学位論文

The Effective Masses of Scalar Fields in the Radiation Dominated Universe

(輻射優勢宇宙におけるスカラー場の有効質量)

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Abstract

Effective mass plays important roles in the particle physics models of the very early Universe. For example, thermal effective mass may be the origin of some cosmological phase transitions like electroweak symmetry breaking. Another example is that the so-called Hubble-induced mass, which is generated by supergravity effects during inflation, is a key for the Affleck-Dine baryogenesis, the adiabatic solution for the cosmological moduli problem and so on.

In this thesis, we consider the effective masses of scalar fields in the radiation dominated Universe. We in particular pay attention to the effective mass of a weakly coupled scalar field ϕ which interacts with the thermal plasma via Planck-suppressed interactions. Such a Planck-suppressed interacting scalar field ϕ often appears in particle physics models. However, what magnitude of the effective mass of ϕ arises in the radiation dominated era has not been clarified so far. We investigate this issue, for the first time, by using the techniques of thermal field theory which is the most reliable method for treating the finite temperature system. At first, we consider a toy model in which scalar fields or fermion fields consist of the thermal bath and these thermal fields generate the effective mass for ϕ. Despite the fact that we use thermal field theory, we face some difficulties in the analysis. To overcome the difficulties, we propose a solid and more transparent strategy for the analysis of the effective mass of ϕ. Finally, we apply the improved method to the analysis of the effective mass of ϕ which is generated by the minimal supersymmetric standard model (MSSM) plasma. The resultant effective mass for ϕ is of the order of the Hubble scale times some powers of the coupling constants of the thermalized fields.

Contents

Chapter 1

Introduction

Effective mass plays important roles in particle physics models of the early Universe. Before the electroweak phase transition, the zero-temperature masses of particles we observe today, which are generated by the Higgs scalar vacuum expectation value (VEV) [1–4], are absent. However, there are possibilities that some fields have VEV or finite energy density in that era (for instance the inflationary era [5–9]). Then, such VEV or finite energy density serve as sources for the effective masses of fields.

The effective masses of fields can be generated by the thermal effect (radiation energy density). (See for example Refs. [10–12]. We will review the thermal field theory in Chap. 2.) In principle, if there is a thermal bath, all fields can be affected by thermal effect and the dynamics of the fields can be changed qualitatively. For example, the symmetry preservation occurs by a thermal potential, which eventually causes the phase transition [13–15]. Also, the reheating process after inflation would be affected by the thermal dissipation of energy [16–22].

The effective mass can also be provided by the Planck-suppressed interactions with some field VEV or finite energy density. Such a Planck-suppressed interaction is provided for example by supergravity (SUGRA) effect (with, for instance, the inflation energy) [23–27]. Supergravity is the local version of supersymmetry (SUSY) [28], which provides many interesting phenomenology. Here, SUSY is an attractive candidate for the physics beyond the standard model of particle physics. One of the most interesting feature of SUSY for cosmology is that there are naturally many scalar fields. In particular, there are many flat directions in the field space in the minimal supersymmetric standard model (MSSM) [29]. These scalar fields may be responsible for the important phenomena in the early Universe like inflation, baryogenesis and so on (for review see for example Refs. [30]).

In this thesis, we consider the effective mass of a scalar field ϕ in the radiation dominated (RD) era. We in particular pay attention to the case in which the scalar field ϕ is not directly coupled to the thermal bath. Even in this case, the scalar field ϕ would have some interactions with the bath thorough supergravity effects (Plank-suppressed interactions). Whether or not this supergravity effect in the RD era provides the Hubble scale effective mass for ϕ was an issue [31,32]. Furthermore, the magnitude of such an effective mass in the RD era is important for some models of the early Universe. However, these issues on the effective mass have not been resolved so far. Thus, it is important to clarify what magnitude of the effective mass the Planck-suppressed interacting field ϕ acquires by using a reliable method. This is the motivation of this thesis. Namely, the purpose of this thesis is to investigate the effective mass of ϕ in the RD era by using thermal field theory which is the most reliable technique for treating the finite temperature system [33–35]. As a result, we for the first time clarify the magnitude of the effective mass of ϕ in the RD era, which is of the order of the Hubble scale times the coupling constants of the thermalized fields.

The rest of this thesis is organized as follows. In Chap. 2, we briefly overview the RD era after inflation. In particular, we will see the standard picture of the cosmic reheating process. Then in Chap. 3, we review thermal field theory. There, we see how the Green functions are defined in the so-called imaginary-time and real-time formalism. We also summarize the self-energy, dispersion relation and free energy for a scalar field and QED fields. In Chap. 4, we summarize the form of the kinetic terms and interaction terms in the supergravity framework. Then, we briefly review the effects of the Hubble scale effective mass in the inflaton dominated era. In Chap. 5, we review what has been done in the previous studies [31,32] for the evaluation of the effective mass of the Planck-suppressed interacting scalar field ϕ. Here, we point out the problem of the previous studies and we attempt to improve the situation by using thermal field theory. However, there are some difficulties in the procedure. To overcome the difficulties, in Chap. 6, we propose a solid and more transparent procedure for the evaluation of the effective mass in which we have only to evaluate the free energy density of the system. As a demonstration, we apply this improved method to the case in which the thermal bath consists of the MSSM particles. Finally, Chap. 7 is devoted to conclusions.

Chapters 5 and 6 are based on Refs. [33,34] and Ref. [35], respectively.

Notation

In this thesis, we adopt the following notations:

- The signs of metric $g_{\mu\nu}$ are defined as $(+---)$.
- For coordinate vectors and momentums, Greek indices $\mu, \nu, \cdots$ represent 4-dimensional spacetime coordinates, running over $(0,1,2,3)$ in the Minkowski spacetime or $(4,1,2,3)$ in the Euclidean spacetime. On the other hand, Latin indices $i, j, \cdots$ represent 3-

dimensional space coordinate, $(1, 2, 3)$.

- We use the natural unit $\hbar = c = k_B = 1$.
- ϵ is a positive infinitesimal parameter $\epsilon \to +0$.

Chapter 2

The early radiation dominated Universe

In this chapter, we briefly review the standard picture of the early Universe [36] in particular the cosmic reheating after inflation. The overview of the thermal history of the Universe is as follows. Let us start with the epoch of the cosmic reheating after inflation where the inflaton converts its energy to the radiation. Then the Universe is dominated by the cosmic plasma with temperature T which decreases as the Universe expands. When the temperature drops to the electroweak energy scale, $T \sim \mathcal{O}(100)$ GeV, the electroweak phase transition occurs and the standard model particles acquire the zero-temperature masses by the Higgs mechanism. Furthermore, when the temperature becomes the QCD energy scale, $T \sim \mathcal{O}(100)$ MeV, the QCD phase transition occurs and the quarks and gluons become confined in hadrons. At the temperature $T \sim 1$ MeV, neutrinos decouple from the photon thermal bath and then electron annihilates. After that the Big Bang Nucleosynthesis occurs. Finally, at the temperature $T \sim 1$ eV, the RD era ends (the matter-radiation equality) and the matter dominated (MD) era begins. The recombination takes place at the temperature $T \sim 0.1$ eV.

The rest of this chapter is organized as follows. In Sec. 2.1, we summarize the Friedman-Robertson-Walker (FRW) Universe. Then in Sec. 2.2, we see the standard picture of the cosmic reheating after inflation. We will derive the reheating temperature which is the initial condition for the RD era.

2.1 The flat Universe with the FRW metric

The dynamics of the Universe is governed by the Einstein equation:

$$\mathcal{R}_{\mu\nu} - g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu}, \tag{2.1}$$

where $\mathcal{R}_{\mu\nu}$ is the Ricci tensor, $\mathcal{R}$ is the Ricci scalar , $g_{\mu\nu}$ is the metric, G is the Newton constant and $T_{\mu\nu}$ is the energy-momentum tensor. Here, we consider the flat Universe with the FRW metric and thus the line element is given by

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\mathrm{d}\mathbf{x}^2, \tag{2.2}$$

where $a(t)$ is the scale factor. In this case, the $(0,0)$ component of the Einstein equation (2.1) leads to the following Friedman equation:

$$H^2 = \frac{1}{3M_\mathrm{P}^2}\rho, \tag{2.3}$$

where H is the Hubble parameter, $M_\mathrm{P} \simeq 2.4 \times 10^{18}$ GeV is the reduced Planck mass and ρ is the total energy density of the Universe. Since we assume that the Universe is homogeneous and isotropic, the energy-momentum tensor of the Universe is taken to be the perfect fluid form: $T^\mu{}_\nu = \mathrm{diag}(\rho, -p, -p, -p)$ (p is the pressure). Thus, the energy conservation law $T^{0\nu}{}_{;\nu} = 0$ (the symbol " ; " represents the covariant derivative) leads to the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -3H(1+w)\rho, \tag{2.4}$$

where w represents the equation of state for the energy density and the pressure: $p = w\rho$. Note that Eq. (2.4) is equivalent to the first law of thermodynamics with $\mathrm{d}S = 0$ (S is the total entropy):

$$\mathrm{d}(\rho a^3) = -p\mathrm{d}(a^3). \tag{2.5}$$

(The other conservation law $T^{i\nu}{}_{;\nu} = 0$ is automatically satisfied for the FRW metric and the perfect fluid form of T^μ_ν.) From Eq. (2.4), if w is constant, the total energy density ρ scales as

$$\rho \propto a^{-3(1+w)}. \tag{2.6}$$

Corresponding to the energy dominating the Universe, w takes the following values:

$$w = \begin{cases} 1/3 & \text{(radiation-dominated)}, \\ 0 & \text{(matter-dominated)}, \\ -1 & \text{(vacuum energy-dominated)}. \end{cases} \tag{2.7}$$

Thus, from Eq. (2.6), we obtain the scaling law of ρ as

$$\rho \propto \begin{cases} a^{-4}, \ a \propto t^{1/2} & \text{(radiation-dominated)}, \\ a^{-3}, \ a \propto t^{2/3} & \text{(matter-dominated)}, \\ a^0\ , \ a \propto \mathrm{e}^{Ht} & \text{(vacuum energy-dominated)}, \end{cases} \tag{2.8}$$

2.2 The cosmic reheating after inflation

The standard picture of the cosmic reheating after inflation is as follows [36]. After inflation, the inflaon field oscillates around the potential minimum. We here consider the case in which the inflaton oscillates with the quadratic potential around the potential minimum. In this case, the equation of state for the inflaton is $w = 0$ and thus behaves like a pressureless matter. The inflaton dissipates its energy through the decay process into the radiation. Below, we neglect the back reaction by the radiation, namely the scattering process between the inflaon and the radiation. From the energy conservation and the Friedman equation, the evolution equations for ρ_I and ρ_r (the energy densities of the inflaton and radiation, respectively) are given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\rho_I &= -3H\rho_I - \Gamma_I \rho_I, \\ \frac{\mathrm{d}}{\mathrm{d}t}\rho_r &= -4H\rho_r + \Gamma_I \rho_I, \\ H^2 &= \frac{1}{3M_{\mathrm{P}}^2}\left(\rho_I + \rho_r\right), \end{aligned} \tag{2.9}$$

where Γ_I is the inflaton decay rate (here we assume it as a constant). Note that Eq. (2.9) assumes that the decay products of the inflaton thermalize instantaneously and form the radiation. Thus, the energy density of the radiation ρ_r has the following form determined by the thermodynamics:

$$\rho_r = \frac{\pi^2 g_*}{30} T^4, \tag{2.10}$$

where g_* is the relativistic degrees of freedom in the thermal bath and T is the temperature of the radiation.

In the sudden decay approximation, the inflaton is assumed to decay suddenly at $H = \Gamma_I$ and inflaton energy is completely converted to the radiation energy. In this case, the relation $3H^2 M_{\mathrm{P}}^2 = \rho_r$ gives the following reheating temperature T_{RH}:

$$T_{\mathrm{RH}} = \left(\frac{90}{\pi^2 g_*}\right)^{1/4} \sqrt{M_{\mathrm{P}} \Gamma_I}. \tag{2.11}$$

Considering the very early times in which the inflaton coherent oscillation dominates the Universe (the MD era), we can estimate the maximum temperature as following. The formal solution to Eq. (2.9) is given by

$$\begin{aligned} \rho_I(t) &= \rho_I(t_0)\left(\frac{a(t_0)}{a(t)}\right)^3 \mathrm{e}^{-\Gamma_I (t-t_0)}, \\ \rho_r(t) &= \rho_r(t_0)\left(\frac{a(t_0)}{a(t)}\right)^4 + \Gamma_I \rho_I(t_0)\left(\frac{a(t_0)}{a(t)}\right)^4 \int_{t_0}^{t} \mathrm{d}t' \left(\frac{a(t')}{a(t_0)}\right) \mathrm{e}^{-\Gamma_I (t'-t_0)}, \end{aligned} \tag{2.12}$$

where t_0 is an initial time. Since we consider the MD era with $\Gamma_I(t - t_0) \ll 1$, the scale factor changes as $a(t) \propto t^{2/3}$ and we obtain

$$\rho_r(t) \simeq \Gamma_I \rho_I(t_0) \left(\frac{t_0}{t}\right)^{8/3} t \left(\frac{t}{t_0}\right)^{2/3} = \frac{\Gamma_I \rho_I(t_0) t_0^2}{t}. \tag{2.13}$$

Here, we have assumed that the radiation does not exist at the initial time ($\rho_r(t_0) = 0$). Now, if we write $\rho_I(t_0) = M^4$ and $t \simeq t_0 \simeq \frac{2}{3H} \simeq \frac{2M_{\rm P}}{\sqrt{3}M^2}$, we obtain

$$\rho_r(t) \simeq \frac{2}{\sqrt{3}} M^2 \Gamma_I M_{\rm P} = \sqrt{\frac{2\pi^2 g_*}{45}} M^2 T_{\rm RH}^2 \tag{2.14}$$

for the very early times. Thus, using $\rho_r = (\pi^2 g_*/30)\ T_{\rm max}^4$, the maximum temperature $T_{\rm max}$ is estimated as

$$T_{\rm max} \simeq \left(\frac{40}{\pi^2 g_*}\right)^{1/8} \sqrt{M T_{\rm RH}}. \tag{2.15}$$

Since $M > (\pi^2 g_*/30)^{1/4}\ T_{\rm RH}$, $T_{\rm max} > T_{\rm RH}$ is verified. From this result, we can see that even if Γ_I is suppressed and thus $T_{\rm RH}$ is low, the Universe experiences much higher temperature $T_{\rm max}$.

Before we close this chapter, let us comment on the studies beyond the above simple picture of the cosmic reheating. In Refs. [37–40], the non-perturbative decay of the inflaton, which is called preheating, is studied. In Refs. [16–22], the thermal dissipation effect to the reheating process is investigated. These effects are very interesting itself and important to the reheating process once we want to look it closely. However, we do not pursue these subjects in this thesis and we simply take Eq. (2.11) as the reheating temperature of the Universe which is the highest temperature in the RD era.

Chapter 3

Review of thermal field theory

In this chapter, we review thermal field theory. (See for example Refs. [10–12].) The most interesting feature of thermal field theory is that there is a typical energy scale, *i.e.*, the temperature T. The existence of the temperature T and thus thermal effects make differences from the zero-temperature field theory. For example, the dispersion relation of a thermalized field generically has complicated structure, fields acquire effective masses characterized by T, the least-*free energy* state is realized instead of the least-*energy* state and so on. We will see the basis of the thermal field theory below and we will use the techniques in later chapters.

The organization of this chapter is as follows. In Sec. 3.1, we introduce the Green function and the spectral function which are the most important ingredients in thermal field theory. Also, we will see two kinds of formalism which are necessary for concrete calculations: the imaginary-time (Matsubara) formalism and the real-time formalism. Then, in Sec. 3.2, we investigate the property of the self-energy. In particular, we carefully consider the analytic continuation of the self-energy. We also calculate the self-energies of a real scalar field and QED fields. In Sec. 3.3, we derive the dispersion relations for a real scalar field and QED fields. Finally, in Sec. 3.4, we evaluate the free energies for a yukawa plasma and a QED plasma, which will be the basis for a later chapter.

3.1 Elements of thermal field theory

3.1.1 Path-integral formulation of thermal field theory

In this subsection, we formulate thermal field theory of a real scalar field by using the path-integral method. Assigning the time-contour of the path-integral, we will introduce both the imaginary-time formalism and real-time formalism. For a fermionic field, we briefly derive the properties of propagators in Appendix. B.

A physical quantity A in a finite temperature system is defined as the thermal average:

$$\langle \hat{A} \rangle \equiv \frac{1}{Z} \mathrm{tr} \left(\mathrm{e}^{-\beta \hat{\mathcal{H}}} \hat{A} \right), \tag{3.1}$$

where the quantities with ˆ are the quantum operators, $\beta = 1/T$ is the inverse temperature, $\hat{\mathcal{H}}$ is the Hamiltonian of the system and Z is the partition function given by

$$Z = \mathrm{tr} \left(\mathrm{e}^{-\beta \hat{\mathcal{H}}} \right). \tag{3.2}$$

Here and hereafter, we neglect the chemical potential for simplicity. As for the thermalized system of a real scalar field φ, using the complete set of state $\{|\varphi(\mathbf{x}), t_i\rangle\}$ at an initial time t_i, the partition function Z can be written as

$$\begin{aligned} Z &= \int \Pi_{\mathbf{x}} \mathrm{d}\varphi(\mathbf{x}) \ \langle \varphi(\mathbf{x}), t_i | \ \mathrm{e}^{-\beta \hat{\mathcal{H}}} \ | \varphi(\mathbf{x}), t_i \rangle \\ &= \int \Pi_{\mathbf{x}} \mathrm{d}\varphi(\mathbf{x}) \ \langle \varphi(\mathbf{x}), t_i - i\beta | \varphi(\mathbf{x}), t_i \rangle \\ &= \mathcal{N} \int_C \Pi_{t,\mathbf{x}} \mathrm{d}\varphi(t, \mathbf{x}) \ \exp \left\{ i \int_C \mathrm{d}^4x \ \mathcal{L} \right\} \Big|_{\text{periodic}} \\ &\equiv \int_C \mathcal{D}\varphi \ \exp \left\{ i \int_C \mathrm{d}^4x \ \mathcal{L} \right\} \Big|_{\text{periodic}}, \end{aligned} \tag{3.3}$$

where $|\varphi(\mathbf{x}), t\rangle$ $(= \mathrm{e}^{i\hat{\mathcal{H}}t} |\varphi(\mathbf{x}), 0\rangle)$ is the eigenstate of the Heisenberg field operator $\hat{\varphi}(t, \mathbf{x})$. In the second line in Eq. (3.3), we have identified $\mathrm{e}^{-\beta \hat{\mathcal{H}}}$ as the (inverse) time evolution operator $\mathrm{e}^{-i\hat{\mathcal{H}}t}$ with the imaginary time $t = -i\beta$. Also, we have expressed the transition matrix $\langle \varphi(\mathbf{x}), t_i - i\beta | \varphi(\mathbf{x}), t_i \rangle$ by the path-integral in the third line. $\mathcal{N}$ is a constant and $\mathcal{L}$ is the Lagrangian of the system. C is a time-contour in the complex time plain, which starts from an initial time $t = t_i$ and ends at a final time $t = t_i - i\beta$. The above path-integral must satisfy the periodic boundary condition [41, 42]:

$$\varphi(t = t_i - i\beta, \mathbf{x}) = \varphi(t = t_i, \mathbf{x}), \tag{3.4}$$

which is shown in Eq. (3.3) by "periodic". This is due to the fact that the trace in Eq. (3.2) sandwiches the operator between the same states.

The generating functional for the n-point Green functions $Z_C[J]$, with $Z_C[0] = Z$, is given by

$$\begin{aligned} Z_C[J] &= \int_C \mathcal{D}\varphi \ \exp \left\{ i \int_C \mathrm{d}^4x \ (\mathcal{L} + J\varphi) \right\} \Big|_{\text{periodic}} \\ &= Z \ \langle \hat{T}_C \ \exp \left\{ i \int_C \mathrm{d}^4x \ J(x) \hat{\varphi}(x) \right\} \rangle, \end{aligned} \tag{3.5}$$

where $\hat{T}_C$ is the time ordering operator along the time-contour C and $J(x)$ is a classical source function. Note that the full propagator $D_C(x - x')$ (here $x, x' \in C$ and we write

$D_C(x,x') = D_C(x-x')$ by the translational invariance of the thermal equilibrium system) is given by the functional derivative of $Z[J]$:

$$\begin{aligned} D_C(x-x') &= \frac{1}{Z}\frac{\delta^2 Z_C[J]}{i\delta J(x) i\delta J(x')}\bigg|_{J=0} \\ &= \frac{1}{Z}\int_C \mathcal{D}\varphi\ \varphi(x)\varphi(x')\ \exp\left\{i\int_C \mathrm{d}^4x\ \mathcal{L}\right\} \\ &= \langle \hat{T}_C\ \hat{\varphi}(x)\hat{\varphi}(x')\rangle. \end{aligned} \tag{3.6}$$

Thus, we can evaluate the full propagator $D_C(x-x')$ by the thermal averaged two-point function. Note that, from Eq. (3.4), the full propagator $D_C(x-x')$ satisfies the periodic boundary condition [41, 42]:

$$D_C(x-i\beta) = D_C(x). \tag{3.7}$$

Writing the Lagrangian as $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + \mathcal{L}_{\mathrm{int}}(\varphi)$, the generating functional $Z_C[J]$ can be written as

$$Z_C[J] = \exp\left\{i\int_C \mathrm{d}^4x\ \mathcal{L}_{\mathrm{int}}\left(\frac{\delta}{i\delta J}\right)\right\} Z_C^F[J]. \tag{3.8}$$

Here, $Z_C^F[J]$, with $Z_C^F[0] = Z^F$ (the free partition function), is the free generating functional given by

$$\begin{aligned} Z_C^F[J] &= \int_C \mathcal{D}\varphi\ \exp\left\{i\int_C \mathrm{d}^4x\ \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + J\varphi\right)\right\}\bigg|_{\mathrm{periodic}} \\ &= Z^F\ \exp\left\{\int_C \mathrm{d}^4x\ \int_C \mathrm{d}^4x'\ \frac{-1}{2}J(x)D_C^F(x-x')J(x')\right\}, \end{aligned} \tag{3.9}$$

where $D_C^F(x-x')$ is the free propagator given by

$$D_C^F(x-x') = \frac{1}{Z^F}\frac{\delta^2 Z_C^F[J]}{i\delta J(x) i\delta J(x')}\bigg|_{J=0}. \tag{3.10}$$

Note that, from Eq. (3.4), the free propagator $D_C^F(x-x')$ satisfies the periodic boundary condition [41, 42]:

$$D_C^F(x-i\beta) = D_C^F(x). \tag{3.11}$$

In the following two subsections, we briefly summarize two formalisms, *i.e.*, the imaginary-time and real-time formalisms. In the imaginary-time formalism, the time-contour C is taken to be the simplest one: $C = [0, -i\beta]$ along the imaginary axis. On the other hand, in the real-time formalism, the time-contour C is taken to be more complicated one which includes the real axis.

3.1.2 Imaginary-time formalism

In this subsection, we take the contour as $C_I = [0, -i\beta]$ along the imaginary axis. This is the case for the so-called imaginary-time (Matsubara) formalism [43]. In this formalism, the time coordinate is taken to be pure imaginary and the metric is now the Euclidean one:

$$
\begin{aligned}
x_0 &= -ix_4 \quad (0 \leq x_4 \leq \beta), \\
g_{\mu\nu} &= -\delta_{\mu\nu} \quad (\mu, \nu = 4, 1, 2, 3).
\end{aligned}
\tag{3.12}
$$

Also, we denote the coordinate four-vector in the Euclidean spacetime as

$$
x_{E\mu} = (x_4, \mathbf{x}). \tag{3.13}
$$

Accordingly, we have

$$
\begin{aligned}
\int_{C_I} \mathrm{d}^4x &= -i \int_0^\beta \mathrm{d}^4x_E = -i \int_0^\beta \mathrm{d}x_4 \int \mathrm{d}^3\mathbf{x}, \\
\{\gamma_\mu, \gamma_\nu\} &= -2\delta_{\mu\nu} \quad (\mu, \nu = 4, 1, 2, 3), \\
\gamma_0 &= -i\gamma_4,
\end{aligned}
\tag{3.14}
$$

where γ_μ is the Dirac gamma matrix.

Below, we consider a real scalar field φ as the simplest example. For other types of field like fermion or gauge field, the basic procedure is almost the same as the one in the real scalar field case. First of all, let us write down the imaginary-time action for the real scalar field φ. The action for φ in the Minkowski spacetime is given by

$$
\begin{aligned}
iS &= i \int \mathrm{d}^4x \ \mathcal{L}(\varphi) \\
&= i \int \mathrm{d}^4x \ \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + \mathcal{L}_{\text{int}}(\varphi) \right).
\end{aligned}
\tag{3.15}
$$

Here, we assume that the interaction term $\mathcal{L}_{\text{int}}(\varphi)$ does not include derivative interactions. In order to go to the imaginary-time formalism, we need to replace the time coordinate as $x^0 \to -ix_4$ and

$$
\partial_\mu\varphi\partial^\mu\varphi \to -\partial_\mu\varphi\partial_\mu\varphi. \tag{3.16}
$$

Thus, the transition of the action (3.15) to the imaginary-time formalism one, S_E, is as follows

$$
\begin{aligned}
iS &\to i(-i) \int_0^\beta \mathrm{d}^4x_E \ \left(-\frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 + \mathcal{L}_{\text{int}}(\varphi) \right) \\
&= -\int_0^\beta \mathrm{d}^4x_E \ \left(\frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi + \frac{1}{2}m^2\varphi^2 - \mathcal{L}_{\text{int}}(\varphi) \right) \\
&\equiv -S_E.
\end{aligned}
\tag{3.17}
$$

The generating functional for the n-point Green functions in the imaginary-time formalism $Z_{C_I}[J]$, with $Z_{C_I}[0] = Z$, is given by

$$\begin{aligned} Z_{C_I}[J] &= \int_{C_I} \mathcal{D}\varphi \exp\left\{ -\int_0^\beta \mathrm{d}^4 x_E \; (\mathcal{L}(\varphi) - J\varphi) \right\} \Big|_{\text{periodic}} \\ &= \exp\left\{ \int_0^\beta \mathrm{d}^4 x_E \; \mathcal{L}_{\text{int}} \left(\frac{\delta}{\delta J} \right) \right\} Z_{C_I}^{(F)}[J], \end{aligned} \tag{3.18}$$

where $Z_{C_I}^F[J]$ is the free generating functional with the time-contour C_I as given in Eq. (3.9). The propagator in the imaginary-time formalism, $\Delta(x_E - x'_E)$ (here we use the short-hand notation $\Delta(x_E) = \Delta(x_4, \mathbf{x})$), can be obtained from the functional derivative of $Z_{C_I}[J]$ as

$$\begin{aligned} \Delta(x_E - x'_E) &= \frac{1}{Z} \frac{\delta^2 Z_{C_I}[J]}{\delta J(x_E) \delta J(x'_E)} \Bigg|_{J=0} \\ &= \frac{1}{Z} \int_{C_I} \mathcal{D}\varphi \; \varphi(x_E)\varphi(x'_E) \exp\{-S_E\} \\ &= \langle \hat{T}_{C_I} \; \hat{\varphi}(x_E)\hat{\varphi}(x'_E) \rangle. \end{aligned} \tag{3.19}$$

From the generating functional (3.18), we can obtain the Feynman rule in the imaginary-time formalism which we summarize in Appendix C. The advantage of this formalism is that the calculation is simple. However, since the energy is pure imaginary in this formalism, respecting the periodic (anti-periodic for fermions) boundary condition (3.4), we have to make analytic continuation for physical quantities in order to have the real energy.

Before we go to the real-time formalism, let us discuss more about the propagator in the imaginary-time formalism, $\Delta(x_E - x'_E)$, for later convenience. From Eq. (3.19), we have

$$\begin{aligned} \Delta(x_E - x'_E) &= \langle \hat{T}_{C_I} \; \hat{\varphi}(x_E)\hat{\varphi}(x'_E) \rangle \\ &= \theta_{C_I}(x_4 - x'_4) \; \Delta^{>}(x_E - x'_E) + \theta_{C_I}(x'_4 - x_4) \; \Delta^{<}(x_E - x'_E), \end{aligned} \tag{3.20}$$

where $\theta_{C_I}(x_4 - x'_4)$ is the step function on the time-contour C_I. Here, we have defined $\Delta^{>(<)}(x_E - x'_E)$ as

$$\begin{aligned} \Delta^{>}(x_E - x'_E) &= \langle \hat{\varphi}(x_E)\hat{\varphi}(x'_E) \rangle, \\ \Delta^{<}(x_E - x'_E) &= \langle \hat{\varphi}(x'_E)\hat{\varphi}(x_E) \rangle. \end{aligned} \tag{3.21}$$

Note that, from Eq. (3.4), the full propagator in the imaginary-time formalism, $\Delta(x_E)$, satisfies the periodic boundary condition [41, 42]:

$$\Delta(x_E + \beta) = \Delta(x_E), \tag{3.22}$$

which is a special case of Eq. (3.7). For taking into account of the periodicity (3.22), we have only to represent $\Delta(x_E)$ by the Fourier series and integral:

$$\Delta(x_E) = T \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \mathrm{e}^{+iK_\mu x_{E\mu}} \Delta(i\omega_n, \mathbf{k}), \quad k_4 = -\omega_n = -\frac{2n\pi}{\beta}, \tag{3.23}$$

where n runs all integers ($n = \cdots, -1, 0, 1, \cdots$). Here and hereafter, we denote the Euclidean four-momentum by capital letter as[3-1]

$$K_\mu = (k_4, \mathbf{k}) = (-\omega_n, \mathbf{k}) \tag{3.24}$$

and $K_\mu x_{E\mu} = k_4 x_4 + \mathbf{k} \cdot \mathbf{x} = -\omega_n x_4 + \mathbf{k} \cdot \mathbf{x}$ (we have defined $x_{E\mu} = (x_4, \mathbf{x})$), while we use small letter like k_μ for the four-momentum in the Minkowski spacetime. The inverse transformation of Eq. (3.23) is given by

$$\Delta(K) = \Delta(i\omega_n, \mathbf{k}) = \int_0^\beta \mathrm{d}x_4 \int \mathrm{d}^3\mathbf{x}\ \mathrm{e}^{-iK_\mu x_{E\mu}} \Delta(x_E). \tag{3.25}$$

Here and hereafter, we use the notation $\Delta(K) = \Delta(i\omega_n, \mathbf{k})$.

The discussion so far is valid for the full propagator in the imaginary-time formalism, $\Delta(x_E)$. Before we go, let us see the consequence for the free propagator $\Delta^F(x_E)$. (Below, we use the superscript " F " for the functions of free fields.) The equation of motion for the free propagator is given by

$$(-\partial_{x\mu}\partial_{x\mu} + m^2)\Delta^F(x_E - x'_E) = \delta^{(4)}(x_E - x'_E). \tag{3.26}$$

From Eqs. (3.23) and (3.26), the Fourier component $\Delta^F(i\omega_n, \mathbf{k})$ satisfies the following equation:

$$(K^2 + m^2)\Delta^F(i\omega_n, \mathbf{k}) = 1, \tag{3.27}$$

where $K^2 = K_\mu K_\mu = k_4^2 + \mathbf{k}^2 = \omega_n^2 + \mathbf{k}^2$. Thus, we arrive at the following free propagator in the imaginary-time formalism:

$$\Delta^F(K) = \frac{1}{K^2 + m^2} = \frac{1}{\omega_n^2 + \mathbf{k}^2 + m^2}. \tag{3.28}$$

3.1.3 Real-time formalism

In this subsection, we take the time-contour C to be more complicated than in the imaginary-time formalism. Namely, we take $C = C_R = C_1 \cup C_2 \cup C_3 \cup C_4$, where $C_1 = [t_i = -\infty, t_f = +\infty]$ along the real axis, $C_2 = [+\infty - i\sigma, -\infty - i\sigma]$ along the horizontal straight

[3-1]The minus sign of $-\omega_n$ is just a convention.

line, $C_3 = [+\infty, +\infty - i\sigma]$ along the vertical straight line and $C_4 = [-\infty - i\sigma, -\infty - i\beta]$ along the vertical straight line (here, $0 \le \sigma \le \beta$). This is the standard choice of the path in the so-called real-time formalism. Note that we have chosen the initial and the final time as $t_i = -\infty$ and $t_f = +\infty$, respectively. In this formalism, the time coordinate is allowed to be complex, but real time actually has the major role. Also, we work with the Minkowski metric in this formalism.

Below, we consider the real scalar field φ as the simplest example. For other types of field like fermion or gauge field, the basic procedure is almost the same as the one in the real scalar field case. First, let us write down the action for φ in the real-time formalism:

$$\begin{aligned} iS &= i\int_{C_R} \mathrm{d}^4x \; \mathcal{L}(\varphi) \\ &= i\int_{C_R} \mathrm{d}^4x \; \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + \mathcal{L}_{\mathrm{int}}(\varphi)\right), \end{aligned} \tag{3.29}$$

where we assume that $\mathcal{L}_{\mathrm{int}}(\varphi)$ does not include derivative interactions. The generating functional for the n-point Green functions in the real-time formalism is given by

$$\begin{aligned} Z_{C_R}[J] &= \int_{C_R} \mathcal{D}\varphi \; \exp\left\{i\int_{C_R} \mathrm{d}^4x \; (\mathcal{L} + J\varphi)\right\}\Big|_{\mathrm{periodic}} \\ &= \exp\left\{i\int_{C_R} \mathrm{d}^4x \; \mathcal{L}_{\mathrm{int}}\left(\frac{\delta}{i\delta J}\right)\right\} Z^F_{C_R}[J]. \end{aligned} \tag{3.30}$$

Here, $Z^F_{C_R}[J]$ is the free generating functional given by

$$\begin{aligned} Z^F_{C_R}[J] &= \int_{C_R} \mathcal{D}\varphi \; \exp\left\{i\int_{C_R} \mathrm{d}^4x \; \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + J\varphi\right)\right\}\Big|_{\mathrm{periodic}} \\ &= Z^F \; \exp\left\{\int_{C_R} \mathrm{d}^4x \; \int_{C_R} \mathrm{d}^4x' \; \frac{-1}{2} J(x) D^F_{C_R}(x-x') J(x')\right\}, \end{aligned} \tag{3.31}$$

where Z^F is the free partition function.

In order to obtain the relevant form of the full generating functional $Z_{C_R}[J]$ for the Green functions with real time, let us consider the free generating functional $Z^F_{C_R}[J]$ for a moment. Since we have chosen the initial and the final time as $t_i = -\infty$ and $t_f = +\infty$, respectively, the free propagator $D^F_{C_R}(x-x') = \int_{-\infty}^{\infty} \frac{\mathrm{d}k_0}{2\pi} \int \frac{\mathrm{d}^3\mathrm{k}}{(2\pi)^3} \; \mathrm{e}^{-ik\cdot(x-x')} D^F_{C_R}(k)$ drops for $x_0 \in C_{12}$, $x'_0 \in C_{34}$ (here $C_{ij} = C_i \cup C_j$) and $x_0 \in C_{34}$, $x'_0 \in C_{12}$, which is the consequence of the Riemann-Lebesgue lemma [10][3-2]. Here, we have used the short-hand notation: $k\cdot(x-x') = k_0(x_0 - x'_0) - \mathbf{k}\cdot(\mathbf{x} - \mathbf{x}')$. Thus we can separate the free generating functional as

$$Z^F_{C_R}[J] = Z^F_{C_{12}}[J] \times Z^F_{C_{34}}[J], \tag{3.32}$$

[3-2] Here, $D^F_{C_R}(k)$ needs to be integrable. Also, for $x_0 \in C_{12}(C_{34})$ and $x'_0 \in C_{34}(C_{12})$, $x_0 \neq \pm\infty$ is needed. In this case, we require $J(x) \to 0$ ($x_0 \to \pm\infty$).

where $Z^F_{C_{ij}}[J]$ is given by

$$Z^F_{C_{ij}}[J] = \mathcal{N}_{C_{ij}} \exp\left\{\int_{C_{ij}} \mathrm{d}^4x \int_{C_{ij}} \mathrm{d}^4x' \, \frac{-1}{2} J(x) D^F_{C_{ij}}(x-x') J(x')\right\}. \tag{3.33}$$

Here, $\mathcal{N}_{C_{ij}}$ is a constant. Since we are interested in the Green functions which have the time coordinates on the real axis C_1, $Z^F_{C_{34}}$ can be considered as a multiple constant in Eq. (3.32):

$$Z^F_{C_R}[J] = \mathcal{N}' \, Z^F_{C_{12}}[J]. \tag{3.34}$$

where N' is a irrelevant constant for $Z_{C_{12}}[J]$ (namely, $J(x)$ here lives only on C_{12}).

Now, let us return to our subject. From Eqs. (3.30) and (3.34), we effectively have the following full generating functional which contains real time:

$$\begin{aligned} Z_{C_R}[J] &= \mathcal{N}' \exp\left\{i \int_{C_{12}} \mathrm{d}^4x \, \mathcal{L}_{\mathrm{int}}\left(\frac{\delta}{i\delta J}\right)\right\} Z^F_{C_{12}}[J] \\ &= \int \mathcal{D}\varphi_1 \mathcal{D}\varphi_2 \exp\left\{i \int_{-\infty}^{\infty} \mathrm{d}^4x \, (\mathcal{L}[\varphi_1] - \mathcal{L}[\varphi_2] + J_1\varphi_1 - J_2\varphi_2)\right\}\Big|_{\mathrm{periodic}} \\ &= Z \, \langle \hat{T}_{C_{12}} \exp\left\{i \int_{C_{12}} \mathrm{d}^4x \, J(x)\hat{\varphi}(x)\right\}\rangle, \end{aligned} \tag{3.35}$$

where, we have used the short-hand notations $J_1 = J(x_0, \mathbf{x})$, $J_2 = J(x_0 - i\sigma, \mathbf{x})$ (we use the same notation for φ_i) and $\int_{-\infty}^{\infty} \mathrm{d}^4x = \int_{-\infty}^{\infty} \mathrm{d}x_0 \int \mathrm{d}^3\mathbf{x}$. From Eq. (3.35), the full propagator in the real-time formalism, $D_{ij}(x-x')$, is now given by

$$\begin{aligned} D_{ij}(x-x') &= \frac{1}{Z} \frac{\delta^2 Z_{C_R}[J]}{i\delta J_i(x) i\delta J_j(x')}\Bigg|_{J=0} \\ &= \begin{pmatrix} D_{11}(x-x') & D_{12}(x-x') \\ D_{21}(x-x') & D_{22}(x-x') \end{pmatrix} \\ &= \begin{pmatrix} \langle \hat{T} \, \hat{\varphi}(x)\hat{\varphi}(x')\rangle & \langle \hat{\varphi}(x')\hat{\varphi}(x)\rangle \\ \langle \hat{\varphi}(x)\hat{\varphi}(x')\rangle & \langle \hat{\bar{T}}\hat{\varphi}(x)\hat{\varphi}(x')\rangle \end{pmatrix}, \end{aligned} \tag{3.36}$$

where $\hat{T}$ and $\hat{\bar{T}}$ are the time-ordering and anti time-ordering operators, respectively, and we have used the matrix notation (the (i,j) component corresponds to $x_0 \in C_i$, $x'_0 \in C_j$ $(i,j = 1,2)$). We have also used Eq. (3.6) in the last line. Thus, we have four types of propagator in the real-time formalism. For the convenience in the next subsection, let us write down the full propagator $D_{ij}(x-x')$ without the matrix form:

$$\begin{aligned} D_{ij}(x-x') &= \langle \hat{T}_{C_{12}} \hat{\varphi}(x)\hat{\varphi}(x')\rangle \\ &= \theta_{C_{12}}(x_0 - x'_0)\langle \hat{\varphi}(x)\hat{\varphi}(x')\rangle + \theta_{C_{12}}(x'_0 - x_0)\langle \hat{\varphi}(x')\hat{\varphi}(x)\rangle \\ &= \theta_{C_{12}}(x_0 - x'_0) D^{>}(x-x') + \theta_{C_{12}}(x'_0 - x_0) D^{<}(x-x'), \end{aligned} \tag{3.37}$$

where $\theta_{C_{12}}(x_0-x_0')$ is the step function on the time-contour C_{12}, namely, $\theta_{C_{12}}(x_0-x_0') = 1$ (when x_0 is after x_0' on C_{12}) and $\theta_{C_{12}}(x_0 - x_0') = 0$ (when x_0 is before x_0' on C_{12}). Also, we have defined $D^{>(<)}(x - x')$ as

$$\begin{aligned} D^{>}(x - x') &= \langle \hat{\varphi}(x)\hat{\varphi}(x')\rangle, \\ D^{<}(x - x') &= \langle \hat{\varphi}(x')\hat{\varphi}(x)\rangle. \end{aligned} \tag{3.38}$$

From the generating functional (3.35), we can obtain the Feynman rule in the real-time formalism. The Feynman rule in this formalism is "almost" the same as in the usual zero-temperature formulation (so we do not show it in this thesis). The difference is that we have two types of vertex, $\mathcal{L}_{\rm int}(\varphi_1)$ and $-\mathcal{L}_{\rm int}(\varphi_2)$, since we have now two degrees of freedom φ_1 and φ_2. One of the degrees of freedom φ_2 can be interpreted as an unphysical degrees of freedom living on the contour C_2. This unphysical degrees of freedom inevitably appears when we include the real axis in the time-contour since we have to "come back" along the real axis in the path-integral in this case. We note that the physical quantities have only the φ_1 external lines. The advantage of the real-time formalism is that we do not need any analytic continuation for Green functions which is needed in the imaginary-time formalism to obtain the real energy. However, the price of this formalism is the complexity of the calculation which is originated by the existence of the unphysical degrees of freedom φ_2.

3.1.4 Spectral function

The spectral function $\rho(k)$ of the thermalized field φ is the most important function for describing the state of φ, which is defined by

$$\rho(k) \equiv D^{>}(k) - D^{<}(k). \tag{3.39}$$

Here, $D^{>(<)}(k)$ is defined as the Fourier component of $D^{>(<)}(x-x')$ (defined in Eq. (3.38)) in the following equation:

$$D^{>(<)}(x - x') = \int_{-\infty}^{\infty} \frac{\mathrm{d}k_0}{2\pi} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ \mathrm{e}^{-ik\cdot(x-x')} D^{>(<)}(k), \tag{3.40}$$

where $k \cdot (x - x') = k_0(x_0 - x_0') - \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')$. Note that the spectral function $\rho(k)$ is defined in the real-time formalism. Since the spectral function $\rho(k)$ is nothing but the two-point Green function, $\rho(k)$ contains the information about the state of φ, *i.e.*, the dispersion relation, the width and the residue of the quasi-particle pole of φ and so on (see for example Refs. [11, 12]).

In this subsection, according to Ref. [15], we connect the spectral function $\rho(k)$ defined in Eq. (3.39) to the imaginary-time formalism. First, let us apply the analytic continuation to $D^{>(<)}(x)$ (which is defined by Eq. (3.38)) into the imaginary-time. On the imaginary-time contour C_I, the analytically-continued quantity $D^{>(<)}(x)$ coincides with $\Delta^{>(<)}(x_E)$ (which is defined by Eq. (3.21)) as[3-3]

$$D^{>(<)}(-ix_4, \mathbf{x}) = \Delta^{>(<)}(x_4, \mathbf{x}). \tag{3.41}$$

Furthermore, since the periodic boundary condition (3.22) can be rewritten as $\Delta(x_4-\beta) = \Delta(x_4)$, we have the following equation:

$$\Delta^{>}(x_4, \mathbf{x}) = \Delta^{<}(x_4 - \beta, \mathbf{x}). \tag{3.42}$$

Putting together Eqs. (3.41) and (3.42), and using the analytic continuation $x_4 = ix^0$ (here x^0 is real), we have

$$D^{>}(x^0, \mathbf{x}) = D^{<}(x^0 + i\beta, \mathbf{x}). \tag{3.43}$$

Thus, writing the two-point Green functions $D^{>(<)}$ by the Fourier integral

$$D^{>(<)}(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \mathrm{e}^{-ik\cdot x} D^{>(<)}(k), \tag{3.44}$$

we obtain, from Eq. (3.43), the following important relation between the Fourier components $D^{>(<)}(k)$ as

$$D^{>}(k) = \mathrm{e}^{\beta k_0} D^{<}(k). \tag{3.45}$$

Furthermore, defining a function $D^{+}(k)$[3-4] as

$$D^{+}(k) \equiv \frac{1}{2}(D^{>}(k) + D^{<}(k)) \tag{3.46}$$

and using the definition of the spectral function (3.39) and the relation (3.45), we obtain the following relations:

$$\begin{aligned} D^{<}(k) &= f_B(k_0)\rho(k), \\ D^{>}(k) &= (1 + f_B(k_0))\rho(k), \\ D^{+}(k) &= \frac{1}{2}\coth\left(\frac{\beta k_0}{2}\right)\rho(k), \end{aligned} \tag{3.47}$$

where $f_B(k_0) = 1/(\mathrm{e}^{\beta k_0} - 1)$ is the Bose-Einstein distribution function.

[3-3]Do not confuse the symbol “Δ” with “D”. We have denoted the propagator in the imaginary-time formalism by the symbol “Δ” which should be distinguished from the one in the real-time formalism, “D”. When we have to explicitly show the time-contour C of a propagator, we will denote the propagator by the symbol “D_C” as in Sec. 3.1.1.

[3-4]$D^{+}(k)$ is sometimes called as statistical propagator.

Now, let us write down the full propagator in the imaginary-time formalism, $\Delta(i\omega_n, \mathbf{k})$, in terms of the spectral function $\rho(k)$. Using the inverse Fourier transformation (3.25), the relation $\Delta(x_E) = \Delta^>(x_E)$ and Eqs. (3.41) and (3.44), we obtain

$$\begin{aligned}
\Delta(i\omega_n, \mathbf{k}) &= \int_0^\beta \mathrm{d}x_4 \ \mathrm{e}^{i\omega_n x_4} \ \Delta^>(x_4 - 0, \mathbf{k}) \\
&= \int_0^\beta \mathrm{d}x_4 \ \mathrm{e}^{i\omega_n x_4} \ D^>(-ix_4, \mathbf{k}) \\
&= \int_0^\beta \mathrm{d}x_4 \ \mathrm{e}^{i\omega_n x_4} \int_{-\infty}^{\infty} \frac{\mathrm{d}k_0'}{2\pi} \mathrm{e}^{-ik_0'(-ix_4)} \ D^>(k_0', \mathbf{k}) \\
&= -\int_{-\infty}^{\infty} \frac{\mathrm{d}k_0'}{2\pi} \frac{\rho(k_0', \mathbf{k})}{i\omega_n - k_0'}.
\end{aligned} \tag{3.48}$$

Applying the analytic continuation to Eq. (3.48) as $i\omega_n \to k_0 \pm i\epsilon$, we obtain

$$\begin{aligned}
\Delta(k_0 + i\epsilon, \mathbf{k}) - \Delta(k_0 - i\epsilon, \mathbf{k}) &= -\int_{-\infty}^{\infty} \frac{\mathrm{d}k_0'}{2\pi} \left(\frac{1}{k_0 - k_0' + i\epsilon} - \frac{1}{k_0 - k_0' - i\epsilon} \right) \rho(k_0', \mathbf{k}) \\
&= i\rho(k),
\end{aligned} \tag{3.49}$$

where we have used the relation $\frac{1}{k_0 - k_0' \pm i\epsilon} = \hat{P}\frac{1}{k_0 - k_0'} \mp i\pi\delta(k_0 - k_0')$ ($\hat{P}\frac{1}{k_0 - k_0'}$ is the principal value of $\frac{1}{k_0 - k_0'}$). Thus, we obtain the expression for the spectral function $\rho(k)$, which is defined in the real-time formalism (3.39), by the propagators in the imaginary-time formalism, $\Delta(K)$, which are now analytically-continued ones as

$$\begin{aligned}
\rho(k) &= D^>(k) - D^<(k) \\
&= (-i\Delta(k_0 + i\epsilon, \mathbf{k})) - (-i\Delta(k_0 - i\epsilon, \mathbf{k})).
\end{aligned} \tag{3.50}$$

This is a quite useful equation since, in the thermal equilibrium system, it is often more convenient to evaluate quantities in the imaginary-time formalism than in the real-time one.

Before we go to the next section, let us write down the Fourier components $D_{ij}(k)$ of the real-time propagator $D_{ij}(x - x') = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \ \mathrm{e}^{-ik\cdot(x-x')} D_{ij}(k)$ by using Eq. (3.37). Since the time arguments of $D_{11}(x - x')$ and $D_{22}(x - x')$ are real, we can use the Fourier transform $D_{11(22)}(k) = \int \mathrm{d}^4x \ \mathrm{e}^{ik\cdot(x-x')} D_{11(22)}(x - x')$. Then, using the expression for the step function $\theta(t) = \int_{-\infty}^{\infty} \frac{i\mathrm{d}z}{2\pi} \frac{1}{z + i\epsilon} \mathrm{e}^{-izt}$ for real t, we obtain the expression for $D_{11(22)}(k)$. On the other hand, for $D_{12(21)}(k)$, it is convenient to express $D_{12(21)}(x - y)$ by the Fourier integral and then read off the Fourier component $D_{12(21)}(k)$. The result is as follows

$$
\begin{aligned}
D_{11}(k) &= \int_{-\infty}^{\infty} \frac{i\mathrm{d}k_0'}{2\pi} \frac{\rho(k_0',\mathbf{k})}{k_0 - k_0' + i\epsilon} + f_B(k_0)\rho(k) \\
&= \hat{\mathrm{P}} \int_{-\infty}^{\infty} \frac{i\mathrm{d}k_0'}{2\pi} \frac{\rho(k_0',\mathbf{k})}{k_0 - k_0'} + \left(\frac{1}{2} + f_B(k_0)\right)\rho(k), \\
D_{22}(k) &= \left(D_{11}(k)\right)^*, \\
D_{12}(k) &= \mathrm{e}^{\sigma k_0} f_B(k_0)\rho(k), \\
D_{21}(k) &= \mathrm{e}^{-\sigma k_0}(1 + f_B(k_0))\rho(k).
\end{aligned}
\tag{3.51}
$$

The discussion so far is valid for the full spectral function $\rho(k)$ and the propagators $D_{ij}(k)$ and $\Delta(K)$. Here, let us see the consequences for the free spectral function $\rho^F(k)$ and the free propagator $D_{ij}^F(k)$ in the real-time formalism (the free propagator in the imaginary-time formalism, $\Delta^F(K)$, is given by Eq. (3.28)). First, from Eqs. (3.28) and (3.50), we obtain the free spectral function $\rho^F(k)$ as

$$
\begin{aligned}
\rho^F(k) &= \frac{i}{(k_0 + i\epsilon)^2 - \mathbf{k}^2 - m^2} - \frac{i}{(k_0 - i\epsilon)^2 - \mathbf{k}^2 - m^2} \\
&= 2\pi \ \mathrm{sign}(k_0)\ \delta(k^2 - m^2),
\end{aligned}
\tag{3.52}
$$

where we have used the short-hand notation $k^2 = k\cdot k = (k_0)^2 - \mathbf{k}^2$. Then, from Eqs. (3.51) and (3.52), we obtain the free propagator in the real-time formalism, $D_{ij}^F(k)$, as

$$
\begin{aligned}
D_{11}^F(k) &= \frac{i}{k^2 - m^2 + i\epsilon} + f_B(|k_0|)2\pi\delta(k^2 - m^2), \\
D_{22}^F(k) &= \left(D_{11}(k)\right)^*, \\
D_{12}^F(k) &= \mathrm{e}^{\sigma k_0} f_B(k_0)\rho^F(k), \\
D_{21}^F(k) &= \mathrm{e}^{-\sigma k_0}(1 + f_B(k_0))\rho^F(k).
\end{aligned}
\tag{3.53}
$$

3.2 Self-energy

3.2.1 Analytic continuation of the self-energy

Here, we derive the following important equation for the self-energy:

$$
\bar{\Pi}(k_0, \mathbf{k}) = \Pi(k_0 + i\epsilon k_0, \mathbf{k}),
\tag{3.54}
$$

where $\bar{\Pi}(k_0,\mathbf{k})$ $(= \bar{\Pi}(k))$ is the component of the diagonalized self-energy defined below in the real-time formalism and $\Pi(k_0 + i\epsilon k_0, \mathbf{k})$ is the self-energy analytically-continued from the one in the imaginary-time formalism, $\Pi(i\omega_n, \mathbf{k})$ $(= \Pi(K))$. Namely, $\bar{\Pi}(k_0, \mathbf{k})$ is connected with $\Pi(i\omega_n, \mathbf{k})$ by the analytic continuation $i\omega_n \to k_0 + i\epsilon k_0$. We will explicitly calculate $\Pi(i\omega_n, \mathbf{k})$ for some kinds of field in Secs. 3.2.3, 3.2.4 and 3.2.5.

First, let us consider the Dyson equation in the imaginary-time formalism:

$$\Delta(i\omega_n,\mathbf{k}) = \Delta^F(i\omega_n,\mathbf{k}) + \Delta^F(i\omega_n,\mathbf{k})(-\Pi(i\omega_n,\mathbf{k}))\Delta(i\omega_n,\mathbf{k}), \tag{3.55}$$

where $\Delta(i\omega_n,\mathbf{k})$ and $\Delta^F(i\omega_n,\mathbf{k})$ are the full and the free propagators in the imaginary-time formalism, respectively. From this equation, analytically-continued propagator $D'(k)$ defined by

$$D'(k) \equiv -i\Delta(k_0 + i\epsilon k_0,\mathbf{k}) \tag{3.56}$$

satisfies the following equation:

$$D'(k) = D'^F(k) + D'^F(k)(-i\Pi(k_0 + i\epsilon k_0,\mathbf{k}))D'(k). \tag{3.57}$$

From Eqs. (3.48) and (3.56), the analytically-continued quantity $D'(k)$ can be expressed by the spectral function $\rho(k)$ as

$$\begin{aligned} D'(k) &= \int_{-\infty}^{\infty} \frac{i\mathrm{d}k_0'}{2\pi}\frac{\rho(k_0',\mathbf{k})}{k_0 - k_0' + i\epsilon k_0} \\ &= \hat{\mathrm{P}}\int_{-\infty}^{\infty} \frac{i\mathrm{d}k_0'}{2\pi}\frac{\rho(k_0',\mathbf{k})}{k_0 - k_0'} + \frac{1}{2}\mathrm{sign}(k_0)\rho(k). \end{aligned} \tag{3.58}$$

In particular, for the free propagator $D'^F(k)$, we have the following expression from Eqs. (3.52) and (3.58):

$$D'^F(k) = \frac{i}{k^2 - m^2 + i\epsilon}, \tag{3.59}$$

which is equal to the zero-temperature part of $D_{11}^F(k)$ given in Eq. (3.53).

Now, we note that the matrix form[3-5] of the propagator in the real-time formalism, $\hat{D}(k)$ ($= D_{ij}(k)$ in the component notation), with the symmetric time-path ($\sigma = \beta/2$) can be diagonalized by the symmetric matrix $\hat{U}^{-1}(k)$ given by

$$\hat{U}(k) = \begin{pmatrix} \sqrt{1+f_B(|k_0|)} & \sqrt{f_B(|k_0|)} \\ \sqrt{f_B(|k_0|)} & \sqrt{1+f_B(|k_0|)} \end{pmatrix}, \quad \hat{U}^{-1}(k) = \begin{pmatrix} \sqrt{1+f_B(|k_0|)} & -\sqrt{f_B(|k_0|)} \\ -\sqrt{f_B(|k_0|)} & \sqrt{1+f_B(|k_0|)} \end{pmatrix}. \tag{3.60}$$

[3-5]Here, we show matrix quantities with " hat ". Do not consider it as an operator for which we have also used " hat ".

In fact, from Eqs. (3.51), $\hat{D}(k)$ is diagonalized as follows

$$\begin{aligned}\hat{U}^{-1}(k)\hat{D}(k)\hat{U}^{-1}(k) &= U^{-1}\begin{pmatrix}\hat{\mathrm{P}}\int_{-\infty}^{\infty}\frac{i\mathrm{d}k_0'}{2\pi}\frac{\rho(k_0',\mathbf{k})}{k_0-k_0'} & 0\\ 0 & -\hat{\mathrm{P}}\int_{-\infty}^{\infty}\frac{i\mathrm{d}k_0'}{2\pi}\frac{\rho(k_0',\mathbf{k})}{k_0-k_0'}\end{pmatrix}U^{-1}\\ &\quad + U^{-1}\begin{pmatrix}\left(\frac{1}{2}+f_B(k_0)\right)\rho(k) & \mathrm{e}^{\beta k_0/2}f_B(k_0)\rho(k)\\ \mathrm{e}^{\beta k_0/2}f_B(k_0)\rho(k) & \left(\frac{1}{2}+f_B(k_0)\right)\rho(k)\end{pmatrix}U^{-1}\\ &= \begin{pmatrix}\hat{\mathrm{P}}\int_{-\infty}^{\infty}\frac{i\mathrm{d}k_0'}{2\pi}\frac{\rho(k_0',\mathbf{k})}{k_0-k_0'} & 0\\ 0 & -\hat{\mathrm{P}}\int_{-\infty}^{\infty}\frac{i\mathrm{d}k_0'}{2\pi}\frac{\rho(k_0',\mathbf{k})}{k_0-k_0'}\end{pmatrix} + \begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}\mathrm{sign}(k_0)\rho(k)\\ &= \begin{pmatrix}D'(k) & 0\\ 0 & D'(k)^*\end{pmatrix}\\ &\equiv \hat{D}_{\mathrm{diag.}}(k),\end{aligned} \tag{3.61}$$

where in the third line we have used Eq. (3.58). Thus, we can see that the matrix $\hat{U}(k)$ (or $\hat{U}^{-1}(k)$) connects the analytically-continued quantity $D'(k)$ defined in Eq. (3.56) with the real-time propagator $D_{ij}(k)$. In particular, in order to transform the free field propagator $\hat{D}^F(k)$ ($= D^F_{ij}(k)$ in the component notation) by the matrix $\hat{U}^{-1}(k)$, we need only to replace the spectral function $\rho(k)$ by $\rho_F(k)$ given in Eq. (3.52). As a result, $\hat{D}^F(k)$ is diagonalized as follows

$$\hat{U}^{-1}(k)\hat{D}^F(k)\hat{U}^{-1}(k) = \begin{pmatrix}D'^F(k) & 0\\ 0 & D'^F(k)^*\end{pmatrix} \equiv \hat{D}^F_{\mathrm{diag.}}(k). \tag{3.62}$$

Furthermore, we note that the matrix form of the real-time propagator $\hat{D}(k)$ satisfies the Dyson equation given by

$$\hat{D}(k) = \hat{D}^F(k) + \hat{D}^F(k)(-i\hat{\Pi}(k))\hat{D}(k), \tag{3.63}$$

where $\hat{\Pi}(k)$ is the matrix form of the self-energy in the real-time formalism ($\Pi_{ij}(k)$ in the component notation). From Eqs. (3.61) and (3.62), multiplying the transformation matrix $\hat{U}^{-1}(k)$ from left and right in Eq. (3.63), we obtain the diagonalized form of the Dyson equation in the real-time formalism as follows

$$\hat{D}_{\mathrm{diag.}}(k) = \hat{D}^F_{\mathrm{diag.}}(k) + \hat{D}^F_{\mathrm{diag.}}(k)\begin{pmatrix}-i\bar{\Pi}(k) & 0\\ 0 & (-i\bar{\Pi}(k))^*\end{pmatrix}\hat{D}_{\mathrm{diag.}}(k). \tag{3.64}$$

Here, since the propagators are diagonalized by the transformation matrix $\hat{U}^{-1}(k)$, the self-energy is automatically diagonalized by $\hat{U}(k)$: $\hat{U}(k)\hat{\Pi}(k)\hat{U}(k) = \mathrm{diag}(\bar{\Pi}(k), \bar{\Pi}(k)^*)$ (we have denoted its (1,1) component as $\bar{\Pi}(k)$). Eventually, we obtain the following equation:

$$\begin{pmatrix} D'(k) & 0 \\ 0 & D'(k)^* \end{pmatrix} = \begin{pmatrix} D'^F(k) + D'^F(k)(-i\bar{\Pi}(k))D'(k) & 0 \\ 0 & D'^F(k)^* + D^F(k)^*(-i\bar{\Pi}(k))^* D(k)^* \end{pmatrix}. \tag{3.65}$$

Comparing this equation with Eq. (3.57), we arrive at the analytic continuation (3.54). From Eq. (3.54), we can calculate the diagonalized self-energy $\bar{\Pi}(k)$ by the self-energy in the imaginary-time formalism, $\Pi(i\omega_n, \mathbf{k})$, vial the analytic continuation $i\omega_n \to k_0 + i\epsilon k_0$. In the next subsection, we will connect $\bar{\Pi}(k)$ with the self-energy in the real-time formalism, $\Pi_{ij}(k)$.

3.2.2 Bogoliubov matrix

In this subsection, we at first derive the analytic continuation of the self-energy (3.54) with a generic contour parameter $\sigma (0 \le \sigma \le \beta)$. Then, we will connect $\bar{\Pi}(k)$ with the self-energy in the real-time formalism, $\Pi_{ij}(k)$. As we have done in the symmetric case ($\sigma = \beta/2$) in the previous subsection, it is useful to diagonalize the propagators. The diagonalization can be done by the so-called Bogoliubov matrix [44–46]. One of the important conclusion of this subsection is that the diagonal component $\bar{\Pi}(k)$ ($= \Pi(k_0 + i\epsilon k_0, \mathbf{k})$) is independent of the parameter σ.

First, from Eqs. (3.51) and (3.58), we note that the propagator in the real-time formalism, $D_{ij}(k)$, can be written in the following form:

$$\begin{aligned} D_{11}(k) &= D'(k)\cosh^2\theta + D'(k)^* \sinh^2\theta, \\ D_{22}(k) &= D_{11}(k)^*, \\ D_{12}(k) &= \mathrm{e}^{(\sigma-\beta/2)k_0}\left(D'(k) + D'(k)^*\right)\sinh\theta\cosh\theta, \\ D_{21}(k) &= \mathrm{e}^{-(\sigma-\beta/2)k_0}\left(D'(k) + D'(k)^*\right)\sinh\theta\cosh\theta, \end{aligned} \tag{3.66}$$

where $\cosh\theta$ and $\sinh\theta$ are given by

$$\begin{aligned} \cosh^2\theta &= \theta(k_0)(1 + f_B(k_0)) - \theta(-k_0) f_B(k_0), \\ \sinh^2\theta &= \theta(k_0) f_B(k_0) - \theta(-k_0)(1 + f_B(k_0)), \\ \sinh\theta\cosh\theta &= \mathrm{sign}(k_0)\mathrm{e}^{\beta k_0/2} f_B(k_0). \end{aligned} \tag{3.67}$$

From Eqs. (3.66) and (3.67), we can explicitly check that the propagators can be summarized into the matrix form as follows

$$\begin{aligned} \hat{D} &= \begin{pmatrix} D_{11}(k) & D_{12}(k) \\ D_{21}(k) & D_{22}(k) \end{pmatrix} = \hat{V}(k) \begin{pmatrix} D'(k) & 0 \\ 0 & D'(k)^* \end{pmatrix} \hat{V}(k), \\ \hat{V}(k) &= \begin{pmatrix} \cosh\theta & \mathrm{e}^{(\sigma-\beta/2)k_0}\sinh\theta \\ \mathrm{e}^{-(\sigma-\beta/2)k_0}\sinh\theta & \cosh\theta \end{pmatrix}. \end{aligned} \tag{3.68}$$

The matrix $\hat{V}(k)$ here is the Bogoliubov matrix [44–46]. From Eq. (3.68), we can see that the inverse Bogoliubov matrix $\hat{V}^{-1}(k)$ converts the matrix form of the real-time propagator $\hat{D}(k)$ into the diagonalized matrix $\mathrm{diag}(D'(k), D'(k)^*)$. Therefore, as we have done in the previous subsection, we can diagonalize the Dyson equation of the real-time propagator (3.63) by the Bogoliubov matrix $\hat{V}^{-1}(k)$. Namely, the matrix form of the Dyson equation multiplied by the Bogoliubov matrix

$$\begin{aligned}\hat{V}^{-1}(k)\hat{D}(k)\hat{V}^{-1}(k) &= \hat{V}^{-1}(k)\hat{D}^F(k)\hat{V}^{-1}(k)\\ &+ \hat{V}^{-1}(k)\hat{D}^F(k)\hat{V}^{-1}(k) \times \hat{V}(k)(-i\hat{\Pi}(k))\hat{V}(k) \times \hat{V}^{-1}(k)\hat{D}(k)\hat{V}^{-1}(k),\end{aligned} \tag{3.69}$$

leads to the following diagonalized equation:

$$\begin{pmatrix} D'(k) & 0 \\ 0 & D'(k)^* \end{pmatrix} = \begin{pmatrix} D'^F(k) & 0 \\ 0 & D'^F(k)^* \end{pmatrix} \begin{pmatrix} (-i\bar{\Pi}(k)) & 0 \\ 0 & (-i\bar{\Pi}(k))^* \end{pmatrix} \begin{pmatrix} D'(k) & 0 \\ 0 & D'(k)^* \end{pmatrix}. \tag{3.70}$$

Consequently, we can conclude that Eq. (3.54) is verified for a generic contour parameter σ. Note that Eq. (3.60) is the Bogoliubov matrix with $\sigma = \beta/2$.

From Eqs. (3.69) and (3.70), the diagonalized self-energy $\bar{\Pi}(k)$ is related to the self-energy in the real-time formalism, $\Pi_{ij}(k)$, as follows

$$\begin{aligned}&\begin{pmatrix} -i\Pi_{11}(k) & -i\Pi_{12}(k) \\ -i\Pi_{21}(k) & -i\Pi_{22}(k) \end{pmatrix}\\ =&\hat{V}^{-1}(k)\begin{pmatrix} -i\bar{\Pi}(k) & 0 \\ 0 & (-i\bar{\Pi}(k))^* \end{pmatrix}\hat{V}^{-1}(k)\\ =&\begin{pmatrix} -i\left[\bar{\Pi}\cosh^2\theta - \bar{\Pi}^*\sinh^2\theta\right] & -i\left[-(\bar{\Pi}-\bar{\Pi}^*)\mathrm{e}^{(\sigma-\beta/2)k_0}\sinh\theta\cosh\theta\right] \\ -i\left[-(\bar{\Pi}-\bar{\Pi}^*)\mathrm{e}^{-(\sigma-\beta/2)k_0}\sinh\theta\cosh\theta\right] & (-i\left[\bar{\Pi}\cosh^2\theta - \bar{\Pi}^*\sinh^2\theta\right])^* \end{pmatrix}.\end{aligned} \tag{3.71}$$

Thus, we obtain the following equations:

$$\begin{aligned}&\mathrm{Re}\ \Pi_{11}(k) = \mathrm{Re}\ \bar{\Pi}(k),\\ &\mathrm{Im}\ \Pi_{11}(k) = \mathrm{sign}(k_0)(1+2f_B(k_0))\mathrm{Im}\ \bar{\Pi}(k),\\ &\Pi_{22}(k) = -(\Pi_{11}(k))^*,\\ &\Pi_{12}(k) = -2i\mathrm{e}^{\sigma k_0}\mathrm{sign}(k_0)f_B(k_0)\mathrm{Im}\ \bar{\Pi}(k),\\ &\Pi_{21}(k) = \mathrm{e}^{-2\sigma k_0}\mathrm{e}^{\beta k_0}\Pi_{12}(k) \quad (\Pi_>(k) = \mathrm{e}^{\beta k_0}\Pi_<(k) \ \text{ for } \sigma = 0).\end{aligned} \tag{3.72}$$

We note that the self-energy $\Pi_{ij}(k)$ is the Fourier component of the two-point function in the real-time formalism, $\Pi(x^{(i)} - x^{(j)})$, with $x_0^{(1)} \in C_1$ and $x_0^{(2)} \in C_2$:

$$\Pi(x^{(i)} - x^{(j)}) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-ik_0\mathrm{Re}(x_0^{(i)}-x_0^{(j)})+i\mathbf{k}\cdot(\mathbf{x}^{(i)}-\mathbf{x}^{(j)})}\Pi_{ij}(k). \tag{3.73}$$

Thus, we have $\Pi_{12}(k) \propto \mathrm{e}^{\sigma k_0}$ and $\Pi_{21}(k) \propto \mathrm{e}^{-\sigma k_0}$. Combining this fact with Eq. (3.72), we conclude that $\bar{\Pi}(k)$ $(= \Pi(k_0 + i\epsilon k_0, \mathbf{k}))$ is independent of the contour parameter σ. Therefore, we can obtain the self-energy by using any σ in the real-time formalism. Now, we have only to evaluate in more convenient formalism (either the imaginary-time one or real-time one) and then, if necessary, use Eqs. (3.54) and (3.72) to obtain the self-energy in the formalism under consideration.

3.2.3 Real scalar field

In the following three subsections, we summarize the self-energy of thermalized fields. For details of the evaluation, see Appendix D.

Here, we consider a real scalar field φ which has the following yukawa interaction:

$$\mathcal{L}_{\text{int.}} = -\frac{y^2}{2}\varphi^2\tilde{\psi}^*\tilde{\psi}, \tag{3.74}$$

where y is the yukawa coupling constant and $\tilde{\psi}$ is a complex scalar field. We assume that $\tilde{\psi}$ is in thermal equilibrium. Below, we neglect the zero-temperature mass of $\tilde{\psi}$ compared with the temperature T of the thermal bath. From the interaction term (3.74), we obtain the following self-energy Π of φ at the one-loop level by using the imaginary-time formalism:

$$\Pi = \frac{y^2T^2}{6}, \tag{3.75}$$

which is real and momentum independent. Note that the dissipative (imaginary) part of the self-energy of φ does not arise at the one-loop level with the interaction (3.74).

3.2.4 QED electron

Here, we consider the self-energy of electron in the plasma of QED. As we will see below, the self-energy of electron has rather complicated structure compared with the case in the previous subsection.

The QED interaction is given by

$$\mathcal{L}_{\text{int.}} = eA_\mu\bar{\psi}\gamma^\mu\psi, \tag{3.76}$$

where e is the QED coupling constant, ψ is the electron field and A_μ is the photon field. Below, we neglect the electron zero-temperature mass compared with the temperature T of the thermal bath. In order to evaluate the self-energy of electron at the one-loop level, we use the Hard-Thermal-Loop (HTL) approximation [47–49]. Here, the HTL approximation is an approximation for the diagrammatic calculation, in which the internal lines of the loops are assumed to be dominated by the momentum of the order of T. Namely, the momentum of the external line is assumed to be less than the temperature T in the HTL approximation. Using the this approximation, we obtain the following QED electron self-energy $\Sigma(P)$ at the one-loop level in the imaginary-time formalism (see Appendix D for details):

$$\Sigma(P) = m_f^2 \int \frac{\mathrm{d}\Omega}{4\pi} \frac{\hat{\not{K}}}{P_\mu \hat{K}_\mu}, \tag{3.77}$$

where we have used the short-hand notation for the angular integral of (θ, ϕ) as $\int \mathrm{d}\Omega = \int_0^\pi \mathrm{d}\theta \ \sin\theta \ \int_0^{2\pi} \mathrm{d}\phi$ (θ is the zenith angle and ϕ is the azimuth angle with the direction of $\mathbf{p}$ taken to be along the z-axis). $P_\mu = (-\omega_n', \mathbf{p})$, $\hat{K}_\mu = (-i, \hat{\mathbf{k}})$, $\hat{\not{K}} = \hat{K}_\mu \gamma_\mu = -i\gamma_4 + \hat{\mathbf{k}} \cdot \boldsymbol{\gamma}$ and $P_\mu \hat{K}_\mu = i\omega_n' + \mathbf{p} \cdot \hat{\mathbf{k}}$ [3-6] [3-7]. Here, $\omega_n' = (2n+1)\pi/\beta$ (n is integer) is the imaginary-time discrete energy for electron. We have to use the odd integer $2n+1$ for fermions since fermions have the anti-periodic boundary condition in thermal field theory [41, 42] (see Appendix B). In Eq. (3.77), m_f is the electron thermal mass given by

$$m_f^2 = \frac{e^2 T^2}{8}. \tag{3.78}$$

3.2.5 QED photon

Here, we consider the photon self-energy in the QED plasma with the interaction (3.76).

Using the HTL approximation, we obtain the following QED photon self-energy $\Pi_{\mu\nu}(Q)$ at the one-loop level in the imaginary-time formalism (see Appendix D for details):

$$\Pi_{\mu\nu}(Q) = 2m_\gamma^2 \int \frac{\mathrm{d}\Omega}{4\pi} \left(\frac{i\omega_n}{Q_\rho \hat{K}_\rho} \hat{K}_\mu \hat{K}_\nu + \delta_{\mu 4}\delta_{\nu 4} \right), \tag{3.79}$$

where $Q_\mu = (-\omega_n, \mathbf{q})$[3-8], $\hat{K}_\mu = (-i, \hat{\mathbf{k}})$ and m_γ is the photon (asymptotic) thermal mass-squared (which is identified by the dispersion relation of photon) given by

$$m_\gamma^2 = \frac{e^2 T^2}{6}. \tag{3.80}$$

The so-called plasma frequency is given by $\omega_P = \frac{1}{3} eT = \sqrt{\frac{2}{3}} m_\gamma$.

3.3 Dispersion relations of quasi-particle poles

3.3.1 Real scalar field

Here, we consider again the real scalar field φ in Sec. 3.2.3 and derive the dispersion relation of φ.

[3-6] Here, we use the symbol " ˆ " for the light-like four-vector $\hat{K}_\mu$ (with $\hat{K}^2 = 0$) and the three-dimensional unit-vector $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$.

[3-7] As a convention, we have used the notation $\Sigma(P) = \Sigma(i\omega_n', \mathbf{p})$ for the imaginary-time self-energy.

[3-8] As a convention, we have used the notation $\Pi_{\mu\nu}(Q) = \Pi_{\mu\nu}(i\omega_n, \mathbf{p})$ for the imaginary-time self-energy.

From the Dyson equation $\Delta(K)^{-1} = \Delta^F(K)^{-1} + \Pi$ and Eqs. (3.28) and (3.75), we obtain the imaginary-time propagator, $\Delta(K)$, at the one-loop level as

$$\Delta(K) = \frac{1}{K^2 + \Pi} = \frac{1}{\omega_n^2 + |\mathbf{k}|^2 + \frac{y^2T^2}{6}}, \tag{3.81}$$

where $K_\mu = (k_4, \mathbf{k}) = (-\omega_n, \mathbf{k})$ and $K^2 = \omega_n^2 + |\mathbf{k}|^2$ (remember that $\omega_n = 2\pi n/\beta$ and n is an integer). Here, the notation $\Delta(K) = \Delta(i\omega_n, \mathbf{k})$ is used.

Now, let us consider a general argument as follows. In the real-time formalism, the retarded propagator $D^R(k)$ is an important quantity, which is defined by the analytic continuation $i\omega_n \to k_0 + i\epsilon$ as $D^R(k) \equiv -i\Delta(k_0 + i\epsilon, \mathbf{k})$. This is because $D^R(k)$ is the propagator of the physical propagating particle. (Here and hereafter, we use the superscript " R " for the retarded functions.) The definition of $D^R(k)$ is equivalent to the definition in the coordinate space as $D^R(x-x') \equiv \theta(x_0-x'_0)\langle[\hat{\varphi}(x), \hat{\varphi}(x')]\rangle$ with $x_0, x'_0 \in C_1$ which gives the same $D^R(k)$ by the Fourier transform of $D^R(x-x')$. The retarded self-energy $\Pi^R(k)$ is also important, which is defined by $\Pi^R(k) \equiv \Pi(k_0 + i\epsilon, \mathbf{k})$ ($\Pi(i\omega_n, \mathbf{k})$ is the self-energy in the imaginary-time formalism), since the propagating particle pole of $D^R(k)$ is determined by $\Pi^R(k)$[3-9]. Returning to our subject, since Eq. (3.75) is real and momentum independent, the retarded self-energy with the interaction (3.74) is equal to $\Pi^R(k) = \mathrm{Re}\ \Pi^R(k) = \mathrm{Re}\ \Pi(k_0 + i\epsilon, \mathbf{k}) = $ Eq. (3.75).

From Eq. (3.81) and the analytic continuation $i\omega_n \to q_0 + i\epsilon$, the retarded propagator of φ at the one-loop level in the Minkowski spacetime, $D^R(k) \equiv -i\Delta(k_0 + i\epsilon, \mathbf{k})$, is given by

$$D^R(k) = \frac{i}{(k_0 + i\epsilon)^2 - |\mathbf{k}|^2 - \frac{y^2T^2}{6}}, \tag{3.83}$$

In Eq. (3.83), the zero of the real part of the denominator determines the quasi-particle pole of φ [3-10]. Thus, the dispersion relation $\omega_{\text{th}}^2(\mathbf{k})$, which is equal to k_0^2 at the pole, is given by

$$\omega_{\text{th}}^2(\mathbf{k}) = |\mathbf{k}|^2 + \frac{y^2T^2}{6}. \tag{3.84}$$

From this equation, we find that φ has the following thermal mass-squared in the yukawa plasma:

$$m_{\text{th}}^2 = \frac{y^2T^2}{6}, \tag{3.85}$$

[3-9] This can be understood by the Dyson equation in the imaginary-time formalism (3.55). In fact, after the analytic continuation $i\omega_n \to k_0 + i\epsilon$, Eq. (3.55) leads to the following Dyson equation for $D^R(k)$:

$$D^R(k) = D^{RF}(k) + D^{RF}(k)(-i\Pi^R(k))D^R(k), \tag{3.82}$$

where $D^{RF}(k) \equiv -i\Delta^F(k_0 + i\epsilon, \mathbf{k})$ is the free retarded propagator.

[3-10] We call the pole of the retarded propagator (or equivalently the spectral function) as the quasi-particle pole which includes the thermal corrections.

which is equal to $\omega_{\rm th}^2(\mathbf{0})$ at rest ($\mathbf{k}=\mathbf{0}$).

We can also derive the dispersion relation of the quasi-particle pole by using the spectral function. From Eqs. (3.50) and (3.81), we obtain the following spectral function of φ, $\rho(k)$:

$$\begin{aligned}\rho(k) &= \frac{i}{(k_0+i\epsilon)^2-|\mathbf{k}|^2-\frac{y^2T^2}{6}} - \frac{i}{(k_0-i\epsilon)^2-|\mathbf{k}|^2-\frac{y^2T^2}{6}} \\ &= 2\pi\ \mathrm{sign}(k_0)\ \delta(k_0^2-|\mathbf{k}|^2-\frac{g^2T^2}{6}).\end{aligned} \tag{3.86}$$

Then, we can read the dispersion relation of the quasi-particle pole, $k_0^2 = \omega_{\rm th}^2(\mathbf{k})$, off Eq. (3.86), which is the same as the one given in Eq. (3.84).

3.3.2 QED electron

Here, we derive the spectral function of QED electron and then read the quasi-particle poles off the spectral function. (The spectral function of a fermion is defined in Appendix B.) For this purpose, we explicitly evaluate the self-energy $\Sigma(P)$ in the imaginary-time formalism given by Eq. (3.77):

$$\begin{aligned}\Sigma(P) &= m_f^2 \int \frac{\mathrm{d}\Omega}{4\pi}\frac{\hat{\not{K}}}{P_\mu \hat{K}_\mu} \\ &= m_f^2\left\{-i\gamma_4\int\frac{\mathrm{d}\Omega}{4\pi}\frac{1}{i\omega'+|\mathbf{p}|\cos\theta} + \gamma_i\int\frac{\mathrm{d}\Omega}{4\pi}\frac{\hat{k}_i}{i\omega'+|\mathbf{p}|\cos\theta}\right\} \\ &= -i\gamma_4\frac{m_f^2}{|\mathbf{p}|}Q_0\left(\frac{i\omega'}{|\mathbf{p}|}\right) + \gamma_i\hat{p}_i\frac{m_f^2}{|\mathbf{p}|}\left(1-\frac{i\omega'}{|\mathbf{p}|}Q_0\left(\frac{i\omega'}{|\mathbf{p}|}\right)\right),\end{aligned} \tag{3.87}$$

where $P_\mu = (p_4,\mathbf{p}) = (-\omega',\mathbf{p})$, the notation $\Sigma(P)=\Sigma(i\omega',\mathbf{p})$ is used, $\hat{K}_\mu = (-i,\hat{\mathbf{k}})$, $\hat{\not{K}} = \hat{K}_\mu\gamma_\mu = -i\gamma_4+\hat{k}_i\gamma_i$, $\hat{k}_i = k_i/|\mathbf{k}|$, $\hat{p}_i = p_i/|\mathbf{p}|$ and m_f is the electron thermal mass (3.78). ω' is the fermionic discrete imaginary-time energy (though we do not show the integer subscript). Also, $Q_0(x) = \frac{1}{2}\ln\frac{x+1}{x-1}$ is the Legendre function of degree 0 of the second kind.

The Dyson equation for electron in the imaginary-time formalism is given by $S(P)^{-1} = S^F(P)^{-1}+\Sigma(P)$, where $S(P)$ is the (resumed) electron propagator and $S^F(P) = \not{P}$ (we neglect the zero-temperature electron mass) is the free one given in Eq. (C.10). From this Dyson equation and Eq. (3.87), $S(P)^{-1}$ is given by

$$\begin{aligned}S(P)^{-1} &= \not{P}+\Sigma(P) \\ &= i\gamma_4\left(i\omega'-\frac{m_f^2}{|\mathbf{p}|}Q_0\left(\frac{i\omega'}{|\mathbf{p}|}\right)\right)+\gamma_i\hat{p}_i\left(|\mathbf{p}|+\frac{m_f^2}{|\mathbf{p}|}\left(1-\frac{i\omega'}{|\mathbf{p}|}Q_0\left(\frac{i\omega'}{|\mathbf{p}|}\right)\right)\right) \\ &= i\gamma_4 A_0(P)+\gamma_i\hat{p}_i A_s(P) \\ &= \frac{1}{2}(i\gamma_4+\gamma_i\hat{p}_i)(A_0(P)+A_s(P))+\frac{1}{2}(i\gamma_4-\gamma_i\hat{p}_i)(A_0(P)-A_s(P)),\end{aligned} \tag{3.88}$$

where we have defined $A_0(P)(= A_0(i\omega', \mathbf{p}))$ and $A_s(P)(= A_s(i\omega', \mathbf{p}))$ as

$$\begin{aligned} A_0(P) &= i\omega' - \frac{m_f^2}{|\mathbf{p}|} Q_0\left(\frac{i\omega'}{|\mathbf{p}|}\right), \\ A_s(P) &= |\mathbf{p}| + \frac{m_f^2}{|\mathbf{p}|}\left(1 - \frac{i\omega'}{|\mathbf{p}|} Q_0\left(\frac{i\omega'}{|\mathbf{p}|}\right)\right). \end{aligned} \tag{3.89}$$

Here, we note that $\frac{1}{2}(i\gamma_4 + \gamma_i \hat{p}_i)$ and $\frac{1}{2}(i\gamma_4 - \gamma_i \hat{p}_i)$ satisfy the following relations:

$$\begin{aligned} &\frac{1}{2}(i\gamma_4 + \gamma_i\hat{p}_i)\ \frac{1}{2}(i\gamma_4 + \gamma_j\hat{p}_j) = 0, \\ &\frac{1}{2}(i\gamma_4 - \gamma_i\hat{p}_i)\ \frac{1}{2}(i\gamma_4 - \gamma_j\hat{p}_j) = 0, \\ &\frac{1}{2}(i\gamma_4 + \gamma_i\hat{p}_i)\ \frac{1}{2}(i\gamma_4 - \gamma_i\hat{p}_i) = i\gamma_4\ \frac{1}{2}(i\gamma_4 - \gamma_i\hat{p}_i) = \frac{1}{2}(i\gamma_4 + \gamma_i\hat{p}_i)\ i\gamma_4, \\ &\frac{1}{2}(i\gamma_4 - \gamma_i\hat{p}_i)\ \frac{1}{2}(i\gamma_4 + \gamma_i\hat{p}_i) = i\gamma_4\ \frac{1}{2}(i\gamma_4 + \gamma_i\hat{p}_i) = \frac{1}{2}(i\gamma_4 - \gamma_i\hat{p}_i)\ i\gamma_4. \end{aligned} \tag{3.90}$$

From Eqs. (3.88) and (3.90), we can check that the electron propagator in the imaginary-time formalism, $S(P)$, is given by

$$S(P) = \frac{1}{2}(i\gamma_4 + \gamma_i\hat{p}_i)\Delta_+(P) + \frac{1}{2}(i\gamma_4 - \gamma_i\hat{p}_i)\Delta_-(P), \tag{3.91}$$

where we have defined $\Delta_\pm(P)$ as

$$\Delta_\pm(P) = (A_0(P) \mp A_s(P))^{-1} = (i\omega' \mp |\mathbf{p}| - \Pi_\pm(P))^{-1}. \tag{3.92}$$

Here, $\Pi_\pm(P)(= \Pi_\pm(i\omega', \mathbf{p}))$ is given by

$$\Pi_\pm(P) = \frac{m_f^2}{2|\mathbf{p}|}\left[\left(1 \mp \frac{i\omega'}{|\mathbf{p}|}\right)\ln\left(\frac{i\omega'/|\mathbf{p}| + 1}{i\omega/|\mathbf{p}| - 1}\right) \pm 2\right]. \tag{3.93}$$

Now, from Eqs. (3.91) and (B.25), the spectral function of electron, $\tilde{\rho}(p)$, is given by

$$\begin{aligned} \tilde{\rho}(p) &= (-iS(p_0 + i\epsilon, \mathbf{p})) - (-iS(p_0 - i\epsilon, \mathbf{p})) \\ &= \frac{1}{2}(-\gamma_0 + \gamma_i\hat{p}_i)((-i\Delta_+(p_0 + i\epsilon, \mathbf{p})) - (-i\Delta_+(p_0 - i\epsilon, \mathbf{p}))) \\ &\quad + \frac{1}{2}(-\gamma_0 - \gamma_i\hat{p}_i)((-i\Delta_-(p_0 + i\epsilon, \mathbf{p})) - (-i\Delta_-(p_0 - i\epsilon, \mathbf{p}))) \\ &= \frac{1}{2}(-\gamma_0 + \gamma_i\hat{p}_i)\rho_+(p) + \frac{1}{2}(-\gamma_0 - \gamma_i\hat{p}_i)\rho_-(p), \end{aligned} \tag{3.94}$$

where we have defined $\rho_\pm(p)$ as

$$\rho_\pm(p) = (-i\Delta_\pm(p_0 + i\epsilon, \mathbf{p})) - (-i\Delta_\pm(p_0 - i\epsilon, \mathbf{p})). \tag{3.95}$$

From Eqs. (3.92), (3.93) and (3.95), $\rho_\pm(p)$ is evaluated as follows

$$\begin{aligned}
&\rho_\pm(p)\\
=&-2\ \mathrm{Im}\frac{1}{(p_0+i\epsilon)\mp|\mathbf{p}|-\Pi_\pm(p_0+i\epsilon,\mathbf{p})}\\
=&-2\ \mathrm{Im}\frac{1}{p_0\mp|\mathbf{p}|-\mathrm{Re}\ \Pi_\pm(p_0,\mathbf{p})+i\left(\pi m_f^2\frac{|\mathbf{p}|\mp p_0}{2|\mathbf{p}|^2}\theta(|\mathbf{p}|^2-p_0^2)+\epsilon\left(1-\frac{\partial\mathrm{Re}\Pi_\pm(p_0,\mathbf{p})}{\partial p_0}\right)\right)}\\
=&\theta(|\mathbf{p}|^2-p_0^2)\beta_\pm(p)+\theta(p_0^2-|\mathbf{p}|^2)2\pi\mathrm{sign}\left(1-\frac{\partial\mathrm{Re}\Pi_\pm(p_0,p)}{\partial p_0}\right)\delta(p_0\mp|\mathbf{p}|-\mathrm{Re}\Pi_\pm(p_0,\mathbf{p})).
\end{aligned}\tag{3.96}$$

Here, we have defined the functions $\beta_\pm(p)$ as

$$\beta_\pm(p)=\frac{\pi\frac{m_f^2}{|\mathbf{p}|}(1\mp x)}{\left[|\mathbf{p}|(x\mp1)-\frac{m_f^2}{2|\mathbf{p}|}\left((1\mp x)\ln\left|\frac{x+1}{x-1}\right|\pm2\right)\right]^2+\frac{\pi^2m_f^4}{4|\mathbf{p}|^2}(1\mp x)^2},\tag{3.97}$$

where $x=p_0/|\mathbf{p}|$. From Eqs. (3.92) and (3.96), the equations

$$\begin{aligned}
\mathrm{Re}\left(\Delta_+(p_0,\mathbf{p})^{-1}\right)&=p_0-|\mathbf{p}|-\mathrm{Re}\ \Pi_+(p_0,\mathbf{p})=0,\\
\mathrm{Re}\left(\Delta_-(p_0,\mathbf{p})^{-1}\right)&=p_0+|\mathbf{p}|-\mathrm{Re}\ \Pi_-(p_0,\mathbf{p})=0
\end{aligned}\tag{3.98}$$

(with $p_0^2-|\mathbf{p}|^2>0$) determine the quasi-particle poles of electron. Since Re $\Delta_\pm(p_0,\mathbf{p})$ have the parity property[3-11]

$$\mathrm{Re}\ \Delta_+(p_0,\mathbf{p})=-\mathrm{Re}\ \Delta_-(-p_0,\mathbf{p}),\tag{3.99}$$

denoting the poles of Re $\Delta_+(p)$ as $p_0=\omega_+(\mathbf{p}),-\omega_-(\mathbf{p})$ ($\omega_+(\mathbf{p})$ and $\omega_-(\mathbf{p})$ can be taken as positive), the poles of Re $\Delta_-(p)$ are given by $p_0=\omega_-(\mathbf{p}),-\omega_+(\mathbf{p})$. From Eqs. (3.93) and (3.98), these dispersion relations $\omega_\pm(\mathbf{p})$ are determined by the following equations:

$$\begin{aligned}
\omega_+(\mathbf{p})-|\mathbf{p}|-\frac{m_f^2}{2|\mathbf{p}|}\left[\left(1-\frac{\omega_+(\mathbf{p})}{|\mathbf{p}|}\right)\ln\left(\frac{\omega_+(\mathbf{p})+|\mathbf{p}|}{\omega_+(\mathbf{p})-|\mathbf{p}|}\right)+2\right]&=0,\\
\omega_-(\mathbf{p})+|\mathbf{p}|-\frac{m_f^2}{2|\mathbf{p}|}\left[\left(1+\frac{\omega_-(\mathbf{p})}{|\mathbf{p}|}\right)\ln\left(\frac{\omega_-(\mathbf{p})+|\mathbf{p}|}{\omega_-(\mathbf{p})-|\mathbf{p}|}\right)-2\right]&=0.
\end{aligned}\tag{3.100}$$

To solve these equations, we need numerical calculation. However, we can write down analytically the limiting forms of $\omega_\pm(\mathbf{p})$ as follows [11]

$$\begin{aligned}
\omega_+(\mathbf{p})&\simeq\begin{cases}m_f+\frac{1}{3}|\mathbf{p}| & (|\mathbf{p}|\ll m_f),\\ |\mathbf{p}|+\frac{m_f^2}{|\mathbf{p}|} & (|\mathbf{p}|\gg m_f),\end{cases}\\
\omega_-(\mathbf{p})&\simeq\begin{cases}m_f-\frac{1}{3}|\mathbf{p}| & (|\mathbf{p}|\ll m_f),\\ |\mathbf{p}|+2|\mathbf{p}|\exp\left(-1-\frac{2|\mathbf{p}|^2}{m_f^2}\right) & (|\mathbf{p}|\gg m_f).\end{cases}
\end{aligned}\tag{3.101}$$

[3-11] Also, we have Im $\Delta_+(p_0,\mathbf{p})=$ Im $\Delta_-(-p_0,\mathbf{p})$ (for $p_0^2-|\mathbf{p}|^2<0$).

Now, using Eq. (3.100) for the last line of Eq. (3.96), we arrive at the expression for the spectral functions $\rho_\pm(p)$ as follows[3-12]

$$\rho_\pm(p) = \theta(|\mathbf{p}|^2 - p_0^2)\ \beta_\pm(p) + 2\pi\left(Z_\pm(\mathbf{p})\delta(p_0 - \omega_\pm(\mathbf{p})) + Z_\mp(\mathbf{p})\delta(p_0 + \omega_\mp(\mathbf{p}))\right). \quad (3.103)$$

Note that, for $p^2 = p_0^2 - |\mathbf{p}|^2 > 0$, the imaginary-parts of $\Pi_\pm(p)$ given in Eq. (3.93) vanish and thus the spectral functions $\rho_\pm(p)$ have the zero-width poles at $p_0 = \omega_\pm(\mathbf{p}), -\omega_\mp(\mathbf{p})$. Here, the residues at the poles, $Z_\pm(\mathbf{p})$, are given by

$$Z_\pm(\mathbf{p}) = \left(1 - \frac{\partial \mathrm{Re}\ \Pi_\pm(\omega_\pm, \mathbf{p})}{\partial \omega_\pm}\right)^{-1} = \frac{\omega_\pm^2 - |\mathbf{p}|^2}{2m_f^2}. \quad (3.104)$$

We can write down analytically the limiting forms of $Z_\pm(\mathbf{p})$ as follows [11]

$$\begin{aligned} Z_+(\mathbf{p}) &\simeq \begin{cases} \frac{1}{2} + \frac{|\mathbf{p}|}{3m_f} & (|\mathbf{p}| \ll m_f), \\ 1 + \frac{m_f^2}{2|\mathbf{p}|^2}\left(1 - \ln\frac{2|\mathbf{p}|^2}{m_f^2}\right) \sim 1 & (|\mathbf{p}| \gg m_f), \end{cases} \\ Z_-(\mathbf{p}) &\simeq \begin{cases} \frac{1}{2} - \frac{|\mathbf{p}|}{3m_f} & (|\mathbf{p}| \ll m_f), \\ \frac{2|\mathbf{p}|^2}{m_f^2}\ \exp\left(-1 - \frac{2|\mathbf{p}|^2}{m_f^2}\right) \sim 0 & (|\mathbf{p}| \gg m_f). \end{cases} \end{aligned} \quad (3.105)$$

3.3.3 QED photon

Here, we derive the spectral function of QED photon and then read the quasi-particle poles off the spectral function. For this purpose, we explicitly evaluate the self-energy (3.79):

$$\Pi_{\mu\nu}(Q) = 2m_\gamma^2 \int \frac{\mathrm{d}\Omega}{4\pi}\left(\frac{i\omega}{Q_\rho \hat{K}_\rho}\hat{K}_\mu \hat{K}_\nu + \delta_{\mu 4}\delta_{\nu 4}\right),$$

where $Q_\mu = (q_4, \mathbf{q}) = (-\omega, \mathbf{q})$, the notation $\Pi_{\mu\nu}(Q) = \Pi_{\mu\nu}(i\omega, \mathbf{q})$ is used, $\hat{K}_\mu = (-i, \hat{\mathbf{k}})$ and m_γ is the photon thermal mass (3.80). Here, ω is the bosonic discrete imaginary-time energy (though we do not show the integer subscript).

First, let us decompose the photon self-energy $\Pi_{\mu\nu}$ (Q). Since $\Pi_{\mu\nu}(Q)$ is orthogonal to the four-vector Q_μ:

$$Q_\mu \Pi_{\mu\nu}(Q) = 2m_\gamma^2 \int \frac{\mathrm{d}\Omega}{4\pi}\left(i\omega \hat{K}_\nu + Q_4 \delta_{\nu 4}\right) = 0, \quad (3.106)$$

[3-12] From Eqs. (C.10) and (B.25), the spectral function of free electron, $\tilde{\rho}^F(p)$, is given by

$$\begin{aligned} \tilde{\rho}^F(p) &= (-iS^F(p_0 + i\epsilon, \mathbf{p})) - (-iS^F(p_0 - i\epsilon, \mathbf{p})) \\ &= \pi\left\{\delta(p_0 - |\mathbf{p}|)\left(\gamma^0 - \hat{\mathbf{p}}\cdot\boldsymbol{\gamma}\right) + \delta(p_0 + |\mathbf{p}|)\left(\gamma^0 + \hat{\mathbf{p}}\cdot\boldsymbol{\gamma}\right)\right\} \\ &= 2\pi\ \mathrm{sign}(p_0)\ \delta(p^2)\ \not{p}. \end{aligned} \quad (3.102)$$

Comparing this equation with Eqs. (3.94) and (3.103), one can see that free electron satisfies $Z_+(\mathbf{p}) = 1$, $Z_-(\mathbf{p}) = 0$ and $\omega_+(\mathbf{p}) = |\mathbf{p}|$ (massless).

we can decompose $\Pi_{\mu\nu}(Q)$ into the longitudinal mode (denoted by L) and the transverse mode (denoted by T) by using the projection operators $P_{\mu\nu}^{L,T}(Q)$ as

$$\Pi_{\mu\nu}(Q) = \Pi_L(Q)P_{\mu\nu}^L(Q) + \Pi_T(Q)P_{\mu\nu}^T(Q). \tag{3.107}$$

Here, the projection operators, $P_{\mu\nu}^{L,T}(Q)$, are defined as follows

$$\begin{aligned} &P_{44}^T(Q) = P_{4i}^T(Q) = 0,\ P_{ij}^T(Q) = \delta_{ij} - \frac{q_i q_j}{|\mathbf{q}|^2}, \\ &P_{\mu\nu}^L(Q) = \delta_{\mu\nu} - \frac{Q_\mu Q_\nu}{Q^2} - P_{\mu\nu}^T(Q), \\ &P_{44}^L(Q) = \frac{|\mathbf{q}|^2}{Q^2},\ P_{4i}^L = \frac{\omega q_i}{Q^2},\ P_{ij}^L = \frac{\omega^2}{Q^2}\frac{q_i q_j}{|\mathbf{q}|^2}. \end{aligned} \tag{3.108}$$

We note that $P_{\mu\nu}^{L,T}(Q)$ satisfy the projection properties as follows

$$\begin{aligned} &Q_\mu P_{\mu\nu}^L = Q_\mu P_{\mu\nu}^T = 0, \\ &P_{\mu\nu}^L P_{\nu\rho}^T = P_{\mu\nu}^T P_{\nu\rho}^L = 0, \\ &P_{\mu\nu}^L P_{\nu\rho}^L = P_{\mu\rho}^L,\ P_{\mu\nu}^T P_{\nu\rho}^T = P_{\mu\nu}^T. \end{aligned} \tag{3.109}$$

From Eqs. (3.107) and (3.108), the longitudinal and transverse self-energies, $\Pi_L(Q)$ and $\Pi_T(Q)$, are given by

$$\begin{aligned} &\Pi_L(Q) = \frac{Q^2}{\omega|\mathbf{q}|}\Pi_{43}(Q), \\ &\Pi_T(Q) = \Pi_{11}(Q)(= \Pi_{22}(Q)), \end{aligned} \tag{3.110}$$

where we have taken the 3-axis parallel to the three-vector $\mathbf{q}$.

From Eqs. (3.79) and (3.110), we can evaluate $\Pi_L(Q)$ and $\Pi_T(Q)$ (we take the 3-axis parallel to the three-vector $\mathbf{q}$ as in Eq. (3.110)) as follows

$$\begin{aligned} \Pi_L(Q) &= \frac{Q^2}{\omega|\mathbf{q}|} \times 2m_\gamma^2 \int \frac{\mathrm{d}\Omega}{4\pi} \frac{i\omega}{i\omega + |\mathbf{q}|\cos\theta}(-i)\cos\theta \\ &= \frac{Q^2}{|\mathbf{q}|^2} \times 2m_\gamma^2 \left(1 - \frac{i\omega}{|\mathbf{q}|}Q_0\left(\frac{i\omega}{|\mathbf{q}|}\right)\right) \\ &\equiv \frac{Q^2}{|\mathbf{q}|^2}\tilde{\Pi}_L(Q), \end{aligned} \tag{3.111}$$

where we have defined $\tilde{\Pi}_L(Q) = 2m_\gamma^2 \left(1 - \frac{i\omega}{|\mathbf{q}|}Q_0\left(\frac{i\omega}{|\mathbf{q}|}\right)\right)$. Also,

$$\begin{aligned} \Pi_T(Q) &= \frac{1}{2}(\Pi_{11}(Q) + \Pi_{22}(Q)) \\ &= m_\gamma^2 \int \frac{\mathrm{d}\Omega}{4\pi} \frac{i\omega}{i\omega + |\mathbf{q}|\cos\theta}\sin^2\theta \\ &= m_\gamma^2 \frac{i\omega}{|\mathbf{q}|}\left(\frac{i\omega}{|\mathbf{q}|} + \frac{q^2}{|\mathbf{q}|^2}Q_0\left(\frac{i\omega}{|\mathbf{q}|}\right)\right). \end{aligned} \tag{3.112}$$

The Dyson equation for photon in the imaginary-time formalism is given by $\Delta^{-1}_{\mu\nu}(Q) = \Delta^{F\ -1}_{\mu\nu}(Q) + \Pi_{\mu\nu}(Q)$, where $\Delta_{\mu\nu}(Q)$ is the (resumed) electron propagator and $\Delta^{F}_{\mu\nu}(Q) = \delta_{\mu\nu}/Q^2$ (Feynman gauge) is the free one given in Eq. (C.14). From this Dyson equation and Eqs. (3.107) and (3.108), $\Delta^{-1}_{\mu\nu}(Q)$ is given by

$$\begin{aligned}
\Delta^{-1}_{\mu\nu}(Q) &= \delta_{\mu\nu}Q^2 + \Pi_{\mu\nu}(Q) \\
&= \left(P^L_{\mu\nu}(Q) + P^T_{\mu\nu}(Q) + \frac{Q_\mu Q_\nu}{Q^2}\right) Q^2 + \Pi_L(Q)P^L_{\mu\nu}(Q) + \Pi_T(Q)P^T_{\mu\nu}(Q) \\
&= \left(Q^2 + \Pi_L(Q)\right) P^L_{\mu\nu}(Q) + \left(Q^2 + \Pi_T(Q)\right) P^T_{\mu\nu}(Q) + Q_\mu Q_\nu.
\end{aligned} \tag{3.113}$$

From Eqs. (3.109) and (3.113), we obtain the following photon propagator $\Delta_{\mu\nu}(Q)$:

$$\Delta_{\mu\nu}(Q) = P^L_{\mu\nu}(Q)\Delta_L(Q) + P^T_{\mu\nu}(Q)\frac{|\mathbf{q}|^2}{Q^2}\Delta_T(Q) + \frac{1}{Q^2}\frac{Q_\mu Q_\nu}{Q^2}, \tag{3.114}$$

where $\Delta_L(Q)$ and $\Delta_T(Q)$ are the propagator of the longitudinal and the transverse mode photons which are projected by the operators $P^L_{\mu\nu}$ and $P^T_{\mu\nu}$:

$$\begin{aligned}
\Delta_L(Q) &= \frac{1}{|\mathbf{q}|^2 + \tilde{\Pi}_L(Q)}, \\
\Delta_T(Q) &= \frac{1}{Q^2 + \Pi_T(Q)}.
\end{aligned} \tag{3.115}$$

($\tilde{\Pi}_L(Q)$ is defined in Eq. (3.111).)

Now, let us define the spectral function of QED photon for the longitudinal and transverse modes, $\rho_L(q)$ and $\rho_T(q)$, as follows[3-13]

$$\begin{aligned}
\rho_L(q) &\equiv (-i\Delta_L(q_0 + i\epsilon, \mathbf{q})) - (-i\Delta_L(q_0 - i\epsilon, \mathbf{q})), \\
\rho_T(q) &\equiv (-i\Delta_T(q_0 + i\epsilon, \mathbf{q})) - (-i\Delta_T(q_0 - i\epsilon, \mathbf{q})).
\end{aligned} \tag{3.116}$$

For the longitudinal mode, the spectral function $\rho_L(q)$ can be written as follows

$$\begin{aligned}
&\rho_L(q) \\
&= -2\ \mathrm{Im}\frac{1}{|\mathbf{q}|^2 + \tilde{\Pi}_L(q_0 + i\epsilon, \mathbf{q})} \\
&= -2\ \mathrm{Im}\frac{1}{|\mathbf{q}|^2 + \mathrm{Re}\ \tilde{\Pi}_L(q_0, \mathbf{q}) + i\left(\pi m_\gamma^2 \frac{q_0}{|\mathbf{q}|}\theta(|\mathbf{q}|^2 - q_0^2) + \epsilon\frac{\partial \tilde{\Pi}_L(q_0,\mathbf{q})}{\partial q_0}\right)} \\
&= \theta(|\mathbf{q}|^2 - q_0^2)\ \beta_L(q_0, \mathbf{q}) + \theta(q_0^2 - |\mathbf{q}|^2)\ 2\pi\ \mathrm{sign}\left(\frac{\partial \mathrm{Re}\ \tilde{\Pi}_L(q_0, \mathbf{q})}{\partial q_0}\right)\delta(|\mathbf{q}|^2 + \mathrm{Re}\ \tilde{\Pi}_L(q_0, \mathbf{q}))
\end{aligned} \tag{3.117}$$

[3-13]Note that the zeros of the retarded propagators $\Delta_{L,T}(q_0 + i\epsilon, \mathbf{q})$ are equivalent to the poles of $\rho_{L,T}(q)$.

Here, we have defined the function $\beta_L(q)$ as

$$\beta_L(q) = \frac{2\pi m_\gamma^2 x}{\left[|\mathbf{q}|^2 + 2m_\gamma^2\left(1 - \frac{x}{2}\ln\left|\frac{x+1}{x-1}\right|\right)\right]^2 + \pi^2 m_\gamma^4 x^2}, \tag{3.118}$$

where $x = q_0/|\mathbf{q}|$. From Eqs. (3.111) and (3.117), the equation

$$|\mathbf{q}|^2 + 2m_\gamma^2\left[1 - \frac{q_0}{2|\mathbf{q}|}\ln\left(\frac{q_0/|\mathbf{q}|+1}{q_0/|\mathbf{q}|-1}\right)\right] = 0 \tag{3.119}$$

(with $q_0^2 - |\mathbf{q}|^2 > 0$) determines the quasi-particle pole of the longitudinal photon. To solve Eq. (3.119), we need numerical calculation. However, we can write down analytically the limiting form of $\omega_L(\mathbf{q})$ as follows [11]

$$\omega_L(\mathbf{q}) \simeq \begin{cases} \omega_P^2 + \frac{3}{5}|\mathbf{q}|^2 & (|\mathbf{q}| \ll m_\gamma), \\ |\mathbf{q}|^2\left(1 + 2\exp\left(-\frac{|\mathbf{q}|^2+m_\gamma^2}{m_\gamma^2}\right)\right)^2 & (|\mathbf{q}| \gg m_\gamma), \end{cases} \tag{3.120}$$

where $\omega_P = \frac{1}{3}eT = \sqrt{\frac{2}{3}}m_\gamma$ is the plasma frequency. Using Eq. (3.119) in the last line of Eq. (3.117), we arrive at the following expression for the spectral function $\rho_L(q)$:

$$\rho_L(q) = \theta(|\mathbf{q}|^2 - q_0^2)\,\beta_L(q_0,\mathbf{q}) + 2\pi Z_L(\mathbf{q})\left(\delta(q_0 - \omega_L(\mathbf{q})) - \delta(q_0 + \omega_L(\mathbf{q}))\right), \tag{3.121}$$

Note that, for $q^2 = q_0^2 - |\mathbf{q}|^2 > 0$, the imaginary-part of $\Pi_L(q)$ given in Eq. (3.111) vanishes and thus the spectral function $\rho_L(q)$ has the zero-width poles at $q_0 = \pm\omega_L(\mathbf{q})$. Here, the residue $Z_L(\mathbf{q})$ is given by

$$Z_L(\mathbf{q}) = \left(\frac{\partial \mathrm{Re}\,\tilde{\Pi}_L(\omega_L,\mathbf{q})}{\partial\omega_L}\right)^{-1} = \frac{\omega_L(\omega_L^2 - |\mathbf{q}|^2)}{|\mathbf{q}|^2(|\mathbf{q}|^2 + 2m_\gamma^2 - \omega_L^2)}. \tag{3.122}$$

We can write down analytically the limiting form of $Z_L(\mathbf{q})$ as follows [11]

$$Z_L(\mathbf{q}) \simeq \begin{cases} \frac{\omega_P}{2|\mathbf{q}|^2}\left(1 - \frac{3}{10}\frac{|\mathbf{q}|^2}{\omega_P^2}\right) & (|\mathbf{q}| \ll m_\gamma), \\ \frac{2|\mathbf{q}|}{m_\gamma^2}\exp\left(-\frac{|\mathbf{q}|^2+m_\gamma^2}{m_\gamma^2}\right) & (|\mathbf{q}| \gg m_\gamma). \end{cases} \tag{3.123}$$

Note that, for $|\mathbf{q}| \gg m_\gamma$, $Z_L(\mathbf{q})$ vanishes exponentially. This means that the longitudinal mode of photon disappears when the thermal effect is negligible.

On the other hand, for the transverse mode, the spectral function, $\rho_T(q)$, can be written as

$$
\begin{aligned}
&\rho_T(q)\\
=&(-i\Delta_T(q_0+i\epsilon,\mathbf{q}))-(-i\Delta_T(q_0-i\epsilon,\mathbf{q}))\\
=&-2\ \mathrm{Im}\frac{1}{(q_0+i\epsilon)^2-|\mathbf{q}|^2-\Pi_T(q_0+i\epsilon,\mathbf{q})}\\
=&-2\ \mathrm{Im}\frac{1}{q_0^2-|\mathbf{q}|^2-\mathrm{Re}\ \Pi_T(q_0,\mathbf{q})+i\left(\pi m_\gamma^2\ \frac{q_0(q^2-q_0^2)}{2q^3}\theta(q^2-q_0^2)+\epsilon\left(2q_0-\frac{\partial\Pi_T(q_0,q)}{\partial q_0}\right)\right)}\\
=&\theta(|\mathbf{q}|^2-q_0^2)\ \beta_T(q)+\theta(q_0^2-|\mathbf{q}|^2)2\pi\mathrm{sign}\left(2q_0-\frac{\partial\mathrm{Re}\Pi_T(q_0,\mathbf{q})}{\partial q_0}\right)\delta(q_0^2-|\mathbf{q}|^2-\mathrm{Re}\Pi_T(q_0,\mathbf{q})).
\end{aligned}
\tag{3.124}
$$

Here, we have defined the function $\beta_T(q)$ as

$$
\beta_T(q)=\frac{\pi m_\gamma^2 x(1-x^2)}{\left[|\mathbf{q}|^2(x^2-1)-m_\gamma^2\left(x^2+\frac{x(1-x^2)}{2}\ln\left|\frac{x+1}{x-1}\right|\right)\right]^2+\pi^2 m_\gamma^4\ \frac{x^2(1-x^2)^2}{4}}, \tag{3.125}
$$

where $x=q_0/|\mathbf{q}|$. From Eqs. (3.112) and (3.124), the equation

$$
\frac{q_0^2}{|\mathbf{q}|^2}-1-\frac{m_\gamma^2}{|\mathbf{q}|^2}\left[\frac{q_0^2}{|\mathbf{q}|^2}+\frac{q_0(1-q_0^2/|\mathbf{q}|^2)}{2|\mathbf{q}|}\ln\left(\frac{q_0/|\mathbf{q}|+1}{q_0/|\mathbf{q}|-1}\right)\right]=0 \tag{3.126}
$$

(with $q_0^2-|\mathbf{q}|^2>0$) determines the quasi-particle pole of the transverse photon. To solve Eq. (3.126), we need numerical calculation. However, we can write down the limiting form of $\omega_T(\mathbf{q})$ analytically [11]:

$$
\omega_T(\mathbf{q})\simeq\begin{cases}\omega_P^2+\frac{6}{5}|\mathbf{q}|^2 & (|\mathbf{q}|\ll m_\gamma),\\ |\mathbf{q}|^2+m_\gamma^2 & (|\mathbf{q}|\gg m_\gamma).\end{cases} \tag{3.127}
$$

From Eqs. (3.120) and (3.127), for $|\mathbf{q}|\to 0$, we cannot distinguish the frequencies of the transverse and longitudinal modes (they are both ω_P). On the other hand, for $|\mathbf{q}|\gg m_\gamma$, the dispersion relation of the transverse mode is the parabolic one with the "asymptotic" thermal mass m_γ, which is quite different from the longitudinal mode's dispersion relation. Using Eqs. (3.126) in the last line of (3.124), we arrive at the following expression for the spectral function $\rho_T(q)$:

$$
\rho_T(q)=\theta(|\mathbf{q}|^2-q_0^2)\ \beta_T(q)+2\pi Z_T(\mathbf{q})\left(\delta(q_0-\omega_T(\mathbf{q}))-\delta(q_0+\omega_T(\mathbf{q}))\right). \tag{3.128}
$$

Note that, for $q^2=q_0^2-|\mathbf{q}|^2>0$, the imaginary-part of $\Pi_T(q)$ given in Eq. (3.112) vanishes and thus the spectral function $\rho_T(q)$ has the zero-width poles at $q_0=\pm\omega_T(\mathbf{q})$. Here, the residue $Z_T(\mathbf{q})$ is given by

$$
Z_T(\mathbf{q})=\left(2\omega_T-\frac{\partial\mathrm{Re}\ \Pi_T(\omega_T,\mathbf{q})}{\partial\omega_T}\right)^{-1}=\frac{\omega_T(\omega_T^2-|\mathbf{q}|^2)}{2m_\gamma^2\omega_T^2-(\omega_T^2-|\mathbf{q}|^2)^2}. \tag{3.129}
$$

We can write down analytically the limiting form of $Z_T(q)$ as follows [11]

$$Z_T(\mathbf{q}) \simeq \begin{cases} \frac{1}{2\omega_P}\left(1-\frac{4}{5}\frac{|\mathbf{q}|^2}{\omega_P^2}\right) & (|\mathbf{q}| \ll m_\gamma), \\ \frac{1}{2|\mathbf{q}|} & (|\mathbf{q}| \gg m_\gamma). \end{cases} \tag{3.130}$$

For $|\mathbf{q}| \gg m_\gamma$, $Z_T(\mathbf{q})$ reaches the zero-temperature form. Namely, in this limit, the thermal bath does not affect the propagating transverse photon.

3.4 Free energy

In this section, we evaluate the free energy of thermal equilibrium systems. This will be the basis for Chap. 6.

For the preparation in this section, we define the following four dimensional integral in the imaginary-time formalism:

$$\begin{aligned} b_n &\equiv \int \frac{[\mathrm{d}Q]}{(Q^2)^n} = T\sum_{n=-\infty}^{\infty}\int\frac{\mathrm{d}^3\mathbf{q}}{(Q^2)^n} \ \text{(for the bosonic discrete energy } \ \omega_n = 2n\pi/\beta), \\ f_n &\equiv \int \frac{\{\mathrm{d}Q\}}{(Q^2)^n} = T\sum_{n=-\infty}^{\infty}\int\frac{\mathrm{d}^3\mathbf{q}}{(Q^2)^n} \ \text{(for the fermionic discrete energy } \ \omega'_n = (2n+1)\pi/\beta). \end{aligned} \tag{3.131}$$

Here, $Q_\mu = (q_4, \mathbf{q}) = (-\omega_n, \mathbf{q})$ (for boson), $(-\omega'_n, \mathbf{q})$ (for fermion). b_n and f_n are the loop-integrals over the massless bosonic and fermionic propagators, respectively. For example, b_1, f_1 can be evaluated as

$$\begin{aligned} b_1 &= \frac{T^2}{12}, \\ f_1 &= -\frac{T^2}{24} = -\frac{1}{2}b_1, \end{aligned} \tag{3.132}$$

where we have removed the divergences which arise from the zero-point energies. Since these are the temperature independent quadratic divergences, we can remove them by the zero-temperature counter-terms [10, 50–54]. b_1 and f_1 are nothing but the one-loop integrals of massless boson and fermion, respectively.

3.4.1 Yukawa plasma

In this subsection, we evaluate the free energy of the thermalized system consisting of yukawa interacting fields:

$$\mathcal{L}_{\text{int.}} = -y\chi\overline{\psi_R}\psi_L + h.c., \tag{3.133}$$

where χ is a complex scalar field, $\psi_{R/L} = P_{R/L}\psi$ is the Dirac fermion projected by the operator $P_{R/L} = (1 \pm \gamma_5)/2$. Below, we neglect the zero-temperature masses of the boson and fermion compared with the temperature.

The free energy Ω can be expressed by the partition function Z (all the chemical potentials are assumed to be zero) as

$$\Omega = -T \ln Z. \tag{3.134}$$

For the free field case, the free energy Ω^F is given by

$$\Omega^F/V = -\frac{\pi^2 T^4}{90} \times \left(2 + 4 \times \frac{7}{8}\right) = -\frac{11\pi^2 T^4}{180}, \tag{3.135}$$

where V is the spatial volume of the system. Here, the factor 2 (4) in the bracket comes from the number of the physical degrees of freedom of the complex scalar field (fermions) and the factor 7/8 originates from the Fermi statistics. The leading order correction to the free energy comes from the two-loop contribution, Ω_2, which is evaluated as

$$\begin{aligned} \Omega_2/V &= y^2 \int [\mathrm{d}Q]\{\mathrm{d}K\} \frac{\mathrm{tr}\,(P_L(\not{K} - \not{Q})P_R \not{K})}{K^2 Q^2 (K-Q)^2} \\ &= -y^2 f_1(2b_1 - f_1) \\ &= \frac{5y^2 T^4}{576}. \end{aligned} \tag{3.136}$$

For higher order correction to the free energy, see for example Ref. [12].

3.4.2 QED plasma

In this subsection, we derive the free energy of the QED plasma. The interaction term is given by Eq. (3.76). The free energy Ω can be expressed by the partition function Z (all the chemical potentials are assumed to be zero) as

$$\Omega = -T \ln Z. \tag{3.137}$$

For the free field case, the free energy Ω^F is given by

$$\Omega^F/V = -\frac{\pi^2 T^4}{90} \times \left(2 + 4 \times \frac{7}{8}\right) = -\frac{11\pi^2 T^4}{180}, \tag{3.138}$$

where V is the spatial volume of the system. Here, the factor 2 (4) in the bracket comes from the number of the physical degrees of freedom of photon (electron) and the factor 7/8 originates from the Fermi statistics. The leading order correction to the free energy comes from the two-loop contribution, Ω_2, which is evaluated as

$$\begin{aligned}\Omega_2/V &= \frac{e^2}{2}\int[\mathrm{d}Q]\{\mathrm{d}K\}\frac{\mathrm{tr}\,(\gamma_\mu(\not{K}-\not{Q})\gamma_\mu\not{K})}{K^2Q^2(K-Q)^2} \\ &= -2e^2\,(f_1b_1+f_1b_1-f_1f_1) \\ &= \frac{5e^2T^4}{288}.\end{aligned} \tag{3.139}$$

For higher order correction to the free energy, see for example Ref. [12].

Chapter 4

The Hubble-induced mass in the inflaton dominated Universe

Supersymmetry (SUSY) is an attractive candidate for the physics beyond the standard model (SM). Its local version, supergravity, leads to various phenomena in cosmology. In particular, when the inflaton dominates the Universe during and after inflation, the supergravity effect induces an effective mass of order H, the Hubble expansion rate, for a general scalar field ϕ coupled to the inflaton sector by Planck-suppressed interactions [23–27], unless its mass is protected by some symmetry. Such an effective mass of order H is called a *Hubble-induced mass* and plays an important role in many cosmological scenarios. For example, a negative Hubble-induced mass enables the Affleck-Dine baryogenesis mechanism [55–57]. The enhanced Hubble-induced mass also is a key for solving the cosmological moduli problem [58, 59]. On the other hand, the Hubble-induced mass will be a main obstacle for implementing the curvaton mechanism [60–62] in supergravity and it must be suppressed at least by about one order of magnitude.

In this chapter, we will see the effects of the Hubble-induced mass in the inflaton dominated era. In Sec. 4.1, we summarize the supergravity effects for kinetic term, scalar potential and fermion interaction, which will be the basis for Chap. 6. Next, in Sec. 4.2, we briefly review how the Hubble-induced mass play the important role in the early Universe. We will see the effect for the eta problem of inflation models in supergravity, Affleck-Dine baryogenesis, the adiabatic solution for the cosmological moduli problem and the eta problem for curvaton models in supergravity.

4.1 Scalar potential and fermion interactions in supergravity

In this section, we summarize the kinetic term, scalar potential and fermion interaction in supergravity. In supergravity, the kinetic terms for scalar fields χ_i are given by [28]

$$\mathcal{L}_{\text{kin.}} = K_{i\bar{j}} \partial_\mu \chi_i^* \partial^\mu \chi_j, \tag{4.1}$$

where K is the Kähler potential, and the subscripts i and $\bar{j}$ of $K_{i\bar{j}}$ represent the derivatives by a scalar field χ_i and χ_j^*, respectively. Also, the kinetic terms for fermions $\tilde{\chi}_i$ are given by [28]

$$\mathcal{L}^{\tilde{\chi}}_{\text{kin.}} = K_{i\bar{j}} \tilde{\chi}_i i \sigma^\mu \partial_\mu \tilde{\chi}_j^*, \tag{4.2}$$

where $\sigma^\mu = (1, \sigma^i)$ (σ^i are the Pauli matrices). Next, the F-term scalar potential is given by the following formula [28]:

$$V_F = \mathrm{e}^{K/M_{\mathrm{P}}^2} \left(D_i W K^{i\bar{j}} \overline{D_j W} - \frac{3|W|^2}{M_{\mathrm{P}}^2} \right), \tag{4.3}$$

where W is the superpotential, $D_i W = W_i + K_i W / M_{\mathrm{P}}^2$ and $K^{i\bar{j}}$ is the inverse of $K_{i\bar{j}}$. Also, the D-term scalar potential is given by [28]

$$V_D = \frac{g^2}{2} \mathrm{Re}\ f_{ab}^{-1} D^a D^{b*}, \tag{4.4}$$

where g is the gauge coupling, f_{ab} is the gauge kinetic function and D^a is given by

$$D^a = K_i (T^a)_{ij} \chi_j, \quad D^{a*} = \chi_i^* (T^a)_{ij} K_{\bar{j}}. \tag{4.5}$$

Here, T^a is the generator of the gauge group. In this thesis, we consider the case $f_{ab} = \delta_{ab}$ only. Finally, the fermion interaction term in supergravity is given by the following formula [28]:

$$\mathcal{L}_f = -\frac{1}{2} \mathrm{e}^{K/(2M_{\mathrm{P}}^2)} \left(\mathcal{D}_i D_j W \right) \tilde{\chi}^i \tilde{\chi}^j + h.c. + \cdots, \tag{4.6}$$

where $\tilde{\chi}^i$ are two-component fermionic fields with the label of the field species i, and $\cdots$ includes interactions between ξ^i and gauge, gravity superfields. Here, $\mathcal{D}_i D_j W = W_{ij} + K_{ij} W / M_{\mathrm{P}}^2 + K_i D_j W / M_{\mathrm{P}}^2 + K_j D_i W / M_{\mathrm{P}}^2 - K_i K_j W / M_{\mathrm{P}}^4 - \Gamma_{ij}^k D_k W / M_{\mathrm{P}}^4$ and $\Gamma_{ij}^k = K^{i\bar{l}} (K_{j\bar{l}})_i$.

4.2 Effects of the Hubble-induced mass in the inflationary Universe

In this section, we consider the effects of the Hubble-induced mass in the inflaton dominated era, which is generated by the supergravity effect and the inflaton energy density. First of all, let us see how the Hubble-induced mass arises. If the Kähler potential includes the minimal part

$$K = |\phi|^2 + \cdots \tag{4.7}$$

and if the inflation potential is given by the F-term potential, the F-term potential contains the following effective mass term for ϕ during inflation [23–25]:

$$\begin{aligned} V_F &\supset \frac{|\phi|^2}{M_{\rm P}^2}\rho_{\rm inf} \\ &\simeq 3H_I^2|\phi|^2, \end{aligned} \tag{4.8}$$

where H_I is the Hubble parameter during inflation and $\rho_{\rm inf}$ is the inflaton energy density. Here, we have used the Friedman equation $3H_I^2 M_{\rm P}^2 \simeq \rho_{\rm inf}$ during inflation. Thus, the scalar field ϕ acquires the Hubble scale effective mass and this is the so-called Hubble-induced mass. Now, it should be noted that the contribution to the Hubble-induced mass can also arise from the non-minimal part of the Kähler potential. In this case, the Hubble-induced mass depends on model-dependent parameters in the non-minimal Kähler potential. In fact, we will see such example in Sec. 4.2.2, in which the Hubble-induced mass has negative sign. Also, the Hubble-induced mass is generated in the inflaton oscillation dominated era after inflation.

4.2.1 The eta problem for inflation models

Here, we identify $\phi_1 = \sqrt{2}\,{\rm Re}\,\phi$ as a inflaton field. Since the Hubble-induced mass term in Eq. (4.8) is generically generated for ϕ_1 during inflation, the slow-roll condition for ϕ_1 is easily broken. Assuming that the slow-roll condition is satisfied unless the Hubble-induced mass exists, we have

$$\epsilon = \frac{1}{2}M_{\rm P}^2\left(\frac{V'}{V}\right)^2 \simeq \frac{\phi_1^2}{2M_{\rm P}^2}, \tag{4.9}$$

while

$$\eta = M_{\rm P}^2\frac{V''}{V} \simeq 1, \tag{4.10}$$

where V is the inflaton potential and the prime denotes the derivative by ϕ_1. This is the eta problem for the inflation model-building in supergravity [23–25].

A solution for this problem is to adopt a superpotential linear in the inflaton [26,27,63] which is realized by imposing the R-symmetry. Another solution is to assign the shift symmetry of the inflaton for the Kähler potential [64,65]. The former leads to the Hybrid or new inflation type potential, and the latter the chaotic or topological inflation type potential. Furthermore, we can avoid the eta problem if the inflation is realized by the D-term potential. For recent review, see for example Ref. [66].

4.2.2 The Affleck-Dine baryogenesis

Here, we will see that the (negative) Hubble-induced mass provides the initial condition for the Affleck-Dine baryogenesis mechanism [55–57].

Let us consider the following potential for the Affleck-Dine field ϕ:

$$V = \left(m_{3/2}^2 - cH^2\right)|\phi|^2 + am_{3/2}\left(\frac{\phi^n}{M^{n-3}} + h.c.\right) + \frac{|\phi|^{2(n-1)}}{M^{2n-6}}, \tag{4.11}$$

where $c > 0$ and a are model-dependent parameter, $m_{3/2}$ is the gravitino mass of order the SUSY breaking scale, $n \geq 4$ is an integer and M is a cut-off scale. The Hubble-induced mass here has negative sign, which arises from the non-minimal Kähler potential like $K = |\phi|^2 + |I|^2 + (1+c/3)|\phi|^2|I|^2/M_{\rm P}^2 + \cdots$ (I is the inflaton). The Affleck-Dine field ϕ is originally a flat direction with baryon (or lepton) charge in the field space, but is lifted by the SUSY breaking effect and the non-renormalizable term. During inflation, the Affleck-Dine field ϕ is stabilized at the potential minimum ϕ_0 where the negative Hubble-induced mass and the non-renormalizable term is balanced:

$$\phi_0 \simeq \left(H_I M^{n-2}\right)^{1/(n-1)}. \tag{4.12}$$

After the inflation, the Universe is dominated by the inflaton coherent oscillation which behaves as matter ($\rho_m \propto a^{-3}$). The Hubble-induced mass now decreases as $H^2(t) \propto t^{-2}$ and the Affleck-Dine filed ϕ traces around the time-dependent potential minimum $\phi_0(t) \simeq \left(H(t)M^{n-2}\right)^{1/(n-1)}$. Eventually, the SUSY breaking mass $m_{3/2}$ dominates over the potential when $H(t) < m_{3/2}$ and then the Affleck-Dine field ϕ goes to the origin.[4-1]

[4-1] So far is the dynamics of the radial component of ϕ. Let us see the angular component to investigate the baryogenesis. Assuming that the Affleck-Dine field ϕ has the baryon charge $+1$, the total baryon number is given by the Noether theorem as

$$\begin{aligned} N_B &= \int \mathrm{d}^3\mathbf{x}\; i\left(\phi\dot{\phi}^* - \phi^*\dot{\phi}\right) \\ &= \int \mathrm{d}^3\mathbf{x}\; 2r \times r\dot{\theta} \end{aligned} \tag{4.13}$$

where the integral is over the co-moving coordinate. In the above second line, we have decomposed the Affleck-Dine field ϕ as $\phi = re^{i\theta}, \quad r > 0$. From Eq. (4.13), we see that the baryon number is generated by the "torque" $r \times r\dot{\theta}$. In Eq. (4.11), this torque comes from the A-term: $am_{3/2}(\phi^n + h.c.)/M^{n-3} =$

4.2.3 The adiabatic solution for the cosmological moduli problem

In this subsection, we will see that the Hubble-induced mass can solve the cosmological moduli problem, which is known as the adiabatic solution [58, 59].

First, let us overview the cosmological moduli problem [73–75]. A moduli field is originally a flat direction in the field space and has no charge. Since the moduli field ϕ_1 is light (its mass is lifted by SUSY breaking effect $m \ll H_I$) and has no charge, ϕ_1 can have any field value during inflation which is estimated to be around the Planck mass scale $M_{\rm P}$. Because of this large field value, the moduli field ϕ_1 stores huge energy density. If the reheating temperature T_{RH} is high enough ($T_{RH} > T_{\rm osc.}$) ($T_{\rm osc.}$ is defined in Eq. (4.15)), the moduli field ϕ_1 begins to oscillate when $H(t_{\rm osc.}) = m$ in the RD era[4-2], where $t_{\rm osc.}$ is given by

$$t_{\rm osc.} \simeq 3 \times 10^{-29} \text{ sec} \left(\frac{10^4 \text{ GeV}}{m} \right). \tag{4.14}$$

At the onset of the moduli oscillation $t_{\rm osc.}$, $H = m$, the Friedman equation leads to

$$\frac{\pi^2 g_*}{30} T_{\rm osc.}^4 = 3 m^2 M_{\rm P}^2. \tag{4.15}$$

Using Eq. (4.15), the ratio of the energy density of the moduli ϕ_1 and radiation is given by

$$\frac{\rho_{\phi_1}}{\rho_{\rm rad}} = \frac{\phi_1^2(t_{\rm osc.})}{6 M_{\rm P}^2} \left(\frac{a(t)}{a(t_{\rm osc.})} \right). \tag{4.16}$$

Since ρ_{ϕ_1} and $\rho_{\rm rad}$ scale as a^{-3} and a^{-4}, respectively, the energy density of the coherent oscillation of the moduli field eventually dominates over the radiation at the time t_* when $\rho_{\phi_1} = \rho_{\rm rad}$. Here, t_* is estimated to be

$$t_* \simeq 36 \; t_{\rm osc.} \left(\frac{M_{\rm P}}{\phi_1(t_{\rm osc.})} \right)^4. \tag{4.17}$$

Since the moduli field has no symmetry, the typical field value of the moduli field at the on set of the oscillation, $\phi_1(t_{\rm osc.})$, is typically expected to be the Planck scale $\phi_1(t_{\rm osc.}) \sim M_{\rm P}$. If this is the case, Eqs. (4.14) and (4.17) lead to $t_* \sim 10^{-27}$sec (10^4 GeV$/m$). On the other hand, the lifetime of the moduli field ϕ_1 is estimated as

$$\tau_\phi \simeq \frac{M_{\rm P}^2}{m^3} \simeq 1 \text{ sec} \left(\frac{10^4 \text{ GeV}}{m} \right)^3, \tag{4.18}$$

$2 a m_{3/2} (r^n / M^{n-3}) \cos(n\theta)$. Finally, the parameters for N_B are restricted to give the correct size of the baryon to photon ratio today $n_B/n_\gamma \sim 10^{-9}$. We note that the scenario of the Affleck-Dine baryogenesis is much altered if the Q-ball formation occurs [67–72].

[4-2] For the case with $T_{RH} < T_{\rm osc.}$, the moduli field begins to oscillate during inflation. Even in this case, the following discussion does not change significantly.

where we have assumed that the moduli interaction is given by a dimension 5 operator which is suppressed by $M_{\rm P}$. If the moduli interaction is suppressed by $M_{\rm P}^2$ or higher, the lifetime of the moduli is longer. Thus, for the wide range of the initial value $\phi_1(t_{\rm osc.})$, the moduli field ϕ_1 decays and produces huge entropy at around the Big-Bang Nucleosynthesis (BBN), which spoils the success of BBN. This is the notorious cosmological moduli problem [73–75].

It has been pointed out that the Hubble-induced mass is useful for solving the moduli problem [58, 59]. Denoting the moduli field value during inflation as ϕ_{1*}, the moduli field is stabilized at ϕ_{1*} during inflation by the positive Hubble-induced mass. Then, in the inflaton oscillation dominated era the moduli filed potential is here given by

$$V = \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}cH^2(t)\left(\phi_1 - \phi_{1*}\right)^2, \tag{4.19}$$

where $c > 0$ and $H(t) = 2/(3t)$ (the inflaton MD era). For this potential, the time-dependent potential minimum $\bar{\phi}_1(t)$ is given by

$$\bar{\phi}_1(t) = \frac{cH^2(t)}{m^2 + cH^2(t)}\phi_{1*}. \tag{4.20}$$

If the parameter $c > 0$ is sufficiently large ($\sim 10^2$ or more), the minimum $\bar{\phi}_1(t)$ goes to the origin with the time scale H^{-1} which is very slow compared with the time scale of the moduli field ϕ, m^{-1}, after $cH^2 \sim m^2$. In this case, the moduli field ϕ_1 follows the time-dependent potential minimum $\bar{\phi}_1(t)$ [58,59]. Eventually, the amplitude of the moduli oscillation is significantly suppressed and the entropy production is highly suppressed. This is the adiabatic solution for the cosmological moduli problem.

4.2.4 The eta problem in the curvaton scenario

Here, we will see that the Hubble-induced mass during inflation is an obstacle for the curvaton mechanism [60–62] in supergravity.

Below, we assume that the Kähler potential for the curvaton is the minimal form. Then the curvaton acquires the positive Hubble-induced mass-squared $\tilde{m}^2 = 3H_I^2$ through supergravity effect as we have explained in Eq. (4.8). Thus the curvaton model with quadratic potential has the following potential during inflation in supergravity:

$$V = \frac{1}{2}\left(m_\sigma^2 + 3H_I^2\right)\sigma^2, \tag{4.21}$$

where the curvaton zero-temperature mass is assumed to be much small: $m_\sigma \ll H_I$. Since the curvaton has a large (effective) mass compared with H_I, the power spectrum of the curvaton fluctuation $\delta\sigma_k(t)$ is given by[4-3]

$$\begin{aligned}\mathcal{P}_{\delta\sigma}(k,t) &= \frac{k^3}{2\pi^2}|\delta\sigma_k(t)|^2 \\ &\simeq \left(\frac{H_I}{2\pi}\right)^2\left(\frac{k}{a(t)H_I}\right)^3,\end{aligned} \tag{4.22}$$

where k is the co-moving momentum, t is the cosmic time, and the scale factor $a(t)$ grows exponentially: $a(t) = \exp(H_I t)$. Thus, the curvaton cannot acquire large super-horizon fluctuation during inflation. This spoils the curvaton mechanism. The large $\eta = M_{\rm P}V''/V \simeq 3$ is originated from the large Hubble scale effective mass term in Eq. (4.21). Thus, in a word, this large η leads to the suppression of the curvaton fluctuation during inflation. So, let us call this problem as the eta problem for the curvaton. If we choose a non-minimal Kähler potential for the curvaton, the Hubble-induced mass can be suppressed. However, in this case we have to tune the parameter in the non-minimal Kähler potential. The curvaton eta problem may be solved by imposing the shift-symmetry on the curvaton Kähler potential or using the D-term inflation model.

[4-3]Here, we have not removed the zero-point fluctuation. If we remove it, the power spectrum will be suppressed more.

Chapter 5

Issues on the Hubble-induced mass in the RD era

As we have seen in the previous chapter, supergravity effects play important roles in the inflaton dominated era in which scalar fields generally acquire the Hubble-induced effective mass through the Planck-suppressed interactions. In a word, this Hubble-induced mass is originated from the energy of the inflaton which dominates the Universe. After the inflationary era, reheating process occurs and the radiation-dominated (RD) era follows. Then a question may arise: is there any source for the effective mass of the order of H after inflation? There is a possibility that the thermal plasma in the RD era provides the source as expected in Ref. [56], since the inflaton energy seems to have converted to the energy of the plasma through reheating process.

In the following two chapters, we investigate the question about whether or not scalar fields acquire an effective mass of the order of the Hubble scale H in the RD era through Planck-suppressed interactions. The starting point we adopt in this chapter is the same as in Refs. [31, 32] in which the thermal expectation value of the kinetic terms of thermalized fields was discussed. These authors claimed opposite answers to the question, *i.e.*, the Hubble scale effective mass does arise and does not from the kinetic terms of the thermalized fields. The crucial point was what dispersion relations we should use for the thermal fields. Since their procedure was naive and they did not use reliable formulation for the thermal fields, their claims could not be justified. In order to treat thermal fields, we here use the techniques of thermal field theory which is the most reliable approach to the issue [33, 34]. This procedure clarifies what dispersion relation the thermal fields have.

However, this is not the end of the story. Even though we apply thermal field theory, there is some problems, as we will see later, in the analysis in this chapter. Namely, this procedure is not so transparent when we proceed analytical calculations, and even worse the analysis suffers from the temperature dependent quadratic divergence. To overcome these difficulties, we propose a solid and more transparent strategy [35] in the next chapter. For the sake of completeness of this thesis, in this chapter we review what has been developed on the issue before the solid and transparent strategy appears.

The organization of this chapter is as follows. In Sec. 5.1, we briefly review the previous studies on the issue and point out the problems of the naive estimations. In Sec. 5.2, we consider the effective mass of ϕ which arises from the thermal expectation value of the scalar field kinetic term in the thermal bath. This thermal expectation value is expressed by the spectral function of the thermalized scalar field, which is evaluated by thermal field theory. In Sec. 5.3, we consider the effective mass of ϕ which arises from the thermal expectation value the fermion kinetic term in the thermal bath. Again, this thermal expectation value is expressed by the spectral function of the thermalized fermion field, which is evaluated by using thermal field theory with some approximations to proceed the analytical calculation. In the course of these analysis, we clarify some difficulties in the procedure we employ here.

5.1 Set-up and naive estimations

Here and hereafter, we consider two complex scalar fields ϕ and χ, whose masses are originally (*i.e.*, at zero temperature) much smaller than the Hubble scale H, in supergravity framework. Here, ϕ is assumed to be decoupled from the thermal bath, whereas χ is in equilibrium with the bath in the RD era. It is assumed that these two fields ϕ and χ interact with each other via the non-minimal Kähler potential given by

$$K = |\phi|^2 + |\chi|^2 + c\,\frac{|\phi|^2|\chi|^2}{M_{\rm P}^2}, \tag{5.1}$$

where $c = \mathcal{O}(1)$ is a model-dependent parameter. Here, ϕ and χ are chiral superfields which include the scalar ϕ and the scalar χ (and fermion $\tilde{\chi}$) as component fields, respectively[5-1]. Even if there are higher order corrections in terms of $M_{\rm P}^{-1}$ in Eq. (5.1), these corrections does not change the following discussions. Then, from Eqs. (4.1) and (5.1), the kinetic term of χ has the following form:

$$\mathcal{L}^{\chi}_{\rm kin.} = \left(1 + c\,\frac{|\phi|^2}{M_{\rm P}^2}\right)\partial_\mu\chi^*\partial^\mu\chi. \tag{5.2}$$

Below, we consider the effecive mass-squared of the scalar field ϕ, $\tilde{m}_\phi^2$, especially in the RD era after the inflationary era. In the RD era, where the Hubble-induced mass due to the inflaton potential disappear, we are interested in what value the effective mass-squared

[5-1] Here and hereafter, we use the same symbols ϕ and χ for both the superfields and the component scalar fields, unless otherwise stated.

$\tilde{m}_\phi^2$ takes. From Eq. (5.2), the kinetic term contribution is given by

$$\tilde{m}_\phi^2|_{\rm kin.} = -\frac{c}{M_{\rm P}^2}\langle \partial_\mu \hat{\chi}^* \partial^\mu \hat{\chi}\rangle. \tag{5.3}$$

Here and hereafter, $\langle \cdots \rangle \equiv {\rm tr}({\rm e}^{-\beta\hat{\mathcal{H}}}\cdots)/{\rm tr}({\rm e}^{-\beta\hat{\mathcal{H}}})$ represents the thermal expectation value, where $\hat{\mathcal{H}}$ is the Hamiltonian of the thermal bath. In Ref. [32], it is insisted that the effective mass-squared which arises from the χ's kinetic term takes a value much smaller than the Hubble scale: $\tilde{m}_\phi^2|_{\rm kin.} \sim \frac{m_\chi^2}{T^2}H^2 \ll H^2$, where m_χ is the zero temperature mass of χ and T is the temperature of the thermal bath. However, the following argument seems possible [31]. The effective mass-squared of ϕ originated from Eq. (5.3) is determined by the thermal average $\langle \partial_\mu \hat{\chi}^* \partial^\mu \hat{\chi}\rangle$ in the RD era. Using an equation of motion for χ, we naively estimate the effective mass-squared as[5-2]

$$\begin{aligned}\tilde{m}_\phi^2|_{\rm kin.} &\simeq \frac{c}{M_{\rm P}^2}\langle \hat{\chi}^* \Box \hat{\chi}\rangle \\ &\simeq -\frac{c m_{\rm th}^2}{M_{\rm P}^2}\langle \hat{\chi}^* \hat{\chi}\rangle \\ &\simeq -c y^2 H^2,\end{aligned} \tag{5.4}$$

where y is a coupling strength of the scalar field χ to the thermal bath (see Eq. (**??**)). Here, thermal mass $m_{\rm th} \simeq yT$ for χ, $\langle \hat{\chi}^* \hat{\chi}\rangle \simeq T^2$, and $T^4 \simeq H^2 M_{\rm p}^2$ are used. In Eq. (5.4), the nontrivial equalities are the first and second line, namely, it is ambiguous whether or not we can use the equation of motion and the thermal mass in the equalities.

There is another naive estimation. Let us directly evaluate $\langle \partial_\mu \hat{\chi}^* \partial^\mu \hat{\chi}\rangle$ by using the expansion of the scalar field χ:

$$\hat{\chi}(x) = \int \frac{{\rm d}^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\left(\hat{a}_{\mathbf{k}}{\rm e}^{-ik\cdot x} + \hat{a}_{\mathbf{k}}^\dagger {\rm e}^{ik\cdot x}\right) \tag{5.5}$$

where $\omega_{\mathbf{k}} = \sqrt{|\mathbf{k}|^2 + M_\chi^2}$, M_χ is a kinetic mass of χ, and $\hat{a}_{\mathbf{k}}(\hat{a}_{\mathbf{k}}^\dagger)$ is the annihilation (creation) operator. Then,

$$\begin{aligned}\tilde{m}_\phi^2|_{\rm kin.} &= -\frac{c}{M_{\rm P}^2}\langle \partial_\mu \hat{\chi}^* \partial^\mu \hat{\chi}\rangle \\ &= -\frac{c M_\chi^2}{M_{\rm P}^2}\int \frac{{\rm d}^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\frac{1}{{\rm e}^{\beta\omega_{\mathbf{k}}} - 1},\end{aligned} \tag{5.6}$$

where we have used $k^2 = M_\chi^2$. One may consider M_χ is the zero temperature mass $M_\chi = m_\chi \ll T$, leading to $\tilde{m}_\phi^2 \ll H^2$. On the other hand, as the scalar field χ is in the thermal bath, another may insist χ acquire a thermal mass and $M_\chi = m_{\rm th} \simeq$

[5-2] Such an estimate seems to have been raised in Ref. [56].

yT, leading to $\tilde{m}_\phi^2 \simeq H^2$. These considerations should be confirmed by using a reliable formulation. Therefore, the main purpose of this chapter is to investigate the question: whether $\langle \partial_\mu \hat{\chi}^* \partial^\mu \hat{\chi} \rangle \simeq y^2 T^4$ is correct or not. The essence is what dispersion relation we should use for the thermalized field. The strategy here is to express $\langle \partial_\mu \hat{\chi}^* \partial^\mu \hat{\chi} \rangle$ by the spectral function. By using the spectral function, we can investigate the above question quantitatively by thermal field theory as we will see below. In the course of the analysis, we will face some difficulties which will be resolved in the next chapter.

5.2 Scalar field contributions

In this section, we consider the effective mass of the scalar field ϕ which arises from the kinetic term of the scalar field χ in the thermal bath. We assume here spatial homogeneity and isotropy of the background metric. We also take the zero-temperature mass of χ as $m_\chi = 0$ for simplicity, although the following argument can be applied for nonzero m_χ with $m_\chi \ll H$.

Below, we decompose the complex scalar field as $\chi = \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2)$ ($\chi_{1,2}$ are real scalar fields). Then, the expectation value can be evaluated by $\langle \partial_\mu \hat{\chi}^* \partial^\mu \hat{\chi} \rangle = \langle \partial_\mu \hat{\chi}_1 \partial^\mu \hat{\chi}_1 \rangle$ as long as χ_1 and χ_2 are equivalent.

Now, let us evaluate the expectation value $\langle \partial_\mu \hat{\chi}_1 \partial^\mu \hat{\chi}_1 \rangle$ when the real scalar field χ_1 is in equilibrium with the thermal bath. The thermalization of χ_1 is assumed to take much less time than the Hubble expansion time scale, and the effect of Hubble expansion rate is effectively included in the plasma temperature T. Moreover, assuming the relativistic degrees of freedom in the thermal bath is large enough, we neglect the effect of ϕ-χ_1 interaction Eq. (5.1) to the bath. The smallness of the back reaction of ϕ to the thermal bath is also verified by the tiny coupling between them.

First of all, we express this expectation value by the so-called statistical propagator for the real scalar field χ_1. For this purpose, we note the following equation:

$$\langle \partial_\mu \hat{\chi}_1(x) \partial^\mu \hat{\chi}_1(x) \rangle = \partial_\mu^{x_1} \partial^{x_2 \mu} D^+(x_1, x_2)|_{x_1 = x_2 = x}, \tag{5.7}$$

where we have introduced the two-point function (statistical propagator) $D^+(x_1, x_2)$ as

$$D^+(x_1, x_2) = \frac{1}{2} \langle \{ \hat{\chi}_1(x_1), \hat{\chi}_1(x_2) \} \rangle. \tag{5.8}$$

Since χ_1 is in thermal equilibrium, two-point functions for χ_1 depend only on the difference of the two points: $D^+(x_1, x_2) = D^+(x_1 - x_2)$. We firstly use the spatial Fourier transform as

$$
\begin{aligned}
\langle \partial_\mu \hat{\chi}_1(x) \partial^\mu \hat{\chi}_1(x) \rangle &= \partial_\mu^{x_1} \partial^{x_2 \mu} \int \frac{\mathrm{d}^3 \mathbf{k}}{(2\pi)^3} \mathrm{e}^{+i\mathbf{k}\cdot(\mathbf{x_1}-\mathbf{x_2})} D^+(t_1 - t_2, \mathbf{k}) \Big|_{x_1 = x_2 = x} \\
&= \int \frac{\mathrm{d}^3 \mathbf{k}}{(2\pi)^3} (-\partial_y^2 - |\mathbf{k}|^2) D^+(y, \mathbf{k}) \Big|_{y=0},
\end{aligned}
\tag{5.9}
$$

where $y = t_1 - t_2$.

Next, we have to know the expression for the Fourier component $D^+(y, \mathbf{k})$. Since the real scalar field χ_1 is in thermal equilibrium, we can use the KMS relation [41, 42] (see Eq. (3.47)):

$$
D^+(\omega, \mathbf{k}) = \frac{1}{2} \coth\left(\frac{\beta\omega}{2}\right) \rho_{\chi_1}(\omega, \mathbf{k}), \tag{5.10}
$$

where $\rho_{\chi_1}(\omega, \mathbf{k})$ is the spectral function of χ_1. Then, we obtain the following expression:

$$
\begin{aligned}
D^+(y, \mathbf{k}) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-i\omega y} D^+(\omega, \mathbf{k}) \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{4\pi} \mathrm{e}^{-i\omega y} \coth\left(\frac{\beta\omega}{2}\right) \rho_{\chi_1}(\omega, \mathbf{k}).
\end{aligned}
\tag{5.11}
$$

Now, the problem is reduced to what form the spectral function $\rho_{\chi_1}(\omega, \mathbf{k})$ takes. Once the interaction of the scalar field χ_1 is given, we can calculate the self-energy and the spectral function of χ_1 at the leading order of the coupling constants. Here, let us assume that the self-energy of χ_1, Π_χ has the following form:

$$
\Pi_\chi = \frac{\kappa y^2 T^2}{12} \equiv m_{\mathrm{th}}^2, \tag{5.12}
$$

which is real and here we parameterize the magnitude of the self-energy or thermal mass-squared m_{th}^2 by $\kappa = \mathcal{O}(1)$. Such a self-energy can be realized at the leading order of the coupling if the interaction of the real scalar field χ_1 is dominated by, for example, quartic interactions or yukawa interactions[5-3]. As we have done in Eq. (3.86), from Eq. (5.12), we can obtain the spectral function of χ_1, $\rho_{\chi_1}(\omega, \mathbf{k})$, at the leading order of the coupling y as

$$
\begin{aligned}
\rho_{\chi_1}(\omega, \mathbf{k}) &= \frac{i}{(\omega + i\epsilon)^2 - |\mathbf{k}|^2 - m_{\mathrm{th}}^2} - \frac{i}{(\omega - i\epsilon)^2 - |\mathbf{k}|^2 - m_{\mathrm{th}}^2} \\
&= 2\pi \ \mathrm{sign}(\omega) \delta(\omega^2 - \omega_{\mathrm{th}}^2(\mathbf{k})),
\end{aligned}
\tag{5.13}
$$

[5-3]For example, if the interaction term is given by $\mathcal{L}_{\mathrm{int}} = -\frac{y}{\sqrt{2}}\chi_1 \bar{\psi}\psi - \frac{y^2}{2}\chi_1^2(\tilde{\psi}_1^* \tilde{\psi}_1 + \tilde{\psi}_2^* \tilde{\psi}_2)$, the parameter is $\kappa = 3$ in the one-loop HTL approximation. Here, $\psi, \tilde{\psi}_i$ $(i = 1, 2)$ are massless Dirac Fermion and complex scalar field, respectively, and both in the thermal bath. For finite-temperature system, a fermionic loop has not only the same factor (-1) as in the zero temperature system but also has another factor (-1) arising from the anti-periodicity of fermionic field. Thus, the bosonic and fermionic contributions to the thermal mass of χ_1 do not cancel out each other at least in the one-loop HTL approximation.

where $\omega_{\rm th}(\mathbf{k}) = \sqrt{|\mathbf{k}|^2 + m_{\rm th}^2}$ is the dispersion relation of the thermal field χ_1.

Then, it is easy to obtain $D^+(y, \mathbf{k})$ from Eqs. (5.11) and (5.13) as

$$D^+(y, \mathbf{k}) = \frac{\mathrm{Re}\left(\mathrm{e}^{-i\omega_{\rm th}(\mathbf{k})y}\right)}{\omega_{\rm th}(\mathbf{k})} \left(\frac{1}{2} + f_B(\omega_{\rm th}(\mathbf{k}))\right). \tag{5.14}$$

Now, we are in a position to evaluate the expectation value $\langle \partial_\mu \chi_1(x) \partial^\mu \chi_1(x) \rangle$. Substituting Eq. (5.14) into Eq. (5.9), we obtain the following expression:

$$\langle \partial_\mu \chi_1(x) \partial^\mu \chi_1(x) \rangle = \frac{1}{2\pi^2} \int_0^\infty \mathrm{d}|\mathbf{k}|\ |\mathbf{k}|^2\ \frac{\omega_{\rm th}^2(\mathbf{k}) - |\mathbf{k}|^2}{\omega_{\rm th}(\mathbf{k})} \left(\frac{1}{2} + f_B(\omega_{\rm th}(\mathbf{k}))\right), \tag{5.15}$$

where we have used spatial isotropy for $\omega_{\rm th}(\mathbf{k})$ since χ_1 is in thermal equilibrium. We note that the time derivative in Eq. (5.9) picks up the thermally corrected poles, which make sure the validity of substituting thermal mass in our naive estimates in Eqs. (5.4) and (5.6).

Now, let us write down more explicit expression for $\langle \partial_\mu \hat{\chi}_1(x) \partial^\mu \hat{\chi}_1(x) \rangle$. Since $\omega_{\rm th}(k)$ obey the dispersion relation $\omega_{\rm th}(k) = \sqrt{|\mathbf{k}|^2 + m_{\rm th}^2}$, the factor in Eq. (5.15) becomes

$$\frac{\omega_{\rm th}^2(\mathbf{k}) - |\mathbf{k}|^2}{\omega_{\rm th}(\mathbf{k})} = \frac{m_{\rm th}^2}{\sqrt{|\mathbf{k}|^2 + m_{\rm th}^2}} \tag{5.16}$$

and we obtain

$$\begin{aligned} \langle \partial_\mu \hat{\chi}_1(x) \partial^\mu \hat{\chi}_1(x) \rangle &\simeq \frac{m_{\rm th}^2}{2\pi^2} \int_0^\infty \mathrm{d}|\mathbf{k}|\ \frac{|\mathbf{k}|^2}{\sqrt{|\mathbf{k}|^2 + m_{\rm th}^2}} \left(\frac{1}{2} + f_B\left(\sqrt{|\mathbf{k}|^2 + m_{\rm th}^2}\right)\right) \\ &= m_{\rm th}^2 \left(\langle \hat{\chi}_1^2(x) \rangle_{\rm vac} + \langle \hat{\chi}_1^2(x) \rangle_{\rm T}\right). \end{aligned} \tag{5.17}$$

Here, $\langle \chi_1^2(x) \rangle_{\rm vac}$ and $\langle \chi_1^2(x) \rangle_{\rm T}$ are given by

$$\begin{aligned} \langle \hat{\chi}_1^2(x) \rangle_{\rm vac} &= \frac{1}{4\pi^2} \int_0^\infty \mathrm{d}|\mathbf{k}|\ \frac{|\mathbf{k}|^2}{\sqrt{|\mathbf{k}|^2 + m_{\rm th}^2}}, \\ \langle \hat{\chi}_1^2(x) \rangle_{\rm T} &= \frac{1}{2\pi^2} \int_0^\infty \mathrm{d}|\mathbf{k}|\ \frac{|\mathbf{k}|^2}{\sqrt{|\mathbf{k}|^2 + m_{\rm th}^2}}\ f_B\left(\sqrt{|\mathbf{k}|^2 + m_{\rm th}^2}\right) = \frac{T^2}{2\pi^2} J\left(\beta m_{\rm th}\right), \end{aligned} \tag{5.18}$$

and $J(\alpha)$ is defined by

$$J(\alpha) \equiv \int_\alpha^\infty \mathrm{d}x\ \frac{\sqrt{x^2 - \alpha^2}}{\mathrm{e}^x - 1}. \tag{5.19}$$

The "vacuum" contribution $\langle \hat{\chi}_1^2(x) \rangle_{\rm vac}$ can be evaluated if we adopt the momentum cut-off $M_{\rm P}$ which is the cut-off scale in the supergravity framework. Unfortunately, this vacuum contribution leads to the temperature dependent quadratic divergence for $\tilde{m}_\phi^2$ and we cannot remove this divergence by the procedure itself we have employed here. We return to this difficulty in the end of this section. Here, in order to proceed the analysis, we simply neglect the vacuum contribution $\langle \hat{\chi}_1^2(x)\rangle_{\text{vac}}$. Then, we obtain

$$\langle \partial_\mu \hat{\chi}_1(x) \partial^\mu \hat{\chi}_1(x)\rangle \simeq \frac{m_{\text{th}}^2 T^2}{12} = \frac{\kappa y^2 T^4}{144}, \tag{5.20}$$

where we have used an approximation $J(\beta m_{\text{th}}) = J(\kappa y) \sim J(0) = \pi^2/6$ in the last line assuming the coupling y is small enough. Eq. (5.20) leads to the conclusion that the statement "$\langle \partial_\mu \chi_1(x) \partial^\mu \chi_1(x)\rangle \sim y^2 T^4$" for the real scalar field χ_1 is verified. It is easy to extent Eq. (5.20) to the kinetic term for the complex scalar field χ:

$$\begin{aligned}\langle \partial_\mu \hat{\chi}^*(x) \partial^\mu \hat{\chi}(x)\rangle &= \langle \partial_\mu \hat{\chi}_1(x) \partial^\mu \hat{\chi}_1(x)\rangle \\ &\simeq \frac{\kappa y^2 T^4}{144}.\end{aligned} \tag{5.21}$$

Then, the effective mass-squared for the scalar field ϕ arising from the kinetic term of the thermalized field χ is given by [33]

$$\begin{aligned}\tilde{m}_\phi^2|_{\text{kin.}} &\simeq -\frac{c\kappa}{144}\frac{y^2 T^4}{M_{\text{P}}^2} \\ &= -\frac{15 c\kappa}{24\pi^2 g_*} y^2 H^2,\end{aligned} \tag{5.22}$$

where the relation $3M_{\text{P}}^2 H^2 = \frac{\pi^2 g_*}{30} T^4$ in the RD era is used. Here, g_* is the effective number of the relativistic degrees of freedom in the thermal bath. This is the result which answer the question we raise in Sec. 5.1, namely, the thermal plasma in the early universe provide a source for the Hubble-induced mass-squared $\simeq y^2 H^2/g_*$ under the Kähler potential Eq. (5.1). We note that if there are N complex scalar fields like χ in the thermal bath, the Hubble-induced mass-squared Eq. (5.22) would be enhanced by the factor N.

As we have pointed out, we have neglected the vacuum contribution in Eq. (5.20). Since supergravity framework has a cutoff scale M_{P}, we can write down the divergent vacuum contribution $\langle \hat{\chi}_1^2(x)\rangle_{\text{vac}}$. Unfortunately, there is a temperature-dependent quadratic divergence $\langle \partial_\mu \chi_1(x) \partial^\mu \chi_1(x)\rangle \supset m_{\text{th}}^2 M_{\text{P}}^2/(8\pi^2)$ and we have simply neglect it in Eq. (5.20). In order to remove this temperature dependent divergence, we may have to redefine the effective mass-squared $\tilde{m}_\phi^2$ rather than using Eq. (5.3). However we do not pursue this problem, since we have another strategy for the analysis of the effective mass of ϕ, which is a solid and transparent procedure. We will investigate the detail of this strategy in the next chapter.

5.3 Fermion contributions

So far, we have considered the scalar field (χ) contribution to the effective mass-squared $\tilde{m}_\phi^2$. In supersymmetry framework, however, there is also fermionic counterpart $\tilde{\chi}$. It is expected that the kinetic term and the interaction term of $\tilde{\chi}$ also contribute to $\tilde{m}_\phi^2$. Using again the spectral function, we evaluate the contribution from fermion as well as the scalar field.

In the following, we consider the scalar field ϕ and a chiral fermion $\tilde{\chi}$ (we use two-component notation) in supergravity framework. We assume that masses of ϕ and $\tilde{\chi}$ are originally (*i.e.*, at zero temperature) much smaller than the Hubble scale H, and that ϕ and $\tilde{\chi}$ are interacting only via the non-minimal Kähler potential given by Eq. (5.1):

$$K = |\phi|^2 + |\chi|^2 + c\,\frac{|\phi|^2|\chi|^2}{M_{\rm P}^2}.$$

Here, ϕ and χ are superfields which include the scalar ϕ and the chiral fermion $\tilde{\chi}$, respectively. Then, from Eq. (4.2) the kinetic term of $\tilde{\chi}$ is given by

$$\mathcal{L}_{\rm kin.}^{\tilde{\chi}} = \left(1 + c\frac{|\phi|^2}{M_{\rm P}^2}\right)\hat{\tilde{\chi}}(x) i\sigma^\mu\partial_\mu\tilde{\chi}^*(x). \tag{5.23}$$

Below, we consider the effective mass-squared for the scalar field ϕ, $\tilde{m}_\phi^2$, in the RD era. From the kinetic term Eq. (5.23), the effective mass-squared $\tilde{m}_\phi^2$ from the ϕ - $\tilde{\chi}$ Planck-suppressed interaction is given by

$$\tilde{m}_\phi^2|_{\rm kin.}^{\rm fermion} = -\frac{c}{M_{\rm P}^2}\langle\hat{\tilde{\chi}}(x) i\sigma^\mu\partial_\mu\hat{\tilde{\chi}}^*(x)\rangle. \tag{5.24}$$

In this chapter, we assume that the chiral fermion $\tilde{\chi}$ is in thermal equilibrium. Here and hereafter, again $\langle\cdots\rangle$ represents the thermal expectation value.

Below, we assume the chiral fermion zero-temperature mass $m_0 = 0$ for simplicity, although the following argument can be applied for $m_0 \ll m_f$ (m_f is the thermal mass for the fermion $\tilde{\chi}$)[5-4]. Since the thermalization rate of the fermion is much larger than the Hubble expansion rate, we evaluate the expectation value in Eq. (5.24) in Minkowski spacetime in the following discussion. The Hubble expansion rate relates to the evaluation only through the thermal bath temperature T. Moreover, assuming the relativistic degrees of freedom in the thermal bath is large enough, we neglect the back reaction of ϕ - $\tilde{\chi}$ interaction to the bath. The smallness of the back reaction of ϕ to the thermal bath is also verified by the tiny coupling between them.

First of all, we note the following equation:

$$\langle\hat{\tilde{\chi}}(x) i\sigma^\mu\partial_\mu\hat{\tilde{\chi}}^*(x)\rangle = -i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu^{x_1}\left(\bar{G}^{(+)\dot{\alpha}\alpha}(x_1,x_2) + \frac{i}{2}\bar{G}^{(-)\dot{\alpha}\alpha}(x_1,x_2)\right)\Bigg|_{x_1=x_2=x}, \tag{5.25}$$

[5-4]When the zero-temperature mass is relatively large, $m_0 \simeq m_f (\ll T)$, we have to reconsider the following discussion. On the other hand, as we assume the chiral fermion is in thermal equilibrium, $m_0 \gtrsim T$ case is irrelevant here.

where $\alpha, \dot{\alpha}$ are the spinor indices. Here, we have defined the Green functions $\bar{G}^{(\pm)\dot{\alpha}\alpha}(x_1, x_2)$ as

$$\begin{aligned} \bar{G}^{(+)\dot{\alpha}\alpha}(x_1, x_2) &= \frac{1}{2}\langle\left[\hat{\tilde{\chi}}^{*\dot{\alpha}}(x_1), \hat{\tilde{\chi}}^{\alpha}(x_2)\right]\rangle, \\ \bar{G}^{(-)\dot{\alpha}\alpha}(x_1, x_2) &= i\langle\left\{\hat{\tilde{\chi}}^{*\dot{\alpha}}(x_1), \hat{\tilde{\chi}}^{\alpha}(x_2)\right\}\rangle. \end{aligned} \tag{5.26}$$

Since the chiral fermion $\tilde{\chi}$ is in thermal equilibrium, the Green functions depend only on the difference $x_1 - x_2$: $\bar{G}^{(\pm)\dot{\alpha}\alpha}(x_1, x_2) = \bar{G}^{(\pm)\dot{\alpha}\alpha}(x_1 - x_2)$. Thus, applying spatial Fourier transform, we obtain the following expression:

$$\begin{aligned} &\langle\hat{\tilde{\chi}}(x) i\sigma^\mu \partial_\mu \hat{\tilde{\chi}}^*(x)\rangle \\ &= -i\sigma^\mu_{\alpha\dot{\alpha}}\partial^{x_1}_\mu \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\, \mathrm{e}^{i\mathbf{p}\cdot(\mathbf{x_1}-\mathbf{x_2})} \left(\bar{G}^{(+)\dot{\alpha}\alpha}(t_1 - t_2, \mathbf{p}) + \frac{i}{2}\bar{G}^{(-)\dot{\alpha}\alpha}(t_1 - t_2, \mathbf{p})\right)\Bigg|_{x_1 = x_2 = x} \\ &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\, \mathrm{tr}\left\{(-i\partial_y + \mathbf{p}\cdot\boldsymbol{\sigma})\left(\bar{G}^{(+)}(y, \mathbf{p}) + \frac{i}{2}\bar{G}^{(-)}(y, \mathbf{p})\right)\right\}\Bigg|_{y=0}, \end{aligned} \tag{5.27}$$

where $y = t_1 - t_2$. To go further, we can use the following KMS relation between the Green functions for the chiral fermion $\tilde{\chi}$ [41, 42]:

$$\bar{G}^{(+)}(\omega, \mathbf{p}) = -\frac{i}{2}\tanh\left(\frac{\beta\omega}{2}\right)\bar{G}^{(-)}(\omega, \mathbf{p}). \tag{5.28}$$

From this relation, we obtain

$$\bar{G}^{(+)}(y, \mathbf{p}) + \frac{i}{2}\bar{G}^{(-)}(y, \mathbf{p}) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{4\pi}\, \mathrm{e}^{-i\omega y}\left(\tanh\left(\frac{\beta\omega}{2}\right) - 1\right)\tilde{\rho}_\chi(\omega, \mathbf{p}), \tag{5.29}$$

where we have used the relation $\bar{G}^{(-)}(\omega, \mathbf{p}) = i\bar{\rho}_\chi(\omega, \mathbf{p})$.

Now, we are in a position to use the formula for the spectral function $\bar{\rho}_\chi(\omega, \mathbf{p})$ under quasi-particle approximation. In this approximation, the interactions are assumed to be included in the thermally corrected effective masses of quasiparticles [76–78]. Then, quasi-particles interact only weakly, and the imaginary parts of poles of the spectral function are assumed to be much smaller than the real counterparts. Namely, in this approximation, we neglect the continuum (or multi-particle) state contribution. Assuming that the width of the pole is negligible and using the quasi-particle approximation, the spectral function $\bar{\rho}_\chi(\omega, \mathbf{p})$ is given by [11]

$$\begin{aligned} \bar{\rho}_\chi(\omega, \mathbf{p}) = &\ \pi\left[Z_+(\mathbf{p})\delta(\omega - \omega_+(\mathbf{p})) + Z_-(\mathbf{p})\delta(\omega + \omega_-(\mathbf{p}))\right](1 + \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}) \\ &+ \pi\left[Z_-(\mathbf{p})\delta(\omega - \omega_-(\mathbf{p})) + Z_+(\mathbf{p})\delta(\omega + \omega_+(\mathbf{p}))\right](1 - \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}), \end{aligned} \tag{5.30}$$

where $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ and $Z_\pm(\mathbf{p})$ are the residues of the poles $\omega_\pm(\mathbf{p})$. The brief derivation of Eq. (5.30) is given in Appendix E[5-5]. Using Eqs. (5.29) and (5.30), Eq. (5.25) is reduced to a relatively simple form as follows

$$\langle \hat{\tilde{\chi}}(x) i\sigma^\mu \partial_\mu \hat{\tilde{\chi}}^*(x) \rangle = \frac{1}{\pi^2} \int_0^\infty \mathrm{d}|\mathbf{p}|\ |\mathbf{p}|^2 \Bigg\{ Z_+(\mathbf{p})\left(\omega_+(\mathbf{p}) - |\mathbf{p}|\right)\left(f_F(\omega_+(\mathbf{p})) - \frac{1}{2}\right) + Z_-(\mathbf{p})\left(\omega_-(\mathbf{p}) + |\mathbf{p}|\right)\left(f_F(\omega_-(\mathbf{p})) - \frac{1}{2}\right)\Bigg\}, \tag{5.31}$$

where $f_F(\omega) = 1/(\mathrm{e}^{\beta\omega} + 1)$ is the Fermi-Dirac distribution function. For the evaluation of Eq. (5.31), basically we need the numerical integration since $Z_\pm$ and $\omega_\pm$ have rather complicated form. Here, under a reasonable approximation, we evaluate Eq. (5.31) analytically.

To proceed the analysis further, let us approximate $f_F(\omega_\pm(\mathbf{p}))$. The approximation we use here is based on the following three observations. First, $f_F(\omega_\pm(\mathbf{p}))$ has a cutoff around $\omega_\pm(\mathbf{p}) \simeq T$. Second, when $\omega_\pm(\mathbf{p})$ is small (when $|\mathbf{p}| \ll T$), we can neglect $\omega_\pm(\mathbf{p})$ dependence of $f_F(\omega_\pm(\mathbf{p}))$. Finally, $\omega_\pm(|\mathbf{p}|) \simeq |\mathbf{p}|$ for $|\mathbf{p}| \gg m_f$, where $m_f^2 = \kappa' y^2 T^2/12$ is the thermal mass for the chiral fermion $\tilde{\chi}$. Here, y is a coupling constant and $\kappa' = \mathcal{O}(1)$ is a model-dependent constant.[5-6]. From the above observations, it is reasonable to use the following approximation formula for $f_F(\omega_\pm(\mathbf{p}))$ for all intervals of $|\mathbf{p}|$:

$$f_F(\omega_\pm(\mathbf{p})) \simeq \frac{1}{\mathrm{e}^{\beta|\mathbf{p}|} + 1}. \tag{5.32}$$

Then, the expectation value for the kinetic term Eq. (5.31) becomes

$$\langle \hat{\tilde{\chi}}(x) i\sigma^\mu \partial_\mu \hat{\tilde{\chi}}^*(x) \rangle = \frac{1}{\pi^2} \int_0^\infty \mathrm{d}|\mathbf{p}|\ |\mathbf{p}|^2\ \left\{ Z_+(\mathbf{p})\left(\omega_+(\mathbf{p}) - |\mathbf{p}|\right) + Z_-(\mathbf{p})\left(\omega_-(\mathbf{p}) + |\mathbf{p}|\right)\right\} \frac{1}{\mathrm{e}^{\beta|\mathbf{p}|} + 1}, \tag{5.33}$$

where we have neglected the "vacuum" contribution, which is independent of the distribution function. This vacuum contribution leads to the temperature dependent quadratic divergence for $\tilde{m}_\phi^2$ and we cannot remove this divergence by the procedure itself we have employed here. We return to the difficulty accompanied with this vacuum contribution at the end of this section. Here, to proceed the analysis, we simply neglect this vacuum contribution. The contributions from the integration intervals $[m_f, T]$ and $[0, m_f]$ in Eq. (5.33) give the leading and the next leading order contributions in terms of the

[5-5]We use the symbol $\bar{G}$ for real-time propagator, while we use G for the imaginary-time one in Appendix E.

[5-6]For example, if the interaction term is given by $\mathcal{L}_{\text{int.}} = -y\varphi\tilde{\chi}\tilde{\lambda} + h.c.$ (φ is a complex scalar field and $\tilde{\lambda}$ is a chiral fermion), we obtain $\kappa' = 3/4$ under the one-loop HTL approximation.

coupling y, respectively. Thus, assuming the coupling y is small enough, we apply the approximation formulae $\omega_+(\mathbf{p}) \simeq |\mathbf{p}| + m_f^2/|\mathbf{p}|$, $\omega_-(\mathbf{p}) \simeq |\mathbf{p}|$, $Z_+ \simeq 1$, $Z_- \simeq 0$ (see Eqs. (3.101) and (3.105)) to the whole interval $[0, \infty]$ although these formulae are valid only within the interval $[m_f, T]$:

$$\begin{aligned}\langle \hat{\tilde{\chi}}(x) i\sigma^\mu \partial_\mu \hat{\tilde{\chi}}^*(x)\rangle &\simeq \frac{1}{\pi^2}\int_0^\infty \mathrm{d}|\mathbf{p}|\ |\mathbf{p}|^2 \times \frac{m_f^2}{|\mathbf{p}|}\frac{1}{\mathrm{e}^{\beta|\mathbf{p}|}+1} \\ &= \frac{m_f^2 T^2}{12} = \frac{\kappa' y^2 T^4}{144}.\end{aligned} \tag{5.34}$$

From Eq. (5.34), the effective mass of the scalar field ϕ originated from the kinetic term of the chiral fermion is given by [34]

$$\begin{aligned}\tilde{m}_\phi^2|_{\text{kin.}}^{\text{fermion}} &\simeq -\frac{c\kappa'}{144}\frac{y^2 T^4}{M_{\mathrm{P}}^2} \\ &= -\frac{15 c\kappa'}{24\pi^2 g_*} y^2 H^2,\end{aligned} \tag{5.35}$$

where $m_f^2 = \kappa' y^2 T^2/12$ and the relation $3M_{\mathrm{P}}^2 H^2 = \frac{\pi^2 g_*}{30} T^4$ in the RD era is used, and g_* is the effective number of the relativistic degrees of freedom in the thermal bath.

Eq. (5.35) is the result of this section. This effective mass of ϕ has almost the same magnitude as the one from scalar field kinetic term in the thermal bath [33]. Thus, we find that the chiral fermion kinetic term gives the effective mass-squared $\sim y^2 H^2/g_*$ to the scalar field ϕ under the quasi-particle approximation for $\tilde{\chi}$. As in the scalar field kinetic term case, if there are N chiral fermions, the effective mass-squared (5.35) would be enhanced by the factor N.

We have neglected the "vacuum" contribution in Eq. (5.33). The "vacuum" contribution may lead to $\langle \hat{\tilde{\chi}}(x) i\sigma^\mu \partial_\mu \hat{\tilde{\chi}}^*(x)\rangle_{\text{vac}} \simeq \frac{-1}{2\pi^2}\int_0^{M_{\mathrm{P}}} \mathrm{d}|\mathbf{p}|\ |\mathbf{p}|^2 \times \frac{m_f^2}{|\mathbf{p}|} = -\frac{m_f^2 M_{\mathrm{P}}^2}{4\pi^2}$, where we have introduced the cut-off scale M_{P}. Unfortunately, this is the temperature dependent quadratic divergence, which we have simply neglect it in Eq. (5.33). As in the case with the scalar field contribution, in order to remove this temperature dependent divergence, we may have to redefine the effective mass-squared $\tilde{m}_\phi^2$ rather than using Eq. (5.24). However, again we do not pursue this problem here, since we have another strategy for the analysis of the effective mass of ϕ, which is a solid and more transparent procedure. We will investigate the detail of this anticipated upgrade in the next chapter.

Chapter 6

Analysis of the Hubble-induced mass in the RD era

In the previous chapter, the effective mass-squared of ϕ was expressed in terms of the thermal expectation value of the kinetic term of the coupled (scalar or fermion) fields in the thermal bath. There, the thermalized fields were implicitly assumed to be gauge singlets and the thermal expectation value was evaluated based on thermal field theory. However, the procedure given in the previous chapters would become complicated if we considered all the contributions to the effective mass of ϕ from a realistic thermal bath like SUSY SM particle plasma. This is because the thermal expectation values of the gauge covariant kinetic terms would have to be evaluated. Also, we have to use the quasi-particle approximation and some non-trivial approximations for the analytical evaluation in the procedure employed in the previous chapter. These makes the analysis less transparent. Even worse the analysis in the previous chapter was suffered from the temperature dependent quadratic divergence and we have simply neglected it.

In this chapter, we propose a solid and more transparent procedure for evaluating the effective mass of ϕ [35]. Our observation is as follows: the evaluation will become simple and transparent, if we first rescale the chiral superfields so that the ϕ-dependence is absorbed into yukawa and gauge couplings. This enables us to read off the effective mass term for ϕ from the free energy density calculated with the rescaled couplings, since the free energy density plays the role of the potential term in the equation of motion for ϕ as we will see later.

The main purpose of this chapter is to propose a systematic evaluation of the effective mass of the Planck-suppressed interacting scalar field ϕ. For the first time, the magnitude of the effective mass of such a scalar field in the RD era is clarified with a reliable procedure. The procedure employed in this chapter is free from the complexity, non-trivial approximation and the temperature dependent quadratic divergence. As a demonstration, we show an example calculation with the MSSM plasma. As a first estimate, we give a complete analytic expression for the leading order (in terms of couplings) contribution to the effective mass of ϕ. The rest of this chapter is organized as follows. In Sec. 6.1, we explain our strategy for evaluating the effective mass of ϕ from yukawa couplings. In Sec. 6.2, we also explain how to incorporate the contributions of the gauge couplings. Then, in Sec. 6.3, we give an analytic expression for the effective mass of ϕ arising from MSSM plasma. We also show the numerical result for the temperature dependence of the effective mass of ϕ. In Sec. 6.4, we discuss about the dynamics of the scalar field ϕ which is decoupled from the thermal bath. Finally, Sec. 6.5 is devoted to a brief summary of this chapter.

6.1 Contribution from a yukawa coupling

In this section, we consider a scalar field χ and a fermion $\tilde{\chi}$ in the thermal bath in SUSY. For the moment, we omit the gauge fields for simplicity, though the following procedures can be applied directly to the case with the gauge fields. The scalar field ϕ, which is decoupled from the thermal bath, is assumed to have a coupling with χ and $\tilde{\chi}$ only through the following non-minimal Kähler potential given by Eq. (5.1):

$$K = |\phi|^2 + |\chi|^2 + c\frac{|\phi|^2|\chi|^2}{M_{\rm P}^2}.$$

In the following subsections, we evaluate the contributions to the effective mass of the scalar field ϕ, $\tilde{m}_\phi$, from the thermalized fields χ and $\tilde{\chi}$. In the evaluation, we neglect the zero-temperature masses of χ and $\tilde{\chi}$ for simplicity[6-1].

6.1.1 Scalar field contributions

In this subsection, we will take into account the supergravity effects that appear both in the kinetic term of the scalar field χ and in the F-term potential. We note that the latter effects were neglected in Chap. 5.

From Eqs. (5.1) and (4.1), the scalar field χ has the following kinetic term:

$$\mathcal{L}_{\rm kin.}^{\chi} = \left(1 + \frac{c|\phi|^2}{M_{\rm P}^2}\right)\partial_\mu\chi^*\partial^\mu\chi. \tag{6.1}$$

On the other hand, the F-term potential in supergravity is given by Eq. (4.3):

$$V_F = {\rm e}^{K/M_{\rm P}^2}\left(D_i W K^{i\bar{j}}\overline{D_j W} - \frac{3|W|^2}{M_{\rm P}^2}\right).$$

[6-1]The following argument is valid when the zero-temperature masses of χ and $\tilde{\chi}$ are much smaller than m_s and m_f, respectively. Here, m_s (m_f) is the thermal mass of χ ($\tilde{\chi}$). When the zero-temperature masses of $\chi, \tilde{\chi}$ are comparable to the thermal masses m_s, m_f, we have to include contributions from the zero-temperature masses to $\tilde{m}_\phi$.

Assuming that the superpotential is independent of ϕ, i.e., $W = W(\chi)$, we can extract the ϕ-dependent term from Eq. (4.3) as

$$V_F|_{\phi\text{-dep.}} = \left(1 + \frac{(1-c)|\phi|^2}{M_{\rm P}^2}\right)|W_\chi|^2 + \mathcal{O}\left(M_{\rm P}^{-4}\right). \tag{6.2}$$

Below, in order to regard the scalar field ϕ as a quasi-static external field for χ (and $\tilde{\chi}$), we assume that the zero-temperature mass of ϕ is much smaller than the thermalization rate of χ (and $\tilde{\chi}$). Then, we have the canonical kinetic term for the scalar field χ by rescaling:

$$\mathcal{L}^{\chi}_{\rm kin.} = \partial_\mu \chi'^* \partial^\mu \chi', \quad \chi' \equiv \left(1 + \frac{c|\phi|^2}{M_{\rm P}^2}\right)^{1/2}\chi. \tag{6.3}$$

Now, we consider the following yukawa interaction in the superpotenital for χ:

$$W = \frac{y}{3!}\chi^3. \tag{6.4}$$

Then, from Eqs. (6.2) and (6.3), we obtain

$$V_F|_{\phi\text{-dep.}} = \left(1 + \frac{(1-c)|\phi|^2}{M_{\rm P}^2}\right)\left(1 + \frac{c|\phi|^2}{M_{\rm P}^2}\right)^{-2}\frac{y^2}{4}\left(|\chi'|^2\right)^2 = \frac{y'^2}{4}\left(|\chi'|^2\right)^2, \tag{6.5}$$

where we have replaced the coupling y^2 with y'^2 defined by

$$y'^2 \equiv y^2\left(1 + \frac{(1-3c)|\phi|^2}{M_{\rm P}^2}\right). \tag{6.6}$$

Here and hereafter, we neglect $\mathcal{O}(M_{\rm P}^{-4})$ terms. In Eq. (6.6), the kinetic term contribution and the scalar potential contribution are corresponding to $y'^2 \supset y^2 \times \frac{-2c|\phi|^2}{M_{\rm P}^2}$ and $y'^2 \supset y^2 \times \frac{(1-c)|\phi|^2}{M_{\rm P}^2}$, respectively. Note that, using the canonically normalized scalar field χ', the supergravity effects in the kinetic term (6.1) and the F-term potential (6.2) are eventually absorbed into the rescaled yukawa coupling y'^2. What we have to do for evaluating $\tilde{m}_\phi$ is, then, to extract the effective mass term for ϕ from the free energy density generated by the rescaled yukawa coupling y'^2, since the free energy density is nothing but the potential term in the equation of motion for ϕ (we will see in Sec. 6.4). We note that since the scalar field kinetic term and the interaction terms are now canonical and renormalizable, respectively, in the sense of the zero-temperature system, the free energy density is free from the temperature dependent divergence once we prepare the zero-temperature counter terms [10, 50–54]. Thus, the 4-point interaction (6.5) gives rise to the 2-loop contribution to the (renormalized) free energy density of the system, $\tilde{\Omega}_2$, which is given by

$$\tilde{\Omega}_2 = \frac{y'^2 T^4}{288} = \frac{y^2 T^4}{288} - \frac{\left(c - \frac{1}{3}\right) y^2 |\phi|^2}{96}\frac{T^4}{M_{\rm P}^2}. \tag{6.7}$$

From this free energy density, we obtain the following effective mass-squared $\tilde{m}_\phi^2$ from the 4-point yukawa interaction (6.5):

$$\tilde{m}_\phi^2 = -\frac{c - \frac{1}{3}}{96}\frac{y^2 T^4}{M_{\rm P}^2}. \tag{6.8}$$

In Eq. (6.8), the kinetic term contribution is corresponding to $\tilde{m}_\phi^2 \supset -\frac{c}{144}\frac{y^2T^4}{M_{\rm P}^2}$, which is the same as the one in Eq. (5.22) with $\kappa = 1$ which is the case here (4-point scalar interaction $V = \frac{y^2}{4}|\chi|^4$ in Eq. (6.5) and without the fermion interactions). Note that $\tilde{m}_\phi^2$ naturally vanishes for $c = 1/3$ in Eq. (6.8), since $c = 1/3$ corresponds to the sequestered Kähler potential form with which the Planck-suppressed interaction between ϕ and χ is essentially absent. As a remark of this subsection, when we consider contributions to $\tilde{m}_\phi$ from MSSM plasma in Sec. 6.3, the above procedure is applied to 4-point interactions which consist of squarks, sleptons and Higgs fields.

6.1.2 Fermion contributions

In this subsection, we will take into account the supergravity effects in a yukawa interaction involving the fermion $\tilde{\chi}$ as well as in the kinetic terms for χ and $\tilde{\chi}$. We note that the supergravity effects in the yukawa interaction were neglected in Chap. 5.

From Eqs. (5.1) and (4.2), the fermionic field $\tilde{\chi}$ has the following kinetic term:

$$\mathcal{L}_{\rm kin.}^{\tilde{\chi}} = \left(1 + \frac{c|\phi|^2}{M_{\rm P}^2}\right)\tilde{\chi} i \sigma^\mu \partial_\mu \tilde{\chi}^*. \tag{6.9}$$

On the other hand, the fermion interaction term in supergravity is given by Eq. (4.6):

$$\mathcal{L}_f = -\frac{1}{2}{\rm e}^{K/(2M_{\rm P}^2)}\left(\mathcal{D}_i D_j W\right)\xi^i\xi^j + h.c. + \cdots .$$

Since ϕ is treated as a quasi-static external field for the fermion $\tilde{\chi}$, we have the canonical kinetic term for $\tilde{\chi}$ by rescaling:

$$\mathcal{L}_{\rm kin.}^{\tilde{\chi}} = \tilde{\chi}' i\sigma^\mu\partial_\mu\tilde{\chi}'^*, \quad \tilde{\chi}' = \left(1 + \frac{c|\phi|^2}{M_{\rm P}^2}\right)^{1/2}\tilde{\chi}. \tag{6.10}$$

The rescaling factor coincides with the scalar field case (see Eq. (6.3)), since it is actually possible to rescale the superfield χ to absorb the ϕ-dependence.

Assuming the non-minimal Kähler potential (5.1) and the superpotential (6.4), Eq. (4.6) gives rise to the following ϕ - $\tilde{\chi}$ interaction:

$$\mathcal{L}_f = -\left(1 + \frac{|\phi|^2}{2M_{\rm P}^2}\right)\left(1 + \frac{c|\phi|^2}{M_{\rm P}^2}\right)^{-3/2}\frac{y}{2}\chi'\tilde{\chi}'\tilde{\chi}' + h.c. = -\frac{y'}{2}\chi'\tilde{\chi}'\tilde{\chi}' + h.c., \tag{6.11}$$

where y' is identical to the one given in Eq. (6.6)[6-2]. In Eq. (6.11), the scalar field kinetic term contribution, the fermion kinetic term contribution and the scalar-fermion-fermion interaction contribution are corresponding to $y' \supset y \times \frac{-c|\phi|^2}{2M_\mathrm{P}^2}$, $y' \supset y \times \frac{-c|\phi|^2}{M_\mathrm{P}^2}$ and $y' \supset y \times \frac{|\phi|^2}{2M_\mathrm{P}^2}$, respectively. Note that the supergravity effects in the kinetic terms (6.3), (6.10) and the scalar-fermion-fermion interaction (6.11) are absorbed into the rescaled yukawa coupling y'. Then, all we need to do is to extract the effective mass term for ϕ from the free energy density arising from the rescaled yukawa coupling. This is because the free energy density is nothing but the potential term in the equation of motion for ϕ (we will see in Sec. 6.4). We note that since the kinetic terms and the interaction terms are now canonical and renormalizable, respectively, in the sense of the zero-temperature system, the free energy density is free from the temperature dependent divergence once we prepare the zero-temperature counter terms [10,50–54]. The scalar-fermion-fermion interaction (6.11) generates the 2-loop contribution to the (renormalized) free energy density of the system, $\tilde{\Omega}_2$, which is given by

$$\tilde{\Omega}_2 = \frac{5y'^2T^4}{1152} = \frac{5y^2T^4}{1152} - \frac{(c-\frac{1}{3})5y^2|\phi|^2}{384}\frac{T^4}{M_\mathrm{P}^2}. \tag{6.12}$$

From this free energy density, we obtain the following effective mass-squared $\tilde{m}_\phi^2$ from the yukawa interaction (6.11):

$$\tilde{m}_\phi^2 = -\frac{5(c-\frac{1}{3})}{384}\frac{y^2T^4}{M_\mathrm{P}^2}. \tag{6.13}$$

In Eq. (6.13), the fermion kinetic term contribution is corresponding to $\tilde{m}_\phi^2 \supset -\frac{5c}{576}\frac{y^2T^4}{M_\mathrm{P}^2}$. The comparison with Eq. (5.35), however, is difficult since in the previous chapter we have made some approximations (including the quasi-particle approximation) in order to obtain the analytical expression (5.35). Before closing this subsection, we note that when we consider contributions to $\tilde{m}_\phi$ from MSSM plasma in Sec. 6.3, the above procedure is applied to the quark-(s)quark-Higgs(ino) and lepton-(s)lepton-Higgs(ino) yukawa interactions originated from the MSSM superpotential.

6.2 Contribution from a gauge coupling

We have seen in Sec. 6.1 that the ϕ-dependences are absorbed into the yukawa couplings by the rescaling (6.3) and (6.10). In this section, we will see that, if the coupled field is charged under gauge symmetry, the rescaling of the chiral fermion generates ϕ-dependent corrections in the gauge coupling at one-loop level. As we shall see later, the numerical

[6-2] The superficial gap of order $\mathcal{O}\left(M_\mathrm{P}^{-4}\right)$ is due to the approximation employed here.

coefficient of this correction turns out to be relatively large especially for the $SU(3)_c$, which partially cancels the one-loop suppression.

In this section, we assume that there are chiral supermultiplets χ_i which have gauge charges, and that the corresponding gauge supermultiplet $V = V^a T^a$ (T^a is the generator) in the thermal bath interacts with ϕ only through the Kähler potential.[6-3] The non-minimal Kähler potential (5.1) is now modified to

$$K = |\phi|^2 + \sum_i \left(1 + \frac{c_i|\phi|^2}{M_{\rm P}^2}\right) \chi_i^\dagger e^{2gV} \chi_i, \tag{6.14}$$

where the sum runs over all the chiral supermultiplets χ_i. In order to obtain the canonical kinetic term, we rescale the chiral supermultiplets χ_i as

$$\chi_i' \equiv \left(1 + \frac{c_i|\phi|^2}{M_{\rm P}^2}\right)^{1/2} \chi_i. \tag{6.15}$$

We note that this rescaling also absorbs the ϕ-dependences in the D-terms and the scalar-fermion-gaugino interaction. Namely, at the tree level, we do not have to rescale the gauge couplings. However, since the chiral supermultiplets χ_i have the gauge charge, the rescaling (6.15) gives rise to the following rescaling anomaly [79, 80]:

$$\prod_i \mathcal{D}\chi_i \mathcal{D}\chi_i^\dagger = \prod_i \mathcal{D}\chi_i' \mathcal{D}\chi_i'^\dagger \exp\left\{ i \int d^4x \sum_i \frac{-1}{16} \int d^2\theta \, \frac{t_2(\chi_i)}{8\pi^2} \frac{c_i|\phi|^2}{M_{\rm P}^2} W_\alpha^a(V_h) W^{\alpha a}(V_h) + h.c. \right\}, \tag{6.16}$$

where $t_2(\chi_i)$ is the Dynkin index and is equal to $1/2$ when χ_i belongs to the fundamental representation, and V_h is the gauge supermultiplet with holomorphic gauge coupling. (V and g are the canonically normalized gauge supermultiplet and coupling before the rescaling.) Here and hereafter, we neglect the $\mathcal{O}(M_{\rm P}^{-4})$ terms. Then, the gauge supermultiplet has the following kinetic term:

$$\mathcal{L}_{\rm kin.}^{\rm gauge} = \frac{1}{16} \int d^2\theta \, \frac{1}{g'^2} W_\alpha^a(g'V) W^{\alpha a}(g'V) + h.c., \tag{6.17}$$

where we have defined the rescaled gauge coupling g'^2 as

$$g'^2 = g^2 \left(1 - g^2 \sum_i \frac{t_2(\chi_i)}{8\pi^2} \frac{c_i|\phi|^2}{M_{\rm P}^2}\right)^{-1} \simeq g^2 \left(1 + \sum_i \frac{t_2(\chi_i)}{2\pi} \frac{g^2}{4\pi} \frac{c_i|\phi|^2}{M_{\rm P}^2}\right). \tag{6.18}$$

From Eq. (6.18), we see that the rescaled gauge coupling g'^2 has the ϕ dependence but with an extra loop-suppression factor compared to the yukawa coupling contribution (6.6). We note that since all the kinetic terms and the interaction terms are now canonical and

[6-3] In particular, no dilatonic coupling is assumed.

renormalizable in the sense of the zero-temperature system, the free energy density is free from the temperature dependent divergence once we prepare the zero-temperature counter terms [10, 50–54]. Thus, when g is the gauge coupling constant of an $SU(N_c)$ SUSY Yang-Mills theory, the $SU(N_c)$ gauge interactions give rise to the 2-loop contribution to the (renormalized) free energy density of the system, $\tilde{\Omega}_2$, which is give by [81]

$$\begin{aligned}\tilde{\Omega}_2 &= N_g\left(N_c + 3\sum_i t_2(\chi_i)\right)\frac{g'^2T^4}{64}\\ &= \left(N_c^2-1\right)\left(N_c + 3\sum_i t_2(\chi_i)\right)\left(1+\sum_i\frac{t_2(\chi_i)}{2\pi}\frac{g^2}{4\pi}\frac{c_i|\phi|^2}{M_{\rm P}^2}\right)\frac{g^2T^4}{64},\end{aligned} \tag{6.19}$$

where we have used $N_g = N_c^2 - 1$. Thus, we obtain the following effective mass-squared $\tilde{m}_\phi^2$ generated by the gauge coupling g:

$$\tilde{m}_\phi^2 = \left(N_c^2-1\right)\left(N_c + 3\sum_i t_2(\chi_i)\right)\frac{\sum_i c_i t_2(\chi_i)}{128\pi}\frac{g^2}{4\pi}\frac{g^2T^4}{M_{\rm P}^2}. \tag{6.20}$$

Note that the numerical coefficient, $(N_c^2-1)(N_c + 3\sum_i t_2(\chi_i))$, can be large, partially canceling the the one-loop suppression factor. Therefore we cannot simply neglect the contribution to $\tilde{m}_\phi^2$ from the gauge coupling. In the next section, we evaluate all the 2-loop free energy density generated by the rescaled yukawa and gauge couplings in MSSM. There, we will see that the gauge coupling contributions to $\tilde{m}_\phi$ can be large corrections to the top yukawa coupling contribution.

6.3 Hubble-induced mass from MSSM plasma

In this section, we provide an analytic expression for the effective mass $\tilde{m}_\phi$ from the yukawa and gauge couplings in MSSM. We also estimate the temperature dependence of $\tilde{m}_\phi^2/H^2$ numerically.

We have explained in the previous sections how to evaluate $\tilde{m}_\phi$ for the given non-minimal Kähler potential and superpotential. In the following, we assume the non-minimal Kähler potential (6.14) where i is now regarded as the MSSM chiral supermultiplet and we replace gV with the MSSM gauge superfields (times gauge couplings). We also assume sufficiently high temperature of the plasma and neglect all the zero-temperature (soft SUSY-breaking) masses of MSSM particles[6-4].

First, let us evaluate the contribution to $\tilde{m}_\phi$ from the MSSM yukawa couplings. We

[6-4] We neglect the soft SUSY-breaking masses in the analytic expression for $\tilde{m}_\phi$, while we take it into account in the renormalization group running of the couplings.

consider the following MSSM superpotential:

$$W_{\rm MSSM} = y_t\left(\bar{t}_R t_L H_u^0 - \bar{t}_R b_L H_u^+\right) + y_b\left(\bar{b}_R b_L H_d^0 - \bar{b}_R t_L H_d^-\right) + y_\tau\left(\bar{\tau}_R \tau_L H_d^0 - \bar{\tau}_R \nu_\tau H_d^-\right), \tag{6.21}$$

where $t_L, b_L, \tau_L, \nu_\tau, H_u^+, H_u^0, H_d^0$ and H_d^- are the $SU(2)_L$ charged chiral superfields, and $\bar{t}_R, \bar{b}_R$ and $\bar{\tau}_R$ are the $SU(2)_L$ singlet anti-particle chiral superfields. Here, we have omitted the 1st and 2nd generation yukawa couplings since they are are much smaller than the 3rd generation ones. Now, we include the supergravity effect which we have discussed in section 6.1. Namely, we rescale all the chiral supermultiplets and yukawa couplings in order to absorb the supergravity effects in the kinetic terms, F-term potential and fermion interactions into the yukawa couplings y_t, y_b and y_τ. As a consequence, we find that the rescaling results in the following replacement for the yukawa couplings $|y|^2 \to |y'|^2$ ($y = y_t, y_b, y_\tau$):

$$|y'|^2 = |y|^2\left(1 + \frac{(1 - c_i - c_j - c_k)|\phi|^2}{M_{\rm P}^2}\right), \tag{6.22}$$

where c_i, c_j and c_k are the coefficients in the non-minimal Kähler potential (5.1) for the corresponding chiral fields. As an illustration, let us consider the interactions arising from the term $W = y_\tau \bar{\tau}_R \tau_L H_d^0$ in Eq. (6.21). From this term, we obtain a 4-point yukawa interaction $|\partial W/\partial \tau_L|^2 = y_\tau^2 |\tilde{\bar{\tau}}_R|^2 |H_d^0|^2$ ($\tilde{\bar{\tau}}_R, H_d^0$ are the scalar component of the superfields $\bar{\tau}_R, H_d^0$). In this case, the coefficients in Eq. (6.22) are determined as $c_i = c_{\tau_L},\ c_j = c_{\bar{\tau}_R},\ c_k = c_{H_d^0}$. On the other hand, one of the yukawa interactions involving fermions we obtain from the term W is $-y_\tau \bar{\tau}_R \tau_L H_d^0 + h.c.$ ($\bar{\tau}_R, \tau_L$ are fermions and H_d^0 is a scalar). For this interaction, we determine the coefficients in Eq. (6.22) as $c_i = c_{\tau_L},\ c_j = c_{\bar{\tau}_R},\ c_k = c_{H_d^0}$ which are identical to the above 4-point yukawa interaction contribution. Now, taking into account of Eq. (6.22), the sum of the 2-loop contributions to the free energy density, $\tilde{\Omega}_2$, from the 3rd generation yukawa couplings are summarized as following:

$$\tilde{\Omega}_2|_{\rm yukawa} = \frac{9\pi T^4}{8} \sum_{i=t,b,\tau} \gamma_i \alpha_{y_i}\left(1 - \frac{3(\bar{c}_i - \frac{1}{3})|\phi|^2}{M_{\rm P}^2}\right). \tag{6.23}$$

where we have defined $\alpha_{y_i} \equiv |y_i|^2/(4\pi)$, $\gamma_t = \gamma_b = 1$ and $\gamma_\tau = 1/3$. Here, $\bar{c}_t, \bar{c}_b$ and $\bar{c}_\tau$ are defined by $\bar{c}_t = \frac{1}{3}\left(c_{\bar{t}_R} + c_{t_L} + c_{H_u}\right)$, $\bar{c}_b = \frac{1}{3}\left(c_{\bar{b}_R} + c_{t_L} + c_{H_d}\right)$ and $\bar{c}_\tau = \frac{1}{3}\left(c_{\bar{\tau}_R} + c_{\tau_L} + c_{H_d}\right)$, respectively. Since the chiral superfields which are included in the same gauge multiplet should have the same coefficient c_i, we have set $c_{b_L} = c_{t_L}, c_{H_u^0} = c_{H_u^+}, c_{H_d^0} = c_{H_d^-}, c_{\nu_\tau} = c_{\tau_L}$. Each contribution to Ω_2 is briefly described in Appendix F. From Eq. (6.23), we can extract the contribution to $\tilde{m}_\phi^2$ from the yukawa couplings y_t, y_b and y_τ.

Next, let us evaluate the contribution to $\tilde{m}_\phi$ from the MSSM gauge couplings. Using the formula Eq. (6.18), we obtain the rescaled gauge couplings in MSSM as

$$\alpha_s' = \alpha_s \left(1 + \frac{3}{\pi}\frac{\bar{c}_s \alpha_s |\phi|^2}{M_{\rm P}^2}\right), \ \alpha_2' = \alpha_2 \left(1 + \frac{7}{2\pi}\frac{\bar{c}_2 \alpha_2 |\phi|^2}{M_{\rm P}^2}\right), \ \alpha_Y' = \alpha_Y \left(1 + \frac{11}{2\pi}\frac{\bar{c}_Y \alpha_Y |\phi|^2}{M_{\rm P}^2}\right). \tag{6.24}$$

And, from Eq. (6.20), the resultant contribution to $\tilde{m}_\phi^2|_{\text{2-loop}}$ is summarized as follows

$$\tilde{m}_\phi^2|_{SU(3)_c} = \frac{63}{2}\frac{\bar{c}_s \alpha_s^2 T^4}{M_{\rm P}^2}, \quad \tilde{m}_\phi^2|_{SU(2)_L} = \frac{483}{32}\frac{\bar{c}_2 \alpha_2^2 T^4}{M_{\rm P}^2}, \quad \tilde{m}_\phi^2|_{U(1)_Y} = \frac{363}{32}\frac{\bar{c}_Y \alpha_Y^2 T^4}{M_{\rm P}^2}, \tag{6.25}$$

where $\alpha_i = g_i^2/(4\pi)$, and g_s, g_2 and g_Y are the gauge couplings of $SU(3)_c, SU(2)_L$ and $U(1)_Y$, respectively. Here, we have defined $\bar{c}_s = \frac{1}{12}\sum_i^{SU(3)_c\text{triplet}} c_i$, $\bar{c}_2 = \frac{1}{14}\sum_i^{SU(2)_L\text{doublet}} c_i$ and $\bar{c}_Y = \sum_i Y_i^2 c_i / \sum_i Y_i^2$. In the definition of $\bar{c}_Y$, i runs all the $U(1)_Y$ chiral supermultiplets.

Now, we are in a position to evaluate the total amount of the effective mass of the Planck-suppressed interacting scalar field ϕ. From Eqs. (6.23) and (6.25), the total contribution to $\tilde{m}_\phi^2$ from the MSSM plasma is given by [35]

$$\begin{aligned}
\tilde{m}_\phi^2 &= -\frac{27\pi}{8}\sum_{i=t,b,\tau}\gamma_i\left(\bar{c}_i - \frac{1}{3}\right)\alpha_{y_i}\frac{T^4}{M_{\rm P}^2} + \left(\frac{63}{2}\bar{c}_s\alpha_s^2 + \frac{483}{32}\bar{c}_2\alpha_2^2 + \frac{363}{32}\bar{c}_Y\alpha_Y^2\right)\frac{T^4}{M_{\rm P}^2} \\
&= \left\{-\frac{81}{61\pi}\sum_{i=t,b,\tau}\gamma_i\left(\bar{c}_i - \frac{1}{3}\right)\alpha_{y_i} + \frac{756}{61\pi^2}\bar{c}_s\alpha_s^2 + \frac{1449}{244\pi^2}\bar{c}_2\alpha_2^2 + \frac{1089}{244\pi^2}\bar{c}_Y\alpha_Y^2\right\}H^2,
\end{aligned} \tag{6.26}$$

where, in the second line, we have used the Friedmann equation in the RD era $3M_{\rm P}^2H^2 = \frac{\pi^2 g_*}{30}T^4$ and $g_* = 228.75 = 915/4$ for the MSSM plasma as the relativistic degrees of freedom in the thermal bath. Here, the thermalization rate of the MSSM particles are much larger than the Hubble expansion rate, and thus the Hubble expansion rate is involved in the above evaluation only through the temperature of the thermal bath. Furthermore, we neglect the back reaction of ϕ to the thermal bath since the coupling between them is tiny (suppressed by $M_{\rm P}^2$). Eq. (6.26) is the analytic expression for $\tilde{m}_\phi^2$ and is one of the main result of this chapter. Note that the largest contributions to $\tilde{m}_\phi^2$ come from the top yukawa coupling y_t and $SU(3)_c$ gauge coupling g_s in typical temperature (see figures below).

Let us comment on higher-loop contributions to $\tilde{m}_\phi^2$. In QCD at finite temperature, it has been recognized that the higher-loop contributions to the free energy density are important and the convergence is poor in the ordinary perturbation theory. A lot of effort has been paid to the calculation of the higher-loop contributions to the free energy density and even the $\mathcal{O}(g_s^6 \ln g_s)$ result was obtained [82]. On the other hand, improved perturbation theories are also investigated and the resultant free energies are found to have good convergence [83–85] (for reviews see Ref. [86–88]). From the results in these studies, we observe that the leading order result in the ordinary perturbation theory is different from the convergence-improved result at most by a factor of order unity. Returning to our subject, the poor convergence of the free energy density evaluation in the ordinary perturbation theory would be true also in MSSM at finite temperature. In fact, we evaluate the next-to-leading order contributions to $\tilde{m}_\phi^2$ in Appendix F, and find that the next-to-leading order contribution is comparable to the leading order (2-loop) one. However, from the observation in the QCD results, even if we include the higher-loop contributions, the magnitude of $\tilde{m}_\phi^2$ would change from the leading order one at most by a factor of order unity. Thus, the leading order result (6.26) can serve as the first estimate of $\tilde{m}_\phi^2$ from the MSSM plasma. Since our main purpose of this paper is to propose a systematic evaluation of $\tilde{m}_\phi$ and show an example calculation with the MSSM plasma, we do not pursue the effect of the higher-loop contributions on $\tilde{m}_\phi$ here.

Lastly we show the numerical results for the temperature dependence of $\tilde{m}_\phi^2/H^2$. In all the figures, we use the public code SOFTSUSY [89] in order to evolve the coupling constants according to the renormalization group equations. For the sake of simplicity, we apply the boundary condition of the minimal supergravity model. However, it should be emphasized that the resultant value of $\tilde{m}_\phi$ does not change significantly even if we impose other boundary condition like the minimal gauge-mediated SUSY breaking or minimal anomaly-mediated SUSY breaking model one. Below, we take $m_0 = m_{1/2} =$ 3 TeV, $A_0 = 0$, $\tan\beta = 20$ (and 40), $\mathrm{sign}(\mu) = +1$ in the minimal supergravity model. Here, $m_0, m_{1/2}$ and A_0 are the unified scalar mass, gaugino mass, trilinear scalar coupling at the GUT scale, respectively, and $\mathrm{sign}(\mu)$ is the sign of the supersymmetric μ term. Also, $\tan\beta = \langle H_u^0 \rangle / \langle H_d^0 \rangle$ is the ratio of the Higgs field vacuum expectation values at the weak scale. $\tilde{m}_\phi^2/H^2$ has only small dependence on the parameter choice as long as the soft SUSY-breaking masses are $\mathcal{O}(1 \sim 10)$ TeV.

Fig. 6.1 shows $\tilde{m}_\phi^2/H^2$ for $\tan\beta = 20, 40$. Here, we set $c_i = 1$ for all the chiral superfields i. The black solid line is the total (yukawa + gauge) contributions to $\tilde{m}_\phi^2/H^2$. The red dashed line, blue dotted line are the sum of the yukawa, gauge coupling contributions to $\tilde{m}_\phi^2/H^2$, respectively. From Fig. 6.1, one can see that $|\tilde{m}_\phi^2|$ is about $H^2/100$, though $\tilde{m}_\phi^2$ mildly depends on the plasma temperature and $\tan\beta$. Although we do not show here, we have checked that the largest contributions to $\tilde{m}_\phi^2$ come from y_t and g_s in most of the temperature range.

In Fig. 6.2, we set $c_i = 0$ (minimal Kähler potential case) for all the chiral superfields i. Here, we again choose $\tan\beta = 20, 40$ cases. The black solid line, red dashed line, green

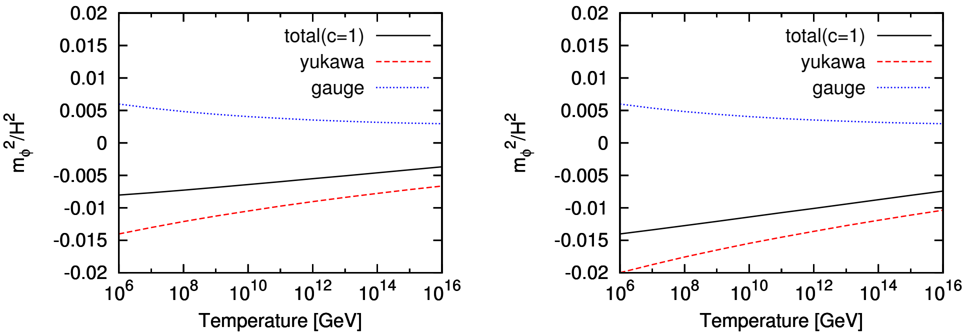

Figure 6.1: Hubble-induced mass-squared $\tilde{m}_\phi^2$ for $\tan\beta = 20$ (left panel), 40 (right panel) [35]. We set $c_i = 1$ for all chiral superfields i. The black solid line is the total (yukawa + gauge) contribution to $\tilde{m}_\phi^2/H^2$. The red dashed line, blue dotted line are the sum of the yukawa, gauge coupling contributions to $\tilde{m}_\phi^2/H^2$, respectively.

dotted line and blue dash-dotted line are the total, y_t, y_b and y_τ contributions to $\tilde{m}_\phi^2/H^2$, respectively. Note that the gauge coupling contributions vanish since no rescaling of the coupled fields is required. From Figs. 6.2, one can see that $\tilde{m}_\phi^2$ is about $H^2/100$. We note that $\tilde{m}_\phi^2$ is always positive in this minimal Kähler potential case ($c_i = 0$).

Finally, in Fig. 6.3, we set $c_i = 1/3$ (sequestered Kähler potential case) for all the chiral superfields i. The black solid line, red dashed line, green dotted line and blue dash-dotted line are the total, g_s, g_2 and g_Y contributions to $\tilde{m}_\phi^2/H^2$, respectively. The yukawa coupling contributions vanish in this case since the chiral superfields are essentially decoupled from the scalar ϕ. Nevertheless the gauge coupling contributions appear because of the rescaling anomaly. From Fig. 6.3, one can see that $\tilde{m}_\phi^2$ is about $H^2/1000 \sim H^2/500$. We note that $\tilde{m}_\phi^2$ is independent of $\tan\beta$ and is always positive in this sequestered Kähler potential case ($c_i = 1/3$).

6.4 The scalar field dynamics and the effective mass

Here, we discuss the dynamics of the Planck-suppressed interacting scalar field ϕ which is decoupled from the thermal bath. In the above evaluations, we have mentioned that the free energy density serves as the potential term in the equation of motion for the scalar field ϕ. However, it is not a trivial fact since the free energy density of the thermal bath itself seems nothing to do with the decoupled scalar field ϕ. In order to discuss about the dynamics of ϕ and its relation to the free energy density, we have to use the knowledge of the non-equilibrium field theory (for a review, see for example Ref. [90]) since the scalar field ϕ is in non-equilibrium.

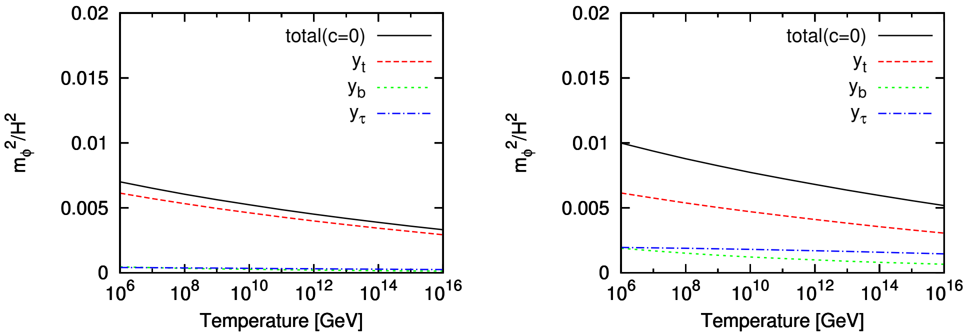

Figure 6.2: Hubble-induced mass-squared $\tilde{m}_\phi^2$ for $\tan\beta = 20$ (left panel), 40 (right panel) [35]. We set $c_i = 0$ (minimal Kähler potential case) for all chiral superfields i. Here, all the gauge coupling contributions are absent because of the choice $c_i = 0$. The black solid line, red dashed line, green dotted line and blue dash-dotted line are the total, y_t, y_b and y_τ contributions to $\tilde{m}_\phi^2/H^2$, respectively.

Let us start the discussion with the generating functional for the Green functions of ϕ and the thermalized field χ, which is given by [91] (see also [17])

$$\begin{aligned} Z[J_\phi, J_\chi] &= \int \mathcal{D}\phi_i \mathcal{D}\phi_i' (\rho_\phi)_{\phi_i \phi_i'} \int_{C;\phi_i}^{\phi_i'} \mathcal{D}\phi \, \exp\left\{ i \int_C \mathrm{d}^4x \, (\mathcal{L}_\phi(\phi) + J_\phi \phi) \right\} \\ &\quad \times \int_{C+C_I;\chi_i}^{\chi_i} \mathcal{D}\chi \, \exp\left\{ i \int_{C+C_I} \mathrm{d}^4x \, (\mathcal{L}_\chi(\chi) + J_\chi \chi + \mathcal{L}_{\mathrm{int.}}(\phi,\chi)) \right\} \\ &= \langle T_{C_I} \left\{ \mathrm{e}^{i\int_{C_I} \mathrm{d}^4x \, J_\chi \hat{\chi}} \right\} T_C \left\{ \mathrm{e}^{i\int_C \mathrm{d}^4x \, (J_\phi \hat{\phi} + J_\chi \hat{\chi})} \right\} \rangle_{\mathrm{full}}, \end{aligned} \tag{6.27}$$

where $\mathcal{L}_\phi(\phi), \mathcal{L}_\chi(\chi)$ and $\mathcal{L}_{\mathrm{int.}}(\phi,\chi)$ are the Lagrangian density for ϕ, χ and their interaction, respectively. C and C_I stand for the time contour along the real axis: $[t_i \to +\infty \to t_i]$ (the Keldysh contour [92]) and parallel with the imaginary axis: $[t_i, t_i - i\beta]$, respectively. T_C and T_{C_I} are the time-ordering operator along the path C and C_I, respectively. For a moment, we omit the effect of the cosmic expansion since we at first integrate out the thermalized field χ which is governed by the bath temperature T ($\gg H$). Here, we have defined $\langle \cdots \rangle_{\mathrm{full}} = \mathrm{tr}(\hat{\rho}_{\mathrm{full}} \cdots)$ and we have assumed that the full density matrix of the system $\hat{\rho}_{\mathrm{full}}$ is given by $\hat{\rho}_{\mathrm{full}} = \hat{\rho}_\phi \bigotimes \hat{\rho}_\chi^{(\mathrm{eq})}$ at the initial time t_i. Also, we have used the notation $\phi_i = \phi(t_i, \mathbf{x})$ and $\langle \phi_i | \, \hat{\rho}_\phi \, | \phi_i' \rangle = (\rho_\phi)_{\phi_i, \phi_i'}$. The path integral $\int_{C;\phi_i}^{\phi_i'} \mathcal{D}\phi$ means that the time contour is C and the initial and final field configurations are ϕ_i and ϕ_i', respectively. Since we are interested in the dynamics of the scalar field ϕ, let us set $J_\chi = 0$. Then, integrating out the thermalized field χ, we obtain the reduced generating functional for ϕ as

$$Z[J_\phi, 0] = \int \mathcal{D}\phi_i \mathcal{D}\phi_i' (\rho_\phi)_{\phi_i \phi_i'} \int_{C;\phi_i}^{\phi_i'} \mathcal{D}\phi \, \exp\left\{ i \int_C \mathrm{d}^4x \, (\mathcal{L}_\phi(\phi) + J_\phi \phi) + iS_{\mathrm{infl.}}[\phi] \right\}, \tag{6.28}$$

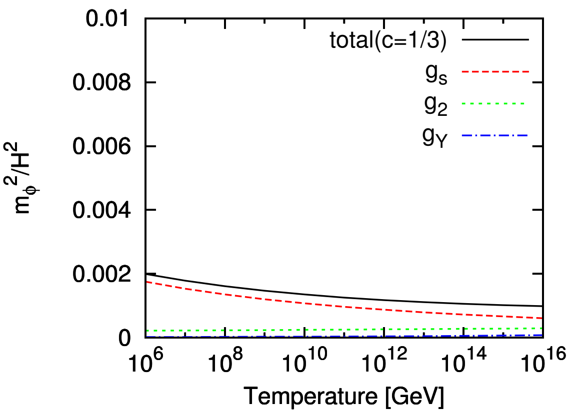

Figure 6.3: Hubble-induced mass-squared $\tilde{m}_\phi^2$ [35]. We have set $c_i = 1/3$ (sequestered Kähler potential case) for all chiral superfields i. In this case, all the yukawa coupling contributions vanish and only gauge couplings contribute to $\tilde{m}_\phi^2$. Here, $\tilde{m}_\phi^2$ has a negligible dependence on $\tan\beta$. The black solid line, red dashed line, green dotted line and blue dash-dotted line are the total, g_s, g_2 and g_Y contributions to $\tilde{m}_\phi^2/H^2$, respectively.

where $S_{\text{infl.}}[\phi]$ is the influence functional [93]. We note that $S_{\text{infl.}}[\phi]$ depends on the thermal bath temperature T since χ is in the bath. In the case we have considered in this chapter, the influence functional $S_{\text{infl.}}[\phi]$ is nothing but the free energy $F[\phi]$:

$$\begin{aligned} S_{\text{infl.}}[\phi] &= -\int_C \mathrm{d}^4x \, \left(\tilde{\Omega}_0 + \tilde{\Omega}_1[\phi] + \tilde{\Omega}_2[\phi] + \cdots\right) \\ &= -\int_C \mathrm{d}^4x \, \left(\tilde{\Omega}_0 + \tilde{\Omega}_2[0] + \tilde{m}_\phi^2 \phi^*(x)\phi(x) + \cdots\right) \\ &= -F[\phi], \end{aligned} \tag{6.29}$$

where $\tilde{\Omega}_n[\phi]$ is the n-loop free energy density with the rescaled couplings, $\tilde{\Omega}_0$ is independent of ϕ and "$\cdots$" represents the higher loop contributions to the free energy. In the second line in Eq. (6.29), we have used the results of the previous sections: $\tilde{\Omega}_2[\phi] = \tilde{\Omega}_2[0] + \tilde{m}_\phi^2\phi^*\phi$. Also, we have neglected $\tilde{\Omega}_1[\phi]$ in the second line in Eq. (6.29) since $\tilde{\Omega}_1[\phi]$ is the higher order correction than $\tilde{\Omega}_2[\phi]$ in terms of the couplings. The effective action for $\varphi = \langle\hat{\phi}\rangle_{\text{full}}$, $\Gamma[\varphi]$, is obtained from the Legendre transformation of $W[J_\phi] = -i\ln Z[J_\phi, 0]$:

$$\Gamma[\varphi] = W[J_\phi] - \int_C \mathrm{d}^4x \, J_\phi(x)\varphi(x), \qquad \left(\varphi(x) = \frac{\delta W[J_\phi]}{\delta\phi(x)} = \langle\hat{\phi}(x)\rangle_{\text{full}}\right). \tag{6.30}$$

For the tree level, we obtain the following effective action for φ:

$$\Gamma[\varphi]|_{\text{tree}} = \int_C \mathrm{d}^4x \, \mathcal{L}_\phi(\varphi) - F[\varphi]. \tag{6.31}$$

The equation of motion for φ is obtained by the variational principle. Since we now have the Keldysh contour C, there are two kinds of fields φ_+ and φ_-, corresponding to each branch of the path ($C_+ = [t_i \to +\infty]$ and $C_- = [+\infty \to t_i]$). However, one of the degrees of freedom is unphysical since we have only one value $\varphi(x)$ for each space-time coordinate x. Thus, the center of mass coordinate $\varphi_c = \frac{1}{2}(\varphi_+ + \varphi_-)$ and the relative coordinate $\varphi_\Delta = \varphi_+ - \varphi_-$ is often used and $\varphi_\Delta = 0$ is taken after variating $\Gamma[\varphi_c, \varphi_\Delta]$ by φ^*_Δ. Namely, the equation of motion for $\varphi_c(x)$ is given by the following formula:

$$\left.\frac{\delta \Gamma[\varphi_c, \varphi_\Delta]}{\delta \varphi^*_\Delta(x)}\right|_{\varphi_\Delta = 0} = 0. \tag{6.32}$$

For our cases in this chapter, $S_{\text{infl.}}[\varphi_c, \varphi_\Delta] = F[\varphi_c, \varphi_\Delta]$ is given by Eq. (6.29). Thus the equation of motion for $\varphi_c(x)$ is given by

$$\ddot{\varphi}_c(x) + 3H\dot{\varphi}_c(x) - \frac{\nabla^2}{a^2(t)}\varphi_c(x) = -\left.\frac{\delta F[\varphi_c, \varphi_\Delta]}{\delta \varphi^*_\Delta(x)}\right|_{\varphi_\Delta = 0}, \tag{6.33}$$

where we have inserted the effect of the cosmic expansion. Assuming that the effective mass term for $\varphi_\pm$, $F[\varphi_c, \varphi_\Delta] \supset \int_{-\infty}^{\infty} \mathrm{d}^4x\ (\tilde{m}^2_\phi \varphi^*_c(x)\varphi_\Delta(x) + h.c.)$, dominates over $S_{\text{infl.}}$ $(= -F)$, the equation of motion for φ_c is given by[6-5]

$$\ddot{\varphi}_c(x) + 3H\dot{\varphi}_c(x) - \frac{\nabla^2}{a^2(t)}\varphi_c(x) \simeq -\tilde{m}^2_\phi \varphi_c(x). \tag{6.34}$$

Now, it is clear from Eq. (6.33) that the free energy $F[\varphi_c, \varphi_\Delta]$ is nothing but the potential term in the equation of motion for $\varphi_c(x)$. This is what we would like to show in this section. In particular, Eq. (6.34) clearly shows that the effective mass term with $\tilde{m}_\phi$ serves as the mass term for the equation of motion for φ_c. In other words, the effective mass $\tilde{m}_\phi$ is an important quantity for the dynamics of φ_c.

Before we discuss the effect of the $\tilde{m}_\phi$ on the dynamics of ϕ, let us mention about the back reaction of ϕ to the thermal bath. Since the coupling between ϕ and the thermal bath is suppressed by M_{P}^2, the relaxation rate of ϕ, Γ_ϕ, is about $\Gamma_\phi/H \sim (H/M_{\text{P}})^4 \ll 1$ as long as the Hubble scale effective mass dominates the ϕ's potential. Thus, ϕ is always decoupled from the thermal bath. The back reaction of ϕ to the thermal bath is also characterised by Γ_ϕ which is much smaller than the bath temperature T. This means that the back reaction of ϕ to the thermal bath is always negligible in our cases.

Now, we are in a position to discuss the impact of the "Hubble-induced mass" in the RD era (6.26) on cosmology. As for the adiabatic solution of the cosmological moduli

[6-5] In general, the influence functional $S_{\text{infl.}}[\phi]$ (in our case the free energy $F[\phi]$) includes the non-local terms which entail the dissipative coefficient (friction term) and the noise term in the equation of motion (see for example Refs. [17–22, 91]).

problem, our findings show that, even if the couplings between the modulus and the MSSM sector are enhanced by two orders of magnitude, i.e., $|c| = \mathcal{O}(100)$, the Hubble-induced mass for the modulus is not sufficient to suppress the modulus abundance when it starts to oscillate after reheating. This results in a rather robust upper bound on the reheating temperature [59, 94] for the adiabatic solution to work. Also, the Hubble-induced mass in the RD era may be useless for the Affleck-Dine baryogenesis scenario since there is often much larger thermal correction of order $\alpha_s^2 T^4 \log(\phi^2/T^2)$ [95]. On the other hand, it may be good for the model building of the curvaton scenario in supergravity framework, since the curvaton does not suffer from the Hubble-induced mass at least in the RD era.

6.5 Brief summary and discussion

In this chapter, we have proposed a systematic evaluation of the effective mass of a Planck-suppressed interacting scalar field ϕ. The virtue of this procedure is that the analysis is solid and transparent. In fact, we have overcome the difficulties we faced in the previous chapter. The strategy we have used is as follows. First, we have rescaled the chiral superfields so that the supergravity effects in the kinetic terms, F-term potential and fermion yukawa interactions are absorbed into the rescaled yukawa and gauge couplings. The gauge couplings receive ϕ-dependent corrections from the rescaling anomaly, which is accompanied by a one-loop suppression factor (see Eq. (6.18)) compared to the yukawa couplings (6.22). However, there are relatively large numerical factors in the rescaled gauge couplings and thus we have to include the gauge coupling contributions in the evaluation of $\tilde{m}_\phi$. Then invoking the free energy density with the rescaled couplings, we have read off the expression for $\tilde{m}_\phi$. As a concrete and realistic example, we have shown the calculation of $\tilde{m}_\phi$ arising from the MSSM plasma through Planck-supressed interactions in the non-minimal Kähler potential like Eq. (6.14). The resultant $\tilde{m}_\phi^2$ arising from the sufficiently high temperature MSSM plasma is given in Eq. (6.26), which is about $10^{-3}H^2 \sim 10^{-2}H^2$ for typical parameter sets.

Chapter 7

Conclusions

In this thesis, we have investigated the effective mass of the weakly coupled scalar filed ϕ which interacts with the thermal bath via the Planck-suppressed interaction in the RD era. Such a very weakly coupled scalar filed often appears in the particle physics models in the early Universe. The scalar field ϕ does not have the ordinary coupling to the thermal field and does not acquire the usual thermal mass of order gT (g is the coupling). However, through the Planck-suppressed interaction to the thermal bath, ϕ is expected to have an effective mass of order $T^2/M_{\rm P} \sim H$ in the RD era. Whether or not such a Hubble scale effective mass really arises was an issue. Also, clarifying the magnitude of this effective mass is important for some scenarios of the early Universe.

In this thesis, in order to obtain a reliable result, we have based on thermal field theory. The procedure we have used in Chap. 5 is evaluating the thermal expectation value in terms of the spectral function. This enables us to see how the Hubble scale effective mass is generated and resolves the difficulty the previous studies faced (what dispersion we should use for the thermalized fields). However, this procedure is not so transparent, needs some approximations and even worse suffers from the temperature dependent quadratic divergence. To overcome these difficulties, in Chap. 6, we have proposed a solid and more transparent procedure in which we have only to evaluate the free energy density of the system. The strategy is that we first rescale the chiral fields and the couplings of the thermal bath in order to obtain the canonical form of the kinetic terms and the renormalizable form of the interaction terms for the thermalized fields. Then, we evaluate the free energy density of the system and read off the effective mass of ϕ. The virtue of this procedure is that the analysis is systematic and free from the temperature dependent quadratic divergence. As a demonstration for the complete analysis at leading order of coupling constants, we consider the MSSM plasma and evaluate the effective mass of ϕ. For the first time, we clarify the magnitude of the effective mass-squared of ϕ in the RD era and the result is $|\tilde{m}_\phi^2| = \mathcal{O}(10^{-2} \sim 10^{-3})\ H^2$ for typical parameter sets. This is rather small value in the sense that we cannot use this Hubble scale effective mass (we call it Hubble-induced mass) for the adiabatic solution of the cosmological moduli problem as well as the Affleck-Dine baryogenesis. On the other hand, it may be good for the model building of the curvaton scenario in supergravity framework, since the curvaton does not suffer from the Hubble-induced mass at least in the RD era.

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Appendix A

Statistical mechanics

In this appendix, according to Ref. [96], we briefly summarize the consequences of statistical mechanics as the basis of thermal field theory. For definiteness of our argument, we consider quantum statistical mechanics. Below, we consider a thermal equilibrium system surrounded by a thermal bath. This (sub)system consists of sufficiently many particles but is rather few compared with its environment. The system is characterized by some thermodynamical variables, *e.g.*, internal energy E, entropy S, temperature T, pressure P and volume V. These thermodynamical variables are related each other and we need only two variables (or more if chemical potentials exist) to characterize the system.

A.1 Entropy

Let us consider a thermal equilibrium system surrounded by a thermal bath. We denote the energy of the (macro) system as E_n (n is the label of the state) which is basically fluctuated by the interaction with the thermal bath. We also denote the distribution function of the system as ρ_{mn} which is nothing but the density operator $\hat{\rho}$ in the energy basis. The thermal average of a physical quantity A is defined by

$$\begin{aligned}\langle \hat{A} \rangle &= \mathrm{tr}\left(\hat{\rho}\hat{A}\right) \\ &= \sum_{mn} \rho_{mn} A_{nm}.\end{aligned} \tag{A.1}$$

The interaction between thermal equilibrium (macro) systems can be neglected and thus we can take the thermal equilibrium density matrix ρ_{mn} as diagonal in the energy basis: $\rho_{mn} = \rho_n(E_n)\delta_{mn}$. Since $\hat{\rho}$ depends only on the (almost) conserved quantities[A-1] and $\ln \hat{\rho}$

[A-1] This fact comes from the Liouville's theorem. For the quantum mechanical case, the Liouville's theorem says that $\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = i[\hat{\rho}, \hat{\mathcal{H}}] = 0$ ($\hat{\mathcal{H}}$ is the Hamiltonian of the system).

has the addition property[A-2] we have

$$\hat{\rho} = \exp\left\{a + b\hat{\mathcal{H}}\right\}, \tag{A.2}$$

where a and b are constants. Here, we assume that the system is rest as a whole. Thus, the thermal average of $\ln\hat{\rho}$, $\langle\ln\hat{\rho}\rangle = \sum_n \rho_n \ln\rho_n$, is given by

$$\begin{aligned}\langle\ln\hat{\rho}\rangle &= a + b\langle\hat{\mathcal{H}}\rangle \\ &= \ln\langle\hat{\rho}\rangle.\end{aligned} \tag{A.3}$$

Furthermore, the distribution of the system has a peak with a width $\Delta\Gamma$ (number of states), whose position is corresponding to the energy $\langle\hat{\mathcal{H}}\rangle$. We define the width $\Delta\Gamma$ by the following equation[A-3]:

$$\langle\hat{\rho}\rangle \times \Delta\Gamma = 1. \tag{A.4}$$

Now, let us define the entropy S as

$$S \equiv \ln\Delta\Gamma, \tag{A.5}$$

which is positive by this definition. Using Eq. (A.4), the entropy S can be written as

$$\begin{aligned}S &= -\ln\langle\hat{\rho}\rangle \\ &= -\langle\ln\hat{\rho}\rangle \\ &= -\sum_n \rho_n \ln\rho_n.\end{aligned} \tag{A.6}$$

From this equation, we can evaluate the entropy of the system once we know the distribution function $\rho_n(E_n)$.

A.2 Free energies and chemical potential

In this section, we review the definitions of various free energies. Here, we start with the first law of thermodynamics. The total differential of the energy E ($= \langle E_n\rangle$ in the notation of the previous section) of the system is given by

$$\mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V + \mu\mathrm{d}N, \tag{A.7}$$

where we have introduced the chemical potential μ for the particle number N in the system.

[A-2]This property comes from the statistical independence of macro system.

[A-3]Note that, in the classical mechanics case, ρ_n corresponds to the distribution function in the phase space and $\Delta\Gamma$ corresponds to the phase space volume $\Delta q\Delta p$.

Using $T\mathrm{d}S = \mathrm{d}(TS) - S\mathrm{d}T$ in Eq. (A.7), we obtain the following equation for the Helmholtz free energy $F = E - TS$:

$$\mathrm{d}F = -S\mathrm{d}T - P\mathrm{d}V + \mu\mathrm{d}N. \tag{A.8}$$

Furthermore, using $\mu\mathrm{d}N = \mathrm{d}(\mu N) - N\mathrm{d}\mu$ in Eq. (A.8), we have the following equation for the free energy $\Omega = F - \mu N$:

$$\mathrm{d}\Omega = -S\mathrm{d}T - P\mathrm{d}V - N\mathrm{d}\mu. \tag{A.9}$$

On the other hand, if we use $-P\mathrm{d}V = \mathrm{d}(-PV) + V\mathrm{d}P$ in Eq. (A.8), we obtain the following equation for the Gibbs free energy $\Phi = F + PV$:

$$\mathrm{d}\Phi = -S\mathrm{d}T + V\mathrm{d}P + \mu\mathrm{d}N. \tag{A.10}$$

Now, since Φ is a function of T, P and N, the addition property of the free energy[A-4] leads to the following expression:

$$\Phi = Nf(T, P), \tag{A.11}$$

where $f(T, P)$ is a function of T and P. From Eqs. (A.10) and (A.11), we have

$$\mu = \left(\frac{\partial \Phi}{\partial N}\right)_{T,P} = f(T, P) \tag{A.12}$$

and thus

$$\Phi = \mu N \ (= F + PV). \tag{A.13}$$

From this equation, we can conclude that the chemical potential μ is nothing but the Gibbs energy Φ per particle. Also, from the above equation, we can express the free energy $\Omega = F - \mu N$ as

$$\Omega = -PV. \tag{A.14}$$

Before we close this section, let us consider the chemical potential. The expression for the chemical potential μ can be obtained from Eqs. (A.7), (A.8) and (A.10) as

$$\mu = \left(\frac{\partial E}{\partial N}\right)_{S,V} = \left(\frac{\partial F}{\partial N}\right)_{T,V} = \left(\frac{\partial \Phi}{\partial N}\right)_{T,P}. \tag{A.15}$$

[A-4] Since E, S, V and N have the addition property, the free energies $F = E - TS$, $\Omega = F - \mu N$ and $\Phi = F + PV$ also have the addition property (T, P and μ are the same for any subsystem in a thermal bath).

Thus, as is the case for photon, if the particle number N is not under control, N is automatically determined by the least-free energy principle for the situation under consideration[A-5]. Namely, if T, V = constant, the chemical potential for photon (or some non-charged particles) is determined as

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} = 0. \tag{A.16}$$

On the other hand, the average particle number of the system N can be obtained from Eq. (A.9) as

$$N = -\left(\frac{\partial \Omega}{\partial \mu}\right)_{T,V}. \tag{A.17}$$

A.3 Canonical distribution

In this section, we derive an important relation between the free energy F and the partition function for a canonical ensemble. The canonical distribution is given by

$$\rho_n(E_n) = \frac{1}{Z}\mathrm{e}^{-E_n/T}, \tag{A.18}$$

where Z is the partition function given by

$$Z = \sum_n \mathrm{e}^{-E_n/T}. \tag{A.19}$$

From Eqs. (A.6) and (A.18), the entropy for the canonical ensemble has the following expression:

$$\begin{aligned} S &= -\sum_n \left(\frac{1}{Z}\mathrm{e}^{-E_n/T}\right) \ln\left(\frac{1}{Z}\mathrm{e}^{-E_n/T}\right) \\ &= \frac{E}{T} + \ln Z. \end{aligned} \tag{A.20}$$

Thus, the partition function Z can be written by the Helmholtz free energy $F = E - TS$ as

$$Z = \mathrm{e}^{-F/T}. \tag{A.21}$$

It is interesting to compare this expression with the definition of Z (Eq. (A.19)). In a word, F is an effective potential of the system and thus should be minimized in a thermal equilibrium state (this is nothing but the least-free energy principle). The free energy F can be obtained from the partition function Z as

$$F = -T \ln Z. \tag{A.22}$$

This is the important relation we would like to derive here.

[A-5] The state of the least-free energy is equivalent to the thermal equilibrium state. If the free energy is not the least value, the state is not in thermal equilibrium.

A.4 Grand canonical distribution

Here, we derive an important relation between the free energy Ω and the partition function for the grand canonical ensemble. The result is similar to the one in the previous section. The grand canonical distribution is given by

$$\rho_n = \frac{1}{\Xi}\mathrm{e}^{-(E_n-\mu N_n)/T}, \tag{A.23}$$

where Ξ is the grand partition function given by

$$\Xi = \sum_n \mathrm{e}^{-(E_n-\mu N_n)/T}. \tag{A.24}$$

From Eqs. (A.6) and (A.23), the entropy for the grand canonical ensemble has the following expression:

$$\begin{aligned} S &= -\sum_n \left(\frac{1}{\Xi}\mathrm{e}^{-(E_n-\mu N_n)/T}\right) \ln\left(\frac{1}{\Xi}\mathrm{e}^{-(E_n-\mu N_n)/T}\right) \\ &= \frac{E-\mu N}{T} + \ln\Xi. \end{aligned} \tag{A.25}$$

Thus, the grand partition function Ξ can be written by the free energy $\Omega = F - \mu N$ as

$$\Xi = \mathrm{e}^{-\Omega/T}. \tag{A.26}$$

Again, It is interesting to compare this expression with the definition of Ξ (Eq. (A.24)). In a word, Ω is an effective potential of the system and thus should be minimized in a thermal equilibrium state (this is nothing but the least-free energy principle). The free energy Ω can be obtained from the partition function Ξ as

$$\Omega = -T\ln\Xi. \tag{A.27}$$

This is the important relation we would like to derive in this section.

Appendix B

Fermion propagator

In this appendix, we derive the propagator of a fermion. We start with the full thermal propagator of a fermion $\psi(x_E)$ in the imaginary-time formalism, $S(x_E - x'_E)$, defined by

$$\begin{aligned} S(x_E - x'_E) &\equiv \frac{1}{Z}\mathrm{tr}\left(\mathrm{e}^{-\beta\hat{\mathcal{H}}}\hat{T}_{C_I}\ \hat{\psi}(x_E)\hat{\bar{\psi}}(x'_E)\right) \\ &= \langle \hat{T}_{C_I}\ \hat{\psi}(x_E)\hat{\bar{\psi}}(x'_E)\rangle, \end{aligned} \tag{B.1}$$

where the quantities with $\hat{}$ are the quantum operators, $\bar{\psi} = \psi^\dagger\gamma_0$, $\beta = 1/T$ is the inverse temperature, $\hat{\mathcal{H}}$ is the Hamiltonian of the system and Z is the partition function given by

$$Z = \mathrm{tr}\left(\mathrm{e}^{-\beta\hat{\mathcal{H}}}\right). \tag{B.2}$$

Also, $\hat{T}_{C_I}$ is the time-ordering operator on the time contour $C_I = [0, -i\beta]$ along the imaginary-time axis. Here and hereafter, we neglect the chemical potential for simplicity.

Let us consider the property of $S(x_E - x'_E)$ with the imaginary-time x_4 and x'_4 $(0 \leq x_4, x'_4 \leq \beta)$ in order to investigate the boundary condition of $S(x_E - x'_E)$. First, we define $S^>(x_E - x'_E)$ and $S^<(x_E - x'_E)$ as

$$S(x_E - x'_E) = \begin{cases} \langle\hat{\psi}(x_E)\hat{\bar{\psi}}(x'_E)\rangle = S^>(x_E - x'_E), & (\text{for } x_4 \geq x'_4), \\ -\langle\hat{\bar{\psi}}(x'_E)\hat{\psi}(x_E)\rangle = S^<(x_E - x'_E), & (\text{for } x'_4 \geq x_4). \end{cases} \tag{B.3}$$

Then, $S^>(x_E - x'_E)$ and $S^<(x_E - x'_E)$ satisfy the following relation at the boundary $x_4 = 0, \beta$ (here, $0 \leq x'_4 \leq \beta$):

$$\begin{aligned} S^<(x_E - x'_E)|_{x_4=0} &= S(x_E - x'_E)|_{x_4=0}, \\ S^>(x_E - x'_E)|_{x_4=\beta} &= S(x_E - x'_E)|_{x_4=\beta}. \end{aligned} \tag{B.4}$$

From this relation, we obtain

$$\begin{aligned} S^{<}(x_E - x'_E)|_{x_4=0} &= \frac{1}{Z}\mathrm{tr}\left(\mathrm{e}^{-\beta\hat{\mathcal{H}}}(-\hat{\bar{\psi}}(x'_4,\mathbf{x}')\hat{\psi}(0,\mathbf{x}))\right) \\ &= -\frac{1}{Z}\mathrm{tr}\left(\hat{\bar{\psi}}(x'_4,\mathbf{x}')\mathrm{e}^{-\beta\hat{\mathcal{H}}}\hat{\psi}(\beta,\mathbf{x})\right) \\ &= -\frac{1}{Z}\mathrm{tr}\left(\mathrm{e}^{-\beta\hat{\mathcal{H}}}\hat{\psi}(\beta,\mathbf{x})\hat{\bar{\psi}}(x'_4,\mathbf{x}')\right) \\ &= -S^{>}(x_E - x'_E)|_{x_4=\beta}. \end{aligned} \tag{B.5}$$

Here, we have cyclicly moved the operators in the trace. Namely, we do not have additional minus sign in the third line in Eq. (B.5). Thus, we obtain the following anti-periodic boundary condition for the fermion $\psi(x_E)$ [41, 42][B-1]:

$$S(x_E - x'_E)|_{x_4=0} = -S(x_E - x'_E)|_{x_4=\beta}. \tag{B.6}$$

For taking this anti-periodicity into account, we have only to represent $S(x_E)$ by the Fourier series and integral:

$$S(x_E) = T\sum_{n=-\infty}^{\infty}\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\mathrm{e}^{iP_\mu x_{E\mu}}S(i\omega'_n,\mathbf{p}),\quad \omega'_n = \frac{(2n+1)\pi}{\beta}, \tag{B.7}$$

where n runs all integers ($n = \cdots, -1, 0, 1, \cdots$). In this thesis, we denote the fermionic discrete energy by the symbol with prime as ω'_n. Here and hereafter, we denote the Euclidean four-momentum by capital letter as[B-2]

$$P_\mu = (p_4, \mathbf{p}) = (-\omega'_n, \mathbf{p}) \tag{B.8}$$

and $P_\mu x_{E\mu} = p_4 x_4 + \mathbf{p}\cdot\mathbf{x} = -\omega'_n x_4 + \mathbf{p}\cdot\mathbf{x}$ (we have defined $x_{E\mu} = (x_4, \mathbf{x})$), while we use small letter like p_μ for the four-momentum in the Minkowski spacetime. The inverse transformation of Eq. (B.7) is given by

$$S(P) = S(i\omega'_n, \mathbf{p}) = \int_0^\beta \mathrm{d}x_4 \int \mathrm{d}^3\mathbf{x}\ \mathrm{e}^{-iP_\mu x_{E\mu}}\Delta(x_E). \tag{B.9}$$

The discussion so far is valid for the full propagator in the imaginary-time formalism, $S(x_E)$. Before we go, let us see the consequence for the free propagator $S^F(x_E)$. (Below, we use the superscript " F " for the functions of free fields.) The equation of motion for the free propagator is given by

$$(-i\partial\!\!\!/ + m)S^F(x_E - x'_E) = \delta^{(4)}(x_E - x'_E), \tag{B.10}$$

[B-1] This is quite different from the bosonic case: the bosonic field obeys the periodic boundary condition.
[B-2] The minus sign of $-\omega'_n$ is just a convention.

where $\partial\!\!\!/ = \partial_\mu \gamma_\mu = \partial_4 \gamma_4 + \partial_i \gamma_i$ in the imaginary-time formalism with $\gamma_4 = i\gamma_0$ [B-3]. From Eqs. (B.7) and (B.10), the Fourier component $S^F(P)$ satisfies the following equation:

$$(P\!\!\!/ + m) S^F(P) = 1, \tag{B.11}$$

where $P\!\!\!/ = P_\mu \gamma_\mu = -\omega'_n \gamma_4 + \mathbf{p} \cdot \boldsymbol{\gamma}$. Thus, we arrive at the following free propagator for the fermion ψ in the imaginary-time formalism:

$$S^F(P) = \frac{1}{P\!\!\!/} = \frac{-P\!\!\!/ + m}{P^2 + m^2}, \tag{B.12}$$

where $P^2 = P_\mu P_\mu = \omega'^2_n + |\mathbf{p}|^2 + m^2$ [B-4].

Next, we apply the analytic continuation to the full imaginary-time propagator $S(P)$ in order to investigate real-time propagators. The propagator on a contour C, $\bar{S}_C(x - x')$, can be written as

$$\begin{aligned} \bar{S}_C(x - x') &= \langle \hat{T}_C \ \hat{\psi}(x) \hat{\bar{\psi}}(x') \rangle \\ &= \theta_C(x_0 - x'_0) \ \bar{S}^>(x - x') + \theta_C(x'_0 - x_0) \ \bar{S}^<(x - x'), \end{aligned} \tag{B.14}$$

where we have allowed the complex values of x_0 and x'_0 and defined $\bar{S}^{>(<)}(x - x')$ as

$$\begin{aligned} \bar{S}^>(x - x') &= \langle \hat{\psi}(x) \hat{\bar{\psi}}(x') \rangle, \\ \bar{S}^<(x - x') &= -\langle \hat{\psi}(x') \hat{\bar{\psi}}(x) \rangle. \end{aligned} \tag{B.15}$$

On the imaginary-time contour C_I, $\bar{S}^{>(<)}(x)$ coincides with $S^{>(<)}(x_E)$ defined in Eq. (B.3) as

$$\bar{S}^{>(<)}(-ix_4, \mathbf{x}) = S^{>(<)}(x_4, \mathbf{x}). \tag{B.16}$$

Furthermore, using Eq. (B.6) which is equivalent to $S(x_4 - 0, \mathbf{x}) = -S(x_4 - \beta, \mathbf{x})$ and Eq. (B.3), we have

$$S^>(x_4 - 0, \mathbf{x}) = -S^<(x_4 - \beta, \mathbf{x}). \tag{B.17}$$

Putting together Eqs. (B.16) and (B.17), and using the analytic continuation $x_4 = ix_0$ (here x_0 is real), we have

$$\bar{S}^>(x^0, \mathbf{x}) = -\bar{S}^<(x^0 + i\beta, \mathbf{x}). \tag{B.18}$$

[B-3] In this thesis, we use the Dirac slash notation in both the Minkowski spacetime $p\!\!\!/ = p_\mu \gamma^\mu$ and the Euclidean spacetime (the imaginary-time formalism) $P\!\!\!/ = P_\mu \gamma_\mu$.

[B-4] There is a useful formula in the imaginary-time formalism:

$$A\!\!\!/ A\!\!\!/ = -A^2. \tag{B.13}$$

This is the consequence of $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$.

Let us the two-point Green functions $\bar{S}^{>(<)}(x)$ by the Fourier integral

$$\bar{S}^{>(<)}(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \bar{S}^{>(<)}(p), \tag{B.19}$$

where $p \cdot x = p_\mu x^\mu = p_0 x_0 - \mathbf{p} \cdot \mathbf{x}$. Then, we obtain the following important relation between the Fourier components $\bar{S}^{>(<)}(p)$ from Eq. (B.18) as

$$\bar{S}^{>}(p) = -e^{\beta p_0} \bar{S}^{<}(p). \tag{B.20}$$

If we define the spectral function $\tilde{\rho}(p)$ and a function $\bar{S}^{+}(p)$[B-5] as

$$\begin{aligned} \tilde{\rho}(p) &\equiv \bar{S}^{>}(p) - \bar{S}^{<}(p), \\ \bar{S}^{+}(p) &\equiv \frac{1}{2}(\bar{S}^{>}(p) + \bar{S}^{<}(p)), \end{aligned} \tag{B.21}$$

we obtain the following relations:

$$\begin{aligned} \bar{S}^{<}(p) &= -f_F(p_0)\tilde{\rho}(p), \\ \bar{S}^{>}(p) &= (1 - f_F(p_0))\tilde{\rho}(p), \\ \bar{S}^{+}(p) &= \frac{1}{2} \tanh\left(\frac{\beta p_0}{2}\right) \tilde{\rho}(p). \end{aligned} \tag{B.22}$$

Now, let us write down the full propagator in the imaginary-time formalism, $S(i\omega'_n, \mathbf{p})$, in terms of the spectral function $\tilde{\rho}(p)$. The basic strategy to relate the propagator and the spectral function is the same as in the scalar field case. Using the inverse Fourier transformation (B.9), the relation $S(x_E) = S^{>}(x_E)$ and Eqs. (B.16) and (B.19), we obtain

$$\begin{aligned} S(i\omega'_n, \mathbf{p}) &= \int_0^\beta dx_4 \; e^{i\omega'_n x_4} \; S^{>}(x_4 - 0, \mathbf{p}) \\ &= \int_0^\beta dx_4 \; e^{i\omega'_n x_4} \; \bar{S}^{>}(-ix_4, \mathbf{p}) \\ &= \int_0^\beta dx_4 \; e^{i\omega'_n x_4} \int_{-\infty}^{\infty} \frac{dp'_0}{2\pi} e^{-ip'_0(-ix_4)} \; \bar{S}^{>}(p'_0, \mathbf{p}) \\ &= -\int_{-\infty}^{\infty} \frac{dp'_0}{2\pi} \frac{\tilde{\rho}(p'_0, \mathbf{k})}{i\omega_n - p'_0}. \end{aligned} \tag{B.23}$$

Applying the analytic continuation to Eq. (B.23) as $i\omega'_n \to p_0 \pm i\epsilon$, we obtain

$$\begin{aligned} S(p_0 + i\epsilon, \mathbf{p}) - S(p_0 - i\epsilon, \mathbf{p}) &= -\int_{-\infty}^{\infty} \frac{dp'_0}{2\pi} \left(\frac{1}{p_0 - p'_0 + i\epsilon} - \frac{1}{p_0 - p'_0 - i\epsilon} \right) \tilde{\rho}(p'_0, \mathbf{p}) \\ &= i\tilde{\rho}(p), \end{aligned} \tag{B.24}$$

[B-5] $\bar{S}^{+}(p)$ is sometimes called as statistical propagator.

where we have used the relation $\frac{1}{p_0-p_0'\pm i\epsilon}=\hat{P}\frac{1}{p_0-p_0'}\mp i\pi\delta(p_0-p_0')$ ($\hat{P}\frac{1}{p_0-p_0'}$ is the principal value of $\frac{1}{p_0-p_0'}$). Thus, we can express the spectral function $\tilde{\rho}(k)$, which is defined in the real-time formalism (B.21), by the propagator $S(P)$ defined in the imaginary-time formalism:

$$\begin{aligned}\tilde{\rho}(p) &= \bar{S}^{>}(p)-\bar{S}^{<}(p)\\ &= (-iS(p_0+i\epsilon,\mathbf{p}))-(-iS(p_0-i\epsilon,\mathbf{p})).\end{aligned} \tag{B.25}$$

This is a quite useful equation since it is often more convenient to evaluate quantities in the imaginary-time formalism than in the real-time one.

Appendix C

Feynman rule

Here, we write down the Lagrangians for a scalar field, QED and QCD in the imaginary-time formalism for the convenience of diagrammatic calculation.

C.1 Real scalar field

Here, let us write down the imaginary-time action for a real scalar field φ. The action for φ in the Minkowski spacetime is given by

$$\begin{aligned} iS &= i\int \mathrm{d}^4x\ \mathcal{L}(\varphi) \\ &= i\int \mathrm{d}^4x\ \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + \mathcal{L}_{\text{int.}}(\varphi)\right). \end{aligned} \tag{C.1}$$

Here, we assume that the interaction term $\mathcal{L}_{\text{int.}}(\varphi)$ does not include derivative interactions. In order to go to the imaginary-time formalism, we need to replace the time coordinate as $x^0 \to -ix_4$ and

$$\partial_\mu\varphi\partial^\mu\varphi \to -\partial_\mu\varphi\partial_\mu\varphi. \tag{C.2}$$

Thus, the transition of the action (C.1) to the imaginary-time formalism one, S_E, is as follows

$$\begin{aligned} iS &\to i(-i)\int_0^\beta \mathrm{d}^4x_E\ \left(-\frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 + \mathcal{L}_{\text{int.}}(\varphi)\right) \\ &= -\int_0^\beta \mathrm{d}^4x_E\ \left(\frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi + \frac{1}{2}m^2\varphi^2 - \mathcal{L}_{\text{int.}}(\varphi)\right) \\ &\equiv -S_E. \end{aligned} \tag{C.3}$$

From this action, we obtain the free propagator of φ in the imaginary-time formalism as follows

$$\Delta^F(K) = \frac{1}{K^2+m^2} = \frac{1}{\omega_n^2+|\mathbf{k}|^2+m^2}, \tag{C.4}$$

where $K_\mu = (k_4, \mathbf{k}) = (-\omega_n, \mathbf{k})$ is the four-momentum in the imaginary-time formalism, $K^2 = K_\mu K_\mu = \omega_n^2 + |\mathbf{k}|^2$ and $\omega_n = 2\pi n/\beta$ (n is an integer).

C.2 QED

Next, we consider the QED action. Let us start with the QED action in the Minkowski spacetime:

$$iS = i\int \mathrm{d}^4x \, \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi + eA_\mu\bar{\psi}\gamma^\mu\psi\right). \tag{C.5}$$

where A_μ is the photon field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the photon field strength and ψ is the electron field. Also, e is the QED coupling constant. For the transition to the imaginary-time formalism, we need the following replacements:

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &\to F_{\mu\nu}F_{\mu\nu}, \\ \bar{\psi}\psi = \psi^\dagger\gamma^0\psi &\to \psi^\dagger(-i\gamma_4)\psi = \psi^\dagger\gamma^0\psi = \bar{\psi}\psi, \\ \not{A} = \gamma^\mu A_\mu &\to -\gamma_\mu A_\mu = -\not{A}, \end{aligned} \tag{C.6}$$

where $\mu = 0,1,2,3$ on the left-hand side and $\mu = 4,1,2,3$ on the right-hand side and we have used $\gamma_4 \equiv i\gamma_0$. In particular, the transition of the momentum is given by

$$(p_0, \mathbf{p}) = p_\mu = i\partial_\mu \to -i\partial_\mu = P_\mu = (p_4, \mathbf{p}) = (-\omega_n, \mathbf{p}). \tag{C.7}$$

and thus

$$\begin{aligned} \not{\partial} = \gamma^\mu\partial_\mu &\to -\gamma_\mu(-\partial_\mu) = \not{\partial}, \\ \not{p} = \gamma^\mu p_\mu &\to -\gamma_\mu P_\mu = -\not{P}, \end{aligned} \tag{C.8}$$

where again $\mu = 0,1,2,3$ on the left-hand side and $\mu = 4,1,2,3$ on the right-hand side. Eqs. (C.7) and (C.8) are the consequences of the transition $-iX_\mu P^\mu \to +iX_\mu P_\mu$ in the exponential in the Fourier transformation. From Eqs. (C.6) and (C.8), the transition of the action (C.5) to the imaginary-time formalism one, S_E, is as follows

$$\begin{aligned} iS &\to i(-i)\int \mathrm{d}^4x_E \, \left(-\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi - eA_\mu\bar{\psi}\gamma_\mu\psi\right) \\ &= -\int \mathrm{d}^4x_E \, \left(\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \bar{\psi}(-i\not{\partial} + m)\psi + eA_\mu\bar{\psi}\gamma_\mu\psi\right) \\ &\equiv -S_E. \end{aligned} \tag{C.9}$$

From this action, we obtain the free propagators for the photon filed, $\Delta^F_{\mu\nu}(K)$, and for the electron field, $S^F(P)$, in the imaginary-time formalism as follows

$$\begin{aligned} \Delta^F_{\mu\nu}(K) &= \frac{\delta_{\mu\nu}}{K^2} = \delta_{\mu\nu}\Delta^F(K) \quad \text{(Feynmann gauge)}, \\ S^F(P) &= \frac{1}{\not{P} + m} = \frac{-\not{P} + m}{P^2 + m^2} = (-\not{P} + m)\tilde{\Delta}^F(P), \end{aligned} \tag{C.10}$$

where $K_\mu = (k_4, \mathbf{k}) = (-\omega_n, \mathbf{k})$, $\omega_n = 2n\pi/\beta$, $P_\mu = (p_4, \mathbf{p}) = (-\omega'_m, \mathbf{p})$ and $\omega'_m = (2m+1)\pi/\beta$. The even and odd numbers, $2n$ and $2m+1$, are originated from the periodicity (for boson) and the anti-periodicity (for fermion) of the boundary conditions, respectively [41, 42]. In Eq. (C.10), $\Delta^F(K)$ is the same as the one given in Eq. (C.4) and we have defined $\tilde{\Delta}^F(P)$ as

$$\tilde{\Delta}^F(P) = \frac{1}{P^2 + m^2} = \frac{1}{\omega'^2_m + \mathbf{p}^2 + m^2}. \tag{C.11}$$

We note that $\Delta^F(K)$ and $\tilde{\Delta}^F(P)$ have the same form at the first sight, though these have the different types of the imaginary-time discrete energies, ω_n and ω'_m, respectively.

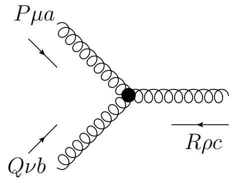
C.3 QCD

Let us move to the QCD action. The QCD action in the covariant gauge is given by

$$\begin{aligned} iS &= i\int \mathrm{d}^4x\, \mathcal{L} \\ &= i\int \mathrm{d}^4x \left(-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar{\eta}^a\partial^\mu\partial_\mu\eta^a + g_s f^{abc}\bar{\eta}_a\partial^\mu(A^c_\mu\eta^b) \right. \\ &\qquad \left. + \bar{\psi}(i\partial\!\!\!/ - m)\psi + g_s A^a_\mu T^a\bar{\psi}\gamma^\mu\psi \right), \end{aligned} \tag{C.12}$$

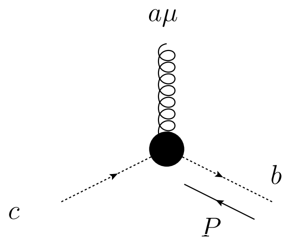
where A^a_μ is the gluon field, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc}A^b_\mu A^c_\nu$ is the gluon field strength, ψ is the quark field, η^a is the ghost field. Also, g_s is the QCD coupling constant, f^{abc} is the structure constant of QCD and the superscripts $a, b, \cdots$ are the color indices. Using the same procedure in the QED case in the previous section, the transition of the action (C.12) to the imaginary-time formalism one, S_E, is as follows

$$\begin{aligned} iS &\to i(-i)\int \mathrm{d}^4x_E \left(-\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 - \bar{\eta}^a\partial_\mu\partial_\mu\eta^a + g_s f^{abc}\bar{\eta}_a\partial_\mu(A^c_\mu\eta^b) \right. \\ &\qquad \left. + \bar{\psi}(i\partial\!\!\!/ - m)\psi - g_s A^a_\mu T^a\bar{\psi}\gamma_\mu\psi \right) \\ &= -\int \mathrm{d}^4x_E \left(\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 + \bar{\eta}^a\partial_\mu\partial_\mu\eta^a - g_s f^{abc}\bar{\eta}_a\partial_\mu(A^c_\mu\eta^b) \right. \\ &\qquad \left. + \bar{\psi}(-i\partial\!\!\!/ + m)\psi + g_s A^a_\mu T^a\bar{\psi}\gamma_\mu\psi \right) \\ &\equiv -S_E, \end{aligned} \tag{C.13}$$



$$= ig_s f_{abc}(\delta_{\mu\nu}(P-Q)_\rho + \delta_{\nu\rho}(Q-R)_\mu + \delta_{\rho\mu}(R-P)_\nu)$$

Figure C.1: 3-gluon vertex



$$= ig_s f_{abc} P_\mu$$

Figure C.2: ghost-gluon vertex

From this action, we obtain the free propagators for the gluon field, $\Delta^{Fab}_{\mu\nu}(K)$, the quark field, $S^F(P)$, and the ghost field, $\Delta^{Fab}(Q)$, as follows

$$\begin{aligned} \Delta^{Fab}_{\mu\nu}(K) &= \frac{\delta_{\mu\nu}\delta^{ab}}{K^2} = \delta_{\mu\nu}\delta^{ab}\Delta^F(K) \quad \text{(Feynmann gauge)}, \\ S^F(P) &= \frac{1}{\not{P} + m} = \frac{-\not{P} + m}{P^2 + m^2} = (-\not{P} + m)\tilde{\Delta}^F(P), \\ \Delta^{Fab}(Q) &= \frac{\delta^{ab}}{Q^2} = \delta^{ab}\Delta^F(Q), \end{aligned} \tag{C.14}$$

where $K = (k_4, \mathbf{k}) = (-\omega_n, \mathbf{k})$, $P = (p_4, \mathbf{p}) = (-\omega'_m, \mathbf{p})$ and $Q = (q_4, \mathbf{q}) = (-\omega_l, \mathbf{q})$. Note that the gluon and the ghost propagators have the even number imaginary-time discrete energies, while the quark propagator has the odd number one. From Eq. (C.13), we also obtain the QCD vertices as follows

$$\begin{aligned} -S_E|_{\text{quark-gluon}} &= -\int d^4x_E \; g_S\bar{\psi}\gamma_\mu A^a_\mu T^a \psi, \\ -S_E|_{\text{ghost-gluon}} &= +\int d^4x_E \; ig_s f_{abc}(P_\mu \bar{\eta}_b) A_{a\mu}\eta_c, \\ -S_E|_{\text{3-gluon}} &= -\int d^4x_E \; g_s f_{abc}(\partial_\mu A^a_\nu) A^b_\mu A^c_\nu, \\ -S_E|_{\text{4-gluon}} &= -\int d^4x_E \; \frac{1}{4} g_s^2 f_{abc} f_{ade} A^b_\mu A^c_\nu A^d_\mu A^e_\nu. \end{aligned} \tag{C.15}$$

Appendix D

Self-energy

In this appendix, we explicitly evaluate the self-energies of a real scalar field, QED electron and QED photon in the imaginary-time formalism, which we have discussed in Secs. 3.2.3, 3.2.4 and 3.2.5.

D.1 Real scalar field

In this section, we evaluate the self-energy of a real scalar field φ. Here, we consider the following yukawa interaction:

$$\mathcal{L}_{\text{int.}} = -\frac{y^2}{2}\varphi^2\tilde{\psi}^*\tilde{\psi}, \tag{D.1}$$

where y is the coupling constant, $\tilde{\psi}$ is a complex scalar field. Below, we neglect the zero-temperature mass of $\tilde{\psi}$ compared with the temperature T. The self-energy of φ, Π, at the one-loop level which arises from Eq. (D.1) is evaluated as follows

$$\begin{aligned} -\Pi &= -\frac{y^2}{2}\int\frac{\mathrm{d}^4K}{(2\pi)^4}\Delta^F(K)\times 4 \\ &= -\frac{y^2}{2}\times 4\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}T\sum_{n=-\infty}^{\infty}\frac{1}{\omega_n^2+|\mathbf{k}|^2}, \end{aligned} \tag{D.2}$$

where $\Delta^F(K)$ is the free propagator given by Eq. (C.4). Here and hereafter, we use the short-hand notation:

$$\int\frac{\mathrm{d}^4K}{(2\pi)^4} = T\sum_{n=-\infty}^{\infty}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}. \tag{D.3}$$

The Euclidean four-momentum K_μ is given as $K_\mu = (k_4, \mathbf{k}) = (-\omega_n, \mathbf{k})$ and $\omega_n = 2\pi n/\beta$ (n is an integer and $\beta = 1/T$).

In order to perform the summation $T\sum_n \frac{1}{\omega_n^2+|\mathbf{k}|^2}$, we use the technique of the complex integration as (see Ref. [11])

$$\begin{aligned} T\sum_n \frac{1}{\omega_n^2+|\mathbf{k}|^2} &= T\oint_C \frac{\mathrm{d}z}{2\pi i}\ \frac{1}{-z^2+|\mathbf{k}|^2}\ \frac{\mathrm{e}^{\beta z}+1}{\mathrm{e}^{\beta z}-1}\ \frac{\beta}{2} \\ &= \frac{1}{|\mathbf{k}|}\left(\frac{1}{2}+f_B(|\mathbf{k}|)\right), \end{aligned} \tag{D.4}$$

where $\oint_C \mathrm{d}z$ shows the complex integration with the complex variable z along the contour $C = [+\epsilon - i\infty, +\epsilon + i\infty] \cup [-\epsilon + i\infty, -\epsilon - i\infty]$ parallel with the imaginary-axis: $\oint_C \mathrm{d}z = \int_{+\epsilon-i\infty}^{+\epsilon+i\infty} \mathrm{d}z + \int_{-\epsilon+i\infty}^{-\epsilon-i\infty} \mathrm{d}z$. Also, $f_B(|\mathbf{k}|) = 1/(\mathrm{e}^{\beta|\mathbf{k}|}-1)$ is the Bose-Einstein distribution function. The zero-point energy contribution $\frac{1}{|\mathbf{k}|}\times\frac{1}{2}$ in Eq. (D.4) gives the divergent contribution to the self-energy Π. Since this is a temperature independent quadratic divergence, we can remove it by the zero-temperature mass counter-term [10, 50–54]. Thus we have the renormalized self-energy Π as follows

$$\begin{aligned} -\Pi &= -2y^2\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{f_B(k)}{k} \\ &= -\frac{y^2T^2}{6}, \end{aligned} \tag{D.5}$$

Consequently, we obtain the following self-energy Π of φ at the one-loop level in the imaginary-time formalism:

$$\Pi = \frac{y^2T^2}{6}, \tag{D.6}$$

which is real and momentum independent.

D.2 QED electron

Here, we evaluate the self-energy of electron in the QED plasma. As we will see below, the self-energy of electron has rather complicated structure compared with the case in the previous section.

Since the QED interaction in the imaginary-time formalism is given by

$$-S_E|_{\text{int.}} = -\int \mathrm{d}^4x_E\ eA_\mu\bar{\psi}\gamma_\mu\psi, \tag{D.7}$$

the self-energy of electron, $\Sigma(P)$, can be evaluated at the one-loop level in the imaginary-time formalism as follows

$$-\Sigma(P) = e^2\int \frac{\mathrm{d}^4K}{(2\pi)^4}\left\{\gamma_\mu S^F(P-K)\Delta^F_{\mu\nu}(K)\gamma_\nu\right\}, \tag{D.8}$$

where we have used the short-hand notation (D.3), $K_\mu = (-\omega_n, \mathbf{k})$, $P_\mu = (-\omega', \mathbf{p})$ and $\Sigma(P) = \Sigma(i\omega', \mathbf{p})$. Here, ω_n is the bosonic imaginary-time discrete energy. $P_\mu = (-\omega', \mathbf{p})$ is the external electron momentum and ω' is the fermionic imaginary-time discrete energy (though we do not show the integer subscript). $S^F(P-K)$ and $\Delta^F_{\mu\nu}(K)$ are the free electron and the (Feynman gauge) free photon propagators, respectively, which are given in Eq. (C.10). Neglecting the electron mass and using the relation $S^F(P-K) = (\not{K} - \not{P})\tilde{\Delta}^F(P-K)$ (see Eq. (C.10)), we obtain

$$\Sigma(P) = -e^2 \int \frac{\mathrm{d}^4K}{(2\pi)^4} \{\gamma_\mu(\not{K} - \not{P})\gamma_\mu\} \, \Delta^F(K)\tilde{\Delta}^F(P-K). \tag{D.9}$$

Let us apply the HTL approximation to our evaluation, in which we assume that the internal lines are dominated by the momentum of the order of the temperature T and the momentum of the external line is less than T. In our case here, this approximation means that $\not{K} - \not{P} \simeq \not{K}$ in Eq. (D.9) and thus $\Sigma(P)$ reduces to

$$\begin{aligned}\Sigma(P) &= -2e^2 \int \frac{\mathrm{d}^4K}{(2\pi)^4} \not{K} \, \Delta^F(K)\tilde{\Delta}^F(P-K) \\ &= -2e^2 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left\{ -T\sum_n \omega_n \Delta^F(K)\tilde{\Delta}^F(P-K)\gamma_4 + T\sum_n k_l \Delta^F(K)\tilde{\Delta}^F(P-K)\gamma_l \right\},\end{aligned} \tag{D.10}$$

where $\hat{k}_l = k_l/|\mathbf{k}|$.

In order to evaluate Eq. (D.10), we use the HTL approximation more and the frequency sum shown below. Let us define $\Delta^F_s(K)$ and $\tilde{\Delta}^F_s(K)$ by

$$\begin{aligned}\Delta^F(K) &= \sum_{s=\pm1} \Delta^F_s(K) = \sum_{s=\pm1} \frac{-s}{2E_k}\frac{1}{i\omega_n - sE_k}, \quad (\omega_n = 2n\pi/\beta), \\ \tilde{\Delta}^F(K) &= \sum_{s=\pm1} \tilde{\Delta}^F_s(K) = \sum_{s=\pm1} \frac{-s}{2E_k}\frac{1}{i\omega'_n - sE_k}, \quad (\omega'_n = (2n+1)\pi/\beta),\end{aligned} \tag{D.11}$$

where $E_k = \sqrt{|\mathbf{k}|^2 + m^2}$ (for a moment, we leave the zero-temperature mass of electron, m). Then, we have the following frequency sum:

$$\begin{aligned}T\sum_n \Delta_{s_1}(K)\tilde{\Delta}_{s_2}(P-K) &= T\sum_n \Delta_{s_1}(i\omega_n, \mathbf{k})\tilde{\Delta}_{s_2}(i(\omega' - \omega_n), \mathbf{p} - \mathbf{k}) \\ &= \frac{-s_1 s_2}{4E_1E_2}\frac{1 + f_B(s_1E_1) - f_F(s_2E_2)}{i\omega' - s_1E_1 - s_2E_2},\end{aligned} \tag{D.12}$$

where $E_1 = \sqrt{|\mathbf{k}|^2 + m^2}$ and $E_2 = \sqrt{|\mathbf{p} - \mathbf{k}|^2 + m^2}$. Here, $f_B(E) = 1/(\mathrm{e}^{\beta E} - 1)$ and $f_F(E) = 1/(\mathrm{e}^{\beta E} + 1)$ are the Bose-Einstein and Fermi-Dirac distribution functions, respectively. Carrying out the summation over $s_1, s_2 = \pm 1$ in Eq. (D.12), we have

$$\begin{aligned} T\sum_{ns_1s_2} \Delta_{s_1}(K)\tilde{\Delta}_{s_2}(P-K) &= \frac{-1}{4E_1E_2}\left\{\frac{1+f_B(E_1)-f_F(E_2)}{i\omega'-E_1-E_2} - \frac{1+f_B(E_1)-f_F(-E_2)}{i\omega'-E_1+E_2}\right. \\ &\qquad \left. - \frac{1+f_B(-E_1)-f_F(E_2)}{i\omega'+E_1-E_2} + \frac{1+f_B(-E_1)-f_F(-E_2)}{i\omega'+E_1+E_2}\right\} \\ &\simeq \frac{f_B(|\mathbf{k}|)+f_F(|\mathbf{k}|)}{4|\mathbf{k}|^2}\left\{\frac{-1}{i\omega'+|\mathbf{p}|\cos\theta} + \frac{1}{i\omega'-|\mathbf{p}|\cos\theta}\right\} \\ &= \frac{f_B(|\mathbf{k}|)+f_F(|\mathbf{k}|)}{4|\mathbf{k}|^2}\left\{\frac{-1}{P_\mu\hat{K}_\mu} + \frac{1}{P_\mu\hat{K}'_\mu}\right\}, \end{aligned} \tag{D.13}$$

where we have defined $\hat{K}_\mu = (-i, \hat{\mathbf{k}}), \hat{K}'_\mu = (-i, -\hat{\mathbf{k}})$ and $\mathbf{p}\cdot\hat{\mathbf{k}} = |\mathbf{p}|\cos\theta$ (θ is the angle between the two three-vectors $\mathbf{p}$ and $\hat{\mathbf{k}}$). Also, we have $P_\mu\hat{K}_\mu = i\omega + |\mathbf{p}|\cos\theta$ and $P_\mu\hat{K}'_\mu = i\omega - |\mathbf{p}|\cos\theta$. In Eq. (D.13), we have set $m = 0$ again and used the HTL approximation in the second line. Namely, $f_B(E_1) = f_B(|\mathbf{k}|)$ and we have approximated as $f_F(E_2) \simeq f_F(|\mathbf{k}|)$ and $i\omega' \pm E_1 \mp E_2 \simeq i\omega' \pm |\mathbf{p}|\cos\theta$. Also, we have neglected the contributions which have the denominator $i\omega' \pm (E_1+E_2) \simeq i\omega' \pm 2|\mathbf{k}|$ since these do not lead to the leading order contribution (proportional to T^2) to the self-energy. Likewise, we can check the following frequency sums [11]:

$$\begin{aligned} T\sum_{ns_1s_2} \omega_n\Delta_{s_1}(K)\tilde{\Delta}_{s_2}(P-K) &\simeq \frac{f_B(|\mathbf{k}|)+f_F(|\mathbf{k}|)}{4|\mathbf{k}|}\left\{\frac{\hat{K}_4}{P_\mu\hat{K}_\mu} + \frac{\hat{K}'_4}{P_\mu\hat{K}'_\mu}\right\}, \\ T\sum_{ns_1s_2} k_l\Delta_{s_1}(K)\tilde{\Delta}_{s_2}(P-K) &\simeq \frac{f_B(|\mathbf{k}|)+f_F(|\mathbf{k}|)}{4|\mathbf{k}|}\left\{\frac{-\hat{k}_l}{P_\mu\hat{K}_\mu} + \frac{\hat{k}_l}{P_\mu\hat{K}'_\mu}\right\}. \end{aligned} \tag{D.14}$$

From Eqs. (D.10) and (D.14), we obtain the following electron self-energy in the HTL approximation at the one-loop level in the imaginary-time formalism:

$$\Sigma(P) = m_f^2 \int \frac{\mathrm{d}\Omega}{4\pi}\frac{\hat{K}\!\!\!/}{P_\mu\hat{K}_\mu}, \tag{D.15}$$

where

$$m_f^2 = \frac{e^2T^2}{8} \tag{D.16}$$

is the electron thermal mass-squared.

D.3 QED photon

In this section, let us evaluate the photon self-energy in the QED plasma. The photon self-energy, $\Pi_{\mu\nu}(Q)$, at the one-loop level is given by

$$-\Pi_{\mu\nu}(Q) = -e^2 \int \frac{\mathrm{d}^4K}{(2\pi)^4}(8K_\mu K_\nu - 4K^2\delta_{\mu\nu})\tilde{\Delta}^F(K)\tilde{\Delta}^F(Q-K), \tag{D.17}$$

where we have used the notation (D.3), $K_\mu = (-\omega'_n, \mathbf{k})$, $Q_\mu = (-\omega, \mathbf{q})$ and $\Pi_{\mu\nu}(Q) = \Pi_{\mu\nu}(i\omega, \mathbf{q})$. Here, ω'_n is the fermionic imaginary-time discrete energy. $Q_\mu = (-\omega, \mathbf{q})$ is the external photon momentum and ω is the bosonic imaginary-time discrete energy (though we do not show the integer subscript). $\tilde{\Delta}^F(K)$ is given by Eq. (C.11).

First of all, let us rewrite Eq. (D.17) as

$$\Pi_{\mu\nu}(Q) = 8e^2 \int \frac{\mathrm{d}^4K}{(2\pi)^4} K_\mu K_\nu \tilde{\Delta}^F(K)\tilde{\Delta}^F(Q-K) - 4e^2\delta_{\mu\nu} \int \frac{\mathrm{d}^4K}{(2\pi)^4}\tilde{\Delta}^F(K), \tag{D.18}$$

where we have used the relation $K^2\tilde{\Delta}^F(K) = 1$ and $\int \frac{\mathrm{d}^4K}{(2\pi)^4}\tilde{\Delta}^F(Q-K) = \int \frac{\mathrm{d}^4K}{(2\pi)^4}\tilde{\Delta}^F(K)$ in the second term. Neglecting the electron zero-temperature mass, the second term is evaluated as

$$\int \frac{\mathrm{d}^4K}{(2\pi)^4}\tilde{\Delta}^F(K) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, T \sum_{n=-\infty}^{\infty} \frac{1}{\omega'^2_n + |\mathbf{k}|^2}. \tag{D.19}$$

In order to perform the summation $T\sum_n \frac{1}{\omega'^2_n+|\mathbf{k}|^2}$ (here $\omega'_n = (2n+1)\pi/\beta$), we use the technique of the complex integration as (see Ref. [11])

$$\begin{aligned} T\sum_n \frac{1}{\omega'^2_n + |\mathbf{k}|^2} &= T \oint_C \frac{\mathrm{d}z}{2\pi i}\, \frac{1}{-z^2 + |\mathbf{k}|^2}\, \frac{\mathrm{e}^{\beta z} - 1}{\mathrm{e}^{\beta z} + 1}\, \frac{-\beta}{-2} \\ &= \frac{1}{|\mathbf{k}|}\left(\frac{1}{2} - f_F(|\mathbf{k}|)\right), \end{aligned} \tag{D.20}$$

where $\oint_C \mathrm{d}z$ shows the complex integration with the complex variable z along the contour $C = [+\epsilon - i\infty, +\epsilon + i\infty] \cup [-\epsilon + i\infty, -\epsilon - i\infty]$ parallel with the imaginary-axis: $\oint_C \mathrm{d}z = \int_{+\epsilon-i\infty}^{+\epsilon+i\infty} \mathrm{d}z + \int_{-\epsilon+i\infty}^{-\epsilon-i\infty} \mathrm{d}z$. Also, $f_F(|\mathbf{k}|) = 1/(\mathrm{e}^{\beta|\mathbf{k}|} + 1)$ is the Fermi-Dirac distribution function. The zero-point energy contribution $\frac{1}{|\mathbf{k}|} \times \frac{1}{2}$ in Eq. (D.20) gives the divergent contribution to the self-energy $\Pi_{\mu\nu}(Q)$. Since this is a temperature independent quadratic divergence, we can remove it by the zero-temperature counter-term [10, 50–54]. Thus, from Eqs. (D.19) and (D.20), we obtain

$$\int \frac{\mathrm{d}^4K}{(2\pi)^4}\tilde{\Delta}^F(K) = -\frac{T^2}{24}, \tag{D.21}$$

where we have removed the zero-point energy as mentioned above. Then, from Eqs. (D.18) and (D.21), the self-energy of photon, $\Pi_{\mu\nu}(Q)$, to be evaluated becomes

$$\begin{aligned}\Pi_{\mu\nu}(Q) &= I_{\mu\nu}(Q) + \frac{e^2T^2}{6}\delta_{\mu\nu}, \\ I_{\mu\nu}(Q) &= 8e^2\int \frac{\mathrm{d}^4K}{(2\pi)^4}K_\mu K_\nu \tilde{\Delta}^F(K)\tilde{\Delta}^F(Q-K).\end{aligned} \tag{D.22}$$

Below, we evaluate $I_{\mu\nu}(Q)$.

First, we evaluate the following quantity:

$$J_{ij}(Q) = \int \frac{\mathrm{d}^4K}{(2\pi)^4}k_ik_j\tilde{\Delta}^F(K)\tilde{\Delta}^F(Q-K). \tag{D.23}$$

Using Eqs.(D.3) and (D.11) to Eq. (D.23), we obtain

$$J_{ij}(Q) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, k_ik_j \sum_{s_1,s_2=\pm1}\frac{s_1s_2}{4E_1E_2}\, T\sum_{n=-\infty}^{\infty}\frac{1}{i\omega'_n - s_1E_1}\frac{1}{i(\omega-\omega'_n)-s_2E_2}, \tag{D.24}$$

where we have defined $E_1 = |\mathbf{k}|$ and $E_2 = |\mathbf{q}-\mathbf{k}|$ (we neglect the zero-temperature mass of electron). In order to perform the summation $T\sum_n \frac{1}{i\omega'_n-s_1E_1}\frac{1}{i(\omega-\omega'_n)-s_2E_2}$ (remember that ω'_n is fermionic imaginary-time discrete energy, while ω is the bosonic one), we use the technique of the complex integration as (see Ref. [11])

$$\begin{aligned}T\sum_{n=-\infty}^{\infty}\frac{1}{i\omega'_n - s_1E_1}\frac{1}{i(\omega-\omega'_n)-s_2E_2} &= T\oint_C \frac{\mathrm{d}z}{2\pi i}\,\frac{1}{z-s_1E_1}\frac{1}{i\omega - z - s_2E_2}\,\frac{\mathrm{e}^{\beta z}-1}{\mathrm{e}^{\beta z}+1}\,\frac{-\beta}{-2} \\ &= -\frac{1-f_F(s_1E_1)-f_F(s_2E_2)}{i\omega - s_1E_1 - s_2E_2},\end{aligned} \tag{D.25}$$

where $\oint_C \mathrm{d}z$ shows the complex integration with the complex variable z along the contour $C = [+\epsilon - i\infty, +\epsilon + i\infty] \cup [-\epsilon + i\infty, -\epsilon - i\infty]$ parallel with the imaginary-axis: $\oint_C \mathrm{d}z = \int_{+\epsilon-i\infty}^{+\epsilon+i\infty}\mathrm{d}z + \int_{-\epsilon+i\infty}^{-\epsilon-i\infty}\mathrm{d}z$. Using this sum rule for Eq. (D.24), we obtain the following equation:

$$J_{ij}(Q) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, k_ik_j \sum_{s_1,s_2=\pm1}\frac{-s_1s_2}{4E_1E_2}\frac{1-f_F(s_1E_1)-f_F(s_2E_2)}{i\omega - s_1E_1 - s_2E_2}. \tag{D.26}$$

Carrying out the summation over $s_1, s_2 = \pm 1$, $J_{ij}(Q)$ reduces to

$$\begin{aligned}
J_{ij}(Q) &= \int \frac{\mathrm{d}\Omega}{4\pi} \int \frac{\mathrm{d}|\mathbf{k}|\ |\mathbf{k}|^2}{2\pi^2} \frac{-|\mathbf{k}|^2 \hat{k}_i \hat{k}_j}{4E_1E_2} \left\{ \frac{1 - f_F(E_1) - f_F(E_2)}{i\omega - E_1 - E_2} - \frac{1 - f_F(E_1) - f_F(-E_2)}{i\omega - E_1 + E_2} \right. \\
&\qquad \left. - \frac{1 - f_F(-E_1) - f_F(E_2)}{i\omega + E_1 - E_2} + \frac{1 - f_F(-E_1) - f_F(-E_2)}{i\omega + E_1 + E_2} \right\} \\
&\simeq \int \frac{\mathrm{d}\Omega}{4\pi} \int \frac{\mathrm{d}|\mathbf{k}|\ |\mathbf{k}|^2}{2\pi^2} \frac{-|\mathbf{k}|^2 \hat{k}_i \hat{k}_j}{4|\mathbf{k}|(|\mathbf{k}| - \mathbf{q}\cdot\hat{\mathbf{k}})} \\
&\qquad \times \left\{ \frac{1 - f_F(|\mathbf{k}|) - f_F(|\mathbf{k}| - \mathbf{q}\cdot\hat{\mathbf{k}})}{i\omega - 2|\mathbf{k}| + \mathbf{q}\cdot\hat{\mathbf{k}}} - \frac{f_F(|\mathbf{k}|) - f_F(|\mathbf{k}| - \mathbf{q}\cdot\hat{\mathbf{k}})}{i\omega - \mathbf{q}\cdot\hat{\mathbf{k}}} \right. \\
&\qquad \left. - \frac{f_F(|\mathbf{k}|) - f_F(|\mathbf{k}| - \mathbf{q}\cdot\hat{\mathbf{k}})}{i\omega + \mathbf{q}\cdot\hat{\mathbf{k}}} + \frac{1 - f_F(|\mathbf{k}|) - f_F(|\mathbf{k}| - \mathbf{q}\cdot\hat{\mathbf{k}})}{i\omega + 2|\mathbf{k}| - \mathbf{q}\cdot\hat{\mathbf{k}}} \right\},
\end{aligned} \tag{D.27}$$

where we have used the relation $1 - f_F(E) - f_F(-E) = 0$, $E_1 = |\mathbf{k}|$ and $E_2 = |\mathbf{q} - \mathbf{k}| \simeq |\mathbf{k}| - \mathbf{q}\cdot\hat{\mathbf{k}}$ (the HTL approximation). Using the HTL approximation furthermore as we have done in Eq. (D.13), we obtain

$$\begin{aligned}
J_{ij}(Q) &\simeq \int \frac{\mathrm{d}\Omega}{4\pi} \int \frac{\mathrm{d}|\mathbf{k}|\ |\mathbf{k}|^2}{2\pi^2} \frac{-\hat{k}_i \hat{k}_j}{4} \left\{ \frac{1 - 2f_F(|\mathbf{k}|)}{-2|\mathbf{k}|} + \frac{\mathbf{q}\cdot\hat{\mathbf{k}}}{Q_\mu \hat{K}'_\mu} \frac{\partial f_F(|\mathbf{k}|)}{\partial|\mathbf{k}|} \right. \\
&\qquad \left. - \frac{\mathbf{q}\cdot\hat{\mathbf{k}}}{Q_\mu \hat{K}_\mu} \frac{\partial f_F(|\mathbf{k}|)}{\partial|\mathbf{k}|} - \frac{1 - 2f_F(|\mathbf{k}|)}{2|\mathbf{k}|} \right\} \\
&\simeq -\frac{3}{4\pi^2} \int_0^\infty \mathrm{d}|\mathbf{k}|\ |\mathbf{k}| f_F(|\mathbf{k}|) \int \frac{\mathrm{d}\Omega}{4\pi} \hat{k}_i \hat{k}_j + \frac{1}{2\pi^2} \int_0^\infty \mathrm{d}|\mathbf{k}|\ |\mathbf{k}| f_F(|\mathbf{k}|) \int \frac{\mathrm{d}\Omega}{4\pi} \frac{i\omega}{Q_\mu \hat{K}_\mu} \hat{k}_i \hat{k}_j \\
&= -\frac{T^2}{48} \delta_{ij} + \frac{T^2}{24} \int \frac{\mathrm{d}\Omega}{4\pi} \frac{i\omega}{Q_\mu \hat{K}_\mu} \hat{k}_i \hat{k}_j,
\end{aligned} \tag{D.28}$$

where $\hat{K}_\mu = (-i, \hat{\mathbf{k}})$ and $\hat{K}'_\mu = (-i, -\mathbf{k})$ and we have removed the zero-point energy (the temperature independent quadratic divergence) in the second line. Also, we have used $\int \frac{\mathrm{d}\Omega}{4\pi} \hat{k}_i \hat{k}_j = \frac{1}{3}\delta_{ij}$ in the last line. Now, the (i,j) component of $I_{\mu\nu}(Q)$ in Eq. (D.22) is given by

$$I_{ij}(Q) = 8e^2 J_{ij}(Q) = -\frac{e^2T^2}{6}\delta_{ij} + \frac{e^2T^2}{3} \int \frac{\mathrm{d}\Omega}{4\pi} \frac{i\omega}{Q_\mu \hat{K}_\mu} \hat{k}_i \hat{k}_j. \tag{D.29}$$

Next, we evaluate I_{4i}. For this purpose, we consider the following quantity:

$$J_{4i}(Q) = \int \frac{\mathrm{d}^4K}{(2\pi)^4} (-\omega'_n) k_i \tilde{\Delta}^F(K) \tilde{\Delta}^F(Q - K). \tag{D.30}$$

Here, there is a useful relation for dealing with the ω_n' factor:

$$\omega_n' \tilde{\Delta}^F(K) = \sum_{s=\pm 1} (-isE_k) \tilde{\Delta}_s^F(K), \tag{D.31}$$

where $\tilde{\Delta}^F(K) = \sum_{s=\pm 1} \tilde{\Delta}_s^F(K)$ (see Eq. (D.11))[D-1D-2]. Now, using the same procedure in the evaluation of $J_{ij}(Q)$, $J_{4i}(Q)$ is evaluated as

$$\begin{aligned} J_{4i}(Q) &= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} k_i \sum_{s_1 s_2 = \pm 1} \frac{is_2}{4E_2} T \sum_{n=-\infty}^{\infty} \frac{1}{i\omega_n' - s_1 E_1} \frac{1}{i(\omega - \omega_n') - s_2 E_2} \\ &= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} k_i \sum_{s_1 s_2 = \pm 1} \frac{-is_2}{4E_2} \frac{1 - f_F(s_1 E_1) - f_F(s_2 E_2)}{i\omega - s_1 E_1 - s_2 E_2} \\ &\simeq \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{-i|\mathbf{k}|\hat{k}_i}{4(|\mathbf{k}| - \mathbf{q}\cdot\hat{\mathbf{k}})} \left\{ \frac{1 - f_F(|\mathbf{k}|) - f_F(|\mathbf{k}| - \mathbf{q}\cdot\hat{\mathbf{k}})}{i\omega - 2|\mathbf{k}| + \mathbf{q}\cdot\hat{\mathbf{k}}} + \frac{f_F(|\mathbf{k}|) - f_F(|\mathbf{k}| - \mathbf{q}\cdot\hat{\mathbf{k}})}{i\omega + \mathbf{q}\cdot\hat{\mathbf{k}}} \right. \\ &\qquad \left. + \frac{f_F(|\mathbf{k}|) - f_F(|\mathbf{k}| - \mathbf{q}\cdot\hat{\mathbf{k}})}{i\omega - \mathbf{q}\cdot\hat{\mathbf{k}}} + \frac{1 - f_F(|\mathbf{k}) - f_F(|\mathbf{k}| - \mathbf{q}\cdot\hat{\mathbf{k}})}{i\omega + 2|\mathbf{k}| - \mathbf{q}\cdot\hat{\mathbf{k}}} \right\} \end{aligned} \tag{D.33}$$

where we have used the relation $1 - f_F(E) - f_F(-E) = 0$, $E_1 = |\mathbf{k}|$ and $E_2 = |\mathbf{q} - \mathbf{k}| \simeq |\mathbf{k}| - \mathbf{q}\cdot\hat{\mathbf{k}}$ (the HTL approximation). Using the HTL approximation furthermore as we have done in Eq. (D.13), we obtain

$$\begin{aligned} J_{4i}(Q) &\simeq \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{(-i)\hat{k}_i}{4} \left\{ \frac{\mathbf{q}\cdot\hat{\mathbf{k}}}{Q_\mu \hat{K}_\mu} + \frac{\mathbf{q}\cdot\hat{\mathbf{k}}}{Q_\mu \hat{K}_\mu'} \right\} \frac{\partial f_F(|\mathbf{k}|)}{\partial |\mathbf{k}|} \\ &= \frac{1}{4\pi^2} \int_0^\infty \mathrm{d}|\mathbf{k}|\ |\mathbf{k}|^2 \frac{\partial f_F(|\mathbf{k}|)}{\partial |\mathbf{k}|} \int \frac{\mathrm{d}\Omega}{4\pi} \frac{\mathbf{q}\cdot\hat{\mathbf{k}}}{Q_\mu \hat{K}_\mu} (-i)\hat{k}_i \\ &= \frac{T^2}{24} \int \frac{\mathrm{d}\Omega}{4\pi} \frac{i\omega}{Q_\mu \hat{K}_\mu} (-i)\hat{k}_i, \end{aligned} \tag{D.34}$$

where we have used $\int \frac{\mathrm{d}\Omega}{4\pi} \hat{k}_i = 0$ in the last line. Thus, we obtain

$$I_{4i}(Q) = I_{i4}(Q) = 8e^2 J_{4i}(Q) = \frac{e^2 T^2}{3} \int \frac{\mathrm{d}\Omega}{4\pi} \frac{i\omega}{Q_\mu \hat{K}_\mu} (-i)\hat{k}_i. \tag{D.35}$$

[D-1] From Eq. (D.11), we can easily prove Eq. (D.31) as

$$\begin{aligned} \omega_n' \tilde{\Delta}^F(K) &= \sum_{s=\pm 1} \frac{is}{2E_k} \frac{i\omega_n'}{i\omega_n' - sE_k} \\ &= \sum_{s=\pm 1} \frac{is}{2E_k} + \sum_{s=\pm 1} \frac{-s}{2E_k} \frac{-isE_k}{i\omega_n' - sE_k} \\ &= \sum_{s=\pm 1} (-isE_k) \tilde{\Delta}_s^F(K). \end{aligned} \tag{D.32}$$

[D-2] However, this relation does *not* mean the replacement $\omega_n'^2 \to (-isE_k)^2$. For the $\omega_n'^2$ factor, it is convenient to use the relation $\omega_n'^2 = K^2 - |\mathbf{k}|^2$.

Finally, we evaluate $I_{44}(Q)$. Using the relation $\omega_n'^2 = K^2 - |\mathbf{k}|^2$, we have

$$\begin{aligned} I_{44}(Q) &= 8e^2 \int \frac{\mathrm{d}^4K}{(2\pi)^4} \omega_n'^2 \tilde{\Delta}^F(K) \tilde{\Delta}^F(Q-K) \\ &= 8e^2 \left\{ \int \frac{\mathrm{d}^4K}{(2\pi)^4} \tilde{\Delta}^F(K) - J_{ii}(Q) \right\} \\ &\simeq \frac{e^2T^2}{6} + \frac{e^2T^2}{3} \int \frac{\mathrm{d}\Omega}{4\pi} \frac{i\omega}{Q_\mu \hat{K}_\mu} (-i)(-i). \end{aligned} \tag{D.36}$$

where we have used the relation $K^2\tilde{\Delta}^F(K) = 1$, $\int \frac{\mathrm{d}^4K}{(2\pi)^4} \tilde{\Delta}^F(Q-K) = \int \frac{\mathrm{d}^4K}{(2\pi)^4} \tilde{\Delta}^F(K)$ and Eq. (D.23) in the second line. Also, we have used Eqs. (D.21) and (D.28) in the last line.

The results in the above, (D.29), (D.35) and (D.36), can be summarized as follows

$$I_{\mu\nu}(Q) = \frac{e^2T^2}{6}(\delta_{\mu4}\delta_{\nu4} - \delta_{ij}) + \frac{e^2T^2}{3} \int \frac{\mathrm{d}\Omega}{4\pi} \frac{i\omega}{Q_\rho \hat{K}_\rho} \hat{K}_\mu \hat{K}_\nu. \tag{D.37}$$

Now, we are in a position to obtain the resultant QED photon self-energy, $\Pi_{\mu\nu}(Q)$, at the one-loop level. From Eqs. (D.22) and (D.37), the result is given by

$$\Pi_{\mu\nu}(Q) = 2m_\gamma^2 \int \frac{\mathrm{d}\Omega}{4\pi} \left(\frac{i\omega}{Q_\rho \hat{K}_\rho} \hat{K}_\mu \hat{K}_\nu + \delta_{\mu4}\delta_{\nu4} \right), \tag{D.38}$$

where

$$m_\gamma^2 = \frac{1}{6} e^2 T^2 \tag{D.39}$$

is the photon (asymptotic) thermal mass-squared, as we can identify from the dispersion relation[D-3]. We note that the so-called plasma frequency, ω_P, is given by $\omega_P = \frac{1}{3}eT = \sqrt{\frac{2}{3}} m_\gamma$.

[D-3] Note that the existence of m_γ does not mean the spontaneous breaking of the gauge symmetry [47–49]. In fact, the gauge symmetry breaking term does not arise in the effective Lagrangian for photon.

Appendix E

Spectral function of chiral fermion

In this appendix, we derive the spectral function for a chiral fermion $\tilde{\chi}$. Let us start with the following thermally corrected self-energy of the chiral fermion, $\bar{\Sigma}(P)$ (we here use the two-component notation):

$$\bar{\Sigma}(P) = a + b\,\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}, \tag{E.1}$$

where the notation $\bar{\Sigma}(P) = \bar{\Sigma}(i\omega, \mathbf{p})$ is used, $P_\mu = (p_4, \mathbf{p}) = (-\omega, \mathbf{p})$ is the fermion external momentum, $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ and we have neglected the chiral fermion zero-temperature mass. For the one-loop Hard Thermal Loop approximation with yukawa and/or gauge interactions, the parameters a and b have the following form:

$$\begin{aligned} a &= \frac{m_f^2}{2|\mathbf{p}|} \ln\left(\frac{i\omega + |\mathbf{p}|}{i\omega - |\mathbf{p}|}\right), \\ b &= \frac{m_f^2}{|\mathbf{p}|}\left(1 - \frac{i\omega}{2|\mathbf{p}|} \ln\left(\frac{i\omega + |\mathbf{p}|}{i\omega - |\mathbf{p}|}\right)\right), \end{aligned} \tag{E.2}$$

where m_f is the fermion thermal mass. For instance, if we assume an interaction term $\mathcal{L}_{\text{int.}} = -g\varphi\tilde{\chi}\tilde{\lambda} + h.c.$ (φ is a complex scalar field and $\tilde{\lambda}$ is a chiral fermion), we obtain $m_f^2 = g^2T^2/16$.

The inverse propagator for $\tilde{\chi}$ including the thermally corrected self-energy, $G(P)^{-1}$, is evaluated from the Dyson equation as

$$\begin{aligned} G(P)^{-1} &= P_\mu \sigma_\mu - \bar{\Sigma}(P) \\ &= (i\omega - |\mathbf{p}| - a - b)\frac{-1 - \hat{\mathbf{p}} \cdot \boldsymbol{\sigma}}{2} + (i\omega + |\mathbf{p}| - a + b)\frac{-1 + \hat{\mathbf{p}} \cdot \boldsymbol{\sigma}}{2}, \end{aligned} \tag{E.3}$$

where $\sigma_\mu = (i, \boldsymbol{\sigma})$ in the Euclidean spacetime (in the imaginary-time formalism). Here,

$\frac{-1\pm\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}{2}$ are the projection operators which satisfy the following equations:

$$\begin{aligned} \frac{-1+\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}{2}\,\frac{-1+\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}{2} &= \frac{-1+\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}{2}, \\ \frac{-1-\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}{2}\,\frac{-1-\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}{2} &= \frac{-1-\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}{2}, \\ \frac{-1+\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}{2}\,\frac{-1-\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}{2} &= \frac{-1-\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}{2}\,\frac{-1+\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}{2} = 0. \end{aligned} \tag{E.4}$$

Thus, from Eqs. (E.3) and (E.4), the propagator for $\tilde{\chi}$, $G(P)$, can be written as following:

$$G(P) = (i\omega - |\mathbf{p}| - a - b)^{-1}\frac{-1-\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}{2} + (i\omega + |\mathbf{p}| - a + b)^{-1}\frac{-1+\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}{2}. \tag{E.5}$$

Now, the spectral function for the chiral fermion $\tilde{\chi}$, $\bar{\rho}(p)$, is given by

$$\begin{aligned} \bar{\rho}(p) &= (-iG(p_0+i\epsilon,\mathbf{p})) - (-iG(p_0-i\epsilon,\mathbf{p})) \\ &= \bar{\rho}_+(p)\frac{1+\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}{2} + \bar{\rho}_-(p)\frac{1-\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}}{2}, \end{aligned} \tag{E.6}$$

where

$$\bar{\rho}_\pm(p) = -2\ \mathrm{Im}\frac{1}{(p_0+i\epsilon-a)\mp(|\mathbf{p}|+b)}. \tag{E.7}$$

We note that, from Eqs. (E.2) and (3.93), the spectral functions $\bar{\rho}_\pm(p)$ given by Eq. (E.7) have the same form as in Eq. (3.96). This is what we would like to derive here. The limiting formulae for the dispersion relations for the poles $\omega_\pm(\mathbf{p})$ and the residues $Z_\pm(\mathbf{p})$ under the one-loop HTL approximation are given by Eqs. (3.101) and (3.105), respectively. In the originally (*i.e.*, at zero-temperature) massless limit, Dirac and Majorana fields have the same form of the dispersion relations as in Eqs. (3.101) and (3.105) [97].

Appendix F

The free energy density of the MSSM plasma

F.1 The 2-loop contributions

In this appendix, we show each contribution to the 2-loop free energy density from the yukawa couplings (6.23) and the gauge couplings (6.19).

First, we write down the contributions to Eq. (6.23):

$$\begin{aligned} \tilde{\Omega}_2^{ssss} &= \frac{|y|^2 T^4}{144} \times \begin{cases} 3 & \text{(with squark)}, \\ 1 & \text{(without squark)}, \end{cases} \\ \tilde{\Omega}_2^{sff} &= \frac{|y|^2 T^4}{144} \times \begin{cases} \frac{15}{4} & \text{(with squark)}, \\ \frac{5}{4} & \text{(without squark)}, \end{cases} \end{aligned} \tag{F.1}$$

where, $\tilde{\Omega}_2^{ssss}$, $\tilde{\Omega}_2^{sff}$ are the 2-loop contributions to the free energy density which are generated by the 4-point scalar interaction $ssss$, the scalar-fermion-fermion interaction sff, respectively. Here, we have used s, f as symbols for the scalars and fermions in the relevant interactions. There are six 2-loop diagrams from 4-point scalar interactions for each yukawa coupling ($|y_t|^2, |y_b|^2, |y_\tau|^2$). Also, there are six 2-loop diagrams from scalar-fermion-fermion interactions for each yukawa coupling. After taking the sum of these contributions, we finally obtain Eq. (6.23).

Next, we write down each 2-loop contribution to Eq. (6.19) from SUSY $SU(N_c)$ theory:

$$\begin{aligned}
&\tilde{\Omega}_2^{Aff} = N_g \left(\sum_i t_2(i)\right) \times \frac{5}{4} \times \frac{g^2T^4}{144}, \quad \tilde{\Omega}_2^{sf\lambda} = N_g \left(\sum_i t_2(i)\right) \times \frac{5}{2} \times \frac{g^2T^4}{144}, \\
&\tilde{\Omega}_2^{ssss} = N_g \left(\sum_i t_2(i)\right) \times \frac{1}{2} \times \frac{g^2T^4}{144}, \quad \tilde{\Omega}_2^{ssA} = N_g \left(\sum_i t_2(i)\right) \times \frac{-3}{2} \times \frac{g^2T^4}{144}, \\
&\tilde{\Omega}_2^{ssAA} = N_g \left(\sum_i t_2(i)\right) \times 4 \times \frac{g^2T^4}{144}, \\
&\tilde{\Omega}_2^{A\lambda\lambda} = N_g N_c \times \frac{5}{4} \times \frac{g^2T^4}{144}, \quad \tilde{\Omega}_2^{Acc} = N_g N_c \times \frac{1}{4} \times \frac{g^2T^4}{144}, \\
&\tilde{\Omega}_2^{AAA} = N_g N_c \times \frac{-9}{4} \times \frac{g^2T^4}{144}, \quad \tilde{\Omega}_2^{AAAA} = N_g N_c \times 3 \times \frac{g^2T^4}{144}.
\end{aligned} \tag{F.2}$$

where, $\tilde{\Omega}_2^{\hat{\mathcal{O}}}$ is the 2-loop contribution to the free energy density which is generated by the interaction $\hat{\mathcal{O}}$. Here, we have used s, f, A, λ and c as symbols for the chiral scalar, chiral fermion, gauge field, gaugino and ghost field in the relevant interactions, respectively. Summing up the contributions in Eq. (F.2), we eventually obtain Eq. (6.19). Here, the summation $\sum_i$ runs all the $SU(N_c)$ chiral supermultiplet i. The Dynkin index $t_2(i) = 1/2$ when the chiral supermultiplet i belongs to the fundamental representation. We note that for SUSY $U(1)_Y$ theory, we can apply the formula (6.19) with $N_g = 1, N_c = 0, t_2(i) = Y_i^2$.

F.2 The next-to-leading order contributions

In this appendix, we derive an analytic expression for the next-to-leading order contribution to $\tilde{m}_\phi$ from the MSSM plasma. To do this, we evaluate the contribution to the free energy density from the ring diagrams [12, 98] generated by the rescaled couplings (6.22) and (6.24). Then, from the expression for the ring diagram free energy density, we read off the effective mass $\tilde{m}_\phi$ at next-to-leading order.

First, we consider the contribution from the ring diagrams of gluon, W-boson and B-boson in MSSM to the free energy density, $\tilde{\Omega}_3^{\text{gauge}}$. These gauge field ring diagrams can be evaluated by the usual method in thermal field theory [12] and is given by [81]

$$\tilde{\Omega}_3^{\text{gauge}} = -\frac{T}{12\pi}\left(8m'^3_{D,g} + 3m'^3_{D,W} + m'^3_{D,B}\right), \tag{F.3}$$

where $m'_{D,g}, m'_{D,W}$ and $m'_{D,B}$ are the Debye masses of gluon, W-boson and B-boson, respectively, and are given by [99]

$$m'^2_{D,g} = \frac{9}{2}g'^2_s T^2, \quad m'^2_{D,W} = \frac{9}{2}g'^2_2 T^2, \quad m'^2_{D,B} = \frac{11}{2}g'^2_Y T^2. \tag{F.4}$$

Note that we have already rescaled the chiral superfields. Now, using Eqs. (6.24) and (F.4), the gauge field ring diagram contribution (F.3) reduces to

$$\begin{aligned}\tilde{\Omega}_3^{\text{gauge}} &= -\frac{T^4}{12\pi}\left\{54\sqrt{2}g_s'^3 + \frac{81}{2\sqrt{2}}g_2'^3 + \frac{11\sqrt{11}}{2\sqrt{2}}g_Y'^3\right\} \\ &= -\left\{\frac{162\sqrt{2}}{\sqrt{\pi}}\bar{c}_s\alpha_s^{5/2} + \frac{567}{4\sqrt{2\pi}}\bar{c}_2\alpha_2^{5/2} + \frac{121\sqrt{11}}{4\sqrt{2\pi}}\bar{c}_Y\alpha_Y^{5/2}\right\}\frac{T^4}{M_{\rm P}^2}|\phi|^2 + (\phi\text{-indep.}).\end{aligned} \tag{F.5}$$

Thus, we obtain the following contribution to $\tilde{m}_\phi^2$ from the gauge field ring diagrams in MSSM:

$$\begin{aligned}\tilde{m}_\phi^2|_{\text{ring}}^{\text{gague}} &= -\left\{\frac{162\sqrt{2}}{\sqrt{\pi}}\bar{c}_s\alpha_s^{5/2} + \frac{567}{4\sqrt{2\pi}}\bar{c}_2\alpha_2^{5/2} + \frac{121\sqrt{11}}{4\sqrt{2\pi}}\bar{c}_Y\alpha_Y^{5/2}\right\}\frac{T^4}{M_{\rm P}^2} \\ &= -\left\{\frac{3888\sqrt{2}}{61\pi^{5/2}}\bar{c}_s\alpha_s^{5/2} + \frac{1701\sqrt{2}}{61\pi^{5/2}}\bar{c}_2\alpha_2^{5/2} + \frac{363\sqrt{22}}{61\pi^{5/2}}\bar{c}_Y\alpha_Y^{5/2}\right\}H^2,\end{aligned} \tag{F.6}$$

where we have used the Friedmann equation in the RD era $3M_{\rm P}^2H^2 = \frac{\pi^2 g_*}{30}T^4$ and $g_* = 228.75 = 915/4$ as in Eq. (6.26). We note that the numerical coefficients of the gauge couplings in Eq. (F.6) are about five times larger than the ones in Eq. (6.25) and have opposite sign.

Next, let us evaluate the contribution from the ring diagrams of the MSSM chiral scalar fields to the free energy density, $\tilde{\Omega}_3^{\text{scalar}}$. The usual method in thermal field theory [12] can be applied for the evaluation of $\tilde{\Omega}_3^{\text{scalar}}$ and the result is given by [81][F-1]

$$\tilde{\Omega}_3^{\text{scalar}} = -\frac{T}{6\pi}\sum_i^{\text{scalar}} m_i^3, \tag{F.7}$$

where i runs all the chiral scalar fields in MSSM. Here, m_i is the thermal mass of the scalar field i and is summarized in Ref. [99]. From Eqs. (6.22), (6.24) and (F.7), after rescaling the MSSM chiral superfields, the ring diagrams of the chiral scalar fields contribute to the free energy density as

$$\tilde{\Omega}_3^{\text{scalar}} = -\sum_i^{\text{scalar}}\frac{\xi_i}{4\pi}\frac{m_i^3 T}{M_{\rm P}^2}|\phi|^2 + (\phi\text{-indep.}), \tag{F.8}$$

where $\mathcal{O}(M_{\rm P}^{-4})$ terms are neglected and ξ_i is defined by

$$\xi_i\frac{|\phi|^2}{M_{\rm P}^2} = \frac{m_i'^2|_{|\phi|^2}}{m_i^2}. \tag{F.9}$$

[F-1] In Ref. [81], the factor $\frac{N_g}{12N_c}$ should be replaced by $\frac{N_g}{4N_c}$ in Eqs.(5) and (6) . This corrected factor $\frac{N_g}{4N_c}$ agrees with Ref. [99].

Here, $m_i'^2|_{|\phi|^2}$ is the ϕ-dependent part of the thermal mass-squared $m_i'^2$ in which the couplings are replaced by the rescaled ones (6.22) and (6.24). From Eq. (F.8), we obtain the following contribution to $\tilde{m}_\phi^2$ from the scalar field ring diagrams in MSSM:

$$\tilde{m}_\phi^2|_{\rm ring}^{\rm scalar} = -\sum_i^{\rm scalar} \frac{\xi_i}{4\pi}\frac{m_i^3 T}{M_{\rm P}^2} = -\sum_i^{\rm scalar} \frac{6\xi_i}{61\pi^3}\frac{m_i^3}{T^3}H^2. \tag{F.10}$$

Here, we have used the Friedmann equation in the RD era as in Eq. (6.26).

Now, we are in a position to sum up the ring diagram contributions. From Eqs. (F.6) and (F.10), the total ring diagram contribution $\tilde{m}_\phi^2|_{\rm ring}$ is obtained as follows

$$\tilde{m}_\phi^2|_{\rm ring} = \left\{ \sum_i^{\rm scalar} \frac{-6\xi_i}{61\pi^3}\frac{m_i^3}{T^3} - \frac{3888\sqrt{2}}{61\pi^{5/2}}\bar{c}_s\alpha_s^{5/2} - \frac{1701\sqrt{2}}{61\pi^{5/2}}\bar{c}_2\alpha_2^{5/2} - \frac{363\sqrt{22}}{61\pi^{5/2}}\bar{c}_Y\alpha_Y^{5/2} \right\} H^2. \tag{F.11}$$

We note that the ring diagram contribution (F.11) is rather significant compared with the leading order (2-loop) one given in Eq. (6.26). This would be the signature of the poor convergence of the ordinary perturbation theory as mentioned below Eq. (6.26). Thus in order to obtain a reliable result for $\tilde{m}_\phi^2$, we have to proceed the evaluation up to sufficiently higher-loop order or apply the improved perturbation theory. However, since the leading order result in ordinary perturbation theory would be different from the convergence-improved result at most by a factor of order unity as we observe in literatures like Ref. [83], the leading order result (6.26) can serve as the first estimate of the systematic evaluation of $\tilde{m}_\phi^2$ from the MSSM plasma.

Bibliography

[1] Peter W. Higgs. *Broken symmetries, massless particles and gauge fields. Phys.Lett.*, 12:132–133, 1964.

[2] Peter W. Higgs. *Broken Symmetries and the Masses of Gauge Bosons. Phys.Rev.Lett.*, 13:508–509, 1964.

[3] F. Englert and R. Brout. *Broken Symmetry and the Mass of Gauge Vector Mesons. Phys.Rev.Lett.*, 13:321–323, 1964.

[4] G.S. Guralnik, C.R. Hagen, and T.W.B. Kibble. *Global Conservation Laws and Massless Particles. Phys.Rev.Lett.*, 13:585–587, 1964.

[5] Alan H. Guth. *The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems. Phys.Rev.*, D23:347–356, 1981.

[6] Alexei A. Starobinsky. *A New Type of Isotropic Cosmological Models Without Singularity. Phys.Lett.*, B91:99–102, 1980.

[7] K. Sato. *First Order Phase Transition of a Vacuum and Expansion of the Universe. Mon.Not.Roy.Astron.Soc.*, 195:467–479, 1981.

[8] Andrei D. Linde. *A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems. Phys.Lett.*, B108:389–393, 1982.

[9] Andreas Albrecht and Paul J. Steinhardt. *Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking. Phys.Rev.Lett.*, 48:1220–1223, 1982.

[10] N.P. Landsman and C.G. van Weert. *Real and Imaginary Time Field Theory at Finite Temperature and Density. Phys.Rept.*, 145:141, 1987.

[11] M. Le Bellac. *Thermal field theory. Cambridge University Press*, 2000.

[12] J. I. Kapsuta and C. Gale. *Finite temperature Field Theory. Cambridge University Press, second edition*, 2006.

[13] D.A. Kirzhnits. *Weinberg model in the hot universe. JETP Lett.*, 15:529–531, 1972.

[14] D.A. Kirzhnits and Andrei D. Linde. *Macroscopic Consequences of the Weinberg Model. Phys.Lett.*, B42:471–474, 1972.

[15] L. Dolan and R. Jackiw. *Symmetry Behavior at Finite Temperature. Phys.Rev.*, D9:3320–3341, 1974.

[16] Edward W. Kolb, Alessio Notari, and Antonio Riotto. *On the reheating stage after inflation. Phys.Rev.*, D68:123505, 2003.

[17] Jun'ichi Yokoyama. *Fate of oscillating scalar fields in the thermal bath and their cosmological implications. Phys.Rev.*, D70:103511, 2004.

[18] Jun'ichi Yokoyama. *Can oscillating scalar fields decay into particles with a large thermal mass? Phys.Lett.*, B635:66–71, 2006.

[19] Marco Drewes. *On the Role of Quasiparticles and thermal Masses in Nonequilibrium Processes in a Plasma.* 2010.

[20] Marco Drewes and Jin U. Kang. *The Kinematics of Cosmic Reheating. Nucl.Phys.*, B875:315–350, 2013.

[21] Kyohei Mukaida and Kazunori Nakayama. *Dynamics of oscillating scalar field in thermal environment. JCAP*, 1301:017, 2013.

[22] Kyohei Mukaida and Kazunori Nakayama. *Dissipative Effects on Reheating after Inflation. JCAP*, 1303:002, 2013.

[23] Burt A. Ovrut and Paul J. Steinhardt. *Supersymmetry and Inflation: A New Approach. Phys.Lett.*, B133:161, 1983.

[24] Michael Dine, Willy Fischler, and Dennis Nemeschansky. *Solution of the Entropy Crisis of Supersymmetric Theories. Phys.Lett.*, B136:169, 1984.

[25] G.D. Coughlan, R. Holman, Pierre Ramond, and Graham G. Ross. *Supersymmetry and the Entropy Crisis. Phys.Lett.*, B140:44, 1984.

[26] Edmund J. Copeland, Andrew R. Liddle, David H. Lyth, Ewan D. Stewart, and David Wands. *False vacuum inflation with Einstein gravity. Phys.Rev.*, D49:6410–6433, 1994.

[27] Ewan D. Stewart. *Inflation, supergravity and superstrings. Phys.Rev.*, D51:6847–6853, 1995.

[28] J. Wess and J. Bagger. *Supersymmetry and supergravity. Princeton, USA: Univ. Pr.*, 1992.

[29] Tony Gherghetta, Christopher F. Kolda, and Stephen P. Martin. *Flat directions in the scalar potential of the supersymmetric standard model. Nucl.Phys.*, B468:37–58, 1996.

[30] David H. Lyth and Antonio Riotto. *Particle physics models of inflation and the cosmological density perturbation. Phys.Rept.*, 314:1–146, 1999.

[31] T. Asaka, M. Kawasaki, and Masahide Yamaguchi. *Initial condition for new inflation in supergravity. Phys.Rev.*, D61:027303, 2000.

[32] David H. Lyth and Takeo Moroi. *The Masses of weakly coupled scalar fields in the early universe. JHEP*, 0405:004, 2004.

[33] Masahiro Kawasaki and Tomohiro Takesako. *Hubble Induced Mass in Radiation Dominated Universe. Phys.Lett.*, B711:173–177, 2012.

[34] Masahiro Kawasaki and Tomohiro Takesako. *Remarks on Hubble Induced Mass from Fermion Kinetic Term. Phys.Lett.*, B718:522–525, 2012.

[35] Masahiro Kawasaki, Fuminobu Takahashi, and Tomohiro Takesako. *Hubble-induced mass from MSSM plasma. JCAP*, 1304:008, 2013.

[36] Edward W. Kolb and Michael S. Turner. *The Early Universe. Front.Phys.*, 69:1–547, 1990.

[37] Lev Kofman, Andrei D. Linde, and Alexei A. Starobinsky. *Reheating after inflation. Phys.Rev.Lett.*, 73:3195–3198, 1994.

[38] Y. Shtanov, Jennie H. Traschen, and Robert H. Brandenberger. *Universe reheating after inflation. Phys.Rev.*, D51:5438–5455, 1995.

[39] Lev Kofman, Andrei D. Linde, and Alexei A. Starobinsky. *Towards the theory of reheating after inflation. Phys.Rev.*, D56:3258–3295, 1997.

[40] Gary N. Felder, Lev Kofman, and Andrei D. Linde. *Instant preheating. Phys.Rev.*, D59:123523, 1999.

[41] Ryogo Kubo. *Statistical mechanical theory of irreversible processes. 1. General theory and simple applications in magnetic and conduction problems. J.Phys.Soc.Jap.*, 12:570–586, 1957.

[42] Paul C. Martin and Julian S. Schwinger. *Theory of many particle systems. 1. Phys.Rev.*, 115:1342–1373, 1959.

[43] Takeo Matsubara. A New approach to quantum statistical mechanics. *Prog.Theor.Phys.*, 14:351–378, 1955.

[44] N.P. Landsman. *Consistent Real Time Propagators for Any Spin, Mass, Temperature and Density. Phys.Lett.*, B172:46, 1986.

[45] Hideki Matsumoto. *The Causal Function in Many Body Problems. Fortsch.Phys.*, 25:1, 1977.

[46] H. Umezawa, H. Matsumoto, and M. Tachiki. *Thermo Field Dynamics and Condensed States. North-Holland, Amsterdam*, 1982.

[47] Eric Braaten and Robert D. Pisarski. *Soft Amplitudes in Hot Gauge Theories: A General Analysis. Nucl.Phys.*, B337:569, 1990.

[48] Eric Braaten and Robert D. Pisarski. *Deducing Hard Thermal Loops From Ward Identities. Nucl.Phys.*, B339:310–324, 1990.

[49] J. Frenkel and J.C. Taylor. *High Temperature Limit of Thermal QCD. Nucl.Phys.*, B334:199, 1990.

[50] R.E. Norton and J.M. Cornwall. *On the Formalism of Relativistic Many Body Theory. Annals Phys.*, 91:106, 1975.

[51] M.B. Kislinger and P.D. Morley. *Collective Phenomena in Gauge Theories. 2. Renormalization in Finite Temperature Field Theory. Phys.Rev.*, D13:2771, 1976.

[52] P.D. Morley and M.B. Kislinger. *Relativistic Many Body Theory, Quantum Chromodynamics and Neutron Stars/Supernova. Phys.Rept.*, 51:63, 1979.

[53] H. Matsumoto, I. Ojima, and H. Umezawa. *Perturbation and Renormalization in Thermo Field Dynamics. Annals Phys.*, 152:348, 1984.

[54] Antti J. Niemi and Gordon W. Semenoff. *Thermodynamic Calculations in Relativistic Finite Temperature Quantum Field Theories. Nucl.Phys.*, B230:181, 1984.

[55] Ian Affleck and Michael Dine. *A New Mechanism for Baryogenesis. Nucl.Phys.*, B249:361, 1985.

[56] Michael Dine, Lisa Randall, and Scott D. Thomas. *Supersymmetry breaking in the early universe. Phys.Rev.Lett.*, 75:398–401, 1995.

[57] Michael Dine, Lisa Randall, and Scott D. Thomas. *Baryogenesis from flat directions of the supersymmetric standard model. Nucl.Phys.*, B458:291–326, 1996.

[58] Andrei D. Linde. *Relaxing the cosmological moduli problem. Phys.Rev.*, D53:4129–4132, 1996.

[59] Kazunori Nakayama, Fuminobu Takahashi, and Tsutomu T. Yanagida. *On the Adiabatic Solution to the Polonyi/Moduli Problem. Phys.Rev.*, D84:123523, 2011.

[60] David H. Lyth and David Wands. *Generating the curvature perturbation without an inflaton. Phys.Lett.*, B524:5–14, 2002.

[61] Takeo Moroi and Tomo Takahashi. *Effects of cosmological moduli fields on cosmic microwave background. Phys.Lett.*, B522:215–221, 2001.

[62] Kari Enqvist and Martin S. Sloth. *Adiabatic CMB perturbations in pre - big bang string cosmology. Nucl.Phys.*, B626:395–409, 2002.

[63] Kazuya Kumekawa, Takeo Moroi, and Tsutomu Yanagida. *Flat potential for inflaton with a discrete R invariance in supergravity. Prog.Theor.Phys.*, 92:437–448, 1994.

[64] M. Kawasaki, Masahide Yamaguchi, and T. Yanagida. *Natural chaotic inflation in supergravity. Phys.Rev.Lett.*, 85:3572–3575, 2000.

[65] M. Kawasaki, Masahide Yamaguchi, and T. Yanagida. *Natural chaotic inflation in supergravity and leptogenesis. Phys.Rev.*, D63:103514, 2001.

[66] Masahide Yamaguchi. *Supergravity based inflation models: a review. Class.Quant.Grav.*, 28:103001, 2011.

[67] Alexander Kusenko and Mikhail E. Shaposhnikov. *Supersymmetric Q balls as dark matter. Phys.Lett.*, B418:46–54, 1998.

[68] Kari Enqvist and John McDonald. *Q balls and baryogenesis in the MSSM. Phys.Lett.*, B425:309–321, 1998.

[69] Kari Enqvist and John McDonald. *B - ball baryogenesis and the baryon to dark matter ratio. Nucl.Phys.*, B538:321–350, 1999.

[70] S. Kasuya and M. Kawasaki. *Q ball formation through Affleck-Dine mechanism. Phys.Rev.*, D61:041301, 2000.

[71] S. Kasuya and M. Kawasaki. *Q Ball formation in the gravity mediated SUSY breaking scenario. Phys.Rev.*, D62:023512, 2000.

[72] S. Kasuya and M. Kawasaki. *Q ball formation: Obstacle to Affleck-Dine baryogenesis in the gauge mediated SUSY breaking?* *Phys.Rev.*, D64:123515, 2001.

[73] G.D. Coughlan, W. Fischler, Edward W. Kolb, S. Raby, and Graham G. Ross. *Cosmological Problems for the Polonyi Potential.* *Phys.Lett.*, B131:59, 1983.

[74] John R. Ellis, Dimitri V. Nanopoulos, and M. Quiros. *On the Axion, Dilaton, Polonyi, Gravitino and Shadow Matter Problems in Supergravity and Superstring Models.* *Phys.Lett.*, B174:176, 1986.

[75] A. S. Goncharov, Andrei D. Linde, and M. I. Vysotsky. *COSMOLOGICAL PROBLEMS FOR SPONTANEOUSLY BROKEN SUPERGRAVITY.* *Phys.Lett.*, B147:279, 1984.

[76] A. Peshier, Burkhard Kampfer, O.P. Pavlenko, and G. Soff. *A Massive quasiparticle model of the SU(3) gluon plasma.* *Phys.Rev.*, D54:2399–2402, 1996.

[77] Peter Levai and Ulrich W. Heinz. *Massive gluons and quarks and the equation of state obtained from SU(3) lattice QCD.* *Phys.Rev.*, C57:1879–1890, 1998.

[78] A. Peshier, Burkhard Kampfer, and G. Soff. *The Equation of state of deconfined matter at finite chemical potential in a quasiparticle description.* *Phys.Rev.*, C61:045203, 2000.

[79] Ken-ichi Konishi and Ken-ichi Shizuya. *Functional Integral Approach to Chiral Anomalies in Supersymmetric Gauge Theories.* *Nuovo Cim.*, A90:111, 1985.

[80] Nima Arkani-Hamed and Hitoshi Murayama. *Holomorphy, rescaling anomalies and exact beta functions in supersymmetric gauge theories.* *JHEP*, 0006:030, 2000.

[81] J. Grundberg, T.H. Hansson, and U. Lindstrom. *Thermodynamics of N=1 supersymmetric QCD.* 1995.

[82] K. Kajantie, M. Laine, K. Rummukainen, and Y. Schroder. *The Pressure of hot QCD up to g6 ln(1/g).* *Phys.Rev.*, D67:105008, 2003.

[83] Jens O. Andersen, Eric Braaten, and Michael Strickland. *Hard thermal loop resummation of the free energy of a hot gluon plasma.* *Phys.Rev.Lett.*, 83:2139–2142, 1999.

[84] Jens O. Andersen, Eric Braaten, Emmanuel Petitgirard, and Michael Strickland. *HTL perturbation theory to two loops.* *Phys.Rev.*, D66:085016, 2002.

[85] Jens O. Andersen, Michael Strickland, and Nan Su. *Gluon Thermodynamics at Intermediate Coupling.* *Phys.Rev.Lett.*, 104:122003, 2010.

[86] Jean-Paul Blaizot, Edmond Iancu, and Anton Rebhan. *Thermodynamics of the high temperature quark gluon plasma.* 2003.

[87] Ulrike Kraemmer and Anton Rebhan. *Advances in perturbative thermal field theory. Rept.Prog.Phys.*, 67:351, 2004.

[88] Jens O. Andersen and Michael Strickland. *Resummation in hot field theories. Annals Phys.*, 317:281–353, 2005.

[89] B.C. Allanach. *SOFTSUSY: a program for calculating supersymmetric spectra. Comput.Phys.Commun.*, 143:305–331, 2002.

[90] Juergen Berges. *Introduction to nonequilibrium quantum field theory. AIP Conf.Proc.*, 739:3–62, 2005.

[91] D. Boyanovsky, K. Davey, and C.M. Ho. *Particle abundance in a thermal plasma: Quantum kinetics vs. Boltzmann equation. Phys.Rev.*, D71:023523, 2005.

[92] L.V. Keldysh. *Diagram technique for nonequilibrium processes. Zh.Eksp.Teor.Fiz.*, 47:1515–1527, 1964.

[93] R.P. Feynman and Jr. Vernon, F.L. *The Theory of a general quantum system interacting with a linear dissipative system. Annals Phys.*, 24:118–173, 1963.

[94] Fuminobu Takahashi and Tsutomu T. Yanagida. *Why have supersymmetric particles not been observed? Phys.Lett.*, B698:408–410, 2011.

[95] Alexey Anisimov and Michael Dine. *Some issues in flat direction baryogenesis. Nucl.Phys.*, B619:729–740, 2001.

[96] L. D. Landau and E. M. Lifshitz. *Statistical Physics. Pergamon Press*, 1969.

[97] Antonio Riotto and Iiro Vilja. *Propagation of Majorana fermions in hot plasma. Phys.Lett.*, B402:314–319, 1997.

[98] Murray Gell-Mann and Keith A. Brueckner. *Correlation Energy of an Electron Gas at High Density. Phys.Rev.*, 106:364–368, 1957.

[99] D. Comelli and J.R. Espinosa. *Bosonic thermal masses in supersymmetry. Phys.Rev.*, D55:6253–6263, 1997.