学位論文

Localization of Five Dimensional Super Yang-Mills Theory and 4D/2D Duality

(5次元超対称ヤンミルズ理論の局所化と 4D/2D双対性)

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東京大学大学院理学系研究科物理学専攻
松宮就章

Ph.D. Thesis

Localization of Five Dimensional Super Yang-Mills Theory and 4D/2D Duality

Nariaki Matsumiya

Department of Physics, Faculty of Science, University of Tokyo

Abstract

In this thesis, we investigate a 4D/2D duality and its relation to M5-branes by using the five-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. It is conjectured that the four-dimensional $\mathcal{N} = 2$ gauge theory on $S^1 \times S^3$ is dual to the two-dimensional q-deformed Yang-Mills theory on a Riemann surface Σ. This duality is related to M5-branes on $S^1 \times S^3 \times \Sigma$, and the five-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory describes M5-branes on S^1. Therefore, we compute the partition function and correlation functions of the five-dimensional theory, and it turns out that the five-dimensional theory is equivalent to the q-deformed Yang-Mills theory. This result is agreement with the 4D/2D duality. This thesis is based on these papers [1] [2].

Contents

1 Introduction

Since the discovery, M-theory has been a fascinating research area in string theory. It is introduced at the strong coupling of type IIA string theory, and has the eleven-dimensional supergravity as the low energy limit. It has two types of branes, M2-brane and M5-brane as its elementary objects. The study of branes is important for understanding M-theory. In particular, when we consider N coincident branes, it is known that they have exotic statistical properties.

Such statistical properties of branes are studied by the AdS/CFT correspondence (gauge/gravity correspondence)[3]. In the case of string theory, there exist fundamental strings, NS5-branes and D-branes in various dimensions as basic components. When we consider a large number of D-branes, its bound state describes a black hole (brane). The world volume theory of N coincident D-branes is described by open strings which intertwine D-branes, and its low energy limit is described by the supersymmetric $U(N)$ gauge theory. The AdS/CFT correspondence implies that the Bekenstein-Hawking entropy of the black hole is related to the degree of freedoms in the gauge theory.[4] Indeed, both of them are equal each other.

The statistical properties of M2/M5-branes can be studied by the AdS/CFT correspondence, similarly. In the eleven-dimensional supergravity, black hole solutions which correspond to multiple M2 or M5-branes were discovered. The Bekenstein-Hawking entropies of them were computed. [5] Curiously, the entropy of the black hole solution for N coincident M2-branes grows as $N^{3/2}$. For N coincident M5-branes, the entropy grows as N^3.[5] This consequence indicates that the world volume theories of M2/M5-branes are not usual $U(N)$ gauge theories, since the entropy of the usual gauge theory grows as N^2 as in D-branes.

For M2-branes, ABJM model [6] is a successful model which describes the microscopic theory. This model is defined as $\mathcal{N} = 6$ Chern-Simons theory with matters in three-dimensions. Perturbatively, the degree of freedoms grows as N^2. However, there exists non perturbative method called the localization method, which enables us to compute the partition function and correlation functions for BPS operators. It turns out that the degree of freedoms for ABJM model grows as $N^{3/2}$ and this result is consistent with the prediction of the AdS/CFT correspondence.[7]

The localization method for functional integrals is not very new, and it has been studied from the last century. The path integral is a standard tool for the quantization, and it contains infinite-dimensional integrals. In some special case, such an infinite-dimensional integral can be reduced to a finite-dimensional integral by using the localization method. Let us explain the idea of the method, briefly. We consider a system which has a symmetry described by an operator Q and a functional V which satisfies $Q^2V = 0$. It may be possible to deform the action by adding a Q-exact term

$$S \to S - tQV,$$

with a deformation parameter t. Since $Q^2V = 0$, the partition function for the deformed action is independent from the parameter t. In the $t \to \infty$ limit, the dominant contribution of the path integral comes from the set of fixed points which are solutions of the equation $QV = 0$. The measure in the integral over the set of the fixed points comes from the integration over the fluctuations around the fixed points. Indeed, the integration over the fluctuations is a Gaussian integral, then we can perform the integral exactly.

This method was successfully applied to four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory on S^4.[8] The expectation value of the Wilson loop was computed. In this case, the expectation value became the finite-dimensional integral. In addition, the partition function of the $\mathcal{N} = 2$ theory on an ellipsoid S^4 [9] and the one on a squashed S^4 [10] were computed. These calculations play important roles in 4D/2D duality that we discuss later.

This localization method was applied to not only four-dimensional theories but also other gauge theories in various dimensions. The partition function of three-dimensional supersymmetric Chern-Simons theory on S^3 was computed [11] [12], and it became a zero-dimensional matrix integral. Therefore, the partition function of ABJM model was also computed, and it became a zero-dimensional matrix integral. Consequently, the $N^{3/2}$ behavior was derived from the matrix integral[7]. There are a lot of applications of the localization method. Three-dimensional theories on an ellipsoid and a squashed S^3 [13][14], a Seifert manifold [15]. Two-dimensional theories on S^2 [16] [17], five-dimensional theories on S^5 [18][19], and other five-dimensional manifolds [1][2][20][21][22][23][24].

We would like to find the microscopic theory for M5-branes. The world volume theory of them is the six-dimensional $\mathcal{N} = (2,0)$ theory, and the bosonic part contains a self-dual 2-form $B_{\mu\nu}$ and five scalars. However, it is difficult to construct the covariant action for the six-dimensional $\mathcal{N} = (2,0)$ theory. In order to explain its difficulty, let us consider the action for the 2-form. The action for a gauge field A_μ is given by

$$-\frac{1}{4}\int d^6x F_{\mu\nu}F^{\mu\nu}.$$

In analogy with this, an action for the Abelian two-form $B_{\mu\nu}$ is naively given by

$$S = \int d^6x H_{\mu\nu\rho}H^{\mu\nu\rho},$$

with the field strength $H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]}$. Because of the self-duality condition $H_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma\tau\lambda}H^{\sigma\tau\lambda}$,

$$\begin{aligned} S &= \int d^6x \; \epsilon_{\mu\nu\rho\sigma\tau\lambda}H^{\sigma\tau\lambda}H^{\mu\nu\rho} \\ &= -\int d^6x H^{\sigma\tau\lambda}H_{\sigma\tau\lambda} = 0. \end{aligned}$$

This action has to be zero. This is the reason why the construction of the action is difficult.

By introducing an auxiliary scalar field, an action for one M5-brane can be obtained. This model is called PST model [25][26]. It has the Lorentz symmetry and describes one M5-brane well. However, for multiple M5-branes, it is difficult to obtain the covariant action, then it is hard to study M5-branes directly.

One way to study multiple M5-branes is by 4D/2D duality. Let us consider M5-branes on $M_4 \times \Sigma$ with a four-dimensional closed manifold M_4 and a Riemann surface Σ. By compactifying M5-branes on Σ, we obtain a four-dimensional theory on M_4. On the other hand, by compactifying M5-branes on M_4, we obtain a two-dimensional theory on Σ. There is a correspondence between them, and it is called 4D/2D duality. These two theories and the 4D/2D duality have to reproduce properties of M5-branes, therefore the study of them has been an active research area.

The four-dimensional theory on M_4 is $\mathcal{N} = 2$ gauge theory with matters.[27] Four-dimensional $\mathcal{N} = 2$ gauge theory has been studied for a long time. Seiberg and Witten solved the low energy $\mathcal{N} = 2$ theory with matters and the Coulomb branch moduli space is specified by the Seiberg-Witten curve.[28][29] Indeed, this curve has a relation with the Riemann surface Σ.

For $M_4 = S^4$, the two-dimensional theory on Σ is believed to be Toda-Liouville theory. Toda-Liouville theory is dual to four-dimensional $\mathcal{N} = 2$ gauge theory on S^4, and this duality is called AGT conjecture.[30] The Nekrasov partition function [31][32] of the four-dimensional $\mathcal{N} = 2$ gauge theory is identified with the correlation function of the Toda-Liouville theory.

In analogy with AGT conjecture, it was proposed that the four-dimensional $\mathcal{N} = 2$ gauge theory on $S^1 \times S^3$ is dual to the two-dimensional q-deformed Yang-Mills theory on Σ [33]. ($M_4 = S^1 \times S^3$)The evidence for the validity comes from the fact that the superconformal index of the four-dimensional theory is equal to the partition function of the two-dimensional q-deformed Yang-Mills theory in the zero area limit of Σ. Two-dimensional Yang-Mills theory is almost topological. We will discuss two-dimensional Yang-Mills theories and this 4D/2D duality in section 2.

In this thesis, we focus on the latter conjecture. This conjecture is related to the M5-branes theory on $S^1 \times S^3 \times \Sigma$. By compactifying M5-branes on Σ, it yields the four-dimensional theory. On the other hand, when we compactify the M5-branes on $S^1 \times S^3$, we will obtain the two-dimensional q-deformed Yang-Mills theory. We discuss the derivation of the two-dimensional theory, and it is the main result of this thesis.[2] In order to do this, a five-dimensional theory plays an important role. In fact, it is believed that the M5-branes theory on S^1 is equivalent to the five-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. [34][35] By the S^1 compactification, the 2-form $B_{\mu\nu}$ in six-dimension becomes the gauge field $A_M = B_{M5}$ in five-dimension, thus the six-dimensional $\mathcal{N} = (2, 0)$ theory becomes a five-dimensional gauge theory. In fact, the Kaluza-Klein modes (KK modes) are expected to correspond to the five-dimensional instanton solutions, and it is conjectured that the five-dimensional $\mathcal{N}=2$ supersymmetric Yang-Mills theory describes the M5-branes on S^1. Therefore, we study the five-dimensional $\mathcal{N}=2$ supersymmetric Yang-Mills theory on $S^3 \times \Sigma$ and discuss the relation between the five-dimensional theory and the two-dimensional q-deformed Yang-Mills theory on Σ. The relation between the four theories is depicted in the Figure.

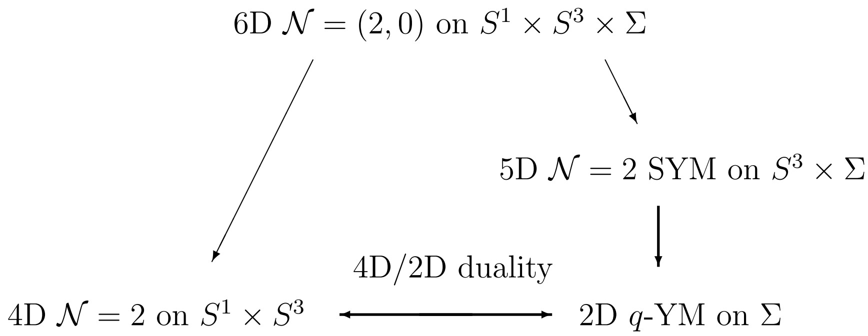

Figure: Relation between four theories

If the five-dimensional $\mathcal{N}=2$ supersymmetric Yang-Mills theory describes M5-branes on S^1, the five-dimensional theory should reproduce properties of M5-branes. This is the reason why to study the five-dimensional theory is active. The statistical property for N coincident M5-branes is that the degree of freedoms is proportional to N^3. The degree of freedoms for the $SU(N)$ five-dimensional $\mathcal{N}=2$ supersymmetric Yang-Mills theory on S^5 was computed, and it turned out that it grows as N^3 [36] [37]. This result is consistent with the above conjecture. There is a lot of research about the relation between the five-dimensional theory and M5-branes.

In gauge theories, there are non-local operators such as Wilson loops. In the six-dimensional $\mathcal{N}=(2,0)$ theory, there are surface operators. Surface operators are originated from M2-branes. M2-branes can attach to M5-branes, and the surface operators are defined by the boundaries. Upon compactification, surface operators become either surface operators, loop operators or defect operators. For example, let us consider M5-branes on $S^4 \times \Sigma$. When a surface operator is wrapped on a one-cycle in S^4 and a one-cycle in Σ, this surface operator becomes a loop operator in the four-dimensional theory on S^4. On the other hand, in the Toda-Liouville theory on Σ, the surface operator becomes another loop operator. There is a correspondence between the loop operator in the four-dimensional theory and the loop operator in the two-dimensional theory, and this correspondence is studied in terms of AGT conjecture. [38] [39]

The organization is as follows. In section 2, we review a 4D/2D duality which insists that the four-dimensional $\mathcal{N}=2$ gauge theory on $S^1 \times S^3$ is dual to the two-dimensional q-deformed Yang-Mills theory on a Riemann surface.[33] We compute the superconformal index of the four-dimensional theory, and we will find that it is equal to the partition function of the two-dimensional theory in the zero area limit of the Riemann surface. It is believed that this duality comes from M5-branes on $S^1 \times S^3 \times \Sigma$. In section 3, we investigate the five-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory on $S^3 \times \Sigma$, since this five-dimensional theory describes M5-branes on S^1. We compute the partition function of the five-dimensional theory, and it turns out that the partition function is equal to the partition function of the two-dimensional q-deformed Yang-Mills theory. This result is agreement with the above conjecture [33].

2 Review of 4D/2D Duality

In this section, we will explain a 4D/2D duality. By Gaiotto's argument [27], a four-dimensional $\mathcal{N}=2$ gauge theory is derived from the compactification of the six-dimensional $\mathcal{N}=(2,0)$ theory on a Riemann surface Σ. On the other hand, when we compactify the six-dimensional theory on a four-dimensional manifold, we obtain a two-dimensional theory on the Riemann surface. In [33], the two-dimensional theory which is dual to the four-dimensional $\mathcal{N}=2$ theory on $S^1 \times S^3$ is identified with a two-dimensional q-deformed Yang-Mills theory by comparing the superconformal index of the four-dimensional theory with the partition function of the q-deformed Yang-Mills theory. In this section, we review the conjecture by computing the superconformal index and the partition function of the q-deformed Yang-Mills theory.

2.1 4D $\mathcal{N}=2$ Gauge Theory

2.1.1 Quiver Diagram

A four-dimensional $\mathcal{N}=2$ theory consists of the vector multiplet and hypermultiplets. The $\mathcal{N}=2$ vector multiplet consists of the $\mathcal{N}=1$ vector multiplet $V=(A_\mu,\lambda)$ in the adjoint representation of the gauge group G and an $\mathcal{N}=1$ chiral multiplet $\Phi=(\phi,\tilde{\lambda})$ in the adjoint representation. In order to demand $\mathcal{N}=2$ supersymmetry, we demand this theory is invariant under the $SU(2)_R$ rotation which acts on $(\lambda,\tilde{\lambda})$.

The Lagrangian for the $\mathcal{N}=2$ vector multiplet is given by

$$\frac{\mathrm{Im}\tau}{4\pi}\int d^4\theta \mathrm{tr}\Phi^\dagger e^{[V,\,\cdot\,]}\Phi + \int d^2\theta \frac{-i\tau}{8\pi}\mathrm{tr}W_\alpha W^\alpha + c.c\ , \tag{1}$$

with the field strength W_α and the gauge coupling τ.

An $\mathcal{N}=2$ hypermultiplet in the representation R consists of an $\mathcal{N}=1$ chiral multiplet $Q=(q,\psi)$ in the representation R and an $\mathcal{N}=1$ chiral multiplet $\tilde{Q}=(\tilde{q},\tilde{\psi})$ in the representation $\bar{R}$. The $SU(2)_R$ rotation acts on complex scalars $(q,\tilde{q})$, and the theory is invariant under the rotation for the $\mathcal{N}=2$ supersymmetry. Let us consider the representation R is the fundamental representation of the gauge group $G=SU(N)$, and N_f hypermultiplets $(Q_i,\tilde{Q}^i)$ in the fundamental representation of N_f flavor symmetries.$(i=1,2,\cdots,N_f)$ The Lagrangian of the hypermultiplets is given by

$$\sum_i \int d^4\theta \left[Q^{\dagger i}e^V Q_i + \tilde{Q}^i e^V \tilde{Q}_i^\dagger\right] + \sum_i \int d^2\theta \left[\tilde{Q}^i \Phi Q_i + \mu_i \tilde{Q}^i Q_i + c.c\right], \tag{2}$$

with the mass μ_i.

In particular, we will discuss four-dimensional $\mathcal{N}=2$ superconformal gauge theories [1] which are obtained by M5-branes wrapped on a Riemann surface.[27] These theories are called class $\mathcal{S}$ theory. This four-dimensional theory is specified by a quiver diagram, and in type A_1 case ($G=SU(2)$), one can find the Lagrangian of the four-dimensional theory. For simplicity, we consider the A_1 theory. A quiver diagram is denoted by cubic vertices, internal lines and external lines. We will explain the correspondence between fields and components of a quiver diagram.

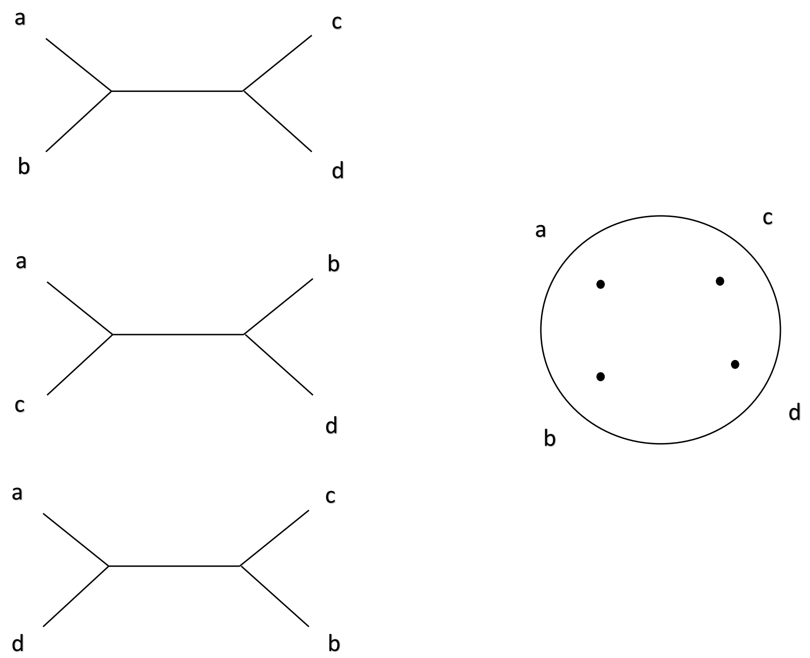

Figure 1: four flavors

Let us consider that the Riemann surface is a four-punctured sphere. (Figure 1) The corresponding quiver diagram consists of two cubic vertices, an internal line and 4 external lines. Each vertex corresponds to a trifundamental chiral multiplet, the internal line corresponds to an $SU(2)_G$ vector multiplet and each external line corresponds to an $SU(2)$ flavor symmetry. Thus, this four-dimensional theory consists of the $SU(2)_G$ vector multiplet and two chiral multiplets in the trifundamental representation of $SU(2)^3$. We explain the relation between four hypermultiplets $(Q_i, \tilde{Q}^i)$ $(i=1,2,3,4)$ and the two trifundamental chiral multiplets.

The superpotential W for the four hypermultiplets is given by

$$W=\sum_{i=1}^{4}\mu_i \tilde{Q}_a{}^i Q^a{}_i+\sum_{i=1}^{4}\tilde{Q}^{ai}\Phi_{ab}Q^b{}_i+c.c, \tag{3}$$

where $a,b=1,2$ are indices of the fundamental representation of the gauge group $SU(2)_G$. The fundamental representation of $SU(2)$ is pseudo real, then the flavor symmetry is

[1] I would like to thank Yuji Tachikawa for his lecture on $\mathcal{N}=2$ theories [40].

enhanced to the $SO(8)$ symmetry which rotates eight chiral multiplets. We combine $(Q_i, \tilde{Q}^i)$ into q_I $(I = 1, 2, \cdots, 8)$, and the superpotential W which is invariant under the $SO(8)$ flavor symmetry is given by

$$W = \sum_{I,J} \mu^{IJ} q_I^a q_J^b \epsilon_{ab} + q_I^a q_J^b \Phi_{ab} \delta^{IJ}, \tag{4}$$

with

$$\mu^{IJ} = \begin{pmatrix} & -\mu_1 \\ \mu_1 & \end{pmatrix} \oplus \begin{pmatrix} & -\mu_2 \\ \mu_2 & \end{pmatrix} \oplus \begin{pmatrix} & -\mu_3 \\ \mu_3 & \end{pmatrix} \oplus \begin{pmatrix} & -\mu_4 \\ \mu_4 & \end{pmatrix}. \tag{5}$$

Under the following decomposition

$$SO(8) \supset SO(4) \times SO(4) \simeq SU(2)_a \times SU(2)_b \times SU(2)_c \times SU(2)_d, \tag{6}$$

the q_I is decomposed into two trifundamental chiral multiplets $q_{aiu}, \tilde{q}_{akx}$. The a, i, u, k, x are indices of the fundamental representation for $SU(2)_G, SU(2)_a, SU(2)_b, SU(2)_c, SU(2)_d$, respectively. Then, the superpotential for the q_{aiu} is given by

$$W_q = \Phi^{ab} q_{aiu} q_{bjv} \epsilon^{ij} \epsilon^{uv} + \mu^{ij} q_{aiu} q_{bjv} \epsilon^{ab} \epsilon^{uv} + \hat{\mu}^{uv} q_{aiu} q_{bjv} \epsilon^{ij} \epsilon^{ab}, \tag{7}$$

with

$$\mu^{ij} = \frac{\mu_1 - \mu_2}{2} \mathrm{diag}(1, -1) \quad , \quad \hat{\mu}^{uv} = \frac{\mu_1 + \mu_2}{2} \mathrm{diag}(1, -1). \tag{8}$$

As for the $\tilde{q}_{akx}$, the superpotential is given by

$$W_{\tilde{q}} = \Phi^{ab} \tilde{q}_{akx} \tilde{q}_{bly} \epsilon^{kl} \epsilon^{xy} + \tilde{\mu}^{kl} \tilde{q}_{akx} \tilde{q}_{bly} \epsilon^{ab} \epsilon^{xy} + \hat{\tilde{\mu}}^{xy} \tilde{q}_{akx} \tilde{q}_{bly} \epsilon^{kl} \epsilon^{ab}, \tag{9}$$

with

$$\tilde{\mu}^{kl} = \frac{\mu_3 - \mu_4}{2} \mathrm{diag}(1, -1) \quad , \quad \hat{\tilde{\mu}}^{xy} = \frac{\mu_3 + \mu_4}{2} \mathrm{diag}(1, -1). \tag{10}$$

Then, one can obtain the explicit Lagrangian for this upper quiver diagram (Figure 1).

We can take other decompositions. The central diagram in the Figure 1 shows the decomposition of q_I into q'_{aik} and $\tilde{q}'_{aux}$, the lower diagram shows the decomposition of q_I into q''_{aix} and $\tilde{q}''_{aku}$. These theories are equivalent each other by the S-duality.

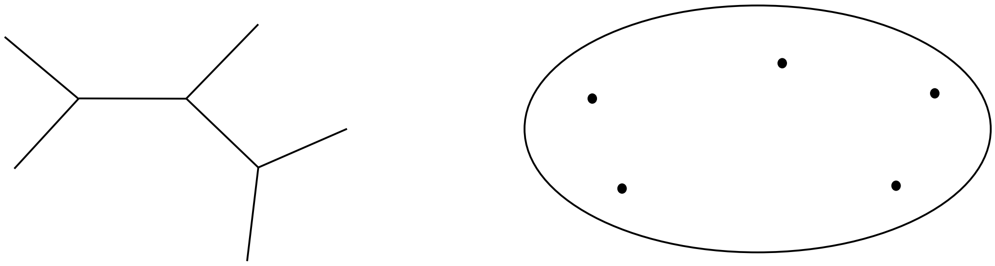

Figure 2: five flavors

Next, let us consider a five-punctured sphere. (Figure 2) There are three cubic vertices and they are connected as shown in the Figure. The internal lines correspond to the $SU(2)_1 \times SU(2)_2$ vector multiplet. The left trifundamental chiral multiplet carries indices of $SU(2)^2$ flavor symmetries and the $SU(2)_1$ gauge symmetry, and the right trifundamental chiral multiplet carries indices of other $SU(2)^2$ flavor symmetries and the $SU(2)_2$ gauge symmetry. One can obtain their superpotentials like (9). In addition, we have the central trifundamental chiral multiplet q_{aiu} where a, i are indices of the gauge symmetries $SU(2)_1, SU(2)_2$, respectively. The remaining index u is an index for the $SU(2)$ flavor symmetry. We can denote the q_{aiu} as a hypermultiplet $(Q^i_a, \tilde{Q}^a_i)$, which is defined by

$$Q^i_a = q_{aiu=1}\epsilon^{ij} \quad , \quad \tilde{Q}^a_i = q_{biu=2}\epsilon^{ab}. \tag{11}$$

This hypermultiplet $(Q^i_a, \tilde{Q}^a_i)$ is in the bifundamental representation of the $SU(2)_1 \times SU(2)_2$. The superpotential for the bifundamental hypermultiplet is given by

$$\tilde{Q}^i_a \left((\Phi_1)_i{}^j \delta^a{}_b + \delta_i{}^j (\Phi_2)^a{}_b\right) Q^j_b + \mu \tilde{Q}^i_a Q^a_i, \tag{12}$$

with the mass μ. Φ_1 is the $\mathcal{N} = 1$ chiral multiplet in the $\mathcal{N} = 2$ $SU(2)_1$ vector multiplet, and Φ_2 is the $\mathcal{N} = 1$ chiral multiplet in the $\mathcal{N} = 2$ $SU(2)_2$ vector multiplet.

Next, consider the one-punctured torus.(Figure 3). There is a trifundamental chiral multiplet q_{aiu}, and the $SU(2)$ vector multiplet is coupled to the indices a and i. The (a, i) is in the tensor product of the two fundamental representations $\mathbf{2}$ of $SU(2)$, then we can decompose it into a singlet and a triplet,

$$q_{aiu} \to \begin{cases} \text{triplet } q'_{\alpha u} \\ \text{singlet } q''_u, \end{cases} \tag{13}$$

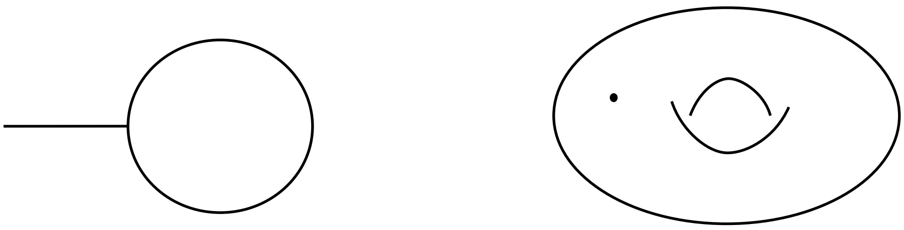

Figure 3: one-punctured torus

where the α is the index of the $\mathbf{3}$ of the $SU(2)$ gauge symmetry. The $(q''_{u=1}, q''_{u=2})$ is the free hypermultiplet with the mass μ, and the $(q'_{\alpha u=1}, q''_{\alpha u=2})$ is the hypermultiplet in the triplet of the gauge symmetry $SU(2)$ with the mass μ. One can obtain the superpotential for them.

The last example is the genus-two Riemann surface (Figure 4). The gauge symmetry is $SU(2)_1 \times SU(2)_2 \times SU(2)_3$. In the upper quiver diagram, the left trifundamental chiral multiplet is decomposed into q'_a in $\mathbf{1} \otimes \mathbf{2} \otimes \mathbf{1}$ and q''_{ai} in $\mathbf{3} \otimes \mathbf{2} \otimes \mathbf{1}$. The a is an index of $\mathbf{2}$ for $SU(2)_1$, the i is an index of $\mathbf{3}$ for $SU(2)_2$. The right trifundamental chiral multiplet is decomposed into $\tilde{q}'_a$ in $\mathbf{1} \otimes \mathbf{2} \otimes \mathbf{1}$ and $\tilde{q}''_{ax}$ in $\mathbf{1} \otimes \mathbf{2} \otimes \mathbf{3}$. The x is an index of $\mathbf{3}$ for $SU(2)_3$. Since there are no external lines, they have no mass terms.

In the lower quiver diagram, the two trifundamental chiral multiplets are in $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2}$ of the gauge group $SU(2)^3$. They have also no mass terms. And the lower theory is equivalent to the upper theory by the S-duality.

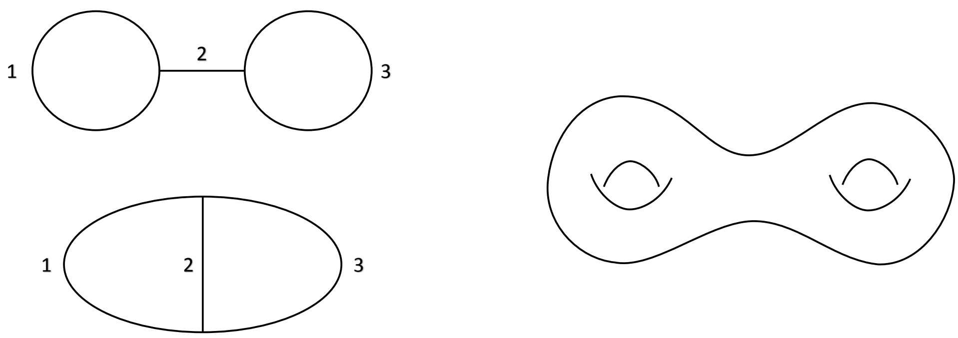

Figure 4: genus-two torus

In general, when a quiver diagram consists of N_V cubic vertices, N_G internal lines and N_F external lines, there are N_V trifundamental chiral multiplets, $SU(2)^{N_G}$ gauge symmetries and $SU(2)^{N_F}$ flavor symmetries in the four-dimensional theory.

2.1.2 Partition Function

Before computing the superconformal index, we study the partition function of a four-dimensional theory on $S^1 \times S^3$.

Suppose we have bosonic modes with the energy E_i in the representation R_i of the gauge group and fermionic modes with the energy E'_i in the representation R'_i, the partition function is given by

$$\begin{aligned} Z &= \text{tr } x^E \\ &= \prod_i \left(\sum_{n_i=0}^{\infty} x^{n_i E_i} \right) \prod_j \left(\sum_{n'_j=0}^{\infty} x^{n'_j E'_j} \right) \\ &\times \left[\#\text{of singlet in } sym^{n_1}(R_1) \otimes sym^{n_2}(R_2) \otimes \cdots \otimes anti^{n'_1}(R'_1) \otimes \cdots \right], \end{aligned} \tag{14}$$

where the trace is over the states of the theory on S^3, $x = \exp(-\beta)$ and β is the radius of the S^1. The $sym^n(R)$ ($anti^n(R)$) is the symmetrization (antisymmetrization) of the product $R \otimes R \otimes \cdots \otimes R$. The physical states are in the singlet representation of $sym^{n_1}(R_1) \otimes sym^{n_2}(R_2) \otimes \cdots \otimes anti^{n'_1}(R'_1) \otimes \cdots$, because they are invariant under the gauge symmetry. By using the characters of these representations, we can rewrite the partition function as the integral over the gauge group.[41]

The character $\chi_R(U) = \text{tr}_R\ (U)$ satisfies the following property

$$\int [dU] \bar{\chi}_R(U) \chi_{R'}(U) = \delta_{RR'}, \tag{15}$$

where U is an element of the gauge group G, and $[dU]$ is the Haar measure. In particular, when R =trivial, $R' = R_1 \otimes R_2 \otimes \cdots \otimes R_n$,

$$\int [dU] \prod_{i=1}^{n} \chi_{R_i}(U) = [\#\text{of singlet in } R_1 \otimes R_2 \otimes \cdots \otimes R_n]. \tag{16}$$

Then, the partition function becomes

$$\begin{aligned} Z &= \int [dU] \prod_i \left(\sum_{n_i=0}^{\infty} x^{n_i E_i} \chi_{sym^{n_i}(R_i)}(U) \right) \prod_j \left(\sum_{n'_j=0}^{\infty} x^{n'_j E'_j} \chi_{anti^{n'_j}(R'_j)}(U) \right) \\ &= \int [dU] \prod_i \exp \left(\sum_{n_i=1}^{\infty} \frac{x^{n_i E_i}}{n_i} \chi_{R_i}(U^{n_i}) \right) \prod_j \exp \left(\sum_{n'_j=1}^{\infty} \frac{(-1)^{n'_i+1} x^{n'_j E'_j}}{n'_j} \chi_{R'_j}(U^{n'_j}) \right). \end{aligned} \tag{17}$$

Here, we used the following properties

$$\begin{aligned}\chi_{sym^n(R)}(U) &= (U_R)^{a_1}{}_{[a_1}(U_R)^{a_2}{}_{a_2}\cdots(U_R)^{a_n}{}_{a_n]_+},\\ \chi_{anti^n(R)}(U) &= (U_R)^{a_1}{}_{[a_1}(U_R)^{a_2}{}_{a_2}\cdots(U_R)^{a_n}{}_{a_n]_-}.\end{aligned} \tag{18}$$

The $[\cdots]_\pm$ indicates the symmetrization or the antisymmetrization.

When we take $z_B^R(x) = \sum_{R_i=R} x^{E_i}$, $z_F^R(x) = \sum_{R_i'=R} x^{E_i'}$, we can obtain the simple expression,

$$Z = \int [dU] \exp\left[\sum_R \sum_{n_R=1}^{\infty} \frac{1}{n_R} f_{n_R}(x^{n_R})\chi_R(U^{n_R})\right], \tag{19}$$

with $f_{n_R}(x^{n_R}) = z_B^R(x^{n_R}) + (-1)^{n_R+1} z_F^R(x^{n_R})$.

We defined the partition function by the trace, since we consider the four-dimensional theory on S^1. The partition function can be written by the integration over the gauge group.[41] This expression is useful to consider the four-dimensional quiver gauge theory.

2.1.3 Superconformal Index

From now on, we consider a four-dimensional $\mathcal{N} = 2$ gauge theory on $S^3 \times S^1$ specified by a quiver diagram with no flavor symmetries, and we will compute the superconformal index.[42][43] For simplicity, we consider the case that the gauge group is $SU(2)$. The general superconformal index is parametrized by three chemical potentials, since there are global symmetries, $U(1) \times SO(4) \simeq U(1) \times SU(2) \times SU(2)$ isometry and $SU(2) \times U(1)$ R-symmetry. The general superconformal index is computed in [42][43], however it is enough to discuss a special index with one parameter here. The one parameter superconformal index is defined as

$$Z_{SCI} = \mathrm{tr}(-1)^F q^{\Delta - r}, \tag{20}$$

with Δ is the conformal dimension, and r is the R-charge of $SU(2)$ R-symmetry. q is a chemical potential, and we will find its value later. By applying the method discussed in the previous subsection, we can obtain the following expression for a quiver gauge theory,

$$Z_{SCI} = \int \prod_{l\in\mathcal{G}} [dU_l] \exp\left[\sum_{n=1}^{\infty} \frac{1}{n}\left\{\sum_{i\in\mathcal{G}} f(q^n)\chi_{adj}(U_i^n) + \sum_{(i,j,k)\in\mathcal{V}} g(q^n)\chi_{3f}(U_i^n, U_j^n, U_k^n)\right\}\right], \tag{21}$$

with U_i is an $SU(2)$ matrix. $\mathcal{G}$ is a set of internal lines which correspond to the gauge group. $\mathcal{V}$ is a set of cubic vertices, each vertex is labeled by three internal lines. χ_{adj}

and χ_{3f} are the character of the adjoint representation and the one of the trifundamental representation, respectively. $f(q^n)$ and $g(q^n)$ are "single-letter partition functions" for gauge fields and trifundamental chiral multiplets, respectively. $f(q)$ and $g(q)$ are computed in [43], they are given by

$$\begin{aligned} f(q) &= \frac{-2q}{1-q}, \\ g(q) &= \frac{q^{\frac{1}{2}}}{1-q}. \end{aligned} \tag{22}$$

The character $\chi_R(U_i)$ depends on a single variable α_i for $U_i \in SU(2)$, since we can write

$$U_i = V_i^\dagger \begin{pmatrix} e^{i\alpha_i} & 0 \\ 0 & e^{-i\alpha_i} \end{pmatrix} V_i \ , \qquad (V_i \in SU(2)) \tag{23}$$

then,

$$\chi_{R_n}(U_i) = \frac{a_i^n - a_i^{-n}}{a_i - a_i^{-1}} (\equiv \chi_{R_n}(a_i)). \tag{24}$$

$a_i = \exp(i\alpha_i)$ and R_n denotes the irreducible representation of $SU(2)$ and its dimension is n. R_3 is the adjoint representation, and R_2 is the fundamental representation. As for the trifundamental representation,

$$\begin{aligned} \chi_{3f}(U_i, U_j, U_k) &= \chi_{R_2}(U_i)\chi_{R_2}(U_j)\chi_{R_2}(U_k) \\ &= \chi_{R_2}(a_i)\chi_{R_2}(a_j)\chi_{R_2}(a_k), \end{aligned} \tag{25}$$

thus the superconformal index is written by

$$Z_{SCI} = \int \prod_{i\in\mathcal{G}} [da_i] \left[\prod_{i,j\in\mathcal{G}} \eta(a_i, a_j) \right] \left[\prod_{(i,j,k)\in\mathcal{V}} I(a_i, a_j, a_k) \right], \tag{26}$$

where

$$\begin{aligned} \eta(a_1) &= \exp\left[-2 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1-q^n} \chi_{adj}(a_1^n) \right], \\ \eta(a_1, a_2) &= \eta(a_1) \sum_{n=1}^{\infty} \chi_{R_n}(a_1)\chi_{R_n}(a_2), \\ I(a_1, a_2, a_3) &= \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{q^{n/2}}{1-q^n} \chi_{R_2}(a_1^n)\chi_{R_2}(a_2^n)\chi_{R_2}(a_3^n) \right]. \end{aligned} \tag{27}$$

The index $I(a_1, a_2, a_3)$ for the hypermultiplet is an elementary constituent. The index $\eta(a_1, b_1)$ for the gauge field glues $I(a_1, a_2, a_3)$ to $I(b_1, b_2, b_3)$ by integration over a_1 and b_1. Therefore, we can obtain the total superconformal index by this gluing. In fact, $I(a_1, a_2, a_3)$ corresponds to a three-point correlation function (three-punctured sphere), and $\eta(a_1, a_2)$ corresponds to a two-point correlation function (annulus) in the dual two-dimensional theory. We will confirm this correspondence later.

Let us proceed to compute. To do this, we use the following equality,

$$\frac{I(a_1, a_2, a_3)\eta(a_1)^{1/2}\eta(a_2)^{1/2}\eta(a_3)^{1/2}}{(q;q)^3(q^2;q)} = \sum_{n=1}^{\infty} \frac{\chi_{R_n}(a_1)\chi_{R_n}(a_2)\chi_{R_n}(a_3)}{\dim_q R_n}, \tag{28}$$

with R_n running over the n-dimensional irreducible representation,

$$\begin{aligned} (a;q) &= \prod_{i=0}^{\infty}(1 - aq^i), \\ \dim_q R_n &= [n]_q, \\ [x]_q &= \frac{q^{-x/2} - q^{x/2}}{q^{-1/2} - q^{1/2}}. \end{aligned} \tag{29}$$

The $\dim_q R$ is the quantum dimension of the irreducible representation R. We can prove the equality (28) by comparing the analytic properties. [33] [44] We discuss the poles and the residues.

Let us consider the left hand side. Since $\chi_{R_2}(a) = (a^2 - a^{-2})/(a - a^{-1})$,

$$I(a_1, a_2, a_3) = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n}\frac{q^{n/2}}{1 - q^n}\left(x^n + y^n + z^n + u^n + \frac{1}{x^n} + \frac{1}{y^n} + \frac{1}{z^n} + \frac{1}{u^n}\right)\right], \tag{30}$$

with $x = a_1/(a_2a_3), y = a_2/(a_1a_3), z = a_3/(a_1a_2), u = a_1a_2a_3$ and $xyzu = 1$. By using following equalities

$$\begin{aligned} \frac{1}{1-p} &= \sum_{m=0}^{\infty} p^m, \\ \log(1-p) &= -\sum_{m=0}^{\infty}\frac{p^m}{m}, \end{aligned} \tag{31}$$

then,

$$\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^{n/2}}{1-q^n}x^n\right] = \frac{1}{(\sqrt{q}x;q)}. \tag{32}$$

Therefore, we obtain the following expression of the left hand side of (28),

$$L.H.S = \frac{(1-q)(q;q)^2(qxy;q)(qzu;q)(qxz;q)(qyu;q)(qyz;q)(qxu);q)}{(\sqrt{q}x;q)(\sqrt{q}/x;q)(\sqrt{q}y;q)(\sqrt{q}/y;q)(\sqrt{q}z;q)(\sqrt{q}/z;q)(\sqrt{q}u;q)(\sqrt{q}/u;q)}. \quad (33)$$

This expression is symmetric in x, y, z, u. Therefore, we focus on the analytic properties as a function of x. We have poles when $x = q^{1/2-l}$ ($l \in \mathbb{Z}$). In particular, we consider l is positive. By using the following equalities,

$$\begin{aligned} \frac{(q^{3/2-l}y;q)(q^{1/2+l}y;q)}{(\sqrt{q}y;q)(\sqrt{q}/y;q)} &= \frac{(-y)^l}{q^{l^2/2}(1-yq^{1/2-l})}, \\ \frac{(qyz;q)(q/(yz);q)}{(q^{1-l}yz;q)(q^l/(yz);q)} &= \frac{(-yz)^{1-l}}{1-yz}q^{l(l-1)/2}, \end{aligned} \quad (34)$$

we obtain the residue of the left hand side,

$$\begin{aligned} \mathrm{Res}_{LHS} &= -\frac{q^{-1/2}-q^{1/2}}{A(q^{1/2-l},y,z)}, \\ A(x,y,z) &= x+y+z+u-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}-\frac{1}{u}. \end{aligned} \quad (35)$$

Next, we study the right hand side of (28),

$$R.H.S = \frac{q^{-1/2}-q^{1/2}}{A(x,y,z)} \sum_{n=1}^{\infty} \frac{q^{n/2}}{1-q^n}(x^n+y^n+z^n+u^n-\frac{1}{x^n}-\frac{1}{y^n}-\frac{1}{z^n}-\frac{1}{u^n}). \quad (36)$$

This is symmetric in x, y, z, u. In order to see poles as a function of x, use the following equality,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{q^{1/2}}{1-q^n}x^n &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(q^{1/2+m}x)^n \\ &= \sum_{m=0}^{\infty} \frac{q^{1/2+m}x}{1-q^{1/2+m}x}. \end{aligned} \quad (37)$$

Then, we have poles at $x = q^{1/2-l}$ ($l \in \mathbb{Z}$) in (36). When l is positive, the $1/x^n$ term only contributes to the residue, and its value is

$$\mathrm{Res}_{R.H.S} = -\frac{q^{-1/2}-q^{1/2}}{A(q^{1/2-l},y,z)}. \quad (38)$$

The L.H.S and R.H.S have same poles and same residues, therefore the equality (28) is proved.

By using (28), it is convenient to define rescaled indices

$$\begin{aligned}\hat{I}(a_1,a_2,a_3) &= I(a_1,a_2,a_3)\eta(a_1)^{1/2}\eta(a_2)^{1/2}\eta(a_3)^{1/2}\\ &= N(q)\sum_{n=1}^{\infty}\frac{\chi_{R_n}(a_1)\chi_{R_n}(a_2)\chi_{R_n}(a_3)}{\dim_q R_n},\\ \hat{\eta}(a_1,a_2) &= \eta(a_1)^{-1}\eta(a_1,a_2)\\ &= \sum_{n=1}^{\infty}\chi_{R_n}(a_1)\chi_{R_n}(a_2),\\ N(q) &= (q;q)^3(q^2;q).\end{aligned} \tag{39}$$

One can check these values up to overall constant when q is small. As for a trifundamental chiral multiplet,

$$\sum_{n=1}^{\infty}\frac{\chi_{R_n}(a_1)\chi_{R_n}(a_2)\chi_{R_n}(a_3)}{\dim_q R_n} = 1 + q^{1/2}\chi_{R_2}(a_1)\chi_{R_2}(a_2)\chi_{R_2}(a_3) + \cdots . \tag{40}$$

The first term of the right hand side comes from the contribution of the vacuum. The second term comes from the contribution of the scalar in the trifundamental chiral multiplet $Q_{aiu} = (q_{aiu}, \psi_{aiu})$. Their conformal dimensions are given by $(1, 3/2)$ and R-charges of $SU(2)$ R-symmetry are given by $(1/2, 0)$. Therefore, the index $\mathrm{tr}(-1)^F q^{\Delta - r}$ for the scalar q_{aiu} is given by $q^{1/2}$.

The total index is given by

$$Z_{SCI} = \int \prod_{i\in\mathcal{G}}[da_i]\left[\prod_{i,j\in\mathcal{G}}\hat{\eta}(a_i,a_j)\right]\left[\prod_{(i,j,k)\in\mathcal{V}}\hat{I}(a_i,a_j,a_k)\right]. \tag{41}$$

Let us proceed to compute the superconformal index. The index which corresponds to the upper diagram (Figure 5) is given by

$$\begin{aligned}Z^{(1)}_{SCI}(a) &= \int [db_1][db_2]\hat{\eta}(b_1,b_2)\hat{I}(a,b_1,b_2)\\ &= N(q)\sum_{n=1}^{\infty}\frac{\chi_{R_n}(a)}{\dim_q R_n},\end{aligned} \tag{42}$$

and the index which corresponds to the lower diagram (Figure 5) is given by

$$\begin{aligned}Z^{(2)}_{SCI}(a,b) &= \int [dc_1][dc_2][dc_3][dc_4]\hat{\eta}(c_1,c_2)\hat{\eta}(c_3,c_4)\hat{I}(a,c_1,c_3)\hat{I}(b,c_2,c_4)\\ &= N(q)^2\sum_{n=1}^{\infty}\frac{\chi_{R_n}(a)\chi_{R_n}(b)}{(\dim_q R_n)^2}.\end{aligned} \tag{43}$$

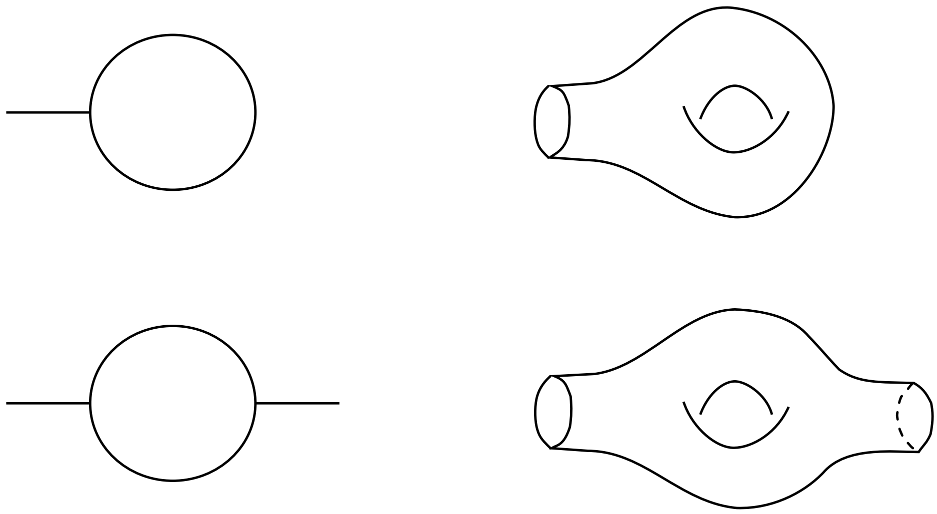

Figure 5: decomposition of Riemann surface

Then, we can obtain the total index which corresponds to a Riemann surface with no punctures by gluing the above indices,

$$\begin{aligned} Z_{SCI} &= \int \left(\prod_{i=1}^{g-1} [da_i] \right) Z_{SCI}^{(1)}(a_1) Z_{SCI}^{(2)}(a_1, a_2) \cdots Z_{SCI}^{(2)}(a_{g-2}, a_{g-1}) Z_{SCI}^{(1)}(a_{g-1}) \\ &\propto \sum_{n=1}^{\infty} (\dim_q R_n)^{\chi(\Sigma)}, \end{aligned} \tag{44}$$

where $\chi(\Sigma) = 2 - 2h$ is the Euler number of the Riemann surface Σ with the genus h. In terms of the quiver diagram, the genus h is the number of the loops in the diagram. We will see that this is equal to the partition function of the two-dimensional q-deformed Yang-Mills theory in the zero area limit up to overall constant.

2.1.4 What's q?

In the index, one can see that the parameter q is found in the form for a scalar in a vector multiplet

$$q^{\Delta} = e^{-\beta E}, \tag{45}$$

where Δ is the conformal weight of states over which the index have the summation, and the energy E can be obtained through the state-operator mapping in conformal field theories. $\beta = 2\pi R$, where R is the radius of S^1.

For a four-dimensional massless scalar of conformal weight $\Delta = 1$, the conformal coupling term of it with the scalar curvature in the Lagrangian gives it a mass

$$E = m = \frac{1}{l}, \tag{46}$$

on $\mathbb{R} \times S^3$, where the radius of the S^3 is l. Therefore, the state-operator mapping in conformal field theories suggests that

$$E = \frac{\Delta}{l}. \tag{47}$$

Then,

$$e^{-\beta E} = e^{-\frac{2\pi R \Delta}{l}}, \tag{48}$$

therefore, the parameter q can be read as

$$q = \exp\left(-\frac{2\pi R}{l}\right). \tag{49}$$

2.2 2D Yang-Mills Theory

In this subsection, we discuss two-dimensional Yang-Mills theories. They are almost topological theories, and they become topological in the zero area limit. We can find this property by computing the partition function.

First, we study the ordinary Yang-Mills theory on a Riemann surface and compute its partition function. After that, we define the q-deformed Yang-Mills theory and one can find its partition function is equal to the four-dimensional index.

2.2.1 Ordinary Yang-Mills Theory

The Lagrangian of the ordinary Yang-Mills theory on a Riemann surface Σ is given by

$$\mathcal{L} = -\frac{1}{4}\mathrm{tr} F_{\mu\nu}F^{\mu\nu} \quad (\mu, \nu = 1, 2), \tag{50}$$

where the field strength $F_{\mu\nu}$ is defined by $\partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$. We define the local complex coordinate z as $z = x^1 + ix^2$. By introducing a scalar ϕ in the adjoint representation of the gauge group G, the path integral that we would like to calculate is [45]

$$
\begin{aligned}
&\int \mathcal{D}A \exp\left(-\frac{i}{4}\int_\Sigma d^2z\sqrt{\det g_{\mu\nu}}\mathrm{tr}(g^{\bar{z}z}F_{\bar{z}z})^2\right) \\
&= \int \mathcal{D}A\mathcal{D}\phi \exp\left(-\frac{i}{4}\int_\Sigma d^2z\sqrt{\det g_{\mu\nu}}\mathrm{tr}\left[\phi^2 - 2i\phi g^{\bar{z}z}F_{\bar{z}z}\right]\right).
\end{aligned}
\tag{51}
$$

This Lagrangian is invariant under the gauge transformation

$$
A_z \to A_z - D_z\omega \quad , \quad \phi \to \phi + ig[\omega, \phi]. \tag{52}
$$

In order to compute the partition function, we have to introduce ghost fields $b, c, \bar{c}$ and a BRST operator Q, which is given by

$$
\begin{aligned}
&Q\phi = ig[c, \phi] \ , \ QA_\mu = -D_\mu c, \\
&Qc = \frac{ig}{2}[c, c] \ , \ Q\bar{c} = ib \ , \ Qb = 0.
\end{aligned}
\tag{53}
$$

The BRST operator Q is nilpotent. For gauge fixing, we add a Q-exact term $\mathcal{L}_G$ to the Lagrangian

$$
\begin{aligned}
\mathcal{L}_G &= Q(i\sum_{\alpha\in\Lambda}\bar{c}^{-\alpha}\phi^\alpha + i\sum_i \bar{c}^i\nabla^\mu A^i_\mu) \\
&= \sum_{\alpha\in\Lambda}\left(b^{-\alpha}\phi^\alpha + g\bar{c}^{-\alpha}[c, \phi]^\alpha\right) + \sum_i\left(b^i\nabla^\mu A^i_\mu + \bar{c}^i\nabla^\mu D_\mu c^i\right),
\end{aligned}
\tag{54}
$$

where $i = 1, 2, \cdots, r(= \mathrm{rank}G)$ is an index of Cartan generators of the gauge group G, and Λ is the set of all roots of G. The Cartan generators H_i and the root generators E_α satisfy the following conditions,

$$
\begin{aligned}
&[H_i, E_\alpha] = \alpha_i E_\alpha \ , \ [E_\alpha, E_{-\alpha}] = \sum_i \alpha_i H_i, \\
&\mathrm{tr}[H_i H_j] = \delta_{ij} \ , \ \mathrm{tr}[E_{-\alpha}E_\alpha] = 1.
\end{aligned}
\tag{55}
$$

Then, the partition function is given by

$$
Z = \int \mathcal{D}A\mathcal{D}\phi\mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}b \exp\left(-\frac{i}{4}\int_\Sigma d^2z\sqrt{\det g_{\mu\nu}}\mathrm{tr}\left[\phi^2 - 2i\phi g^{\bar{z}z}F_{\bar{z}z}\right] - \int_\Sigma d^2z\sqrt{\det g_{\mu\nu}}\mathcal{L}_G\right).
\tag{56}
$$

The integration over the ghost b^α, b^i yields gauge fixing conditions $\phi^\alpha = 0$ and $\nabla^\mu A^i_\mu = 0$. The integration over $c^\alpha, \bar{c}^\alpha$ yields

$$
\prod_{\alpha\in\Lambda_+} \mathrm{Det}_{(0,0)}\left(g(\alpha\cdot\phi)\right)^2, \tag{57}
$$

with $(\alpha \cdot \phi) = \sum_i \alpha_i \phi^i$, and Λ_+ is the set of positive roots of G. $\mathrm{Det}_{(p,q)}$ is the functional determinant over the space of the (p,q)-forms on the Riemann surface Σ. The integration over $A_z^\alpha, A_{\bar{z}}^\alpha$ yields

$$\prod_{\alpha\in\Lambda_+} \mathrm{Det}_{(1,0)} \left(g(\alpha\cdot\phi)\right)^{-2}. \tag{58}$$

The $U(1)^r$ gauge field A^i is decomposed into a classical part A_c^i and a quantum fluctuation A_q^i. The classical part A_c^i is labeled by the first Chern number (monopole number) [45]

$$\int dA_c^i = \frac{2\sqrt{2}\pi}{g} m^i \quad . \quad (m^i \in \mathbb{Z}) \tag{59}$$

We will explain this equality in section 3.4.

The integration over the fluctuation A_q^i indicates that the scalar ϕ^i is constant and it yields the determinant $\mathrm{Det}_{(0,0)}[\nabla_\mu \nabla^\mu]^{-1}$. The integration over the c^i and $\bar{c}^i$ yields the determinant $\mathrm{Det}_{(0,0)}[\nabla_\mu \nabla^\mu]$. Then, these two determinants are canceled.

By using the condition that ϕ^i is constant, one can simplify the determinants (57) and (58). By the Hodge decomposition [45], a 1-form ω on the Σ can be written by

$$\omega = d\eta + *d\tilde{\eta} + \gamma, \tag{60}$$

where η and $\tilde{\eta}$ are 0-forms and γ is a harmonic 1-form on Σ. The 0-form η enters as $d\eta$ in (60), then the harmonic 0-form (constant 0-form) does not enter. The dimension for the space of the harmonic 1-forms is $2h$ with the genus h of Σ. Therefore, (58) becomes

$$\prod_{\alpha\in\Lambda_+} \left[\frac{\mathrm{Det}_{(0,0)} \left(g(\alpha\cdot\phi)\right)^{-2}}{\left(g(\alpha\cdot\phi)\right)^{-2}} \right] \left(g(\alpha\cdot\phi)\right)^{-2h}, \tag{61}$$

with constant ϕ^i. Wrapping up (57) and (61), the determinant over $c^\alpha, \bar{c}^\alpha$ and A^α becomes

$$\prod_{\alpha\in\Lambda_+} \left(g(\alpha\cdot\phi)\right)^{\chi(\Sigma)}. \tag{62}$$

Therefore, the partition function becomes

$$\begin{aligned} Z &= \sum_{m^i\in\mathbb{Z}} \int d\phi^i \prod_{\alpha\in\Lambda_+} \left(g(\alpha\cdot\phi)\right)^{\chi(\Sigma)} \exp\left(\sum_i \left[\frac{\mathrm{Area}(\Sigma)}{2} (\phi^i)^2 - \frac{2\sqrt{2}\pi}{g} m^i \phi^i \right] \right) \\ &= \sum_{m^i\in\mathbb{Z}} \prod_{\alpha\in\Lambda_+} \left(\frac{ig^2}{\sqrt{2}} (\alpha\cdot m) \right)^{\chi(\Sigma)} \exp\left(-\frac{g^2 \mathrm{Area}(\Sigma)}{4} \sum_i (m^i)^2 \right). \end{aligned} \tag{63}$$

The partition function depends on the area of the Riemann surface Σ, thus the theory is not topological. When the area of the Σ is zero, the two-dimensional Yang-Mills theory becomes a topological theory.

In fact, we need to regularize the partition function. For simplicity, let us consider $G = SU(2)$. The partition function becomes

$$Z \propto \sum_{m\in\mathbb{Z}} (g^2 m)^{\chi(\Sigma)=2-2h} \exp\left(-\frac{g^2 \mathrm{Area}(\Sigma)}{4} m^2\right). \tag{64}$$

When $\chi(\Sigma) < 0$, this partition function is divergent from the $m = 0$ term. Then, we need some regularization and exclude the $m = 0$ term. We have argued that the m^2 comes from the integration over the ghost fields and the m^{-2h} comes from the gauge fields. One way to regularize the theory is to add a small mass term μ of the gauge field A^α.[45] By this deformation, the partition function is given by

$$\begin{aligned} Z &\propto \sum_{m\in\mathbb{Z}} g^{2\chi(\Sigma)} \frac{m^2}{(m+\mu)^{2h}} \exp\left(-\frac{g^2 \mathrm{Area}(\Sigma)}{4} m^2\right) \\ &= \sum_{m\in\mathbb{Z}, m\neq 0} g^{2\chi(\Sigma)} \frac{m^2}{(m+\mu)^{2h}} \exp\left(-\frac{g^2 \mathrm{Area}(\Sigma)}{4} m^2\right). \end{aligned} \tag{65}$$

Therefore, this is not divergent in the limit $\mu \to 0$.

2.2.2 q-deformed Yang-Mills Theory

One way to define the two dimensional q-deformed Yang-Mills theory is to make the scalar ϕ^i periodic. [46]

In the case of $G = SU(2)$, we impose the periodicity $\sqrt{2}\phi \to \sqrt{2}\phi + n/(2g\tilde{l})$ ($n \in \mathbb{Z}$). The measure in (63) is replaced by

$$(g\sqrt{2}\phi)^{\chi(\Sigma)} \to \left[\sin(4\sqrt{2}\pi g\tilde{l}\phi)\right]^{\chi(\Sigma)}, \tag{66}$$

with a deformation parameter $\tilde{l}$. The partition function is given by,

$$\begin{aligned} Z^{SU(2)}_{q-YM} &= \sum_{m\in\mathbb{Z}} \int d\phi \left[\sin(4\sqrt{2}\pi g\tilde{l}\phi)\right]^{\chi(\Sigma)} \exp\left(\left[\frac{\mathrm{Area}(\Sigma)}{2}\phi^2 - \frac{2\sqrt{2}\pi}{g} m\phi\right]\right) \\ &\propto \sum_{n\neq 0} [n]_q^{\chi(\Sigma)} \exp\left(-\frac{n^2 g^2 \mathrm{Area}(\Sigma)}{4}\right) \end{aligned} \tag{67}$$

with $q = \exp(-2\pi g^2 \tilde{l})$. In the zero area limit, one finds

$$Z_{q-YM}^{SU(2)}\Big|_{\text{Area}(\Sigma)\to 0} \propto \sum_{n\neq 0}[n]_q^{\chi(\Sigma)}. \tag{68}$$

For the general gauge group G, the measure in (63) is replaced by

$$\prod_{\alpha\in\Lambda_+}(g(\alpha\cdot\phi))^{\chi(\Sigma)} \to \prod_{\alpha\in\Lambda_+}\left[\sin(4\pi g\tilde{l}(\alpha\cdot\phi))\right]^{\chi\Sigma}. \tag{69}$$

The partition function (68) is written by the quantum dimension of the irreducible representation. This is the reason why this theory is called q-deformed Yang-Mills theory. Up to overall constant, this partition function is equal to the index (44). Because of this equality, it is expected that the two-dimensional q-deformed Yang-Mills theory is dual to the four-dimensional $\mathcal{N}=2$ gauge theory. In the four-dimensional theory, the parameter q is determined by l, R (see (49)) which are radii of the S^3 and the S^1, respectively.

More precisely, we will discuss the duality. In the quiver diagram, a cubic vertex denotes a trifundamental chiral multiplet and an internal line denotes a gauge field. In terms of the two-dimensional theory, a cubic vertex corresponds to a three-punctured sphere and an internal line corresponds to an annulus. Therefore, it is expected that the index $\hat{I}$ for the chiral multiplet and the index $\hat{\eta}$ for the gauge field are equal to the amplitude for a three-punctured sphere (three-point correlation function) and the amplitude for an annulus (two-point correlation function), respectively. We can find the correspondence by calculating these amplitudes in the two-dimensional theory.

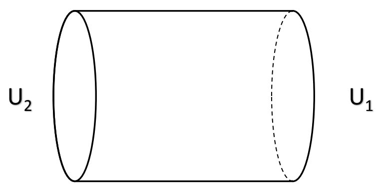

Figure 6: annulus

In order to compute the amplitudes for the ordinary Yang-Mills theory [46][47][48], let us consider the quantization on the cylinder. We can take $\chi_{R_n}(U)$ as basis of the Hilbert space, where

$$U = \mathcal{P}\exp\left(i\oint A\right), \tag{70}$$

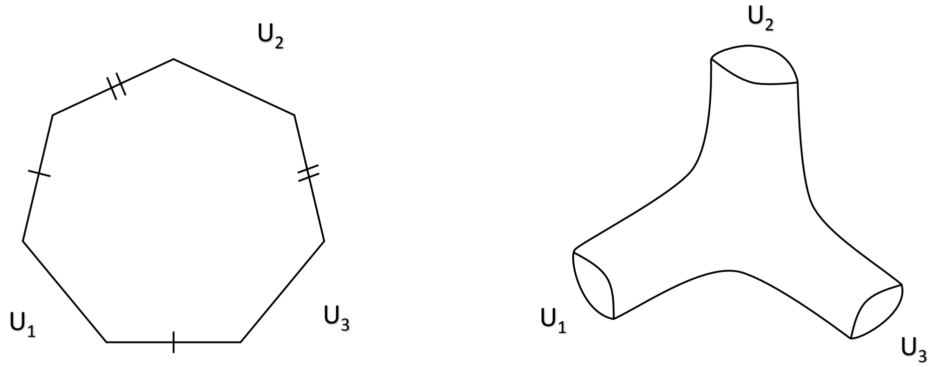

Figure 7: pants

since the U is the solution of the Gauss law constraint equation. $\mathcal{P}$ denotes the usual path-ordering operator. Therefore, the amplitude of an annulus of length T (Figure 6) is given by

$$\begin{aligned} Z(U_1, U_2) &= \langle U_1 | \exp(-LT\mathcal{H}) | U_2 \rangle \\ &= \sum_{n=1}^{\infty} \chi_{R_n}(U_1) \chi_{R_n}(U_2) \exp(-LT\mathcal{H}_{R_n}), \end{aligned} \tag{71}$$

with the Hamiltonian density $\mathcal{H}$. L is the circumference of the annulus. Then, in the zero area limit, it yields

$$Z(U_1, U_2) \to \sum_{n=1}^{\infty} \chi_{R_n}(U_1) \chi_{R_n}(U_2). \tag{72}$$

When $U_2 = 1$, this amplitude becomes a disk amplitude,

$$Z(U) = \sum_{n=1}^{\infty} (\mathrm{dim} R_n)\, \chi_{R_n}(U) \exp\left(-\mathcal{H}_{R_n} \times \mathrm{Area}\right). \tag{73}$$

In order to obtain an amplitude for a sphere with three punctures (or a pants diagram), we glue pieces of the same boundary like the Figure 7. By using the property of characters (see appendix A)

$$\int [dU] \chi_R(UVU^\dagger W) = \frac{\chi_R(V)\chi_R(W)}{\mathrm{dim} R}, \tag{74}$$

We can find the amplitude

$$
\begin{aligned}
Z(U_1,U_2,U_3) &= \sum_{n=1}^{\infty}\int [dW_1][dW_2]\ \mathrm{dim}R_n\ \chi_{R_n}(W_1U_1W_1^{\dagger}W_2U_2W_2^{\dagger}U_3) \\
&\qquad\qquad \times \exp\left(-\mathcal{H}_{R_n}\times \mathrm{Area}\right) \\
&= \sum_{n=1}^{\infty}\frac{\chi_{R_n}(U_1)\chi_{R_n}(U_2)\chi_{R_n}(U_3)}{\mathrm{dim}R_n}\exp\left(-\mathcal{H}_{R_n}\times \mathrm{Area}\right) \\
&\to \sum_{n=1}^{\infty}\frac{\chi_{R_n}(U_1)\chi_{R_n}(U_2)\chi_{R_n}(U_3)}{\mathrm{dim}R_n}. \qquad \text{(zero area limit)}
\end{aligned}
\tag{75}
$$

Correlation functions for the q-deformed theory are obtained by the replacement $\mathrm{dim}R \to \mathrm{dim}_q R$. Then, we see that the amplitude of the pants and the amplitude of the annulus are equal to the index $\hat{I}$ for the hypermultiplet and the index $\hat{\eta}$ for the gauge field up to overall constant factors, respectively.

In this section, we found that the superconformal index of the four-dimensional theory is equal to the partition function of the q-deformed Yang-Mills theory for $G = SU(2)$. Because of the equality, the 4D/2D duality between the two theories was proposed. This conjecture was generalized for the case of $G = SU(N)$.[33] Therefore, we expect that the two-dimensional q-deformed Yang-Mills theory is derived from the M5-branes on $S^1 \times S^3 \times \Sigma$. In the next section, we will discuss the derivation.

3 Localization of 5D $\mathcal{N} = 2$ Super Yang-Mills Theory

In this section, we compute the partition function of the five dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory on $S^3 \times \Sigma$ by the localization method. It is believed that the five dimensional theory describes the M5-branes on $S^1 \times S^3 \times \Sigma$. Therefore, if the 4D/2D duality discussed in the previous section is correct, the partition function of the five-dimensional theory is equal to the partition function of the two-dimensional q-deformed Yang-Mills theory.

In order to put the five-dimensional theory on $S^3 \times \Sigma$, we will explain the Killing spinors on the round S^3 and the "partial twisting" [49][50] on the surface Σ, and give the supersymmetry transformations and the Lagrangian. After that, we will curry out the localization procedure to compute the partition function of the five-dimensional theory. It will turn out that the result of summing up the one-loop determinants yields the partition function of the two-dimensional q-deformed Yang-Mills theory.

3.1 Super Yang-Mills Theory on $\mathbb{R}^5$

The $\mathcal{N} = 2$ supersymmetric Yang-Mills theory on $\mathbb{R}^5$ consists of an $\mathcal{N} = 1$ vector multiplet and an $\mathcal{N} = 1$ hypermultiplet in the adjoint representation of the gauge group G. Although the R-symmetry of the $\mathcal{N} = 2$ theory is $SO(5)$, its subgroup $\simeq SU(2) \times SU(2)$ is manifest in terms of the $\mathcal{N} = 1$ supermultiplets, we will respect only one of the two $SU(2)$s.

The vector multiplet consists of a gauge field $A_M (M = 1, 2, \cdots, 5)$, a real scalar σ, auxiliary fields $D^{\dot{\alpha}}{}_{\dot{\beta}}$, and a symplectic Majorana spinor $\Psi^{\dot{\alpha}}$. The indices $\dot{\alpha}, \dot{\beta}$ label the components of the fundamental representation of $SU(2)$ R-symmetry. The symplectic Majorana spinor $\Psi^{\dot{\alpha}}$ satisfies the following condition,

$$\bar{\Psi}_{\dot{\alpha}} = (\Psi^{\dot{\alpha}})^\dagger = (\Psi^{\dot{\beta}})^T C_5 \epsilon_{\dot{\beta}\dot{\alpha}}, \tag{76}$$

where T denotes the transpose and $\epsilon_{\dot{\alpha}\dot{\beta}}$ is the invariant tensor of the $SU(2)$ R-symmetry. The auxiliary fields $D^{\dot{\alpha}}{}_{\dot{\beta}}$ are anti-Hermitian and in the adjoint representation of the $SU(2)$ R-symmetry, and obeys the condition

$$D^{\dot{\alpha}}{}_{\dot{\beta}} = -(D^{\dot{\beta}}{}_{\dot{\alpha}})^\dagger \ , \ D^{\dot{\alpha}}{}_{\dot{\gamma}}\epsilon^{\dot{\gamma}\dot{\alpha}} = D^{\dot{\beta}}{}_{\dot{\gamma}}\epsilon^{\dot{\gamma}\dot{\alpha}} \ , \ D^{\dot{\alpha}}{}_{\dot{\alpha}} = 0. \tag{77}$$

Our notations for the charge conjugation matrix C_5 and the gamma matrices Γ^M are explained in appendix B. Since all the fields are in the adjoint representation of the gauge group G, they are denoted in the matrix notation as

$$\Phi = \Phi^A T^A, \tag{78}$$

with the normalization $\mathrm{tr}[T^A T^B] = \delta^{AB}$.

On $\mathbb{R}^5$, the Lagrangian of the vector multiplet is given by,

$$\mathcal{L}_V^{(0)} = \mathrm{tr}\left(-\frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}D_M\sigma D^M\sigma + i\bar{\Psi}_{\dot{\alpha}}\Gamma^M D_M\Psi^{\dot{\alpha}} - g\bar{\Psi}_{\dot{\alpha}}[\sigma,\Psi^{\dot{\alpha}}] - \frac{1}{4}D^{\dot{\alpha}}{}_{\dot{\beta}}D^{\dot{\beta}}{}_{\dot{\alpha}}\right). \tag{79}$$

Where the field strength F_{MN} is defined by

$$F_{MN} = \partial_M A_N - \partial_N A_M + ig[A_M, A_N], \tag{80}$$

and the covariant derivative is defined by

$$D_M\Phi = \partial_M\Phi + ig[A_M,\Phi]. \tag{81}$$

The supersymmetric transformation is given by

$$\begin{aligned}
&\delta_\Sigma^{(0)}A_M = -i\bar{\Sigma}_{\dot{\alpha}}\Gamma_M\Psi^{\dot{\alpha}}\ ,\ \delta_\Sigma^{(0)}\sigma = i\bar{\Sigma}_{\dot{\alpha}}\Psi^{\dot{\alpha}},\\
&\delta_\Sigma^{(0)}\Psi^{\dot{\alpha}} = -\frac{1}{2}\left(\frac{1}{2}F_{MN}\Gamma^{MN}\Sigma^{\dot{\alpha}} + \Gamma^M D_M\sigma\Sigma^{\dot{\alpha}} + D^{\dot{\alpha}}{}_{\dot{\beta}}\Sigma^{\dot{\beta}}\right),\\
&\delta_\Sigma^{(0)}D^{\dot{\alpha}}{}_{\dot{\beta}} = i\left[D_M\bar{\Psi}_{\dot{\beta}}\Gamma^M\Sigma^{\dot{\alpha}} + \bar{\Sigma}_{\dot{\beta}}\Gamma^M D_M\Psi^{\dot{\alpha}} + ig([\sigma,\bar{\Psi}_{\dot{\beta}}]\Sigma^{\dot{\alpha}} + \bar{\Sigma}_{\dot{\beta}}[\sigma,\Psi^{\dot{\alpha}}])\right],
\end{aligned} \tag{82}$$

where the parameter $\Sigma^{\dot{\alpha}}$ satisfies the symplectic Majorana condition (76) , and it is constant. $\mathcal{L}_V^{(0)}$ is invariant under this transformation.

The $\mathcal{N}=1$ hypermultiplet in the adjoint representation consists of a complex scalar $H_{\dot{\alpha}}$, a Dirac spinor Ξ and auxiliary fields $F_{H\alpha}$, where the index α is distinct from the one $\dot{\alpha}$ of the $SU(2)_R$ symmetry. ($\alpha = 1,2$) Since they are in the adjoint representation, they are also denoted in the matrix notation. The Lagrangian of the hypermultiplet and the interaction term between the vector multiplet and the hypermultiplet are given by

$$\begin{aligned}
\mathcal{L}_H^{(0)} =&\mathrm{tr}\left[-D^M\bar{H}^{\dot{\alpha}}D_M H_{\dot{\alpha}} - i\bar{\Xi}\Gamma^M D_M\Xi + \bar{F}_H{}^\alpha F_{H\alpha}\right],\\
\mathcal{L}_{int}^{(0)} =&\mathrm{tr}\big[g^2[\sigma,\bar{H}^{\dot{\alpha}}][\sigma,H_{\dot{\alpha}}] + igD^{\dot{\alpha}}{}_{\dot{\beta}}[\bar{H}^{\dot{\beta}},H_{\dot{\alpha}}] - g\bar{\Xi}[\sigma,\Xi]\\
&- 2g\bar{\Xi}[H_{\dot{\alpha}},\Psi^{\dot{\alpha}}] + 2g[\bar{H}^{\dot{\alpha}},\bar{\Psi}_{\dot{\alpha}}]\Xi\big],
\end{aligned} \tag{83}$$

where $\bar{H}^{\dot{\alpha}}$ is the complex conjugate of $H_{\dot{\alpha}}$, and $\bar{\Xi} = \Xi^\dagger$. They are invariant under the following supersymmetric transformation,

$$
\begin{aligned}
&\delta_\Sigma^{(0)} H_{\dot{\alpha}} = -i\bar{\Sigma}_{\dot{\alpha}}\Xi, \\
&\delta_\Sigma^{(0)} \Xi = \left(\Gamma^M D_M H_{\dot{\alpha}} + ig[\sigma, H_{\dot{\alpha}}]\right) + F_{H\alpha}\check{\Sigma}^\alpha, \\
&\delta_\Sigma^{(0)} F_{H\alpha} = i\bar{\check{\Sigma}}_\alpha \left[\Gamma^M D_M \Xi - ig[\sigma, \Xi] - 2ig[H_{\dot{\beta}}, \Psi^{\dot{\beta}}]\right],
\end{aligned}
\tag{84}
$$

if accompanied by the transformation (82). The parameters $\check{\Sigma}^\alpha$ are linearly independent spinors of $\Sigma^{\dot{\alpha}}$ and also obey the symplectic Majorana condition,

$$
\bar{\check{\Sigma}}_\alpha = (\check{\Sigma}^\beta)^T C_5 \epsilon_{\beta\alpha}. \tag{85}
$$

The supersymmetry transformation (82) (84) yields a closed algebra on any field Φ for the supersymmetry parameters $\Sigma^{\dot{\alpha}}, \Theta^{\dot{\alpha}}, \check{\Sigma}^\alpha, \check{\Theta}^\alpha$ (They are specified later.) as

$$
\left[\delta_\Theta^{(0)}, \delta_\Sigma^{(0)}\right] \Phi = \xi^M \partial_M \Phi + \delta_G \Phi, \tag{86}
$$

with the Killing vector

$$
\xi^M = i\bar{\Theta}_{\dot{\alpha}} \Gamma^M \Sigma^{\dot{\alpha}}, \tag{87}
$$

where δ_G is a gauge transformation, and the parameter ω is given by

$$
\omega = \xi^M A_M + i\bar{\Theta}_{\dot{\alpha}} \Sigma^{\dot{\alpha}} \sigma. \tag{88}
$$

3.2 Supersymmetry on $S^3 \times \Sigma$

At the beginning, we consider a supersymmetry on $S^3 \times \mathbb{R}^2$. We pick up the Killing spinor ϵ on the unit round S^3 obeying the Killing spinor equation

$$
\nabla_m \epsilon = \partial_m \epsilon + \frac{1}{4}\omega_m^{ab}\gamma^{ab}\epsilon = \frac{i}{2}\gamma_m \epsilon, \qquad (m = 1, 2, 3) \tag{89}
$$

with the spin connection ω_m^{ab} $(a, b = 1, 2, 3)$ of the unit S^3. The supersymmetry parameter $\Sigma^{\dot{\alpha}}$ is defined by

$$
\begin{aligned}
\Sigma^{\dot{\alpha}=1} &= \epsilon \otimes \zeta_+, \\
\Sigma^{\dot{\alpha}=2} &= C_3^{-1}\epsilon^* \otimes \zeta_-,
\end{aligned}
\tag{90}
$$

where $*$ denotes the complex conjugation, and two-dimensional Weyl spinors on $\mathbb{R}^2$,

$$\zeta_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}, \tag{91}$$

and the $\zeta_{\pm}$ satisfy

$$i\Gamma^4\Gamma^5 \left(1 \otimes \zeta_{\pm}\right) = \pm 1 \otimes \zeta_{\pm}. \tag{92}$$

One of important properties of $\Sigma^{\dot{\alpha}}$ is obeying the following condition,

$$\Gamma^{45}\Sigma^{\dot{\alpha}} = -2iN^{\dot{\alpha}}{}_{\dot{\beta}}\Sigma^{\dot{\beta}} \equiv -2\tilde{\Sigma}^{\dot{\alpha}}, \tag{93}$$

where the matrix $N^{\dot{\alpha}}{}_{\dot{\beta}}$ is defined by

$$N^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{2}(\sigma_3)^{\dot{\alpha}}{}_{\dot{\beta}}. \tag{94}$$

By using the equation (89), the $\Sigma^{\dot{\alpha}}$ obeys the condition

$$\nabla_m\Sigma^{\dot{\alpha}} = iN^{\dot{\alpha}}{}_{\dot{\beta}}\Sigma^{\dot{\beta}} = \Gamma_m\tilde{\Sigma}^{\dot{\alpha}}. \tag{95}$$

The linearly independent parameter $\check{\Sigma}^{\alpha}$ is defined by

$$\begin{aligned} \check{\Sigma}^{\alpha=1} &= \epsilon \otimes \zeta_{-}, \\ \check{\Sigma}^{\alpha=2} &= C_3^{-1}\epsilon^* \otimes \zeta_{+}, \end{aligned} \tag{96}$$

and obeys the conditions

$$\begin{aligned} \Gamma^{45}\check{\Sigma}^{\alpha} &= 2iN^{\alpha}{}_{\beta} = -2\tilde{\check{\Sigma}}^{\alpha}, \\ \nabla_m\check{\Sigma}^{\alpha} &= -iN^{\alpha}{}_{\beta}\Gamma_m\check{\Sigma}^{\beta} = \Gamma_m\tilde{\check{\Sigma}}^{\alpha}, \end{aligned} \tag{97}$$

where

$$N^{\alpha}{}_{\beta} = \frac{1}{2}(\sigma_3)^{\alpha}{}_{\beta}. \tag{98}$$

When we replace $\mathbb{R}^2$ in $S^3 \times \mathbb{R}^2$ by a Riemann surface Σ, we would like to keep using the spinor $\Sigma^{\dot{\alpha}}{}_{\dot{\beta}}$ as a supersymmetry transformation parameter to define a supersymmetry on $S^3 \times \Sigma$. In order to do that, we need to introduce a background gauge field by gauging the $SU(2)_R$ symmetry. The covariant derivative on a Weyl spinor $\zeta_\pm$ on the Riemann surface Σ is given by

$$\nabla_z \zeta_\pm = \left(\partial_z \mp \frac{i}{2}\omega_z\right)\zeta_\pm, \tag{99}$$

where, $\omega_z = \omega_z^{45}$ is the spin connection on the Riemann surface Σ. We have identified the local complex coordinates $z, \bar{z}$ on Σ with $z = x^4 + ix^5, \bar{z} = x^4 - ix^5$. The covariant derivative with background fields is defined by

$$\begin{aligned} \nabla_z \Phi^{\dot{\alpha}} &= \partial_z \Phi^{\dot{\alpha}} + \frac{1}{2}\omega_z \Gamma^{45}\Phi^{\dot{\alpha}} + i\omega_z N^{\dot{\alpha}}{}_{\dot{\beta}}\Phi^{\dot{\beta}}, \\ \nabla_z \Phi^{\alpha} &= \partial_z \Phi^{\alpha} + \frac{1}{2}\omega_z \Gamma^{45}\Phi^{\alpha} - i\omega_z N^{\alpha}{}_{\beta}\Phi^{\beta}. \end{aligned} \tag{100}$$

The third terms $i\omega_z N^{\dot{\alpha}}{}_{\dot{\beta}}$ and $-i\omega_z N^{\alpha}{}_{\beta}$ are background fields. Because of the background, the supersymmetric parameters obey the following equations,

$$\begin{aligned} \nabla_z \Sigma^{\dot{\alpha}} &= 0, \\ \nabla_z \check{\Sigma}^{\alpha} &= 0. \end{aligned} \tag{101}$$

This "partial twisting" [49][50] affects the spin representation of the fields which carry the indices of the $SU(2)_R$ symmetry or the indices α, β of another $SU(2)$ symmetry. In order to see this, it is convenient to give the gauge field and spinors in terms of three-dimensional fields. As for the vector multiplet,

$$\begin{aligned} &A_m \ \ (m = 1,2,3) \ \ , \ A_z = \frac{1}{2}(A_4 - iA_5), \\ &\Psi^{\dot{\alpha}=1} = \lambda \otimes \zeta_+ + \psi \otimes \zeta_- \ \ , \ \Psi^{\dot{\alpha}=2} = C_3^{-1}\psi^* \otimes \zeta_+ + C_3^{-1}\lambda^* \otimes \zeta_-, \\ &D = D^1{}_1 + g^{\bar{z}z}F_{\bar{z}z} \ \ , \ F_z = \frac{1}{2}D^1{}_2 \ \ , \ \bar{F}_{\bar{z}} = \frac{1}{2}D^2{}_1. \end{aligned} \tag{102}$$

As for the hypermultiplet,

$$\begin{aligned} \tilde{H} = H_{\dot{\alpha}=1} \ \ &, \ \ H = (H_{\dot{\alpha}=2})^*, \\ \Xi = \tilde{\chi} \otimes \zeta_+ &+ C_3^{-1}\chi^* \otimes \zeta_-. \end{aligned} \tag{103}$$

The covariant derivative on $\Psi^{\dot{\alpha}}$,

$$\nabla_z \Psi^{\dot{\alpha}=1} = (\partial_z \lambda) \otimes \zeta_+ + (\partial_z + i\omega_z)\psi \otimes \zeta_-. \tag{104}$$

Therefore, λ becomes a scalar field on Σ, ψ becomes a (1,0)-form ψ_z on Σ. In the same way, the auxiliary fields D and F_z become a scalar and a (1,0)-form, respectively. The scalars $\tilde{H}$ and H become Weyl spinors of positive chirality. The auxiliary fields $F_{H\alpha=1}$, $F_{H\alpha=2}$ become Weyl spinors of negative chirality and positive chirality , respectively.

We obtained the Killing spinor equation on $S^3 \times \Sigma$ by introducing the background fields. The supersymmetric parameters are covariantly constant along the Σ, but they are not covariantly constant along the S^3. Then, the supersymmetry transformation (82) (84) does not yield a closed algebra and the Lagrangian is not invariant under the transformation. (appendix C)

In order to obtain a closed algebra and a supersymmetric Lagrangian, we have to modify the supersymmetric transformation. As for the vector multiplet, the modification is given by

$$\delta'_\Sigma D^{\dot{\alpha}}{}_{\dot{\beta}} = -2i\left(\bar{\tilde{\Sigma}}_{\dot{\beta}}\Psi^{\dot{\alpha}} + \bar{\Psi}_{\dot{\beta}}\tilde{\Sigma}^{\dot{\alpha}}\right). \tag{105}$$

As for the hypermultiplet,

$$\begin{aligned} \delta'_\Sigma \Xi &= 2H_{\dot{\alpha}}\tilde{\Sigma}^{\dot{\alpha}}, \\ \delta'_\Sigma F_\alpha &= \frac{i}{2}\bar{\tilde{\Sigma}}_\alpha \Gamma^{45}\Xi. \end{aligned} \tag{106}$$

The modified transformation $\delta = \delta^{(0)} + \delta'$ gives a closed algebra,

$$[\delta_\Theta, \delta_\Sigma]\Phi = \pounds_\xi \Phi + \delta_G \Phi + \delta_R \Phi, \tag{107}$$

with

$$\begin{aligned} \Theta^{\dot{\alpha}=1} &= \eta \otimes \zeta_+, \\ \check{\Theta}^{\alpha=1} &= \eta \otimes \zeta_-, \end{aligned} \tag{108}$$

where η is a solution of the Killing spinor equation (89).

The first term in the right hand side of (107) is the Lie derivative $\pounds_\xi$. $\pounds_\xi$ on a five-dimensional spinor is defined by

$$\pounds_\xi \Psi = \xi^M \nabla_M \Psi + \frac{1}{4}(\nabla_M \xi_N)\Gamma^{MN}\Psi. \tag{109}$$

Note that the translation in $\mathbb{R}^5$ is extended to the infinitesimal diffeomorphism with the parameter ξ^M on the curved space $S^3 \times \Sigma$, and the diffeomorphism also transforms the background vielbein non-trivially. Therefore, the Lorentz transformation in the second term of the above definition is needed to compensate the diffeomorphism in order to keep the background vielbein intact.

The second term in the right hand side of (107) is the same gauge transformation as before. The third term δ_R is the transformation of a $U(1)$ subgroup of the $SU(2)_R$ symmetry and the other $SU(2)$ symmetry, which is given by

$$\begin{aligned}
\delta_R \Phi^{\dot{\alpha}} &= 2i\left(\bar{\tilde{\Theta}}_{\dot{\beta}}\Sigma^{\dot{\alpha}} - \bar{\tilde{\Sigma}}_{\dot{\beta}}\Theta^{\dot{\alpha}}\right)\Phi^{\dot{\beta}}, \\
\delta_R \Phi_{\alpha} &= -2i\Phi_{\beta}\left(\bar{\tilde{\Theta}}_{\alpha}\bar{\tilde{\Sigma}}^{\beta} - \bar{\tilde{\Sigma}}_{\alpha}\bar{\tilde{\Theta}}^{\beta}\right).
\end{aligned} \tag{110}$$

In terms of the three dimensional fields, the supersymmetric transformation of the vector multiplet on $S^3 \times \Sigma$ is given by

$$\begin{aligned}
&\delta_\Sigma A_m = -i\left[\bar{\epsilon}\gamma_m\lambda - \bar{\lambda}\gamma_m\epsilon\right] \quad , \quad \delta_\Sigma A_z = -\bar{\epsilon}\psi_z, \\
&\delta_\Sigma \sigma = i\left[\bar{\epsilon}\lambda - \bar{\lambda}\epsilon\right], \\
&\delta_\Sigma \lambda = -\frac{1}{2}\left[\frac{1}{2}F_{mn}\gamma^{mn} + \gamma^m D_m\sigma + D\right]\epsilon, \\
&\delta_\Sigma \psi_z = \left[iF_{mz}\gamma^m\epsilon - iD_z\sigma\epsilon - F_z C_3^{-1}\epsilon^*\right], \\
&\delta_\Sigma D = i\left[D_m\bar{\lambda}\gamma^m\epsilon + \bar{\epsilon}\gamma^m D_m\lambda + ig([\sigma,\bar{\lambda}]\epsilon + \bar{\epsilon}[\sigma,\lambda]) + \frac{i}{2}(\bar{\epsilon}\lambda - \bar{\lambda}\epsilon)\right], \\
&\delta_\Sigma F_z = i\left[-\epsilon^T C_3\gamma^m D_m\psi_z + 2i\epsilon^T C_3 D_z\lambda + ig\epsilon^T C_3[\sigma,\psi_z] - \frac{i}{2}\epsilon^T C_3\psi_z\right],
\end{aligned} \tag{111}$$

for the hypermultiplet,

$$\begin{aligned}
&\delta_\Sigma \tilde{H} = -i\bar{\epsilon}\tilde{\chi} \quad , \quad \delta_\Sigma H = -i\bar{\epsilon}\chi, \\
&\delta_\Sigma \tilde{\chi} = \left[D_m\tilde{H}\gamma^m + ig[\sigma,\tilde{H}] + i\tilde{H}\right]\epsilon + \left[-2i(D_z H)^* + F_{H2}\right]C_3^{-1}\epsilon^*, \\
&\delta_\Sigma \chi = \left[D_m H\gamma^m + ig[\sigma,H] + iH\right]\epsilon + \left[2i(D_z\tilde{H})^* - F_{H1}^*\right]C_3^{-1}\epsilon^*, \\
&\delta_\Sigma F_{H1}^* = i\left[(D_m\chi)^T C_3\gamma^m + 2i(D_z\tilde{\chi})^\dagger + \frac{i}{2}\chi^T C_3 + ig[\sigma,\chi^T]C_3 + 2ig\left([\tilde{H}^*,\bar{\psi}] + [H,\lambda^T]C_3\right)\right]\epsilon, \\
&\delta_\Sigma F_{H2}^* = -i\left[(D_m\tilde{\chi})^\dagger\gamma^m + 2i(D_z\chi)^T C_3 + \frac{i}{2}\tilde{\chi}^\dagger - ig[\sigma,\tilde{\chi}^\dagger]C_3 - 2ig\left([\tilde{H}^*,\bar{\lambda}] + [H,\psi^T]C_3\right)\right]C_3^{-1}\epsilon^*.
\end{aligned} \tag{112}$$

We modified the supersymmetry and obtained the closed algebra. In addition to that, we also have to modify the Lagrangian. The Lagrangian $\mathcal{L}_V^{(0)}, \mathcal{L}_H^{(0)}, \mathcal{L}_{int}^{(0)}$ in the $\mathbb{R}^5$ need to be covariantized in order to put them on the curved space $S^3 \times \Sigma$. In addition, the gauging of $SU(2)_R$ symmetry and the other $SU(2)$ symmetry in them will be done to consider the supersymmetry transformation. However, these are not enough to obtain invariant Lagrangian under the transformation (111) (112). One needs additional terms to the covariantized Lagrangians. In fact, it turns out that the additional terms to $\mathcal{L}_V^{(0)}$ of the vector multiplet are given by

$$\begin{aligned}\mathcal{L}'_V &= \mathrm{tr}\left[N^{\dot\alpha}{}_{\dot\beta}\bar\Psi_{\dot\alpha}\Psi^{\dot\beta} + iN^{\dot\alpha}{}_{\dot\beta}D^{\dot\beta}{}_{\dot\alpha}\sigma - \frac{1}{2}\varepsilon^{ij}F_{ij}\sigma + \sigma^2 + \frac{1}{2}\omega_{c.s}\right] \\ &= \mathrm{tr}\left[\frac{1}{2}g^{\bar zz}\bar\psi_{\bar z}\psi_z + \bar\lambda\lambda + \sigma\sigma + i\sigma(D - 2g^{\bar zz}F_{\bar zz}) + \frac{1}{2}\omega_{c.s}\right],\end{aligned} \tag{113}$$

with the Chern-Simons term $\omega_{c.s}$

$$\omega_{c.s} = \varepsilon^{mnk}\left(A_m\partial_n A_k + \frac{i}{3}gA_m[A_n, A_k]\right). \tag{114}$$

As for the hypermultiplet,

$$\begin{aligned}\mathcal{L}'_H &= \mathrm{tr}\left[-\frac{i}{2}\bar\Xi\Gamma^{45}\Xi - \bar H^{\dot\alpha}H_{\dot\alpha}\right] \\ &= -\mathrm{tr}\left[\frac{1}{2}(\bar{\tilde\chi}\tilde\chi + \bar\chi\chi) + \tilde H^*\tilde H + H^*H\right].\end{aligned} \tag{115}$$

Then, the total Lagrangian is invariant under the supersymmetry transformation (111) (112), and it is given by

$$\mathcal{L} = \mathcal{L}_V^{(0)} + \mathcal{L}_{int}^{(0)} + \mathcal{L}_H^{(0)} + \mathcal{L}'_V + \mathcal{L}'_H. \tag{116}$$

3.3 Computation of Partition Function

In this subsection, we will compute the partition function of the $\mathcal{N} = 2$ five-dimensional supersymmetric Yang-Mills theory by the localization method. In the Lagrangian, the kinetic terms of the scalar σ and auxiliary fields D, F_z have the wrong sign. In order to make the path-integral well-defined, they are need to be analytically continued. We replace the scalar σ by $i\sigma$, and we regard D as a real field and $\bar F_{\bar z} = (F_z)^*$.

3.3.1 Localization Method

At the beginning, we explain the localization method which is convenient to compute the partition function and the correlation functions of BPS states. In order to do that, we need a BRST transformation which is nilpotent. The BRST transformation is defined by using the supersymmetry transformation.

To define the BRST transformation, we regard that the parameter ϵ is independent from the parameter $\bar\epsilon$ and we take $\bar\epsilon = 0$. Next, we denote $\epsilon = a\zeta$, $\eta = b\zeta$ by Grassmann odd numbers a, b and a Grassmann even spinor ζ. Thus, we define the BRST transformation δ_Q as

$$a\delta_Q \Phi = \delta_\epsilon \Phi \Big|_{\bar{\epsilon}=0}. \tag{117}$$

The closed algebra of the supersymmetric transformation becomes

$$[\delta_\epsilon, \delta_\eta]\Phi \Big|_{\bar{\epsilon}=0,\bar{\eta}=0} = -2ab\delta_Q^2 \Phi. \tag{118}$$

The left hand side is equal to 0, because

$$\xi^M \Big|_{\bar{\epsilon}=\bar{\eta}=0} = \omega \Big|_{\bar{\epsilon}=\bar{\eta}=0} = 0. \tag{119}$$

Therefore the transformation δ_Q is nilpotent.

In order to compute the partition function, we add a Q-exact term $\delta_Q V$ to the Lagrangian. We consider the partition function of the deformed theory with the parameter t,

$$Z_t = \int \mathcal{D}\Phi \exp\left[\int d^D x \big(\mathcal{L} - t\delta_Q V\big)\right]. \tag{120}$$

Since the δ_Q is nilpotent and $\mathcal{L}$ is invariant under the δ_Q, for any V,

$$\begin{aligned} \frac{d}{dt} Z_t &= -\int \mathcal{D}\Phi \ \delta_Q \left(V \exp\left[\int d^D x \big(\mathcal{L} - t\delta_Q V\big)\right]\right) \\ &= 0. \end{aligned} \tag{121}$$

Therefore, the deformed partition function Z_t is independent on t, and we can take the limit $t \to \infty$. In this limit, solutions of $\delta_Q V = 0$ (fixed points) mainly contribute to the partition function. We decompose the field Φ into the fixed points $\Phi_{F.P}$ and the fluctuations $\tilde{\Phi}$ around the fixed points as

$$\Phi = \Phi_{F.P} + \frac{1}{\sqrt{t}}\tilde{\Phi}. \tag{122}$$

Substituting this Φ into the partition function,

$$\begin{aligned} Z &= Z_t = Z_{t\to\infty} \\ &= \int \mathcal{D}\Phi_{F.P}\mathcal{D}\tilde{\Phi} \exp\left[\int d^D x \big(\mathcal{L}(\Phi = \Phi_{F.P}) + (\text{quadratic terms of } \tilde{\Phi})\big)\right]. \end{aligned} \tag{123}$$

The integration over the fluctuations is a Gaussian integral. After integrating over the fluctuations, the partition function is reduced to the integration over the fixed points.

By following the same procedure, we can compute expectation values of BPS operators. Let us consider an operator $\mathcal{O}$ which satisfies

$$\delta_Q \mathcal{O} = 0. \tag{124}$$

As for this operator,

$$\begin{aligned}\langle \mathcal{O} \rangle &= \frac{1}{Z} \int \mathcal{D}\Phi_{F.P} \mathcal{D}\tilde{\Phi} \ \mathcal{O}(\Phi = \Phi_{F.P}) \ \exp\left[\int d^D x \big(\mathcal{L}(\Phi = \Phi_{F.P}) + (\text{quadratic terms of } \tilde{\Phi}) \big) \right] \\ &= \frac{1}{Z} \int \mathcal{D}\Phi_{F.P} \ \mathcal{O}(\Phi = \Phi_{F.P}) \ \mathcal{L}_{\text{1-loop}}(\Phi_{F.P}) \exp\left[\int d^D x \big(\mathcal{L}(\Phi = \Phi_{F.P}) \big) \right],\end{aligned} \tag{125}$$

where the 1-loop determinant $\mathcal{L}_{\text{1-loop}}$ is given by

$$\mathcal{L}_{\text{1-loop}}(\Phi_{F.P}) = \int \mathcal{D}\tilde{\Phi} \ \exp\left[\int d^D x \big(\text{quadratic terms of } \tilde{\Phi} \big) \right]. \tag{126}$$

3.3.2 BRST Transformation

Let us compute the partition function of the five-dimensional theory.

The BRST transformation for the vector multiplet is given by

$$\begin{aligned}&\delta_Q A_m = -i\bar{\lambda}\gamma_m \epsilon \ \ , \ \ \delta_Q A_z = 0 \ \ , \ \ \delta_Q A_{\bar{z}} = \bar{\psi}_{\bar{z}} \epsilon, \\ &\delta_Q \sigma = \bar{\lambda}\epsilon, \\ &\delta_Q \lambda = -\frac{1}{2}\left[\frac{1}{2} F_{mn}\gamma^{mn} + i\gamma^m D_m \sigma + D \right] \epsilon \ \ , \quad \delta_Q \bar{\lambda} = 0, \\ &\delta_Q \psi_z = [iF_{mz}\gamma^m + D_z \sigma]\epsilon \ \ , \ \ \delta_Q \bar{\psi}_{\bar{z}} = \bar{F}_{\bar{z}} \epsilon^T C_3, \\ &\delta_Q D = \left[-iD_m \bar{\lambda}\gamma^m \epsilon + ig[\sigma, \bar{\lambda}]\epsilon - \frac{1}{2}\bar{\lambda}\epsilon \right], \\ &\delta_Q F_z = \epsilon^T C_3 \left[-i\gamma^m D_m \psi_z - 2D_z \lambda - ig[\sigma, \psi_z] + \frac{1}{2}\psi_z \right] \ , \ \ \delta_Q \bar{F}_{\bar{z}} = 0,\end{aligned} \tag{127}$$

where ϵ is a Grassmann even spinor. As for the hypermultiplet,

$$
\begin{aligned}
&\delta_Q \tilde{H} = 0 \ , \quad \delta_Q H = 0, \\
&\delta_Q (\tilde{H})^* = -i(\tilde{\chi})^\dagger \epsilon \ , \quad \delta_Q (H)^* = -i(\chi)^\dagger \epsilon, \\
&\delta_Q \tilde{\chi} = \Big[D_m \tilde{H} \gamma^m - g[\sigma, \tilde{H}] + i\tilde{H}\Big] \epsilon \ , \quad \delta_Q (\tilde{\chi})^\dagger = \epsilon^T C_3 [2iD_z H + (F_{H2})^*], \\
&\delta_Q \chi = [D_m H \gamma^m - g[\sigma, H] + iH]\epsilon \ , \quad \delta_Q (\chi)^\dagger = -\epsilon^T C_3 \Big[2iD_z \tilde{H} + F_{H1}\Big] \ , \\
&\delta_Q F_{H1} = 0 \ , \quad \delta_Q (F_{H2})^* = 0, \\
&\delta_Q (F_{H1})^* = i\Big[- (D_m \chi)^T C_3 \gamma^m - 2i(D_z \tilde{\chi})^\dagger - \frac{i}{2}\chi^T C_3 \\
&\qquad\qquad + g[\sigma, \chi^T] C_3 - 2ig\Big([\tilde{H}^*, \psi^\dagger] + [H, \lambda^T] C_3\Big)\Big]\epsilon, \\
&\delta_\Sigma F_{H2} = i\epsilon^T C_3 \Big[\gamma^m D_m \tilde{\chi} - 2iC_3^{-1}(D_z \chi)^* - \frac{i}{2}\tilde{\chi} \\
&\qquad\qquad + g[\sigma, \tilde{\chi}] - 2ig\Big([\tilde{H}, \lambda] + C_3^{-1}[H^*, \psi^*]\Big)\Big].
\end{aligned} \tag{128}
$$

The BRST transformation obeys $\delta_Q^2 = 0$.

3.3.3 Contribution From Vector Multiplet

Let us compute the contribution from the vector multiplet. In order to compute the partition function, we add a Q-exact term $\delta_Q V_V$ to the Lagrangian. We take the V_V as

$$
V_V = \mathrm{tr}\left[(\delta_Q \lambda)^\dagger \lambda + \frac{1}{2} g^{\bar{z}z} (\delta_Q \psi_z)^\dagger \psi_z + \frac{1}{2} g^{\bar{z}z} \bar{\psi}_{\bar{z}} (\delta_Q \bar{\psi}_{\bar{z}})^\dagger\right]. \tag{129}
$$

In particular, the bosonic part of the Q-exact term is given by

$$
\delta_Q V_V \Big|_B = \mathrm{tr}\left[|\delta_Q \lambda|^2 + \frac{1}{2} g^{\bar{z}z} \left|\delta_Q \psi_z\right|^2 + \frac{1}{2} g^{\bar{z}z} \left|\delta_Q \bar{\psi}_{\bar{z}}\right|^2\right], \tag{130}
$$

then, this part is positive definite. Therefore, the fixed point equation $\delta_Q V_V = 0$ becomes

$$
\delta_Q \lambda = -\frac{1}{2}\left[\frac{1}{2} F_{mn} \gamma^{mn} + i\gamma^m D_m \sigma + D\right]\epsilon = 0, \tag{131}
$$

$$
\delta_Q \psi_z = [iF_{mz} \gamma^m + D_z \sigma]\epsilon = 0, \tag{132}
$$

$$
\delta_Q \bar{\psi}_{\bar{z}} = \bar{F}_{\bar{z}} \epsilon^T C_3 = 0, \tag{133}
$$

the solution is given by

$$
A_m = 0 \ , \quad D = 0 \ , \quad F_z = 0 \ , \quad A_z = A_z(z, \bar{z}) \ , \quad \sigma = \sigma(z, \bar{z}) \ , \quad D_z \sigma = 0. \tag{134}
$$

Substituting the fixed point into the Lagrangian of the vector multiplet,

$$\mathcal{L}_{YM} = \mathrm{tr}\left[-\sigma^2 + 2\sigma g^{\bar{z}z} F_{\bar{z}z}\right]. \tag{135}$$

Naively, this is the action of the two-dimensional Yang-Mills theory by integrating out the scalar σ. However, because of the condition $D_z\sigma = 0$, we can not perform the integration here.

The bosonic part and the fermionic part of the deformation $\delta_Q V_V$ are given by

$$\begin{aligned}
\delta_Q V_V\Big|_B &= \frac{1}{2}\mathrm{tr}\Big[\frac{1}{4}F_{mn}F^{mn} + g^{\bar{z}z}g^{mn}F_{mz}F_{n\bar{z}} + \frac{1}{2}D^m\sigma D_m\sigma + g^{\bar{z}z}D_{\bar{z}}\sigma D_z\sigma \\
&\quad + \frac{1}{2}D^2 + g^{\bar{z}z}\bar{F}_{\bar{z}}F_z + ik^m g^{\bar{z}z}\left(F_{mz}D_{\bar{z}}\sigma - F_{m\bar{z}}D_z\sigma\right) + ik_m\epsilon^{mnk}g^{\bar{z}z}F_{n\bar{z}}F_{kz}\Big], \\
\delta_Q V_V\Big|_F &= i\mathrm{tr}\Big[-\bar{\lambda}\gamma^m D_m\lambda - \frac{i}{2}\bar{\lambda}\lambda + g\bar{\lambda}[\sigma,\lambda] + \frac{1}{2}g^{z\bar{z}}\bar{\psi}_{\bar{z}}\gamma^m D_m\psi_z - \frac{i}{4}g^{z\bar{z}}\bar{\psi}_{\bar{z}}\psi_z \\
&\quad + \frac{1}{2}g^{\bar{z}z}\left(g\bar{\psi}_{\bar{z}}[\sigma,\psi_z] - ik_m\bar{\psi}_{\bar{z}}\gamma^m\psi_z + 2i\bar{\lambda}D_{\bar{z}}\psi_z - i\bar{\psi}_{\bar{z}}D_z\lambda + ik_m\bar{\psi}_{\bar{z}}\gamma^m D_z\lambda\right)\Big],
\end{aligned} \tag{136}$$

where the Killing vector

$$k_m = \bar{\epsilon}\gamma_m\epsilon, \tag{137}$$

and the normalization $\bar{\epsilon}\epsilon = 1$.

Around the fixed points, we will perform the path integral over the fluctuation. Since the bosonic fields σ and A_z have a nontrivial background as the fixed point, we expand the fields as

$$\sigma = \sigma(z,\bar{z}) + \frac{1}{\sqrt{t}}\tilde{\sigma}(x^m,z,\bar{z}) \quad , \quad A_z = A_z(z,\bar{z}) + \frac{1}{\sqrt{t}}\tilde{A}_z(x^m,z,\bar{z}), \tag{138}$$

while other fields $\Phi \to (1/\sqrt{t})\tilde{\Phi}$.

One also needs the gauge-fixing procedure for the calculation of the path integral. We will follow [11] and add the gauge-fixing term and the ghost term

$$\mathrm{tr}\left[\bar{C}\nabla_m D^m C + B\nabla^m A_m\right]. \tag{139}$$

There remains the residual gauge symmetry,

$$\sigma \to \sigma + ig[\omega(z,\bar{z}),\sigma] \quad , \quad A_z \to A_z - D_z\omega(z,\bar{z}). \tag{140}$$

The gauge transformation parameter ω is independent on the coordinate of the S^3. The symmetry is the redundancy of the background fields, but not of the fluctuations.

And the gauge fixing procedure can be carried out in a similar way to the two-dimensional Yang-Mills theory. [45][46][51] One can make use of the residual symmetry (140) to put the background $\sigma(z,\bar{z})$ in the Cartan subalgebra of the Lie algebra of G, such that

$$\sigma(z,\bar{z}) = \sum_{i=1}^{r} \sigma_i H_i, \tag{141}$$

where $H_i (i = 1, 2, \cdots r)$ are the generators of the Cartan subalgebra of G of rank r. Then, the condition $D_z\sigma = 0$ is given by

$$D_z\sigma = \partial_z\sigma + ig[A_z, \sigma] = 0. \tag{142}$$

Thus, the background field $A_z(z,\bar{z})$ should also be in the Cartan subalgebra of the Lie algebra of G such that

$$A_z(z,\bar{z}) = \sum_{i=1}^{r} A_z^i(z,\bar{z}) H_i, \tag{143}$$

and furthermore $\sigma_i (i = 1, 2, \cdots, r)$ are constant with respect to the local coordinates $z, \bar{z}$ of the Σ.

Therefore, to impose the gauge fixing-condition (141) (this means $\sigma^\alpha = 0$), we will follow the same BRST quantization procedure as for the two-dimensional Yang-Mills theory. Introducing another auxiliary field

$$b(z,\bar{z}) = \sum_{\alpha\in\Lambda} b^\alpha(z,\bar{z}) E_\alpha, \tag{144}$$

and another pair of ghosts fields

$$\begin{aligned} c(z,\bar{z}) &= \sum_{\alpha\in\Lambda} c^\alpha(z,\bar{z}) E_\alpha, \\ \bar{c}(z,\bar{z}) &= \sum_{\alpha\in\Lambda} \bar{c}^\alpha(z,\bar{z}) E_\alpha, \end{aligned} \tag{145}$$

where Λ is the set of all root of the Lie algebra of G, and the root generators E_α satisfy the algebra

$$[H_i, E_\alpha] = \alpha_i E_\alpha \quad , \quad [E_\alpha, E_{-\alpha}] = \sum_{i=1}^{r} \alpha_i H_i \equiv \alpha \cdot H. \tag{146}$$

we will add another gauge-fixing term and the ghost term

$$\sum_{\alpha\in\Lambda}\left[ib^{-\alpha}\sigma^{\alpha}-ig(\alpha\cdot\sigma)\bar{c}^{-\alpha}c^{\alpha}\right]. \tag{147}$$

After the integration over the root part σ^α, the auxiliary field b^α and the ghosts $\bar{c}^\alpha, c^\alpha$, they give the one-loop determinant

$$\prod_{\alpha\in\Lambda_+}\mathrm{Det}_{(0,0)}\left[g(\alpha\cdot\sigma)\right]^2, \tag{148}$$

which is the same contribution as the ghosts do in the two-dimensional Yang-Mills theory.

Let us proceed to compute the integration over the fluctuation around the fixed points. We will follow the same procedure as in [11][12]. We perform that by expanding all the fields in terms of the spherical harmonics on S^3. To do that , it is convenient to represent the Lagrangian in the differential forms. On the unit round S^3, we use the vielbein $e^a = e^a{}_m dx^m$ which obeys

$$de^a = \epsilon^{abc}e^b\wedge e^c, \tag{149}$$

The spin connection is given by $\omega^{ab}=\epsilon^{abc}e^c$. We define the Hodge duality as

$$\begin{aligned} *\, e^a &= \frac{1}{2}\epsilon^{abc}e^b\wedge e^c,\\ *\,(e^a\wedge e^c) &= \epsilon^{abc}e^c, \end{aligned} \tag{150}$$

with $*1$ the volume form, and we define two operators ι_k and S^a as

$$\begin{aligned} \iota_k e^a &= e^a{}_m k^m = k^a,\\ S^a e^b &= i\epsilon^{abc}e^c, \end{aligned} \tag{151}$$

where $k^m=\bar{\epsilon}\gamma^m\epsilon$ is the Killing vector.

Expanding the Lagrangian $\delta_Q V_V$ in terms of the fluctuations with $t\to\infty$, we obtain the quadratic Lagrangian. The bosonic part of the quadratic Lagrangian $\mathcal{L}^{(B)}_{VQ}$ in a differential form notation gives

$$\begin{aligned} &\frac{1}{2}\mathrm{tr}\Big[\frac{1}{2}d\tilde{A}\wedge *d\tilde{A}+\frac{1}{2}D\tilde{\sigma}\wedge *D\tilde{\sigma}+g^{\bar{z}z}\left(d\tilde{A}_{\bar{z}}-D_{\bar{z}}\tilde{A}\right)\wedge *\left(d\tilde{A}_z-D_z\tilde{A}\right)\\ &\quad+g^{\bar{z}z}\left(D_{\bar{z}}\tilde{\sigma}-ig[\sigma,\tilde{A}_{\bar{z}}]\right)\left(D_z\tilde{\sigma}-ig[\sigma,\tilde{A}_z]\right)*1+ig^{\bar{z}z}\left(D_{\bar{z}}\tilde{\sigma}-ig[\sigma,\tilde{A}_{\bar{z}}]\right)\iota_k\left(d\tilde{A}_z-D_z\tilde{A}\right)*1\\ &\quad-ig^{\bar{z}z}\left(D_z\tilde{\sigma}-ig[\sigma,\tilde{A}_z]\right)\iota_k\left(d\tilde{A}_{\bar{z}}-D_{\bar{z}}\tilde{A}\right)*1-g^{\bar{z}z}\left(d\tilde{A}_{\bar{z}}-D_{\bar{z}}\tilde{A}\right)\wedge *[(k\cdot S)(d\tilde{A}_z-D_z\tilde{A})]\Big], \end{aligned} \tag{152}$$

with $(k\cdot S)=k^aS^a$, where the form notation denotes

$$
\begin{aligned}
\tilde{A} &= \tilde{A}_m dx^m, \\
D\tilde{\sigma} &= d\tilde{\sigma} - ig[\sigma, \tilde{A}].
\end{aligned}
\tag{153}
$$

The gauge fields in the covariant derivatives D_z and $D_{\bar{z}}$ are the background $A_z(z, \bar{z})$ and $A_{\bar{z}}(z, \bar{z})$, respectively. Note that the integration over the auxiliary fields has already been done, and their contributions to the partition function is just an overall constant. For brevity, we omit the tilde $\tilde{}$ for the fermion fluctuations , because all fermionic fields are fluctuations.

The fermionic part of the deformation Lagrangian $\mathcal{L}_{VQ}^{(F)}$ is given by

$$
\begin{aligned}
\mathrm{tr}\Big[&- i\bar{\lambda}\gamma^m \nabla_m \lambda + \frac{1}{2}\bar{\lambda}\lambda + ig\bar{\lambda}[\sigma, \lambda] + \frac{i}{2} g^{z\bar{z}} \bar{\psi}_{\bar{z}} \gamma^m \nabla_m \psi_z + \frac{1}{4}\bar{\psi}_{\bar{z}}\psi_z \\
&+ \frac{i}{2} g g^{z\bar{z}} \bar{\psi}_{\bar{z}} [\sigma, \psi_z] + \frac{1}{2} g^{z\bar{z}} k_m \bar{\psi}_{\bar{z}} \gamma^m \psi_z + g^{z\bar{z}} D_{\bar{z}} \bar{\lambda} \psi_z + \frac{1}{2} g^{z\bar{z}} \bar{\psi}_{\bar{z}} (1 - k_m \gamma^m) D_z \lambda \Big].
\end{aligned}
\tag{154}
$$

In order to compute the integral over the fluctuations, we introduce the scalar spherical harmonics $\varphi_{l,m,\tilde{m}}$ on the S^3. $(l = 0, 1/2, 1, 3/2, \cdots \ ; \ m, \tilde{m} = -l, -l+1, \cdots, l)$ The harmonics have the following properties,

$$
\begin{aligned}
&- *d * d\varphi_{l,m,\tilde{m}} = 4l(l+1)\varphi_{l,m,\tilde{m}}, \\
&(\varphi_{l,m,\tilde{m}})^* = \varphi_{l,-m,-\tilde{m}}, \\
&\int_{S^3} (\varphi_{l',m',\tilde{m}'})^* \varphi_{l,m,\tilde{m}} * 1 = \delta_{l,l'} \delta_{m,m'} \delta_{\tilde{m},m'}.
\end{aligned}
\tag{155}
$$

We can expand $\tilde{\sigma}$ and $\tilde{A}_z$ by the harmonics,

$$
\begin{aligned}
\tilde{\sigma} &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{\tilde{m}=-l}^{l} \tilde{\sigma}_{l,m,\tilde{m}}(z, \bar{z}) \varphi_{l,m,\tilde{m}}(x^m), \\
\tilde{A}_z &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{\tilde{m}=-l}^{l} \tilde{A}_{z,l,m,\tilde{m}}(z, \bar{z}) \varphi_{l,m,\tilde{m}}(x^m).
\end{aligned}
\tag{156}
$$

The vector spherical harmonics are combined by the scalar spherical harmonics and the vielbein. We take the vielbein as the eigenstate of the operators $S^a S^a$ and S^3 as

$$
e^{\pm 1} = \mp \frac{1}{\sqrt{2}} (e^1 \pm ie^2) \quad , \quad e^0 = e^3, \tag{157}
$$

and the vector spherical harmonics on the S^3 are given by

$$
E_{J,M;l,\tilde{m}} = \sum_{m=-l}^{l} \sum_{s=\pm 1,0} \langle l, m; 1, s | J, M \rangle\rangle \, \varphi_{l,m,\tilde{m}} e^s. \tag{158}
$$

The $\langle l,m;1,s|J,M\rangle\rangle$ are the Clebsch-Gordan coefficients of the spin l representation and the spin 1 representation of the $SU(2)$ group into the spin J representation. ($J = l-1, l, l+1$) The Clebsch-Gordan coefficients are listed in appendix D. These harmonics have the properties

$$\begin{aligned}
&* dE_{l+1,M;l,\tilde{m}} = 2(l+1)E_{l+1,M;l,\tilde{m}} \ , \quad d * E_{l+1,M;l,\tilde{m}} = 0 \ , \qquad (l = 0, \frac{1}{2}, 1, \cdots) \\
&* dE_{l-1,M;l,\tilde{m}} = -2lE_{l-1,M;l,\tilde{m}} \ , \quad d * E_{l-1,M;l,\tilde{m}} = 0 \ , \qquad (l = 1, \frac{3}{2}, 2, \cdots) \\
&* dE_{l,M;l,\tilde{m}} = 0 \ , \quad E_{l,M;l,\tilde{m}} = -\frac{i}{2}\sqrt{\frac{1}{l(l+1)}}d\varphi_{l,m,\tilde{m}} \ , \qquad (l = \frac{1}{2}, 1, \frac{3}{2}, \cdots)
\end{aligned} \tag{159}$$

$$\int_{S^3} (E_{J,M;l,\tilde{m}})^* \wedge *E_{J',M';l',\tilde{m}'} = \delta_{J,J'}\delta_{M,M'}\delta_{l,l'}\delta_{\tilde{m},\tilde{m}'}. \tag{160}$$

In terms of the basis, the gauge field is expanded as $\tilde{A} = \tilde{A}_+ + \tilde{A}_- + \tilde{A}_L$,

$$\begin{aligned}
\tilde{A}_+ &= \sum_{l=0}^{\infty} \sum_{M=-l-1}^{l+1} \sum_{\tilde{m}=-l}^{l} \tilde{A}_{l+1,M;l,\tilde{m}}(z,\bar{z})E_{l+1,M;l,\tilde{m}}(x), \\
\tilde{A}_- &= \sum_{l=1}^{\infty} \sum_{M=-l+1}^{l-1} \sum_{\tilde{m}=-l}^{l} \tilde{A}_{l-1,M;l,\tilde{m}}(z,\bar{z})E_{l-1,M;l,\tilde{m}}(x), \\
\tilde{A}_L &= \sum_{l=1/2}^{\infty} \sum_{M=-l}^{l} \sum_{\tilde{m}=-l}^{l} \tilde{A}_{l,M;l,\tilde{m}}(z,\bar{z})E_{l,M;l,\tilde{m}}(x) \\
&= -\frac{i}{2}d\left[\sum_{l=1/2}^{\infty} \sum_{M=-l}^{l} \sum_{\tilde{m}=-l}^{l} \sqrt{\frac{1}{l(l+1)}}\tilde{A}_{l,M;l,\tilde{m}}(z,\bar{z})\varphi_{l,m,\tilde{m}}(x)\right] = d\tilde{u}_L,
\end{aligned} \tag{161}$$

where $\tilde{A}_\pm$ are the transverse modes and $\tilde{A}_L$ is the longitudinal mode. Substituting these expansions into the Lagrangian, we can find that the longitudinal mode $\tilde{u}_L$ can be eliminated in the Lagrangian by the shifting

$$\begin{aligned}
\tilde{A}_z &\to \tilde{A}_z + D_z\tilde{u}_L, \\
\tilde{\sigma} &\to \tilde{\sigma} + ig[\sigma, \tilde{u}_L].
\end{aligned} \tag{162}$$

Therefore, the longitudinal mode only appears in the gauge fixing term

$$\bar{C}d * dC + Bd * \tilde{A} = \bar{C}d * dC + Bd * \tilde{u}_L. \tag{163}$$

The one-loop determinant from the integral over B and $\tilde{u}_L$ cancels the one-loop determinant from the integral over C and $\bar{C}$.

As for the operators ι_k and $(k\cdot S)$ with the Killing vector k_a, we take the Killing spinor ϵ as constant, and then the Killing vector $k^a=\bar{\epsilon}\gamma^a\epsilon$ is also constant. Since $k^ak_a=1$ with the normalization $\bar{\epsilon}\epsilon=1$, we will choose it as $k^a=\delta^a{}_3$. Therefore, one obtains the formulas

$$\int_{S^3}(\varphi_{l',m',\tilde{m}'})^*\iota_k E_{J,M;l,\tilde{m}}*1=\int_{S^3}(\varphi_{l',m',\tilde{m}'})^*e^3\wedge *E_{J,M;l,\tilde{m}}=\delta_{l,l'}\delta_{\tilde{m},\tilde{m}'}\left\langle l,m';1,s=0|J,M\right\rangle\rangle$$
$$\int_{S^3}E_{J',M';l',\tilde{m}'}\wedge *[(k\cdot S)E_{J,M;l,\tilde{m}}]=\delta_{l,l'}\delta_{\tilde{m},\tilde{m}'}\left\langle\langle J',M'|S^3|J,M\rangle\right\rangle. \tag{164}$$

For the coefficients $\langle l,m';1,s=0|J,M\rangle\rangle$, $\langle\langle J',M'|S^3|J,M\rangle\rangle$, see appendix D.

We are not interested in the overall constant of the partition function, but its dependence on the background A_z and σ. The Cartan part $\tilde{\Phi}^i$ of the fluctuation is decoupled from the root part $\tilde{\Phi}^\alpha$ of the fluctuation and the background A_z and σ. Therefore, the integral over the $\tilde{\Phi}^i$ only contributes to the overall constant. We ignore the contribution of the Cartan part, and focus on the integral over $\tilde{\Phi}^\alpha$.

In the action $S^{(B)}_{VQ}=\int_{S^3}d^3x\mathcal{L}^{(B)}_{VQ}$ given in terms of the modes of the fluctuations, after completing the square by shifting, we find that,

$$S^{(B)}_{VQ}=\sum_{\alpha\in\Lambda_+}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{\tilde{m}=-l}^{l}S^{(B)}_{VQ:\alpha,l,m,\tilde{m}}+S^{(B)}_{VQ:H}, \tag{165}$$

where $S^{(B)}_{VQ:H}$ consists of the modes in the Cartan subalgebra of G. The modes in $S^{(B)}_{VQ;\alpha,l,m,\tilde{m}}$ decouple from the modes in the rest of $S^{(B)}_{VQ}$. The action $S^{(B)}_{VQ:\alpha,l,m,\tilde{m}}$ of the modes for $l\geq 1$, $-l+1\leq m\leq l-1$, $-l\leq\tilde{m}\leq l$ is given by

$$\begin{aligned}S^{(B)}_{VQ;\alpha,l,m,\tilde{m}}=&\frac{1}{2}\Big[K^\alpha_{l,m}g^{z\bar{z}}\left|\tilde{A}^\alpha_{z,l,m,\tilde{m}}\right|^2+K^\alpha_{l,-m}g^{z\bar{z}}\left|\tilde{A}^\alpha_{\bar{z},l,m,\tilde{m}}\right|^2+(\tilde{\sigma}^\alpha_{l,m,\tilde{m}})^*\Delta^\alpha_{l,m}\tilde{\sigma}^\alpha_{l,m,\tilde{m}}\\&+\begin{pmatrix}(\tilde{A}^\alpha_{l+1,m;l,\tilde{m}})^* & (\tilde{A}^\alpha_{l-1,m;l,\tilde{m}})^*\end{pmatrix}\begin{pmatrix}A^\alpha_{l,m} & B^\alpha_{l,m}\\ C^\alpha_{l,m} & D^\alpha_{l,m}\end{pmatrix}\begin{pmatrix}\tilde{A}^\alpha_{l+1,m;l,\tilde{m}}\\ \tilde{A}^\alpha_{l-1,m;l,\tilde{m}}\end{pmatrix}\Big],\end{aligned} \tag{166}$$

where the operators are defined by

$$\begin{aligned}K^\alpha_{l,m}&=\left[4l(l+1)-4m+g^2(\alpha\cdot\sigma)^2\right],\\ \Delta^\alpha_{l,m}&=-\frac{4(l-m)(l+m+1)}{K^\alpha_{l,m}}g^{z\bar{z}}D_{\bar{z}}D_z-\frac{4(l+m)(l-m+1)}{K^\alpha_{l,-m}}g^{z\bar{z}}D_zD_{\bar{z}}+4l(l+1),\\ A^\alpha_{l,m}&=U^\alpha_{l,m}\frac{1}{\Delta^\alpha_{l,m}}V^\alpha_{l,m}+u_{l,m}v_{l,m}\quad,\quad B^\alpha_{l,m}=U^\alpha_{l,m}\frac{1}{\Delta^\alpha_{l,m}}\tilde{V}^\alpha_{l,m}+u_{l,m}\tilde{v}_{l,m},\\ C^\alpha_{l,m}&=\tilde{U}^\alpha_{l,m}\frac{1}{\Delta^\alpha_{l,m}}V^\alpha_{l,m}+\tilde{u}_{l,m}v_{l,m}\quad,\quad D^\alpha_{l,m}=\tilde{U}^\alpha_{l,m}\frac{1}{\Delta^\alpha_{l,m}}\tilde{V}^\alpha_{l,m}+\tilde{u}_{l,m}\tilde{v}_{l,m},\end{aligned} \tag{167}$$

with

$$
\begin{aligned}
U^\alpha_{l,m} &= 2(l+m)[2(l+1)+ig(\alpha\cdot\sigma)]\sqrt{\frac{(l+1)(l+m+1)}{(2l+1)(l-m+1)}}\left[\frac{2}{K^\alpha_{l,-m}}g^{z\bar{z}}D_zD_{\bar{z}}-\frac{l}{l+m}\right],\\
\tilde{U}^\alpha_{l,m} &= 2(l+m+1)[2l-ig(\alpha\cdot\sigma)]\sqrt{\frac{l(l+m)}{(2l+1)(l-m)}}\left[\frac{2}{K^\alpha_{l,-m}}g^{z\bar{z}}D_zD_{\bar{z}}-\frac{l+1}{l-m+1}\right],\\
V^\alpha_{l,m} &= 2(l-m)[2(l+1)-ig(\alpha\cdot\sigma)]\sqrt{\frac{(l+1)(l-m+1)}{(2l+1)(l+m+1)}}\left[\frac{2}{K^\alpha_{l,m}}g^{z\bar{z}}D_{\bar{z}}D_z-\frac{l}{l-m}\right],\\
\tilde{V}^\alpha_{l,m} &= 2(l-m+1)[2l+ig(\alpha\cdot\sigma)]\sqrt{\frac{l(l-m)}{(2l+1)(l+m)}}\left[\frac{2}{K^\alpha_{l,-m}}g^{z\bar{z}}D_{\bar{z}}D_z-\frac{l+1}{l+m+1}\right],
\end{aligned}
\tag{168}
$$

$$
\begin{aligned}
u_{l,m} &= \sqrt{\frac{(l+1)(l-m+1)}{(2l+1)(l+m+1)}}[2(l+1)+ig(\alpha\cdot\sigma)],\\
v_{l,m} &= \sqrt{\frac{(l+1)(l+m+1)}{(2l+1)(l-m+1)}}[2(l+1)-ig(\alpha\cdot\sigma)],\\
\tilde{u}_{l,m} &= -\sqrt{\frac{l(l+m)}{(2l+1)(l-m)}}[2l-ig(\alpha\cdot\sigma)],\\
\tilde{v}_{l,m} &= -\sqrt{\frac{l(l-m)}{(2l+1)(l+m)}}[2l+ig(\alpha\cdot\sigma)].
\end{aligned}
\tag{169}
$$

Note here that the covariant derivatives acting on the root part of a field gives

$$
D_z\Phi^\alpha = \partial_z\Phi^\alpha + ig\sum_{i=1}^{r}\alpha_i A^i_z\Phi^\alpha. \tag{170}
$$

We can see that the one-loop determinant from these modes yields

$$
\begin{aligned}
&\prod_{\alpha\in\Lambda_+}\prod_{l=1}^{\infty}\prod_{m=-l+1}^{l-1}\prod_{\tilde{m}=-l}^{l}\Big[\frac{1}{\mathrm{Det}_{(1,0)}K^{\alpha}_{l,m}\mathrm{Det}_{(0,1)}K^{\alpha}_{l,-m}\mathrm{Det}_{(0,0)}\Delta^{\alpha}_{l,m}}\\
&\qquad\times\frac{\mathrm{Det}_{(0,0)}\Delta^{\alpha}_{l,m}}{\mathrm{Det}_{(0,0)}\left[\tilde{u}_{l,m}U^{\alpha}_{l,m}-u_{l,m}\tilde{U}^{\alpha}_{l,m}\right]\mathrm{Det}_{(0,0)}\left[\tilde{v}_{l,m}V^{\alpha}_{l,m}-v_{l,m}\tilde{V}^{\alpha}_{l,m}\right]}\Big]\\
&=\prod_{\alpha\in\Lambda_+}\prod_{l=1}^{\infty}\prod_{m=-l+1}^{l-1}\prod_{\tilde{m}=-l}^{l}\Big[\frac{1}{\mathrm{Det}_{(0,0)}[4l(l+1)]}\frac{\mathrm{Det}_{(0,0)}K^{\alpha}_{l,m}}{\mathrm{Det}_{(1,0)}K^{\alpha}_{l,m}}\frac{\mathrm{Det}_{(0,0)}K^{\alpha}_{l,-m}}{\mathrm{Det}_{(0,1)}K^{\alpha}_{l,-m}}\\
&\qquad\times\frac{\mathrm{Det}_{(0,0)}[4l^2+g^2(\alpha\cdot\sigma)^2]\mathrm{Det}_{(0,0)}[4(l+1)^2+g^2(\alpha\cdot\sigma)^2]}{\mathrm{Det}_{(0,0)}[2g^{z\bar{z}}D_{\bar{z}}D_z-K^{\alpha}_{l,m}]\mathrm{Det}_{(0,0)}[2g^{z\bar{z}}D_zD_{\bar{z}}-K^{\alpha}_{l,-m}]}\Big]\\
&=\prod_{\alpha\in\Lambda_+}\prod_{l=1}^{\infty}\prod_{m=-l+1}^{l-1}\prod_{\tilde{m}=-l}^{l}\Big[\frac{\mathrm{Det}_{(0,0)}K^{\alpha}_{l,m}}{\mathrm{Det}_{(1,0)}K^{\alpha}_{l,m}}\frac{1}{\mathrm{Det}_{(0,0)}[2g^{z\bar{z}}D_{\bar{z}}D_z-K^{\alpha}_{l,m}]}\\
&\qquad\times\frac{\mathrm{Det}_{(0,0)}[2(l+1)+ig(\alpha\cdot\sigma)]\mathrm{Det}_{(0,0)}[-2l+ig(\alpha\cdot\sigma)]}{\sqrt{\mathrm{Det}_{(0,0)}[4l(l+1)]}}\Big].
\end{aligned}
\tag{171}
$$

Further, after some similar algebra, the action $S^{(B)}_{VQ;\alpha,l,m,\tilde{m}}$ of the modes for $l\geq 1/2,\ m=-l,\ l-\leq\tilde{m}\leq l$ can be read as

$$
\begin{aligned}
&\frac{1}{2}\Big[K^{\alpha}_{l,-l}g^{z\bar{z}}\left|\tilde{A}^{\alpha}_{z,l,-l,\tilde{m}}\right|^2+K^{\alpha}_{l,l}g^{z\bar{z}}\left|\tilde{A}^{\alpha}_{\bar{z},l,-l,\tilde{m}}\right|^2+\frac{4l}{K^{\alpha}_{l,-l}}(\tilde{\sigma}_{l,-l,\tilde{m}})^*[-2g^{z\bar{z}}D_{\bar{z}}D_z+(l+1)K^{\alpha}_{l,-l}]\tilde{\sigma}^{\alpha}_{l,-l,\tilde{m}},\\
&\qquad+(l+1)(\tilde{A}^{\alpha}_{l+1,-l,l,\tilde{m}})^*\frac{K^{\alpha}_{l,-l-1}}{-2g^{z\bar{z}}D_{\bar{z}}D_z+(l+1)K^{\alpha}_{l,-l}}[-2g^{z\bar{z}}D_{\bar{z}}D_z+K^{\alpha}_{l,-l}]\tilde{A}^{\alpha}_{l+1,-l,l,\tilde{m}}\Big],
\end{aligned}
\tag{172}
$$

and the one for $l\geq 1/2,\ m=l,\ -l\leq\tilde{m}\leq l$ as

$$
\begin{aligned}
&\frac{1}{2}\Big[K^{\alpha}_{l,l}g^{z\bar{z}}\left|\tilde{A}^{\alpha}_{z,l,l,\tilde{m}}\right|^2+K^{\alpha}_{l,-l}g^{z\bar{z}}\left|\tilde{A}^{\alpha}_{\bar{z},l,l,\tilde{m}}\right|^2+\frac{4l}{K^{\alpha}_{l,-l}}(\tilde{\sigma}_{l,l,\tilde{m}})^*[-2g^{z\bar{z}}D_zD_{\bar{z}}+(l+1)K^{\alpha}_{l,-l}]\tilde{\sigma}^{\alpha}_{l,l,\tilde{m}}\\
&\qquad+(l+1)(\tilde{A}^{\alpha}_{l+1,l;l,\tilde{m}})^*\frac{K^{\alpha}_{l,-l-1}}{-2g^{z\bar{z}}D_zD_{\bar{z}}+(l+1)K^{\alpha}_{l,-l}}[-2g^{z\bar{z}}D_zD_{\bar{z}}+K^{\alpha}_{l,-l}]\tilde{A}^{\alpha}_{l+1,l;l,\tilde{m}}\Big].
\end{aligned}
\tag{173}
$$

These sectors with $l\geq 1/2,\ m=\pm l,\ -l\leq\tilde{m}\leq l$ gives the one-loop determinant

$$
\begin{aligned}
&\prod_{\alpha\in\Lambda_+}\prod_{l=1/2}^{\infty}\prod_{\tilde{m}=-l}^{l}\frac{1}{\mathrm{Det}_{(0,0)}[l^2(l+1)^2]}\left(\frac{\mathrm{Det}_{(0,0)}K^{\alpha}_{l,-l}}{\mathrm{Det}_{(1,0)}K^{\alpha}_{l,l}\mathrm{Det}_{(0,1)}K^{\alpha}_{l,-l}\mathrm{Det}_{(0,0)}K^{\alpha}_{l,-l-1}}\right)^2\\
&\qquad\times\frac{1}{\mathrm{Det}_{(0,0)}[-2g^{z\bar{z}}D_zD_{\bar{z}}+K^{\alpha}_{l,-l}]\mathrm{Det}_{(0,0)}[-2g^{z\bar{z}}D_{\bar{z}}D_z+K^{\alpha}_{l,-l}]}\\
&=\prod_{\alpha\in\Lambda}\prod_{l=1/2}^{\infty}\prod_{\tilde{m}=-l}^{l}\frac{1}{\mathrm{Det}_{(0,0)}[l(l+1)]}\frac{\mathrm{Det}_{(0,0)}K^{\alpha}_{l,-l}}{\mathrm{Det}_{(1,0)}K^{\alpha}_{l,l}\mathrm{Det}_{(0,1)}K^{\alpha}_{l,-l}\mathrm{Det}_{(0,0)}K^{\alpha}_{l,-l-1}}\\
&\qquad\times\frac{1}{\mathrm{Det}_{(0,0)}[-2g^{z\bar{z}}D_{\bar{z}}D_z+K^{\alpha}_{l,-l}]}.
\end{aligned}
\tag{174}
$$

Since the actions $S^{(B)}_{VQ;\alpha,l,\pm(l+1),\tilde{m}}$ of the modes with $l\geq 0$ take simple forms, we will give the sum

$$
\begin{aligned}
&S^{(B)}_{VQ;\alpha,l,l+1,\tilde{m}}+S^{(B)}_{VQ;\alpha,l,-(l+1),\tilde{m}}\\
=&\frac{1}{2}(\tilde{A}^{\alpha}_{l+1,l+1;l,\tilde{m}})^*[-2g^{z\bar{z}}D_zD_{\bar{z}}+K^{\alpha}_{l,-l-1}]\tilde{A}^{\alpha}_{l+1,l+1;l,\tilde{m}}\\
&+\frac{1}{2}(\tilde{A}^{\alpha}_{l+1,-(l+1);l,\tilde{m}})^*[-2g^{z\bar{z}}D_{\bar{z}}D_z+K^{\alpha}_{l,-l-1}]\tilde{A}^{\alpha}_{l+1,-(l+1);l,\tilde{m}},
\end{aligned}
\tag{175}
$$

to yield the one-loop determinant

$$
\prod_{\alpha\in\Lambda_+}\prod_{l=0}^{\infty}\prod_{\tilde{m}=-l}^{l}\frac{1}{\mathrm{Det}_{(0,0)}[-2g^{z\bar{z}}D_zD_{\bar{z}}+K^{\alpha}_{l,-l-1}]\mathrm{Det}_{(0,0)}[-2g^{z\bar{z}}D_{\bar{z}}D_z+K^{\alpha}_{l,-l-1}]}.
\tag{176}
$$

Finally, one can find the action $S^{(B)}_{VQ;\alpha,0,0,0}$

$$
\frac{1}{2}\left[K^{\alpha}_{0,-1}\left|\tilde{A}^{\alpha}_{1,0;0,0}\right|^2+g^{z\bar{z}}\left|D_z\tilde{A}^{\alpha}_{1,0;0,0}+g(\alpha\cdot\sigma)\tilde{A}^{\alpha}_{z,0,0,0}\right|^2+g^{z\bar{z}}\left|D_{\bar{z}}\tilde{A}^{\alpha}_{1,0;0,0}-g(\alpha\cdot\sigma)\tilde{A}^{\alpha}_{\bar{z},0,0,0}\right|^2\right],
\tag{177}
$$

of the modes with $l=0$, and it gives the one-loop determinant

$$
\prod_{\alpha\in\Lambda_+}\frac{1}{\mathrm{Det}_{(0,0)}K^{\alpha}_{0,-1}\mathrm{Det}_{(1,0)}[g(\alpha\cdot\sigma)]\mathrm{Det}_{(0,1)}[g(\alpha\cdot\sigma)]}.
\tag{178}
$$

Let us turn to the one-loop determinant from the fermionic fluctuations. To this end, we will identify the spin operator $\mathcal{S}^a$ ($a=1,2,3$) with the gamma matrix $(1/2)\gamma^a$ ($a=1,2,3$), respectively, and one can easily verify that they obeys the $SU(2)$ algebra

$$
\left[\mathcal{S}^a,\mathcal{S}^b\right]=i\epsilon^{abc}\mathcal{S}^c.
\tag{179}
$$

One can easily see that the left-invariant vector fields $L_a = (-i/2)e_a{}^m\nabla_m$ $(a = 1, 2, 3)$ also satisfy the $SU(2)$ algebra

$$\left[L^a, L^b\right] = i\epsilon^{abc}L^c. \tag{180}$$

Therefore, on the spinors one finds that

$$\begin{aligned}\gamma^m\nabla_m\lambda &= \gamma^a\left(2iL_a + \frac{1}{4}\omega_a{}^{bc}\gamma^{bc}\right)\lambda = 2i\left[(L_a + \mathcal{S}_a)^2 - L_aL_a\right]\lambda,\\ \gamma^m\nabla_m\psi &= \gamma^a\left(2iL_a + \frac{1}{4}\omega_a{}^{bc}\gamma^{bc}\right)\psi = 2i\left[(L_a + \mathcal{S}_a)^2 - L_aL_a\right]\psi.\end{aligned} \tag{181}$$

In order to obtain the spherical harmonics expansion of the spinors, it is useful to introduce the eigenspinors $\eta_{J,M;l,\tilde{m}}$ of the operator $\gamma^m\nabla_m$ by

$$\eta_{J,M;l,\tilde{m}} = \sum_{m=-l}^{l}\sum_{s=\pm 1/2}\langle l, m; \frac{1}{2}, s|J, M\rangle\rangle\varphi_{l,m,\tilde{m}}\zeta'_s, \tag{182}$$

with $\langle l, m; \frac{1}{2}, s|J, M\rangle\rangle$ the Clebsch-Gordan coefficients of the spin l representation and the spin 1/2 representation into the spin $J = l \pm 1/2$ representation, where the spinors $\zeta'_\pm$ satisfy that $\mathcal{S}^3\zeta'_\pm = \pm(1/2)\zeta'_\pm$.

They have their eigenvalues

$$\begin{aligned}\gamma^m\nabla_m\eta_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} &= i(2l + \frac{3}{2})\eta_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}},\\ \gamma^m\nabla_m\eta_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} &= -i(2l + \frac{1}{2})\eta_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}},\end{aligned} \tag{183}$$

and form the orthonormalized basis

$$\int_{S^3}(\eta_{J',M';l',\tilde{m}'})^\dagger\eta_{J,M;l,\tilde{m}} * 1 = \delta_{J,J'}\delta_{M,M'}\delta_{l,l'}\delta_{m,m'}. \tag{184}$$

Substituting the spherical harmonics expansion of the spinors

$$\begin{aligned}\lambda &= \sum_{l=0}^{\infty}\sum_{m=-l-1}^{l}\sum_{\tilde{m}=-l}^{l}\lambda_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}\eta_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} + \sum_{l=1/2}^{\infty}\sum_{m=-l}^{l-1}\sum_{\tilde{m}=-l}^{l}\lambda_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}\eta_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}},\\ \psi &= \sum_{l=0}^{\infty}\sum_{m=-l-1}^{l}\sum_{\tilde{m}=-l}^{l}\psi_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}\eta_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} + \sum_{l=1/2}^{\infty}\sum_{m=-l}^{l-1}\sum_{\tilde{m}=-l}^{l}\psi_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}\eta_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}},\end{aligned} \tag{185}$$

into the Lagrangian $\mathcal{L}_{VQ}^{(F)}$, one obtains the action $S_{VQ}^{(F)} = \int_{S^3} \mathcal{L}_{VQ}^{(F)} * 1$. There one finds the terms including

$$\int_{S^3} (\eta_{J',M';l',\tilde{m}'})^\dagger (1 - k_m \gamma^m) \eta_{J,M;l,\tilde{m}} * 1 = \langle\langle J', M' \left| 1 - 2\mathcal{S}^3 \right| J, M \rangle\rangle \delta_{l,l'} \delta_{m,m'}, \tag{186}$$

with our choice $k^a = \delta_3^a$. For the coefficients $\langle\langle J', M' | 1 - 2\mathcal{S}^3 | J, M \rangle\rangle$, see appendix D.

Similarly to the bosonic part, the modes $\lambda_{l\pm\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}, \psi_{l\pm\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}$ in each sector $(\alpha, l, m, \tilde{m})$ decouple from the modes in the other sectors, and therefore the action $S_{VQ}^{(F)}$ can be divided into the actions $S_{VQ;\alpha,l,m,\tilde{m}}^{(F)}$ of each sector $(l, m, \tilde{m})$ as

$$S_{VQ}^{(F)} = \sum_{\alpha\in\Lambda} \sum_{l=0}^{\infty} \sum_{m=-l-1}^{l} \sum_{\tilde{m}=-l}^{l} S_{VQ;\alpha,l,m,\tilde{m}}^{(F)} + S_{VQ;H}^{(F)}, \tag{187}$$

where $S_{VQ;H}^{(F)}$ consists of the modes in the Cartan subalgebra of G.

After completing the square and shifting the fields properly, the action $S_{VQ;\alpha,l,m,\tilde{m}}^{(F)}$ of the modes for $l \geq 1/2$, $-l \leq m \leq l-1$, $-l \leq \tilde{m} \leq l$, is given by

$$\begin{aligned} &\left(\left(\psi^\alpha_{z;l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}\right)^\dagger \quad \left(\psi^\alpha_{z;l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}\right)^\dagger \right) g^{z\bar{z}} \mathcal{K}^\alpha_{l,m} \begin{pmatrix} \psi^\alpha_{z;l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} \\ \psi^\alpha_{z;l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} \end{pmatrix} \\ &+ \left(\left(\lambda^\alpha_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}\right)^\dagger \quad \left(\lambda^\alpha_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}\right)^\dagger \right) \mathcal{M}^\alpha_{l,m} \begin{pmatrix} \lambda^\alpha_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} \\ \lambda^\alpha_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} \end{pmatrix}, \end{aligned} \tag{188}$$

where

$$\begin{aligned} \mathcal{K}^\alpha_{l,m} &= \begin{pmatrix} -\frac{2l(l+1)-m}{2l+1} + \frac{i}{2} g(\alpha\cdot\sigma) & -\frac{1}{2l+1}\sqrt{(l+m+1)(l-m)} \\ -\frac{1}{2l+1}\sqrt{(l+m+1)(l-m)} & \frac{2l(l+1)-m}{2l+1} + \frac{i}{2} g(\alpha\cdot\sigma) \end{pmatrix}, \\ \mathcal{M}^\alpha_{l,m} &= \begin{pmatrix} 2(l+1) + ig(\alpha\cdot\sigma) & \\ & -2l + ig(\alpha\cdot\sigma) \end{pmatrix} \\ &\quad \times \begin{pmatrix} -\frac{l-m}{2l+1}\frac{2}{K^\alpha_{l,m}} g^{z\bar{z}} D_{\bar{z}} D_z + 1 & -\frac{\sqrt{(l+m+1)(l-m)}}{2l+1}\frac{2}{K^\alpha_{l,m}} g^{z\bar{z}} D_{\bar{z}} D_z \\ -\frac{\sqrt{(l+m+1)(l-m)}}{2l+1}\frac{2}{K^\alpha_{l,m}} g^{z\bar{z}} D_{\bar{z}} D_z & -\frac{l+m+1}{2l+1}\frac{2}{K^\alpha_{l,m}} g^{z\bar{z}} D_{\bar{z}} D_z + 1 \end{pmatrix}, \end{aligned} \tag{189}$$

and one can see that to the one-loop determinant, they yield the contributions

$$\begin{aligned} &\prod_{\alpha\in\Lambda} \prod_{l=1/2}^{\infty} \prod_{m=-l}^{l-1} \prod_{\tilde{m}=-l}^{l} \frac{\mathrm{Det}_{(1,0)} K^\alpha_{l,m}}{\mathrm{Det}_{(0,0)} K^\alpha_{l,m}} \mathrm{Det}_{(0,0)}[2 g^{z\bar{z}} D_{\bar{z}} D_z - K^\alpha_{l,m}] \\ &\qquad \times \mathrm{Det}_{(0,0)} \left[(2(l+1) + ig(\alpha\cdot\sigma))(-2l + ig(\alpha\cdot\sigma)) \right], \end{aligned} \tag{190}$$

up to an overall normalization constant.

For the remaining fermionic modes with $l \geq 0,\ m = -l-1, l;\ -l \leq \tilde{m} \leq l$, after some similar algebra, one obtains

$$\begin{aligned}
&\left(\lambda^{\alpha}_{l+\frac{1}{2},l+\frac{1}{2};l,\tilde{m}}\right)^{\dagger}\left[2(l+1)+ig(\alpha\cdot\sigma)\right]\lambda^{\alpha}_{l+\frac{1}{2},l+\frac{1}{2};l,\tilde{m}} \\
&+\left(\lambda^{\alpha}_{l+\frac{1}{2},-l-\frac{1}{2};l,\tilde{m}}\right)^{\dagger}\frac{1}{\left[-2(l+1)+ig(\alpha\cdot\sigma)\right]}\left[2g^{z\bar{z}}D_{\bar{z}}D_z-K^{\alpha}_{l,-l-1}\right]\lambda^{\alpha}_{l+\frac{1}{2},-l-\frac{1}{2};l,\tilde{m}} \\
&+\frac{1}{2}g^{z\bar{z}}\left(\psi^{\alpha}_{z;l+\frac{1}{2},l+\frac{1}{2};l,\tilde{m}}\right)^{\dagger}\left[-2l+ig(\alpha\cdot\sigma)\right]\psi^{\alpha}_{z;l+\frac{1}{2},l+\frac{1}{2};l,\tilde{m}} \\
&+\frac{1}{2}g^{z\bar{z}}\left(\psi^{\alpha}_{z;l+\frac{1}{2},-l-\frac{1}{2};l,\tilde{m}}\right)^{\dagger}\left[-2(l+1)+ig(\alpha\cdot\sigma)\right]\psi^{\alpha}_{z;l+\frac{1}{2},-l-\frac{1}{2};l,\tilde{m}},
\end{aligned} \tag{191}$$

and finds the one-loop determinant

$$\begin{aligned}
\prod_{\alpha\in\Lambda}\prod_{l=0}^{\infty}\prod_{\tilde{m}=-l}^{l}&\mathrm{Det}_{(1,0)}\left[2(l+1)+ig(\alpha\cdot\sigma)\right]\mathrm{Det}_{(1,0)}\left[-2l+ig(\alpha\cdot\sigma)\right] \\
&\times\mathrm{Det}_{(0,0)}\left[-2g^{z\bar{z}}D_{\bar{z}}D_z+K^{\alpha}_{l,-l-1}\right],
\end{aligned} \tag{192}$$

up to an overall constant.

Wrapping up the contributions from the bosonic fluctuations and the fermionic fluctuations to the one-loop determinant, one obtains

$$\prod_{\alpha\in\Lambda_+}\frac{1}{\mathrm{Det}_{(1,0)}K^{\alpha}_{0,0}}\prod_{l=1/2}^{\infty}\left(\frac{\mathrm{Det}_{(0,0)}K^{\alpha}_{l,l}}{\mathrm{Det}_{(1,0)}K^{\alpha}_{l,l}}\right)^2, \tag{193}$$

up to an overall normalization constant. As shown in section 2.2

$$\mathrm{Det}_{(1,0)}K^{\alpha}_{l,m}=\frac{\left[\mathrm{Det}_{(0,0)}K^{\alpha}_{l,m}\right]\left(K^{\alpha}_{l,m}\right)^h}{K^{\alpha}_{l,m}}, \tag{194}$$

with the genus h of the Σ. Taking account of this result and the contribution from the ghost, one finds that the total one-loop determinant is given by

$$\begin{aligned}
&\left(\prod_{\alpha\in\Lambda_+}g\alpha\prod_{l=1/2}^{\infty}(2l+ig\alpha)(2l-ig\alpha)\right)^{\chi(\Sigma)} \\
\propto&\left(\prod_{\alpha\in\Lambda_+}\sin\left(i\pi g(\alpha\cdot\sigma)\right)\right)^{\chi(\Sigma)}.
\end{aligned} \tag{195}$$

This is the main result.

We are left to perform the path integral over the background gauge fields $\tilde{A}^i_z(z,\bar{z})$, along with the finite dimensional integral over the background σ^i. While the background of the gauge fields $\tilde{A}^i_z(z,\bar{z})$ obeying $\int_\Sigma F^i_{z\bar{z}} d\bar{z}\wedge dz \neq 0$ in the classical action $\mathcal{L}_{cl}$ can contribute to the path integral. Upon the integral over the gauge fields $\tilde{A}^i_z(z,\bar{z})$, the fluctuations of the gauge fields around the background do not appear in the rest of the path integral. One therefore needs to divide the integral over the fluctuations, along with the other possible constant factors.

3.3.4 Contribution From Hypermultiplet

Let us proceed to the hypermultiplet. In order to carry out the localization procedure, we will add the Lagrangian

$$\mathcal{L}_{HQ} = \delta_Q \left[(\delta_Q\tilde{\chi})^\dagger \tilde{\chi} + (\tilde{\chi})^\dagger \left(\delta_Q(\tilde{\chi})^\dagger\right)^\dagger + (\delta_Q\chi)^\dagger\chi + (\chi)^\dagger \left(\delta_Q(\chi)^\dagger\right)^\dagger \right], \tag{196}$$

to the Lagrangian $\mathcal{L}_{VQ} = \delta_Q V_V$. The total Lagrangian will thus be shifted as $\mathcal{L} \to \mathcal{L} - t(\mathcal{L}_{VQ} + \mathcal{L}_{HQ})$.

The fixed point is given by a solution to $\delta_Q\chi = 0,\ \delta_Q\tilde{\chi} = 0$ meaning that

$$\begin{aligned} D_m\tilde{H}\gamma^m + i\tilde{H} - g\left[\sigma,\tilde{H}\right] &= 0, \\ D_m H\gamma^m + iH - g\left[\sigma,H\right] &= 0, \end{aligned} \tag{197}$$

and to $\delta_Q\chi^\dagger = 0,\ \delta_Q\tilde{\chi}^\dagger = 0$. Since the solution to the former equations is given by $\tilde{H} = 0,\ H = 0$, substituting it into the latter equations, one obtains the solution $F_{H1} = F_{H2} = 0$. One thus finds no non-trivial backgrounds.

Then, up to quadratic order of the fluctuations, the bosonic part $\mathcal{L}^{(B)}_{HQ}$ of the Lagrangian $\mathcal{L}_{HQ}$ is given by

$$\begin{aligned} \mathrm{tr}\Big[& \left(D_m\tilde{H}\right)^\dagger D^m\tilde{H} + (D_mH)^\dagger D^m H \\ & + \left(\tilde{H} + ig\left[\sigma,\tilde{H}\right]\right)^\dagger \left(\tilde{H} + ig\left[\sigma,\tilde{H}\right]\right) + (H + ig[\sigma,H])^\dagger (H + ig[\sigma,H]) \\ & + \left(F_{H1} + 2iD_z\tilde{H}\right)^\dagger \left(F_{H1} + 2iD_z\tilde{H}\right) + (F_{H2} - 2i(D_zH)^*)^\dagger (F_{H2} - 2i(D_zH)^*) \Big]. \end{aligned} \tag{198}$$

The fermionic part $\mathcal{L}^{(F)}_{HQ}$ is given by

$$\mathrm{tr}\left[\tilde{\chi}^\dagger k_n\gamma^n \left(i\gamma^m D_m\tilde{\chi} + \frac{1}{2}\tilde{\chi} - ig[\sigma,\tilde{\chi}]\right) + \chi^\dagger k_n\gamma^n \left(i\gamma^m D_m\chi + \frac{1}{2}\chi - ig[\sigma,\chi]\right)\right]. \tag{199}$$

We will carry out similar calculations to what we have done for the vector multiplet by substituting the spherical harmonic expansions of the fluctuations

$$
\begin{aligned}
H &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{\tilde{m}=-l}^{l} H_{l,m,\tilde{m}}\varphi_{l,m,\tilde{m}} \quad , \quad \tilde{H} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{\tilde{m}=-l}^{l} \tilde{H}_{l,m,\tilde{m}}\varphi_{l,m,\tilde{m}}, \\
\chi &= \sum_{l=0}^{\infty}\sum_{m=-l-1}^{l}\sum_{\tilde{m}=-l}^{l} \chi_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}\eta_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} + \sum_{l=1/2}^{\infty}\sum_{m=-l}^{l-1}\sum_{\tilde{m}=-l}^{l} \chi_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}\eta_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}, \\
\tilde{\chi} &= \sum_{l=0}^{\infty}\sum_{m=-l-1}^{l}\sum_{\tilde{m}=-l}^{l} \tilde{\chi}_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}\eta_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} + \sum_{l=1/2}^{\infty}\sum_{m=-l}^{l-1}\sum_{\tilde{m}=-l}^{l} \tilde{\chi}_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}\eta_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}},
\end{aligned}
\tag{200}
$$

into $\mathcal{L}_{HQ}$. Recalling that $k_n\gamma^n = 2\mathcal{S}_3$ and using the Clebsch-Gordan coefficients in appendix D, one obtains the root part of $\mathcal{L}_{HQ}^{(B)}$

$$
\sum_{\alpha\in\Lambda}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{\tilde{m}=-l}^{l}\left[4l(l+1)+1+g^2(\alpha\cdot\sigma)^2\right]\left(\left|\tilde{H}^{\alpha}_{l,m,\tilde{m}}\right|^2+\left|H^{\alpha}_{l,m,\tilde{m}}\right|^2\right), \tag{201}
$$

and the root part of $\mathcal{L}_{HQ}^{(F)}$ is given by the sum of

$$
\begin{aligned}
\sum_{\alpha\in\Lambda}\Bigg(\sum_{l=0}^{\infty}\sum_{\tilde{m}=-l}^{l}\Big[&-\tilde{\chi}^{\alpha\dagger}_{l+\frac{1}{2},l+\frac{1}{2};l,\tilde{m}}[2l+1+ig(\alpha\cdot\sigma)]\tilde{\chi}^{\alpha}_{l+\frac{1}{2},l+\frac{1}{2};l,\tilde{m}} \\
&+\tilde{\chi}^{\alpha\dagger}_{l+\frac{1}{2},-l-\frac{1}{2};l,\tilde{m}}[2l+1+ig(\alpha\cdot\sigma)]\tilde{\chi}^{\alpha}_{l+\frac{1}{2},-l-\frac{1}{2};l,\tilde{m}}\Big] \\
-\sum_{l=1/2}^{\infty}\sum_{m=-l}^{l-1}\sum_{\tilde{m}=-l}^{l}\Bigg[&\begin{pmatrix}\tilde{\chi}^{\alpha\dagger}_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} & \tilde{\chi}^{\alpha\dagger}_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}\end{pmatrix}\begin{pmatrix}\mathcal{A}^{\alpha}_{l,m} & \mathcal{B}^{\alpha}_{l,m}\\ \mathcal{C}^{\alpha}_{l,m} & \mathcal{D}^{\alpha}_{l,m}\end{pmatrix}\begin{pmatrix}\tilde{\chi}^{\alpha}_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}\\ \tilde{\chi}^{\alpha}_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}\end{pmatrix}\Bigg]\Bigg),
\end{aligned}
\tag{202}
$$

and the same terms with $\tilde{\chi}$'s replaced by χ's , where

$$
\begin{aligned}
\mathcal{A}^{\alpha}_{l,m} &= \frac{2m+1}{2l+1}\left(2l+1+ig(\alpha\cdot\sigma)\right), \\
\mathcal{B}^{\alpha}_{l,m} &= -2\frac{\sqrt{(l+m+1)(l-m)}}{2l+1}\left(2l+1+ig(\alpha\cdot\sigma)\right), \\
\mathcal{C}^{\alpha}_{l,m} &= 2\frac{\sqrt{(l+m+1)(l-m)}}{2l+1}\left(2l+1-ig(\alpha\cdot\sigma)\right), \\
\mathcal{D}^{\alpha}_{l,m} &= \frac{2m+1}{2l+1}\left(2l+1-ig(\alpha\cdot\sigma)\right).
\end{aligned}
\tag{203}
$$

Note that the integration over the auxiliary fields has already been done, and their contributions to the partition function is just an overall constant.

We are not interested in the overall normalization constant of the partition function, to which the Cartan part contributes. Therefore, we will focus on the root part, as for the vector multiplet.

One can easily see that the one-loop determinant from the bosonic fluctuations yields

$$\prod_{\alpha\in\Lambda_+}\prod_{l=0}^{\infty}\prod_{m=-l}^{l}\prod_{\tilde{m}=-l}^{l}\left(\frac{1}{\mathrm{Det}_{(\frac{1}{2},0)}\left[(2l+1)^2+g^2(\alpha\cdot\sigma)^2\right]}\right)^2, \tag{204}$$

and the one from the fermionic fluctuations,

$$\prod_{\alpha\in\Lambda_+}\prod_{l=0}^{\infty}\prod_{m=-l}^{l}\prod_{\tilde{m}=-l}^{l}\left(\mathrm{Det}_{(\frac{1}{2},0)}\left[(2l+1)^2+g^2(\alpha\cdot\sigma)^2\right]\right)^2, \tag{205}$$

up to an overall constant.

Wrapping up them, one finds that the hypermultiplet contributes just a constant to the total partition function.

3.4 q-deformed Yang-Mills Theory

Wrapping up the above results, the total partition function of the $\mathcal{N}=2$ theory reduces to the following finite dimensional integral

$$Z_{5D\,SYM}=\mathcal{N}_{5D\,SYM}\sum_m\prod_{i=1}^{r}\int d\sigma^i\left[\prod_{\alpha\in\Lambda_+}2\sin\left(i\pi g(\alpha\cdot\sigma)\right)\right]^{\chi(\Sigma)}\exp\left[\int\mathcal{L}_{YM}d(\mathrm{vol})\right], \tag{206}$$

with m the first Chern number of the two-dimensional gauge field A_z on Σ, and the normalization constant $\mathcal{N}_{5D\,SYM}$ may be different from the normalization in the $\mathcal{N}=1$ theory.

Let us find the parameter q for the comparison with the prediction from the conjecture [34][35] for the six-dimensional $\mathcal{N}=(2,0)$ theory. We will replace the radius of the unit S^3 by l. For brevity, we will take $G=SU(2)$.

Then,

$$\left[\prod_{\alpha\in\Lambda_+}2\sin\left(i\pi g(\alpha\cdot\sigma)\right)\right]^{\chi(\Sigma)} \tag{207}$$

reduces into

$$\left[2\sin\left(i\sqrt{2}\pi gl\sigma\right)\right]^{\chi(\Sigma)}. \tag{208}$$

From the Lagrangian

$$\mathcal{L}_{YM} = 2\pi^2 l^3 \mathrm{tr}\left[-\left(\frac{\sigma}{l}\right)^2 + 2\frac{\sigma}{l} g^{z\bar{z}} F_{z\bar{z}}\right], \tag{209}$$

the classical action is given by

$$\int_\Sigma \mathcal{L}_{YM} d(\mathrm{vol}) = -2\pi^2 l \int_\Sigma d(\mathrm{vol})\sigma^2 - 4i\pi^2 l^2 \sigma \int_\Sigma F_{\bar{z}z} d\bar{z} \wedge dz. \tag{210}$$

We need the summation over the first Chern numbers of the two-dimensional gauge field on Σ. Here, let us explain the fact that the normalization of the first Chern number is given by

$$\int_\Sigma F_{\bar{z}z} d\bar{z} \wedge dz = \frac{2\sqrt{2}\pi}{g} m \quad , \quad m \in \mathbb{Z}. \tag{211}$$

Let us recall that the Cartan subalgebra of the $SU(2)$ gauge group is generated by $H = \sigma_3/\sqrt{2}$, with the normalization $\mathrm{tr}[HH] = 1$. The Lie algebra of the $SU(2)$ gauge group in fact is generated by H and $E_\pm$, which obey that

$$[H, E_\pm] = \pm\sqrt{2} E_\pm \quad , \quad [E_+, E_-] = \sqrt{2} H, \tag{212}$$

in our convention. Therefore, a field ψ in the fundamental representation of the gauge group can be decomposed into the eigenstates of H as

$$H\psi_\pm = \pm\frac{1}{\sqrt{2}}\psi_\pm, \tag{213}$$

and the covariant derivative gives

$$D\psi_+ = d\psi_+ + \frac{i}{\sqrt{2}} g A \psi_+. \tag{214}$$

Under a gauge transformation, the two-dimensional gauge field A_z transforms in a differential form notation as

$$A \to A - \frac{\sqrt{2}}{g} d\Omega, \tag{215}$$

and then the field ψ_+ transforms as

$$\psi_+ \to e^{i\Omega}\psi_+. \tag{216}$$

For brevity, let us take $\Sigma = S^2$ and consider two patches $U_N = \{(\theta,\phi)|0 \le \theta \le \pi/2\}$ and $U_S = \{(\theta,\phi)|\pi/2 \le \theta \le \pi\}$, covering the S^2 with the polar coordinates (θ,ϕ). On $U_N \bigcap U_S$, suppose that the section ψ_N of ψ_+ on U_N is related to the section ψ_S on U_S as $\psi_S = e^{i\Omega}\psi_N$. Then, the requirement that ψ_S be single-valued is satisfied if

$$\Omega = m\phi \quad , \quad (m \in \mathbb{Z}) \tag{217}$$

on the $U_N \bigcap U_S$. Then, the flux is determined as

$$\int_\Sigma dA = \int_{U_N \bigcap U_S} (A_N - A_S) = \int_{U_N \bigcap U_S} \frac{\sqrt{2}}{g} d\Omega = \frac{\sqrt{2}}{g} 2\pi m. \tag{218}$$

Substituting the gauge field configurations into the partition function and summing up over the Chern number m , one can see that the dominant contribution from the integration over σ is given by the points

$$\sigma = \frac{g}{\sqrt{2}} \frac{n}{4\pi^2 l^2}, \tag{219}$$

with $n \in \mathbb{Z}$. Since the measure at the dominant points of σ yields

$$\left(2\sin\left(i\frac{g^2}{4\pi l}n\right)\right)^{\chi(\Sigma)} = \left[e^{-\frac{g^2}{4\pi l}n} - e^{\frac{g^2}{4\pi l}n}\right]^{\chi(\Sigma)} = [n]_q^{\chi(\Sigma)} \left(q^{-1/2} - q^{1/2}\right)^{\chi(\Sigma)}, \tag{220}$$

we obtain the parameter

$$q = \exp\left(-\frac{g^2}{2\pi l}\right). \tag{221}$$

Following [34][35] , instanton solutions in the five-dimensional $\mathcal{N} = 2$ theory correspond to the Kaluza-Klein modes in the six-dimensional $\mathcal{N} = (2,0)$ theory compactified on S^1. In a four-dimensional $SU(2)$ gauge theory, the one-instanton solution gives the classical action

$$\int d^4x F'^a_{mn} F'^a_{mn} = 32\pi^2, \tag{222}$$

where $m, n = 1, \cdots, 4$, with the normalization

$$F'^a_{mn} = \partial_m A'^a_n - \partial_n A'^a_m + \epsilon^{abc} A'^b_m A'^c_n. \tag{223}$$

Therefore, for our convention, identifying

$$A^a_m = -\frac{1}{\sqrt{2}g} A'^a_m, \tag{224}$$

we can see that

$$F^a_{mn} = -\frac{1}{\sqrt{2}g} F'^a_{mn}, \tag{225}$$

thus in the five-dimensional theory, one finds the classical action

$$\int d^5X - \frac{1}{4} F^a_{MN} F^a_{MN} = -\frac{4\pi^2}{g^2} \int dX_5, \tag{226}$$

for the instanton solution of unit instanton charge.

For the six-dimensional $\mathcal{N} = (2,0)$ theory on S^1 of radius R, the instanton solution of unit instanton charge corresponds to the first KK modes, and so one obtains the relation

$$\frac{1}{R} = \frac{4\pi^2}{g^2}. \tag{227}$$

Therefore the parameter q (221) can be read as

$$q = \exp\left(-\frac{2\pi R}{l}\right), \tag{228}$$

which is in perfect agreement with the conjecture (49) reviewed in the section2.

Thus, we have seen that the partition function of the five-dimensional theory yields the partition function of the two-dimensional q-deformed Yang-Mills theory. It is consistent with the proposal [33]. Furthermore, in order to see that the parameter q found in the five-dimensional theory is identical to the q in the superconformal index in the four-dimensional theory, we identified the five-dimensional coupling g as the radius R of the S^1. This identification is also consistent with the prediction of the conjecture[34][35].

The hypermultiplet does not contribute to the partition function. However, the existence of the corresponding six-dimensional theory is not clear for us up to this point.

3.5 Wilson Loop

In the above subsections, we computed the partition function of the five-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory on $S^3 \times \Sigma$ by the localization method. In addition, we can compute correlation functions of BPS operators by this method. There are supersymmetric Wilson loops and we can compute the expectation values of the loops in a similar way to [11].

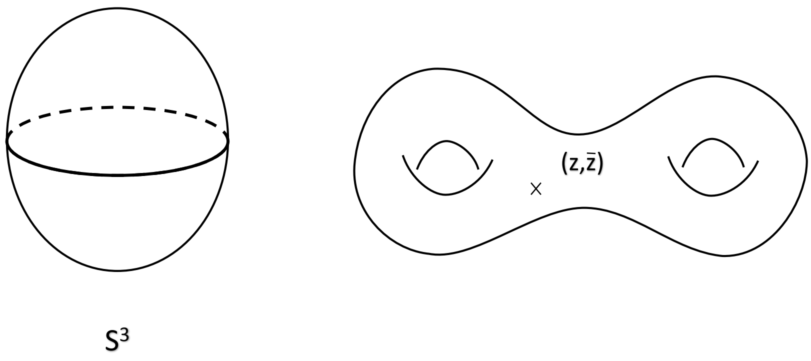

Figure 8: Wilson loop wrapped on the equator in S^3

We consider a loop wrapped on a one-cycle in the S^3. The Wilson loop is defined by

$$W_R(\gamma) = \frac{1}{Z_{5DSYM}} \mathrm{tr}_R \left[\mathcal{P} \exp \left(ig \oint_\gamma d\tau \left(A_m(x^m, z, \bar{z}) \dot{x}(\tau)^m + i\sigma(x^m, z, \bar{z}) \left| \dot{x}(\tau) \right| \right) \right) \right], \tag{229}$$

where γ is a closed path wrapped on a one-cycle in the S^3, and τ is a parameter of the path γ. $\mathcal{P}$ denotes the usual path-ordering operator. The scalar σ is already analytically continued, and it takes the real value. When we pick up an appropriate closed path, the Wilson loop operator is invariant under the supersymmetric transformation

$$\begin{aligned} \delta_\epsilon A_m &= -i \left[\bar{\epsilon} \gamma_m \lambda - \bar{\lambda} \gamma_m \epsilon \right], \\ \delta_\epsilon \sigma &= \bar{\epsilon} \lambda - \bar{\lambda} \epsilon. \end{aligned} \tag{230}$$

ϵ is a constant spinor which is a solution of the Killing spinor equation (89).

The variation of the Wilson loop under the transformation is given by

$$\delta_\epsilon W_R(\gamma) = \frac{1}{Z_{5DSYM}} \mathrm{tr}_R \left[\mathcal{P} \exp \left(g \oint_\gamma d\tau \left(\bar{\epsilon} \left[\gamma_m \dot{x}(\tau)^m - |\dot{x}(\tau)| \right] \lambda - \bar{\lambda} \left[\gamma_m \dot{x}(\tau)^m - |\dot{x}(\tau)| \right] \epsilon \right) \right) \right]. \tag{231}$$

Then, the condition that the operator is invariant gives

$$\left(\gamma_m \dot{x}(\tau)^m - |\dot{x}(\tau)| \right) \epsilon = 0. \tag{232}$$

If we pick up τ as the length of the path, $|\dot{x}(\tau)| = 1$. Then, $\gamma_m \dot{x}(\tau)^m$ has to be independent on τ and we pick up the closed path as a great circle of S^3. When we take $\gamma_m \dot{x}(\tau)^m = \gamma_3$, the condition becomes

$$(\gamma_3 - 1)\,\epsilon = 0. \tag{233}$$

For this closed path, the Wilson loop preserves half of the supersymmetries.

By using the above supersymmetries, we can compute its expectation value by the localization method. The expectation value is given by

$$\begin{aligned}\langle W_R(\gamma)\rangle &= \frac{1}{Z_{5DSYM}} \int \mathcal{D}\Phi \; W_R(\gamma) \exp\left[\int_{S^3\times\Sigma} d^5x \; \mathcal{L}\right] \\ &= \frac{\mathcal{N}_{5D\,SYM}}{Z_{5DSYM}} \sum_m \prod_{i=1}^{r} \int d\sigma^i \mathrm{tr}(e^{-2\pi g l \sigma}) \left[\prod_{\alpha\in\Lambda_+} 2\sin\left(i\pi g l(\alpha\cdot\sigma)\right)\right]^{\chi(\Sigma)} \exp\left[\int \mathcal{L}_{YM} d(\mathrm{vol})\right],\end{aligned} \tag{234}$$

with $\sigma = \sum_i \sigma_i H_i$ and σ_i is constant.

For $G = SU(2)$, the dominant contribution from the integration over σ is given by (219), then the expectation value becomes

$$\begin{aligned}\langle W_R(\gamma)\rangle &= \frac{\sum_{n'\in\mathbb{N}} [n']_q^{\chi(\Sigma)} \mathrm{tr}_R(q^{n'\frac{\sigma_3}{2}})}{\sum_{n'\in\mathbb{N}} [n']_q^{\chi(\Sigma)}} \\ &= \left(\frac{1}{\sum_{n'\in\mathbb{N}} [n']_q^{\chi(\Sigma)}}\right) \sum_{n'\in\mathbb{N}} [n']_q^{\chi(\Sigma)} \frac{[n'k]_q}{[n']_q} \;,\end{aligned} \tag{235}$$

where k is the dimension of the representation R and we took the zero area limit of the Riemann surface Σ. If we believe the 4D/2D duality, there is a corresponding operator in the four-dimensional theory on $S^1 \times S^3$. This is an open question.

4 Summary and Discussion

In this paper, we studied the five-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory on $S^3 \times \Sigma$ and clarified the relation between it and the two-dimensional q-deformed Yang-Mills theory on Σ. This result is compatible with the 4D/2D duality proposed in [33].

However, we do not understand the 4D/2D duality and the M5-branes underlying this duality, completely. There remain open questions.

Surface Operator

In the six-dimensional $\mathcal{N} = (2, 0)$ theory, there are surface operators which are supported on two-dimensional surfaces. The study of them is important to understand the six-dimensional theory. Let us consider surface operators in terms of the 4D/2D duality.

We consider the six-dimensional $\mathcal{N} = (2, 0)$ theory on $S^1 \times S^3 \times \Sigma$. Let us consider a surface operator which is wrapped on the S^1 and a one-cycle in the S^3. We obtain the surface operator in the four-dimensional theory on $S^1 \times S^3$, and the loop operator in the five-dimensional theory on $S^3 \times \Sigma$. The expectation values of them have to be equal each other. In the five-dimensional theory, we computed the Wilson loop wrapped on the great circle of the S^3. Comparing the Wilson loop with the surface operator in the four-dimensional theory is an interesting topic.

Let us consider another surface operator wrapped on the S^1 and a one-cycle in the Σ. In the four-dimensional theory on $S^1 \times S^3$, this operator becomes a loop operator wrapped on the S^1. In the five-dimensional theory on $S^3 \times \Sigma$, the surface operator becomes another loop operator wrapped on the one-cycle in the Σ. By computing the two operators, we will be able to see the correspondence between them.

General Superconformal Index

We considered the special superconformal index parameterized by q in the four-dimensional theory, and the index is equal to the partition function of the q-deformed Yang-Mills theory. In general, the superconformal index is parametrized by three chemical potentials (p, q, t).[43][44] It is expected that the general index gives rise to the (p, q, t)-deformed Yang-Mills theory. It is interesting to extend the localization technique to this case.

N^3 Behavior

The free energy for N coincident M5-branes grows as N^3.[5] The five-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory describes the M5-branes theory on S^1, then the free energy for the five-dimensional theory has to grow as N^3. For example, the free energy for the five-dimensional $SU(N)$ supersymmetric Yang-Mills theory on S^5 grows as N^3.[37]

In the case of the five-dimensional theory on $S^3 \times \Sigma$, its partition function is equal to the partition function of the two-dimensional q-deformed Yang-Mills theory. Therefore, the free energy for the two-dimensional theory has to grow as N^3. In the gravity side, the M5-branes wrapped on a Riemann surface are studied.[52] It is interesting to compare the free energy for the two-dimensional theory with the result in the gravity side.

Other Dualities

4D/2D duality is related to M5-branes on a closed manifold $M_4 \times \Sigma$. In analogy with this duality, 3D/3D duality was proposed.[53] [54] The 3D/3D duality comes from M5-branes on $M_3 \times X_3$ with two three-dimensional closed manifolds M_3 and X_3. It is expected that the three-dimensional theory on X_3 is dual to another three-dimensional theory on M_3. In some special case, this duality was also investigated via the five-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory.[23][24][55] We would be able to apply this method to study other dualities.

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A Character

In this appendix, we explain important properties of a character. A representation R of a Lie group G is defined by a map

$$\rho_R \ : \ G \ \rightarrow \ GL(V_R), \tag{236}$$

with the representation space V_R. The character χ_R is defined by

$$\chi_R(U) = \mathrm{tr}_R(U) \ , \ U \in G \ , \tag{237}$$

and satisfies the following properties

$$\begin{aligned} &\chi_{R\oplus R'}(U) = \chi_R(U) + \chi_{R'}(U), \\ &\chi_{R\otimes R'}(U) = \chi_R(U)\chi_{R'}(U), \\ &\chi_R(1) = \mathrm{dim}R = \mathrm{dim}V_R, \\ &\int [dU]\bar{\chi}_R(U)\chi_{R'}(U) = \delta_{R,R'}, \end{aligned} \tag{238}$$

with the Haar measure $[dU]$. Let $\mathrm{Hom}(V_R, V_R)$ denote the set of all linear transformations from V_R to V_R. We define an action of $U \in G$ on $f \in \mathrm{Hom}(V_R, V_R)$ by

$$(U \cdot f)(v) \equiv \rho_R(U) f(\rho_R(U)^{-1} v) \ . \qquad (v \in V_R) \tag{239}$$

We consider R is an irreducible representation. Let us consider a map

$$F \equiv \int [dU](U \cdot f) \ \ : V_R \rightarrow V_R. \tag{240}$$

For any $V \in G$, $v \in V_R$,

$$\begin{aligned} \rho_R(V) F\, v &= \int [dU] \rho_R(V) \rho_R(U) f(\rho_R(U)^{-1} v) \\ &= \int [dW] \rho_R(W) f(\rho_R(W)^{-1} \rho_R(V) v) = F \rho_R(V) v. \end{aligned} \tag{241}$$

Therefore, by Schur's lemma, this map is proportional to the identity map,

$$\int [dU](U \cdot f) = c \times id, \tag{242}$$

with $c \in \mathbb{C}$. Then,

$$\mathrm{tr}_R \left(\int [dU](U \cdot f) \right) = c \ \mathrm{dim} R. \tag{243}$$

The left hand side becomes

$$L.H.S = \mathrm{tr}_R \left(\int [dU] \rho_R(U) f \rho_R(U)^{-1} \right) = \mathrm{tr}_R(f). \tag{244}$$

Thus, $c = \mathrm{tr}_R(f)/\mathrm{dim}R$,

$$\int [dU](U \cdot f) = \frac{\mathrm{tr}_R(f)}{\mathrm{dim}R} \ . \tag{245}$$

When $f = \rho_R(W)$, this equality becomes

$$\int [dU] \rho_R(UWU^{-1}) = \frac{\chi_R(W)}{\mathrm{dim}R}. \tag{246}$$

Multiplying both sides by $\rho_R(V)$, the trace of them gives

$$\int [dU] \chi_R(UWU^{-1}V) = \frac{\chi_R(W)\chi_R(V)}{\mathrm{dim}R}. \tag{247}$$

B Gamma Matrix

The five-dimensional gamma matrices Γ^M satisfy

$$\left\{ \Gamma^M, \Gamma^N \right\} = 2\delta^{MN}, \tag{248}$$

and they are given in terms of the three-dimensional gamma matrices $\gamma^m = \sigma_m$ ($m = 1, 2, 3$) as

$$\Gamma^m = \gamma^m \otimes \sigma_2 \quad , \quad \Gamma^4 = 1 \otimes \sigma_1 \quad , \quad \Gamma^5 = 1 \otimes \sigma_3, \tag{249}$$

where σ_m are the Pauli matrices.

The five-dimensional charge conjugation matrix C_5 satisfies

$$\begin{aligned} (\Gamma^M)^T &= C_5 \Gamma^M C_5^{-1}, \\ (C_5)^T &= -C_5. \end{aligned} \tag{250}$$

It is given in terms of the three-dimensional charge conjugate matrix $C_3 = i\sigma_2$ as

$$C_5 = C_3 \otimes 1. \tag{251}$$

For five-dimensional spinors Ψ and Ξ, the Fierz identity is given by

$$\Xi\bar{\Psi} = -\frac{1}{4}\bar{\Psi}\Xi - \frac{1}{4}\left(\bar{\Psi}\Gamma_M\Xi\right)\Gamma^M + \frac{1}{8}\left(\bar{\Psi}\Gamma_{MN}\Xi\right)\Gamma^{MN}. \tag{252}$$

C Calculation of SUSY on $S^3 \times \Sigma$

In this appendix, we compute the algebra of the supersymmetry and the variation of the Lagrangian in detail.

C.1 Algebra

$\Sigma^{\dot{\alpha}}$ and $\Theta^{\dot{\alpha}}$ are defined by (90) and (108), respectively. The commutation relations of $\delta^{(0)}$ are computed as

$$\begin{aligned}
[\delta^{(0)}_\Theta, \delta^{(0)}_\Sigma]A_M &= \xi^K\nabla_K A_M - D_M\omega + (\nabla_M\xi^K)A_K - i\sigma\nabla_M(\bar{\Sigma}_{\dot{\alpha}}\Theta^{\dot{\alpha}}),\\
[\delta^{(0)}_\Theta, \delta^{(0)}_\Sigma]\sigma &= \xi^K\nabla_K\sigma + ig[\omega,\sigma],\\
[\delta^{(0)}_\Theta, \delta^{(0)}_\Sigma]\Psi^{\dot{\alpha}} &= \xi^M\nabla_M\Psi^{\dot{\alpha}} + ig[\omega,\Psi^{\dot{\alpha}}],\\
&\quad + \frac{i}{2}\left[\Gamma^{MN}\Sigma^{\dot{\alpha}}(\nabla_M\bar{\Theta}_{\dot{\beta}})\Gamma_N\Psi^{\dot{\beta}} - \Gamma^M\Sigma^{\dot{\alpha}}(\nabla_M\bar{\Theta}_{\dot{\beta}})\Psi^{\dot{\beta}} - (\Theta\leftrightarrow\Sigma)\right],\\
[\delta^{(0)}_\Theta, \delta^{(0)}_\Sigma]D^{\dot{\alpha}}{}_{\dot{\beta}} &= \xi^M\nabla_M D^{\dot{\alpha}}{}_{\dot{\beta}} + ig[\omega, D^{\dot{\alpha}}{}_{\dot{\beta}}]\\
&\quad - \frac{i}{2}\Big[F_{KL}(-(\nabla_M\bar{\Theta}_{\dot{\beta}})\Gamma_{KL}\Gamma^M\Sigma^{\dot{\alpha}} - \bar{\Theta}_{\dot{\beta}}\Gamma^M\Gamma^{KL}\nabla_M\Sigma^{\dot{\alpha}})\\
&\qquad + 2D_K\sigma((\nabla_M\bar{\Theta}_{\dot{\beta}})\Gamma^K\Gamma^M\Sigma^{\dot{\alpha}} - \bar{\Theta}_{\dot{\beta}}\Gamma^M\Gamma^K\nabla_M\Sigma^{\dot{\alpha}})\\
&\qquad - D^{\dot{\gamma}}{}_{\dot{\beta}}(\nabla_M\bar{\Theta}_{\dot{\gamma}})\Gamma^M\Sigma^{\dot{\alpha}} - D^{\dot{\alpha}}{}_{\dot{\gamma}}\bar{\Theta}_{\dot{\beta}}\Gamma^M\nabla_M\Sigma^{\dot{\gamma}} - (\Theta\leftrightarrow\Sigma)\Big],
\end{aligned} \tag{253}$$

with $\xi^K = i\bar{\Theta}_{\dot{\alpha}}\Gamma^K\Sigma^{\dot{\alpha}}$, $\omega = \xi^M A_M + i\bar{\Theta}_{\dot{\alpha}}\Sigma^{\dot{\alpha}}\sigma$. Therefore, the algebras are not closed. By using the Fierz identity and the Killing spinor equations (95), the right hand sides are simplified as

$$
\begin{aligned}
[\delta_\Theta^{(0)}, \delta_\Sigma^{(0)}]\Psi^{\dot\alpha} = {}& \xi^M \nabla_M \Psi^{\dot\alpha} + ig[\omega, \Psi^{\dot\alpha}] \\
& + \frac{i}{8}\Big\{ \Big[15\bar{\tilde{\Theta}}_{\dot\beta}\Sigma^{\dot\alpha} + 3\bar{\tilde{\Theta}}_{\dot\beta}\Gamma^i\Sigma^{\dot\alpha}\Gamma_i - \bar{\tilde{\Theta}}_{\dot\beta}\Gamma^m\Sigma^{\dot\alpha}\Gamma_m \\
& - \frac{3}{2}\bar{\tilde{\Theta}}_{\dot\beta}\Gamma^{ij}\Sigma^{\dot\alpha}\Gamma_{ij} + \bar{\tilde{\Theta}}_{\dot\beta}\Gamma^{im}\Sigma^{\dot\alpha}\Gamma_{im} + \frac{5}{2}\bar{\tilde{\Theta}}_{\dot\beta}\Gamma^{mn}\Sigma^{\dot\alpha}\Gamma_{mn}\Big]\Psi^{\dot\beta} - (\Theta \leftrightarrow \Sigma)\Big\},
\end{aligned}
\tag{254}
$$

$$
\begin{aligned}
[\delta_\Theta^{(0)}, \delta_\Sigma^{(0)}]D^{\dot\alpha}{}_{\dot\beta} = {}& \xi^M \nabla_M D^{\dot\alpha}{}_{\dot\beta} + ig[\omega, D^{\dot\alpha}{}_{\dot\beta}] \\
& + \frac{i}{2}\Big\{ \Big[-\frac{1}{2}F_{mn}\left(\bar{\tilde{\Theta}}_{\dot\beta}\Gamma^{mn}\Sigma^{\dot\alpha} - \bar{\Sigma}_{\dot\beta}\Gamma^{mn}\tilde{\Theta}^{\dot\alpha}\right) + \frac{3}{2}F_{ij}\left(\bar{\tilde{\Theta}}_{\dot\beta}\Gamma^{ij}\Sigma^{\dot\alpha} - \bar{\Sigma}_{\dot\beta}\Gamma^{ij}\tilde{\Theta}^{\dot\alpha}\right) \\
& + F_{im}\left(\bar{\tilde{\Theta}}_{\dot\beta}\Gamma^{im}\Sigma^{\dot\alpha} - \bar{\Sigma}_{\dot\beta}\Gamma^{mi}\tilde{\Theta}^{\dot\alpha}\right) + D_m\sigma\left(\bar{\tilde{\Theta}}_{\dot\beta}\Gamma^{m}\Sigma^{\dot\alpha} + \bar{\Sigma}_{\dot\beta}\Gamma^{m}\tilde{\Theta}^{\dot\alpha}\right) \\
& + 3D_i\sigma\left(\bar{\tilde{\Theta}}_{\dot\beta}\Gamma^{i}\Sigma^{\dot\alpha} + \bar{\Sigma}_{\dot\beta}\Gamma^{i}\tilde{\Theta}^{\dot\alpha}\right) + 3D^{\dot\gamma}{}_{\dot\beta}\bar{\tilde{\Theta}}_{\dot\gamma}\Sigma^{\dot\alpha} - 3D^{\dot\alpha}{}_{\dot\gamma}\bar{\Sigma}_{\dot\beta}\tilde{\Theta}^{\dot\gamma}\Big] - (\Theta \leftrightarrow \Sigma)\Big\}.
\end{aligned}
\tag{255}
$$

In order to simplify them, we use the following properties for symplectic Majorana spinors $\Psi^{\dot\alpha}, \Lambda^{\dot\alpha}$,

$$
\begin{aligned}
\bar{\Psi}_{\dot\alpha}\Lambda^{\dot\beta} &= \bar{\Lambda}_{\dot\alpha}\Psi^{\dot\beta} - \bar{\Lambda}_{\dot\gamma}\Psi^{\dot\gamma}\delta_{\dot\alpha}{}^{\dot\beta}, \\
\bar{\Psi}_{\dot\alpha}\Gamma^M\Lambda^{\dot\beta} &= \bar{\Lambda}_{\dot\alpha}\Gamma^M\Psi^{\dot\beta} - \bar{\Lambda}_{\dot\gamma}\Gamma^M\Psi^{\dot\gamma}\delta_{\dot\alpha}{}^{\dot\beta}, \\
\bar{\Psi}_{\dot\alpha}\Gamma^{MN}\Lambda^{\dot\beta} &= -\bar{\Lambda}_{\dot\alpha}\Gamma^{MN}\Psi^{\dot\beta} + \bar{\Lambda}_{\dot\gamma}\Gamma^{MN}\Psi^{\dot\gamma}\delta_{\dot\alpha}{}^{\dot\beta}.
\end{aligned}
\tag{256}
$$

One can prove them by the condition (76). In addition, because of the conditions (90) and (108), one can obtain,

$$
\begin{aligned}
&\bar{\tilde{\Theta}}_{\dot\beta}\Gamma^m\Sigma^{\dot\alpha} - \bar{\tilde{\Sigma}}_{\dot\beta}\Gamma^m\Theta^{\dot\alpha} = 0, \qquad\qquad (\dot\alpha \neq \dot\beta) \\
&\bar{\tilde{\Theta}}_{\dot\beta}\Gamma^i\Sigma^{\dot\alpha} - \bar{\tilde{\Sigma}}_{\dot\beta}\Gamma^i\Theta^{\dot\alpha} = 0, \\
&\bar{\tilde{\Theta}}_{\dot\beta}\Gamma^{mi}\Sigma^{\dot\alpha} - \bar{\tilde{\Sigma}}_{\dot\beta}\Gamma^{mi}\Theta^{\dot\alpha} = \begin{cases} 0, & (\dot\alpha = \dot\beta) \\ 2\bar{\tilde{\Theta}}_{\dot\beta}\Gamma^{mi}\Sigma^{\dot\alpha}, & (\dot\alpha \neq \dot\beta) \end{cases} \\
&\bar{\tilde{\Theta}}_{\dot\beta}\Gamma^{mn}\Sigma^{\dot\alpha} - \bar{\tilde{\Sigma}}_{\dot\beta}\Gamma^{mn}\Theta^{\dot\alpha} = \bar{\tilde{\Theta}}_{\dot\gamma}\Gamma^{mn}\Sigma^{\dot\gamma}\delta^{\dot\alpha}{}_{\dot\beta}, \\
&\bar{\tilde{\Theta}}_{\dot\beta}\Gamma^{ij}\Sigma^{\dot\alpha} - \bar{\tilde{\Sigma}}_{\dot\beta}\Gamma^{ij}\Theta^{\dot\alpha} = \bar{\tilde{\Theta}}_{\dot\gamma}\Gamma^{ij}\Sigma^{\dot\gamma}\delta^{\dot\alpha}{}_{\dot\beta}.
\end{aligned}
\tag{257}
$$

We use above properties and add the modification (105) to the supersymmetric transformation, and we obtain the closed algebra (107),

$$
\begin{aligned}
&[\delta_\Theta, \delta_\Sigma]A_M = \xi^K\nabla_K A_M + (\nabla_M\xi^K)A_K - D_M\omega, \\
&[\delta_\Theta, \delta_\Sigma]\Psi^{\dot\alpha} = \xi^M\nabla_M\Psi^{\dot\alpha} + ig[\omega, \Psi^{\dot\alpha}] + 2i\left(\bar{\tilde{\Theta}}_{\dot\beta}\Sigma^{\dot\alpha} - \bar{\tilde{\Sigma}}_{\dot\beta}\Theta^{\dot\alpha}\right)\Psi^{\dot\beta} + \frac{i}{4}(\bar{\tilde{\Theta}}_{\dot\gamma}\Gamma^{mn}\Sigma^{\dot\gamma} - \bar{\tilde{\Sigma}}_{\dot\gamma}\Gamma^{mn}\Theta^{\dot\gamma})\Gamma_{mn}\Psi^{\dot\alpha}, \\
&[\delta_\Theta, \delta_\Sigma]D^{\dot\alpha}{}_{\dot\beta} = \xi^M\nabla_M D^{\dot\alpha}{}_{\dot\beta} + ig[\omega, D^{\dot\alpha}{}_{\dot\beta}] \\
&\qquad + 2i\left(\bar{\tilde{\Theta}}_{\dot\gamma}\Sigma^{\dot\alpha} - \bar{\tilde{\Sigma}}_{\dot\gamma}\Theta^{\dot\alpha}\right)D^{\dot\gamma}{}_{\dot\beta} - 2i\left(\bar{\tilde{\Theta}}_{\dot\beta}\Sigma^{\dot\gamma} - \bar{\tilde{\Sigma}}_{\dot\beta}\Theta^{\dot\gamma}\right)D^{\dot\alpha}{}_{\dot\gamma}.
\end{aligned}
\tag{258}
$$

As for the hypermultiplet,

$$
\begin{aligned}
[\delta^{(0)}_\Theta, \delta^{(0)}_\Sigma] H_{\dot\alpha} &= \xi^K \nabla_K H_{\dot\alpha} + ig[\omega, H_{\dot\alpha}], \\
[\delta^{(0)}_\Theta, \delta^{(0)}_\Sigma] \Xi &= \xi^K \nabla_K \Xi + ig[\omega, \Xi] - i\Gamma^M \left[\Sigma^{\dot\alpha} \nabla_M \bar\Theta_{\dot\alpha} \Xi - (\Theta \leftrightarrow \Sigma)\right] \\
&= \xi^K \nabla_K \Xi + ig[\omega, \Xi] - \frac{i}{8}\left(\bar\Theta_{\dot\gamma}\Gamma^{mn}\tilde\Sigma^{\dot\gamma} - \bar\Sigma_{\dot\gamma}\Gamma^{mn}\tilde\Theta^{\dot\gamma}\right)\gamma_{mn}\Xi + \frac{3}{4} i\bar\Theta_{\dot\gamma}\Sigma^{\dot\gamma}\Gamma_{45}\Xi, \\
[\delta^{(0)}_\Theta, \delta^{(0)}_\Sigma] F_{H\alpha} &= \xi^K \nabla_K F_{H\alpha} + ig[\omega, F_{H\alpha}] \\
&\quad - i\Big[F_{H\beta}\bar{\check\Theta}_\alpha \Gamma^M \nabla_M \check\Sigma^\beta + (D_N H_{\dot\beta})\bar{\check\Theta}_\alpha \Gamma^M \Gamma^N \nabla_M \Sigma^{\dot\beta} \\
&\qquad + ig[\sigma, H_{\dot\beta}]\bar{\check\Theta}_\alpha \Gamma^M \nabla_M \Sigma^{\dot\beta} - (\Theta \leftrightarrow \Sigma)\Big] \\
&= \xi^K \nabla_K F_{H\alpha} + ig[\omega, F_{H\alpha}] - i\left(\bar{\check\Sigma}_\alpha \Gamma^m \tilde\Theta^{\dot\beta} - \bar{\check\Theta}_\alpha \Gamma^m \tilde\Sigma^{\dot\beta}\right) D_m H_{\dot\beta} \\
&\quad - 3i\left(\bar{\check\Sigma}_\alpha \Gamma^i \tilde\Theta^{\dot\beta} - \bar{\check\Theta}_\alpha \Gamma^i \tilde\Sigma^{\dot\beta}\right) D_i H_{\dot\beta} - 3\bar\Theta_{\dot\gamma}\Sigma^{\dot\gamma} N^\beta{}_\alpha F_\beta.
\end{aligned}
\tag{259}
$$

By adding the modification (106), we obtain the closed algebra

$$
\begin{aligned}
[\delta_\Theta, \delta_\Sigma] H_{\dot\alpha} &= \xi^K \nabla_K H_{\dot\alpha} + ig[\omega, H_{\dot\alpha}] - 2(\bar\Theta_{\dot\gamma}\Sigma^{\dot\gamma}) H_{\dot\beta} N^{\dot\beta}{}_{\dot\alpha}, \\
[\delta_\Theta, \delta_\Sigma] \Xi &= \xi^K \nabla_K \Xi + ig[\omega, \Xi] + \frac{i}{4}(\bar{\check\Theta}_{\dot\gamma}\Gamma^{mn}\Sigma^{\dot\gamma} - \bar{\check\Sigma}_{\dot\gamma}\Gamma^{mn}\Theta^{\dot\gamma})\Gamma_{mn}\Xi, \\
[\delta_\Theta, \delta_\Sigma] F_{H\alpha} &= \xi^K \nabla_K F_{H\alpha} + ig[\omega, F_{H\alpha}] - 2(\bar\Theta_{\dot\gamma}\Sigma^{\dot\gamma}) F_\beta N^\beta{}_\alpha.
\end{aligned}
\tag{260}
$$

C.2 Lagrangian

As for the vector multiplet,

$$
\begin{aligned}
\delta^{(0)}_\Sigma \mathcal{L}^{(0)}_V =& -i\mathrm{tr}\Big[F^{MN}\bar\Psi_{\dot\alpha}\Gamma_N \nabla_M \Sigma^{\dot\alpha} + \frac{1}{2}F_{KL}\bar\Psi_{\dot\alpha}\Gamma^{MKL}\nabla_M \Sigma^{\dot\alpha} \\
&\qquad + (D^M\sigma)\bar\Psi_{\dot\alpha}\nabla_M \Sigma^{\dot\alpha} + (D_N\sigma)\bar\Psi_{\dot\alpha}\Gamma^{MN}\nabla_M \Sigma^{\dot\alpha}\Big] \\
=& i\mathrm{tr}\Big[\frac{1}{2}F_{mn}\bar\Psi_{\dot\alpha}\Gamma^{mn}\tilde\Sigma^{\dot\alpha} - F_{mi}\bar\Psi_{\dot\alpha}\Gamma^{mi}\tilde\Sigma^{\dot\alpha} - \frac{3}{2}F_{ij}\bar\Psi_{\dot\alpha}\Gamma^{ij}\tilde\Sigma^{\dot\alpha} \\
&\qquad + (D_m\sigma)\bar\Psi_{\dot\alpha}\Gamma^m\tilde\Sigma^{\dot\alpha} + 3(D_i\sigma)\bar\Psi_{\dot\alpha}\Gamma^i\tilde\Sigma^{\dot\alpha}\Big], \\
\delta'_\Sigma \mathcal{L}^{(0)}_V =& i\bar\Psi_{\dot\alpha}\tilde\Sigma^{\dot\beta} D^{\dot\alpha}{}_{\dot\beta},
\end{aligned}
\tag{261}
$$

and, as for the hypermultiplet,

$$
\begin{aligned}
\delta_\Sigma^{(0)}(\mathcal{L}_H^{(0)}+\mathcal{L}_{int}^{(0)}) =&\mathrm{tr}\Big[i(D_mH_{\dot\beta})\bar\Xi\Gamma^m\tilde\Sigma^{\dot\beta}-i(D_m\bar H^{\dot\beta})\bar{\tilde\Sigma}_{\dot\beta}\Gamma^m\Xi+3i(D_iH_{\dot\beta})\bar\Xi\Gamma^i\tilde\Sigma^{\dot\beta}-3i(D_i\bar H^{\dot\beta})\bar{\tilde\Sigma}_{\dot\beta}\Gamma^i\Xi \\
&+3g\sigma\left([H_{\dot\alpha},\bar\Xi]\tilde\Sigma^{\dot\alpha}-[\bar H^{\dot\alpha},\bar{\tilde\Sigma}_{\dot\alpha}\Xi]\right)+3g[\bar H^{\dot\alpha},H_{\dot\beta}]\left(\bar\Psi_{\dot\alpha}\tilde\Sigma^{\dot\beta}+\bar{\tilde\Sigma}_{\dot\alpha}\Psi^{\dot\beta}\right)\Big], \\
\delta'_\Sigma(\mathcal{L}_H^{(0)}+\mathcal{L}_{int}^{(0)}) =&\mathrm{tr}\Big[-2i(D_MH_{\dot\beta})\bar\Xi\Gamma^M\tilde\Sigma^{\dot\beta}+2i(D_M\bar H^{\dot\beta})\bar{\tilde\Sigma}_{\dot\beta}\Gamma^M\Xi-2g\sigma\left([H_{\dot\alpha},\bar\Xi]\tilde\Sigma^{\dot\alpha}-[\bar H^{\dot\alpha},\bar{\tilde\Sigma}_{\dot\alpha}\Xi]\right) \\
&-3g[\bar H^{\dot\alpha},H_{\dot\beta}]\left(\bar\Psi_{\dot\alpha}\tilde\Sigma^{\dot\beta}+\bar{\tilde\Sigma}_{\dot\alpha}\Psi^{\dot\beta}\right)+\frac{3}{2}iH_{\dot\alpha}\bar\Xi\Sigma^{\dot\alpha}-\frac{3}{2}i\bar H^{\dot\alpha}\bar\Sigma_{\dot\alpha}\Xi+i\bar{\tilde\Sigma}_\alpha\Xi\bar F_H^\alpha-i\bar\Xi\bar{\tilde\Sigma}^\alpha F_{H\alpha}\Big],
\end{aligned}
\tag{262}
$$

then, one can see that the above terms are canceled by $\delta_\Sigma(\mathcal{L}'_V+\mathcal{L}'_H)$.

D Clebsch-Gordan Coefficients

D.1 The spin l representation $\otimes$ the spin 1 representation into the spin $J=l,l\pm1$ representations

· the spin $J=l+1$ representation

$$
\begin{aligned}
&|\,J=l+1,M=m\rangle\rangle \\
=&\sqrt{\frac{1}{2(l+1)(2l+1)}}\Big[\sqrt{(l+m)(l+m+1)}\,|l,m-1\rangle\,|1,1\rangle \\
&+\sqrt{2(l+m+1)(l-m+1)}\,|l,m\rangle\,|1,0\rangle+\sqrt{(l-m)(l-m+1)}\,|l,m+1\rangle\,|1,-1\rangle\Big].
\end{aligned}
\tag{263}
$$

· the spin $J=l$ representation

$$
\begin{aligned}
&|\,J=l,M=m\rangle\rangle \\
=&\sqrt{\frac{1}{2l(l+1)}}\Big[-\sqrt{(l+m)(l-m+1)}\,|l,m-1\rangle\,|1,1\rangle \\
&+\sqrt{2}m\,|l,m\rangle\,|1,0\rangle+\sqrt{(l-m)(l+m+1)}\,|l,m+1\rangle\,|1,-1\rangle\Big].
\end{aligned}
\tag{264}
$$

· the spin $J=l-1$ representation

$$
\begin{aligned}
&|\,J=l-1,M=m\rangle\rangle \\
=&\sqrt{\frac{1}{2l(2l+1)}}\Big[\sqrt{(l-m)(l-m+1)}\,|l,m-1\rangle\,|1,1\rangle \\
&-\sqrt{2(l+m)(l-m)}\,|l,m\rangle\,|1,0\rangle+\sqrt{(l+m)(l+m+1)}\,|l,m+1\rangle\,|1,-1\rangle\Big].
\end{aligned}
\tag{265}
$$

Furthermore, the action of the spin operator S_3 on the state $|\,J,M\rangle\rangle$ is obtained as

$$
\begin{aligned}
&S_3\,|\,l+1,M\rangle\rangle \\
=&\frac{M}{l+1}\,|\,l+1,M\rangle\rangle-\frac{1}{l+1}\sqrt{\frac{l}{2l+1}}\sqrt{(l+M+1)(l-M+1)}\,|\,l,M\rangle\rangle\,, \\
&S_3\,|\,l,M\rangle\rangle \\
=&\sqrt{\frac{1}{l(l+1)}}\Big[-l\sqrt{\frac{(l-M+1)(l+M+1)}{(l+1)(2l+1)}}\,|\,l+1,M\rangle\rangle \\
&+M\sqrt{\frac{1}{l(2l+1)}}\,|\,l,M\rangle\rangle-(l+1)\sqrt{\frac{(l-M)(l+M)}{l(2l+1)}}\,|\,l-1,M\rangle\rangle\Big], \\
&S_3\,|\,l-1,M\rangle\rangle \\
=&-\frac{M}{l}\,|\,l-1,M\rangle\rangle-\frac{1}{l}\sqrt{\frac{l+1}{2l+1}}\sqrt{(l+M)(l-M)}\,|\,l,M\rangle\rangle\,.
\end{aligned}
\tag{266}
$$

D.2 The spin l representation $\otimes$ the spin 1/2 representation into the spin $J = l \pm 1/2$ representations

· the spin $J = l + 1/2$ representation ($m = -l-1, -l, \cdots, l-1, l$)

$$
|J=l+\frac{1}{2},M=m+\frac{1}{2}\ \rangle\rangle=\sqrt{\frac{l-m}{2l+1}}\,|l,m+1\rangle\,|\frac{1}{2},-\frac{1}{2}\ \rangle+\sqrt{\frac{l+m+1}{2l+1}}\,|l,m\rangle\,|\frac{1}{2},\frac{1}{2}\ \rangle\,.
\tag{267}
$$

· the spin $J = l - 1/2$ representation ($m = -l, \cdots, l-1$)

$$
|J=l-\frac{1}{2},M=m+\frac{1}{2}\ \rangle\rangle=\sqrt{\frac{l+m+1}{2l+1}}\,|l,m+1\rangle\,|\frac{1}{2},-\frac{1}{2}\ \rangle-\sqrt{\frac{l-m}{2l+1}}\,|l,m\rangle\,|\frac{1}{2},\frac{1}{2}\ \rangle\,.
\tag{268}
$$

The action of the spin operator S_3 on the state $|\,J,M\rangle\rangle$ is obtained as

$$
\begin{aligned}
&S_3|l+\frac{1}{2},m+\frac{1}{2}\ \rangle\rangle \\
=&\frac{1}{2}\frac{2m+1}{2l+1}|l+\frac{1}{2},m+\frac{1}{2}\ \rangle\rangle-\frac{\sqrt{(l+m+1)(l-m)}}{2l+1}|l-\frac{1}{2},m+\frac{1}{2}\ \rangle\rangle\,, \\
&S_3|l-\frac{1}{2},m+\frac{1}{2}\ \rangle\rangle \\
=&-\frac{\sqrt{(l+m+1)(l-m)}}{2l+1}|l+\frac{1}{2},m+\frac{1}{2}\ \rangle\rangle-\frac{1}{2}\frac{2m+1}{2l+1}|l-\frac{1}{2},m+\frac{1}{2}\ \rangle\rangle\,.
\end{aligned}
\tag{269}
$$

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