (2.2) for the vector multiplet. For the background (2.5), we also have

$$F = ig^2\vartheta \frac{B}{16\pi^2}\frac{d\tau\wedge dr}{r^2} - \frac{B}{2}\sin\theta d\theta\wedge d\varphi\,, \tag{2.32}$$

$$*F = -\frac{B}{2r^2}d\tau\wedge dr + ig^2\vartheta\frac{B}{16\pi^2}\sin\theta d\theta\wedge d\varphi\,. \tag{2.33}$$

Our orientation is such that the volume form is $d\tau\wedge dx^1\wedge dx^2\wedge dx^3$. The action (2.2) is divergent in the presence of such a singular dyonic background. We can render the action finite by cutting off the spacetime at $\Sigma_3 \equiv \{r=\delta\}$ and by adding the boundary term [29, 10]

$$S_{\rm bdry} = \frac{2}{g^2}\int_{\Sigma_3} {\rm Tr}\,(\Phi_9 F - i\Phi_0 * F)\wedge d\tau\,. \tag{2.34}$$

We find that

$$
\begin{aligned}
S_{\rm vec} =& \frac{1}{g^2\delta}\left(4\pi^2 R + \frac{g^2\vartheta^2 R}{16\pi^2}\right)\mathrm{Tr}B^2 - i\vartheta R\,\mathrm{Tr}\left(A_\tau^{(\infty)}B\right)\,,\\
S_{\rm bdry} =& -\frac{1}{g^2\delta}\left(4\pi^2 R + \frac{g^2\vartheta^2 R}{16\pi^2}\right)\mathrm{Tr}B^2 - \frac{8\pi^2 R}{g^2}\mathrm{Tr}\left(\Phi_9^{(\infty)}B\right) + \vartheta R\mathrm{Tr}\left(\Phi_0^{(\infty)}B\right)\,.
\end{aligned}
\tag{2.35}
$$

Thus the classical on-shell action is given by

$$
S_{\rm cl}(B) \equiv S_{\rm vec} + S_{\rm bdry} \;=\; -\frac{8\pi^2 R}{g^2}\mathrm{Tr}\left[\Phi_9^{(\infty)}B\right] - i\vartheta R\;\mathrm{Tr}\left[\left(A_\tau^{(\infty)} + i\Phi_0^{(\infty)}\right)B\right]\,. \tag{2.36}
$$

The on-shell action nicely combines with the weight $e^{iB\cdot\Theta}$ for the magnetic charge in (2.10) so that

$$
\langle T_B\rangle \sim e^{iB\cdot\Theta}e^{-S_{\rm cl}(B)} = e^{2\pi i B\cdot\mathbf{b}}\,, \tag{2.37}
$$

where $\mathbf{b}$ was define in (2.13). This is the leading classical approximation to the 't Hooft operator vev. We will compute one-loop and non-perturbative corrections in the following sections.

2.3 One-loop determinants

Having computed the classical contribution to the 't Hooft operator vev, in this section we will compute the one-loop correction following [9] and in parallel with [10]. As we saw in the previous section, the path integral reduces to a sum over saddle points. For each saddle point we need to compute the fluctuation determinants. The methods here will also be used in Section 2.4 for the computation of such non-perturbative corrections.

2.3.1 Gauge fixing

The gauge fixing action in the R_ξ-gauge is

$$
S_{\rm gf} = \int d^4x\,\mathrm{Tr}\left(-i\,\widetilde{c}\sum_{M=1,2,3,9} D_{(0)}^M D_M c + \widetilde{b}\left(i\sum_{M=1,2,3,9} D_{(0)}^M \widetilde{A}_M + \frac{\xi}{2}\widetilde{b}\right)\right)\,. \tag{2.38}
$$

We have defined $\widetilde{A}_M \equiv A_M - A_{(0)M}$ where $A_{(0)M}$ is the background configuration given in (2.5). The ghost fields $c, \widetilde{c}$ are fermionic, and $\widetilde{b}$ is bosonic. By defining the BRST transformations[8]

$$
\begin{aligned}
&Q_{\rm B}\cdot A_M = -[c, D_M]\,,\quad Q_{\rm B}\cdot\Psi = -i[c,\Psi]\,,\quad Q_{\rm B}\cdot K_i = -i[c,K_i]\,,\\
&Q_{\rm B}\cdot c = -\frac{i}{2}[c,c]\,,\quad Q_{\rm B}\cdot\widetilde{c} = \widetilde{b}\,,\quad Q_{\rm B}\cdot\widetilde{b} = 0\,,
\end{aligned}
\tag{2.39}
$$

[8]To compare with Pestun's formalism in [9], set $\widetilde{a}_0, b_0, c_0, \widetilde{c}_0$ to zero. Then separate his BRST transformation δ into our $Q_{\rm B}$ and the part δ_0 proportional to a_0: $\delta = Q_{\rm B} + \delta_0$. Then our Q can be written as $s + \delta_0$ with $a_0 = -\Phi_{(0)0}$, where s denotes the supersymmetry transformation in [9].

we can write

$$S_{\rm gf} = Q_{\rm B}\cdot V_{\rm gh}\,, \quad V_{\rm gh} \equiv \int d^4x {\rm Tr}\left(\widetilde{c}\left(i\sum_{M=1,2,3,9} D^M_{(0)}\widetilde{A}_M + \frac{\xi}{2}\widetilde{b}\right)\right)\,. \tag{2.40}$$

The BRST transformation squares to zero, $\{Q_{\rm B}, Q_{\rm B}\} = 0$. Unlike the case of S^4 [9] where the spacetime is compact, we do not need to introduce ghosts-for-ghosts to deal with constant gauge transformations.

We define the action of the supercharge Q on the ghosts by

$$\begin{aligned} &Q\cdot c = -v^M\widetilde{A}_M \equiv -\widetilde{\Phi} = -i\widetilde{\Phi}_0 - \widetilde{A}_\tau\,, \quad Q\cdot\widetilde{c} = 0\,,\\ &Q\cdot\widetilde{b} = -v^M D_M\widetilde{c} = -\partial_\tau\widetilde{c} - i[A_\tau + i\Phi_0, \widetilde{c}]\,. \end{aligned} \tag{2.41}$$

In the background Q annihilates all the fermions, therefore the background is supersymmetric. We have $\{Q, Q\}(\text{ghost}) = 0$.

2.3.2 One-loop determinants and the index theorem

After gauge fixing, the total fermionic symmetry we use for localization is

$$\widehat{Q} \equiv Q + Q_{\rm B}\,. \tag{2.42}$$

While Q^2 in (2.22) involves a gauge transformation G_Λ with a dynamical gauge parameter $\Lambda = A_\tau + i\Phi_0$, the gauge transformation that appears in $\widehat{Q}^2 = Q^2 + \{Q, Q_{\rm B}\}$ turns out to have a fixed parameter $\Lambda = A_{(0)\tau} + i\Phi_{(0)0} = A^{(\infty)}_\tau + i\Phi^{(\infty)}_0$:[9]

$$\widehat{Q}^2 = -\partial_\tau - i(A^{(\infty)}_\tau + i\Phi^{(\infty)}_0) + MF\,. \tag{2.43}$$

Saddle points of the path integral remain the same after we replace $Q\cdot V$ by $\widehat{Q}\cdot\widehat{V}$. Recall that $M \equiv M_0$ is one of the mass parameters defined below (2.3) and that F is the flavor symmetry generator. The path integral to consider is

$$\int \mathcal{D}A\mathcal{D}\Psi\mathcal{D}K\mathcal{D}\widetilde{b}\mathcal{D}c\mathcal{D}\widetilde{c}\, e^{-S-t\widehat{Q}\cdot\widehat{V}}\,, \tag{2.44}$$

where

$$\widehat{V} = \left\langle \Psi\,, \overline{\widehat{Q}\cdot\Psi}\right\rangle + V_{\rm gh}\,. \tag{2.45}$$

In order to evaluate the path integral in the limit $t\to\infty$, we need to compute the superdeterminant of the kinetic operator in $\widehat{Q}_{(0)}\cdot\widehat{V}^{(2)}$, where $\widehat{Q}_{(0)}$ is the linearization of $\widehat{Q}$, and $\widehat{V}^{(2)}$ is the quadratic part of $\widehat{V}$. Following [9] let us define

$$X_0 = (\widetilde{A}_M)^9_{M=1}\,, \quad X_1 = (\Upsilon_i, c, \widetilde{c}) \tag{2.46}$$

[9] For the gauge field $\widehat{Q}^2\cdot A_M = -\partial_\tau\widetilde{A}_M - i[A^{(\infty)}_\tau + i\Phi^{(\infty)}_0, \widetilde{A}_M]$.

and their partners

$$\begin{aligned} X_0' \equiv & \widehat{Q}_{(0)} \cdot X_0 = (\Psi_M - [c, D_{(0)M}])_{M=1}^9 \,, \\ X_1' \equiv & \widehat{Q}_{(0)} \cdot X_1 = \Big(K_i - i(\overline{\nu}_i \Gamma^{MN} \epsilon) D_{(0)M} \widetilde{A}_N \,, -\widetilde{\Phi}, b \Big) \,. \end{aligned} \tag{2.47}$$

Now $\widehat{V}^{(2)}$ takes the form

$$V^{(2)} = \left\langle \begin{pmatrix} X_0' & X_1 \end{pmatrix} , \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} X_0 \\ X_1' \end{pmatrix} \right\rangle \,, \tag{2.48}$$

where D_{00} and others are certain differential operators. Then $\widehat{Q}_{(0)} \cdot V^{(2)}$ is given by

$$\begin{aligned} \widehat{Q}_{(0)} \cdot V^{(2)} = & \left\langle \begin{pmatrix} X_0, X_1' \end{pmatrix} \begin{pmatrix} -\mathcal{R}_{00} & \\ & 1 \end{pmatrix} , \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} X_0 \\ X_1' \end{pmatrix} \right\rangle \\ & + \left\langle \begin{pmatrix} X_0', X_1 \end{pmatrix} , \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} -1 & \\ & -\mathcal{R}_{11} \end{pmatrix} \begin{pmatrix} X_0' \\ X_1 \end{pmatrix} \right\rangle \,, \end{aligned} \tag{2.49}$$

where $\widehat{Q}_{(0)}^2 \cdot X_0 = \mathcal{R}_{00} \cdot X_0$ and $\widehat{Q}_{(0)}^2 \cdot X_1 = \mathcal{R}_{11} \cdot X_1$. Thus the one-loop determinant is given by

$$\begin{aligned} Z_{\text{1-loop}} \quad = & \quad \frac{\det^{1/2} \left[\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} -1 & \\ & -\mathcal{R}_{11} \end{pmatrix} \right]}{\det^{1/2} \left[\begin{pmatrix} -\mathcal{R}_{00} & \\ & 1 \end{pmatrix} \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \right]} = \frac{\det^{1/2} \mathcal{R}_{11}}{\det^{1/2} \mathcal{R}_{00}} \\ = & \quad \frac{\det^{1/2}_{\mathrm{Coker} D_{10}} \mathcal{R}}{\det^{1/2}_{\mathrm{Ker} D_{10}} \mathcal{R}} \,. \end{aligned} \tag{2.50}$$

In the final line we have introduced notation $\mathcal{R} = \widehat{Q}_{(0)}^2$ and used the fact that $\mathcal{R}$ commutes with D_{10} as guaranteed by $\mathcal{R}$-invariance of $\widehat{V}$. Thus we only need the differential operator D_{10}, which can be obtained by explicitly computing $\widehat{V}^{(2)}$. It is easy to see what to expect from the results in Section 2.2.2. There we saw that the localization equations are given by the Bogomolny and Dirac-Higgs equations. In Appendix A.2, we will show that D_{10} involves the linearization of these equations as well as the dual of the gauge transformation.

The symmetry generator $\mathcal{R} = \widehat{Q}_{(0)}^2$ is given in (2.43). In a general $\mathcal{N} = 2$ theory, we replace that last term MF by $\sum_f M_f F_f$, where F_f are the flavor symmetry generators in (2.11). We also perform the shift (2.12) of the τ derivative. It is also useful to rescale $\mathcal{R}$ as $\mathcal{R} \to -R\mathcal{R}$. This does not affect the value of the one-loop determinant (2.50) due to cancellations between the numerator and the denominator. Then $\mathcal{R}$ takes a simple expression

$$\mathcal{R} = \varepsilon R \partial_\tau - i\lambda(J_3 + I_3) + i\mathbf{a} + i \sum_{f=1}^{N_f} m_f F_f \,. \tag{2.51}$$

We have introduced a formal parameter ε that should be set to one at the end of calculation. A Fourier mode $e^{in\tau/R}$ along S^1 contributes $in\varepsilon$ to $\mathcal{R}$.

The form (2.50) of the one-loop determinant implies that it can be obtained from the equivariant index of the operator D_{10}

$$\mathrm{ind}\, D_{10} \equiv \mathrm{Tr}_{\mathrm{Ker} D_{10}} e^{2\pi\mathcal{R}} - \mathrm{Tr}_{\mathrm{Coker} D_{10}} e^{2\pi\mathcal{R}} \,. \tag{2.52}$$

Indeed if it is given in terms of weights w_j and multiplicities c_j as

$$\mathrm{ind}\, D_{10} = \sum c_j e^{w_j} \,, \tag{2.53}$$

the one-loop determinant is given by $Z_{\text{1-loop}} = \left(\prod_j w_j^{c_j}\right)^{-1/2}$. In the following we will separately define the indices for differential operators acting on vector and hypermultiplets. We will also adopt a normalization for ind that corresponds to $\mathrm{ind}(D_{10}) \to -\frac{1}{2}\mathrm{ind}(D_{10})$, so that the translation from the index to the one-loop determinant is simply given by the rule $\sum_j c_j e^{w_j} \to \prod_j w_j^{c_j}$. Then

$$Z_{\text{1-loop}} = \prod_j w_j^{c_j} \,. \tag{2.54}$$

Thus we need to compute the weights under the gauge transformation with parameter $\mathbf{a} := R(A_\tau^{(\infty)} + i\Phi_0^{(\infty)})$, a time translation by ε, and a spatial rotation along the 3-axis with angle $2\pi\lambda$, and flavor transformations with parameters m_f.

2.3.3 Correspondence between singular monopoles and instantons

Let us review a correspondence between a singular monopole on $\mathbb{R}^3$ and an instanton. We have to treat monopoles with the singularities in (2.5) to calculate 't Hooft loop operators by definition but we do not know how to obtain directly the contribution from such singular monopoles to partition functions or indecies. Instead instanton partition functions are well known. Therefore we use this correspondence for the calculation of 't Hooft loop operators. This subsubsection is relevent for Section 2.3.4 and Section 2.4.

This correspondence was found by Kronheimer in [30] and the correspondence between singular monopoles on $\mathbb{R}^3$ and $U(1)_K$-invariant instantons on a Taub-NUT space is relevent for our calculation. A Taub-NUT space is labeled by the coordinates $(\vec{x} = (x_1, x_2, x_3), \psi)$ and has the following metric

$$ds^2 = V d\vec{x}^2 + V^{-1}(d\psi + \omega)^2 \,, \qquad V = l + \frac{1}{2|\vec{x}|} \,, \qquad d\omega = - *_3 dV \,, \tag{2.55}$$

where $l > 0$ is a constant. The reason why we should choose this space as a four-dimensional spacetime where instantons live will become clear in (2.63). The important fact is that this space has a noncontractible cycle in the asymptotic region and the gauge field can have a holonomy with arbitrary value along this cycle.

To show that there is a map from a singular monopole to an instanton, we will construct a four-dimensional gauge connection from the three-dimensional fields $(A(\vec{x}), \Phi(\vec{x}))$ with singularities

$$A \sim \frac{B}{2}\cos\theta d\varphi\,, \quad \Phi \sim \frac{B}{2r} \qquad \text{near } \vec{x}=0\,, \tag{2.56}$$

where $(r := |\vec{x}|, \theta, \varphi)$ are the spherical coordinates on a 3-ball centered at $\vec{x}=0$. The above Φ corresponds to Φ_9 in (2.5). From the above three-dimensional fields, we can construct the following smooth four-dimenisonal configuration

$$\mathcal{A}(\vec{x},\psi) \equiv g\left(A(\vec{x}) + \Phi(\vec{x})\frac{d\psi+\omega}{V}\right)g^{-1} - igdg^{-1}\,. \tag{2.57}$$

We denote its curvature by $\mathcal{F} = d\mathcal{A} + i\mathcal{A}\wedge\mathcal{A}$. The singularities in A and Φ cancel in (2.57) to define a smooth four-dimensional gauge field $\mathcal{A}$. Here g is a suitable singular gauge transformation that locally behaves as $g \sim e^{iB\psi}$ near $\vec{x}=0$ so that $\mathcal{A}$ is smooth there. The four-dimensional field $\mathcal{A}$ is invariant under the $U(1)_K$ action $\psi \to \psi - \nu$, which rotates the circle fiber in the spacetime as well as acts on the gauge bundle as a gauge transformation $e^{iB\nu}$. Vice versa, we can also construct a three-dimensional field configuration (A,Φ) with the singularities (2.56) from a $U(1)_K$-invariant smooth four-dimensional gauge field $\mathcal{A}$.

In order to show that there is a one-to-one correspondence between singular monopoles and $U(1)_K$-invariant instantons, we have to show that the condition that the above (A,Φ) satisfies the Bogomolny equation

$$*_3\, F = D\Phi \tag{2.58}$$

is equivalent to the condition that the the above $U(1)_K$-invariant $\mathcal{A}$ satisfies the anti-self-dual equation

$$*_4\, \mathcal{F} + \mathcal{F} = 0\,. \tag{2.59}$$

To show this, let us use the fact that $\mathcal{A}$ is obtained by a singular gauge transformation from

$$\widetilde{\mathcal{A}} = A + \Phi\frac{d\psi+\omega}{V}\,, \tag{2.60}$$

therefore $\mathcal{F} = g\widetilde{\mathcal{F}}g^{-1}$. Then, for the orientation (volume form)$\propto (d\psi+\omega)dx^1dx^2dx^3$,

$$\mathcal{F} \;=\; g\left(F + D\Phi\wedge\frac{d\psi+\omega}{V} - \Phi\frac{*_3 dV}{V} + \Phi(d\psi+\omega)\wedge\frac{dV}{V^2}\right)g^{-1}\,, \tag{2.61}$$

and

$$*_4\,\mathcal{F} \;=\; g\left(-*_3\,F\wedge\frac{d\psi+\omega}{V} - *_3 D\Phi - \Phi\frac{(d\psi+\omega)\wedge dV}{V^2} + \Phi\frac{*_3 dV}{V}\right)g^{-1}\,, \tag{2.62}$$

so $\mathcal{F} + *_4\mathcal{F} = 0$ if and only if $F = *_3 D\Phi$.

The holonomy of the four-dimensional field at infinity $|\vec{x}| = \infty$ is related to the scalar expectation value as

$$Pe^{-i\oint\mathcal{A}} \to e^{-2\pi i\Phi^{(\infty)}/l} \text{ as } |\vec{x}| \to \infty \tag{2.63}$$

up to conjugation. Since we have to set the vev of Φ to $\Phi_9^{(\infty)}$ in (2.5) in our problem, we have to give the corresponding holonomy to $\mathcal{A}$. Indeed we can do so since $\mathcal{A}$ lives on the Taub-NUT space, which has a noncontractible cycle S^1 as a fiber. However our results do not depend on this holonomy since only the local behavior around the origin of the space contributes.

Let us see the local region around the origin of this space. The metric (2.55) approaches that of $\mathbb{C}^2$ in the limit $l \to 0$, $ds^2 \to ds^2_{\mathbb{C}^2}$, where

$$\begin{aligned} ds^2_{\mathbb{C}^2} =& (2r)^{-1}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right] + 2r(d\psi + \omega)^2 \\ =& |dz_1|^2 + |dz_2|^2 \end{aligned} \tag{2.64}$$

and

$$z_1 = r^{1/2}\cos\frac{\theta}{2}e^{-i\psi+i\varphi/2}, \quad z_2 = r^{1/2}\sin\frac{\theta}{2}e^{i\psi+i\varphi/2}, \quad \omega = -\frac{1}{2}\cos\theta d\varphi\,. \tag{2.65}$$

For general l, Taub-NUT space is isomorphic as a complex manifold to $\mathbb{C}^2 = \{(z_1, z_2)\}$ with the same parametrization.

Later we will have to consider the action $\varphi \to \varphi + 2\pi\lambda$ on the space $\mathbb{R}^3$ labeled by $(r := |\vec{x}|, \theta, \varphi)$ caused by $i\lambda J_3$ in (2.51) when we calculate $\operatorname{Tr} e^{2\pi\mathcal{R}}$ in (2.52). According to (2.65), this action corresponds to $(z_1, z_2) \to e^{\pi i\lambda}(z_1, z_2)$ on the instanton side.

2.3.4 Calculation of the equivalent index

Before we delve into the details of the calculations, let us summarize our methodology that extends the techniques developed in [10], listing at the same time the relevant complexes and their interrelations. We showed above that the vector multiplet contribution to the one-loop determinant can be computed from the index of the complex that linearizes the Bogomolny equations in $\mathbb{R}^3$

$$D_{\text{Bogo}} : 0 \to \Omega^0(\operatorname{ad} E) \xrightarrow{(D\,,[i\Phi_9,\,\bullet])} \Omega^1(\operatorname{ad} E) \oplus \Omega^0(\operatorname{ad} E) \to \Omega^1(\operatorname{ad} E) \to 0\,, \tag{2.66}$$

where $\operatorname{ad} E$ is the adjoint gauge bundle. The second arrow is the gauge transformation whose conjugate[10] appeared in (A.12), and the third is the map $(\delta A, \delta\Phi_9) \mapsto *D\delta A - D\delta\Phi_9 + i[\Phi_9, \delta A]$ in (A.11). As reviewed in Section 2.3.3, the Bogomolny equations with

[10] The equivariant index remains the same when we "fold" (2.66) into $0 \to \Omega^0 \oplus \Omega^1 \to \Omega^1 \oplus \Omega^0 \to 0$, where twisting by $\operatorname{ad} E$ is implicit, and the second arrow is the linearized Bogomolny equations plus the dual of a gauge transformation (A.12). The same remark applies to the self-dual complex (2.67). It is the folded form of the complexes that naturally arises from gauge-fixing.

a single singularity on $\mathbb{R}^3$ are equivalent to the anti-self-duality equations on the (single-centered) Taub-NUT space with invariance under the action of the group that we call $U(1)_K$. Linearizing the correspondence, we will obtain the index of the Bogomolny complex[11] (2.66) from the index of the self-dual complex

$$D_{\mathrm{SD}} : 0 \to \Omega^0(\mathrm{ad}\, E) \xrightarrow{D} \Omega^1(\mathrm{ad}\, E) \xrightarrow{(1+*)D} \Omega^{2+}(\mathrm{ad}\, E) \to 0 \tag{2.67}$$

on the four-dimensional space by taking an invariant part under the $U(1)_K$ action [31, 10]. Similarly the hypermultiplet contribution will be derived from index of the complex

$$D_{\mathrm{DH},R} : 0 \to \Gamma(S \otimes R(E)) \xrightarrow{\sigma^j D_j + \Phi_9} \Gamma(S \otimes R(E)) \to 0\,, \tag{2.68}$$

where S is the spinor bundle over $\mathbb{R}^3$, and Φ_9 acts on $q \in \Gamma(S \otimes R(E))$ in the matter representation R. Its index will be obtained from the $U(1)_K$ invariant part of the index of the twisted Dirac complex [10]

$$D_{\mathrm{Dirac},R} : 0 \to \Gamma(S^+ \otimes R(E)) \xrightarrow{\overline{\sigma}^\mu D_\mu} \Gamma(S^- \otimes R(E)) \to 0 \tag{2.69}$$

in four dimensions.

Both the self-dual and Dirac complexes are related to the Dolbeault complex

$$\overline{D}_R : 0 \to \Omega^{0,0}(R(E)) \to \Omega^{0,1}(R(E)) \to \Omega^{0,2}(R(E)) \to 0\,. \tag{2.70}$$

To see this note that upon complexification we have $\Omega^0_{\mathbb{C}} = \Omega^{0,0}$, $\Omega^1_{\mathbb{C}} = \Omega^{1,0} \oplus \Omega^{0,1}$ and $\Omega^{2+}_{\mathbb{C}} = \Omega^{2,0} \oplus \Omega^{0,0}\omega \oplus \Omega^{0,2}$, where ω is the Kähler form. See, e.g., [32]. Since by Hodge duality $\Omega^{2,2} = \Omega^{0,0}$ and $\Omega^{2,1} = \Omega^{1,0}$, the complexification of the self-dual complex (2.67) is isomorphic to the Dolbeault complex (2.70) with $R = \mathrm{ad}$ twisted by $\Omega^{0,0} \oplus \Omega^{2,0}$. For spinors recall that $\Omega^{p,q} = \Gamma(\Lambda^{p,q})$ and that $K = \Lambda^{2,0}$ is the canonical line bundle. We have

$$S^+ = K^{1/2} \otimes (\Lambda^{0,0} \oplus \Lambda^{0,2})\,, \quad S^- = K^{1/2} \otimes \Lambda^{0,1}\,. \tag{2.71}$$

Thus the Dirac complex (2.69) is isomorphic to the Dolbeault complex (2.70) twisted by $(\Omega^{2,0})^{1/2}$.

Let us now review the index of the Dolbeault complex. We will compute the index of the Dolbeault complex on Taub-NUT space by applying the Atiyah-Bott fixed point formula. Taub-NUT space is holomorphically isomorphic to flat $\mathbb{C}^2$ with local coordinates (z_1, z_2), for which the $U(1) \times U(1)$-equivariant index of the (untwisted) Dolbeault complex is given by

$$\mathrm{ind}(\overline{\partial}) = \frac{t_1 t_2}{(1-t_1)(1-t_2)}\,. \tag{2.72}$$

[11]We will refer to (2.66) and (2.68) as the Bogomolny and Dirac-Higgs (DH) complexes.

Let us denote by $U(1)_{J+R}$ the group generated by $J_3 + I_3$, the simultaneous spatial and R-symmetry rotations. The action of (t_1, t_2) on $\mathbb{C}^2$ is standard, $(z_1, z_2) \mapsto (t_1 z_1, t_2 z_2)$, and is related to $U(1)_K \times U(1)_{J+R}$ as

$$t_1 = e^{-2\pi i\nu + \pi i\lambda}\,, \qquad t_2 = e^{2\pi i\nu + \pi i\lambda}\,. \tag{2.73}$$

Here $e^{2\pi i\nu}$ parametrizes $U(1)_K$, while $2\pi\lambda$ is the angle of rotation along the 3-axis of $\mathbb{R}^3$, which corresponds to $\pi\lambda$ in the exponent acting on (z_1, z_2) as explained in the sentences below (2.65).

For our purposes the best way to understand the formula (2.72) is to consider the group action on the basis of sections. For example an element of $\Omega^{0,0}$ can be expanded as

$$\sum_{k,l,m,n} c_{klmn} z_1^k \overline{z}_1^l z_2^m \overline{z}_2^n\,, \tag{2.74}$$

where $k, l, m, n \in \mathbb{Z}_{\geq 0}$ and the coefficients transform as $c_{klmn} \mapsto t_1^{-k+l} t_2^{-m+n} c_{klmn}$. Elements of $\Omega^{0,1}$ and $\Omega^{0,2}$ admit similar expansions. Summing up the weights with appropriate signs determined by the degrees in the complex, we obtain

$$\begin{aligned} \mathrm{ind}_\delta(\overline{\partial}) &= \sum_{k,l,m,n\geq 0} (1 - t_1 - t_2 + t_1 t_2) t_1^{-k+l} t_2^{-m+n} \\ &= \frac{(1-t_1)(1-t_2)}{(1-e^{-\delta}t_1^{-1})(1-e^{-\delta}t_1)(1-e^{-\delta}t_2^{-1})(1-e^{-\delta}t_2)}\,. \end{aligned} \tag{2.75}$$

Factors $e^{-\delta}$ with small $\delta > 0$ are inserted to keep track of how we expand the numerator. We obtain (2.72) from the regularized index (2.75) by taking the limit $\delta \to 0$. Including the gauge group action, we obtain the index for the Dolbeault operator twisted by $R(E)$

$$\mathrm{ind}_\delta(\overline{D}_R) = \frac{(1-t_1)(1-t_2)}{(1-e^{-\delta}t_1^{-1})(1-e^{-\delta}t_1)(1-e^{-\delta}t_2^{-1})(1-e^{-\delta}t_2)} \sum_{w\in R} e^{2\pi i w\cdot \mathbf{a}}\,. \tag{2.76}$$

The relationships of the self-dual and Dirac complexes to the Dolbeault complex described above imply that

$$\mathrm{ind}_\delta(D_{\mathrm{SD},\mathbb{C}}) = (1 + t_1^{-1} t_2^{-1})\,\mathrm{ind}_\delta(\overline{D}_{\mathrm{adj}})\,, \tag{2.77}$$

$$\mathrm{ind}_\delta(D_{\mathrm{Dirac},R}) = t_1^{-1/2} t_2^{-1/2}\,\mathrm{ind}_\delta(\overline{D}_R)\,. \tag{2.78}$$

Furthermore, the indices of the Bogomolny and Dirac-Higgs complexes are obtained by taking the $U(1)_K$-invariant parts. This can be implemented by substituting (2.73) and $\mathbf{a} \to \mathbf{a} + B\nu$ and then integrating over ν:

$$\mathrm{ind}(D_{\mathrm{Bogo},\mathbb{C}}) = \lim_{\delta\to 0} \int_0^1 d\nu\; \mathrm{ind}_\delta(D_{\mathrm{SD},\mathbb{C}})\big|_{\mathbf{a}\to\mathbf{a}+B\nu}\,, \tag{2.79}$$

$$\mathrm{ind}(D_{\mathrm{DH},R}) = \lim_{\delta\to 0} \int_0^1 d\nu\; \mathrm{ind}_\delta(D_{\mathrm{Dirac},R})\big|_{\mathbf{a}\to\mathbf{a}+B\nu}\,. \tag{2.80}$$

The factors $e^{-\delta}$ in the integrands specify which poles to pick in the contour integrals. We also need to take into account the Fourier modes on S^1 that give rise to an infinite sum $\sum_n e^{in\varepsilon}$. The formal parameter ε for time translation should be set to one at the end of the calculation.

Finally, the one-loop determinant $Z^{\text{vm}}_{\text{1-loop}}$ for the vector multiplet is obtained by the rule $\sum_j c_j e^{w_j} \to \prod_j w_j^{c_j}$ from

$$\text{ind}(D^{\text{vm}}) = \frac{1}{2}\sum_{n\in\mathbb{Z}} e^{2\pi i n\varepsilon}\text{ind}(D_{\text{Bogo},\mathbb{C}})\,. \tag{2.81}$$

The factor of 1/2 in (2.81) accounts for the complexification of the Bogomolny complex.

For the hypermultiplet, the one-loop determinant $Z^{\text{hm}}_{\text{1-loop}}$ arises if the same rule is applied to [10]

$$\text{ind}(D^{\text{hm}}_R) = -\frac{1}{2}\sum_{n\in\mathbb{Z}} e^{2\pi i n\varepsilon}\sum_{f=1}^{N_f}\left(e^{-2\pi i m_f}\text{ind}(D_{\text{DH},R}) + e^{2\pi i m_f}\text{ind}(D_{\text{DH},R})|_{\mathbf{a}\to-\mathbf{a}}\right). \tag{2.82}$$

Let us explain the meaning of this expression (2.82). The precise flavor symmetry of a massless theory is best described in terms of half-hypermultiplets. If an irreducible representation R is real, half-hypermultiplets can only appear in an even number $2N_f$, and the flavor symmetry G_{F} is $Sp(2N_f)$. The symplectic group $Sp(2N_f)$ has rank N_f in our convention. For a complex irreducible representation R, half-hypermultiplets always appear in conjugate pairs $R \oplus \overline{R}$. With N_f such pairs, the flavor symmetry is $U(N_f)$. When an irreducible representation R is pseudo-real, the theory is anomalous unless an even number $2N_f$ of half-hypermultiplets are present [33]. The flavor symmetry group in this case is $SO(2N_f)$. Parameters m_f in (2.82) are the equivariant parameters for the flavor group G_{F} of the massless theory, and are related to the physical masses M_f and the flavor chemical potentials μ_f as

$$m_f = -\mu_f + iRM_f\,. \tag{2.83}$$

The particular combination of terms in (2.82) was derived in [10] based on Higgsing which produces various types of matter representations.

The indices $\text{ind}(D_{\text{Bogo},\mathbb{C}})$ and $\text{ind}(D_{\text{DH},R})$ were computed in [10]:

$$\begin{aligned}
\text{ind}(D_{\text{Bogo},\mathbb{C}}) &= -\frac{e^{\pi i\lambda}+e^{-\pi i\lambda}}{2}\sum_{\alpha} e^{2\pi i\alpha\cdot\mathbf{a}}\left(e^{(|\alpha\cdot B|-1)\pi i\lambda}+e^{(|\alpha\cdot B|-3)\pi i\lambda}+\ldots+e^{-(|\alpha\cdot B|-1)\pi i\lambda}\right),\\
\text{ind}(D_{\text{DH},R}) &= -\frac{1}{2}\sum_{w\in R} e^{2\pi i w\cdot\mathbf{a}}\left(e^{(|w\cdot B|-1)\pi i\lambda}+e^{(|w\cdot B|-3)\pi i\lambda}+\ldots+e^{-(|w\cdot B|-1)\pi i\lambda}\right).
\end{aligned} \tag{2.84}$$

By applying the rule to (2.81) and (2.82), we find the one-loop determinant

$$\begin{aligned}
&\prod_{n\in\mathbb{Z}}\prod_{\alpha}\prod_{k=0}^{|\alpha\cdot B|-1}\left[n\varepsilon+\frac{1}{2}\lambda+\alpha\cdot\mathbf{a}+\left(\frac{|\alpha\cdot B|-1}{2}-k\right)\lambda\right]^{-1/2} \\
\sim\ &\prod_{\alpha>0}\prod_{k=0}^{|\alpha\cdot B|-1}\prod_{\pm}\sin^{-1/2}\left[\pi\left(\alpha\cdot\mathbf{a}\pm\left(\frac{|\alpha\cdot B|}{2}-k\right)\lambda\right)\right] \\
=:\ &Z^{\text{vm}}_{\text{1-loop}}(\mathbf{a},\lambda;B)\,, \qquad (2.85)
\end{aligned}$$

for the vector multiplet and

$$\begin{aligned}
&\prod_{n\in\mathbb{Z}}\prod_{f=1}^{N_f}\prod_{w\in R}\prod_{k=0}^{|w\cdot B|-1}\left[n\varepsilon+w\cdot\mathbf{a}-m_f+\left(\frac{|w\cdot B|-1}{2}-k\right)\lambda\right]^{1/2} \\
\sim\ &\prod_{f=1}^{N_f}\prod_{w\in R}\prod_{k=0}^{|w\cdot B|-1}\sin^{1/2}\left[\pi\left(w\cdot\mathbf{a}-m_f+\left(\frac{|w\cdot B|-1}{2}-k\right)\lambda\right)\right] \\
=:\ &Z^{\text{hm}}_{\text{1-loop}}(\mathbf{a},m_f,\lambda;B) \qquad (2.86)
\end{aligned}$$

for the hypermultiplet. In the final expressions we set ε to one. When there is more than one matter irreducible representation we need to take a product over them. Combining the vector multiplet and hypermultiplet contributions, the one-loop factor is given by

$$Z_{\text{1-loop}}(\mathbf{a},m_f,\lambda;B):=Z^{\text{vm}}_{\text{1-loop}}(\mathbf{a},\lambda;B)Z^{\text{hm}}_{\text{1-loop}}(\mathbf{a},m_f,\lambda;B)\,. \qquad (2.87)$$

2.4 Contributions from monopole screening

In this section we calculate the contributions from non-perturbative saddle points of the localization action $Q\cdot V$. Since the bosonic part of $Q\cdot V$ is given by $||Q\cdot\Psi||^2$, these saddle points are the solutions of the equation $Q\cdot\Psi=0$. As we saw in Section 2.2.2, the solutions of $Q\cdot\Psi=0$ are the fixed points of the Bogomolny equations with a prescribed singularity.

2.4.1 Definition of Z_{mono}

The moduli space of the solutions of the Bogomolny equations with a singularity prescribed by B has infinitely many components. For example, even for $B=0$ there exist the components whose elements are smooth monopoles with charges labeled by all $v\in\Lambda_{cr}$. In our localization calculation only the components that contain fixed points of the $U(1)_{J+R}\times T$-action are relevant, where T is the maximal torus of the gauge group. Invariance under $U(1)_{J+R}\times T$-action is a strong constraint, because the T-invariance for generic $a\in\mathfrak{t}$ requires the adjoint fields to be Abelian, *i.e.*, that they belong to $\mathfrak{t}$. The only Abelian solutions to the Bogomolny equations are the singular Dirac monopole solutions, and the singularity must be located at the point where the 't Hooft operator is inserted. This argument almost shows that the background configuration (2.5) is the only saddle point of the path integral. Abelian solutions of the Dirac form (2.5), where B is replaced by some other coefficient $v \in \Lambda_{cr} + B$, can however arise as a limit in the family of solutions whose singularity has coefficient B [34].

Such solutions represent the situations where a smooth monopole appears infinitely near from the Dirac monopole with charge B and it screen the charge B. Due to this smooth monopole, the total magnetic charge is changed from B into v.

We denote by $M(B, v)$ the moduli space of such solutions to the Bogomolny equation.

A generic point of $M(B, v)$ is a solution that approaches the background (2.5) near the origin, and the same expression with B replaced by v asymptotically at infinity. It can be shown that we need $||v|| \leq ||B||$ for $M(B, v)$ to be non-empty in [10].

In order to obtain the indices for $M(B, v)$, we use Kronheimer's correspondence, reviewed in Section 2.3.3. In this correspondence, a solution to Bogomolny equations with Dirac singularities can be mapped to an instanton located at the point on Taub-NUT space where the S^1 fiber degenerates. Instead of treating the moduli space of the above monopoles directly, we will calculate the partition functions for instantons corresponding to the monopoles.

Since this calculation needs only the local behavior of the fields near the origin of the Taub-NUT space, we can replace this space by $\mathbb{C}^2$. A more satisfying justification for this replacement is the fact that such a small instanton solution belongs to a component of the instanton moduli space that is isomorphic as a complex variety to a component of the instanton moduli space for $\mathbb{C}^2$ [35]. See also [36].

Since all the fixed points in (B, v) take the form of the 't Hooft background (2.5) except that B is replaced by v, each contributes a factor $e^{S_{\rm cl}(v)}$ computed in Section 2.2.3. This classical contribution $Z_{\rm cl}$ to the indices depends only on v and is universal among the fixed points in $M(B, v)$. The non-perturbative contribution $Z_{\rm mono}(B, v)$ from the screening monopoles will be obtained from Nekrasov formulae of instanton partition functions, reviewed in Section 1.2, in the following subsubsections.

The total expressions of the indices in presence of the 't Hooft loop operator with charge B should be written

$$\sum_v e^{S_{\rm cl}(v)}\, Z_{\text{1-loop}}(v)\, Z_{\rm mono}(B, v)\,. \tag{2.88}$$

2.4.2 Monopole moduli space for $G = U(N)$

In order to compute $Z_{\rm mono}(B, v)$ explicitly, we need a method to describe the component $M(B, v)$ of the monopole moduli space and their fixed points. Let us now review the ADHM construction of $M(B, v)$ in the case $G = U(N)$ [34].

A monopole solution in $M(B, v)$ descends from a $U(1)_K$-invariant instanton on the Taub-NUT space as we saw in Section 2.3.3. Since our calculation needs only the local behavior of the fields near this point, we can replace the Taub-NUT space by $\mathbb{C}^2$ parametrized by coordinates $z = (z_1, z_2)$, which are the same as the ones in (2.65). Let us set $W := \mathbb{C}^N$ and $V := \mathbb{C}^k$. The instanton bundle over $\mathbb{C}^2$ with instanton number k is described by a family of complexes

$$V \xrightarrow{\alpha(z)} \mathbb{C}^2 \otimes V \oplus W \xrightarrow{\beta(z)} V \,, \tag{2.89}$$

where the maps depend on z as

$$\alpha(z) = \begin{pmatrix} z_2 - B_2 \\ -z_1 + B_1 \\ -J \end{pmatrix} \,, \quad \beta(z) = \begin{pmatrix} z_1 - B_1 & z_2 - B_2 & -I \end{pmatrix} \,. \tag{2.90}$$

When the complex ADHM equation

$$[B_1, B_2] + IJ = 0 \tag{2.91}$$

which is equivalent to $\beta(z)\alpha(z) = 0$ is satisfied, the cohomology groups

$$H_z^0 = \mathrm{Ker}[\alpha(z)] \,, \quad H_z^1 = \mathrm{Ker}[\beta(z)]/\mathrm{Im}[\alpha(z)] \,, \quad H_z^2 = V/\mathrm{Im}[\beta(z)] \tag{2.92}$$

can be defined. If $H_z^0 = H_z^2 = 0$, $E_z = H_z^1$ describes the fiber of a smooth irreducible instanton bundle over $\mathbb{C}^2$. We are also interested in singular configurations that arise as a limit of smooth ones, therefore we set $E_z = H_z^1 - H_z^0 - H_z^2$ in general. The Euler characteristic $\dim H_z^0 - \dim H_z^1 + \dim H_z^2 = -\dim E_z = -N$ is independent of z.

Next let us see how the $U(1)_K$ action $\psi \to \psi - \nu$ acts on the moduli space. It acts on (z_1, z_2) as $(z_1, z_2) \mapsto (e^{2\pi i\nu} z_1, e^{-2\pi i\nu} z_2)$. Since (B_1, B_2) represent the positions of the instantons, they transform as $(B_1, B_2) \mapsto (e^{2\pi i\nu} B_1, e^{-2\pi i\nu} B_2)$. The group $U(1)_K$ also acts on the gauge bundle. The fiber E_0 at $z = 0$ if mapped to itself, and its character for $U(1)_K$ is given by $e^{2\pi i B\nu}$ where $e^{2\pi\nu} \in U(1)_K$ and the charge B of the 't Hooft operator is regarded as an $N \times N$ diagonal matrix. The group $U(1)_K$ also acts on W and V. Since W represents the fiber E_∞ at $z = \infty$, its character is $\mathrm{Tr} e^{2\pi i v\nu}$. The character of V can be written as $e^{2\pi i K\nu}$ with a $k \times k$ diagonal matrix K. The identification of E_z with $H_z^1 - H_z^0 - H_z^2$ implies that K is determined by[12]

$$\mathrm{Tr} e^{2\pi i B\nu} = \mathrm{Tr} e^{2\pi i v\nu} + (e^{2\pi i\nu} + e^{-2\pi i\nu} - 2)\mathrm{Tr} e^{2\pi i K\nu} \qquad e^{2\pi i\nu} \in U(1)_K \tag{2.93}$$

up to conjugation. We can read the instanton number k corresponding to $M(B, v)$ and the eigenvalues of the matrix K from the above expression. In the last part in the next subsubsection, we will do this in case where $B = (1, -1), v = (0, 0)$ in $SU(2)$ guage theory as an example.

[12] A warning on notation. The "K" in $U(1)_K$ stands for Kronheimer. The matrix K is the weight of $U(1)_K$ acting on the k-dimensional vector space on which B_1 and B_2 act as endomorphisms.

To describe $M(B,v)$, we impose $U(1)_K$ invariance on the ADHM data. Namely the ADHM data must satisfy the conditions

$$B_1 + [K, B_1] = 0\,, \quad -B_2 + [K, B_2] = 0\,, \quad KI - Iv = 0\,, \quad vJ - JK = 0\,. \tag{2.94}$$

For the instanton moduli space, one would take a quotient by $GL(k, \mathbb{C})$. The matrix K breaks the $GL(k, \mathbb{C})$ into its commutant $\prod_r GL(k_r, \mathbb{C})$, where k_r is the number of entries of the r-th largest integer in the eigenvalues of K.

The ADHM data are considered equivalent if they are related by an action of $\prod_r GL(k_r, \mathbb{C})$:

$$(B_1, B_2, I, J) \sim (gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1}) \qquad g \in \prod_r GL(k_r, \mathbb{C})\,. \tag{2.95}$$

Thus the complex variety $M(B,v)$ is given by the holomorphic quotient

$$M(B,v) = \left\{ (B_1, B_2, I, J) \left| \begin{array}{ccc} B_1 + [K, B_1] & = & 0 \\ -B_2 + [K, B_2] & = & 0 \\ KI - IM & = & 0 \\ MJ - JK & = & 0 \end{array} \right. \right\} \Big/ \prod_r GL(k_r, \mathbb{C})\,. \tag{2.96}$$

2.4.3 Fixed points and their contributions

Next we turn to the description of fixed points. We need to know which fixed point $\vec{Y}$ on the instanton moduli space descends to the specific component $M(B,v)$ of the monopole moduli space. The fixed points are given by the ADHM data (B_1, B_2, I, J) that satisfy

$$\begin{aligned} -\varepsilon_1 B_1 + [\phi, B_1] = 0\,, \quad -\varepsilon_2 B_2 + [\phi, B_2] = 0\,, \\ \phi I - Ia = 0\,, \quad -(\varepsilon_1 + \varepsilon_2)J + aJ - J\phi = 0 \end{aligned} \tag{2.97}$$

for any $(\varepsilon_1, \varepsilon_2, a) \in \mathrm{Lie}\,[U(1) \times U(1) \times T]$ for some $\phi = \mathrm{diag}(\phi_1, \ldots, \phi_k)$ parametrizing the Cartan subalgebra of $\prod_r U(k_r) \subset U(k)$. Solutions to these equations are known and are expressed in terms of Young tableaux $\vec{Y}$. The sets $\{\phi_s\}_{s=1,\ldots,k}$ of the eigenvalues of ϕ are

$$\phi_s = (i_s - 1)\varepsilon_1 + (j_s - 1)\varepsilon_2 + a_{\alpha(s)}\,, \tag{2.98}$$

where s labels each box in $\vec{Y} = (Y_1, \cdots, Y_N)$ and $\alpha(s) \in \{1, \ldots, N\}$ is such that $s \in Y_{\alpha(s)}$. (i_s, j_s) are the coordinates of the box s in $Y_{\alpha(s)}$.

If we substitute

$$\varepsilon_1 \to -\nu\,, \quad \varepsilon_2 \to \nu\,, \quad a_\alpha \to v_\alpha \nu\,, \tag{2.99}$$

into the equations (2.97) and replace ϕ with K, it becomes the same equations as (2.94). So the fixed point $\vec{Y}$ in the instanton moduli space also satisfies the $U(1)_K$ invariance condition

(2.94) if K is the same as ϕ with (2.98) where we substitute (2.99). Therefore the $U(1)_K$-invariant fixed points correspond to

$$\vec{Y} \quad \text{such that} \quad K_s = v_{\alpha(s)} - j_s + i_s \tag{2.100}$$

up to a permutation of $s \in \{1, \ldots, k\}$. On the left-hand side, K_s is the s-th eigenvalue of the matrix K. On the right-hand side s labels each box in the N-tuple of Young tableaux $\vec{Y}$.

Next we turn to weights. After explaining general procedures, we will demonstrate it in a exmaple. When we obtained the instanton partition functions in Section 1.2, we considered the weights at each fixed point in the instanton moduli spaces. Here we have to focus on $U(1)_K$ invariant parts in the instanton moduli space, so we will extract the factors invariant under the following action from those weights.

$$\varepsilon_1 \to \varepsilon_1 - \nu\,, \quad \varepsilon_2 \to \varepsilon_2 + \nu\,, \quad a_\alpha \to a_\alpha + v_\alpha \nu\,. \tag{2.101}$$

To the index (2.11), only the field configurations that are invariant under the action $(x_1 + ix_2) \to e^{2\pi i\lambda}(x_1 + x_2)$ contribute. If we map this action to the instanton side according to Kronheimer's correspondence, it coresponds to the field configurations invariant under $(z_1, z_2) \to e^{\pi i\lambda}(z_1, z_2)$ as explained below (2.65). Therefoer we will consider the instanton partition functions where the regularization parameters are set to $\epsilon_1 = \epsilon_2 = \frac{\lambda}{2}$. Thus we can obtain the partition functions of the screening monopoles on $\mathbb{R}^3$.

Since the geometry in our problem is $S^1 \times \mathbb{R}^3$, those weights must be entered in sine function.

Let us see an example. We will consider the case where $B = (1, -1)$, $v = (0, 0)$ in $\mathcal{N} = 2^*$ $SU(2)$ gauge theory. First we have to identify the fixed points of $M(B, v)$. Substituing this B and v into (2.93), we get $\mathrm{Tr}\, e^{iK\nu} = 1$. It means that K is 1×1 matrix and the eigenvalue is 0, i.e, $K = (0)$. So we have to consider the moduli space where the instanton number is 1 and each fixed point is labeled by a pair of Young tableaux $\vec{Y} = (Y_1, Y_2)$ where $|\vec{Y}| = 1$. We can see that both of the Young pairs $(\{1\}, \{\})$ and $(\{\}, \{1\})$ satisfy the condition (2.93). There is one box in both pairs and it is labeled as $s = 1$. In the first pair, it is in the first row and first column in Y_1 so $\alpha(s) = 1$, $(i_s, j_s) = (1, 1)$. Now we consider the case with $v = (0, 0)$ so $v_1 = v_2 = 0$. Therefore $v_{\alpha(s)} + j_s - i_s = 0$ and it is consistent with $K = (0)$. The same is true in the second pair. Therefore both of these pairs descend to the fixed points in $M(B, v)$.

The weights at each Young pair in the instanton partition function were obtained in Section 1.2.

$$\begin{aligned}
Z^{\text{inst}}_{(\{1\},\{\})} &= \frac{(-2a - m)(\varepsilon_1 - m)(\varepsilon_2 - m)(2a + \varepsilon_1 + \varepsilon_2 - m)}{(-2a)\varepsilon_1\varepsilon_2(2a + \varepsilon_1 + \varepsilon_2)} \\
Z^{\text{inst}}_{(\{\},\{1\})} &= \frac{(2a - m)(\varepsilon_1 - m)(\varepsilon_2 - m)(-2a + \varepsilon_1 + \varepsilon_2 - m)}{(2a)\varepsilon_1\varepsilon_2(-2a + \varepsilon_1 + \varepsilon_2)}\,.
\end{aligned}$$

Then we extract invariant factors under the action (2.101) from the above expressoins.

$$Z^{\text{inst}}_{\{\{1\},\{\}\}} \to \frac{(-2a-m)(2a+\varepsilon_1+\varepsilon_2-m)}{(-2a)(2a+\varepsilon_1+\varepsilon_2)}\,, \quad Z^{\text{inst}}_{\{\{\},\{1\}\}} \to \frac{(2a-m)(-2a+\varepsilon_1+\varepsilon_2-m)}{(2a)(-2a+\varepsilon_1+\varepsilon_2)}\,.$$

Thus we got the partition functions of the monopoles relevant for our present case on $\mathbb{R}^3$. Then we set $\epsilon_1 = \epsilon_2 = \lambda/2$. Since our spacetime is $S^1 \times \mathbb{R}^3$, the above factors must be entered in sine functions. Note that the physical mass, which we are using, and Nekrasov mass are different by $(\epsilon_1+\epsilon_2)/2$ so we have to substitute $m \to m + \lambda/2$. Finally we obtain the contribution from the screening monopoles to the index with $B = (1, -1)$ as follows

$$Z_{\text{mono}}(B, v) = \frac{\sin\pi(2\mathbf{a}+m+\frac{\lambda}{2})\sin\pi(2\mathbf{a}-m+\frac{\lambda}{2})}{\sin(2\pi\mathbf{a})\sin\pi(2\mathbf{a}+\lambda)} + \frac{\sin\pi(2\mathbf{a}+m-\frac{\lambda}{2})\sin\pi(2\mathbf{a}-m-\frac{\lambda}{2})}{\sin(2\pi\mathbf{a})\sin\pi(2\mathbf{a}-\lambda)}\,. \tag{2.102}$$

2.5 Gauge theory results

For a Wilson operator in an arbitrary representation R, the on-shell action vanishes. The only saddle point in the path integral is the trivial one, and the one-loop determinant is 1 due to Bose-Fermi cancellations. Thus the expectation value is given by evaluating the holonomy (2.4) in the background:

$$\langle W_R \rangle = \text{Tr}_R \exp\left[2\pi i R\left(A_\tau^{(\infty)} + i\Phi_0^{(\infty)}\right)\right] = \text{Tr}_R e^{2\pi i \mathbf{a}}\,, \tag{2.103}$$

where $\mathbf{a}$ was defined in (2.13).

For the 't Hooft operator, we combine the classical, one-loop, and monopole screening contributions from the previous sections:

$$\langle T_B \rangle = \sum_v e^{2\pi i v\cdot\mathbf{b}} Z_{\text{1-loop}}(v) Z_{\text{mono}}(B, v)\,. \tag{2.104}$$

2.5.1 $SU(2)$ $\mathcal{N} = 2^*$

For $SU(2)$, it is convenient to substitute

$$\mathbf{a} \to \begin{pmatrix} \mathbf{a} & \\ & -\mathbf{a} \end{pmatrix}, \qquad \mathbf{b} \to \begin{pmatrix} \mathbf{b} & \\ & -\mathbf{b} \end{pmatrix} \tag{2.105}$$

with the understanding that in the following the symbols $\mathbf{a}$ and $\mathbf{b}$ are complex numbers rather than matrices. For this gauge group we can label the loop operators $L_{p,q}$ by a pair of integers (p, q), where p and q are magnetic and electric charges respectively [37, 38, 23], and they are related to the coweight and the highest weight of the representation as

$$\begin{aligned} &B = (p/2, -p/2) \equiv \text{diag}(p/2, -p/2) \in \Lambda_{cw}\,, \\ &(q/2, -q/2) \equiv \text{diag}(q/2, -q/2) \in \Lambda_w \quad \leftrightarrow \quad \text{spin } q/2 \text{ representation}\,. \end{aligned} \tag{2.106}$$

The most basic Wilson operator $W_{1/2} = L_{0,1}$ corresponding to spin 1/2 has an expectation value

$$\langle W_{1/2}\rangle = \langle L_{0,1}\rangle = e^{2\pi i\mathbf{a}} + e^{-2\pi i\mathbf{a}}\,. \tag{2.107}$$

For the minimal 't Hooft operator $T_{1/2} = L_{1,0}$ that is S-dual to $W_{1/2}$, we find

$$\langle T_{1/2}\rangle = \langle L_{1,0}\rangle = (e^{2\pi i\mathbf{b}} + e^{-2\pi i\mathbf{b}})\left(\frac{\sin(2\pi\mathbf{a}+\pi m)\sin(2\pi\mathbf{a}-\pi m)}{\sin\left(2\pi\mathbf{a}+\frac{\pi}{2}\lambda\right)\sin\left(2\pi\mathbf{a}-\frac{\pi}{2}\lambda\right)}\right)^{1/2}\,. \tag{2.108}$$

Monopole screening does not occur in this case and it only includes $e^{2\pi i v\cdot\mathbf{b}}Z_{\text{1-loop}}(v)|_{v=(1/2,-1/2)}$ and $e^{2\pi i v\cdot\mathbf{b}}Z_{\text{1-loop}}(v)|_{v=(-1/2,1/2)}$.

For the minimal dyonic loops $L_{1,\pm1}$,

$$\langle L_{1,\pm1}\rangle = (e^{2\pi i(\mathbf{b}\pm\mathbf{a})} + e^{-2\pi i(\mathbf{b}\pm\mathbf{a})})\left(\frac{\sin(2\pi\mathbf{a}+\pi m)\sin(2\pi\mathbf{a}-\pi m)}{\sin\left(2\pi\mathbf{a}+\frac{\pi}{2}\lambda\right)\sin\left(2\pi\mathbf{a}-\frac{\pi}{2}\lambda\right)}\right)^{1/2}\,. \tag{2.109}$$

The simplest example with monopole screening contribution is given by

$$\begin{aligned}\langle L_{2,0}\rangle =&(e^{4\pi i\mathbf{b}} + e^{-4\pi i\mathbf{b}})\frac{\prod_{s_1,s_2=\pm1}\sin^{1/2}\left(2\pi\mathbf{a}+s_1\pi m+s_2\frac{\pi}{2}\lambda\right)}{\sin^{1/2}(2\pi\mathbf{a}+\pi\lambda)\sin^{1/2}(2\pi\mathbf{a}-\pi\lambda)\sin(2\pi\mathbf{a})}\\ &+\sum_{s=\pm}\frac{\prod_{\pm}\sin\pi(2\mathbf{a}\pm m+s\lambda/2)}{\sin(2\pi\mathbf{a})\sin\pi(2\mathbf{a}+s\lambda)}\,.\end{aligned} \tag{2.110}$$

The first line includes $e^{2\pi i v\cdot b}Z_{\text{1-loop}}(v)|_{v=(1,-1)}$ and $e^{2\pi i v\cdot\mathbf{b}}Z_{\text{1-loop}}(v)|_{v=(-1,1)}$. The second line corresponds to $e^{2\pi i v\cdot\mathbf{b}}Z_{\text{1-loop}}(v)Z_{\text{mono}}(B,v)|_{v=(0,0)}$. $Z_{\text{1-loop}}(v)$ is obtained from (2.87) and $Z_{\text{mono}}(B,v)$ in this case is (2.102) itself. We observe that this is the Moyal product of the minimal 't Hooft operator vev with itself,

$$\langle L_{2,0}\rangle = \langle L_{1,0}\rangle * \langle L_{1,0}\rangle\,. \tag{2.111}$$

In the $SU(2)$ case $*$ is defined by

$$(f*g)(\mathbf{a},\mathbf{b}) \equiv e^{i\frac{\lambda}{8\pi}(\partial_{\mathbf{b}}\partial_{\mathbf{a}'}-\partial_{\mathbf{a}}\partial_{\mathbf{b}'})}f(\mathbf{a},\mathbf{b})g(\mathbf{a}',\mathbf{b}')|_{\mathbf{a}'=\mathbf{a},\mathbf{b}'=\mathbf{b}} \tag{2.112}$$

with a different coefficient due to the factor of 2 in the inner product

$$\mathbf{a}\cdot\mathbf{b} \to \mathrm{Tr}[\mathrm{diag}(\mathbf{a},-\mathbf{a})\cdot\mathrm{diag}(\mathbf{b},-\mathbf{b})] = 2\mathbf{a}\mathbf{b}\,. \tag{2.113}$$

In Section 2.6, we will explain how the Moyal product appears from the structure of the path integral.

The precise choice of signs and relative numerical normalizations among terms is difficult to fix purely in gauge theory without additional assumptions. In the examples considered in this thesis we choose to be pragmatic and make the choice by assuming physically reasonable structures such as Moyal multiplication, correspondence with the Verlinde operators, as well as agreement with classical $SL(2,\mathbb{C})$ holonomies in the $\lambda \to 0$ limit.

2.5.2 $U(N)$ $\mathcal{N}=2^*$

For the gauge group $U(N)$, the minimal 't Hooft operators, with charges[13] $B=(\pm 1, 0^{N-1})$ (the power indicates the number of repeated entries) corresponding to the fundamental and anti-fundamental representations of the Langlands dual group, have the expectation values

$$\left\langle T_{B=(\pm 1,0^{N-1})}\right\rangle = \sum_{l=1}^{N} e^{\pm 2\pi i \mathbf{b}_l} \left(\prod_{\pm} \prod_{j\neq l} \frac{\sin\pi(\mathbf{a}_l - \mathbf{a}_j \pm m)}{\sin\pi(\mathbf{a}_l - \mathbf{a}_j \pm \lambda/2)} \right)^{1/2} . \tag{2.114}$$

For the magnetic charge $B=(1,-1,0^{N-2})$, corresponding to the adjoint representation,

$$\begin{aligned} &\left\langle T_{B=(1,-1,0^{N-2})}\right\rangle \\ &= \sum_{k\neq l} e^{2\pi i(\mathbf{b}_k - \mathbf{b}_l)} \left[\frac{\left[\prod_{\pm,\pm} \sin\pi\left(\mathbf{a}_{kl} \pm m \pm \lambda/2\right)\right] \left[\prod_{\pm}\prod_{j\neq k,l} \sin\pi\left(\mathbf{a}_{kj} \pm m\right) \sin\pi\left(\mathbf{a}_{lj} \pm m\right)\right]}{\sin^2\pi\mathbf{a}_{kl} \prod_{\pm} \sin\pi\left(\mathbf{a}_{kl} \pm \lambda\right) \left[\prod_{\pm}\prod_{j\neq k,l} \sin\pi\left(\mathbf{a}_{kj} \pm \lambda/2\right) \sin\pi\left(\mathbf{a}_{lj} \pm \lambda/2\right)\right]} \right]^{1/2} \\ &\qquad + \sum_{l=1}^{N} \prod_{j\neq l} \frac{\prod_{\pm} \sin\pi(\mathbf{a}_{lj} \pm m + \lambda/2)}{\sin\pi\mathbf{a}_{lj} \sin\pi(\mathbf{a}_{lj} + \lambda)} . \end{aligned} \tag{2.115}$$

From (2.114) and (2.115) we find that

$$\left\langle T_{B=(1,-1,0^{N-2})}\right\rangle = \left\langle T_{B=(-1,0^{N-1})}\right\rangle * \left\langle T_{B=(1,0^{N-1})}\right\rangle . \tag{2.116}$$

For $B=(2,0^{N-1})$,

$$\begin{aligned} \left\langle T_{B=(2,0^{N-1})}\right\rangle = \sum_{k=1}^{N} e^{4\pi i \mathbf{b}_k} \left(\prod_{j\neq k} \frac{\prod_{\pm,\pm} \sin\pi(\mathbf{a}_{kj} \pm m \pm \lambda/2)}{\left[\sin^2\pi\mathbf{a}_{kj} \prod_{\pm} \sin\pi(\mathbf{a}_{kj} \pm \lambda)\right]} \right)^{1/2} \\ + \sum_{k\neq l} \frac{\prod_{\pm} \sin\pi(\mathbf{a}_{kl} \pm m + \lambda/2)}{\sin\pi(\mathbf{a}_{kl} + \lambda) \sin\pi\mathbf{a}_{kl}} . \end{aligned} \tag{2.117}$$

For this we find

$$\left\langle T_{B=(2,0^{N-1})}\right\rangle = \left\langle T_{B=(1,0^{N-1})}\right\rangle * \left\langle T_{B=(1,0^{N-1})}\right\rangle . \tag{2.118}$$

Results for the gauge group $SU(N)$ can be obtained by taking $\mathbf{a}$ and $\mathbf{b}$ traceless.

[13] The Cartan subalgebra of $U(N)$ is spanned by real diagonal matrices. For $SU(N)$ they must be traceless. We often drop "diag" in $a = \mathrm{diag}(a_1, \ldots, a_N)$ to simplify notation. The inner product is defined by the trace $a \cdot a' = \mathrm{Tr}\, aa'$, and this is used to identify the Cartan algebra with its dual.

2.5.3 $U(2)$ $N_f = 4$

The minimal 't Hooft operator in this theory is the one with $B = (1, -1)$. Its vev is given as

$$\begin{aligned}\langle T_B \rangle =& e^{2\pi i \mathbf{b}_{12}} \left(\frac{\prod_{f=1}^{4} \sin\pi(\mathbf{a}_1 - m_f) \sin\pi(\mathbf{a}_2 - m_f)}{\sin^2 \pi\mathbf{a}_{12} \prod_{\pm} \sin\pi(\mathbf{a}_{12} \pm \lambda)} \right)^{1/2} \\ &+ e^{-2\pi i \mathbf{b}_{12}} \left(\frac{\prod_{f=1}^{4} \sin\pi(\mathbf{a}_1 + m_f) \sin\pi(\mathbf{a}_2 + m_f)}{\sin^2 \pi\mathbf{a}_{12} \prod_{\pm} \sin\pi(\mathbf{a}_{12} \pm \lambda)} \right)^{1/2} \\ &+ \frac{\prod_{f=1}^{4} \sin\pi\left(\mathbf{a}_1 - m_f + \frac{\lambda}{2}\right)}{\sin\pi\mathbf{a}_{12} \sin\pi\left(-\mathbf{a}_{12} - \lambda\right)} + \frac{\prod_{f=1}^{4} \sin\pi\left(\mathbf{a}_2 - m_f + \frac{\lambda}{2}\right)}{\sin\pi\mathbf{a}_{21} \sin\pi\left(-\mathbf{a}_{21} - \lambda\right)} .\end{aligned} \tag{2.119}$$

We have defined $\mathbf{a}_{jk} = \mathbf{a}_j - \mathbf{a}_k$. The last line includes the contribution from the screening monopoles.

2.5.4 $U(N)$ $N_f = 2N$

For the minimal 't Hooft operator given by the magnetic charge $B = \text{diag}(1, -1, 0^{N-2})$ corresponding to the adjoint representation, we obtain

$$\begin{aligned}&\langle T_B \rangle \\ &= \sum_{\substack{1\le k,l \le N \\ k \neq l}} e^{2\pi i(\mathbf{b}_k - \mathbf{b}_l)} \frac{\left[\prod_{f=1}^{N} \sin\pi(\mathbf{a}_k - m_f) \sin\pi(\mathbf{a}_l - m_f)\right]^{1/2}}{\sin\pi\mathbf{a}_{kl} \prod_{\pm} \left[\sin\pi(\mathbf{a}_{kl} \pm \lambda) \prod_{j \neq k,l} \sin\pi(\mathbf{a}_{kj} \pm \lambda/2) \sin\pi(\mathbf{a}_{jl} \pm \lambda/2) \right]^{1/2}} \\ &\quad + \sum_{l=1}^{N} \frac{\prod_{f=1}^{2N} \sin\pi\left(\mathbf{a}_l - m_f + \frac{\lambda}{2}\right)}{\prod_{j \neq l} \sin\pi\mathbf{a}_{lj} \sin\pi\left(-\mathbf{a}_{lj} - \lambda\right)} .\end{aligned} \tag{2.120}$$

We have introduced the notation $\mathbf{a}_{jk} := \mathbf{a}_j - \mathbf{a}_k$.

We emphasize that (2.120) and (2.119) are the vev of the 't Hooft operator in the $U(N)$ and $U(2)$ theories, not in the $SU(N)$ and $SU(2)$ theories. We will compare (2.120) and (2.119) with the Verlinde operators in Toda and Liouville theories in Section 2.7 that we will propose to be related to the loop operators in the $SU(N)$ and $SU(2)$ theories. While we do not have a computational method intrinsic to $SU(N)$, we will see that (2.120) and (2.119), when a is restricted to be traceless, do reproduce a-dependent terms in the CFT results.

2.5.5 Relation to the vev on S^4

As written in Section 1.3, the expression of the vev of the 't Hooft loop on S^4 has the following form

$$\langle T_B \rangle_{S^4} = \int da \sum_v Z_{\text{south}}(a + v/2, \bar{q})\, Z_{\text{equator}}(a, m; B, v)\, Z_{\text{north}}(a - v/2, q)\,. \tag{2.121}$$

On the other hand, the expression of the vev of the 't Hooft loop on $S^1 \times \mathbb{R}^3$ has the following form

$$\langle T_B \rangle_{S^1 \times \mathbb{R}^3} = \sum_v e^{2\pi i v \cdot \mathbf{b}}\, Z_L(\mathbf{a}, m, \lambda; B, v)\,. \tag{2.122}$$

If we set the twisting parameter $\lambda = 1$ on $S^1 \times \mathbb{R}^3$, there is a following relation

$$Z_{\text{equator}}(\mathbf{a}, m; B, v) = Z_L(\mathbf{a}, m + \frac{1}{2}, \lambda = 1; B, v)\,. \tag{2.123}$$

It can be understood by using the following intuition. If we look at the place where the loop operator is inserted in the equator of S^4, the geometry around there is locally the same as $S^1 \times \mathbb{R}^3$. The S^1 in the latter corresponds to the S^1 where the loop operator exists in S^4. Therefore the vev on $S^1 \times \mathbb{R}^3$ can reproduce only the contribution from the equator to the vev on S^4. There are no poles on $S^1 \times \mathbb{R}^3$ so it cannot reproduce the contribution Z_{north} and Z_{south} from the north and south poles on S^4 to the vev of the loop on S^4.

2.6 Noncommutative algebra and quantization

By using the structure of the path integral we have found, in this section we show that the vevs of the loop operators on $S^1 \times \mathbb{R}^3$, inserted on the 3-axis ($x^1 = x^2 = 0$), form a non-commutative algebra, when the axis is considered as time and the operators are time-ordered. We will begin with the $U(1)$ case and then discuss the general gauge group.

2.6.1 Maxwell theory

Let us explain how non-commutativity arises in the algebra of Wilson-'t Hooft operators in Maxwell theory on $S^1 \times \mathbb{R}^3$ upon twisting by a spatial rotation along the S^1.

We begin with an intuitive explanation based on classical fields [26]. By taking S^1 as time, the expectation value of the product of Wilson (W) and 't Hooft (T) operators can be thought of as the trace

$$\langle W \cdot T \rangle = \text{Tr}_{\mathcal{H}(W \cdot T)} (-1)^F e^{-2\pi R H} e^{2\pi i \lambda J_3} \tag{2.124}$$

taken in the Hilbert space $\mathcal{H}(W \cdot T)$ defined by the loop operators. The space $\mathcal{H}(W \cdot T)$ differs from the simple product $\mathcal{H}(W) \otimes \mathcal{H}(T)$ because when both W and T are present, their electric and magnetic fields produce the Poynting vector $\vec{E}\times\vec{B}$ that carries a non-zero angular momentum. The orientation of the Poyinting vector, and therefore the phase $e^{2\pi i\lambda J_3}$, depends on the relative positions of the operators on the 3-axis.

Next we present an approach suitable for localization. For simplicity let us turn off the theta angle. The loop operator $L_{p,q}$ with magnetic and electric charges (p,q) at the origin $\vec{x}=0$ is defined by the path integral over the fluctuations around the singular background

$$A = A_\tau^{(\infty)} d\tau + p\frac{\cos\theta}{2} d\varphi \tag{2.125}$$

with the insertion of the holonomy

$$e^{-iq\oint_{S^1} A}\,. \tag{2.126}$$

We note here that the expression for the monopole field in (2.125) has Dirac strings in two directions ($\theta=0,\pi$). The expectation value $\langle L_{p,q}\rangle$ is a function of (a,b), which are normalized electric and magnetic background Wilson lines

$$\mathbf{a} := RA_\tau^{(\infty)}\,, \qquad \mathbf{b} := \frac{\Theta}{2\pi}\,. \tag{2.127}$$

We claim that the path integral yields the expectation value

$$\langle L_{p,q}\rangle = e^{-2\pi i(q\mathbf{a}+p\mathbf{b})}\,. \tag{2.128}$$

The magnetic part is essentially the definition of the magnetic Wilson line Θ, which is defined as the chemical potential for the magnetic charge at infinity. The electric part arises because the holonomy (2.126) is evaluated against the background Wilson line.

Let us introduce a twist along the S^1. If we think of the circle as the time direction, we can write

$$\langle L_{p,q}\rangle = \mathrm{Tr}_{\mathcal{H}(L_{p,q})}(-1)^F e^{-2\pi RH} e^{2\pi i\lambda J_3}\,, \tag{2.129}$$

where J_3 is the Cartan generator of the spatial rotation group $SU(2)$. The twist by J_3 means that we rotate the system by angle $2\pi\lambda$ as we go along S^1, i.e., we introduce the identification

$$(\tau + 2\pi R, \varphi) \sim (\tau, \varphi + 2\pi\lambda)\,. \tag{2.130}$$

In terms of the new coordinates $(\tau',\varphi') = (\tau, \varphi + \frac{\lambda}{R}\tau)$, the identification is simply

$$(\tau' + 2\pi R, \varphi') \sim (\tau', \varphi')\,. \tag{2.131}$$

The components of the gauge field are related as

$$A_{\tau'} = A_\tau - \frac{\lambda}{R} A_\varphi\,, \quad A_{\varphi'} = A_\varphi\,. \tag{2.132}$$

Note that A_φ represents a holonomy around the Dirac strings. In our choice of local trivialization $A_\varphi(\theta = \pi/2) = 0$, so we have a simple relation

$$\oint_{S^1} A = 2\pi \mathbf{a} \quad \text{at} \quad \theta = \pi/2. \tag{2.133}$$

Thus the monopole field does not contribute to the holonomy as claimed above, in fact even after twisting. The holonomies at $\theta \neq \pi/2$ are, however, shifted from a. Indeed we find

$$\oint_{S^1} A = \int_0^{2\pi R} d\tau' A_{\tau'} = 2\pi \left(\mathbf{a} \mp \frac{p}{2}\lambda\right) \quad \text{at} \quad \theta = \left\{ \begin{array}{l} 0\,, \\ \pi\,. \end{array} \right. \tag{2.134}$$

One can picture the shift as arising from the holonomy winding around the Dirac strings. Then for the product of Wilson and 't Hooft operators $W \equiv L_{0,1}, T \equiv L_{1,0}$

$$W(\vec{x} = (0,0,z)) \cdot T(\vec{x} = 0)\,, \tag{2.135}$$

its expectation value is given by

$$\langle W(\vec{x} = (0,0,z)) \cdot T(\vec{x} = 0)\rangle = e^{-2\pi i(\mathbf{a}\mp\frac{1}{2}\lambda)} e^{-2\pi i \mathbf{b}} \text{ for } \left\{ \begin{array}{l} z > 0\,, \\ z < 0\,. \end{array} \right. \tag{2.136}$$

The Wilson loop operator (2.126) for $z > 0$ is evaluated at $\theta = 0$, and for $z < 0$ at $\theta = \pi$. The difference λ between the shifts in a at $z > 0$ and $z < 0$ is independent of the choice of local trivialization. We can also see that the expectation value of the product of operators is given by the Moyal product of the expectation values:

$$\langle W(z) \cdot T(0)\rangle = \left\{ \begin{array}{ll} \langle W\rangle * \langle T\rangle & \text{for } z > 0\,, \\ \langle T\rangle * \langle W\rangle & \text{for } z < 0\,, \end{array} \right. \tag{2.137}$$

where the Moyal product $*$ is defined by

$$(f * g)(\mathbf{a}, \mathbf{b}) := \lim_{\mathbf{a}'\to\mathbf{a},\, \mathbf{b}'\to\mathbf{b}} e^{i\frac{\lambda}{4\pi}(\partial_{\mathbf{b}}\partial_{\mathbf{a}'} - \partial_{\mathbf{a}}\partial_{\mathbf{b}'})} f(\mathbf{a}, \mathbf{b}) g(\mathbf{a}', \mathbf{b}')\,. \tag{2.138}$$

This is the special case of the more general result for an arbitrary gauge group that we now turn to.

2.6.2 Non-Abelian gauge theories

Here we consider a general $\mathcal{N} = 2$ gauge theory with arbitrary matter content. Let us suppose that we have multiple loop operators $L_i \equiv L_{B_i,R_i}(\vec{x} = (0,0,z_i))$ located at various points $\vec{x} = (0,0,z_i)$ on the 3-axis, ordered so that

$$z_1 > z_2 > \ldots > z_n\,. \tag{2.139}$$

In the localization calculation, it suffices to consider the Abelian configurations with magnetic charges v_i associated with B_i as only these contribute to the path integral. As is clear from the Maxwell case, the holonomy at z_i around S^1 is shifted by the magnetic fields $\propto v_j$ created by L_j for $j \neq i$:

$$\mathbf{a} \to \mathbf{a} + \frac{\lambda}{2}\left(\sum_{j<i} v_j - \sum_{j>i} v_j\right) . \tag{2.140}$$

Let us assume that the individual operator vevs are given by

$$\langle L \rangle = \sum_{v,w} Z_{L,\text{total}}(\mathbf{a}, \mathbf{b}; v, w) \equiv \sum_{v,w} e^{2\pi i(w\cdot\mathbf{a}+v\cdot\mathbf{b})} Z_L(\mathbf{a}, m_f, \lambda; v, w) \tag{2.141}$$

for some functions $Z_L(\mathbf{a}, m_f, \lambda; v, w)$. Then localization calculation yields

$$\langle L_1 \cdot L_2 \cdot \ldots \cdot L_n \rangle = \prod_{i=1}^{n} \sum_{w_i} \sum_{v_i} Z_{L_i,\text{total}}\left(\mathbf{a} + \frac{\lambda}{2}\left(\sum_{j<i} v_j - \sum_{j>i} v_j\right), \mathbf{b}\,; v_i, w_i\right) , \tag{2.142}$$

One can easily see that (2.142) is the Moyal product of the expectation values of individual operators

$$\langle L_1 \cdot L_2 \cdot \ldots \cdot L_n \rangle = \langle L_1 \rangle * \langle L_2 \rangle * \ldots * \langle L_n \rangle , \tag{2.143}$$

where $*$ is defined by

$$(f * g)(\mathbf{a}, \mathbf{b}) := e^{i\frac{\lambda}{4\pi}(\partial_{\mathbf{b}}\cdot\partial_{\mathbf{a}'} - \partial_{\mathbf{a}}\cdot\partial_{\mathbf{b}'})} f(\mathbf{a}, \mathbf{b}) g(\mathbf{a}', \mathbf{b}')\Big|_{\mathbf{a}'=\mathbf{a},\mathbf{b}'=\mathbf{b}} \tag{2.144}$$

with the natural product $\cdot$ between the derivatives inside the exponential.

As a concrete example, let us consider $SU(2)$ $\mathcal{N} = 2^*$ theory. We computed the vev of the charge-two 't Hooft operator in (2.110). As explained in [10], this operator corresponds to the product of two minimal 't Hooft operators. This is because the resolution of the singular moduli space corresponds to separating the charge-two 't Hooft operator into two minimal ones [34]. Indeed one can check that the expression (2.110) is precisely the Moyal product of (2.108) with itself.

2.6.3 Deformation quantization of the Hitchin moduli space

We are now going to explain that the noncommutative algebra structure given by the Moyal multiplication above realizes a deformation quantization of the Hitchin moduli space associated with the gauge theory.

In [12], a correspondence between certain $\mathcal{N} = 2$ gauge theories and punctured Riemann surfaces C was discovered. The correspondence is a main ingredient of the relation [5] between gauge theories and two-dimensional conformal field theories. The correspondence is also manifested in the relation between the gauge theories and the Hitchin systems on the

Riemann surfaces. This made it possible to study the integrable structure [39, 40, 41] as well as the low-energy dynamics of these theories using the Hitchin system on the Riemann surfaces [42], generalizing [43].

Let $A = A_z dz + A_{\bar{z}} d\bar{z}$ be a connection of a G-bundle over C, and $\varphi = \varphi_z dz + \overline{\varphi}_{\bar{z}} d\bar{z}$ an adjoint-valued 1-form. They are assumed to possess prescribed singularities at the punctures. The Hitchin moduli space is the space of solutions to

$$\begin{aligned} F_{z\bar{z}} &= [\varphi_z, \overline{\varphi}_{\bar{z}}] \,, \\ D_{\bar{z}}\varphi_z = 0 \,, &\quad D_z \overline{\varphi}_{\bar{z}} = 0 \,, \end{aligned} \tag{2.145}$$

up to G-gauge transformations. The Hitchin moduli space is hyperKähler , and therefore has a $\mathbb{CP}^1$ of complex structures $\mathcal{J}$, each being a linear combination of three complex structures $\mathcal{J} = I, J$, and K. Each complex structure $\mathcal{J}$ is associated with a real symplectic form $\omega_{\mathcal{J}} := g\mathcal{J}$, as well as a holomorphic symplectic form $\Omega_{\mathcal{J}}$. For $\mathcal{J} = I, J, K$, these are given by $\Omega_I = \omega_J + i\omega_K, \Omega_J = \omega_K + i\omega_I, \Omega_K = \omega_I + i\omega_J$.

In the original assignment of I, J, K by Hitchin [44], we are particularly interested in the complex structure J. The combination $\mathcal{A} \equiv A + i\varphi$ is then holomorphic, and (2.145) implies that[14] $\mathcal{A}$ is a flat $G_{\mathbb{C}}$ connection. In terms of $\mathcal{A}$, Ω_J is given by

$$\Omega_J \propto \int_C \mathrm{Tr}\, \delta\mathcal{A} \wedge \delta\mathcal{A} \,. \tag{2.146}$$

The $U(1)$ R-symmetry rotates the phases of $\varphi_z, \overline{\varphi}_{\bar{z}}$, and $\Phi_0 + i\Phi_9$, and Ω_J transforms accordingly [42].

We focus on the one-punctured torus, which corresponds to $SU(2)$ $\mathcal{N} = 2^*$ theory. Let us define generators of the first homology so that the holonomy matrices (A, B, M) along them satisfy the relation

$$AB = MBA \,. \tag{2.147}$$

Here M is the holonomy around a small circle surrounding the puncture, and A and B are the holonomy matrices for the usual A- and B-cycles. Dehn's theorem [45, 46] allows us to label the non-self-intersecting closed curves by two integers (p, q) with equivalence $(p, q) \sim (-p, -q)$. They can be naturally identified with the charges of loop operators in (2.106) [23]. In particular, we have the correspondence

$$\langle L_{0,1} \rangle \quad \leftrightarrow \quad \mathrm{Tr} A \,, \tag{2.148}$$

$$\langle L_{1,0} \rangle \quad \leftrightarrow \quad \mathrm{Tr} B \,, \tag{2.149}$$

$$\langle L_{1,\pm 1} \rangle \quad \leftrightarrow \quad \mathrm{Tr} A^{\pm 1} B \,. \tag{2.150}$$

[14] More precisely, the first of (2.145) combined with the difference of the second and the third is equivalent to the flatness of $\mathcal{A}$. The J-holomorphic structure of the Hitchin moduli space can be described by dropping the sum of the second and the third equations, and by taking the quotient with respect to $G_{\mathbb{C}}$ gauge transformations.

Let us consider the case $\lambda = 0$. From (2.107-2.109) we find that

$$\langle L_{0,1}\rangle_{\lambda=0} = e^{2\pi i\mathbf{a}} + e^{-2\pi i\mathbf{a}}, \tag{2.151}$$

$$\langle L_{1,0}\rangle_{\lambda=0} = (e^{2\pi i\mathbf{b}} + e^{-2\pi i\mathbf{b}})\left(\frac{\sin(2\pi\mathbf{a}+\pi m)\sin(2\pi\mathbf{a}-\pi m)}{\sin^2(2\pi\mathbf{a})}\right)^{1/2}, \tag{2.152}$$

$$\langle L_{1,\pm 1}\rangle_{\lambda=0} = (e^{2\pi i(\mathbf{b}\pm\mathbf{a})} + e^{-2\pi i(\mathbf{b}\pm\mathbf{a})})\left(\frac{\sin(2\pi\mathbf{a}+\pi m)\sin(2\pi\mathbf{a}-\pi m)}{\sin^2(2\pi\mathbf{a})}\right)^{1/2}. \tag{2.153}$$

Replacing the arrows in (2.150) by equalities, these expressions were exactly given as the definition of the Darboux coordinates[15] $(\mathbf{a},\mathbf{b})$ on the Hitchin moduli space with respect to the symplectic structure Ω_J! Later in [48], $(\mathbf{a},\mathbf{b})$ were identified with the complexification of the Fenchel-Nielsen coordinates of Teichmüller space. Here we see that both the coordinates $(\mathbf{a},\mathbf{b})$ and the symplectic structure Ω_J arise naturally from the gauge theory on $S^1\times\mathbb{R}^3$.

For $SU(2)$ $N_f = 4$ theory, our gauge theory calculation of the 't Hooft and dyonic operator vevs is not complete due to the difficulty with monopole screening contributions. The relation with Liouville theory and the formula (2.195) below suggests, however, that $(\mathbf{a},\mathbf{b})$ are the complexified Fenchel-Nielsen coordinates on the Hitchin moduli space associated with the four-punctured sphere [49].

2.7 Gauge theory results and Liouville/Toda theories

In this subsection we propose a precise relation between the loop operator vevs on $S^1\times\mathbb{R}^3$ and the corresponding difference operators that act on the conformal blocks of Liouville and Toda field theories.

Let us consider the Liouville theory on a genus g Riemann surface with n punctures $C_{g,n}$. In this thesis we only consider the two cases $C_{1,1}$ and $C_{0,4}$. The correlation function of primary fields V_{α_e} $(e = 1,\ldots,n)$ with momenta α_e, inserted at the punctures $z = z_e$, takes the form

$$\left\langle \prod_e V_{\alpha_e}(z_e)\right\rangle_{C_{g,n}} = \int d\alpha\, \mathcal{C}_E(\alpha)|\mathcal{F}_E(\alpha)|^2, \tag{2.154}$$

where α is the internal momentum and E is the set of external momentum. In this subsection, we use the notation $E = \alpha_1$ in case of $C_{1,1}$ and $E = \{\alpha_4,\alpha_3,\alpha_2,\alpha_1\}$ in case of $C_{0,4}$. The function $\mathcal{C}_E(\alpha)$ is a product of three-point functions. The conformal block $\mathcal{F}_E(\alpha)$ depends on α, E, and the gluing parameters q holomorphically. The central charge c of Liouville theory is parametrized as

$$c = 1 + 6Q^2, \qquad Q = b + b^{-1}. \tag{2.155}$$

[15] In [47] the Darboux coordinates were denoted by (α,β), and are related to our $(\mathbf{a},\mathbf{b})$ by a trivial rescaling. We also have $\mathrm{Tr}M = 2\cos\pi m$.

The parameter b is related to the equivariant parameters $\varepsilon_1, \varepsilon_2$ of the Omega background as $b^2 = \varepsilon_1/\varepsilon_2$.

As written in Section 1.5, there exists a difference operator Λ_γ, the Verlinde operator, whose action on $\mathcal{F}_E(\alpha)$ we denote by

$$\mathcal{F}_E(\alpha) \to [\Lambda_\gamma \cdot \mathcal{F}_E](\alpha) = \sum_v H_{v\,;\,E}(\alpha)\mathcal{F}_E(\alpha - vb)\,. \tag{2.156}$$

Its vev is defined as

$$\int d\alpha\, \mathcal{C}(\alpha\,;E)\, \overline{\mathcal{F}_E(\alpha)}\, [\Lambda_\gamma \cdot \mathcal{F}_E](\alpha)\,. \tag{2.157}$$

We change the normalization of the conformal block and define

$$\mathcal{B}_E(\alpha) \equiv \mathcal{C}(\alpha\,;E)^{1/2}\mathcal{F}_E(\alpha) \tag{2.158}$$

using the square root of the function $\mathcal{C}(\alpha\,;E)$ that appears in the correlation function (2.154). In this normalization, the Liouville correlation function is simply given by

$$\Big\langle \prod V_{\alpha_e} \Big\rangle_{C_{g,n}} = \int d\alpha\, |\mathcal{B}_E(\alpha)|^2\,, \tag{2.159}$$

where we used the fact that in the physical range of Liouville momenta, the function $\mathcal{C}(\alpha\,;E)$ is real. The Verlinde operator acts on $\mathcal{B}_E(\alpha)$ as the difference operator defined by

$$[\mathcal{L}_\gamma \cdot \mathcal{B}_E](\alpha) \equiv \mathcal{C}(\alpha\,;E)^{1/2}[\Lambda_\gamma \cdot \mathcal{F}_E](\alpha)\,. \tag{2.160}$$

Its vev is then given by

$$\int d\alpha\; \overline{\mathcal{B}_E(\alpha)}\, [\mathcal{L}_\gamma \cdot \mathcal{B}_E](\alpha)\,. \tag{2.161}$$

To see how it is related to the vevs of the loop operators on $S^1 \times \mathbb{R}^3$, we shift the dummy variables α_i to $\alpha_i + \frac{1}{2}v\mathsf{b}$ and define Z_v as follows

$$(2.161) = \int d\alpha \overline{\mathcal{B}}_E(\alpha + \frac{1}{2}vb) \sum_v Z_v(\alpha\,;E)\, \mathcal{B}_E(\alpha - \frac{1}{2}vb)\,, \tag{2.162}$$

where the function $\overline{\mathcal{B}}_E$ is defined so that $\overline{\mathcal{B}}_E(\alpha) = \overline{\mathcal{B}_E(\alpha)}$ and

$$Z_v(\alpha\,;E) := \left[\frac{\mathcal{C}(\alpha + \frac{1}{2}vb\,;E)}{\mathcal{C}(\alpha - \frac{1}{2}vb\,;E)}\right]^{\frac{1}{2}} H_{v\,;\,E}(\alpha + \frac{1}{2}vb)\,. \tag{2.163}$$

The operator algebra of $\mathcal{L}_\gamma$ is isomorphic to that of Λ_γ. $\mathcal{L}_\gamma$ can be written as

$$\mathcal{L}_\gamma = \sum_v e^{-(b/2)v\cdot\partial_\alpha} Z_v(\alpha\,,E)\, e^{-(b/2)v\cdot\partial_\alpha} \tag{2.164}$$

up to an overall constant. In the case γ is purely magnetic, we conjecture that the above Verlinde operator $\mathcal{L}_\gamma$ is related to the 't Hooft loop on $S^1 \times \mathbb{R}^3$ as

$$\langle L \rangle_{S^1\times\mathbb{R}^3} = \sum_v e^{2\pi i (v\cdot \mathbf{b})} Z_v(\mathbf{a}, m_f)\,, \tag{2.165}$$

if we identify parameters as written in the below subsubsections. Note that v and α in (2.164) are variables with one component while v and $\mathbf{b}$ in (2.165) are variables with two components, $v = (v, -v)$ and $\mathbf{b} = (\mathbf{b}, -\mathbf{b})$.

We have focused so far on the correspondence [5] between the gauge theories whose gauge group is $SU(2)$ and Liouville theory on the corresponding Riemann surface, but we also propose that the above relation should hold for more general gauge groups $SU(N)$ and A_{N-1} Toda theories [5, 6]. Some examples of Verlinde operators in Toda theories were computed as written in 1.5. Verlinde operators act on conformal blocks as follows

$$\mathcal{F}_E(\alpha) \to [\Lambda_\gamma \cdot \mathcal{F}_E](\alpha) = \sum_{\mathbf{k}} H_{\mathbf{k}\,;\,\mathbf{E}}(\alpha)\, \mathcal{F}_E(\alpha - b h_{\mathbf{k}})\,, \tag{2.166}$$

where $\mathbf{k} = 1, \ldots, N$ in case of the Verlinde operator (1.100) on $C_{1,1}$ and $\mathbf{k} = (l,k), (1 \leq l, k \leq N)$ in case of the Verlinde operator (1.104) on $C_{0,4}$. Here $h_{(l,k)} := h_{l,k} = h_l - h_k$ and the notation of h_k is reviewed in Section 1.4.4. We define the following quantities that are generalization of (2.163) to Toda theory

$$Z_{\mathbf{k}}(\alpha\,;E) := \left[\frac{\mathcal{C}(\alpha + \frac{1}{2} b h_{\mathbf{k}}\,;E)}{\mathcal{C}(\alpha - \frac{1}{2} b h_{\mathbf{k}}\,;E)}\right]^{\frac{1}{2}} H_{\mathbf{k};\mathbf{E}}(\alpha + \frac{1}{2} b h_{\mathbf{k}})\,. \tag{2.167}$$

$\mathcal{L}_\gamma$ can be written as

$$\mathcal{L}_\gamma = \sum_{\mathbf{k}} e^{-(b/2) h_{\mathbf{k}}\cdot\partial_\alpha} Z_{\mathbf{k}}(\alpha\,, E)\, e^{-(b/2) h_{\mathbf{k}}\cdot\partial_\alpha} \tag{2.168}$$

We conjecture that the Verlinde operators in Toda theories are precisely related to the loop operator vevs on $S^1 \times \mathbb{R}^3$ as

$$\langle L \rangle_{S^1\times\mathbb{R}^3} = \sum_{\mathbf{k}} e^{2\pi i (\mathbf{b}\cdot h_{\mathbf{k}})} Z_{\mathbf{k}}(\mathbf{a}, \{m_f\}_f)\,. \tag{2.169}$$

As seen in Section 1.5.1, $Z_{\mathbf{k}}$ in (2.167) for $b = 1$ corresponds to the contribution $Z_{\text{1-loop,equator}}$ from the equator to the vev of the 't Hooft loop operator on S^4. As written in Section 2.5.5, the vevs of the 't Hooft loop operators on $S^1 \times \mathbb{R}^3$ for $\lambda = 1$ are related to the equator contributions $Z_{\text{1-loop,equator}}$ to the vevs on S^4.

The above conjecture is consitent with these facts. The new aspect in the conjecture is that this relation between $Z_{\mathbf{k}}$ and the vevs on $S^1 \times \mathbb{R}^3$ holds not only for $b = \lambda = 1$ but also for general values of $b^2 = \lambda$.

This relation is also described in terms of the Weyl transform (ordering) [16]

$$\langle L\rangle_{S^1\times\mathbb{R}^3} \quad \overset{\text{Weyl}}{\Longrightarrow} \quad \mathcal{L}\,. \tag{2.170}$$

The parameter $\mathbf{b}$ plays the role of the canonical momentum:

$$\begin{aligned} \mathbf{b} &\leftrightarrow i\frac{\lambda}{2\pi}\frac{\partial}{\partial\mathbf{a}} && \text{in general,} \\ \mathbf{b}_i &\leftrightarrow i\frac{\lambda}{4\pi}\frac{\partial}{\partial\mathbf{a}_i} && \text{for } SU(2) \text{ and Liouville,} \end{aligned} \tag{2.171}$$

where $\mathbf{a}$ on the Toda theory side appears in the expression of α so that $\partial/(\partial\alpha) = -b\,\partial/(\partial a)$ as we will see in the following subsubsections. Thus our proposal implies that the Verlinde operators are the Weyl transform of the loop operator vevs on $S^1\times\mathbb{R}^3$, when the gauge theory has a Lagrangian description. It is very natural to conjecture that this relation should hold even when the gauge theory does not admit a Lagrangian description [51, 12]. Below we demonstrate our proposal with several examples.

2.7.1 $SU(2)$ $\mathcal{N}=2^*$

This theory corresponds to the Liouville theory on the one-punctured torus as written in Section 1.4.3. Let $C(\alpha_1,\alpha_2,\alpha_3)$ be the three-point function of Liouville theory. We denote the internal and external Liouville momenta by α and α_1 respectively. The Verlinde loop operator that corresponds to the minimal 't Hooft operators acts on the conformal block as

$$[\mathcal{L}_\gamma\cdot\mathcal{F}_{\alpha_1}](\alpha) = \sum_{\pm} H_{\pm\,;\,\alpha_1}(\alpha)\mathcal{F}_{\alpha_1}\left(\alpha\pm b/2\right)\,. \tag{2.172}$$

$H_{\pm\,;\,\alpha_1}$ in the above expression corresponds to $H_{v=\mp\frac{1}{2}\,;\,E}$ in (2.156). Its vev is

$$\langle\,\mathcal{L}_\gamma\,\rangle = \int_{Q/2+i\mathbb{R}} d\alpha\, C(\alpha,\alpha_1,Q-\alpha)\sum_{\pm}\overline{\mathcal{F}_{\alpha_1}(\alpha)}H_{\pm\,;\,\alpha_1}(\alpha)\mathcal{F}_{\alpha_1}\left(\alpha\pm b/2\right)\,. \tag{2.173}$$

The map between the Liouville theory parameters α,α_1 and the gauge theory parameters $\mathbf{a}, m$ is given by

$$\alpha = \frac{Q}{2}+\frac{\mathsf{a}}{b},\quad \alpha_1 = \frac{Q}{2}+\frac{\mathsf{m}}{b}\,, \tag{2.174}$$

where $\mathsf{m} := m+1/2$. Substituting these parameters into (1.101), we get the coefficients $H_{\pm\,;\,\alpha_1}$ as

$$H_{\pm\,;\,\alpha_1}(\alpha) = \frac{\Gamma(\pm2\mathbf{a})\Gamma(\pm2\mathbf{a}+bQ)}{\Gamma(\pm2\mathbf{a}+\mathsf{m}+bQ/2)\Gamma(\pm2\mathbf{a}-\mathsf{m}+bQ/2)}. \tag{2.175}$$

[16] For a 2-dimensional phase space parametrized by (q,p), the operator $\mathcal{O}$ and its inverse Weyl transform f are related by $f(q,p) = \int d\sigma e^{-\frac{i}{\hbar}p\sigma}\langle q|e^{\frac{i}{2\hbar}\sigma\widehat{p}}\mathcal{O}(\widehat{q},\widehat{p})e^{\frac{i}{2\hbar}\sigma\widehat{p}}|q\rangle\,, \mathcal{O} = \frac{1}{(2\pi)^2\hbar}\int d\sigma d\tau dq dp e^{-i\tau(\widehat{q}-q)-\frac{i}{\hbar}\sigma(\widehat{p}-p)}f(q,p)\,,$ where $[\widehat{q},\widehat{p}] = i\hbar$, $\widehat{q}|q\rangle = q|q\rangle$, $\langle q|q'\rangle = \delta(q-q')$ [50].

By performing the manipulations explained above, we shift the argument in (2.173) as $\alpha \to \alpha \mp b/4$. Then the expectation value of the Verlinde operator for the minimal 't Hooft loop $T = L_{1,0}$ becomes

$$\langle \mathcal{L} \rangle = \sum_{\pm} \int d\alpha\, C(\alpha \mp b/4, \alpha_1, Q - \alpha \pm b/4) \overline{\mathcal{F}_E(\alpha \pm b/4)} H_{\pm\,;E}(\alpha \mp b/4) \mathcal{F}_E\,(\alpha \pm b/4)\ . \tag{2.176}$$

The **a**-dependent part of the three-point function $C(\alpha, \alpha_1, Q - \alpha)$ reads

$$C(\alpha, \alpha_1, Q - \alpha) \propto \frac{\prod_{s_1, s_2 = \pm} \Gamma_b(Q/2 + 2s_1\, \mathbf{a}/b + s_2\, \mathsf{m}/b)}{\prod_{s=\pm} \Gamma_b(Q + 2s\, \mathbf{a}/b) \Gamma_b(2s\, \mathbf{a}/b)}\,, \tag{2.177}$$

where $\Gamma_b(z)$ is defined so that $\Gamma_b(z) := \Gamma_2(z|b, 1/b)$ where Γ_2 is described in (1.59). For the present purpose we only need the relations

$$\Gamma_b(z) = \Gamma_{1/b}(z)\,, \qquad \Gamma_b(z + b) = \frac{\sqrt{2\pi} b^{bz-1/2}}{\Gamma(bx)} \Gamma_b(z)\,. \tag{2.178}$$

As seen in (2.162), the factor Z_v, which is related the vev of the loop operators on $S^1 \times \mathbb{R}^3$ directly, is defined as follows

$$\begin{aligned}\langle \mathcal{L} \rangle = &\sum_{\pm} \int d\alpha\, C(\alpha \mp b/4, \alpha_1, Q - \alpha \pm b/4)^{\frac{1}{2}}\, \overline{\mathcal{F}_E(\alpha \pm b/4)}\, Z_{\pm}(\alpha, \alpha_1) \\ &\times C(\alpha \pm b/4, \alpha_1, Q - \alpha \mp b/4)^{\frac{1}{2}} \mathcal{F}_E\,(\alpha \pm b/4)\ .\end{aligned} \tag{2.179}$$

Comparing (2.176) with (2.179), we get

$$\begin{aligned}Z_{\pm}(\alpha, E) &= \left(\frac{C(\alpha \mp b/4, \alpha_1, Q - \alpha \pm b/4)}{C(\alpha \pm b/4, \alpha_1, Q - \alpha \mp b/4)} \right)^{1/2} H_{\pm\,;\alpha_1}(\alpha \mp b/4) \\ &= \left(\frac{\prod_{\pm} \cos(2\pi \mathbf{a} \pm \pi \mathsf{m})}{\prod_{\pm} \sin(2\pi \mathbf{a} \pm \pi b^2/2)} \right)^{1/2}\ .\end{aligned} \tag{2.180}$$

It means $Z_+(\alpha, \alpha_1) = Z_-(\alpha, \alpha_1)$. Thus the Verlinde operator (2.164) acting on $\mathcal{B}_{\alpha_1}(\alpha)$ is given as

$$\mathcal{L}_\gamma = \sum_{\pm} e^{\pm \frac{1}{4} b^2 \partial_{\mathbf{a}}} \left(\prod_{\pm} \frac{\cos(2\pi \mathbf{a} \pm \pi \mathsf{m})}{\sin(2\pi \mathbf{a} \pm \frac{\pi}{2} b^2)} \right)^{1/2} e^{\pm \frac{1}{4} b^2 \partial_{\mathbf{a}}}\,. \tag{2.181}$$

This is indeed related to the 't Hooft operator vev (2.108) by the Weyl transform above.

2.7.2 $SU(N)$ $\mathcal{N}=2^*$

The Verlinde operator corresponding to the 't Hooft operator with charge $B=(1,0^{N-1})$, acting on the Toda conformal block for the one-punctured torus, was computed in [11] in the standard normalization. The notation in Toda theory is reviewed in Section 1.4.4. As written in (1.87) and (1.88), the three-point function with two generic momenta $\alpha_1,\ \alpha_2$ and one semi-degenerate momentum $\alpha_3=\kappa\omega_{N-1}$ is given by

$$C(\alpha_1,\alpha_2,\alpha_3=\kappa\omega_{N-1})\propto C^{(1)}(\alpha_1,\alpha_2,\kappa):=\frac{\prod_{i<j}\Upsilon(\langle Q\rho-\alpha_1,h_{ij}\rangle)\Upsilon(\langle Q\rho-\alpha_2,h_{ij}\rangle)}{\prod_{i,j=1}^{N}\Upsilon(\kappa/N+\langle\alpha_1-Q\rho,h_i\rangle+\langle\alpha_2-Q\rho,h_j\rangle)} \tag{2.182}$$

and the three-point function where $\alpha_1,\ \alpha_2$ are generic momentum and $\alpha_3=\kappa\omega_1$ is given by

$$C(\alpha_1,\alpha_2,\alpha_3=\kappa\omega_1)\propto C^{(2)}(\alpha_1,\alpha_2,\kappa):=\frac{\prod_{i<j}\Upsilon(\langle Q\rho-\alpha_1,h_{ij}\rangle)\Upsilon(\langle Q\rho-\alpha_2,h_{ij}\rangle)}{\prod_{i,j=1}^{N}\Upsilon(\kappa/N-\langle\alpha_1-Q\rho,h_i\rangle-\langle\alpha_2-Q\rho,h_j\rangle)}\,. \tag{2.183}$$

The definition of $\Upsilon(x)$ is written in (1.58).

The two-dimensional theory corresponding to $\mathcal{N}=2^*$ is the $SU(N)$ Toda theory on the torus with one semi-degenerate puncture. With the parametrization

$$\alpha=Q+i\widehat{a},\qquad \alpha_1=\left(\frac{Q}{2}+i\widehat{m}\right)N\omega_{N-1}\,, \tag{2.184}$$

the vev of the Verlinde operator corresponding to the minimal 't Hooft operator $T=T_{B=(1,0^{N-1})}$ is

$$\langle\mathcal{L}\rangle=\int d\alpha\, C(2Q\rho-\alpha,\alpha_1,\alpha)\,\overline{\mathcal{F}_{\alpha_1}(\alpha)}\sum_{k=1}^{N}H_{k\,;\alpha_1}(\alpha)\mathcal{F}_{\alpha_1}(\alpha-bh_k)\,, \tag{2.185}$$

where

$$H_{k\,;\alpha_1}(\alpha)=\prod_{j\neq k}\frac{\Gamma(ib\widehat{a}_{jk})\Gamma(bQ+ib\widehat{a}_{jk})}{\Gamma(bQ/2+ib\widehat{a}_{jk}-ib\widehat{m})\Gamma(bQ/2+ib\widehat{a}_{jk}+ib\widehat{m})}\,, \tag{2.186}$$

as obtained in (1.100). The relation between the gauge thoery parameters a,m and the Toda theory parameters α,α_1 is given as

$$\alpha=Q\rho-\frac{\mathbf{a}}{b}\,,\qquad \alpha_1=\left(\frac{Q}{2}-\frac{\mathsf{m}}{b}\right)N\omega_{N-1}\,, \tag{2.187}$$

where $\mathsf{m}:=m+1/2$. Let us define

$$\widetilde{\Upsilon}(x):=\frac{\Upsilon(x+b)}{\Upsilon(x)}=\frac{\Gamma(bx)}{\Gamma(1-bx)}b^{1-2bx}\,. \tag{2.188}$$

Let us rewrite (2.185) into the expression using $\mathcal{B}_{\alpha_1}(\alpha) = C(2Q\rho - \alpha, \alpha_1, \alpha)^{1/2}\mathcal{F}_{\alpha_1}(\alpha)$ in the same way as the Liouville theory case.

$$\langle \mathcal{L} \rangle = \int d\alpha\, \overline{\mathcal{B}}(\alpha + \frac{bh_k}{2}) \sum_{k=1}^{N} Z_k(\alpha, \alpha_1)\, \mathcal{B}_{\alpha_1}(\alpha - \frac{bh_k}{2})\,. \tag{2.189}$$

We obtain Z_k as follows

$$Z_k(\alpha, \alpha_1) = \left(\frac{C(\alpha + bh_k/2, \alpha_1, 2Q\rho - \alpha - bh_k/2)}{C(\alpha - bh_k/2, \alpha_1, 2Q\rho - \alpha + bh_k/2)} \right)^{1/2} H_{\alpha\,;\,\alpha_1}(\alpha + bh_k/2)$$

$$= \left(\prod_{j<l,\pm} \frac{\Upsilon\Big(\pm (\mathbf{a}_{jl}/b - b(\delta_{jk} - \delta_{kl})/2)\Big)}{\Upsilon\Big(\pm (\mathbf{a}_{jl}/b + b(\delta_{jk} - \delta_{kl})/2)\Big)} \prod_{j\neq l} \frac{\Upsilon\left(Q/2 - \mathsf{m}/b - \mathbf{a}_{jl}/b - b(\delta_{jk} - \delta_{kl})/2\right)}{\Upsilon\left(Q/2 - \mathsf{m}/b - \mathbf{a}_{jl}/b + b(\delta_{jk} - \delta_{kl})/2\right)} \right)^{1/2}$$
$$\times \prod_{j\neq k} \frac{\Gamma(-\mathbf{a}_{jk} - b^2/2)\Gamma(bQ - \mathbf{a}_{jk} - b^2/2)}{\prod_{\pm} \Gamma(bQ/2 - \mathbf{a}_{jk} - b^2/2 \pm \mathsf{m})}$$

$$= \left(\prod_{j<k} \frac{\widetilde{\Upsilon}(\mathbf{a}_{jk}/b - b/2)}{\widetilde{\Upsilon}(-\mathbf{a}_{jk}/b - b/2)} \prod_{k<l} \frac{\widetilde{\Upsilon}(-\mathbf{a}_{kl}/b - b/2)}{\widetilde{\Upsilon}(\mathbf{a}_{kl}/b - b/2)} \frac{\prod_{j\neq k} \widetilde{\Upsilon}(Q/2 - \mathsf{m}/b - \mathbf{a}_{jk}/b - b/2)}{\prod_{k\neq l} \widetilde{\Upsilon}(Q/2 - \mathsf{m}/b - \mathbf{a}_{kl}/b - b/2)} \right)^{1/2}$$
$$\times \prod_{j\neq k} \frac{\Gamma(-\mathbf{a}_{jk} - b^2/2)\Gamma(1 - \mathbf{a}_{jk} + b^2/2)}{\prod_{\pm} \Gamma(1/2 - \mathbf{a}_{jk} \pm \mathsf{m})}$$
$$= \left(\prod_{j\neq k} \frac{\widetilde{\Upsilon}(\mathbf{a}_{jk}/b - b/2)}{\widetilde{\Upsilon}(-\mathbf{a}_{jk}/b - b/2)} \frac{\widetilde{\Upsilon}(Q/2 - \mathsf{m}/b - \mathbf{a}_{jk}/b - b/2)}{\widetilde{\Upsilon}(Q/2 - \mathsf{m}/b + \mathbf{a}_{jk}/b - b/2)} \right)^{1/2}$$
$$\times \prod_{j\neq k} \frac{\Gamma(-\mathbf{a}_{jk} - b^2/2)\Gamma(1 - \mathbf{a}_{jk} + b^2/2)}{\prod_{\pm} \Gamma(1/2 - \mathbf{a}_{jk} \pm \mathsf{m})}$$
$$= \left(\prod_{j\neq k} \frac{\Gamma(\mathbf{a}_{jk} - b^2/2)\Gamma(1 + \mathbf{a}_{jk} + b^2/2) \prod_{\pm} \Gamma(1/2 \pm \mathsf{m} - \mathbf{a}_{jk})}{\Gamma(1 - \mathbf{a}_{jk} + b^2/2)\Gamma(-\mathbf{a}_{jk} - b^2/2) \prod_{\pm} \Gamma(1/2 \pm \mathsf{m} + \mathbf{a}_{jk})} \right)^{1/2}$$

$$\times \prod_{j\neq k} \frac{\Gamma(-\mathbf{a}_{jk} - b^2/2)\Gamma(1 - \mathbf{a}_{jk} + b^2/2)}{\prod_{\pm} \Gamma(1/2 - \mathbf{a}_{jk} \pm \mathsf{m})}$$
$$= \left(\prod_{j\neq k} \frac{\Gamma(\mathbf{a}_{jk} - b^2/2)\Gamma(1 + \mathbf{a}_{jk} + b^2/2)\Gamma(-\mathbf{a}_{jk} - b^2/2)\Gamma(1 - \mathbf{a}_{jk} + b^2/2)}{\prod_{s_1, s_2 = \pm} \Gamma(1/2 + s_1\mathsf{m} + s_2\mathbf{a}_{jk})} \right)^{1/2}$$
$$= \left(\prod_{j\neq k} \prod_{\pm} \frac{\cos\pi(\mathbf{a}_{jk} \pm \mathsf{m})}{\sin\pi(\mathbf{a}_{jk} \pm b^2/2)} \right)^{1/2}\,. \tag{2.190}$$

Thus we find

$$\mathcal{L}_{B=(1,0^{N-1})} = \sum_{l=1}^{N} e^{-\frac{b^2}{2}h_l\cdot\partial_{\mathbf{a}}} \left(\prod_{\pm}\prod_{j\neq l} \frac{\cos\pi(\mathbf{a}_{lj}\pm \mathsf{m})}{\sin\pi(\mathbf{a}_{lj}\pm\lambda/2)}\right) e^{-\frac{b^2}{2}h_l\cdot\partial_{\mathbf{a}}} . \tag{2.191}$$

Note that h_l are the coweights that correspond to the weights in the fundamental representation of the Langlands dual group. The Verlinde operator (2.191) is the Weyl transform of the vev (2.114) on $S^1\times\mathbb{R}^3$ as expected.

2.7.3 $SU(2)$ $N_f = 4$

To compare with gauge theory calculations, we relate the gauge theory parameters $\mathbf{a}$ and $\{m_f\}_{f=1}^4$ to the Liouville theory parameters α and $\{\alpha_e\}_{e=1}^4$ by

$$\alpha = \frac{Q}{2}+\frac{\mathsf{a}}{b}, \quad \alpha_1 = \frac{Q}{2}+\frac{\mathsf{m}_1-\mathsf{m}_2}{2b}, \quad \alpha_2 = \frac{Q}{2}+\frac{\mathsf{m}_1+\mathsf{m}_2}{2b}, \tag{2.192}$$

$$\alpha_3 = \frac{Q}{2}+\frac{\mathsf{m}_3+\mathsf{m}_4}{2b}, \quad \alpha_4 = \frac{Q}{2}+\frac{\mathsf{m}_3-\mathsf{m}_4}{2b}, \tag{2.193}$$

where $\mathsf{m}_f := m_f + 1/2$. In the next subsubsection, we will calculate the Verlinde operator corresponding to the minimal 't Hooft operator in A_{N-1} Toda theory. In case of $N=2$, it will become the Liouville theory result

$$\begin{aligned}\mathcal{L}_\gamma = &\sum_{\pm} e^{\pm\frac{1}{2}b^2\partial_{\mathbf{a}}} \left(\frac{\prod_{f=1}^4\prod_{s=\pm}\cos(\pi\mathbf{a}+s\pi\mathsf{m}_f)}{\sin(2\pi\mathbf{a}+\pi b^2)\sin^2(2\pi\mathbf{a})\sin(2\pi\mathbf{a}-\pi b^2)}\right)^{1/2} e^{\pm\frac{1}{2}b^2\partial_{\mathbf{a}}}\\ &-\frac{1}{2}\cos\pi(b^2-\sum_f \mathsf{m}_f) + \sum_{s=\pm}\frac{\prod_{f=1}^4\cos\pi\left(-b^2/2+s\mathbf{a}+\mathsf{m}_f\right)}{\sin(2\pi\mathbf{a})\sin\pi(sb^2-2\mathbf{a})} .\end{aligned} \tag{2.194}$$

In view of (2.164) and (2.165), this is related to the 't Hooft operator vev (2.119) in the $U(2)$ theory with $\mathbf{a}_1 = -\mathbf{a}_2 = \mathbf{a}$ by the Weyl transform, except the a-independent term $-\frac{1}{2}\cos\pi(b^2-\sum\mathsf{m}_f)$. This subtle mismatch suggests that there should be another 't Hooft loop that deals with singularities of the monopole moduli space in a different way from this section and its vev is related to the above expression exactly. We will discuss this 't Hooft loop in Section 4.

The whole expression of (2.194) is invariant under $\mathbf{a}\to-\mathbf{a}$ as well as under the action of the $SO(8)$ Weyl group.[17] By the argument given above, then, Liouville theory predicts that

[17] Four generators of the Weyl group of the $SO(8)$ flavor group act on the masses as $\mathsf{m}_1\leftrightarrow\mathsf{m}_2$, $\mathsf{m}_2\leftrightarrow\mathsf{m}_3$, $\mathsf{m}_3\leftrightarrow\mathsf{m}_4$, and $\mathsf{m}_3\leftrightarrow-\mathsf{m}_4$ respectively. Agreement up to an additive constant is almost as much as one can hope for. Without $SO(8)$ Weyl invariance, however, (2.119) with $\mathbf{a}_1=\mathbf{a}=-\mathbf{a}_2$ cannot be the answer for $SU(2)$ gauge theory.

there should be another 't Hooft loop operators in the $SU(2)$ theory with $N_f = 4$ where the contribution from screening monopole is as follows has

$$Z_{\text{mono}}(\mathbf{a}, \{m_f\}_f; 2, 0) = -\frac{1}{2}\cos\pi(\lambda - \sum_f m_f) - \sum_{s=\pm} \frac{\prod_{f=1}^4 \sin\pi\,(s\mathbf{a} - m_f + \lambda/2)}{\sin(2\pi\mathbf{a})\sin\pi(s\lambda + 2\mathbf{a})}\,. \tag{2.195}$$

Thus the whole expression of this 't Hooft operator vev should be[18]

$$\begin{aligned}\langle L_{2,0}\rangle =&(e^{4\pi i\mathbf{b}} + e^{-4\pi i\mathbf{b}})\left(\frac{\prod_\pm \prod_{f=1}^4 \sin\pi(\mathbf{a} \pm m_f)}{\sin^2 2\pi\mathbf{a}\prod_\pm \sin\pi(2\mathbf{a} \pm \lambda)}\right)^{1/2} \\ &- \frac{1}{2}\cos\pi(\lambda - \sum_f m_f) - \sum_{s=\pm} \frac{\prod_{f=1}^4 \sin\pi\,(s\mathbf{a} - m_f + \lambda/2)}{\sin(2\pi\mathbf{a})\sin\pi(s\lambda + 2\mathbf{a})}\,.\end{aligned} \tag{2.196}$$

In Appendix A.3, we show that the $\lambda = 0$ limit of this expression coincides with the classical holonomy on the four-punctured sphere written in terms of the complexified Fenchel-Nielsen coordinates.

2.7.4 $SU(N)$ $N_f = 2N$

The Verlinde operator corresponding to the 't Hooft operator with $B = (1, -1, 0^{N-2})$ in the $SU(N)$ superconformal QCD was computed in [11] and the result is written in (1.104) and (1.105).

The relation between the gauge thoery parameters $\mathbf{a}, \{m_f\}_{f=1}^{2N}$ and the Toda theory parameters $\alpha, \{\alpha_e\}_{e=1}^4$ is given as

$$\begin{aligned} &\alpha = Q\rho - \frac{\mathbf{a}}{b}\,, \\ &\alpha_2 = \left(\frac{Q}{2} - \frac{\mathsf{m}_2}{b}\right) N\omega_{N-1}\,, \quad \alpha_3 = \left(\frac{Q}{2} - \frac{\mathsf{m}_3}{b}\right) N\omega_1\,, \\ &\alpha_1 = Q\rho - \frac{\mathsf{m}_1}{b}\,, \qquad\qquad \alpha_4 = Q\rho - \frac{\mathsf{m}_4}{b}\,, \\ &\mathsf{m}_f = \begin{cases} \mathsf{m}_2 + \mathsf{m}_1 \cdot h_i \equiv \mathsf{m}_2 + \mathsf{m}_{1,i} & \text{for } \ f \leq N\,, \\ \mathsf{m}_3 - \mathsf{m}_4 \cdot h_{f-N} \equiv \mathsf{m}_3 - \mathsf{m}_{4,f-N} & \text{for } \ f \geq N+1\,, \end{cases} \end{aligned} \tag{2.197}$$

where $\mathsf{m}_f := m_f + 1/2$. The vev of Verlinde operator is given as

$$\langle \mathcal{L} \rangle = \int d\alpha\, C(\alpha_4, \alpha_3, \alpha) C(2Q\rho - \alpha, \alpha_2, \alpha_1)\overline{\mathcal{F}_E(\alpha)} \sum_{l\neq k} H_{(l,k)\,;\,E}(\alpha)\mathcal{F}_E(\alpha - bh_{lk})\,,$$

where $E = \{\alpha_4, \alpha_3, \alpha_2, \alpha_1\}$. Let us calculate $Z_{(l,k)\,;\,E}(\alpha)$ by using (1.104) and (1.105). We use the following notation $(h_{lk})_{ij} := (h_l - h_k) \cdot (h_i - h_j) = \delta_{li} - \delta_{lj} - \delta_{ki} + \delta_{kj}$ and $\widetilde{\Upsilon}^{(2)} :=$

[18] Essentially the same expression has been obtained purely from quantization of the Hitchin system [52].

$\frac{\Upsilon(x+2b)}{\Upsilon(x)} = \frac{\Gamma(bx)\Gamma(bx+b^2)}{\Gamma(1-bx)\Gamma(1-bx-b^2)}$, which is analogous to $\widetilde{\Upsilon}(x) := \frac{\Upsilon(x+b)}{\Upsilon(x)} = \frac{\Gamma(bx)}{\Gamma(1-bx)}$.

$$
\begin{aligned}
Z_{(l,k)\,;\,E}(\alpha) &= \left(\frac{C(\alpha_4,\alpha_3,\alpha+bh_{lk}/2)C(2Q\rho-\alpha-bh_{lk}/2,\alpha_2,\alpha_1)}{C(\alpha_4,\alpha_3,\alpha-bh_{lk}/2)C(2Q\rho-\alpha+bh_{lk}/2,\alpha_2,\alpha_1)}\right)^{1/2} H_{(l,k)\,;\,E}(\alpha+bh_{lk}/2)\\
&= \left(\frac{C^{(2)}(Q\rho-\mathsf{m}_4/b,\ Q\rho-\mathbf{a}/b+bh_{lk}/2,\ \kappa=Q\rho/2-\mathsf{m}_3/b)}{C^{(2)}(Q\rho-\mathsf{m}_4/b,\ Q\rho-\mathbf{a}/b-bh_{lk}/2,\ \kappa=Q\rho/2-\mathsf{m}_3/b)}\right.\\
&\qquad \left.\times\frac{C^{(1)}(Q\rho-\mathsf{m}_1/b,\ Q\rho+\mathbf{a}/b-bh_{lk}/2,\ \kappa=Q/2-\mathsf{m}_2/b)}{C^{(1)}(Q\rho-\mathsf{m}_1/b,\ Q\rho+\mathbf{a}/b+bh_{lk}/2,\ \kappa=Q/2-\mathsf{m}_2/b)}\right)^{1/2} H_{(l,k)\,;\,E}(\alpha+bh_{lk}/2)
\end{aligned}
$$

$$
\begin{aligned}
&= \pi^2\left(\prod_{i<j}\frac{\Upsilon(\mathbf{a}_{ij}/b-b(h_{lk})_{ij}/2)}{\Upsilon(\mathbf{a}_{ij}/b+b(h_{lk})_{ij}/2)}\prod_{i,j}\frac{\Upsilon(Q/2-\mathsf{m}_3/b+\mathsf{m}_{4,i}/b+\mathbf{a}_j/b+b(\delta_{lj}-\delta_{kj})/2)}{\Upsilon(Q/2-\mathsf{m}_3/b+\mathsf{m}_{4,i}/b+\mathbf{a}_j/b-b(\delta_{lj}-\delta_{kj})/2)}\right.\\
&\qquad\left.\times\prod_{i<j}\frac{\Upsilon(-\mathbf{a}_{ij}/b+b(h_{lk})_{ij}/2)}{\Upsilon(-\mathbf{a}_{ij}/b-b(h_{lk})_{ij}/2)}\prod_{i,j}\frac{\Upsilon(Q/2-\mathsf{m}_2/b-\mathsf{m}_{1,i}/b+\mathbf{a}_j/b+b(\delta_{lj}-\delta_{kj})/2)}{\Upsilon(Q/2-\mathsf{m}_2/b-\mathsf{m}_{1,i}/b+\mathbf{a}_j/b-b(\delta_{lj}-\delta_{kj})/2)}\right)^{1/2}\\
&\qquad\times\frac{\prod_{j\neq l}\Gamma(-\mathbf{a}_{jl}-b^2(1+\delta_{kj})/2)\Gamma(bQ-\mathbf{a}_{jl}-b^2(1+\delta_{kj})/2)}{\prod_f\Gamma\left(bQ/2+\mathbf{a}_l-b^2/2-\mathsf{m}_f\right)\Gamma\left(bQ/2-\mathbf{a}_k-b^2/2+\mathsf{m}_f\right)}\\
&\qquad\times\prod_{j\neq k}\Gamma(-\mathbf{a}_{kj}+b^2(\delta_{lj}-1)/2)\Gamma(bQ-\mathbf{a}_{kj}+b^2(\delta_{jl}-1)/2))\\
&= \pi^2\left(\prod_{i\neq j}\frac{\Upsilon(\mathbf{a}_{ij}/b-b(h_{lk})_{ij}/2)}{\Upsilon(\mathbf{a}_{ij}/b+b(h_{lk})_{ij}/2)}\prod_{i,j}\frac{\Upsilon(Q/2-\mathsf{m}_3/b+\mathsf{m}_{4,i}/b+\mathbf{a}_j/b+b(\delta_{lj}-\delta_{kj})/2)}{\Upsilon(Q/2-\mathsf{m}_3/b+\mathsf{m}_{4,i}/b+\mathbf{a}_j/b-b(\delta_{lj}-\delta_{kj})/2)}\right.\\
&\qquad\left.\times\prod_{i,j}\frac{\Upsilon(Q/2-\mathsf{m}_2/b-\mathsf{m}_{1,i}/b+\mathbf{a}_j/b+b(\delta_{lj}-\delta_{kj})/2)}{\Upsilon(Q/2-\mathsf{m}_2/b-\mathsf{m}_{1,i}/b+\mathbf{a}_j/b-b(\delta_{lj}-\delta_{kj})/2)}\right)^{1/2}\\
&\qquad\times\frac{\Gamma(-\mathbf{a}_{kl}-b^2)\Gamma(bQ-\mathbf{a}_{kl}-b^2)\Gamma(-\mathbf{a}_{kl})\Gamma(bQ-\mathbf{a}_{kl})}{\prod_f\Gamma\left(1/2+\mathbf{a}_l-\mathsf{m}_f\right)\Gamma\left(1/2-\mathbf{a}_k+\mathsf{m}_f\right)}\\
&\qquad\times\prod_{j\neq l,k}\Gamma(-a_{jl}-b^2/2)\Gamma(bQ-\mathbf{a}_{jl}-b^2/2)\Gamma(-\mathbf{a}_{kj}-b^2/2)\Gamma(bQ-\mathbf{a}_{kj}-b^2/2)
\end{aligned}
$$

$$
\begin{aligned}
&= \pi^2\left(\frac{\Upsilon(\mathbf{a}_{lk}/b-b)\Upsilon(\mathbf{a}_{kl}/b+b)}{\Upsilon(\mathbf{a}_{lk}/b+b)\Upsilon(\mathbf{a}_{kl}/b-b)}\right.\\
&\qquad\times\prod_{i\neq l,k}\frac{\Upsilon(\mathbf{a}_{il}/b+b/2)\Upsilon(\mathbf{a}_{ik}/b-b/2)}{\Upsilon(\mathbf{a}_{il}/b-b/2)\Upsilon(\mathbf{a}_{ik}/b+b/2)}\prod_{j\neq l,k}\frac{\Upsilon(\mathbf{a}_{lj}/b-b/2)\Upsilon(\mathbf{a}_{kj}/b+b/2)}{\Upsilon(\mathbf{a}_{lj}/b+b/2)\Upsilon(\mathbf{a}_{kj}/b-b/2)}\\
&\qquad\times\prod_i\frac{\Upsilon\left(Q/2-\mathsf{m}_3/b+\mathsf{m}_{4,i}/b+\mathbf{a}_l/b+b/2\right)\Upsilon\left(Q/2-\mathsf{m}_3/b+\mathsf{m}_{4,i}/b+\mathbf{a}_k/b-b/2\right)}{\Upsilon\left(Q/2-\mathsf{m}_3/b+\mathsf{m}_{4,i}/b+\mathbf{a}_l/b-b/2\right)\Upsilon\left(Q/2-\mathsf{m}_3/b+\mathsf{m}_{4,i}/b+\mathbf{a}_k/b+b/2\right)}
\end{aligned}
$$

$$
\times \frac{\Upsilon\left(Q/2-\mathsf{m}_2/b-\mathsf{m}_{1,i}/b+\mathbf{a}_l/b+b/2\right)\Upsilon\left(Q/2-\mathsf{m}_2/b-\mathsf{m}_{1,i}/b+\mathbf{a}_k/b-b/2\right)}{\Upsilon\left(Q/2-\mathsf{m}_2/b-\mathsf{m}_{1,i}/b+\mathbf{a}_l/b-b/2\right)\Upsilon\left(Q/2-\mathsf{m}_2/b-\mathsf{m}_{1,i}/b+\mathbf{a}_k/b+b/2\right)}\Bigg)^{1/2}
$$

$$
\times \frac{\Gamma(-\mathbf{a}_{kl}-b^2)\Gamma(1-\mathbf{a}_{kl})\Gamma(-\mathbf{a}_{kl})\Gamma(b\mathsf{Q}-\mathbf{a}_{kl})}{\prod_f \Gamma\left(1/2+\mathbf{a}_l-\mathsf{m}_f\right)\Gamma\left(1/2-\mathbf{a}_k+\mathsf{m}_f\right)} \prod_{j\neq l,k}\prod_{\pm}\Gamma(-\mathbf{a}_{jl}\pm b^2/2)\Gamma(-\mathbf{a}_{kj}\pm b^2/2)
$$

$$
=\pi^2\Bigg(\frac{\widetilde{\Upsilon}^{(2)}(\mathbf{a}_{kl}/b-b)}{\widetilde{\Upsilon}^{(2)}(\mathbf{a}_{lk}/b-b)} \prod_{i\neq l,k}\frac{\widetilde{\Upsilon}(\mathbf{a}_{il}/b-b/2)}{\widetilde{\Upsilon}(\mathbf{a}_{ik}/b-b/2)}\prod_{j\neq l,k}\frac{\widetilde{\Upsilon}(\mathbf{a}_{kj}/b-b/2)}{\widetilde{\Upsilon}(\mathbf{a}_{lj}/b-b/2)}
$$

$$
\times\prod_i \frac{\widetilde{\Upsilon}\left(Q/2-\mathsf{m}_3/b+\mathsf{m}_{4,i}/b+\mathbf{a}_l/b-b/2\right)}{\widetilde{\Upsilon}\left(Q/2-\mathsf{m}_3/b+\mathsf{m}_{4,i}/b+\mathbf{a}_k/b-b/2\right)}\frac{\widetilde{\Upsilon}\left(Q/2-\mathsf{m}_2/b-\mathsf{m}_{1,i}/b+\mathbf{a}_l/b-b/2\right)}{\widetilde{\Upsilon}\left(Q/2-\mathsf{m}_2/b-\mathsf{m}_{1,i}/b+\mathbf{a}_k/b-b/2\right)}\Bigg)^{1/2}
$$

$$
\times \frac{\Gamma(-\mathbf{a}_{kl}-b^2)\Gamma(1-\mathbf{a}_{kl})\Gamma(-\mathbf{a}_{kl})\Gamma(bQ-\mathbf{a}_{kl})}{\prod_f \Gamma\left(1/2+\mathbf{a}_l-\mathsf{m}_f\right)\Gamma\left(1/2-\mathbf{a}_k+\mathsf{m}_f\right)} \prod_{j\neq l,k}\prod_{\pm}\Gamma(-\mathbf{a}_{jl}\pm b^2/2)\Gamma(-\mathbf{a}_{kj}\pm b^2/2)
$$

$$
=\pi^2\Bigg(\frac{\Gamma(\mathbf{a}_{kl}-b^2)\Gamma(\mathbf{a}_{kl})\Gamma(1-\mathbf{a}_{lk}+b^2)\Gamma(1-\mathbf{a}_{lk})}{\Gamma(1-\mathbf{a}_{kl}+b^2)\Gamma(1-\mathbf{a}_{kl})\Gamma(\mathbf{a}_{lk}-b^2)\Gamma(\mathbf{a}_{lk})}\prod_f\frac{\Gamma\left(1/2-\mathsf{m}_f+\mathbf{a}_l\right)\Gamma\left(1/2+\mathsf{m}_f-\mathbf{a}_k\right)}{\Gamma\left(1/2+\mathsf{m}_f-\mathbf{a}_l\right)\Gamma\left(1/2-\mathsf{m}_f+\mathbf{a}_k\right)}
$$

$$
\times\prod_{i\neq l,k}\frac{\Gamma(\mathbf{a}_{il}-b^2/2)\Gamma(1-\mathbf{a}_{ik}+b^2/2)\Gamma(\mathbf{a}_{ki}-b^2/2)\Gamma(1-\mathbf{a}_{li}+b^2/2)}{\Gamma(1-\mathbf{a}_{il}+b^2/2)\Gamma(\mathbf{a}_{ik}-b^2/2)\Gamma(1-\mathbf{a}_{ki}+b^2/2)\Gamma(\mathbf{a}_{li}-b^2/2)}\Bigg)^{1/2}
$$

$$
\times \frac{\Gamma(-\mathbf{a}_{kl}-b^2)\Gamma(1-\mathbf{a}_{kl})\Gamma(-\mathbf{a}_{kl})\Gamma(bQ-\mathbf{a}_{kl})}{\prod_f \Gamma\left(1/2+\mathbf{a}_l-\mathsf{m}_f\right)\Gamma\left(1/2-\mathbf{a}_k+\mathsf{m}_f\right)}
$$

$$
\times\prod_{j\neq l,k}\Gamma(-\mathbf{a}_{jl}-b^2/2)\Gamma(1-\mathbf{a}_{jl}+b^2/2)\Gamma(-\mathbf{a}_{kj}-b^2/2)\Gamma(1-\mathbf{a}_{kj}+b^2/2)
$$

$$
=\frac{\prod_f[\cos\pi(\mathbf{a}_l-\mathsf{m}_f)\cos\pi(\mathbf{a}_k-\mathsf{m}_f)]^{\frac{1}{2}}}{\prod_{\pm}\Big[\sin\pi(\pm\mathbf{a}_{lk})\sin\pi(\pm\mathbf{a}_{lk}-b^2)\prod_{j\neq l,k}\sin\pi(\pm\mathbf{a}_{jl}-b^2/2)\sin\pi(\pm\mathbf{a}_{jk}-b^2/2)\Big]^{\frac{1}{2}}}\,. \qquad (2.198)
$$

Thus we get

$$
\mathcal{L}_\gamma
$$

$$
=\sum_{\substack{1\leq j,k\leq N\\ j\neq k}} e^{-\frac{b^2}{2}e_{jk}\cdot\partial_{\mathbf{a}}}\frac{\Big[\prod_{f=1}^N\cos\pi(\mathbf{a}_j-\mathsf{m}_f)\cos\pi(\mathbf{a}_k-\mathsf{m}_f)\Big]^{\frac{1}{2}}}{\sin\pi\mathbf{a}_{jk}\prod_{\pm}\Big[\sin\pi(\mathbf{a}_{jk}\pm b^2)\prod_{i\neq j,k}\sin\pi(\mathbf{a}_{ji}\pm\frac{b^2}{2})\sin\pi(\mathbf{a}_{ik}\pm\frac{b^2}{2})\Big]^{\frac{1}{2}}}e^{-\frac{b^2}{2}e_{jk}\cdot\partial_{\mathbf{a}}}
$$

$$
+\sum_{k=1}^N\frac{\prod_{f=1}^{2N}\cos\pi\left(\mathbf{a}_k-\mathsf{m}_f+\frac{b^2}{2}\right)}{\prod_{i\neq k}\sin\pi\mathbf{a}_{ki}\sin\pi\left(-\mathbf{a}_{ki}-b^2\right)} \qquad (2.199)
$$

$$
+(-1)^{N-1}\frac{e^{N\pi i(\sum_{f>N}\mathsf{m}_f-\sum_{f\leq N}\mathsf{m}_f)/N}\sin\pi b^2}{\sin(\pi(N-2)b^2)}-\frac{1}{2}\cos\pi(b^2-\sum_f\mathsf{m}_f)\,.
$$

This implies that there is an certain 't Hooft operator in the $SU(N)$ theory with $N_f = 2N$ whose vev on $S^1 \times \mathbb{R}^3$ is given by

$$
\begin{aligned}
&\langle T_{B=(1,-1,0^{N-2})} \rangle \\
&= \sum_{\substack{1\le j,k\le N \\ j\neq k}} e^{2\pi i(\mathbf{b}_j-\mathbf{b}_k)} \frac{\left[\prod_{f=1}^{N} \sin\pi(\mathbf{a}_j - m_f)\sin\pi(\mathbf{a}_k - m_f)\right]^{1/2}}{\sin\pi\mathbf{a}_{jk}\prod_{\pm}\left[\sin\pi(\mathbf{a}_{jk}\pm\lambda)\prod_{i\neq j,k}\sin\pi(\mathbf{a}_{ji}\pm\lambda/2)\sin\pi(\mathbf{a}_{ik}\pm\lambda/2)\right]^{1/2}} \\
&\qquad + \sum_{k=1}^{N} \frac{\prod_{f=1}^{2N}\sin\pi\left(\mathbf{a}_k - m_f + \lambda/2\right)}{\prod_{i\neq k}\sin\pi\mathbf{a}_{ki}\sin\pi\left(-\mathbf{a}_{ki}-\lambda\right)} \\
&\qquad + (-1)^{N-1}\frac{e^{\pi i(\sum_{f>N} m_f - \sum_{f\le N} m_f)}\sin\pi\lambda}{\sin(\pi(N-2)\lambda)} + \frac{(-1)^{N-1}}{2}\cos\pi(\lambda - \sum_f m_f)\,.
\end{aligned}
\tag{2.200}
$$

This is identical to the $U(N)$ result (2.120) except the last two terms. This subtle mismatch suggests that there is another 't Hooft loop operator whose vev is given as above.

2.8 Discussion

Let us conclude with remarks on future directions and related works.

We focused on conformal $\mathcal{N} = 2$ gauge theories because localization calculations are the cleanest for them. Loop operators in non-conformal asymptotically free theories also exhibit rich dynamics [26] and the spectrum of BPS states is often simpler. The easiest way to compute correlation functions in such theories would be to start with a conformal theory and decouple some matter fields by sending their mass to infinity. It would be interesting to study this limit in detail.

Our calculation of the loop operator vevs, or the supersymmetric index (2.11), in terms of the complexified Fenchel-Nielsen coordinates made use of the equivariant index theorem. It is amusing to note that the calculation of the supersymmetric index in terms of the Fock-Goncharov coordinates can also be formulated in terms of an index theorem, but applied to the moduli space constructed from the Seiberg-Witten prepotential governing the IR dynamics [53, 54] .

In our computational scheme for the monopole screening contributions $Z_{\rm mono}(B, v)$ in $\mathcal{N} = 2^*$ theory, the 't Hooft operator is S-dual to the Wilson operator in a product of fundamental representations. As such the 't Hooft operator is reducible, i.e., it can be written as a linear combination of other loop operators with positive coefficients. Related to this is the fact that the 't Hooft operator vev (2.119) in the $G = U(2)$ theory with $N_f = 4$ fundamental hypermultiplets becomes $SO(8)_{\rm F}$ Weyl-invariant not just by substituting $(a_1, a_2) \to (a, -a)$, but only after adding an a-independent term in (2.194). It is important to develop a method intrinsic to irreducible loop operators for gauge group $SU(N)$ rather than $U(N)$. This may involve decomposing the cohomology of the monopole moduli space into irreducible representations of the Langlands dual gauge group [55] and incorporate the computation of operator product expansions [34, 38, 56].

We found that the loop operators in $\mathcal{N} = 2$ theories on $S^1 \times \mathbb{R}^3$ realize a deformation quantization of the Hitchin moduli space. We expect that this can be explained in the framework of [57], by dimensionally reducing the theory on the circle parametrized by τ as well as the one parametrized by the polar angle in the 12-plane. We would obtain a $(4,4)$ sigma model on a half plane whose target space is the Hitchin moduli space, and the boundary condition would correspond to the canonical coisotropic brane. Liouville/Toda conformal blocks arise as open string states by including another boundary mapped to the brane of opers. It would be interesting to study these systems in more detail and understand the appearance of the Weyl transform.

Some of the results in [42] obtained by the wall-crossing formula can be reproduced from our results that are obtained directly by localization calculations. It would be interesting to further explore the relation between the UV and IR theories as well as the integrability aspects of the loop operators.

3 Instantons on A_1 ALE

This section is based on my paper [2]. S-duality is an equivalence of two quantum field theories. This enables us to understand strongly coupled regimes of theories from perturbative analysis of the weakly coupled theories. Some of four-dimensional $\mathcal{N} = 2$ superconformal gauge theories have S-duality and they are important examples since in these theories the instanton partition functions, which are nonperturbative objects, are exactly known and we can see how they behave under the S transformation.

In [5] it was proposed that four-dimensional $\mathcal{N} = 2$ superconformal gauge theories on $\mathbb{C}^2$ and S^4 with direct products of $SU(2)$ gauge symmetries are related to the Liouville field theory on the corresponding Riemann surfaces. S-duality is realized as a modular transformation of the two-dimensional conformal field theory. This correspondence was extended in [12] to non-conformal $SU(2)$ gauge theories with $N_f < 4$ fundamental hypermultiplets. The instanton partition functions of these theories are shown to be equal to the norms or the inner products of the states called the Whittaker vectors in the Virasoro Verma module in Liouville theory.

More recently, in [13] this correspondence was further generalized to the correspondence between the $\mathcal{N} = 2$ $SU(2)$ pure gauge theory on $\mathbb{C}^2/\mathbb{Z}_2$ and the Neveu-Schwarz sector of $\mathcal{N} = 1$ super Liouville theory. Since there are noncontractible cycles in the asymptotic region of $\mathbb{C}^2/\mathbb{Z}_2$, the gauge field can have nontrivial holonomies $U = \exp(i \oint A)$ around them. Thus $U(2)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_2$ have four sectors, where $U = \exp(i \oint A) =$

$(1,1),(-1,-1),(1,-1)$ and $(-1,1)$[19]. The last two sectors do not exist in $SU(2)$ gauge theories. We will denote the instanton partition function of the sector with the holonomy $U = (e^{\pi i q_1},\ e^{\pi i q_2})$ by $Z^{\rm inst}_{q_1,q_2}$. In this case q_1 and q_2 can take values 0 and 1. In [13] it was shown that the instanton partition functions of the sectors with the holonomies $U = (1,1)$ and $U = (-1,-1)$ are equal to the norms of the Whittaker vectors of the Neveu-Schwarz sector as follows

$$\begin{aligned} Z^{\rm pure,\,inst}_{0,0} &= {}_{\rm NS,even}\langle\Lambda|\Lambda\rangle_{\rm NS,even} && (3.1)\\ Z^{\rm pure,\,inst}_{1,1} &= {}_{\rm NS,odd}\langle\Lambda|\Lambda\rangle_{\rm NS,odd}\,, && (3.2)\end{aligned}$$

where $|\Lambda\rangle_{\rm NS,even}$ and $|\Lambda\rangle_{\rm NS,odd}$ are the sums of states with integer and half-integer levels in the Verma module, respectively.

In [14] the gauge theory side of the correspondence was formulated on the A_1 ALE space, which is a resolution of $\mathbb{C}^2/\mathbb{Z}_2$. In [15, 16] the correspondence between the instanton partition functions of $SU(2)$ superconformal gauge theories on $\mathbb{C}^2/\mathbb{Z}_2$ or the A_1 ALE space and the conformal blocks in the Neveu-Schwarz sector of the super Liouville theory on the corresponding Riemann surfaces was also checked.

In [58] it was checked explicitly that the $SU(2)$ pure gauge theory on $\mathbb{C}^2/\mathbb{Z}_4$ corresponds to the parafermionic field theory with the spin 4/3-fractional supercurrent by showing that the instanton partition functions with $SU(2)$ holonomies coincide with the norms of the Whittaker vectors in the sectors analogous to the Neveu-Schwarz sector.

It is a natural question what is a counterpart of the Ramond sector in the gauge theories. We propose that the sectors with the holonomies $U = (1,-1)$ and $U = (-1,1)$ in $U(2)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_2$ correspond to the Ramond sector in super Liouville theory.[20] As strong evidence in favor of this proposal, we will check the following relations

$$Z^{\rm pure,\,inst}_{0,1} = Z^{\rm pure,\,inst}_{1,0} = {}_{\rm R+}\langle\Lambda^2|\Lambda^2\rangle_{\rm R+} = (-i)\times{}_{\rm R-}\langle\Lambda^2|\Lambda^2\rangle_{\rm R-}\,, \qquad (3.3)$$

in the case of the $U(2)$ pure gauge theory. The Whittaker vectors in the Ramond sector $|\Lambda^2\rangle_{\rm R\pm}$ are defined in section 3.2. In addition, we will check the following relations

$$\begin{aligned} Z^{N_f=1,\,\rm inst}_{0,1\,;\,1} &= Z^{N_f=1,\,\rm inst}_{1,0\,;\,0} = {}_{\rm R+}\langle\Lambda^2/2|\Lambda,m\rangle^{(1)}_{\rm R+} = (-i)\times{}_{\rm R-}\langle\Lambda^2/2|\Lambda,m\rangle^{(1)}_{\rm R-} && (3.4)\\ Z^{N_f=1,\,\rm inst}_{0,1\,;\,0} &= Z^{N_f=1,\,\rm inst}_{1,0\,;\,1} = {}_{\rm R+}\langle\Lambda^2/2|\Lambda,m\rangle^{(2)}_{\rm R+} = (-i)\times{}_{\rm R-}\langle\Lambda^2/2|\Lambda,m\rangle^{(2)}_{\rm R-}\,, && (3.5)\end{aligned}$$

in the case of $U(2)$ gauge theory with a fundamental hypermultiplet. The states $|\Lambda,m\rangle^{(1)}_{\rm R\pm}$ and $|\Lambda,m\rangle^{(2)}_{\rm R\pm}$ are also defined in section 3.2 and r in $Z^{N_f=1,\ \rm inst}_{q_1,q_2\,;\,r}$ denotes a flavor Wilson line.

[19] Holonomy $U = {\rm diag}(e^{\pi i q_1}, e^{\pi i q_2})$ is denoted just by $U = (e^{\pi i q_1}, e^{\pi i q_2})$ in this thesis.

[20] In the pure gauge theory the $U(1)$ part of $U = \mp i(i,-i)$ is decoupled since the fields are in the adjoint representation. In the theory with a fundamental hypermultiplet, the $U(1)$ part may be thought of as the Wilson line for the flavor symmetry. In either case, the total background holonomy acting on the fields gives a well-defined representation of $\pi_1(S^3/\mathbb{Z}_2) = \mathbb{Z}_2$.

As we will explain in section 3.1, the instanton partition functions are computed for theories with not only the holonomies of the gauge field but also the flavor Wilson line around the cycle at infinity.

As a natural extension of [5], we expect that S-duality of $\mathcal{N} = 2$ $U(2)$ superconformal gauge theories on $S^4/\mathbb{Z}_2$ corresponds to modular invariance of the super Liouville theory on the corresponding Riemann surfaces. For example, if we transform the coupling constant τ in the $\mathcal{N} = 2^*$ $U(2)$ gauge theory into $-1/\tau$, the complex structure τ of the one-punctured torus is transformed into $-1/\tau$ on the super Liouville theory side. This modular transformation exchanges the two cycles in the torus and therefore mixes the Neveu-Schwarz and the Ramond sectors. Thus the correspondence between each gauge theory sector and each super Liouville theory sector suggests that in $U(2)$ gauge theories, the S transformation $\tau \to -1/\tau$ closes only when we consider not only the sectors with the holonomies $U = (1, 1)$ and $U = (-1, -1)$ but also the sectors with the non-$SU(2)$ holonomies $U = (1, -1)$ and $U = (-1, 1)$.

The rest of this section is organized as follows. In section 3.1 we calculate the instanton partition functions of the $\mathcal{N} = 2$ $U(2)$ gauge theories with and without a fundamental hypermultiplet on $\mathbb{C}^2/\mathbb{Z}_2$. In section 3.2 we define the Whittaker vectors in the Ramond sector and show that the norms or the inner products of the vectors equal the instanton partition functions.

3.1 Gauge theory on the ALE spaces

In this section, we calculate the instanton partition functions of the sectors with the holonomies $U = (1, -1)$ and $U = (-1, 1)$ in the case of $\mathcal{N} = 2$ $U(2)$ gauge theories with and without a fundamental hypermultiplet on $\mathbb{C}^2/\mathbb{Z}_2$. Before we perform this calculation in section 3.1.2, we review how to calculate instantons in a general $\mathcal{N} = 2$ $U(N)$ gauge theory on $\mathbb{C}^2/\mathbb{Z}_p$ following [59] in section 3.1.1 .

3.1.1 Instanton counting in the general case

Now let us consider an $\mathcal{N} = 2$ $U(N)$ gauge theory on $\mathbb{C}^2/\mathbb{Z}_p$, where $\mathbb{Z}_p$ acts on $(z_1, z_2) \in \mathbb{C}^2$ as $(z_1, z_2) \to (e^{2\pi i/p} z_1, e^{-2\pi i/p} z_2)$. Since there are noncontractible cycles in the asymptotic region of $\mathbb{C}^2/\mathbb{Z}_p$, the gauge field can have a nontrivial holonomy $U = \exp(i \oint A) = (e^{2\pi i q_1/p}, \cdots, e^{2\pi i q_N/p})$,where $0 \le q_1, \cdots, q_N \le p-1$, around them. If it has the configuration with the holonomy at a saddle point in the path integral and all the fields are covariantly constant with respect to the gauge field with this configuration, then the periodicities of the fields are twisted by the holonomy. That is, running once around the noncontractible cycle on $\mathbb{C}^2/\mathbb{Z}_p$, the fields receive the gauge transformation U. Since the $\mathbb{Z}_p$ action rotates the fields once around the noncontractible cycle, this action causes not only the transformation

$(z_1, z_2) \to (e^{2\pi i/p} z_1, e^{-2\pi i/p} z_2)$ but also the gauge transformation U. In order to get the partition functions of the $U(N)$ gauge theory on $\mathbb{C}^2/\mathbb{Z}_p$, we must perform path integral over the space of the field configurations on $\mathbb{C}^2$ that are invariant under this $\mathbb{Z}_p$ action.

Now we recall how to calculate the instanton partition function of the $\mathcal{N} = 2$ $U(N)$ gauge theory on $\mathbb{C}^2$. Later we will extract the contribution of $\mathbb{Z}_p$-invariant field configurations from this partition function. To regularize the volume of the moduli space of k instantons, we perform a group action $U(1)^2 \times U(1)^N$ on the moduli space and consider only the contributions from the instantons corresponding to the fixed points under this group action. Choosing an element of the group $U(1)^2 \times U(1)^N$ whose generator is labeled by $\xi = (\epsilon_1, \epsilon_2, a_1, \cdots, a_N)$, we can write the instanton partition function as

$$Z(\xi) = \sum_{\text{fixed points}} \frac{1}{\prod_i w_i(\xi)|_{\text{fixed points}}} \,. \tag{3.6}$$

The right hand side is the sum over the fixed points and each summand is the contributions from the fluctuations around the instanton configuration corresponding to the fixed point in the moduli space. We denote the set of the weights of the action labeled by ξ on the tangent space of the moduli space by $\{w_i(\xi)\}$. The equivariant parameters $a_1, \cdots, a_N$ are related to the vacuum expectation value of the adjoint scalar field in the vector multiplet as $\langle \phi \rangle = \mathrm{diag}(a_1, \cdots, a_N)$.

The moduli space of k instantons can be constructed from the two complex vector spaces, $V = \mathbb{C}^k$ and $W = \mathbb{C}^N$. The fixed points are labeled by N-tuples of Young tableaux $(Y_1, ..., Y_N)$ with k boxes. At the fixed points the vector space V becomes a representation of the action. The weights of this representation are described by the Young tableaux, i.e. each box (i, j) in the diagram Y_m corresponds to the weight $a_m + (i-1)\epsilon_1 + (j-1)\epsilon_2$. The explicit formulae of the instanton partition functions of $U(N)$ gauge theories are written in Section 1.2.

Then let us consider the instanton partition function on $\mathbb{C}^2/\mathbb{Z}_p$. We have to perform two modifications to that on $\mathbb{C}^2$ (3.6). First we have to choose the fixed points that are invariant under the $\mathbb{Z}_p$ action and sum over only invariant fixed points. Secondly we have to choose the contributions from the fluctuations whose configurations are invariant under the $\mathbb{Z}_p$ action.

We explain which fixed point is invariant under the $\mathbb{Z}_p$ action. As explained above, each fixed point is labeled by a N-tuple of Young tableaux and then we will assign a charge of $\mathbb{Z}_p$ to each box in the Young tableaux. To do it, we determine a charge of each parameter of $\epsilon_1, \epsilon_2, a_1, \cdots$ and a_N. The parameters ϵ_1 and ϵ_2 are related to two subspaces $\mathbb{C}$ of $\mathbb{C}^2$

(a) $(Y_1, Y_2) = (\{2,1\},\{1\})$ $U = (1,1)$ (b) $(Y_1, Y_2) = (\{2,1\},\{1\})$ $U = (1,-1)$

Figure 11: Examples of charge assignment to boxes in the Young tableaux

respectively and they transform under the $\mathbb{Z}_p$ action as follows

$$\begin{aligned}\epsilon_1 &\rightarrow \epsilon_1 + \frac{2\pi}{p} \\ \epsilon_2 &\rightarrow \epsilon_2 - \frac{2\pi}{p}\,.\end{aligned} \tag{3.7}$$

Thus we think of the charge of ϵ_1 and ϵ_2 as 1 and -1 respectively. On the other hand, since $\{a_m\}$ $(1 \leq m \leq N)$ is the set of the weights on the space W, which is a fundamental representation space of $U(N)$, they transform as follows under the gauge transformation $U = (e^{\pi i q_1/p}, \cdots, e^{\pi i q_N/p})$ caused by the $\mathbb{Z}_p$ action:

$$a_m \rightarrow a_m + \frac{2\pi q_m}{p}\,. \tag{3.8}$$

Thus the charge of a_m is q_m. As explained above, the box (i,j) in the Young tableau Y_m corresponds to a one-dimensional subspace the weight on which is $a_m + (i-1)\epsilon_1 + (j-1)\epsilon_2$. Therefore the charge of this box is $q_m + (i-1) - (j-1)$.

For example, let us think of a $U(2)$ gauge theory on $\mathbb{C}^2/\mathbb{Z}_2$ and the 2-tuple of the Young tableaux $(Y_1, Y_2) = (\{2,1\},\{1\})$ in the following two cases. In the case of the holonomy $U = (1,1)$, we determine the charge of each box in these Young tableaux as in (a) in figure 1 and in the case of the holonomy $U = (1,-1)$, the charge of each box is as in (b). For the holonomy $U = (e^{i\pi q_1}, e^{i\pi q_2})$, the box at the bottom left $(i,j) = (1,1)$ in each of Y_1 and Y_2 has charge q_1 and charge q_2. As described in Figure 11, where the charges are shown modulo $p = 2$, every time we move to the right by one step, the charge increases by one and every time we move up by one step, the charge decreases by one. We denote the number of the boxes with charge 0 and 1 by k_0 and k_1 respectively. In the case (a), $(k_0, k_1) = (2,2)$ and in the case (b), $(k_0, k_1) = (1,3)$.

Returning to the general $U(N)$ gauge theory on $\mathbb{C}^2/\mathbb{Z}_p$, we can define $\{k_q\}$ $(0 \leq q \leq p-1)$ similarly. There is a relation between the holonomy and $\{k_q\}$ [59, 60]. Let us look at the N diagonal elements of the $N \times N$ matrix U representing the holonomy. We denote the number of the diagonal elements with the value $\exp(2\pi i q/p)$ by N_q $(0 \leq q \leq p-1)$. The relation is as follows

$$c_1(E) = \sum_{q=0}^{p-1} \left(N_q - 2k_q + k_{q+1} + k_{q-1}\right) c_1(T_q)\,, \tag{3.9}$$

where $c_1(E)$ is the first Chern class of the gauge bundle E and $c_1(T_q)$ is the first Chern class of a vector bundle T_q whose base space is $\mathbb{C}^2/\mathbb{Z}_p$ and whose fiber space is a complex one-dimensional space. This fiber bundle keeps in account the fact that parallel transporting a section of the bundle along a noncontractible cycle on the base space gives the holonomy $e^{\pi i q/p}$. The vector bundle T_0 has a trivial connection and therefore $c_1(T_0) = 0$. In this thesis, we only consider the case where $N = p = 2$ and $c_1(E)/c_1(T_1) = 0, 1$. In the case $c_1(E)/c_1(T_1) = 0$, the relation (3.9) becomes $0 = N_1 - 2(k_1 - k_0)$ with $N_1 = 0, 2$ and in the case $c_1(E)/c_1(T_1) = 1$, the relation (3.9) becomes $1 = N_1 - 2(k_1 - k_0)$ with $N_1 = 1$. That is, we consider the following three cases in this thesis.

$$\begin{array}{lll} \text{Case 1} & N_1 = 0, & k_1 - k_0 = 0 \\ \text{Case 2} & N_1 = 2, & k_1 - k_0 = 1 \\ \text{Case 3} & N_1 = 1, & k_1 - k_0 = 0 \end{array} \tag{3.10}$$

Therefore we must choose the appropriate pairs of Young tableaux when we calculate the instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_2$ with the specified holonomy. For example, for the holonomy $U = (1, 1)$, or equivalently $N_1 = 0$, we must choose the pairs of the Young tableaux where $k_1 - k_0 = 0$. The pair of the Young tableaux $(Y_1, Y_2) = (\{2, 1\}, \{1\})$ in figure 1 satisfies $k_1 - k_0 = 0$ in this case. On the other hand, for the holonomy $U = (1, -1)$, or equivalently $N_1 = 1$, we must choose the pairs of the Young tableaux where $k_1 - k_0 = 0$ again. The pair of the Young tableaux in figure 1 does not satisfy $k_1 - k_0 = 0$ in this case. Therefore this pair of the Young tableaux contributes to $Z^{\text{inst}}_{0,0}$ in (3.1) but does not contribute to $Z^{\text{inst}}_{0,1}$ in (3.3).

In addition to the restriction on N-tuples of Young tableaux, we must restrict the weights in each summand of (3.6) . For example, in the case of the $U(2)$ pure gauge theory on $\mathbb{C}^2$, the summand corresponding to the fixed point labeled by the pair of the Young tableaux $(Y_1, Y_2) = (\{1, 1\}, \{0\})$ is the product of the following eight weights

$$\begin{aligned} Z^{U(2)\ \text{pure}}_{\{1,1\}\{0\}} = & \frac{1}{(-a_1 + a_2)(-a_1 + a_2 - \epsilon_1)(\epsilon_1)(2\epsilon_1)(\epsilon_2)(-\epsilon_1 + \epsilon_2)(a_1 - a_2 + \epsilon_1 + \epsilon_2)} \\ & \times \frac{1}{(a_1 - a_2 + 2\epsilon_1 + \epsilon_2)} , \end{aligned} \tag{3.11}$$

according to the Nekrasov formulae (1.36) . In order to get the instanton partition function on $\mathbb{C}^2/\mathbb{Z}_2$ from that on $\mathbb{C}^2$, we must choose the fluctuations that have the appropriate periodicity on $\mathbb{C}^2/\mathbb{Z}_2$. In the case of the trivial holonomy, in order for the fields to be single-valued on $\mathbb{C}^2/\mathbb{Z}_2$, we should choose the fluctuations whose configurations have even parity on $\mathbb{C}^2$. The weights corresponding to the fluctuations that satisfy this condition are ones that are invariant under (3.7) with $p = 2$ modulo 2π. In the example above, $(-a_1+a_2)$, $(2\epsilon_1)$, $(-\epsilon_1+\epsilon_2)$ and $(a_1 - a_2 + \epsilon_1 + \epsilon_2)$ satisfy the condition. Then the contribution from the pair of the

Young tableaux $(Y_1, Y_2) = (\{1,1\}, \{0\})$ to the instanton partition function on $\mathbb{C}^2/\mathbb{Z}_2$ is

$$\frac{1}{(-a_1 + a_2)(2\epsilon_1)(-\epsilon_1 + \epsilon_2)(a_1 - a_2 + \epsilon_1 + \epsilon_2)} \,. \tag{3.12}$$

For a nontrivial holonomy U, the periodicity condition is twisted. Then the weights corresponding to the fluctuations with the appropriate periodicity are those invariant modulo 2π under the simultaneous transformation of (3.7) and (3.8). For example, for the holonomy $U = (1, -1)$, or equivalently $(q_1, q_2) = (0, 1)$, we must extract the weights invariant modulo 2π under

$$\begin{array}{rcl} a_1 & \rightarrow & a_1 \\ a_2 & \rightarrow & a_2 + \pi \\ \epsilon_1 & \rightarrow & \epsilon_1 + \pi \\ \epsilon_2 & \rightarrow & \epsilon_2 - \pi \,, \end{array} \tag{3.13}$$

from (3.11). Therefore the contribution from the pair of the Young tableaux $(Y_1, Y_2) = (\{1,1\}, \{0\})$ to the instanton partition function on $\mathbb{C}^2/\mathbb{Z}_2$ is

$$\frac{1}{(-a_1 + a_2 - \epsilon_1)(2\epsilon_1)(-\epsilon_1 + \epsilon_2)(a_1 - a_2 + 2\epsilon_1 + \epsilon_2)} \,. \tag{3.14}$$

To summarize there are two rules in calculating the instanton partition functions with the holonomy $U = (e^{\pi i q_1}, \cdots, e^{\pi i q_N})$ of $U(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_p$. One is which N-tuple of Young tableaux we should choose. We should assign charge $q_m + (i-1) - (j-1)$ to the box in th i-th column and the j-the row in a Young tableaux Y_m $(1 \leq m \leq N)$ and choose the N-tuples of the Young tableaux that satisfy the relation (3.9), where k_q denotes the number of the boxes with charge q and N_q denotes the number of the diagonal elements with the value $e^{\pi i q}$ in the holonomy U. The other rule is which weights we should extract in the contribution from each N-tuple of Young tableaux in (1.36). The weights that are invariant modulo 2π under the transformations (3.7) and (3.8) contribute to the instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_p$.

In the next subsection, we will apply the rules to calculate the instanton partition functions of the $U(2)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_2$ with and without a fundamental hypermultiplet for the holonomy $U = (1, -1)$ and $(-1, 1)$.

3.1.2 Instanton contributions corresponding to the Ramond sector

Now we calculate the instanton partition function of the sector with the holonomy $U = (1, -1)$ in the $\mathcal{N} = 2$ $U(2)$ pure gauge theory on $\mathbb{C}^2/\mathbb{Z}_2$. As explained below the relation (3.9), this relation becomes $1 = N_1 - 2(k_1 - k_0)$ with $N_1 = 1$, that is, $k_0 = k_1$ in this case and we must choose the pairs of the Young tableaux that satisfy $k_0 = k_1$. Since the instanton number is $k = k_0 + k_1$, we must consider only an even number of instantons. While the 1-instanton factor $\exp(-S^{\text{inst}}_{k=1})$ in the $U(2)$ pure gauge theory on $\mathbb{C}^2$ is Λ^4 as written in Section 1.2, that in the gauge theory on $\mathbb{C}^2/\mathbb{Z}_2$ is Λ^2 since the volume of $\mathbb{C}^2/\mathbb{Z}_2$ is half that of $\mathbb{C}^2$. Therefore the instanton partition function in this case can be written as follows

$$Z^{\text{pure inst}}_{0,1} = \sum_{N\in\mathbb{Z}_{\geq 0}} \Lambda^{4N} Z^{\text{pure},\,(2N)}_{0,1}\,, \tag{3.15}$$

where $Z^{\text{pure},\,(2N)}_{0,1}$ is the contribution from $2N$ instantons with the holonomy labeled by $(q_1, q_2) = (0, 1)$. First we calculate the 2-instantons partition function. In this case all the pairs of the Young tableaux (Y_1, Y_2) that have two boxes

$$\begin{aligned}&(\{1,1\},\{0\}),\ (\{2\},\{0\}),\ (\{0\},\{1,1\}),\ (\{0\},\{2\}),\\ &(\{1\},\{1\})\end{aligned}$$

satisfy the condition (3.9).

<u>Contribution from ({1,1},{0})</u>

Substituting $a_1 = -a_2 = a$ into (3.14), the contribution from $(\{1,1\},\{0\})$ to the partition function is

$$z^{(\{1,1\},\{0\})}_{0,1} = \frac{1}{(-2a-\epsilon_1)(2\epsilon_1)(-\epsilon_1+\epsilon_2)(2a+2\epsilon_1+\epsilon_2)}, \tag{3.16}$$

where $z^{(Y_1,Y_2)}_{q_1,q_2}$ denotes the contribution from the pair of the Young tableaux (Y_1, Y_2) when the holonomy is $U = (e^{\pi i q_1}, e^{\pi i q_2})$. The contributions $z^{(\{2\},\{0\})}$, $z^{(\{0\},\{1,1\})}$ and $z^{(\{0\},\{2\})}$ can be obtained from (3.16) through the transformations $(\epsilon_1 \leftrightarrow \epsilon_2), (a \leftrightarrow -a)$ and $(\epsilon_1 \leftrightarrow \epsilon_2,\ a \leftrightarrow -a)$. The transformation $(a \leftrightarrow -a)$, or equivalently $(a_1 \leftrightarrow a_2)$ not only changes the pair of the Young tableaux $(Y_1, Y_2) = (\{1,1\},\{0\})$ to the pair $(Y_1, Y_2) = (\{0\},\{1,1\})$ but also changes the $\mathbb{Z}_2$ transformation of a_1 and a_2. Since we would like to fix the holonomy now, the change of the $\mathbb{Z}_2$ transformation may give different results from what we want in a general case. However, in the case of the pure gauge theory, all the fields are in the adjoint representation and the periodicities of them are the same whether $U = (1, -1)$ or $U = (-1, 1)$. Therefore even if we change which of a_1 and a_2 shifts by π under the $\mathbb{Z}_2$ transformation, the partition function does not change.

<u>Contribution from ({1},{1})</u>

Similarly extracting weights invariant modulo 2π under the $\mathbb{Z}_2$ transformation (3.13) from the summand corresponding to $(\{1\},\{1\})$ in the Nekrasov formulae (1.36) and substituting $a_1 = -a_2 = a$ into them

$$\begin{aligned}&(\epsilon_1)^2(a_1-a_2+\epsilon_1)(a_1-a_2-\epsilon_1)(\epsilon_2)^2(a_1-a_2+\epsilon_2)(a_1-a_2-\epsilon_2)\\ \to\ &(2a+\epsilon_1)(2a-\epsilon_1)(2a+\epsilon_2)(2a-\epsilon_2),\end{aligned}$$

we can get the contribution from the pair of the Young tableaux as follows

$$z_{0,1}^{(\{1\},\{1\})} = \frac{1}{(2a+\epsilon_1)(2a-\epsilon_1)(2a+\epsilon_2)(2a-\epsilon_2)}. \tag{3.17}$$

Then the total contribution from 2 instantons for the holonomy $U = (1, -1)$ is

$$\begin{aligned} Z_{0,1}^{\text{pure},\,(2)} &= z_{0,1}^{(\{1,1\},\{0\})} + (\epsilon_1 \leftrightarrow \epsilon_2) + (a \leftrightarrow -a) + (\epsilon_1 \leftrightarrow \epsilon_2, a \leftrightarrow -a) \\ &\quad + z_{0,1}^{(\{1\},\{1\})} \\ &= -\frac{2(2a^2 - 2\epsilon_1^2 - 5\epsilon_1\epsilon_2 - 2\epsilon_2^2)}{\epsilon_1\epsilon_2(-2a+2\epsilon_1+\epsilon_2)(2a+2\epsilon_1+\epsilon_2)(-2a+\epsilon_1+2\epsilon_2)(2a+\epsilon_1+2\epsilon_2)}\,. \end{aligned} \tag{3.18}$$

Next we calculate the contribution from 4 instantons. The pairs of the Young tableaux that satisfy $k_0 = k_1$ are as follows

$$\begin{array}{llll} (\{1,1,1,1\},\{0\}), & (\{4\},\{0\}), & (\{0\},\{1,1,1,1\}), & (\{0\},\{4\}), \\ (\{2,1,1\},\{0\}), & (\{3,1\},\{0\}), & (\{0\},\{2,1,1\}), & (\{0\},\{3,1\}), \\ (\{2,2\},\{0\}), & & (\{0\},\{2,2\}), & \\ (\{1,1,1\},\{1\}), & (\{3\},\{1\}), & (\{1\},\{1,1,1\}), & (\{1\},\{3\}), \\ (\{1,1\},\{1,1\}), & (\{2\},\{2\}), & & \\ (\{1,1\},\{2\}, & (\{2\},\{1,1\})\,. & & \end{array} \tag{3.19}$$

Then we can calculate the contributions from these pairs of the Young tableaux in the same way as the case of 2 instantons. They are written in appendix A.5.1. Summing them up, we get the total contribution from 4 instantons as follows

$$\begin{aligned} Z_{0,1}^{\text{pure},\,(4)} =& \frac{2\big(16\epsilon_1^4 + 108\epsilon_1^3\epsilon_2 + 202\epsilon_1^2\epsilon_2^2 + 108\epsilon_1\epsilon_2^3 + 16\epsilon_2^4 - a^2(20\epsilon_1^2 + 33\epsilon_1\epsilon_2 + 20\epsilon_2^2) + 4a^4\big)}{\epsilon_1^2\epsilon_2^2(-2a+2\epsilon_1+\epsilon_2)(2a+2\epsilon_1+\epsilon_2)(-2a+4\epsilon_1+\epsilon_2)(2a+4\epsilon_1+\epsilon_2)} \\ &\times \frac{1}{(-2a+\epsilon_1+2\epsilon_2)(2a+\epsilon_1+2\epsilon_2)(-2a+\epsilon_1+4\epsilon_2)(2a+\epsilon_1+4\epsilon_2)}\,. \end{aligned} \tag{3.20}$$

We will compare these results (3.18) and (3.20) with super Liouville theory in the next section.

In the case of the holonomy $U = (-1, 1)$, we have to change the $\mathbb{Z}_2$ transformation (3.13). As written below (3.16), in the pure gauge theory the value of the partition function is the same whether $U = (1, -1)$ or $U = (-1, 1)$, that is, $Z_{0,1}^{\text{pure},\,(k)} = Z_{1,0}^{\text{pure},\,(k)}$.

If we calculate the instanton partition functions of theories with fundamental hypermultiplets, we can see the difference between the values for the holonomies $U = (1, -1)$ and $U = (-1, 1)$. For example, in the case of the theory with $N_f = 1$ fundamental hypermultiplet, the instanton partition functions with these holonomies can be written as follows

$$Z_{q_1,q_2\,;\,r}^{N_f=1} = \sum_{N\in\mathbb{Z}_{\geq 0}} \Lambda^{3N}\ Z_{q_1,q_2\,;\,r}^{N_f=1,\,(2N)}\,, \tag{3.21}$$

where q_1 and q_2 label the holonomy of the gauge field, $U = (e^{\pi i q_1}, e^{\pi i q_2})$, and r labels the Wilson line for the flavor symmetry around the noncontractible cycle. In the case where $r = 0$ or 1, we keep invariant weights up to 2π under the shifts (3.13) together with the shift of the mass of the hypermultiplet, $m \to m + r\pi$. This shift corresponds to the fact that the hypermultiplet fields receive the flavor symmetry transformation when we rotate the fields around the cycle. Using the Nekrasov formulae in Section 1.2, we can calculate the 2-instanton partition functions as follows

$$\begin{aligned} Z^{N_f=1,(2)}_{0,1;0} &= Z^{N_f=1,(2)}_{1,0;1} \\ &= \frac{2(\epsilon_1^3+\epsilon_2^3)+7\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2)-m(4\epsilon_1^2+10\epsilon_1\epsilon_2+4\epsilon_2^2)+3a\epsilon_1\epsilon_2-2a^2(\epsilon_1+\epsilon_2-2m)}{\epsilon_1\epsilon_2(-2a+2\epsilon_1+\epsilon_2)(2a+2\epsilon_1+\epsilon_2)(-2a+\epsilon_1+2\epsilon_2)(2a+\epsilon_1+2\epsilon_2)} \end{aligned} \tag{3.22}$$

$$\begin{aligned} Z^{N_f=1,(2)}_{1,0;0} &= Z^{N_f=1,(2)}_{0,1;1} \\ &= \frac{2(\epsilon_1^3+\epsilon_2^3)+7\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2)-m(4\epsilon_1^2+10\epsilon_1\epsilon_2+4\epsilon_2^2)-3a\epsilon_1\epsilon_2-2a^2(\epsilon_1+\epsilon_2-2m)}{\epsilon_1\epsilon_2(-2a+2\epsilon_1+\epsilon_2)(2a+2\epsilon_1+\epsilon_2)(-2a+\epsilon_1+2\epsilon_2)(2a+\epsilon_1+2\epsilon_2)}\,. \end{aligned} \tag{3.23}$$

The 4-instanton partition functions are given in (A.28). If we change the holonomy of the gauge field as $U \to -U$ and the flavor Wilson line, the periodicities of all the fields including the fundamental fields are the same. Therefore the relations $Z^{N_f=1}_{0,1;0} = Z^{N_f=1}_{1,0;1}$ and $Z^{N_f=1}_{1,0;0} = Z^{N_f=1}_{0,1;1}$ hold.

3.2 Whittaker vectors in the Ramond sector

In a general conformal field theory, eigenstates of the Virasoro generators L_1 and L_2 in a Verma module are called Whittaker vectors. Now we will consider the Ramond sector in $\mathcal{N} = 1$ super Liouville theory corresponding to $\mathcal{N} = 2$ $U(2)$ gauge theories with the non-$SU(2)$ holonomies on $\mathbb{C}^2/\mathbb{Z}_2$. The Lagrangian of $\mathcal{N} = 1$ super Liouville theory, the super-Virasoro algebra and Ramond primary states are written in appendix A.4. Then we consider the Verma module for the primary states $|\alpha\rangle_{\mathrm{R}\pm}$ in the Ramond sector where $\alpha = a + Q/2$. We propose that the norms of certain Whittaker vectors $|\Lambda^2\rangle_{\mathrm{R}\pm}$ coincide with the instanton partition functions with the scalar vev $(a, -a)$ and the non-$SU(2)$ holonomies of the $U(2)$ pure gauge theory. The Whittaker vectors $|\Lambda^2\rangle_{\mathrm{R}\pm}$ are defined by the following conditions

$$L_1|\Lambda^2\rangle_{\mathrm{R}\pm} = \frac{\Lambda^2}{2}|\Lambda^2\rangle_{\mathrm{R}\pm} \tag{3.24}$$

$$G_1|\Lambda^2\rangle_{\mathrm{R}\pm} = 0\,. \tag{3.25}$$

Since the second condition implies that $L_2|\Lambda^2\rangle_{\mathrm{R}\pm} = 0$, these states are Whittaker vectors. The vectors $|\Lambda^2\rangle_{\mathrm{R}\pm}$ can be written as a following power series in Λ^2

$$|\Lambda^2\rangle_{\mathrm{R}\pm} = \sum_{N\in\mathbb{Z}_{\geq 0}} \Lambda^{2N}|N\rangle_{\mathrm{R}\pm}\,, \tag{3.26}$$

where the states $|N\rangle_{\mathrm{R}\pm}$ have a level N in the Verma module and satisfy the following conditions

$$L_1|N\rangle_{\mathrm{R}\pm} = \frac{1}{2}|N-1\rangle_{\mathrm{R}\pm} \tag{3.27}$$
$$G_1|N\rangle_{\mathrm{R}\pm} = 0\,. \tag{3.28}$$

The states $|0\rangle_{\mathrm{R}\pm}$, $|1\rangle_{\mathrm{R}\pm}$ and $|2\rangle_{\mathrm{R}\pm}$ in the Whittaker vectors (3.26) can be written as follows

$$|0\rangle_{\mathrm{R}\pm} \equiv |\alpha\rangle_{\mathrm{R}\pm} \tag{3.29}$$
$$|1\rangle_{\mathrm{R}\pm} = x_1^{(1,\pm)}\, L_{-1}|0\rangle_{\mathrm{R}\pm} + x_2^{(1,\pm)}\, G_{-1}|0\rangle_{\mathrm{R}\mp} \tag{3.30}$$
$$\begin{aligned}|2\rangle_{\mathrm{R}\pm} &= x_1^{(2,\pm)}\, L_{-1}^2|0\rangle_{\mathrm{R}\pm} + x_2^{(2,\pm)}\, L_{-2}|0\rangle_{\mathrm{R}\pm}\\ &\quad + x_3^{(2,\pm)}G_{-2}|0\rangle_{\mathrm{R}\mp} + x_4^{(2,\pm)}L_{-1}G_{-1}|0\rangle_{\mathrm{R}\mp}\,.\end{aligned} \tag{3.31}$$

The coefficients $x_1^{(1,\pm)}$ and $x_2^{(1,\pm)}$ in (3.30) are determined by the conditions (3.27) and (3.28) as follows

$$x_1^{(1,\pm)} = \frac{3c+16\Delta}{4\,(3c\Delta+16\Delta^2+9\beta^2)} \tag{3.32}$$
$$x_2^{(1,\pm)} = \frac{\mp(3\pm 3i)\beta}{\sqrt{2}\,(3c\Delta+16\Delta^2+9\beta^2)}\,, \tag{3.33}$$

where Δ denotes the conformal dimension of the Ramond primary states $\Delta_\alpha^{\mathrm{R}\pm}$ in (A.26), c denotes the central charge (A.21) and $\beta = -a/\sqrt{2}$. The coefficients in (3.31) are determined as (A.29). Then if we use BPZ conjugates to determine the norms and choose the convention where ${}_{\mathrm{R}+}\langle\alpha|\alpha\rangle_{\mathrm{R}+} = (-i)\times{}_{\mathrm{R}-}\langle\alpha|\alpha\rangle_{\mathrm{R}-} = 1$, the norms of the states $|1\rangle_{\mathrm{R}\pm}$ and $|2\rangle_{\mathrm{R}\pm}$ are as follows

$${}_{\mathrm{R}+}\langle 1|1\rangle_{\mathrm{R}+} = (-i)\times{}_{\mathrm{R}-}\langle 1|1\rangle_{\mathrm{R}-} = \frac{9c^2\Delta+96c\Delta^2+256\Delta^3+54c\beta^2-288\Delta\beta^2-432\beta^4}{8(3c\Delta+16\Delta^2+9\beta^2)^2} \tag{3.34}$$

and as (A.30). Comparing these norms with the instanton partition functions $Z_{0,1}^{\text{pure},\,(2)}$ (3.18) and $Z_{0,1}^{\text{pure},\,(4)}$ (3.20) and using the fact that $Z_{0,1}^{\text{pure},\,(N)} = Z_{1,0}^{\text{pure},\,(N)}$, we find the following relation

$$Z_{0,1}^{\text{pure},\,(2N)} = Z_{1,0}^{\text{pure},\,(2N)} = {}_{\mathrm{R}+}\langle N|N\rangle_{\mathrm{R}+} = (-i)\times{}_{\mathrm{R}-}\langle N|N\rangle_{\mathrm{R}-}\,. \tag{3.35}$$

This relation is equivalent to (3.3) because of (3.15) and (3.26).

Next we introduce another set of Whittaker vectors $|\Lambda, m\rangle^{(s)}_{\mathrm{R}\pm}$ and show that the inner products of them and the vectors (3.26) equal to the instanton partition functions of the $U(2)$ gauge theory with $N_f = 1$. The Whittaker vectors $|\Lambda, m\rangle^{(s)}_{\mathrm{R}\pm}$ $(s = 1, 2)$ are defined in terms of $c^{(1)}_{\pm} := (\pm 1 + i)/2$ and $c^{(2)}_{\pm} := (\mp 1 - i)/2$ as follows

$$|\Lambda, m\rangle^{(s)}_{\mathrm{R}\pm} = \sum_{N \in \mathbb{Z}_{\geq 0}} \Lambda^N |N, m\rangle^{(s)}_{\mathrm{R}\pm}, \tag{3.36}$$

where the states $|N, m\rangle^{(s)}_{\mathrm{R}\pm}$ are level-N states in the same Verma module as the states $|N\rangle_{\mathrm{R}\pm}$ in (3.26) and satisfy

$$L_1 |N, m\rangle^{(s)}_{\mathrm{R}\pm} = -\left(m - \frac{Q}{2}\right) |N-1, m\rangle^{(s)}_{\mathrm{R}\pm} \tag{3.37}$$

$$G_1 |N, m\rangle^{(s)}_{\mathrm{R}\pm} = c^{(s)}_{\pm} \, |N-1, m\rangle^{(s)}_{\mathrm{R}\mp}, \tag{3.38}$$

for $N \geq 1$ and $|0, m\rangle^{(s)}_{\mathrm{R}\pm} \equiv |0\rangle_{\mathrm{R}\pm}$. The parameter m is the mass of the fundamental hypermultiplet in the corresponding gauge theory and transformed into $Q - m$ by Weyl symmetry[21].

The states $|1, m\rangle^{(s)}_{\mathrm{R}\pm}$ and $|2, m\rangle^{(s)}_{\mathrm{R}\pm}$ are determined to be as follows

$$|1, m\rangle^{(s)}_{R\pm} = y^{(1,\pm,s)}_1 \, L_{-1}|0\rangle_{R\pm} + y^{(1,\pm,s)}_2 \, G_{-1}|0\rangle_{R\mp} \tag{3.39}$$

$$\begin{aligned} |2, m\rangle^{(s)}_{\mathrm{R}\pm} &= y^{(2,\pm,s)}_1 \, L^2_{-1}|0\rangle_{\mathrm{R}\pm} + y^{(2,\pm,s)}_2 \, L_{-2}|0\rangle_{\mathrm{R}\pm} \\ &\quad + y^{(2,\pm,s)}_3 G_{-2}|0\rangle_{\mathrm{R}\mp} + y^{(2,\pm,s)}_4 L_{-1}G_{-1}|0\rangle_{\mathrm{R}\mp}, \end{aligned} \tag{3.40}$$

where

$$y^{(1,\pm,s)}_1 = -\frac{(3c + 16\Delta)(m - Q/2) \mp 6(1 \mp i)\sqrt{2} c^{(s)}_{\pm} \, \beta}{2(3c\,\Delta + 16\Delta^2 + 9\beta^2)} \tag{3.41}$$

$$y^{(1,\pm,s)}_2 = \frac{8\Delta \, c^{(s)}_{\pm} \pm 3(1 \pm i)\sqrt{2} \, (m - Q/2)\beta}{3c\,\Delta + 16\Delta^2 + 9\beta^2} \tag{3.42}$$

and the coefficients in (3.40) are given in (A.31), (A.32), (A.33) and (A.34). Substituting the values of $c^{(s)}_{+}$ and $c^{(s)}_{-}$, we get the following inner products

$${}_{\mathrm{R}+}\langle 1|1, m\rangle^{(1)}_{\mathrm{R}+} = (-i) \times {}_{\mathrm{R}-}\langle 1|1, m\rangle^{(1)}_{\mathrm{R}-} = -\frac{3c(m - Q/2) + 16\Delta(m - Q/2) - 6\sqrt{2}\beta}{4(3c\Delta + 16\Delta^2 + 9\beta^2)} \tag{3.43}$$

$${}_{\mathrm{R}+}\langle 1|1, m\rangle^{(2)}_{\mathrm{R}+} = (-i) \times {}_{\mathrm{R}-}\langle 1|1, m\rangle^{(2)}_{\mathrm{R}-} = -\frac{3c(m - Q/2) + 16\Delta(m - Q/2) + 6\sqrt{2}\beta}{4(3c\Delta + 16\Delta^2 + 9\beta^2)} \tag{3.44}$$

[21] In [12], m is defined to be the mass parameter that is transformed into $-m$ by Weyl symmetry.

and (A.35). Comparing (3.22), (3.23) and (A.28) with (3.43), (3.44) and (A.35), we find the following relations

$$Z^{N_f=1,\,(2N)}_{0,1\,;\,1} = Z^{N_f=1,\,(2N)}_{1,0\,;\,0} = \frac{1}{2^N}\,{}_{\mathrm{R}+}\langle N|N,m\rangle^{(1)}_{\mathrm{R}+} = -\frac{i}{2^N}\times{}_{\mathrm{R}-}\langle N|N,m\rangle^{(1)}_{\mathrm{R}-} \tag{3.45}$$

$$Z^{N_f=1,\,(2N)}_{0,1\,;\,0} = Z^{N_f=1,\,(2N)}_{1,0\,;\,1} = \frac{1}{2^N}\,{}_{\mathrm{R}+}\langle N|N,m\rangle^{(2)}_{\mathrm{R}+} = -\frac{i}{2^N}\times{}_{\mathrm{R}-}\langle N|N,m\rangle^{(2)}_{\mathrm{R}-}\,. \tag{3.46}$$

These relations are equivalent to (3.4) and (3.5) because of (3.21), (3.26) and (3.36).

4 Discussion

In Section 2, we saw that the loop operators along S^1 in 4d $\mathcal{N}=2$ supersymmetric gauge theories on $S^1\times\mathbb{R}^3$ correspond to the Verlinde operators in Liouville or Toda theory.

In case of $SU(N)$ $N_f=2N$ gauge theory, our 't Hooft loop operators does not match the corresponding Verlinde operator completely. For $N=2$, only the first term in the second line in (2.194) cannot be reproduced from gauge theory calculation (2.119). It suggests that there is another 't Hooft loop operator where the singularities of the monopole moduli space are dealt differently and its vev matches the Verlinde operator. Although we do not know how to define the second 't Hooft loop operators so far, there is also a similar subtlety in the instanton partition functions on ALE spaces. There are two definitions of these partition functions that deal differently with the singularities in the instanton moduli space and the results of these two instanton partition functions do not coincide in case of $SU(N)$ $N_f=2N$ gauge theory as observed in [61], where these two instanton partition functions are called "orbifolded instantons" and "instantons in the resolved spaces". Furthermore the authors of [62] proposed and checked quantitatively that the orbifold instantons of $SU(2)$ gauge theories on A_1 ALE space correspond to our 't Hooft loop operators with magnetic charge $B=(1,-1)$ of the same theories on $S^1\times\mathbb{R}^3$. Therefore we expect that the other instantons partition functions on A_1 ALE space will give us a hint about how to define the second 't Hooft loop operators.

Let us turn to the origin of 4d-2d correspondence. We twisted the periodicity of the fields along the S^1 and the parameter of twisting corresponds to the parameter b in Liouville theory. As a future work, we want to derive it from 6d (2,0) theory. We put 6d (2,0) theory on $S^1\times\mathbb{R}^3\times C$ with surface operator along $S^1\times\rho$, where $\rho\subset C$. If we compactify it on S^1, we will obtain 5d SYM on $\mathbb{R}^3\times C$ with a loop operator along ρ. If we further compactify it on $\mathbb{R}^3$, we expect that we will obtain Verlinde operators in Liouville theory on C. We want to check it. Since we twisted the periodicity along the S^1 in the original 6d theory, we have to consider how the reduced 5d SYM is influenced by it and how the twisting parameter will become the parameter b in Liouville theory after compactification.

In Section 3 we saw that the instanton partition function on $M_4 := \mathbb{R}^4/\mathbb{Z}_2$ corresponds to super Liouville theory.

As a future work, we want to derive that super Liouville theory appears from M5-brane theory and show that each gauge theory sector corresponds to each super Liouville sector from the 6d theory. In 4d theory, the gauge field A_μ has holonomy along a non-contractible cycle s in the asymptotic region of $\mathbb{R}^4/\mathbb{Z}_2$. The gauge field A_μ in 4d comes from the self-dual 2-form $B_{\mu\nu}$ in the 6d theory.

In order to construct the 6d theory that will be reduced to the above 4d theory with holonomy, the self-dual 2-form B should have holonomy along $s \times \rho$ where ρ is in C. We want to show that compactification of this 6d theory on M_4 gives super Liouville theory but it is impossible to demonstrate it directly since we do not know the Lagrangian of the 6d theory so far.

We need S^1 in a four-dimensional manifold where the gauge theory lives, since compactification of the 6d theory on S^1 gives the well-known 5d SYM theory. Then we want the geometry that includes S^1 and that yields super Liouville theory. Before considering it, let us recall the relation among instantons on $\mathbb{R}^4$, the vev of the loop operators on S^4, the vev of the loop operators on $S^1 \times \mathbb{R}^3$. The instanton partition function on $\mathbb{R}^4$ corresponds to the conformal blocks in Liouville theory. Next we considered S^4. It can be made from two patches that are locally $\mathbb{R}^4$. The gauge theory on this manifold corresponds to the correlation functions in Liouville theory. Furthermore if we insert the loop operators along the equator on S^4, it corresponds to the vev of Verlinde operators in Liouville theory. If we consider $S^1 \times \mathbb{R}^3$ and insert the loop operators along S^1, we can extract the contribution to the vev on S^4 from the equator as explained in Section 2.5.5. So it also corresponds to the Verlinde operators in Liouville theory.

If we replace the geometry $\mathbb{R}^4$ in the first step with $\mathbb{R}^4/\mathbb{Z}_2$, we will see a similar correspondence. The instanton partition functions on $\mathbb{R}^4/\mathbb{Z}_2$ correspond to super Liouville theory. As S^4 can be made from two patches that are locally $\mathbb{R}^4$ and correspond to the northern and southern hemispheres respectively, we consider the four-dimensional compact manifold $M_4^{(1)}$ that can be made from two copies of $\mathbb{R}^4/\mathbb{Z}_2$. There is a noncontractible cycle in the equator of this manifold and the gauge field has nontrivial holonomy along it. Furthermore we can insert the loop operators along this equator. Next we consider the geometry $M_4^{(2)}$ that is locally the same as the above manifold but does not have the north and south poles and extends to the infinities. We expect that the vevs of the loop operators on this manifold corresponds to Verlinde operators in super Liouville theory. In this case, we expect that each gauge theory sector corresponds to each super Liouville sector again.

This four-dimensional manifold $M_4^{(2)}$ has a S^1 fiber on which we put the holonomy and the loop operators. Then after putting 6d theory on $M_4^{(2)} \times C$, we can compactify the 6d theory on this S^1 and we will obtain 5d SYM. If we further compactify it on the remaining three-dimensional manifold in $M_4^{(2)}$, we might obtain Verlinde operators in super Liouville theory.

The dimensional reduction from 6d to 5d is nontrivial since we have nontrivial holonomy along the S^1. We do not know how the reduced 5d SYM is influenced by this holonomy. The reduced 5d SYM should depend on the value of the holonomy. I hope we can predict the influence on 5d SYM from the assumption that this 5d SYM gives the Neveu-Schwarz and Ramond sectors of super Liouville theory depending on the holonomy and it will help us understand a new aspect of the 6d theory.

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A Appendix

A.1 Spinors and gamma matrices

Chiral spinors Ψ and ϵ transform in a representation of $Spin(10)$, whose generators are constructed from 32×32 matrices $\mathbf{\Gamma}^M$ obeying

$$\{\mathbf{\Gamma}^M, \mathbf{\Gamma}^N\} = 2\eta^{MN} \qquad M = 0, 1, \dots, 9\,. \tag{A.1}$$

We use the Euclidean signature $\eta^{MN} = \delta^{MN}$. We can take $\mathbf{\Gamma}^M$ in the form

$$\mathbf{\Gamma}^M = \begin{pmatrix} 0 & \widetilde{\Gamma}^M \\ \Gamma^M & 0 \end{pmatrix}, \tag{A.2}$$

where $\Gamma^M \equiv (\Gamma^0, \Gamma^1, \dots, \Gamma^9)$ and $\widetilde{\Gamma}^M \equiv (-\Gamma^0, \Gamma^1, \dots, \Gamma^9)$ are 16×16 matrices that satisfy

$$\widetilde{\Gamma}^M \Gamma^N + \widetilde{\Gamma}^N \Gamma^M = 2\delta^{MN}\,, \qquad \Gamma^M \widetilde{\Gamma}^N + \Gamma^N \widetilde{\Gamma}^M = 2\delta^{MN}\,. \tag{A.3}$$

We also use notation $\Gamma^{MN} \equiv \widetilde{\Gamma}^{[M}\Gamma^{N]}$, $\widetilde{\Gamma}^{MN} \equiv \Gamma^{[M}\widetilde{\Gamma}^{N]}$, and $\Gamma^{MNPQ} \equiv \widetilde{\Gamma}^{[M}\Gamma^N\widetilde{\Gamma}^P\Gamma^{Q]}$. Our spinors have positive chirality with respect to the chirality matrix

$$\mathbf{\Gamma} \equiv -i\mathbf{\Gamma}^1 \dots \mathbf{\Gamma}^9\mathbf{\Gamma}^0 = \begin{pmatrix} -i\widetilde{\Gamma}^1\Gamma^2 \dots \widetilde{\Gamma}^9\Gamma^0 & 0 \\ 0 & -i\Gamma^1\widetilde{\Gamma}^2 \dots \Gamma^9\widetilde{\Gamma}^0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.4}$$

In ten dimensions with Euclidean signature the chiral spinor representation is complex. We take $\Gamma^1, \ldots, \Gamma^9$ to be real and $\Gamma^0 = i$ pure imaginary. As in [10], for the explicit expressions we use matrices as defined in appendix A of [9] with a permutation of spacetime indices. Let $\underline{\Gamma}^M$ be the gamma matrices in [9]. Then our Γ^M are given by

$$\begin{aligned} \Gamma^M = \underline{\Gamma}^{M+1} \quad \text{for} \quad M = 1,2,3,5,6,7\,, \\ \Gamma^4 = \underline{\Gamma}^1\,, \quad \Gamma^8 = \underline{\Gamma}^5\,, \quad \Gamma^9 = \underline{\Gamma}^9\,, \quad \Gamma^0 = i\underline{\Gamma}^0. \end{aligned} \tag{A.5}$$

The factor of i in the relation to Γ^0 arises because our present conventions use the Euclidean metric $\eta^{MN} = \delta^{MN}$, while [9] used the Lorentz metric with $\eta^{00} = -1$.

For off-shell supersymmetry, we need a set of spinors ν^i ($i = 1, \ldots, 7$) that satisfy the relations [9, 63]

$$\begin{aligned} \epsilon\Gamma^M \nu_i &= 0\,, \\ \frac{1}{2}(\epsilon\Gamma_N\epsilon)\widetilde{\Gamma}^N_{\alpha\beta} &= \nu^i_\alpha \nu^i_\beta + \epsilon_\alpha\epsilon_\beta\,, \\ \nu_i\Gamma^M\nu_j &= \delta_{ij}\epsilon\Gamma^M\epsilon\,. \end{aligned} \tag{A.6}$$

Explicitly, we take

$$\begin{aligned} \nu_j =& \Gamma^{8,j+4}\epsilon \qquad j = 1,2,3\,, \\ \nu_4 =& \Gamma^{89}\epsilon\,, \\ \nu_j =& \Gamma^{8,j-4}\epsilon \qquad j = 5,6,7\,. \end{aligned} \tag{A.7}$$

We also use the standard Pauli matrices σ^i, $i = 1,2,3$, defined as

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\,, \quad \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}\,, \quad \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\,. \tag{A.8}$$

A.2 Differential operators for the one-loop determinants

In this appendix we derive the differential operators whose indices enter the one-loop calculations.

We will need the relations inverse to (2.46) and (2.47):

$$\begin{aligned} \widetilde{A}_M &= X_{0M} \qquad M = 1, \ldots, 9\,, \\ \widetilde{\Phi}_0 &\equiv \widetilde{A}_0 = iX'_{18} + iX_{04} \\ \Psi_M &= X'_{0M} - D_{(0)M}X_{18} \qquad M = 1, \ldots, 9\,, \\ K_j &= X'_{1j} + i\sum_{M=1}^{9}\sum_{N=1}^{9}(\overline{\nu}_j\Gamma^{MN}\epsilon)D_{(0)M}X_{0N} \qquad j = 1, \ldots, 7\,, \\ \Upsilon_j &= X_{1j} \qquad j = 1, \ldots, 7 \\ c &= X_{18}\,, \qquad \widetilde{c} = X_{19}\,, \quad \widetilde{b} = X'_{19}\,. \end{aligned} \tag{A.9}$$

Then the quadratic part of $\widehat{V}$ is given by

$$\widehat{V}^{(2)} = \int d^4x \,\mathrm{Tr} \left(\sum_{M=1,2,3,9} (X'_{0M} - D_{(0)M} X_{18})(-D_{(0)\tau} X_{0M} + D_{(0)M} X_{04}) \right.$$
$$\left. + \sum_{j=1}^{7} X_{1j} \left(X'_{1j} + 2i(\nu_j \widetilde{\Gamma}^{NP} \overline{\epsilon}) D_{(0)N} X_{0P} \right) + X_{19} \left(i \sum_{M=1,2,3,9} D^M_{(0)} X_{0M} + \frac{\xi}{2} X'_{19} \right) \right) . \tag{A.10}$$

From this we read off D_{10}:

$$\begin{aligned} &(D_{10} \cdot X_0)^7_{j=1} \\ =&2i \sum_{k=1}^{3} \sum_{l=1}^{3} (\nu_j \widetilde{\Gamma}^{kl} \overline{\epsilon}) D_{(0)k} X_{0l} + 2i \sum_{k=1}^{3} \sum_{l=5}^{8} (\nu_j \widetilde{\Gamma}^{kl} \overline{\epsilon}) D_{(0)k} X_{0l} + 2i \sum_{k=1}^{3} (\nu_j \widetilde{\Gamma}^{k9} \overline{\epsilon}) D_{(0)k} X_{09} \\ &+ 2i \sum_{l=1}^{3} (\nu_j \widetilde{\Gamma}^{9l} \overline{\epsilon}) D_{(0)9} X_{0l} + 2i \sum_{l=5}^{8} (\nu_j \widetilde{\Gamma}^{9l} \overline{\epsilon}) D_{(0)9} X_{0l} \, . \end{aligned}$$

The differential operator D_{10} splits into the vector and hypermultiplet parts. Let us begin with the vector multiplet. For $j = 1, 2, 3$, we have

$$\begin{aligned} &(D_{10} \cdot X_0)^3_{j=1} \\ =& -2i\epsilon_{jkl} D_{(0)k} X_{0l} + 2i D_{(0)j} X_{09} - 2i D_{(0)9} X_{0j} \\ =& -2i(D_{\mathrm{Bogo}} \cdot X_0)_j \end{aligned} \tag{A.11}$$

where we used that $\nu_j \widetilde{\Gamma}^{kl} \overline{\epsilon} = -\epsilon_{jkl}$ for $j, k, l \in \{1, 2, 3\}$, $\nu_j \widetilde{\Gamma}^{kl} \overline{\epsilon} = 0$ for $j, k \in \{1, 2, 3\}$ and $l \in \{5, 6, 7, 8\}$, $\nu_j \widetilde{\Gamma}^{k9} \overline{\epsilon} = \delta_{jk}$ for $j, k \in \{1, 2, 3\}$, $\nu_j \widetilde{\Gamma}^{9l} \overline{\epsilon} = 0$ for $j \in \{1, 2, 3\}$ and $l \in \{5, 6, 7, 8\}$. The differential operator D_{Bogo} is the linearization of the Bogomolny equations. For $j = 9$, we get

$$(D_{10} \cdot X_0)_9 = i \sum_{M=1,2,3,9} D^M_{(0)} X_{0M} \, . \tag{A.12}$$

This is the conjugate of the linearized gauge transformation and has its origin in the gauge-fixing condition. We also have

$$(D_{10} \cdot X_0)_8 = \sum_{M=0}^{9} D^M_{(0)} \left(D_{(0)M} X_{04} - D_{(0)\tau} X_{0M} \right) \, . \tag{A.13}$$

As in [9], the computation of the symbol shows that (A.13) can be dropped by neglecting X_{04} and X_{18}, and that D_{10} acting on the vector multiplet fails to be elliptic, though we have checked that D_{10} is transversally elliptic, i.e., it is elliptic in the directions other than τ. Since we work in a non-compact space, our application of the localization formula for the index is done formally, as in the calculation of the instanton partition function.

For the hypermultiplet, we need to consider the components $j = 4, 5, 6, 7$ of (A.11):

$$\begin{aligned}&(D_{10} \cdot X_0)_{j=4}^{7}\\ =&2i \sum_{k=1}^{3} \sum_{l=5}^{8} (\nu_j \widetilde{\Gamma}^{kl} \overline{\epsilon}) D_{(0)k} X_{0l} + 2i \sum_{l=5}^{8} (\nu_j \widetilde{\Gamma}^{9l} \overline{\epsilon}) D_{(0)9} X_{0l} \,.\end{aligned}$$

This differential operator is the "realification" of the Dirac-Higgs operator

$$D_{\rm DH} \equiv \sigma^i D_{(0)i} + [\Phi_{(0)9}, \cdot\,] \tag{A.14}$$

acting on the "spinor" $2^{-1/2}(X_{05} - iX_{06} + iX_{07} + X_{08}, iX_{05} - X_{06} - X_{07} - iX_{08})^T$ and mapping to another $2^{-1/2}(iX_{11} + iX_{12} + X_{13} - X_{14}, X_{11} - X_{12} - iX_{13} - iX_{14})^T$.

A.3 $SU(2)$ holonomies on the four-punctured sphere

The Hitchin moduli space on the four-punctured sphere as a complex manifold is described by four $SL(2, \mathbb{C})$ holonomy matrices M_e ($e = 1, \ldots, 4$) satisfying $M_1 M_2 M_3 M_4 = 1$ up to conjugation with fixed conjugacy classes for M_e. We set

$$W = {\rm Tr} M_1 M_2, \,, \quad T = {\rm Tr} M_1 M_4 \,, \qquad D = {\rm Tr} M_1 M_3 \,. \tag{A.15}$$

They satisfy the identity

$$\begin{aligned}0 =&D^2 + (WT - {\rm Tr} M_1 {\rm Tr} M_3 - {\rm Tr} M_2 {\rm Tr} M_4) D\\ &+ (W - {\rm Tr} M_1 {\rm Tr} M_2)(W - {\rm Tr} M_3 {\rm Tr} M_4) + (T - {\rm Tr} M_2 {\rm Tr} M_3)(T - {\rm Tr} M_1 {\rm Tr} M_4)\\ &+ \sum_{e=1}^{4} ({\rm Tr} M_e)^2 - \prod_{e=1}^{4} {\rm Tr} M_e - 4 \,.\end{aligned} \tag{A.16}$$

We expect that the quantities W, T, and D correspond to Wilson, 't Hooft, and dyonic operators [23] in the $SU(2)$ theory with $N_f = 4$ fundamental hypermultiplets. Anticipating a match with the results of localization, we make an ansatz

$$W = x + 1/x \,, \quad T = -(y^2 + 1/y^2) Z(x) + C_1 \,, \quad D = \left(xy^2 + \frac{1}{xy^2} \right) Z(x) + C_2 \,, \tag{A.17}$$

where $Z(x)$ is a function of $x \equiv e^{2\pi i a}$, and C_1 and C_2 are independent of $y \equiv e^{2\pi i b}$. The ansatz is motivated by the localization computation, where we expect a common one-loop factor $Z(x)$ for T and D. Let us substitute these into (A.16) and organize the equation in powers of y. The minus sign in the first term in T was put by hand to ensure that there are no terms proportional to y^4 or $1/y^4$. We can choose C_1 and C_2 such that terms proportional to y^2 and $1/y^2$ also vanish. Then y drops out of the equation (A.16), which can then be solved for Z. The result is

$$\begin{aligned} Z =& 4\frac{\prod_{\pm}\prod_{i=1}^{4}\sin^{1/2}\pi(a\pm m_f)}{\sin^2 2\pi a}\,, \\ C_1 =& 2\frac{\prod_{f=1}^{4}\cos\pi m_f}{\cos^2\pi a}+2\frac{\prod_{f=1}^{4}\sin\pi m_f}{\sin^2\pi a}\,, \\ C_2 =& 2\frac{\prod_{f=1}^{4}\cos\pi m_f}{\cos^2\pi a}-2\frac{\prod_{f=1}^{4}\sin\pi m_f}{\sin^2\pi a}\,, \end{aligned} \tag{A.18}$$

where $\mathrm{Tr}M_e = e^{2\pi i\gamma_e}+e^{-2\pi i\gamma_e}$ and

$$2\gamma_1 = m_1-m_2\,,\quad 2\gamma_2 = m_1+m_2\,,\quad 2\gamma_3 = m_3+m_4\,,\quad 2\gamma_4 = m_3-m_4\,. \tag{A.19}$$

Then $-T/4$ is precisely the $\lambda = 0$ limit of (2.196).[22] These expressions for W, T, and D were given in [47] as the definition of Darboux coordinates a and b.

A.4 Super Liouville theory

The Lagrangian of $\mathcal{N}=1$ super Liouville theory is as follows

$$\mathcal{L} = \frac{1}{8\pi}(\partial_a\phi)^2+\frac{1}{2\pi}(\psi\overline{\partial}\psi+\overline{\psi}\partial\overline{\psi})+2i\mu b^2\overline{\psi}\psi e^{b\phi}+2\pi b^2\mu^2 e^{2b\phi}\,. \tag{A.20}$$

The central charge of this conformal field theory is

$$c = 1+2Q^2,\quad Q = b+\frac{1}{b}\,. \tag{A.21}$$

The algebra is as follows

$$[L_n, L_m] = (n-m)L_{n+m}+\frac{c}{8}(n^3-n)\delta_{n+m}\,, \tag{A.22}$$

$$\{G_k, G_l\} = 2L_{k+l}+\frac{c}{2}(k^2-\frac{1}{4})\delta_{k+l}\,, \tag{A.23}$$

$$[L_n, G_k] = \left(\frac{1}{2}n-k\right)G_{n+k}\,. \tag{A.24}$$

In the Neveu-Schwarz sector, n and m take integer values and k and l take half-integer values. In the Ramond sector, all of them take integer values.

In the Neveu-Schwarz sector, the primary fields can be written as $V_\alpha^{\mathrm{NS}} = e^{\alpha\phi}$ where $\alpha = \frac{Q}{2}+a$ ($a\in i\mathbb{R}$). The conformal dimensions of them are

$$\Delta_\alpha^{\mathrm{NS}} = \frac{1}{2}\alpha(Q-\alpha)\,. \tag{A.25}$$

[22] It would be nice to understand the origin of several minus signs that seem unavoidable.

In the Ramond sector, the primary fields are written as $V_\alpha^{\mathrm{R}\pm} = \sigma^\pm e^{\alpha\phi}$ with the spin field $\sigma^\pm$ with the dimension 1/16. The parameter α can be written as $\alpha = \frac{Q}{2} + a$ ($a \in i\mathbb{R}$) as in the Neveu-Schwarz sector. Both of $V_\alpha^{\mathrm{R}\pm}$ have the conformal dimension

$$\Delta_\alpha^{\mathrm{R}} = \frac{1}{16} + \frac{1}{2}\alpha(Q-\alpha)\,, \tag{A.26}$$

and G_0 acts on them as follows

$$G_0\, V_\alpha^{\mathrm{R}\pm} \;=\; i\beta\,\exp(\mp i\pi/4)\; V_\alpha^{\mathrm{R}\mp}\,,$$

where $\beta = -a/\sqrt{2}$. In section 3.2 we consider the Ramond primary states $|\alpha\rangle_{\mathrm{R}\pm}$ corresponding to the Ramond primary fields $V_\alpha^{\mathrm{R}\pm}$.

A.5 Explicit calculation

A.5.1 Instanton partition function

The contribution to the 4-instanton partition function $Z_{0,1}^{\mathrm{pure},(4)}$ (3.20) from each pair of the Young tableaux is as follows

$$\begin{aligned}
z_{0,1}^{(\{1,1,1,1\},\{0\})} &= \frac{1}{(-2a-3\epsilon_1)(-2a-\epsilon_1)(2\epsilon_1)(4\epsilon_1)(-3\epsilon_1+\epsilon_2)(-\epsilon_1+\epsilon_2)} \\
&\quad\times\frac{1}{(2a+2\epsilon_1+\epsilon_2)(2a+4\epsilon_1+\epsilon_2)} \\
z_{0,1}^{(\{2,1,1\},\{0\})} &= \frac{1}{(-2a-\epsilon_1)(2\epsilon_1)(-2a-\epsilon_2)(3\epsilon_1-\epsilon_2)(-\epsilon_1+\epsilon_2)(2a+2\epsilon_1+\epsilon_2)} \\
&\quad\times\frac{1}{(-2\epsilon_1+2\epsilon_2)(2a+\epsilon_1+2\epsilon_2)}
\end{aligned}$$

$$\begin{aligned}
z_{0,1}^{(\{2,2\},\{0\})} &= \frac{1}{(-2a-\epsilon_1)(2\epsilon_1)(-2a-\epsilon_2)(\epsilon_1-\epsilon_2)(2\epsilon_2)(-\epsilon_1+\epsilon_2)} \\
&\quad\times\frac{1}{(2a+2\epsilon_1+\epsilon_2)(2a+\epsilon_1+2\epsilon_2)} \\
z_{0,1}^{(\{1,1,1\},\{1\})} &= \frac{1}{(-2a-\epsilon_1)(2\epsilon_1)(-2a+\epsilon_1)(2a+3\epsilon_1)(2a+\epsilon_2)(-2a-2\epsilon_1+\epsilon_2)} \\
&\quad\times\frac{1}{(-\epsilon_1+\epsilon_2)(2a+2\epsilon_1+\epsilon_2)}
\end{aligned}$$

$$\begin{aligned}
z_{0,1}^{(\{1,1\},\{1,1\})} &= \frac{1}{(-2a+\epsilon_1)(2a+\epsilon_1)(2\epsilon_1)^2(-2a+\epsilon_2)(2a+\epsilon_2)(-\epsilon_1+\epsilon_2)^2} \\
z_{0,1}^{(\{2\},\{1,1\})} &= \frac{1}{(2a+\epsilon_1)(2\epsilon_1)(\epsilon_1-\epsilon_2)(-2a+2\epsilon_1-\epsilon_2)(2\epsilon_2)(-2a+\epsilon_2)} \\
&\quad\times\frac{1}{(-\epsilon_1+\epsilon_2)(2a-\epsilon_1+2\epsilon_2)}\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
Z^{\text{pure},(4)}_{0,1} &= z^{(\{1,1,1,1\},\{0\})}_{0,1} + (\epsilon_1 \leftrightarrow \epsilon_2) + (a \leftrightarrow -a) + (\epsilon_1 \leftrightarrow \epsilon_2, a \leftrightarrow -a) \\
&\quad + z^{(\{2,1,1\},\{0\})}_{0,1} + (\epsilon_1 \leftrightarrow \epsilon_2) + (a \leftrightarrow -a) + (\epsilon_1 \leftrightarrow \epsilon_2, a \leftrightarrow -a) \\
&\quad + z^{(\{2,2\},\{0\})}_{0,1} + (a \leftrightarrow -a) \\
&\quad + z^{(\{1,1,1\},\{1\})}_{0,1} + (\epsilon_1 \leftrightarrow \epsilon_2) + (a \leftrightarrow -a) + (\epsilon_1 \leftrightarrow \epsilon_2, a \leftrightarrow -a) \\
&\quad + z^{(\{1,1\},\{1,1\})}_{0,1} + (\epsilon_1 \leftrightarrow \epsilon_2) \\
&\quad + z^{(\{2\},\{1,1\})}_{0,1} + (\epsilon_1 \leftrightarrow \epsilon_2) \\
&= \frac{2\big(16\epsilon_1^4 + 108\epsilon_1^3\epsilon_2 + 202\epsilon_1^2\epsilon_2^2 + 108\epsilon_1\epsilon_2^3 + 16\epsilon_2^4 - a^2(20\epsilon_1^2 - 20\epsilon_2^2 - 33\epsilon_1\epsilon_2) + 4a^4\big)}{\epsilon_1^2\epsilon_2^2(-2a+2\epsilon_1+\epsilon_2)(2a+2\epsilon_1+\epsilon_2)(-2a+4\epsilon_1+\epsilon_2)(2a+4\epsilon_1+\epsilon_2)} \\
&\quad \times \frac{1}{(-2a+\epsilon_1+2\epsilon_2)(2a+\epsilon_1+2\epsilon_2)(-2a+\epsilon_1+4\epsilon_2)(2a+\epsilon_1+4\epsilon_2)} \,. \qquad \text{(A.27)}
\end{aligned}$$

In the case of the $U(2)$ gauge theory with $N_f = 1$, the 4-instanton partition functions $Z^{(4)}_{q_1,q_2,;r}$ with the holonomies of the gauge field $U = (e^{\pi i q_1}, e^{\pi i q_2})$ and the flavor Wilson line labeled by r, are as follows

$$\begin{aligned}
Z^{(4)}_{0,1\,;\,1} &= Z^{(4)}_{1,0\,;\,0} \\
&= \Big(32\epsilon_1^6 + 304\epsilon_1^5\epsilon_2 + 1030\epsilon_1^4\epsilon_2^2 + 1543\epsilon_1^3\epsilon_2^3 + 1030\epsilon_1^2\epsilon_2^4 + 304\epsilon_1\epsilon_2^5 + 32\epsilon_2^6 \\
&\quad + m(-128\epsilon_1^5 - 992\epsilon_1^4\epsilon_2 - 2480\epsilon_1^3\epsilon_2^2 - 2480\epsilon_1^2\epsilon_2^3 - 992\epsilon_1\epsilon_2^4 - 128\epsilon_2^5) \\
&\quad + m^2(128\epsilon_1^4 + 864\epsilon_1^3\epsilon_2 + 1616\epsilon_1^2\epsilon_2^2 + 864\epsilon_1\epsilon_2^3 + 128\epsilon_2^4) \\
&\quad + a\big(-96\epsilon_1^4\epsilon_2 - 294\epsilon_1^3\epsilon_2^2 - 294\epsilon_1^2\epsilon_2^3 - 96\epsilon_1\epsilon_2^4 + m(192\epsilon_1^3\epsilon_2 + 396\epsilon_1^2\epsilon_2^2 + 192\epsilon_1\epsilon_2^3)\big) \\
&\quad + a^2\big(-40\epsilon_1^4 - 248\epsilon_1^3\epsilon_2 - 410\epsilon_1^2\epsilon_2^2 - 248\epsilon_1\epsilon_2^3 - 40\epsilon_2^4 \\
&\qquad + m(160\epsilon_1^3 + 424\epsilon_1^2\epsilon_2 + 424\epsilon_1\epsilon_2^2 + 160\epsilon_2^3) + m^2(-160\epsilon_1^2 - 264\epsilon_1\epsilon_2 - 160\epsilon_2^2)\big) \\
&\quad + a^3\big(24\epsilon_1^2\epsilon_2 + 24\epsilon_1\epsilon_2^2 - 48m\epsilon_1\epsilon_2\big) \\
&\quad + a^4\big(8\epsilon_1^2 + 40\epsilon_1\epsilon_2 + 8\epsilon_2^2 + m(-32\epsilon_1 - 32\epsilon_2) + 32m^2\big)\Big) \\
&\times \Big(4\epsilon_1^2\epsilon_2^2(-2a+2\epsilon_1+\epsilon_2)(2a+2\epsilon_1+\epsilon_2)(-2a+4\epsilon_1+\epsilon_2)(2a+4\epsilon_1+\epsilon_2) \\
&\quad (-2a+\epsilon_1+2\epsilon_2)(2a+\epsilon_1+2\epsilon_2)(-2a+\epsilon_1+4\epsilon_2)(2a+\epsilon_1+4\epsilon_2)\Big)^{-1} . \qquad \text{(A.28)}
\end{aligned}$$

If we transform a into $-a$, we get the value of the instanton partition functions $Z^{(4)}_{1,0\,;\,1} = Z^{(4)}_{0,1\,;\,0}$.

A.5.2 Whittaker vectors

We denote $\Delta^{\rm R}_\alpha$ just by Δ in this subsection. Note that we choose the convention where the coefficient of the central charge in (A.22) is 1/8 rather than 1/12. The coefficients in (3.31) are as follows

$$
\begin{aligned}
x_1^{(2,\pm)} &= \frac{45c^2+24c\,(7+12\Delta)+32\,(28\Delta+8\Delta^2+45\beta^2)}{(3c\Delta+16\Delta^2+9\beta^2)\,(15c(7+32\Delta)+8\,(49+238\Delta+64\Delta^2+450\beta^2))}\\
x_2^{(2,\pm)} &= -\frac{3(168c+45c^2+896\Delta+288c\Delta+256\Delta^2+2112\beta^2-720c\beta^2-768\Delta\beta^2)}{8(3c\Delta+16\Delta^2+9\beta^2)(392+105c+1904\Delta+480c\Delta+512\Delta^2+3600\beta^2)}\\
x_3^{(2,\pm)} &= \frac{\mp(3\pm 3i)\beta\,(-75c+8(21+46\Delta+90\beta^2))}{\sqrt{2}(3c\Delta+16\Delta^2+9\beta^2)\,(15c\,(7+32\Delta)+8\,(49+238\Delta+64\Delta^2+450\beta^2))}\\
x_4^{(2,\pm)} &= \frac{\mp(12\pm 12i)\sqrt{2}(-14+15c+16\Delta)\beta}{(3c\Delta+16\Delta^2+9\beta^2)(15c(7+32\Delta)+8(49+238\Delta+64\Delta^2+450\beta^2))}\,. \qquad (A.29)
\end{aligned}
$$

Then the norms of the states $|2\rangle_{\mathrm{R}\pm}$ (3.31) are as follows

$$
\begin{aligned}
{}_{\mathrm{R}+}\langle 2|2\rangle_{\mathrm{R}+} &= (-i)\times {}_{\mathrm{R}-}\langle 2|2\rangle_{\mathrm{R}-}\\
&= \big(1008c^2\Delta+270c^3\Delta+10752c\Delta^2+3168c^2\Delta^2+28672\Delta^3+10752c\Delta^3+8192\Delta^4\\
&\qquad +6048c\beta^2+405c^2\beta^2-32256\Delta\beta^2+31968c\Delta\beta^2-135936\Delta^2\beta^2-22464\beta^4\\
&\qquad +25920c\beta^4-497664\Delta\beta^4-311040\beta^6\big)\\
&\quad \times\big(8(3c\Delta+16\Delta^2+9\beta^2)^2(392+105c+1904\Delta+480c\Delta+512\Delta^2+3600\beta^2)\big)^{-1}\,.
\end{aligned}
\qquad (A.30)
$$

The coefficients in (3.40) are as follows. We denote $m-Q/2$ by m_{sL} .

$$
\begin{aligned}
&y_1^{(2,\pm,s)}\\
&= 4\Big(45c^2m_{sL}^2+128\Delta^2(2m_{sL}^2-3c_+^{(s)}c_-^{(s)})+64\Delta\big(14m_{sL}^2-21c_+^{(s)}c_-^{(s)}\mp 3(1\mp i)\sqrt{2}m_{sL}c_\pm^{(s)}\beta\big)\\
&\qquad +12c\big(2(7+12\Delta)m_{sL}^2-30\Delta c_+^{(s)}c_-^{(s)}\mp 15(1\mp i)\sqrt{2}m_{sL}c_\pm^{(s)}\beta\big)\\
&\qquad\quad +24\beta\big(\pm 7\sqrt{2}(1\mp i)m_{sL}c_\pm^{(s)}+60m_{sL}^2\beta-75c_+^{(s)}c_-^{(s)}\beta\big)\Big)\\
&\quad\times\Big((3c\Delta+16\Delta^2+9\beta^2)\big(15c(7+32\Delta)+8(49+238\Delta+64\Delta^2+450\beta^2)\big)\Big)^{-1}
\end{aligned}
\qquad (A.31)
$$

$$
\begin{aligned}
&y_2^{(2,\pm,s)}\\
&= \Big(c\,\big(1920\Delta(1+2\Delta)c_+^{(s)}c_-^{(s)}\pm 540(1\mp i)\sqrt{2}m_{sL}c_\pm^{(s)}\beta-72m_{sL}^2(7+12\Delta-30\beta^2)\big)\\
&\qquad -135c^2m_{sL}^2+4096\Delta^3c_+^{(s)}c_-^{(s)}-256\Delta^2(3m_{sL}^2-64c_+^{(s)}c_-^{(s)})\\
&\qquad +48\beta\big(-132m_{sL}^2\beta+165c_+^{(s)}c_-^{(s)}\beta\mp(5\mp 5i)\sqrt{2}m_{sL}c_\pm^{(s)}(7+36\beta^2)\big)\\
&\qquad +16\Delta\big(300(\mp 1+i)\sqrt{2}m_{sL}c_\pm^{(s)}\beta+24m_{sL}^2(-7+6\beta^2)+32c_+^{(s)}c_-^{(s)}(14+45\beta^2)\big)\Big)\\
&\quad\times\Big(2(3c\Delta+16\Delta^2+9\beta^2)(15c(7+32\Delta)+8(49+238\Delta+64\Delta^2+450\beta^2))\Big)^{-1}
\end{aligned}
\qquad (A.32)
$$

$$
\begin{aligned}
y_3^{(2,\pm,s)} & \\
= 2(1\pm i)\Big(& 3c\, m_{sL}\big(4(1\mp i)(7+32\Delta)c_\pm^{(s)} \pm 75\sqrt{2}m_{sL}\beta\big) \\
& \pm 256\Delta^2 c_\pm^{(s)}\big(4(\mp 1+i)m_{sL} - 3\sqrt{2}c_\mp^{(s)}\beta\big) \\
& \mp 24\beta\big(-14\sqrt{2}c_+^{(s)}c_-^{(s)} \mp 24(1\mp i)m_{sL}c_\pm^{(s)}\beta + 3\sqrt{2}m_{sL}^2(7+30\beta^2)\big) \\
& \mp 16\Delta\big(69\sqrt{2}m_{sL}^2\beta + 27\sqrt{2}c_+^{(s)}c_-^{(s)}\beta \pm 2(1\mp i)m_{sL}c_\pm^{(s)}(7+72\beta^2)\big)\Big) \\
\times \Big(& (3c\Delta + 16\Delta^2 + 9\beta^2)\big(15c(7+32\Delta) + 8(49+238\Delta+64\Delta^2+450\beta^2)\big)\Big)^{-1} \qquad (A.33)
\end{aligned}
$$

$$
\begin{aligned}
y_4^{(2,\pm,s)} & \\
= -4\Big(& 15c\, m_{sL}\big((7+32\Delta)c_\pm^{(s)} \pm 12(1\pm i)\sqrt{2}m_{sL}\beta\big) \\
& + 2(1\pm i)\Big(128(1\mp i)\Delta^2 m_{sL}c_\pm^{(s)} + 3\beta\big(\mp 28\sqrt{2}m_{sL}^2 \pm 35\sqrt{2}c_+^{(s)}c_-^{(s)} + 60(1\mp i)m_{sL}c_\pm^{(s)}\beta\big) \\
& \qquad + \Delta\big(28(1\mp i)m_{sL}c_\pm^{(s)} \pm 96\sqrt{2}m_{sL}^2\beta \mp 240\sqrt{2}c_+^{(s)}c_-^{(s)}\beta\big)\Big)\Big) \\
\times \Big(& (3c\Delta + 16\Delta^2 + 9\beta^2)\big(15c(7+32\Delta) + 8(49+238\Delta+64\Delta^2+450\beta^2)\big)\Big)^{-1}. \qquad (A.34)
\end{aligned}
$$

The inner products in (3.45) for $N = 2$ are as follows

$$
\begin{aligned}
& {}_{\mathrm{R}+}\langle 2|2,m\rangle_{\mathrm{R}+}^{(1)} = (-i)\times {}_{\mathrm{R}-}\langle 2|2,m\rangle_{\mathrm{R}-}^{(2)} \\
& = \Big(129024c\Delta^2 + 34560c^2\Delta^2 + 688128\Delta^3 + 221184c\Delta^3 + 196608\Delta^4 + m_{sL}^2(32256c^2\Delta \\
& \qquad + 8640c^3\Delta + 344064c\Delta^2 + 101376c^2\Delta^2 + 917504\Delta^3 + 344064c\Delta^3 + 262144\Delta^4) \\
& \quad + \beta\, m_{sL}(-4536\sqrt{2}c^2 - 1215\sqrt{2}c^3 + 145152\sqrt{2}c\Delta - 31536\sqrt{2}c^2\Delta - 473088\sqrt{2}\Delta^2 \\
& \qquad + 283392\sqrt{2}c\Delta^2 - 4067328\sqrt{2}\Delta^3) \\
& \\
& + \beta^2\big(18144c + 4860c^2 + 96768\Delta + 127872c\Delta + 25920c^2\Delta + 2165760\Delta^2 - 387072c\Delta^2 \\
& \qquad - 442368\Delta^3 + m_{sL}^2(193536c + 12960c^2 - 1032192\Delta + 1022976c\Delta - 4349952\Delta^2)\big) \\
& + \beta^3 m_{sL}\,(96768\sqrt{2} - 88128\sqrt{2}c + 38880\sqrt{2}c^2 + 156672\sqrt{2}\Delta + 387072\sqrt{2}c\Delta \\
& \qquad - 10838016\sqrt{2}\Delta^2) \\
& + \beta^4\big(228096 - 77760c + 1133568\Delta - 414720c\Delta - 442368\Delta^2 + m_{sL}^2(-718848 \\
& \qquad + 829440c - 15925248\Delta)\big)
\end{aligned}
$$

$$
+\beta^5 m_{sL}(912384\sqrt{2}-311040\sqrt{2}c-6967296\sqrt{2}\Delta)-9953280\beta^6 m_{sL}^2\Big)
\times\big(64(3c\Delta+16\Delta^2+9\beta^2)^2(392+105c+1904\Delta+480c\Delta+512\Delta^2+3600\beta^2)\big)^{-1}. \tag{A.35}
$$

If we transform β into $-\beta$, we get the expression of ${}_{\mathrm{R}+}\langle 2|2,m\rangle^{(2)}_{\mathrm{R}+}=(-i)\times\ {}_{\mathrm{R}-}\langle 2|2,m\rangle^{(2)}_{\mathrm{R}-}$.

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