学位論文

Properties and phenomenology of non-Abelian vortices in dense QCD

(高密度QCDにおける非可換量子渦の性質と現象論)

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Properties and phenomenology
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Abstract

QCD matter at high densities and low temperatures is expected to be a color superconductor. At extremely high densities, the ground state is the color-flavor locked (CFL) phase, that exhibits superfluidity as well as color superconductivity. It is known that QCD matter in the CFL phase hosts topologically stable vortices. An important feature of these vortices is that they have internal degrees of freedom, that are bosonic and fermionic. These modes propagate along the vortices. The bosonic modes are called orientational zero modes, and they are the Nambu-Goldstone modes associated with the breaking of the color-flavor locked symmetry because of the presence of a vortex. On the other hand, the fermionic modes are "Majorana" fermions, which emerge as a result of the particle-hole symmetry.

This thesis consists of two investigations on the non-Abelian vortices in the CFL phase.

Firstly, we discuss the interaction of vortices with quasiparticles, such as phonons, gluons, CFL mesons, and photons. The interaction Lagrangian with phonons and gluons is derived via a dual transformation. It turns out the interaction with gluons is dependent on the orientation of a vortex. This gives rise to an orientation-dependent interaction energy between two vortices. We also discuss the interaction of vortices with CFL mesons. We extend the chiral Lagrangian, and derive the Lagrangian of CFL mesons under the background of a vortex solution. We also investigate the interaction of vortices with photons and its phenomenological consequences. The orientational zero modes localized on vortices are charged with respect to $U(1)_{\mathrm{EM}}$ symmetry. The interaction Lagrangian is determined by symmetry consideration. Based on the low-energy Lagrangian, we discuss the scattering of photons off a vortex. We discuss an optical property of a vortex lattice. It is expected that a vortex lattice is formed if CFL matter exists inside the core of a rotating dense star. We show that a lattice of vortices serves as a polarizer of photons.

Secondly, we analyze the non-Abelian statistics of vortices, which is brought about by the existence of Majorana fermions inside vortices. We consider the exchange statistics of vortices each of which traps an odd number of Majorana fermions. Exchange of two vortices turns out to be non-Abelian, and the corresponding operator is further decomposed into two parts: a part that is essentially equivalent to the exchange operator of vortices having a single Majorana fermion in each vortex, and a part representing the Coxeter group. We obtain the basis of the Hilbert space by using the Dirac fermions defined by combining two Majorana fermions trapped in separate vortices, and we find the matrix representation of the exchange operators in the Hilbert space. We show that the decomposition of the exchange operator implies tensor product structure in its matrix representation.

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Chapter 1

Introduction

Vortices appear in diverse fields of physics. Most familiar ones are the vortices in hydrodynamics. For example in the Naruto Strait, one can find big tidal whirlpools, that have been fascinated people in past and present. Big whirlpools are depicted in a ukiyo-e picture by Hiroshige Utagawa, a famous ukiyo-e artist in the Edo period. Vortices exist in an exotic kind of fluid called a superfluid. Superfluids are fluids without viscosity and they can be seen in various physical systems such as liquid helium [1] and ultra-cold atomic gasses [2, 3, 4]. A superfluid accommodates *quantized* vortices, that can be viewed as topological defects in a Bose-Einstein condensate. The quantization of circulation of a superfluid vortex is first predicted by Lars Onsager [5], and later confirmed experimentally. The vortices are important degrees of freedom in the dynamics of superfluids [6, 1, 7, 8]. A rotating superfluid is threaded with numerous vortices and they form a vortex lattice. Vortices are also created at phase transitions by the Kibble-Zurek mechanism [9, 10, 11, 12]. Superfluid vortices also play pivotal roles in quantum turbulence in liquid helium and atomic BEC [13, 8]. In two dimensions, vortices are considered to be responsible for the Berezinskii-Kosterlitz-Thouless (BKT) transitions [14, 15, 16].

Topological solitons like quantized vortices also manifest themselves in the condensed matter physics of quantum chromodynamics (QCD), which is the theory of the strong interaction. QCD matter exhibits a rich variety of phases at finite temperatures and/or baryon densities [17]. The stability of topological solitons is closely related to the vacuum structure, or the residual symmetry of the ground state. Depending on the symmetry of the ground state, QCD matter accommodates various kinds of topological solitons such as vortices, skyrmions, and vortex-domain wall composites (see Ref. [18]). Topological solitons have fascinated many researchers, and it is an important problem to reveal the properties of the topological solitons from the phenomenological view point. They could affect the properties of the matter realized inside compact stars or the matter created in heavy-ion collisions.

A rough sketch of QCD phase diagram is shown in Fig. 1.1. At low temperatures and low baryon chemical potentials, quarks are confined in color-singlet states called hadrons. At very high temperatures, the deconfined plasma of quarks and gluons, which is called the quark-gluon plasma (QGP), is expected to be realized [19]. The early Universe is a natural place for the QGP to appear. Recently, it has become possible to create hot matter experimentally by means of the relativistic nucleus-nucleus
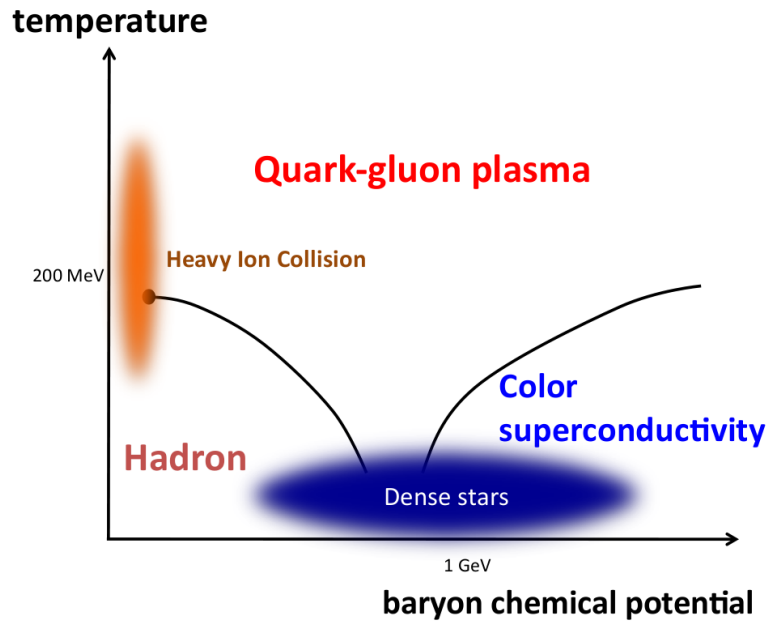
Figure 1.1: Schematic phase diagram of QCD matter as a function of temperature (horizontal axis) and baryon chemical potential (vertical axis). At low temperatures and low densities, QCD matter takes the form of hadrons, that we can find around us. At high temperatures, the matter is in quark-gluon plasma phase, which would be realized at heavy-ion collisions or in the early Universe. Color superconducting phase would be realized at high densities and low temperatures like inside compact stars.

collisions at the Relativistic Heavy Ion Collider (RHIC) in BNL and at the Large Hadron Collider (LHC) in CERN.

Theoretical studies suggest that dense and cold quark matter is a color superconductor [20, 21, 22, 23, 24], in which two quarks form a Cooper pair and make a Bose-Einstein condensation. Such a state of matter can be realized in the core of neutron stars. It is a theoretical and experimental challenge to reveal the properties of a color superconductor and to verify its existence experimentally or observationally. At asymptotically high densities, perturbation theory become reliable and we can perform controlled calculations. It is believed that the phase in which all of the three light quarks make pairs symmetrically and form condensates is realized. This is called the color-flavor locked (CFL) phase [24, 25]. At lower densities, the effect of strange quarks comes into play, and other pairing patterns can be favored. Various phases of color superconductivity are discussed, such as the kaon condensation (CFL-K) phase [26], the two-flavor superconducting (2SC) phase [21], the crystalline superconducting phase [27, 28], and the magnetic CFL (MCFL) phase [29, 30, 31]. For reviews of the phase structure of QCD matter, see Refs. [32, 33, 34, 35, 17].

Among the various ground states of QCD matter the CFL phase is an important phase, since it is realized at asymptotically high densities. The CFL phase has an interesting property that it

exhibits both superfluidity and superconductivity. In the CFL phase, the original symmetry of QCD, $G = U(1)_\mathrm{B} \times SU(3)_\mathrm{C} \times SU(3)_\mathrm{L} \times SU(3)_\mathrm{R}$, is spontaneously broken down to the color-flavor locked symmetry $H = SU(3)_\mathrm{C+L+R} \equiv SU(3)_\mathrm{C+F}$, apart from discrete symmetry. The system exhibits superfluidity due to the breaking of the global $U(1)_\mathrm{B}$ symmetry, as well as color superconductivity because of broken color $SU(3)_\mathrm{C}$ symmetry. As in a superfluid helium, we can expect the existence of topological vortices from the consideration of the ground-state symmetry [36, 37]. Vortices in the CFL phase would be created if the matter is rotated, as is observed in rotating superfluids in condensed matter systems. Thus, if the CFL phase is realized in the cores of dense stars, vortices are inevitably created since the stars rotate rapidly. The superfluid vortices discussed in Refs. [36, 37] have integral winding numbers with respect to $U(1)_\mathrm{B}$ symmetry. Later it turned out that such vortices are not minimally winding vortices. Energetically, it is more favorable for a vortex with an integral winding number to decay into a set of vortices with lower energies. The precise structure of minimally-winding vortices is first pointed out by Balachandran, Digal, and Matsuura [38]. The stable vortices are so-called non-Abelian vortices, which are superfluid vortices as well as color magnetic flux tubes. Since they carry 1/3 quantized $U(1)_\mathrm{B}$ circulations, an integer $U(1)_\mathrm{B}$ vortex decays into three non-Abelian vortices with different color fluxes that are canceled in total [39]. Color magnetic flux tubes were studied before [37], but their stability is not assured by topology. The properties of non-Abelian vortices have been studied using the Ginzburg-Landau theory [40, 39, 41, 42, 43, 44, 45] or the Bogoliubov–de Gennes equation [46, 47].

A remarkable property of non-Abelian vortices is that both bosonic and fermionic zero energy modes are localized in the cores of non-Abelian vortices and propagate along them as gapless excitations. The bosonic zero modes are the Kelvin modes and orientational modes. The Kelvin modes are the Nambu-Goldstone modes associated with the breaking of two translational symmetries transverse to the vortex, and they also appear in the vortices in superfluids in condensed matter physics. Orientational zero modes are also Nambu-Goldstone modes, and they originates from the breaking of the CFL symmetry $SU(3)_\mathrm{C+F}$ into its subgroup $[SU(2) \times U(1)]_\mathrm{C+F}$ inside the vortex core [39, 43]. The low-energy effective field theory of the orientational zero modes is written as the $\mathbb{C}P^2$ model inside the $1 + 1$ dimensional vortex worldsheet, where $\mathbb{C}P^2 \simeq SU(3)/[SU(2) \times U(1)]$ is the target space spanned by the Nambu-Goldstone modes [43]. On the other hand, Majorana fermion zero modes, which belong to a triplet of the core symmetry $SU(2)_\mathrm{C+F}$, have been found in the Bogoliubov-de Gennes equation and the low-energy effective theory in the $1 + 1$ dimensional vortex worldsheet has been derived [46]. The existence of these fermion zero modes is ensured by topology, which can be seen as the index theorem [47].

In this thesis, we investigate phenomenological consequences caused by the existence of these internal degrees of freedom. Firstly, we investigate the interaction of vortices with quasiparticles in the CFL phase. The interaction is fundamental physical information for discussing the phenomena related to vortices, since the quasiparticles mediates the forces between vortices.

Secondly, we study the non-Abelian statistics, which is a remarkable phenomenon caused by the fermion zero modes inside vortices. There has been considerable interest recently in zero-energy

fermion modes trapped inside vortices in superconductors [48]. Vortices in a chiral p-wave super-conductor are endowed with non-Abelian statistics [49, 50] because of the zero-energy Majorana fermions inside them [51]. Excitations which obey non-Abelian statistics are called non-Abelian anyons. They are expected to form the basis of topological quantum computations [52, 53] and have been investigated intensively [54, 55, 56]. In particular, a vortex in the CFL phase has multiple Majorana fermions. This results in a quite interesting structure. Because of the fermionic zero modes, degenerate ground states appears. These states are mixed under the exchange of vortices. We will see that the representation of a Coxeter group appears in the transformation matrix, due to the multiple fermions in a vortex.

The thesis is organized as follows. In Chap. 2, we summarize basic aspects of color superconductivity. Special emphasis is put on the color-flavor locked phase in which non-Abelian vortices appear. We also discuss the effective theories to describe the low-energy dynamics of color-superconducting media. In Chap. 3, we review the basic properties of topological vortices in the CFL phase. We show that the appearance of vortices is guaranteed by the topology of the CFL ground state. The explicit form of a vortex solution is given by solving the equations of motion derived from the Ginzburg-Landau Lagrangian. We also explain why the orientational zero modes appear. The effective low-energy theory of the orientational zero modes is derived and it turns out to be the $\mathbb{C}P^2$ model. We then discuss the effects of finite strange quark mass on the effective theory. In Chap. 4, we study the interaction of non-Abelian vortices with quasiparticles. In Sec. 4.1, the interactions between a non-Abelian vortex and phonons or gluons are discussed. The interaction with phonons is obtained by a dual transformation in which the phonon field is dualized to a Abelian two-form field, while the interaction with gluons is achieved by non-Abelian dual transformation in which the gluon field is dualized to a non-Abelian massive two-form field. The latter affects the interaction between two non-Abelian vortices at short distances. We give the orientation dependence of the intervortex force, that is mediated by the exchange of massive gluons. In Sec. 4.2, we derive the chiral Lagrangian of the CFL mesons in the presence of a non-Abelian vortex. In Sec. 4.3, we study the interaction between non-Abelian vortices and photons. Because of the orientational modes that are electromagnetically charged, vortices interact with photons. The coupling of photons the orientational modes is described by the $U(1)$ gauged $\mathbb{C}P^2$ model. We solve the scattering problem of photons off a vortex and we obtain the scattering cross section per unit length of a vortex. As an interesting consequence of the vortex-photon interaction, we show that a lattice of non-Abelian vortices behaves as a polarizer [58]. In Chap. 5, we discuss the non-Abelian statistics, which is brought about by the existence of Majorana zero modes inside vortices. We first explain why Majorana zero modes appears inside vortices in a superconductor in terms of the Bogoliubov–de Gennes equation, which is an eigenvalue equation of the meanfield Hamiltonian. We then discuss exchange statistics of non-Abelian vortices in $d = 2 + 1$ dimensions. It is shown that multiple zero-mode Majorana fermions inside vortices lead to a novel kind of non-Abelian anyons [59, 60]. We show that the representation of exchange operations is decomposed into two parts: a part that is essentially equivalent to the exchange of vortices

having a single Majorana fermion in each vortex, and a part representing the Coxeter group. We also show that the decomposition of the exchange operator implies tensor product structure in its matrix representation. Chapter 6 is devoted to the summary and discussion of future perspectives.

The original results in this thesis is based on the papers [58, 60, 18].

Chapter 2

Basics of color superconductivity in dense QCD

The main subject of the thesis is the topological vortices in color superconductors. In the following, we briefly review the basic properties of color superconductivity. In Sec. 2.1, we discuss why color superconductivity occurs and what kind of pairing is favored. After reviewing the symmetry of QCD, we discuss the color-flavor locked phase in detail, which hosts topologically stable non-Abelian vortices. For a comprehensive review on color superconductivity, see Ref. [35] and references therein. In Sec. 2.2, we give a review on an effective description of color superconductors near the critical temperature, which is the Ginzburg-Landau theory. The effect of a finite strange quark mass is also taken into account.

2.1 Color superconductivity

2.1.1 Why does color superconductivity occur?

The understanding of superconductivity is revolutionized by the BCS theory [61]. According to it, the existence of a well defined Fermi surface and net attractive interaction is sufficient for the Cooper instability to occur. This condition is satisfied for the QCD matter at low temperatures and high baryon densities. At asymptotically high densities (large μ), perturbation theory is reliable and we can perform controlled calculations[1]. Quarks can be regarded as degenerate Fermi liquids and they have well-defined Fermi surfaces at low temperatures. The interaction between quarks through gluon exchange has an attractive channel, which is sufficient for the Cooper instability to occur. This imply

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[1] At large μ, the typical momentum exchanged in a collision becomes large, since the states inside the large Fermi sea are filled. For collisions with large momentum transfers, the effective coupling $g(\mu)$ is small, and we can use weak coupling calculations. The small-angle collisions, in which momentum transfers are small, can be potentially dangerous, but it turns out to be safe thanks to the medium effects. The propagator of exchanged gluons gets modifications from the medium. Electric gluons are Debye-screened, and magnetic interactions are screened by Laudau-damping and color-Meissner effect. Thus, small-angle scatterings can also be dealt with perturbatively.

the realization of color superconductivity at low temperature and high density goes back to the works of Barrois, Bailin and Love [20, 21].

There are two origins of attractive interactions in QCD. One is from the perturbative point of view. Quarks have an attractive channel in one-gluon exchange interaction, which is the leading order contribution in perturbative expansion. Interaction energy of two quarks has the following structure like $(\bar{q}_1 T^a \gamma_\mu q_1)(\bar{q}_2 T^a \gamma^\mu q_2)$ where q_1 and q_2 are quark fields and T^a are the generators of $SU(N_C)$ color symmetry. The color generators satisfy

$$(T^a)^{\alpha\beta}(T^a)^{\gamma\delta} = -\frac{N_C + 1}{N_C}(T_A)^{\alpha\gamma}(T_A)^{\delta\beta} + \frac{N_C - 1}{N_C}(T_S)^{\alpha\gamma}(T_S)^{\delta\beta}, \tag{2.1}$$

where T_S (T_A) are the (anti)symmetric generators. This corresponds to the decomposition of the tensor product into irreducible representations, $3 \otimes 3 = \bar{3}_A \oplus 6_S$. The minus sign of the first term in Eq. (2.1) indicates that the interaction is attractive. Intuitively, this means that combining two quarks that are each in the color-3 representation to obtain a diquark that is in the $\bar{3}$ representation lowers the total energy due to color flux. In the color superconductivity, the fundamental color interaction itself has attractive channel, while in the metal superconductivity the electromagnetic interaction between electrons is repulsive.

The other source of attraction is the instanton-induced interaction. Instantons are the solutions of the classical equations of motion which correspond to the tunneling events between degenerate classical vacua. The instantons give rise to a vertex called the *'t Hooft interaction* [62] which involves $2N_F$ quarks where N_F is the number of flavors. If one considers three light flavors, the interaction involves six quarks. The instanton-antiinstanton pairs lead to an effective four-quark vertex. Incorporating these interactions to the BCS analysis, Rapp *et al.* has shown that the BCS gap is of order 100 MeV [63] at intermediate density where instantons are abundant [2]. While one-gluon exchange interactions do not distinguish positive and negative parity states, instanton-induced interactions favor the positive parity pairing. So the parity of the ground state is positive, if the system under consideration is an eigenstate of parity.

Regarding the Lorentz indices, pairing with zero angular momentum is favored since it allows condensation which utilize the entire Fermi surface to lower the energy. The antisymmetric property in color and Lorentz indices require the flavor to be antisymmetric via the Pauli principle. Therefore, the ground state of a color superconductor is characterized by the following diquark condensate,

$$\Phi_L^{\alpha i} = \epsilon^{\alpha\beta\gamma}\epsilon^{ijk}\langle (q_L)_{\beta j}^T C(q_L)_{\gamma k}\rangle, \tag{2.2}$$

$$\Phi_R^{\alpha i} = \epsilon^{\alpha\beta\gamma}\epsilon^{ijk}\langle (q_R)_{\beta j}^T C(q_R)_{\gamma k}\rangle, \tag{2.3}$$

where $\alpha, \beta, \gamma \cdots$ and $i, j, k \cdots$ are color and flavor indices respectively, and C is the charge conjugation matrix. Here the transpose is with respect to spinor indices. For the positive parity pairing,

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[2] The instanton-induced interaction is strong at intermediate densities $\mu \simeq 400 - 800$ MeV since it gives rise to the constituent masses of quarks of order 100 MeV. In this case whether the quarks can be regarded as degenerate Fermi liquids or not is a nontrivial question. Therefore the applicability of the BCS theory may have to be reexamined in this region.

$\Phi_{\mathrm{L}} = -\Phi_{\mathrm{R}}$. This pairing pattern is in the $\bar{3}$ representation in both color and flavor symmetry, and $J^P = 0^+$ under Lorentz transformation [3] as denoted above.

2.1.2 Symmetry of QCD

Let us recall the symmetry of QCD. QCD matter consists of quarks and gluons. We regard quarks as in the antifundamental representation [4] of both of the color $SU(3)_{\mathrm{C}}$ symmetry and flavor $SU(3)_{\mathrm{L(R)}}$ symmetry. The fields $q_{\mathrm{L(R)}}$ for left-handed and right-handed quarks transform as

$$q_{\mathrm{L}i}^{\alpha} \to e^{i\theta_{\mathrm{B}}/2 + i\theta_{\mathrm{A}}/2} \left(\bar{U}_{\mathrm{C}}\right)^{\alpha}_{\ \beta} \left(\bar{U}_{\mathrm{L}}\right)_{i}^{\ j} q_{\mathrm{L}j}^{\beta}, \quad q_{\mathrm{R}i}^{\alpha} \to e^{i\theta_{\mathrm{B}}/2 - i\theta_{\mathrm{A}}/2} \left(\bar{U}_{\mathrm{C}}\right)^{\alpha}_{\ \beta} \left(\bar{U}_{\mathrm{L}}\right)_{i}^{\ j} q_{\mathrm{R}j}^{\beta}, \tag{2.5}$$

where $\bar{U}_{\mathrm{C}} \in SU(3)_{\mathrm{C}}$ and $\bar{U}_{\mathrm{L(R)}} \in SU(3)_{\mathrm{L(R)}}$ are representation matrices of each symmetry group in the antifundamental representations, and $e^{i\theta_{\mathrm{B(A)}}/2} \in U(1)_{\mathrm{B(A)}}$. In the case of massless three-flavor case, the QCD Lagrangian is invariant under $SU(3)_{\mathrm{L}}$, $SU(3)_{\mathrm{R}}$, $U(1)_{\mathrm{B}}$, and $U(1)_{\mathrm{A}}$ symmetry, although the axial symmetry $U(1)_{\mathrm{A}}$ is broken the quantum anomaly[5]. The flavor symmetry becomes approximate when the mass of quarks are considered.

It is evident from the definition that the diquark condensate fields are in the fundamental representations of color and flavor groups. The color $SU(3)_{\mathrm{C}}$ group, the flavor group $SU(3)_{\mathrm{L}} \times SU(3)_{\mathrm{R}}$, the $U(1)_{\mathrm{B}}$ group and the $U(1)_{\mathrm{A}}$ group act on Φ_{L} and Φ_{R} as

$$\Phi_{\mathrm{L}} \to e^{i\theta_{\mathrm{B}} + i\theta_{\mathrm{A}}} U_{\mathrm{C}} \Phi_{\mathrm{L}} U_{\mathrm{L}}, \quad \Phi_{\mathrm{R}} \to e^{i\theta_{\mathrm{B}} - i\theta_{\mathrm{A}}} U_{\mathrm{C}} \Phi_{\mathrm{R}} U_{\mathrm{R}}, \tag{2.6}$$

where $U_{\mathrm{C}} \in SU(3)_{\mathrm{C}}$, and $U_{\mathrm{L(R)}} \in SU(3)_{\mathrm{L(R)}}$ are representation matrices in the fundamental representations, and $e^{i\theta_{\mathrm{B(A)}}} \in U(1)_{\mathrm{B(A)}}$.

2.1.3 Phases of color superconductivity

Phases of color superconductivity are characterized by the expectation value of the Cooper pair (2.3). Since the order parameter is a matrix, there can be several patterns of symmetry breaking depending on the parameters such as density, temperature, and masses of quarks.

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[3] Under the parity transformation, the Dirac fermion q transforms as $q \to \eta_P \gamma_0 q$ where $\eta_P = \pm 1$ is the intrinsic parity of the particle. The condensate wave function (2.3) transforms as

$$\langle q^T C \gamma_5 q \rangle \longrightarrow \eta_P^2 \langle q^T \gamma_0 C \gamma_5 \gamma_0 q \rangle = \langle q^T C \gamma_5 q \rangle \tag{2.4}$$

where we have used anticommuting property of γ matrices. Therefore the pairing (2.3) is a scalar under parity transformation.

[4] Here we assign the antifundamental representation to quarks so that the diquark field be in the fundamental representation. The distinction between fundamental and antifundamental representations are just matter of definition and not essential.

[5] The axial $U(1)$ symmetry is effectively restored at higher densities or temperatures at which the instanton density gets lower [64].

Color-flavor locked phase

At asymptotically high densities where up, down and strange quarks can be regarded as massless[6], particularly symmetric phase called the color-flavor locked (CFL) [25] phase is expected to be realized. All the three light flavors take part in the pairing symmetrically in the CFL phase. The order parameter takes the form [7]

$$\Phi_i^\alpha = \epsilon^{\alpha\beta\gamma}\epsilon_{ijk}\langle (q^T)^j_\beta C\gamma_5 q^k_\gamma \rangle \propto \delta_i^\alpha \Delta_{\mathrm{CFL}}, \tag{2.7}$$

where Δ_{CFL} is the gap parameter. The condensation (2.7) leads to the following symmetry breaking,

$$SU(3)_{\mathrm{C}} \times SU(3)_{\mathrm{L}} \times SU(3)_{\mathrm{R}} \times U(1)_{\mathrm{B}} \longrightarrow SU(3)_{\mathrm{C+L+R}}. \tag{2.8}$$

Equation (2.7) is called "color-flavor locked" pairing since it transforms nontrivially under separate rotation of color/flavor and the ground state is invariant under simultaneous "locked" rotation of both. The similar "locking" is also observed, for example, in the B phase of the superfluid ^{3}He, where the order parameter is not invariant under separate rotation of the spin and orbital degree of freedom, but is unchanged if one rotates both simultaneously.

Matter in the CFL phase is a superconductor since the local color symmetry is completely broken. It is also a superfluid due to the broken $U(1)_{\mathrm{B}}$ symmetry. Note also that the chiral symmetry is broken in the CFL phase. The condensate is not invariant under separate global flavor rotation of left and right handed quarks.

Let us examine the topology of the order parameter space in the spontaneous symmetry breaking to the CFL phase in detail, paying attention to the discrete groups. Since we consider the positive-parity states, $\Phi_{\mathrm{L}} = -\Phi_{\mathrm{R}} \equiv \Phi$. The order parameter field Φ is transformed under the actions of the symmetries of QCD as

$$\Phi \to e^{i\theta_{\mathrm{B}}} U_{\mathrm{C}} \Phi U_{\mathrm{F}}, \tag{2.9}$$

where $e^{i\theta_{\mathrm{B}}} \in U(1)_{\mathrm{B}}$, $U_{\mathrm{C}} \in SU(3)_{\mathrm{C}}$, and $U_{\mathrm{F}} \in SU(3)_{\mathrm{L+R}}$. Since there is a redundancy in the action of the symmetries, the actual symmetry group is given by

$$G = \frac{SU(3)_{\mathrm{C}} \times SU(3)_{\mathrm{F}} \times U(1)_{\mathrm{B}}}{(\mathbb{Z}_3)_{\mathrm{C+B}} \times (\mathbb{Z}_3)_{\mathrm{F+B}}}, \tag{2.10}$$

where the discrete groups are defined by

$$(\mathbb{Z}_3)_{\mathrm{C+B}} : \left(\omega^k \mathbf{1}_3, \mathbf{1}_3, \omega^{-k}\right) \in SU(3)_{\mathrm{C}} \times SU(3)_{\mathrm{F}} \times U(1)_{\mathrm{B}}, \tag{2.11}$$

$$(\mathbb{Z}_3)_{\mathrm{F+B}} : \left(\mathbf{1}_3, \omega^k \mathbf{1}_3, \omega^{-k}\right) \in SU(3)_{\mathrm{C}} \times SU(3)_{\mathrm{F}} \times U(1)_{\mathrm{B}}, \tag{2.12}$$

with $k = 0, 1, 2$ and ω is defined by

$$\omega \equiv e^{2\pi i/3}. \tag{2.13}$$

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[6]The masses of up and down quarks are about $5 - 7$ MeV while the mass of the strange quark is about 100 MeV. We consider the case in which μ is much larger than these masses.

[7] All the degenerate pairing pattern of the CFL phase is obtained by applying the transformation of the original symmetry of QCD on (2.7).

Note that the discrete groups can be rearranged as

$$(\mathbb{Z}_3)_{\mathrm{C+B}} \times (\mathbb{Z}_3)_{\mathrm{F+B}} \simeq (\mathbb{Z}_3)_{\mathrm{C+F}} \times (\mathbb{Z}_3)_{\mathrm{C-F+B}}, \tag{2.14}$$

with

$$(\mathbb{Z}_3)_{\mathrm{C+F}} \quad : \quad \left(\omega^k \mathbf{1}_3, \omega^{-k} \mathbf{1}_3, 1\right) \in SU(3)_{\mathrm{C}} \times SU(3)_{\mathrm{F}} \times U(1)_{\mathrm{B}}, \tag{2.15}$$

$$(\mathbb{Z}_3)_{\mathrm{C-F+B}} \quad : \quad \left(\omega^k \mathbf{1}_3, \omega^k \mathbf{1}_3, \omega^{-2k}\right) \in SU(3)_{\mathrm{C}} \times SU(3)_{\mathrm{F}} \times U(1)_{\mathrm{B}}. \tag{2.16}$$

In the CFL phase, we can take the diquark condensate field as

$$\Phi = \Delta_{\mathrm{CFL}} \mathbf{1}_3, \tag{2.17}$$

without loss of generality. In the ground state of the CFL phase, the full symmetry group G is spontaneously broken down to the following group,

$$H = \frac{SU(3)_{\mathrm{C+F}} \times (\mathbb{Z}_3)_{\mathrm{C-F+B}}}{(\mathbb{Z}_3)_{\mathrm{C+B}} \times (\mathbb{Z}_3)_{\mathrm{F+B}}} \simeq \frac{SU(3)_{\mathrm{C+F}}}{(\mathbb{Z}_3)_{\mathrm{C+F}}}. \tag{2.18}$$

Thus, we find the order parameter space of the ground state as

$$\frac{G}{H} \simeq \frac{SU(3)_{\mathrm{C-F}} \times U(1)_{\mathrm{B}}}{(\mathbb{Z}_3)_{\mathrm{C-F+B}}} \simeq U(3)_{\mathrm{C-F+B}}. \tag{2.19}$$

This $U(3)$ manifold is parametrized by 9 would-be Nambu-Goldstone (NG) modes, among which 8 are eaten by the gluons via the Higgs mechanism and only one massless scalar field (referred to as the $U(1)_{\mathrm{B}}$ phonon) associated with the $U(1)_{\mathrm{B}}$ symmetry breaking remains in the physical spectrum[8].

2.1.4 Elementary excitations in the CFL phase

Elementary excitations in the CFL phase are summarized as follows.

- *Nambu-Goldstone bosons.* There appear light eight NG bosons from the chiral symmetry breaking and one massless phonon from $U(1)_{\mathrm{B}}$ breaking. Diquarks are in 3 representation of color $SU(3)_{\mathrm{C}}$. Gauge invariant excitations can be constructed by combining the diquark qq to form color singlets, which corresponds to integrating out the gauge fields. Although the $U(1)_{\mathrm{A}}$ symmetry is explicitly broken in the vacuum by the instanton effect, when the baryon density is high and instantons are dilute, η' mesons become nearly massless and we can regard them as the NG boson due to the breaking of $U(1)_{\mathrm{A}}$ symmetry.

- *Quarks.* All of them acquire the mass gap of order Δ_{CFL}.

- *Gluons.* All eight gluons are massive since $SU(3)_{\mathrm{C}}$ symmetry is completely broken. Their electric screening masses (the Debye mass) are of order $g_{\mathrm{s}}\mu$ [65] where μ is the baryon chemical potential, while their pole masses are of order Δ_{CFL} [66, 67, 68].

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[8] To be precise, the $U(1)_{\mathrm{B}}$ is broken to $\mathbb{Z}_2$ which flips the signs of left and right quarks ($q_{\mathrm{L}} \to -q_{\mathrm{L}}$ and $q_{\mathrm{R}} \to -q_{\mathrm{R}}$). This $\mathbb{Z}_2$ cannot be described in the GL theory.
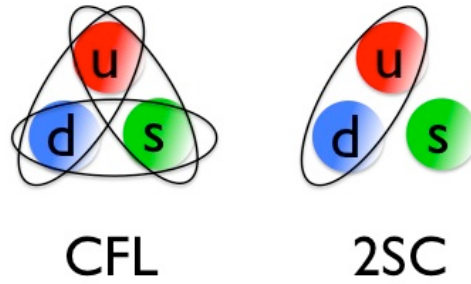
Figure 2.1:   Schematic figure of pairing patterns in the CFL and 2SC phases. In the CFL phase up, down and strange flavors take part in the pairing, while in the 2SC phase only light two flavors form a condensate.

2.1.5   Other phases of color superconductivity

There can be several other phases of color superconductivity depending on parameters such as mass, density and temperature. Inhomogeneous superconducting phases are also discussed. For discussions on other possible phases of color superconductivity, see Ref. [35] and references therein.

We do not discuss the other phases of color superconductivity in detail here since they do not admit topological vortices. However, let us see another well-studied pairing pattern for comparison with the CFL phase. The two flavor color superconductivity (2SC) phase is characterized by the condensate [21]

$$\langle (q^T)_i^\alpha C\gamma_5 q_j^\beta \rangle \propto \epsilon_{ij3}\epsilon^{\alpha\beta3}\Delta_{2SC}. \tag{2.20}$$

In the 2SC phase, quarks with two colors and two flavors participate in the condensation. There is no condensate which involves strange quarks. The 2SC phase is expected to be realized at densities where the mass of the strange quark cannot be neglected and the pairing including the strange quarks are suppressed.

Symmetry breaking pattern is[9]

$$SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_B \longrightarrow SU(2)_C \times SU(2)_L \times SU(2)_R \times U(1)_{\tilde{B}}, \tag{2.21}$$

where $U(1)_{\tilde{B}}$ is the combination of $U(1)_B$ and $U(1)_8 \in SU(3)_C$. The baryon number is conserved by the condensate as a mixing of the original baryon number and one color generator T^8 which is broken. We can say that matter in the 2SC phase is a superconductor, but not a superfluid since no global $U(1)$ symmetry is broken. This phase does not admit topologically stable vortices.

---

[9]We have assumed to start with massless up and down quarks.

2.2 Low-energy description of a color superconductor

2.2.1 Ginzburg-Landau theory

A general tool to discuss the phase structure and low-energy excitations is the Ginzburg-Landau (GL) theory [69, 70, 71], which is applicable near the transition temperature. GL theory is basically the expansion in terms of the order parameter and its derivatives. Due to the instanton effects, the state with positive parity is favored compared to the one with negative parity as a ground state [25, 23]. Thus we take the positive-parity state[10],

$$\Phi_{\mathrm{L}} = -\Phi_{\mathrm{R}} \equiv \Phi. \tag{2.22}$$

The GL Lagrangian consists of the order parameter Φ and its derivatives. We add up possible terms consistent with the symmetry of QCD, which is given by

$$
\begin{aligned}
\mathcal{L} &= \mathrm{Tr}\left[-\frac{\varepsilon_3}{2}F_{0i}F^{0i} - \frac{1}{4\lambda_3}F_{ij}F^{ij}\right] - \frac{\varepsilon_0}{2}(F^{\mathrm{EM}})_{0i}(F^{\mathrm{EM}})^{0i} - \frac{1}{4\lambda_0}(F^{\mathrm{EM}})_{ij}(F^{\mathrm{EM}})^{ij} \\
&\quad + \mathrm{Tr}\left[2i\gamma K_0(\mathcal{D}_0\Phi^\dagger\Phi - \Phi^\dagger\mathcal{D}_0\Phi) + K_0\mathcal{D}_0\Phi^\dagger\mathcal{D}^0\Phi + K_3\mathcal{D}_i\Phi^\dagger\mathcal{D}^i\Phi\right] - V,
\end{aligned} \tag{2.23}
$$

$$V = \alpha\mathrm{Tr}\left(\Phi^\dagger\Phi\right) + \beta_1\left[\mathrm{Tr}(\Phi^\dagger\Phi)\right]^2 + \beta_2\mathrm{Tr}\left[(\Phi^\dagger\Phi)^2\right] + \frac{3\alpha^2}{4(\beta_1 + 3\beta_2)}, \tag{2.24}$$

where λ_3 is a magnetic permeability, and ε_3 is a dielectric constant for gluons, $i, j = 1, 2, 3$ are indices for space coordinates, and the covariant derivative and the field strength of gluons are defined by

$$\mathcal{D}_\mu\Phi = \partial_\mu\Phi - ig_{\mathrm{s}}A_\mu^a T^a\Phi, \tag{2.25}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_{\mathrm{s}}[A_\mu, A_\nu]. \tag{2.26}$$

Here, μ, ν are indices for spacetime coordinates and g_{s} stands for the $SU(3)_{\mathrm{C}}$ coupling constant. Note that the Lagrangian (2.23) no longer has the Lorentz symmetry, which is explicitly broken by the existence of superconducting matter. At sufficiently high densities where perturbative calculations are valid, the parameters $\alpha, \beta_{1,2}, K_{0,3}$ are calculated as [69, 71]

$$\alpha = 4N(\mu)\log\frac{T}{T_{\mathrm{c}}}, \tag{2.27}$$

$$\beta_1 = \beta_2 = \frac{7\zeta(3)}{8(\pi T_{\mathrm{c}})^2}N(\mu) \equiv \beta, \tag{2.28}$$

$$K_3 = \frac{7\zeta(3)}{12(\pi T_{\mathrm{c}})^2}N(\mu), \tag{2.29}$$

$$\lambda_0 = \epsilon_0 = 1, \quad \lambda_3 = \epsilon_3 = 1, \tag{2.30}$$

$$K_0 = K_3/3, \tag{2.31}$$

---

[10] In general, the left-handed and right-handed diquark condensates can be different by a $SU(3)$ phase, $\Phi_{\mathrm{L}} = -\Phi_{\mathrm{R}}U$ with $U \in SU(3)$.

where μ is the baryon chemical potential and $N(\mu) \equiv \mu^2/2\pi^2$ is the density of state at the Fermi surface. The parameter K_0 has not been calculated in the literature, but can be derived following the same procedure in Ref. [72, 73].

The masses of the elementary excitations can be written in terms of the parameters above as

$$m_{\rm g}^2 = 2\lambda_3 g_{\rm s}^2 \Delta_{\rm CFL}^2 K_3, \quad m_1^2 = -\frac{2\alpha}{K_3}, \quad m_8^2 = \frac{4\beta\Delta_{\rm CFL}^2}{K_3}. \tag{2.32}$$

where $m_{\rm g}$ is the gluon mass, and $m_{1(8)}$ is the mass of the massive mesons in singlet (adjoint) representation of color-flavor locked symmetry $SU(3)_{\rm C+F}$. From this together with Eq. (2.31), we find the following relation

$$m_{\rm g} \sim \sqrt{\lambda_3} g_{\rm s}\mu, \quad m_1 = 2m_8 \sim 2\Delta_{\rm CFL}. \tag{2.33}$$

Since $g_{\rm s}\mu \gg \Delta_{\rm CFL}$ at the high density limit, we have

$$\kappa_{1,8} \equiv \frac{m_{1,8}}{m_{\rm g}} \ll 1. \tag{2.34}$$

This implies that the CFL phase is in the type I superconductor [11] [74].

Effects of a strange quark mass

So far, we have considered the CFL phase at the asymptotically high densities where all the quark masses $m_{\rm u,d,s}$ are negligible compared to the baryon chemical potential μ. In this subsection, let us consider the effect of a finite non-zero strange quark mass. We consider the situation where

$$0 \simeq m_{\rm u,d} < m_{\rm s} \ll \mu. \tag{2.35}$$

The effects of non-zero quark masses become important at smaller baryon chemical potentials. It was found [75] that the non-zero quark mass together with the β-equilibrium and the electric charge neutrality changes the CFL phase to the modified CFL (mCFL) phase where the color-flavor locking symmetry is further broken as

$$SU(3)_{\rm C+L+R} \to U(1)^2. \tag{2.36}$$

An important difference between the CFL and mCFL phases is that the quark chemical potentials $\mu_{\rm u}, \mu_{\rm d}, \mu_{\rm s}$ take different values. Hence, there appear difference between the Fermi momenta. This makes $\Delta_{\rm ud}$ different from $\Delta_{\rm ds} = \Delta_{\rm us}$. If one further imposes the electric charge neutrality, the gaps of the diquark condensation take different values as [75]

$$\Delta_{\rm ud} > \Delta_{\rm ds} > \Delta_{\rm us}. \tag{2.37}$$

---

[11] This does not mean that a state with vortices is unstable in the CFL phase. NG bosons for the $U(1)_{\rm B}$ symmetry breaking induce repulsive force between vortices, which stabilize multivortex states.

This is responsible for the symmetry breaking in Eq. (2.36). The correction to quadratic order to the GL potential in Eq. (2.23) was obtained as [76, 75]

$$\delta V \;=\; \frac{2}{3}\varepsilon \, \mathrm{tr}\, \left[\Phi^\dagger \Phi\right] + \varepsilon \, \mathrm{tr}\, \left[\Phi^\dagger X_3 \Phi\right], \tag{2.38}$$

$$\varepsilon \;=\; N(\mu)\frac{m_{\mathrm{s}}^2}{\mu^2} \log \frac{\mu}{T_c}, \tag{2.39}$$

$$X_3 \;=\; \mathrm{diag}\left(0, \frac{1}{2}, -\frac{1}{2}\right). \tag{2.40}$$

The first term in Eq. (2.38) can be absorbed into the definition of α as

$$\alpha' \equiv \alpha + \frac{2}{3}\varepsilon. \tag{2.41}$$

If we ignore the second term in Eq. (2.38), all the results in the previous sections are still valid under the understanding of the replacement α with α' in all equations. Therefore, an essential difference from the massless case is given rise to by the second term in Eq. (2.38). Since the term is sufficiently small if $m_{\mathrm{s}} \ll \mu$, we will treat it as a perturbation in Sec. 3.4.3.

Chapter 3

Vortices in color superconductors

In this section, we review the basic properties of vortices in the CFL phase. In Sec. 3.1, we review the topological defects in general. We discuss the relation between the stability of topological defects and the topology of order parameter spaces. In Sec. 3.2, we discuss the properties of $U(1)_{\mathrm{B}}$ superfluid vortices. In Sec. 3.3, we introduce the non-Abelian vortices (also called as the semi-superfluid vortices). An interesting feature of the non-Abelian vortices is the existence of the internal orientational zero modes. In Sec. 3.4 we give the low-energy effective theory for the orientational zero modes and its derivation. The effects of the strange quark mass are also taken into account.

3.1 Topological defects at work

Existence of topological defects is closely related to the degeneracy of ground states. Topological defects are solutions in field theories which are "winding" around some structures in the space of degenerate ground states. Degeneracy of ground states typically happens together with a spontaneous symmetry breaking. Here we consider the topological defects appearing in a spontaneous symmetry breaking.

Before embarking on the discussion of vortices, let us look at some examples of topological defects. They appear in various materials and they play important roles in determining the properties. We list some examples below.

- *Vortices in the two-dimensional XY model* play a vital role in the transition from low temperature to a high temperature disordered phase [16]. A vortex and an anti-vortex form a bound state at low temperature. When the temperature is increased above some critical value, vortex pairs dissociate and the phase coherence is broken.

- *Vortices in atomic and helium superfluids* have been extensively studied both theoretically and experimentally. In the two-fluid model [77, 78], which is an effective theory for superfluids, the super and normal components are basically independent on each other. However, their motion becomes correlated in the presence of vortices. The vortices in superfluids are known to form

a lattice when a superfluid is rotated. The dynamics of vortices is believed to play a key role in determining the properties of quantum turbulence [79].

- *Skyrmions* are topologically nontrivial solutions in non-linear sigma models. They were first proposed as a model of baryons [80]. A quantum Hall ferromagnet, whose effective theory is described by $O(3)$ sigma model in two spatial dimensions, accommodates skyrmions. A crystal of skyrmions is experimentally observed [81].

- *Domain walls and vortices* appear in ferromagnetic materials. Magnetic domain walls can be moved by an electric current [82] and they are expected to be applied to memory devices like magnetoresistive random access memory (MRAM). Domain-wall racetrack memory proposed by Parkin is one possibility for high-density storage device [83, 84].

3.1.1 Classification of topological defects

We briefly comment on how to argue the stability of topological defects.

Let us consider a symmetry breaking in which the original symmetry group G is spontaneously broken to a smaller group H. The *order parameter space* is defined as the set of all possible values of the order parameter field Φ. The order parameter is the label to distinguish degenerate ground states. Since all the degenerate order parameter fields in the same phase are obtained by acting G on a particular Φ_0, the order parameter space $\mathcal{M}$ is expressed as

$$\mathcal{M} = \{\rho(g)\Phi_0 | g \in G\}, \tag{3.1}$$

where ρ is some representation of G. The *isotropy group* H_Φ at $\Phi \in \mathcal{M}$ is defined by $H_\Phi = \{h \in G | \rho(h)\Phi = \Phi\}$. Two isotropy groups H_Φ, H_Ψ for $\Phi \in \mathcal{M}, \Psi \in \mathcal{M}$ are isomorphic and related by the conjugation $gH_\Phi g^{-1} = H_\Psi$ with $g \in G$ such that $\rho(g)\Phi = \Psi$. So we will omit the subscript of H which denotes the point in $\mathcal{M}$. Since the order parameter Φ is invariant under the action of H, the order parameter space can be identified as $\mathcal{M} \simeq G/H$.

Topological defects are solutions of equations of motion in a field theory, which are topologically distinct from a trivial ground state. The theory of the homotopy group is a natural language to discuss the topology of the order parameter space. Topological defects are classified according their homotopy classes. A solution cannot be deformed continuously into another one which belongs to other homotopy class. If the n-th homotopy group of the order parameter space $\pi_n(\mathcal{M})$ consists of only unit element, all the solutions are topologically equivalent and all of them can be continuously deformed into trivial solutions. In this case there is no stable topological excitation. Existence of stable topological defects corresponds to a nontrivial homotopy group of the order parameter space.

Various topological defects in ordered media in three dimensions are classified by homotopy groups as in Table 3.1.1. The dimensionality of a topological defect and the order of homotopy group to classify the defect is related as shown in Table 3.1.1. A general discussion on topological defects can be found in Refs. [85, 86, 87].

Topological defect	Dimension of the defect	Classification
Domain wall	2	$\pi_0(G/H)$
Vortex	1	$\pi_1(G/H)$
Monopole	0	$\pi_2(G/H)$
Texture	3	$\pi_3(G/H)$

Table 3.1: Classification of topological solitons in three spatial dimensions

3.2 Abelian vortices

As discussed above, topologically stable vortices appear when the first fundamental group of the order parameter space $\pi_1(G/H)$ is nontrivial. The simplest case in which vortices appear is when $G = U(1)$ and $H = \{e\}$ where e is the unit element. These vortices are called *Abelian vortices*. Here we review basic properties of Abelian vortices.

QCD has two independent phase rotations, $U(1)_\mathrm{B}$ and $U(1)_\mathrm{A}$ symmetries, under which the quark field is transformed as

$$q \rightarrow e^{i\theta_\mathrm{A}\gamma_5 + i\theta_\mathrm{B}}q. \tag{3.2}$$

Accordingly, the diquark condensates are transformed as

$$\Phi_\mathrm{L} = e^{i\theta_\mathrm{A} + i\theta_\mathrm{B}}\Phi_\mathrm{L}, \quad \Phi_\mathrm{R} \rightarrow e^{-i\theta_\mathrm{A} + i\theta_\mathrm{B}}\Phi_\mathrm{R}. \tag{3.3}$$

Therefore, the formation of diquark condensates breaks both of the $U(1)_\mathrm{B}$ and $U(1)_\mathrm{A}$ symmetries. The $U(1)_\mathrm{A}$ symmetry is also broken explicitly by the chiral anomaly. We here discuss the $U(1)_\mathrm{B}$ global vortices [36, 37], each of which has integral winding number with respect to the $U(1)_\mathrm{B}$ symmetry. They turned out to be energetically unstable, and a $U(1)_\mathrm{B}$ vortex should decay into vortices with minimal energy [38]. There also exist vortices associated with the $U(1)_\mathrm{A}$ symmetry, although the $U(1)_\mathrm{A}$ symmetry is explicitly broken by the quantum anomaly. As a result of this explicit breaking, a $U(1)_\mathrm{A}$ vortex is always accompanied by a domain wall. The properties of vortex-domain wall composites are investigated in Ref. [88].

3.2.1 $U(1)_\mathrm{B}$ superfluid vortices

Here we discuss the properties of $U(1)_\mathrm{B}$ superfluid vortices. We call a vortex with an integral winding number with respect to $U(1)_\mathrm{B}$ as a $U(1)_\mathrm{B}$ vortex. The $U(1)_\mathrm{B}$ vortices are basically similar to the vortices in atomic or He superfluids in condensed matter physics. A $U(1)_\mathrm{B}$ vortex does not carry a color magnetic flux and there is no internal degree of freedom, which are not the case for non-Abelian vortices discussed later.

We can find the solutions of $U(1)_\mathrm{B}$ vortices by solving the equation of motion derived from the GL Lagrangian, as follows. If one circles around a $U(1)_\mathrm{B}$ vortex, the overall phase of the condensate matrix is rotated once. Thus, in order to find such a solution, it is natural to take the condensate matrix
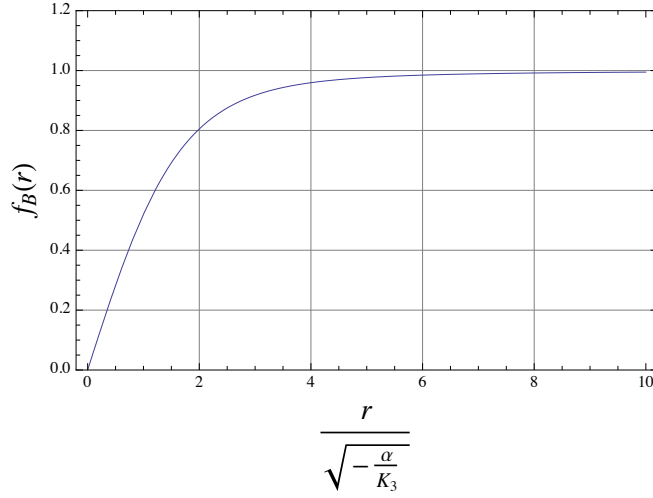
Figure 3.1: Profile function $f_B(r)$ of a $U(1)_B$ vortex with $k = 1$.

to be proportional to the unit matrix, $\Phi(x) = \phi(x)\mathbf{1}_3/\sqrt{3}$. Then, the Lagrangian is reduced to the following form:

$$\mathcal{L}_{U(1)_B} = K_0|\partial_0\phi|^2 - K_3|\partial_i\phi|^2 - \left[\alpha|\phi|^2 + \frac{4\beta}{3}|\phi|^4 + \frac{3\alpha^2}{16\beta}\right]. \tag{3.4}$$

This is the Lagrangian of the Goldstone model, that describes a relativistic superfluid. In the ground state, $\phi = \sqrt{-3\alpha/8\beta} \equiv \sqrt{3}\Delta_{\text{CFL}}$, and the $U(1)_B$ symmetry is broken. Topologically stable vortices are characterized by an integer, since the first homotopy group of the order parameter space is given by $\pi_1[U(1)_B] \simeq \mathbb{Z}$. Let us find an axially-symmetric solution. A vortex placed along the z-axis can be parametrized as

$$\phi(r, \theta) = \sqrt{3}\Delta_{\text{CFL}}f_B(r)e^{ik\theta}, \tag{3.5}$$

where (r, θ, z) is the cylindrical coordinates with $x + iy \equiv re^{i\theta}$, $f_B(r)$ is a vortex profile function, and $k \in \mathbb{Z}$ is the winding number of a vortex, which characterizes the strength of a vortex. Substituting the ansatz (3.5), the equation of motion derived from the Lagrangian (3.4) reduces to an ordinary differential equation,

$$f_B'' + \frac{f_B'}{r} - \frac{k^2}{r^2}f_B + \frac{\alpha}{K_3}\left(f_B^2 - 1\right)f_B = 0. \tag{3.6}$$

As a boundary condition, we take $f_B(0) = 0$ so that the field is regular at the origin. At far distances from the vortex core, we take $f_B(\infty) = 1$, which assures that the value of the field goes to the ground-state value. The numerical solution of $f_B(r)$ is shown in Fig. 3.1. The function $f_B(r)$ approaches the ground-state value at distances $r > \Delta_{\text{CFL}}^{-1}$. The asymptotic behavior at $r \to 0$ and $r \to \infty$ can be

obtained analytically,

$$f_{\rm B}(r) \;=\; c_k \left(\frac{r}{\sqrt{-\alpha/K_3}} \right)^k + \cdots, \qquad\qquad r \to 0, \tag{3.7}$$

$$f_{\rm B}(r) \;=\; 1 - \mathcal{O}\left[\left(\frac{r}{\sqrt{-\alpha/K_3}} \right)^{-2} \right] + \cdots, \qquad r \to \infty. \tag{3.8}$$

The constants c_k are obtained by numerical methods, for example, $c_1 = 0.58\cdots$ for $k = 1$ [89]. The tension of a vortex, which is the energy of a vortex per unit length, is logarithmically divergent,

$$T_{U(1)_{\rm B}} = 6\pi\Delta_{\rm CFL}^2 K_3 k^2 \log \frac{L}{\xi} + \cdots, \tag{3.9}$$

where L is an IR cutoff scale representing the system size, and ξ is a UV cutoff of the order of the inverse of the vortex core size, and $\cdots$ indicates finite contributions to the tension.

3.3 Non-Abelian vortices in the CFL phase

In this section, we introduce non-Abelian vortices, which are energetically the most stable vortices in the CFL phase. The non-Abelian vortices are first introduced in Ref. [38].

Let us first discuss the existence of non-Abelian vortices in the CFL phase from the topological point of view. Recall that, topological solitons, like vortices, are a type of solutions which are topologically distinct from the vacuum solutions. If one circles around a topological vortex, the value of the order parameter also makes a loop in the order parameter space. When the loop in the order parameter space cannot be continuously deformed to a point, the solution is topologically stable, which means that the solution cannot be continuously deformed to the vacuum solution. In the case of the CFL phase, the order parameter space is given by

$$\frac{G}{H} \simeq \frac{SU(3)_{\rm C-F} \times U(1)_{\rm B}}{(\mathbb{Z}_3)_{\rm C-F+B}} \simeq U(3)_{\rm C-F+B}, \tag{3.10}$$

as explained in Chap. 2. The space $U(3)$ accommodates non-Abelian vortices, as explained below. Let us consider a closed loop in $U(3)$ that is nontrivial. The loop in $U(3)$ can also be described by a path in $SU(3) \times U(1)$, which becomes closed by $\mathbb{Z}_3$ identification. Consider a curve that starts from $(\mathbf{1}, \mathbf{1})$ and end at (ω^{-1}, ω) in $SU(3) \times U(1)$, where $\omega \equiv e^{-i2\pi/3}$, and $U(1)$ is rotated in the clockwise way. This path is nontrivial in $SU(3)$ space, then, it is also a nontrivial closed loop l in $U(3)$. Let us denote the set of loops which are homotopically equivalent to l by $[l]$. Namely, $[l]$ is the homotopy class represented by l.

The associated loop in the order parameter space for a non-Abelian vortex take is in the class $[l]^{3n+1}$ or $[l]^{3n+2}$ where n is an integer. In this case, the projection of a loop into the $SU(3)$ space is always a path, that cannot be contracted to a point. Accordingly, a color flux is present inside a vortex. We can see that an Abelian vortex correspond to $[l]^{3n}$. Take $[l]^3$ for example. The loop $[l]^3$ is

associated with a closed loop in both $SU(3)$ and $U(1)$. A closed loop in $SU(3)$ can be continuously deformed to a point, since $SU(3)$ is a simply connected space. Thus, $[l]^3$ is a nontrivial loop only in $U(1)$ and corresponds to the Abelian vortices. In this case a vortex does not have a color flux, since the order parameter is not transformed color generators along a loop.

Let us explicitly find the loops in the order parameter space for minimally winding loops $[l]$ and $[l]^{-1}$. The projections of these loops in $SU(3)$ and $U(1)$ spaces are given by

$$l \quad : \quad (\mathbf{1}, \mathbf{1}) \to (\omega^{-1}, \omega) \in SU(3) \times U(1), \tag{3.11}$$

$$l^{-2} \quad : \quad (\mathbf{1}, \mathbf{1}) \to (\omega^2, \omega^{-2}) = (\omega^{-1}, \omega^{-2}) \in SU(3) \times U(1). \tag{3.12}$$

In $U(1)_{\mathrm{B}}$ space, the loop is generated by the baryon charge,

$$Q_{\mathrm{B}} = \frac{2}{3}\mathbf{1}. \tag{3.13}$$

The $U(1)_{\mathrm{B}}$ parts of these loops can be expressed as

$$l \quad : \quad e^{i\phi Q_{\mathrm{B}}/2}, \tag{3.14}$$

$$l^{-2} \quad : \quad e^{-i\phi Q_{\mathrm{B}}}, \tag{3.15}$$

with $\phi \in [0, 2\pi]$, which is a parameter of the circle. On the other hand, the $SU(3)$ part is generated by the following generator,

$$T_8 \equiv \frac{1}{2\sqrt{3}} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.16}$$

The projections of loops in $SU(3)$ are the same, as is evident from Eqs. (3.11) and (3.12). Hence, we obtain the parametrization of the loops l and l^{-2} as

$$l \quad : \quad M_1(\phi) = \Delta_{\mathrm{CFL}} e^{2i\phi T_8/\sqrt{3}} \cdot e^{i\phi Q_{\mathrm{B}}/2} = \Delta_{\mathrm{CFL}} \begin{pmatrix} e^{i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.17}$$

$$l^{-2} \quad : \quad M_2(\phi) = \Delta_{\mathrm{CFL}} e^{-2i\phi T_8/\sqrt{3}} \cdot e^{-i\phi Q_{\mathrm{B}}} = \Delta_{\mathrm{CFL}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\phi} & 0 \\ 0 & 0 & e^{-i\phi} \end{pmatrix}. \tag{3.18}$$

These are the loops in the order parameter space. The vortices that correspond to the loops above are named as M_1 and M_2 vortices, respectively [38].

3.3.1 M_1 vortices

Let us find the solution of equations of motion which corresponds to the loop l in the order parameter space. We consider the solutions of the Euler-Lagrange equations derived from the following Ginzburg-Landau free energy,

$$F = \mathrm{Tr}\left[-K_3 \mathcal{D}_i \Phi^\dagger \mathcal{D}_i \Phi - \frac{F_{ij}^2}{4\lambda_3} \right] - \alpha \mathrm{Tr}\left[\Phi^\dagger \Phi \right] - \beta \left(\left[\mathrm{Tr}(\Phi^\dagger \Phi) \right]^2 - \mathrm{Tr}\left[(\Phi^\dagger \Phi)^2 \right] \right) + \frac{3\alpha^2}{16\beta}, \tag{3.19}$$

where we discarded the terms irrelevant for the discussions below. We take $\lambda_3 = 1$ in the rest of this section.

Let a vortex be placed along the z-axis. We can parametrize the solution which corresponds to the loop given by Eq. (3.17) in the order parameter space as

$$\Phi(r, \theta) = \Delta_{\text{CFL}} \begin{pmatrix} e^{i\theta} f(r) & 0 & 0 \\ 0 & g(r) & 0 \\ 0 & 0 & g(r) \end{pmatrix}, \tag{3.20}$$

$$A_i(r, \theta) = \frac{\epsilon_{ij} x^j}{g_{\text{s}} r^2} (1 - h(r)) \begin{pmatrix} -\frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \tag{3.21}$$

where $i = 1, 2$, (r, θ) is the radial coordinate in xy-plane, and $f(r)$, $g(r)$, and $h(r)$ are the profile functions that characterize the vortex solution. Note that this is not the only lowest-energy solution. The solutions with of the following form,

$$\Phi(r, \theta) = \Delta_{\text{CFL}} \begin{pmatrix} g(r) & 0 & 0 \\ 0 & e^{i\theta} f(r) & 0 \\ 0 & 0 & g(r) \end{pmatrix}, \tag{3.22}$$

$$A_i(r, \theta) = \frac{\epsilon_{ij} x^j}{g_{\text{s}} r^2} (1 - h(r)) \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \tag{3.23}$$

or

$$\Phi(r, \theta) = \Delta_{\text{CFL}} \begin{pmatrix} g(r) & 0 & 0 \\ 0 & g(r) & 0 \\ 0 & 0 & e^{i\theta} f(r) \end{pmatrix}, \tag{3.24}$$

$$A_i(r, \theta) = \frac{\epsilon_{ij} x^j}{g_{\text{s}} r^2} (1 - h(r)) \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}, \tag{3.25}$$

have the same energy as that of the solution like Eqs. (3.20) and (3.21). In fact, there are infinitely-many degenerate vortices with the same energy, as is discussed later in Sec. 3.4. We impose the following boundary condition at spatial infinities,

$$f(\infty) = 1, \quad g(\infty) = 1, \quad h(\infty) = 0. \tag{3.26}$$

The last condition means that the gauge field is pure gauge at spatial infinity. The condition (3.26) is consistent with the loop (3.17) in the order parameter space. For $r \to 0$, we take

$$f(0) = 0, \quad g'(0) = 0, \quad h(0) = 1. \tag{3.27}$$

The decomposition of the action of the $U(3)$ group in Eq. (3.20) to the $U(1)_B$ and $SU(3)_C$ actions can be found as

$$\Phi = \Delta_{\text{CFL}} \begin{pmatrix} e^{i\theta} f(r) & 0 & 0 \\ 0 & g(r) & 0 \\ 0 & 0 & g(r) \end{pmatrix} = \Delta_{\text{CFL}} e^{\frac{i\theta}{3}} \begin{pmatrix} e^{\frac{2i\theta}{3}} f(r) & 0 & 0 \\ 0 & e^{-\frac{i\theta}{3}} g(r) & 0 \\ 0 & 0 & e^{-\frac{i\theta}{3}} g(r) \end{pmatrix}. \quad (3.28)$$

If one circles around the vortex along a loop, the $SU(3)$ and $U(1)$ contributions becomes (ω^{-1}, ω), which is consistent with Eq. (3.11).

Let us determine the behavior of the profile functions f, g, h. The Euler-Lagrange equations that the profile functions should satisfy are given by

$$f'' + \frac{f'}{r} - \frac{(2h+1)^2}{9r^2} f - \frac{m_1^2}{6} f \left(f^2 + 2g^2 - 3 \right) - \frac{m_8^2}{3} f(f^2 - g^2) = 0, \quad (3.29)$$

$$g'' + \frac{g'}{r} - \frac{(h-1)^2}{9r^2} g - \frac{m_1^2}{6} g \left(f^2 + 2g^2 - 3 \right) - \frac{m_8^2}{6} g(f^2 - g^2) = 0, \quad (3.30)$$

$$h'' - \frac{h'}{r} - \frac{m_g^2}{3} \left(g^2(h-1) + f^2(2h+1) \right) = 0. \quad (3.31)$$

Perturbatively calculated values of the masses m_1, m_8, and m_g are given in Eq. (2.32). Vortex configurations can be obtained by solving the coupled equations above of the functions $f(r)$, $g(r)$, and $h(r)$.

Let us find the asymptotic behaviors of the profile functions at large distances from the vortex core, which can be studied analytically. We consider small fluctuations around the asymptotic values $(f, g, h) = (1, 1, 0)$ and define

$$\delta F(r) = (f(r) + 2g(r)) - 3, \quad \delta G(r) = f(r) - g(r), \quad \delta h(r) = h(r). \quad (3.32)$$

Here, $\delta F(r)$ is the fluctuation of the trace part of Φ and $\delta G(r)$ is the one for the traceless part proportional to T_8. In terms of the functions defined in Eq. (3.32), the order parameter is written as

$$\Phi = \Delta_{\text{CFL}} \mathbf{1}_3 + \Delta_{\text{CFL}} \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \delta F(x) + \Delta_{\text{CFL}} \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \delta G(r) + \cdots. \quad (3.33)$$

At far distances from the vortex core, we can linearize the field equations in $\delta F(r)$, $\delta G(r)$, and $\delta h(r)$ as

$$\left[\left(\frac{d}{dr} \right)^2 + \frac{1}{r} \frac{d}{dr} - m_1^2 - \frac{1}{9r^2} \right] \delta F = \frac{1}{3r^2}, \quad (3.34)$$

$$\left[\left(\frac{d}{dr} \right)^2 + \frac{1}{r} \frac{d}{dr} - m_8^2 - \frac{1}{9r^2} \right] \delta G = \frac{2}{3r^2} \delta h, \quad (3.35)$$

$$\left[\left(\frac{d}{dr} \right)^2 - \frac{1}{r} \frac{d}{dr} - m_g^2 \right] \delta h = \frac{2}{3} m_g^2 \delta G. \quad (3.36)$$

The approximate solution of the first equation is given by [40]

$$\delta F = q_1 \sqrt{\frac{\pi}{2m_1 r}} e^{-m_1 r} - \frac{1}{3m_1^2 r^2} + O\left((m_1 r)^{-4}\right). \tag{3.37}$$

The first term is much smaller than the others which is usually neglected as in the case of the $U(1)$ global vortex. The dominant terms decrease polynomially. At the high baryon density where $m_{\rm g} \gg m_{1,8}$, the solutions of Eqs. (3.35) and (3.36) are given by

$$\delta G = q_8 \sqrt{\frac{\pi}{2m_8 r}} e^{-m_8 r}, \quad \delta h = -\frac{2}{3} \frac{m_{\rm g}^2}{m_{\rm g}^2 - m_8^2} \delta G. \tag{3.38}$$

Here $q_{1,8}$ are numerical constants.

The behavior of δh is interesting. The gluon has the magnetic mass $m_{\rm g}$ by the Higgs mechanism in the CFL phase, so naively one expects the tail of the gauge field is determined by $m_{\rm g}$ like $\delta h \sim e^{-m_{\rm g} r}$. However, the function δh actually decays as $\delta h \sim e^{-m_8 r}$, namely the asymptotic behavior is determined by the mass of adjoint scalar fields, m_8. The intervortex forces of the non-Abelian vortices in CFL could be quite different from those in conventional metallic superconductors, since the asymptotic behaviors of the vortex string is deeply related to the intervortex forces[1]. In the opposite case $m_{\rm g} \ll m_8$, the asymptotic behaviors are changed from Eq. (3.38) as

$$\delta G = -\frac{2q_{\rm g}}{3} \frac{1}{(m_8^2 - m_{\rm g}^2) r^2} \sqrt{\frac{\pi}{2m_8 r}} e^{-m_{\rm g} r}, \quad \delta h = q_{\rm g} \sqrt{\frac{\pi m_{\rm g} r}{2}} e^{-m_{\rm g} r}. \tag{3.39}$$

In this case the asymptotic behaviors are governed by $m_{\rm g}$ which is smaller than m_8. The coefficients q_8 and $q_{\rm g}$ depend on the masses m_1, m_8 and $m_{\rm g}$ and can be determined numerically [40].

The full numerical solutions for Eqs. (3.29), (3.30) and (3.31) were obtained in Ref. [40]. The behaviors of the profile functions depend on the mass parameters $m_{1,8,{\rm g}}$. In particular, the value of $g(0)$ at the origin is quite sensitive to the ratio m_1/m_8. The value $g(0)$ becomes lager than 1 for $m_1 > m_8$, and is smaller than 1 for $m_1 < m_8$, see Fig. 3.2. Since $f(r) - g(r)$ plays a role of an order parameter for the breaking of $SU(3)_{\rm C+F}$, information of the profile functions $f(r)$ and $g(r)$ are important. Perturbative calculations indicate that $m_1 = 2m_8 \ll m_{\rm g}$. In this case, $g(0)$ is convex near the center of a vortex. So in the case of a realistic non-Abelian vortex in the CFL phase, $g(0)$ is greater than 1 as shown in the right panel of Fig.3.2. The profile functions $f(r), g(r), h(r)$, the energy and the color-magnetic flux densities for a particular choice of the parameter ratio $(m_1, m_8, m_{\rm g}) = (2, 1, 10)$ are shown in Fig. 3.3. In this case $m_{\rm g}$ is much greater than $m_{1,8}$, the color-magnetic flux has almost same width as the scalar density distribution. This is consistent with the asymptotic behavior discussed before. The asymptotic tail of the gluon field behave like $\sim e^{-m_8 r}$, which depends only on m_8 when the scalar mass is lighter than the mass of gluons, $m_8 < m_{\rm g}$.

---

[1] In the case of vortices in the Abelian-Higgs model[90, 91], the tail of the scalar field behave like $\exp(-m_{\rm H} r)$ with the Higgs mass $m_{\rm H}$, while that of magnetic flux behaves like $\exp(-m_{\rm e} r)$ where $m_{\rm e}$ is the photon mass. However, when $(m_{\rm e} > 2m_{\rm H})$, the tail of the gauge boson becomes $\exp(-2m_{\rm H} r)$ [92, 93, 94].
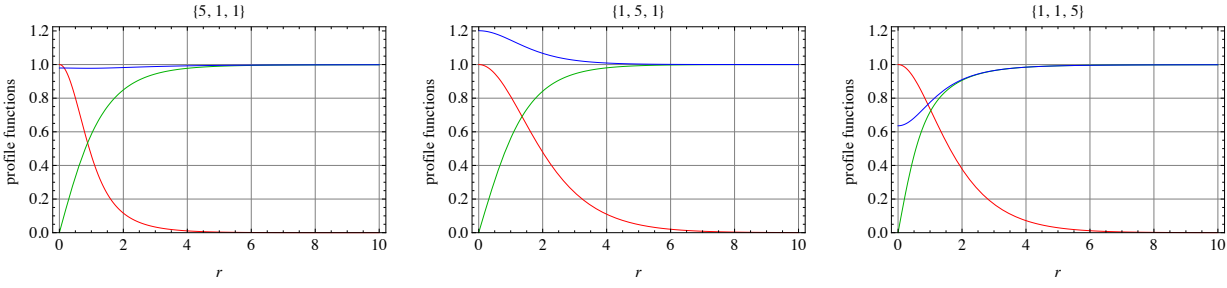
Figure 3.2: Profile functions $\{f(r), g(r), h(r)\} = \{\text{green}, \text{blue}, \text{red}\}$ of a non-Abelian vortex in the CFL phase (taken from Ref. [18]). The ratios of the masses set to $\{m_\text{g}, m_1, m_8\} = \{5, 1, 1\}, \{1, 5, 1\}, \{1, 1, 5\}$. The radial coordinate is in the unit of $m_1(= m_8)$, $m_\text{g}(= m_8)$, $m_g(= m_1)$ for the three pictures, respectively. $g(r)$ is almost flat when $m_1 \simeq m_8$.

Let us compare the non-Abelian vortices explained in this section with the Abelian vortices in the previous section Sec. 3.2. A $U(1)_\text{B}$ vortex is characterized by an element of the first homotopy group, $\pi_1[U(1)_\text{B}]$. As can be seen from Eq. (3.5), a $U(1)_\text{B}$ vortex has an integer winding number. A non-Abelian vortex is also characterized by an element of $\pi_1[U(1)_\text{B}]$, but its winding number can be a fractional number. The winding number of a non-Abelian vortex is a multiple of $1/3$. A minimal vortex has the winding number $1/3$. This can be explicitly seen by writing the asymptotic behavior of Φ given in Eq. (3.28) at $r \to \infty$ as

$$\Phi \sim \Delta_\text{CFL} e^{\frac{i\theta}{3}} \mathbf{1}_3. \tag{3.40}$$

Here "$\sim$" stands for the equivalence under the gauge transformation. This explicitly shows that the minimally winding non-Abelian vortex has only 1/3 winding number in the $U(1)_\text{B}$ space.

Finding a minimally winding solution is important, because such a solution is most stable. A non-Abelian vortex is a global vortex, so its tension consists of two parts: a logarithmically divergent part and a finite part.

$$\mathcal{T}_{\text{M}_1} = \mathcal{T}_{\text{div};\text{M}_1} + \mathcal{T}_{\text{fin};\text{M}_1}. \tag{3.41}$$

The dominant contribution to the tension (logarithmically divergent) comes from the kinetic term,

$$\mathcal{T}_{\text{div};\text{M}_1} \simeq K_3 \int d^2x \, \text{tr} \, \mathcal{D}_i \Phi (\mathcal{D}_i \Phi)^\dagger = \frac{1}{9} \times 6\pi \Delta_\text{CFL}^2 K_3 \log \frac{L}{\xi}. \tag{3.42}$$

This should be compared with the tension of $U(1)_\text{B}$ integer vortex given in Eq. (3.9). The tension of a vortex is proportional to the square of the winding number. Since the minimal winding number of a non-Abelian vortex is $1/3$, the tension of the non-Abelian vortex is $1/3^2 = 1/9$ times that of a $U(1)_\text{B}$ integer vortex. That is why a non-Abelian vortex is most stable in the CFL phase.

3.3.2   M$_2$ vortices

Here let us briefly discuss a vortex configuration called the M$_2$ vortices. This solutions corresponds to the loop l^{-2} in the order parameter space. The ansatz for the M$_2$ vortex is similar to that in Eqs. (3.20)
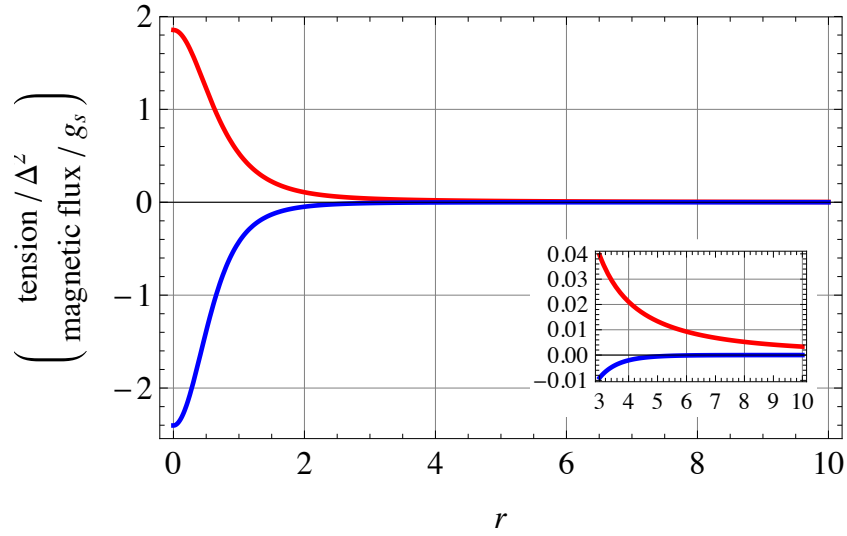
Figure 3.3: Energy density (red) and the non-Abelian magnetic flux (blue) for a minimally winding non-Abelian vortex as a function of the distance from the center of the vortex (taken from Ref. [18]). We chose the ratio of parameters as $(m_\mathrm{g}, m_1, m_8) = (10, 2, 1)$. The radial coordinate is in the unit of m_8^{-1}. While the energy density has a long tail, the magnetic flux exponentially goes to zero at $r \sim m_8^{-1}$.

and (3.21),

$$\Phi(r,\theta) = \Delta_\mathrm{CFL} \begin{pmatrix} q(r) & 0 & 0 \\ 0 & e^{-i\theta}p(r) & 0 \\ 0 & 0 & e^{-i\theta}p(r) \end{pmatrix}, \tag{3.43}$$

$$A_i(r,\theta) = \frac{\epsilon_{ij}x^j}{g_\mathrm{s}r^2}(1 - h(r)) \begin{pmatrix} -\frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}. \tag{3.44}$$

The ansatz for Φ can be rewritten as

$$\Phi(r,\theta) = \Delta_\mathrm{CFL} e^{-i\frac{2\theta}{3}} \begin{pmatrix} e^{i\frac{2\theta}{3}} & 0 & 0 \\ 0 & e^{-i\frac{\theta}{3}} & 0 \\ 0 & 0 & e^{-i\frac{\theta}{3}} \end{pmatrix} \begin{pmatrix} q(r) & 0 & 0 \\ 0 & p(r) & 0 \\ 0 & 0 & p(r) \end{pmatrix}. \tag{3.45}$$

This shows that an M_2 vortex has the winding number $-2/3$ with respect to $U(1)_\mathrm{B}$, whose magnitude is twice larger than that of an M_1 vortex. For the rest part, the ansatz goes into the $SU(3)_\mathrm{C}$ orbit, more explicitly, $S^1 \subset SU(3)_\mathrm{C}$ which is generated by T_8. Like in the case of an M_1 vortex, the tension of the M_2 vortex consists of a divergent part and a finite part. The divergent part is given by

$$\mathcal{T}_{\mathrm{div};M_2} = \frac{4}{9} \times 6\pi^2 \Delta_\mathrm{CFL}^2 K_3 \log \frac{L}{\xi}. \tag{3.46}$$

This is four times larger than the tension of an M_1 vortex. This implies that an M_2 vortex with a red flux decays into two M_1 vortices with green and blue colors. The contributions to the tension from the color-magnetic flux are the same as that of an M_1 vortex.

3.4   Orientational zero modes of non-Abelian vortices

3.4.1   What are orientational zero modes?

We here explain the orientational zero modes, which are a kind of Nambu-Goldstone modes. Inside a vortex, the symmetry of the CFL ground states is further broken to a smaller group. This results in a Nambu-Goldstone mode, which propagates only along a vortex.

Let us identify the symmetry breaking pattern. A vortex of the lowest energy placed along the z axis takes the form,

$$\Phi_0(x) = \begin{pmatrix} f(r)\mathrm{e}^{i\theta} & 0 & 0 \\ 0 & g(r) & 0 \\ 0 & 0 & g(r) \end{pmatrix}, \quad A_i(x) = \frac{1}{g}\frac{\epsilon_{ij}x^j}{r^2}[1-h(r)]\begin{pmatrix} -2/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}, \quad (3.47)$$

where $\{f(r),\ g(r), h(r)\}$ are the profile functions. The orientational zero modes are identified as follows. The solution Φ_0 can be gauge-transformed into the form,

$$\Phi_0 = \Delta_{\mathrm{CFL}}\mathrm{diag}(f(r)\mathrm{e}^{i\theta}, g(r), g(r)) \sim \Delta_{\mathrm{CFL}}e^{\frac{i\theta}{3}}\left(\frac{F(r)}{3}\mathbf{1}_3 + \sqrt{\frac{2}{3}}G(r)T_8\right), \quad (3.48)$$

Because of the term proportional to T_8 in Eq. (3.48), the $SU(3)_{\mathrm{C+F}}$ symmetry is now broken to

$$H' = U(1)_{\mathrm{C+L+R}} \times SU(2)_{\mathrm{C+L+R}}, \quad (3.49)$$

where $U(1)_{\mathrm{C+L+R}}$ is the Abelian subgroup generated by T_8 and $SU(2)_{\mathrm{C+L+R}}$ is $SU(2)$ subgroup which commutes with $U(1)_{\mathrm{C+L+R}}$.

As a consequence, there appear further Nambu-Goldstone modes [39] that parametrize the coset space,

$$\frac{H}{H'} = \frac{SU(3)_{\mathrm{C+L+R}}}{U(1)_{\mathrm{C+L+R}} \times SU(2)_{\mathrm{C+L+R}}} \simeq \mathbb{C}P^2. \quad (3.50)$$

This space is known as the two-dimensional complex projective space. There are degenerate family of vortex solutions with different color magnetic fluxes[2]. Each solution corresponds to a point in the $\mathbb{C}P^2$ space. The general solutions of the same energy can be obtained by applying the $SU(3)_{\mathrm{C+F}}$ transformations on Φ_0 as

$$\Phi \to \Delta_{\mathrm{CFL}}e^{\frac{i\theta}{3}}\left(\frac{F}{3}\mathbf{1}_3 + \sqrt{\frac{2}{3}}GUT_8U^\dagger\right), \quad U \in SU(3)_{\mathrm{C+L+R}}. \quad (3.51)$$

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[2]The degeneracy is broken if we include quantum effects [95][96].

$$\Phi = U\Phi_0'U^{-1} = e^{i\theta/3}\left\{F(f,g) + G(f,g)\left(\phi\phi^\dagger - \frac{1}{3}\mathbf{1}_3\right)\right\}, \tag{3.52}$$

where ϕ is the parameter of the orientational moduli, and we have defined

$$F \equiv f + 2g, \quad G \equiv f - g. \tag{3.53}$$

The order parameter of the breaking of $SU(3)_{\mathrm{C+L+R}}$ is $|G| = |f(r) - g(r)|$. Since both $f(r)$ and $g(r)$ asymptotically reach 1 at spatial infinities, the color-flavor locked symmetry is restored at far distances from the vortex core. As can be seen in Fig. 3.3, $|G|$ takes a maximum value at the center of the non-Abelian vortex. Hence, the orientational NG modes are localized on the non-Abelian vortex.

As is derived in the next section, the low-energy effective dynamics on the vortex is described by a $\mathbb{C}P^2$ model [40], whose Lagrangian is given by

$$\mathcal{L}_{\mathrm{low}} = C \sum_{\mu=0,3} K_\mu \left[\partial^\mu\phi^\dagger\partial_\mu\phi + (\phi^\dagger\partial^\mu\phi)(\phi^\dagger\partial_\mu\phi)\right], \tag{3.54}$$

where C and K_μ are constants and $\phi(t,z)$ is a complex three-component field normalized as $\phi^\dagger\phi = 1$. ϕ transforms as the fundamental representation under $SU(3)_{\mathrm{C+F}}$. At first sight, the parametrization of $\mathbb{C}P^2$ by ϕ has $6 - 1 = 5$ dimensions, since ϕ is 3 dimensional complex vector which satisfies the condition $\phi^\dagger\phi = 1$. However, the action (3.54) has a local $U(1)$ invariance. The Lagrangian (3.54) is invariant under the local transformation $\phi \to e^{i\alpha(x)}\phi$ [97, 98]. One can gauge away one component, so ϕ correctly has 4 degrees of freedom.

3.4.2 Low-energy effective theory of orientational zero modes

As discussed above, a non-Abelian vortex has internal $\mathbb{C}P^2$ orientational modes, that are associated with the spontaneous breaking of the color-flavor locked symmetry. The orientational modes are massless and they propagate along non-Abelian vortices. Here we derive the low-energy theory that describes the orientational modes. We neglect the massive modes in the rest of this subsection, since we are interested in the low-energy dynamics.

Let us first identify the orientational zero modes in the background fields Φ and A_μ. We start with a particular non-Abelian vortex solution,

$$\Phi^\star(x,y) = \Delta_{\mathrm{CFL}}e^{\frac{i\theta}{3}}\left(\frac{F(r)}{3}\mathbf{1}_3 + G(r)\sqrt{\frac{2}{3}}\,T_8\right), \quad A_i^\star(x,y) = -\frac{1}{g_{\mathrm{s}}}\frac{\epsilon_{ij}x^j}{r^2}h(r)\sqrt{\frac{2}{3}}\,T_8. \tag{3.55}$$

A general solution can be obtained by transforming this solution by $SU(3)_{\mathrm{C+F}}$ transformation,

$$\Phi(x,y) = U\Phi^\star U^\dagger = \Delta_{\mathrm{CFL}}e^{\frac{i\theta}{3}}\left(\frac{F(r)}{3}\mathbf{1}_3 - G(r)\left\langle\phi\phi^\dagger\right\rangle\right), \tag{3.56}$$

$$A_i(x,y) = U A_i^\star U^\dagger = \frac{1}{g_{\mathrm{s}}}\frac{\epsilon_{ij}x^j}{r^2}h(r)\left\langle\phi\phi^\dagger\right\rangle, \tag{3.57}$$

where $\langle \mathcal{O} \rangle$ is the traceless part of a square matrix $\mathcal{O}$, and we introduced a complex 3-component vector ϕ that satisfies the following relation

$$-\sqrt{\frac{2}{3}}\, U T_8 U^\dagger = \phi\phi^\dagger - \frac{1}{3}\mathbf{1}_3 \equiv \left\langle \phi\phi^\dagger \right\rangle . \tag{3.58}$$

Taking the trace of Eq. (3.58) gives a constraint on ϕ,

$$\phi^\dagger \phi = 1. \tag{3.59}$$

Thus, we find that ϕ represents the homogeneous coordinates of $\mathbb{C}P^2$ space.

Now we promote the moduli parameters ϕ to the fields depending on the coordinates (t, z) of the vortex worldsheet by using the so-called moduli space approximation. This method is first introduced by Manton for BPS monopoles [99, 100], and is widely used for obtaining the effective low-energy theory of topological solitons. Since we describe the low-energy dynamics of the orientational moduli fields, we restrict ourselves to the study of the dynamics with a typical energy scale much lower than the mass scales of the original theory, m_1, m_8, and m_g. Namely, we consider the situation where

$$|\partial_\alpha \phi(z, t)| \ll \min\{m_1, m_8, m_g\}, \quad (\alpha = 0, 3). \tag{3.60}$$

Neglecting the higher derivative terms with respect to z and t, the low-energy effective theory for the orientational moduli field can be obtained by plugging $\Phi(x, y; \phi(z, t))$ and $A_i(x, y; \phi(z, t))$ into the original Lagrangian (2.23). Note that $\Phi(x, y; \phi(z, t))$ and $A_i(x, y; \phi(z, t))$ depend on the full spacetime coordinates, which are obtained by replacing the moduli parameter ϕ with the moduli field $\phi(z, t)$ in Eq. (3.57). In order to construct the Lagrangian of the effective theory, we also have to determine the x^α dependence of the gauge fields A_α, which are zero in the background solution. For this purpose, we make an ansatz following [101],

$$A_\alpha = \frac{i\rho_\alpha(r)}{g_s}\left[\left\langle \phi\phi^\dagger \right\rangle, \partial_\alpha \left\langle \phi\phi^\dagger \right\rangle\right], \tag{3.61}$$

where $\rho_\alpha(r)$ ($\alpha = 0, 3$) are the functions of the radial coordinate r. The functions $\rho_\alpha(r)$ are determined later.

Substituting all the fields Φ and A_μ into Eq. (2.23), we find the effective Lagrangian at low energies,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}^{(0)} + \mathcal{L}_{\text{eff}}^{(3)}, \tag{3.62}$$

where

$$\mathcal{L}_{\text{eff}}^{(0)} = \int dx^1 dx^2\, \text{tr}\left[-\frac{\varepsilon_3}{2}F_{0i}F^{0i} + K_0 \mathcal{D}_0 \Phi^\dagger \mathcal{D}^0 \Phi\right], \tag{3.63}$$

$$\mathcal{L}_{\text{eff}}^{(3)} = \int dx^1 dx^2\, \text{tr}\left[-\frac{1}{2\lambda_3}F_{3i}F^{3i} + K_3 \mathcal{D}_3 \Phi^\dagger \mathcal{D}^3 \Phi\right]. \tag{3.64}$$

The terms in the Lagrangian can be written in the following way,

$$\mathcal{L}_{\text{eff}}^{(\alpha)} = C^\alpha \mathcal{L}_{\mathbb{C}P^2}^{(\alpha)}, \tag{3.65}$$

$$\mathcal{L}_{\mathbb{C}P^2}^{(\alpha)} = \partial^\alpha \phi^\dagger \partial_\alpha \phi + \left(\phi^\dagger \partial^\alpha \phi\right)\left(\phi^\dagger \partial_\alpha \phi\right), \tag{3.66}$$

where no summation is taken for α. The coefficient C^α in Eq. (3.65) can be determined as follows. After straightforward calculations, we find that the coefficients are expressed as

$$C^0 = \frac{4\pi\alpha_3}{\lambda_3 g_s^2} \int dr \frac{r}{2} \left[\rho_0'^2 + \frac{h^2}{r^2}(1 - \rho_0)^2 + \frac{\beta_3 m_g^2}{\alpha_3}\left((1 - \rho_0)(f - g)^2 + \frac{f^2 + g^2}{2}\rho_0^2\right) \right], \quad (3.67)$$

$$C^3 = \frac{4\pi}{\lambda_3 g_s^2} \int dr \frac{r}{2} \left[\rho_3'^2 + \frac{h^2}{r^2}(1 - \rho_3)^2 + m_g^2\left((1 - \rho_0)(f - g)^2 + \frac{f^2 + g^2}{2}\rho_3^2\right) \right], \quad (3.68)$$

where $\alpha_3 \equiv \varepsilon_3\lambda_3$ and $\beta_3 \equiv K_0/K_3$. The coefficients $C^{0,3}$ should be determined in such a way that the energy is minimized. To this end, we regard the coefficients as "Lagrangian" for the undetermined scalar fields ρ_α. Namely, we solve the following the Euler-Lagrange equations

$$\rho_0'' + \frac{\rho_0'}{r} + (1 - \rho_0)\frac{h^2}{r^2} - \frac{\beta_3 m_g^2}{2\alpha_3}\left[(f^2 + g^2)\rho_0 - (f - g)^2\right] = 0, \quad (3.69)$$

$$\rho_3'' + \frac{\rho_3'}{r} + (1 - \rho_3)\frac{h^2}{r^2} - \frac{m_g^2}{2}\left[(f^2 + g^2)\rho_3 - (f - g)^2\right] = 0, \quad (3.70)$$

with the boundary conditions

$$\rho_{0,3}(0) = 1, \qquad \rho_{0,3}(\infty) = 0. \quad (3.71)$$

Numerical solutions of $\rho_0(r)$, $\rho_3(r)$ and the Kähler class density $c_0(r)$ and $c_3(r)$ $(C_i = (4\pi/g_s^2)\int dr\, rc_i(r))$ with $\lambda_i = \varepsilon_i = 1$ $(i = 0, 3)$ for the mass choices $\{m_1, m_8, m_g\} = \{1, 5, 1\}, \{5, 1, 1\}, \{1, 1, 5\}$ are shown in Fig. 3.4. The corresponding background solutions are shown in Fig. 3.2.
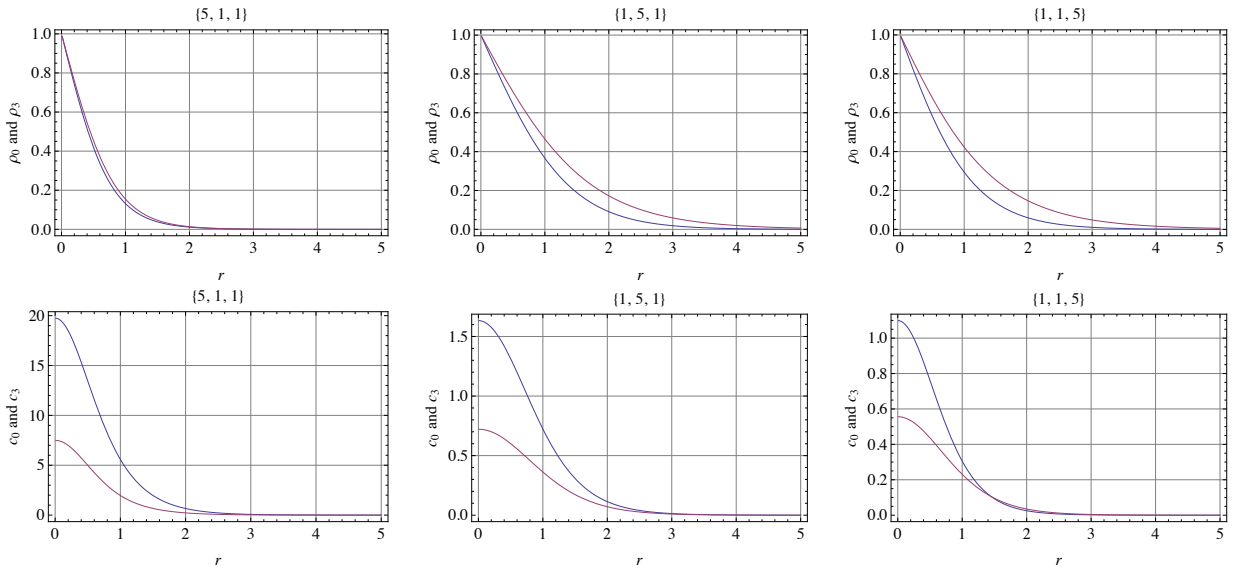

Figure 3.4: Numerical solutions of $\rho_0(r)$ (blue) and $\rho_3(r)$ (purple) for $\{m_g, m_1, m_8\} = \{1, 5, 1\}, \{5, 1, 1\}, \{1, 1, 5\}$ (taken from Ref. [18]). The Kähler class density $c_0(r)$ (blue) and $c_3(r)$ (purple) are shown in the second row.

3.4.3   Effects of a strange quark mass

At lower densities, the effects of a finite strange quark mass is not negligible. Let us consider what happens to the effective theory orientational zero modes as a result of a finite strange quark mass. The results here are obtained in Ref. [44]. We assume that the chemical potential is still substantially larger than m_{s} and we treat the effect as a perturbation. We can asborb the term $\varepsilon \propto m_{\mathrm{s}}^2$ given in Eq. (2.39) into α as Eq. (2.41), and the background non-Abelian vortex solution is given by solving the same equations as Eqs. (3.29)–(3.31) with $\alpha \to \alpha'$. Furthermore, the leading order terms in the Lagrangian for the orientational fields is the same as those given in Eqs. (3.62), (3.65), (3.66) with the replacement $\alpha \to \alpha'$.

Let us find the contribution of the second term in Eq. (2.38) to the low-energy effective Lagrangian of orientational zero modes. By substituting Eq. (3.57) into the second term in Eq. (2.38), we get the following potential term in the $\mathbb{C}P^2$ nonlinear sigma model,

$$V_{\mathbb{C}P^2} \;=\; D\left(|\phi_3|^2 - |\phi_2|^2\right), \tag{3.72}$$

$$D \;\equiv\; \pi\varepsilon\Delta^2 \int_0^\infty dr\, r\left(g^2 - f^2\right), \tag{3.73}$$

where (ϕ_1, ϕ_2, ϕ_3) is the homogeneous coordinate of $\mathbb{C}P^2$ space. Note that Δ, f, and g are obtained from the GL Lagrangian where α is replaced with α'. The quantity D is positive and finite, since $g - f$ is positive for any r and is exponentially small at far distances from the vortex (see Eq. (3.38)). If we take $\varepsilon \to 0$, the low-energy effective theory reduces to the massless $\mathbb{C}P^2$ model where all the points $\phi \in \mathbb{C}P^2$ are degenerate. In other words, the non-Abelian vortices with different orientations have the same tensions. Once non-zero ε is turned on, almost all the points of $\mathbb{C}P^2$ are lifted and only one point $(\phi_1, \phi_2, \phi_3) = (0, 1, 0)$ becomes the global minimum of the effective potential, see Fig. 3.4.3. This means that the non-Abelian vortex with the specific orientation with color "blue-red" and flavor "s-u" is energetically favored,

$$\Phi_{\mathrm{su}} \to \mathrm{diag}(\Delta_{\mathrm{ds}},\, \Delta_{\mathrm{su}}e^{i\theta},\, \Delta_{\mathrm{ud}}), \qquad \text{as} \quad r \to \infty. \tag{3.74}$$

As shown below, all configurations except for the $(0, 1, 0)$ vortex are no longer stable, including the $(1, 0, 0)$ and $(0, 0, 1)$ vortices.

Let us estimate the lifetime of unstable vortices. For simplicity, we consider the decay of a $(1, 0, 0)$ vortex into a $(0, 1, 0)$ vortex (from the left-bottom vertex to the right-bottom vertex of the triangle in Fig. 3.4.3. Since the effective potential is lifted for $|\phi_3| \neq 0$ direction, it is reasonable to set $\phi_3 = 0$ from the beginning. Then we only have to the $\mathbb{C}P^1$ submanifold in the $\mathbb{C}P^2$. It is useful to introduce an inhomogeneous coordinate $u(t) \in \mathbb{C}$, which is related to ϕ as

$$\phi_1 = \frac{1}{\sqrt{1 + |u|^2}}, \quad \phi_2 = \frac{u}{\sqrt{1 + |u|^2}}. \tag{3.75}$$

Then the low-energy effective Lagrangian can be rewritten as

$$\mathcal{L}_{\mathbb{C}P^1} = C^0 \frac{|\dot{u}|^2}{(1 + |u|^2)^2} + D\frac{|u|^2}{(1 + |u|^2)}. \tag{3.76}$$
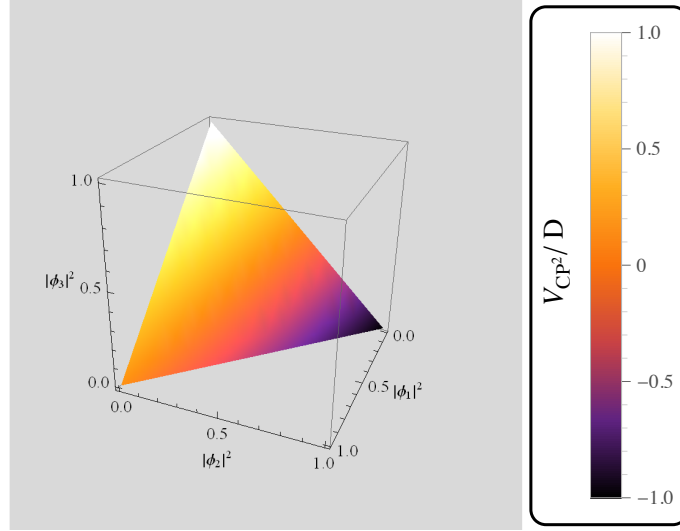
Figure 3.5: Contour plot of the potential (3.72) for the $\mathbb{C}P^2$ modes in the $|\phi_1|^2$-$|\phi_2|^2$-$|\phi_3|^2$ space (taken from Ref. [18]). The colors represent the values of the potential.

The typical time scale appearing in this Lagrangian is

$$\tau = \sqrt{\frac{C^0 K_0}{D}}. \tag{3.77}$$

Let us make an estimation of τ. Since the profile function $f(r)$ ($g(r)$, $h(r)$, and $\rho(r)$) increases (decrease) with a typical scale $r \sim \Delta_{\text{CFL}}^{-1}$ for $m_{\text{g}} \gg m_{1,8}$ as shown in Eqs. (3.37) and (3.38), we find from Eq. (3.67)

$$C^0 \sim \frac{m_{\text{g}}^2}{g_{\text{s}}^2 \Delta_{\text{CFL}}^2 \lambda_3} \sim \left(\frac{\mu}{\Delta_{\text{CFL}}}\right)^2 \frac{1}{\lambda_3}. \tag{3.78}$$

Furthermore, D is estimated from Eq. (3.73) as

$$D \sim \varepsilon \sim m_{\text{s}}^2 \log \frac{\mu}{\Delta_{\text{CFL}}}. \tag{3.79}$$

Thus, the typical time scale of a $(1, 0, 0)$ vortex change into a $(0, 1, 0)$ vortex is estimated as

$$\tau \sim \frac{1}{m_{\text{s}} \sqrt{\lambda_3}} \left(\frac{\mu}{\Delta_{\text{CFL}}}\right)^2 \left(\log \frac{\mu}{\Delta_{\text{CFL}}}\right)^{\frac{1}{2}}. \tag{3.80}$$

In the limit $m_{\text{s}} \to 0$, $\tau \to \infty$ as anticipated.

The results above can be relevant to the state inside the core of neutron stars. When the core of a neutron star cools down below the critical temperature of the CFL phase, a network of non-Abelian vortices will be formed by the Kibble mechanism. Remarkably, the extrapolation of the formula (3.80) to the intermediate density regime relevant to the core of neutron stars ($\mu \sim 500$ MeV) with $\Delta_{\text{CFL}} \sim 10$ MeV and $m_{\text{s}} \simeq 150$ MeV suggests that all the vortices decay radically with the lifetime

of order $\tau \sim 10^{-21}$ second. Although this result should be taken with some care due to the uncertainty of numerical factor in Eq. (3.80), it is reasonable to expect that only one type of non-Abelian vortices, which correspond to the point $(0, 1, 0)$ in the $\mathbb{C}P^2$ space, survive as a response to the rotation of neutron stars. The other decaying non-Abelian vortices will emit Nambu-Goldstone bosons, quarks, gluons, or photons during thermal evolution of neutron stars.

Chapter 4

Interactions of non-Abelian vortices with quasiparticles

In this section, we discuss the interaction of non-Abelian vortices with quasiparticles in the color-superconducting medium. It is necessary to determine the interaction to discuss physical phenomena such as scattering or radiation of quasiparticles by vortices. We can also investigate the interaction between vortices using vortex-quasiparticle interaction, since the intervortex force is mediated by quasiparticles. In Sec. 4.1, we discuss the interaction of vortices with phonons, which are the Nambu-Goldstone mode associated with the breaking of the $U(1)_{\mathrm{B}}$ symmetry, and gluons. In particular, the interaction with gluons is dependent on the orientation of a vortex. This gives rise to an orientation-dependent interaction energy between two vortices. In Sec. 4.2, we discuss the interaction of vortices with CFL mesons. The CFL mesons are the Nambu-Goldstone bosons for the breaking of chiral symmetry. In Sec. 4.3, we investigate the interaction of vortices with photons and its phenomenological consequences. The orientational zero modes localized on vortices are charged with respect to $U(1)_{\mathrm{EM}}$ symmetry. The interaction Lagrangian is determined by symmetry consideration. Based on the interaction, we discuss the scattering of photons off a vortex. We also discuss the optical property of a vortex lattice, which is expected to be formed if CFL matter exists inside the core of a rotating dense star (see Appendix B). We show that a lattice of vortices serves as a polarizer of photons.

4.1   Interaction with phonons and gluons

Here we discuss the interaction of vortices with gluons and phonons. For this purpose, we use a method called dual transformation. This method is useful in dealing with topological defects, since topological defects in an original theory are described as particles in its dual theory. After a dual transformation, we can deal with the interaction of topological defects by the methods of ordinary field theories. The action of the dual theory is derived by using the method of path integration. For example, let us take phonons in three spatial dimensions, which are described by a massless scalar field. In the dual action, phonons are described by a massless antisymmetric tensor field $B_{\mu\nu}$ [102, 103]. Antisymmetric tensor fields have been utilized in describing vortices in superfluids or

perfect fluids [104, 105, 106]. In a dual formulation, the field $B_{\mu\nu}$ is introduced via the method of path integration. On the other hand, the gluons, which are massive because of the Higgs mechanism, are described by massive antisymmetric tensor fields in the dual theory [57].

4.1.1   Topological defects and dual transformations

Here we briefly review what a dual transformation is. A dual transformation relates theories that have different Lagrangians and variables but possess equivalent equations of motion. Under this transformation, particles and solitons typically interchange their roles, i.e. Noether currents are interchanged with topological currents, and vice-versa. The advantages of using a dual transformation against using the original theory are as follows:

1. Under a dual transformation, coupling constant of the original theory g, is replaced by $1/g$ in the dual theory. So one can map a strong coupling theory into a weakly coupled one in which perturbative calculations are reliable.

2. Topological defects in the original theory are described as particles in the dual theory. After a dual transformation, it is possible to deal with the interaction of topological defects by the methods of ordinary field theories. See Fig. 4.1

For example, in three spatial dimensions, a massless scalar field ϕ is dually related to a massless two-form field $B_{\mu\nu}$. Some examples are shown in Table 4.1.1. The duality relation can be generalized to h-forms in d-spatial dimensions, which is also summarized in Table 4.1.1.

spatial dimension	original field	dual field	number of d.o.f
2	massless scalar ϕ	massless vector B_μ	1
3	massless scalar ϕ	massless two-form $B_{\mu\nu}$	1
3	massless vector A_μ	massless vector B_μ	2
3	massive vector A_μ	massive two-form $B_{\mu\nu}$	3
d	massless h-form	massless $d-h-1$	${}_{d-1}C_h$
d	massive h-form	massless $d-h-1$	${}_dC_h$

Table 4.1: Correspondence under the dual transformation. The field ϕ is a real field.

One might think that there can be a mismatch in the number of degrees of freedom between the original and dual fields. In fact, the number of degree of freedom in the dual field is identical to that in the original one, as shown in Table 4.1.1. For example, let us take a massless real scalar field in three spatial dimensions. This field has one degree of freedom. A massless two-form field, which is the dual field to a massless scalar field, is an antisymmetric tensor, so it has six components. The action of a massless two-form field $B_{\mu\nu}$ is given by

$$S = \frac{1}{12} \int d^4x\, H_{\mu\nu\sigma} H^{\mu\nu\sigma}, \tag{4.1}$$
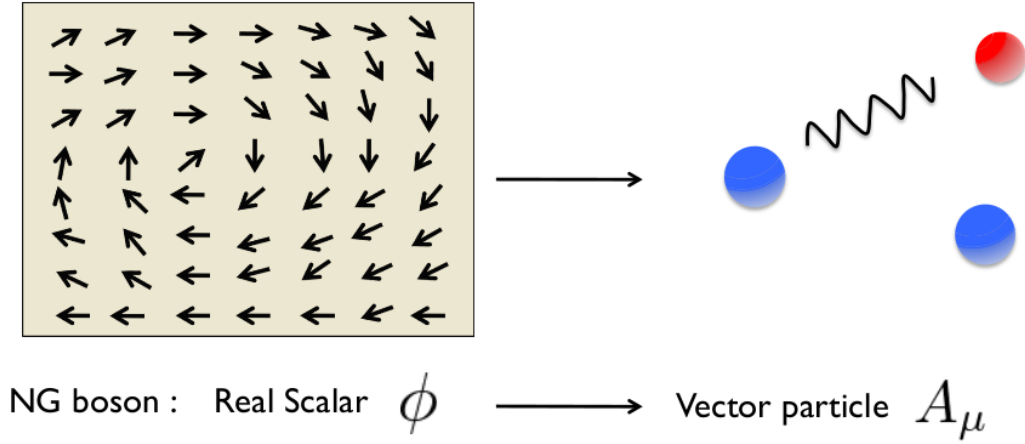
Figure 4.1: Schematic picture of a dual transformation. The left figure shows a vortex-like solution in the XY model in 2+1 dimensions. After changing the variable of the Nambu-Goldstone mode from a real scalar field to a vector field, the vortex appears as a charged particle for the vector field (right figure). The duality in the XY model is first discussed in Ref. [107].

where $H_{\mu\nu\sigma} \equiv \partial_\mu B_{\nu\sigma} + \partial_\nu B_{\sigma\mu} + \partial_\sigma B_{\mu\nu}$ is the field strength of $B_{\mu\nu}$. Among the six components of $B_{\mu\nu}$, three are not dynamical, because the conjugate momentum fields of B_{0i} automatically vanish,

$$\Pi_{0i} = \frac{\partial \mathcal{L}}{\partial(\partial_0 B_{0i})} = 0. \tag{4.2}$$

There remain three components. Two are in fact redundant, since they can be removed by making use of a gauge symmetry. The action (4.1) is invariant under the following local transformation,

$$B_{\mu\nu} \to B'_{\mu\nu} = B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \tag{4.3}$$

where Λ_μ is a massless vector. Thus, since Λ_μ has two degrees of freedom, a massless two-form $B_{\mu\nu}$ correctly has only one degree of freedom.

A dual transformation is written simply in the language of differential forms at least when the gauge group is Abelian [108]. Let us consider the case in which $\pi_h(G/H) = \mathbb{Z}$ in $d + 1$ dimensional space. In this case there is a topological invariant which can be written as

$$\int_{S_h} \omega_h, \tag{4.4}$$

where S_h is an h-dimensional sphere surrounding the defect and ω_h is an h-form which is exact outside S_h, i. e. there exists an $(h-1)$-form ϕ_{h-1} such that $\omega_h = d\phi_{h-1}$. For example, in the case of global vortices in $3 + 1$ dimensions, ϕ_{h-1} is a zero-form (a scalar field), which is the phase of the order parameter field. The action for ϕ_{h-1} is written as

$$S = \int (-)^{h-1} d\phi_{h-1} \wedge *d\phi_{h-1}, \tag{4.5}$$

where $*$ is the Hodge dual operator. A dual transformation corresponds to rewriting the action as

$$S = (-)^{h-1} \int d\phi_{h-1} \wedge *d\phi_{h-1}$$
$$\equiv (-1)^{h-1} \int *(dB) \wedge *(*(dB)), \tag{4.6}$$

where B is an $(d-h)$-form, which is the dual field of ϕ_{h-1}. The fields ϕ_{h-1} and B_{d-h} are related by

$$d\phi_{h-1} = *(dB_{d-h}). \tag{4.7}$$

For example, if one considers the global vortices in (3+1)-dimensions, the above relation is written explicitly as

$$\partial_\mu \phi = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu B^{\rho\sigma}. \tag{4.8}$$

4.1.2   Dual action and vortex-quasiparticle interaction

In order to obtain the dual action, the starting point is the time-dependent Ginzburg-Landau effective Lagrangian in the CFL phase (2.23). After changing variables via the method of path integration, gluons and phonons are described by antisymmetric tensors, $B^a_{\mu\nu}$ and $B^0_{\mu\nu}$, respectively. The low-energy action for phonons and gluons interacting with vortices is given by [45]

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_{\text{int}}, \tag{4.9}$$

where the free part $\mathcal{S}_0$ is defined as

$$\mathcal{S}_0 = \int d^4x \left[-\frac{1}{12\tilde{K}_{\mu\nu\sigma}} \left(H^a_{\mu\nu\sigma}H^{a,\mu\nu\sigma} + H^0_{\mu\nu\sigma}H^{0,\mu\nu\sigma} \right) - \frac{1}{4}m_{\text{g}}^2 \left(B^a_{\mu\nu} \right)^2 \right]. \tag{4.10}$$

See Appendix A for the derivation. In the equation above, $H^a_{\mu\nu\sigma} \equiv \partial_\mu B^a_{\nu\sigma} + \partial_\nu B^a_{\sigma\mu} + \partial_\sigma B^a_{\mu\nu}$ and $H^0_{\mu\nu\sigma} \equiv \partial_\mu B^0_{\nu\sigma} + \partial_\nu B^0_{\sigma\mu} + \partial_\sigma B^0_{\mu\nu}$ are field strength tensors of gluons and phonons, and m_{g} is the mass of the gluons, and m^0 is a space-dependent function written by the profile functions of a vortex. The factor $\tilde{K}_{\mu\nu\sigma} \equiv \epsilon_{\rho\mu\nu\sigma}K^\rho$ comes from the lack of Lorentz invariance, where we have defined $K_\mu = (K_0, K_1, K_1, K_1)^T$. The first(second) term is the kinetic term for gluons(phonons). The third one is the mass term for gluons, which is induced via the Higgs mechanism. In the presence of a vortex, the mass of gluons is dependent on the distance from the center of the vortex according to the change of the values of diquark condensates.

The interaction part $\mathcal{S}_{\text{int}}$ is written as

$$\mathcal{S}_{\text{int}} = - \int d^4x \left[2\pi m^0 B^0_{\mu\nu}\omega^{0,\mu\nu} + \frac{m_{\text{g}}}{g_{\text{s}}} B^a_{\mu\nu}\omega^{a,\mu\nu} \right], \tag{4.11}$$

where $\omega^0_{\mu\nu}$ and $\omega^a_{\mu\nu}$ are vorticity tensors, which depend on the vortex configuration. Their specific forms are discussed later. In the original description, the interaction of vortices with phonons and

gluons are complicated and defies intuitive understanding. In the dual theory, the information of interactions is captured in the vorticity tensors. The vorticity tensors have finite values only around the core of a vortex. Gluons and phonons propagate in the four-dimensional spacetime, and the interaction is localized around the vortex.

Now we discuss the properties of the interaction (4.11). First, let us look at the interaction of vortices with $U(1)_{\mathrm{B}}$ phonons. This part is essentially the same as the case of vortices in a superfluid. For a general vortex configuration we can write the Abelian component of the vorticity tensor as

$$\left(\omega^0\right)_{\rho\sigma}(x) \equiv \frac{1}{2\pi}\epsilon_{\mu\nu\rho\sigma}\partial^\nu\partial^\mu\pi_{\mathrm{MV}}(x). \tag{4.12}$$

where $\pi_{\mathrm{MV}}(x)$ is the multivalued part of the phase of the order parameter fields. The multivalued part is in general allowed, since it is a phase. Equation (4.12) appears to automatically vanish, but in fact it does not, since the two derivatives do not commute, which reflects the multivaluedness of $\pi_{\mathrm{MV}}(x)$. For a general vortex configuration, the vorticity tensor can be written as

$$(\omega^0)^{\mu\nu}(x) = \frac{1}{N}\int d\tau d\sigma \frac{\partial(X^\mu, X^\nu)}{\partial(\tau, \sigma)}\delta^{(4)}(x - X^\mu(\tau, \sigma)), \tag{4.13}$$

where N is the number of colors ($N = 3$) and $X^\mu(\tau, \sigma)$ is the the spacetime position of the vortex parametrized by worldsheet coordinates τ and σ. The interaction of vortices with $U(1)_{\mathrm{B}}$ phonons is rewritten as

$$\mathcal{S}_{\mathrm{int}}^{\mathrm{Ph}} = -\frac{2\pi m^0}{N}\int d\sigma^{\mu\nu}B_{\mu\nu}^0, \tag{4.14}$$

where $d\sigma^{\mu\nu} \equiv \frac{\partial(X^\mu, X^\nu)}{\partial(\tau, \sigma)}d\tau d\sigma$ is an area element of the vortex world sheet. The interaction (4.14) is a natural generalization of the gauge interaction of a point particle,

$$\mathcal{S} = \int dx^\mu A_\mu. \tag{4.15}$$

The factor $1/N$, which is equal to the $U(1)_{\mathrm{B}}$ winding number of vortices with the lowest energy, that reflects the fact that the strength of the interaction is proportional to the winding number with respect to $U(1)_{\mathrm{B}}$ symmetry. We also note that $U(1)_{\mathrm{B}}$ phonons $B_{\mu\nu}^0$ do not couple to the orientational zero modes, namely $U(1)_{\mathrm{B}}$ phonons are blind to the orientation of a vortex.

Next, let us look at the interaction of vortices with gluons. The non-Abelian vorticity tensor $\omega_{\mu\nu}^a$ is written as

$$
\begin{aligned}
\omega_{\lambda\sigma}^a = \epsilon_{\lambda\sigma\mu\nu}\Bigg\{ & \partial_\nu\Big\{-\frac{16}{N}\gamma(r)\left(\partial_\mu\theta + 2N\gamma\delta_{\mu 0}\right)\phi^\dagger T^a\phi \\
& + i\alpha(r)(1 + \beta(r))\left(\phi^\dagger T^a\partial_\mu\phi - \partial_\mu\phi^\dagger T^a\phi + 2\phi^\dagger T^a\phi\partial_\mu\phi^\dagger\phi\right)\Big\} \\
& - \frac{4}{N}\alpha(r)\gamma(r)(1 + \beta(r))\left(\partial_{[\mu}\phi^\dagger T^a\phi + \phi^\dagger T^a\partial_{[\mu}\phi\right)\left(\partial_{\nu]}\theta + \frac{NK_0'}{2K_0}\delta_{\nu]0}\right) \\
& \qquad - \frac{i}{2}\alpha(r)^2(1 + \beta(r)^2) \\
& \times \left[\phi^\dagger T^a\phi\partial_{[\mu}\phi^\dagger\partial_{\nu]}\phi + \partial_{[\mu}\phi^\dagger T^a\partial_{\nu]}\phi + \phi^\dagger T^a\partial_{[\mu}\phi\partial_{\nu]}\phi^\dagger\phi + \partial_{[\mu}\phi^\dagger T^a\phi\partial_{\nu]}\phi^\dagger\phi\right]\Bigg\}. \quad (4.16)
\end{aligned}
$$

where $\alpha(r), \beta(r)$ and $\gamma(r)$ are functions of the distance from the vortex core and are written in terms of vortex solutions, and the parameter γ is the coefficient of the term with one time derivative in Eq. (2.23). The leading-order part in the deviation of the order parameter from the ground-state value is given by

$$\omega_{\lambda\sigma}^a = \epsilon_{\lambda\sigma\mu\nu}\partial^\nu \left[-\frac{16}{N}\gamma(r) \left\{ \partial^\mu\theta + 2N\gamma\delta^{\mu 0} \right\} \phi^\dagger T^a \phi \right]. \tag{4.17}$$

As can be seen in Eq. (4.16), gluons couple to the orientational zero modes on the vortex. As a result, gluons are emitted through the interaction (4.16) when a wave of the $\mathbb{C}P^2$ orientational modes propagates along a vortex-line. By using the interaction derived above (4.16), we can estimate the amount of radiated gluons from a propagating wave in $\mathbb{C}P^2$ orientational space.

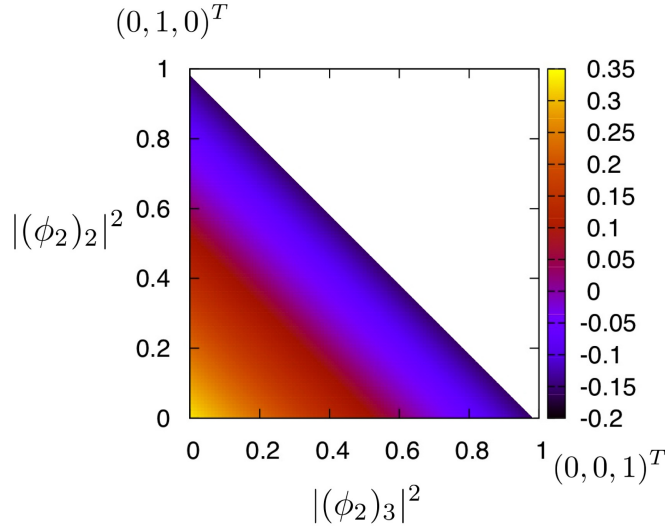### 4.1.3 Orientation dependence of the vortex-vortex interaction

Figure 4.2: Values of $G(\phi_1, \phi_2)$ as a function of $|(\phi_2)_2|^2$ and $|(\phi_2)_3|^2$ (taken from Ref. [18]). If the orientation ϕ_2 is in the red zone the interaction is repulsive, while in the blue zone the interaction is attractive.

As an application of the dual Lagrangian obtained above, let us discuss the orientation dependence of the interaction energy of two parallelly-placed vortices. We assume that the orientation of each vortex is constant along the vortex. The interaction energy due to the gluon exchange is proportional to

$$G(\phi_1, \phi_2) \equiv \phi_1^\dagger T^a \phi_1 \, \phi_2^\dagger T^a \phi_2, \tag{4.18}$$

where ϕ_1 and ϕ_2 denote the orientations of the first and second vortices respectively. We have shown in Fig. 4.2 the value of $G(\phi_1, \phi_2)$ as a function of ϕ_2. We have taken ϕ_1 as $\phi_1 = (1, 0, 0)^T$ without loss of generality. Figure 4.2 indicates that, if the two orientations are close in the $\mathbb{C}P^2$ space, the

interaction through gluon exchanges is repulsive, while if the orientations are far apart, the interaction is attractive. This orientation dependent interaction is expected to be important when the distance between two vortices is small, for example when two vortices cross.

4.2 Interaction with mesons

The interactions with phonons and gluons are topological, in the sense that the interaction term does not involve the metric. In contrast, the interaction with photons is not topological. Here, we discuss the interaction with mesons, which is also non-topological. In this case, one cannot use a dual transformation to obtain the interaction Lagrangian.

First, let us recall the effective action of mesons, which is the chiral perturbation theory. The effective degree of freedom is the following gauge invariant quantity,

$$\Sigma = \Phi_{\mathrm{L}}^{\dagger}\Phi_{\mathrm{R}}, \tag{4.19}$$

which transforms under the flavor symmetry $U(1)_{\mathrm{A}} \times SU(3)_{\mathrm{L}} \times SU(3)_{\mathrm{R}}$ as

$$\Sigma \to e^{i\alpha}g_{\mathrm{L}}^{\dagger}\Sigma g_{\mathrm{R}}, \quad e^{i\alpha} \in U(1)_{\mathrm{A}}, \quad g_{\mathrm{L}} \in SU(3)_{\mathrm{L}}, \quad g_{\mathrm{R}} \in SU(3)_{\mathrm{R}}. \tag{4.20}$$

The chiral symmetry is broken to the vector symmetry $SU(3)_{\mathrm{L+R}}$ with $g_{\mathrm{L}} = g_{\mathrm{R}}$ in the CFL ground state $\Sigma = \Delta_{\mathrm{CFL}}\mathbf{1}_3$. The mesons are $\Sigma = \Delta_{\mathrm{CFL}}^2 g_{\mathrm{L}}^{\dagger}g_{\mathrm{R}} = \Delta_{\mathrm{CFL}}^2 g^2(x) = \Delta_{\mathrm{CFL}}^2 U(x)$ with $g_{\mathrm{L}}^{\dagger} = g_{\mathrm{R}} = g(x)$.

In a non-Abelian vortex background, $\Phi_L = -\Phi_R = \mathrm{diag}(f(r)e^{i\theta}, g(r), g(r))$, the gauge invariant Σ takes the form of

$$\Sigma_v = \mathrm{diag}(f^2(r), g^2(r), g^2(r)) \quad [\to \mathrm{diag}(0, g_0^2, g_0^2) \text{ at } r = 0], \tag{4.21}$$

with a constant $g_0 = g(r = 0)$. In the presence of the vortex, the gauge invariant Σ becomes

$$\Sigma = \sqrt{U}^{\dagger}\Sigma_v\sqrt{U} \tag{4.22}$$

Then, the chiral Lagrangian can be written as

$$\mathcal{L} = \sum_{\mu} f_{\mu}\mathrm{tr}\left(\partial_{\mu}\Sigma^{\dagger}\partial^{\mu}\Sigma\right) = \sum_{\mu} f_{\mu}\mathrm{tr}\left[2\alpha_{\mathrm{L}}\Sigma_v\alpha_{\mathrm{R}}\Sigma_v - (\alpha_{\mathrm{L}}^2 + \alpha_{\mathrm{R}}^2)\Sigma_v^2\right] \tag{4.23}$$

with the decay constants $f_{\mu} = (1/2)(f_{\pi}, f_{\pi}v_{\pi})$, and the left and right Maurer-Cartan forms

$$\alpha_{\mathrm{R}} \equiv \sqrt{U}^{\dagger}\partial_{\mu}\sqrt{U}, \quad \alpha_{\mathrm{L}} \equiv \sqrt{U}\partial_{\mu}\sqrt{U}^{\dagger}. \tag{4.24}$$

Far apart from the vortex core, it reduces to

$$\mathcal{L} = -\Delta_{\mathrm{CFL}}^2 \sum_{\mu} K_{\mu}\mathrm{tr}\left(U^{\dagger}\partial_{\mu}U\right)^2, \tag{4.25}$$

which is expected because the effect of a vortices should vanish at very far from the vortex. The Lagrangian (4.23) describes the mesonic excitations in the presence of a vortex.

4.3   Interaction with electromagnetic fields

Here we discuss the electromagnetic properties of non-Abelian vortices in the CFL phase, and their phenomenological consequences. Although the bulk CFL matter is electromagnetically neutral, the orientational zero modes are charged, as discussed later. So, the vortices interact with photons. In order to deal with the photon-vortex interaction, we consider the low-energy effective action of orientational zero modes interacting with photons. Based on the action, we discuss the scattering of photons off a vortex.

In the following analysis, we neglect the mixing of photons and gluons. The gauge field, A'_μ, which remains massless in the CFL phase, is a mixture of the photon A_μ and a part of gluons A^8_μ, $A'_\mu = -\sin\zeta A_\mu + \cos\zeta A^8_\mu$. Here, the mixing angle ζ is given by $\tan\zeta = \sqrt{3}g_s/2e$ [25], where g_s and e are the strong and electromagnetic coupling constants. At accessible densities ($\mu \sim 1\text{GeV}$), the fraction of the photon is given by $\sin\zeta \sim 0.999$, and so, the massless field A'_μ consists mostly of the ordinary photon and includes a small amount of the gluon. As a first approximation, we neglect the mixing of the gluon to the massless field.

We denote the orientational zero modes by a complex three-component vector $\phi \in \mathbb{C}P^2$, which satisfies $\phi^\dagger\phi = 1$. When we neglect the electromagnetic interaction, the low-energy effective theory on the vortex that is placed along the z axis is described by the following $\mathbb{C}P^2$ nonlinear sigma model [43],

$$\mathcal{L}_{\mathbb{C}P^2} = \sum_{\alpha=0,3} C_\alpha \left[\partial^\alpha \phi^\dagger \partial_\alpha \phi + (\phi^\dagger \partial^\alpha \phi)(\phi^\dagger \partial_\alpha \phi)\right], \tag{4.26}$$

where the orientational moduli ϕ are promoted to dynamical fields, and C_α are numerical constants. Under the color-flavor locked transformation, the $\mathbb{C}P^2$ fields ϕ transform as

$$\phi \to U\phi, \tag{4.27}$$

with $U \in SU(3)_{\text{C+F}}$.

4.3.1   Coupling of orientation modes with electromagnetic fields

Now, let us consider the electromagnetic interactions. The electromagnetic $U(1)_{\text{EM}}$ group is a subgroup of the flavor group $SU(3)_{\text{F}}$. The generator of $U(1)_{\text{EM}}$ is $T_8 = \frac{1}{\sqrt{6}}\text{diag}(-2,1,1)$ in our choice of basis. The effect of electromagnetic interaction is incorporated by gauging the corresponding symmetry. The low-energy effective action on the vortex should be modified to the following gauged $\mathbb{C}P^2$ model,

$$\mathcal{L}_{g\mathbb{C}P^2} = \sum_{\alpha=0,3} C_\alpha \left[\mathcal{D}^\alpha \phi^\dagger \mathcal{D}_\alpha \phi + (\phi^\dagger \mathcal{D}^\alpha \phi)(\phi^\dagger \mathcal{D}_\alpha \phi)\right], \tag{4.28}$$

where the covariant derivative is defined by

$$\mathcal{D}_\alpha \phi = \left(\partial_\alpha - ie\sqrt{6}A_\alpha T_8\right)\phi. \tag{4.29}$$

Thus, the low-energy behavior is described by the $\mathbb{C}P^2$ modes localized on the vortex and photons propagating in three-dimensional space. Hence, the effective action is given by

$$S = \int \left(\frac{\epsilon_0}{2} \boldsymbol{E}^2 - \frac{1}{2\lambda_0} \boldsymbol{B}^2 \right) d^4x + \int \mathcal{L}_{g\mathbb{C}P^2} dz dt, \tag{4.30}$$

where ϵ_0 and λ_0 are the dielectric constant and permeability of the CFL matter, respectively. We can formally recover the Lorentz invariance in the kinetic terms of the photons by the following rescaling,

$$A'_0 = \sqrt{\epsilon_0} A_0, \quad A'_i = \frac{1}{\sqrt{\lambda_0}} A_i, \quad t' = vt, \tag{4.31}$$

where $v \equiv 1/\sqrt{\epsilon_0 \lambda_0}$. By further rescaling the parameters as

$$e' = \sqrt{\lambda_0} e, \quad C'_0 = C_0 v, \quad C'_3 = \frac{C_3}{v}, \tag{4.32}$$

we can write the Lagrangian in the following form,

$$vS = -\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} d^4x + \int \mathcal{L}'_{g\mathbb{C}P^2} dz dt, \tag{4.33}$$

where

$$\mathcal{L}'_{g\mathbb{C}P^2} = \sum_{\alpha=0,3} C'_\alpha \left[\mathcal{D}'^\alpha \phi^\dagger \mathcal{D}'_\alpha \phi + (\phi^\dagger \mathcal{D}'^\alpha \phi)(\phi^\dagger \mathcal{D}'_\alpha \phi) \right], \quad \mathcal{D}'_\alpha \phi = \left(\partial_\alpha - ie' \sqrt{6} A_\alpha T_8 \right) \phi. \tag{4.34}$$

In the discussions below, primes are omitted for notational simplicity.

4.3.2 Scattering of photons off a vortex

We can discuss the consequences of the charged degrees of freedom on the vortex using the low-energy action (4.33). For example, let us consider the photon scattering off a vortex. The equation of motion of the photon field derived from the effective action is given by

$$\begin{aligned}
\partial^\mu F_{\mu\nu} = -C_\nu ie\sqrt{6}\, \delta(x_\perp)(\delta_{0\nu} + \delta_{3\nu}) \\
\times \left\{ \phi^\dagger T_8 \mathcal{D}_\nu \phi - (\mathcal{D}_\nu \phi)^\dagger T_8 \phi - 2\phi^\dagger \mathcal{D}_\nu \phi \phi^\dagger T_8 \phi \right\},
\end{aligned} \tag{4.35}$$

where $\delta(x_\perp) \equiv \delta(x)\delta(y)$ is the delta function in the transverse plane. We consider a situation where a linearly polarized photon is normally incident on the vortex. We assume that the electric field of the photon is parallel to the vortex. Then, the problem is z-independent and we can set $\theta = \theta(t)$, $A_t = A_x = A_y = 0$, and $A_z = A_z(t, x, y)$. The equation of motion is rewritten as

$$\begin{aligned}
(\partial_t^2 - \partial_x^2 - \partial_y^2) A_z(t, x, y) \\
= 12 C_3 e^2 \left\{ \phi^\dagger (T_8)^2 \phi + (\phi^\dagger T_8 \phi)^2 \right\} A_z(t, x, y)\, \delta(x_\perp) \\
\equiv 12 C_3 e^2 f(\phi) A_z(t, x, y)\, \delta(x_\perp),
\end{aligned} \tag{4.36}$$

where we have defined

$$f(\phi) \equiv \phi^\dagger (T_8)^2 \phi + (\phi^\dagger T_8 \phi)^2. \tag{4.37}$$

Equation (4.36) is the same as the one discussed by Witten in the context of superconducting strings [109], except for the orientation-dependent factor, $f(\phi)$.

If we take $A_z(t, x, y) = A_z(x, y)e^{-i\omega}$, Eq. (4.36) reduces to

$$\left(-(\partial_x^2 + \partial_y^2) + 12C_3 e^2 f(\phi)\, \delta(x_\perp)\right) A_z(x, y) = \omega^2 A_z(x, y). \tag{4.38}$$

This is the non-relativistic Schrödinger equation for scattering from a delta-function potential in two dimensions. The scattering solution obeys

$$A_z(x, y) = e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp} - \int dx' dy'\, G(\boldsymbol{x}_\perp, \boldsymbol{x}'_\perp) 12C_3 e^2 f(\phi)\, \delta(x'_\perp)$$
$$= e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp} - 12C_3 e^2 f(\phi) G(\boldsymbol{x}_\perp, \boldsymbol{0}), \tag{4.39}$$

where $G(\boldsymbol{x}_\perp, \boldsymbol{x}'_\perp)$ is the Green function, which is written as

$$G(\boldsymbol{x}_\perp, \boldsymbol{x}'_\perp) = \int \frac{d^2 \boldsymbol{k}_\perp}{(2\pi)^2} \frac{e^{i\boldsymbol{k}_\perp \cdot (\boldsymbol{x}_\perp - \boldsymbol{x}'_\perp)}}{\boldsymbol{k}_\perp^2 - \omega^2 - i\epsilon}. \tag{4.40}$$

Equations (4.39) and (4.40) imply that

$$A_z(\boldsymbol{0}) = \frac{1}{1 + 12C_3 e^2 f(\phi) G(\boldsymbol{0}, \boldsymbol{0})}. \tag{4.41}$$

This needs to be interpreted with care, since $G(\boldsymbol{0}, \boldsymbol{0})$ is infinite. The integral in Eq. (4.40) is ultra-violed divergent, and is written with a cutoff Λ as

$$\frac{1}{2\pi} \ln(\Lambda/\omega). \tag{4.42}$$

This divergence comes from the fact that we considered an infinitely-thin vortex.

So we interpret Eq. (4.40) to mean

$$A_z(\boldsymbol{0}) = \frac{1}{1 + 12C_3 e^2 f(\phi) \ln(\Lambda/\omega)/2\pi}. \tag{4.43}$$

The electric field at the core of the vortex is proportional to $A_z(\boldsymbol{0})$. Since we took the incident wave to be $e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp}$, the incident wave corresponded to $A_z(\boldsymbol{0}) = 1$. The fields induced by the currents excited on the string reduce $A_z(\boldsymbol{0})$ by a factor

$$\eta \equiv \frac{1}{1 + 12C_3 e^2 f(\phi) \ln(\Lambda/\omega)/2\pi}. \tag{4.44}$$

Thus, the solution of the scattering problem is obtained as

$$A_z(x, y) = e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp} - 12C_3 e^2 f(\phi)\eta G(\boldsymbol{x}_\perp, \boldsymbol{0}), \tag{4.45}$$

The asymptotic behavior of the Green function $G(\boldsymbol{x}_\perp, \boldsymbol{0})$ at large distances from the vortex core is given by

$$G(\boldsymbol{x}_\perp, \boldsymbol{0}) \longrightarrow \sqrt{\frac{i}{8\pi\omega|\boldsymbol{x}_\perp|}} \, e^{i\omega|\boldsymbol{x}_\perp|}, \quad \text{as} \quad |\boldsymbol{x}_\perp| \to \infty. \tag{4.46}$$

Using the asymptotic expression, the scattering amplitude $\mathcal{M}$ can be written as

$$\mathcal{M} = -12C_3 e^2 f(\phi)\eta \sqrt{\frac{i}{8\pi\omega|\boldsymbol{x}_\perp|}}. \tag{4.47}$$

We obtain the the cross section per unit length of a vortex, $d\sigma/dz$, as

$$\frac{d\sigma}{dz} = \frac{\left(12C_3 e^2 f(\phi)\right)^2 \eta^2}{8\pi}\lambda = 288\pi \left(C_3 \alpha \eta f(\phi)\right)^2 \lambda, \tag{4.48}$$

where $\lambda \equiv 2\pi/\omega$ is the wavelength of the incident photon, and α is the fine structure constant.

On the other hand, if the electric field of the wave is perpendicular to the vortex, the photon is not scattered, since current can flow only along the vortex.

4.3.3 Vortex lattice as cosmic polarizer

The electromagnetic property of vortices can be phenomenologically important as it may lead to some observable effects. As an illustration of such an effect, we show that a lattice of vortices works as a polarizer of photons. The rotating CFL matter is expected to be threaded with quantum vortices along the axis of rotation, resulting in the formation of a vortex lattice, as discussed in Appendix B [39, 41, 42]. This is basically the same phenomenon as when one rotates atomic superfluids. Suppose that a linearly polarized photon is incident on a vortex lattice as shown in Fig. 4.3. If the electric field of the photon is parallel to the vortices, it induces currents along the vortices, which results in the attenuation of the photon. On the other hand, waves with electric fields perpendicular to the vortices are not affected. This is exactly what a polarizer does. A lattice passes electromagnetic waves of a specific polarization and blocks waves of other polarizations. This phenomenon, resulting from the electromagnetic interaction of vortices, may be useful for finding observational evidence of the existence of CFL matter.

We consider a situation where electromagnetic waves of some intensity normally enter the vortex lattice. We first assume that the electric fields of the waves are parallel to the vortices. The fraction of the loss of intensity when the wave passes through the lattice for a distance dx is

$$\left\langle \frac{d\sigma}{dz} \right\rangle n_v dx \equiv \frac{dx}{L}, \tag{4.49}$$

where n_v is the number of vortices per unit area. We defined the length L by

$$L \equiv 1/\left(n_v \left\langle \frac{d\sigma}{dz} \right\rangle\right) = \ell^2/\left\langle \frac{d\sigma}{dz} \right\rangle, \tag{4.50}$$
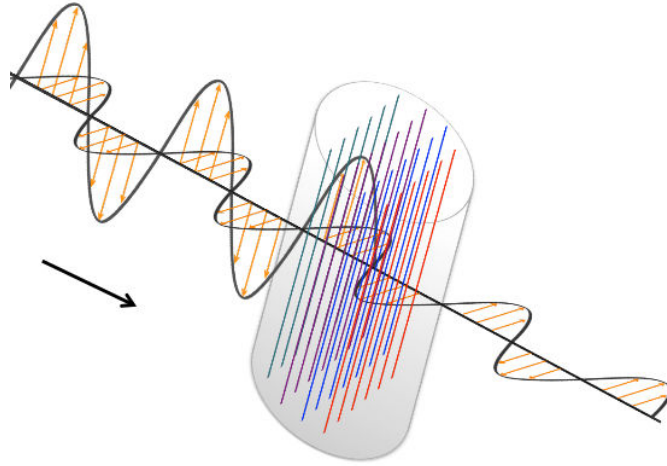
Figure 4.3: Schematic figure of two linearly polarized photons entering a vortex lattice (taken from Ref. [18]). Photons propagate in the direction of the big arrow. The small arrows indicate the electric field vector. The waves whose electric fields are parallel to the vortices are attenuated inside the lattice, while the ones with perpendicular electric fields passes through it.

with the inter-vortex spacing ℓ. As the cross section depends on the internal state (value of ϕ) of the vortex, we have introduced the averaged scattering cross section $\langle d\sigma/dz \rangle$ over the ensemble of the vortices. Let us denote the intensity of waves at distance x from the surface of the lattice as $I(x)$. $I(x)$ satisfies

$$\frac{I(x + dx)}{I(x)} = 1 - \frac{dx}{L}. \tag{4.51}$$

Therefore, the x dependence of $I(x)$ is characterized by the following equation

$$\frac{I'(x)}{I(x)} = -\frac{1}{L}. \tag{4.52}$$

This equation is immediately solved as $I(x) = I_0 e^{-x/L}$, where I_0 is the initial intensity. Hence, the waves are attenuated with the characteristic length L.

Let us make a rough estimate of the attenuation length. The total number of vortices can be estimated, as in Ref. [37], as

$$N_v \simeq 1.9 \times 10^{19} \left(\frac{1\text{ms}}{P_{\text{rot}}} \right) \left(\frac{\mu/3}{300\text{MeV}} \right) \left(\frac{R}{10\text{km}} \right)^2, \tag{4.53}$$

where P_{rot} is the rotation period, μ is the baryon chemical potential and R is the radius of the CFL matter inside dense stars. We have normalized these quantities by typical values. The intervortex spacing is then written as

$$\ell \equiv \left(\frac{\pi R^2}{N_v} \right)^{1/2} \simeq 4.0 \times 10^{-6} \text{ m} \left(\frac{P_{\text{rot}}}{1\text{ms}} \right)^{1/2} \left(\frac{300\text{MeV}}{\mu/3} \right)^{1/2}. \tag{4.54}$$

Therefore, the characteristic decay length of the electromagnetic waves is estimated as

$$L = \frac{\ell^2}{288\pi \left(C_3 \alpha \eta\right)^2 \left\langle f(\phi)^2 \right\rangle \lambda} \simeq \frac{1.2 \times 10^{-11} \text{ m}^2}{\lambda}. \tag{4.55}$$

Here we have determined the value of $f(\phi)$ by considering the effect of a finite strange quark mass m_{s}. The finite strange quark mass breaks the flavor $SU(3)$ symmetry and gives rise to a potential in the $\mathbb{C}P^2$ space, as discussed in Ref. [44]. When m_{s} is larger than the typical kinetic energy of the $\mathbb{C}P^2$ modes, which is given by the temperature $T \leq T_{\text{c}} \sim 10^1$ MeV, and is small enough that the description by the Ginzburg-Landau theory based on chiral symmetry is valid, the orientations of vortices fall into $\phi_0^T = (0, 1, 0)$. This assumption is valid for the realistic value of $m_{\text{s}} \sim 10^2$ MeV. The orientation dependence of the cross section is encapsulated in the function $f(\phi)$ defined in Eq. (4.37). Since $f(\phi_0) = 1/3 \neq 0$, photons interact with a vortex with this orientation. Therefore, we have taken $\langle f(\phi)^2 \rangle = f(\phi_0)^2 = 1/9$. We have also taken $\eta = 1$, $\mu = 900$ MeV and $\Delta_{\text{CFL}} = 100$ MeV, from which the values of C_3 is determined[44].

If we adopt $R \sim 1$ km for the radius of the CFL core, the condition that the intensity is significantly decreased within the core is written as $L \leq 1$ km. The condition can also be stated in terms of the wavelength of the photon as

$$\lambda \geq 1.2 \times 10^{-14} \text{ m} \equiv \lambda_{\text{c}}. \tag{4.56}$$

Hence, a lattice of vortices serves as a wavelength-dependent filter of photons. It filters out the waves with electric fields parallel to the vortices, if the wavelength λ is larger than λ_{c}. The waves that pass through the lattice are linearly polarized ones with the direction of their electric fields perpendicular to the vortices, as shown schematically in Fig. 4.3.

One may wonder why a vortex lattice with a mean vortex distance ℓ can block photons with wavelength many-orders smaller than ℓ. It is true the probability that a photon is scattered during its propagation for a small distance ($\sim \ell$, for example) is small. However, while the photon travels through the lattice, the scattering probability is accumulated and the probability that a photon remains unscattered decreases exponentially. Namely, the small scattering probability is compensated by the large number of vortices through which a photon passes. This is why the vortex mean distance and the wavelength of the attenuated photons can be different.

4.4 Brief summary

In this chapter, we have discussed the interaction of non-Abelian vortices with quasiparticles in the color-superconducting medium. In the first subsection, we discuss the interaction of vortices with phonons, which are the Nambu-Goldstone mode associated with the breaking of the $U(1)_{\text{B}}$ symmetry, and gluons. The interaction Lagrangian is obtain by using the dual transformation. We have found that the interaction with gluons is dependent on the orientation of a vortex. We studied the orientation-dependent interaction energy between two vortices. In the second subsection, we discussed the interaction of vortices with CFL mesons, which are the Nambu-Goldstone bosons for the

breaking of chiral symmetry. In the third subsection, we investigated the interaction of vortices with photons and its phenomenological consequences. The orientational zero modes localized on vortices are charged with respect to $U(1)_{\mathrm{EM}}$ symmetry. The interaction Lagrangian is determined by symmetry consideration. Based on the effective Lagrangian, we discuss the scattering of photons off a vortex. We also discussed the optical property of a vortex lattice (see Appendix. B), which is expected to be formed if CFL matter exists inside the core of a rotating dense star. We have shown that a lattice of vortices serves as a polarizer of photons.

Chapter 5

Non-Abelian statistics of vortices

We consider the exchange statistics of vortices, each of which traps an odd number (N) of Majorana fermions. We assume that the fermions in a vortex transform in the vector representation of the $SO(N)$ group. Exchange of two vortices turns out to be non-Abelian, and the corresponding operator is further decomposed into two parts: a part that is essentially equivalent to the exchange operator of vortices having a single Majorana fermion in each vortex, and a part representing the Coxeter group. Similar decomposition was already found in the case with $N = 3$, and the result shown here is a generalization to the case with an arbitrary odd N. We can obtain the matrix representation of the exchange operators in the Hilbert space that is constructed by using Dirac fermions non-locally defined by Majorana fermions trapped in separated vortices. We also show that the decomposition of the exchange operator implies tensor product structure in its matrix representation.

In Sec. 5.1, we give an introduction to the studies of Majorana fermions inside vortices and explain our motivations. In Sec. 5.3.1, we review how the non-Abelian statistics emerges in the case of vortices each of which contains one Majorana fermion. In Sec. 5.3.2, summarizing the case in which a vortex captures three Majorana fermions [59], we present the generalization to the case of multiple (N) Majorana fermions. We explicitly show factorization of the exchange operators into the known part similar to the case with a single Majorana fermion, and the part corresponding to the Coxeter group. We also show an interesting decomposition of the Majorana fermion operators, which clarifies the action of the Coxeter group on Majorana fermions. In Sec. 5.5 we discuss the relation between the decomposition of the exchange operator and its matrix representation. Section 5.7 is devoted to summary.

5.1   Introduction

There has been considerable interest recently in zero-energy fermion modes trapped inside vortices in superconductors [48]. Vortices in a chiral p-wave superconductor are endowed with non-Abelian statistics [49, 50] because of the zero-energy Majorana fermions inside them [51]. Excitations which obey non-Abelian statistics are called non-Abelian anyons. They are expected to form the basis of topological quantum computations [52, 53] and have been investigated intensively [54, 55, 56]. A

recent classification of topological superconductors clarifies the condition that vortices have zero-energy Majorana (or Dirac) fermions in their cores [110, 111]. It has been argued that, in three spatial dimensions non-Abelian anyons [112, 113, 114] can also be realized by monopoles with Majorana fermions trapped inside their cores, and give a new non-Abelian statistics, the projective ribbon permutation statistics [115].

More recently, the non-Abelian statistics of vortices with multiple Majorana fermions has been investigated and shown to have a novel structure [59]. In this case, the Majorana fermions form the vector representation of the $SO(3)$ group. It is shown that the representation under the exchange of two vortices can be written as the tensor product of two matrices. One matrix is identical to the exchange matrix for vortices with a single Majorana fermion in each core, found by Ivanov [50] modulo trivial change of basis, and the other matrix is shown to be a generator of the Coxeter group, which is a symmetry group of certain polytopes in high dimensions [116, 117, 118] .

The non-Abelian vortex in the CFL phase, discussed so far, is a physical example of such vortices. Color superconductivity is induced by condensation of diquarks, which are pairs of two quarks. The order parameter of a color superconductor is given by a 3×3 matrix,

$$\Phi_{\alpha i} = \epsilon_{\alpha\beta\gamma}\epsilon_{ijk}\langle\left(q^T\right)^j_\beta C\gamma_5\left(q\right)^k_\gamma\rangle, \tag{5.1}$$

where q is the quark field, $\alpha, \beta, \gamma = \mathrm{r, g, b}$ $(i, j, k = \mathrm{u, d, s})$ are color (flavor) indices, and the transpose is employed with respect to the spinor index. At asymptotically high densities, the ground states are expected to be in the CFL phase, in which the diquark acquires an expectation value like $\Phi_{\alpha i} = \Delta\delta_{\alpha i}$, where Δ is a BCS gap function. The original color $SU(3)_\mathrm{C}$ and flavor $SU(3)_\mathrm{F}$ symmetry break down to an $SU(3)_\mathrm{C+F}$ symmetry, the elements of which are given by "locked" rotations of color and flavor, $\Phi \to U\Phi U^{-1}$. This structure is similar to that of the Balian-Werthamerer (BW) state of the ^{3}He superfluids, in which the order parameter is invariant under the locked rotations of spin and orbit states. It is shown that there exist topologically (and energetically) stable vortices [38] in the CFL phase, by examining the symmetry-breaking pattern. The vortices in the CFL phase are color flux tubes as well as superfluid vortices, since both local and global symmetries are broken in the ground state. In the presence of a vortex, the order parameter takes the value like $\Phi(r) = \Delta\mathrm{diag}\{f(r)e^{i\theta}, g(r), g(r)\}$ where r is the radial coordinate and $f(r)$ and $g(r)$ are functions of r. This kind of vortex solution breaks the $SU(3)_\mathrm{C+F}$ symmetry down to $SU(2)_\mathrm{C+F} \times U(1)$ symmetry inside the core [39, 119, 40, 43, 44, 45, 96, 95] and fermion zero modes belong to representations of $SU(2)_\mathrm{C+F}$. It has been shown[46, 47] that one CFL vortex has triplet and singlet Majorana fermions inside it [1]. Thus vortices in a color superconductor provide an example of the system with fermion zero modes in the vector representation of $SO(3)$ in their cores, since the triplet of $SU(2)$ is equivalent to the vector representation of $SO(3)$. However, it should be emphasized that the results obtained in Ref. [59] do not depend on details of specific models. The only assumption adopted there is that a vortex has Majorana fermions which transform in the vector representation of $SO(3)$, and

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[1] The singlet Majorana fermion found in [46] was in fact shown to diverge at the origin and consequently to be non-normalizable [47].
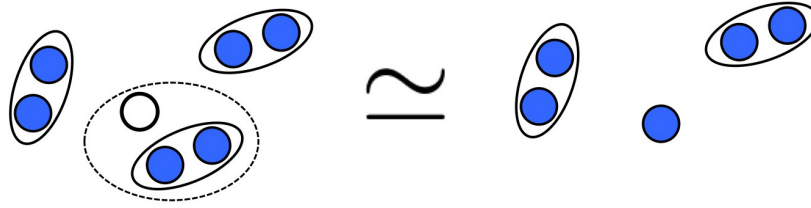
Figure 5.1: Schematic figure of how Majorana fermions appear in a superconductor. Take a hole for example (denoted by a white circle). Because of the interaction with Cooper pairs, a hole can attract and bind to a Cooper pair, , and acquire negative charge. A cluster of Cooper pairs is formed around a hole, and in such a way that no rigorous distinction between a particle and a hole remains.

therefore we expect that such a system can be found in condensed matter systems such as exotic superconductors or ultra-cold atomic gasses.

In this chapter, we generalize the results of Ref. [59] obtained for $SO(3)$ to the case of $SO(N)$ where N is an arbitrary odd integer $N \geq 3$. We discuss the exchange statistics of two vortices having N Majorana fermions trapped in their cores, which are transformed in an $SO(N)$ group. Notice that the $SO(N)$ symmetry is the maximum symmetry in the presence of N Majorana fermions which are real fields. We discuss the exchange statistics at both the operator and representation levels. We find that both the operator and the matrix representation of the exchange operation generally have factorized structures for arbitrary odd N. In particular, the matrix representation is written as a tensor product of two matrices, as previously found in the $SO(3)$ case. One of the two matrices is the one discussed in Ref. [50] in the case that a single fermion is trapped in each vortex. We show that the other is again a generator of the Coxeter group, as in the case of $SO(3)$. When one Majorana fermion is topologically protected in the vortex core such as in class-D topological superconductors, it is robust against perturbations and remains zero energy under the exchange operation. In addition to that, N Majorana fermions remain zero energy as far as the $SO(N)$ symmetry remains intact.

5.2 Majorana fermions in a vortex in superconducting medium

Let us first explain why "Majorana" modes appear inside vortices of a superconductor, even though the original fermions are charged. First let us give an intuitive explanation (see Fig. 5.1). Take a hole for example. A hole interact with the sea Cooper pairs. Because of this interaction, a hole can attract and bind to a Cooper pair, , and acquire negative charge. A composite of a hole and a Cooper pair acquire a negative charge, and look like an particle. This is the reason why there is no rigorous distinction between a particle and a hole.

The emergence of Majorana fermions can be formally described by the BdG equation, which is the

eigenvalue equation of the mean-field Hamiltonian of a superconductor. The essential ingredient is the particle-hole symmetry. We here take a superconducting system with a single species of fermions for simplicity. The BdG equation for this system is given by

$$[\mathcal{H}, \psi^\dagger(\mathcal{E})] = \mathcal{E}\gamma^\dagger(\mathcal{E}), \tag{5.2}$$

where $\mathcal{E}$ is the energy measured from the chemical potential, and the fermion operator γ is a superposition of the creation and annihilation operator of the original fermions,

$$\psi = \int d^2x \left(u(x)\hat{q}^\dagger(x) + v(x)\hat{q}(x) \right). \tag{5.3}$$

Suppose that the Hamiltonian has a particle-hole symmetry, by which we mean that there is an operator C that anticommutes with the Hamiltonian,

$$[C, \mathcal{H}] = 0. \tag{5.4}$$

This indicates that the energy eigenfunctions satisfy the following relation,

$$\psi^\dagger(\mathcal{E}) = \psi(-\mathcal{E}). \tag{5.5}$$

Hence, the states with $\mathcal{E} = 0$ becomes self-conjugate,

$$\psi^\dagger(0) = \psi(0). \tag{5.6}$$

This is nothing but the property of Majorana fermions.

Let us elaborate on the emergence of Majorana fermions in more detail. We here derive the effective theory of Majorana fermions inside vortices. For a superconducting system made of a single species of fermions, the BdG equation in the Nambu-Gor'kov representation $\Psi = (\varphi, \eta)^\mathrm{T}$ (particle in the upper component and hole in the lower component) is given by

$$\mathcal{H}\Psi = \mathcal{E}\Psi, \tag{5.7}$$

where $\mathcal{H}$ is the Hamiltonian in the mean-field approximation,

$$\mathcal{H} \equiv \begin{pmatrix} -i\gamma_0\vec{\gamma}\cdot\vec{\nabla} - \mu & \Delta(\vec{x})\gamma_0\gamma_5 \\ -\Delta^*(\vec{x})\gamma_0\gamma_5 & -i\gamma_0\vec{\gamma}\cdot\vec{\nabla} + \mu \end{pmatrix}. \tag{5.8}$$

The gap profile function $\Delta(\vec{x})$ (three dimensional coordinate $\vec{x} = (x, y, z)$) is given as $\Delta(\vec{x}) \propto \langle\Psi^\mathrm{T}\Psi\rangle$, where the expectation value is given by the sum over all the fermion states in the ground state. This Hamiltonian has the particle-hole symmetry,

$$C^{-1}\mathcal{H}C = -\mathcal{H}^*, \quad C \equiv \begin{pmatrix} 0 & \gamma_2 \\ \gamma_2 & 0 \end{pmatrix}. \tag{5.9}$$

Thus, $\mathcal{H}\Psi = \mathcal{E}\Psi$ implies $\mathcal{H}(C\Psi) = -\mathcal{E}(C\Psi)$ and the spectrum is symmetric above and below the Fermi sea.

If one considers a vortex solution, because of the translational invariance along the vortex (z) axis, the gap is a function of the distance $r = \sqrt{x^2 + y^2}$ from the center of the vortex and θ an angle around the vortex; $\Delta(\vec{x}) = |\Delta(r)|e^{i\theta}$. The gap profile function also satisfies the following boundary conditions; $|\Delta(r = 0)| = 0$ at the center of the vortex and $|\Delta(r = \infty)| = |\Delta_{\text{CFL}}|$ at the position far from the vortex with a bulk gap Δ_{CFL}. Since the system is translationally invariant along the vortex axis, we can always take the fermion states to be eigenstates of the momentum in the z-direction k_z,

$$\Psi^{k_z}_{\pm,m}(r, \theta, z) = \Psi_{\pm,m}(r, \theta)\, e^{ik_z z}, \tag{5.10}$$

where $\Psi_{\pm,m}(r, \theta)$ is the wave function on the x-y plane. Here $\pm$ is for chirality, left and right, of the fermion and an integer m is related to the z component of the total angular momentum J_z. In the Nambu-Gor'kov formalism, the wave function $\Psi_{\pm,m}(r, \theta)$ is given as

$$\Psi_{\pm,m}(r, \theta) = \begin{pmatrix} \varphi_{\pm,m}(r, \theta) \\ \eta_{\mp,m-1}(r, \theta) \end{pmatrix}, \tag{5.11}$$

with the particle component $\varphi_{\pm,m}(r, \theta)$ and the hole component $\eta_{\mp,m-1}(r, \theta)$. The z component of J_z is $m + 1/2$ for the particle and $(m - 1) + 1/2$ for the hole, respectively. Note that the chirality $\pm$ of the particle is assigned in opposite to that of the hole.

The solution of the BdG equation (5.7) gives all the fermion modes in the vortex. They include the scattering states with energies $|\mathcal{E}| > |\Delta_{\text{CFL}}|$ as well as the bound states with energies $|\mathcal{E}| < |\Delta_{\text{CFL}}|$. It is a nontrivial problem to obtain all the fermion solutions. In the present discussion, we concentrate on the fermion states with the lowest energy inside the vortex, which are the most important degrees of freedom at low energies. Furthermore, we here regard the gap profile function $|\Delta(r)|$ as a background field and do not analyze the self-consistent solution for the gap profile function and the fermion wave functions. Such study will be left for future works.

As a result of the particle-hole symmetry, we find that the state with $\mathcal{E} = 0$ is a "Majorana fermion", which has a special property that a particle and a hole are equivalent. The explicit solution of the wave function of the Majorana fermion is given as, for the right mode ($+$ for a particle, $-$ for a hole)

$$\varphi_{+,0}(r, \theta) = C\, e^{-\int_0^r |\Delta(r')|\, dr'} \begin{pmatrix} J_0(\mu r) \\ iJ_1(\mu r)\, e^{i\theta} \end{pmatrix}, \tag{5.12}$$

$$\eta_{-,-1}(r, \theta) = C\, e^{-\int_0^r |\Delta(r')|\, dr'} \begin{pmatrix} -J_1(\mu r)\, e^{-i\theta} \\ iJ_0(\mu r) \end{pmatrix}, \tag{5.13}$$

and for the left mode ($-$ for a particle, $+$ for a hole)

$$\varphi_{-,0}(r, \theta) = C'\, e^{-\int_0^r |\Delta(r')|\, dr'} \begin{pmatrix} J_0(\mu r) \\ -iJ_1(\mu r)\, e^{i\theta} \end{pmatrix}, \tag{5.14}$$

$$\eta_{+,-1}(r, \theta) = C'\, e^{-\int_0^r |\Delta(r')|\, dr'} \begin{pmatrix} J_1(\mu r)\, e^{-i\theta} \\ iJ_0(\mu r) \end{pmatrix}, \tag{5.15}$$

where C and C' are normalization constants, and $J_n(x)$ is the Bessel function. We have represented the solutions in the Weyl (2-component) spinors.

The solutions above satisfy a "Majorana-like" condition ($\kappa = \pm 1$)[2]

$$\Psi = \kappa\, C\, \Psi^*, \tag{5.16}$$

which physically implies the equivalence between a particle and a hole. We note the Majorana fermion is localized at around the center of the vortex. This can bee seen by setting $|\Delta(r)|$ a constant $|\Delta_{\mathrm{CFL}}|$, because the exponential functions in Eqs. (5.12) and (5.13) or Eqs. (5.14) and (5.15) exhibit the behavior like $e^{-|\Delta_{\mathrm{CFL}}|r}$. Fermion zero modes (Majorana fermions) in relativistic theories were found in the (Abelian) vortex in the vacuum [48], where the number of zero modes is determined by the index theorem to be $2n$ for vortices with winding number n [120]. The Majorana-fermion solution in a vortex in a p-wave superconductor was found first by Fukui [121] for non-relativistic fermions, and later in Refs. [122, 46] for relativistic fermions. Although there exist Majorana-like solutions in vortices in p-wave superconductors both for non-relativistic and relativistic fermions, they are absent for non-relativistic fermions in vortices in s-wave superconductors [123]. We leave a comment that the fermions bound in the vortex are intuitively understood also from a view of the Andreev reflection. When the fermions meet the interface between the normal phase (inside of the vortex) and the superconducting phase (outside of the vortex), there appear the Cooper pairs created in the superconducting phase and the holes reflected in the normal phase. This is called the Andreev reflection [124]. The Andreev reflection was also considered in the CFL phase [125]. The multiple number of reflections of the fermions (particle and holes) at the interface make the bound state inside the vortex.

The discussion above is generalized to the non-Abelian vortices in the CFL phase [46]. Important difference is that, in the case of a vortex with minimal winding number, there appear three Majorana fermions propagating along the vortex.

5.3   Vortices and non-Abelian statistics

We here explain the basics of the non-Abelian statistics. We show how the non-Abelian statistics emerges from Majorana fermions trapped in vortices.

5.3.1   Vortices with $N = 1$

We briefly review how non-Abelian statistics emerges for a set of n vortices, each of which contains a single Majorana fermion in its core [50]. This provides the simplest example of non-Abelian statistics of vortices, but as we will see later we can always identify the same structure even for the case with multiple Majorana fermions as far as the number of fermions N is odd.

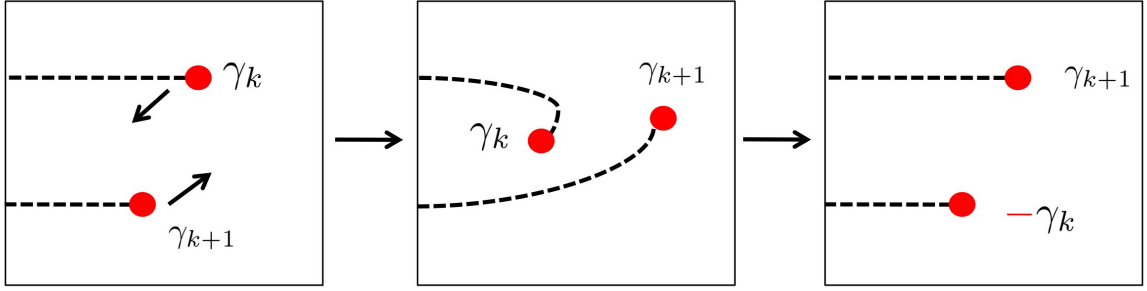[2] $\kappa = 1$ is for the right mode and $\kappa = -1$ is for the left mode.

Figure 5.2: Schematic picture of an exchange of two vortices. When a vortex goes across a cut, it acquires a minus sign.

Consider an exchange of two vortices in a system of $n = 2m$ vortices [3]. Each vortex has a single Majorana fermion localized at the core, and one can specify the position of a vortex on a two-dimensional plane (we label the vortices $k = 1, \cdots, n$). Notice that the trapped Majorana fermion has zero energy, which gives rise to degeneracy of the ground states. Since the existence of Majorana fermions are topologically guaranteed, the degeneracy is not disturbed by small perturbations, and hence we treat the exchange of vortices as an adiabatic process.

Let an operation T_k be defined as an exchange of the k-th and $(k + 1)$-th vortices, in which the $(k + 1)$-th vortex turns around the k-th vortex in an anticlockwise way. All the exchanges of two vortices are realized by successive application of the exchanges of adjacent vortices T_k, and they form a braid group B_n. They indeed satisfy the braid relations:

$$T_k T_\ell T_k = T_\ell T_k T_\ell \quad \text{for} \quad |k - \ell| = 1, \tag{5.17}$$

$$T_k T_\ell = T_\ell T_k \quad \text{for} \quad |k - \ell| > 1. \tag{5.18}$$

Recall that the vortices are accompanied by Majorana fermions, and one can express the action of T_k on Majorana fermions as a transformation. To this end, we define a Majorana operator γ_k corresponding to the Majorana fermion in the k-th vortex [50], satisfying the self-conjugate condition $(\gamma_k)^\dagger = \gamma_k$ and the anticommutation relation $\{\gamma_k, \gamma_\ell\} = 2\delta_{k\ell}$ (the Clifford algebra). Then, the operation T_k induces the following transformation [50]:

$$T_k : \begin{cases} \gamma_k \to \gamma_{k+1} \\ \gamma_{k+1} \to -\gamma_k \end{cases}, \tag{5.19}$$

with the rest γ_ℓ ($\ell \neq k$, $k + 1$) unchanged. One can explicitly check that the transformation (5.19) satisfies the braid relations (5.17) and (5.18). There is a minus sign in the second line, which is essential for the non-Abelian statistics because it gives $T_k^4 = 1$ (the Bose-Einstein or Fermi-Dirac statistics give just $T_k^2 = 1$). This express the fact that the wave function of the Majorana fermion is a

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[3] The structure of the Hilbert space in the presence of an odd number vortices is recently discussed in Ref. [126].
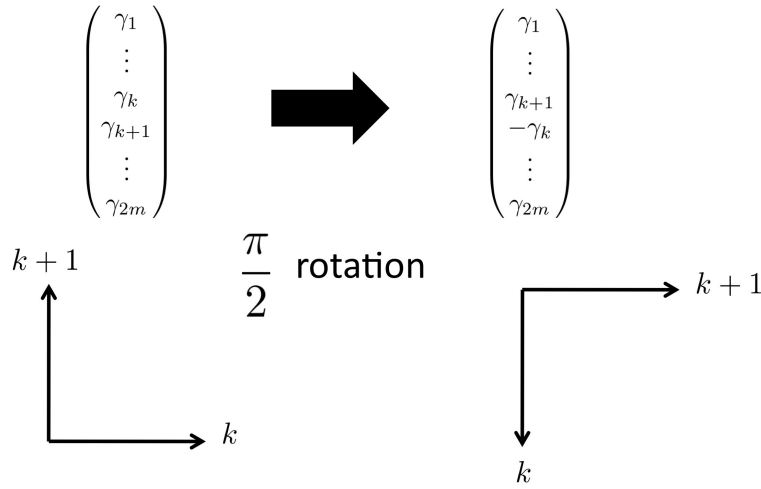
Figure 5.3: Schematic figure of the transformation of γ_i under the exchange of k-th and $k + 1$-th vortices. The γ_i can be regarded as a vector, and the exchange of vortices act on γ_i as "spatial" rotations in k-$k + 1$ plane.

double-valued function for the angle around the vortex axis This can be inferred from the form of the BdG equations; when we shift the phase winding of a vortex as $\theta \to \theta + \alpha$, it can be canceled if the phases of particle and hole wave functions are shifted by $\alpha/2$ and $-\alpha/2$, respectively. It implies that when quasiparticles travel around a vortex at $\alpha = 2\pi$, both particle and hole wave functions receive minus sign. In order to regard the wave function as a singlevalued function, we need to introduce a cut from the center of the vortex to infinitely far from the vortex. The directions of cuts are arbitrary and gauge dependent. It means that the wave function acquires a minus sign when the Majorana fermion goes across the cut (see Fig.5.2).

The transformation (5.19) is realized by the unitary operator

$$\tau_k \equiv \exp\left(\frac{\pi}{4}\gamma_{k+1}\gamma_k\right) = \frac{1}{\sqrt{2}}\left(1 + \gamma_{k+1}\gamma_k\right). \tag{5.20}$$

By explicit calculations, one can check

$$\tau_k \gamma_k \tau_k^{-1} = \gamma_{k+1}, \tag{5.21}$$

$$\tau_k \gamma_{k+1} \tau_k^{-1} = -\gamma_k, \tag{5.22}$$

$$\tau_k \gamma_\ell \tau_k^{-1} = \gamma_\ell \quad (\ell \neq k, k+1). \tag{5.23}$$

The expression (5.20) allows for an intuitive geometrical interpretation. If one regards the index i as that of a cartesian coordinate, the transformation (5.19) is a $\pi/2$-rotation in the k-$k + 1$ plane (see Fig. 5.3). The Equation (5.20) is in fact of the form of an exponentiated rotational generator. To see this, first note that the anticommutations of fermion operators γ_i constitute the Clifford algebra. It is known that, out of Clifford algebra, we can form the representation of the generators of rotational

group $SO(2m)$ [127],

$$M_{ij} = \frac{1}{4i}[\gamma_i, \gamma_j]. \tag{5.24}$$

It can also shown that, under the action of the rotational generator M_{ij}, the operators γ_i transform in the vector representation,

$$[M_{ij}, \gamma_k] = i\left(\delta_{ik}\gamma_j - \delta_{jk}\gamma_i\right) = [M_{ij}^{D=2m}]_{kl}\gamma_l, \tag{5.25}$$

where $M_{ij}^{D=2m}$ is the $2m$-dimensional representation matrix of the generator M_{ij}. From these facts, it is now obvious that the operator τ_k the representation of the group element which expresses the $\pi/2$-rotation in k-$k+1$ plane,

$$\tau_k = \exp\left\{\frac{\pi}{2}i \cdot \frac{1}{4i}[\gamma_k, \gamma_k + 1]\right\} = \exp\left\{\frac{\pi}{2}iM_{k,k+1}\right\}. \tag{5.26}$$

Let us call the operator τ_k for this single Majorana case the "Ivanov operator," since it was first found by Ivanov [50]. One can explicitly see that this transformation is indeed non-Abelian in the matrix representation of τ_k. To construct the Hilbert space on which the operator τ_k acts, we define Dirac fermions [4] by using two Majorana fermions at adjacent vortices $\Psi_K = (\gamma_{2K-1} + i\gamma_{2K})/2$ with $K = 1, \cdots, m$. These Dirac fermion operators satisfy the usual anticommutation relations,

$$\{\Psi_K, \Psi_L^\dagger\} = \delta_{KL}, \quad \{\Psi_K, \Psi_L\} = \{\Psi_K^\dagger, \Psi_L^\dagger\} = 0. \tag{5.27}$$

If one defines Ψ_K and $\Psi_K^\dagger$ as the annihilation and creation operators, respectively, then one can construct the Hilbert space by acting the creation operators $\Psi_K^\dagger$'s on the "Fock vacuum-state" $|0\rangle$ defined by $\Psi_K|0\rangle = 0$ for all K. Within this Hilbert space, the operators τ_k's are now expressed as matrices which we call the Ivanov matrices. The Ivanov matrices contain off-diagonal elements representing the non-Abelian statistics.

Let us find the explicit form of the matrix representations [50]. In the case of two vortices, the two Majorana fermions are combined to a Dirac fermion, whose annihilation and creation operators are given by $\Psi = (\gamma_1 + i\gamma_2)/2$ and $\Psi^\dagger = (\gamma_1 - i\gamma_2)/2$. We define the Fock vacuum by $\Psi|0\rangle = 0$, and we take the states $\{|0\rangle, \Psi^\dagger|0\rangle\}$ as a basis. There is only one exchange operation T_1, whose matrix representation can be written as

$$\tau_1 = \exp\left(\frac{\pi}{4}\gamma_2\gamma_1\right) = \exp\left[\frac{\pi}{4}i\left(2\Psi^\dagger\Psi - 1\right)\right] = \exp\left(\frac{\pi}{4}i\sigma_z\right), \tag{5.28}$$

where σ_z is a Pauli matrix. In the case of four vortices, we can define two Dirac fermions as $\Psi_1 = (\gamma_1 + i\gamma_2)/2$ and $\Psi_2 = (\gamma_3 + i\gamma_4)/2$. There are three elementary exchange operations, T_1, T_2, and T_3. Let us take the states $\{|0\rangle, \Psi_1^\dagger\Psi_2^\dagger|0\rangle, \Psi_1^\dagger|0\rangle, \Psi_2^\dagger|0\rangle\}$ as a basis. Then, the exchange operations are

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[4]We use $k, \ell = 1, \cdots, n$ to label the vortices and the trapped Majorana fermions, while $K, L = 1, \cdots, n/2 = m$ to label the Dirac fermions which are constructed from two Majorana fermions.

represented as the following matrices:

$$\tau_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1-i & 0 & 0 & 0 \\ 0 & 1+i & 0 & 0 \\ 0 & 0 & 1+i & 0 \\ 0 & 0 & 0 & 1-i \end{pmatrix}, \tag{5.29}$$

$$\tau_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i & 0 & 0 \\ -i & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & -i & 1 \end{pmatrix}, \tag{5.30}$$

$$\tau_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1-i & 0 & 0 & 0 \\ 0 & 1+i & 0 & 0 \\ 0 & 0 & 1-i & 0 \\ 0 & 0 & 0 & 1+i \end{pmatrix}. \tag{5.31}$$

The matrix τ_2 has off-diagonal components, which makes the exchange operations non-Abelian.

5.3.2 Vortices with multiple Majorana fermions $N \geq 3$

Now let us turn to the case with vortices having multiple Majorana fermions in their cores. We consider the situation that each vortex traps N Majorana fermions γ_k^a ($a = 1, \cdots, N$) which transform in the vector representation of $SO(N)$ symmetry. We take N to be an arbitrary odd number, and this is a generalization of the simplest nontrivial case $N = 3$ in Ref. [59].

The non-Abelian Majorana operators γ_k^a satisfy the self-conjugate conditions and the anticommutation relations:

$$(\gamma_k^a)^\dagger = \gamma_k^a, \quad \{\gamma_k^a, \gamma_\ell^b\} = 2\delta^{ab}\delta_{k\ell}. \tag{5.32}$$

We define the exchange of the k-th and $(k+1)$-th vortices so that the Majorana fermion operator with each a transforms in the same way as the case with a single Majorana fermion (see Eq. (5.19)) [5]:

$$T_k : \begin{cases} \gamma_k^a & \to \gamma_{k+1}^a \\ \gamma_{k+1}^a & \to -\gamma_k^a \end{cases} \quad \text{for all } a, \tag{5.33}$$

with the rest γ_ℓ^a ($\ell \neq k, k+1$) unchanged. The exchange operations T_k's satisfy the braid relations (5.17) and (5.18). Their action on γ_ℓ^a's can be represented in terms of non-Abelian Majorana operators γ_k^a's in the following way. Since the transformation (5.33) for each a is equivalent to the single Majorana case, one can use the same expression for the unitary operator which induces the transformation for each a:

$$\tau_k^a \equiv \exp\left(\frac{\pi}{4}\gamma_{k+1}^a \gamma_k^a\right) = \frac{1}{\sqrt{2}}\left(1 + \gamma_{k+1}^a \gamma_k^a\right). \tag{5.34}$$

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[5]One may allow for mixture of indices a under the exchange of two vortices, but here we discuss the simplest case where such mixing does not occur.

Thus, the exchange operator for the vortices having multiple fermions should be represented as the product of them:

$$\tau_k^{[N]} \equiv \prod_{a=1}^{N} \tau_k^a \,. \tag{5.35}$$

This exchange operator is $SO(N)$ invariant as shown in the next section. One can check that the operator $\tau_k^{[N]}$ applied to γ_ℓ^a indeed generates the desired transformation (5.33):

$$\tau_k^{[N]} \gamma_k^a (\tau_k^{[N]})^{-1} = \gamma_{k+1}^a \,, \tag{5.36}$$

$$\tau_k^{[N]} \gamma_{k+1}^a (\tau_k^{[N]})^{-1} = -\gamma_k^a \,, \tag{5.37}$$

$$\tau_k^{[N]} \gamma_\ell^a (\tau_k^{[N]})^{-1} = \gamma_\ell^a \qquad (\ell \neq k, \, k+1) \,. \tag{5.38}$$

This transformation is again non-Abelian, which is explicitly seen in the matrix representation.

To obtain the matrix representation, we can perform the same procedure as in the case with single Majorana fermions. Namely, by defining the Dirac fermion operators

$$\Psi_K^a \equiv \frac{1}{2}(\gamma_{2K-1}^a + i\gamma_{2K}^a), \qquad \Psi_K^{a\dagger} \equiv \frac{1}{2}(\gamma_{2K-1}^a - i\gamma_{2K}^a) \tag{5.39}$$

which satisfy $(K, L = 1, \cdots, m)$

$$\{\Psi_K^a, \Psi_L^{b\dagger}\} = \delta_{KL}\delta^{ab}, \quad \{\Psi_K^a, \Psi_L^b\} = \{\Psi_K^{a\dagger}, \Psi_L^{b\dagger}\} = 0, \tag{5.40}$$

we can construct the Hilbert space. Then, we can find matrix representation of $\tau_k^{[N]}$. In Ref. [59], three of us explicitly constructed the Hilbert space for the case of $N = 3$ and $n = 2, 4$, and found matrix expression of $\tau_k^{[3]}$ according to different representations of $SO(3)$. The matrices have off-diagonal elements and thus they are non-Abelian. In principle, one can do the same thing for an arbitrary odd N. In the present paper, however, we first look into interesting structure of the operator $\tau_k^{[N]}$, which was also suggested in the previous paper [59]. Namely, the operator $\tau_k^{[N]}$ itself can be decomposed into two parts. Then, we discuss the relation between the decomposition of $\tau_k^{[N]}$ and the matrix representation of $\tau_k^{[N]}$ in Sec. 5.5.

5.4 The Coxeter group for multiple Majorana fermions $N \geq 3$

In the previous analysis [59] with $N = 3$, it is found that the matrix representation of $\tau_k^{[3]}$ in the four-vortex sector can be decomposed into a tensor product of two matrices: one is the same as the Ivanov matrix for the single Majorana fermion case, and the other is identified with generators of the Coxeter group. We also found that the similar decomposition is possible at the operator level [59]. Namely, one can express $\tau_k^{[3]}$ as a product of two distinct operators which give rise to the corresponding matrices in the matrix representation. In this section, we discuss that such a decomposition at the operator level holds even for an arbitrary odd number of N.

Before we go into details, let us define some useful notations. We first define a composite operator Γ_k^a by

$$\Gamma_k^a \equiv \gamma_{k+1}^a \gamma_k^a, \tag{5.41}$$

which have the following properties,

$$\{\Gamma_k^a, \Gamma_l^b\} = -2\delta_{kl} \quad \text{for} \quad a = b \text{ and } |k - l| \leq 1, \tag{5.42}$$

$$[\Gamma_k^a, \Gamma_l^b] = 0 \qquad \text{for} \quad a \neq b, \text{ or } (a = b \text{ and } |k - l| > 1). \tag{5.43}$$

For later convenience, we define another composite operator $(1 \leq n \leq N)$

$$\Gamma_k^{(n)} \equiv \frac{1}{(n!)^2} \, \mathrm{e}^{\frac{\pi}{2}i(n-1)} \epsilon^{a_1 \cdots a_N} \epsilon^{b_1 \cdots b_N} \, \delta_{b_{n+1}}^{a_{n+1}} \cdots \delta_{b_N}^{a_N} \gamma_{k+1}^{a_1} \cdots \gamma_{k+1}^{a_n} \gamma_k^{b_1} \cdots \gamma_k^{b_n}, \tag{5.44}$$

where $\epsilon^{a_1 \cdots a_N}$ is the completely antisymmetric tensor. It is evident that the operators $\Gamma_k^{(n)}$'s are $SO(N)$ invariants for all n. For example, for $N = 3$,

$$\Gamma_k^{(1)} = \Gamma_k^1 + \Gamma_k^2 + \Gamma_k^3, \quad \Gamma_k^{(2)} = \Gamma_k^1\Gamma_k^2 + \Gamma_k^2\Gamma_k^3 + \Gamma_k^3\Gamma_k^1, \quad \Gamma_k^{(3)} = \Gamma_k^1\Gamma_k^2\Gamma_k^3. \tag{5.45}$$

With those composite operators, the exchange operator $\tau_k^{[N]}$ can be represented as

$$\tau_k^{[N]} = \left(\frac{1}{\sqrt{2}}\right)^N \prod_{a=1}^N (1 + \Gamma_k^a) = \exp\left\{\frac{\pi}{4}\sum_a \Gamma_k^a\right\} = \left(\frac{1}{\sqrt{2}}\right)^N \sum_{n=1}^N \Gamma_k^{(n)}. \tag{5.46}$$

In the final expression, we confirm that the operator $\tau_k^{[N]}$ is $SO(N)$-invariant.

5.4.1   Coxeter group

Here let us introduce Coxeter groups, which are introduced by H.S.M Coxeter in 1934, as abstractions of reflections group. The finite Coxeter groups are equivalent to the finite Euclidean reflection groups.

A Coxeter group S is defined as a group with distinct generators $s_i \in S$ $(i = 1, 2, 3, \cdots)$ satisfying the following two conditions [116, 117, 118]:

(a) $s_i^2 = 1$

(b) $(s_i \, s_j)^{m_{i,j}} = 1$ with a positive integer $m_{i,j} \geq 2$ for $i \neq j$.

The property of the Coxeter group is characterized by the elements of the Coxeter matrix, $(\mathsf{M})_{ij} = m_{i,j}$. The condition (a) gives $m_{i,i} = 1$ for any i. As is shown later, in the case of the $n = 2m$ vortices, the Coxeter matrix is given by a $(2m-1) \times (2m-1)$ matrix whose elements are 1 (diagonal elements, $m_{i,i} = 1$), 3 (adjacent elements, $m_{i,i+1} = m_{i+1,i} = 3$) and 2 (all the others), which can be explicitly written as

$$\mathsf{M}_{2m-1} = \begin{pmatrix} 1 & 3 & 2 & 2 & \cdots \\ 3 & 1 & 3 & 2 & \\ 2 & 3 & 1 & 3 & \\ 2 & 2 & 3 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}. \tag{5.47}$$

This Coxeter group is classified as A_{2m-1}. It is also known as the symmetric group S_{2m}, which is the symmetry group of a regular $(2m - 1)$-simplex.

5.4.2 The case of $SO(3)$

The simplest nontrivial case with $N = 3$ provides us with useful information which is helpful in discussing the decomposition for the general case N. Let us first recall that the operator $\tau_k^{[N=3]}$ has been found to be decomposed into two parts (see Appendix B in Ref. [59]):

$$\tau_k^{[3]} = \sigma_k^{[3]} h_k^{[3]} , \tag{5.48}$$

where both of the operators $\sigma_k^{[3]}$ and $h_k^{[3]}$ are given in terms of the Majorana operators γ_k^a as

$$\sigma_k^{[3]} = \frac{1}{2} \left(1 - \gamma_{k+1}^1 \gamma_{k+1}^2 \gamma_k^1 \gamma_k^2 - \gamma_{k+1}^2 \gamma_{k+1}^3 \gamma_k^2 \gamma_k^3 - \gamma_{k+1}^3 \gamma_{k+1}^1 \gamma_k^3 \gamma_k^1 \right) , \tag{5.49}$$

and

$$h_k^{[3]} = \frac{1}{\sqrt{2}} \left(1 - \gamma_{k+1}^1 \gamma_{k+1}^2 \gamma_{k+1}^3 \gamma_k^1 \gamma_k^2 \gamma_k^3 \right) . \tag{5.50}$$

One can also rewrite them compactly in new notations as (see Eqs. (5.44) and (5.45))

$$\sigma_k^{[3]} = \frac{1}{2} \left(1 + \Gamma_k^{(2)} \right) , \quad h_k^{[3]} = \frac{1}{\sqrt{2}} \left(1 + \Gamma_k^{(3)} \right) . \tag{5.51}$$

Note that $\sigma_k^{[3]}$ and $h_\ell^{[3]}$ are commutative for any pair of k and ℓ. Thus, $\tau_k^{[3]}$ is the product of $\sigma_k^{[3]}$ and $h_k^{[3]}$. It was shown that the operators $\sigma_k^{[3]}$'s satisfy the Coxeter relations,

$$(\sigma_k^{[3]})^2 = 1, \tag{5.52}$$
$$(\sigma_k^{[3]} \sigma_\ell^{[3]})^3 = 1 \quad \text{for} \quad |k - \ell| = 1, \tag{5.53}$$
$$(\sigma_k^{[3]} \sigma_\ell^{[3]})^2 = 1 \quad \text{for} \quad |k - \ell| > 1. \tag{5.54}$$

This Coxeter group for the $n = 2m$ vortices is classified as A_{2m-1}, which is also known as the symmetric group S_{2m}.

In contrast, the other part $h_k^{[3]}$ works in the same way as the Ivanov operator τ_k defined in Eq. (5.20), although $h_k^{[3]}$ has a more complicated structure. This is naturally understood if one notices that the operator $\gamma_k^1 \gamma_k^2 \gamma_k^3$ is $SO(3)$ invariant (singlet), and thus it treats three Majorana fermions together as if it does not have any $SO(3)$ index. This picture is very useful when we consider the general case with N.

5.4.3 The case of $SO(N)$ with arbitrary odd N

The previous analysis suggests that it is possible to decompose the full exchange operator $\tau_k^{[N]}$ into two parts. This is indeed the case. We find that the operator $\tau_k^{[N]}$ can be decomposed as

$$\tau_k^{[N]} = \sigma_k^{[N]} h_k^{[N]} , \tag{5.55}$$

where $\sigma_k^{[N]}$ and $h_k^{[N]}$ are defined by using the notation introduced before as

$$\sigma_k^{[N]} \equiv \left(\frac{1}{\sqrt{2}}\right)^{N-1} \left(1 + \Gamma_k^{(2)} + \Gamma_k^{(4)} + \cdots + \Gamma_k^{(N-1)}\right), \tag{5.56}$$

$$h_k^{[N]} \equiv \frac{1}{\sqrt{2}} \left(1 + \Gamma_k^{(N)}\right). \tag{5.57}$$

Note that $\sigma_k^{[N]}$ and $h_k^{[N]}$ are $SO(N)$ invariant, and $\sigma_k^{[N]}$ and $h_\ell^{[N]}$ are commutative for any k and ℓ. One can readily verify the decomposition (5.55). By using Eqs. (5.56) and (5.57), one can check that the product $\sigma_k^{[N]} h_k^{[N]}$ is equal to the last equation in Eq. (5.46) if one uses Eqs. (5.42) and (5.43).

First of all, let us discuss the properties of $h_k^{[N]}$. The analysis for the case $N = 3$ suggests that if one treats multiple Majorana fermions in a vortex in a unit, then the action of $\tau_k^{[N]}$ will be essentially equivalent to the Ivanov operator. This motivates us to introduce the following "singlet Majorana operator" locally defined on the k-th vortex:

$$\overline{\gamma}_k \equiv \frac{1}{N!} e^{i\frac{\pi}{4}(N-1)} \epsilon^{a_1 a_2 \cdots a_N} \gamma_k^{a_1} \gamma_k^{a_2} \cdots \gamma_k^{a_N}, \tag{5.58}$$

which is manifestly invariant under the $SO(N)$ transformation. The phase factor is included so that the operator becomes self-conjugate $(\overline{\gamma}_k)^\dagger = \overline{\gamma}_k$, and satisfies the Clifford algebra $\{\overline{\gamma}_k, \overline{\gamma}_\ell\} = 2\delta_{k\ell}$. Notice that these properties of $\overline{\gamma}_k$ are the same as those of a single Majorana operator. For $N = 3$, one finds $\overline{\gamma}_k = i\gamma_k^1 \gamma_k^2 \gamma_k^3$, and the operator $h_k^{[3]}$ can be compactly expressed as $h_k^{[3]} = \frac{1}{\sqrt{2}}(1 + \overline{\gamma}_{k+1}\overline{\gamma}_k)$, which has the same structure as the Ivanov operator (5.20). For arbitrary odd N, we find that $h_k^{[N]}$ can also be expressed as

$$h_k^{[N]} = \exp\left(\frac{\pi}{4}\overline{\gamma}_{k+1}\overline{\gamma}_k\right) = \frac{1}{\sqrt{2}}\left(1 + \overline{\gamma}_{k+1}\overline{\gamma}_k\right), \tag{5.59}$$

by noting the relation $\Gamma_k^{(N)} = \overline{\gamma}_{k+1}\overline{\gamma}_k$. Then, $h_k^{[N]}$ works on $\overline{\gamma}_\ell$ as

$$h_k^{[N]} \overline{\gamma}_k (h_k^{[N]})^{-1} = \overline{\gamma}_{k+1}, \tag{5.60}$$

$$h_k^{[N]} \overline{\gamma}_{k+1} (h_k^{[N]})^{-1} = -\overline{\gamma}_k, \tag{5.61}$$

$$h_k^{[N]} \overline{\gamma}_\ell (h_k^{[N]})^{-1} = \overline{\gamma}_\ell \qquad (\ell \neq k,\, k+1), \tag{5.62}$$

for an arbitrary odd $N \geq 3$. Interestingly, this is the same as the transformation (5.21) – (5.23) induced by the Ivanov operator with $N = 1$. Therefore, the operator $h_k^{[N]}$ for the singlet Majorana operator $\overline{\gamma}_k$ is equivalent to the Ivanov operator τ_k for the single Majorana operator γ_k.

Next, we discuss the properties of the operators $\sigma_k^{[N]}$. Similarly to the $N = 3$ case, $\sigma_k^{[N]}$ are generators of the Coxeter group. Indeed, it can be shown that $\sigma_k^{[N]}$ satisfy

$$(\sigma_k^{[N]})^2 = 1, \tag{5.63}$$

$$(\sigma_k^{[N]} \sigma_\ell^{[N]})^3 = 1 \quad \text{for} \quad |k - \ell| = 1, \tag{5.64}$$

$$(\sigma_k^{[N]} \sigma_\ell^{[N]})^2 = 1 \quad \text{for} \quad |k - \ell| > 1, \tag{5.65}$$

and these relations induce the same Coxeter group for $n = 2m$ vortices.

Let us give a proof that the operators $\sigma_k^{[N]}$ defined in Eq. (5.56) satisfy the Coxeter relations in Eqs. (5.64) and (5.65) for arbitrary odd number N.

Proof of $(\sigma_k^{[N]} \sigma_\ell^{[N]})^3 = 1$ **for** $|k - \ell| = 1$

Below we show Eqs. (5.63) and (5.64). Let us first note that the cube of the product of $\tau_k^{[N]}$ and $\tau_{k+1}^{[N]}$ is written as

$$(\tau_k^{[N]} \tau_{k+1}^{[N]})^3 = (\sigma_k^{[N]} h_k^{[N]} \sigma_{k+1}^{[N]} h_{k+1}^{[N]})^3. \tag{5.66}$$

Since $\sigma_k^{[N]}$ commutes with $h_k^{[N]}$ and $h_{k+1}^{[N]}$, the relation above is written as

$$(\tau_k^{[N]} \tau_{k+1}^{[N]})^3 = (\sigma_k^{[N]} \sigma_{k+1}^{[N]})^3 (h_k^{[N]} h_{k+1}^{[N]})^3. \tag{5.67}$$

The left hand side of Eq. (5.67) is equal to -1, which can be shown as follows.

$$
\begin{aligned}
(\tau_k^{[N]} \tau_{k+1}^{[N]})^3 &= \prod_{a=1}^{N} \left\{ \frac{1}{\sqrt{2}}(1 + \Gamma_k^a) \frac{1}{\sqrt{2}}(1 + \Gamma_{k+1}^a) \right\}^3 \\
&= \prod_{a=1}^{N} \left\{ \frac{1}{2} \left(1 + \Gamma_k^a + \Gamma_{k+1}^a + \Gamma_k^a \Gamma_{k+1}^a \right) \right\}^3 \\
&= \prod_{a=1}^{N} \frac{1}{2} \left(-1 + \Gamma_k^a + \Gamma_{k+1}^a + \Gamma_k^a \Gamma_{k+1}^a \right) \frac{1}{2} \left(1 + \Gamma_k^a + \Gamma_{k+1}^a + \Gamma_k^a \Gamma_{k+1}^a \right) \\
&= (-1)^N \\
&= -1,
\end{aligned}
\tag{5.68}
$$

where in the third line we have used the relation,

$$\left\{ \frac{1}{2} \left(1 + \Gamma_k^a + \Gamma_{k+1}^a + \Gamma_k^a \Gamma_{k+1}^a \right) \right\}^2 = \frac{1}{2} \left(-1 + \Gamma_k^a + \Gamma_{k+1}^a + \Gamma_k^a \Gamma_{k+1}^a \right), \tag{5.69}$$

which follows from the anticommuting property of Γ_k^a and Γ_{k+1}^a. In the fourth line, we have again used the anticommuting property.

On the other hand, we can also show that $(h_k^{[N]} h_{k+1}^{[N]})^3$ is equal to -1 as

$$
\begin{aligned}
(h_k^{[N]} h_{k+1}^{[N]})^3 &= \left\{ \frac{1}{2} \left(1 + \Gamma_k^{(N)} + \Gamma_{k+1}^{(N)} + \Gamma_k^{(N)} \Gamma_{k+1}^{(N)} \right) \right\}^3 \\
&= \frac{1}{2} \left(-1 + \Gamma_k^{(N)} + \Gamma_{k+1}^{(N)} + \Gamma_k^{(N)} \Gamma_{k+1}^{(N)} \right) \frac{1}{2} \left(1 + \Gamma_k^{(N)} + \Gamma_{k+1}^{(N)} + \Gamma_k^{(N)} \Gamma_{k+1}^{(N)} \right) \\
&= -1,
\end{aligned}
\tag{5.70}
$$

where we have used the relation

$$\left\{ \frac{1}{2} \left(1 + \Gamma_k^{(N)} + \Gamma_{k+1}^{(N)} + \Gamma_k^{(N)} \Gamma_{k+1}^{(N)} \right) \right\}^2 = \frac{1}{2} \left(-1 + \Gamma_k^{(N)} + \Gamma_{k+1}^{(N)} + \Gamma_k^{(N)} \Gamma_{k+1}^{(N)} \right). \tag{5.71}$$

From Eqs. (5.67), (5.68), and (5.70), we can conclude $(\sigma_k^{[N]} \sigma_\ell^{[N]})^3 = 1$ for $|k - \ell| = 1$.

Proof of $(\sigma_k^{[N]})^2 = 1$ **and** $(\sigma_k^{[N]}\sigma_\ell^{[N]})^2 = 1$ **for** $|k - \ell| > 1$

To prove Eqs. (5.63) and (5.65), we first note that by squaring both sides of Eq. (5.55) and using the relation

$$\left(\frac{1}{\sqrt{2}}(1 + \Gamma_k^a)\right)^2 = \frac{1}{2}(1 + 2\Gamma_k^a - 1) = \Gamma_k^a, \tag{5.72}$$

one finds

$$\Gamma_k^{(N)} = (\sigma_k^{[N]})^2 \, \Gamma_k^{(N)}. \tag{5.73}$$

Then, by multiplying $(\Gamma_k^{(N)})^{-1}$ on both sides from right, we obtain

$$(\sigma_k^{[N]})^2 = 1. \tag{5.74}$$

It follows that $(\sigma_k^{[N]}\sigma_\ell^{[N]})^2 = 1$ for $|k - \ell| > 1$ since $\sigma_k^{[N]}$ and $\sigma_\ell^{[N]}$ with $|k - \ell| > 1$ commute.

We have thus shown that, for an arbitrary odd N, the operators $\sigma_k^{[N]}$ again obey the Coxeter relations of A_{2m-1}. Therefore, for an arbitrary odd N, the exchange operator $\tau_k^{[N]}$ is expressed as a product of a generator of the Coxeter group A_{2m-1} (for the vortex number $n = 2m$) and the Ivanov operator for a single Majorana fermion.

5.4.4   Decomposition of the Majorana operators

Let us recall that the exchange of the Majorana operators γ_ℓ^a's is originally defined as the operation T_k in Eq. (5.33). It is not apparently clear how the decomposed structure of the operator $\tau_k^{[N]}$ indeed works in the exchange operation T_k. To understand this, it is instructive to notice that the Majorana operator γ_k^a can be rewritten as

$$\gamma_k^a = \widetilde{\gamma}_k^a \, \overline{\gamma}_k \,, \tag{5.75}$$

where $\widetilde{\gamma}_k^a$ is a composite operator in the vector representation of $SO(N)$ defined locally on the k-th vortex as

$$\widetilde{\gamma}_k^a \equiv \frac{1}{(N-1)!} \, e^{i\frac{\pi}{4}(N-1)} \epsilon^{aa_1 \cdots a_{N-1}} \gamma_k^{a_1} \cdots \gamma_k^{a_{N-1}}, \tag{5.76}$$

and $\overline{\gamma}_k$ is the singlet Majorana operator defined in Eq. (5.58). The two operators $\widetilde{\gamma}_k^a$ and $\overline{\gamma}_\ell$ commute with each other for any pair of k and ℓ. This expression allows us to extract, from the Majorana operator γ_k^a, the part of a singlet Majorana fermion $\overline{\gamma}_k$ whose properties are well understood. Notice that $\overline{\gamma}_k$ ($\widetilde{\gamma}_k^a$) is composed of an odd number N (an even number $N-1$) of Majorana fermion operators.

Since $\widetilde{\gamma}_k^a$ and $\overline{\gamma}_k$ are expressed in terms of the original Majorana operator γ_k^a, one can immediately find how they are transformed in the exchange T_k. Namely, we find the transformation of $\widetilde{\gamma}_k^a$ and $\overline{\gamma}_k$ by T_k as

$$T_k : \begin{cases} \widetilde{\gamma}_k^a & \to \widetilde{\gamma}_{k+1}^a \\ \widetilde{\gamma}_{k+1}^a & \to \widetilde{\gamma}_k^a \end{cases}, \quad \text{for all } a, \tag{5.77}$$

without a minus sign, and

$$T_k : \begin{cases} \overline{\gamma}_k & \to \overline{\gamma}_{k+1} \\ \overline{\gamma}_{k+1} & \to -\overline{\gamma}_k \end{cases} , \quad \text{for all } a , \tag{5.78}$$

with a minus sign, while the rest $\widetilde{\gamma}_\ell^a$ and $\overline{\gamma}_\ell$ ($\ell \neq k$ and $k+1$) are unchanged [6]. It is easily checked that the simultaneous transformation of $\widetilde{\gamma}_k^a$ and $\overline{\gamma}_k$ reproduces the transformation of γ_k^a in Eq. (5.33). Therefore, we observe from Eq. (5.77) that $\widetilde{\gamma}_\ell^a$'s are transformed by a symmetric group S_{2m}, or the Coxeter group of A_{2m-1} (for the vortex number $n = 2m$), and from Eq. (5.78) that $\overline{\gamma}_\ell$'s are transformed as in the same way as the Ivanov operators for single Majorana fermions [7].

The exchange properties of $\widetilde{\gamma}_k^a$ and $\overline{\gamma}_k$ in Eqs. (5.77) and (5.78) can be discussed at the operator level. Because $\sigma_k^{[N]}$ and $\overline{\gamma}_l$ ($h_k^{[N]}$ and $\widetilde{\gamma}_l^a$) are commutative for any pair of k and ℓ,

$$[\sigma_k^{[N]}, \overline{\gamma}_\ell] = [h_k^{[N]}, \widetilde{\gamma}_\ell^a] = 0 , \tag{5.79}$$

the transformation $\tau_k^{[N]} \gamma_\ell^a (\tau_k^{[N]})^{-1}$ is decomposed as

$$\tau_k^{[N]} \gamma_\ell^a (\tau_k^{[N]})^{-1} = \left\{ \sigma_k^{[N]} \widetilde{\gamma}_\ell^a (\sigma_k^{[N]})^{-1} \right\} \left\{ h_k^{[N]} \overline{\gamma}_\ell (h_k^{[N]})^{-1} \right\} . \tag{5.80}$$

Therefore, $\widetilde{\gamma}_l^a$ and $\overline{\gamma}_l$ are transformed by $\sigma_k^{[N]}$ and $h_k^{[N]}$, respectively. From Eqs. (5.56) and (5.76), $\widetilde{\gamma}_k^a$ is transformed as

$$\sigma_k^{[N]} \widetilde{\gamma}_k^a (\sigma_k^{[N]})^{-1} = \widetilde{\gamma}_{k+1}^a, \tag{5.81}$$

$$\sigma_k^{[N]} \widetilde{\gamma}_{k+1}^a (\sigma_k^{[N]})^{-1} = \widetilde{\gamma}_k^a, \tag{5.82}$$

$$\sigma_k^{[N]} \widetilde{\gamma}_\ell^a (\sigma_k^{[N]})^{-1} = \widetilde{\gamma}_\ell^a \qquad (\ell \neq k, \, k+1) , \tag{5.83}$$

without a minus sign. Thus, the operator $\sigma_k^{[N]}$ acting on $\widetilde{\gamma}_l^a$ reproduces the transformation (5.77). We note that $\sigma_k^{[N]}$ can be expressed in terms of $\widetilde{\gamma}_k^a$ only. On the other hand, $\overline{\gamma}_k$ is transformed by the operator $h_k^{[N]}$ like a single Majorana fermion as demonstrated in Eqs. (5.60)-(5.62), and hence $h_k^{[N]}$ reproduces the transformation (5.78).

To summarize this subsection, in correspondence to the product of $\tau_k^{[N]} = \sigma_k^{[N]} h_k^{[N]}$, the Majorana operator γ_k^a is also expressed by the product of the two parts, $\widetilde{\gamma}_k^a$ obeying the Coxeter group given by $\sigma_k^{[N]}$ and $\overline{\gamma}_k$ obeying Ivanov's exchange given by $h_k^{[N]}$.

---

[6] In general, a composite operator made by an even (odd) number of the Majorana operators is transformed as in Eq. (5.77) (Eq. (5.78)).

[7] If one considers the case when N is an even number, one may define a composite operator by $\overline{\gamma}_k' \equiv \frac{1}{N!} e^{i \frac{\pi}{4} N} \gamma_k^1 \gamma_k^2 \cdots \gamma_k^N$. The operator is self-conjugate $(\overline{\gamma}_k')^\dagger = \overline{\gamma}_k'$, but does not satisfy the Clifford algebra. Furthermore, since N is even, the operation T_k gives the transformation, $\overline{\gamma}_k' \to \overline{\gamma}_{k+1}'$, $\overline{\gamma}_{k+1}' \to \overline{\gamma}_k'$ and the rest $\overline{\gamma}_\ell'$ ($\ell \neq k$ and $k+1$) unchanged. Hence the composite operator $\overline{\gamma}_k'$ for even N does not transform like a singlet Majorana fermion operator.

5.5 Operator decomposition and matrix representation

So far, we have been discussing the factorized structure of the exchange operation of two vortices at the operator level. Everything was written in terms of the Majorana operators, and the decomposition into the Coxeter and Ivanov parts was naturally understood by using the Majorana operators. In contrast, in order to obtain the matrix representation, the usual procedure is to define Dirac fermion operators and use them in constructing the Hilbert space. Since the Dirac fermion operators are constructed from two Majorana fermions located separately at different vortices, it is not trivial if the factorized structure at the operator level is preserved in the matrix representation. For example, the Dirac fermion operator defined in Eq. (5.39) can not be decomposed similarly as the Majorana fermion operator as shown in Eq. (5.75). In this section, we are going to show that the decomposition holds even in the matrix representation in a suitable basis, and discuss the relation between the the decompositions in the operator- and matrix-representation levels.

In the following, we discuss the case with $N = 3$ for simplicity, but the results below can be easily generalized to any odd N.

5.5.1 Construction of the Hilbert space

Let us consider an even number $n = 2m$ of vortices. Then we can construct m Dirac fermion operators Ψ_K^a ($a = 1, 2, 3$; $K = 1, \cdots, m$), given in Eq. (5.39), in the vector representation of $SO(3)$. The Fock vacuum $|0\rangle$ is defined by the Dirac fermion operators as

$$\Psi_K^a |0\rangle = 0, \quad \text{for all } K \text{ and } a. \tag{5.84}$$

One can construct the basis of the Hilbert space by acting the Dirac fermion operators $\Psi_K^{a\dagger}$ on the Fock vacuum $|0\rangle$. The explicit forms of the basis were given in Ref. [59] for $N = 3$ and $n = 2, 4$. Now let us construct the Hilbert space in a way different from Ref. [59]. To this end, we define the number operator for the triplet Dirac fermions of the K-th pair of vortices by

$$\mathcal{N}_K^a \equiv \Psi_K^{a\dagger} \Psi_K^a. \tag{5.85}$$

A generic state for the K-th pair of vortices can be expressed in terms of the eigenvalues of this number operator as

$$|\mathcal{N}_K^1, \mathcal{N}_K^2, \mathcal{N}_K^3\rangle_K \tag{5.86}$$

where $\mathcal{N}_K^a = 0$ or 1 is the occupation number of the fermion created by $\Psi_K^{a\dagger}$ (here we use the same character for the operator and its eigenvalues). Then the basis of the whole Hilbert space is composed of the tensor product of the states for each K,

$$\left\{ \bigotimes_{K=1}^m |\mathcal{N}_K^1, \mathcal{N}_K^2, \mathcal{N}_K^3\rangle_K \right\}. \tag{5.87}$$

5.5.2 Singlet Dirac operators

When $N = 3$, the singlet Majorana operators $\overline{\gamma}_k$ defined in Eq. (5.58) are given by

$$\overline{\gamma}_k = \frac{1}{3!} i\epsilon^{abc} \gamma_k^a \gamma_k^b \gamma_k^c. \qquad (5.88)$$

By using these operators, we define singlet Dirac operators

$$\overline{\Psi}_K \equiv \frac{1}{2} \left(\overline{\gamma}_{2K-1} + i\overline{\gamma}_{2K} \right), \quad \overline{\Psi}_K^\dagger \equiv \frac{1}{2} \left(\overline{\gamma}_{2K-1} - i\overline{\gamma}_{2K} \right). \qquad (5.89)$$

These operators play a particular role when they act on the states. Let us examine the action of the singlet Dirac operators $\overline{\Psi}_K$'s and $\overline{\Psi}_K^\dagger$'s on the Hilbert space (5.87), which is constructed by acting the triplet Dirac operators $\Psi_K^{a\dagger}$'s and Ψ_K^a's on the Fock vacuum $|0\rangle$. To this end, we express the singlet Dirac operators $\overline{\Psi}_K$ and $\overline{\Psi}_K^\dagger$ in terms of the triplet Dirac operators Ψ_K^a and $\Psi_K^{a\dagger}$. For that purpose, we note that the singlet Majorana operators $\overline{\gamma}_{2K-1}$ and $\overline{\gamma}_{2K}$ defined in Eq. (5.58) can be written in two ways:

$$\begin{aligned}
\overline{\gamma}_{2K-1} &= \overline{\Psi}_K + \overline{\Psi}_K^\dagger \\
&= \frac{1}{3!} i\epsilon^{abc} (\Psi_K^a + \Psi_K^{a\dagger})(\Psi_K^b + \Psi_K^{b\dagger})(\Psi_K^c + \Psi_K^{c\dagger}), \\
\overline{\gamma}_{2K} &= (\overline{\Psi}_K - \overline{\Psi}_K^\dagger)/i \\
&= \frac{1}{3!} i\epsilon^{abc} \left(\frac{1}{i} \right)^3 (\Psi_K^a - \Psi_K^{a\dagger})(\Psi_K^b - \Psi_K^{b\dagger})(\Psi_K^c - \Psi_K^{c\dagger}),
\end{aligned} \qquad (5.90)$$

respectively. From these two relations, we can express the singlet Dirac operators $\overline{\Psi}_K$ and $\overline{\Psi}_K^\dagger$ in terms of the triplet Dirac operators Ψ_K^a and $\Psi_K^{a\dagger}$ as

$$\begin{aligned}
\overline{\Psi}_K &= \frac{1}{3!} i\epsilon^{abc} \left(\Psi_K^a \Psi_K^b \Psi_K^{c\dagger} + \Psi_K^a \Psi_K^{b\dagger} \Psi_K^c + \Psi_K^{a\dagger} \Psi_K^b \Psi_K^c + \Psi_K^{a\dagger} \Psi_K^{b\dagger} \Psi_K^{c\dagger} \right), \\
\overline{\Psi}_K^\dagger &= -\frac{1}{3!} i\epsilon^{abc} \left(\Psi_K^{a\dagger} \Psi_K^{b\dagger} \Psi_K^c + \Psi_K^{a\dagger} \Psi_K^b \Psi_K^{c\dagger} + \Psi_K^a \Psi_K^{b\dagger} \Psi_K^{c\dagger} + \Psi_K^a \Psi_K^b \Psi_K^c \right).
\end{aligned} \qquad (5.91)$$

For the later convenience, we define the fermion number operator for the K-th pair of vortices in terms of the singlet Dirac operators $\overline{\Psi}_K^\dagger$ and $\overline{\Psi}_K$:

$$\overline{\mathcal{N}}_K \equiv \overline{\Psi}_K^\dagger \overline{\Psi}_K. \qquad (5.92)$$

The meaning of this number operator will be clarified below.

Now let us see how the singlet Dirac operators $\overline{\Psi}_K$ and $\overline{\Psi}_K^\dagger$ act on the Fock states (5.86) defined by the triplet Dirac operators Ψ_K^a and $\Psi_K^{a\dagger}$, which has a particular intuitive meaning. The action of the singlet Dirac operators $\overline{\Psi}_K$ and $\overline{\Psi}_K^\dagger$ on $|\mathcal{N}_K^1, \mathcal{N}_K^2, \mathcal{N}_K^3\rangle_K$ can be read off from Eqs. (5.91) as

$$\begin{aligned}
&\overline{\Psi}_K |0, 0, 0\rangle_K = |1, 1, 1\rangle_K, \\
&\overline{\Psi}_K |1, 0, 0\rangle_K = \overline{\Psi}_K |0, 1, 0\rangle_K = \overline{\Psi}_K |0, 0, 1\rangle_K = 0, \\
&\overline{\Psi}_K |0, 1, 1\rangle_K = |1, 0, 0\rangle_K, \quad \overline{\Psi}_K |1, 0, 1\rangle_K = |0, 1, 0\rangle_K, \quad \overline{\Psi}_K |1, 1, 0\rangle_K = |0, 0, 1\rangle_K, \\
&\overline{\Psi}_K |1, 1, 1\rangle_K = 0,
\end{aligned} \qquad (5.93)$$

for $\overline{\Psi}_K$, and

$$\overline{\Psi}_K^\dagger |0,0,0\rangle_K = 0,$$
$$\overline{\Psi}_K^\dagger |0,0,1\rangle_K = |1,1,0\rangle_K, \quad \overline{\Psi}_K^\dagger |0,1,0\rangle_K = |1,0,1\rangle_K, \quad \overline{\Psi}_K^\dagger |1,0,0\rangle_K = |0,1,1\rangle_K,$$
$$\overline{\Psi}_K^\dagger |0,1,1\rangle_K = \overline{\Psi}_K^\dagger |1,0,1\rangle_K = \overline{\Psi}_K^\dagger |1,1,0\rangle_K = 0,$$
$$\overline{\Psi}_K^\dagger |1,1,1\rangle_K = |0,0,0\rangle_K, \tag{5.94}$$

for $\overline{\Psi}_K^\dagger$.

Let us define the total fermion number operator for the K-th pair of vortices as the sum of the number operator $\mathcal{N}_K^a$ in Eq. (5.85) of the triplet Dirac fermions:

$$\mathcal{F}_K \equiv \sum_{a=1}^{N} \mathcal{N}_K^a. \tag{5.95}$$

Then we can deduce that the states annihilated by $\overline{\Psi}_K$ are those with odd eigenvalues of $\mathcal{F}_K$ (see the second and fourth lines of Eq. (5.93)), while those with even eigenvalues of $\mathcal{F}_K$ are annihilated by $\overline{\Psi}_K^\dagger$ (see the first and third lines of Eq. (5.94)):

$$\overline{\Psi}_K^\dagger |\text{even}\rangle_K = 0, \quad \overline{\Psi}_K |\text{odd}\rangle_K = 0, \tag{5.96}$$

where $|\text{even}\rangle_K$ and $|\text{odd}\rangle_K$ are eigenstates of the parity operator

$$\mathcal{P}_K \equiv (-1)^{\mathcal{F}_K}, \tag{5.97}$$

namely,

$$\mathcal{P}_K |\text{even}\rangle_K = +1 |\text{even}\rangle_K, \quad \mathcal{P}_K |\text{odd}\rangle_K = -1 |\text{odd}\rangle_K. \tag{5.98}$$

It should also be noted that, when $\overline{\Psi}_K^\dagger$ and $\overline{\Psi}_K$ do not annihilate the state, they create a state with opposite parity:

$$\overline{\Psi}_K^\dagger |\text{odd}\rangle_K = |\text{even}\rangle_K, \quad \overline{\Psi}_K |\text{even}\rangle_K = |\text{odd}\rangle_K. \tag{5.99}$$

These facts imply that the parity operator $\mathcal{P}_K$ anticommutes with $\overline{\Psi}_K$ and $\overline{\Psi}_K^\dagger$:

$$\{\mathcal{P}_K, \overline{\Psi}_K\} = 0, \quad \{\mathcal{P}_K, \overline{\Psi}_K^\dagger\} = 0. \tag{5.100}$$

We claim that the fermion number operator $\overline{\mathcal{N}}_K$ is related with the parity operator $\mathcal{P}_K$ as

$$\mathcal{P}_K = 1 - \overline{\mathcal{N}}_K. \tag{5.101}$$

Namely, the fermion number operator $\overline{\mathcal{N}}_K$ expresses the parity of the total fermion number for each index K. This relation results from Eq. (5.96), which is obvious from the structure of the operators shown in Eqs. (5.91).

By repeating the same argument, the above relation can be generalized to an arbitrary odd N as

$$\mathcal{P}_K = 1 - \overline{\mathcal{N}}_K \quad \text{for} \quad N = 4\ell + 3, \tag{5.102}$$
$$\mathcal{P}_K = \overline{\mathcal{N}}_K \quad \text{for} \quad N = 4\ell + 1. \tag{5.103}$$

5.5.3 The tensor product structure of the Hilbert space

From Eqs. (5.93) and (5.94), we can say that the action of the singlet Dirac operators $\overline{\Psi}_K$ or $\overline{\Psi}_K^\dagger$ is a kind of "NOT" operation, which "flips" the fermion number for each component of the $SO(3)$ vector. Namely, under the action of $\overline{\Psi}_K$, one finds $|\mathcal{N}_K^1, \mathcal{N}_K^2, \mathcal{N}_K^3\rangle_K \to |1 - \mathcal{N}_K^1, 1 - \mathcal{N}_K^2, 1 - \mathcal{N}_K^3\rangle_K$. We can divide the states into pairs, in each of which the two states are related by the NOT operation. When $N = 3$, there are four pairs:

$$|0,0,0\rangle_K \leftrightarrow |1,1,1\rangle_K, \qquad |0,0,1\rangle_K \leftrightarrow |1,1,0\rangle_K,$$
$$|0,1,1\rangle_K \leftrightarrow |1,0,0\rangle_K, \qquad |1,0,1\rangle_K \leftrightarrow |0,1,0\rangle_K. \tag{5.104}$$

It is important to note that the singlet Dirac operators $\overline{\Psi}_K$ and $\overline{\Psi}_K^\dagger$ induce the transition only within these pairs when they act on states (e.g. a transition between $|0,0,0\rangle_K$ and $|1,1,1\rangle_K$), but they never induce an inter-pair transition. This fact is essential in the following discussion.

Now we are ready to discuss how the decomposition of the exchange operator $\tau_k^{[N]}$ results in the tensor-product structure in a matrix representation. From the analysis above, we can take the basis of the Hilbert space which is labeled by the parity $\mathcal{P}_K$ of the number of fermions with index K, and an additional index m_K which labels the choice of a pair in Eq. (5.104). We denote the states by

$$|\mathcal{P}_K, m_K\rangle_K \equiv |\mathcal{P}_K\rangle_K \otimes |m_K\rangle_K. \tag{5.105}$$

Let us consider the matrix elements $_K\langle \mathcal{P}_K, m_K | \tau_k^{[N]} | \mathcal{P}_K', m_K'\rangle_K$ of the exchange operator $\tau_k^{[N]}$ in this basis. Now we show that these matrix elements can be written as the tensor product of the Ivanov matrix and the Coxeter matrix:

$$_K\langle \mathcal{P}_K, m_K | \tau_k^{[N]} | \mathcal{P}_K', m_K'\rangle_K = {}_K\langle m_K | \sigma_k^{[N]} | m_K'\rangle_K \cdot {}_K\langle \mathcal{P}_K | h_k^{[N]} | \mathcal{P}_K'\rangle_K. \tag{5.106}$$

Namely, the Coxeter operator $\sigma_k^{[N]}$ acts only on $|m_K\rangle_K$, while the Ivanov operator $h_k^{[N]}$ acts only on $|\mathcal{P}_K\rangle_K$. To prove this, we show the following two statements for any k and K:

(i) $h_k^{[N]}$ acts as an identity on $|m_K\rangle_K$.

(ii) $\sigma_k^{[N]}$ acts as an identity on $|\mathcal{P}_K\rangle_K$.

Proof of (i).

We can see that the statement (i) is true in the following way. The Ivanov operator $h_k^{[N]}$ is written by the singlet Majorana operators $\overline{\gamma}_k$ (see Eq. (5.59)), and hence by the singlet Dirac operators $\overline{\Psi}_L$ and $\overline{\Psi}_L^\dagger$, see Eq. (5.89). When the singlet Dirac operators $\overline{\Psi}_L$ or $\overline{\Psi}_L^\dagger$ act on states, the only state that can be created is the one in which the fermion numbers are flipped. So, the action of the singlet Dirac operators $\overline{\Psi}_L$ or $\overline{\Psi}_L^\dagger$ never induce an inter-pair transition. Therefore, the Ivanov operator $h_k^{[N]}$ does not change the index m_K, which labels the pair in Eq. (5.104).

Proof of (ii).

The statement (ii) is equivalent to the statement that the Coxeter operator $\sigma_k^{[N]}$ and the total fermion number operator $\overline{\mathcal{N}}_K$ commutes. The total fermion number operator $\overline{\mathcal{N}}_K$ is written by the singlet Majorana operators $\overline{\gamma}_{2K-1}$ and $\overline{\gamma}_{2K}$, which commute with the Coxeter operator $\sigma_k^{[N]}$ as in Eq. (5.79). We thus find that the Coxeter operator $\sigma_k^{[N]}$ and the total fermion number operator $\overline{\mathcal{N}}_K$ also commute.

By showing the statements (i) and (ii), we have shown that the factorization of the exchange operator $\tau_k^{[N]}$ of vortices into the Ivanov operator $h_k^{[N]}$ and the Coxeter operator $\sigma_k^{[N]}$ results in the tensor-product structure in the matrix representation.

5.6   Comments

The $SO(N)$ symmetry considered in this paper is the largest symmetry group in the presence of N Majorana fermions. Whether the symmetry is $SO(N)$ or its subgroups depends on the details of the systems. For instance, a higher (pseudo-)spin S representation of $SO(3)$ contains $2S + 1$ Majorana fermions, but the symmetry acting on them does not have to be $SO(2S + 1)$. Also the representation does not have to be irreducible; for instance four Majorana fermions may be decomposed into one singlet and one triplet of $SO(3)$, but the symmetry group does not have to be $SO(4)$. It will be interesting to extend our results to general representations including reducible representations. An extension to general groups also remains as an interesting future problem to be explored.

For vortices with an even number N of Majorana fermions with the $SO(N)$ symmetry, we have not found any meaningful factorization of the exchange operator $\tau_k^{[N]}$ so far. It remains as a future problem to identify the non-Abelian statistics for even N Majorana fermions. When the symmetry group inside the vortex core is restricted to the unitary subgroup $U(N/2) \subset SO(N)$, N Majorana fermions can be rearranged into $N/2$ complex Dirac fermions in each vortex. In this case, the situation is rather different because Dirac fermions are locally defined, and we do not need to define Dirac fermions non-locally by using two spatially separated vortices. Nevertheless we found a rather different kind of non-Abelian statistics in the case of $N = 2$ with the $U(1)$ symmetry [128] and $N = 4$ with the $U(2)$ symmetry [129]. Identifying the statistics for the general even N of $N/2$ Dirac fermions also remains as a future problem.

Finally, it will be important to look for actual condensed matter systems realizing multiple Majorana fermions in the vortex core, and to study an impact of our results on applications to topological quantum computations.

5.7   Brief summary

We have considered non-Abelian statistics of vortices, each of which has N Majorana zero-energy states inside its core on which an $SO(N)$ symmetry acts. We have investigated how the degenerate states induced by zero modes are transformed under an exchange of neighboring vortices. We have

shown that, for an arbitrary odd N, the exchange operator $\tau_k^{[N]}$ defined in Eq. (5.35), generating the exchange of two neighboring vortices, can be factorized into two parts $\tau_k^{[N]} = \sigma_k^{[N]} h_k^{[N]}$ as seen in Eq. (5.55). The part which is given by $h_k^{[N]}$ defined in Eq. (5.57) is essentially equivalent to the exchange operator introduced by Ivanov. If it is expressed in terms of the composite singlet Majorana operator $\overline{\gamma}_k$ defined in Eq. (5.58), then it has the same form as the exchange operator τ_k in the case of the single Majorana fermion. The other operator $\sigma_k^{[N]}$ defined in Eq. (5.56) is a new part. In Sec. 5.5, we have given a proof that they constitute the Coxeter group of the type A_{2m-1} (the symmetric group S_{2m}) for $n = 2m$ vortices. We have also shown in Sec. 5.5 that the factorization of the exchange operators results in the tensor-product structure in its matrix representation in a suitable basis.

Chapter 6

Summary and concluding remarks

In this thesis, we have studied the dynamics of non-Abelian vortices, that appear in a color supercon-ductor. Summary of the original results is described below.

In Chap. 4, we have discussed the interaction of non-Abelian vortices with quasiparticles in the color-superconducting medium. In Sec. 4.1, we discussed the interaction of vortices with phonons, which are the Nambu-Goldstone mode associated with the breaking of the $U(1)_\mathrm{B}$ symmetry, and gluons. The interaction Lagrangian is obtain by using the dual transformation. We have found that the interaction with gluons is dependent on the orientation of a vortex. We studied the orientation-dependent interaction energy between two vortices. In Sec. 4.2, we discussed the interaction of vor-tices with CFL mesons, which are the Nambu-Goldstone bosons for the breaking of chiral symmetry. In Sec. 4.3, we investigated the interaction of vortices with photons and its phenomenological con-sequences. The orientational zero modes localized on vortices are charged with respect to $U(1)_\mathrm{EM}$ symmetry. The interaction Lagrangian is determined by symmetry consideration. Based on the ef-fective Lagrangian, we discuss the scattering of photons off a vortex. We also discussed the optical property of a vortex lattice , which is expected to be formed if CFL matter exists inside the core of a rotating dense star. We have shown that a lattice of vortices serves as a polarizer of photons.

In Chap. 5, we have discussed the non-Abelian statistics of vortices, which occurs because of the Majorana zero modes inside vortices. After an introduction about the fermion zero modes in Sec. 5.1, we discussed why Majorana fermions appear in a superconductor in Sec. 5.2. In Sec. 5.3, we discussed basics of the non-Abelian statistics. We have investigated how the degenerate states induced by zero modes are transformed under an exchange of neighboring vortices in the case where a vortex core captures one Majorana fermion. We have also extended the discussion of non-Abelian statistics to the case where an odd number (N) of fermions reside inside a vortex. In Sec. 5.4 We have shown that the exchange operator $\tau_k^{[N]}$ defined in Eq. (5.35), generating the exchange of two neighboring vortices, can be factorized into two parts $\tau_k^{[N]} = \sigma_k^{[N]} h_k^{[N]}$ as seen in Eq. (5.55). The part which is given by $h_k^{[N]}$ defined in Eq. (5.57) is essentially equivalent to the exchange operator introduced by Ivanov. If it is expressed in terms of the composite singlet Majorana operator $\overline{\gamma}_k$ defined in Eq. (5.58), then it has the same form as the exchange operator τ_k in the case of the single Majorana fermion. The other operator $\sigma_k^{[N]}$ defined in Eq. (5.56) is a new part. In Sec. 5.5, we have discussed relation between

the operator decomposition and the tensor-product structure in the Matrix representation. We have given a proof that they constitute the Coxeter group of the type A_{2m-1} (the symmetric group S_{2m}) for $n = 2m$ vortices. We have also shown that the factorization of the exchange operators results in the tensor-product structure in its matrix representation in a suitable basis.

Before closing this thesis, let us comment on the future directions on this topic. There are lots of problems regarding the non-Abelian vortices to be solved as stated in the discussion of Ref. [18]. Several fundamental problems on the properties of vortices are left, such as reconnection of vortices, breaking up of a $U(1)_B$ vortex into three non-Abelian vortices, dynamics of a non-Abelian vortex loop, or effects of $U(1)_A$ anomaly on the vortices. From phenomenological view point, one important direction is the application of the results here to the neutron star physics. We believe that the vortices can be the witness of the quark matter. Despite the extensive theoretical studies on the color super-conductivity, there has been no observational evidence that quark matter exists inside dense stars. If color superconducting phase is realized inside the core of a neutron star, it should be threaded with numerous vortices since the star is rapidly rotating. It is important to investigate phenomena caused by vortices that could give some observational clue. For example, neutron stars are known to have strong magnetic fields on their surface. Because of the electromagnetic charge of the orientational modes, vortex interacts with the magnetic fields. The vortices and magnetic fields pushes each other. That could affect the electromagnetic radiations from the stars, that are observable from the earth. Another example would be pulsar glitches. A pulsar glitch is a rapid increase in the rotational period of neutron stars observed intermittently. It is suggested by Anderson and Itoh [130] that unpinning of vortices in neutron superfluid is triggering the glitches. If non-Abelian vortices exist inside neutron stars, the discussion of glitches would be modified. For example, non-Abelian vortices in CFL matter should be connected to superfluid vortices in the neutron superfluid in such a way that the circulations remain the same. This would affect the suggested glitch mechanism. Regarding the non-Abelian statistics of vortices, it is interesting to find condensed-matter realizations of vortices with multiple Majorana fermions, since the discussion in Chap. 5 is based on the fact that a vortex hosts multiple Majorana fermions and does not depend on the details of the system. Identifying the topological field theory to describe such systems is an interesting problem. It is also useful to find a way to utilize such systems to actual procedure of topological quantum computation.

Acknowledgments

Appendix A

Derivation of the dual Lagrangians for phonons and gluons

In this appendix, we derive the dual Lagrangians for phonons and gluons [45]. After a dual transformation, massive gluons are described by massive non-Abelian antisymmetric tensor fields [57] and $U(1)_B$ phonons are described by massless antisymmetric tensor fields. In the dual description vortices appear as sources which can absorb or emit these particles.

Low-energy effective theory of the CFL phase

We start with a time-dependent Ginzburg-Landau(GL) effective Lagrangian for the CFL phase, which is given in Eq. (2.23),

$$
\mathcal{L}(x) = \frac{\varepsilon_3}{2} (\mathbf{E}^a)^2 - \frac{1}{2\lambda_3} (\mathbf{B}^a)^2 + K_0 \text{Tr} \left[(D_0 \Phi)^\dagger D^0 \Phi \right] + K_3 \text{Tr} \left[(D_i \Phi)^\dagger D^i \Phi \right]
$$
$$
- 4i\gamma \text{Tr} \left[\Phi^\dagger D_0 \Phi \right] - V(\Phi), \tag{A1}
$$

where $E_i^a = F_{0i}^a$, $B_i^a = \frac{1}{2} \epsilon_{ijk} F_{jk}^a$. The effect of $U(1)_{\text{EM}}$ electromagnetism is neglected here. The parameters ε_3 and λ_3 are the color dielectric constant and the color magnetic permeability. The Lorentz symmetry does not have to be maintained in general since superconducting matter exists. However, the kinetic term of gluons has a modified Lorentz symmetry in which the speed of light is replaced by $1/\sqrt{\varepsilon_3 \lambda_3}$. It is always possible to restore the Lorentz invariance of the kinetic term of gauge fields by rescaling x^0, A_0^a, K_0, γ and K_3. Therefore we can start with the Lagrangian in which ϵ and λ are taken to be unity. For notational convenience, we introduce a vector $K_\mu \equiv (K_0, K_3, K_3, K_3)$. Thus our starting point is the following GL Lagrangian,

$$
\mathcal{L}(x) = -\frac{1}{4} \left(F_{\mu\nu}^a \right)^2 + K_\mu \text{Tr} \left[(D_\mu \Phi)^\dagger D^\mu \Phi \right] - 4i\gamma \text{Tr} \left[\Phi^\dagger D_0 \Phi \right] - V(\Phi). \tag{A2}
$$

The dual transformation

Here we perform dual transformations within the path integral formalism to derive a dual Lagrangian for the CFL phase. After the transformation, massive gluons are described by massive non-Abelian

antisymmetric tensor fields [57] and $U(1)_B$ phonons are described by massless antisymmetric tensor fields. We show that in the dual description vortices appear as sources which can absorb or emit these particles.

The dual transformation of massive gluons

The partition function of the CFL phase can be written as

$$Z = \int \mathcal{D}A_\mu^a(x)\mathcal{D}\Phi(x) \exp\left\{i\int d^4x \mathcal{L}(x)\right\}, \tag{A3}$$

with the Lagrangian defined in Eq. (A2). We shall impose the gauge fixing condition on the field Φ rather than on the gauge fields since they are integrated out in the end. The gauge fixing condition is taken care of when we consider a concrete vortex solution.

We introduce non-Abelian antisymmetric tensor fields $B_{\mu\nu}^a$ by a Hubbard-Stratonovich transformation,

$$\exp\left[i\int d^4x\left\{-\frac{1}{4}(F_{\mu\nu}^a)^2\right\}\right] \propto \int \mathcal{D}B_{\mu\nu}^a \exp\left[i\int d^4x\left\{-\frac{1}{4}\left[m^2(B_{\mu\nu}^a)^2 - 2m\tilde{B}_{\mu\nu}^a F^{a,\mu\nu})\right]\right\}\right], \tag{A4}$$

where $\tilde{B}_{\mu\nu}^a \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}B^{a,\rho\sigma}$. The parameter m introduced above is a free parameter at this stage. We will choose m later so that the kinetic term of $B_{\mu\nu}^a$ is canonically normalized.

Substituting (A4) into (A3), we can now perform the integration over the gauge fields A_μ^a. The degrees of freedom of gluons are expressed by $B_{\mu\nu}^a$ after this transformation. Each term in the Lagrangian is transformed as follows:

$$
\begin{aligned}
& K_\mu \text{Tr}\{(D_\mu\Phi)^\dagger(D^\mu\Phi)\} - 4i\gamma\text{Tr}\left[\Phi^\dagger D_0\Phi\right] \\
&= K_\mu \text{Tr}\left\{\Phi^\dagger(\overleftarrow{\partial}_\mu + ig_s A_\mu^a T^a)(\overrightarrow{\partial}^\mu - ig_s A^{b,\mu}T^b)\Phi\right\} - 4i\gamma\text{Tr}\left[\Phi^\dagger(\partial_0 - ig_s A_\mu^a T^a)\Phi\right] \\
&= K_\mu \text{Tr}\{(\partial_\mu\Phi)^\dagger(\partial^\mu\Phi)\} - 4i\gamma\text{Tr}\left[\Phi^\dagger\partial_0\Phi\right] + g_s A_\mu^a J^{a,\mu} \\
&\quad + g_s^2 g_{\mu\nu}\sqrt{K_\mu K_\nu}A^{a,\mu}A^{b,\nu}\text{Tr}\left[\Phi^\dagger T^a T^b\Phi\right],
\end{aligned} \tag{A5}
$$

with $J_\mu^a \equiv -iK_\mu\text{Tr}\left[\Phi^\dagger(\overleftarrow{\partial}_\mu - \overrightarrow{\partial}_\mu)T^a\Phi\right] + 4\gamma\text{Tr}\left[\Phi^\dagger T^a\Phi\right]$, and

$$
\begin{aligned}
-\frac{1}{2}m\tilde{B}_{\mu\nu}^a F^{a,\mu\nu} &= -\frac{1}{2}m\tilde{B}_{\mu\nu}^a(2\partial_\nu A_\mu^a + g_s f^{abc}A_\mu^b A_\nu^c) \\
&= mA_\mu^a\partial_\nu\tilde{B}_{\mu\nu}^a + \frac{1}{2}mg_s f^{abc}A_\mu^a A_\nu^b \tilde{B}_{\mu\nu}^c.
\end{aligned} \tag{A6}
$$

Performing the integration over A_μ^a, the following part of the partition function is rewritten as

$$
\begin{aligned}
&\int \mathcal{D}A_\mu^a \exp\left\{i\int d^4x\left[\frac{1}{2}g_s^2 A^{a,\mu}K_{\mu\nu}^{ab}A^{b,\nu} - m\left(\partial^\nu\tilde{B}_{\mu\nu}^a - \frac{g_s}{m}J_\mu^a\right)A^{a,\mu}\right]\right\} \\
&\propto (\det K_{\mu\nu}^{ab})^{-1/2}\exp\left\{i\int d^4x\left[-\frac{1}{2}\left(\frac{m}{g_s}\right)^2\left(\partial_\rho\tilde{B}^{a,\mu\rho} - \frac{g_s}{m}J^{a,\mu}\right)(K^{-1})_{\mu\nu}^{ab}\left(\partial_\sigma\tilde{B}^{b,\nu\sigma} - \frac{g_s}{m}J^{b,\nu}\right)\right]\right\},
\end{aligned} \tag{A7}
$$

where $K_{\mu\nu}^{ab}$ is defined by

$$K_{\mu\nu}^{ab} = \frac{1}{2}g_{\mu\nu}\sqrt{K_\mu K_\nu}\,\mathrm{Tr}\left[\Phi^\dagger T^a T^b \Phi\right] - \frac{m}{g_\mathrm{s}}f^{abc}\tilde{B}_{\mu\nu}^c$$
$$\equiv \boldsymbol{\Phi}_{\mu\nu}^{ab} - \frac{m}{g_\mathrm{s}}\hat{B}_{\mu\nu}^{ab}, \tag{A8}$$

with $\boldsymbol{\Phi}_{\mu\nu}^{ab} \equiv \frac{1}{2}g_{\mu\nu}\sqrt{K_\mu K_\nu}\,\mathrm{Tr}\left[\Phi^\dagger T^a T^b \Phi\right]$ and $\hat{B}_{\mu\nu}^{ab} \equiv f^{abc}\tilde{B}_{\mu\nu}^c$. We define the inverse of $K_{\mu\nu}^{ab}$ by the power-series expansion in $1/g_\mathrm{s}$

$$K^{-1} = \left(\boldsymbol{\Phi} - \frac{m}{g_\mathrm{s}}\hat{B}\right)^{-1} = \boldsymbol{\Phi}^{-1}\sum_{n=0}^{\infty}\left(\frac{m}{g_\mathrm{s}}\hat{B}\boldsymbol{\Phi}^{-1}\right)^n. \tag{A9}$$

As a result, we obtain the following partition function

$$Z \propto \int \mathcal{D}B_{\mu\nu}^a (\det K_{\mu\nu}^{ab})^{-1/2}\exp\left\{i\int d^4x \mathcal{L}_\mathrm{G}^*(x)\right\}, \tag{A10}$$

where $\mathcal{L}_\mathrm{G}^*$ denotes the gluonic part of the dual Lagrangian

$$\mathcal{L}_\mathrm{G}^* = -\frac{1}{2}\left(\frac{m}{g_\mathrm{s}}\right)^2\left(\partial_\rho\tilde{B}^{a,\mu\rho} - \frac{g_\mathrm{s}}{m}J^{a,\mu}\right)\left(K^{-1}\right)_{\mu\nu}^{ab}\left(\partial_\sigma\tilde{B}^{b,\nu\sigma} - \frac{g_\mathrm{s}}{m}J^{b,\nu}\right) - \frac{1}{4}m^2(B_{\mu\nu}^a)^2. \tag{A11}$$

Now we define the non-Abelian vorticity tensor $\omega_{\mu\nu}^a$ as the coefficient of the term linearly proportional to $B_{\mu\nu}^a$. Collecting relevant terms in the above Lagrangian, the coupling between massive gluons and the vorticity is given by

$$\mathcal{L}_\mathrm{G}^* \supset \frac{1}{2}\frac{m}{g_\mathrm{s}}\left[\partial_\rho\tilde{B}^{a,\mu\rho}(\boldsymbol{\Phi}^{-1})_{\mu\nu}^{ab}J^{b,\nu} + J^{a,\mu}(\boldsymbol{\Phi}^{-1})_{\mu\nu}^{ab}\partial_\rho\tilde{B}^{b,\nu\rho}\right] - \frac{1}{2}\left(\frac{m}{g_\mathrm{s}}\right)J^{a,\mu}[\boldsymbol{\Phi}^{-1}\hat{B}\boldsymbol{\Phi}^{-1}]_{\mu\nu}^{ab}J^{b,\nu}$$
$$\equiv -\frac{1}{2}\left(\frac{m}{g_\mathrm{s}}\right)B_{\lambda\sigma}^a\omega^{a,\lambda\sigma}, \tag{A12}$$

where we have defined the vorticity tensor $\omega_{\mu\nu}^a$ as

$$\omega^{a,\lambda\sigma} \equiv \epsilon^{\lambda\sigma\mu\nu}\left[\partial_\nu\left\{(\boldsymbol{\Phi}^{-1})\binom{ab}{\mu\rho}J^{b,\rho}\right\} + J^{e,\alpha}(\boldsymbol{\Phi}^{-1})_{\alpha\mu}^{ec}f^{cda}(\boldsymbol{\Phi}^{-1})_{\nu\beta}^{db}J^{b,\beta}\right]. \tag{A13}$$

Here $A\binom{ab}{\mu\nu}$ is a symmetrized summation defined by $A\binom{ab}{\mu\nu} \equiv A_{\mu\nu}^{ab} + A_{\nu\mu}^{ba}$. This expression for the non-Abelian vorticity is valid for general vortex configurations. The information of vortex configuration is included in Φ and J_μ^a.

The dual transformation of $U(1)_\mathrm{B}$ phonons

In the following, we perform a dual transformation of the NG boson associated with the breaking of $U(1)_\mathrm{B}$ symmetry. This mode corresponds to the fluctuation of the overall phase of Φ which can be parametrized as $\Phi(x) = e^{i\pi(x)}\psi(x)$, where $\pi(x)$ is a real scalar field. Substituting this into the following part in the Lagrangian (A2) leads to [1]

$$K_\mu\mathrm{Tr}\{(\partial_\mu\Phi)^\dagger(\partial^\mu\Phi)\} - 4i\gamma\mathrm{Tr}\{\Phi^\dagger\partial_0\Phi\} =$$
$$K_\mu\left(\partial_\mu\pi\right)^2 M^2 - \partial^\mu\pi J_\mu^0 + K_\mu\mathrm{Tr}(\partial_\mu\psi)^2 - 4i\gamma\mathrm{Tr}\{\psi^\dagger\partial_0\psi\}, \tag{A14}$$

[1] The term $\mathrm{Tr}\left[\partial_\mu\psi^\dagger\psi - \psi^\dagger\partial_\mu\psi\right]$ automatically vanishes since ψ can be decomposed as $\psi = (\Delta + \rho)\mathbf{1}_N + (\chi^a + i\zeta^a)T^a$ and the modes ζ^a are absorbed by gluons.

with $J_\mu^0 \equiv -4\delta_{\mu 0}\gamma M^2$ and $M^2 \equiv \mathrm{Tr}\left[\psi^\dagger\psi\right]$. We will transform the $U(1)_{\mathrm{B}}$ phonon field $\pi(x)$ into a massless two-form field $B_{\mu\nu}^0$. Note that the field $\pi(x)$ has a multivalued part in general; since $\pi(x)$ is the phase degree of freedom, $\pi(x)$ can be multivalued without violating the single-valuedness of $\Phi(x)$. In fact the multivalued part of $\pi(x)$ corresponds to a vortex. Let us denote the multivalued part of $\pi(x)$ as $\pi_{\mathrm{MV}}(x)$.

The dual transformation of this $U(1)_{\mathrm{B}}$ phonon field is essentially the same as the case of a superfluid. We basically follow the argument of [104]. Let us introduce an auxiliary field C_μ by linearizing the kinetic term of $\pi(x)$ in the partition function as follows

$$
\begin{aligned}
Z &\propto \int \mathcal{D}\pi \mathcal{D}\pi_{\mathrm{MV}} \exp i \left[\int d^4 x \left(M^2 K_\mu \{\partial_\mu(\pi + \pi_{\mathrm{MV}})\}^2 - \partial^\mu(\pi + \pi_{\mathrm{MV}})J_\mu^0\right)\right] \\
&\propto \int \mathcal{D}\pi \mathcal{D}\pi_{\mathrm{MV}} \mathcal{D}C_\mu \exp i \left[\int d^4 x \left(-\frac{C_\mu^2}{M^2} - 2C_\mu \sqrt{K_\mu}\partial^\mu(\pi + \pi_{\mathrm{MV}}) - \partial^\mu(\pi + \pi_{\mathrm{MV}})J_\mu^0\right)\right].
\end{aligned}
$$
(A15)

Integration over $\pi(x)$ gives a delta function

$$
\int \mathcal{D}\pi \exp i \left[\int d^4 x \left(-2C_\mu \sqrt{K_\mu}\partial^\mu \pi + \pi\partial^\mu J_\mu^0\right)\right] = \delta\left\{\partial^\mu\left(2C_\mu\sqrt{K_\mu} + J_\mu^0\right)\right\}.
$$
(A16)

Then let us introduce the dual antisymmetric tensor field $B_{\mu\nu}^0$ by

$$
\int \mathcal{D}C_\mu \delta\left\{\partial^\mu\left(2C_\mu\sqrt{K_\mu} + J_\mu^0\right)\right\}\cdots = \int \mathcal{D}C_\mu \mathcal{D}B_{\mu\nu}^0 \delta\left(2C_\mu\sqrt{K_\mu} + J_\mu^0 - m^0\partial^\nu\tilde{B}_{\mu\nu}^0\right)\cdots
$$
(A17)

where the dots denote the rest of the integrand and m^0 is a parameter. By this change of variables we have introduced an infinite gauge volume, corresponding to the transformation $\delta B_{\mu\nu}^0 = \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu$ with a massless vector field Λ_μ. This can be taken care of by fixing the gauge later. There is no nontrivial Jacobian factor as the change of variables is linear. Integrating over C_μ, and transforming a resultant term in the Lagrangian as

$$
\begin{aligned}
m^0\partial^\nu\tilde{B}_{\mu\nu}^0\partial^\mu\pi_{\mathrm{MV}} &= -m^0 B^{0,\rho\sigma}\epsilon_{\mu\nu\rho\sigma}\partial^\nu\partial^\mu\pi_{\mathrm{MV}} \\
&\equiv -2\pi m^0 B^{0,\rho\sigma}\omega_{\rho\sigma}^0,
\end{aligned}
$$
(A18)

where the first equality holds up to a total derivative and we have defined

$$
\omega_{\rho\sigma}^0 \equiv \frac{1}{2\pi}\epsilon_{\mu\nu\rho\sigma}\partial^\nu\partial^\mu\pi_{\mathrm{MV}}.
$$
(A19)

We thus obtain the dual Lagrangian for the $U(1)_{\mathrm{B}}$ phonon part

$$
\mathcal{L}_{\mathrm{Ph}}^* = -\left(\frac{1}{2M}\right)^2 K_\mu(m^0\partial_\nu\tilde{B}_{\mu\nu}^0 - J_\mu^0)^2 - 2\pi m^0 B^{0,\mu\nu}\omega_{\mu\nu}^0.
$$
(A20)

Note that the term linear in $B_{\mu\nu}^0$ coming from the first term of (A20) is a total derivative and does not contribute to the equation of motion. The partition function is proportional to

$$
Z \propto \int \mathcal{D}\pi_{\mathrm{MV}}\mathcal{D}B_{\mu\nu}^0 \exp i \left[\int d^4 x \mathcal{L}_{\mathrm{Ph}}^*\right].
$$
(A21)

The $U(1)_{\mathrm{B}}$ phonons are now described by a massless two-form field $B_{\mu\nu}^0$ and vortices appear as sources for $B_{\mu\nu}^0$.

The dual Lagrangian

In summary, we have shown that the partition function Z of the CFL phase is proportional to Z^* with the dual Lagrangian $\mathcal{L}^*$:

$$Z \propto Z^* = \int \mathcal{D}B_{\mu\nu}^a \mathcal{D}\pi_{\mathrm{MV}} \mathcal{D}B_{\mu\nu}^0 \mathcal{D}\psi \, (\det K_{\mu\nu}^{ab})^{-1/2} \exp\left\{ i \int d^4x \mathcal{L}^*(x) \right\}, \tag{A22}$$

where

$$\mathcal{L}^* = \mathcal{L}_{\mathrm{G}}^* + \mathcal{L}_{\mathrm{Ph}}^* + K_\mu \mathrm{Tr}(\partial_\mu \psi)^2 - 4i\gamma \mathrm{Tr}\{\psi^\dagger \partial_0 \psi\} - V(\psi). \tag{A23}$$

Here $\mathcal{L}_{\mathrm{G}}^*$ and $\mathcal{L}_{\mathrm{Ph}}^*$ are given in (A11) and (A20), respectively. The result above is valid for general vortex configurations. We can discuss the interaction between vortices and quasiparticles in terms of the dual Lagrangian. Vortices are expected to appear as a source term for gluons and $U(1)_{\mathrm{B}}$ phonons.

Appendix B

Vortex formation and vortex lattices as a response to rotation

In this appendix, we discuss the response of CFL matter to rotation. Before discussing the CFL matter, let us first recall what happens if one rotates an superfluid. Suppose one has a superfluid in a vessel and let us rotate the vessel with angular velocity Ω. The ground state of the system can be determined by minimizing the free energy. In a rotating system, the free energy is modified as $F' = F - \boldsymbol{\Omega} \cdot \boldsymbol{L}$, where $\boldsymbol{L}$ is the angular momentum vector. At low temperatures, the entropy term can be neglected and we just have to minimize $H' = H - \boldsymbol{\Omega} \cdot \boldsymbol{L}$, where H and H' are the Hamiltonian in the rest and rotating frames.

The time-evolution of a rotating system is generated by H'. Let us first recall the reason, in a simple example, a non-relativistic point particle in a rotating frame. The Lagrangian of a point particle with mass m is written as $L = m\boldsymbol{v}^2/2$, where $\boldsymbol{v}$ is the velocity in the rest frame. The conjugate momentum and the Hamiltonian is given by

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = m\boldsymbol{v}, \quad H = \boldsymbol{p} \cdot \boldsymbol{v} - L = \frac{m\boldsymbol{v}^2}{2}. \tag{A1}$$

Now let us move to the description in the rotating frame at an angular velocity $\boldsymbol{\Omega}$. Let $\boldsymbol{v}'$ be the velocity of the particle in the rotating system. The two velocities are related by

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\Omega} \times \boldsymbol{r}. \tag{A2}$$

Since the Lagrangian mechanics is covariant under general coordinate transformation, we can switch to the rotating frame just by substituting $\boldsymbol{v}'$ into $\boldsymbol{v}$,

$$L = \frac{m}{2} \left(\boldsymbol{v}' + \boldsymbol{\Omega} \times \boldsymbol{r} \right)^2, \tag{A3}$$

The conjugate momentum and Hamiltonian in the rotating system is given by

$$\boldsymbol{p}' = \frac{\partial L}{\partial \boldsymbol{v}'} = m(\boldsymbol{v}' + \boldsymbol{\Omega} \times \boldsymbol{r})\,(=\boldsymbol{p})\,, \quad H' = \boldsymbol{p}' \cdot \boldsymbol{v}' - L = \frac{\boldsymbol{p}'^2}{2m} - \boldsymbol{\Omega} \cdot (\boldsymbol{r} \times \boldsymbol{p}')\,, \tag{A4}$$

where we have used the cyclic property of the cross product, $\boldsymbol{p}' \cdot (\boldsymbol{\Omega} \times \boldsymbol{r}) = \boldsymbol{\Omega} \cdot (\boldsymbol{r} \times \boldsymbol{p}')$. Noting that $\boldsymbol{p}' = \boldsymbol{p}$, we can rewrite the Hamiltonian in the rotating frame as

$$
\begin{aligned}
H' &= \frac{\boldsymbol{p}'^2}{2m} - \boldsymbol{\Omega} \cdot (\boldsymbol{r} \times \boldsymbol{p}') \\
&= \frac{\boldsymbol{p}^2}{2m} - \boldsymbol{\Omega} \cdot (\boldsymbol{r} \times \boldsymbol{p}) \\
&= H - \boldsymbol{\Omega} \cdot \boldsymbol{L},
\end{aligned}
\tag{A5}
$$

which connects the Hamiltonians in the rest and rotating frames. The discussion above can be straight-forwardly extended to many-body systems, as long as the interaction potential is invariant under rotation.

Now let us discuss the response of a superfluid to rotation. We consider the situation where a superfluid is filled in a cylinder with radius R and there is one vortex in the center of the vessel. Then the superfluid velocity is given by $\boldsymbol{v} = \frac{n}{2\pi r} \boldsymbol{e}_\theta$ with n the winding number. The energy of a vortex per unit length is

$$
E = \int d^2 x \frac{1}{2} \rho \, \boldsymbol{v}^2 = \frac{\rho n^2}{4\pi} \log\left(\frac{R}{a}\right),
\tag{A6}
$$

where ρ is the superfluid density and a is the core radius ($R \gg a$). The angular velocity per unit length is written as

$$
L = \int d^2 x \, (\boldsymbol{r} \times (\rho \boldsymbol{v}))_z = \int_a^R 2\pi r \cdot r \rho |\boldsymbol{v}| \simeq \frac{1}{2} n \rho R^2,
\tag{A7}
$$

where the term proportional to a^2 is neglected. Thus, the energy in the rotating system is given by

$$
E' = \frac{\rho n^2}{4\pi} \log\left(\frac{R}{a}\right) - \frac{1}{2} \Omega n \rho R^2.
\tag{A8}
$$

If this energy is less than 0, one vortex state is favored compared to the state without a vortex. We can define the critical angular velocity $\Omega_{\rm c}$,

$$
\Omega_{\rm c} \equiv \frac{n}{2\pi R} \log\left(\frac{R}{a}\right).
\tag{A9}
$$

Thus, if $\Omega > \Omega_{\rm c}$, the state with vortex is more favorable than the trivial state.

If one further increases the rotational speed, multiple vortices are generated along the rotational axis. They all have the same winding number, so the inter-vortex force is repulsive. All the vortices repel each other, resulting in the formation of a vortex lattice. In the end, the vortex lattice corotates with the vessel, which means that the superfluid velocity at the edge coincides with the speed of wall. Then the circulation Γ can be calculated as $2\pi R \cdot R\Omega$. This should be equal to the sum of the circulations of all the vortices inside the vessel, $\Gamma = nN$, where N is the total number of vortices. Therefore, the total number of vortices

$$
N = \frac{2S\Omega}{n},
\tag{A10}
$$

where S is the cross section of the vessel. In the case of ordinary superfluid, the circulation of each vortex equals to one, $n = 1$.

Let us see the creation of vortices from another point of view. We here consider a BEC and denote the condensate by a complex scalar field $\Phi(x)$. In a rotating frame, the gradient term of the energy functional can be written as (see Eq. (A5))

$$E_{\text{grad}} = \Phi^* \left(\frac{\hat{\mathbf{p}}^2}{2m} - \mathbf{\Omega} \cdot \hat{\mathbf{L}} \right) \Phi = \frac{1}{2m} |[\partial_i - im(\mathbf{r} \times \mathbf{\Omega})_i] \Phi|^2 - \frac{1}{2} m r^2 \Omega^2 |\Phi|^2 \tag{A11}$$

where $\hat{L}_i = i\epsilon_{ijk} r_j \partial_k$ and $\hat{p}_i = \partial_i$, and the equality is meant up to a total derivative. The combination $D_i = \partial_i - im(\mathbf{r} \times \mathbf{\Omega})_i$ can be seen as a covariant derivative on the field Φ. Then $A_i \equiv m\,\mathbf{r} \times \mathbf{\Omega}$ can be seen as a gauge field and this system looks like a charged field under a constant magnetic field $2\mathbf{\Omega}$. Therefore, just as a type II superconductor under an external magnetic field, vortices come into a rotating superfluid. Let us make a comment on the trapping potential. One should add an external potential to trap the condensate, such as $V_{\text{trap}} = \frac{1}{2} \omega^2 r^2$ for BECs. This term and the last term in Eq. (A11) can be combined as $-\frac{1}{2} r^2 \Omega^2 |\Phi|^2 + V_{\text{trap}} = \frac{1}{2} (\omega^2 - \Omega^2) r^2 |\Phi|^2$. When the rotation speed Ω is less than ω, the condensates can be trapped.

Colorful vortex lattices

Then, what happens if one rotates CFL matter? We can repeat the energetical argument. When the vessel is large enough, the energy of a vortex is dominated by the superfluid velocity part and the contribution from the color flux is negligible, Thus, what is modified in the argument above is that n should equal to one third, $n = 1/3$. For realistic neutron stars, the number of vortices can be estimated as

$$N_v \simeq 1.9 \times 10^{19} \left(\frac{1\,\text{ms}}{P_{\text{rot}}} \right) \left(\frac{\mu/3}{300\,\text{MeV}} \right) \left(\frac{R}{10\,\text{km}} \right)^2, \tag{A12}$$

where P_{rot} is the rotational period, μ is the baryon chemical potential and the parameters are normalized by typical values. The corresponding intervortex spacing is given by

$$\ell \equiv \left(\frac{\pi R^2}{N_v} \right)^{1/2} \simeq 4.0 \times 10^{-6}\,\text{m} \left(\frac{P_{\text{rot}}}{1\text{ms}} \right)^{1/2} \left(\frac{300\text{MeV}}{\mu/3} \right)^{1/2}. \tag{A13}$$

Since the intervortex spacing is far larger than the size of a vortex core, which is given by inverse gluon/meson masses, gluons and mesons would not affect the force between two vortices. The intervortex force is dominated by the exchange of $U(1)_{\text{B}}$ phonons. This justifies the treatment above in which we have only considered the contribution of $U(1)_{\text{B}}$ circulations.

In the discussion using the free energy, we can only determine the ground state of the system. The dynamical process of vortex generation can be nontrivial especially for non-Abelian vortices. Basically, as one increases the speed of rotation gradually, vortices enter one by one from the edge of the superfluid. However, in the case of the non-Abelian vortices, it has a color flux and one vortex can not be created because of the color conservation. A vortex with unit $U(1)_{\text{B}}$ winding number does

not have a color flux, but it is energetically unstable. So, one plausible idea is that, a $U(1)_\mathrm{B}$ vortex is created first and then it decays into three non-Abelian vortices. Such two-fold vortex generation is seen in the simulation of rotating three-component BEC [131], in which integer quantized vortices are created first and they decay into fractional vortices.

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