博　士　論　文

Boundaries and domain walls in two-dimensional supersymmetric theories

（二次元超対称理論における境界とドメインウォール）

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Abstract

We apply supersymmetric localization to $\mathcal{N} = (2, 2)$ gauge theories on a hemisphere, with boundary conditions, *i.e.*, D-branes, preserving B-type supersymmetries. We explain how to compute the hemisphere partition function for each object in the derived category of equivariant coherent sheaves, and argue that it depends only on its K theory class. The hemisphere partition function computes exactly the central charge of the D-brane, completing the well-known formula obtained by an anomaly inflow argument. We also formulate supersymmetric domain walls as D-branes in the product of two theories. In particular four-dimensional line operators bound to a surface operator, corresponding via the AGT relation to certain defects in Toda CFT's, are constructed as domain walls. Moreover we exhibit domain walls that realize the $sl(2)$ affine Hecke algebra.

Contents

1. Introduction

Superstring theory is a strong candidate of the unified theory which can describe all of the four fundamental interactions of nature; the electromagnetic interaction, the weak interaction, the strong interaction, and the gravitational interaction. The fundamental objects of the superstring theory are strings, whose dynamics draw two-dimensional surfaces in a space-time. More precisely, a string is described by a map from a world-sheet, *i.e.*, a Riemann surface to a space-time manifold. The space-time manifold is also called the target space in the context of the sigma models and the string theory. The space-time must be ten-dimensional to construct the superstring theory consistently [1]. The oscillation modes of strings describe various particles such as gravitons, gauge bosons, and so on. D-branes are also important ingredients of the superstring theory. D-branes are branes in the ten-dimensional space-time on which the world-sheet with boundaries have ends [2]. D-branes have Ramond-Ramond charges and are coupled with Ramond-Ramond fields which appear in the massless spectrum of the type II superstring theory [3]. D-branes have many interesting properties and enrich the superstring theory. (For reviews of the superstring theory, see the textbooks [4,5] for example.)

Our space-time is usually considered as four-dimensional; one time-dimension and three space-dimensions. Then, it seems that the ten-dimensional space-time is unnatural. To reconcile the idea of the superstring theory with our nature, we consider the compactification of the superstring theory. We assume that the space-time is a direct product $\mathbb{R}^{1,3} \times X$, where $\mathbb{R}^{1,3}$ is the Minkowski space-time and X is a certain six-dimensional manifold. We think that X is very small compared to $\mathbb{R}^{1,3}$ and we can not usually detect the existence of the extra dimensions. The property of X determines the physics of the four-dimensional theory we can detect. Although this might sound a good idea, we have not been able to determine which X should be chosen to describe our nature yet. This is an extremely difficult unsolved problem which is inevitable to complete the superstring theory.

To understand this problem, we should treat the simple examples firstly. For example, if we take a compact Calabi-Yau manifold as X,[1] the four-dimensional theory has $\mathcal{N} = 2$ supersymmetry [6]. Although the sting theory on Calabi-Yau manifolds and the four-dimensional $\mathcal{N} = 2$ theory are too simple to describe the complex physics of nature, the

[1] A compact Calabi-Yau manifold is a compact Kähler manifold which admits a Ricci-flat metric.

simpleness clarifies the physical and mathematical structure of the theory and enables us to analyze them by some exact methods. Therefore those theories have been thoroughly investigated by many physicists and mathematicians for a long time.

The low energy behavior of the four-dimensional physics has topological features [7,4]. Then, we do not need all the information of the superstring theory. The four-dimensional physics is captured by the topological string theories or the topological sigma models, which are obtained by certain deformations of the superstring theory called the topological twists [8,9]. They have some topological nature, *i.e.*, they are independent of the world-sheet metric nor the space-time metric. They depend on the Kähler structure or the complex structure of the space-time Calabi-Yau manifold. Then, the four-dimensional physics is determined from the geometric information of the Calabi-Yau manifold and the BPS D-branes wrapped on its special cycles. For example, the numbers of the vector multiplets and the hyper multiplets in four-dimensional $\mathcal{N}=2$ theories are determined by the hodge numbers of the Calabi-Yau manifold. The Yukawa coupling is obtained by computing the three-point functions of the topological sigma models. The Seiberg-Witten theory [10,11], which solves the low energy behavior of the four-dimensional physics exactly, is also understood in the language of the Calabi-Yau compactification and the BPS states in the four-dimensional theory are described by D-branes [12] (see also the review [13] and references therein). This line of research led to the modern description of the four-dimensional $\mathcal{N}=2$ theories in terms of M-theory [14,15,16].

An important phenomenon in topological sigma models is the mirror symmetry. This is motivated by the T-duality between the type IIA and the type IIB superstring theories. If we compactify the type IIA theory on a Calabi-Yau manifold X and obtain a four-dimensional theory, the T-duality implies the existence of the mirror manifold $X^\vee$ on which we compactify the type IIB theory and obtain the same four-dimensional theory. The existence of the pairs of two manifold X and $X^\vee$ and two equivalent theories on them is called the mirror symmetry. One side of the calculation involves non-perturbative corrections coming from world-sheet instantons and is very difficult, but on the other side the calculation is classical and exactly calculable. The power of the mirror symmetry was shown by the calculation of the Yukawa coupling and the Gromov-Witten invariant, which is roughly speaking the number of holomorphic curves, for nontrivial Calabi-Yau manifolds [17]. This can be restated as the equivalence of two types of topological sigma models (A-, B-model) on X and $X^\vee$ [18,9]. The mirror symmetry is generalized to the case of more general manifolds. Then, we will not restrict our attention to complex three-dimensional Calabi-Yau manifold. (For reviews of the mirror symmetry, see the textbook [19] for example.)

In the context of the mirror symmetry, the equivalence of two types of categories, one is the Fukaya category and the other is the derived category of coherent sheaves was conjectured [20]. This is called the homological mirror symmetry and understood as the mirror symmetry of D-branes. The D-branes in the topological A-, B-model (or the D-branes which preserves the A-, B-type supersymmetry) is called the A-, B-branes. The A-branes are described by the Fukaya categories and the B-branes are described by the derived categories of coherent sheaves. (For reviews of the homological mirror symmetry, see the reviews and textbooks [19,21,22] for example.)

The mirror symmetry turns out to connect physics and various field of mathematics such as algebraic geometry, symplectic geometry, knot theory, representation theory, number theory, and so on. In particular, it is fascinating that the geometric Langlands program, which involves many field of mathematics such as number theory, algebraic geometry, representation theory and is so-called "the grand unified theory of mathematics", can be understood in the framework of the mirror symmetry. (For reviews of the physics and the geometric Langlands program, see the review [23] and the references therein.)

In the analysis of the non-linear sigma models, the two-dimensional $\mathcal{N} = (2,2)$ gauge theories, which are also called the gauged linear sigma models (GLSM) play important roles. At low energy, such gauge theories reduce to non-linear sigma models whose target spaces are Kähler manifolds, in particular Calabi-Yau manifolds under certain conditions [24]. When we put $\mathcal{N} = (2,2)$ gauge theories on a surface with boundary, the boundary conditions describe D-branes at low energy. A boundary condition in the product of two theories can be regarded as a domain wall that connects two regions where the two theories live. It is often useful to consider the corresponding gauge theories instead of the non-linear sigma models. For example, the mirror symmetry in many cases can be understood by the two dual description of the gauge theories [25,26]. The space-time topology changing phenomena such as Ginzburg-Landau/Calabi-Yau correspondence and the flop transition [27,28] are naturally understood by the change of the phase of gauge theories [24]. The phase of gauge theories are described by the FI parameters which parametrize the geometric information of the target space appeared at the low energy.

The virtue to consider the gauge theories is partly in the powerful methods of exact analysis. In particular, we notice the supersymmetric localization method which enables us to compute the partition functions or the correlation functions of BPS observables exactly. By using the localization method, the path integral reduces to the integral over certain locus of field configurations. For good theories, the path integral reduce to a finite dimensional integral or a summation over discrete points. The localization method were first introduced in the context of topological field theories which are obtained by the twisting of the supersymmetric theories [29,8,9,30]. Since the twisting does not change the Lagrangian in flat spaces, the partition functions of the four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories in the omega deformed background were computed by using the localization method [31]. Furthermore, it is found that the localization method can be applied to four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories on $\mathbb{S}^4$ without twisting [32]. After that the localization method has been applied to various supersymmetric gauge theories on various geometries and produced a lot of exact results. The supersymmetric localization plays a pivotal role in the "Golden Age of Exact Results in SUSY QFT" [33].

Recently the supersymmetric localization was applied to $\mathcal{N} = (2, 2)$ gauge theories on a two-sphere and their partition functions, which is called the "sphere partition functions," are obtained [34,35]. From the sphere partition function, we can extract the information of low energy non-linear sigma models, such as quantum Kähler potentials and Gromov-Witten invariants [36,37,38,39]. This provides a new method to compute the Gromov-Witten invariants without using the mirror symmetry. The justification of this method was partially done by considering the theories on a deformed sphere [40]. In addition, there are many usage of the sphere partition functions. We can use the sphere partition functions to check the mirror symmetry quantitatively [40,41]. The sphere partition function is also useful to investigate the phases of the gauge theories, *i.e.*, the topology change of the target space of non-linear sigma model at the low energy [42,43]. The sphere partition functions contains the equivariant Gromov-Witten invariants and is related to the Nekrasov partition functions and Donaldson-Thomas invariants [44,45]. Via the Calabi-Yau compactification, the Seiberg-Witten Kähler potential of four-dimensional $\mathcal{N} = 2$ $SU(2)$ pure gauge theory can be deduced from a certain sphere partition function [46]. The connection between the mirror symmetry in three-dimensional theory and that in the two-dimensional theory is discussed by reducing the $\mathbb{S}^1 \times \mathbb{S}^2$ partition function to the sphere partition function [47].

The localization calculation of two-dimensional supersymmetric gauge theories on a torus yields the (equivariant) indices or, in other words, the (flavored) elliptic genera [48,49,50,51,52]. In particular, the connection between the two-dimensional $\mathcal{N} = (0, 2)$ gauge theories and the four-dimensional geometry, which is suggested from the two ways of compactification of six-dimensional $\mathcal{N} = (0,2)$ theory as the effective theory of M5-branes, is a very interesting topic [53]. In this context, the dynamics of the two-dimensional $\mathcal{N} = (0,2)$ gauge theories was studied by using the equivariant indices [54].

Now we arrive at the theme of this thesis. In the paper [55], the author of this thesis studied boundaries and domain walls in $\mathcal{N} = (2,2)$ gauge theories using supersymmetric localization. Note that two other papers [56,57], which contain some overlapped material with our paper, appeared almost at the same time. The supersymmetric localization of $\mathcal{N} = (2,2)$ gauge theories on $\mathbb{RP}^2$ and its relation to orientifold was also discussed in [58].

This thesis is based on the paper [55]. From now on, we give the more detailed explanation of our results. We consider the supersymmetric localization of $\mathcal{N} = (2,2)$ gauge theories on the hemisphere geometry, which has a single boundary component. The resulting partition function, which is called the "hemisphere partition function", is roughly a half of the sphere partition function [34,35].

There are two broad motivations for studying the hemisphere partition function. The first is the study of D-branes in Calabi-Yau manifolds, with applications to mirror symmetry, Gromov-Witten invariants, D-brane stability, string phenomenology, etc., as we have discussed so far. In such contexts the two dimensional theory describes the worldsheet of a superstring, and one is especially interested in theories that flows to a non-linear sigma model with target space a compact Calabi-Yau. Generically such a theory possesses no flavor symmetries. The hemisphere partition function depends analytically on the complexified FI parameters, which we collectively denote as t and use to parametrize the Kähler moduli space. The second motivation, the main one for us, is to study the dynamics of the two-dimensional quantum field theory in its own right. It is known that $\mathcal{N} = (2,2)$ theories are closely related to integrable models [59,60]. Such a theory also arises as the defining theory for a surface operator embedded in a four-dimensional theory [61]. It is natural to turn on twisted masses $m = (m_a)$, or equivariant parameters for flavor symmetries, in these contexts. Boundaries are interesting ingredients in the physics of the theory, while domain walls ($\simeq$ line operators in two dimensions) provide a natural example of non-local disorder operators, and are akin to 't Hooft loops [62,63,64,65], vortex loops [66,67,68], surface operators [69,70], and domain walls [71,72] in higher dimensions.

The type of boundary conditions $\mathcal{B}$ we study preserve B-type supersymmetries [9]. For abelian gauge theories general B-type boundary conditions were formulated in [73]. We extend these boundary conditions, in a straightforward way, to theories with non-abelian gauge groups and twisted masses.

We will argue that the hemisphere partition function $Z_{\text{hem}}(\mathcal{B};t;m)$ is the overlap $\langle\mathcal{B}|\mathtt{1}\rangle$ of two states, where both the boundary state $\langle\mathcal{B}|$ and the state $|\mathtt{1}\rangle$ created by a topological twist [74] are zero-energy states in the Hilbert space for the Ramond-Ramond sector.

When the gauge theory flows to a non-linear sigma model with a smooth target space, there are coarse and refined classifications of B-branes:

$$\begin{aligned}\{\text{B-branes}\} &\simeq \text{derived category of coherent sheaves}\\ \{\text{topological charges}\} &\simeq \text{K theory}\end{aligned}$$

The latter amounts to classifying B-branes up to dynamical creation and annihilation (tachyon condensation [75]) processes. For details and precise treatments on these mathematical concepts, see for example [21,22,76]. In type II string theory compactified on a Calabi-Yau, such topological charges of branes determine the central charges [77] of the extended supersymmetry algebra in non-compact dimensions. This central charge is given precisely by the overlap $\langle\mathcal{B}|\mathtt{1}\rangle$ [78]. We will argue that the hemisphere partition function $Z_{\text{hem}}(\mathcal{B})$ indeed depends only on the K theory class of the brane. The known formula for the central charge, which is valid in the large volume limit and was obtained by an anomaly inflow argument [79], provides a useful check of our result and is completed by our exact formula.

More generally, our localization computation yields a pairing $\langle\mathcal{B}|\mathtt{f}\rangle$ between the boundary state $\langle\mathcal{B}|$ and an arbitrary element $\mathtt{f}$ of the quantum cohomology ring. With twisted masses for the flavor symmetry group G_{F} turned on, the sheaves, K theories, and quantum/classical cohomologies are replaced by their G_{F}-equivariant versions. Related works that emphasize G_{F}-equivariance include [48,45]. It was found by Nekrasov and Shatashvili [59,60] that the relations in the equivariant quantum cohomology of certain models are precisely the Bethe ansatz equations of spin chains. Our work is thus related to, and in fact most directly motivated by, the study of integrable structures in supersymmetric gauge theories. Integrability suggests the presence of infinite-dimensional quantum group symmetries, whose generators are expected to be realized as domain walls. As mentioned domain walls are D-branes in product theories, and the quantum group symmetries are known to be realized geometrically as so-called convolution algebras in equivariant K theories and derived categories [76]. In this work we take a modest step in this direction by realizing the $sl(2)$ affine Hecke algebra as the domain wall algebra.[2]

Relatedly, the two-dimensional $\mathcal{N} = (2,2)$ theories can also be embedded in a four-dimensional $\mathcal{N} = 2$ theory to define a surface operator [61]. Domain walls in the two-dimensional theory can then be regarded as four-dimensional line operators bound to the surface operator, and via the AGT correspondence [80] is related to certain defects in Toda conformal field theories [81]. We use our results to identify the precise domain walls that correspond to the defects.

We also study Seiberg-like dualities. In some dual pairs of theories, the hemisphere partition functions are found to be identical, while in the others they turn out to differ by a simple overall factor. Such dualities also serve as nice checks of our result.

This thesis is organized as follows. In section 2, we will review the basic facts of $\mathcal{N} = (2,2)$ non-linear sigma models, topological sigma models and the supersymmetric localization methods. In section 3, we will review the A-, B-branes in $\mathcal{N} = (2,2)$ non-linear sigma models, the categorical description of B-branes and the brane amplitudes. In section 4, we will review how $\mathcal{N} = (2,2)$ gauge theories reduce to non-linear sigma models at low energy. We will see the examples of $\mathcal{N} = (2,2)$ gauge theories which will appear in this thesis. In section 5, we will explain the set-up of [55] by specifying the geometry and the physical actions. We will analyze the symmetries of the set-up, and define the boundary conditions that preserve B-type supersymmetries. In particular, we will review two basic sets of boundary conditions for a chiral multiplet, which we call Neumann and Dirichlet conditions (for the entire multiplet). These elementary boundary conditions are combined with the boundary interactions to provide more general boundary conditions. In section 6, we will perform localization and obtain the hemisphere partition function as an integral over scalar zero-modes. We will also provide its alternative expression as a linear combination of certain blocks given as infinite power series. The geometric interpretation of the hemisphere partition function will be explained in section 7. In particular, we will explain how to compute the hemisphere partition function for a given object in the derived category. We will give examples of the hemisphere partition function in section 8. We will match the hemisphere partition function in the large volume limit with the large-volume

[2] The connection between the domain wall and convolution algebras was suggested to us by N. Nekrasov and S. Shatashvili.

formula for the central charges of D-branes in the quintic Calabi-Yau (and for more general complete intersection Calabi-Yau's in Appendix E). Section 9 will be devoted to the study of Seiberg-like dualities. In section 10, we will study domain walls realized as D-branes in a product theory. Such domain walls can be regarded as operators that act on a hemisphere partition function. The action of certain walls are identified with monodromies of the partition function. We will also show that they realize certain defect operators of Toda theories in one case, and the $sl(2)$ affine Hecke algebra in another. Appendices collect useful formulas and detailed computations.

2. $\mathcal{N} = (2,2)$ non-linear sigma models

In this section, we briefly review some basic facts about $\mathcal{N} = (2,2)$ non-linear sigma models. First, we define the $\mathcal{N} = (2,2)$ non-linear sigma model whose target space is a Kähler manifold. Then, we see some properties of this model. The anomaly and the renormalization property which are closely related to each other. The ground states of the models are described as the cohomology of some supercharges. We can also consider the cohomology in the algebra of local operators. The state-operator correspondence between them is clarified after defining the topological twist. We also review the basic facts about the Witten type topological field theory, topological A-, B-model and the supersymmetric localization technique. We write the review of this section in reference to [19,25,82,83].

2.1. $\mathcal{N} = (2,2)$ supersymmetry on flat spaces

First, we summarize the notation used in section 2, 3, 4. We consider theories on $\mathbb{R}^2$ with metric $ds^2 = (dx^1)^2 + (dx^2)^2$. By default, we think of a spinor $\psi = (\psi_\alpha)_{\alpha=1,2}$ as a column vector. We also use the fermion index $\alpha = \pm$ instead of $\alpha = 1,2$ respectively. The indices are raised and lowered by the charge conjugation matrix

$$C = (C^{\alpha\beta}) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C^{-1} = (C_{\alpha\beta}) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

as $\psi^\alpha = C^{\alpha\beta}\psi_\beta$, $\psi_\alpha = C_{\alpha\beta}\psi^\beta$. When the upper index of ψ is contracted with the lower index of λ, we write

$$\psi\lambda := \psi^\alpha\lambda_\alpha = \psi^T C^T \lambda\,,$$

where T indicates the transpose. In flat spaces, we define the gamma matrices as the usual Pauli matrices.

$$\gamma_1 := \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \gamma_2 := \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \gamma_3 := \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

The gamma matrices γ_m ($m = 1,2,3$) have the index structure $\gamma_m = ({\gamma_m}_\alpha{}^\beta)$. A spinor bilinear is defined as

$$\psi\gamma_{m_1}\ldots\gamma_{m_n}\lambda := \psi^T C^T \gamma_{m_1}\ldots\gamma_{m_n}\lambda\,.$$

$\mathcal{N} = (2,2)$ SUSY algebra is generated by the bosonic generators $P_1, P_2, J, R, R_\mathrm{A}$, which are the generators of the translations $P_i := \nabla_i$, the rotation $U(1)_\mathrm{J}$, $J := x^2\nabla_1 + x^1\nabla_2$, the

R-symmetry $U(1)_{\mathrm{R}}$ and the axial R-symmetry $U(1)_{\mathrm{A}}$ respectively, and the supercharges $Q_{\pm}, \bar{Q}_{\pm}$. They satisfy the following algebraic relations.

$$
\begin{aligned}
&Q_\alpha^2 = \bar{Q}_\alpha^2 = 0\,, \{Q_\pm, \bar{Q}_\pm\} = \mp P_1 + iP_2\,,\\
&\{\bar{Q}_+, \bar{Q}_-\} = Z\,, \{Q_+, Q_-\} = \bar{Z}\,, \{Q_-, \bar{Q}_+\} = \tilde{Z}\,, \{Q_+, \bar{Q}_-\} = \bar{\tilde{Z}}\,,\\
&[J, Q_\pm] = \pm Q_\pm\,, [J, \bar{Q}_\pm] = \pm \bar{Q}_\pm\,,\\
&[R, Q_\alpha] = +Q_\alpha\,, [R, \bar{Q}_\alpha] = -\bar{Q}_\alpha\,, [R_{\mathrm{A}}, Q_\pm] = \pm Q_\pm\,, [R_{\mathrm{A}}, \bar{Q}_\pm] = \mp \bar{Q}_\pm\,,
\end{aligned}
$$

where $Z, \tilde{Z}$ are central charges. The supercharges satisfy the hermiticity condition

$$
Q_\alpha^\dagger = \bar{Q}_\alpha\,.
$$

2.2. $\mathcal{N} = (2,2)$ non-linear sigma models on a flat space

We review the $\mathcal{N} = (2,2)$ non-linear sigma models with the world-sheet $\mathbb{R}^2$ and the target space X which is a Kähler manifold. Let ϕ be a map $\phi : \mathbb{R}^2 \to X$. Choosing local holomorphic coordinates $z := x^1 + ix^2$ on $\mathbb{R}^2$ and ϕ^i on X, the map ϕ is described as $\phi^i(z, \bar{z})$ locally and these are the bosonic field of the non-linear sigma models. Let $\phi^* TX$ be the pullback of the complexified tangent bundle $TX = TX^{(1,0)} \oplus TX^{(0,1)}$, where $T_pX^{(1,0)}, T_pX^{(0,1)}$ at some point $p \in X$ are spanned by $\{\partial_i := \partial/\partial\phi^i, \partial_{\bar{i}} := \partial/\partial\bar{\phi}^i\}_i$ respectively. K and $\bar{K}$ are the canonical and anti canonical line bundles of $\mathbb{R}^2$, *i.e.*, their fiber at a point are spanned by dz and $d\bar{z}$ respectively. Then, the fermionic fields are defined by sections,

$$
\begin{aligned}
&\psi_+^i \in \Gamma(\phi^* TX^{(1,0)} \otimes \sqrt{K})\,, \qquad \psi_-^i \in \Gamma(\phi^* TX^{(1,0)} \otimes \sqrt{\bar{K}})\,,\\
&\bar{\psi}_+^i \in \Gamma(\phi^* TX^{(0,1)} \otimes \sqrt{K})\,, \qquad \bar{\psi}_-^i \in \Gamma(\phi^* TX^{(0,1)} \otimes \sqrt{\bar{K}})\,.
\end{aligned}
$$

The chiral multiplet consists of $(\phi^i, \psi^i, \mathrm{F}^i)$, where F^i are auxiliary fields. The R-charges of the fields $(\phi^i, \psi^i, \mathrm{F}^i)$ are (q^i, q^i+1, q^i+2), where q is a real parameter. The axial R-charges of $(\phi^i, \psi_+^i, \psi_-^i \mathrm{F}^i)$ are $(q_{\mathrm{A}}^i, q_{\mathrm{A}}^i + 1, q_{\mathrm{A}}^i - 1, q_{\mathrm{A}}^i)$, where q_{A} is a real parameter.

We take the SUSY parameters ϵ and $\bar{\epsilon}$ to be bosonic constant spinors. The SUSY variation is given by $\delta = \bar{\epsilon}Q + \epsilon\bar{Q}$. In this convention, field contents transform under SUSY as

$$
\begin{aligned}
\delta\phi^i =&\bar{\epsilon}_-\psi_+^i - \bar{\epsilon}_+\psi_-^i\,, \qquad \delta\bar{\phi}^i = \epsilon_-\bar{\psi}_+^i - \epsilon_+\bar{\psi}_-^i\,,\\
\delta\psi_+^i =& -2i\epsilon_-\partial_z\phi^i + \bar{\epsilon}_+\mathrm{F}^i\,, \qquad \delta\psi_-^i = -2i\epsilon_+\partial_{\bar{z}}\phi^i + \bar{\epsilon}_-\mathrm{F}^i\,,\\
\delta\bar{\psi}_+^i =& -2i\bar{\epsilon}_-\partial_z\bar{\phi}^i - \epsilon_+\bar{\mathrm{F}}^i\,, \qquad \delta\bar{\psi}_-^i = -2i\bar{\epsilon}_+\partial_{\bar{z}}\bar{\phi}^i - \epsilon_-\bar{\mathrm{F}}^i\,,\\
\delta\mathrm{F}^i =&2i\epsilon_+\nabla_{\bar{z}}\psi_+^i - 2i\epsilon_-\nabla_z\psi_-^i\,, \qquad \delta\bar{\mathrm{F}}^i = -2i\bar{\epsilon}_+\nabla_{\bar{z}}\bar{\psi}_+^i + 2i\bar{\epsilon}_-\nabla_z\bar{\psi}_-^i\,.
\end{aligned}
$$

These form a representation of the $\mathcal{N} = (2,2)$ SUSY algebra with $Z = \tilde{Z} = 0$. The covariant derivative on the spinors is defined as

$$\nabla_z \psi^i := \partial_z \psi^i + \partial_z \phi^l \Gamma^i_{lj} \psi^j, \; \nabla_{\bar{z}} \psi^i := \partial_{\bar{z}} \psi^i + \partial_{\bar{z}} \phi^l \Gamma^i_{lj} \psi^j \,,$$
$$\nabla_z \bar{\psi}^i := \partial_{\bar{z}} \bar{\psi}^i + \partial_z \bar{\phi}^l \Gamma^{\bar{i}}_{\bar{l}\bar{j}} \bar{\psi}^j, \; \nabla_{\bar{z}} \bar{\psi}^i := \partial_{\bar{z}} \bar{\psi}^i + \partial_{\bar{z}} \bar{\phi}^l \Gamma^{\bar{i}}_{\bar{l}\bar{j}} \bar{\psi}^j$$

where $\Gamma^i_{lj} := g^{i\bar{k}} \partial_l g_{j\bar{k}}, \Gamma^{\bar{i}}_{\bar{l}\bar{j}} := g^{\bar{i}k} \partial_{\bar{l}} g_{\bar{j}k}$ are the Christoffel symbols.

The Lagrangian of the $\mathcal{N} = (2,2)$ non-linear sigma model is described as follows. The D-term of the Lagrangian which is determined from the Kähler potential and the B-field is

$$\begin{aligned}\mathcal{L}_D := \; & 2g_{i\bar{j}}(\partial_z \phi^i \partial_{\bar{z}} \bar{\phi}^j + \partial_{\bar{z}} \phi^i \partial_z \bar{\phi}^j) + i g_{i\bar{j}} (\nabla_{\bar{z}} \bar{\psi}^j_+ \psi^i_+ - \bar{\psi}^j_+ \nabla_{\bar{z}} \psi^i_+) \\ & - i g_{i\bar{j}} (\nabla_z \bar{\psi}^j_- \psi^i_- - \bar{\psi}^j_- \nabla_z \psi^i_-) - R_{i\bar{j}k\bar{l}} \psi^i_+ \psi^k_- \bar{\psi}^j_- \bar{\psi}^l_+ \\ & - g_{i\bar{j}} \left(\mathrm{F}^i + \Gamma^i_{kl} \psi^k_+ \psi^l_- \right) \left(\bar{\mathrm{F}}^j + \Gamma^{\bar{j}}_{\bar{k}\bar{l}} \bar{\psi}^k_- \bar{\psi}^l_+ \right) \,,\end{aligned}$$

where $g_{i\bar{j}}$ is the Kähler metric of X and $R_{i\bar{j}k\bar{l}} := g_{k\bar{m}} \partial_i \Gamma^{\bar{m}}_{\bar{j}\bar{l}}$. is the curvature tensor of X. The Kähler potential $\mathcal{K}(\phi^i, \bar{\phi}^i)$ is related to the metric as $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \mathcal{K}$. In general, the D-term preserves R-symmetries if we assign $\mathcal{K}$ neither the R-charges nor axial charges. However, if $\mathcal{K}$ is a function of $|\phi^i| = \phi^i \bar{\phi}^i$, the D-term preserves R-symmetries under any R-charge assignment.

The F-term of the Lagrangian is

$$\mathcal{L}_F = -\frac{i}{2} \Big(\mathrm{F}^i \partial_i W - \psi^i_+ \psi^j_- \partial_i \partial_j W \Big) - \frac{i}{2} \Big(\bar{\mathrm{F}}^i \partial_{\bar{i}} \bar{W} - \bar{\psi}^i_+ \bar{\psi}^j_- \partial_{\bar{i}} \partial_{\bar{j}} \bar{W} \Big) \,,$$

where the superpotential W is a holomorphic function on X. To make this action invariant under R-symmetries, W should have R-charge -2 and axial R-charge 0. Therefore W should be a quasi-homogeneous function

$$W(e^{i\alpha q^i} \phi^i) = e^{2i\alpha} W(\phi^i) \,.$$

Adding the D-term and F-term and integrating out the auxiliary fields F^i, we obtain the action

$$\begin{aligned}\mathcal{L} := \; & 2g_{i\bar{j}}(\partial_z \phi^i \partial_{\bar{z}} \bar{\phi}^j + \partial_{\bar{z}} \phi^i \partial_z \bar{\phi}^j) + i g_{i\bar{j}} (\nabla_{\bar{z}} \bar{\psi}^j_+ \psi^i_+ - \bar{\psi}^j_+ \nabla_{\bar{z}} \psi^i_+) \\ & - i g_{i\bar{j}} (\nabla_z \bar{\psi}^j_- \psi^i_- - \bar{\psi}^j_- \nabla_z \psi^i_-) - R_{i\bar{j}k\bar{l}} \psi^i_+ \psi^k_- \bar{\psi}^j_- \bar{\psi}^l_+ \\ & - \frac{1}{4} g^{i\bar{j}} \partial_i W \partial_{\bar{j}} \bar{W} + \frac{i}{2} \psi^i_+ \psi^j_- \nabla_i \partial_j W + \frac{i}{2} \bar{\psi}^i_+ \bar{\psi}^j_- \nabla_{\bar{i}} \partial_{\bar{j}} \bar{W} \,.\end{aligned}$$

We often call a non-linear sigma model with a superpotential a Landau-Ginzburg model. The path integral measure from the Lagrangian is defined as

$$e^{-S}, \ S := \int_{\mathbb{R}^2} d^2x\mathcal{L} \, .$$

We can add the B-field contribution to the action,

$$S = \int_{\mathbb{R}^2} d^2x\mathcal{L} - 2\pi i \int_{\mathbb{R}^2} \phi^* B \, ,$$

where B is a closed two-form.

If X has a isometry generated by commuting holomorphic vector fields $\{V_a\}_a$, the non-linear sigma model can be deformed by

$$\mathcal{L}_{\mathcal{V}} = \frac{1}{2}\left|\sum_a \mathrm{m}_a V_a\right|^2 + \frac{1}{2}\left|\sum_a \bar{\mathrm{m}}_a V_a\right|^2 + \frac{i}{2}\sum_a \left(g_{i\bar{i}}\partial_j V_a^i - g_{j\bar{j}}\partial_{\bar{i}}\bar{V}^j\right)\left(\mathrm{m}_a\bar{\psi}_-^i\psi_+^j + \bar{\mathrm{m}}_a\bar{\psi}_+^i\psi_-^j\right) .$$

The supersymmetry is modified by the following terms

$$\begin{aligned}\Delta Q_-\psi_+ &= -i\sum_a \mathrm{m}_a V_a^i \, , \Delta Q_+\psi_- = -i\sum_a \bar{\mathrm{m}}_a V_a^i \, ,\\ \Delta\bar{Q}_-\bar{\psi}_+ &= i\sum_a \bar{\mathrm{m}}_a \bar{V}_a^i \, , \Delta\bar{Q}_+\bar{\psi}_- = i\sum_a \mathrm{m}_a \bar{V}_a^i \, .\end{aligned}$$

This deformation turns on the central charge $\tilde{Z} = i\sum_a \mathrm{m}_a\mathcal{L}_{V_a}$, where $\mathcal{L}_{V_a}$ acts on the fields as $\mathcal{L}_{V_a}\phi^i = V_a^i, \mathcal{L}_{V_a}\psi^i = \partial_j V_a^i \psi_\pm^j$.

2.3. Anomaly, renormalization and Calabi-Yau

In this subsection, we argue the anomaly and the renormalization of the $\mathcal{N} = (2,2)$ non-linear sigma model. The existence of the axial anomaly is closely related to the renormalization property. Especially we focus on the properties which depend on whether the target space X is Calabi-Yau or not.

Anomaly

In quantum theory, the R-symmetry $U(1)_{\mathrm{R}}$ suffers from no anomaly but the axial R-symmetry $U(1)_{\mathrm{A}}$ may be anomalous. For a given map ϕ, using the Fujikawa method, we find that $U(1)_{\mathrm{A}}$ is broken to $\mathbb{Z}_{2k}$ where

$$k = \int_{\mathbb{R}^2} c_1(\phi^* TX^{(1,0)}) = \langle c_1(TX), \phi_*[\mathbb{R}^2]\rangle \, .$$

$\phi_*[\mathbb{R}^2]$ is a homology class obtained by the push forward of $\mathbb{R}^2$ by the map ϕ. Under the axial R rotation, the path integral measure transforms by the factor $e^{2ik\alpha}, \alpha \in \mathbb{R}$. Since the B-field contribution is described as

$$\exp\left(2\pi i \int_{\mathbb{R}^2} \phi^* B\right) ,$$

the axial rotation is equivalent to the shift in the cohomology class

$$[B] \to [B] + \frac{\alpha}{\pi} c_1(TX) .$$

The class $2\pi[B]$ takes values in $H^2(X,\mathbb{R})/H^2(X,\mathbb{Z})$. If k is divisible by $p \in \mathbb{N}$ for any map ϕ, $U(1)_{\mathrm{A}}$ is broken to $\mathbb{Z}_{2p}$. This happens when $c_1(TX)$ is p times some element of $H^2(X,\mathbb{Z})$. To preserve the axial R-symmetry, $c_1(TX)$ should be zero, *i.e.*, X should be a Calabi-Yau manifold.

Renormalization

In general, the Kähler metric is renormalized in quantum theory. The beta function for the metric is

$$\beta_{i\bar{j}} = \frac{1}{2\pi} R_{i\bar{j}}$$

at 1-loop level. $R_{i\bar{j}}$ is the Ricci tensor of X, which is related to $c_1(TX)$ as

$$c_1(TX) = \frac{i}{2\pi} R_{i\bar{j}} dz^i \wedge d\bar{z}^j .$$

Let $g_{0i\bar{j}}$ be the bare metric and $g_{i\bar{j}}$ be the metric at the scale μ. If $R_{i\bar{j}} > 0$, the bare metric is

$$g_{0i\bar{j}} = g_{i\bar{j}} + \frac{1}{2\pi} \log\left(\frac{\Lambda_{\mathrm{UV}}}{\mu}\right) R_{i\bar{j}} ,$$

where Λ_{UV} is the UV cut-off scale. If we take the continuum limit $\Lambda_{\mathrm{UV}} \to \infty$, the metric becomes very large, and the perturbation theory becomes better in this region. Then, a sigma model on a Ricci positive Kähler manifold is asymptotic free. If $R_{i\bar{j}} = 0$, i .e ., the target space is a Calabi-Yau manifold, the sigma model is scale invariant at 1-loop level but receives the higher loop corrections. It is known that the beta function is non vanishing at 4-loop level [84]. If $R_{i\bar{j}} < 0$ the perturbation theory breaks down at high energy. Then, the sigma model on a Ricci negative Kähler manifold is not a well-defined theory.

Although the Kähler metric receives the higher loop correction, the renormalization behavior of the Kähler class $[\omega]$ is exactly described as

$$[\omega](\mu) = [\tilde{\omega}] + \log\left(\frac{\mu}{\Lambda}\right) c_1(TX)\,,$$

where $[\tilde{\omega}] \in H^2(X,\mathbb{R})\backslash(\mathbb{R}\cdot c_1(TX))$ and Λ is a scale parameter. Let $k = \dim H^2(X,\mathbb{R})$, there are k Kähler parameters which parametrize the Kähler class $[\omega]$. If $c_1(TX) \neq 0$, the dimensional transmutation occurs, *i.e.*, the scale parameter Λ replaces one of the Kähler parameters which form a coordinate system of $H^2(X,\mathbb{R})$. In this case, the shift of the class $[B]$ in the direction of $c_1(TX)$ can be cancelled by the axial rotation. The class $[B]$ is also parametrized by k B-class parameters. Then, one of the B-class parameters is unphysical and replaced by the physical parameter Λ. If $c_1(TX) = 0$, the Kähler class $[\omega]$ does not run and there is no axial anomaly. Therefore, all of the Kähler parameters and the B-class parameters are marginal.

2.4. Supersymmetric ground states and (twisted) chiral ring

In this subsection, we argue the cohomology defined by some supercharges acting on the states or the local operators. The former describes the supersymmetric ground states and the latter describes the (twisted) chiral ring.

Supersymmetric ground states

First, we consider the case $Z = \tilde{Z} = 0$. Define $Q_A := Q_+ + \bar{Q}_-, Q_B := \bar{Q}_+ + \bar{Q}_-$. From the SUSY algebra, we can deduce the following formulae.

$$\begin{aligned}
&\{Q_A, Q_A^\dagger\} = \{Q_B, Q_B^\dagger\} = 2H\,,\ H := iP_2\,,\\
&Q_A^2 = Q_B^2 = 0\,.
\end{aligned}$$

If H has a discrete spectrum, there is a one to one correspondence between the cohomology classes of Q_A or Q_B and the supersymmetric ground states. The index of Q is the Witten index which is invariant under the small perturbation of the theory.

In the case that we turn on the central charge $\tilde{Z} = i\sum_a \mathrm{m}_a \mathcal{L}_{V_a}$, the second line of the above formulae is modified as

$$Q_A^2 = \bar{\tilde{Z}} = -i\sum_a \mathrm{m}_a \mathcal{L}_{V_a}\,, \quad Q_B^2 = Z = 0\,.$$

We consider the equivariant cohomology, *i.e.*, the cohomology of Q_A restricted on the states which are invariant under the action of $\sum_a \mathrm{m}_a \mathcal{L}_{V_a}$. A continuous symmetry never breaks in two dimensions according to the Coleman's theorem [85]. Therefore, there is a one to one correspondence between the equivariant cohomology classes of Q_A and the supersymmetric ground states. Since turning on the central charge is just a small perturbation, the Witten index is the same as in the $\tilde{Z} = 0$ case.

(Twisted) Chiral ring

A chiral operator $\mathcal{O}$ is an operator which satisfies $[Q_B, \mathcal{O}] = 0$, and a twisted chiral operator $\mathcal{O}$ is an operator which satisfies $[Q_A, \mathcal{O}] = 0$. For example, the chiral multiplet scalar ϕ is a chiral operator.[3] The (twisted) chiral operators are invariant under the world-sheet translation up to $Q_B(Q_A)$-exact terms. Then, the $Q_B(Q_A)$-cohomology classes are invariant under the world-sheet translation. For two (twisted) chiral operators $\mathcal{O}_1, \mathcal{O}_2$, their product $\mathcal{O}_1\mathcal{O}_2$ is also a (twisted) chiral operator. Therefore the cohomology classes form a ring. The ring which consists of Q_B-cohomology classes is called the chiral ring, and the ring which consists of Q_A-cohomology classes is called the twisted chiral ring. When we turn on the central charge, we consider the equivariant (twisted) chiral ring.

From the state-operator correspondence, we expect that there exists a one to one correspondence between the set of supersymmetric ground states and the (twisted) chiral ring. We will come back to this subject after defining the topological A-, B-twist in the next subsection.

2.5. Topological A-, B-model

In this section, we first review the definition of the Witten type topological field theory. Then, we introduce two types of topological twisting which make topological field theories from $\mathcal{N} = (2,2)$ non-linear sigma models.

Witten type topological field theory

On a Riemannian manifold with metric $g_{\mu\nu}$, we consider a quantum field theory with a collection Φ of fields which are Grassmannian graded. Assume that this theory has a

[3] The scalar component in the twisted chiral multiplet, which we have not discussed so far, is a twisted chiral operator.

symmetry generated by a Grassmann-odd scalar generator Q which acts as a derivation and satisfies the following properties.
(1) $Q^2 = 0$.
(2) The action S is Q-exact up to topological terms, *i.e.*, $S = \{Q, V\} +$(topological terms) for some V.
(3) The energy momentum tensor $T_{\mu\nu}$ is Q-exact, *i.e.*, $T_{\mu\nu} = \{Q, G_{\mu\nu}\}$, where $G_{\mu\nu} = \delta V/\delta g^{\mu\nu}$.
A theory which satisfies the properties noted above is called a Witten type topological field theory. If Q is nilpotent up to the global symmetry of the theory, we can consider a equivariant version.

The partition function of this theory

$$Z = \int D\phi e^{-S} \,,$$

where ϕ denotes the fields of this theory, does not depend on the metric g,[4] because

$$\frac{\delta Z}{\delta g^{\mu\nu}} = -\int D\phi \{Q, G_{\mu\nu}\} e^{-S} = \langle \{Q, G_{\mu\nu}\} \rangle = 0 \,.$$

$\langle \mathcal{O} \rangle$ is the unnormalized vacuum expectation value of $\mathcal{O}$. Since Q is a symmetry of the theory, the vacuum expectation values of Q-exact operators vanish. Furthermore, the vacuum expectation values of Q-closed operators do not depend on the metric g, because

$$\frac{\delta}{\delta g^{\mu\nu}} \langle \mathcal{O} \rangle = \langle \mathcal{O} T_{\mu\nu} \rangle = \langle \{Q, \mathcal{O} G_{\mu\nu}\} \rangle = 0 \,.$$

Therefore, we consider the Q-cohomology classes as observables in topological field theory.

We also note that the semiclassical calculation of the partition function or the correlation functions is exact in Witten type topological field theory. By introducing a parameter t as

$$\langle \mathcal{O} \rangle = \int D\phi \mathcal{O} e^{-\mathrm{t}\{Q,V\} + (\text{topological terms})} \,,$$

we find that this does not depend on this parameter, because

$$\frac{\partial}{\partial \mathrm{t}} \langle \mathcal{O} \rangle = -\langle \{Q, \mathcal{O} V\} \rangle = 0 \,.$$

[4] The strategy used in this proof needs the assumption that we can do the functional version of the partial integration without yielding boundary terms. If boundary terms appear, Q-symmetry becomes anomalous. This is the origin of the holomorphic anomaly equations in topological string theory [86,87].

We assume that $\{Q, V\}$ is a positive semidefinite functional with respect to the fields ϕ. Taking the $\mathrm{t} \to \infty$ limit, the functional integral boils down to the 1-loop calculation around the saddle points ϕ^* which satisfy $\{Q, V\}[\phi^*] = 0$. This strategy for the Witten type topological field theory is the origin of the recent supersymmetric localization method.

Topological A-, B-twist

Now we introduce the topological twist. Suppose that the theory have either the R-symmetry $U(1)_{\mathrm{R}}$ or the axial R-symmetry $U(1)_{\mathrm{A}}$ with all (axial) R-charges are integer-valued. We replace $U(1)_{\mathrm{J}}$ by $U(1)'_{\mathrm{J}}$ generated by $J' := J - R$ or $J' := J + R_{\mathrm{A}}$. The former replacement is called the A-twist and the latter is called the B-twist. Since the B-twist need the axial R-symmetry, the target space of a B-twistable theory is Calabi-Yau. The twisted theory on a curved world-sheet is obtained by gauging $U(1)'_{\mathrm{J}}$ by the spin connection, which cause the modification of the spins of the fields. The energy momentum tensor is also modified as

$$T'_{\mu\nu} := T_{\mu\nu} + \frac{1}{4}(\epsilon_{\mu\lambda}\partial^{\lambda} J_{\nu} + \epsilon_{\nu\lambda}\partial^{\lambda} J_{\mu})\,,$$

where J_μ is the current of (axial) R-symmetry.

We consider the $q_{\mathrm{R}} = q_{\mathrm{A}} = 0$ case. The spins (J-charges), the R-charges, the axial R-charges and the twisted J'-charges of the fields are listed as follows.

	ϕ	ψ_+	ψ_-	$\bar{\psi}_+$	$\bar{\psi}_-$	F	$\bar{\mathrm{F}}$
J-charge	0	1	-1	1	-1	0	0
R-charge	0	1	1	-1	-1	2	-2
axial R-charge	0	1	-1	-1	1	0	0
A-twisted J'-charge	0	0	-2	2	0	-2	2
B-twisted J'-charge	0	2	-2	0	0	0	0

After the A-twist, $\phi, \psi_+, \bar{\psi}_-$ become the scalar fields, ψ_-, F become the sections of $\phi^* TX^{(1,0)} \otimes \bar{K}$ and $\bar{\psi}_+, \bar{\mathrm{F}}$ become the sections of $\phi^* TX^{(0,1)} \otimes K$. Furthermore, $Q_A := \bar{Q}_+ + Q_-$ becomes the Grassman-odd scalar operator and then can be globally defined on any world-sheet. After the B-twist, $\phi, \bar{\psi}_+, \bar{\psi}_-, \mathrm{F}, \bar{\mathrm{F}}$ become the scalar fields and ψ_+, ψ_- are the sections of $\phi^* TX^{(1,0)} \otimes K, \phi^* TX^{(0,1)} \otimes \bar{K}$ respectively. Furthermore, $Q_B := \bar{Q}_+ + \bar{Q}_-$ becomes the Grassman-odd scalar operator, and then, can be globally defined on any world-sheet.

After the A(B)-twist, the action without superpotential becomes $Q_A(Q_B)$-exact up to topological terms. The energy momentum tensor also becomes $Q_A(Q_B)$-exact. Therefore, these twists make Witten type topological field theories from $\mathcal{N} = (2,2)$ non-linear sigma models. We call these theories topological A-, B-models. It is well known that a topological A-model depends on the Kähler form of its target space but not depends on the Kähler metric and the complex structure. On the other hand, a topological B-model depends on the complex structure but not depends on the the Kähler metric and Kähler form.

For B-twisted Landau-Ginzburg model, the action is neither Q_B-exact nor topological. But the energy momentum tensor is shown to be Q_B-exact and the correlation functions are still topological. We cannot use the supersymmetric localization but can use the analogous technique [88]. On the other hand it is nontrivial to define the A-twist for Landau-Ginzburg models. The A-twisted Landau-Ginzburg models defined in [89] has Q_A-exact action and become Witten type topological field theories.

We will give more detailed explanation about the topological A-, B-models in the next subsection.

Supersymmetric ground states and (twisted) chiral ring revisited

Now we come back to the state-operator correspondence between the set of supersymmetric ground states and the (twisted) chiral ring. The topological A-, B-models has nilpotent charges Q_A, Q_B. Then, the (twisted) chiral ring elements are good observable in the topological B(A)-models.

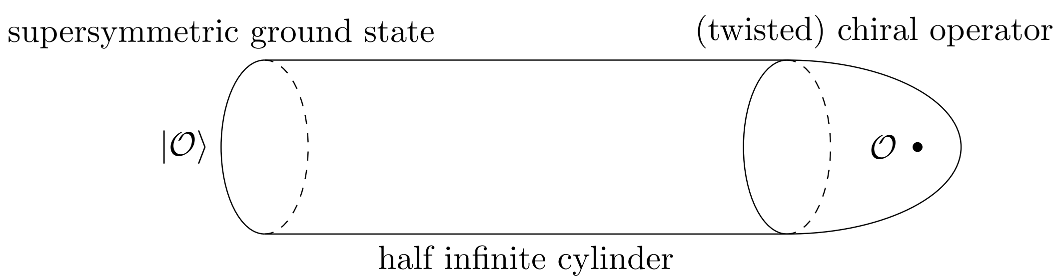

Figure 1 *A supersymmetric ground state and a (twisted) chiral operator.*

According to [74], we consider the topological A-, B-models on a half-infinite cylinder capped-off at infinity (see Figure 1). We insert a twisted chiral operator or a chiral operator at the tip of this cigar-like hemisphere. Note that this is a vertex operator in the NS-NS sector. The state corresponding to this vertex operator propagates through the half-infinite cylinder to the boundary. At the flat region the twisted theory is equivalent to the untwisted theory but there is a (axial) R-symmetry flux, which makes the boundary conditions for fermions periodic. Then, the Ramond-Ramond state appears at the boundary. After the infinite propagation through the cylinder, states become ground states of the Hamiltonian. Therefore, this half-infinite cylinder causes a spectral flow from a NS-NS vertex operator to a supersymmetric ground state in the Ramond-Ramond sector. In summary, for a theory which can be A(B)-twisted, the space of Ramond-Ramond ground states is isomorphic to the twisted chiral ring (chiral ring) as a vector space. For a theory which can be both A-twisted and B-twisted, the space of Ramond-Ramond ground states, the twisted chiral ring and the chiral ring are isomorphic to each other as a vector space.

Topological correlation functions and (twisted) chiral ring

We consider the three point function of the topological B-model. We take the world sheet $\Sigma = S^2$ and three chiral operators $\mathcal{O}_i, \mathcal{O}_j, \mathcal{O}_k$ on Σ. Here the basis $\{\mathcal{O}_i\}_i$ generates the chiral ring. We calculate the correlation function $C_{ijk} := \langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle$. Since the theory is topological, this correlation function does not depend on the metric of Σ, the insertion points of the operators, the metric of the target space, Kähler (twisted chiral) parameters. This only depends on the complex structure (chiral) parameters holomorphically. Let the identity operator has the index 0, $\mathcal{O}_0 = \mathrm{id}$, then we define

$$\eta_{ij} := C_{ij0} = \langle \mathcal{O}_i \mathcal{O}_j \rangle.$$

This is called the topological metric. We assume that this is an invertible matrix and denote the inverse matrix as η^{ij}, then $\eta^{ij}\eta_{jk} = \delta^i_k$. From the topological property of the correlation function, we can consider that $\mathcal{O}_j$ is in the vicinity of $\mathcal{O}_k$. From the chiral ring structure, we can expand the product

$$\mathcal{O}_j \mathcal{O}_k = \mathcal{O}_l C^l_{jk} + [Q_B, \Lambda] \, .$$

Then, we obtain the relation,

$$C_{ijk} = \langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle = \langle \mathcal{O}_i (\mathcal{O}_l C^l_{jk} + [Q_B, \Lambda]) \rangle = \eta_{il} C^l_{jk} \, .$$

Therefore, the chiral ring is determined by the three-point functions of the topological B-model. In the same way, the twisted chiral ring s determined by the three-point functions of the topological A-model, and its structure constant only depends on the Kähler (twisted chiral) parameters holomorphically.

In the next subsection, we compute the three-point functions of the topological A-, B-model.

2.6. More on topological A-model

In this subsection, we give brief explanation of the A-twisted non-linear sigma models. Remember that, after the A-twist, $\phi, \psi_+, \bar{\psi}_-$ become the scalar fields, ψ_-, F become the sections of $\phi^* TX^{(1,0)} \otimes \bar{K}$ and $\bar{\psi}_+, \bar{\mathrm{F}}$ become the sections of $\phi^* TX^{(0,1)} \otimes K$. We rename the field $\psi, \bar{\psi}$ as

$$\chi^i := \psi_+^i\,,\ \bar{\chi}^i := \bar{\psi}_-^i\,,\ \rho_{\bar{z}}^i := \psi_-^i\,,\ \bar{\rho}_z^i := \bar{\psi}_+^i\,.$$

The action is

$$\begin{aligned}\mathcal{L}_D := &\, 2g_{i\bar{j}}(\partial_z\phi^i\partial_{\bar{z}}\bar{\phi}^j + \partial_{\bar{z}}\phi^i\partial_z\bar{\phi}^j) + ig_{i\bar{j}}(\nabla_{\bar{z}}\bar{\rho}_z^j\chi^i - \bar{\rho}_z^j\nabla_{\bar{z}}\chi^i) \\ &- ig_{i\bar{j}}(\nabla_z\bar{\chi}^j\rho_{\bar{z}}^i - \bar{\chi}^j\nabla_z\rho_{\bar{z}}^i) - R_{i\bar{j}k\bar{l}}\chi^i\rho_{\bar{z}}^k\bar{\chi}^j\bar{\rho}_z^l \\ &- g_{i\bar{j}}\left(\mathrm{F}_{\bar{z}}^i + \Gamma_{kl}^i\chi^k\rho_{\bar{z}}^l\right)\left(\bar{\mathrm{F}}_z^j + \Gamma_{\bar{k}\bar{l}}^{\bar{j}}\bar{\chi}^k\bar{\rho}_z^l\right)\,,\end{aligned}$$

The supersymmetry transformation is generated by $\epsilon = \bar{\epsilon}_- = -\epsilon_+$, *i.e.*,

$$\begin{aligned}\delta\phi^i =&\epsilon\chi^i\,, \qquad \delta\bar{\phi}^i = \epsilon\bar{\chi}^i\,, \\ \delta\chi^i =&0\,, \qquad \delta\rho_{\bar{z}}^i = 2i\epsilon\partial_{\bar{z}}\phi^i + \epsilon\mathrm{F}_{\bar{z}}^i\,, \\ \delta\bar{\rho}_z^i =&-2i\epsilon\partial_z\bar{\phi}^i + \epsilon\bar{\mathrm{F}}_z^i\,, \qquad \delta\bar{\chi}^i = 0 \\ \delta\mathrm{F}_{\bar{z}}^i =&-2i\epsilon\nabla_{\bar{z}}\chi^i\,, \qquad \delta\bar{\mathrm{F}}_z^i = 2i\epsilon\nabla_z\bar{\chi}^i\,.\end{aligned}$$

To make local operators, we only use the scalar fields $\phi, \bar{\phi}, \chi, \bar{\chi}$. From the SUSY algebra, we can find the correspondence between the fields $\chi, \bar{\chi}$ and the differential forms on X,

$$\chi^i \leftrightarrow d\phi^i\,,\ \bar{\chi}^i \leftrightarrow d\bar{\phi}^i\,,\ Q_+ \leftrightarrow \partial\,, \bar{Q}_- \leftrightarrow \bar{\partial}\,,\ Q_A = Q_+ + \bar{Q}_- \leftrightarrow d = \partial + \bar{\partial}\,.$$

Then, there is a one to one correspondence between the Q_A-cohomology classes and the de Rahm cohomology classes. Associated to a homology class D, we can consider a local operator $\mathcal{O}_D(z), z \in \Sigma$ which is defined by the Poincaré dual, *i.e.*, a differential form whose support is D.

We consider the correlation function

$$\langle\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n\rangle := \int D\phi D\chi D\rho e^{-S}\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n\,.$$

We classify the map ϕ by the cohomology class

$$\beta = \phi_*[\Sigma] \in H_2(X, \mathbb{Z}),$$

and decompose the path integral

$$\langle \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n\rangle = \sum_{\beta\in H_2(X,\mathbb{Z})}\langle \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n\rangle_\beta\,,$$
$$\langle \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n\rangle_\beta := \int_{\phi_*[\Sigma]=\beta} D\phi D\chi D\rho e^{-S}\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n\,.$$

Now we consider the R-charges and the axial R-charges of the operators. Let $\mathcal{O}_i$ corresponds to the element $\omega_i\in H^{p_i,q_i}(X)$, which has the R-charge p_i-q_i and the axial R-charge $-p_i-q_i$. Since the R-symmetry is not anomalous, the correlation function is non-vanishing only when $\sum_{i=1}^n p_i=\sum_{i=1}^n q_i$. The axial symmetry is anomalous and

$$\text{(The number of }\chi\text{ zero modes)} - \text{(The number of }\rho\text{ zero modes)} = 2k$$
$$k := c_1(TX)\cdot\beta + \dim_{\mathbb{C}} X(1-g)$$

The correlation function is non-vanishing only when $\sum_{i=1}^n(p_i+q_i)=2k$. In summary, the correlation function is non-vanishing only when

$$\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = k\,. \tag{2.1}$$

Now we calculate the correlation function by the supersymmetric localization technique. First, we note that the action can be written as

$$S = S_{\text{exact}} + S_{\text{top}}\,,$$
$$S_{\text{exact}} := \int_\Sigma d^2z\{Q_A, V\}\,,\ S_{\text{top}} := \int_\Sigma \phi^*(\omega-2\pi iB) = (\omega-2\pi iB)\cdot\beta\,,$$
$$V := g_{i\bar{j}}\left\{\bar{\rho}^{\bar{j}}_z\left(i\partial_{\bar{z}}\phi^i - \frac{1}{2}(\mathrm{F}^i_{\bar{z}}+\Gamma^i_{kl}\chi^k\rho^l_{\bar{z}})\right) - \left(i\partial_z\bar{\phi}^{\bar{j}} + \frac{1}{2}(\bar{\mathrm{F}}^{\bar{j}}_z + \Gamma^{\bar{j}}_{\bar{k}\bar{l}}\bar{\chi}^{\bar{k}}\bar{\rho}^{\bar{l}}_z)\right)\rho^i_{\bar{z}}\right\}\,.$$

Therefore, the A-twisted non-linear sigma model is a Witten type topological field theory. As mentioned before, this model only depends on the Kähler form of X and other information of X is included in the Q_A-exact term. Since the bosonic part of the Q_A-exact term is

$$S_{\text{exact}}\big|_{\text{bos}} = 4|\partial_{\bar{z}}\phi^i|^2 + \left|\mathrm{F}^i_{\bar{z}}+\Gamma^i_{lk}\chi^l\rho^k_{\bar{z}}\right|^2\,,$$

the saddle points satisfy

$$\partial_{\bar{z}}\phi^i = 0\,,\qquad \mathrm{F}^i_{\bar{z}} = -\Gamma^i_{lk}\chi^l\rho^k_{\bar{z}}\,.$$

We consider the path integral

$$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle_\beta = e^{-(\omega - 2\pi i B)\cdot\beta} \int_{\phi_*[\Sigma]=\beta} D\phi D\chi D\rho e^{-\mathrm{t}S_{\mathrm{exact}} - S_{\mathrm{top}}} \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \,.$$

Since this does not depend on t, we take $\mathrm{t} \to \infty$ limit. Then, the path integral reduces to the integral over the saddle locus, *i.e.*, the moduli space of holomorphic maps

$$\mathcal{M}_\Sigma(X,\beta) = \left\{ \phi : \Sigma \to X \middle| \phi \text{ is holomorphic, } \phi_*[\Sigma] = \beta \right\} \,.$$

We assume that this moduli space is a smooth manifold. Furthermore, we assume that
(1) $k \in \mathbb{Z}_{\geq 0}$,
(2) there is no ρ zero mode.
Then, the tangent space $T\mathcal{M}_\Sigma(X,\beta)$ is identified with the space of χ zero modes and

$$\dim_{\mathbb{C}} \mathcal{M}_\Sigma(X,\beta) = k \,.$$

As usual in the topological field thories, the 1-loop determinant becomes 1 since the bosonic determinant cancels the fermonic determinant. The operator $\mathcal{O}_i(z_i)$ can be identified with the pullback $[\omega_i] \in H^{p_i,q_i}(X)$ by the evaluation map at z_i,

$$\mathrm{ev}_i : \mathcal{M}_\Sigma(X,\beta) \ni \phi \mapsto \phi(z_i) \in X \,.$$

Then,

$$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle_\beta = e^{-(\omega - 2\pi i B)\cdot\beta} \int_{\mathcal{M}_\Sigma(X,\beta)} \mathrm{ev}_1^* \omega_1 \wedge \cdots \wedge \mathrm{ev}_n^* \omega_n \,.$$

The non-vanishing condition (2.1) coincides with the condition that $\mathrm{ev}_1^*\omega_1 \wedge \cdots \wedge \mathrm{ev}_n^*\omega_n$ becomes the top form of the integral on $\mathcal{M}_\Sigma(X,\beta)$. If $[\omega_i]$ is the Poincaré dual of the cycles $[D_i]$, it is known that

$$\begin{aligned} n_{\beta,D_1,\ldots,D_n} &:= \int_{\mathcal{M}_\Sigma(X,\beta)} \mathrm{ev}_1^*\omega_1 \wedge \cdots \wedge \mathrm{ev}_n^*\omega_n \\ &= \# \left\{ \phi : \Sigma \to X \middle| \phi \text{ is holomorphic, } \phi_*[\Sigma] = \beta,\ \phi(z_i) \in D_i \forall i \right\} \,. \end{aligned}$$

The correlation function becomes

$$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle = \sum_{\beta \in H_2(X,\mathbb{Z})} e^{-(\omega - 2\pi i B)\cdot\beta} n_{\beta,D_1,\ldots,D_n}$$

Here we note that $\omega \cdot \beta \geq 0$ and this summation is well-defined. This is because the Kähler form ω restricted on $\phi_*[\Sigma]$ is positive definite. The inequality is satisfied only when $\beta = 0$, *i.e.*, $\phi_*[\Sigma]$ is a point in X.

In the large volume limit where ω is very large, the correlation function is dominated by the $\beta = 0$ contribution. For $\beta = 0$, $\mathcal{M}_\Sigma(X, 0) \simeq X$. Since $\dim_\mathbb{C} X = k$, the non-vanishing condition (2.1) is satisfied only when the genus $g = 0$, *i.e.*, Σ is the Riemann sphere. Since $\mathrm{ev}_i = \mathrm{id}_X$, the correlation function becomes the intersection number;

$$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle_0 = n_{0,D_1,\ldots,D_n} = \int_X \omega_1 \wedge \cdots \wedge \omega_n = \#(D_1 \cap D_2 \cap \cdots \cap D_n)\,.$$

It is known that the topological metric does not receive world-sheet-instanton corrections and $\beta = 0$ result is exact.[5]

$$\eta_{ij} = \langle \mathcal{O}_i \mathcal{O}_j \rangle = \int_X \omega_1 \wedge \omega_2 = \#(D_1 \cap D_2)\,.$$

Now we consider the more general case where the assumption (2) is not imposed, *i.e.*, there are some ρ zero modes. A ρ zero mode is a solution of

$$\partial_{\bar{z}} \rho_{zi} = 0\,, \qquad \rho_{zi} := g_{i\bar{j}} \rho_z^{\bar{j}}\,.$$

The space of the ρ zero mode is identified with the space of holomorphic sections $H^0(\Sigma, K \otimes \phi^* T^* X^{(1,0)})$. Assume that the dimension of this space is a constant l at any point of $\mathcal{M}_\Sigma(X, \beta)$. Then, $\dim_\mathbb{C} \mathcal{M}_\Sigma(X, \beta) = k + l$ and the space $H^0(\Sigma, K \otimes \phi^* T^* X_{(1,0)})$ defines a rank l vector bundle $\mathcal{V}$ over $\mathcal{M}_\Sigma(X, \beta)$. From the path integral with respect to ρ, we obtain the Euler class $e(\mathcal{V})$ and the correlation function can be written as

$$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle_\beta = e^{-(\omega - 2\pi i B)\cdot\beta} \int_{\mathcal{M}_\Sigma(X,\beta)} e(\mathcal{V}) \wedge \mathrm{ev}_1^* \omega_1 \wedge \cdots \wedge \mathrm{ev}_n^* \omega_n\,.$$

Since, $e(\mathcal{V})$ is an (l, l)-form, this integral is well-defined.

As an example we consider the case of the complex projective space $X = \mathbb{P}^1$. The cohomology group of $\mathbb{P}^1$ is

$$H^i(\mathbb{P}^1, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0, 2\,, \\ 0 & \text{for } i = 1\,. \end{cases}$$

[5] See the argument in section 7 of [9].

$H^0(\mathbb{P}^1)$ is generated by 1 and $H^2(\mathbb{P}^n)$ is generated by x which is a $(1,1)$-form and Poincaré dual to a point;

$$\int_{\mathbb{P}^1} x = 1,$$

We denote the operators corresponding to $1, x$ as P, Q respectively. P is an identity operator. Since the topological metric does not receive the world-sheet instant on correction, we can compute the following three-point functions.

$$\langle P\mathcal{O}_i\mathcal{O}_j\rangle = \eta_{ij} = \begin{cases} 1 & (i,j) = (P,Q) \text{ or } (Q,P)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

The nontrivial three point function is

$$\langle QQQ\rangle = \sum_{n\in\mathbb{Z}} \langle QQQ\rangle_n\,.$$

Since the first Chern class $c_1(T\mathbb{P}^1) = 2x$, for the map ϕ with $\beta = n[\mathbb{P}^1]$ we have $k = c_1(T\mathbb{P}^1)\cdot\beta + \dim_{\mathbb{C}} \mathbb{P}^1(1-0) = 2n+1$. From the non-vanishing condition, only the mapping with $n = 1$ contribute. Let the three Q operators correspond to the Poincaré dual of the three distinct point $x_1, x_2, x_3 \in \mathbb{P}^1$ respectively. The number of maps which map the three distinct point $z_1, z_2, z_3 \in \Sigma = \mathbb{P}^1$ to the three distinct point $x_1, x_2, x_3 \in X = \mathbb{P}^1$ is only one, then $n_{1,y_1,y_2,y_3} = 1$. Therefore

$$\langle QQQ\rangle = \langle QQQ\rangle_1 = e^{-t}\,,\ t := (\omega - 2\pi iB)\cdot[\mathbb{P}^1] = \int_{\mathbb{P}^1}(\omega - 2\pi iB)\,.$$

From the above result, the twisted chiral ring relation is

$$PP = P\,,\ PQ = QP = Q\,,\ QQ = e^{-t}P\,.$$

If we take the large volume limit $t \to \infty$, the twisted chiral ring reduces to the cohomology ring of $\mathbb{P}^1$. Therefore the twisted chiral ring is called the quantum cohomology in the literature of mathematics.

The twisted chiral ring of $\mathbb{P}^n$ is described as

$$PP = P\,,\ PQ = QP = Q\,,\ Q^n = e^{-t}P\,. \tag{2.2}$$

Here Q corresponds to the $(1,1)$-form which is Poincaré dual to a hyper surface. A hyper surface is a locus where one of the homogenous coordinates becomes zero.

2.7. More on topological B-model

In this subsection, we give brief explanation of the B-twisted non-linear sigma models. As we have noted, to define the B-twisted theory, the target space X should be a Calabi-Yau manifold. Remember that after the B-twist $\phi, \bar{\psi}_+, \bar{\psi}_-, \mathrm{F}, \bar{\mathrm{F}}$ become the scalar fields, ψ_+, ψ_- become the sections of $\phi^* TX^{(1,0)} \otimes K$ and $\phi^* TX^{(0,1)} \otimes \bar{K}$ respectively. We rename the field $\psi, \bar{\psi}$ as

$$\bar{\eta}^i := \bar{\psi}^i_+ + \bar{\psi}^i_- \,, \ \theta_i := g_{i\bar{j}}(\bar{\psi}^j_+ - \bar{\psi}^j_-) \,, \ \rho^i_z := \psi^i_+ \,, \ \rho^i_{\bar{z}} := \psi^i_- \,.$$

The action is

$$\begin{aligned}\mathcal{L}_D := & 2g_{i\bar{j}}(\partial_z \phi^i \partial_{\bar{z}} \bar{\phi}^j + \partial_{\bar{z}} \phi^i \partial_z \bar{\phi}^j) + i g_{i\bar{j}} \left(\nabla_{\bar{z}} (\bar{\eta}^j + \bar{\theta}^j) \rho^i_z - (\bar{\eta}^j + \bar{\theta}^j) \nabla_{\bar{z}} \rho^i_z \right) \\ & - i g_{i\bar{j}} \left(\nabla_z (\bar{\eta}^j - \bar{\theta}^j) \rho^i_{\bar{z}} - (\bar{\eta}^j - \bar{\theta}^j) \nabla_z \rho^i_{\bar{z}} \right) - R_{i\bar{j}k\bar{l}} \rho^i_z \rho^k_{\bar{z}} \bar{\eta}^j \bar{\theta}^l - g_{i\bar{j}} \mathrm{F}^j \bar{\mathrm{F}}^j \,,\end{aligned}$$

where $\bar{\theta}^i := g^{\bar{i}j}\theta_j$.[6] The supersymmetry transformation is generated by $\epsilon = -\epsilon_+ = \epsilon_-$, *i.e.*,

$$\begin{aligned}\delta\phi^i =& 0\,, \qquad \delta\bar{\phi}^i = \epsilon\bar{\eta}^i \,, \\ \delta\rho^i_z =& -2i\epsilon\partial_z\phi^i \,, \qquad \delta\rho^i_{\bar{z}} = 2i\epsilon\partial_{\bar{z}}\phi^i \,, \\ \delta\bar{\eta}^i =& 0\,, \qquad \delta\theta_i = 2\epsilon g_{i\bar{j}}\bar{\mathrm{F}}^j \,, \\ \delta\mathrm{F}^i =& 2i\epsilon\left(\nabla_{\bar{z}}\rho^i_z + \nabla_z\rho^i_{\bar{z}}\right) - R^i_{j\bar{k}l}\bar{\eta}^k\rho^j_z\rho^l_{\bar{z}} \,, \qquad \delta\bar{\mathrm{F}}^i = -\Gamma^{\bar{i}}_{\bar{j}\bar{k}}\bar{\eta}^j\bar{\mathrm{F}}^k \,.\end{aligned}$$

To make local operators, we only use the scalar fields $\phi, \bar{\phi}, \bar{\eta}, \theta$. From the SUSY algebra, we can find the correspondence between the fields η, θ and the geometric objects on X,

$$\begin{aligned}& \bar{\eta}^i \leftrightarrow d\bar{\phi}^i \,, \ \theta_i \leftrightarrow \frac{\partial}{\partial\phi^i} \,, \ Q_B \leftrightarrow \bar{\partial} \,, \\ & \omega^{j_1,j_2,\ldots,j_q}_{\bar{i}_1,\bar{i}_2,\ldots\bar{i}_p}\bar{\eta}^{i_1}\bar{\eta}^{i_2}\cdots\bar{\eta}^{i_p}\theta_{j_1}\theta_{j_2}\cdots\theta_{j_q} \leftrightarrow \omega^{j_1,j_2,\ldots,j_q}_{\bar{i}_1,\bar{i}_2,\ldots\bar{i}_p} d\bar{\phi}^{i_1} d\bar{\phi}^{i_2}\cdots d\bar{\phi}^{i_p}\frac{\partial}{\partial\phi^{j_1}}\frac{\partial}{\partial\phi^{j_2}}\cdots\frac{\partial}{\partial\phi^{j_q}}\end{aligned}$$

Then, there is a one to one correspondence between the Q_B-cohomology classes and the Dolbeault cohomology classes in

$$\bigoplus_{p,q=0} H^{0,p}(M, \wedge^q TX^{(1,0)}) \,,$$

We consider the correlation function

$$\langle\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n\rangle := \int D\phi D\chi D\rho e^{-S}\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n \,.$$

[6] Here we did the redefinition of the auxiliary fields $\mathrm{F}, \bar{\mathrm{F}}$.

Let $\mathcal{O}_i$ corresponds to the element $[\omega_i] \in H^{0,p_i}(M, \wedge^{q_i} TX^{(1,0)})$. From the argument of the R symmetry and the axial R symmetry, the correlation function is non-vanishing only when

$$\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = \dim_{\mathbb{C}} X(1-g)\,.$$

For $g = 0$, $\sum_i p_i = \sum_i q_i = \dim_{\mathbb{C}} X$. For $g = 1$, all p_i, q_i should be zero. For $g > 1$, the above condition is never satisfied. Now we calculate the correlation functions by the supersymmetric localization technique. First, we note that the action can be written as

$$S = S_{\text{exact}}\,, \qquad S_{\text{exact}} := \int_\Sigma d^2 z \{Q_B, V\}\,,$$
$$V := i g_{i\bar{j}} (\rho^i_z \partial_{\bar{z}} \bar{\phi}^j - \rho^i_{\bar{z}} \partial_z \bar{\phi}^j) - \frac{1}{2} \theta_i \mathrm{F}^i\,.$$

Therefore, the B-twisted non-linear sigma model is a Witten type topological field theory. Since the bosonic part of the Q_B-exact term is

$$S_{\text{exact}}\big|_{\text{bos}} = 2|\partial_z \phi^i|^2 + 2|\partial_{\bar{z}} \phi^i|^2 + |\mathrm{F}^i|^2\,,$$

the saddle points satisfy

$$\phi^i = \text{constant}\,, \qquad \mathrm{F}^i_{\bar{z}} = 0\,.$$

We consider the path integral

$$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle := \int D\phi D\chi D\rho e^{-\mathrm{t}S} \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n\,.$$

Since this does not depend on t, we take $\mathrm{t} \to \infty$ limit. Then, the path integral reduces to the integral over X. From the non vanishing condition of the correlation function, we only consider the case $[\omega_1 \wedge \cdots \wedge \omega_n] \in H^{0,N}(M, \wedge^N TX^{(1,0)})$, $N := \dim_{\mathbb{C}} X$. If X is a Calabi-Yau manifold, it is known that $H^{0,N}(M, \wedge^N TX^{(1,0)})$ is nonzero and one-dimensional, and $\wedge^N TX^{(1,0)}$ is isomorphic to the space of holomorphic N-forms. Let Ω be a non-vanishing holomorphic N-form. The choice of Ω depends on the complex structure of X and is unique up to overall constant. There is a natural map

$$H^{0,N}(M, \wedge^N TX^{(1,0)}) \ni [\omega] \mapsto [\langle \omega, \Omega \rangle \wedge \Omega] \in H^{(N,N)}(X)\,,$$
$$\langle \omega, \Omega \rangle := \omega^{i_1, i_2, \ldots, i_N}_{\bar{j}_1, \bar{j}_2, \ldots \bar{j}_N} \Omega_{i_1, i_2, \ldots i_N} d\bar{\phi}^{j_1} \wedge d\bar{\phi}^{j_2} \wedge \cdots \wedge d\bar{\phi}^{j_p}\,.$$

The integration over the fermion zero mode causes this mapping and the correlation function is described as follows.

$$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle := \int_X \langle \omega_1 \wedge \cdots \wedge \omega_n, \Omega \rangle \wedge \Omega\,.$$

3. $\mathcal{N}=(2,2)$ non-linear sigma models with D-branes

In this section, we briefly review some basic facts about $\mathcal{N}=(2,2)$ non-linear sigma models with world-sheet boundaries. First, we find the supersymmetric boundary conditions for fields which preserves both A and B-type supersymmetry. Then, we see the properties of submanifolds to which the boundary of the world-sheet can be mapped. If we preserve the A-type supersymmetry at the boundary, the submanifolds should be coisotropic submanifolds. These boundary conditions are called the A-branes. If we preserve the B-type supersymmetry at the boundary, the submanifolds should be complex submanifolds. These boundary conditions are called the B-branes. We concentrate on the analysis of the B-branes in this thesis. We add boundary interactions which correspond to gauge fields on the B-branes. Then, we briefly comment that the B-branes are described by the notion of the derived category of the coherent sheaves. Finally, we consider the boundary states which correspond to the A-, B-branes and the overlap between the boundary states and the supersymmetric ground states of the A-, B-model. We write the review of this section in reference to [19,21,22,26,73].

3.1. Supersymmetric boundary condition

We consider a world-sheet with boundary $I\times\mathbb{R}$, where x^1 parametrizes $I:=[0,\infty]$ or $[0,\pi]$ and x^2 parametrizes $\mathbb{R}$. In the presence of the boundary, we should take care of boundary terms. The variation of the action becomes

$$\delta S=\int_{\Sigma}d^2x\delta\Phi(\text{bulk equation of motion})+\int_{\partial\Sigma}dx^2\delta\Phi(\text{boundary equation of motion}).$$

The bulk equation of motion is the same as that in the theory with boundary. In addition, we impose the boundary equation of motion,

$$\delta\phi^I(g_{IJ}\partial_1\phi^J+2\pi iB_{IJ}\partial_2\phi^J)|_{\partial\Sigma}=0,\ g_{IJ}(\psi_+^I\delta\psi_+^J-\psi_-^I\delta\psi_-^J)|_{\partial\Sigma}=0\,,$$

where I,J is the index of the local real coordinates of the target space X and $B=\frac{1}{2}B_{IJ}dx^I\wedge dx^J$. In the presence of the boundary, the action is invariant under the SUSY transformation up to boundary term,

$$\begin{aligned}\delta S=\int_{\partial\Sigma}dx^2\Bigg\{&\epsilon_+\left(-\frac{g_{i\bar{j}}}{2}\bar{\psi}_-^j\partial_1\phi^i+\frac{i}{2}\left(g_{i\bar{j}}-2\pi B_{i\bar{j}}\right)\bar{\psi}_-^j\partial_2\phi^i+\frac{1}{4}\psi_+^i\partial_iW\right)\\&+\epsilon_-\left(\frac{g_{i\bar{j}}}{2}\bar{\psi}_+^j\partial_1\phi^i+\frac{i}{2}\left(g_{i\bar{j}}+2\pi B_{i\bar{j}}\right)\bar{\psi}_+^j\partial_2\phi^i-\frac{1}{4}\psi_-^i\partial_iW\right)\\&+\bar{\epsilon}_+\left(-\frac{g_{i\bar{j}}}{2}\partial_1\bar{\phi}^j\psi_-^i+\frac{i}{2}\left(g_{i\bar{j}}+2\pi B_{i\bar{j}}\right)\partial_2\bar{\phi}^j\psi_-^i-\frac{1}{4}\bar{\psi}_+^i\partial_{\bar{i}}\bar{W}\right)\\&+\bar{\epsilon}_-\left(\frac{g_{i\bar{j}}}{2}\partial_1\bar{\phi}^j\psi_+^i+\frac{i}{2}\left(g_{i\bar{j}}-2\pi B_{i\bar{j}}\right)\partial_2\bar{\phi}^j\psi_+^i+\frac{1}{4}\bar{\psi}_-^i\partial_{\bar{i}}\bar{W}\right)\Bigg\}\,.\end{aligned}$$

To preserve some supersymmetry, this boundary term should vanish. From this formula, we can extract the supercurrents to the x^1 direction.

$$G^1_\pm = \frac{g_{i\bar{j}}}{2}\partial_1\bar{\phi}^j\psi^i_+ + \frac{i}{2}\left(\pm g_{i\bar{j}} - 2\pi B_{i\bar{j}}\right)\partial_2\bar{\phi}^j\psi^i_+ + \frac{1}{4}\bar{\psi}^i_-\partial_{\bar{i}}\bar{W}\,,$$
$$\bar{G}^1_\pm = \frac{g_{i\bar{j}}}{2}\bar{\psi}^j_+\partial_1\phi^i + \frac{i}{2}\left(\pm g_{i\bar{j}} + 2\pi B_{i\bar{j}}\right)\bar{\psi}^j_+\partial_2\phi^i - \frac{1}{4}\psi^i_-\partial_i W\,.$$

We consider the two cases where the A-type supersymmetry $Q_A := Q_+ + \bar{Q}_-, Q_A^\dagger := \bar{Q}_+ + Q_-$ or the B-type supersymmetry $Q_B := \bar{Q}_+ + \bar{Q}_-, Q_B^\dagger := Q_+ + Q_-$ is preserved. Here we argue the general properties which stand in both cases. Both condition includes an $\mathcal{N}=1$ subalgebra generated by $\epsilon_+ = -\epsilon_- = \bar{\epsilon}_+ = -\bar{\epsilon}_- =: \epsilon \in \mathbb{R}$. Then, the boundary term of the SUSY transformation becomes

$$\delta S = \epsilon\int_{\partial\Sigma} dx^2\left\{-g_{IJ}\partial_1\phi^I(\psi^J_+ + \psi^J_-) - ig_{IJ}\partial_2\phi^I(\psi^J_+ - \psi^J_-)\right.$$
$$\left. -2\pi i B_{IJ}\partial_2\phi^I(\psi^J_+ + \psi^J_-) + \frac{1}{4}(\psi^I_+ + \psi^I_-)\partial_I(W - \bar{W})\right\}\,.$$

We consider a D-brane wrapped on a submanifold $\gamma \subset X$, *i.e.*, the boundary of the worldsheet is mapped to γ. Then, $i\partial_2\phi^I|_{\partial\Sigma}$ and $\delta\phi^I|_{\partial\Sigma}$ are tangent to γ. Imposing the boundary equation of motion and $\delta S = 0$, we find the boundary condition

$$\begin{aligned}
&T^I_b := \partial_2\phi^I|_{\partial\Sigma},\ \text{tangent to }\gamma,\\
&N^I_b := (\partial_1\phi^I + 2\pi i g^{IJ}B_{JK}\partial_2\phi^K)|_{\partial\Sigma},\ \text{normal to }\gamma,\\
&T^I_f := (\psi^I_+ + \psi^I_-)|_{\partial\Sigma},\ \text{tangent to }\gamma,\\
&N^I_f := \{-(\psi^I_+ - \psi^I_-) - 2\pi g^{IJ}B_{JK}(\psi^K_+ + \psi^K_-)\}|_{\partial\Sigma},\ \text{normal to }\gamma,\\
&(W - \bar{W})|_\gamma = \text{constant.}
\end{aligned}$$

Next, we determine the properties of the submanifold γ which preserves the A-, B-type supersymmetry.

3.2. A-brane

We consider the boundary conditions which preserve the A-type supersymmetry, *i.e.*,

$$\begin{aligned}
0 = G^1_+ + \bar{G}^1_- = &-\frac{i}{2}(\omega + 4\pi^2 B\omega^{-1}B)(T_b, T_f) + \frac{i}{2}\omega(N_b, N_f)\\
&- i\pi\omega^{-1}(gN_b, BT_f) - i\pi\omega^{-1}(gN_f, BT_b)\\
&+ \frac{1}{4}N^I_f\partial_I\mathrm{Re}(W) + \frac{\pi}{2}(g^{-1}BT_f)^I\partial_I\mathrm{Re}(W)\,.
\end{aligned}$$

$\omega(N,N)=0$ means that γ is a coisotropic submanifold, *i.e.*, $(T\gamma)^\circ \subset T\gamma$ is satisfied, where $(T\gamma)^\circ \subset TX|_\gamma$ is orthogonal to $T\gamma$ with respect to ω. From $\omega^{-1}(gN, BT)=0$, B should vanish on $(T\gamma)^\circ \times T\gamma$. Since $(\omega+4\pi^2 B\omega^{-1}B)(T,T)=0$, B should be non-degenerate and $\omega+4\pi^2B\omega^{-1}B=0$ on $T\gamma/(T\gamma)^\circ$. To make $N^I\partial_I \mathrm{Re}(W)=0$, the gradient of $\mathrm{Re}(W)$ should be tangent to $T\gamma$. Furthermore, to make $(g^{-1}BT_f)^I\partial_I\mathrm{Re}(W)=0$, the gradient of $\mathrm{Re}(W)$ should be tangent to $(T\gamma)^\circ$. The final two conditions are satisfied from the boundary condition $\mathrm{Im}(W)|_\gamma = \text{constant}$, since $\mathrm{grad}\ \mathrm{Re}(W) = -J\mathrm{grad}\ \mathrm{Im}(W)$ where J is the complex structure of X.

The properties of the submanifolds γ which preserves the A-type supersymmetry is summarized as,

$$
\begin{aligned}
&(T\gamma)^\circ \subset T\gamma\,,\\
&B=0 \text{ on } (T\gamma)^\circ \times T\gamma\,,\\
&\omega+4\pi^2 B\omega^{-1}B=0 \text{ on } T\gamma/(T\gamma)^\circ\,,\\
&\mathrm{Im}(W)|_\gamma=\text{constant}\,.
\end{aligned}
$$

The boundary conditions for non-linear sigma models which satisfy the above conditions are called coisotropic A-branes.

Here we consider the two extreme case. If $(T\gamma)^\circ = T\gamma$, γ is a Lagrangian submanifold and B vanishes on $T\gamma$. This is called a Lagrangian A-brane. If $(T\gamma)^\circ=\emptyset$, $\gamma=X$. This is a space-filling A-brane. In this case, the third condition means that $2\pi\omega^{-1}B$ becomes a complex structure of X. In general, the dimension of γ should be real $2n+\dim_{\mathbb{R}} X/2$, $n\in\mathbb{Z}_{\geq 0}$ dimensional.

3.3. B-brane

We consider the boundary conditions which preserve the B-type supersymmetry, *i.e.*,

$$
\begin{aligned}
0=\bar{G}^1_+ + \bar{G}^1_- = &-\frac{i}{2}\omega(T_b,N_f)+\frac{i}{2}\omega(N_b,T_f)+i\pi(BJ+{}^tJB)(T_b,T_f)\\
&-\frac{1}{4}T_f^I\partial_I\mathrm{Re}(W)\,.
\end{aligned}
$$

First, we should impose $\omega(T,N)=0$. Since $\omega={}^tJg$, $\omega(T,N)=\langle JT, gN\rangle=0$. Then, $gN\in(T\gamma)^\perp$ implies $JT\in T\gamma$. This means that $T\gamma$ is a complex submanifold of TX. The condition $BJ+{}^tJB=0$ reduces to $(B|_{T\gamma})^{(2,0)}=0$, then $B|_{T\gamma}$ has only $(1,1)$-component. To make $T_f^I\partial_I\mathrm{Re}(W)=0$, W should be constant along γ. The boundary conditions for non-linear sigma models which satisfy the above conditions are called B-branes. Hereafter, we concentrate on the B-branes.

3.4. Boundary interactions

We consider the boundary interactions, *i.e.*, the gauge fields on D-branes. In the presence of multiple D-branes, the gauge fields become non-Abelian. If we consider the brane-anti-brane bound system, we should also consider the tacyon fields between them. Taking this into consideration, the gauge fields become superconnections.

First, we consider the flat target space $X = \mathbb{R}^{2n}$. We start from the vertex operators for tachyon and the massless vector boson

$$V_T := k \cdot \psi e^{ik\cdot\phi}\,,\ (k^2 = 1)\,, \quad V_A^\epsilon := (\epsilon \cdot \partial_2 \phi - (\epsilon \cdot \psi)(k \cdot \psi))e^{ik\cdot\phi}\ \ (k^2 = 0, k \cdot \epsilon = 0)\,,$$

The off-shell finite versions of them are written as

$$\frac{i}{2}\psi^I \partial_I \mathbf{T}(\phi)\,, \quad \partial_2 \phi^I A_I(\phi) - \frac{i}{4} F_{IJ}(\phi)\psi^I \psi^J\,.$$

$\mathbf{T}(x)$ is a tachyon field on $\mathbb{R}^{2n}$. A_I is a gauge field and $F_{IJ} := \partial_I A_J - \partial_J A_I + i[A_I, A_J]$ is the field strength. We can construct the boundary interaction, *i.e.*, the boundary Lagrangian from them. Before doing that, we need further explanation about the tachyon field. Naïvely, the tachyon operator is fermionic and cannot be added to the boundary Lagrangian, but if we consider the brane-anti-brane system we can add it. The brane-anti-brane system can be described by the $\mathbb{Z}^2$-graded Chan-Paton space

$$\mathcal{V} = \mathcal{V}^{\mathrm{e}} \oplus \mathcal{V}^{\mathrm{o}}\,,$$

where $\mathcal{V}^{\mathrm{e}}$ is the even subspace and $\mathcal{V}^{\mathrm{o}}$ is the odd subspace. We define the action of fermionic fields ψ^I to be anti-commute with the odd linear operators on $\mathcal{V}$, and define the tachyon field $\mathbf{T}(\phi)$ to be an odd operator on $\mathcal{V}$. Then, the total tachyon operator becomes bosonic and cannot be added to the boundary Lagrangian. To preserve the gauge symmetry and $\mathcal{N} = 1$ subalgebra generated by $\epsilon_+ = -\epsilon_- = \bar{\epsilon}_+ = -\bar{\epsilon}_- =: \epsilon \in \mathbb{R}$, we modify the above operators and obtain the boundary Lagrangian

$$P\exp\left(i\int_{\partial\Sigma} \mathcal{A}_2\right)\,, \quad \mathcal{A}_2 := \partial_2\phi^I A_I(\phi) - \frac{i}{4}F_{IJ}(\phi)\psi^I\psi^J + \frac{i}{2}\psi^I D_I \mathbf{T}(\phi) + \frac{1}{2}\mathbf{T}(\phi)^2\,,$$

where $D_I \mathbf{T} = \partial_I \mathbf{T} + i[A_I, \mathbf{T}]$. The first term $\partial_2\phi^I A_I$is the pullback of the connection of the vector bundle $E = E^{\mathrm{e}} \oplus E^{\mathrm{o}}$ on $\mathbb{R}^{2n}$ with fiber $\mathcal{V} = \mathcal{V}^{\mathrm{e}} \oplus \mathcal{V}^{\mathrm{o}}$. The other three terms are known as the curvature of the Quillen's superconnection.

Now we consider the boundary interaction which preserves B-type supersymmetry. The field strength must satisfy $F_{ij} = F_{\bar{i}\bar{j}} = 0$, then F has only $(1,1)$-component. This means that the vector bundle E is a holomorphic bundle. The tachyon field should be decomposed as

$$\mathbf{T} = i\mathcal{Q}(\phi) - i\bar{\mathcal{Q}}(\bar{\phi})$$

where $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ are holomorphic and anti-holomorphic respectively, *i.e.*,

$$D_{\bar{i}}\mathcal{Q} = D_i\bar{\mathcal{Q}} = 0.$$

Furthermore, they satisfy the following relation;

$$\mathcal{Q}^2 = W(\phi)\cdot \mathrm{id}\,, \quad \bar{\mathcal{Q}}^2 = \bar{W}(\bar{\phi})\cdot \mathrm{id}.$$

This is so called the matrix factorization property. If there is no superpotential, $\mathcal{Q}^2 = \bar{\mathcal{Q}}^2 = 0$. A pair of the odd operators $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ is called the tachyon profile. Finally, the boundary interaction can be written as

$$\mathcal{A}_2 := \partial_2\phi^I A_I - \frac{i}{4}F_{IJ}(\phi)\psi^I\psi^J - \frac{1}{2}\psi^i D_i\mathcal{Q} + \frac{1}{2}\bar{\psi}^i D_{\bar{i}}\bar{\mathcal{Q}} + \frac{1}{2}\{\mathcal{Q},\bar{\mathcal{Q}}\}\,.$$

From this expression, we can see that $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ should have R-charge ± 1 respectively. To be more precise, we introduce a homomorphism $\rho : U(1)_{\mathrm{R}} \to \mathrm{End}(\mathcal{V})$, and require that the tachyon profile satisfies the conditions

$$\rho(e^{i\alpha R})\mathcal{Q}(e^{-i\alpha R}\cdot\phi)\rho(e^{-i\alpha R}) = e^{i\alpha}\mathcal{Q}(\phi)\,,$$
$$\rho(e^{i\alpha R})\bar{\mathcal{Q}}(\bar{\phi}\cdot e^{i\alpha R})\rho(e^{-i\alpha R}) = e^{-i\alpha}\bar{\mathcal{Q}}(\bar{\phi})\,.$$

The generalization of the final result to any Kähler manifold X is straightforward.

3.5. Comments on B-brane categories

Hereafter, we consider the non-linear sigma model on a Kähler manifold X without superpotential. We decompose the Chan-Paton bundle E into the subbundles E^j with R-charge j, $\rho(e^{i\alpha R})E^j = e^{i\alpha j}E^j$. We assume the R-charges j are all integers and the Chan-Paton bundle is decomposed as

$$E^{\mathrm{e}} = \bigoplus_{j:\mathrm{even}} E^j\,, \quad E^{\mathrm{o}} = \bigoplus_{j:\mathrm{odd}} E^j\,.$$

Since Q has the R-charge 1, the Chan-Paton bundle can be described as a complex of holomorphic vector bundles.

$$\cdots \xrightarrow{Q} E^{j-1} \xrightarrow{Q} E^{j} \xrightarrow{Q} E^{j+1} \xrightarrow{Q} \cdots ,$$

where Q becomes a differential of the complex. Then, the set of B-branes is described by the set of complexes of holomorphic vector bundles over X.

To use the language of the derived category, we should generalize the holomorphic vector bundles to the coherent sheaves. Then, the B-branes are described by the derived category of coherent sheaves, which is denoted by $D(X)$. The objects of this category are coherent sheaves, *i.e.*, B-branes. The morphisms are described by the open strings stretching between two B-branes. The objects of the derived category are defined up to the equivalence relation which is called the quasi-isomorphism. If a B-brane is obtained from another B-brane by the D-term deformation or the Brane-anti-brane annihilation, these two B-branes are quasi-isomorphic. Since the D-term deformation and the Brane-anti-brane annihilation do not change the IR behavior, the quasi-isomorphism means the IR equivalence [73]. For detailed explanation of B-brane categories, see [19,21,22,73].

However, as emphasized in [73], this is the technical generalization to use the language of the derived category. We still think that the B-branes are the complexes of holomorphic vector bundles. Actually, it is well known that any coherent sheaf on a reasonable space is quasi-isomorphic to a complex of holomorphic vector bundles. This is the renowned Hilbert's syzygy theorem (see the chapter 5 section 4 of [90] for example).

3.6. Brane amplitudes and central charges

In this subsection, we consider the overlap $\langle \mathcal{B} | \mathcal{O} \rangle$ between the Ramond-Ramond ground states discussed in subsection 2.5 and the boundary states which correspond to the A-, B-branes (see Figure 2). From the view point of closed strings, this overlap is also called the brane amplitude. There are four types of the overlap, that is, combinations of the A-, B-branes and the A-, B-twisted capped region. In this thesis, the combination of the B-branes and the A-twisted capped region is important. We also consider the mirror of this, that is, the combination of the A-branes and the B-twisted capped region. We focus on the case of the Calabi-Yau three-fold. In the context of the Calabi-Yau compactification, the brane amplitudes turn out to play the role of the central charges of the BPS states in four dimensional $\mathcal{N} = 2$ gauge theories.

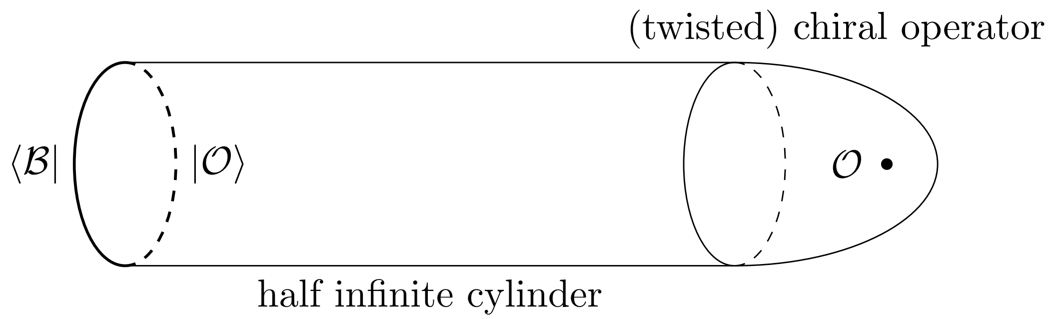

Figure 2 *The overlap between a boundary state and a Ramond-Ramond ground state.*

A-branes with B-twist

Firstly, we consider the overlap between the A-brane and the Ramond-Ramond ground state which corresponds to the chiral operator. If we take a Calabi-Yau three-fold as X, there may exist three-dimensional Lagrangian A-branes and five-dimensional coisotropic A-branes. However, as long as the holonomy is not a proper subgroup of $SU(3)$, $H_5(X,\mathbb{Z}) = 0$ and there exists no five-dimensional brane. Therefore we focus only on the Lagrangian A-branes.

Before computing the brane amplitude, we briefly review the relation between the D-branes and the central charges in the Calabi-Yau compactification (see section 5 of [22] for example). In the context of type IIA Calabi-Yau compactification, the BPS D-branes correspond to the BPS states in four-dimensional $\mathcal{N} = 2$ gauge theories. The BPS D-branes form a subset in the set of A-branes, that is, the BPS condition is stronger than the A-brane condition. The BPS D-branes are wrapped on the special Lagrangian submanifold [91]. Now we define the special Lagrangian submanifold. Let Ω the non-vanishing holomorphic three-form on X. For a Lagrangian submanifold L, the volume form on L can be written as

$$dV_L = Re^{-i\pi\xi(p)}\Omega|_L\,, \quad p \in L\,,$$

where R is a positive real constant and the $\xi(p)$ is a position dependent phase. If the phase

$$\xi = \frac{1}{\pi}\arg\frac{\Omega|_L}{dV_L}$$

is constant along L, L is called the special Lagrangian submanifold. On the special Lagrangian submanifold the phase can be written as

$$\xi(L) := \frac{1}{\pi}\arg\int_L \Omega\,.$$

Using this quantity we can discuss the stability of the D-branes, in terms of four-dimensional theory, the wall-crossing phenomena of BPS states. And the quantity

$$Z(L) := \int_L \Omega$$

plays the role of the central charge. We can see that this result is reasonable from the intuitive discussion. The above quantity measure the volume of L and this describes the mass of the brane on L. For BPS branes, the mass coincides with the central charge.

In the context of type IIB Calabi-Yau compactification, any A-brane corresponds to a BPS state. Let L be a Lagrangian submanifold and consider the A-brane wrapped on it. In general, the central charge of four-dimensional $\mathcal{N} = 2$ supergravity (gauge theory) can be written as

$$Z(L) = Q_i \Pi^i,$$

where Q_i is the electro-magnetic charge vector and Π^i is the coefficient vector. The charge Q_i is determined by the Ramond-Ramond charge (K-theory class) of the A-brane. The A-brane has the charge

$$[L] = \sum Q_i \Sigma^i \in H^3(X, \mathbb{Z})\,,$$

where $\{\Sigma^i\}_i$ form a basis of $H^3(X, \mathbb{Z})$ Π^a is determined by

$$\Pi^i := \int_{\Sigma^i} \Omega\,.$$

Now we come back to the discussion of the brane amplitude. We consider the A-brane wrapped on a Lagrangian submanifold L. As we have discussed in subsection 2.7, the Ramond-Ramond state corresponds to the Dolbeaut cohomology class. From the anomaly argument of the R-symmetry, the Ramond-Ramond state which corresponds to $\omega \in H^{(p,3-p)}(X, \mathbb{Z})$ gives the nontrivial result. It is known that the overlap $\langle L|\omega\rangle$ does not depend on the Kähler deformation. Then the calculation in the large volume limit becomes exact. In this limit, only the constant maps contribute to the path integral. In particular, if we consider the state which correspond to Ω, the overlap becomes

$$\langle L|\Omega\rangle = \int_L \Omega\,.$$

This is nothing but the central charge discussed above.

B-branes with A-twist

Next, we consider the overlap between the B-brane and the Ramond-Ramond ground state which corresponds to the twisted chiral operator. From the mirror symmetry, we expect that this overlap also describes the central charge of the B-brane. However, this overlap depend on the Kähler deformation, and then, the calculation in the large volume limit is not exact. In general, this overlap receives the α' correction and the world-sheet instanton correction, and then, the direct calculation is extremely difficult. If we know the mirror pair, we can calculate the overlap exactly by using the mirror symmetry. Even if we do not know the mirror pair, we find that the hemisphere partition function provides the exact formula for this overlap, as we will discuss in subsection 8.2 and Appendix E.

Here we only consider the calculation with insertion of the identity operator in the large volume limit, that is, we ignore the α' correction and the world-sheet instanton correction. We take the B-brane which is described by a holomorphic vector bundle $E \to X$. Then the Ramond-Ramond charge is obtained from the anomaly inflow argument [79] as

$$\mathrm{ch}(E)\sqrt{\mathrm{Td}(TX)}\,.$$

Since $\mathrm{ch}(E) = \mathrm{Tr}\exp(F/2\pi)$, and F always appears in the form of $2\pi B + F$, the Chern character always appears in the form of $e^B\mathrm{ch}(E)$. From the holomorphy of the supersymmetric theory, B appears always in the form of $B + i\omega$. Then the overlap, *i.e.*, the large volume formula of the central charge should be written as

$$Z(E) := \langle E|1\rangle = \int_X \mathrm{ch}(E)e^{B+i\omega}\sqrt{\hat{A}(TX)}\,,$$

where we used the formula $\mathrm{Td}(TX) = e^{c_1(TX)/2}\hat{A}(TX)$. Since X is Calabi-Yau, $c_1(TX) = 0$. If we consider the complex of holomorphic vector bundles $E_i \to X$;

$$\mathcal{E} : \cdots \longrightarrow E^{i-1} \longrightarrow E^i \longrightarrow E^{i+1} \longrightarrow \cdots\,,$$

the Chern character becomes

$$\mathrm{ch}(\mathcal{E}) = \sum_i (-1)^i \mathrm{ch}(E^i)\,,$$

and then the central charge is described as

$$Z(\mathcal{E}) := \langle \mathcal{E}|1\rangle = \sum_i (-1)^i Z(E^i)\,.$$

4. $\mathcal{N}=(2,2)$ gauge theories and non-linear sigma models at low energy

In this section, we briefly review some basic facts about $\mathcal{N}=(2,2)$ gauge theories. First, we define $\mathcal{N}=(2,2)$ gauge theories on flat spaces. Then, we argue how $\mathcal{N}=(2,2)$ gauge theories reduce to $\mathcal{N}=(2,2)$ non-linear sigma models. The anomaly and the renormalization property of $\mathcal{N}=(2,2)$ gauge theories turn out to be analogous to those of $\mathcal{N}=(2,2)$ non-linear sigma models. Finally, we see some examples of $\mathcal{N}=(2,2)$ gauge theories which will appear in this thesis. We write the review of this section in reference to [19,25].

4.1. $\mathcal{N}=(2,2)$ gauge theories on a flat space

An $\mathcal{N}=(2,2)$ gauge theory in two dimensions can be thought of as a dimensional reduction of an $\mathcal{N}=1$ gauge theory in four dimensions, and in particular contains gauge and chiral multiplets. The gauge multiplet for the gauge group G which is a compact Lie group consists of the gauge field A_μ, real scalars $\sigma_{1,2}$, gauginos λ, $\bar{\lambda}$, and the real auxiliary field D. The R-charges of the gauge multiplet fields $(A_\mu,\sigma_1,\sigma_2,\lambda,\bar{\lambda},\mathrm{D})$ are $(0,0,0,1,-1,0)$. To see the axial R-charge assignment, we have to take the linear combination of the real scalars as $\sigma=\sigma_1-i\sigma_2,\ \bar{\sigma}=\sigma_1+i\sigma_2$. The axial R-charges of $(\sigma,\bar{\sigma},\lambda_+,\lambda_-,\bar{\lambda}_+,\bar{\lambda}_-)$ are $(-2,2,1,-1,-1,1)$ and the R-charges of other fields in the gauge multiplet are 0.

The chiral multiplet in some irreducible representation of G consists of the complex scalar fields ϕ, fermions ψ and complex auxiliary field F. The R-charges of the chiral multiplet fields (ϕ,ψ,F) are $(q,q+1,q+2)$, where q is a real parameter. The axial R-charges of (ϕ,ψ_+,ψ_-,F) are $(q_\mathrm{A},q_\mathrm{A}+1,q_\mathrm{A}-1,q_\mathrm{A})$, where q_A is a real parameter.

We take the SUSY parameters ϵ and $\bar{\epsilon}$ to be bosonic constant spinors. The SUSY variation is given by $\delta=\bar{\epsilon}Q+\epsilon\bar{Q}$. In this convention fields in a vector multiplet transform under SUSY as

$$
\begin{aligned}
&\delta\lambda=(i\mathcal{V}_m\gamma^m-\mathrm{D})\epsilon\,,\qquad \delta\bar{\lambda}=(i\bar{\mathcal{V}}_m\gamma^m+\mathrm{D})\bar{\epsilon},\\
&\delta A_\mu=-\frac{i}{2}\left(\bar{\epsilon}\gamma_\mu\lambda+\bar{\lambda}\gamma_\mu\epsilon\right)\,,\quad \delta\sigma_1=\frac{1}{2}\left(\bar{\epsilon}\lambda+\bar{\lambda}\epsilon\right)\,,\quad \delta\sigma_2=-\frac{i}{2}\left(\bar{\epsilon}\gamma^3\lambda+\bar{\lambda}\gamma^3\epsilon\right)\,,\\
&\delta\mathrm{D}=-\frac{i}{2}\bar{\epsilon}\not{D}\lambda-\frac{i}{2}[\sigma_1,\bar{\epsilon}\lambda]-\frac{1}{2}[\sigma_2,\bar{\epsilon}\gamma^3\lambda]+\frac{i}{2}\epsilon\not{D}\bar{\lambda}+\frac{i}{2}[\sigma_1,\bar{\lambda}\epsilon]+\frac{1}{2}[\sigma_2,\bar{\lambda}\gamma^3\epsilon],
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{V}_m&=(D_1\sigma_1+D_2\sigma_2\,,\ D_2\sigma_1-D_1\sigma_2\,,\ F_{\hat{1}\hat{2}}+i[\sigma_1,\sigma_2])\,,\\
\bar{\mathcal{V}}_m&=(-D_1\sigma_1+D_2\sigma_2\,,\ -D_2\sigma_1-D_1\sigma_2\,,\ F_{\hat{1}\hat{2}}-i[\sigma_1,\sigma_2])\,.
\end{aligned}
$$

For a chiral multiplet, the SUSY transformation laws are given by

$$\begin{aligned}
\delta\phi =&\bar{\epsilon}\psi\,, \qquad \delta\bar{\phi} = \epsilon\bar{\psi}\,,\\
\delta\psi =&+i\gamma^\mu\epsilon D_\mu\phi + i\epsilon\sigma_1\phi + \gamma^3\epsilon\sigma_2\phi + \bar{\epsilon}\mathrm{F}\,,\\
\delta\bar{\psi} =&-i\bar{\epsilon}\gamma^\mu D_\mu\bar{\phi} + i\bar{\epsilon}\bar{\phi}\sigma_1 + \bar{\epsilon}\gamma^3\bar{\phi}\sigma_2 + \epsilon\bar{\mathrm{F}}\,,\\
\delta\mathrm{F} =&\epsilon\Big(i\gamma^\mu D_\mu\psi - i\sigma_1\psi + \gamma^3\sigma_2\psi - i\lambda\phi\Big)\,,\\
\delta\bar{\mathrm{F}} =&\bar{\epsilon}\Big(i\gamma^\mu D_\mu\bar{\psi} - i\bar{\psi}\sigma_1 - \gamma^3\bar{\psi}\sigma_2 + i\bar{\phi}\lambda\Big)\,.
\end{aligned}$$

These form a representation of the $\mathcal{N}=(2,2)$ SUSY algebra with $Z=\tilde{Z}=0$.

Now we describe the action of an $\mathcal{N}=(2,2)$ gauge theory. The Lagrangian for the gauge multiplet can be written as

$$\begin{aligned}
\mathcal{L}_{\text{vec}}^{\text{bulk}} \equiv \frac{1}{2g^2}\mathrm{Tr}\Big[&F_{12}^2 + D_\mu\sigma_1 D^\mu\sigma_1 + D_\mu\sigma_2 D^\mu\sigma_2 - [\sigma_1,\sigma_2]^2 + \mathrm{D}^2\\
&-\frac{i}{2}(D_\mu\bar{\lambda}\gamma^\mu\lambda - \bar{\lambda}\gamma^\mu D_\mu\lambda) + i\bar{\lambda}[\sigma_1,\lambda] + \bar{\lambda}\gamma^3[\sigma_2,\lambda]\Big]\,.
\end{aligned}$$

In general we can introduce a coupling g for each simple or abelian factor in G. The Lagrangian for a chiral multiplet is

$$\begin{aligned}
\mathcal{L}_{\text{chi}}^{\text{bulk}} \equiv \Big[&D_\mu\bar{\phi}D^\mu\phi + \bar{\phi}(\sigma_1^2+\sigma_2^2)\phi + \bar{\mathrm{F}}\mathrm{F} + i\bar{\phi}\mathrm{D}\phi\\
&+\frac{i}{2}(D_\mu\bar{\psi}\gamma^\mu\psi - \bar{\psi}\gamma^\mu D_\mu\psi) + \bar{\psi}\big(i\sigma_1 - \sigma_2\gamma^3\big)\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi\Big]\,,
\end{aligned}$$

If a theory has some flavor symmetry, we can introduce twisted mass for a chiral multiplet. Without superpotential, the twisted mass m can be introduced by the replacement $\sigma_{1,2}\to\sigma_{1,2}+\mathrm{m}_{1,2}$. In general the action involves an arbitrary number of chiral fields ϕ_a with R-charge q_a and twisted mass m_a. Inclusion of the twisted mass turns on the central charge $\tilde{Z}=i\mathrm{m}_a F^a = i(\mathrm{m}_{1a}-i\mathrm{m}_{2a})F^a$, where F^a is the generator of the flavor symmetry acting on the chiral multiplet fields as $F^a\cdot\phi_b=\delta^a_b\phi_b$. We consider how to include the twisted masses in the presence of superpotential in section 5.5.

If the gauge group G contains an abelian factor we should also include the topological term. For $G=U(N)$ this is

$$S_\theta \equiv -i\frac{\theta}{2\pi}\int \mathrm{Tr}\,F\,.$$

This is further supplemented by the Fayet-Iliopoulos (FI) term

$$S_{\text{FI}} \equiv -i\frac{r}{2\pi}\int d^2x\sqrt{h}\,\mathrm{Tr}\mathrm{D}\,.$$

Finally, if the superpotential $W(\phi)$ is non-zero we also have

$$\mathcal{L}_W = -\frac{i}{2}\Big(\mathrm{F}^i\partial_i W - \frac{1}{2}\psi^i\psi^j\partial_i\partial_j W\Big) - \frac{i}{2}\Big(\bar{\mathrm{F}}_i\bar{\partial}^i\bar{W} - \frac{1}{2}\bar{\psi}_i\bar{\psi}_j\bar{\partial}^i\bar{\partial}^j\bar{W}\Big)\,.$$

Here ϕ^i collective denote the components of $\phi = (\phi_a)$. Noting that W is gauge invariant with R-charge -2 and axial R-charge 0.

4.2. Vacuum manifolds and non-linear sigma model

Integrating out the auxiliary fields, we obtain the potential energy

$$\begin{aligned} U = &\sum_{I:\mathrm{abelian}} \frac{g^2}{2}\left(\bar{\phi}T^I\phi - \frac{r_I}{2\pi}\right)^2 + \sum_{I:\mathrm{non-abelian}} \frac{g^2}{2}(\bar{\phi}T^I\phi)^2 \\ &-\frac{1}{2g^2}\mathrm{Tr}[\sigma_1,\sigma_2]^2 + \bar{\phi}(\sigma_1^2+\sigma_2^2)\phi + \frac{1}{4}\sum_a\left|\frac{\partial W}{\partial\phi_a}\right|^2 . \end{aligned}$$

Here T^I are the generators of G acting on V_{mat} which is the space carrying the matter representation R_{mat}; for each irreducible representation R_a in the decomposition

$$R_{\mathrm{mat}} = \oplus R_a\,,$$

we have a chiral multiplet whose scalar component we call ϕ_a. We consider the vacuum manifold, *i.e.*, the zero locus of the potential U modulo gauge transformations.

The vacuum manifold depends on the value of the FI parameters r_I. The FI parameters have the phase structure. Within some phase the vacuum manifold has the same topology. However, if we go to other phases, the topology of the vacuum manifold changes. We only consider the region where there exist solutions of the D-term equations

$$\begin{cases} \bar{\phi}T^I\phi = 0 & \text{for } I \text{ non-abelian}\,, \\ \bar{\phi}T^I\phi - \dfrac{r_I}{2\pi} = 0 & \text{for } I \text{ abelian}\,, \end{cases}$$

and the F-term equations

$$\frac{\partial W}{\partial\phi_a} = 0\,.$$

If $\phi \neq 0$ for all solutions, $\sigma_{1,2}$ must be zero and the vacuum manifold consists only of the Higgs branch. If $r_I = 0$, ϕ may be zero for some solutions and $\sigma_{1,2}$ take values in Cartan subalgebra and the Coulomb branch appears. When the Coulomb branch appears, we can not obtain the non-linear sigma model at low energy. The points where Coulomb branch appears is called singularities. In quantum theory, we have to take the effect of the theta angles θ_I into account. We can derive the singular locus by considering the effective twisted superpotential $\tilde{W}$, which is obtained by integrating out the chiral multiplet fields.

$$\frac{\partial \tilde{W}_{\text{eff}}}{\partial \sigma} = 0$$

determines the singular locus in the (r_I, θ_I) space. We will also find these singularities in the UV partition function.

Now we show that the $\mathcal{N} = (2,2)$ gauge theory reduces to the non-linear sigma model whose target space is the vacuum manifold. For the sake of simplicity, we consider the case without superpotential. The modes of ϕ transverse to the vacuum manifolds obtain the following mass term from the potential,

$$-\sum_{I:\text{abelian}} \frac{g^2 r_I}{2\pi} \bar{\phi}_{\text{tr}} T^I \phi_{\text{tr}} \, .$$

By the Higgs mechanism, the gauge field acquire the same mass by eating the Nambu-Goldstone bosons. The modes of the fermions in chiral multiplets which satisfy

$$\bar{\psi}_{\tan} T^I \phi = \bar{\phi} T^I \psi_{\tan} = 0$$

are massless because the non-derivative fermion bilinear terms are vanishes. These modes are tangent to the manifold determined by the D-term equations. Other fermonic modes including the vector multiplet fermions have masses of the same order. Then, in the effective theory at a scale much smaller than any of $g\sqrt{r_I}$, the massive modes decouple and the effective theory becomes the non-linear sigma model whose target space is the vacuum manifold. Therefore, if we take the IR limit $g^2 \to \infty$, the $\mathcal{N} = (2,2)$ gauge theory reduces to the non-linear sigma model.

If we have a superpotential, some modes have masses depend on the parameter of the superpotential. To obtain the non-linear sigma model we have to take the limit of g and these parameters with appropriate scaling.

The vector multiplet action vanishes in the IR limit $g^2 \to \infty$. Then, the vector multiplet fields become auxiliary fields and can be described in terms of the chiral multiplet fields. The equations of motion imply that in the present limit,

$$A_\mu = M_{IJ}^{-1} \left(i\bar{\phi} T^I (\overleftarrow{\partial} - \overrightarrow{\partial})_\mu \phi + \bar{\psi} T^I \gamma_\mu \psi \right) T^J \, ,$$

$$\sigma_1 = -i M_{IJ}^{-1} (\bar{\psi} T^I \psi) T^J \, , \qquad \sigma_2 = M_{IJ}^{-1} (\bar{\psi} \gamma_3 T^I \psi) T^J \, ,$$

where the derivatives $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ act on $\bar{\phi}$ and ϕ respectively, and M^{-1}_{IJ} is the inverse of the matrix $M^{IJ} = \bar{\phi}\{T^I, T^J\}\phi$. Under $\phi(x) \to g(x)\phi(x)$, we get the correct transformation $d - iA \to g(x)(d - iA)g^{-1}(x)$, etc.

We consider the geometric phases in which the theory reduces to a non-linear sigma model with a smooth target space. We can see that how the theta angle θ and the FI-parameter r are related to the B-field and the Kähler form of the target space, respectively. Since the theta term involves only the abelian part $I = 0$, the discussion is essentially the same as in the abelian case. (See for example [26].) First, note that the matrix M^{IJ} is block-diagonal; the entries with $(I = 0, J \neq 0)$ or $(I \neq 0, J = 0)$ vanish because of the D-term equations. Thus, the $U(1)$ part of the gauge field is given, in the current approximation, by

$$\mathrm{Tr}\, A = \frac{2\pi i}{r}(d\bar{\phi} \cdot \phi - \bar{\phi} \cdot d\phi)\,.$$

The θ-term (5.10) gives a factor $\exp(-\frac{2\theta}{r}\int d\phi \wedge d\bar{\phi})$ in the path integral. This should be identified with the B-field coupling $\exp(2\pi i \int B)$. Thus,

$$B = \frac{i\theta}{\pi r} d\phi \wedge d\bar{\phi}\,,$$

where ϕ and $\bar{\phi}$ are constrained by the D-term equations. On the other hand the Kähler form of the target space is given, in the large volume limit, by

$$\omega = \frac{i}{2\pi} d\phi \wedge d\bar{\phi}\,.$$

In order to understand the natural combinations of parameters, let us temporarily consider the A-model where ϕ is holomorphic on the world-sheet and the kinetic term in the chiral multiplet action gives a factor $\exp(-2\pi \int \omega)$ for a world-sheet instanton. By combining it with the B-field and the boundary interaction for bundle, we get

$$\mathrm{Tr}\, P \exp\big(2\pi i \int_{\mathbb{R}^2} \iota^*(B + i\omega)\big) \tag{4.1}$$

where ι^* is the pullback by the embedding ι from the world-sheet $\mathbb{R}^2$ to the target space, *i.e.*, the vacuum manifold of the UV gauge theory.

4.3. Anomaly, renormalization and Calabi-Yau

In this subsection, we argue the anomaly and the renormalization of the $\mathcal{N} = (2,2)$ gauge theories, which are closely related to those of $\mathcal{N} = (2,2)$ non-linear sigma models.

Anomaly

In quantum theory, the R-symmetry $U(1)_{\mathrm{R}}$ suffers from no anomaly but the axial R-symmetry $U(1)_{\mathrm{A}}$ may be anomalous. Under the axial R rotation $e^{i\alpha}$, the path integral weight transforms as $e^{-2i\alpha K}$

$$K = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{Tr}_{R_{\mathrm{mat}}} F\,.$$

The contribution of the theta term is $e^{-i\theta k}$, where

$$k = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{Tr}F\,.$$

Then, the axial rotation is equivalent to the shift in the theta angle

$$\theta \to \theta + 2\alpha\frac{K}{k}\,.$$

The abelian factor contribute to the trace and the K/k captures the $U(1)$ charge of R_{mat}. Therefore K/k is a integer and $U(1)_{\mathrm{A}}$ is broken to $\mathbb{Z}_{|2K/k|}$. In the IR non-linear sigma models, the axial rotation is equivalent to the shift in the B-field. This is consistent with the fact that the theta angle in the UV gauge theory corresponds to the B-field in the IR theory as we have seen in the last subsection. To preserve the axial R-symmetry, K should be zero.

Renormalization

In general, the FI parameter r is renormalized in quantum theory. Its renormalization behavior is exactly described as

$$r(\mu) = \log\left(\frac{\mu}{\Lambda}\right)\frac{K}{k}\,,$$

where Λ is a scale parameter. In the IR non-linear sigma models, the Kähler class is renormalized. This is consistent with the fact that the FI parameter in the UV gauge theory corresponds to the the Kähler class in the IR theory as we have seem in the last subsection. If $K \neq 0$, the dimensional transmutation occurs, *i.e.*, the scale parameter Λ replaces some of the dimensionless parameters. In this case, the FI parameter runs and the shift of the theta angle θ can be cancelled by the axial rotation. Therefore the FI parameter and the theta angle are unphysical and replaced by the physical parameter Λ. If $K = 0$, the FI parameter and the theta angle become marginal.

Calabi-Yau condition

As we have seen, the discussion of the anomaly and the renormalization of the $\mathcal{N} = (2,2)$ gauge theories parallels to those of $\mathcal{N} = (2,2)$ non-linear sigma models. When the target space is Calabi-Yau, there is no axial anomaly and the Kähler class does not run. In summary, these three conditions are equivalent.
(1) The vacuum manifold is Calabi-Yau.
(2) The axial anomaly does not appear.
(3) The FI parameters are marginal.

4.4. Examples of $\mathcal{N} = (2,2)$ gauge theories

In this subsection we show some examples of $\mathcal{N} = (2,2)$ gauge theories and corresponding non-linear sigma models. In this thesis we are concerned with the geometric phases in which the theory reduces to a non-linear sigma model with a smooth target space. Then, we mainly focus on the geometric phases of theories here. We consider two cases.

Case 1: $W = 0$, target space X

This is the setup where the gauge theory has no superpotential, and flows in the IR to a non-linear sigma model with target space X, which takes the form of a Kähler quotient

$$X = \mu^{-1}(0)/G\,.$$

The moment map $\mu = (\mu^I)_{I=1}^{\dim G} : V_{\mathrm{mat}} \to \mathfrak{g}^*$ is given by

$$\mu^I \equiv \begin{cases} \bar{\phi} T^I \phi & \text{for } I \text{ non-abelian}\,, \\ \bar{\phi} T^I \phi - \dfrac{r_I}{2\pi} & \text{for } I \text{ abelian}\,. \end{cases} \tag{4.2}$$

This moment map generates the action of G. The complex structure of X can also be specified by viewing it as a holomorphic quotient:

$$X = (V_{\mathrm{mat}} \backslash \text{deleted set})/G_{\mathbb{C}}\,. \tag{4.3}$$

Here $G_{\mathbb{C}}$ is the complexification of G, and the deleted set consists of those points whose $G_{\mathbb{C}}$-orbits do not intersect with $\mu^{-1}(0)$. If the gauge group G is abelian, X is a toric variety. Now we show four examples which will appear in this thesis.

Example 1: Complex space

Let us consider the theory of n free chiral multiplets. This theory is a sigma model with target space $X = \mathbb{C}^n$.

Example 2: Projective space

Let us consider the theory with gauge group $U(1)$ and N_{F} fundamental chiral multiplets $\phi_i, i = 1, \ldots, n$. The vacuum manifold is

$$\left\{ \sum_{a=1}^{N_{\mathrm{F}}} |\phi_a|^2 = \frac{r}{2\pi} \right\} \Big/ U(1)\,.$$

We assume that $r > 0$. The complexification of $U(1)$ is $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$, and the deleted set is $\{0\} \in \mathbb{C}^n$. Then, the another description of the vacuum manifold is $(\mathbb{C}^n\backslash\{0\})/\mathbb{C}^*$. This is nothing but the projective space $\mathbb{P}^n$.

In this model, the Calabi-Yau condition is not satisfied and the dimensional transmutation occurs. But if we take an appropriate energy scale μ larger than Λ, this model reduces to the non-linear sigma models with target space $X = \mathbb{P}^n$ in the IR limit.

Example 3: Resolved conifold

Let us consider the theory with gauge group $U(1)$, 2 fundamental chiral multiplets ϕ_1, ϕ_2 and 2 anti-fundamental chiral multiplets $\tilde{\phi}_1, \tilde{\phi}_2$. The vacuum manifold is

$$\left\{ |\phi_1|^2 + |\phi_2|^2 - |\tilde{\phi}_1|^2 - |\tilde{\phi}_1|^2 = \frac{r}{2\pi} \right\} \Big/ U(1)\,.$$

In this model, the Calabi-Yau condition is satisfied and the FI parameter r becomes marginal. For $r \gg 0$, the vacuum manifold can be written as

$$\left\{ (\phi_1, \phi_2, \tilde{\phi}_1, \tilde{\phi}_1) \Big| \phi_1, \phi_2 \neq 0 \right\} \Big/ \mathbb{C}^*\,.$$

This manifold is the resolved conifold $\mathcal{O}(-1)^{\oplus 2} \to \mathbb{P}^1$. (ϕ_1, ϕ_2) parametrize the base coordinate and $(\tilde{\phi}_1, \tilde{\phi}_2)$ parametrize the fiber coordinate. For $r \ll 0$, the vacuum manifold can be written as

$$\left\{ (\phi_1, \phi_2, \tilde{\phi}_1, \tilde{\phi}_1) \Big| \tilde{\phi}_1, \tilde{\phi}_2 \neq 0 \right\} \Big/ \mathbb{C}^*\,.$$

This manifold is also the resolved conifold, but $(\tilde{\phi}_1, \tilde{\phi}_2)$ parametrize the base coordinate and (ϕ_1, ϕ_2) parametrize the fiber coordinate. When moving r from the region $r \gg 0$ to $r = 0$, the base $\mathbb{P}^1$ shrinks to a point and total manifold becomes singular. Further, moving $r = 0$ to the region $r \ll 0$, the singularity is resolved and obtain the manifold with the roles of base coordinates and fiber coordinates are exchanged. This type of topology change is known as the flop transition.

Example 4: Grassmannian

Let us consider the theory with gauge group $U(N)$ and N_{F} fundamental chiral multiplets $Q^i_f, i = 1, \ldots, N, f = 1, \ldots, N_{\mathrm{F}}$. The vacuum manifold is

$$\left\{ \sum_{f=1}^{N} |Q^i_f|^2 = \frac{r}{2\pi}, \forall i = 1, \ldots, N_{\mathrm{F}} \right\} \Big/ U(N)\,.$$

If we assume that $r > 0$, this manifold can be written as

$$\{Q | \mathrm{rk}\ Q = N\} / GL(N, \mathbb{C})$$

This is nothing but the Grassmannian $\mathrm{Gr}(N, N_{\mathrm{F}})$ which parametrize all N-dimensional linear subspaces of a N_{F}-dimensional vector space.

Case 2: $W = P \cdot \mathrm{G}(x)$, target space M

In the second situation we consider, the theory has a superpotential of the form

$$W = P \cdot \mathrm{G}(x) = P_\alpha \mathrm{G}^\alpha(x)\,,$$

where we split the chiral fields ϕ into two groups as $\phi = (x, P_\alpha)$. Assuming that the space

$$M = \mu^{-1}(0) \cap \mathrm{G}^{-1}(0)/G$$

is smooth, the F-term equations $\frac{\partial}{\partial \phi^i} W(\phi) = 0$ reduce to

$$P_\alpha = 0\,, \qquad \mathrm{G}^\alpha(x) = 0\,.$$

Thus, M is the target space of the low-energy theory, and is a submanifold of $X = \mu^{-1}(0)|_{P=0}/G$. If we focus on the complex structure, M is given as

$$M = (V_{\mathrm{mat}} \backslash \text{deleted set}) \cap \mathrm{G}^{-1}(0)/G_{\mathbb{C}}\,. \tag{4.4}$$

Now we show three examples which will appear in this thesis.

Example 5: Quintic Calabi-Yau

Let us consider a $G = U(1)$ theory with chiral fields $(P, \phi_1, \ldots, \phi_5)$ with charges $(-5, 1, 1, 1, 1, 1)$. We assign R-charges $(q_P, q_1, \ldots, q_5) = (-2, 0, \ldots, 0)$ respectively. We include the superpotential $W = P\mathrm{G}(\phi)$, where G is a degree-five polynomial. In this theory, the Calabi-Yau condition is satisfied and the FI parameter r becomes marginal. We consider the case $r \gg 0$. The vacuum manifold is

$$\left\{ \sum_{a=1}^{5} |\phi_a|^2 = \frac{r}{2\pi}, G(\phi) = 0 \right\} \Big/ U(1) \,.$$

This is known as a quintic Calabi-Yau manifold, which is the hypersurface in $\mathbb{P}^4$ given by $\mathrm{G}(\phi) = 0$.

If we consider the case $r \ll 0$, which is not a geometric phase, only P have the vacuum expectation values $|P| = \sqrt{|r|/10\pi}$ classically. If we choose a vacuum expectation values $\langle P \rangle$, the $U(1)$ gauge symmetry breaks to $\mathbb{Z}_5$ since P has the gauge charge -5. Then, the vector multiplet and P-multiplet acquire masses by the Higgs mechanism and they become infinitely massive in the IR limit $g \to \infty$. The theory reduces to the Landau-Ginzburg model with superpotential $W = \langle P \rangle G(\phi)$. The residual gauge group $\mathbb{Z}_5$ acts nontrivially on the chiral multiplet fields ϕ_i. This theory is called a Landau-Ginzburg orbifold. We will not consider Landau-Ginzburg orbifolds in this thesis.

Example 6: Complete intersection Calabi-Yau in a product of projective spaces

Let us consider a gauge theory with gauge group $G = U(1)^m = \prod_{r=1}^{m} U(1)_r$ and the following matter contents: the chiral multiplet fields

$$\phi_{r,1}, \ldots, \phi_{r,N_m}$$

charged only under $U(1)_r$ with charge 1, and

$$P_a,\; a = 1, \ldots, k$$

that have $U(1)^m$ charges $(-l_a^1, \ldots, -l_a^m)$ and R-charge -2. We also include a superpotential $W = \sum_{a=1}^{k} P_a \mathrm{G}_a(\phi)$, where $\mathrm{G}_a(\phi)$ is a degree-l_a^r polynomial with respect to $\phi_{r,1}, \ldots, \phi_{r,N_m}$. The Calabi-Yau condition is

$$\sum_a l_a^r = N_r \,.$$

For $r \gg 0$ the gauge theory flows to the nonlinear sigma model whose target space M is known as a complete intersection Calabi-Yau in a product of projective spaces. M is described as follows. Let us consider a direct product of projective spaces $X = \prod_{r=1}^{m} \mathbb{P}^{N_r - 1}$. We take sections s_a of the line bundles $\mathcal{O}(l_a^1, \ldots, l_a^m)$ for $a = 1, \ldots, k$ which are defined by the polynomials $\mathrm{G}_a(\phi)$. Then, M is defined as the intersection of their zero-loci $s_a^{-1}(0)$. We assume M is a smooth manifold.

Example 7: Cotangent bundle of Grassmannian

Let us consider the theory with gauge group $G = U(N)$, $N_{\rm F}$ fundamentals $Q^i{}_f$ and anti-fundamentals $\tilde{Q}^f{}_i$ and one adjoint $\Phi^i{}_j$ ($i, j = 1, \ldots, N$ and $f = 1, \ldots, N_{\rm F}$). We include the superpotential

$$W = \mathrm{Tr}\, \tilde{Q}\Phi Q\,.$$

The vacuum manifold is

$$\left\{ \sum_{f=1}^{N} (|Q_f^i|^2 - |\tilde{Q}_i^f|^2) = \frac{r}{2\pi}, \sum_{f=1}^{N} Q_f^i \tilde{Q}_j^f = 0, \forall i = 1, \ldots, N_{\rm F} \right\} \Big/ U(N)\,,$$

with Φ playing the role of P. In this model, the Calabi-Yau condition is satisfied and the FI parameter r becomes marginal. For $r \gg 0$, the vacuum manifold can be written as

$$\left\{ (Q, \tilde{Q}) | Q\tilde{Q} = 0, \mathrm{rk}\ Q = N \right\} / GL(N, \mathbb{C})\,.$$

This is the cotangent bundle of the Grassmannian $T^*\mathrm{Gr}(N, N_{\rm F})$. Q_f^i parametrize the base coordinate and $\tilde{Q}$ parametrize the fiber coordinate. For $r \ll 0$, the vacuum manifold can be written as

$$\left\{ (Q, \tilde{Q}) | Q\tilde{Q} = 0, \mathrm{rk}\ \tilde{Q} = N \right\} / GL(N, \mathbb{C})\,.$$

This is also $T^*\mathrm{Gr}(N, N_{\rm F})$, but the roles of base coordinates and fiber coordinates are exchanged. This is also a example of the flop transition.

5. $\mathcal{N} = (2,2)$ gauge theories on a hemisphere

In this section, we consider $\mathcal{N} = (2,2)$ gauge theories on a hemisphere. First, we explain the construction of $\mathcal{N} = (2,2)$ supersymmetry on the curved two-dimensional geometries to consider according to [34,35]. We review the definition of $\mathcal{N} = (2,2)$ theories on a two-sphere by specifying the the physical Lagrangians and modify the set-up by adding a boundary along the equator. We also describe the boundary conditions, both for vector and chiral multiplets, with which we will perform localization. We then review another ingredient, the boundary interactions that involve the Chan-Paton degrees of freedom [73]. Finally, we find that the boundary condition preserves the B-type supersymmetry.

5.1. Bulk data for $\mathcal{N} = (2,2)$ theories

An $\mathcal{N} = (2,2)$ gauge theory in two dimensions can be thought of as a dimensional reduction of an $\mathcal{N} = 1$ gauge theory in four dimensions, and in particular contains gauge and chiral multiplets. Such a theory on the curved geometries we study is specified by the data

$$(G, V_{\mathrm{mat}}, t, W, m)\,.$$

The gauge group G is a compact Lie group, and V_{mat} is the space carrying the matter representation R_{mat}; for each irreducible representation R_a in the decomposition

$$R_{\mathrm{mat}} = \oplus R_a\,,$$

we have a chiral multiplet whose scalar component we call ϕ_a. The symbol t denotes a collection of complexified FI parameters. If the gauge group is $U(N)$, it is given as $t = r - i\theta$, where r is the FI parameter and θ is the theta angle. The superpotential $W(\phi)$ is a gauge invariant holomorphic function of $\phi = (\phi_a)$. The complexified twisted masses $m = (m_a)$ are complex combinations of the real twisted masses m_a and the R-charges q_a:

$$m_a = -\frac{1}{2} q_a - i\ell \mathrm{m}_a\,.$$

Here ℓ is a length parameter of the geometry. The vector R-symmetry group[7] $U(1)_{\mathrm{R}}$, more precisely its Lie algebra $\mathfrak{u}(1)_{\mathrm{R}}$, acts on the fields ϕ_a according to the R-charges q_a. If the superpotential is zero, m_a are arbitrary complex parameters. We can regard m as taking

[7] The axial R-symmetry, which may or may not be anomalous, is broken explicitly by couplings in the action defined on the curved geometries.

values in the complexified Cartan subalgebra of the flavor symmetry group. When W is non-zero, they are constrained by the condition that for each term in the expansion of $W(\phi)$, m_a for all the fields ϕ_a in the term sum to 1. Correspondingly, the flavor symmetry group G_{F} is smaller than in the $W = 0$ case. A relation between (m_a) and the reduced flavor symmetries will be given in (5.25).

5.2. Conformal Killing spinors in two-dimensional geometries with boundary

Our aim is to compute the partition function of an $\mathcal{N} = (2,2)$ theory on a hemisphere. We will argue in section 6.4 that the hemisphere partition function computes the overlap of the D-brane boundary state in the Ramond-Ramond sector and a closed string state corresponding to the identity operator. For this purpose, it is useful to introduce a deformation parameter ($\ell/\tilde{\ell}$ below) that interpolates between a hemisphere with a round metric and a flat semi-infinite cylinder. Let us study the conformal Killing spinors in these geometries.

Round hemisphere

We first consider the hemisphere with the round metric

$$ds^2 = \ell^2(d\vartheta^2 + \sin^2\!\vartheta\, d\varphi^2) \tag{5.1}$$

in the region $0 \leq \vartheta \leq \pi/2$, $0 \leq \varphi \leq 2\pi$. The corresponding vielbein are given by $e^{\hat{1}} = \ell d\vartheta$, $e^{\hat{2}} = \ell \sin\vartheta d\varphi$. We denote by

$$\gamma^{\hat{1}} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \gamma^{\hat{2}} = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \gamma^{\hat{3}} = \gamma^3 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

the usual Pauli matrices. The conformal Killing spinor equations[8]

$$\nabla_\mu \epsilon = \gamma_\mu \tilde{\epsilon}$$

have four independent solutions

$$\epsilon = e^{-s\frac{i}{2}\vartheta\gamma_{\hat{2}}} \begin{pmatrix} e^{\frac{i}{2}\varphi} \\ 0 \end{pmatrix}, \quad e^{-s\frac{i}{2}\vartheta\gamma_{\hat{2}}} \begin{pmatrix} 0 \\ e^{-\frac{i}{2}\varphi} \end{pmatrix}, \tag{5.2}$$

[8] The non-zero component of the spin connection is $\omega_{\hat{1}\hat{2}} = -\cos\vartheta d\varphi$, and the covariant derivatives acting on a spinor are given by $\nabla_\vartheta = \partial_\vartheta$, $\nabla_\varphi = \partial_\varphi - \frac{i}{2}\cos\vartheta\gamma^3$. Note that $\tilde{\epsilon} = (1/2)\gamma^\mu\nabla_\mu\epsilon$.

with $s = \pm 1$. The SUSY transformations on a round sphere were constructed in [34,35]. In our convention, these are obtained by taking $\tilde{\ell} = \ell$ in (A.1) and (A.2). The SUSY parameters ϵ and $\bar{\epsilon}$ that appear there are conformal Killing spinors, each having four independent solutions. They parametrize the superconformal algebra on round $\mathbb{S}^2$, which contains eight fermionic charges. The $\mathcal{N} = 2$ SUSY algebra $SU(2|1)$ on $\mathbb{S}^2$, which does not contain dilatation and is compatible with masses, is generated by the spinors ϵ with $s = 1$ and $\bar{\epsilon}$ with $s = -1$. Thus, $SU(2|1)$ contains four fermionic generators. The boundary at $\vartheta = \pi/2$, however, breaks the isometry from $SU(2)$ to $U(1)$. Thus, we restrict to the subalgebra $SU(1|1)$ generated by two fermionic charges δ_ϵ and $\delta_{\bar{\epsilon}}$ given by

$$\epsilon = e^{-\frac{i}{2}\vartheta\gamma_{\hat{2}}} \begin{pmatrix} e^{\frac{i}{2}\varphi} \\ 0 \end{pmatrix}, \quad \bar{\epsilon} = e^{\frac{i}{2}\vartheta\gamma_{\hat{2}}} \begin{pmatrix} 0 \\ e^{-\frac{i}{2}\varphi} \end{pmatrix}. \tag{5.3}$$

The isometry that appears in $\{\delta_\epsilon, \delta_{\bar{\epsilon}}\}$ shifts φ by a constant and preserves the boundary.

Note that the spinors in (5.3) are anti-periodic in φ. Since bosons are periodic, fermions are all anti-periodic. We will see in section 5.5 that there is a natural field redefinition that makes all the fields periodic in φ along the boundary.

Deformed hemisphere

We will also consider the deformed metric [40]

$$ds^2 \equiv h_{\mu\nu} dx^\mu dx^\nu = f^2(\vartheta) d\vartheta^2 + \ell^2 \sin^2 \vartheta d\varphi^2 , \tag{5.4}$$

where $f^2(\vartheta) = \ell^2 \cos^2 \vartheta + \tilde{\ell}^2 \sin^2 \vartheta$. If we introduce the non-dynamical gauge field

$$V^{\mathrm{R}} = \frac{1}{2}\left(1 - \frac{\ell}{f(\vartheta)}\right) d\varphi \tag{5.5}$$

for $U(1)_{\mathrm{R}}$, the spinors (5.3) satisfy

$$D_\mu \epsilon = \frac{1}{2f}\gamma_\mu \gamma_3 \epsilon , \qquad D_\mu \bar{\epsilon} = -\frac{1}{2f}\gamma_\mu \gamma_3 \bar{\epsilon} , \tag{5.6}$$

where the covariant derivatives act as $D_\mu \epsilon = (\nabla_\mu - i V^{\mathrm{R}}_\mu)\epsilon$, $D_\mu \bar{\epsilon} = (\nabla_\mu + i V^{\mathrm{R}}_\mu)\bar{\epsilon}$. We assigned R-charges $+1$ and -1 to ϵ and $\bar{\epsilon}$ respectively. These spinors generate the superalgebra $SU(1|1)$, which contains the isometry $U(1)$ that is compatible both with the deformed metric and the boundary $\vartheta = \pi/2$. The corresponding fermionic transformations are listed in (A.1) and (A.2).[9]

[9] These formulas are essentially taken from [40] except that we flip the sign of q.

Half-infinite cylinder

In the limit $\tilde{\ell} \to \infty$, the region near $\vartheta = \pi/2$ becomes a half-infinite cylinder; by replacing ϑ with $x = -\tilde{\ell}\cos\vartheta$, the deformed metric becomes

$$ds^2 = dx^2 + \ell^2 d\varphi^2$$

in the limit. This geometry is flat, and the SUSY algebra gets enhanced.

5.3. $\mathcal{N} = (2,2)$ theories on a deformed hemisphere

We now give the precise construction of an $\mathcal{N} = (2,2)$ theory on the deformed hemisphere for the data $(G, V_{\mathrm{mat}}, t, W, m)$ defined in section 5.1.

The gauge multiplet for gauge group G consists of the gauge field A_μ, real scalars $\sigma_{1,2}$, gauginos λ, $\bar{\lambda}$, and the real auxiliary field D. Let us define

$$\delta_Q \equiv \delta_\epsilon + \delta_{\bar{\epsilon}}\,,$$

where the SUSY transformations δ_ϵ and $\delta_{\bar{\epsilon}}$ are given in (A.1) and (A.2). On a full deformed sphere the physical Lagrangian for a vector multiplet is [40]

$$\mathcal{L}_{\mathrm{vec}}^{\mathrm{exact}} \equiv \frac{1}{g^2}\delta_Q \delta_{\bar{\epsilon}} \mathrm{Tr}\left(\frac{1}{2}\bar{\lambda}\gamma^3\lambda - 2i\mathrm{D}\sigma_2 + \frac{i}{f(\vartheta)}\sigma_2^2\right)\,. \tag{5.7}$$

See Appendix A for our spinor conventions. In general we can introduce a coupling g for each simple or abelian factor in G. Noting that δ_Q^2 is a bosonic symmetry one can show that (5.7) is invariant under δ_Q. This Lagrangian can be written, up to total derivative terms, as

$$\begin{aligned}\mathcal{L}_{\mathrm{vec}}^{\mathrm{bulk}} \equiv \frac{1}{2g^2}\mathrm{Tr}\bigg[\left(F_{\hat{1}\hat{2}} + \frac{\sigma_1}{f}\right)^2 + D_\mu\sigma_1 D^\mu\sigma_1 + D_\mu\sigma_2 D^\mu\sigma_2 - [\sigma_1,\sigma_2]^2 + \mathrm{D}^2 \\ - \frac{i}{2}(D_\mu\bar{\lambda}\gamma^\mu\lambda - \bar{\lambda}\gamma^\mu D_\mu\lambda) + i\bar{\lambda}[\sigma_1,\lambda] + \bar{\lambda}\gamma^3[\sigma_2,\lambda]\bigg]\,.\end{aligned}$$

Since we are interested in manifolds with boundary it is important to keep the total derivative terms. After some calculations, we obtain

$$\int d^2x\sqrt{h}\,\mathcal{L}_{\mathrm{vec}}^{\mathrm{exact}} = \int d^2x\sqrt{h}\,\mathcal{L}_{\mathrm{vec}}^{\mathrm{bulk}} + \oint_{\vartheta=\frac{\pi}{2}} d\varphi\,\mathcal{L}_{\mathrm{vec}}^{\mathrm{bdry}}\,,$$

where[10]

$$\mathcal{L}_{\rm vec}^{\rm bdry} = \frac{1}{g^2}\mathrm{Tr}\Big[-\frac{i\ell}{\tilde{\ell}}\sigma_2 D_1\sigma_2 + i\ell\left(F_{\hat{1}\hat{2}} + \frac{1}{\tilde{\ell}}\sigma_1\right)\sigma_2 + \frac{i\ell}{4}(\bar{\lambda}_1\lambda_2 - \bar{\lambda}_2\lambda_1)\Big]\,.$$

A chiral multiplet consists of a complex scalar ϕ, a fermion ψ, a complex auxiliary field F, and their conjugate. If the R-charge of ϕ is q, those of ψ and F are $q+1$ and $q+2$ respectively. The Lagrangian

$$\mathcal{L}_{\rm chi}^{\rm exact} \equiv \delta_Q\delta_{\bar{\epsilon}}\left(-\bar{\psi}\gamma^3\psi + 2\bar{\phi}\left(\sigma_2 - i\frac{q+1}{2f}\right)\phi\right)\,, \tag{5.8}$$

has the structure

$$\int d^2x\sqrt{h}\,\mathcal{L}_{\rm chi}^{\rm exact} = \int d^2x\sqrt{h}\,\mathcal{L}_{\rm chi}^{\rm bulk} + \oint_{\vartheta=\frac{\pi}{2}} d\varphi\,\mathcal{L}_{\rm chi}^{\rm bdry}\,,$$

with

$$\begin{aligned}\mathcal{L}_{\rm chi}^{\rm bulk} \equiv \Big[&D_\mu\bar{\phi}D^\mu\phi + \bar{\phi}\left(\sigma_1^2 + \sigma_2^2 - i\frac{q+1}{f}\sigma_2 - \frac{q^2}{4f^2} - \frac{q}{4}\mathcal{R}\right)\phi + \bar{\mathrm{F}}\mathrm{F} + i\bar{\phi}\mathrm{D}\phi \\ &+ \frac{i}{2}(D_\mu\bar{\psi}\gamma^\mu\psi - \bar{\psi}\gamma^\mu D_\mu\psi) + \bar{\psi}\big(i\sigma_1 - \left(\sigma_2 - \frac{iq}{2f}\right)\gamma^3\big)\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi\Big]\,,\end{aligned} \tag{5.9}$$

and

$$\mathcal{L}_{\rm chi}^{\rm bdry} = \ell\left[\bar{\phi}\,\sigma_1\phi + i\bar{\psi}\Big(1 + \frac{\gamma_{\hat{1}}}{2}\Big)\psi\right]\,,$$

where $\mathcal{R}$ is the scalar curvature. The twisted mass m can be introduced by the replacement $\sigma_2 \to \sigma_2 + \mathrm{m}$. In general the action involves an arbitrary number of chiral fields ϕ_a with R-charge q_a and twisted mass m_a.

If the gauge group G contains an abelian factor we should also include the topological term. For $G = U(N)$ this is $-i(\theta/2\pi)\int \mathrm{Tr}\,F$, which on the hemisphere is a Wilson loop. It should be supersymmetrized into

$$S_\theta \equiv -\frac{\theta}{2\pi}\oint_{\vartheta=\frac{\pi}{2}}\mathrm{Tr}\,(iA_\varphi - \ell\sigma_2)\,d\varphi\,. \tag{5.10}$$

This is further supplemented by the Fayet-Iliopoulos (FI) term

$$S_{\rm FI} \equiv -i\frac{r}{2\pi}\int d^2x\sqrt{h}\,\mathrm{Tr}\left(\mathrm{D} - \frac{\sigma_2}{f}\right)\,. \tag{5.11}$$

[10] For general values of ϑ, $\mathcal{L}_{\rm vec}^{\rm exact} = \mathcal{L}_{\rm vec}^{\rm bulk} + (1/g^2)D_\mu\mathrm{Tr}\big[-i\bar{\epsilon}\gamma^\mu\gamma^m\epsilon\mathcal{V}_m\sigma_2 + (i/2)(\bar{\lambda}\gamma^3\epsilon)\bar{\epsilon}\gamma^\mu\lambda + \varepsilon^{\mu\nu}\sigma_1 D_\nu\sigma_2 + \bar{\epsilon}\gamma^\mu\epsilon\mathrm{D}\sigma_2 - (i/4)\bar{\lambda}\gamma^\mu\lambda\big]$ and $\mathcal{L}_{\rm chi}^{\rm exact} = \mathcal{L}_{\rm chi}^{\rm bulk} + D_\mu\big[i\varepsilon^{\mu\nu}\bar{\epsilon}\epsilon\bar{\phi}D_\nu\phi + \bar{\epsilon}\gamma^3\gamma^\mu\epsilon\bar{\phi}\sigma_1\phi + \bar{\epsilon}\gamma^\mu\epsilon\bar{\phi}\sigma_2\phi - \bar{\epsilon}\gamma^\mu\epsilon(q/2f)\bar{\phi}\phi + i(\epsilon\bar{\psi})\bar{\epsilon}\gamma^\mu\gamma^3\psi - (i/2)\bar{\psi}\gamma^\mu\psi\big]\,.$

Both S_θ and S_{FI} are invariant under δ_Q by themselves.

Finally, if the superpotential $W(\phi)$ is non-zero we also have

$$\mathcal{L}_W = -\frac{i}{2}\Big(F^i\partial_i W - \frac{1}{2}\psi^i\psi^j\partial_i\partial_j W\Big) - \frac{i}{2}\Big(\bar{F}_i\bar{\partial}^i\bar{W} - \frac{1}{2}\bar{\psi}_i\bar{\psi}_j\bar{\partial}^i\bar{\partial}^j\bar{W}\Big)\,. \tag{5.12}$$

Here ϕ^i collective denote the components of $\phi = (\phi_a)$. Noting that W is gauge invariant with R-charge -2, one can show that its variation is a total derivative

$$\delta_Q\mathcal{L}_W = \frac{1}{2}D_\mu\left(\epsilon\gamma^\mu\psi^i\partial_i W + \bar{\epsilon}\gamma^\mu\bar{\psi}_i\bar{\partial}^i\bar{W}\right)\,, \tag{5.13}$$

known as the Warner term [92]. This needs to be cancelled by the SUSY variation of the boundary interaction that we will discuss in section 5.5.

We define our supersymmetric theory by the functional integral of

$$\exp(-S_{\text{phys}}) \times (\text{boundary interaction})$$

with the total physical action

$$S_{\text{phys}} \equiv \int d^2x\sqrt{h}\left(\mathcal{L}^{\text{bulk}}_{\text{vec}} + \mathcal{L}^{\text{bulk}}_{\text{chi}} + \mathcal{L}_W\right) + S_\theta + S_{\text{FI}}\,. \tag{5.14}$$

For the theory to be supersymmetric, the total integrand has to be invariant under supersymmetry transformations. We focus on the supercharge Q of our choice. For the vector multiplet we need to impose such boundary conditions that annihilate $\delta_Q\int\sqrt{h}\,\mathcal{L}^{\text{bulk}}_{\text{vec}} = -\delta_Q\oint d\varphi\,\mathcal{L}^{\text{bdry}}_{\text{vec}}$. Similarly, $\delta_Q\oint d\varphi\,\mathcal{L}^{\text{bdry}}_{\text{chi}}$ must vanish under the boundary conditions for chiral multiplets. In section 5.4 we will see that the boundary conditions introduced in [73] do the job. We will also see there, following [73], that the Warner term (5.13) can be cancelled by a suitable boundary interaction.

5.4. Basic boundary conditions for vector and chiral multiplets

Let us introduce several basic boundary conditions that are compatible with the supercharge Q. These are straightforward generalizations of the boundary conditions found in [73] for abelian gauge groups.

Vector multiplets

The boundary condition for a vector multiplet we consider in this paper[11] consists of the following set of boundary conditions on the component fields at $\vartheta = \pi/2$:

$$\begin{gathered}\sigma_1 = 0\,, \quad D_1\sigma_2 = 0\,, \quad A_1 = 0\,, \quad F_{12} = 0\,,\\ \bar{\epsilon}\lambda = \epsilon\bar{\lambda} = 0\,, \quad D_1(\bar{\epsilon}\gamma_3\lambda) = D_1(\epsilon\gamma_3\bar{\lambda}) = 0\,,\\ D_{\hat{1}}(\mathrm{D} - iD_{\hat{1}}\sigma_1) = 0\,.\end{gathered} \tag{5.15}$$

The term $\mathcal{L}_{\rm vec}^{\rm bdry}$ vanishes with this condition imposed. In particular we have $\delta_Q \oint d\varphi\, \mathcal{L}_{\rm vec}^{\rm bdry} = 0$, as needed for preserving Q.

Chiral multiplets

For a chiral multiplet, we study two sets of boundary conditions for the component fields at $\vartheta = \pi/2$. The *Neumann boundary condition* for a chiral multiplet is given by

$$\begin{gathered}D_1\phi = D_1\bar{\phi} = 0\,,\\ \bar{\epsilon}\gamma_3\psi = \epsilon\gamma_3\bar{\psi} = 0\,, \quad D_1(\bar{\epsilon}\psi) = D_1(\epsilon\bar{\psi}) = 0\,,\\ \mathrm{F} = 0\,.\end{gathered} \tag{5.16}$$

Chiral multiplets with this boundary condition describe the target space directions tangent to a submanifold wrapped by the D-brane. In particular, for space-filling D-branes all the chiral multiplets obey the Neumann boundary condition. The *Dirichlet boundary condition* for a chiral multiplet is given by[12]

$$\begin{gathered}\phi = \bar{\phi} = 0\,,\\ \bar{\epsilon}\psi = \epsilon\bar{\psi} = 0\,, \quad D_1(\bar{\epsilon}\gamma_3\psi) = D_1(\bar{\epsilon}\gamma_3\bar{\psi}) = 0\,,\\ D_1(e^{-i\varphi}\mathrm{F} + iD_{\hat{1}}\phi) = 0\,.\end{gathered} \tag{5.17}$$

The complex scalar field ϕ parametrizes a direction normal to a submanifold. In either case the boundary condition implies that $\mathcal{L}_{\rm chi}^{\rm bdry} = 0$, ensuring that $\delta_Q \oint d\varphi\, \mathcal{L}_{\rm chi}^{\rm bdry} = 0$.

We will see in section 7.2, generalizing an argument in the abelian case studied by [73], that any lower-dimensional D-brane can be described as a bound state of space-filling D-branes carrying Chan-Paton fluxes.

[11] The boundary condition (5.15) preserves the full gauge symmetry G along the boundary. It should also be possible to formulate a boundary condition that preserves a subgroup H, as in [71].

[12] After the field redefinition (5.26), the last line simply reads $D_1(\mathrm{F}^{\rm new} + iD_{\hat{1}}\phi^{\rm new}) = 0$.

5.5. Boundary interactions

Following [73], we now introduce supersymmetric boundary interactions that will play an important role. First, we introduce the Chan-Paton vector space

$$\mathcal{V} = \mathcal{V}^{\mathrm{e}} \oplus \mathcal{V}^{\mathrm{o}} .$$

This is $\mathbb{Z}_2$-graded, and accordingly $\mathrm{End}(\mathcal{V})$ can be given the structure of a superalgebra. The space of fields is also a superalgebra, and (by implicitly taking the tensor product of superalgebras), we can make fermions anti-commute with odd linear operators acting on $\mathcal{V}$. The boundary interaction will be constructed using a conjugate pair of odd operators $\mathcal{Q}(\phi)$ and $\bar{\mathcal{Q}}(\bar{\phi})$, called a *tachyon profile*. These are respectively polynomials of ϕ and $\bar{\phi}$, and must satisfy the conditions we describe below.

Gauge group G, flavor group G_{F}, and the vector R-symmetry group $U(1)_{\mathrm{R}}$ act on the space $\mathcal{V}$. In other words, there is a representation, or equivalently a homomorphism[13]

$$\rho : G \times G_{\mathrm{F}} \times U(1)_{\mathrm{R}} \to \mathrm{End}(\mathcal{V}) .$$

We demand that the tachyon profile is invariant under G and G_{F}:

$$\rho(g)\mathcal{Q}(g^{-1} \cdot \phi)\rho(g)^{-1} = \mathcal{Q}(\phi) , \qquad \rho(g)\bar{\mathcal{Q}}(\bar{\phi} \cdot g)\rho(g)^{-1} = \mathcal{Q}(\bar{\phi}) \tag{5.18}$$

for $g \in G \times G_{\mathrm{F}}$. For the R-symmetry, let us denote the generator by R. It acts on a chiral multiplet ϕ_a, in the notation of section 5.1, as

$$R \cdot \phi_a = q_a \phi_a , \tag{5.19}$$

where q_a is the R-charge. We require that the tachyon profile satisfies the conditions

$$\begin{aligned} \rho(e^{i\alpha R})\mathcal{Q}(e^{-i\alpha R} \cdot \phi)\rho(e^{-i\alpha R}) &= e^{i\alpha}\mathcal{Q}(\phi) , \\ \rho(e^{i\alpha R})\bar{\mathcal{Q}}(\bar{\phi} \cdot e^{i\alpha R})\rho(e^{-i\alpha R}) &= e^{-i\alpha}\bar{\mathcal{Q}}(\bar{\phi}) . \end{aligned} \tag{5.20}$$

We can now define the *boundary interaction* [93,73], an $\mathrm{End}(\mathcal{V})$-valued 1-form along the boundary circle at $\vartheta = \pi/2$:

$$\begin{aligned} \mathcal{A}_{\hat{\varphi}} = \rho_*(A_{\hat{\varphi}} + i\sigma_2) + \frac{\rho_*(R)}{2\ell} + i\rho_*(\mathrm{m}) \\ + \frac{i}{2}\{\mathcal{Q}, \bar{\mathcal{Q}}\} + \frac{1}{2}(\psi_1 - \psi_2)^i \partial_i \mathcal{Q} + \frac{1}{2}(\bar{\psi}_1 - \bar{\psi}_2)_i \partial^i \bar{\mathcal{Q}} . \end{aligned} \tag{5.21}$$

[13] More precisely, we allow ρ to be a projective representation. See sections 5.1 and 7.3. We denote the induced representation of the Lie algebra by ρ_*.

Here the representation ρ_* of the Lie algebra of $G \times G_{\rm F} \times U(1)_{\rm R}$ is induced from ρ. In the path integral we include

$$\mathrm{Str}_{\mathcal{V}}\Big[P\exp\Big(i\oint d\varphi \mathcal{A}_\varphi\Big)\Big]\,. \tag{5.22}$$

As in [94,73], one can show with some calculations that the Q variation of the boundary interaction $\mathcal{A}_{\hat\varphi}$ cancels the Warner term $\delta_Q \mathcal{L}_W$ in (5.13),

$$\begin{aligned}
&\delta_Q \mathrm{Str}_{\mathcal{V}}\Big[Pe^{i\oint d\varphi\mathcal{A}_\varphi}e^{-\int d^2x\sqrt{h}\mathcal{L}_W}\Big]\\
&= \mathrm{Str}_{\mathcal{V}}\Big[Pe^{i\oint d\varphi\mathcal{A}_\varphi}e^{-\int d^2x\sqrt{h}\mathcal{L}_W}\Big(i\oint d\varphi\,\delta_Q\mathcal{A}_\varphi - \int d^2x\sqrt{h}\,\delta_Q\mathcal{L}_W\Big)\Big]\\
&= 0\,,
\end{aligned}$$

if $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ satisfy

$$\mathcal{Q}^2 = W\cdot \mathbf{1}_{\mathcal{V}}\,,\qquad \bar{\mathcal{Q}}^2 = \bar{W}\cdot\mathbf{1}_{\mathcal{V}}\,. \tag{5.23}$$

When the conditions (5.23) are satisfied, we say that the tachyon profile $\mathcal{Q}$ is a *matrix factorization* of the superpotential W. The boundary interaction (5.21) allows us to construct interesting supersymmetric theories on a hemisphere.

In order to compare (5.21) with [73], it is useful to introduce a version of vector R-symmetry group (in general distinct from the original) and perform a field redefinition. This will also be important to understand the target space interpretation in section 6.4.

Consider first the case $W = 0$. Because an R-symmetry mixed with flavor symmetries[14] is also an R-symmetry, we can define a new R-symmetry by

$$R_{\rm deg} = R - q_a F^a\,,$$

where F^a are the flavor generators (for $W = 0$) such that

$$F^a \cdot \phi_b = \delta^a_b \phi_b\,.$$

The R-charges for the new R-symmetry for all ϕ_a vanish, and those of the superpartners ψ_a and F_a are 1 and 2, respectively. The first condition in (5.20) applied to $R_{\rm deg}$ implies that the tachyon profile $\mathcal{Q}$ increases the eigenvalue of $R_{\rm deg}$ by one: $[\rho_*(R_{\rm deg}),\mathcal{Q}] = \mathcal{Q}$. We require that the eigenvalues of $R_{\rm deg}$ in $\mathcal{V}$ are all integers. Then, we can decompose $\mathcal{V}$ into

[14] Mixing with gauge symmetries plays no role, so we exclude the possibility from discussion.

the eigenspaces $\mathcal{V}^i$ of R_{deg} with eigenvalue i. Since $W = 0$, $\mathcal{Q}$ defines a differential of the cochain complex

$$\ldots \longrightarrow \mathcal{V}^i \longrightarrow \mathcal{V}^{i+1} \longrightarrow \ldots$$

Whether W is zero or not, we will require that there is an R-symmetry generator R_{deg} that has only even (odd) integer eigenvalues in $\mathcal{V}^{\text{e}}$ (respectively $\mathcal{V}^{\text{o}}$), and even integer eigenvalues d_a on ϕ_a. Any such generator is related to the previous R-symmetry generator R as

$$R_{\text{deg}} = R - q_\alpha F^\alpha \,, \tag{5.24}$$

where F^α are the Cartan generators of the flavor group G_{F} preserved by W, and q_α take real values. As we will see in section 7.1, there is a natural choice of R_{deg} when the gauge theory flows to a non-linear sigma model. Using d_a, we can parametrize the complexified twisted masses by the Cartan of G_{F} as $m_a = -(1/2)\text{d}_a + m_\alpha (F^\alpha)_a$, where[15]

$$m_\alpha = -\frac{1}{2} q_\alpha - i\ell \text{m}_\alpha \,. \tag{5.25}$$

When the superpotential W breaks all flavor symmetries, m_a are simply R-charges rescaled, $m_a = -\text{d}_a/2$.

Let us consider the simultaneous redefinition

$$\Phi(\vartheta, \varphi) \to \Phi^{\text{new}}(\vartheta, \varphi) = e^{-\frac{i}{2} R_{\text{deg}} \varphi} \cdot \Phi(\vartheta, \varphi) \tag{5.26}$$

of all the bosonic and fermionic fields Φ in the theory. Since we demanded that R_{deg} has even integers as eigenvalues on the scalars ϕ_a, bosonic fields remain periodic while fermions become periodic from anti-periodic.

In the new description, which is valid in the neighborhood of the boundary, the background gauge field (5.5) for (the original) $U(1)_{\text{R}}$ is shifted as

$$V^{\text{R}} \to V^{\text{R,new}} = V^{\text{R}} - \frac{1}{2} d\varphi = -\frac{\ell}{2 f(\vartheta)} d\varphi \,. \tag{5.27}$$

In addition, the field redefinition induces an extra background gauge field for the flavor symmetry:

$$V^{\text{F}} = \frac{1}{2} q_\alpha F^\alpha d\varphi \,. \tag{5.28}$$

[15] The symbols $(q_\alpha, F^\alpha, m_\alpha)$, labeled by the directions α in the Cartan of G_{F}, should be distinguished from (q_a, F^a, m_a) labeled by a parametrizing irreducible matter representations. The term $-(1/2)\text{d}_a$ in m_a is analogous to a shift in the four-dimensional mass on $\mathbb{S}^4$ noticed in [95].

The full covariant derivative

$$D_\mu = \nabla_\mu - iA_\mu - iV_\mu R$$

becomes

$$D_\mu^{\mathrm{new}} = \nabla_\mu - iA_\mu - iV_\mu^{\mathrm{R,new}} R - iV_\mu^{\mathrm{F}} \,.$$

If we apply the redefinition to SUSY parameters, they become at $\vartheta = \pi/2$

$$\epsilon_{\mathrm{flat}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \,, \quad \bar{\epsilon}_{\mathrm{flat}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \,. \tag{5.29}$$

Each spinor gives rise to a linear combination of left- and right-moving, barred or unbarred, supercharges. Thus, they correspond to the B-type supersymmetries [9].

The field redefinition (5.26) removes from $\mathcal{A}_{\hat{\varphi}}$ the R-symmetry background and induces a flavor background (5.28):

$$\mathcal{A}_{\hat{\varphi}}^{\mathrm{new}} = \rho_*(A_{\hat{\varphi}} + i\sigma_2) + \rho_*(V_{\hat{\varphi}}^{\mathrm{F}} + i\mathrm{m}) + \frac{i}{2}\{\mathcal{Q}^{\mathrm{new}}, \bar{\mathcal{Q}}^{\mathrm{new}}\} + \ldots \,. \tag{5.30}$$

This expression agrees with the interaction found in [73] when the flavor part is taken into account.

Let us summarize sections 5.4 and 5.5. Given a theory specified by the bulk data $(G, V_{\mathrm{mat}}, t, W, m)$, we can define a boundary condition $\mathcal{B}$, or a *D-brane*, by the data

$$\mathcal{B} = (\mathbf{Neu}, \mathbf{Dir}, \mathcal{V}, \mathcal{Q}) \,.$$

The vector multiplet obey the boundary condition (5.15). The symbols **Neu** and **Dir** denote that set of chiral multiplets that obey the Neumann and the Dirichlet boundary conditions (5.16) and (5.17), respectively. We will often assume that $\mathbf{Dir} = \emptyset$ and simply write $\mathcal{B} = (\mathcal{V}, \mathcal{Q})$. The Chan-Paton space $\mathcal{V} = \mathcal{V}^{\mathrm{e}} \oplus \mathcal{V}^{\mathrm{o}}$ is $\mathbb{Z}_2$-graded and carries a representation of $G \times G_{\mathrm{F}} \times U(1)_{\mathrm{R}}$. It must admit a new R-symmetry generator R_{deg} that is a mixture of the original R-symmetry (encoded in m) and flavor symmetries, and has integer eigenvalues on $\mathcal{V}$ that descend to the $\mathbb{Z}_2$-grading. The tachyon profile $\mathcal{Q}$ is a matrix factorization of W, *i.e.*, an odd linear operator on $\mathcal{V}$ that squares to $W \cdot \mathbf{1}_{\mathcal{V}}$.

6. Localization on a hemisphere

In this section we perform the localization calculation of $\mathcal{N} = (2,2)$ gauge theories on a hemisphere. The supersymmetric localization we reviewed in subsection 2.5 can be used to analyze supersymmetric gauge theories which are not twisted. We derive the partition function of $\mathcal{N} = (2,2)$ gauge theories on a hemisphere, which we call the "hemisphere partition function." We find the Hilbert space interpretation of the hemisphere partition function, *i.e.*, they can be considered as the overlaps of the states in the BPS Hilbert space. From this argument, we can derive the sphere partition function, *i.e.*, the partition function of the $\mathcal{N} = (2,2)$ gauge theories on a sphere computed in [34,35]. Our derivation of the sphere partition function does not have any ambiguities which exist in the localization calculation in [34,35].

6.1. Localization action and locus

In a supersymmetric quantum field theory, we know a priori that the path integral receives contributions from the field configurations that are annihilated by the supercharges.[16] Moreover, if the locus of such invariant configurations is finite-dimensional, the path integral can be exactly performed by evaluating the one-loop determinant in the normal directions. This statement holds for any action that preserves supersymmetry as long as its behavior for large values of fields is reasonable.

Though the one-loop determinant depends on the choice of the action, there is still redundancy; if the action is modified by adding an exact term, the one-loop determinant does not change by the standard argument. In the following, we will use (5.7) and (5.8) to define the localization action

$$S_{\rm loc} \equiv \int d^2x \sqrt{h} (\mathcal{L}^{\rm exact}_{\rm vec} + \mathcal{L}^{\rm exact}_{\rm chi}) \,. \tag{6.1}$$

Namely, we will consider the path integral

$$Z_{\rm hem} \equiv \int [DA_\mu \dots D\phi \dots] \, {\rm Str}_{\mathcal{V}} \Big[P \exp \Big(i \oint d\varphi \mathcal{A}_\varphi \Big) \Big] \exp\left(-S_{\rm phys} - {\rm t} S_{\rm loc}\right) \,,$$

where the boundary interaction $\mathcal{A}_\varphi$ and the physical action $S_{\rm phys}$ are defined in (5.21) and (5.14), respectively. Since $S_{\rm loc}$ is Q-exact, the path integral is independent of t. We

[16] One of the early references that discusses this explicitly is [30].

evaluate the path integral in the limit t $\to +\infty$; the one-loop determinant can be obtained from the quadratic part of $S_{\rm loc}$.

For a generic assignment of R-charges, the localization locus for the theory on a (deformed) two-sphere was determined in [34,35,40]. On the hemisphere with the symmetry-preserving boundary condition (5.15), we have a further simplification that the flux B vanishes. Then, the only non-vanishing field in the locus is

$$\sigma_2 = \text{const}\,. \tag{6.2}$$

In this locus, the physical action $S_{\rm phys}$ contributes to the path integral

$$e^{-i\ell t \operatorname{Tr}\sigma_2}\,, \tag{6.3}$$

which comes from S_θ in (5.10) and $S_{\rm FI}$ in (5.11). Here we have set $t = r - i\theta$. As part of the classical contribution, we also need to evaluate the supertrace (5.22). It is most cleanly evaluated using the expression (5.30) after the field redefinition (5.26). In the localization locus (6.2), the supertrace becomes

$$\operatorname{Str}_{\mathcal V}\left[e^{-2\pi\ell\rho_*(\sigma_2)}e^{-2\pi i\rho_*(-\frac{1}{2}q_\alpha F^\alpha - i\ell \mathrm{m})}\right] = \operatorname{Str}_{\mathcal V}\left[e^{-2\pi i\rho_*(\sigma + m_\alpha F^\alpha)}\right]\,. \tag{6.4}$$

where we defined $\sigma = -i\ell\sigma_2$. In most of the paper we will simply write (6.4) as

$$\operatorname{Str}_{\mathcal V}\left[e^{-2\pi i(\sigma+m)}\right]\,.$$

6.2. One-loop determinants

In this section we compute the one-loop determinant for the saddle point configuration (6.2). Because the computations are easier for chiral multiplets than for vector multiplets, we first treat the former. For simplicity we work with the round metric (5.1) and suppress ℓ during computations.

Let us consider a chiral multiplet in a representation R of the gauge group. Around the localization locus (6.2), the chiral multiplet part of the localization action (6.1) reads, to the quadratic order,

$$S^{(2)}_{\rm chi} = \int d^2x\sqrt{h}\Big[\bar\phi\Big(\mathrm{M}^2 - i(q+1)\sigma_2 - \frac{q^2+2q}{4}\Big)\phi + \bar{\mathrm{F}}\mathrm{F} - \bar\psi\gamma^3\Big(i\gamma^3\gamma^\mu D_\mu + \sigma_2 - \frac{iq}{2}\Big)\psi\Big]\,,$$

where

$$\mathrm{M}^2 \equiv -D^\mu D_\mu + \sigma_2^2\,.$$

The Gaussian integral over F and $\bar{\mathrm{F}}$ does not depend on any parameter and will be ignored. As we show in Appendix C, the Dirac operator in the particular combination $\gamma^3\gamma^\mu D_\mu$ is self-adjoint on the hemisphere—the naive one $i\gamma^\mu D_\mu$ is not—when the relevant boundary conditions are imposed on the spinors.

Let us denote the weights of R by w. To avoid clutter we assume that each weight w has multiplicity 1; it is trivial to drop the assumption. Each field can be expanded in an orthonormal basis consisting of weight vectors e_w such that $\sigma_2 \cdot e_w = w(\sigma_2)e_w$. We write $\bar{e}^w \equiv (e_w)^\dagger$. Using the scalar spherical harmonics Y_{jm} and the spinor harmonics $\chi^\pm_{jm}(\vartheta,\varphi)$ reviewed in Appendix B, we expand

$$\begin{aligned}
\phi &= \sum_w \sum_{j=0}^{\infty} \sum_{m=-j}^{j}{}' \phi^w_{jm} Y_{jm}(\vartheta,\varphi) e_w \,, & \bar\phi &= \sum_w \sum_{j=0}^{\infty} \sum_{m=-j}^{j}{}' (\phi^w_{jm})^* Y_{jm}(\vartheta,\varphi)^* \bar{e}^w \,, \\
\psi &= \sum_w \sum_{s=\pm} \sum_{j=\frac12}^{\infty} \sum_{m=-j}^{j}{}' \psi^{ws}_{jm} \chi^s_{jm}(\vartheta,\varphi) e_w \,, & \bar\psi &= \sum_w \sum_{s=\pm} \sum_{j=\frac12}^{\infty} \sum_{m=-j}^{j}{}' \bar\psi^s_{wjm} \chi^s_{jm}(\vartheta,\varphi) \bar{e}^w \,.
\end{aligned} \tag{6.5}$$

The symbol Σ' indicates that the sum is restricted to such m that

$$j-m = \begin{cases} \text{even} & \text{for} \quad \phi \text{ and } \bar\phi\,, \\ \text{odd} & \text{for} \quad s=+ \text{ in } \psi \text{ and } \bar\psi\,, \\ \text{even} & \text{for} \quad s=- \text{ in } \psi \text{ and } \bar\psi\,. \end{cases}$$

for the Neumann-type boundary conditions (5.16), and

$$j-m = \begin{cases} \text{odd} & \text{for} \quad \phi \text{ and } \bar\phi\,, \\ \text{even} & \text{for} \quad s=+ \text{ in } \psi \text{ and } \bar\psi\,, \\ \text{odd} & \text{for} \quad s=- \text{ in } \psi \text{ and } \bar\psi\,. \end{cases}$$

for the Dirichlet-type boundary conditions (5.17). Using the mode expansions, the eigenvalues, and the orthogonality relations reviewed in Appendix B, we obtain

$$\begin{aligned}
S^{(2)}_{\text{chi}} = {} & \frac12 \sum_w \sum_{j=0}^{\infty} \sum_{m=-j}^{j}{}' (\phi^w_{jm})^* \left[\left(j+\frac12\right)^2 + \left(w\cdot\sigma_2 - i\frac{q+1}{2}\right)^2 \right] \phi^w_{jm} \\
& + \frac12 \sum_w \sum_{j=1/2}^{\infty} \sum_{m=-j}^{j}{}' (-1)^{m+1/2} s\, \bar\psi^{-s}_{w,j,-m} \left[s\, i \left(j+\frac12\right) + w\cdot\sigma_2 - i\frac{q}{2}\right] \psi^{ws}_{jm}\,.
\end{aligned} \tag{6.6}$$

From this we can calculate the one-loop determinant.

$$
\begin{aligned}
Z_{\text{1-loop}}^{\text{chi}} &= \prod_w \frac{\prod_{j=1/2}^{\infty}\left[\left(j+\frac{1}{2}\right)^2+\left(w\cdot\sigma_2-i\frac{q}{2}\right)^2\right]^{j+1/2}}{\prod_{j=0}^{\infty}\left[\left(j+\frac{1}{2}\right)^2+\left(w\cdot\sigma_2-i\frac{q+1}{2}\right)^2\right]^{(j+1 \text{ or } j)}} \\
&= \prod_w \begin{cases} 1\Big/\prod_{j=0}^{\infty}\left[j-i\left(w\cdot\sigma_2-i\frac{q}{2}\right)\right] & \text{(Neumann)}\,, \\ \prod_{j=0}^{\infty}\left(j+1+i\left(w\cdot\sigma_2-i\frac{q}{2}\right)\right) & \text{(Dirichlet)}\,. \end{cases}
\end{aligned} \tag{6.7}
$$

The twisted mass m can be introduced by replacing $w\cdot\sigma_2 \to w\cdot\sigma_2+\mathrm{m}$. The infinite products can be regularized by the gamma function $\Gamma(1+z)=e^{-\gamma z}\prod_{k=1}^{\infty}e^{z/k}(1+z/k)^{-1}$, where γ is the Euler constant. Even if we use the gamma function so that we get the required zeros and poles, there are ambiguities in the overall z-dependent normalizations. For reasons we explain in sections 6.3 and 8.1, we choose

$$
Z_{\text{1-loop}}^{\text{chi}}(\sigma;m) = \begin{cases} Z_{\text{1-loop}}^{\text{chi, Neu}} \equiv \prod_{w\in R}\Gamma(w\cdot\sigma+m) & \text{(Neumann)}\,, \\ Z_{\text{1-loop}}^{\text{chi, Dir}} \equiv \dfrac{-2\pi i\, e^{\pi i(w\cdot\sigma+m)}}{\prod_{w\in R}\Gamma(1-w\cdot\sigma-m)} & \text{(Dirichlet)}\,, \end{cases} \tag{6.8}
$$

where the product is over all the weights in the representation R, and

$$
\sigma \equiv -i\ell\sigma_2\,, \qquad m \equiv -\frac{q}{2}-i\ell\mathrm{m}\,.
$$

We have recovered ℓ for the definition of σ.

The infinite products require UV regularization and result in the running of the effective FI parameters. As in [35], we take into account the effect of renormalization by replacing the FI parameter with its renormalized value. For each abelian factor in the gauge group G, this gives

$$
t \to t_{\text{ren}} = t - \sum_a Q_a \ln(\ell M_{\text{UV}})\,, \tag{6.9}
$$

where Q_a are the charges of the chiral multiplets, and M_{UV} is the UV cut-off.[17] In the Calabi-Yau case $\sum_a Q_a = 0$, we have $t_{\text{ren}} = t$.

[17] By the same mechanism, effective FI parameters are generated for flavor symmetries [34]. The partition function is then multiplied by the factor $e^{-m\ln(\ell M_{\text{UV}})}$ for each twisted mass m.

We turn to the vector multiplet for the gauge group G. In the R_ξ gauge, the localization action $S_{\rm loc}$ augmented by the ghost action [35], around the locus (6.2), reads

$$S^{(2)}_{\rm vec} = \int d^2x\sqrt{h}\,{\rm Tr}\Big[A^\mu\left({\rm M}^2+1\right)A_\mu + 2\tilde\sigma_1\varepsilon^{\mu\nu}\nabla_\mu A_\nu + \tilde\sigma_1\left({\rm M}^2+1\right)\tilde\sigma_1 + \tilde\sigma_2{\rm M}^2\tilde\sigma_2 + {\rm D}^2 + \bar\lambda\gamma^3\left(i\gamma^3\gamma^\mu D_\mu+\sigma_2\right)\lambda + \bar c\,{\rm M}^2 c\Big] \tag{6.10}$$

up to the quadratic order, where $\tilde\sigma_r$ are the fluctuations of the fields σ_r, and

$${\rm M}^2 := -D^\mu D_\mu + \sigma_2^2.$$

The Gaussian integral over D is trivial and will be neglected.

On the vector multiplet we impose the boundary condition (5.15). Let us denote the basis of $\mathfrak{g}_{\mathbb{C}}$ by H_i ($i=1,\ldots,{\rm rk}\,G$) and E_α, where H_i span the Cartan subalgebra, and α are the roots of G: $[H_i, E_\alpha] = \alpha(H_i)E_\alpha$, $E_\alpha^\dagger = E_{-\alpha}$. We choose a decomposition of the root system into the positive and the negative roots. For $r=1,2$, we expand

$$\tilde\sigma_r = \sum_{\alpha>0}\sum_{j=0}^{\infty}\sum_{m=-j}^{j}{}' \,\tilde\sigma^\alpha_{rjm}Y_{jm}(\vartheta,\varphi)E_\alpha + h.c. + \ldots$$

The ellipses indicate terms in the Cartan subalgebra, whose contributions are independent of physical parameters and will be dropped. Ghosts $(c,\bar c)$ are expanded in a way similar to $(\phi,\bar\phi)$ with coefficients $(c^\alpha_{jm}, \bar c_{\alpha jm})$, respectively. The expansions of the gauginos $(\lambda,\bar\lambda)$ are similar to those of $(\psi,\bar\psi)$, and have respectively the coefficients $(\lambda^{s\alpha}_{jm}, \bar\lambda^s_{\alpha jm})$. For the gauge field,

$$A_\mu = \sum_{\alpha>0}\sum_{\lambda=1}^{2}\sum_{j=1}^{\infty}\sum_{m=-j}^{j}{}' \,A^{\alpha\lambda}_{jm}(C^\lambda_{jm})_\mu E_\alpha + h.c. + \ldots\,,$$

where $(C^\lambda_{jm})_\mu$ are the vector spherical harmonics reviewed in Appendix B. The sums $\sum'_m$ are restricted to those m which satisfy

$$j-m = \begin{cases} \text{even(odd)} & \text{for} \quad \lambda = 1(2) \text{ in } A_\mu\,, \\ \text{odd} & \text{for} \quad \tilde\sigma_1, c, \bar c\,, \\ \text{even} & \text{for} \quad \tilde\sigma_2\,, \\ \text{even(odd)} & \text{for} \quad s=+(-) \text{ in } \lambda \text{ and } \bar\lambda\,. \end{cases}$$

The eigenvalues of the kinetic operators as well as the pairings of the eigenmodes can be found by using the properties of the spherical harmonics reviewed in Appendix B. Let us split the quadratic action (6.10) into the bosonic and the fermionic parts. The bosonic part $S^{(2)\mathrm{b}}_{\mathrm{vec}}$ reads

$$
\begin{aligned}
S^{(2)\mathrm{b}}_{\mathrm{vec}} = \sum_{\alpha>0} \Bigg(& \sum_{\lambda=1}^{2}\sum_{j=1}^{\infty}\sum_{m=-j}^{j}{}' (A^{\alpha\lambda}_{jm})^* \left[j(j+1)+(\alpha\cdot\sigma_2)^2\right] A^{\alpha\lambda}_{jm} \\
& - \sum_{j=1}^{\infty}\sum_{m=-j}^{j}{}' \left[(\tilde\sigma^{\alpha}_{1jm})^* \sqrt{j(j+1)} A^{\alpha 2}_{jm} + c.c.\right] \\
& + \sum_{r=1}^{2}\sum_{j=0}^{\infty}\sum_{m=-j}^{j}{}' (\tilde\sigma^{\alpha}_{rjm})^* \left[j(j+1)+(\alpha\cdot\sigma_2)^2+2-r\right] \tilde\sigma^{\alpha}_{rjm} \Bigg) .
\end{aligned}
\tag{6.11}
$$

The gaugino part is similar to the fermionic part in the chiral multiplet action (6.6). The ghost part is

$$
\sum_{\alpha}\sum_{j=0}^{\infty}\sum_{m=-j}^{j}{}' \bar c_{-\alpha,j,-m} \left[j(j+1)+(\alpha\cdot\sigma_2)^2\right] c^{\alpha}_{jm} .
$$

Let us now calculate the one-loop determinant $Z^{\mathrm{vec}}_{\text{1-loop}}$ for the vector multiplet. The combined contribution from $A^{\alpha 2}_{jm}$ and $\tilde\sigma_1$ to $Z^{\mathrm{vec}}_{\text{1-loop}}$ is

$$
\begin{aligned}
& \prod_{\alpha>0}\prod_{j=1}^{\infty} \begin{vmatrix} j(j+1)+(\alpha\cdot\sigma_2)^2 & \sqrt{j(j+1)} \\ \sqrt{j(j+1)} & j(j+1)+(\alpha\cdot\sigma_2)^2+1 \end{vmatrix}^{j} \\
= & \prod_{\alpha>0}\prod_{j=1}^{\infty} \left[j^2+(\alpha\cdot\sigma_2)^2\right]^j \left[(j+1)^2+(\alpha\cdot\sigma_2)^2\right]^j .
\end{aligned}
$$

The contributions from the other modes can be computed straightforwardly. Combining everything together, we obtain

$$
Z^{\mathrm{vec}}_{\text{1-loop}} \sim \prod_{\alpha>0}\prod_{j=0}^{\infty} \left[j^2+(\alpha\cdot\sigma_2)^2\right] .
$$

Recall the notation $\sigma = -i\ell\sigma_2$. After regularization, we obtain[18]

$$
Z^{\mathrm{vec}}_{\text{1-loop}} = \prod_{\alpha>0} \alpha\cdot\sigma \sin(\pi\alpha\cdot\sigma) . \tag{6.12}
$$

6.3. Results for the hemisphere partition function

We now write down the partition function of the $\mathcal{N}=(2,2)$ theory $(G, V_{\mathrm{mat}}, t, W, m)$ on a hemisphere with boundary condition $\mathcal{B} = (\mathbf{Neu}, \mathbf{Dir}, \mathcal{V}, \mathcal{Q})$. Putting together the

[18] An analogous factor appears in an integral representation of a vortex partition function [96].

calculations in sections 6.1 and 6.2, we obtain the partition function[19]

$$Z_{\text{hem}}(\mathcal{B};t;m) = \frac{1}{|W(G)|}\int_{\sigma\in i\mathfrak{t}} \frac{d^{\text{rk}(G)}\sigma}{(2\pi i)^{\text{rk}(G)}} \text{Str}_{\mathcal{V}}[e^{-2\pi i(\sigma+m)}] e^{t_{\text{ren}}\cdot\sigma} Z_{\text{1-loop}}(\mathcal{B};\sigma;m)\,, \quad (6.13)$$

where the one-loop determinant is

$$\begin{aligned} Z_{\text{1-loop}}(\mathcal{B};\sigma;m) = \Big(\prod_{\alpha>0} \alpha\cdot\sigma \frac{\sin(\pi\alpha\cdot\sigma)}{-\pi}\Big) \prod_{a\in\mathbf{Neu}} \prod_{w\in R_a} \Gamma(w\cdot\sigma + m_a) \\ \times \prod_{a\in\mathbf{Dir}} \prod_{w\in R_a} \frac{-2\pi i e^{\pi i(w\cdot\sigma+m_a)}}{\Gamma(1-w\cdot\sigma - m_a)}\,. \end{aligned} \quad (6.14)$$

Here $W(G)$ is the Weyl group, $\mathfrak{t} = \mathfrak{t}(G)$ is the Cartan subalgebra, and rk denotes the rank. Recall also that $t_{\text{ren}}\cdot\sigma$ with $t_{\text{ren}} = r_{\text{ren}} - i\theta$ denotes the renormalized FI and the topological couplings (6.9) for the abelian factors in the gauge group G.[20] The complexified twisted masses $m = (m_a)$ are defined as the combinations $m_a = -\frac{1}{2}q_a - i\ell \mathrm{m}_a$ of the R-charges q_a and the real twisted masses m_a. In the rest of the paper, we will refer to m_a simply as twisted masses.

In the special case $G = U(1)$, the partition function becomes

$$Z_{\text{hem}} = \int \frac{d\sigma}{2\pi i} e^{t_{\text{ren}}\sigma} \text{Str}_{\mathcal{V}}[e^{-2\pi i(\sigma+m)}] \prod_{a\in\mathbf{Neu}} \Gamma(Q_a\sigma + m_a) \prod_{a\in\mathbf{Dir}} \frac{-2\pi i\, e^{\pi i(Q_a\sigma+m_a)}}{\Gamma(1-Q_a\sigma - m_a)}\,, \quad (6.15)$$

where Q_a is the $U(1)$ charge for the a-th chiral multiplet.

Depending on the representations in which the chiral fields transform, it may be necessary to deform the contour in the asymptotic region so that the integral is convergent. For r deep inside the Kähler cone of a geometric phase, the integral (6.13) can be evaluated explicitly by the residue theorem.

In particular for theories whose axial R-symmetry is non-anomalous in flat space,[21] we can write down a general formula for Z_{hem} using multi-dimensional residues, as in the case of the $\mathbb{S}^2$ partition function [42]. Let H_i, $i = 1, \dots \text{rk}(G)$, be the simple coroots, which we treat as a basis of $\mathfrak{t}_{\mathbb{C}}$. Let us expand

$$\sigma = \sum_j \sigma^j H_j\,, \qquad w\cdot\sigma = \sum_j w_j\sigma^j\,, \qquad t\cdot\sigma = \sum_j t_j\sigma^j \quad (6.16)$$

[19] We divided each sine by $-\pi$, so that the hemisphere partition functions behave better under dualities discusses in section 9.

[20] If $G = U(N)$, $t_{\text{ren}}\cdot\sigma = t_{\text{ren}}\text{Tr}\,\sigma$.

[21] This is equivalent to the condition $\sum_a \sum_{w\in R_a} w = 0$, which makes the asymptotic behavior of the integrand to be determined by $e^{t\cdot\sigma}$.

and write $\vec{\sigma} = (\sigma^j)$, $\vec{w} = (w_j)$, $\vec{t} = \vec{r} - i\vec{\theta} = (t_j = r_j - i\theta_j)$. When G is non-abelian, t_j in (6.16) are not all independent. Let I be a subset of $\{(a,w)|a \in \mathbf{Neu}, w \in R_a\}$ with $|I| = \mathrm{rk}(G)$ such that the weights w that appear are linearly independent. Denote by $\mathfrak{I}$ the set of such subsets I. Each I is associated with gamma function factors $\Gamma(w\cdot\sigma + m_a)$, $(a,w) \in I$. We denote by P_I the set of the points p with $\sigma(p) \in \mathfrak{t}_{\mathbb{C}}$ satisfying

$$(w\cdot\sigma(p) + m_a)_{(a,w)\in I} \in \mathbb{Z}_{\leq 0}^{\mathrm{rk}(G)}\,. \tag{6.17}$$

Following [42], define

$$C(I) := \left\{ \vec{r} = \sum_{(a,w)\in I} r_{aw}\vec{w} \,\middle|\, r_{aw} > 0 \text{ for all } (a,w) \in I \right\}. \tag{6.18}$$

The hemisphere partition function (6.14) is then given as

$$Z_{\mathrm{hem}}(\mathcal{B}) = \frac{1}{|W(G)|} \sum_{\substack{I\in\mathfrak{I}:\\ \vec{r}\in C(I)}} \sum_{p\in P_I} \operatorname*{Res}_{\sigma=\sigma(p)} \Big(\mathrm{Str}_{\mathcal{V}}[e^{-2\pi i(\sigma+m)}] e^{t_{\mathrm{ren}}\cdot\sigma} Z_{\text{1-loop}}(\mathcal{B};\sigma;m) \Big)\,. \tag{6.19}$$

The definition of Res, the multi-dimensional residue [90], will be apparent from the next paragraph.

An elementary way to understand the formula (6.19) goes as follows. For given FI parameters $\vec{r}$, (6.13) can be evaluated in principle by successive integrations over σ^1, σ^2, etc. There are many gamma function factors of which we pick poles, and the combinatorics in such a calculation becomes quite complicated. The combinatorics for the total contribution from the set of factors specified by I, however, is not affected by the presence of other factors, and is in fact captured by a simple change of integration variables. Namely we take $\{w\cdot\sigma + m_a|(a,w) \in I\}$ as new variables to be integrated over along the imaginary axis and compute the residues of the chosen factors. Unless $r_{aw} > 0$ for all $(a,w) \in I$, the contribution vanishes.

Although we do not do this explicitly, it should be possible to obtain the infinite sum expression (6.19) by localization with a different Q-exact action [34,35]. In such a computation, the saddle point configurations correspond to the discrete Higgs vacua, namely the solutions to the D-term and F-term equations satisfying $(w\cdot\sigma + m_a)\phi_a = 0$ for all a. The label I specifies the chiral fields that take non-zero vevs. Indeed, the decomposition $\vec{r} = \sum_{(a,w)\in I} r_{aw}\vec{w}$ implies that the D-term equations[22] can be solved by

[22] The D-term equations read $D^I \propto \mu^I = 0$, where μ^I are given in (4.2).

setting $\phi_a^w = (r_{aw}/2\pi)^{1/2}$ for $(a,w) \in I$ with other $\phi_a^w = 0$. The value of σ is fixed by the condition $w \cdot \sigma + m_a = 0$ for $(a,w) \in I$, corresponding to the tip of the cone determined by (6.17). Each infinite sum specified by I is a power series in the exponentiated FI-parameters, and defines an analog of the 3d holomorphic block [97].

The results above were obtained by explicit localization calculations on a hemisphere with the round metric (5.1). We now argue that they should also be valid for the deformed metric (5.4) by interpreting the one-loop determinants (6.8) and (6.12) using the equivariant index theorem as in [32,64,34]. With an appropriate choice of localization action $S_{\text{loc}} = \delta_Q \mathbb{V}$, the one-loop determinant should be given from the equivariant index by converting a sum into a product according to

$$\operatorname{ind} \mathbb{D} = \sum_j c_j e^{\lambda_j} \to Z_{\text{1-loop}} = \prod_j \lambda_j^{-c_j/2} ,$$

where $\mathbb{D}$ is a differential operator in $\mathbb{V}$, j parametrize the eigenmodes of the bosonic symmetry generator δ_Q^2, $c_j = \pm 1$, and λ_j are the eigenvalues of δ_Q^2. When the geometry has no boundary, the index $\operatorname{ind} \mathbb{D}$ is given as a sum of contributions from the fixed points of δ_Q^2. In the presence of boundary, at least with suitable boundary conditions such as those in [98], the equivariant index is a sum of fixed point contributions and the boundary contributions. Thus, the one-loop determinant $Z_{\text{1-loop}}$ should also factorize into such local contributions.

For a chiral multiplet, it was shown in [34] that the combined contribution from the north and the south poles ($\vartheta = 0$ and π respectively) of the round two-sphere is

$$\prod_w \frac{\Gamma(w \cdot \sigma + m)}{\Gamma(1 - w \cdot \sigma - m)} \sim Z_{\text{1-loop}}^{\text{chi},\mathbb{S}^2} \sim Z_{\text{1-loop}}^{\text{chi, Neu}} Z_{\text{1-loop}}^{\text{chi, Dir}} ,$$

where by $\sim$ we mean the match of zeros and poles. It was also shown in [40] that the full sphere one-loop determinant is independent of the metric deformation (5.4). As in the four-dimensional case [32,64], we interpret the square-root $(Z_{\text{1-loop}}^{\text{chi},\mathbb{S}^2})^{1/2} \sim (Z_{\text{1-loop}}^{\text{chi, Neu}} Z_{\text{1-loop}}^{\text{chi, Dir}})^{1/2}$ as the local contribution from each of the north and the south poles.[23] Then, (6.8) implies, in the case of the round sphere, that the single-boundary contribution to the one-loop determinant is

$$(\sin[\pi(w \cdot \sigma + m)])^{-1/2} \tag{6.20}$$

[23] In [34], $Z_{\text{1-loop}}^{\text{chi, Neu}}$ and $Z_{\text{1-loop}}^{\text{chi, Dir}}$ were assigned to distinct poles.

for the Neumann boundary condition, and

$$(\sin[\pi(w\cdot\sigma+m)])^{1/2} \tag{6.21}$$

for the Dirichlet boundary condition (up to ambiguities in the overall factors). On the other hand, the local approximate form of $\mathbb{D}$ and the action of δ_Q^2 near the boundary is essentially independent of deformation. Thus, we expect that the single-boundary contribution to the one-loop determinant is given by the same formulas (6.20) and (6.21), even after deformation.[24] Then, the formula (6.8) for the one-loop determinant on a hemisphere should also be valid for the deformed metric (5.4). We can apply the same logic to the vector multiplet, recalling that the full sphere one-loop determinant is $\prod_{\alpha>0}(\alpha\cdot\sigma)^2$ [34,35]. It follows that the single-boundary contribution to one-loop determinant is

$$\prod_{\alpha>0}\sin(\pi\alpha\cdot\sigma)\,.$$

The local contributions to the one-loop determinant from the poles and the boundary are determined by δ_Q^2, and cannot be affected by the deformation parameter $\tilde{\ell}$. The classical contributions computed in 6.1 are also independent of $\tilde{\ell}$. These arguments suggest that the expression of the hemisphere partition function (6.13) should also be valid for the deformed metric (5.4).

6.4. Hilbert space interpretation

We argued above that the partition function on the deformed sphere is independent of the parameter $\tilde{\ell}$. In the limit that $\tilde{\ell}\to\infty$, the geometry near the boundary $\vartheta=\pi/2$ becomes flat, and the non-dynamical gauge field $V^{\mathrm{R,new}}$ in (5.27) for $U(1)_{\mathrm{R}}$ vanishes in the frame where all the fields are periodic.

The boundary condition $\mathcal{B}$ on a hemisphere $0\le\vartheta\le\pi/2$ defines the boundary state $\langle\mathcal{B}|$ in the Hilbert space of the theory on a spatial circle. Since all the fields are periodic in the frame with $V^{\mathrm{R,new}}(\tilde{\ell}\to\infty)=0$, $\langle\mathcal{B}|$ is in the *Ramond-Ramond sector.* The hemisphere partition function (6.13) is the overlap $\langle\mathcal{B}|\mathtt{1}\rangle$ between $\langle\mathcal{B}|$ and a state $|\mathtt{1}\rangle$ created by the path integral on the hemisphere with no operator insertion. Let $\mathtt{f}(\sigma)$ be a gauge invariant

[24] As a check, one can compute the one-loop determinant on $\mathbb{S}^1\times$ (interval) by mode expansion and confirm that it is the product of two boundary contributions, for any pair of boundary conditions on the two boundaries.

polynomial of σ. The result (6.13) can be generalized to include a twisted chiral operator $\mathtt{f}(\sigma_1 - i\sigma_2)$:

$$\begin{aligned}\langle\mathcal{B}|\mathtt{f}\rangle &= \int_{\mathcal{B}} \mathcal{D}A\ldots e^{-S_{\rm phys}} \mathrm{Str}_{\mathcal{V}}\Big[P\exp\Big(i\oint d\varphi \mathcal{A}_\varphi\Big)\Big]\,\mathtt{f}(\sigma_1 - i\sigma_2)\\ &= \frac{1}{|W(G)|}\int_{\sigma\in i\mathfrak{t}} \frac{d^{\mathrm{rk}\,G}\sigma}{(2\pi i)^{\mathrm{rk}\,G}} \mathrm{Str}_{\mathcal{V}}[e^{-2\pi i(\sigma+m)}]e^{t_{\rm ren}\cdot\sigma} Z_{\text{1-loop}}(\mathcal{B};\sigma;m)\mathtt{f}(\sigma)\,,\end{aligned} \tag{6.22}$$

where $\int_{\mathcal{B}}$ indicates functional integration with the boundary condition $\mathcal{B}$. The Ramond-Ramond state $|\mathtt{f}\rangle$ is created by the path integral, defined using the physical action (5.14), with the insertion of $\mathtt{f}(\sigma_1 - i\sigma_2)$ at $\vartheta = 0$. By an argument in [40], it should be identified with the state defined by the path integral of the A-twisted theory [74].[25] We will identify the boundary state $\langle\mathcal{B}|$ with its projection to the BPS subspace. The overlap $\langle\mathcal{B}|\mathtt{f}\rangle$ is nothing but the brane amplitude discussed in subsection 3.6.

The partition function on the full sphere $0 \le \vartheta \le \pi$, as computed in [34,35], is the overlap $Z_{\mathbb{S}^2} = \langle\mathtt{1}|\mathtt{1}\rangle$. By generalizing to include $\mathcal{O}_1 \equiv \mathtt{f}(\sigma_1 - i\sigma_2)$ at $\vartheta = 0$, and $\overline{\mathcal{O}}_2 \equiv \mathtt{g}(-\sigma_1 - i\sigma_2)$ at $\vartheta = \pi$, we obtain

$$\begin{aligned}\langle\mathtt{g}|\mathtt{f}\rangle &= \langle\overline{\mathcal{O}}_2(\vartheta=\pi)\mathcal{O}_1(\vartheta=0)\rangle = \int \mathcal{D}A\ldots e^{-S_{\rm phys}}\mathtt{g}(-\sigma_1 - i\sigma_2)\mathtt{f}(\sigma_1 - i\sigma_2)\\ &= \frac{c}{|W(G)|}\sum_{B\in\Lambda_{\rm cochar}}\int_{\sigma\in i\mathfrak{t}}\frac{d^{\mathrm{rk}(G)}\sigma}{(2\pi i)^{\mathrm{rk}(G)}} e^{t_{\rm ren}\cdot(\sigma - B/2)}e^{\bar{t}_{\rm ren}\cdot(\sigma+B/2)}(-1)^{w_0\cdot B}\mathtt{g}\Big(\sigma + \frac{B}{2}\Big)\\ &\quad\times \mathtt{f}\Big(\sigma - \frac{B}{2}\Big)\prod_{\alpha>0}\Big[\frac{(\alpha\cdot B)^2}{4} - (\alpha\cdot\sigma)^2\Big]\prod_a\prod_{w\in R}\frac{\Gamma(w\cdot(\sigma - B/2) + m_a)}{\Gamma(1 - w\cdot(\sigma + B/2) - m_a)}\,.\end{aligned} \tag{6.23}$$

We have included a normalization constant c and used a weight w_0 to parametrize the ambiguity in the normalization of the flux sectors labeled by GNO charges [99] $B \in \Lambda_{\rm cochar}(G)$.[26]

The path integral on the other half of the sphere ($\pi/2 \le \vartheta \le \pi$) gives

$$\begin{aligned}\langle\mathtt{g}|\mathcal{B}\rangle &= \int_{\mathcal{B}} \mathcal{D}A\ldots e^{-S_{\rm phys}} \mathrm{Str}_{\mathcal{V}}\Big[P\exp\Big(i\oint d\varphi \tilde{\mathcal{A}}_\varphi\Big)\Big]\,\mathtt{g}(-\sigma_1 - i\sigma_2)\\ &= \frac{1}{|W(G)|}\int_{\sigma\in i\mathfrak{t}} \frac{d^{\mathrm{rk}\,G}\sigma}{(2\pi i)^{\mathrm{rk}\,G}} \mathrm{Str}_{\mathcal{V}}[e^{2\pi i(\sigma+m)}]e^{\bar{t}_{\rm ren}\cdot\sigma} Z_{\text{1-loop}}(\mathcal{B};\sigma;m)\mathtt{g}(\sigma)\,,\end{aligned} \tag{6.24}$$

[25] The argument was used to justify the proposal that the $\mathbb{S}^2$ partition function is related to the Kähler potential on the Kähler moduli space [36].

[26] The lattice $\Lambda_{\rm cochar}(G)$ consists of the elements of the Cartan subalgebra which have integer pairings with the weights that appear in all the representations of the group G (rather than $\mathfrak{g}$).

where

$$\tilde{\mathcal{A}}_{\hat{\varphi}} = \rho_*(A_{\hat{\varphi}} + i\sigma_2) + \frac{\rho_*(R)}{2\ell} + i\rho_*(\mathrm{m}) - \frac{i}{2}\{\mathcal{Q}, \bar{\mathcal{Q}}\} + \frac{i}{2}\left((\psi_1 - \psi_2)^i \partial_i \mathcal{Q} + (\bar{\psi}_1 - \bar{\psi}_2)_i \partial^i \bar{\mathcal{Q}}\right) .$$

It is also natural to consider the partition function on a cylinder with boundary conditions $\mathcal{B}_{1,2}$ along the two boundaries

$$\langle \mathcal{B}_1 | \mathcal{B}_2 \rangle = \int_{\mathcal{B}_1, \mathcal{B}_2} \mathcal{D}A \ldots e^{-S_{\text{phys}}} \mathrm{Str}_{\mathcal{V}_1}\Big[P \exp\Big(i \oint d\varphi \mathcal{A}_\varphi^+\Big)\Big] \mathrm{Str}_{\mathcal{V}_2}\Big[P \exp\Big(i \oint d\varphi \mathcal{A}_\varphi^-\Big)\Big] , \quad (6.25)$$

with

$$\mathcal{A}_\varphi^\pm = \rho_*(A_{\hat{\varphi}} + i\sigma_2) + \rho_*(V_{\hat{\varphi}}^{\mathrm{F}} + i\mathrm{m}) \pm \frac{i}{2}\{\mathcal{Q}, \bar{\mathcal{Q}}\} + \frac{1}{2} e^{\frac{\pi i}{4}(1 \mp 1)} \left((\psi_1 - \psi_2)^i \partial_i \mathcal{Q} + (\bar{\psi}_1 - \bar{\psi}_2)_i \partial^i \bar{\mathcal{Q}}\right) .$$

This is a supersymmetric index of the theory on a spatial interval. Since it is independent of the width, this quantity can be computed by a supersymmetric quantum mechanics or classical formulas involving characteristic classes, as we will see in section 7.2. In particular there is no ambiguity in this quantity.

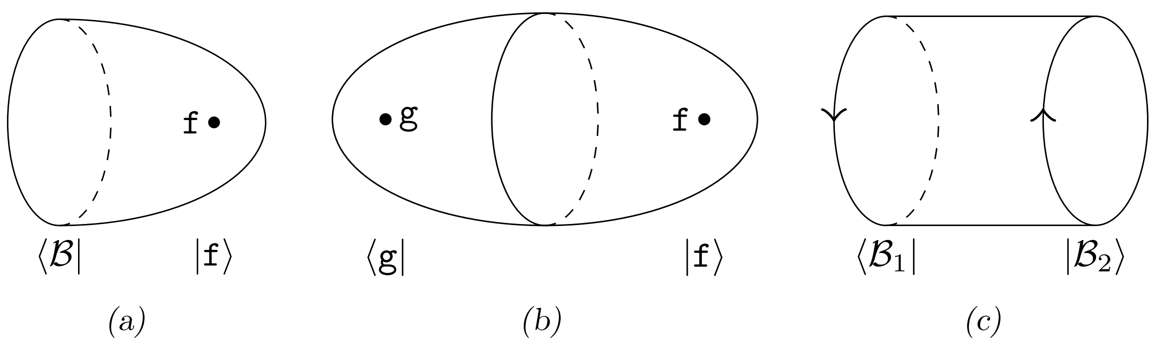

Figure 3 *(a) Hemisphere with an operator insertion. (b) Twisted chiral/twisted anti-chiral 2-point function. (c) Cylinder partition function.*

The Hilbert space interpretation implies that the $\mathbb{S}^2$ partition function (or its generalization (6.23)) is determined by the hemisphere partition functions (or their generalizations) and the cylinder partition function (6.25). Namely, by choosing boundary states $|\mathcal{B}_a\rangle$ that form a basis of the BPS Hilbert space, we set

$$\chi_{ab} = \langle \mathcal{B}_a | \mathcal{B}_b \rangle$$

and denote the inverse matrix by χ^{ab}. Then,

$$\langle \mathtt{g} | \mathtt{f} \rangle = \langle \mathtt{g} | \mathcal{B}_a \rangle \chi^{ab} \langle \mathcal{B}_b | \mathtt{f} \rangle .$$

In some examples with twisted masses, we will introduce another basis $\{|\mathbf{v}\rangle\}$ that is orthonormal. In that case we can write $\langle \mathtt{g}|\mathtt{f}\rangle = \sum_{\mathbf{v}} \langle \mathtt{g}|\mathbf{v}\rangle\langle \mathbf{v}|\mathtt{f}\rangle$. In section 8.4 we will demonstrate such factorizations, and see how they allow us to fix the parameters c and w_0 that parametrize the ambiguities in the $\mathbb{S}^2$ partition function of the $T^*\mathrm{Gr}(N, N_\mathrm{F})$ model studied there.

Let us now show that the twisted chiral operators $\mathtt{f}(\sigma_1 - i\sigma_2)$ inserted at $\vartheta = 0$ satisfy certain relations. Recall that in (6.16), we introduced fictitious FI parameters t_j, $j = 1, \ldots, \mathrm{rk}(G)$. It is useful to consider an auxiliary theory $\hat{\mathcal{T}}$ obtained from the original theory $\mathcal{T}$ as follows.[27] Theory $\hat{\mathcal{T}}$ has gauge group $U(1)^{\mathrm{rk}(G)}$, complexified FI parameters $\hat{t}_j$, and several chiral multiplets. Some of them are ϕ_a^w with gauge charges w_j, with the same twisted masses as in $\mathcal{T}$. The other chiral multiplets, Φ_α, are massless and labeled by all roots α. They have gauge charges $\alpha_j = \alpha(H_j)$, and R-charge 0 for $\alpha > 0$ and -2 for $\alpha < 0$. corresponding complexified FI parameters $\hat{t}_j$. Let us consider the boundary condition $\hat{\mathcal{B}}$ for $\hat{\mathcal{T}}$, consisting of the boundary conditions on ϕ_{aw} determined by $\mathcal{B}$, the Dirichlet boundary condition on Φ_α, as well as the boundary interactions from $\mathcal{B}$. Then we have the relation

$$Z_{\mathrm{hem}}(\mathcal{T}; \mathcal{B}; t; m) \propto \Big(\prod_{\alpha>0} \alpha_j \frac{\partial}{\partial \hat{t}_j}\Big) Z_{\mathrm{hem}}(\hat{\mathcal{T}}; \hat{\mathcal{B}}; \hat{t}; m)\Big|_{\hat{t}=t}.$$

Let $\rho = (1/2)\sum_{\alpha>0} \alpha$ be the Weyl vector. We can derive the differential equations

$$\begin{aligned}\Big[&\prod_a \prod_{\substack{w\in R_a\\ w_j>0}} \prod_{n=0}^{w_j-1} \Big(w_k \frac{\partial}{\partial \hat{t}_k} + m_a + n\Big) \\ &- (-1)^{2\rho_j} e^{-\hat{t}_j} \prod_a \prod_{\substack{w\in R_a\\ w_j<0}} \prod_{n=0}^{|w_j|-1} \Big(w_k \frac{\partial}{\partial \hat{t}_k} + m_a + n\Big)\Big] Z_{\mathrm{hem}}(\hat{\mathcal{T}}; \hat{\mathcal{B}}; \hat{t}; m) = 0\end{aligned} \tag{6.26}$$

for $j = 1, \ldots, \mathrm{rk}(G)$, either by contour deformation or by using the power series representation (6.19). Since each derivative $\partial/\partial\hat{t}_k$ brings down σ^k, the differential equations imply certain relations that hold when inserted in the integral (6.13). By specializing to $\hat{t} = t$, we find that the expressions

$$\mathtt{F}_j(\sigma; m) \equiv \prod_a \prod_{\substack{w\in R_a\\ w_j>0}} \prod_{n=0}^{w_j-1} (w\cdot\sigma + m_a + n) - (-1)^{2\rho_j} e^{-t_j} \prod_a \prod_{\substack{w\in R_a\\ w_j<0}} \prod_{n=0}^{|w_j|-1} (w\cdot\sigma + m_a + n)\,, \tag{6.27}$$

[27] This is the *associated Cartan theory* in [42] with a slightly different R-charge assignment.

or more precisely their Weyl-invariant combinations, vanish when inserted into the hemisphere partition function, $\langle \mathcal{B} | \mathtt{F}_j \rangle = 0$, for any $\mathcal{B}$. Thus we have twisted chiral ring relations (not necessarily fundamental, see subsection 8.2)

$$\mathtt{F}_j(\sigma; m) = 0\,. \tag{6.28}$$

In section 8, we will see some examples of the twisted chiral ring relations. However, the hemisphere partition function does not seem to preserve the ring structure. For example, we can easily see that the hemisphere partition function with the insertion of the operator $\sigma \mathtt{F}_j(\sigma; m)$ does not vanish in general. This is the problem to be understood precisely.

7. Hemisphere partition functions and geometry

In this section, we see the relation between the hemisphere partition functions and the low energy geometry. As we have seen in section 5, the boundary of $N=(2,2)$ gauge theories preserves the B-type supersymmetry. At low energy, the boundary conditions of $N=(2,2)$ gauge theories become the B-branes of the non-linear sigma models. A set of the boundary data determines which B-brane is realized at low energy. Conversely, we can construct the boundary data corresponding to the B-brane at low energy. The hemisphere partition function gives a map from the derived categories of the coherent sheaves which describe B-branes to the functions of the parameters in $N=(2,2)$ gauge theories. More precisely we find that the hemisphere partition function depends only on the K-theory class.

7.1. Target space interpretation of the gauge theory

In this paper we are concerned with the geometric phases in which the theory reduces to a non-linear sigma model with a smooth target space. We consider two cases.

Case 1: $W=0$, target space X

This is the setup where the gauge theory has no superpotential, and flows in the IR to a non-linear sigma model with target space X, which takes the form of a Kähler quotient

$$X=\mu^{-1}(0)/G\,.$$

The moment map $\mu=(\mu^I)_{I=1}^{\dim G}:V_{\mathrm{mat}}\to\mathfrak{g}^*$ is given by

$$\mu^I\equiv\begin{cases}\bar{\phi}T^I\phi & \text{for } I \text{ non-abelian}\,,\\ \bar{\phi}T^I\phi-\dfrac{r_I}{2\pi} & \text{for } I \text{ abelian}\,,\end{cases} \tag{7.1}$$

where T^I are the generators of G which we split into abelian and non-abelian simple factors. The complex structure of X can also be specified by viewing it as a holomorphic quotient:

$$X=(V_{\mathrm{mat}}\backslash\text{deleted set})/G_{\mathbb{C}}\,. \tag{7.2}$$

Here $G_{\mathbb{C}}$ is the complexification of G, and the deleted set consists of those points whose $G_{\mathbb{C}}$-orbits do not intersect with $\mu^{-1}(0)$. If the gauge group G is abelian, X is a toric variety.

Case 2: $W = P \cdot \mathrm{G}(x)$**, target space** M

In the second situation we consider, the theory has a superpotential of the form

$$W = P \cdot \mathrm{G}(x) = P_\alpha \mathrm{G}^\alpha(x) \,,$$

where we split the chiral fields ϕ into two groups as $\phi = (x, P_\alpha)$. Assuming that the space

$$M = \mu^{-1}(0) \cap \mathrm{G}^{-1}(0)/G$$

is smooth, the F-term equations $\frac{\partial}{\partial \phi^i} W(\phi) = 0$ reduce to

$$P_\alpha = 0 \,, \qquad \mathrm{G}^\alpha(x) = 0 \,.$$

Thus, M is the target space of the low-energy theory, and is a submanifold of $X = \mu^{-1}(0)|_{P=0}/G$. If we focus on the complex structure, M is given as

$$M = (V_{\mathrm{mat}} \backslash \text{deleted set}) \cap \mathrm{G}^{-1}(0) \cap \{P_\alpha = 0\}/G_{\mathbb{C}} \,. \tag{7.3}$$

Let us now consider the target space interpretation of the boundary interaction $\mathcal{A}$. For simplicity we turn off the twisted masses, work in the flat limit ($\tilde{\ell} \to \infty$ with finite $x = -\tilde{\ell} \cos \vartheta$), and assume that the gauge group is $G = U(N)$, for which the D-term equations take the form

$$\bar{\phi} T^I \phi - \frac{r}{2\pi} \delta^{I0} = 0 \tag{7.4}$$

with $T^{I=0} = (1/N)\mathbf{1}$ corresponding to the abelian part. We take the FI parameter to be large and positive $r \gg 0$. In the IR limit $g^2 \to \infty$, the gauge theory flows to the non-linear sigma model with the target space X in Case 1 and M in Case 2. We assume that the target space is smooth. The equations of motion that follow from (5.9) imply that in the present limit [73],

$$A_\mu = M^{-1}_{IJ} \left(i \bar{\phi} T^I (\overleftarrow{\partial} - \overrightarrow{\partial})_\mu \phi + \bar{\psi} T^I \gamma_\mu \psi \right) T^J \,,$$

$$\sigma_1 = -i M^{-1}_{IJ} (\bar{\psi} T^I \psi) T^J \,, \qquad \sigma_2 = M^{-1}_{IJ} \left(i \frac{1+q}{f} \bar{\phi} T^I \phi + \bar{\psi} \gamma_3 T^I \psi \right) T^J \,,$$

where the derivatives $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ act on $\bar{\phi}$ and ϕ respectively, and M^{-1}_{IJ} is the inverse of the matrix $M^{IJ} = \bar{\phi}\{T^I, T^J\}\phi$. Under $\phi(x) \to g(x)\phi(x)$, we get the correct transformation $d - iA \to g(x)(d - iA)g^{-1}(x)$, etc. Let R be a representation of G. As noted in the context of an abelian gauge theory in [73], the expression $M^{-1}_{IJ}\left(i\bar{\phi}T^{I}(\overleftarrow{\partial}-\overrightarrow{\partial})_{\mu}\phi\right)$, contracted with the generators T^{J} acting on a vector space V, is the pull-back of a connection on the natural holomorphic vector bundle constructed from V. This bundle is defined as

$$((\text{solutions of the D-term and F-term equations}) \times V)/G\,. \tag{7.5}$$

Thus, the Chan-Paton space $\mathcal{V}$ descends to a collection of holomorphic vector bundles.

We can also see that how the theta angle θ and the FI-parameter r are related to the B-field and the Kähler form of the target space, respectively. Since the theta term involves only the abelian part $I=0$, the discussion is essentially the same as in the abelian case. (See for example [26].) First, note that the matrix M^{IJ} is block-diagonal; the entries with $(I=0, J\neq 0)$ or $(I\neq 0, J=0)$ vanish because of the D-term equations (7.4). Thus, the $U(1)$ part of the gauge field is given, in the current approximation, by

$$\operatorname{Tr} A = \frac{2\pi i}{r}(d\bar{\phi}\cdot\phi - \bar{\phi}\cdot d\phi)\,.$$

The θ-term (5.10) gives a factor $\exp(-\frac{2\theta}{r}\int d\phi\wedge d\bar{\phi})$ in the path integral. This should be identified with the B-field coupling $\exp(2\pi i\int B)$. Thus,

$$B = \frac{i\theta}{\pi r}d\phi\wedge d\bar{\phi}\,,$$

where ϕ and $\bar{\phi}$ are constrained by the D-term equations (7.4). On the other hand the Kähler form of the target space is given, in the large volume limit, by

$$\omega = \frac{i}{2\pi}d\phi\wedge d\bar{\phi}\,.$$

In order to understand the natural combinations of parameters, let us temporarily consider the A-model where ϕ is holomorphic on the world-sheet and the kinetic term in (5.9) gives a factor $\exp(-2\pi\int\omega)$ for a world-sheet instanton. By combining it with the B-field and the boundary interaction for bundle, we get

$$\operatorname{Tr} P\exp\left(i\oint_{\partial\Sigma}\iota^{*}A_{\text{target}}\right)\exp\left(2\pi i\int_{\Sigma}\iota^{*}(B+i\omega)\right) \tag{7.6}$$

where A_{target} is a connection on the bundle and ι^{*} is the pullback by the embedding $\iota:\Sigma\hookrightarrow X$ or M.

7.2. Hemisphere partition function, derived category of coherent sheaves, and K theory

In (6.13) we derived an expression of the hemisphere partition function for arbitrary boundary data $\mathcal{B} = (\mathbf{Neu}, \mathbf{Dir}, \mathcal{V}, \mathcal{Q})$. We assumed that the whole gauge multiplet satisfies the symmetry preserving boundary condition (5.15). The collections of chiral multiplets satisfying the Neumann condition (5.16) and the Dirichlet boundary condition (5.17) are denoted by **Neu** and **Dir**, respectively. The Chan-Paton vector space $\mathcal{V}$ is a representation of $G \times G_{\mathrm{F}} \times U(1)_{\mathrm{R}}$, and its $\mathbb{Z}_2$-grading is given by the $U(1)_{\mathrm{R}}$ charge (weight) modulo 2. The tachyon profile $\mathcal{Q}$ is an odd linear transformation on $\mathcal{V}$.

Suppose that an $\mathcal{N} = (2,2)$ non-linear sigma model has as target space a non-singular algebraic variety. In this paper we are interested in an $\mathcal{N} = (2,2)$ gauge theory that flows at low energy to such a non-linear sigma model. As in section 7.1, we denote the target space as X if it is the quotient of a linear space minus a deleted set, and as M if it is the zero-locus of some section on such X. Two high-energy boundary conditions that give rise to the same boundary condition (D-brane) at low energy should be considered as the same. It is believed that the low-energy branes that preserve B-type supersymmetry form a category equivalent to what is known as the (bounded) derived category of coherent sheaves, which we denote by $D(X)$ or $D(M)$. We argue that the hemisphere partition function gives a well-defined map

$$Z_{\mathrm{hem}} : D(X \text{ or } M) \to \{\text{functions of } (t, m)\} . \tag{7.7}$$

Let us discuss what this means and how to show it.

Physically, a coherent sheaf is a D-brane whose world-volume does not necessarily wrap the whole target space. An object of the derived category is a complex of coherent sheaves, up to an equivalence relation called quasi-isomorphism. An important point is that *any object in the derived category of (non-equivariant) coherent sheaves on a reasonable space X or M is quasi-isomorphic to a complex of holomorphic vector bundles.*[28] Thus, an arbitrary D-brane, even one with lower dimensions, can be represented as a bound state of space-filling branes.

[28] Any equivariant coherent sheaf has a *locally free resolution*, *i.e.*, a representative of the quasi-isomorphism class by a complex of equivariant holomorphic vector bundles. (Proposition 5.1.28 of [76]). Though we personally do not know that every object in the derived category has the property, this seems likely and will be assumed.

Indeed, there is an operation to bind D-branes. Given two complexes $\mathcal{E}, \mathcal{F}$ defined respectively as

$$\ldots \xrightarrow{d_{\mathcal{E}}^{i-1}} \mathcal{E}^i \xrightarrow{d_{\mathcal{E}}^{i}} \mathcal{E}^{i+1} \xrightarrow{d_{\mathcal{E}}^{i+1}} \ldots\,, \qquad d_{\mathcal{E}}^{i+1} d_{\mathcal{E}}^{i} = 0\,,$$

$$\ldots \xrightarrow{d_{\mathcal{F}}^{i-1}} \mathcal{F}^i \xrightarrow{d_{\mathcal{F}}^{i}} \mathcal{F}^{i+1} \xrightarrow{d_{\mathcal{F}}^{i+1}} \ldots\,, \qquad d_{\mathcal{F}}^{i+1} d_{\mathcal{F}}^{i} = 0\,,$$

and a collection f of homomorphisms $f^i : \mathcal{E}^i \to \mathcal{F}^i$ such that $f^{i+1} \cdot d_{\mathcal{E}}^i = d_{\mathcal{F}}^i \cdot f^i$,[29] the mapping cone of f, denoted as $C(f)$, is the complex whose i-th term is $C(f)^i = \mathcal{E}^{i+1} \oplus \mathcal{F}^i$ with differential $d_{C(f)}^i(x,y) = (-d_{\mathcal{E}}^{i+1}(x), f^{i+1}(x) + d_{\mathcal{F}}^i(y))$. The brane $C(f)$ is the bound state of $\mathcal{E}$ and the anti-brane of $\mathcal{F}$. It is known that $f : \mathcal{E} \to \mathcal{F}$ is a quasi-isomorphism if and only if $C(f)$ is exact.

Thus, in order to show that (7.7) is well-defined, we need to i) define a map[30]

$$\text{complex of holomorphic vector bundles} \longmapsto \text{boundary condition } \mathcal{B} \tag{7.8}$$

and then ii) show that an exact complex of vector bundles has a vanishing hemisphere partition function. Part i) will be done in section 7.3. Part ii) will be discussed in section 7.3 and Appendix D. Since vector bundles are carried by space-filling branes, we can assume that all chiral multiplets obey the Neumann boundary condition in (6.13).

The Grothendieck group of the derived category, which is isomorphic to the K theory of the target space, is an additive group generated by $[\mathcal{E}]$ for any complex $\mathcal{E}$ of holomorphic vector bundles, with the relation

$$[C(f)] = [\mathcal{E}] - [\mathcal{F}]$$

for any $f : \mathcal{E} \to \mathcal{F}$. The relation is clearly respected by Z_{hem}. Thus, Z_{hem} depends only on the K theory class.

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[29] Such a collection of homomorphisms is called a cochain map.

[30] In Case 2, *i.e.*, for target space $M \subset X$, our construction, given in section 7.3, of Z_{hem} for an object of $D(M)$ involves resolving the pushforward of the object to X by a complex of vector bundles. Thus the relevant bundles in (7.8) are those on X, not M.

7.3. From complexes of vector bundles to boundary conditions

The aim here is to define the map (7.8) that yields a boundary condition for a given complex of holomorphic vector bundles. We will treat separately Cases 1 and 2.

Case 1

When the target space is a quotient space X of the form (7.2), we have a natural G_{F}-equivariant holomorphic vector bundle for each representation of $(G \times G_{\mathrm{F}})_{\mathbb{C}}$ as in (7.5); if V is the representation space, focusing on the holomorphic structure, the bundle is given as[31]

$$((V_{\mathrm{mat}}\backslash \text{deleted set}) \times \mathcal{V})\,/G_{\mathbb{C}}\,. \tag{7.9}$$

We will assume that any object in $D(X)$ can be represented as a complex of holomorphic vector bundles constructed in this way.

Given a complex $\mathcal{E}$ of vector bundles of the form (7.9), one can construct the corresponding boundary condition $\mathcal{B}$ using a straightforward generalization of a procedure in [73]. Suppose that the i-th term $\mathcal{E}^i$ in the complex arises from the representation V^i of $(G \times G_{\mathrm{F}})_{\mathbb{C}}$. Then, we simply take as the Chan-Paton space $\mathcal{V} = \mathcal{V}^{\mathrm{e}} \oplus \mathcal{V}^{\mathrm{o}}$ with $\mathcal{V}^{\mathrm{e}} = \oplus_{i:\mathrm{even}} V^i$, $\mathcal{V}^{\mathrm{o}} = \oplus_{i:\mathrm{odd}} V^i$. Since the chiral fields serve as target space coordinates, it is natural to choose an R-symmetry R_{deg}, introduced in section 5.5, so that $R_{\mathrm{deg}} \cdot \phi^a = 0$. We let R_{deg} have eigenvalue $i \in \mathbb{Z}$ on V^i. The differential[32] $d_{\mathcal{E}} = (d^i_{\mathcal{E}})$ naturally pulls back to the tachyon profile $\mathcal{Q}$ that squares to zero. Thus, we obtain the map

$$\mathcal{E} \longmapsto \mathcal{B} = (\mathcal{V}, \mathcal{Q})\,. \tag{7.10}$$

In the case that G is abelian and G_{F} is trivial, many examples of this construction were studied in [73]. Non-abelian and equivariant examples will be given in section 8.

In order to show that the map (7.7) is well-defined, we need to show that the hemisphere partition function for an exact complex vanishes. The proof that (7.7) is well-defined amounts to showing that the supertrace in the integrand cancels all the poles that could potentially contribute in (6.19). This is explained in Appendix D, by using the resolved conifold as an example.

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[31] If $G = U(N)$, $V_{\mathrm{mat}} = \{(Q^i{}_f)\} = \mathbf{N}^{\oplus N_{\mathrm{F}}}$, deleted set $= \{Q : \mathrm{rk}(Q) < N\}$, the anti-fundamental representation $\bar{\mathbf{N}}$ gives the tautological bundle over the Grassmannian $\mathrm{Gr}(N, N_{\mathrm{F}})$.

[32] It is a differential in the sense of homological algebra, and is an algebraic operation.

Case 2

The construction of the map (7.8) for target space M in (7.3) is also a generalization of the procedure in the abelian, non-equivariant setting introduced in [73].[33] This is a little more involved than in Case 1.

Recall that the chiral fields x parametrize the ambient space X. The superpotential is

$$W = P \cdot \mathrm{G} = P_\alpha \mathrm{G}^\alpha(x)\,,$$

where $\mathrm{G} = (\mathrm{G}^\alpha)$ represents a section s of a vector bundle E and the field P takes values in the dual E^* by the construction in (7.9). Given an object $\mathcal{E}$ of the derived category $D(M)$, we first push it forward by the inclusion $i : M \to X$. The resulting object of $D(X)$ is quasi-isomorphic to a complex $\hat{\mathcal{E}}$ of vector bundles over X

$$\ldots \xrightarrow{d} \hat{\mathcal{E}}^j \xrightarrow{d} \hat{\mathcal{E}}^{j+1} \xrightarrow{d} \ldots\,. \tag{7.11}$$

In the present case, we define the new R-symmetry R_{deg} in section 5.5 so that

$$R_{\mathrm{deg}} \cdot x = 0\,, \qquad R_{\mathrm{deg}} \cdot P_\alpha = -2P_\alpha\,.$$

As in Case 1, $\hat{\mathcal{E}}$ and d naturally lifts to a Chan-Paton space $\mathcal{V}$ and an odd operator $\mathcal{Q}_{(0)}$ on $\mathcal{V}$, which squares to zero: $\mathcal{Q}_{(0)}^2 = 0$. Since we have a superpotential $\mathcal{W}$, we need a matrix factorization as the boundary interaction in order to cancel the Warner term (5.13) and preserve supersymmetry. This can be constructed by the ansatz

$$\mathcal{Q} = \mathcal{Q}_{(0)} + \sum_\alpha P_\alpha \mathcal{Q}_{(1)}^\alpha + \frac{1}{2!}\sum_{\alpha,\beta} P_\alpha P_\beta \mathcal{Q}_{(1)}^{\alpha\beta} + \ldots \tag{7.12}$$

The equation $\mathcal{Q}^2 = W \cdot \mathbf{1}$ can be used recursively to find $\mathcal{Q}_{(k)}^{\alpha_1 \ldots \alpha_k}$. The existence of a solution to the equation was shown in [73]. Thus, the boundary interaction is purely determined by the geometric consideration, except a subtlety that we now discuss.

In Case 2 we need to shift the assignment, to $\mathcal{V}$, of overall charges for the abelian part of $G \times G_{\mathrm{F}}$. The shift is from the charges specified by the representations V^i. We now argue for the necessity of the shift by generalizing an argument in [73]. First, note that if we know the overall charge assignment for one D-brane on M, then the relative

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[33] Though this construction was referred to as the "compact" case in [73], we adapt it to any manifold M, such as $T^*\mathrm{Gr}(N, N_{\mathrm{F}})$, obtained as the zero-locus $s^{-1}(0)$ of a section s.

charge assignment for other D-branes is automatically determined. Thus, we focus on the simplest D-brane, the space-filling brane carrying no gauge flux. This corresponds to the trivial line bundle over M, or in other words to the structure sheaf $\mathcal{O}_M$. Its pushforward $i_*\mathcal{O}_M$ to the ambient space X is known to be quasi-isomorphic to the so-called Koszul complex

$$\wedge^{\mathrm{r}} E^* \longrightarrow \ldots \longrightarrow \wedge^2 E^* \longrightarrow E^* \longrightarrow \mathcal{O}_X \,,$$

where $\mathrm{r} = \mathrm{rk}\, E$ and the last term has degree zero. The differential is the contraction by the section s that defines M. The natural way to implement the Koszul complex in the gauge theory is to quantize free fermions living along the boundary [100,101]. After quantization we obtain fermionic oscillators η_α, $\bar{\eta}^\alpha$ satisfying the anti-commutation relations $\{\eta_\alpha, \bar{\eta}^\beta\} = \delta_\alpha^\beta$. Let $|0\rangle$ be the Clifford vacuum: $\eta_\alpha|0\rangle = 0$. Then, the Koszul complex is realized by

$$\mathbb{C}\bar{\eta}^1 \ldots \bar{\eta}^{\mathrm{r}}|0\rangle \longrightarrow \ldots \longrightarrow \bigoplus_\alpha \mathbb{C}\bar{\eta}^\alpha|0\rangle \longrightarrow \mathbb{C}|0\rangle$$

with the differentials given by $\mathcal{Q}_{(0)} = \eta_\alpha \mathrm{G}^\alpha(x)$. The recursive procedure above terminates in one step, and simply gives

$$\mathcal{Q} = \eta_\alpha \mathrm{G}^\alpha(x) + \bar{\eta}^\alpha P_\alpha \,. \tag{7.13}$$

This is manifestly a matrix factorization: $\mathcal{Q}^2 = W \cdot \mathbf{1}$.

The question is which amount of abelian charges we should assign to $|0\rangle$. Suppose that the bundle E arises from representation ρ_E of $G \times G_{\mathrm{F}}$. The trivial line bundle $\mathcal{O}_X$, and hence the space $\mathbb{C}|0\rangle$, corresponds to the trivial representation in the construction (7.9). Physically, however, the canonical choice is to assign one-dimensional projective[34] representations to $|0\rangle$ and $\bar{\eta}^1 \ldots \bar{\eta}^{\mathrm{r}}|0\rangle$ symmetrically:

$$\mathbb{C}|0\rangle \leftrightarrow (\det \rho_E)^{1/2} \,, \qquad \mathbb{C}\bar{\eta}^1 \ldots \bar{\eta}^{\mathrm{r}}|0\rangle \leftrightarrow (\det \rho_E)^{-1/2} \,. \tag{7.14}$$

This suggests the map

$$\mathcal{E} \in D(M) \mapsto \mathcal{B} = (\mathcal{V}, \mathcal{Q}) \tag{7.15}$$

defined as follows. For the complex (7.11) quasi-isomorphic to $i_*\mathcal{E}$, suppose that the vector bundle $\hat{\mathcal{E}}^i$ arises via (7.9) from a representation ρ_i of $G \times G_{\mathrm{F}}$. Then, we take

$$\mathcal{V} = \bigoplus_i \mathcal{V}^i \,, \tag{7.16}$$

[34] As in the world-sheet theory of a superstring, these are representations of a covering of $G \times G_{\mathrm{F}}$.

as the Chan-Paton space, where $\mathcal{V}^i$ is the representation space of

$$\rho_i \otimes (\det \rho_E)^{1/2}\,. \tag{7.17}$$

The tachyon profile $\mathcal{Q}$ is determined by the procedure explained around in (7.12).

The validity of (7.15) will be checked by comparing the hemisphere partition function with the large volume formula of the D-brane central charge in section 8.2, as well as by showing that the resulting hemisphere partition functions for the structure sheaf in certain target spaces are invariant under various dualities.

8. Examples

In this section, we see the examples of the hemisphere partition function. We consider the $\mathcal{N} = (2,2)$ theories considered in subsection 4.4. From these examples, we can find many important properties of the hemisphere partition function.

(1) The D0-brane on $\mathbb{C}^n$ is a good example to see that a brane wrapped on a submanifold can be constructed as a bound state of space-filling branes. In terms of the gauge theory, the Dirichlet condition can be realized as the Neumann condition with appropriate boundary interaction.

(2) The hemisphere partition function provides the exact formula of B-brane central charges discussed in subsection 3.6. We compare the hemisphere partition function in the large volume limit with the large volume formula for central charge obtained from the anomaly inflow argument [79]. We check the coincidence of these two objects in the case of D-branes in quintic Calabi-Yau. In the Appendix E, we further check this in the case of complete intersections in products of projective spaces.

(3) We obtain the Higgs branch representation of the hemisphere partition function as discussed for the sphere partition function. We use the examples of the projective spaces and the grassmannians.

(4) By using the Higgs branch representation of the hemisphere partition function, we construct the sphere partition function in the case of the cotangent bundle of Grassmannians $T^*\mathrm{Gr}(N, N_\mathrm{F})$. As we mentioned before, we can fix the ambiguities of the sphere partition function.

8.1. *D0-brane on* $\mathbb{C}^n$

Let us consider the theory of n free chiral multiplets ϕ^i, $i = 1, \ldots, n$, with target space $X = \mathbb{C}^n$. The flavor symmetry $G_\mathrm{F} = U(n)$ allows us to consider equivariant sheaves. In particular, the skyscraper sheaf at the origin, *i.e.*, the D0-brane can be resolved by the Koszul complex

$$\Lambda^{n,0} \longrightarrow \Lambda^{n-1,0} \longrightarrow \ldots \longrightarrow \Lambda^{0,0} = \mathcal{O}\,, \tag{8.1}$$

where $\Lambda^{p,q}$ is the vector bundle of (p,q)-forms, and the differential is the contraction by $\phi^i\partial_i$. The map (7.10) can be described by fermionic oscillators obeying $\{\eta_i, \bar{\eta}^j\} = \delta_i^j$ with $i, j = 1, \ldots, n$, and the Clifford vacuum $|0\rangle$ such that $\eta_i|0\rangle = 0$ for any i. The tachyon profile

$$\mathcal{Q}(\phi) = \phi^i\eta_i\,, \qquad \bar{\mathcal{Q}}(\bar{\phi}) = \bar{\phi}_i\bar{\eta}^i$$

gives a realization of the differential. The boundary contribution (6.4) is $\prod_j (1-e^{2\pi i m_j})$. The one-loop determinant should be computed for the Neumann conditions for all ϕ^i since the D0-brane is constructed as a bound state of space-filling branes. It is simply $\prod_j \Gamma(m_j)$. The hemisphere partition function of the model is therefore

$$Z_{\text{hem}}(\text{D0-brane}) = \prod_j \Gamma(m_j)(1-e^{2\pi i m_j}) = \prod_j \frac{-2\pi i e^{\pi i m_j}}{\Gamma(1-m_j)}\,. \tag{8.2}$$

This reproduces the hemisphere partition function for the full Dirichlet condition.[35]

8.2. Quintic Calabi-Yau

Let us consider a $G = U(1)$ theory with chiral fields $(P, \phi_1, \ldots, \phi_5)$ with charges $(-5, 1, 1, 1, 1, 1)$. We assign R-charges $(q_P, q_1, \ldots, q_5) = (-2, 0, \ldots, 0)$ respectively. If we include the superpotential $W = P\mathrm{G}(\phi)$, where G is a degree-five polynomial, the theory with $r \gg 0$ flows to the non-linear sigma model with target space the quintic M, which is the hypersurface in $\mathbb{P}^4$ given by $\mathrm{G}(\phi) = 0$. Let us consider the line bundle $\mathcal{O}_M(n)$ obtained by pulling $\mathcal{O}_{\mathbb{P}^4}(n)$ back to M. We can apply the map (7.15) to construct the boundary condition $\mathcal{B} = (\mathcal{V}, \mathcal{Q})$. The Chan-Paton space $\mathcal{V}$ is the fermionic Fock space spanned by $|0\rangle$ and $\bar\eta|0\rangle$ with $\{\eta, \bar\eta\} = 1$, and the tachyon profile is given by

$$\mathcal{Q} = \mathrm{G}(\phi)\eta + P\bar\eta\,.$$

Following (7.17) we assign gauge charge $n + 5/2$ to $|0\rangle$. Thus,

$$Z_{\text{hem}}[\mathcal{O}_M(n)] = \int_{i\mathbb{R}} \frac{d\sigma}{2\pi i} e^{-2\pi i n\sigma}(e^{-5\pi i\sigma} - e^{5\pi i\sigma}) e^{t\sigma}\Gamma(\sigma)^5\Gamma(1-5\sigma)\,. \tag{8.3}$$

As mentioned after (6.15), convergence requires a deformation of the contour for large $|\sigma|$. Specifically, we choose the contour to approach straight lines tilted to the left by angle $\delta > 0$ from the imaginary axis, and demand that $r\delta > \theta + 2\pi n$. Deep in the geometric phase where $r \gg 0$, we can choose δ to be small. We also demand that the contour crosses the real axis with positive $\mathrm{Re}\,\sigma$.[36] This integral can be evaluated by the Cauchy theorem, and is expressed as a power series in e^{-t}, together with cubic polynomial terms in t:

$$Z_{\text{hem}}[\mathcal{O}_M(n)] = -\frac{20}{3}\pi^4\Big(\frac{t}{2\pi i} - n\Big)\Big(2\Big(\frac{t}{2\pi i} - n\Big)^2 + 5\Big) - 400\pi i\zeta(3) + \mathcal{O}(e^{-t})\,. \tag{8.4}$$

[35] More precisely, the zeros due to the gamma functions in the denominator of (8.2) coincide with the zeros in (6.7) for the full Dirichlet condition. The relative normalization in (6.8) between the Neumann and the Dirichlet conditions was chosen to agree with (8.2).

[36] One can also realize such a contour as a Lagrangian brane by a boundary condition [57].

We can compare this with the large volume formula for the central charge, which we derived in subsection 3.6, [37]

$$\int_M \mathrm{ch}(\mathcal{O}_M(n))e^{B+i\omega}\sqrt{\hat{A}(TM)}\,. \tag{8.5}$$

Our conventions for B and ω can be found in section 7.1. Let $\mathbf{e}$ be the generator of $H^2(M,\mathbb{Z})$ such that $\int_M \mathbf{e}^3 = 5$. If we make the identification

$$B + i\omega = \frac{it}{2\pi}\mathbf{e} + \mathcal{O}(e^{-t})$$

in the large volume limit $t \to +\infty$, (8.5) becomes

$$\int_M e^{n\mathbf{e}} e^{it\mathbf{e}/2\pi}\Big(1+\frac{5}{6}\mathbf{e}^2\Big)^{1/2} = -\frac{5}{12}\Big(\frac{t}{2\pi i}-n\Big)\Big[2\Big(\frac{t}{2\pi i}-n\Big)^2+5\Big]\,,$$

which agrees with the hemisphere partition function (8.4) up to an overall numerical factor, as well as constant and exponentially suppressed terms. This is the most direct demonstration that our hemisphere partition function computes the central charge of the D-brane, or more precisely the overlap of the D-brane boundary state in the Ramond-Ramond sector and the identity closed string state. We see that the hemisphere partition function also captures the constant term proportional to $\zeta(3)$; it is expected to arise at the four-loop order in the non-linear sigma model [84,17].

In Appendix E, we generalize the results here and exhibit the agreement between the hemisphere partition function and the large volume formula (8.5) for branes in an arbitrary complete intersection Calabi-Yau in a product of projective spaces.

One can also show that Z_{hem} satisfies a differential equation

$$\left(\partial_t^4 - 5^5 e^{-t}\prod_{j=1}^{4}(\partial_t - j/5)\right) Z_{\text{hem}}[\mathcal{O}_M(n)] = 0\,.$$

This is the well-known Picard-Fuchs equation obeyed by the periods of the mirror quintic.

[37] In our convention, $\mathrm{ch}\,E = \mathrm{Tr}\exp(F/2\pi)$, $B+i\omega = -(t/2\pi i)\mathbf{e}$, and $F+2\pi B$ is the gauge invariant combination. See (7.6).

8.3. Projective spaces and Grassmannians

Let us consider the theory with gauge group $G = U(1)$, $N_{\rm F}$ fundamental chiral multiplets Q_f ($f = 1, \ldots, N_{\rm F}$), and without a superpotential. We denote the complexified twisted masses by $-m_f$. For $r \gg 0$ and $m_f = 0$, the classical space of vacua is the complex projective space $X = \mathbb{P}^{N_{\rm F}-1}$. This is the simplest example of Case 1 discussed in section 7.2; the space $V_{\rm mat} = \mathbb{C}^{N_{\rm F}}$ of matter fields carries charge $+1$ under $G = U(1)$ and the anti-fundamental representation $\bar{\mathbf{N}}_{\rm F}$ of the flavor group $G_{\rm F} = U(N_{\rm F})$.

The D-brane carrying n units of the gauge flux is the line bundle $\mathcal{O}(n)$. The derived category of coherent sheaves $D(X)$, as well as the K theory $K(X)$ and their $G_{\rm F}$-equivariant versions, is known to be generated by the Beilinson basis, $\mathcal{O}(n)$ with $0 \le n \le N_{\rm F} - 1$. The hemisphere partition function of $\mathcal{O}(n)$ is given by

$$Z_{\rm hem}(\mathcal{O}(n)) = \int_{-i\infty}^{i\infty} \frac{d\sigma}{2\pi i} e^{-2\pi i n\sigma} e^{t_{\rm ren}\sigma} \prod_{f=1}^{N_{\rm F}} \Gamma(\sigma - m_f)\,. \tag{8.6}$$

If $r \gg 0$, for convergence we tilt the contour in the asymptotic region toward the negative real direction as $\operatorname{Im}\sigma \to \pm\infty$. If $\operatorname{Re} m_f < 0$ we simply close the contour along the imaginary axis to the left and compute the integral by picking up the poles at $\sigma = m_f - k$, $k \in \mathbb{Z}_{\ge 0}$. For other values of m_f we define the integral by analytic continuation, or equivalently by choosing the contour in the intermediate region so that we pick the same poles.

$$Z_{\rm hem}(\mathcal{O}(n)) = \sum_{v=1}^{N_{\rm F}} e^{m_v(t_{\rm ren} - 2\pi i n)} \sum_{k=0}^{\infty} e^{-k t_{\rm ren}} \frac{(-1)^k}{k!} \prod_{f \neq v} \Gamma(m_{vf} - k)\,,$$

where $m_{vf} = m_v - m_f$.

If $\operatorname{Re} m_f < 0$, the hemisphere partition function (8.6) satisfies a differential equation

$$\left(\prod_{f=1}^{N_{\rm F}} \left(\frac{\partial}{\partial t_{\rm ren}} - m_f \right) - e^{-t_{\rm ren}} \right) Z_{\rm hem}(\mathcal{O}(n)) = 0\,.$$

In the $m_f \to 0$ limit, this differential equation implies the relation

$$\sigma^{N_{\rm F}} - e^{-t_{\rm ren}} = 0\,,$$

which is nothing but the twisted chiral ring relation of $\mathbb{P}^{N_{\rm F}-1}$ (2.2).

Next, we consider the theory with gauge group $G = U(N)$, $N_{\rm F}$ fundamental chiral multiplets Q^i_f ($i = 1, \ldots, N$ and $f = 1, \ldots, N_{\rm F}$), and with no superpotential. Again the complexified twisted masses will be denoted by $-m_f$. For $r \gg 0$ and $N \leq N_{\rm F}$ the target space of the low-energy theory is the Grassmannian $X = {\rm Gr}(N, N_{\rm F})$ of N-dimensional subspaces in $\mathbb{C}^{N_{\rm F}}$. The flavor group $G_{\rm F} = U(N_{\rm F})$ acts on X naturally. Let V be a vector space in some representation of $G \times G_{\rm F}$. For the corresponding holomorphic vector bundle E given by (7.9), the hemisphere partition function is given by

$$Z_{\rm hem}(\mathcal{O}(E)) = \frac{1}{N!}\int_{i\mathbb{R}^N} \frac{d^N\sigma}{(2\pi i)^N} {\rm Tr}_V\left[e^{-2\pi i(\sigma+m)}\right] e^{t_{\rm ren}{\rm Tr}\,\sigma} \prod_{i<j} \sigma_{ij} \frac{\sin \pi\sigma_{ji}}{\pi} \prod_{f=1}^{N_{\rm F}} \prod_{j=1}^{N} \Gamma(\sigma_j - m_f)\,.$$

We take the traces by viewing σ as a diagonal matrix, and abbreviate symbols as $\sigma_{ij} = \sigma_i - \sigma_j$, $m_{fg} = m_f - m_g$. Let us assume that $r \gg 0$. The integral can be computed by the residue theorem. We will frequently use the notation

$$\mathbf{v} = \{f_1 < f_2 < \ldots < f_N\} \subseteq \{1, \ldots, N_{\rm F}\} \tag{8.7}$$

to label the sequences of poles. These should correspond to the classical Higgs vacua that are the saddle points in a different localization scheme [34,35]. We also denote the complement sets as

$$\mathbf{v}^\vee = \{1, \ldots, N_{\rm F}\} \backslash \mathbf{v}\,.$$

Let us define $m^{\mathbf{v}} = (m^{\mathbf{v}}_j)$ by

$$m^{\mathbf{v}}_j = m_{f_j}\,. \tag{8.8}$$

Picking up the poles at

$$\sigma_j = m^{\mathbf{v}}_{\vec{k}} \equiv m^{\mathbf{v}}_j - k_j\,, \qquad k_j \in \mathbb{Z}_{\geq 0}\,, \tag{8.9}$$

and using the vortex partition function defined in (F.1), we obtain

$$Z_{\rm hem}(\mathcal{O}(E)) = \sum_{\mathbf{v}} {\rm Tr}_V\left(e^{-2\pi i(m^{\mathbf{v}}+m)}\right) e^{t_{\rm ren}{\rm Tr}\, m^{\mathbf{v}}} \Big(\prod_{f\in\mathbf{v}} \prod_{g\in\mathbf{v}^\vee} \Gamma(m_{fg})\Big) Z^{\mathbf{v}}_{\rm vortex}(t_{\rm ren}; m)\,. \tag{8.10}$$

8.4. Cotangent bundles of Grassmannians $T^*{\rm Gr}(N, N_{\rm F})$

Let us consider the theory with gauge group $G = U(N)$, $N_{\rm F}$ fundamentals ${Q^i}_f$ and anti-fundamentals ${\tilde{Q}^f}_i$ and one adjoint ${\Phi^i}_j$ ($i, j = 1, \ldots, N$ and $f = 1, \ldots, N_{\rm F}$). We include the superpotential

$$W = {\rm Tr}\,\tilde{Q}\Phi Q\,.$$

For $r \gg 0$, the theory flows to the non-linear sigma model with target space the cotangent bundle of the Grassmannian $M = T^*\mathrm{Gr}(N, N_\mathrm{F})$, with Φ playing the role of P in section 7.1. We denote the twisted masses of $(Q_f, \tilde{Q}^f, \Phi)$ by $(-m_f, 1 + m_f - m_\mathrm{ad}, m_\mathrm{ad})$ respectively.

We illustrate the Hilbert space interpretation in section 6.4 using this model. We choose w_0 in the formula (6.23) for the two-point function $\langle \mathtt{g}|\mathtt{f}\rangle$ so that $w_0 \cdot B = (N - 1)\sum B_j$. The integral (6.23) can be evaluated as in [34]. It becomes

$$\begin{aligned}\langle \mathtt{g}|\mathtt{f}\rangle = c\sum_{\mathbf{v}} e^{(t+\bar{t})\operatorname{Tr} m^{\mathbf{v}}} \prod_{f\in\mathbf{v}}\prod_{g\in\mathbf{v}^\vee} \frac{\Gamma(m_{fg})\Gamma(1-m_{fg}-m_\mathrm{ad})}{\Gamma(1-m_{fg})\Gamma(m_{fg}+m_\mathrm{ad})} \\ \times Z^{\mathbf{v}}_\mathrm{vortex}(\bar{t}; m; \mathtt{g}) Z^{\mathbf{v}}_\mathrm{vortex}(t; m; \mathtt{f})\,,\end{aligned} \tag{8.11}$$

where $\mathbf{v}$ and $m^{\mathbf{v}}$ were defined in section 8.3, and $Z^{\mathbf{v}}_\mathrm{vortex}(t; m; \mathtt{f})$ is a generalization of the vortex partition function (F.1)

$$\begin{aligned}&Z^{\mathbf{v}}_\mathrm{vortex}(t; m; \mathtt{f}) \\ &= \sum_{\vec{k}\in\mathbb{Z}^N_{\geq 0}} e^{-|\vec{k}|t}\mathtt{f}(m^{\mathbf{v}}_{\vec{k}}) \prod_i \left(\prod_j \frac{(m_{f_i f_j} + m_\mathrm{ad} - k_i)_{k_j}}{(m_{f_i f_j} - k_i)_{k_j}} \prod_{f\in\mathbf{v}^\vee} \frac{(m_{f_i f} + m_\mathrm{ad} - k_i)_{k_i}}{(m_{f_i f} - k_i)_{k_i}}\right).\end{aligned}$$

By defining

$$\langle \mathbf{v}|\mathtt{f}\rangle = c^{\frac{1}{2}} e^{t\operatorname{Tr} m^{\mathbf{v}}} \left[\prod_{f\in\mathbf{v}}\prod_{g\in\mathbf{v}^\vee} \frac{\Gamma(m_{fg})\Gamma(1-m_{fg}-m_\mathrm{ad})}{\Gamma(m_{fg}+m_\mathrm{ad})\Gamma(1-m_{fg})}\right]^{\frac{1}{2}} Z^{\mathbf{v}}_\mathrm{vortex}(t; m; \mathtt{f}) \tag{8.12}$$

and

$$\langle \mathtt{g}|\mathbf{v}\rangle = c^{\frac{1}{2}} e^{\bar{t}\operatorname{Tr} m^{\mathbf{v}}} \left[\prod_{f\in\mathbf{v}}\prod_{g\in\mathbf{v}^\vee} \frac{\Gamma(m_{fg})\Gamma(1-m_{fg}-m_\mathrm{ad})}{\Gamma(m_{fg}+m_\mathrm{ad})\Gamma(1-m_{fg})}\right]^{\frac{1}{2}} Z^{\mathbf{v}}_\mathrm{vortex}(\bar{t}; m; \mathtt{g})$$

we can write $\langle \mathtt{g}|\mathtt{f}\rangle = \sum_{\mathbf{v}}\langle \mathtt{g}|\mathbf{v}\rangle\langle \mathbf{v}|\mathtt{f}\rangle$.

In order to justify our choice of w_0 and relate c to the normalization of hemisphere partition functions, let us compute the hemisphere partition function $Z_\mathrm{hem}(\mathcal{O}_M) = \langle \mathcal{B}[\mathcal{O}_M]|\mathtt{1}\rangle$ and more generally $\langle \mathcal{B}[\mathcal{O}_M]|\mathtt{f}\rangle$ for the structure sheaf $\mathcal{O}_M$. We can use the matrix factorization (7.13). In the present notation we introduce oscillators $(\eta^i{}_j, \bar{\eta}^i{}_j)$ satisfying $\{\eta^i{}_j, \bar{\eta}^k{}_l\} = \delta^i_l \delta^k_j$, and let $|0\rangle$ be the Clifford vacuum: $\eta^i{}_j|0\rangle = 0$. Then,

$$\mathcal{Q} = Q\tilde{Q}\eta + \Phi\bar{\eta}$$

with the indices contracted. Assigning the abelian charges symmetrically between $|0\rangle$ and $\prod_{i,j}\bar{\eta}^i{}_j|0\rangle$ as in (7.14), we find the contribution $\prod_{i,j=1}^N 2i\sin\pi(\sigma_{ij} + m_\mathrm{ad})$ from the boundary interaction. We will see in section 9.2 that for a geometrically expected duality to hold, we need to multiply the hemisphere partition function (6.13) by an extra N-dependent overall factor, *e.g.*, $(2\pi i)^{-N^2}$. We thus go ahead and include it. Then[38],

$$\begin{aligned} Z_{\text{hem}}(\mathcal{O}_M) = \int_{-i\infty}^{i\infty} \frac{d^N\sigma}{(2\pi i)^N N!} e^{t\operatorname{Tr}\sigma} \prod_{i<j} \sigma_{ij} \frac{\sin\pi\sigma_{ji}}{\pi} \prod_{i,j=1}^{N} \frac{\sin\pi(\sigma_{ij}+m_{\text{ad}})}{\pi} \\ \times \prod_{i,j=1}^{N} \Gamma(\sigma_{ij}+m_{\text{ad}}) \prod_{j=1}^{N}\prod_{f=1}^{N_{\text{F}}} \Gamma(\sigma_j - m_f)\Gamma(1-\sigma_j+m_f-m_{\text{ad}}) \,. \end{aligned} \tag{8.13}$$

By applying (6.19) we find

$$Z_{\text{hem}}(\mathcal{O}_M) = \sum_{\mathbf{v}} e^{t\operatorname{Tr} m^{\mathbf{v}}} \Bigg[\prod_{f\in\mathbf{v}}\prod_{g\in\mathbf{v}^\vee} \Gamma(m_{fg})\Gamma(1-m_{fg}-m_{\text{ad}})\Bigg] Z^{\mathbf{v}}_{\text{vortex}}(t;m) \,. \tag{8.14}$$

Note that the same argument t as in (8.11) appears in the vortex partition function here; this is only possible for our choice of w_0. We can compute $\langle\mathcal{B}[\mathcal{O}_M]|\mathtt{f}\rangle$ similarly. Comparing with (8.12), we find that $\langle\mathcal{B}[\mathcal{O}_M]|\mathtt{f}\rangle = \sum_{\mathbf{v}} \langle\mathcal{B}[\mathcal{O}_M]|\mathbf{v}\rangle\langle\mathbf{v}|\mathtt{f}\rangle$, where

$$\langle\mathcal{B}[\mathcal{O}_M]|\mathbf{v}\rangle = c^{-1/2}\Bigg[\prod_{f\in\mathbf{v}}\prod_{g\in\mathbf{v}^\vee} \frac{\pi}{\sin\pi m_{fg}} \frac{\pi}{\sin\pi(m_{fg}+m_{\text{ad}})}\Bigg]^{1/2} . \tag{8.15}$$

A parallel consideration shows that $\langle\mathcal{B}[\mathcal{O}_M]|\mathbf{v}\rangle = \langle\mathbf{v}|\mathcal{B}[\mathcal{O}_M]\rangle$, giving an expression for the cylinder partition function. It is expected to coincide with the equivariant index of the Dirac operator on M. Indeed, $\langle\mathcal{B}[\mathcal{O}_M]|\mathcal{B}[\mathcal{O}_M]\rangle$ determined by (8.15) agrees with[39]

$$\operatorname{ind}(\not{D}) = \sum_{p\colon\text{fixed points}} \frac{1}{\det_{TM_p}(g^{-1/2}-g^{1/2})} \tag{8.16}$$

if we take $c = (2\pi)^{2N(N_{\text{F}}-N)}$.

It is trivial to generalize these results to a holomorphic vector bundle E, or equivalently the sheaf $\mathcal{O}_M(E)$ of holomorphic sections of E. We assume that E arises via (7.9) from a vector space V carrying a representation of $(G\times G_{\text{F}})_{\mathbb{C}}$. We find

$$\langle\mathcal{B}[\mathcal{O}_M(E)]|\mathbf{v}\rangle = \operatorname{Tr}_V e^{-2\pi i(m^{\mathbf{v}}+m)}\Bigg[\prod_{f\in\mathbf{v}}\prod_{g\in\mathbf{v}^\vee} \frac{1}{2\sin\pi m_{fg}} \frac{1}{2\sin\pi(m_{fg}+m_{\text{ad}})}\Bigg]^{1/2} .$$

[38] Compared with (6.14), we see that the boundary interaction has an effect of changing the boundary condition for Φ from Neumann to Dirichlet.

[39] It is possible to show by localization that the equivariant Dirac index given by (8.16), or more generally by (G.2), is indeed the corresponding partition function on the cylinder.

Another class of natural D-branes are sheaves supported on the zero-section of $T^*\mathrm{Gr}(N, N_\mathrm{F})$. Let us consider a vector bundle over $\mathrm{Gr}(N, N_\mathrm{F})$ and call it E, abusing notation slightly. We assume that E is constructed from a representation V of $(G \times G_\mathrm{F})_\mathbb{C}$. We wish to compute the hemisphere partition function for the sheaf $\iota_* \mathcal{O}_{\mathbf{Gr}}(E)$, where ι is the inclusion. Following the procedure for Case 2 in section 7.3, we further pushforward $\iota_* \mathcal{O}_{\mathbf{Gr}}(E)$ by the inclusion $i : M \to X$, where

$$X = \{(Q, \tilde{Q}) | \mathrm{rk}\, Q = N\}/GL(N)\,. \tag{8.17}$$

Since $\mathbf{Gr}$ is given in X simply by the equations $\tilde{Q}^f = 0$, we have a locally-free resolution of $i_* \iota_* \mathcal{O}_{\mathbf{Gr}}$,

$$\wedge^\mathrm{r} F^* \longrightarrow \ldots \longrightarrow \wedge^2 F^* \longrightarrow F^* \longrightarrow \mathcal{O}_X\,, \tag{8.18}$$

where $\mathrm{r} = N$ is the rank of the equivariant vector bundle F, of which $(\tilde{Q}^f)$ defines a section. A resolution of $i_* \iota_* \mathcal{O}_{\mathbf{Gr}}(E)$ is obtained by tensoring each term in (8.18) with the bundle $\hat{E}$ over X that arises from V via (7.9). The complex (8.18) can be translated into the boundary interaction by introducing oscillators satisfying $\{\eta^i{}_f, \bar{\eta}^g{}_j\} = \delta^i_j \delta^g_f$. The Chan-Paton space $\mathcal{V}$ is obtained by tensoring with V the Fock space built on the vacuum $|0\rangle$ annihilated by $\eta^f{}_j$, and the tachyon profile is given by $\mathcal{Q} = \tilde{Q}^f{}_i \eta^i{}_f + \Phi^i{}_j Q^j{}_f \bar{\eta}^f{}_i$. According to (7.17), we must assign the same abelian charges to $|0\rangle$ as in the $\mathcal{O}_M$ case. Then, $|0\rangle$ contributes the factor $e^{N^2 \pi i m_\mathrm{ad}}$. We find the integral representation

$$\begin{aligned} Z_\mathrm{hem}(\iota_* \mathcal{O}_{\mathbf{Gr}}(E)) = \left[\frac{e^{\pi i m_\mathrm{ad}}}{2\pi i}\right]^{N^2} \int \frac{d^N \sigma}{(2\pi i)^N N!} e^{t \mathrm{Tr}\, \sigma} \prod_{j,f} \left(1 - e^{-2\pi i (\sigma_j - m_f + m_\mathrm{ad})}\right) \\ \times \prod_{i<j} \sigma_{ij} \frac{\sin \pi \sigma_{ji}}{\pi} \mathrm{Tr}_V (e^{-2\pi i (\sigma + m)}) \prod_{i,j} \Gamma(\sigma_{ij} + m_\mathrm{ad}) \\ \times \prod_{j,f} \Gamma(\sigma_j - m_f) \Gamma(1 - \sigma_j + m_f - m_\mathrm{ad})\,. \end{aligned} \tag{8.19}$$

As we see by comparing with (6.14) an effect of the boundary interaction is to modify the boundary condition for $\tilde{Q}^f$ from the Neumann to the Dirichlet condition, as we expect for a brane supported on the zero-section. Only the sequences of poles (8.9) contribute, with other combinations of apparent poles canceled.[40] We then find

$$
\begin{aligned}
Z_{\rm hem}(\iota_* \mathcal{O}_{\mathbf{Gr}}(E)) = e^{N\pi i \sum_f m_f} \sum_{\mathbf{v}} \mathrm{Tr}_V \big(e^{-2\pi i (m^{\mathbf{v}}+m)}\big) e^{(t-N_{\rm F}\pi i)\mathrm{Tr}\, m^{\mathbf{v}}} \\
\times \Big(\prod_{f\in\mathbf{v}} \prod_{g\in\mathbf{v}^\vee} \frac{2\pi i e^{-\pi i m_{\rm ad}} \Gamma(m_{fg})}{\Gamma(m_{fg}+m_{\rm ad})} \Big) Z_{\rm vortex}(t;m)\,.
\end{aligned}
\tag{8.20}
$$

By identifying this with $\sum_{\mathbf{v}} \langle \mathcal{B}[\iota_* \mathcal{O}_{\rm Gr}(E)] | \mathbf{v}\rangle \langle \mathbf{v} | \mathbb{1}\rangle$ and using (8.12), we obtain

$$
\begin{aligned}
\langle \mathcal{B}[\iota_* \mathcal{O}_{\rm Gr}(E)] | \mathbf{v}\rangle = e^{N_{\rm F}\pi i (\sum_f m_f - \mathrm{Tr} m^{\mathbf{v}})} e^{-N(N_{\rm F}-N)\pi i m_{\rm ad}} i^{N(N_{\rm F}-N)} \\
\times \mathrm{Tr}_V \big(e^{-2\pi i (m^{\mathbf{v}}+m)}\big) \prod_{f\in\mathbf{v}} \prod_{g\in\mathbf{v}^\vee} \left[\frac{\sin \pi (m_{fg}+m_{\rm ad})}{\sin \pi m_{fg}} \right]^{1/2} .
\end{aligned}
\tag{8.21}
$$

The matrix element $\langle \mathbf{v} | \mathcal{B}[\iota_* \mathcal{O}_{\rm Gr}(E)]\rangle$ is obtained by replacing i with $-i$ in (8.21).

The $T^*\mathrm{Gr}(N, N_{\rm F})$ models are known to possess integrable structures corresponding to the $SU(2)$ spin 1/2 XXX spin chain. Indeed the quantum cohomology relations (6.28) in this case read

$$
\prod_{f=1}^{N_{\rm F}} \frac{\sigma_i - m_f}{\sigma_i - m_f - 1 + m_{\rm ad}} = (-1)^{N_{\rm F}} e^{-t} \prod_{j\neq i} \frac{\sigma_i - \sigma_j - m_{\rm ad}}{\sigma_i - \sigma_j + m_{\rm ad}}\,.
\tag{8.22}
$$

In the limit $|m_f|, |m_{\rm ad}| \gg 1$,[41] the "1" in a denominator of (8.22) can be neglected. In this limit (8.22) become the Bethe ansatz equations [59,60].

[40] Here $\vec{r}$ in (6.18) is given by $\vec{r} = (r, \ldots, r)$. It is not possible to satisfy the conditions $r_{aw} > 0$ in (6.18) if I involves an anti-fundamental. If I involves the adjoint and fundamentals, the zeros from the product in the first line of (8.19) cancel the poles.

[41] One way to take this limit is to make the 2d curvature small. See (5.25).

9. Seiberg-like dualities

In this section, we discuss the Seiberg-like dualities of the $\mathcal{N} = (2, 2)$ gauge theories. Basically, the hemisphere partition functions for the two theories which are dual of each other coincide. However, we find that the nontrivial duality relation of the hemisphere partition functions for $T^*\mathrm{Gr}(N, N_\mathrm{F})$ model is nontrivial.

9.1. Grassmannian model and the $(N, N_\mathrm{F}) \leftrightarrow (N_\mathrm{F} - N, N_\mathrm{F})$ duality

Recall from section 8.3 that the $U(N)$ theory with $N_\mathrm{F} \geq N$ fundamental chiral multiplets Q_f with $r \gg 0$ is in the geometric phase with target space the Grassmannian $\mathrm{Gr}(N, N_\mathrm{F})$. To simplify equations we can take the flavor symmetry group to be $SU(N_\mathrm{F})$ since the overall $U(1)$ is part of the gauge group. Correspondingly, we require that the twisted masses $-m_f$ of Q_f sum to zero:

$$\sum_{f=1}^{N_\mathrm{F}} m_f = 0\,. \tag{9.1}$$

The hemisphere partition function was computed in (8.10). Let us focus on the structure sheaf $\mathcal{O}$ and consider the map of parameters

$$(N, N_\mathrm{F}, t_\mathrm{ren}, m) \to (N_\mathrm{F} - N, N_\mathrm{F}, t_\mathrm{ren}, -m)\,. \tag{9.2}$$

The exponential factor in (8.10) is invariant because of (9.1). The one-loop determinant is also manifestly invariant under (9.2) and $\mathbf{v} \to \mathbf{v}^\vee$. As shown in [34] the vortex partition function $Z^\mathbf{v}_\mathrm{vortex}$ is also invariant. Thus, we have the equality

$$Z_\mathrm{hem}[\mathrm{Gr}(N, N_\mathrm{F}); \mathcal{O}; t_\mathrm{ren}; m] = Z_\mathrm{hem}[\mathrm{Gr}(N_\mathrm{F} - N, N_\mathrm{F}); \mathcal{O}; t_\mathrm{ren}; -m]$$

for the structure sheaf. This equality extends to D-branes carrying vector bundles

$$Z_\mathrm{hem}[\mathrm{Gr}(N, N_\mathrm{F}); E; t_\mathrm{ren}; m] = Z_\mathrm{hem}[\mathrm{Gr}(N_\mathrm{F} - N, N_\mathrm{F}); E^\vee; t_\mathrm{ren}; -m]$$

if we define the map $E \mapsto E^\vee$, in a way compatible with tensor product, by the assignments

$$\begin{aligned}\text{tautological bundle} &\longmapsto (\mathcal{O}^{N_\mathrm{F}}/\text{tautological bundle})^*\,,\\ \mathcal{O}^{N_\mathrm{F}}/\text{tautological bundle} &\longmapsto (\text{tautological bundle})^*\,.\end{aligned}$$

We denoted by $*$ the dual bundle (in the usual sense), whose fiber is the dual of the fiber for the original bundle. (Somewhat confusingly, the quotient, $\mathcal{O}^{N_\mathrm{F}}$/tautological bundle, is sometimes called the dual tautological bundle.) We also recall that the tautological bundle is constructed from the anti-fundamental representation of $GL(N)$ via (7.9).[42]

[42] The assignment $V \mapsto \mathrm{Tr}_V[\mathrm{diag}(x_1^{-1}, \ldots, x_N^{-1}) \times \mathrm{diag}(x_1^{-1}, \ldots, x_{N_\mathrm{F}}^{-1})]$ defines a map $D(X) \to K^{GL(N_\mathrm{F})}(X) \simeq \mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}; x_{N+1}^{\pm 1}, \ldots, x_{N_\mathrm{F}}^{\pm 1}]^{S_N \times S_{N_\mathrm{F}-N}}$ for $X = \mathrm{Gr}(N, N_\mathrm{F})$ [102].

9.2. $T^*\mathrm{Gr}(N, N_F)$ model

The hemisphere partition function for $\mathcal{O}_{T^*\mathrm{Gr}(N,N_\mathrm{F})}$ was computed in (8.14). We again impose the condition (9.1) on the fundamental masses. Under the map

$$N \to N_F - N\,,\ t \to t\,,\ m_f \to -m_f\,,\ m_{\mathrm{ad}} \to m_{\mathrm{ad}}\,,\ \mathbf{v} \to \mathbf{v}^\vee\,,$$

the exponential factor and the one-loop determinant are invariant. The vortex partition functions $Z^{U(N),\mathbf{v}}_{\mathrm{vortex}}(t; m_f, m_{\mathrm{ad}}) \equiv Z^{\mathbf{v}}_{\mathrm{vortex}}(t; m_f, m_{\mathrm{ad}})$ are not invariant, but we found the relations

$$(1 + (-1)^{N_F} e^{-t})^{(N_F-2N)(m_{\mathrm{ad}}-1)} Z^{U(N),\mathbf{v}}_{\mathrm{vortex}}(t; m_f, m_{\mathrm{ad}}) = Z^{U(N_F-N),\mathbf{v}^\vee}_{\mathrm{vortex}}(t; -m_f, m_{\mathrm{ad}}) \quad (9.3)$$

by comparing the power series expansions in e^{-t}.[43] Since the prefactor on the left hand side is independent of $\mathbf{v}$, we find a similar relation for the hemisphere partition functions.[44] In particular, in the limit $\mathrm{Re}\, t \gg 0$ the hemisphere partition function is invariant. The same relation holds for the hemisphere partition functions of $\iota_* \mathcal{O}_{\mathbf{Gr}}$. It can also be extended to include vector bundles as we did for Grassmannians in section 9.1.

9.3. $U(N)$ gauge group with fundamental and determinant matter fields

Let us consider the Grassmannian model with an extra chiral multiplet in the $(-N_\mathrm{F})$-th power of the determinant representation with twisted mass m_{det}. For simplicity we impose the Dirichlet condition for the determinant matter and the Neumann condition for the fundamentals. Then, the hemisphere partition function is

$$Z_{\mathrm{hem}}(N, N_\mathrm{F}; t; m_f, m_{\mathrm{det}}) = \sum_{\mathbf{v}} e^{t \operatorname{Tr} m^{\mathbf{v}}} Z^{\mathbf{v}}_{\text{1-loop}}(m_f, m_{\mathrm{det}}) Z^{\mathbf{v}}_{\mathrm{vortex}}(t; m_f, m_{\mathrm{det}})$$

with the one-loop determinant given by

$$Z^{\mathbf{v}}_{\text{1-loop}}(m_f, m_{\mathrm{det}}) = \frac{-2\pi i e^{\pi i(-N_\mathrm{F} \operatorname{Tr} m^{\mathbf{v}} + m_{\mathrm{det}})}}{\Gamma\left(1 + N_\mathrm{F} \operatorname{Tr} m^{\mathbf{v}} - m_{\mathrm{det}}\right)} \prod_{f \in \mathbf{v}} \prod_{g \in \mathbf{v}^\vee} \Gamma(m_{fg})$$

and the vortex partition function defined in (F.1). It was found in [48] that the superconformal index of this model is invariant under

$$N \to N_F - N,\ t \to t,\ m_f \to -m_f,\ m_{\mathrm{det}} \to m_{\mathrm{det}},\ \mathbf{v} \to \mathbf{v}^\vee\,.$$

One can show that the vortex partition functions in this case are duality invariant, by noting that they are simply related to those of the Grassmannian model. Thus, the hemisphere partition function is also invariant under the duality map.

[43] Similar relations hold between instanton partition functions computed in different schemes for ALE spaces [103].

[44] A similar relation also holds for the sphere partition functions.

9.4. $SU(N)$ gauge theories

To study Seiberg-like dualities for $SU(N)$ theories, we use a trick introduced in [34]; the hemisphere partition function of the $SU(N)$ gauge theory is related to that of the $U(N)$ gauge theory by

$$Z_{\text{hem}}^{SU(N)}(b) = \int_{-\infty}^{\infty} \frac{dr}{2\pi} e^{-rb} Z_{\text{hem}}^{U(N)}(r, \theta = 0).$$

Then, the duality of the $U(N)$ hemisphere partition function implies a duality of the $SU(N)$ hemisphere partition function.

The $U(1)$ baryonic symmetry is defined by its action on the fundamentals ${Q^i}_f$ ($i = 1, \ldots N$, $f = 1, \ldots, N_{\text{F}}$) and the anti-fundamentals ${\tilde{Q}^{\tilde{f}}}_i$ ($i = 1, \ldots N$, $f = 1, \ldots, N_{\text{A}}$)

$${Q^i}_f \to e^{2\pi i b/N} {Q^i}_f, \quad {\tilde{Q}^{\tilde{f}}}_i \to e^{-2\pi i b/N} {\tilde{Q}^{\tilde{f}}}_i\,.$$

It is the $U(1)$ part of the $U(N)$ gauge group that we ungauge. The baryonic and the anti-baryonic operators

$$B_{f_1,\ldots,f_N} = \varepsilon_{i_1\ldots i_N} Q^{i_1}_{\ f_1} \cdots Q^{i_N}_{\ f_N}, \qquad \tilde{B}^{\tilde{f}_1,\ldots,\tilde{f}_N} = \varepsilon^{i_1\ldots i_N} \tilde{Q}^{\tilde{f}_1}_{\ i_1} \cdots \tilde{Q}^{\tilde{f}_N}_{\ i_N}$$

in the $SU(N)$ theory are charged under this $U(1)$. The pure-imaginary parameter b, which is dual to the FI parameter r, becomes the twisted mass for the baryonic symmetry. Indeed, starting with the Coulomb branch representation (6.13) of $Z_{\text{hem}}^{U(N)}$, the delta function given by the r integral

$$\int_{-\infty}^{\infty} \frac{dr}{2\pi} e^{-rb} e^{r\text{Tr}\sigma} = \delta(ib - i\text{Tr}\sigma)$$

produces the hemisphere partition function for the $SU(N)$ theory.

10. Monodromies and domain walls

In this section, we consider the domain walls in $\mathcal{N} = (2,2)$ gauge theories. The gauge theory on one side of a domain wall is different from the gauge theory on the other side in general. Considering the folding of the full sphere, domain walls can be considered as boundaries in the product of these two theories on the hemisphere. Such domain walls can be regarded as operators which act on the hemisphere partition function. In particular, we consider the monodromy domain walls which are defined by the monodromies with respect to the complexified FI parameter. The hemisphere partition function is in the form of Mellin-Barnes integral, and becomes the generalized hypergeometric function if the gauge group is $U(1)$. It is a well-defined problem to consider the monodromy of the generalized hypergeometric function (see Appendix G). However, to analytically continue the integral form of the hemisphere partition function, we should transform the integrand appropriately. In Appendix H, we discuss that this transformation corresponds to the grade restriction rule, which was proposed in [73].

We consider the two examples of the monodromy domain walls.

The first example is discussed in the context of the AGT relation. We consider certain $\mathcal{N} = (2,2)$ gauge theories which describes the surface operators in the four-dimensional $\mathcal{N} = 2$ gauge theories. In this case, the hemisphere partition function coincide with the Liouville/Toda conformal block with degenerate insertion, *i.e.*, the instanton partition function in the context of the AGT relation. The domain walls define defect operators in Liouville/Toda theory and are considered as the line operators bound to a surface operator [81].

The second example is related the integrability of two-dimensional gauge theories proposed by Nekrasov and Shatashvili [59,60]. The integrability of two-dimensional gauge theories means that the twisted chiral rings coincides with the Bethe ansatz equations of spin chains. This suggests the presence of quantum group symmetry, whose generators are expected to be realized as the domain walls acting on the hemisphere partition function with scalar fields inserted. As a first step, we realized the $sl(2)$ affine Hecke algebra as the domain wall algebra. Such quantum group symmetries are known to be realized geometrically as so-called convolution algebras in equivariant K theories and derived categories. We will describe the domain walls as D-branes in product theories and discuss the relation to the geometric representation of the $sl(2)$ affine Hecke algebra.

In this section we consider supersymmetric localization for theories with domain walls preserving B-type supersymmetries. Let us assume that a domain wall is located along the circle $\vartheta = \pi/2$ of the sphere $\mathbb{S}^2$. The domain wall connects theory $\mathcal{T}_1$ on the first hemisphere $0 \le \vartheta \le \pi/2$ and another theory $\mathcal{T}_2$ on the second hemisphere $\pi/2 \le \vartheta \le \pi$. As we explain below, the theory $\mathcal{T}_2$ can be mapped to another theory $\mathcal{I}[\mathcal{T}_2]$ on the first hemisphere. A domain wall is then defined as a D-brane in the *folded theory* $\mathcal{T}_1 \times \mathcal{I}[\mathcal{T}_2]$ on the first hemisphere $0 \le \vartheta \le \pi/2$. When both $\mathcal{T}_1$ and $\mathcal{T}_2$ are in geometric phases, the BPS domain walls, or line operators, are in a one-to-one correspondence with objects in the derived category of equivariant coherent sheaves in the product of the target spaces.

Let us consider an involution[45] $\mathcal{I}_0$ that acts on a chiral multiplet (ϕ, ψ, F) as

$$\mathcal{I}_0 \cdot \phi(\vartheta,\varphi) = \phi(\pi-\vartheta,\varphi)\,, \quad \mathcal{I}_0 \cdot \psi(\vartheta,\varphi) = -\gamma_{\hat{1}}\psi(\pi-\vartheta,\varphi)\,,$$
$$\mathcal{I}_0 \cdot \bar{\psi}(\vartheta,\varphi) = -\gamma_{\hat{1}}\bar{\psi}(\pi-\vartheta,\varphi)\,, \quad \mathcal{I}_0 \cdot F(\vartheta,\varphi) = -F(\pi-\vartheta,\varphi)\,.$$

On a vector multiplet $(A_\mu, \sigma_{1,2}, \lambda, \mathrm{D})$, we define

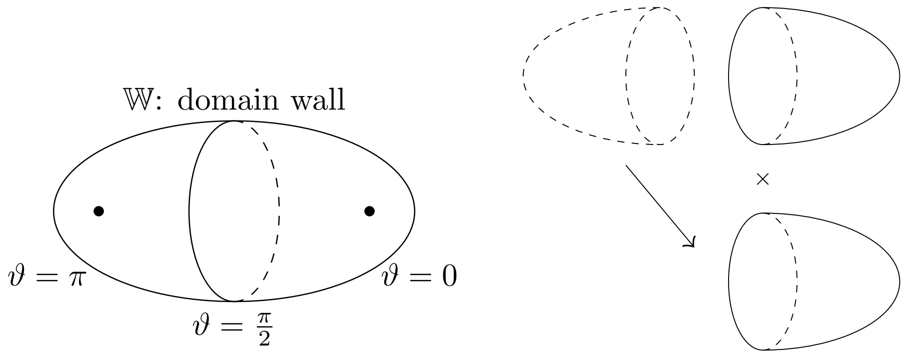
$$\begin{gathered}
\mathcal{I}_0 \cdot A_\vartheta(\vartheta,\varphi) = -A_\vartheta(\pi-\vartheta,\varphi)\,, \quad \mathcal{I}_0 \cdot A_\varphi(\vartheta,\varphi) = A_\varphi(\pi-\vartheta,\varphi)\,,\\
\mathcal{I}_0 \cdot \sigma_1(\vartheta,\varphi) = -\sigma_1(\pi-\vartheta,\varphi)\,, \quad \mathcal{I}_0 \cdot \sigma_2(\vartheta,\varphi) = \sigma_2(\pi-\vartheta,\varphi)\,,\\
\mathcal{I}_0 \cdot \lambda(\vartheta,\varphi) = \gamma_{\hat{1}}\lambda(\pi-\vartheta,\varphi)\,, \quad \mathcal{I}_0 \cdot \bar{\lambda}(\vartheta,\varphi) = \gamma_{\hat{1}}\bar{\lambda}(\pi-\vartheta,\varphi)\,,\\
\mathcal{I}_0 \cdot \mathrm{D}(\vartheta,\varphi) = \mathrm{D}(\pi-\vartheta,\varphi)\,.
\end{gathered}$$

One can define a more general involution $\mathcal{I} \equiv \mathcal{I}_1 \circ \mathcal{I}_0$ by composing $\mathcal{I}_0$ with a discrete flavor symmetry transformation $\mathcal{I}_1$ that acts on each chiral multiplet as multiplication by $+1$ or -1. If the theory has superpotential W, the signs need to be chosen so that $W(\mathcal{I}\cdot\phi) = -W(\phi)$ and $\mathcal{L}_W$ in (5.12) is invariant under $\mathcal{I}$. The theory $\mathcal{I}[\mathcal{T}]$ is obtained from the original theory $\mathcal{T}$ by mapping the fields using $\mathcal{I}$, and by flipping the sign of the theta angle θ.

The trivial domain wall, which we will call the identity domain wall $\mathbb{W}[\mathbf{1}]$, corresponds to a single theory $\mathcal{T}$ with gauge group G on the full sphere $0 \le \vartheta \le \pi$. If we apply $\mathcal{I}$ to

[45] If we regard two-dimensional $\mathcal{N}=(2,2)$ supermultiplets as four-dimensional $\mathcal{N}=1$ multiplets independent of two coordinates (x^3, x^4), the involution $\mathcal{I}_0$ acts as a reflection $(\vartheta,\varphi,x^3,x^4) \mapsto (\pi-\vartheta,\varphi,x^3,-x^4)$ followed by a $U(1)_R$ transformation. The SUSY parameters transform as $\mathcal{I}_0\cdot\epsilon(\vartheta,\varphi) = \gamma_{\hat{1}}\epsilon(\pi-\vartheta,\varphi)$, $\mathcal{I}_0\cdot\bar{\epsilon}(\vartheta,\varphi) = \gamma_{\hat{1}}\bar{\epsilon}(\pi-\vartheta,\varphi)$. Invariant parameters give the supercharges that commute with $\mathcal{I}_0$.

the part of the theory on $\pi/2 \leq \vartheta \leq \pi$, then we get the product theory $\mathcal{T} \times \mathcal{I}[\mathcal{T}]$ with gauge group $G \times G$ on the hemisphere $0 \leq \vartheta \leq \pi/2$. If $\mathcal{T}$ has gauge group G, the product theory has gauge group $G \times G$. Thus, the identity domain wall provides an example of a supersymmetric boundary condition that reduces gauge symmetry; along the boundary the unbroken gauge group is the diagonal subgroup $(G \times G)_{\text{diag}} \simeq G$.



(a) *(b)*

Figure 4 *(a) A sphere with a domain wall. (b) Folding a hemisphere.*

If $\mathcal{T}$ is in a geometric phase with low-energy target space X and if we take $\mathcal{I} = \mathcal{I}_0$, the identity domain wall is realized by the boundary condition corresponding to the diagonal ΔX of $X \times X$:

$$\mathcal{B}[\mathbb{W}(\mathbf{1})] = \mathcal{B}[\mathcal{O}_{\Delta X}]\,.$$

The general pairing (6.23) between the (twisted) chiral and anti-chiral operators can be written as

$$\langle \mathtt{g}|\mathtt{f}\rangle = \langle \mathtt{g}|\mathbb{W}(\mathbf{1})|\mathtt{f}\rangle = \langle \mathcal{B}[\mathcal{O}_{\Delta X}]| \cdot |\mathtt{f}\rangle_1 \otimes |\mathtt{g}\rangle_2\,.$$

In the rest of the section, we will be studying the expectation values of more general domain walls $\mathbb{W}$ on $\mathbb{S}^2$

$$\langle \mathbb{W}\rangle_{\mathbb{S}^2} = \langle \mathtt{1}|\mathbb{W}|\mathtt{1}\rangle = \langle \mathcal{B}[\mathbb{W}]| \cdot |\mathtt{1}\rangle_1 \otimes |\mathtt{1}\rangle_2 \tag{10.1}$$

or more generally the matrix elements (see Figure 4)

$$\langle \mathtt{g}|\mathbb{W}|\mathtt{f}\rangle = \langle \mathcal{B}[\mathbb{W}]| \cdot |\mathtt{f}\rangle_1 \otimes |\mathtt{g}\rangle_2\,.$$

10.2. Monodromy domain walls, four-dimensional line operators, and Toda theories

We now apply the machinery we have developed to find a two-dimensional gauge theory realization of certain four-dimensional line operators bound to a surface operator [69,81,104]. To avoid clutter, details of calculations are relegated to Appendix G.

The relevant four-dimensional theory is the $\mathcal{N} = 2$ theory with gauge group $U(N_\mathrm{F})$ with $2N_\mathrm{F}$ fundamental hypermultiplets. Some of its physical observables are captured by two-dimensional $A_{N_\mathrm{F}-1}$ Toda conformal field theories on a sphere with four punctures of specific types [80,105], via the AGT relation. In particular the basic surface operator of the four-dimensional theory corresponds to a fully degenerate field of the Toda theory [81,106]. It was argued in [81] that four-dimensional line operators bound to a surface operator correspond to monodromies of the conformal blocks, with the insertion point of the degenerate field varied along closed paths. In the limit where the four-dimensional gauge coupling becomes weak, the correlation function of the Toda theory with the degenerate insertion coincides with the $\mathbb{S}^2$ partition function of an $\mathcal{N} = (2, 2)$ gauge theory described below [35]. In this limit, the four-dimensional line operator becomes a two-dimensional line operator, or equivalently a domain wall. Our aim is to find its intrinsic description within the two-dimensional gauge theory.

The the two-dimensional theory in question has gauge group $G = U(1)$, N_F chirals ϕ_f of charge $+1$, and N_F chirals $\tilde{\phi}_f$ of charge -1, with no superpotential. We denote the twisted masses of the chirals by $m = (m_f, \tilde{m}_f)_{f=1}^{N_\mathrm{F}}$. Correspondingly the flavor symmetry group is $G_\mathrm{F} = U(N_\mathrm{F})_1 \times U(N_\mathrm{F})_2$, under which (ϕ_f) and $(\tilde{\phi}_f)$ are in $(\mathbf{N}_\mathrm{F}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{N}_\mathrm{F})$, respectively. For $r \gg 0$, the IR theory has as the target space a toric Calabi-Yau that we denote by X. There are N_F classical vacua $\sigma = -m_v$ labeled by $v = 1, \ldots, N_\mathrm{F}$.

As we show in Appendix G the $\mathbb{S}^2$ partition function takes the form $\langle \mathtt{1}|\mathtt{1}\rangle = \sum_v \langle \mathtt{1}|v\rangle\langle v|\mathtt{1}\rangle$, where

$$\langle v|\mathtt{1}\rangle = (2\pi i)^{N_\mathrm{F}-1/2} e^{-tm_v} \left[\prod_{f\neq v} \frac{\Gamma(m_{fv})}{\Gamma(1-m_{fv})} \prod_f \frac{\Gamma(m_v + \tilde{m}_f)}{\Gamma(1-m_v-\tilde{m}_f)} \right]^{1/2} Z^v_\mathrm{vortex}(t, m)\,,$$

and $\langle \mathtt{1}|v\rangle = \langle v|\mathtt{1}\rangle|_{t\to\bar{t}}$. The vortex partition functions as defined in (F.1) are given in (G.1). Their explicit expressions imply that the matrix elements $\langle v|\mathtt{1}\rangle$ as functions of e^{-t} obey the differential equation

$$\left[e^{-t} \prod_f (\partial_t - \tilde{m}_f) + (-1)^{N_\mathrm{F}-1} \prod_f (\partial_t + m_f) \right] \langle v|\mathtt{1}\rangle = 0\,, \tag{10.2}$$

which has regular singularities at $e^{-t} = 0, (-1)^{N_{\rm F}}, \infty$.[46] The monodromy along a path γ on $\mathcal{M}_K = \mathbb{P}^1 \backslash \{0, (-1)^{N_{\rm F}}, \infty\}$ is given in the form

$$\langle v|\mathtt{1}\rangle \to \sum_{w=1}^{N_{\rm F}} M(\gamma)_{vw} \langle w|\mathtt{1}\rangle \,. \tag{10.3}$$

When z moves along γ and then along γ', the corresponding modnoromy matrix is $M(\gamma')M(\gamma)$.

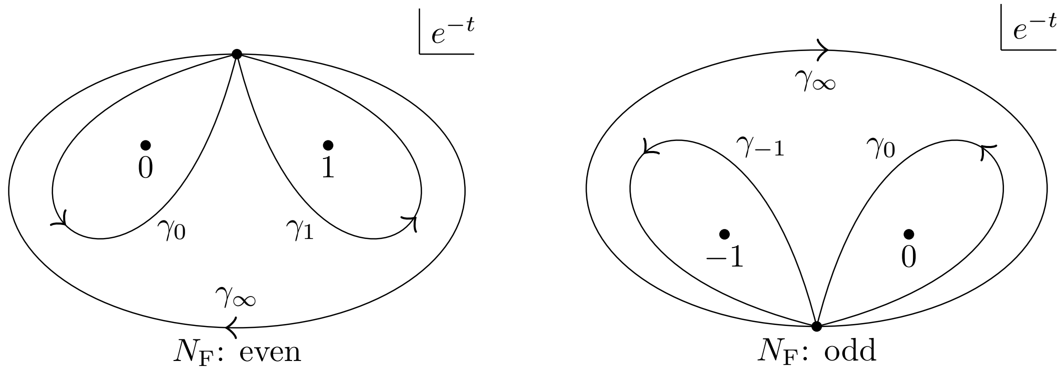


Figure 5 *Paths for monodromies.*

Let us consider the three paths $(\gamma_0, \gamma_{\pm 1}, \gamma_\infty)$ depicted in Figure 5, where we have γ_1 for $N_{\rm F}$ even and γ_{-1} for $N_{\rm F}$ odd. In Appendix G we derive the monodromy matrices

$$\begin{aligned} M(\gamma_0)_{vw} &= \delta_{vw} e^{2\pi i m_v} \,, \\ M(\gamma_{\pm 1})_{vw} &= \delta_{vw} - e^{-\pi i \sum_f (m_f + \tilde{m}_f)} S_{vw} \,, \\ M(\gamma_\infty)_{vw} &= \delta_{vw} e^{-2\pi i m_v} + e^{\pi i \sum_f (m_f + \tilde{m}_f)} e^{-2\pi i m_w} S_{vw} \,, \end{aligned} \tag{10.4}$$

where

$$S_{vw} = \left[\frac{\prod_f 2i \sin \pi (m_v + \tilde{m}_f) 2i \sin \pi (m_w + \tilde{m}_f)}{\prod_{f \neq v} 2i \sin \pi m_{fv} \prod_{f \neq w} 2i \sin \pi m_{fw}} \right]^{1/2} \times \begin{cases} (-1) & \text{for } N_{\rm F} \quad \text{even} \,, \\ e^{\pi i m_{wv}} & \text{for } N_{\rm F} \quad \text{odd} \,. \end{cases}$$

Because of the relation $M(\gamma_0) M(\gamma_{\pm 1}) M(\gamma_\infty) = 1$, only $M(\gamma_0)$ and $M(\gamma_{\pm 1})$ are independent. In view of (10.1) and $\langle \mathsf{g}|v\rangle = \langle v|\mathsf{g}\rangle|_{t \to \bar{t}}$, the monodromy for each path γ should be realized as a domain wall $\mathbb{W}(\gamma)$ such that

$$\langle \mathcal{B}[\mathbb{W}(\gamma)]| \cdot |w\rangle_1 \otimes |v\rangle_2 = \langle v|\mathbb{W}(\gamma)|w\rangle = M(\gamma)_{vw} \,.$$

[46] These are the singularities in the quantum Kähler moduli space $\mathcal{M}_K$ of the non-compact Calabi-Yau X, and the equation (G.5) with $m \to 0$ can be identified with the Picard-Fuchs equation for the periods of the mirror Calabi-Yau manifold, and can be easily obtained from the period integrals of the mirror Langdau-Ginzburg model [25].

It is clear from (10.4) that the domain wall $\mathbb{W}(\gamma_0)$ is simply the gauge Wilson loop with charge $+1$. Geometrically it corresponds to a sheaf supported on the diagonal ΔX.

Denote by L and $\tilde{L}$ the topologically trivial equivariant line bundles constructed from the representations $(\mathbf{det}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{det})$ of $G_{\mathrm{F}} = U(N_{\mathrm{F}})_1 \times U(N_{\mathrm{F}})_2$, respectively. By comparing (10.4) with (G.3) and (G.4), we find for $\gamma_{\pm 1}$[47]

$$
\begin{aligned}
&\langle \mathbb{1}|\mathbb{W}(\gamma_{\pm 1})|\mathbb{1}\rangle = \sum_{v,w} \langle \mathbb{1}|v\rangle M(\gamma_{\pm 1})_{vw} \langle w|\mathbb{1}\rangle \\
&= \langle \mathbb{1}|\mathbb{1}\rangle + (-1)^{N_{\mathrm{F}}-1} \langle \mathcal{B}(L^{-1/2} \otimes \tilde{L}^{1/2} \otimes \mathcal{O}_Y(\lfloor -N_{\mathrm{F}}/2 \rfloor))|\mathbb{1}\rangle \langle \mathcal{B}(\mathcal{O}_Y(-\lfloor N_{\mathrm{F}}/2 \rfloor))|\mathbb{1}\rangle_{t \to \bar{t}}\,,
\end{aligned}
$$

where $\lfloor x \rfloor$ denotes the largest integer not more than x. Thus,

$$
\begin{aligned}
&\big\langle \mathcal{B}[\mathbb{W}(\gamma_{\pm 1})]\big| \\
&= \big\langle \mathcal{B}[\mathcal{O}_\Delta]\big| + (-1)^{N_{\mathrm{F}}-1} \big\langle \mathcal{B}\big[\mathcal{O}_Y\left(\lfloor N_{\mathrm{F}}/2 \rfloor\right) \otimes (L^{-1} \otimes \tilde{L})^{\frac{1}{2}} \boxtimes \mathcal{O}_Y\left(-\lfloor N_{\mathrm{F}}/2 \rfloor\right)\big]\big|\,.
\end{aligned}
\tag{10.5}
$$

Here $\boxtimes$ denotes the external tensor product [76].[48]

We expect that a monodromy in the Kähler moduli space acts on the derived category as a Fourier-Mukai transform. It would be interesting to compare (10.5) with the kernel of the corresponding Fourier-Mukai transform.

We computed the monodromies by first decomposing the hemisphere partition function into the vortex partition functions, and then by computing their monodromies. It is also possible to compute monodromies, or more generally perform analytic continuation from one region to another, using the integral representation (6.13). We given an example of such analytic continuation in Appendix H.

10.3. Monodromy domain walls and the affine Hecke algebra

Next, let us consider the theory realizing $M = T^*\mathbb{P}^1 = T^*\mathrm{Gr}(1,2)$, a special case of the model studied in section 8.4. This is almost identical to the model with $N_{\mathrm{F}} = 2$ considered in section 10.2, but it includes a neutral chiral multiplet Φ with twisted mass m_{ad}, interacting via the superpotential $\mathcal{W} = \tilde{Q}^f \Phi Q_f$. Since the superpotential affects the hemisphere partition function only by constraining the twisted masses, we can recycle the

[47] By the tensor product ($\otimes$) of two sheaves, we mean the tensor product of the complexes corresponding to the sheaves.

[48] If $p_i : X_1 \times X_2 \to X_i$ are the projections and $\mathcal{E}_i$ are complexes of holomorphic vector bundles ($i = 1, 2$), $\mathcal{E}_1 \boxtimes \mathcal{E}_2$ is the complex $p_1^*\mathcal{E}_1 \otimes p_2^*\mathcal{E}_2$ over $X_1 \times X_2$, where p_i^* are the pullbacks by p_i.

computations there. The difference in the conventions in sections 8.4 (and here) and 10.2 (there) requires a replacement $m_f^{\text{there}} = -m_f^{\text{here}}$, $\tilde{m}_f^{\text{there}} = 1 + m_f^{\text{here}} - m_{\text{ad}}^{\text{here}}$. We also demand that $m_1 + m_2 = 0$.

We are interested in the monodromy of the matrix element $\langle v|\mathtt{1}\rangle$ in the $T^*\mathbb{P}^1$ model, computed in (8.12). Thus, the monodromy matrices are identical to (10.4) with the replacement above:

$$\begin{aligned} M(\gamma_0)_{vw} &= \delta_{vw} e^{-2\pi i m_v} \,, \\ M(\gamma_1)_{vw} &= \delta_{vw} - e^{2\pi i m_{\text{ad}}} S_{vw} \,, \\ M(\gamma_\infty)_{vw} &= \delta_{vw} e^{2\pi i m_w} + e^{-2\pi i m_{\text{ad}}} e^{2\pi i m_w} S_{vw} \,, \end{aligned} \tag{10.6}$$

with

$$S_{vw} = -\left[\frac{\prod_f 2i \sin \pi(m_{vf} + m_{\text{ad}}) 2i \sin \pi(m_{wf} + m_{\text{ad}})}{\prod_{f \neq v} 2i \sin \pi m_{vf} \prod_{f \neq w} 2i \sin \pi m_{wf}} \right]^{1/2} . \tag{10.7}$$

Let us set

$$q = e^{2\pi i m_{\text{ad}}} \,, \quad X = M(\gamma_0)^{-1} \,, \quad T = -1 + \frac{q}{1-q} S \,.$$

The relation $M(\gamma_0)M(\gamma_1)M(\gamma_\infty) = 1$ implies that

$$(T+1)(T-q) = 0 \,. \tag{10.8}$$

The explicit expression (10.7) can be used to show another relation

$$T X^{-1} - X T = (1-q) X \,. \tag{10.9}$$

The two relations (10.8) and (10.9) define the so-called $sl(2)$ affine Hecke algebra, and we have followed the notation in [76]. We used the monodromies to motivate and derive the relations, but we can study the domain wall realization of the algebra on its own right. The generator X is simply the gauge charge -1 Wilson loop, and corresponds geometrically to the sheaf $\pi_\Delta^* \mathcal{O}(-1)$, where π_Δ is the projection from the diagonal of $T^*\mathbb{P}^1 \times T^*\mathbb{P}^1$ to the diagonal of the base $\mathbb{P}^1 \times \mathbb{P}^1$:

$$X_{vw} = \langle \mathcal{B}(\pi_\Delta^* \mathcal{O}(-1))| \cdot |w\rangle_1 \otimes |v\rangle_2 \,.$$

For T, or a related operator $c = -T - 1 = -\frac{q}{1-q} S$, we find from (8.21) and (10.7)

$$\begin{aligned} c_{vw} &= -q^{1/2} \langle v|\mathcal{B}(\iota_* \mathcal{O}_{\mathbb{P}^1}(-1)\rangle \langle \mathcal{B}(\iota_* \mathcal{O}_{\mathbb{P}^1}(-1)|w\rangle \\ &= q^{-1/2} \langle \mathcal{B}(\iota_* \mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \iota_* \mathcal{O}_{\mathbb{P}^1}(-1))| \cdot |w\rangle_1 \otimes |v\rangle_2 \,. \end{aligned} \tag{10.10}$$

The $sl(2)$ affine Hecke algebra is a basic example of an algebra that can be constructed geometrically as a convolution algebra [76]. The sheaf we found for X is precisely what appears in the construction. On the other hand, our sheaf for $c = -1 - T$ is slightly different from the one in the convolution algebra, though their supports coincide. It is desirable to understand in more generality the relation between the algebras realized by domain walls and convolution.

11. Conclusion and discussion

In this thesis, we have studied boundaries and domain walls in $\mathcal{N} = (2,2)$ gauge theories using supersymmetric localization. In particular, we have applied supersymmetric localization to $\mathcal{N} = (2,2)$ theories on a hemisphere, with boundary conditions preserving B-type supersymmetries, which become B-branes at low energy, and obtained the hemisphere partition function. We have found various properties of the hemisphere partition function. We also have studied the domain walls in a two-sphere.

In section 5, we have defined the supersymmetric gauge theory on a hemisphere with boundary conditions that preserve B-type supersymmetries. For a chiral multiplet, we have considered two basic sets of boundary conditions, which we have called Neumann and Dirichlet conditions. For a vector multiplet, we have considered the set of boundary which preserves the full gauge symmetry. These elementary boundary conditions are combined with the boundary interactions, which are determined by the $\mathbb{Z}_2$-graded Chan-Paton space $\mathcal{V}$ and the tachyon profile $\mathcal{Q}$, to provide more general boundary conditions.

In section 6, we have performed localization and obtain the hemisphere partition function as an integral over scalar zero-modes. We have provided its alternative expression as a linear combination of certain blocks given as infinite power series. We have argued that the hemisphere partition function is invariant under the deformation of the hemisphere as in the case of the sphere partition function [40]. We have found the Hilbert space interpretation of the hemisphere partition function, *i.e.*, it can be considered as the overlap of the boundary state and the Ramond-Ramond state in the BPS Hilbert space. From this argument, we have derived the sphere partition function without any ambiguity which exists in the localization calculation in [34,35].

In section 7, we have explained the geometric interpretation of the hemisphere partition function. We have related the boundary data of the $\mathcal{N} = (2,2)$ gauge theories and the B-brane, *i.e.*, the objects in the derived category of coherent sheaves. We have explained how to compute the hemisphere partition function for a given object in the derived category. We have found that the hemisphere partition function depends only on the K theory class.

In section 8, we have given the examples of the hemisphere partition functions. From these examples, we have found many important properties of the hemisphere partition function. In particular, the hemisphere partition function provides the exact formula of B-brane central charges. We have matched the hemisphere partition function with the large-volume formula for the central charges in the quintic Calabi-Yau (and for more general complete intersection Calabi-Yau's in Appendix E).

In section 9 we have studied the Seiberg-like dualities. In some example, we have shown that the hemisphere partition functions for the two theories which are dual of each other coincide. However, we have found that the hemisphere partition functions for some duality pairs does not coincide. This is due to the nontrivial duality relation between the vortex partition functions (see also the Appendix F).

In section 10, we have studied domain walls realized as D-branes in a product theory. Such domain walls can be regarded as operators that act on a hemisphere partition function. The action of certain walls, the monodromy domain walls, are identified with monodromies of the partition function with respect to the complexified FI parameteres. We have shown that they realize certain defect operators of Liouville/Toda theories in one case, and the $sl(2)$ affine Hecke algebra in another. The former case has been discussed in the context of the AGT relation. Certain $\mathcal{N}=(2,2)$ gauge theories describe the surface operators in the four-dimensional $\mathcal{N}=2$ gauge theories and their hemisphere partition functions coincide with the Liouville/Toda conformal blocks with degenerate insertion, *i.e.*, the instanton partition function in the context of the AGT relation. In this case the domain walls which define defect operators in Liouville/Toda theory are considered as the line operators bound to a surface operator [81]. In the second example, we have described the domain walls as D-branes in product theories and discussed the relation to the geometric representation of the $sl(2)$ affine Hecke algebra. Such quantum group symmetries are known to be realized geometrically as so-called convolution algebras in equivariant K theories and derived categories. However, our result is slightly different from the one in the convolution algebra.

Now we comment on some future directions of our study. Firstly, one of the most remarkable properties of the hemisphere partition function is that they provide the exact formula of B-brane central charges in Calabi-Yau compactification. Since D-brane central charge is one of the most important objects in Calabi-Yau compactification, mirror symmetry and D-brane stability, hemisphere partition function will play an important role for future study. For this purpose, we would like to derive hemisphere partition function for mirror systems, *i.e.*, $\mathcal{N}=(2,2)$ gauge theories with boundaries which describes A-branes at low energy. The mirror system on a sphere is considered in [41]. Then, we should consider the generalization of this result to the deformed hemisphere geometry. We are also interested in the relation to four-dimensional $\mathcal{N} = 2$ gauge theories obtained by the compactificaition on Calabi-Yau three-folds and their mathematical structure. The D-brane stability is related to the wall-crossing phenomena in four-dimensional $\mathcal{N} = 2$ gauge theories [16] and the mathematical theory of the stability conditions [107]. The study of the equivariant Gromov-Witten invariant is also interesting. While the relation between the sphere partition function and the Givental's formalism [108] in [44,45], we think the hemisphere partition function is more appropriate object to relate with the Givental's formalism. As we have discussed in subsection 6.4, it is also important to understand how the twisted chiral ring relations are realized in the hemisphere partition function more precisely, to consider the various applications of the hemisphere partition function. The discussion in [109] might give a hint to solve this problem. It would be marvelous if the hemisphere partition function could be used for the application of the mirror symmetry such as the knot theory and the geometric Langlands program.

Secondly, we would like to understand the relation between domain walls, the integrability and the geometric representation more precisely. In particular we are interested in how the domain wall algebras act on integrable systems. To clarify the relation to spin chains, we are trying to realize the $sl(n)$ affine Hecke algebra as the domain wall algebra. To understand the relation to the geometric representation, we are also trying to interpret domain walls in terms of the Fourier-Mukai transform. Considering the gauge theoreis which realize Nakajima quiver varieties as target spaces is also interesting in the context of AGT relation [110,111].

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Appendix A. Spinor conventions and supersymmetry transformations

By default we think of a spinor $\psi = (\psi_\alpha)_{\alpha=1,2}$ as a column vector. The indices are raised and lowered by the charge conjugation matrix

$$C = (C^{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad C^{-1} = (C_{\alpha\beta}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

as $\psi^\alpha = C^{\alpha\beta}\psi_\beta$, $\psi_\alpha = C_{\alpha\beta}\psi^\beta$. When the upper index of ψ is contracted with the lower index of λ, we write

$$\psi\lambda = \psi^\alpha\lambda_\alpha = \psi^T C^T \lambda \,,$$

where T indicates the transpose. The gamma matrices γ_m ($m = 1, 2, 3$) have the index structure $\gamma_m = (\gamma_{m\alpha}{}^\beta)$. A spinor bilinear is defined as

$$\psi\gamma_{m_1} \ldots \gamma_{m_n}\lambda = \psi^T C^T \gamma_{m_1} \ldots \gamma_{m_n}\lambda \,.$$

We always take the SUSY parameters ϵ and $\bar\epsilon$ to be bosonic. We assume that they are conformal Killing spinors satisfying (5.6). In this convention fields in a vector multiplet transform under SUSY as

$$\begin{aligned}
&\delta\lambda = (i\mathcal{V}_m\gamma^m - \mathrm{D})\epsilon \,, \qquad \delta\bar\lambda = (i\bar{\mathcal{V}}_m\gamma^m + \mathrm{D})\bar\epsilon, \\
&\delta A_\mu = -\frac{i}{2}\left(\bar\epsilon\gamma_\mu\lambda + \bar\lambda\gamma_\mu\epsilon\right) , \quad \delta\sigma_1 = \frac{1}{2}\left(\bar\epsilon\lambda + \bar\lambda\epsilon\right) , \quad \delta\sigma_2 = -\frac{i}{2}\left(\bar\epsilon\gamma^3\lambda + \bar\lambda\gamma^3\epsilon\right) , \\
&\delta\mathrm{D} = -\frac{i}{2}\bar\epsilon \not{D}\lambda - \frac{i}{2}[\sigma_1, \bar\epsilon\lambda] - \frac{1}{2}[\sigma_2, \bar\epsilon\gamma^3\lambda] + \frac{i}{2}\epsilon\not{D}\bar\lambda + \frac{i}{2}[\sigma_1, \bar\lambda\epsilon] + \frac{1}{2}[\sigma_2, \bar\lambda\gamma^3\epsilon],
\end{aligned} \tag{A.1}$$

where

$$\begin{aligned}
\mathcal{V}_m &= \left(D_1\sigma_1 + \frac{f(\vartheta)}{\ell\sin\vartheta}D_2\sigma_2 \,,\ D_2\sigma_1 - \frac{\ell\sin\vartheta}{f(\vartheta)}D_1\sigma_2 \,,\ F_{\hat1\hat2} + i[\sigma_1, \sigma_2] + \frac{1}{f(\vartheta)}\sigma_1\right) , \\
\bar{\mathcal{V}}_m &= \left(-D_1\sigma_1 + \frac{f(\vartheta)}{\ell\sin\vartheta}D_2\sigma_2 \,,\ -D_2\sigma_1 - \frac{\ell\sin\vartheta}{f(\vartheta)}D_1\sigma_2 \,,\ F_{\hat1\hat2} - i[\sigma_1, \sigma_2] + \frac{1}{f(\vartheta)}\sigma_1\right) .
\end{aligned}$$

For a chiral multiplet of R-charge q, the SUSY transformation laws are given by

$$\begin{aligned}
\delta\phi =& \bar\epsilon\psi \,, \qquad \delta\bar\phi = \epsilon\bar\psi \,, \\
\delta\psi =& + i\gamma^\mu\epsilon D_\mu\phi + i\epsilon\sigma_1\phi + \gamma^3\epsilon\sigma_2\phi - i\frac{q}{2f(\vartheta)}\gamma_3\epsilon\phi + \bar\epsilon\mathrm{F} \\
\delta\bar\psi =& - i\bar\epsilon\gamma^\mu D_\mu\bar\phi + i\bar\epsilon\bar\phi\sigma_1 + \bar\epsilon\gamma^3\bar\phi\sigma_2 - i\frac{q}{2f(\vartheta)}\bar\epsilon\gamma_3\bar\phi + \epsilon\bar{\mathrm{F}} \\
\delta\mathrm{F} =& \epsilon\Big(i\gamma^\mu D_\mu\psi - i\sigma_1\psi + \gamma^3\sigma_2\psi - i\lambda\phi\Big) - i\frac{q}{2}\psi\gamma^\mu D_\mu\epsilon \\
\delta\bar{\mathrm{F}} =& \bar\epsilon\Big(i\gamma^\mu D_\mu\bar\psi - i\bar\psi\sigma_1 - \gamma^3\bar\psi\sigma_2 + i\bar\phi\lambda\Big) - i\frac{q}{2}\bar\psi\gamma^\mu D_\mu\bar\epsilon \,.
\end{aligned} \tag{A.2}$$

The twisted mass m can be introduced by replacing $\sigma_2 \to \sigma_2 + \mathrm{m}$.

Appendix B. Spherical harmonics

We will first review the Jacobi polynomials that appear in the scalar monopole harmonics. Although we only deal with the situations with vanishing fluxes, a special case of monopole harmonics will appear in the construction of spinor spherical harmonics. We will also review the vector spherical harmonics. In this appendix, we take the metric to be that of the round unit sphere

$$ds^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2 \,. \tag{B.1}$$

The symbol $q \in (1/2)\mathbb{Z}$ denotes the monopole charge and should not be confused with the R-charge of a chiral multiplet.

B.1. Jacobi polynomials and scalar monopole harmonics

Jacobi polynomials are defined as [112]

$$P_n^{\alpha\beta}(x) := \frac{(\alpha+1)_n}{n!}\ {}_2F_1\left(-n, 1+\alpha+\beta+n; \alpha+1; \frac{1-x}{2}\right),$$

where ${}_2F_1$ is the hypergeometric function and $(x)_n$ is the Pochhammer symbol

$$(a)_n := a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

The variable x takes values in $[-1,1]$. An alternative definition is known as Rodrigues' formula:

$$P_n^{\alpha\beta}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\},$$

where $n, n+\alpha, n+\beta, n+\alpha+\beta \in \mathbb{Z}_{\geq 0}$. When $n, n+\alpha, n+\beta, n+\alpha+\beta \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}$, we can also write

$$P_n^{\alpha\beta}(x) = \sum_{s=\max\{0,-\beta\}}^{\min\{n,n+\alpha\}} \frac{(n+\alpha)!(n+\beta)!}{s!(n+\alpha-s)!(\beta+s)!(n-s)!}\left(\frac{x-1}{2}\right)^{n-s}\left(\frac{x+1}{2}\right)^{s}.$$

For $\alpha, \beta > -1$, they satisfy the orthogonality relations

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_n^{\alpha\beta}(x)P_m^{\alpha\beta}(x)dx = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}\delta_{nm}\,.$$

The polynomials $\{P_n^{\alpha,\beta}(x)\}_{n=0}^{\infty}$ form a complete orthogonal system in $L^2_{\alpha,\beta}([-1,1])$, *i.e.*, the space of functions which are square integrable with weight $(1-x)^\alpha(1+x)^\beta$.

Let us review the basic properties of the monopole scalar harmonics [113]. When the monopole charge q is non-zero, the scalar harmonics consist of sections of a topologically non-trivial line bundle $\mathcal{O}(2q)$. Since we are most interested in the boundary of a hemisphere, we work in the patch $0 < \vartheta < \pi$.

We define

$$\begin{aligned}
&Y_{qjm}(\vartheta,\varphi) := M_{qjm}(1-x)^{\alpha/2}(1+x)^{\beta/2}P_n^{\alpha\beta}(x)e^{im\varphi}, \\
&M_{qjm} := 2^m\sqrt{\frac{2j+1}{4\pi}\frac{(j-m)!(j+m)!}{(j-q)!(j+q)!}}, \\
&x := \cos\vartheta,\ \alpha := -q-m,\ \beta := q-m,\ n := j+m.
\end{aligned}$$

For $q=0$, $Y_{jm} := Y_{0jm}$ give the usual spherical harmonics. For given $q \in \mathbb{Z}/2$, j and m take values

$$j = |q|, |q|+1, |q|+2, \ldots\,, \qquad m = -j, -j+1, \ldots, j\,.$$

$\{Y_{qjm}\}_{j,m}$ form a complete orthonormal system in the space of square integrable sections of the line bundle $\mathcal{O}(2q)$.

The covariant derivative for the sections of $\mathcal{O}(2q)$ is given by $D_\mu = \partial_\mu - iq\omega_\mu$, where $\omega_\mu = (0, -\cos\vartheta)$ is the spin connection. The monopole scalar harmonics are the eigenfunctions of the Laplacian:

$$\begin{aligned}
-D^\mu D_\mu Y_{qjm} &\equiv \left[-\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta} - \frac{1}{\sin^2\vartheta}\left(\frac{\partial^2}{\partial\varphi^2} + 2iq\cos\vartheta\frac{\partial}{\partial\varphi} - q^2\cos^2\vartheta\right)\right]Y_{qjm} \\
&= [j(j+1)-q^2]Y_{qjm}.
\end{aligned}$$

The monopole harmonics provide an orthonormal basis with respect to the natural inner product:

$$\int_{\mathbb{S}^2} Y_{qjm}(\vartheta,\varphi)^*\, Y_{qj'm'}(\vartheta,\varphi) = \delta_{jj'}\delta_{mm'}\,, \tag{B.2}$$

where the measure is $d\vartheta d\varphi \sin\vartheta$ and the complex conjugate is related to the original harmonics as

$$Y^*_{qjm} = (-1)^{q+m}Y_{-q,j,-m}\,. \tag{B.3}$$

Under $\vartheta \to \pi - \vartheta$, Y_{jm} is even for $j+m$ even, and is odd for $j+m$ odd. In particular

$$\begin{aligned}
\partial_\vartheta Y_{jm}|_{\vartheta=\pi/2} = 0 &\quad \text{if} \quad j+m \text{ is even}\,, \\
Y_{jm}|_{\vartheta=\pi/2} = 0 &\quad \text{if} \quad j+m \text{ is odd}\,.
\end{aligned}$$

The orthogonality relations on the hemisphere can be obtained from (B.2) by doubling the integration region to the full sphere.

B.2. Spinor and vector spherical harmonics

We write $\not{D} \equiv \gamma^\mu D_\mu$. Let us consider the spectral problem with respect to the modified Dirac operator

$$\gamma^3 \not{D} = \begin{pmatrix} & \partial_\vartheta - \dfrac{i}{\sin\vartheta}\partial_\varphi + \dfrac{1}{2}\cot\vartheta \\ -\partial_\vartheta - \dfrac{i}{\sin\vartheta}\partial_\varphi - \dfrac{1}{2}\cot\vartheta & \end{pmatrix} =: \begin{pmatrix} & \underline{D}^\dagger \\ \underline{D} & \end{pmatrix}$$

on $\mathbb{S}^2$. One can check that the eigenspinors are given by

$$\chi^\pm_{jm}(\vartheta,\varphi) := \frac{1}{2}\begin{pmatrix} (1\mp i)Y_{-1/2,jm}(\vartheta,\varphi) \\ (j+1/2)^{-1}(-i\pm 1)\underline{D}Y_{-1/2,jm}(\vartheta,\varphi) \end{pmatrix}, \tag{B.4}$$

which satisfy

$$\gamma^3 \not{D}\chi^\pm_{jm} = \pm(j+1/2)\chi^\pm_{jm} \,.$$

The range of the quantum numbers is given by

$$j = \frac{1}{2}, \frac{3}{2}, \dots, \qquad m = -j, -j+1, \dots, j \,.$$

The eigenspinors form an orthonormal basis on $\mathbb{S}^2$:

$$\int_{\mathbb{S}^2} (\chi^s_{jm})^\dagger \chi^{s'}_{j'm'} = \delta_{ss'}\delta_{jj'}\delta_{mm'} \,.$$

Next, let us review the vector spherical harmonics described *e.g.*, in [114]. We define the one-forms

$$\begin{aligned} (C^1_{jm})_\mu(\vartheta,\varphi) &:= \frac{1}{\sqrt{j(j+1)}}\begin{pmatrix} \partial_\vartheta Y_{jm}(\vartheta,\varphi) \\ imY_{jm}(\vartheta,\varphi) \end{pmatrix}, \\ (C^2_{jm})_\mu(\vartheta,\varphi) &:= \frac{1}{\sqrt{j(j+1)}}\begin{pmatrix} -(im/\sin\vartheta)Y_{jm}(\vartheta,\varphi) \\ \sin\vartheta\,\partial_\vartheta Y_{jm}(\vartheta,\varphi) \end{pmatrix}. \end{aligned} \tag{B.5}$$

With the quantum numbers taking values

$$j = 1, 2, 3, \dots, \qquad m = -j, -j+1, \dots, j \,,$$

the whole sequence $\{C^\lambda_{jm}\}_{\lambda,j,m}$ forms an orthonormal basis of one-forms on $\mathbb{S}^2$. Moreover they are eigenvectors of the vector Laplacian:

$$-D^\mu D_\mu C^{1(2)}_{jm} = [j(j+1)-1]\,C^{1(2)}_{jm} \,.$$

They also have the properties

$$D_\mu (C^1_{jm})^\mu = -\sqrt{j(j+1)} Y_{jm}\,,\ D_\mu (C^2_{jm})^\mu = 0\,,$$
$$\varepsilon^{\mu\nu} D_\mu (C^1_{jm})_\nu = 0\,,\ \varepsilon^{\mu\nu} D_\mu (C^2_{jm})_\nu = -\sqrt{j(j+1)} Y_{jm}\,.$$

Appendix C. Eigenvalue problems on a round hemisphere

In this Appendix we study the eigenvalue problems and their solutions, which we use in section 6.2 to compute the one-loop determinants.

We are interested in the Neumann and the Dirichlet boundary conditions at $\vartheta = \pi/2$:

$$\partial_\vartheta \Phi|_{\vartheta=\pi/2} = 0 \quad \text{(Neumann)} \quad \text{and} \quad \Phi|_{\vartheta=\pi/2} = 0 \quad \text{(Dirichlet)}\,.$$

One can check that the Laplacian $-D^\mu D_\mu$ is self-adjoint on the hemisphere $0 \leq \vartheta \leq \pi/2$ with these boundary conditions. For the harmonics Y_{jm}, the conditions respectively reduce to

$$P^{-m,-m}_{j+m}(0) = 0\,, \quad \text{and} \quad \partial_x P^{-m,-m}_{j+m}(x)|_{x=0} = 0.$$

The property $P^{\alpha,\beta}_n(-x) = (-1)^n P^{\beta,\alpha}_n(x)$ implies that the eigenmodes that survive the boundary conditions are given by

$$Y_{jm},\ j-m = \text{even},\ \text{eigenvalue} = j(j+1) \quad \text{(Neumann)}\,,$$
$$Y_{jm},\ j-m = \text{odd},\ \ \text{eigenvalue} = j(j+1) \quad \text{(Dirichlet)}\,.$$

We have indicated the eigenvalues of the Laplacian $-D^\mu D_\mu$. Since $-D^\mu D_\mu$ is self-adjoint on the hemisphere when either boundary condition is imposed, the surviving modes form an orthogonal system. The precise normalizations can be inferred from the relations among such modes

$$\int_{0\leq\vartheta\leq\pi/2} Y_{jm}(\vartheta,\varphi)^*\, Y_{j'm'}(\vartheta,\varphi) = \frac{1}{2}\delta_{jj'}\delta_{mm'}\,, \tag{C.1}$$

which can be obtained from (B.2) by doubling the integration region to $0 \leq \vartheta \leq \pi$.

Let us consider two types of boundary conditions for a spinor $\psi = (\psi_1, \psi_2)^T$:

$$(\psi_1 + \psi_2)|_{\vartheta=\pi/2} = 0 \quad \text{(A)} \qquad \text{and} \quad (\psi_1 - \psi_2)|_{\vartheta=\pi/2} = 0 \quad \text{(B)}\,.$$

Suppose that another spinor λ obeys the same boundary condition as ψ. Then,

$$\langle \psi, \gamma^3 \not{D} \lambda\rangle \equiv \int_{\vartheta\leq\pi/2} \psi^\dagger \gamma^3 \not{D}\lambda = \langle \gamma^3 \not{D}\psi, \lambda\rangle - \int d\varphi\, \psi^\dagger \gamma^1\gamma^3\lambda|_{\vartheta=\pi/2}\,.$$

For both (A) and (B),

$$\psi^\dagger\gamma^1\gamma^3\lambda|_{\vartheta=\pi/2}\propto\left[(\psi^\dagger)^1\lambda_2-(\psi^\dagger)^2\lambda_1\right]|_{\vartheta=\pi/2}=0.$$

Thus, the Dirac operator $\gamma^3\not{D}$, together with the boundary condition either (A) or (B), is self-adjoint on the hemisphere.

For $\chi^{\pm}_{jm}$ the condition (A) reduces to

$$[(2j+1)\mp(1-2m)]P^{1/2-m,-1/2-m}_{j+m}(0)\pm(j-m+1)P^{3/2-m,1/2-m}_{j+m-1}(0)=0.$$

The modes that survive the condition are

$$\begin{aligned}&\chi^{+}_{jm}\,,\ j-m=\text{odd}\,,\ \text{eigenvalue}=j+1/2\,,\\&\chi^{-}_{jm}\,,\ j-m=\text{even}\,,\ \text{eigenvalue}\ =-(j+1/2)\,.\end{aligned}$$

Similarly, (B) reduces to

$$[(2j+1)\pm(1-2m)]P^{1/2-m,-1/2-m}_{j+m}(0)\mp(j-m+1)P^{3/2-m,1/2-m}_{j+m-1}(0)=0\,,$$

and the surviving modes are

$$\begin{aligned}&\chi^{+}_{jm}\,,\ j-m=\text{even}\,,\ \text{eigenvalue}=j+1/2\,,\\&\chi^{-}_{jm}\,,\ j-m=\text{odd}\,,\ \text{eigenvalue}\ =-(j+1/2)\,.\end{aligned}$$

Among the surviving modes we have

$$\int_{\vartheta\le\pi/2}\chi^{s}_{jm}(\vartheta,\varphi)^\dagger\chi^{s'}_{j'm'}(\vartheta,\varphi)=\frac{1}{2}\delta_{ss'}\delta_{jj'}\delta_{mm'}\,,\tag{C.2}$$

$$\int_{\vartheta\le\pi/2}\chi^{s}_{jm}(\vartheta,\varphi)\gamma_3\chi^{s'}_{j'm'}(\vartheta,\varphi)=\frac{s'(-1)^{m-1/2}}{2}\delta_{s,-s'}\delta_{jj'}\delta_{m,-m'}\,.\tag{C.3}$$

Finally, we consider the boundary condition

$$A_\vartheta|_{\vartheta=\pi/2}=\partial_\vartheta A_\varphi|_{\vartheta=\pi/2}=0\,.$$

for vector harmonics (B.5). The modes that survive are

$$\begin{aligned}&C^1_{jm},\ j-m=\text{even},\ \text{spectrum}\ j(j+1),\text{degeneracy}\ j+1,\\&C^2_{jm},\ j-m=\text{odd},\ \text{spectrum}\ j(j+1),\text{degeneracy}\ j.\end{aligned}$$

Appendix D. Hemisphere partition functions for exact complexes

The aim of this appendix is to argue that the map (7.7) is well-defined. Namely we argue that the hemisphere partition function for each object of the derived category $D(X \text{ or } M)$ does not depend on the choice of a complex of vector bundles used in the construction.

As an example in Case 1, let us consider the resolved conifold. The gauge group is $G = U(1)$, and there are four chiral fields $\phi = (\phi^1, \phi^2, \phi^3, \phi^4)$ with gauge charges $w_a = (+1, +1, -1, -1)$. The flavor group is $G_{\rm F} = U(1)^4 = \prod_{a=1}^{r} U(1)_a$, where ϕ^a has charge $+1$ for $U(1)_a$ and charge zero for $U(1)_{b\neq a}$.

Let $m = (m_a)$ be the complexified twisted masses for ϕ^a. For $r \gg 0$, the model is in the geometric phase and flows to the non-linear sigma model with target space the resolved conifold X. We want to show that for an exact equivariant complex $(\mathcal{E}, d)$ of vector bundles given by

$$0 \longrightarrow \mathcal{E}^1 \longrightarrow \ldots \longrightarrow \mathcal{E}^n \longrightarrow 0\,,$$

the partition function $Z_{\rm hem}(\mathcal{E})$ vanishes. Following the the definition of (7.10), we let V^i be the representation of $G \times G_{\rm F}$ from which the vector bundle $\mathcal{E}^i$ arises via (7.9). We assume that the values of m_a are generic. Under this assumption, the integral

$$Z_{\rm hem}(\mathcal{E}) = \int_{-i\infty}^{i\infty} \frac{d\sigma}{2\pi i} {\rm Str}_V[e^{-2\pi i \rho(\sigma,m)}] e^{t\sigma} \Gamma(\sigma+m_1)\Gamma(\sigma+m_2)\Gamma(-\sigma+m_3)\Gamma(-\sigma+m_4)\,,$$

where we wrote explicitly the representation $\rho_*(\sigma, m)$ of ${\rm Lie}(G \times G_{\rm F})$, is evaluated by residues to give

$$Z_{\rm hem}(\mathcal{E}) = \sum_{v=1}^{2} {\rm Str}_V[e^{-2\pi i \rho_*(-m_v,m)}] e^{-tm_v} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \prod_{a\neq v} \Gamma(w_a(-m_v-k)+m_a)\,.$$

This involves two sequences of poles at $\sigma = -m_v, -m_v - 1, \ldots$ $(v = 1, 2)$. As noted in [34,35], the beginning of each sequence corresponds to a solution of the condition

$$(w_a\sigma + m_a)\phi^a = 0$$

with ϕ^a satisfying the D-term equation

$$\sum_a w_a |\phi^a|^2 = \frac{r}{2\pi}\,.$$

Such values of (σ, ϕ) describe a fixed point in X under the action of the flavor group $G_{\rm F}$.[49] We now recall that the tachyon profile $\mathcal{Q}$ has to satisfy the condition that $\rho(g)\mathcal{Q}(g^{-1}\cdot\phi)\rho(g)^{-1} = \mathcal{Q}(\phi)$ for any $g \in G \times G_{\rm F}$. For $g = (e^{-2\pi i\sigma}, e^{-2\pi i m}) \in G \times G_{\rm F}$ and ϕ under consideration then,

$$\rho(g)\mathcal{Q}(\phi) = \mathcal{Q}(\phi)\rho(g)\,.$$

This relation together with Hodge decomposition shows that there are complete cancellations between $\operatorname{Im} d^i$ and $\operatorname{Ker} d^{i+1}$ so that $\operatorname{Str}_V[e^{-2\pi i\rho_*(\sigma,m)}]$ vanishes at all poles, and hence $Z_{\rm hem} = 0$ for an exact complex $\mathcal{E}$.

For more general X, if a given exact complex can be made equivariant with twisted masses generic enough so that the poles become simple, the same argument can be applied to show that $Z_{\rm hem}$ vanishes.

Next, let us consider the Fermat quintic M as an example of Case 2. The chiral fields are (P, x_a). The fields x^a, $a = 1, \ldots, 5$, parametrize X. The superpotential $W = P(x_1^5 + \ldots + x_5^5)$ does not allow us to introduce real twisted masses. Given an object in $D(M)$, we push it forward to $D(X)$, where $X = \mathbb{P}^4$ and resolve it there.

In order to argue that the map $D(M) \to \mathbb{C}$ is well-defined, suppose that we have two resolutions in X of the same object of $D(M)$. For the resolutions, which are quasi-isomorphic in X, we construct the boundary interactions according to (7.15). The difference of their hemisphere partition functions is clearly the hemisphere partition function of their mapping cone, which is exact. Thus, if $Z_{\rm hem}$ vanishes for any exact complex in X, then the map $Z_{\rm hem} : D(M) \to \mathbb{C}$ is well-defined.

We have not found such a proof yet. As an alternative, we offer an example of exact complex for which $Z_{\rm hem}$ indeed vanishes. Consider the following complex $\mathcal{E}$ of vector bundles over $X = \mathbb{P}^4$:

$$0 \to \mathcal{O}(n) \to \mathcal{O}(n+1)^5 \to \mathcal{O}(n+2)^{10} \to \mathcal{O}(n+3)^{10} \to \mathcal{O}(n+4)^5 \to \mathcal{O}(n+5) \to 0\,.$$

In terms of fermionic oscillators $\{\eta_a, \bar\eta_b\} = \delta_{ab}$, this complex is realized as the Fock space $\mathcal{V}$ built on the vacuum $|0\rangle$ satisfying $\eta_a|0\rangle = 0$. The differential is $\mathcal{Q}_0 = x_a\eta_a$, and the tachyon profile is $\mathcal{Q} = \mathcal{Q}_0 + \sum_a P x_a^4 \bar\eta_a$. This is exact since $\{\mathcal{Q}, \bar{\mathcal{Q}}\}$ is everywhere positive. The boundary interaction $(\mathcal{V}, \mathcal{Q})$ then contributes

$$\operatorname{Str}_{\mathcal{V}}(e^{-2\pi i\sigma}) \propto \sin^5 \pi\sigma\,,$$

[49] For a more general X for which $G_{\rm F}$ is non-abelian, we should consider a fixed point with respect to the maximal torus of $G_{\rm F}$.

which has order 5 zeros at $\sigma \in \mathbb{Z}$. It then follows that the hemisphere partition function vanishes,

$$Z_{\rm hem}(\mathcal{E}) = \int_{-i\infty}^{i\infty} {\rm Str}_{\mathcal{V}}(e^{-2\pi i\sigma}) e^{t\sigma}\Gamma(\sigma)^5\Gamma(1-5\sigma) = 0\,,$$

when the integral is evaluated by closing the contour to the left.

Finally, let us consider another example of Case 2, $M = T^*{\rm Gr}(N, N_{\rm F})$ considered in section 8.4. As in the previous example, we want to show that $Z_{\rm hem}$ vanishes for an exact complex on the ambient space X given as in (8.17). The general result (6.19) with the definition (6.18) of $C(I)$ implies that we need to find decompositions of the vector $\vec{r} = (r, \ldots, r)$ by the weights of fundamental, anti-fundamental, and adjoint representations, with positive coefficients. One can show that anti-fundamental weights can never appear in such decompositions. The poles are associated with fixed points on $T^*{\rm Gr}(N, N_{\rm F})$ with respect to the $U(1)^{N_{\rm F}} (\subset G_{\rm F})$ action. Indeed, the decomposition $\vec{r} = \sum_{(a,w)\in I} r_{aw}\vec{w}$ implies that the D-term equations can be solved by setting $\phi_a^w = (r_{aw}/2\pi)^{1/2}$ for $(a, w) \in I$ (with other $\phi_a^w = 0$), and the poles σ satisfy $e^{-2\pi i(w\cdot\sigma+m_a)} = 1$ for $(a, w) \in I$. Thus, at the poles $\rho(g)$ and $\mathcal{Q}_0(\phi)$ commute with each other, and ${\rm Str}_V[e^{-2\pi i\rho_*(\sigma,m)}]$ vanishes, as in the case of the resolved conifold. Since the poles are simple for generic twisted mass parameters, the hemisphere partition function vanishes.

Appendix E. Complete intersection CYs in a product of projective spaces

In this appendix we generalize the result for the quintic obtained in section 8.2. Let us consider a direct product of projective spaces $X = \prod_{r=1}^m \mathbb{P}^{N_r-1}$. We take sections s_a of the line bundles $\mathcal{O}(l_a^1, \ldots, l_a^m)$ for $a = 1, \ldots, k$ and assume that the intersection M of their zero-loci $s_a^{-1}(0)$ is a smooth manifold. For M to be Calabi-Yau, l_a^r must satisfy

$$\sum_a l_a^r = N_r\,.$$

This geometry is realized by a gauge theory with gauge group $G = U(1)^m = \prod_{r=1}^m U(1)_r$ and the following matter content: the chiral multiplet fields

$$\phi_{r,1}, \ldots, \phi_{r,N_m}$$

charged only under $U(1)_r$ with charge 1, and

$$P_a,\ a = 1, \ldots, k$$

that have $U(1)^m$ charges $(-l_a^1, \ldots, -l_a^m)$ and R-charge -2. We also include a superpotential $W = \sum_{a=1}^k P_a \mathrm{G}_a(\phi)$, where $\mathrm{G}_a(\phi)$ are the polynomials that define the sections s_a. For $r \gg 0$ the gauge theory flows to the nonlinear sigma model whose target space M.

Let us take as the Chan-Paton space $\mathcal{V}$ the fermionic Fock space generated by the Clifford algebra $\{\eta_a, \bar{\eta}_b\} = \delta_{ab}$, $a, b = 1, \ldots, k$ and the Clifford vacuum $|0\rangle$ satisfying $\eta_a|0\rangle = 0$. The tachyon profile is given by $\mathcal{Q} = \mathrm{G}_a \eta_a + P_a \bar{\eta}_a$ and is a matrix factorization, $\mathcal{Q}^2 = W$. Via (7.15) this corresponds to the Koszul resolution

$$\wedge^k E \xrightarrow{i_s} \cdots \xrightarrow{i_s} \wedge^2 E \xrightarrow{i_s} E \xrightarrow{i_s} \mathcal{O}_X(n_1, \ldots, n_m)\,,$$

of the sheaf $\mathcal{O}_M(n_1, \ldots, n_m)$, where

$$E = \bigoplus_{a=1}^k \mathcal{O}_X(n_1 - l_a^1, \ldots, n_m - l_a^m)$$

and i_s is the contraction by the section $s = (s_a)$ of the vector bundle $\bigoplus_{a=1}^k \mathcal{O}_X(l_a^1, \ldots, l_a^m)$. Following the rule (7.17) we assign gauge charges

$$(n_1 + \sum_a l_a^1/2, \ldots, n_m + \sum_a l_a^m/2) = (n_1 + N_1/2, \ldots, n_m + N_m/2)$$

to $|0\rangle$. Thus,

$$\begin{aligned} &Z_{\mathrm{hem}}[\mathcal{O}_M(n_1, \ldots, n_m)] \\ &= \int_{i\mathbb{R}^m} \frac{d\sigma^m}{(2\pi i)^m} e^{-2\pi i n_r \sigma_r} \Big[\prod_{a=1}^k \frac{2}{i} \sin(\pi l_a^r \sigma_r)\Big] e^{t_r \sigma_r} \Big[\prod_{r=1}^m \Gamma(\sigma_r)^{N_r}\Big] \prod_{a=1}^k \Gamma\left(1 - l_a^r \sigma_r\right) \\ &= (-2\pi i)^k \int_{i\mathbb{R}^m} \frac{d\sigma^m}{(2\pi i)^m} e^{(t_r - 2\pi i n_r)\sigma_r} \frac{\prod_r \Gamma(\sigma_r)^{N_r}}{\prod_a \Gamma(l_a^r \sigma_r)}\,. \end{aligned} \tag{E.1}$$

This integral can be evaluated by residues, and is given by the coefficient of $\prod_r \sigma_r^{-1}$ in the Laurent expansion of the integrand, up to exponentially suppressed terms for $\mathrm{Re}\, t \gg 0$.

We wish to compare this with the large volume formula obtained in subsection 3.6,

$$\int_M \mathrm{ch}(\mathcal{E}) e^{B + i\omega} \sqrt{\hat{A}(TM)} \tag{E.2}$$

for the central charge of $\mathcal{E} \in D(M)$. The complexified Kähler form $B + i\omega$ depends linearly on the complexified FI parameters $t = (t_r)$ in the large volume limit. Note the relation

$$\prod_j \sqrt{\frac{x_j}{e^{x_j/2} - e^{-x_j/2}}} - \prod_j \Gamma\left(1 + \frac{i x_j}{2\pi}\right) = \mathcal{O}(x_j^3)\,,$$

which is valid when $\sum_j x_j = 0$. This implies that the polynomial terms in t, appearing in (E.2) with the first three highest orders, also appear in the integral

$$\int_M \mathrm{ch}(\mathcal{E}) e^{B+i\omega} \hat{\Gamma}(TM)\,. \tag{E.3}$$

Here $\hat{\Gamma}$ is the multiplicative characteristic class[50] defined via the splitting principle as

$$\hat{\Gamma}(E) = \prod_j \Gamma\left(1 + \frac{ix_j}{2\pi}\right)\,, \tag{E.4}$$

where x_j are the Chern roots of a vector bundle E. Using the exact sequence

$$0 \longrightarrow TM \longrightarrow TX|_M \longrightarrow \bigoplus_{a=1}^{k} \mathcal{O}(l_a^1, \ldots, l_a^m)|_M \longrightarrow 0$$

and the Euler sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus N_r} \longrightarrow T\mathbb{P}^{N_r - 1} \to 0$$

for each r, we can write

$$\hat{\Gamma}(TM) = \frac{i^*\hat{\Gamma}(TX)}{i^*\hat{\Gamma}(\bigoplus_a \mathcal{O}(l_a^1, \ldots, l_a^m))} = \prod_{r=1}^{m} \Gamma\left(1 + \frac{i\mathbf{e}_r}{2\pi}\right)^{N_r} \Big/ \prod_{a=1}^{k} \Gamma\left(1 + \frac{\sum_r l_a^r \mathbf{e}_r}{2\pi i}\right)\,,$$

where $\mathbf{e}_r = i^* h_r$, and the hyperplane classes $h_r \in H^2(\mathbb{P}^{N_r-1})$ satisfy $\int_X \prod_r h_r^{N_r-1} = 1$. Thus, we can rewrite the large volume formula for the central charge as

$$\begin{aligned}
&\int_M \mathrm{ch}(\mathcal{O}_M(n_1, \ldots, n_m)) e^{B+i\omega} \sqrt{\hat{A}(TM)} \\
&\sim \int_M e^{\frac{i}{2\pi}\sum_r (t_r - 2\pi i n_r)\mathbf{e}_r} \frac{\prod_r \Gamma\left(1 + \frac{i}{2\pi}\mathbf{e}_r\right)^{N_r}}{\prod_a \Gamma\left(1 + \frac{i}{2\pi}\sum_r l_a^r \mathbf{e}_r\right)} \\
&= (-2\pi i)^k \int_X \prod_{r=1}^{m} \left(\frac{ih_r}{2\pi}\right)^{N_r} e^{\frac{i}{2\pi}\sum_r (t_r - 2\pi i n_r) h_r} \frac{\prod_r \Gamma\left(\frac{i}{2\pi} h_r\right)^{N_r}}{\prod_a \Gamma\left(\frac{i}{2\pi}\sum_r l_a^r h_r\right)}\,.
\end{aligned} \tag{E.5}$$

In the last line we used the fact that the Poincaré dual of the homology class $[s_a^{-1}(0)]$ is $c_1(\mathcal{O}(l_a^1, \ldots, l_a^m)) = \sum_r l_a^r h_r$. Comparing (E.5) with (E.1), we see that the hemisphere partition function agrees with the central charge in the large volume limit, up to an overall numerical factor, for the polynomial terms in t with the first three highest orders.

[50] We learned of the relevance of the Gamma class $\hat{\Gamma}$ to the hemisphere partition function in talks by D. Morrison and K. Hori. Our use of the Gamma class was motivated by their talks.

Appendix F. Vortex partition functions

Basic building blocks of the hemisphere partition function for theories with gauge group $G = U(N)$ and $N_{\mathrm{F}} \geq N$ fundamental chiral multiplets are the vortex partition functions [115]. Here we give certain expressions that arise in the sphere and the hemisphere partition functions. We take them as *definitions* of the vortex partition functions in the presence of other matter fields in various representations. Conceptually the vortex partition functions are equivariant integrals on the moduli space of vortex solitons with appropriate integrands, but the first principle derivations have been given only for some of the representations. One may regard the definitions here as predictions.

Let $-m_f$ be the twisted masses of the fundamentals. We define the vortex partition function specified by $\mathbf{v} \equiv \{f_1 < \ldots < f_N\} \subseteq \{1, \ldots, N_F\}$ as

$$Z^{\mathbf{v}}_{\mathrm{vortex}}(t_{\mathrm{ren}}, m) \equiv \sum_{k_1,\ldots,k_N=0}^{\infty} \prod_{j<l} (-1)^{k_{jl}} \Big(1 - \frac{k_{jl}}{m_{f_j f_l}}\Big) \prod_{a \notin \mathbf{v}} Z^{\mathbf{v}}_{R_a}(\vec{k}; m_a; \vec{\beta}) e^{-|\vec{k}| t_{\mathrm{ren}}} . \tag{F.1}$$

In the product, a runs over all chiral multiplets in irreducible representations R_a of $U(N)$, except the fundamentals corresponding to $f \in \mathbf{v}$. Let $(x)_k = x(x+1)\ldots(x+k-1)$ be the Pochhammer symbol. For the fundamental representation $Z^{\mathbf{v}}_{\mathrm{fund}}$ appears in the form

$$Z^{\mathbf{v}}_{\mathrm{fund}}(\vec{k}; -m_f) = \frac{(-1)^{\sum_j k_j}}{\prod_{j=1}^{N} (1 + m_f - m_{f_j})_{k_j}} .$$

For anti-fundamental, adjoint, and $\det^n$ representations, the $Z^{\mathbf{v}}_R$ is given by

$$Z^{\mathbf{v}}_{\mathrm{antifund}}(\vec{k}; m) = \prod_{j=1}^{N} (m - m_{f_j})_{k_j} , \qquad Z^{\mathbf{v}}_{\mathrm{adj}}(\vec{k}; m) = \prod_{i,j=1}^{N} \frac{(m_{f_i f_j} - k_i + m)_{k_j}}{(m_{f_i f_j} - k_i + m)_{k_i}} ,$$

$$Z^{\mathbf{v}}_{\det^n}(\vec{k}; m) = \frac{1}{(1 + m + n \sum_j m_{f_j})_{|\vec{k}|}} .$$

More generally, each infinite sum specified by I in (6.19), normalized so that the series starts with 1, defines an analog of the vortex partition function.

We study several Seiberg-like dualities in section 9. The vortex partition functions for the $T^*\mathrm{Gr}$ models are not duality invariant; rather, they satisfy a non-trivial relation (9.3). We found numerically that similar relations[51] hold for $U(N)$ theories with N_{F} fundamental

[51] For $N_{\mathrm{A}} \leq N_{\mathrm{F}} - 2$, the vortex partition functions are invariant under the duality map $N \to N_F - N$, $t_{\mathrm{ren}} \to t_{\mathrm{ren}} - N_{\mathrm{A}} \pi i$, $m_f \to -m_f - 1/2$, $\tilde{m}_a \to -\tilde{m}_a + 1/2$, $\mathbf{v} \to \mathbf{v}^{\vee}$.

and N_A anti-fundamental matter fields with $N_\mathrm{A} = N_\mathrm{F}, N_\mathrm{F} - 1$. By denoting the vortex partition function as $Z^{(N,N_\mathrm{F},N_\mathrm{A}),\mathbf{v}}_{\text{vortex}}(t_{\text{ren}}; m_f, \tilde{m}_a)$, for $N_\mathrm{A} = N_\mathrm{F}$ we have

$$\begin{aligned}&(1+(-1)^{N_F-N+1}e^{-t_{\text{ren}}})^{-(N_F-N)+\sum_{f=1}^{N_\mathrm{F}} m_f + \sum_{a=1}^{N_\mathrm{A}} \tilde{m}_a} Z^{(N,N_\mathrm{F},N_\mathrm{A}),\mathbf{v}}_{\text{vortex}}(t_{\text{ren}}; m_f, \tilde{m}_a) \\ &\quad = Z^{(N_F-N,N_\mathrm{F},N_\mathrm{A}),\mathbf{v}^\vee}_{\text{vortex}}(t_{\text{ren}} - N_\mathrm{A}\pi i; -m_f - 1/2, -\tilde{m}_a + 1/2)\,,\end{aligned}$$

and for $N_\mathrm{A} = N_\mathrm{F} - 1$,

$$\exp((-1)^{N_\mathrm{F}-N+1}e^{-t_{\text{ren}}}) Z^{U(N),\mathbf{v}}_{\text{vortex}}(t_{\text{ren}}; m_f, \tilde{m}_a) = Z^{U(N_F-N),\mathbf{v}^\vee}_{\text{vortex}}(t_{\text{ren}} - N_\mathrm{A}\pi i; -m_f - 1/2, -\tilde{m}_a + 1/2)\,.$$

Appendix G. Detailed calculations for a $U(1)$ theory

Let us consider the two-dimensional gauge theory in section 10.2. The $\mathbb{S}^2$ partition function is

$$Z_{\mathbb{S}^2}(X) = c\sum_{\mathbf{v}} e^{-(t+\bar{t})m_v} \prod_{f\neq v} \frac{\Gamma(m_{fv})}{\Gamma(1-m_{fv})} \prod_f \frac{\Gamma(m_v + \tilde{m}_f)}{\Gamma(1-m_v-\tilde{m}_f)} Z^v_{\text{vortex}}(t,m) Z^v_{\text{vortex}}(\bar{t},m)\,,$$

where we chose $w_0 = 0$ for the ambiguity w_0 in (6.23), and c is a normalization constant to be determined. The vortex partition function is as defined in (F.1):

$$Z^v_{\text{vortex}}(t,m) = \sum_{k=0}^{\infty} e^{-kt}(-1)^{kN_F} \prod_{f=1}^{N_\mathrm{F}} \frac{(\tilde{m}_f + m_v)_k}{(1-m_{fv})_k}. \tag{G.1}$$

We can write $Z_{\mathbb{S}^2} = \sum_v \langle 1|v\rangle\langle v|1\rangle$ if we set

$$\langle v|1\rangle = c^{1/2} e^{-tm_v} \left[\prod_{f\neq v} \frac{\Gamma(m_{fv})}{\Gamma(1-m_{fv})} \prod_f \frac{\Gamma(m_v+\tilde{m}_f)}{\Gamma(1-m_v-\tilde{m}_f)}\right]^{1/2} Z^v_{\text{vortex}}(t,m)$$

and

$$\langle 1|v\rangle = c^{1/2} e^{-\bar{t}m_v} \left[\prod_{f\neq v} \frac{\Gamma(m_{fv})}{\Gamma(1-m_{fv})} \prod_f \frac{\Gamma(m_v+\tilde{m}_f)}{\Gamma(1-m_v-\tilde{m}_f)}\right]^{1/2} Z^v_{\text{vortex}}(\bar{t},m)\,.$$

We can compute the cylinder partition function $\langle \mathcal{B}(\mathcal{O}_X(n_2))|\mathcal{B}(\mathcal{O}_X(n_1))\rangle$ by a generalization of (8.16),

$$\mathrm{ind}_{F\otimes E^*}(\not{D}) = \sum_{p\text{: fixed points}} \frac{1}{\det_{TX_p}(g^{-1/2} - g^{1/2})} \mathrm{Tr}_{F_p}(g)\mathrm{Tr}_{E_p}(g^{-1})\,. \tag{G.2}$$

We find

$$\langle \mathcal{B}(\mathcal{O}_X(n_2))|\mathcal{B}(\mathcal{O}_X(n_1))\rangle = \sum_v e^{2\pi i n_{21} m_v} \left[\prod_{f\neq v} 2i\sin\pi m_{fv} \prod_f 2i\sin\pi(m_v+\tilde{m}_f)\right]^{-1},$$

where $n_{ab} := n_a - n_b$. This can be written as $\sum_v \langle \mathcal{B}(\mathcal{O}_X(n_2))|v\rangle\langle v|\mathcal{B}(\mathcal{O}_X(n_1))\rangle$ by setting

$$\langle \mathcal{B}(\mathcal{O}_X(n))|v\rangle = e^{2\pi i n m_v} \left[\prod_{f\neq v} 2i\sin\pi m_{fv} \prod_f 2i\sin\pi(m_v+\tilde{m}_f)\right]^{-1/2}$$

and

$$\langle v|\mathcal{B}(\mathcal{O}_X(n))\rangle = e^{-2\pi i n m_v} \left[\prod_{f\neq v} 2i\sin\pi m_{fv} \prod_f 2i\sin\pi(m_v+\tilde{m}_f)\right]^{-1/2}.$$

The hemisphere partition function for $\mathcal{B}(\mathcal{O}_X(n))$ is

$$\begin{aligned} Z_{\text{hem}}(\mathcal{B}(\mathcal{O}_X(n))) &= \int \frac{d\sigma}{2\pi i} e^{-2\pi i n\sigma} e^{t\sigma} \prod_{f=1}^{N_{\mathrm{F}}} \Gamma(\sigma+m_f)\Gamma(-\sigma+\widetilde{m}_f) \\ &= \sum_{v=1}^{N_F} e^{2\pi i n m_v} Z^v_{\text{cl}}(t,m) Z^v_{\text{1-loop}}(m) Z^v_{\text{vortex}}(t,m)\,. \end{aligned}$$

where

$$Z^v_{\text{cl}}(t,m) = e^{-tm_v}\,, \qquad Z^v_{\text{1-loop}}(m) = \prod_{f\neq v}\Gamma(m_{fv})\prod_f \Gamma(\widetilde{m}_f + m_v)\,.$$

We can write

$$Z_{\text{hem}}(\mathcal{B}(\mathcal{O}_X(n))) = \sum_{v=1}^{N_{\mathrm{F}}} \langle \mathcal{B}(\mathcal{O}_X(n))|v\rangle\langle v|\mathtt{1}\rangle = \langle \mathcal{B}(\mathcal{O}_X(n))|\mathtt{1}\rangle\,.$$

if we set $c = (2\pi i)^{2N_{\mathrm{F}}-1}$.

We will also be interested in the brane for the structure sheaf of Y, the submanifold defined by setting to zero the chiral fields $\tilde{\phi}_f$. This corresponds to Case 1 of section 7.3. Let us introduce fermionic oscillators satisfying $\{\eta_f, \bar{\eta}_g\} = \delta_{fg}$, $\eta_f|0\rangle = 0$. A locally free resolution of $\mathcal{O}_Y$ is given by a complex of equivariant vector bundles which corresponds to

$$\mathbb{C}\bar{\eta}_1\ldots\bar{\eta}_{N_{\mathrm{F}}}|0\rangle \to \ldots \to \bigoplus_{f<g}\mathbb{C}\bar{\eta}_f\bar{\eta}_g|0\rangle \to \bigoplus_f \mathbb{C}\bar{\eta}_f|0\rangle \to \underline{\mathbb{C}|0\rangle}$$

with the differential $\mathcal{Q} = \tilde{\phi}_f \eta^f$. The underline indicates the degree-zero location. Including the twist by $\mathcal{O}_X(n)$, we find

$$\begin{aligned}\langle \mathcal{B}(\mathcal{O}_Y(n))|v\rangle &= \prod_f (1 - e^{+2\pi i(m_v+\tilde{m}_f)}) \times \langle \mathcal{B}(\mathcal{O}_X(n))|v\rangle \\ &= (-1)^{N_\mathrm{F}} e^{2\pi i n m_v} e^{N_\mathrm{F}\pi i m_v} e^{\pi i \sum_f \tilde{m}_f} \left[\frac{\prod_f 2i\sin\pi(m_v+\tilde{m}_f)}{\prod_{f\neq v} 2i\sin\pi m_{fv}}\right]^{1/2}\end{aligned} \tag{G.3}$$

and

$$\langle v|\mathcal{B}(\mathcal{O}_Y(n))\rangle = e^{-2\pi i n m_v} e^{-N_\mathrm{F}\pi i m_v} e^{-\pi i \sum_f \tilde{m}_f} \left[\frac{\prod_f 2i\sin\pi(m_v+\tilde{m}_f)}{\prod_{f\neq v} 2i\sin\pi m_{fv}}\right]^{1/2}. \tag{G.4}$$

We wish to derive the monodromies of $\langle v|\mathtt{1}\rangle$ along paths on the (e^{-t})-plane. To simplify the computations let us set $z = (-1)^{N_\mathrm{F}} e^{-t}$. The differential equation (10.2) becomes

$$\left[z \prod_{f=1}^{N_\mathrm{F}} \left(z\frac{d}{dz} + \tilde{m}_f\right) - \prod_{f=1}^{N_\mathrm{F}} \left(z\frac{d}{dz} - m_f\right)\right] G(z) = 0\,, \tag{G.5}$$

which has N_F basic solutions

$$G_v(z) = z^{m_v}\, {}_{N_\mathrm{F}}F_{N_\mathrm{F}-1}\left(\left.\begin{matrix}\{\tilde{m}_f+m_v\}_{f=1}^{N_\mathrm{F}} \\ \{1-m_f+m_v\}_{f\neq v}^{N_\mathrm{F}}\end{matrix}\right| z\right) \tag{G.6}$$

analytic on the complex z-plane minus the branch cuts $(-\infty,0]\cup[1,\infty)$. In terms of the functions G_v and the coefficients

$$A_v = (2\pi i)^{N_\mathrm{F}-1/2} \left[\prod_{f\neq v} \frac{\Gamma(m_{fv})}{\Gamma(1-m_{fv})} \prod_f \frac{\Gamma(m_v+\tilde{m}_w)}{\Gamma(1-m_v-\tilde{m}_w)}\right]^{1/2} \times \begin{cases} 1 & \text{for } N_\mathrm{F} \quad \text{even}\,, \\ e^{-\pi i m_v} & \text{for } N_\mathrm{F} \quad \text{odd}\,, \end{cases}$$

we can write

$$\langle v|\mathtt{1}\rangle = A_v G_v(z)\,. \tag{G.7}$$

On G_v, the monodromy along a path $\tilde{\gamma}$ acts as

$$G_v(z) \to \sum_w \mathbf{M}(\tilde{\gamma})_{vw} G_w(z)$$

for some matrix $\mathbf{M}(\tilde{\gamma})_{vw}$. If a path $\tilde{\gamma}$ on the z-plane corresponds to the path γ on the (e^{-t})-plane, the matrix $\mathbf{M}(\tilde{\gamma})$ is related to $M(\gamma)$ in (10.3) by a diagonal similarity transformation

$$M(\gamma)_{vw} = A_v \mathbf{M}(\tilde{\gamma})_{vw} A_w^{-1}\,. \tag{G.8}$$

For the small loop $\tilde{\gamma}_0$ going around $z = 0$ counterclockwise, the monodromy acts as $G_v(z) \to e^{2\pi i m_v} G_v(z)$. Thus, $\mathbf{M}(\tilde{\gamma}_0)_{vw} = e^{2\pi i m_v}\delta_{vw}$.

In order to obtain monodromies along other paths, let us consider independent solutions of (G.5) around $z = \infty$ [116]

$$\tilde{G}_v(z) := z^{-\tilde{m}_v}\, {}_{N_{\rm F}}F_{N_{\rm F}-1}\left(\left. \begin{matrix} \{m_f+\tilde{m}_v\}_{f=1}^{N_{\rm F}} \\ \{1+\tilde{m}_{vf}\}_{f\neq v}^{N_{\rm F}} \end{matrix} \right| \frac{1}{z} \right), \ v = 1, \dots, N_F.$$

They are analytic on $\mathbb{C}\backslash(-\infty, 1]$. We can relate $G_v(z)$ defined near $z = 0$ and $\tilde{G}_v(z)$ defined near $z = \infty$ by analytic continuation upon choosing a path that connects the two regions. The relation, the connection formula, depends on whether the path goes above ($\epsilon = +1$) or below ($\epsilon = -1$) the singularity at $z = 1$:

$$G_v(z) = \sum_{w=1}^{N_{\rm F}} e^{i\pi\epsilon(m_v+\tilde{m}_w)} \prod_{f\neq v}^{N_{\rm F}} \frac{\Gamma(1+m_{vf})}{\Gamma(1-\tilde{m}_w-m_f)} \prod_{f\neq w}^{N_{\rm F}} \frac{\Gamma(\tilde{m}_{fw})}{\Gamma(\tilde{m}_f+m_v)} \tilde{G}_w(z)\,.$$

By exchanging $z \leftrightarrow z^{-1}$ and $m \leftrightarrow \tilde{m}$ we obtain the inverse formula

$$\widetilde{G}_v(z) = \sum_{w=1}^{N_{\rm F}} e^{i\pi\epsilon(\tilde{m}_v+m_w)} \prod_{f\neq v}^{N_{\rm F}} \frac{\Gamma(1+\tilde{m}_{vf})}{\Gamma(1-m_w-\widetilde{m}_f)} \prod_{f\neq w}^{N_{\rm F}} \frac{\Gamma(m_{fw})}{\Gamma(m_f+\widetilde{m}_v)} G_w(z)\,,$$

where the two regions are connected along a path below ($\epsilon = +1$) or above ($\epsilon = -1$) $z = 1$.

Let us define a path $\tilde{\gamma}_{\epsilon_1\epsilon_2\epsilon_3}$ as follows. It first goes from $z = 0$ to $+\infty$ above or below $z = 1$ for $\epsilon_1 = +1$ or $\epsilon_1 = -1$, respectively. Then, for $\epsilon_2 = +1(-1)$, it moves along a very large circle clockwise(counterclockwise), and does not move for $\epsilon_2 = 0$. Finally, $\epsilon_3 = 1$ or $\epsilon_3 = -1$ if the path goes from $z = +\infty$ back to 0 below or above $z = 1$. The monodromy along $\tilde{\gamma}_{\epsilon_1\epsilon_2\epsilon_3}$ is[52]

$$\begin{aligned} G_v(z) \to & \sum_w \sum_g e^{\pi i\epsilon_1(m_v+\tilde{m}_g)} \prod_{f\neq v} \frac{\Gamma(1+m_{vf})}{\Gamma(1-\tilde{m}_g-m_f)} \prod_{f\neq g} \frac{\Gamma(\tilde{m}_{fg})}{\Gamma(\tilde{m}_f+m_v)} \\ & \times e^{2\pi i\epsilon_2\tilde{m}_g} e^{\pi i\epsilon_3(\tilde{m}_g+m_w)} \prod_{f\neq g} \frac{\Gamma(1+\tilde{m}_{gf})}{\Gamma(1-m_w-\widetilde{m}_f)} \prod_{f\neq w} \frac{\Gamma(m_{fw})}{\Gamma(m_f+\widetilde{m}_g)} G_w(z) \\ = & \sum_w e^{\pi i(\epsilon_1 m_v+\epsilon_3 m_w)} \frac{\prod_{f\neq v}\Gamma(1+m_{vf})\prod_{f\neq w}\Gamma(m_{fw})}{\prod_f \Gamma(\widetilde{m}_f+m_v)\Gamma(1-m_w-\widetilde{m}_f)} \\ & \times \pi \sum_g e^{i\pi(+\epsilon_1+2\epsilon_2+\epsilon_3)\tilde{m}_g} \frac{\prod_{f\neq v,w}\sin\pi(m_f+\widetilde{m}_g)}{\prod_{f\neq g}\sin\pi\widetilde{m}_{fg}} G_w(z)\,. \end{aligned}$$

[52] The expressions of the form $\prod_{f\neq v,w} C_f$ mean $(\prod_f C_f)/C_vC_w$ in this appendix.

If $n=\epsilon_2+(\epsilon_1+\epsilon_3)/2$ satisfies $|n|\leq 1$,[53] we can rewrite the monodromy in the form

$$G_v(z)\to\sum_w \mathcal{M}^{\epsilon_1\epsilon_2\epsilon_3}_{vw}G_w(z)\,,$$

where

$$\begin{aligned}\mathcal{M}^{\epsilon_1\epsilon_2\epsilon_3}_{vw}&=\pi e^{\pi i(\epsilon_1 m_v+\epsilon_3 m_w)}\frac{\prod_{f\neq v}\Gamma(1+m_{vf})\prod_{f\neq w}\Gamma(m_{fw})}{\prod_f\Gamma(\tilde m_f+m_v)\Gamma(1-m_w-\tilde m_f)}\\&\quad\times\left[\delta_{vw}e^{-2\pi i n m_v}\frac{\prod_{f\neq v}\sin\pi m_{fv}}{\prod_f\sin\pi(\tilde m_f+m_v)}+(-1)^{N_{\mathrm{F}}-1}2nie^{n\pi i\left(\sum_f\tilde m_f+\sum_{f\neq v,w}m_f\right)}\right]\\&=\delta_{vw}e^{-2\epsilon_2\pi i m_v}+2n\pi i e^{i\pi[n\sum_f(m_f+\tilde m_f)+(\epsilon_1-n)m_v+(\epsilon_3-n)m_w]}\mathbf{S}_{vw}\,.\end{aligned}$$

The matrix

$$\mathbf{S}_{vw}\equiv(-1)^{N_{\mathrm{F}}-1}\frac{\prod_{f\neq v}\Gamma(1+m_{vf})\prod_{f\neq w}\Gamma(m_{fw})}{\prod_f\Gamma(\tilde m_f+m_v)\Gamma(1-m_w-\tilde m_f)}.$$

satisfies the equations[54]

$$\begin{aligned}\mathbf{S}_{vv}&=\frac{(-1)^{N_{\mathrm{F}}-1}}{\pi}\frac{\prod_f\sin\pi(\tilde m_f+m_v)}{\prod_{f\neq v}\sin\pi m_{fv}},\\\sum_{g=1}^{N_{\mathrm{F}}}\mathbf{S}_{vg}\mathbf{S}_{gw}&=\frac{1}{2i\pi}\left(e^{i\pi\sum_f(m_f+\tilde m_f)}-e^{-i\pi\sum_f(m_f+\tilde m_f)}\right)\mathbf{S}_{vw}\,.\end{aligned}$$

In particular the monodromy matrices for the basic paths in Figure 5 are

$$\begin{aligned}\mathbf{M}(\tilde\gamma_0)_{vw}&=\delta_{vw}e^{2\pi i m_v},\\\mathbf{M}(\tilde\gamma_1)_{vw}&=\mathcal{M}^{-1,0,-1}_{vw}=\delta_{vw}-2\pi i e^{-\pi i\sum_f(m_f+\tilde m_f)}\mathbf{S}_{vw}\,,\\\mathbf{M}(\tilde\gamma_\infty)_{vw}&=\mathcal{M}^{1,1,-1}_{vw}=\delta_{vw}e^{-2\pi i m_v}+2\pi i e^{\pi i\sum_f(m_f+\tilde m_f)}e^{-2\pi i m_w}\mathbf{S}_{vw}.\end{aligned}$$

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[53] For such n we have the identity [116]

$$\begin{aligned}&\sum_g e^{2\pi i n\tilde m_g}\frac{\prod_{f\neq v,w}\sin\pi(m_f+\tilde m_g)}{\prod_{f\neq g}\sin\pi\tilde m_{fg}}\\&=\delta_{vw}e^{-2\pi i n m_v}\frac{\prod_{f\neq v}\sin\pi m_{fv}}{\prod_f\sin\pi(\tilde m_f+m_v)}+(-1)^{N_{\mathrm{F}}-1}2nie^{ni\pi\left[\sum_f\tilde m_f+\sum_{f\neq v,w}m_f\right]}\,.\end{aligned}$$

[54] The second equation can be proved by using the identity

$$\sum_g\frac{\prod_f\sin(\tilde m_f+m_g)}{\prod_{f\neq g}\sin(m_f-m_g)}=\frac{(-1)^{N_{\mathrm{F}}-1}}{2i}\left(e^{i\pi\sum_f(m_f+\tilde m_f)}-e^{-i\pi\sum_f(m_f+\tilde m_f)}\right)\,.$$

One can check that $\mathbf{M}(\tilde{\gamma}_0)\mathbf{M}(\tilde{\gamma}_1)\mathbf{M}(\tilde{\gamma}_\infty) = 1$ as expected.[55] After the similarity transformation (G.8), we obtain the monodromy matrices (10.4).

Appendix H. Grade restriction rule and analytic continuation

In this appendix we explain how to use the integral representation (6.13) to analytically continue a hemisphere partition function from one region to another in the Kähler moduli space. This involves choosing a complex of bundles representing a given object in the derived category so that each bundle satisfies the so-called grade restriction rule [73]. We will use a D2-brane on the resolved conifold as an example.

We first review a derivation of the grade restriction rule from the integral representation of Z_{hem}, as explained in a talk by K. Hori. Let us consider a general $U(1)$ gauge theory with N_F chiral multiplets with gauge charges Q_f and twisted masses m_f, $f = 1, \ldots, N_F$, satisfying $\sum_f Q_f = 0$. We impose the Neumann boundary condition on all chiral fields and include a Wilson loop with gauge charge n. The hemisphere partition function is then

$$\int_{-i\infty}^{i\infty} \frac{d\sigma}{2\pi i} e^{t\sigma} e^{-2\pi i n\sigma} \prod_{f=1}^{N_F} \Gamma(Q_f\sigma + m_f),$$

where $t = r - i\theta$. In the limit $\sigma \to \pm i\infty$, the absolute value of the integrand behaves as $\exp\Big(\big(-\pi\mathcal{S} \pm (2\pi n + \theta)\big)|\sigma|\Big)$, where $\mathcal{S} = \sum_{Q_f>0} Q_f$. When the *grade restriction rule*[56]

$$-\frac{\mathcal{S}}{2} < n + \frac{\theta}{2\pi} < \frac{\mathcal{S}}{2} \tag{H.1}$$

is obeyed, the σ-integral along the imaginary axis is absolutely convergent, and the hemisphere partition function can be analytically continued from $r \gg 0$ to $r \ll 0$.

Let us consider a $U(1)$ gauge theory with chiral multiplet fields (ϕ_1, ϕ_2) with charge $+1$, and $(\tilde{\phi}_1, \tilde{\phi}_2)$ with charge -1. We denote their twisted masses as (m_1, m_2) and $(\tilde{m}_1, \tilde{m}_2)$ respectively. The theory flows to the nonlinear sigma model whose target space X is defined by the equation $|\phi_1|^2 + |\phi_2|^2 - |\tilde{\phi}_1|^2 - |\tilde{\phi}_2|^2 = r/2\pi$ and the $U(1)$ quotient. In the phase $r \gg 0$, this is the resolved conifold, the total space of $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \to \mathbb{P}^1$, where (ϕ_1, ϕ_2) parametrize the base $\mathbb{P}^1$ and $(\tilde{\phi}_1, \tilde{\phi}_2)$ are the fiber coordinates. In the flopped phase $r \ll 0$

[55] We defined $\mathbf{M}(\tilde{\gamma})$ for all $\tilde{\gamma}$ using a base point on a common Riemann sheet. For a discussion on the choice of base point and relations satisfied by monodromy matrices, see [117].

[56] The energy for large $|\sigma_1 - i\sigma_2|$ is bounded from below only if (H.1) is satisfied [73].

the roles of (ϕ_1, ϕ_2) and $(\tilde{\phi}_1, \tilde{\phi}_2)$ are exchanged. Let $i^{\pm} : \mathbb{P}^1 \to X$ be the embeddings in the $\pm r \gg 0$ phases respectively.

We are interested in transporting the sheaf $i_*^+ \mathcal{O}_{\mathbb{P}^1}$ from $r \gg 0$ to $r \ll 0$, through the window $-2\pi < \theta < 0$, for which the grade restriction rule is obeyed only by $n = 0, 1$. In particular, we will perform an analytic continuation of its hemisphere partition function.

To study this problem, let us introduce fermionic oscillators satisfying $\{\eta_f, \bar{\eta}_g\} = \{\tilde{\eta}_f, \bar{\tilde{\eta}}_g\} = \delta_{fg}$ $(f, g = 1, 2)$, with the corresponding Clifford vacua such that $\eta_f|0\rangle = \tilde{\eta}_g|\tilde{0}\rangle = 0$. We assume that $|\tilde{0}\rangle$ is neutral under gauge and flavor symmetries, and identify $|\tilde{0}\rangle = \tilde{\eta}_2\tilde{\eta}_1|0\rangle$. Consider the following two complexes of vector spaces

$$0 \longrightarrow \mathbb{C}\bar{\eta}_1\bar{\eta}_2|0\rangle \longrightarrow \mathbb{C}\bar{\eta}_1|0\rangle \oplus \mathbb{C}\bar{\eta}_2|0\rangle \longrightarrow \underline{\mathbb{C}|0\rangle} \longrightarrow 0\,, \tag{H.2}$$

$$0 \longrightarrow \mathbb{C}\bar{\tilde{\eta}}_1\bar{\tilde{\eta}}_2|\tilde{0}\rangle \longrightarrow \mathbb{C}\bar{\tilde{\eta}}_1|\tilde{0}\rangle \oplus \mathbb{C}\bar{\tilde{\eta}}_2|\tilde{0}\rangle \longrightarrow \underline{\mathbb{C}|\tilde{0}\rangle} \longrightarrow 0\,, \tag{H.3}$$

with the underline indicating degree zero. The differentials are $\mathcal{Q} = \sum_{f=1,2} \phi_f \eta_f$, $\tilde{\mathcal{Q}} = \sum_{f=1,2} \tilde{\phi}_f \tilde{\eta}_f$ respectively. These represent complexes of equivariant vector bundles. In the phase $r \gg 0$, $\{\mathcal{Q}, \bar{\mathcal{Q}}\}$ is positive definite, implying that (H.2) is exact and represents the zero object in the derived category. On the other hand, in the same phase, (H.3) is the Koszul resolution [76] of $i_*^+ \mathcal{O}_{\mathbb{P}^1}$ supported on $\{\tilde{\phi}_1 = \tilde{\phi}_2 = 0\}$, which is the D-brane we are interested in. Again the roles of (H.2) and (H.3) are swapped for $r \ll 0$.

The gauge charges of $|\tilde{0}\rangle$, $\bar{\tilde{\eta}}_f|\tilde{0}\rangle$, $\bar{\tilde{\eta}}_1\bar{\tilde{\eta}}_2|\tilde{0}\rangle$ are $0, 1, 2$ respectively. The last one is outside the range (H.1). As a consequence, the hemisphere partition function for (H.3)

$$(-2\pi i)^2 e^{\pi i(\tilde{m}_1 + \tilde{m}_2)} \int_{-i\infty}^{i\infty} \frac{d\sigma}{2\pi i} e^{(t-2\pi i)\sigma} \frac{\Gamma(\sigma + m_1)\Gamma(\sigma + m_2)}{\Gamma(1 + \sigma - \tilde{m}_1)\Gamma(1 + \sigma - \tilde{m}_2)} \tag{H.4}$$

does not converge absolutely along the imaginary axis. For $r \gg 0$, convergence requires us to choose the σ contour so that asymptotically $\sigma \to \pm i(1 \pm \epsilon)\infty$, and this gives

$$\begin{aligned} Z_{\text{hem}}(i_*^+ \mathcal{O}_{\mathbb{P}^1}) = (-2\pi i)^2 e^{\pi i(\tilde{m}_1 + \tilde{m}_2)} \sum_{v=1}^{2} e^{-m_v(t-2\pi i)} \frac{\prod_{f\neq v}^2 \Gamma(m_f - m_v)}{\prod_{f=1}^2 \Gamma(1 - m_v - \tilde{m}_f)} \\ \times {}_2F_1\left(\left. {\{\tilde{m}_f + m_v\}_{f=1}^2 \atop \{1 - m_f + m_v\}_{f\neq v}^2} \right| e^{-t} \right). \end{aligned} \tag{H.5}$$

For $r \ll 0$ we need $\sigma \to \pm i(1 \mp \epsilon)\infty$, and (H.4) vanishes, as it should for the zero object. The two functions are not related by analytic continuation.

In order to analytically continue $Z_{\rm hem}(i_*\mathcal{O}_{\mathbb{P}^1})$ from $r \gg 0$ to $r \ll 0$, we may evaluate (H.4) by residues and apply the connection formula, as we did in Appendix G. Here we explain an alternative method found in [73].

The problematic term $\mathbb{C}\bar{\bar{\eta}}_1\bar{\bar{\eta}}_2|\tilde{0}\rangle$ can be eliminated from the complex (H.3)by binding the D-brane (H.3) with the other D-brane (H.2), which is empty for $r \gg 0$. Let f be the unique cochain map from (H.3) to (H.2), with degrees shifted for the latter, such that $\mathbb{C}\bar{\bar{\eta}}_1\bar{\bar{\eta}}_2|\tilde{0}\rangle$ in (H.3) is mapped to $\mathbb{C}|0\rangle$ in (H.2) by the identity map. The bound state of the two D-branes is the mapping cone $C(f)$:

$$\begin{tikzcd}
& \mathbb{C}\bar{\bar{\eta}}_1\bar{\bar{\eta}}_2|\tilde{0}\rangle \arrow[r] \arrow[ddr, "\mathbf{1}"] & \mathbb{C}\bar{\bar{\eta}}_1|\tilde{0}\rangle \oplus \mathbb{C}\bar{\bar{\eta}}_2|\tilde{0}\rangle \arrow[r] & \mathbb{C}|\tilde{0}\rangle \\
& \bigoplus & \bigoplus & \\
\mathbb{C}\bar{\eta}_1\bar{\eta}_2|0\rangle \arrow[r] & \mathbb{C}\bar{\eta}_1|0\rangle \oplus \mathbb{C}\bar{\eta}_2|0\rangle \arrow[r] & \mathbb{C}|0\rangle &
\end{tikzcd}$$

The pair, which carries the gauge charge 2 and is connected by the identity map, can be neglected in computing $Z_{\rm hem}$ for $C(f)$.[57] The other terms carry gauge charges 0 or 1. The hemisphere partition function can be written as

$$\begin{aligned}
Z_{\rm hem}(C(f)) = \int \frac{d\sigma}{2\pi i} e^{t\sigma} \Big[1 - e^{-2\pi i\sigma}\left(e^{2\pi i\tilde{m}_1} + e^{2\pi i\tilde{m}_2}\right) + (e^{2\pi i m_1} + e^{2\pi i m_2}) e^{2\pi i(\tilde{m}_1+\tilde{m}_2-\sigma)} \\
- e^{2\pi i(m_1+m_2+\tilde{m}_1+\tilde{m}_2)} \Big] \prod_{f=1}^{2} \Gamma(\sigma+m_f)\Gamma(-\sigma+\tilde{m}_f)\,.
\end{aligned}$$

This integral along the imaginary axis is now absolutely convergent for $-2\pi < \theta < 0$, and interpolates the hemisphere partition functions in the two phases.

In the phase $r \gg 0$, the contribution from (H.2) is trivial, and $Z_{\rm hem}(C(f))$ coincides with $Z_{\rm hem}(i_*^+\mathcal{O}_{\mathbb{P}^1})$ in (H.5). In the phase $r \ll 0$, the contribution from (H.3) becomes trivial and $Z_{\rm hem}(C(f))$ coincides with the hemisphere partition function for (H.2)

$$\begin{aligned}
Z_{\rm hem}(i_*^-\mathcal{O}_{\mathbb{P}^1}(2)[1]) &= -(-2\pi i)^2 e^{\pi i(m_1+m_2+2\tilde{m}_1+2\tilde{m}_2)} \\
&\quad\times \int \frac{d\sigma}{2\pi i} e^{(t-2\pi i)\sigma} \frac{\Gamma(-\sigma+\tilde{m}_1)\Gamma(-\sigma+\tilde{m}_2)}{\Gamma(1-\sigma-m_1)\Gamma(1-\sigma-m_2)} \\
&= -(-2\pi i)^2 e^{\pi i(m_1+m_2+2\tilde{m}_1+2\tilde{m}_2)} \\
&\quad\times \sum_{v=1}^{2} e^{\tilde{m}_v(t-2\pi i)} \frac{\prod_{f\neq v}^{2}\Gamma(\tilde{m}_f-\tilde{m}_v)}{\prod_{f=1}^{2}\Gamma(1-\tilde{m}_v-m_f)} {}_2F_1\left(\begin{matrix} \{m_f+\tilde{m}_v\}_{f=1}^2 \\ \{1-\tilde{m}_f+\tilde{m}_v\}_{f\neq v}^2 \end{matrix} \middle| e^t \right).
\end{aligned}$$

[57] As in [73] one can change the basis to show that $C(f)$ decomposes into a complex $\mathcal{V}^{-3} \to \mathcal{V}^{-2} \to \mathcal{V}^{-1} \to \mathcal{V}^0$ and a trivial pair $\tilde{\mathcal{V}}^{-2} \to \tilde{\mathcal{V}}^{-1}$, where $(\mathcal{V}^{-3}, \mathcal{V}^{-2}, \mathcal{V}^{-1}, \mathcal{V}^0;\ \tilde{\mathcal{V}}^{-2}, \tilde{\mathcal{V}}^{-1})$ carry the same quantum numbers as $(\mathbb{C}\bar{\eta}_1\bar{\eta}_2|0\rangle, \mathbb{C}\bar{\eta}_1|0\rangle \oplus \mathbb{C}\bar{\eta}_2|0\rangle, \mathbb{C}\bar{\bar{\eta}}_1|\tilde{0}\rangle \oplus \mathbb{C}\bar{\bar{\eta}}_2|\tilde{0}\rangle, \mathbb{C}|\tilde{0}\rangle;\ \mathbb{C}\bar{\bar{\eta}}_1\bar{\bar{\eta}}_2|\tilde{0}\rangle, \mathbb{C}|0\rangle)$.

One can check that the relation between $Z_{\mathrm{hem}}(i_*^+\mathcal{O}_{\mathbb{P}^1})$ and $Z_{\mathrm{hem}}(i_*^-\mathcal{O}_{\mathbb{P}^1}(2)[1])$ is consistent with the connection formulas in Appendix G.

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