論文の内容の要旨

論文題目　Multiple-particle diffusion processes from the viewpoint of Dunkl operators: relaxation to the steady state

（ダンクル演算子の視点から見た多粒子拡散過程：定常状態への緩和）

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In this thesis, two families of stochastic interacting particle systems, the interacting Brownian motions and the interacting Bessel processes, are defined as extensions of Dyson's Brownian motion models and the eigenvalue processes of the Wishart and Laguerre processes. This is achieved by considering the parameter β from random matrix theory as a real positive number. These systems consist of several particles which evolve as individual Brownian motions and Bessel processes, and that repel mutually through a logarithmic potential.

The interacting Brownian motions and Bessel processes are realized as special cases of Dunkl processes, which are a broad family of multivariate stochastic processes defined by using the differential-difference operators known as Dunkl operators. More precisely, Dunkl processes are generalizations of multivariate Brownian motions in the sense that, while the transition probability density of Brownian motion obeys the heat equation, the transition probability density that defines Dunkl processes obeys the Dunkl generalization of the heat equation. The Dunkl heat equation is obtained by replacing the spatial partial derivatives by Dunkl operators. Because Dunkl operators are defined based on finite sets of vectors, called root systems, which generate finite reflection groups, there are several kinds of Dunkl operators depending on the type of reflection group under consideration.

A particular feature of Dunkl processes is that, due to the difference terms present in the Dunkl heat equation, they are discontinuous. However, it is possible to extract their continuous, or "radial" part. In consequence, radial Dunkl processes are diffusion processes with a drift that is determined by the kind of Dunkl operators considered for their definition. In particular, the interacting Brownian motions and Bessel processes are realized as the radial Dunkl processes that correspond to the symmetric group and the group generated by all the permutations and sign changes of the components of N-dimensional vectors, respectively.

One of the tools provided by Dunkl operator theory, the intertwining operator, relates spatial partial derivatives with Dunkl operators. It also maps multidimensional Brownian motions into Dunkl processes, in the sense that applying the intertwining operator on the heat equation gives the Dunkl heat equation. The intertwining operator is a very powerful tool, but its explicit form is only known in a few special cases, and there is no systematic way to calculate its effect on arbitrary functions in the present. Due to its defining characteristics, the intertwining operator contains information about the properties of all types of Dunkl processes. Consequently, a better understanding of the intertwining operator leads to a better understanding of Dunkl processes in general and of the interacting Brownian motions and Bessel processes in particular.

In this thesis, the steady state and freezing ($\beta \to \infty$) regimes of the interacting Brownian motions and Bessel processes are studied. It is proved that the steady-state distributions of these processes coincide with the eigenvalue distributions of the β-Hermite and β-Laguerre ensembles of random matrices through numerical simulations and exact calculations. Moreover, it is shown that the scaled final positions of the particles in these processes become fixed at the zeroes of the Hermite and Laguerre polynomials in the freezing limit. These sets of points are known as Fekete points, and they constitute the positions at which a series of charged particles in a background confining potential minimize the total potential energy.

These results are obtained as the consequence of two more general results proved in this thesis. The first of these results is that Dunkl processes in general converge to a steady-state distribution that only depends on the type of Dunkl operators considered and that is independent of the initial distribution. The steady-state distribution takes the form of a Boltzmann factor with a potential energy given by the sum of a background harmonic potential and a weighted sum of logarithmic potentials. This steady-state distribution reduces naturally to the eigenvalue distributions of the β-Hermite and β-Laguerre ensembles of random matrices when the reflection group associated to the Dunkl operators is chosen appropriately (e.g., the symmetric group yields the eigenvalue distribution of the β-Hermite ensembles). A lower bound for the time required to achieve the steady state is given, and this bound is shown to be finite for finite values of β.

The second general result is that in the freezing limit, the scaled final position of the Dunkl processes is fixed to a set of points called the peak set, which is the set of points which maximizes their steady-state distribution. The freezing configuration is achieved instantaneously, and like in the case of the steady-state distribution, it is independent of the initial distribution of the processes. By choosing the correct reflection group, the peak set is reduced to the Fekete points that correspond to either of the two interacting particle systems considered in this work.

In order to obtain these results, previously unknown relations involving the intertwining operator are derived for Dunkl processes in general. One of these general results is the effect of the intertwining operator for any linear function, while another general result concerns the effect of the intertwining operator on the exponential function for any type of Dunkl operators in the freezing limit. In the particular case of the interacting Brownian motions and Bessel processes, the effect of the intertwining operator on symmetric polynomials is derived from the generalized hypergeometric functions. Using these particular cases of the intertwining operator, an alternate derivation of the freezing limit of the intertwining operator applied to the exponential function is obtained for the interacting Brownian motions and Bessel processes.