Spaces of stability conditions on Calabi-Yau categories associated with quivers

(箙に付随するCalabi-Yau圏の安定性条件の空間について)

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Contents

1 Introduction

The notion of a stability condition on a triangulated category $\mathcal{D}$ was introduced by Bridgeland in [Bri07] motivated by Douglas's work on Π-stability for D-branes in string theory ([Dou02]). Bridgeland also showed that the space of stability conditions $\mathrm{Stab}(\mathcal{D})$ has the structure of a complex manifold and there is a local isomorphism map

$$\pi\colon \mathrm{Stab}(\mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$$

where $K(\mathcal{D})$ is the K-group of $\mathcal{D}$. The most important examples of triangulated categories are derived categories of coherent sheaves on varieties or those of modules over algebras. Since the space of stability conditions $\mathrm{Stab}(\mathcal{D})$ provides a geometrical way to study the original category $\mathcal{D}$, to study the spaces of stability conditions on derived categories is important problem.

There is an important class of triangulated categories, called Calabi-Yau N (CY_N) triangulated categories ([Kel08]). A triangulated category $\mathcal{D}$ over a field k is CY_N if there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(E, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(F, E[N])^*, \quad \text{for } E, F \in \mathcal{D}$$

where $[N]$ is the N-th shift and $*$ means the dual k-vector space. Important examples of CY_N categories are given by the derived category $\mathcal{D}_{\mathrm{fd}}(\Gamma_N Q)$ of finite dimensional dg modules over the Ginzburg CY_N dg algebra $\Gamma_N Q$ associated with a quiver Q ([Gin, Kel11]).

In this thesis, we study the spaces of stability conditions on $\mathcal{D}_{\mathrm{fd}}(\Gamma_N Q)$ in two type cases; one is $N = 2$ and Q is a connected quiver without loops, and the other is $N \geq 3$ and Q is the A_n-quiver.

In Part I, we consider the case $N = 2$ and Q is a connected quiver without loops. If Q is not of ADE type, the dg algebra $\Gamma_2 Q$ is quasi-isomorphic to the graded algebra $\Pi(Q)$, called preprojective algebra of Q, and hence the derived category of finite dimensional dg modules over $\Gamma_2 Q$ is triangulated equivalent to the bounded derived category of finite dimensional nilpotent modules over $\Pi(Q)$. Therefore, we study the space of stability conditions on a CY_2 category $\mathcal{D}_Q$ given as the bounded derived category of the preprojective algebra. In geometric setting, the spaces of stability conditions for some CY_2 triangulated categories were studied in [Bri08, Bri09b, IUU10, Oka06]. When $\mathcal{D}$ is the bounded derived category of coherent sheaves of a K3 surface ([Bri08]) or a certain subtriangulated category of coherent sheaves of a resolution of a Kleinian singularity ([Bri09b]), it was shown that the distinguished connected component of $\mathrm{Stab}(\mathcal{D})$ is a covering space of some open subset of $\mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$ related to root systems and how the group of deck transformations acts on it. Further, the connectedness and simply connectedness problems of $\mathrm{Stab}(\mathcal{D})$ for some particular $\mathcal{D}$ were solved in [IUU10, Oka06, ST01].

Following the works in [Bri08, Bri09b, Tho06], we describe the distinguished connected component $\mathrm{Stab}^\circ(\mathcal{D}_Q) \subset \mathrm{Stab}(\mathcal{D}_Q)$ in terms of root systems of Kac-Moody Lie algebras associated with Q, and show that the space $\mathrm{Stab}^\circ(\mathcal{D}_Q)$

becomes the covering space of some open subset $X_{\text{reg}} \subset \text{Hom}_{\mathbb{Z}}(K(\mathcal{D}_Q), \mathbb{C})$. Further, we give the relationship between the group of deck transformations and the autoequivalence group of $\mathcal{D}_Q$. Our results generalize the results for ADE or affine ADE quivers in [Bri09b, Tho06] to all quivers without loops.

In Part II, we consider the case $N \geq 3$ and Q is the A_n-quiver. In [ST01, Tho06], a certain CY_N triangulated category $\mathcal{D}_n^N$ appears in mirror symmetry for the derived Fukaya category of Lagrangian submanifolds consisting of vanishing cycles of the Milnor fiber of the A_n-singularity. The category $\mathcal{D}_n^N$ also described as the derived category of finite dimensional dg modules of the CY_N Ginzburg dg algebra of the A_n-quiver. The main subject of Part II is the space of stability conditions on the category $\mathcal{D}_n^N$.

Recently, Brideland and Smith proved that the moduli space of meromorphic quadratic differentials with simple zeros can be identified to the space of stability conditions on a CY_3 triangulated category defined as the derived category of finite dimensional dg modules over the Ginzburg dg algebra associated with a ideal triangulation of a marked bordered surface. The idea of constructing stability conditions from quadratic differentials comes from the work of physicists Gaiotto-Moore-Neitzke in [GMN]. In [BS], Bridgeland-Smith established many mathematical foundations for the space of quadratic differentials and gave the mathematical understanding of the work in [GMN].

In Part II, we study the space of stability conditions on $\mathcal{D}_n^N$ by using some generalizations of Bridgeland-Smith's theory. By generalizing the assumption simple zeros to zeros of order $(N-2)$ and triangulations to N-angulations, we can treat not only CY_3 categories but CY_N categories, only in the easiest case that the surface is a disk with some marked points on the boundary. As a result, we can show that the distinguished connected component $\text{Stab}^\circ(\mathcal{D}_n^N) \subset \text{Stab}(\mathcal{D}_n^N)$ is isomorphic to the universal covering of the space of polynomials $p_n(z) = z^{n+1} + u_1 z^{n-1} + \cdots + u_n$ $(u_1, \dots, u_n \in \mathbb{C})$ with simple zeros. Further, the central charges of $\text{Stab}^\circ(\mathcal{D}_n^N)$ are described by the periods of quadratic differentials of the form $p_n(z)^{N-2} dz^{\otimes 2}$ on the Riemann sphere $\mathbb{P}^1$.

1.1 Summary of results in Part I

Let Q be a connected finite quiver without loops (1-cycles) and let $\{1, \dots, n\}$ be the vertices of Q. Further, assume that the underlying graph of Q is not of ADE type. For the quiver Q, we can define a $\mathbb{C}$-algebra $A := \Pi(Q)$ called preprojective algebra of Q. A has a natural grading $A = \oplus_{i \geq 0} A_i$, and a right A-module M is called nilpotent if there is some positive integer k such that $MA_l = 0$ for all $l \geq k$. Let $\mathcal{A}_Q$ be an abelian category of finite dimensional nilpotent right A-modules and $\mathcal{D}_Q := D^b(\mathcal{A}_Q)$ be a bounded derived category of $\mathcal{A}_Q$. It is known that the triangulated category $\mathcal{D}_Q$ is a CY_2 category. (If Q is ADE-type, there is a certain CY_2 triangulated category, for which all corresponding results holds. See [Bri09b] for more details.)

The abelian category $\mathcal{A}_Q$ is finite length with finitely many simple modules $\{S_1, \dots, S_n\}$ corresponding to n-vertices of Q, so every object of $\mathcal{A}_Q$ has the Jordan-Holder filtration by these simple modules. The K-group of $\mathcal{D}_Q$ is given by

$$K(\mathcal{D}_Q) \cong \bigoplus_{i=1}^{n} \mathbb{Z}[S_i]$$

and $K(\mathcal{D}_Q)$ has a natural bilinear form $\chi\colon K(\mathcal{D}_Q) \times K(\mathcal{D}_Q) \to \mathbb{Z}$, called the Euler form, defined by

$$\chi(E, F) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{D}_Q}(E, F[i])$$

for $E, F \in \mathcal{D}_Q$. The CY-2 property of $\mathcal{D}_Q$ implies that the Euler form χ is symmetric.

For Q, we introduce $n \times n$ matrix A_Q, called the generalized Cartan matrix (GCM for short), by $(A_Q)_{ij} := 2\delta_{ij} - (q_{ij} + q_{ji})$ where q_{ij} is a number of arrows from i to j in Q. The root lattice L_Q associated with A_Q is defined by

$$L_Q := \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i$$

where α_i's are n-free generators called simple roots of L_Q. We also define a symmetric bilinear form $(\ ,\ )\colon L_Q \times L_Q \to \mathbb{Z}$ by $(\alpha_i, \alpha_j) := (A_Q)_{ij}$. This symmetric bilinear form define reflections $r_1, \dots, r_n\colon L_Q \to L_Q$ with respect to simple roots $\alpha_1, \dots, \alpha_n$ and denote by W the group generated by these reflections, called the Weyl group.

The Euler form χ is computed by $\chi(S_i, S_j) = (A_Q)_{ij}$, therefore we have an isomorphism

$$(K(\mathcal{D}_Q), \chi) \cong (L_Q, (\ ,\ )), \quad [S_i] \mapsto \alpha_i$$

between two $\mathbb{Z}$-lattices with symmetric bilinear forms.

Let $\Delta_+^{\mathrm{re}} \subset L_Q$ ($\Delta_+^{\mathrm{im}} \subset L_Q$) be the set of positive real (imaginary) roots. The imaginary cone $I \subset L_Q \otimes_{\mathbb{Z}} \mathbb{R}$ is a closure of convex hull of $\Delta_+^{\mathrm{im}} \cup \{0\} \subset L_Q \otimes_{\mathbb{Z}} \mathbb{R}$ in natural topology as a finite dimensional vector space. Let $V := \operatorname{Hom}_{\mathbb{Z}}(L_Q, \mathbb{C})$ and introduce the subset $X \subset V$ by

$$X := V \setminus \bigcup_{\lambda \in I \setminus \{0\}} H_\lambda$$

where $H_\lambda := \{\, Z \in V \mid Z(\lambda) = 0 \,\}$. Further, define the regular subset $X_{\mathrm{reg}} \subset V$, on which the Weyl group W acts freely, by

$$X_{\mathrm{reg}} := X \setminus \bigcup_{\alpha \in \Delta_+^{\mathrm{re}}} H_\alpha.$$

Now, consider the space of stability conditions on $\mathcal{D}_Q$. A stability condition on $\mathcal{D}_Q$ ([Bri07]) consists of a full abelian subcategory $\mathcal{A} \subset \mathcal{D}_Q$, called the heart of bounded t-structure, with a group homomorphism

$$Z\colon K(\mathcal{D}_Q) \longrightarrow \mathbb{C}$$

called a central charge which satisfies the condition

$$Z(E) \in \{\, re^{i\pi\phi} \ \in \mathbb{C} \,|\, r \in \mathbb{R}_{>0}, \phi \in (0,1] \,\}$$

for every non-zero object $E \in \mathcal{A}$. Write by $\mathrm{Stab}(\mathcal{D}_Q)$ the set of stability conditions on $\mathcal{D}_Q$ with the additional condition, called the support property (see Definition 2.10).

For the space $\mathrm{Stab}(\mathcal{D}_Q)$, there is a distinguished connected component $\mathrm{Stab}^\circ(\mathcal{D}_Q) \subset \mathrm{Stab}(\mathcal{D}_Q)$ which contains stability conditions with the heart $\mathcal{A}_Q$. By the main result in [Bri07], $\mathrm{Stab}^\circ(\mathcal{D}_Q)$ is a complex manifold of complex dimension n and there is a local isomorphism map

$$\pi \colon \mathrm{Stab}^\circ(\mathcal{D}_Q) \longrightarrow \mathrm{Hom}_{\mathcal{D}_Q}(K(\mathcal{D}_Q), \mathbb{C}).$$

defined by taking central charges from stability conditions. Note that under the identification $K(\mathcal{D}_Q) \cong L_Q$, we have $\mathrm{Hom}_{\mathcal{D}_Q}(K(\mathcal{D}_Q), \mathbb{C}) \cong V$.

In [ST01], P. Seidel and R.P. Thomas defined autoequivalences $\Phi_{S_i} \in \mathrm{Aut}(\mathcal{D}_Q)$, called spherical twists, for spherical objects $S_1, \dots, S_n$, and showed that they satisfy braid relations. Write by $\mathrm{Br}(\mathcal{D}_Q) \subset \mathrm{Aut}(\mathcal{D}_Q)$ the subgroup generated by these spherical twists. The action of $\mathrm{Br}(\mathcal{D}_Q)$ on $\mathrm{Stab}(\mathcal{D}_Q)$ preserves the distinguished connected component $\mathrm{Stab}^\circ(\mathcal{D}_Q)$.

The next theorem is the main result of this paper. This generalizes the results by T. Bridgeland and R.P. Thomas ([Bri09b, Tho06]) from finite or affine type root systems to indefinite type root systems.

Theorem 1.1 *There is a covering map*

$$\underline{\pi} \colon \mathrm{Stab}^\circ(\mathcal{D}_Q) \longrightarrow X_{\mathrm{reg}}/W$$

and the subgroup $\mathbb{Z}[2] \times \mathrm{Br}(\mathcal{D}_Q) \subset \mathrm{Aut}(\mathcal{D}_Q)$ *acts as the group of deck transformations (*$\mathbb{Z}[2] \subset \mathrm{Aut}(\mathcal{D}_Q)$ *is the subgroup generated by the shift functor* $[2] \in \mathrm{Aut}(\mathcal{D}_Q)$*).*

By the van der Lek's result ([vdL83]), the fundamental group of X_{reg}/W is given by

$$\pi_1(X_{\mathrm{reg}}/W) \cong \mathbb{Z}[\gamma] \times G_W$$

where $G_W = \langle \sigma_1, \dots, \sigma_n \rangle$ is the Artin group with generators $\sigma_1, \dots, \sigma_n$ ([BS72]) associated with the Weyl group $W = \langle r_1, \dots, r_n \rangle$. The factor $\mathbb{Z}[\gamma]$ is generated by a loop γ around the orthogonal hyperplanes of the imaginary cone $I \backslash \{0\}$. Theorem 1.1 implies that there is a surjective group homomorphism

$$\widetilde{\rho} \colon \mathbb{Z}[\gamma] \times G_W \to \mathbb{Z}[2] \times \mathrm{Br}(\mathcal{D}_Q).$$

We can show that $\widetilde{\rho}$ sends the generators σ_i to the spherical twists Φ_{S_i} and γ to the shift functor [2].

For a quiver Q, we get an underlying graph $\underline{Q}$ by forgetting orientations of arrows in Q. The automorphism group $\mathrm{Aut}(\underline{Q})$ of the graph $\underline{Q}$ acts on $\mathcal{A}_Q$ by permutating simple modules $S_1, \dots, S_n \in \mathcal{A}_Q$ corresponding to vertices.

Let $\mathrm{Aut}^\circ(\mathcal{D}_Q) \subset \mathrm{Aut}(\mathcal{D}_Q)$ be the subgroup of autoequivalences which preserve the distinguished connected component $\mathrm{Stab}^\circ(\mathcal{D}_Q)$. Further, write by $\mathrm{Aut}^\circ_*(\mathcal{D}_Q) = \mathrm{Aut}^\circ(\mathcal{D}_Q)/\mathrm{Nil}^\circ(\mathcal{D}_Q)$ the quotient of $\mathrm{Aut}^\circ(\mathcal{D}_Q)$ by the autoequivelences $\mathrm{Nil}^\circ(\mathcal{D}_Q)$ which acts trivially on $\mathrm{Stab}^\circ(\mathcal{D}_Q)$. Then, the similar result to Corollary 1.4 in [Bri09b] also holds.

Corollary 1.2 *The group* $\mathrm{Aut}^\circ_*(\mathcal{D}_Q)$ *is given by*

$$\mathrm{Aut}^\circ_*(\mathcal{D}_Q) \cong \mathbb{Z}[1] \times (\mathrm{Br}(\mathcal{D}_Q) \rtimes \mathrm{Aut}(\underline{Q}))$$

where $\mathrm{Aut}(\underline{Q})$ *acts on* $\mathrm{Br}(\mathcal{D}_Q)$ *by permutating the generators* $\Phi_{S_1}, \dots, \Phi_{S_n}$.

Similar to the case for K3 surfaces in [Bri08] and Kleinian singularities in [Bri09b] (which correspond to finite or affine type quivers), we expect the following properties for the space $\mathrm{Stab}(\mathcal{D}_Q)$.

Conjecture 1.3 *(1) The space* $\mathrm{Stab}(\mathcal{D}_Q)$ *is connected. Hence* $\mathrm{Stab}^\circ(\mathcal{D}_Q) = \mathrm{Stab}(\mathcal{D}_Q)$.

(2) The space $\mathrm{Stab}^\circ(\mathcal{D}_Q)$ *is simply connected. In other words, the sujective group homomorphism*

$$\rho \colon G_W \longrightarrow \mathrm{Br}(\mathcal{D}_Q)$$

is injective. (Hence isomorphism.)

Conjecture 1.3 (1) was solved for $\hat{A}_1$-quiver in [Oka06], and for A_n-quivers and $\hat{A}_n$-quivers in [IUU10]. Conjecture 1.3 (2) was solved for A_n-quivers in [ST01], for ADE-quivers in [BT11], and for $\hat{A}_n$-quivers in [IUU10].

Further, the $K(\pi, 1)$ conjecture for Artin groups (see [Par]) together with above two conjectures implies that the space $\mathrm{Stab}(\mathcal{D}_Q)$ is contractible.

Note that if both Conjecture 1.3 (1) and Conjecture 1.3 (2) hold, then the autoequivalence group of $\mathcal{D}_Q$ is given by

$$\mathrm{Aut}(\mathcal{D}_Q) \cong \mathbb{Z}[1] \times (\mathrm{Br}(\mathcal{D}_Q) \rtimes \mathrm{Aut}(\underline{Q})).$$

Next, we consider relationship to stability conditions on derived categories of path algebras. Let Q be an acyclic quiver and $\mathbb{C}Q$ be a path algebra of Q. Let mod-$\mathbb{C}Q$ be an abelian category of finite dimensional $\mathbb{C}Q$ modules and $D^b(\mathbb{C}Q)$ be a derived category of mod-$\mathbb{C}Q$. K-group of $D^b(\mathbb{C}Q)$ is given by

$$K(D^b(\mathbb{C}Q)) \cong \bigoplus_{i=1}^{n} \mathbb{Z}[S_i]$$

where S_i's are simple modules of $\mathbb{C}Q$. By identifying $K(D^b(\mathbb{C}Q))$ with the root lattice L_Q, we have $V \cong \mathrm{Hom}_{\mathbb{Z}}(K(D^b(\mathbb{C}Q)), \mathbb{C})$ as in the previous section. Futher, there is a distinguished connected component $\mathrm{Stab}^\circ(D^b(\mathbb{C}Q)) \subset \mathrm{Stab}(D^b(\mathbb{C}Q))$ which contains all stability conditons on mod-$\mathbb{C}Q$.

The following result was showed in [Shi] for all m-Kronecker quivers (a m-Kronecker quiver consists of two vertices and m pararell arrows). Since these quivers are good examples for all type quivers, finite ($m = 1$), affine ($m = 2$) and indefinite ($m \geq 3$), we naively expect the following property for general acyclic quivers.

Conjecture 1.4 *Let Q be an acyclic quiver. The restriction of a local isomorphism map*

$$\pi\colon \mathrm{Stab}^{\circ}(D^b(\mathbb{C}Q)) \to V$$

on the regular subset $X_{\mathrm{reg}} \subset V$ is a covering map. Further, X_{reg} is the maximal subset of V having this covering property.

We expect that this conjecture contributes to the study of the spaces of stability conditions on derived categories of path algebras.

1.2 Summary of results in Part II

Let M_n be the space of polynomials $p_n(z) = z^{n+1} + u_1 z^{n-1} + \cdots u_n$ ($u_1, \ldots, u_n \in \mathbb{C}$) with simple zeros. Note that the fundamental group of M_n is the Artin group B_{n+1}. The action of $\mathbb{C}^*$ on M_n is given by $(u_1, \ldots, u_n) \mapsto (ku_1, \ldots, k^n u_n)$ for $k \in \mathbb{C}^*$.

We consider z to be a complex variable and $p_n(z)$ to be a meromorphic function on the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. For $p_n(z) \in M_n$, we define the quadratic differential on $\mathbb{P}^1$ by $\phi(z) := p_n(z)^{N-2} dz^{\otimes 2}$ and denote by $Q(N, n)$ the space of such differentials. Note that the space $Q(N, n)$ is isomorphic to M_n as a complex manifold.

We introduce the homology group $H_{\pm}(\phi)$ associated with the differential $\phi(z) \in Q(N, n)$ by the following. If N is even, the homology group is defined by the relative homology group $H_+(\phi) = H_1(\mathbb{C}, \mathrm{Zero}(\phi); \mathbb{Z})$ where $\mathrm{Zero}(\phi) \subset \mathbb{C}$ is the set of zeros of ϕ. If N is odd, the homology group is defined by $H_-(\phi) := H_1(S \backslash \pi^{-1}(\infty); \mathbb{Z})$ where $\pi\colon S \to \mathbb{P}^1$ is the hyperelliptic curve $S = \{y^2 = p_n(z)\}$. These holomogy groups form a local system on $Q(N, n)$.

For $\phi(z) = p_n(z)^{N-2} dz^{\otimes 2} \in Q(N, n)$, consider the meromorphic 1-form by $\psi(z) = p_n(z)^{\frac{N-2}{2}} dz^{\otimes 2}$, which is the square root of $\phi(z)$. Note that the 1-form $\psi(z)$ is holomorphic on $\mathbb{P}^1 \backslash \{\infty\}$ if N is even or $S \backslash \pi^{-1}(\infty)$ if N is odd. Hence, we have the well-defined linear map $Z_\phi\colon H_{\pm}(\phi) \longrightarrow \mathbb{C}$, called the period of $\phi(z)$, by the integration of $\psi(z)$ via the cycle $\gamma \in H_{\pm}(\phi)$:

$$Z_\phi(\gamma) := \int_\gamma \psi.$$

Let Γ be a free abelian group of rank n. An isomorphism of abelian groups $\theta\colon \Gamma \xrightarrow{\sim} H_{\pm}(\phi)$ is called a Γ-framing of $\phi \in Q(N, n)$. Denote by $Q(N, n)^{\Gamma}$ the set of framed differentials. For a framed differential $(\phi, \theta) \in Q(N, n)^{\Gamma}$, the composition of $\theta\colon \Gamma \to \mathcal{H}_{\pm}(\phi)$ and $Z_\phi\colon \mathcal{H}_{\pm}(\phi) \to \mathbb{C}$ gives a linear map $Z_\phi \circ \theta\colon \Gamma \to \mathbb{C}$. We define the period map

$$\mathcal{W}_N\colon Q(N, n)^{\Gamma} \to \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C})$$

by $(\phi, \theta) \mapsto Z_\phi \circ \theta$.

Fix a framed differential $* = (\phi_0, \theta_0) \in Q(N,n)^\Gamma$ and let $Q(N,n)^\Gamma_* \subset Q(N,n)^\Gamma$ be the connected component containing (ϕ_0, θ_0). Depending on the parity of N, there are two type integral representations $\rho_+ \colon B_{n+1} \to GL(n, \mathbb{Z})$ and $\rho_- \colon B_{n+1} \to GL(n, \mathbb{Z})$, which give monodromy representations of $\pi_1(Q(N,n), *) \cong B_{n+1}$ on $H_\pm(\phi_0)$. Let W_+ and W_- be the groups generated by the image of ρ_+ and ρ_- respectively. Then, we have a $W_\pm$-torsor $Q(N,n)^\Gamma_* \to Q(N,n)$ by forgetting framings $(\phi, \theta) \mapsto \phi$. Hence, the universal covering of $Q(N,n)^\Gamma_*$ is isomorphic to the universal covering $\widetilde{M_n}$ of M_n, and we have $Q(N,n)^\Gamma_* \cong \widetilde{M_n}/P_\pm$ where $P_\pm = \operatorname{Ker} \rho_\pm$. Further, we can lift the period map to

$$\mathcal{W}_N \colon \widetilde{M_n} \to \operatorname{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}).$$

Finally, we also mention that the action of $\mathbb{C}^*$ on M_n lifts to the action of $\mathbb{C} \cong \widetilde{\mathbb{C}^*}$ on $\widetilde{M_n}$.

For a quiver Q, we can define a dg algebra $\Gamma_N Q$, called the CY_N Ginzburg dg algebra, introduced by Ginzburg in [Gin]. Let $\mathcal{D}_{\mathrm{fd}}(\Gamma_N Q)$ be the derived category of dg modules over $\Gamma_N Q$ with finite total dimension. It was proved by Keller in [Kel11] that the category $\mathcal{D}_{\mathrm{fd}}(\Gamma_N Q)$ is a CY_N triangulated category. Let $Q = \overrightarrow{A_n}$ be the A_n-quiver and set $\mathcal{D}_n^N := \mathcal{D}_{\mathrm{fd}}(\Gamma_N \overrightarrow{A_n})$. An object $S \in \mathcal{D}_n^N$ is called N-spherical if

$$\operatorname{Hom}_{\mathcal{D}_n^N}(S, S[i]) = \begin{cases} k & \text{if} \quad i = 0, N \\ 0 & \text{otherwise.} \end{cases}$$

There is some full subcategory $\mathcal{H}_\Gamma$, called the standard heart of $\mathcal{D}_n^N$, and there are n spherical objects $S_1, \dots, S_n \in \mathcal{H}_\Gamma$ which generates $\mathcal{H}_\Gamma$ by extensions.

In [ST01], Seidel and Thomas defined an autoequivalence $\Phi_S \in \operatorname{Aut}(\mathcal{D}_n^N)$, called a spherical twist, for a spherical object $S \in \mathcal{D}_n^N$. They also proved that the subgroup $\operatorname{Sph}(\mathcal{D}_n^N) \subset \operatorname{Aut}(\mathcal{D}_n^N)$ generated by $\Phi_{S_1}, \dots, \Phi_{S_n}$ is isomorphic to the braid group B_{n+1}.

A stability condition $\sigma = (Z, \mathcal{H})$ on a triangulated category $\mathcal{D}$ consists of a heart $\mathcal{H} \subset \mathcal{D}$ of a bounded t-structure, and a linear map $Z \colon K(\mathcal{H}) \to \mathbb{C}$ called the central charge, which satisfy some axioms. In [Bri07], Bridgeland showed that there is a suitable topology on the set of stability conditions $\operatorname{Stab}(\mathcal{D})$, and the projection map

$$\mathcal{Z} \colon \operatorname{Stab}(\mathcal{D}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$$

defined by $(Z, \mathcal{H}) \mapsto Z$ is a local isomorphism. As a result, we can say that the space $\operatorname{Stab}(\mathcal{D})$ has the structure of a complex manifold. On $\operatorname{Stab}(\mathcal{D})$, there are two group actions, one is the action of $\mathbb{C} =\cong \widetilde{\mathbb{C}^*}$ and the other is the action of $\operatorname{Aut}(\mathcal{D})$.

We set $\mathcal{D} = \mathcal{D}_n^N$. There is a distinguished connected component $\operatorname{Stab}^\circ(\mathcal{D}_n^N) \subset \operatorname{Stab}(\mathcal{D}_n^N)$ which contains stability conditions with the standard heart $\mathcal{H}_\Gamma$, and the action of the group $\operatorname{Sph}(\mathcal{D}_n^N)$ preserves the connected component $\operatorname{Stab}^\circ(\mathcal{D}_n^N)$.

Since $\mathrm{Sph}(\mathcal{D}_n^N) \cong B_{n+1}$, the space $\mathrm{Stab}^\circ(\mathcal{D}_n^N)$ has the action of the braid group B_{n+1}.

The main result in Part II is the following.

Theorem 1.5 (Theorem 10.15) *There is a B_{n+1}-equivariant and $\mathbb{C}$-equivariant isomorphism of complex manifolds $\widetilde{K}$ such that the diagram*

$$\begin{array}{ccc} \widetilde{M_n} & \xrightarrow{\widetilde{K}} & \mathrm{Stab}^\circ(\mathcal{D}_n^N) \\ & {\scriptstyle \mathcal{W}_N} \searrow \quad \swarrow {\scriptstyle \mathcal{Z}} & \\ & \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}) & \end{array}$$

commutes.

For $N = 2$, the result was proved by Thomas in [Tho06]. For $N = 3$ and $n = 2$, the result was proved by Sutherland. The result for $N = 3$ and arbitrary n follows as the special case of the result of Bridgeland-Smith in [BS], which contains almost all classes of CY_3 categories associated with ideal triangulations of marked bordered surfaces.

Many arguments of this paper are parallel to those of [BS] but we need to generalize CY_3 to CY_N, quivers to $(N-2)$-colored quivers, simple zeros to zeros of order $(N-2)$ and triangulations to N-angulations.

The main difference between this paper and [BS] is can be seen in Section 6. Since the category equivalence of Keller-Yang type ([KY11]) for $N \geq 4$ is not established yet, we need another approach. Fortunately, by using the result of King-Qiu [KQ] concering the hearts of $\mathcal{D}_{\mathrm{fd}}(\Gamma Q)$ for an acyclic quiver togather with the geometric realization of $(N-2)$-cluster category of type A_n by Baur-Marsh in [BM08], we can obtain all necessary results to apply the method in [BS] without using the Keller-Yang type category equivalence.

In [IUU10], Ishii-Ueda-Uehara proved that the space $\mathrm{Stab}(\mathcal{D}_n^2)$ is connected, so $\mathrm{Stab}^\circ(\mathcal{D}_n^2) = \mathrm{Stab}(\mathcal{D}_n^2)$. Together with the result of Thomas in [Tho06], we have $\mathrm{Stab}(\mathcal{D}_n^2) \cong \widetilde{M_n}$ and, in particular we can say that $\mathrm{Stab}(\mathcal{D}_n^2)$ is contractible. For general $N \geq 2$, Qiu proved that $\mathrm{Stab}^\circ(\mathcal{D}_n^N)$ is simply connected (see Corollary 5.5 in [Qiu]). More strongly, our Theorem 1.1 implies the following (see Conjecture 5.7 in [Qiu]).

Corollary 1.6 (Corollary 10.16) *The distinguished connected component* $\mathrm{Stab}^\circ(\mathcal{D}_n^N)$ *is contractible.*

The remaining problem is that the connectedness of the space $\mathrm{Stab}(\mathcal{D}_n^N)$. As in the case $N = 2$, we naively expect that the space $\mathrm{Stab}(\mathcal{D}_n^N)$ is connected, and as a result $\mathrm{Stab}(\mathcal{D}_n^N)$ is contractible.

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Finally, I would like to thank my parents for supporting all my life in the University of Tokyo for ten years.

Notations

We fix the positive integers $N \geq 3$ and $n \geq 1$, and set the positive integer $d_{N,n} := (N-2)(n+1)+2$. We work over the algebraically closed field k. We use the grading of homology for dg algebras and dg modules. All triangulated categories are considered to be k-linear, and assume that the K-groups of triangulated categories are free of finite rank.

2 Background

In this section, we prepare basic definitions we shall use through this paper: bonded t-structure, tilting, stability condition, and spherical twist.

2.1 Bounded t-structure

Let $\mathcal{D}$ be a triangulated category.

Definition 2.1 ([BBD82]) *A t-structure on $\mathcal{D}$ is a full subcategory $\mathcal{F} \subset \mathcal{D}$ satisfying the following conditions:*

(a) $\mathcal{F}[1] \subset \mathcal{F}$,

(b) define $\mathcal{F}^{\perp} := \{G \in \mathcal{D} | \operatorname{Hom}_{\mathcal{D}}(F, G) = 0$ for all $F \in \mathcal{F}\}$, then for every object $E \in \mathcal{D}$, there is an exact triangle $F \to E \to G \to F[1]$ in $\mathcal{D}$ with $F \in \mathcal{F}$ and $G \in \mathcal{F}^{\perp}$.

In addition, a t-structure $\mathcal{F} \subset \mathcal{D}$ is said to be bounded if $\mathcal{F}$ satisfies the condition:

$$\mathcal{D} = \bigcup_{i,j \in \mathbb{Z}} \mathcal{F}^{\perp}[i] \cap \mathcal{F}[j].$$

For a t-structure $\mathcal{P} \in \mathcal{D}$, we define its heart $\mathcal{H}$ by

$$\mathcal{H} := \mathcal{P}^{\perp}[1] \cap \mathcal{P}.$$

It was proved in [BBD82] that the heart $\mathcal{H}$ becomes an abelian category.

The heart of a bounded t-structure is characterized the following properties.

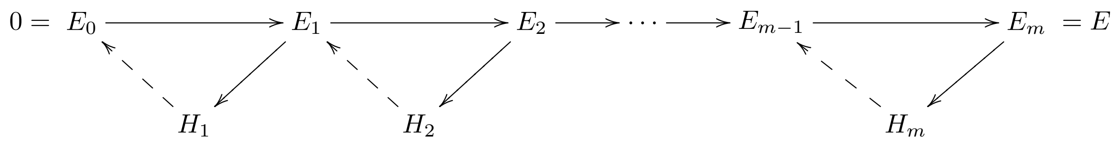
Lemma 2.2 ([Bri07], Lemma 3.2) *A full additive subcategory $\mathcal{H} \subset \mathcal{D}$ becomes the heart of some bounded t-structure $\mathcal{F} \subset \mathcal{D}$ if $\mathcal{H}$ satisfies the following conditions* (1) *and* (2)*.*

(1) For any integers $k_1 > k_2$ and any objects $H_1, H_2 \in \mathcal{H}$, $\operatorname{Hom}_{\mathcal{D}}(H_1(k_1), H_2(k_2)) = 0$.

(2) For any non-zero object $E \in \mathcal{D}$, there is a sequence of integers

$$k_1 > k_2 > \cdots > k_n$$

and exact triangles



with $H_i \in \mathcal{H}[k_i]$ for $i = 1, \ldots, n$.

Lemma 2.2 defines the k-th homology of E with respect to the heart $\mathcal{H}$ by

$$\mathbf{H}_k(E) := \begin{cases} H_i & \text{if } k = k_i \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.3 *By Lemma 2.2, for the heart $\mathcal{H} \subset \mathcal{D}$ of a bounded t-structure, there is a canonical isomorphism of K-groups*

$$K(\mathcal{H}) \cong K(\mathcal{D}).$$

In the following, we only treat bounded t-structures and their hearts. The word heart always refers to the heart of a bounded t-structure.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the hearts of bounded t-structures $\mathcal{P}_1$ and $\mathcal{P}_2$. We write by

$$\mathcal{H}_1 \leq \mathcal{H}_2$$

if they satisfy the condition $\mathcal{P}_2 \subset \mathcal{P}_1$.

2.2 Tilting

Definition 2.4 *Let $\mathcal{A}$ be an abelian category. A pair of full subcategories $(\mathcal{F}, \mathcal{T})$ of $\mathcal{A}$ is called torsion pair if they satisfy the following conditions* (1) *and* (2)*.*

(1) For any $T \in \mathcal{T}$ and $F \in \mathcal{F}$, $\mathrm{Hom}(T, F) = 0$.

(2) For any $E \in \mathcal{A}$, there is a short exact sequence

$$0 \to T \to E \to F \to 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

$\mathcal{T}$ is called a torsion part and $\mathcal{F}$ is called a torsion-free part.

Proposition 2.5 ([HRS96]) *Let $\mathcal{H}$ be a heart of $\mathcal{D}$, and $(\mathcal{F}, \mathcal{T})$ be a torsion pair of $\mathcal{H}$. Then, the full subcategories*

$$\mathcal{H}^\sharp := \{\, E \in \mathcal{D} \mid \boldsymbol{H}_1(E) \in \mathcal{F},\ \boldsymbol{H}_0(E) \in \mathcal{T},\ \boldsymbol{H}_i(E) = 0 \text{ for } i \neq 0, 1 \,\}$$
$$\mathcal{H}^\flat := \{\, E \in \mathcal{D} \mid \boldsymbol{H}_0(E) \in \mathcal{F},\ \boldsymbol{H}_{-1}(E) \in \mathcal{T},\ \boldsymbol{H}_i(E) = 0 \text{ for } i \neq -1, 0 \,\}.$$

are the hearts of bounded t-structures.

The heart $\mathcal{H}^\sharp$ is called the forward tilt of $\mathcal{H}$ with respect to the torsion pair $(\mathcal{F}, \mathcal{T})$, and $\mathcal{H}^\flat$ the backward tilt of $\mathcal{H}$.

For an abelian category $\mathcal{A}$, denote by $\mathrm{Sim}\,\mathcal{A}$ the set of simple objects in $\mathcal{A}$.

Definition 2.6 *An abelian category $\mathcal{A}$ is called finite if the set* $\mathrm{Sim}\,\mathcal{A}$ *is a finite set and they generate $\mathcal{A}$ by means of extensions.*

Note that if an abelian category $\mathcal{A}$ is finite with simple objects $\operatorname{Sim}\mathcal{A} = \{S_1, \ldots, S_n\}$, then we have

$$K(\mathcal{A}) \cong \bigoplus_{i=1}^{n} \mathbb{Z}[S_i]$$

where $[S_i]$'s are K-group classes of the set of simple objects.

Let $\mathcal{H}$ be a finite heart. For a simple object $S \in \mathcal{H}$, Let $\langle S \rangle \subset \mathcal{H}$ be the smallest full subcategory containing S and closed under the extensions.

We introduce subcategories $\langle S \rangle^{\perp}$ and $^{\perp}\langle S \rangle$ by

$$\begin{aligned} \langle S \rangle^{\perp} &:= \{ E \in \mathcal{H} \mid \operatorname{Hom}(S, E) = 0 \} \\ {}^{\perp}\langle S \rangle &:= \{ E \in \mathcal{H} \mid \operatorname{Hom}(E, S) = 0 \}. \end{aligned}$$

Then, they define torsion pairs $(\langle S \rangle, {}^{\perp}\langle S \rangle)$ and $(\langle S \rangle^{\perp}, \langle S \rangle)$. The subcategory $\langle S \rangle$ is the torsion-free part in the former and the torsion part in the latter.

Definition 2.7 *Write by $\mathcal{H}_S^{\sharp}$ the forward tilt with respect to $(\langle S \rangle, {}^{\perp}\langle S \rangle)$, and by $\mathcal{H}_S^{\flat}$ the backward tilt with respect to $(\langle S \rangle^{\perp}, \langle S \rangle)$. We call $\mathcal{H}_S^{\sharp}$ the forward simple tilt by S, and $\mathcal{H}_S^{\flat}$ the backward simple tilt by S.*

Note that by simple tilts, we have $S[1] \in \mathcal{H}_S^{\sharp}$ and $S[-1] \in \mathcal{H}_S^{\flat}$.

We say an object $M \in \mathcal{H}$ is rigid if $\operatorname{Ext}^1(M, M) = 0$.

Proposition 2.8 ([KQ], Proposition 5.2) *Let $\mathcal{D}$ be a triangulated category and $\mathcal{H} \subset \mathcal{D}$ be a finite heart. For any rigid simple object $S \in \mathcal{H}$, the set of simple objects in the simple tilted heart $\mathcal{H}_S^{\sharp}$ or $\mathcal{H}_S^{\flat}$ is*

$$\begin{aligned} \operatorname{Sim}\mathcal{H}_S^{\sharp} &= \{S[1]\} \cup \{ \psi_S^{\sharp}(X) \mid X \in \operatorname{Sim}\mathcal{H},\ X \neq S \} \\ \operatorname{Sim}\mathcal{H}_S^{\flat} &= \{S[-1]\} \cup \{ \psi_S^{\flat}(X) \mid X \in \operatorname{Sim}\mathcal{H},\ X \neq S \} \end{aligned}$$

where

$$\begin{aligned} \psi_S^{\sharp}(X) &= \operatorname{Cone}(X \to S[1] \otimes \operatorname{Hom}^1(X, S)^*)[-1] \\ \psi_S^{\flat}(X) &= \operatorname{Cone}(S[-1] \otimes \operatorname{Hom}^1(S, X) \to X). \end{aligned}$$

Thus, $\mathcal{H}_S^{\sharp}$ and $\mathcal{H}_S^{\flat}$ are also finite.

2.3 The spaces of stability conditions

In this section, we introduce the notion of a stability condition on a triangulated category following Bridgeland in [Bri07], and collect some basic results for the space of stability conditions from [Bri07, Bri08, BS].

Definition 2.9 *Let $\mathcal{D}$ be a triangulated category and $K(\mathcal{D})$ be its K-group. A stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ consists of a group homomorphism $Z \colon K(\mathcal{D}) \to \mathbb{C}$ called central charge and a family of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for $\phi \in \mathbb{R}$ satisfying the following axioms:*

(a) if $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E) = m(E)\exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$,

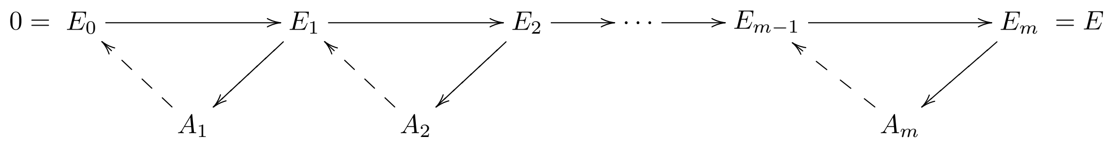
(b) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,

(c) if $\phi_1 > \phi_2$ and $A_i \in \mathcal{P}(\phi_i)$ $(i = 1, 2)$, then $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$,

(d) for $0 \neq E \in \mathcal{D}$, there is a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_m$$

and a collection of exact triangles



with $A_i \in \mathcal{P}(\phi_i)$ for all i.

For a nonzero object $E \in \mathcal{D}$, we define

$$\phi^+_\sigma(E) := \phi_1, \quad \phi^-_\sigma(E) := \phi_m$$

where ϕ_1 and ϕ_n are real numbers determined by the axiom (d).

The mass of a nonzero object $E \in \mathcal{D}$ is defined by

$$m_\sigma(E) := \sum_{i=1}^{m} |Z(A_i)|$$

where A_i's are extension factors determined by the axiom (d).

It follows from the definition that the subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ are abelian categories (see Lemma 5.2. in [Bri07]). The nonzero objects of $\mathcal{P}(\phi)$ are called semistable of phase ϕ in σ, and simple objects in $\mathcal{P}(\phi)$ are called stable of phase ϕ in σ.

For a stability condition $\sigma = (Z, \mathcal{P})$, we introduce the set of semistable classes $\mathcal{C}^{\mathrm{ss}}(\sigma) \subset K(\mathcal{D})$ by

$$\mathcal{C}^{\mathrm{ss}}(\sigma) := \{\, \alpha \in K(\mathcal{D}) \,|\, \text{there exists a semistable object } E \in \mathcal{D} \text{ in } \sigma \text{ such that } [E] = \alpha \,\}.$$

Similarly the set of stable classes $\mathcal{C}^{\mathrm{s}}(\sigma)$ can be defined.

We always assume our stability conditions satisfy the additional assumption called the support property in [KS].

Definition 2.10 *Let $\|\cdot\|$ be some norm on $K(\mathcal{D}) \otimes \mathbb{R}$. A stability condition $\sigma = (Z, \mathcal{P})$ has a support property if there is a some constant $C > 0$ such that*

$$C \cdot |Z(\alpha)| > \|\alpha\|$$

for all $\alpha \in \mathcal{C}^{\mathrm{ss}}(\sigma)$.

Remark 2.11 *For a ray $R = \mathbb{R}_{>0}\alpha \subset K(\mathcal{D})\otimes\mathbb{R}$ (where $\alpha \in K(\mathcal{D})\backslash\{0\}$), define a function f by*

$$f(R) := \frac{|Z(\alpha)|}{\|\alpha\|}.$$

(This doesn't depend on the choice of α, only depend on the ray.)

Let $\mathbb{R}_{>0}\mathcal{C}^{\mathrm{ss}}(\sigma) := \{\, \mathbb{R}_{>0}\alpha \,|\, \alpha \in \mathcal{C}^{\mathrm{ss}}(\sigma) \,\}$ be the set of rays generated by semistable classes of σ. Then, the support property of $\sigma = (Z, \mathcal{P})$ is equivalent to that there is no sequence of rays $R_i \subset \mathbb{R}_{>0}\mathcal{C}^{\mathrm{ss}}(\sigma)\,(i = 1, 2, \dots)$ such that

$$\lim_{i\to\infty} f(R_i) = 0.$$

Let $\mathrm{Stab}(\mathcal{D})$ be the set of all stability conditions on $\mathcal{D}$ with the support property. It is shown in [Bri07] that there is a natural topology on $\mathrm{Stab}(\mathcal{D})$ defined by the metric:

$$d(\sigma_1, \sigma_2) := \sup_{0\neq E\in\mathcal{D}} \left\{ |\phi^-_{\sigma_2}(E) - \phi^-_{\sigma_1}(E)|\,,\, |\phi^+_{\sigma_2}(E) - \phi^+_{\sigma_1}(E)|\,,\, \left|\log\frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)}\right| \right\} \in [0, \infty]$$

for $\sigma_1, \sigma_2 \in \mathrm{Stab}(\mathcal{D})$.

Bridgeland showed the following crucial theorem.

Theorem 2.12 ([Bri07], Theorem 1.2) *The space $\mathrm{Stab}(\mathcal{D})$ has the structure of a complex manifold and the projection map of central charges*

$$\pi\colon \mathrm{Stab}(\mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$$

defined by $(Z, \mathcal{P}) \mapsto Z$ is a local isomorphism onto an open subset of $\mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$.

The next lemma implies local injectivity of the above projection map π.

Lemma 2.13 ([Bri07], Lemma 6.4) *Let $\sigma = (Z, \mathcal{P})$ and $\sigma' = (Z, \mathcal{P}')$ be stability conditions on $\mathcal{D}$ with the same central charge Z. Then, $d(\sigma, \sigma') < 1$ implies $\sigma = \sigma'$.*

The following lemma shall be used in the proof of Proposition 5.4 and Proposition 5.6.

Lemma 2.14 *Fix a class $\alpha \in K(\mathcal{D})$ and let $U \subset \mathrm{Stab}(\mathcal{D})$ be an open subset. If every stability condition $\sigma \in U$ satisfies $\alpha \in \mathcal{C}^{\mathrm{ss}}(\sigma)$, then a stability condition on the boundary $\sigma' \in \partial U$ also satisfies $\alpha \in \mathcal{C}^{\mathrm{ss}}(\sigma')$.*

Proof. This follows from the result for walls and chambers in [Bri08, Section 9] or [BS, Section 7.6].

□

2.4 Central charges on finite abelian categories

In this section, we consider central charges on an abelian category, and give the another description of a stability condition, which consists of a pair of a heart and a central charge on it with the Harder-Narasimhan property. Further, we compute central charges on a finite abelian category.

Definition 2.15 *Let $\mathcal{A}$ be an abelian category and let $K(\mathcal{A})$ be its K-group. A central charge on $\mathcal{A}$ is a group homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{C}$ such that for any nonzero object $0 \neq E \in \mathcal{A}$, the complex number $Z(E)$ lies in semi-closed upper half-plane $H = \{\, re^{i\pi\phi} \ \in \mathbb{C} \mid r \in \mathbb{R}_{>0}, \phi \in (0,1]\,\}$.*

The real number

$$\phi(E) := \frac{1}{\pi} \arg Z(E) \in (0,1]$$

for $0 \neq E \in \mathcal{A}$ is called the phase of E.

A nonzero object $0 \neq E \in \mathcal{A}$ is said to be Z-(semi)stable if for any nonzero proper subobject $0 \neq A \subsetneq E$ satisfies $\phi(A) < (\leq) \phi(E)$.

For a central charge $Z\colon K(\mathcal{A}) \to \mathbb{C}$, the set of semistable classes can be defined similar to the previous section.

A Harder-Narasimhan filtration of an object $0 \neq E \in \mathcal{A}$ is a finite chain of subobjects

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m$$

such that all extension factors $F_i = E_i/E_{i-1}\,(i = 1, \ldots, m)$ are semistable with

$$\phi(F_1) > \cdots > \phi(F_m).$$

The central charge Z is said to have the Harder-Narasimhan property (HN-property) if every nonzero object of $\mathcal{A}$ has a Harder-Narasimhan filtration.

We write $\mathrm{Stab}(\mathcal{A})$ to be the set of central charges on $\mathcal{A}$ with the HN-property and the support property.

In this paper, we shall treat central charges only on finite abelian categories.

Let $\mathcal{A}$ be a finite abelian category with $\mathrm{Sim}\,\mathcal{A} = \{S_1, \ldots, S_n\}$. Recall from Section 2.2 that

$$K(\mathcal{A}) \cong \oplus_{i=1}^{n} \mathbb{Z}[S_i].$$

Any point $(z_1, \ldots, z_n) \in H^n$ defines a central charge $Z\colon K(\mathcal{A}) \to \mathbb{C}$ by $Z(S_i) := z_i$. Conversely, for a given central charge $Z\colon K(\mathcal{A}) \to \mathbb{C}$, the complex number $Z(S_i)$ lies in H for all i. Hence Z determines a point $(z_1, \ldots, z_n) \in H^n$ by $z_i := Z(S_i)$. As a result, the set of central charges on $\mathcal{A}$ is isomorphic to the space H^n.

Lemma 2.16 *Let $Z\colon K(\mathcal{A}) \to \mathbb{C}$ be a central charge on a finite abelian category $\mathcal{A}$. Then Z has the HN-property.*

Proof. This follows from Proposition 2.4. in [Bri07]. □

By this lemma, we have the isomorphism (Lemma 5.2 in [Bri09a])

$$\mathrm{Stab}(\mathcal{A}) \cong H^n.$$

Now, we consider the relationship between stability conditions on a triangulated category and stability conditions on an abelian category derived from a t-structure on a triangulated category.

In [Bri07], Bridgeland gave the following alternative description of a stability condition on $\mathcal{D}$ in terms of a bounded t-structure and a stability condition on a heart.

Proposition 2.17 ([Bri07], Proposition 5.3) *Let $\mathcal{D}$ be a triangulated category. To give a stability condition on $\mathcal{D}$ is equivalent to giving a bounded structure on $\mathcal{D}$ and a central charge with the HN-property on its heart.*

By the above proposition, we have a natural inclusion

$$\mathrm{Stab}(\mathcal{H}) \subset \mathrm{Stab}(\mathcal{D})$$

where $\mathcal{H}$ is a heart of a triangulated category $\mathcal{D}$.

2.5 Simple tilt and gluing

A heart $\mathcal{H} \subset \mathcal{D}$ is called rigid if every simple object S in $\mathcal{H}$ is rigid ($\mathrm{Ext}^1(S,S) = 0$).

Let $\mathcal{H} \subset \mathcal{D}$ be a finite rigid heart with simple objects $\{S_1, \dots, S_n\}$. Recall from the previous section that there is a natural inclusion

$$\mathrm{Stab}(\mathcal{H}) \subset \mathrm{Stab}(\mathcal{D}).$$

From a heart $\mathcal{H}$, we can construct the another hearts $\mathcal{H}^{\sharp}_{S_i}$(or $\mathcal{H}^{\flat}_{S_i}$) by simple tilt as in Section 2.2. The relationship between $\mathrm{Stab}(\mathcal{H})$ and $\mathrm{Stab}(\mathcal{H}^{\sharp}_{S_i})$(or $\mathrm{Stab}(\mathcal{H}^{\flat}_{S_i})$) in $\mathrm{Stab}(\mathcal{D})$ is given by the following lemma.

Lemma 2.18 ([BS], Lemma 7.9) *Let $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$ lies on a unique codimension one boundary of the region $\mathrm{Stab}(\mathcal{H})$ so that $Im\, Z(S_i) = 0$ for a unique simple object S_i. Then, there is a neighborhood $\sigma \in U \subset \mathrm{Stab}(\mathcal{D})$ such that one of the following holds*

(1) $Z(S_i) \in \mathbb{R}_{<0}$ and $U \subset \mathrm{Stab}(\mathcal{H}) \cup \mathrm{Stab}(\mathcal{H}^{\sharp}_{S_i})$,

(2) $Z(S_i) \in \mathbb{R}_{>0}$ and $U \subset \mathrm{Stab}(\mathcal{H}) \cup \mathrm{Stab}(\mathcal{H}^{\flat}_{S_i})$.

2.6 Group actions and spherical twists

For the space $\mathrm{Stab}(\mathcal{D})$, we introduce two commuting group actions.

First, consider the action of $\mathrm{Aut}(\mathcal{D})$ on $\mathrm{Stab}(\mathcal{D})$. Let $\Phi \in \mathrm{Aut}(\mathcal{D})$ be an autoequivalence of $\mathcal{D}$ and $(Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$ be a stability condition on $\mathcal{D}$. Then, the element $\Phi \cdot (Z, \mathcal{P}) = (Z', \mathcal{P}')$ is defined by

$$Z'(E) := Z(\Phi^{-1}(E)), \quad \mathcal{P}'(\phi) := \Phi(\mathcal{P}(\phi)),$$

where $E \in \mathcal{D}$ and $\phi \in \mathbb{R}$.

Second, define the $\mathbb{C}$-action on $\mathrm{Stab}(\mathcal{D})$. For $t \in \mathbb{C}$ and $(Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$, the element $t \cdot (Z, \mathcal{P}) = (Z', \mathcal{P}')$ is defined by

$$Z'(E) := e^{-i\pi t} \cdot Z(E), \quad \mathcal{P}'(\phi) := \mathcal{P}(\phi + \mathrm{Re}\,(t))$$

where $E \in \mathcal{D}$ and $\phi \in \mathbb{R}$. Clearly, this action is free.

Note that by the definition, these two actions are isometries with respect to the distance d on the space $\mathrm{Stab}(\mathcal{D})$.

Definition 2.19 *A triangulated category $\mathcal{D}$ is called Calabi-Yau N (CY_N) category if for any objects $E, F \in \mathcal{D}$ there is a natural isomorphism*

$$\nu_{E,F} \colon \mathrm{Hom}_{\mathcal{D}}(E, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(F, E[N])^*$$

where $E[N]$ is Nth shift of an object E and $*$ means the dual vector space. In other words, the N-th shift functor $[N]$ is a Serre functor.

For $E, F \in \mathcal{D}$, we write $\mathrm{Hom}^i_D(E, F) := \mathrm{Hom}_D(E, F[i])$.

In the following, we assume that a triangulated category $\mathcal{D}$ is algebraic in the sense of Keller (see Section 3.6 in [Kel06]).

We define some autoequivalences of $\mathcal{D}$ which play important role in this paper, called spherical twists, introduced by Seidel and Thomas in [ST01].

An object $S \in \mathcal{D}$ is called N-spherical if

$$\mathrm{Hom}^i_{\mathcal{D}}(S, S) = \begin{cases} k & \text{if} \quad i = 0, N \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.20 ([ST01], Proposition 2.10) *For a spherical object $S \in \mathcal{D}$, there is an autoequivalence $\Phi_S \in \mathrm{Aut}(\mathcal{D})$ such that there is an exact triangle*

$$\mathrm{Hom}^{\bullet}_{\mathcal{D}}(S, E) \otimes S \longrightarrow E \longrightarrow \Phi_S(E)$$

for any object $E \in \mathcal{D}$. The inverse functor $\Phi_S^{-1} \in \mathrm{Aut}(\mathcal{D})$ is given by

$$\Phi_S^{-1}(E) \longrightarrow E \longrightarrow S \otimes \mathrm{Hom}^{\bullet}_{\mathcal{D}}(E, S)^*.$$

The Euler form $\chi \colon K(\mathcal{D}) \times K(\mathcal{D}) \to \mathbb{Z}$ is defined by

$$\chi(E, F) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_k \mathrm{Hom}^i_{\mathcal{D}}(E, F).$$

By the definition, the Euler form χ on a CY_N triangulated category $\mathcal{D}$ is symmetric if N is even, and skew-symmetric if N is odd.

Part I
Stability condition for preprojective algebras

3 Root systems of symmetric Kac-Moody Lie algebras

3.1 Root lattices

In this section, we recall basic notions and results for a root system associated to a generalized Cartan matrix and describe the Weyl group action on it. Further, the set of real roots and imaginary roots are given. We refer to Kac's book and paper [Kac90, Kac78] for more details.

A matrix $A = (a_{ij})_{i,j=1}^n$ is called a generalized Cartan matrix (GCM for short) if

(C1) $a_{ii} = 2$ for $i = 1, \dots, n$,

(C2) $a_{ij} \in \mathbb{Z}_{\le 0}$ for $i \neq j$,

(C3) $a_{ij} = 0 \Rightarrow a_{ji} = 0$.

In this paper, we treat only symmetric GCMs, so the above condition (C3) always holds.

A matrix A is decomposable if A is a block diagonal form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

up to reordering of indices and indecomposable if otherwise.

We associate with A a graph $S(A)$, called Dynkin diagram, as follows. Vertices of the graph $S(A)$ are given by indices $\{1, \dots, n\}$, and distinct two vertices $i \neq j$ are connected by $|a_{ij}|$ edges. It is clear that A is indecomposable if and only if a graph $S(A)$ is connected.

For a real column vector $u = (u_1, u_2, \dots)^T$, the notation $u > 0$ means $u_i > 0$ for all u_i and $u \geq 0$ means $u_i \geq 0$ for all u_i. The next result gives the classification of indecomposable GCMs.

Theorem 3.1 ([Kac90], Theorem 4.3) *Let A be a indecomposable GCM. Then one and only one of the following three possibilities holds:*

(Fin) $\det A \neq 0$; *there exists* $u > 0$ *such that* $Au > 0$; $Au \geq 0 \Rightarrow u > 0$ *or* $u = 0$,

(Aff) $\operatorname{rank} A = n - 1$; *there exists* $u > 0$ *such that* $Au = 0$; $Au = 0 \Rightarrow u = 0$,

(Ind) there exists $u > 0$ *such that* $Au < 0$; $u \geq 0 \Rightarrow u = 0$.

Referring to cases (Fin), (Aff), or (Ind), we shall say that A is of finite, affine, or indefinite type, respectively.

Remark 3.2 *An indecomposable symmetric GCM A is finite type if and only if the Dynkin diagram $S(A)$ is of ADE type, and affine type if and only if $S(A)$ is of $\hat{A}\hat{D}\hat{E}$ type.*

The root lattice associated to A is a free abelian group $L := \oplus_{i=1}^{n}\mathbb{Z}\alpha_i$ generated by n free generators $\Pi := \{\alpha_1, \ldots, \alpha_n\}$ called simple roots. Since A is symmetric, we can define symmetric bilinear form $(\ ,\ ) : L \times L \to \mathbb{Z}$ by $(\alpha_i, \alpha_j) := a_{ij}$.

Define simple reflections $r_i \colon L \to L\ \ (i = 1, \ldots, n)$ by

$$r_i(\lambda) := \lambda - (\lambda, \alpha_i)\alpha_i, \quad \text{for } \lambda \in L.$$

The group generated by these simple reflections $W := \langle r_1, \ldots, r_n\rangle$ is called the Weyl group and satisfies the following relations (see Ch. 3 in [Kac90]):

$$r_i^2 = 1$$

and for $i \neq j$,

$$\begin{aligned} r_i r_j &= r_j r_i && \text{if} \quad a_{ij} = 0 \\ r_i r_j r_i &= r_j r_i r_j && \text{if} \quad a_{ij} = -1. \end{aligned}$$

Note that the symmetric bilinear form $(\ ,\ )$ is invariant under the W-action; $(w(\alpha), w(\beta)) = (\alpha, \beta)$ for any $\alpha, \beta \in L$ and $w \in W$.

For $\alpha = \sum_i k_i\alpha_i$, the support of α is defined to be the full subgraph $\mathrm{supp}(\alpha) \subset S(A)$ with vertices $\{\, i \,|\, k_i \neq 0 \,\} \subset \{\, 1, \ldots, n \,\}$.

Let $L_+ := \sum_{i=1}^{n}\mathbb{Z}_{\geq 0}\alpha_i$ and $L_- := -(L_+) = \sum_{i=1}^{n}\mathbb{Z}_{\leq 0}\alpha_i$. We define the set of (positive or negative) real roots and imaginary roots by using above the W-action on L.

Definition 3.3 *(1) The set of real roots Δ^{re} is defined to be W-orbits of simple roots Π;*

$$\Delta^{\mathrm{re}} := W(\Pi) = \{\, w(\alpha_i) \,|\, w \in W, i = 1, \ldots, n\}.$$

The set of positive real roots Δ_+^{re} (negative real roots Δ_-^{re}) is given by

$$\Delta_+^{\mathrm{re}} := \Delta^{\mathrm{re}} \cap L_+ \quad (\Delta_-^{\mathrm{re}} := \Delta^{\mathrm{re}} \cap L_-\ ).$$

(2) Define the fundamental set of positive imaginary roots K by

$$K := \{\, \alpha \in L_+\backslash\{0\} \,|\, \mathrm{supp}(\alpha) \textit{ is connected in } S(A),\ (\alpha, \alpha_i) \leq 0 \textit{ for } i = 1, \ldots, n \,\}.$$

The set of positive imaginary roots Δ_+^{im} is defined to be W-orbits of K;

$$\Delta_+^{\mathrm{im}} := W(K) = \{\, w(\alpha) \,|\, w \in W, \alpha \in K\}.$$

The set of negative imaginary roots is given by $\Delta_-^{\mathrm{im}} := -\Delta_+^{\mathrm{im}}$ and the set of all imaginary roots is given by $\Delta^{\mathrm{im}} := \Delta_+^{\mathrm{im}} \cup \Delta_-^{\mathrm{im}}$.

Remark 3.4 *(1) For an indecomposable GCM A, by Theorem 3.1 the fundamental set K is non-empty if and only if A is affine or indefinite type. Hence the set of imaginary roots* Δ^{im} *is also non-empty if and only if A is affine or indefinite type.*

(2) Since K is closed under the multiplication of positive integers $\mathbb{Z}_{\geq 1}$ *and the W-action commutes with this multiplication,* Δ^{im}_{+} *is also closed under the multiplication of* $\mathbb{Z}_{\geq 1}$*.*

Let $J \subset \{1, \ldots, n\}$ be a subset of indices of A and consider the submatrix $A_J := (a_{ij})_{i,j \in J}$, which is not necessarily indecomposable. Then we can define a root sublattice $L_J := \oplus_{j \in J} \mathbb{Z}\alpha_j \subset L$ and a Weyl subgroup $W_J :=< r_j \,|\, j \in J >\subset W$ associated to A_J.

Lemma 3.5 *Let* $J \subset \{1, \ldots, n\}$ *be a subset of indices and* $\Delta_J \subset \Delta$ *be the set of roots associated to* A_J*. Then the following inclusion holds:*

$$\Delta_J \subset L_J \cap \Delta.$$

Proof. It immediately follows from the definition of roots. □

Note that the other inclusion also holds, but we don't use in this paper.

3.2 Imaginary cones and the regular subsets X_{reg}

Let A be a GCM and $L = \oplus_{i=1}^{n} \mathbb{Z}\alpha_i$ be a root lattice associated to A. Through this paper, we fix the following notations:

$$\begin{aligned} V_{\mathbb{R}}^* &:= L \otimes_{\mathbb{Z}} \mathbb{R} = \oplus_{i=1}^{n} \mathbb{R}\alpha_i \\ V^* &:= L \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_{i=1}^{n} \mathbb{C}\alpha_i = V_{\mathbb{R}}^* \oplus iV_{\mathbb{R}}^* \\ V_{\mathbb{R}} &:= \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{R}) = \oplus_{i=1}^{n} \mathbb{R}Z^i \\ V &:= \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}) = \oplus_{i=1}^{n} \mathbb{C}Z^i = V_{\mathbb{R}} \oplus iV_{\mathbb{R}} \end{aligned}$$

where $\{ Z^1, \ldots, Z^n \}$ is the dual basis of $\{ \alpha_1, \ldots, \alpha_n \}$ with canonical pairings $\langle Z^i, \alpha_j \rangle = \delta_{ij}$ for $i, j = 1, \ldots, n$. We fix real forms of complex vector spaces V and V^* as in above notations. The space of $\mathbb{C}$-linear maps V will be identified with the space of central charges in section 4.2.

The action of the Weyl group W on L is naturally extended to the action on V^* (or $V_{\mathbb{R}}^*$) and the contragradient action of W on V (or $V_{\mathbb{R}}$) is given by $\langle w(Z), \lambda \rangle := \langle Z, w^{-1}(\lambda) \rangle$ for $Z \in V$ and $\lambda \in V^*$.

We fix the norms for these vector spaces by the following:

$$\text{for } \lambda = \sum_i \lambda_i \alpha_i \in V^* \,(\,\text{or } V_{\mathbb{R}}^*\,), \quad \|\lambda\| := \sqrt{\sum_i |\lambda_i|^2}$$

$$\text{for } Z = \sum_i z_i Z^i \in V \,(\,\text{or } V_{\mathbb{R}}\,), \quad \|Z\| := \sqrt{\sum_i |z_i|^2}.$$

These norms give the natural topology as finite dimensional vector spaces.

Next we introduce the notion of an imaginary cone which plays central role in this paper.

Definition 3.6 *An imaginary cone I is defined by the closure of the convex hull of the set $\Delta_+^{\mathrm{im}} \cup \{0\}$ in $V_{\mathbb{R}}^*$ and an imaginary cone without zero is written by $I_0 := I \backslash \{0\}$ (we also call I_0 an imaginary cone).*

Note that by Remark 3.4, the imaginary cone I_0 associated to an indecomposable GCM A is non-empty if and only if A is affine or indefinite type.

Lemma 3.7 *Assume that the imaginary cone I_0 is non-empty. Then, I is a convex cone supported on $\sum_{i=1}^n \mathbb{R}_{\geq 0}\alpha_i$.*

Proof. The convexity of I is clear by the definition of I. Since $\Delta_+^{\mathrm{im}} \cup \{0\}$ is closed under the multiplication of $\mathbb{Z}_{\geq 0}$ (see Remark 3.4), I is closed under the multiplication of $\mathbb{R}_{\geq 0}$. Futher $\Delta_+^{\mathrm{im}} \cup \{0\}$ is supported on $\sum_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i$, therefore I is supported on $\sum_{i=1}^n \mathbb{R}_{\geq 0}\alpha_i$. □

The next result from [Kac78] plays important role in the proof of main theorem in section 6.

Proposition 3.8 *The set of rays of imaginary roots $\{ \mathbb{R}_{>0}\alpha \mid \alpha \in \Delta_+^{\mathrm{im}} \}$ is dense in the imaginary cone I_0.*

For the compatibility of the notations in later sections, we write the canonical pairing of $Z \in V$ and $\lambda \in V^*$ by $Z(\lambda) := \langle Z, \lambda \rangle$.

The open subset X_{reg} defined as the following corresponds to the space of central charges treated in this paper.

Definition 3.9 *Let $\lambda \in V^*$ and $H_\lambda := \{ Z \in V \mid Z(\lambda) = 0 \}$ denotes a complex orthogonal hyperplane with respect to λ. A subset $X \subset V$ is defined by*

$$X := V \backslash \bigcup_{\lambda \in I_0} H_\lambda$$

and a regular subset $X_{\mathrm{reg}} \subset X \subset V$ is defined by

$$X_{\mathrm{reg}} := X \backslash \bigcup_{\alpha \in \Delta_+^{\mathrm{re}}} H_\alpha.$$

Since the W-action on V^* preserves real roots Δ^{re} and the imaginary cone I_0, the W-action on the subsets X and X_{reg} is well-defined.

The following lemma is used to prove that X_{reg} is an open subset of V.

Lemma 3.10 ([Kac90], Lemma 5.8) *In $V^* \backslash \{0\}$, the limit rays for the set of rays $\{ \mathbb{R}_{>0}\alpha \mid \alpha \in \Delta_+^{\mathrm{re}} \}$ lie in I_0.*

Define an imaginary convex disk by

$$D := I \cap \{\, k_1\alpha_1 + \cdots + k_n\alpha_n \in V_{\mathbb{R}}^* \,|\, k_1 + \cdots + k_n = 1 \,\}.$$

Lemma 3.7 implies that D is a compact convex subset of $V_{\mathbb{R}}^*$.

Note that X can be written by

$$X := V \backslash \bigcup_{\lambda \in D} H_\lambda$$

by using D.

Lemma 3.11 *The subsets $X \subset V$ and $X_{\mathrm{reg}} \subset V$ are open in V.*

Proof. The result that $\bigcup_{\lambda\in D} H_\lambda$ is a closed subset of V follows from the compactness of D.

Let $Z \in X_{\mathrm{reg}}$ and B be a small open ball centered at Z. By Lemma 3.10, accumulated points of hyperplanes $\{H_\alpha\}_{\alpha\in\Delta_+^{\mathrm{re}}}$ are contained in $\bigcup_{\lambda\in D} H_\lambda$. Hence if we take B to be sufficiently small, then B does not intersect any hyperplanes $\{H_\alpha\}_{\alpha\in\Delta_+^{\mathrm{re}}}$, and this implies that X_{reg} is open. □

Lemma 3.12 *Assume that $I_0 \subset V^*$ is non-empty. Let $Z \in X$ and consider the linear map $Z\colon V^* \to \mathbb{C}$. Then the image of the imaginary cone $Z(I_0) \subset \mathbb{C}$ takes the following form:*

$$Z(I_0) = \{\, re^{i\pi\phi} \,|\, r > 0 \ \text{ and } \ \phi_1 \le \phi \le \phi_2 \,\}$$

where $\phi_1, \phi_2 \in \mathbb{R}$ (determined up to modulo $2\mathbb{Z}$) with $0 \le \phi_2 - \phi_1 < 1$.

Proof. Let D be an imaginary convex disk defined above. Since $Z\colon V^* \to C$ is $\mathbb{R}$-linear and D is compact convex, the image $Z(D) \subset \mathbb{C}$ is also compact convex and $Z(I_0) = \mathbb{R}_{>0} Z(D)$. The assumption $Z \in X$ means $0 \notin Z(I_0)$. Thus $Z(D)$ is a compact convex subset of $\mathbb{C}\backslash\{0\}$.

First, assume that $Z(D) \cap \mathbb{R}_{>0} = \phi$ and consider the principal argument $\operatorname{Arg} z \in (0, 2\pi)$ for $z \in \mathbb{C}\backslash\mathbb{R}_{>0}$. Then, by the compactness of $Z(D)$, we can define the maximum and the minimum argument of $Z(D)$ by

$$\phi_1 := \min \{\, (1/\pi) \operatorname{Arg} z \in (0,2) \,|\, z \in Z(D) \}$$
$$\phi_2 := \max \{\, (1/\pi) \operatorname{Arg} z \in (0,2) \,|\, z \in Z(D) \}\,.$$

Clearly $0 \le \phi_2 - \phi_1$, and $\phi_2 - \phi_1 < 1$ easily follows from the convexity of $Z(D)$ and $0 \notin Z(D)$.

If the other case that $Z(D) \cap \mathbb{R}_{>0} \neq \phi$, the convexity of $Z(D)$ and $0 \notin Z(D)$ imply that $Z(D) \cap \mathbb{R}_{<0} = \phi$. Therefore we can do the similar argument for the principal argument $\operatorname{Arg} z \in (-\pi, \pi)$ for $z \in \mathbb{C}\backslash\mathbb{R}_{<0}$. □

By using Lemma 3.12, for $Z \in X$ we introduce the phase of the imaginary cone $\phi^I(Z)$, up to modulo $2\mathbb{Z}$, by

$$\phi^I(Z) := \frac{\phi_2 + \phi_1}{2}$$

where ϕ_1 and ϕ_2 are phases determined in Lemma 3.12.

In the case I_0 is non-empty, define a normalized regular subset X^N_{reg} by

$$X^N_{\mathrm{reg}} := \{ Z \in X_{\mathrm{reg}} \,|\, \phi^I(Z) = 1/2 \}.$$

Normalizing $\phi^I(Z)$ to $1/2$ by the action of $S^1 := \{z \in \mathbb{C} \,|\, |z| = 1\}$ on X_{reg}, we have

$$X_{\mathrm{reg}} \cong S^1 \times X^N_{\mathrm{reg}}.$$

Note that the W-action on X^N_{reg} is also well-defined becase the S^1-action and the W-action commute.

3.3 The Weyl group action on X_{reg}

In order to describe the regular subset X_{reg} in terms of the W and $\mathbb{C}^*$ actions on V, we recall some basic properties of Tits cones.

Throughout this section, we assume that a GCM A is indecomposable.

Consider the restriction of $X \subset V$ to the real subspace $V_{\mathbb{R}} \subset V$; $X_{\mathbb{R}} := X \cap V_{\mathbb{R}}$. If A is finite type, then $I = \phi$, therefore $X = V$ and $X_{\mathbb{R}} = V_{\mathbb{R}}$. If A is affine or indefinite type, then $X_{\mathbb{R}}$ decomposes to two connected components

$$\begin{aligned} X^+_{\mathbb{R}} &= \{ Z_R \in X_{\mathbb{R}} \,|\, Z_{\mathbb{R}}(\lambda) > 0 \quad \text{for all} \quad \lambda \in I_0 \} \\ X^-_{\mathbb{R}} &= \{ Z_R \in X_{\mathbb{R}} \,|\, Z_{\mathbb{R}}(\lambda) < 0 \quad \text{for all} \quad \lambda \in I_0 \}. \end{aligned}$$

To describe these sets in terms of the W-action on $V_{\mathbb{R}}$, we introduce a notion of the Tits cone as follows.

Definition 3.13 ([Kac90], Section 3.12) *Define a subset of $V_{\mathbb{R}}$, called Weyl chamber, by $C_{\mathbb{R}} := \{ Z_R \in V_R \,|\, Z_{\mathbb{R}}(\alpha_i) > 0$ for $i = 1, \dots, n\} \cong \mathbb{R}^n_{>0}$ and let $\overline{C_{\mathbb{R}}} = \{ Z_R \in V_{\mathbb{R}} \,|\, Z_R(\alpha_i) \geq 0$ for $i = 1, \dots, n\} \cong \mathbb{R}^n_{\geq 0}$ be a closure of $C_{\mathbb{R}}$ in $V_{\mathbb{R}}$.*

The Tits cone $T_{\mathbb{R}}$ is defined by

$$T_{\mathbb{R}} := \bigcup_{w \in W} w(\overline{C_{\mathbb{R}}}).$$

Lemma 3.14 ([Kac90], Section 5.8) *The following equality holds:*

$$T_{\mathbb{R}} = \begin{cases} X_{\mathbb{R}} & \textit{if } A \textit{ is finite type} \\ X^+_{\mathbb{R}} \cup \{0\} & \textit{if } A \textit{ is affine or indefinite type.} \end{cases}$$

From this result and the properties of $X^+_{\mathbb{R}}$ together, it turns out that the Tits cone is a convex cone.

Let us introduce the regular subset of $T_{\mathbb{R}}$ by

$$T_{\mathbb{R},\mathrm{reg}} := T_{\mathbb{R}} \setminus \bigcup_{\alpha \in \Delta^{\mathrm{re}}_+} H_\alpha.$$

It is easy to see that $T_{\mathbb{R},\text{reg}}$ is given by

$$T_{\mathbb{R},\text{reg}} = \bigcup_{w \in W} w(C_{\mathbb{R}}).$$

It is known that the closure of the Weyl chamber $\overline{C_{\mathbb{R}}}$ is a fundamental domain of the action of the Weyl group W on $T_{\mathbb{R}}$, and W acts freely on $T_{\mathbb{R},\text{reg}}$.

The next lemma shall be used in the proof of Proposition 3.16.

Lemma 3.15 *Assume that a GCM A is finite type. Then for any $Z \in T_{\mathbb{R},\text{reg}}$, there is an unique element $w \in W$ such that $w \cdot Z$ satisfies $(w \cdot Z)(\alpha_i) < 0$ for all $i = 1, \dots, n$.*

Proof. In the situation that A is finite type,

$$T_{\mathbb{R},\text{reg}} = V_{\mathbb{R}} \backslash \bigcup_{\alpha \in \Delta_+^{\text{re}}} H_\alpha.$$

Therefore the chamber $-C_{\mathbb{R}} = \{ Z_R \in V_R \,|\, Z_{\mathbb{R}}(\alpha_i) < 0 \text{ for } i = 1, \dots, n\}$ is contained in $T_{\mathbb{R},\text{reg}}$, so the result follows. □

In the rest of this section, we assume that I_0 is non-empty, namely a GCM A is affine or indefinite type. Let $\mathbb{H} := \{ z \in \mathbb{C} \,|\, \operatorname{Im} z > 0 \}$ be an upper half plane and introduce the semi-closed upper half plane by $H := \mathbb{H} \cup \mathbb{R}_{<0}$. H is also written by

$$H = \{ re^{i\pi\phi} \in \mathbb{C} \,|\, r > 0,\, \phi \in (0, 1] \}.$$

Define a normalized complexified Weyl chamber by $C^N := \{ Z \in X_{\text{reg}}^N \,|\, Z(\alpha_i) \in H \quad \text{for} \quad i = 1, \dots, n \}$. The following proposition is main result of this section.

Proposition 3.16 *For any $Z \in X_{\text{reg}}^N$, there is an element $w \in W$ such that $w \cdot Z$ lies in $C^N \subset X_{\text{reg}}^N$.*

Proof. Let $Z \in X_{\text{reg}}$. Using the decomposition $V = V_{\mathbb{R}} \oplus iV_{\mathbb{R}}$, we write $Z = Z_R + iZ_I$ where $Z_R, Z_I \in V_{\mathbb{R}}$. Since $\phi^I(Z) = 1/2$, the image $Z(I_0)$ is contained in $\mathbb{H}$ and this implies that $Z_I(\lambda) > 0$ for all $\lambda \in I_0$. Hence we have $Z_I \in X_{\mathbb{R}}^+ = T_{\mathbb{R}} \backslash \{0\}$.

By Lemma 3.14, there is some $w' \in W$ such that $w' \cdot Z_I \in \overline{C_{\mathbb{R}}} \cong \mathbb{R}_{\geq 0}^n$.

Hence $Z' := w' \cdot Z = w' \cdot Z_R + iw' \cdot Z_I$ lies in $\overline{\mathbb{H}}^n \cong \mathbb{R}^n + i\,\mathbb{R}_{\geq 0}^n$ where $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}$.

Depending on $Z' \in \overline{\mathbb{H}}^n$, we define a subset of indices $J \subset \{1, \dots, n\}$ by

$$J := \{ j \,|\, Z'(\alpha_j) \in \mathbb{R} \}.$$

As in Section 2.1 we consider the submatrix A_J and the associated subgraph $S(A_J) \subset S(A)$. Then J is divided as $J = J_1 \cup \cdots J_l$ corresponding to a decomposition of the subgraph $S(A_J)$ into the connected components $S(A_{J_1}), \dots, S(A_{J_l})$, therefore the submatrices $A_{J_1}, \dots, A_{J_l}$ are all indecomposable.

We can say that all indecomposable GCMs $A_{J_1}, \dots, A_{J_l}$ are finite type by the following reason. If A_{J_m} $(m = 1, \dots, l)$ is affine or indefinite type, the image of the corresponding imaginary subcone $I_0^{J_m} \subset I_0$ by Z' is also contained in $\mathbb{H}$. But by the definition of J, the image of all elements in $L_J \otimes \mathbb{R}$ by Z' lies in $\mathbb{R}$, hence the image of $I_0^{J_m} \subset L_J \otimes \mathbb{R}$ is also contained in $\mathbb{R}$ and this gives the contradiction.

Consider the restriction of the linear map $Z' \colon L \to \mathbb{C}$ to the root sublattice $L_{J_m} \subset L$ corresponding to A_{J_m}:

$$Z'|_{L_{J_m}} \colon L_{J_m} \longrightarrow \mathbb{C}.$$

Since $J_m \subset J$, for any α_j $(j \in J_m)$ we have $Z'(\alpha_j) \in \mathbb{R}$, therefore we can regard $Z'|_{L_{J_m}}$ as an element of the regular subset of the Tits subcone $T_{\mathbb{R},\mathrm{reg}}^{J_m}$ corresponding to A_{J_m}. Then we can take an element $w_{J_m} \in W_{J_m}$ to satisfy $(w_{J_m} \cdot Z')(\alpha_j) \in \mathbb{R}_{<0}$ for all $j \in J_m$ as mentioned in Lemma 3.15.

Collect such elements $w_{J_1}, w_{J_2}, \dots, w_{J_l}$ for all $m = 1, \dots, l$ and define $w := w_{J_1} \cdot w_{J_2} \cdots w_{J_l} \cdot w'$ (it isn't depend on the order of $w_{J_1}, w_{J_2}, \dots, w_{J_l}$ since these elements commute). Then w is the desired element. □

Corollary 3.17 *Define $C \subset X_{\mathrm{reg}}$ by*

$$C := \{\, Z \in X_{\mathrm{reg}} \,|\, Z(\alpha_i) \in \mathbb{H} \quad \textit{for } i = 1, \dots, n \,\} \,(\cong \mathbb{H}^n\,).$$

Then, for any $Z \in X_{\mathrm{reg}}$, there are elements $w \in W$ and $k \in \mathbb{C}^*$ (both w and k are not necessarily unique) such that $w \cdot k \cdot Z$ lies in C.

Proof. Since $X_{\mathrm{reg}} \cong S^1 \times X_{\mathrm{reg}}^N$, for $Z \in X_{\mathrm{reg}}$ we can take $w \in W$ and $k \in \mathbb{C}^*$ such that $Z' := w \cdot k \cdot Z$ lies in $C^N \subset H^n$ and clearly we can rotate $Z' \in H^n$ to hold $l \cdot Z' \in \mathbb{H}^n$ by some $l \in \mathbb{C}^*$. Non-uniqueness of them comes from the choice of $\phi^I(k \cdot Z)$ and there are many possibilities of $w \in W$ depending on this choice. □

Proposition 3.18 *The W-action on X_{reg} is free and properly discontinuous. Further, the fundamental domain of this action is given by*

$$S^1 \times C^N \subset S^1 \times X_{\mathrm{reg}}^N \cong X_{\mathrm{reg}}.$$

Proof. First we prove the freeness of the W-action on X_{reg}. By Corollary 3.17, we only need to show it for $Z \in X_{\mathrm{reg}}$ with $Z(\alpha_i) \in \mathbb{H}$ for all $i = 1, \dots, n$. Such an element Z satisfies that $Z_I \in C_{\mathbb{R}} \subset T_{\mathbb{R},\mathrm{reg}}$ where $Z = Z_R + iZ_I$. Note that since the W-action on $T_{\mathbb{R},\mathrm{reg}}$ is free by Lemma 3.14, if $w \cdot Z_I = Z_I$ for $w \in W$ then $w = 1$. Hence if $w \cdot Z = Z$ for $w \in W$ then $w = 1$.

Similarly by Corollary 3.17, the properly discontinuity of the W-action is also reduced to the following statement: for any $Z \in X_{\mathrm{reg}}$ with $Z(\alpha_i) \in \mathbb{H}$ for all $i = 1, \dots, n$, there is an open neighborhood $Z \in U \subset X_{\mathrm{reg}}$ such that $U \cap w(U) = \phi$ for all $1 \neq w \in W$. This is easily shown from the identification $\mathbb{H}^n \cong \mathbb{R}^n + C_{\mathbb{R}}$ $(C_{\mathbb{R}} \cong \mathbb{R}_{>0}^n)$ and $C_{\mathbb{R}} \cap w(C_{\mathbb{R}}) = \phi$ for $1 \neq w \in W$ by Lemma 3.14.

The last statement immediately follows from Proposition 3.16. □

3.4 Walls and chambers in $X_{\rm reg}^N$

Here, we introduce the walls in $X_{\rm reg}^N$. This structure shall be used in Section 4.3 to study the action of a braid group on the space of stability conditions.

Let $\overline{C^N}$ be a closure of C^N in $X_{\rm reg}^N$. For $i = 1, \ldots, n$, introduce the walls $W_{i,\pm} \subset \overline{C^N}$ by

$$W_{i,+} := \{ Z \in X_{\rm reg}^N \,|\, Z(\alpha_i) \in \mathbb{R}_{>0}, Z(\alpha_j) \in \mathbb{H} \text{ for } j \neq i \}$$
$$W_{i,-} := \{ Z \in X_{\rm reg}^N \,|\, Z(\alpha_i) \in \mathbb{R}_{<0}, Z(\alpha_j) \in \mathbb{H} \text{ for } j \neq i \}$$

Note that $W_{i,-} \subset C^N$, but $W_{i,+} \cap C^N = \phi$. However $r_i(W_{i,\pm}) = W_{i,\mp}$ and hence $W_{i,+} \subset r_i(C^N)$.

By using the W-action on $X_{\rm reg}^N$, the set of all walls is defined by

$$\{ w(W_{i,\pm}) \,|\, w \in W,\, i = 1, \ldots, n \}.$$

They correspond to the walls of second kind in terms of [KS].

Lemma 3.19 *For any $Z \in W_{i,\pm}$, there is a neighborhood $Z \in U \subset X_{\rm reg}^N$ such that*
$$U \subset C^N \cup r_i(C^N).$$

Proof. Since $r_i(C^N \cup r_i(C^N)) = C^N \cup r_i(C^N)$ and $r_i(W_{i,\pm}) = W_{i,\mp}$, we only need to consider the case that $Z \in W_{i,-}$.

Let $Z \in W_{i,-}$. Take an open disk $D_i \subset \mathbb{C}$ centered at $Z(\alpha_i) \in \mathbb{R}_{<0}$ satisfying $0 \notin D_i$ and divide it into two pieces

$$D_{i,+} := \{ z \in D_i \,|\, z \in H \}, \quad D_{i,-} := \{ z \in D_i \,|\, z \in -\mathbb{H} \}.$$

Define a neighborhood of Z by $U := X_{\rm reg}^N \cap D_1 \times \cdots \times D_n$ where D_j $(j \neq i)$ is an open disk centered at $Z(\alpha_j) \in \mathbb{H}$ taken sufficiently small to satisfy $D_j \subset \mathbb{H}$. Then, it is easy to check that divided pieces $U_\pm := X_{\rm reg}^N \cap D_1 \times \cdots \times D_{i,\pm} \times \cdots \times D_n$ satisfy $U_+ \subset C^N$ and $U_- \subset r_i(C^N)$. Hence $U = U_+ \cup U_-$ is the desired one. □

3.5 The fundamental group of $X_{\rm reg}$

In this section, we give a fundamental group of $X_{\rm reg}$ by using the result of the van der Lek [vdL83]. This is described in terms of an Artin group associated to a Coxter system of the Weyl group W derived from A ([BS72]).

Definition 3.20 ([BS72]) *An Artin group G_W associated to the Weyl group W (derived from A) is defined to be the group generated by n generators $\sigma_1, \ldots, \sigma_n$ with the following relations: for $i \neq j$,*

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if} \quad a_{ij} = 0$$
$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if} \quad a_{ij} = -1.$$

Recall the decomposition $X_{\mathrm{reg}} \cong S^1 \times X^N_{\mathrm{reg}}$ in Section 3.2. The Weyl group W acts trivially on the first factor S^1, hence

$$X_{\mathrm{reg}}/W \cong S^1 \times \left(X^N_{\mathrm{reg}}/W\right).$$

Take a point $*$ in the interior of the fundamental domain $S^1 \times C^N$ of the W-action (see Proposition 10.11) and $[*]$ be the base point of X_{reg}/W.

Theorem 3.21 ([vdL83], see also [Par]) *Assume that a GCM A is affine or indefinite type. Then, the fundamental group of X_{reg}/W is given by*

$$\pi_1(X_{\mathrm{reg}}/W, [*]) \cong \mathbb{Z}[\gamma] \times G_W.$$

The generator γ of the first factor $\mathbb{Z}[\gamma]$ is given by the S^1-orbit of $[*]$. The generator σ_i of the second factor G_W is given by the path connecting to $*$ and $r_i(*)$ passing the wall $W_{i,\pm}$ in X^N_{reg} just once, which is a loop in X^N_{reg}/W.

Proof. Since

$$X_{\mathrm{reg}}/W \cong S^1 \times \left(X^N_{\mathrm{reg}}/W\right),$$

and $\pi(S^1) \cong \mathbb{Z}$, we prove that

$$\pi_1(X^N_{\mathrm{reg}}/W) \cong G_W.$$

Define a regular subset of the complexified Tits cone by

$$T_{\mathrm{reg}} := \{\, Z \in X_{\mathrm{reg}} \mid \operatorname{Im} Z \in T_{\mathbb{R}} \}.$$

The van der Lek's result in [vdL83] implies that $\pi_1(T_{\mathrm{reg}}/W) \cong G_W$. Therefore we show that X^N_{reg} is homotopic to T_{reg}.

For $Z \in X_{\mathrm{reg}}$, $\operatorname{Im} Z \in T_{\mathbb{R}} \backslash \{0\} = X^+_{\mathbb{R}}$ is equivalent to $Z(I_0) \subset \mathbb{H}$ (see Section 2.3), hence we can write

$$T_{\mathrm{reg}} = \{\, Z \in X_{\mathrm{reg}} \mid Z(I_0) \subset \mathbb{H} \,\}.$$

Construct a deformation retract $h_t \colon T_{\mathrm{reg}} \to T_{\mathrm{reg}}$ by $h_t(Z) := Z \cdot e^{i\pi t(1/2 - \phi^I(Z))}$ where $0 \le t \le 1$. Then, it is easy to check $h_1(T_{\mathrm{reg}}) = X^N_{\mathrm{reg}}$ and $h_1 = \mathrm{id}$ on $X^N_{\mathrm{reg}} \subset T_{\mathrm{reg}}$, and as a result it gives a homotopy equivalence $T_{\mathrm{reg}} \sim X^N_{\mathrm{reg}}$. $\square$

4 Derived categories of preprojective algebras

4.1 Preprojective algebras of quivers

Let Q be a finite quiver without loops. We denote by Q_0 its set of vertices and Q_1 its set of arrows. A opposite quiver Q^{op} is obtained by reversing the orientation of arrows of Q. For an arrow $a \colon i \to j \in Q_1$, we denote the opposite arrow by $a^* \colon j \to i \in Q^{\mathrm{op}}$.

A double quiver $\overline{Q}$ is defined by adding all opposite arrows to Q, so $\overline{Q}_1 = Q_1 \cup Q_1^{\mathrm{op}}$. For a quiver Q, define an adjacent matrix (q_{ij}) of Q by

$$q_{ij} := |\{\text{ arrows from } i \text{ to } j\ \}|.$$

We associate with Q to a GCM A_Q by

$$(A_Q)_{ij} := 2\delta_{ij} - (q_{ij} + q_{ji}).$$

For a connected quiver Q, we say that Q is finite, affine or indefinite type if the corresponding GCM A_Q is finite, affine or indefinite type respectively (see Section 2.1).

Note that if we forget the directions of arrows in Q, then the underlying graph of Q coincides to the graph $S(A_Q)$ defined in section 3.1.

Let us denote by $\mathbb{C}Q$ a path algebra of Q over k. We put a gradation on kQ by using the length of paths.

Definition 4.1 *The preprojective algebra $\Pi(Q)$ associated to Q is defined by*

$$\Pi(Q) := \mathbb{C}\overline{Q}/(\rho)$$

where (ρ) is an ideal of $\mathbb{C}\overline{Q}$ generated by the element

$$\rho := \sum_{a \in Q_1} (aa^* - a^*a).$$

Since ρ is a homogeneous element in $\mathbb{C}\overline{Q}$, the preprojective algebra $\Pi(Q)$ is also a graded algebra by the length of paths.

Let $\mathcal{A}_Q := \text{mod-}\,\Pi(Q)$ be an abelian category of finite dimensional nilpotent right $\Pi(Q)$-modules.

Proposition 4.2 ([Kel08], Section 4**)** *Let $D^b(\mathcal{A}_Q)$ be a bounded derived category of $\mathcal{A}_Q$. If Q is not a finite type quiver, then $D^b(\mathcal{A}_Q)$ is a CY-2 triangulated category.*

For simplicity, we write $\mathcal{D}_Q := D^b(\mathcal{A}_Q)$.

Consider the set of simple modules $\operatorname{Sim} \mathcal{A}_Q := \{ S_1, \dots, S_n \}$ corresponding to the vertices $Q_0 = \{1, \dots, n\}$. Then, the K-group $K(\mathcal{A}_Q)$ is given by

$$K(\mathcal{A}_Q) \cong \oplus_{i=1}^n \mathbb{Z}\,[S_i].$$

We also note that since $\mathcal{D}_Q$ is bounded, we have $K(\mathcal{D}_Q) \cong K(\mathcal{A}_Q)$.

For the K-group classes of simple modules $[S_i], [S_j] \in K(\mathcal{D}_Q)$, the Euler form is computed by $\chi(S_i, S_j) = a_{ij}$ where a_{ij}'s are entries of the GCM A_Q.

Hence we can identify $(K(\mathcal{D}_Q), \chi)$ with the root lattice $(L_Q, (\ ,\ ))$ associated to A_Q by the map $[S_i] \mapsto \alpha_i$ as a $\mathbb{Z}$-lattice with a symmetric bilinear form.

4.2 Seidel-Thomas braid groups

By the CY_N property of $\mathcal{D}_Q$, simple modules $\operatorname{Sim}\mathcal{A}_Q = \{S_1, \ldots, S_n\}$ of $\mathcal{A}_Q$ are 2-spherical in $\mathcal{D}_Q$. Hence they define spherical twists $\Phi_{S_1}, \ldots, \Phi_{S_n} \in \operatorname{Aut}(\mathcal{D}_Q)$. The Seidel-Thomas braid group $\operatorname{Br}(\mathcal{D}_Q)$ is defined to be the subgroup of $\operatorname{Aut}(\mathcal{D}_Q)$ generated by these spherical twists:

$$\operatorname{Br}(\mathcal{D}_Q) := \langle \Phi_1, \ldots, \Phi_n \rangle .$$

Proposition 4.3 ([ST01], Theorem 1.2) *For the group* $\operatorname{Br}(\mathcal{D}_Q)$*, the following relations hold :*

$$\begin{aligned} \Phi_{S_i}\Phi_{S_j} &= \Phi_{S_j}\Phi_{S_i} && \textit{if} \quad \chi(S_i, S_j) = 0 \\ \Phi_{S_i}\Phi_{S_j}\Phi_{S_i} &= \Phi_{S_j}\Phi_{S_i}\Phi_{S_j} && \textit{if} \quad \chi(S_i, S_j) = -1. \end{aligned}$$

Corollary 4.4 *There is a surjective group homomorphism*

$$\rho\colon G_W \to \operatorname{Br}(\mathcal{D}_Q)$$

defined by $\sigma_i \mapsto \Phi_{S_i}$*.*

Note that at the K-group level, a spherical twist Φ_S induces a reflection $[\Phi_S]\colon K(\mathcal{D}_Q) \to K(\mathcal{D}_Q)$ given by

$$[\Phi_S]([E]) = [E] - \chi(S, E)[S]$$

and inverse is to be $[\Phi_S^{-1}] = [\Phi_S]$. In particular, under the identification of $(K(\mathcal{D}_Q), \chi) \cong (L_Q, (\ ,\ ))$, $\operatorname{Br}(\mathcal{D}_Q)$ is reduced to the Weyl group W by the projection map $\Phi_{S_i} \mapsto r_i$.

Recall from Section 3.5 that the fundamental group of X_{reg}/W is isomorphic to $\mathbb{Z}[\gamma] \times G_W$. We can extend the above group homomorphism ρ to the following group homomorphism

$$\widetilde{\rho}\colon \mathbb{Z}[\gamma] \times G_W \longrightarrow \mathbb{Z}[2] \times \operatorname{Br}(\mathcal{D}_Q)$$

which sends the $[\gamma]$ to the twice shift functor $[2] \in \operatorname{Aut}(\mathcal{D}_Q)$.

4.3 Group actions on $\operatorname{Stab}^\circ(\mathcal{D}_Q)$

Consider the space of stability conditions $\operatorname{Stab}(\mathcal{D}_Q)$. For this space, there is a unique distinguished connected component $\operatorname{Stab}^\circ(\mathcal{D}_Q) \subset \operatorname{Stab}(\mathcal{D}_Q)$ which contains $\operatorname{Stab}(\mathcal{A}_Q)$ consisting of stability conditions with the heart $\mathcal{A}_Q$. We consider some group actions on $\operatorname{Stab}^\circ(\mathcal{D}_Q)$ and prove the lifted version of Proposition 3.16 and Corollary 3.17 via the restriction of the projection map

$$\pi\colon \operatorname{Stab}^\circ(\mathcal{D}_Q) \longrightarrow V$$

to the regular subset $X_{\mathrm{reg}} \subset V$. (Though, we see just $\pi(\operatorname{Stab}^\circ(\mathcal{D}_Q)) = X_{\mathrm{reg}}$ later in Proposition 5.6.)

At the K-group level, we can easily see the following remark about semistable classes.

Remark 4.5 *By the action of an autoequivalence* $\Phi \in \mathrm{Aut}(\mathcal{D})$*, the set of semistable classes* $\mathcal{C}^{\mathrm{ss}}(\sigma)$ *for* $\sigma \in \mathrm{Stab}(\mathcal{D})$ *changes to*

$$\mathcal{C}^{\mathrm{ss}}(\Phi \cdot \sigma) = [\Phi](\mathcal{C}^{\mathrm{ss}}(\sigma)).$$

On the other hand, the $\mathbb{C}$*-action on* $\mathrm{Stab}(\mathcal{D})$ *doesn't change semistable classes* $\mathcal{C}^{\mathrm{ss}}(\sigma)$*.*

In the rest of this section, we study the action of the Seidel-Thomas braid group $\mathrm{Br}(\mathcal{D}_Q)$ on $\mathrm{Stab}^\circ(\mathcal{D}_Q)$. Recall the identification $K(\mathcal{D}_Q) \cong L_Q$ and consider the projection map

$$\pi \colon \mathrm{Stab}^\circ(\mathcal{D}_Q) \longrightarrow V$$

where $V = \mathrm{Hom}_{\mathbb{Z}}(L_Q, \mathbb{C})$ as in Section 3.2.

Then, the connected subset $H^n \cong \mathrm{Stab}(\mathcal{A}_Q) \subset \mathrm{Stab}^\circ(\mathcal{D}_Q)$ is isomorphically mapped onto the subset

$$H^n \cong \{\, Z \in X_{\mathrm{reg}} \,|\, Z(\alpha_i) \in H \quad \text{for } i = 1, \dots, n \,\} \subset V.$$

Corresponding to the normalized regular subset $X^N_{\mathrm{reg}} \subset V$, we introduce the space of normalized stability conditions $\mathrm{Stab}(\mathcal{D}_Q)^N$ by

$$\mathrm{Stab}(\mathcal{D}_Q)^N := \{\, \sigma \in \mathrm{Stab}^\circ(\mathcal{D}_Q) \,|\, \pi(\sigma) \in X^N_{\mathrm{reg}} \,\},$$

and normalized stability conditions on $\mathcal{A}_Q$ by

$$\mathrm{Stab}(\mathcal{A}_Q)^N := \{\, \sigma \in \mathrm{Stab}(\mathcal{A}_Q) \,|\, \pi(\sigma) \in X^N_{\mathrm{reg}} \,\}.$$

Note that by the projection

$$\pi \colon \mathrm{Stab}(\mathcal{D}_Q)^N \to X^N_{\mathrm{reg}},$$

$\mathrm{Stab}(\mathcal{A}_Q)^N$ is mapped isomorphically onto the chamber $C^N \subset X^N_{\mathrm{reg}}$.

Define the walls $\widetilde{W_{i,\pm}} \subset \overline{\mathrm{Stab}(\mathcal{A}_Q)^N}$ $(\, i = 1, \dot{,} n \,)$ which are lifts of walls $W_{i,\pm} \subset \overline{C^N}$ $(\, i = 1, \dot{,} n \,)$ on $\mathrm{Stab}(\mathcal{A}_Q)^N$ by

$$\widetilde{W_{i,+}} := \{\, \sigma = (Z, \mathcal{P}) \in \overline{\mathrm{Stab}(\mathcal{A}_Q)^N} \,|\, Z(S_i) \in \mathbb{R}_{>0}, Z(S_j) \in \mathbb{H} \text{ for } j \neq i \,\}$$
$$\widetilde{W_{i,-}} := \{\, \sigma = (Z, \mathcal{P}) \in \overline{\mathrm{Stab}(\mathcal{A}_Q)^N} \,|\, Z(S_i) \in \mathbb{R}_{<0}, Z(S_j) \in \mathbb{H} \text{ for } j \neq i \,\}$$

Note that as in Section 3.4, $\widetilde{W_{i,-}} \subset \mathrm{Stab}(\mathcal{A}_Q)^N$ but $\widetilde{W_{i,+}} \cap \mathrm{Stab}(\mathcal{A}_Q)^N = \phi$. However, $\Phi^{-1}_{S_i}(\widetilde{W_{i,-}}) = \widetilde{W_{i,+}}$ and $\widetilde{W_{i,+}} \subset \Phi^{-1}_{S_i}(\mathrm{Stab}(\mathcal{A}_Q)^N)$.

Under these notations, the lifted version of Lemma 3.19 also holds.

Lemma 4.6 *Let* $\sigma \in \widetilde{W_{i,\pm}} \subset \overline{\mathrm{Stab}(\mathcal{A}_Q)^N}$*. Then, there is a neighborhood* $\sigma \in U \subset \mathrm{Stab}(\mathcal{D}_Q)^N$ *such that one of the following holds*

(1) $U \subset \mathrm{Stab}(\mathcal{A}_Q)^N \cup \Phi^{-1}_{S_i}(\mathrm{Stab}(\mathcal{A}_Q)^N)$ *if* $\sigma \in \widetilde{W_{i,+}}$*,*

(2) $U \subset \mathrm{Stab}(\mathcal{A}_Q)^N \cup \Phi_{S_i}(\mathrm{Stab}(\mathcal{A}_Q)^N)$ *if* $\sigma \in \widetilde{W_{i,-}}$.

Proof. Note that in a CY_2 category $\mathcal{D}_Q$, simple tilted categories $(\mathcal{A}_Q)^{\sharp}_{S_i}$ and $(\mathcal{A}_Q)^{\flat}_{S_i}$ correspond to $\Phi^{-1}_{S_i}(\mathcal{A}_Q)$ and $\Phi_{S_i}(\mathcal{A}_Q)$. Hence it follows from Lemma 2.18. □

Lemma 4.7 *The image of the projection map* $\pi\colon \mathrm{Stab}^{\circ}(\mathcal{D}_Q) \to V$ *contains* X_{reg}.

Proof. Recall that by Corollary 3.17, the orbit of $H^n \subset V$ under the action of $\mathbb{C}^*$ and W coincides to X_{reg}. Since the action of $\mathbb{C}$ and $\mathrm{Br}(\mathcal{D}_Q)$ reduced to the action of $\mathbb{C}^*$ and W on the base space V, the orbit of $\mathrm{Stab}(\mathcal{A}_Q) \subset \mathrm{Stab}^{\circ}(\mathcal{D}_Q)$ under the action of $\mathbb{C}$ and $\mathrm{Br}(\mathcal{D}_Q)$ mapped to the subset $X_{\mathrm{reg}} \subset V$. □

Let $\mathrm{Stab}^{\circ}(\mathcal{D}_Q)^N$ be the connected component of $\mathrm{Stab}(\mathcal{D}_Q)^N$ which contains $\mathrm{Stab}(\mathcal{A}_Q)^N$. Now, we lift Proposition 3.16 and Corollary 3.17 via the restricted projection map $\pi\colon \pi^{-1}(X_{\mathrm{reg}}) \to X_{\mathrm{reg}}$.

Proposition 4.8 *For any* $\sigma \in \mathrm{Stab}^{\circ}(\mathcal{D}_Q)^N$*, there is an autoequivalence* $\Phi \in \mathrm{Br}(\mathcal{D}_Q)$ *such that* $\Phi \cdot \sigma$ *lies in* $\mathrm{Stab}(\mathcal{A}_Q)^N$.

Proof. Let $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}^{\circ}(\mathcal{D}_Q)^N$ and take a path $\gamma\colon [0,1] \to \mathrm{Stab}^{\circ}(\mathcal{D}_Q)^N$ such that $\gamma(0) \in \mathrm{Stab}(\mathcal{A}_Q)^N$ and $\gamma(1) = \sigma$. Since $\pi(\sigma) \in X^N_{\mathrm{reg}}$ lies in some chamber $w(C^N)\,(w \in W)$ and by Lemma 3.19, we can deform γ to satisfy that for $t \in (0,1)$ the path $\pi(\gamma) \subset X^N_{\mathrm{reg}}$ passes the walls $\{\, w(W_{i,\pm}) \mid w \in W, i = 1, \dots, n \,\}$ only at $t_1, \dots, t_m \in (0,1)$ with $0 < t_1 < \cdots < t_m < 1$. (Perhaps, the start point $\pi(\gamma(0))$ and end point $\pi(\gamma(1))$ may lie on the walls or higher codimension walls.)

Since $\gamma(\,[0,t_1)\,) \subset \mathrm{Stab}(\mathcal{A}_Q)^N$ and $\pi(\gamma(t_1)) \in W_{i,\pm}$ for some i, the stability $\gamma(t_1) \in \mathrm{Stab}^{\circ}(\mathcal{D}_Q)^N$ lies in $\widetilde{W_{i,\pm}}$. If $\gamma(t_1) \in \widetilde{W_{i_1,+}}$, define $\Phi_1 := \Phi_{S_{i_1}}$, and if $\gamma(t_1) \in \widetilde{W_{i,-}}$, define $\Phi_1 := \Phi^{-1}_{S_{i_1}}$. Then, by Lemma 4.6, $(\Phi_1 \cdot \gamma)(\,(t_1,t_2)\,) \subset \mathrm{Stab}(\mathcal{A}_Q)^N$ and $\pi(\Phi_1 \cdot \gamma(t_2)) \in W_{i_2,\pm}$. Hence $\Phi_1 \cdot \gamma(t_2) \in \widetilde{W_{i_2,\pm}}$, and we can define similarly $\Phi_2 := \Phi_{S_{i_2}} \Phi_1$ or $\Phi_2 := \Phi^{-1}_{S_{i_2}} \Phi_1$.

Repeating this process, we get an autoequivalence $\Phi_m \in \mathrm{Br}(\mathcal{D}_Q)$ such that $(\Phi_m \cdot \gamma)(\,(t_m, 1]\,) \subset \mathrm{Stab}(\mathcal{A}_Q)^N$. □

Let $\pi^{-1}(X_{\mathrm{reg}})^{\circ}$ be the connected component of $\pi^{-1}(X_{\mathrm{reg}})$ which contains $\mathrm{Stab}(\mathcal{A}_Q)$. (In Section 5.2, we show that $\pi^{-1}(X_{\mathrm{reg}})$ is connected and coincides to $\mathrm{Stab}^{\circ}(\mathcal{D}_Q)$.)

Corollary 4.9 *For* $\sigma \in \pi^{-1}(X_{\mathrm{reg}})^{\circ} \subset \mathrm{Stab}^{\circ}(\mathcal{D}_Q)$*, there are elements* $\Phi \in \mathrm{Br}(\mathcal{D}_Q)$ *and* $k \in \mathbb{C}$ *such that* $\Phi \cdot k \cdot \sigma \in \mathrm{Stab}(\mathcal{A}_Q)^N$.

Proof. Let $\sigma \in \pi^{-1}(X_{\mathrm{reg}})^{\circ}$ and take a path $\gamma\colon [0,1] \to \pi^{-1}(X_{\mathrm{reg}})^{\circ}$ such that $\gamma(0) \in \mathrm{Stab}(\mathcal{A}_Q)$ and $\gamma(1) = \sigma$. By the $\mathbb{C}$-action on $\pi^{-1}(X_{\mathrm{reg}})^{\circ}$, we can normalize γ to the path $\gamma' = k \cdot \gamma$ which lies in $\mathrm{Stab}^{\circ}(\mathcal{D}_Q)^N$ where $k\colon [0,1] \to \mathbb{C}$ and $\gamma'(t) = k(t) \cdot \gamma(t)$. Then, the result follows from Proposition 4.8. □

5 Proof of main theorem in part I

5.1 Indivisible roots and semistable classes

Following in [Kin94], we introduce a King's stability condition for the space of nilpotent $\Pi(Q)$-modules with a fixed K-group class $\alpha \in K(\mathcal{A}_Q)$. This stability condition is identified with the Bridgeland's stability condition in Lemma 5.3. As in previous sections, we identify $K(\mathcal{A}_Q)$ with the corresponding root lattice L_Q.

Let $\alpha \in L_Q$ and write the set of nilpotent modules (representations) of $\Pi(Q)$ with the class α by $\mathrm{Rep}(\Pi(Q),\alpha)^{\mathrm{nil}}$. A King's stability condition on $\mathcal{A}_Q$ is defined to be a $\mathbb{R}$-linear map $\lambda\colon L_Q \to \mathbb{R}$ satisfying $\lambda(\alpha) = 0$. A module $M \in \mathrm{Rep}(\Pi(Q),\alpha)^{\mathrm{nil}}$ is said to be λ-(semi)stable if any proper nonzero submodule $0 \neq N \subsetneq M$ satisfies $\lambda(N) > (\geq) 0$.

We write the subset consisting of λ-stable nilpotent modules with the class α by

$$\mathrm{Rep}(\Pi(Q),\alpha)^{\lambda,\mathrm{nil}} \subset \mathrm{Rep}(\Pi(Q),\alpha)^{\mathrm{nil}}.$$

Following [CBVdB04], we introduce a notion of generic stability conditions.

Definition 5.1 *A King's stability condition $\lambda\colon L_Q \to \mathbb{R}$ is said to be generic with respect to α if $\lambda(\beta) \neq 0$ for all $0 < \beta < \alpha$.*

A root $\alpha \in \Delta$ is called indivisible if there is no $\beta \in \Delta$ satisfying $\alpha = m\beta$ for $|m| > 1$. The following result in [CBVdB04] was made to prove Kac's conjecture for indivisible roots in geometric way by using the moduli space of stable modules of preprojective algebras. Here, we use this result to understand the relationship between semistable classes of stability conditions and indivisible roots.

Proposition 5.2 ([CBVdB04], Proposition 1.2) *Let $\alpha \in \Delta_+$ be a positive indivisible root and suppose that λ is generic with respect to α. Then, the number of irreducible components of $\mathrm{Rep}(\Pi(Q),\alpha)^{\lambda,\mathrm{nil}}$ is equal to a root multiplicity $\dim \mathfrak{g}_\alpha$.*

From this result it turns out that if λ is generic with respect to a positive indivisible root $\alpha \in \Delta_+$, then the moduli space of λ-stable nilpotent modules with the class α is non-empty.

As stated in Section 5.2 of [BT09], we can identify King's stability conditions with Bridgeland's stability conditions by the following way.

Lemma 5.3 *Fix $\alpha \in L_{Q,+}$. Let $Z \in \mathrm{Stab}(\mathcal{A}_Q)$ be a central charge on $\mathcal{A}_Q$ and define a $\mathbb{R}$-linear map $\lambda_\alpha\colon L_Q \to \mathbb{R}$ by*

$$\lambda_\alpha(\beta) := -Im\,\frac{Z(\beta)}{Z(\alpha)}.$$

for $\beta \in L_Q$. A module $M \in \mathcal{A}_Q$ with a class $[M] = \alpha$ is λ_α-(semi)stable in the sense of King if and only if M is Z-(semi)stable in the sense of Bridgeland as in Section 4.2.

The next proposition is main result of this section and to be used in the next section.

Proposition 5.4 *Let $\sigma = (Z, \mathcal{P}) \in \pi^{-1}(X_{\mathrm{reg}})^{\circ}$. Then, the set of indivisible roots Δ is contained in the set of σ-semistable classes $\mathcal{C}^{\mathrm{ss}}(\sigma)$:*

$$\{ \alpha \in \Delta \mid \alpha \text{ is indivisible} \} \subset \mathcal{C}^{\mathrm{ss}}(\sigma).$$

Proof. First note that the set of all indivisible roots are invariant by the action of the Weyl group W. Since $\sigma \in \pi^{-1}(X_{\mathrm{reg}})^{\circ}$, by Remark 4.5 and Corollary 4.9, it is sufficient to prove that any stability condition $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{A}_Q)$ contains all indivisible roots as semistable classes.

Fix a positive indivisible root $\alpha \in \Delta_+$ and consider an dense open subset of $\mathrm{Stab}(\mathcal{A}_Q)$ consisting of stability conditions which is generic with respect to α:

$$\{ Z \in \mathrm{Stab}(\mathcal{A}_Q) \mid \lambda_\alpha(\beta) \neq 0 \text{ for all } 0 < \beta < \alpha \}$$

where $\lambda_\alpha(\beta) = -\mathrm{Im}\,(Z(\beta)/Z(\alpha))$, defined in Lemma 5.3.

By Proposition 5.2, any stability condition in this subset contains α as an stable class. Since this subset is dense in $\mathrm{Stab}(\mathcal{A}_Q)$, by Lemmma 2.14 we can say that any stability condition in $\mathrm{Stab}(\mathcal{A}_Q)$ contains α at least as a semistable class. □

5.2 Projection of central charges

In this section, we determine the image of central charges via the projection map

$$\pi \colon \mathrm{Stab}^{\circ}(\mathcal{D}_Q) \to V$$

by using Proposition 5.4 proved in the last section.

Lemma 5.5 *Let $\partial X_{\mathrm{reg}}$ be a boundary of X_{reg} and assume that $Z \in \partial X_{\mathrm{reg}}$. Then, there is at least one ray $R \subset I_0 \cup \mathbb{R}_{>0}\Delta_+^{\mathrm{re}}$ such that $Z(R) = 0$.*

Proof. It immediately follows from the definition of the open subset $X_{\mathrm{reg}} \subset V$ (see Definition 3.9):

$$X_{\mathrm{reg}} = V \setminus \bigcup_{\lambda \in I_0 \cup \Delta_+^{\mathrm{re}}} H_\lambda \quad (H_\lambda = \{ Z \in V \mid Z(\lambda) = 0 \}).$$

□

Proposition 5.6 *The projection map*

$$\pi \colon \mathrm{Stab}^{\circ}(\mathcal{D}_Q) \to V$$

maps $\mathrm{Stab}^{\circ}(\mathcal{D}_Q)$ onto the subset $X_{\mathrm{reg}} \subset V$.

Proof. One inclusion $X_{\mathrm{reg}} \subset \pi(\mathrm{Stab}^\circ(\mathcal{D}_Q))$ is Lemma 4.7. Here we prove the other inclusion $\pi(\mathrm{Stab}^\circ(\mathcal{D}_Q)) \subset X_{\mathrm{reg}}$, which is equivalent to that $\pi^{-1}(X_{\mathrm{reg}})^\circ = \mathrm{Stab}^\circ(\mathcal{D}_Q)$.

Since $\mathrm{Stab}^\circ(\mathcal{D}_Q)$ is the connected component which contains $\pi^{-1}(X_{\mathrm{reg}})^\circ$, it is sufficient to prove that $\pi^{-1}(X_{\mathrm{reg}})^\circ$ is open and closed.

First note that since X_{reg} is open, the connected component $\pi^{-1}(X_{\mathrm{reg}})^\circ$ is also open. Hence, the closedness of $\pi^{-1}(X_{\mathrm{reg}})^\circ$ is equivalent to that it has no boundary points.

Assume that $\pi^{-1}(X_{\mathrm{reg}})^\circ$ has the boundary point $\sigma = (Z, \mathcal{P})$. Then, $\sigma = (Z, \mathcal{P})$ is projected on $\partial X_{\mathrm{reg}}$, therefore by Lemma 5.5, there is a ray $R \subset I_0 \cup \mathbb{R}_{>0}\Delta_+^{\mathrm{re}}$ such that $Z(R) = 0$.

For the ray $R \subset I_0 \cup \mathbb{R}_{>0}\Delta_+^{\mathrm{re}}$, by Proposition 3.8, we can take a sequence of rays $R_i \, (i = 1, 2, \dots,)$ satisfying

$$\lim_{i \to \infty} R_i = R$$

where $R_i = \mathbb{R}_{>0}\alpha_i$ and each $\alpha_i \in \Delta_+$ is an indivisible positive root.

On the other hand, since σ lies in the closure of $\pi^{-1}(X_{\mathrm{reg}})^\circ$, by Lemma 2.14 and Proposition 5.4, σ contains all indivisible roots as semistable classes. In particular, the above rays $R_i \, (i = 1, 2, \dots)$ are contained in $\mathbb{R}_{>0}\mathcal{C}^{\mathrm{ss}}(\sigma)$.

Since $Z(R) = 0$, we have

$$\lim_{i \to \infty} f(R_i) = f(R) = 0$$

where f is a function defined in Remark 2.10. But this contradicts to the support property of σ (see Remark 2.10).

□

5.3 Covering structures

Proposition 5.7 *The action of* $\mathbb{Z}[2] \times \mathrm{Br}(\mathcal{D}_Q) \subset \mathrm{Aut}(\mathcal{D}_Q)$ *on* $\mathrm{Stab}^\circ(\mathcal{D}_Q)$ *is free and properly discontinuous.*

Proof. These two properties are clearly satisfied for the action of $\mathbb{Z}[2]$, hence we prove them for the action of $\mathrm{Br}(\mathcal{D}_Q)$.

We first prove the freeness of the action of $\mathrm{Br}(\mathcal{D}_Q)$. By Corollary 4.9, it is sufficient to prove that for $\sigma \in \mathrm{Stab}(\mathcal{A}_Q)$ and $\Phi \in \mathrm{Br}(\mathcal{D}_Q)$, if $\sigma = \Phi \cdot \sigma$ then $\Phi \cong \mathrm{id}$. Let $\sigma = (Z, \mathcal{P})$ and suppose $(Z, \mathcal{P}) = ([\Phi]^{-1} \cdot Z, \Phi(\mathcal{P}))$. Since any object in $\mathcal{D}_Q$ is generated by finite extensions of objects $\{S_1, \dots, S_n\}$ and shifts of them, the isomorphism $\Phi(S_i) \cong S_i$ for all $i = 1, \dots, n$ implies $\Phi \cong \mathrm{id}$. Assume that $S_i \in \mathcal{P}(\phi_i) \subset \mathcal{A}_Q$, then also $\Phi(S_i) \in \mathcal{P}(\phi_i) \subset \mathcal{A}_Q$ since $\Phi(\mathcal{P}(\phi_i)) = \mathcal{P}(\phi_i)$. At the K-group level, $[\Phi]^{-1} \cdot Z = Z$ implies that $[\Phi] = \mathrm{id}$ as an element of the Weyl group W, therefore we have $[\Phi(S_i)] = [S_i]$. Both $\Phi(S_i)$ and S_i are objects in $\mathcal{A}_Q$ with the same K-group class $[S_i]$, and such an object is unique up to isomorphism in $\mathcal{A}_Q$. Hence $\Phi(S_i) \cong S_i$.

The last part to prove is that the action of $\mathrm{Br}(\mathcal{D}_Q)$ is properly discontinuous. We prove that for any $\sigma \in \mathrm{Stab}^\circ(\mathcal{D}_Q)$, there is some open subset $\sigma \in U$ such that $U \cap \Phi \cdot U = \phi$ for any $\Phi \in \mathrm{Br}(\mathcal{D}_Q)$, which is not isomorphic to the identity. Fix $\sigma \in \mathrm{Stab}^\circ(\mathcal{D}_Q)$. There are two cases either $[\Phi] = \mathrm{id}$ or not. Recall from Proposition 10.11 that the W-action on X_{reg} is free and properly discontinuous. Hence, by using it together with the local isomorphism property of $\pi \colon \mathrm{Stab}^\circ(\mathcal{D}_Q) \to X_{\mathrm{reg}}$ from Theorem 2.12, we have such an open subset $\sigma \in U$ for $\Phi \in \mathrm{Br}(\mathcal{D}_Q)$ with $[\Phi] \neq \mathrm{id}$. Further, we take U to be sufficiently small to satisfy that $d(\sigma, \sigma') < 1/2$ for all $\sigma' \in U$. Note that by Lemma 2.13, $d(\sigma, \Phi \cdot \sigma) \geq 1$ for $\Phi \in \mathrm{Br}(\mathcal{D}_Q)$ with $[\Phi] = \mathrm{id}$ but $\Phi \ncong \mathrm{id}$, hence $d(\sigma, \Phi \cdot \sigma') > 1/2$ and in particular $U \cap \Phi \cdot U = \phi$. Therefore we have $U \cap \Phi \cdot U = \phi$ for all $\Phi \in \mathrm{Br}(\mathcal{D}_Q)$ with $\Phi \ncong \mathrm{id}$. □

Write by $\underline{\pi}$ the composition of two maps $\pi \colon \mathrm{Stab}^\circ(\mathcal{D}_Q) \to X_{\mathrm{reg}}$ and $X_{\mathrm{reg}} \to X_{\mathrm{reg}}/W$. Now, we prove Theorem 1.1.

Theorem 5.8 *The projection map*

$$\underline{\pi} \colon \mathrm{Stab}^\circ(\mathcal{D}_Q) \longrightarrow X_{\mathrm{reg}}/W$$

is a covering map and the subgroup $\mathbb{Z}[2] \times \mathrm{Br}(\mathcal{D}_Q) \subset \mathrm{Aut}(\mathcal{D}_Q)$ acts as the deck transformation group. In particular, there is a surjective group homomorphism from the fundamental group of X_{reg} to the deck transformation group defined by

$$\widetilde{\rho} \colon \mathbb{Z}[\gamma] \times G_W \longrightarrow \mathbb{Z}[2] \times \mathrm{Br}(\mathcal{D}_Q).$$

Proof. The remaining part to show is that the quotient of $\mathrm{Stab}^\circ(\mathcal{D}_Q)$ by $\mathbb{Z}[2] \times \mathrm{Br}(\mathcal{D}_Q)$ is coincides to X_{reg}/W. This statement is equivalent to that for any $\sigma_1, \sigma_2 \in \mathrm{Stab}^\circ(\mathcal{D}_Q)$, if $\underline{\pi}(\sigma_1) = \underline{\pi}(\sigma_2)$, then there are elements $[2n] \in \mathbb{Z}[2]$ and $\Phi \in \mathrm{Br}(\mathcal{D}_Q)$ such that $\sigma_1 = \Phi \cdot [2n] \cdot \sigma_2$.

First note that $\underline{\pi}(\sigma_1) = \underline{\pi}(\sigma_2)$ in X_{reg}/W implies there is an unique element $w \in W$ such that $\pi(\sigma_1) = w \cdot \pi(\sigma_2)$ in X_{reg}.

By Corollary 4.9, we can assume that $\sigma_1 \in \mathrm{Stab}(\mathcal{A}_Q)^N$, and there are elements $k \in \mathbb{C}$ and $\Phi \in \mathrm{Br}(\mathcal{D}_Q)$ such that $\sigma_2' := \Phi \cdot k \cdot \sigma_2$ lies in $\mathrm{Stab}(\mathcal{A}_Q)^N$. By projecting σ_2' on X_{reg}, we have

$$\pi(\sigma_2') = [\Phi] \cdot e^{-i\pi k} \cdot \pi(\sigma_2) = [\Phi] \cdot e^{-i\pi k} \cdot w^{-1} \cdot \pi(\sigma_1).$$

Since $\mathrm{Stab}(\mathcal{A}_Q)^N$ is mapped isomorphically onto the normalized fundamental domain $C^N \subset X_{\mathrm{reg}}^N$ of the W-action on X_{reg}^N, the equality $\pi(\sigma_2') = [\Phi] \cdot e^{-i\pi k} \cdot w^{-1} \cdot \pi(\sigma_1)$ implies that $e^{-i\pi k} = 1$, $[\Phi] \cdot w^{-1} = 1$ and $\sigma_2' = \sigma_1$. Hence, $k = 2n \in 2\mathbb{Z}$ and $\Phi \cdot [2n] \cdot \sigma_2 = \sigma_1$.

□

The proof of Corollary 1.2 is completely the same way for the proof of Corollary 1.4 in [Bri09b].

Part II

Stability conditions for CY_N algebras of the A_n-quivers

6 Quadratic differentials associated with polynomials

6.1 Braid group and representation

Definition 6.1 *The Artin braid group B_{n+1} is the group generated by n generators $\sigma_1, \sigma_2, \ldots, \sigma_n$ and the relations*

$$\begin{aligned} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \quad & \textit{if } |i-j| = 1 \\ \sigma_i\sigma_j = \sigma_i\sigma_j \quad & \textit{if } |i-j| \geq 2. \end{aligned}$$

We consider two type representations of B_{n+1} on a lattice associated with an A_n-quiver. Such representations are known reduced Burau representations with parameters ± 1 (See Section 3 in [KT08]).

An A_n-quiver $\overrightarrow{A_n}$ is a quiver obtained by giving directions to edges of a Dynkin diagram of type A_n. Let $\{1, \ldots, n\}$ be the set of vertices of $\overrightarrow{A_n}$. We also assume that the vertex i is adjacent to vertices $i-1$ and $i+1$. Let q_{ij} the number of arrows from i to j.

Define L to be a free abelian group of rank n generated by $\alpha_1, \ldots, \alpha_n$:

$$L := \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i.$$

We define a symmetric bilinear form $<\,,\,>_+$ and a skew-symmetric form $<\,,\,>_-$ on L by

$$\begin{aligned} < \alpha_i, \alpha_j >_+ &:= \delta_{ij} + \delta_{ji} - (q_{ij} + q_{ji}) \\ < \alpha_i, \alpha_j >_- &:= \delta_{ij} - \delta_{ji} - (q_{ij} - q_{ji}) \end{aligned}$$

where δ_{ij} is the Kronecker delta.

We introduce the reflections via the bilinear form $<\,,\,>_\pm$ by

$$r_i^\pm(\alpha_j) := \alpha_j - < \alpha_i, \alpha_j >_\pm \alpha_i \quad \text{for } i = 1, \ldots, n.$$

Let $W_\pm$ be the group generated by these reflections $r_1^\pm, \ldots, r_n^\pm$.

The two type representations $\rho_\pm \colon B_{n+1} \to \mathrm{GL}(L, \mathbb{Z})$ are defined by

$$\rho_\pm(\sigma_i) := r_i^\pm.$$

We put the kernel of $\rho_\pm$ by $P_\pm$. Then, we have a exact sequence

$$1 \longrightarrow P_\pm \longrightarrow B_{n+1} \longrightarrow W_\pm \longrightarrow 1.$$

Note that the group W_+ is a symmetric group of degree $n+1$ and the group P_+ is a pure braid group.

6.2 Space of polynomials

Let $p_n(z)$ be a polynomial of the following form

$$p_n(z) = z^{n+1} + u_1 z^{n-1} + u_2 z^{n-2} + \cdots + u_n, \quad u_1, \ldots, u_n \in \mathbb{C}.$$

We denote by $\Delta(p_n)$ the discriminant of p_n. (If $p_n(z) = \prod_{i=1}^{n+1}(z - a_i)$, then $\Delta(p_n) = \prod_{i<j}(a_i - a_j)^2$.)

Denote by M_n the space of such polynomials with simple zeros

$$M_n := \{ p_n(z) \mid \Delta(p_n) \neq 0 \}.$$

Define the free $\mathbb{C}^*$-action on M_n by

$$(k \cdot p_n)(z) := z^{n+1} + k u_1 z^{n-1} + k^2 u_2 z^{n-2} + \cdots + k^n u_n \quad (k \in \mathbb{C}^*).$$

We note that for this action, the equation

$$(k \cdot p_n)(kz) = k^{n+1} p_n(z)$$

holds.

Consider the description of this space by using the configuration space on $\mathbb{C}$. Let $C_{n+1}(\mathbb{C})$ be the configuration space of $(n+1)$ distinct points in $\mathbb{C}$ and $C^0_{n+1}(\mathbb{C})$ be the subspace consists of configurations with center of mass at the origin

$$C^0_{n+1}(\mathbb{C}) := \{ (a_1, \ldots, a_{n+1}) \in \mathbb{C}^{n+1} \mid \sum_{i=1}^{n+1} a_i = 0, \, a_i \neq a_j \text{ for } i \neq j \}.$$

Note that $\mathbb{C}$ acts on $C_{n+1}(\mathbb{C})$ by parallel transportations and $C^0_{n+1}(\mathbb{C}) = C_{n+1}(\mathbb{C})/\mathbb{C}$. The degree $(n+1)$ symmetric group S_{n+1} acts freely on $C^0_{n+1}(\mathbb{C})$ by permutating $n+1$ points. Consider the covering map $C^0_{n+1}(\mathbb{C}) \to M_n$ defined by

$$(a_1, \ldots, a_{n+1}) \mapsto \prod_{i=1}^{n+1} (z - a_i),$$

then we have $C^0_{n+1}(\mathbb{C})/S_{n+1} \cong M_n$.

Take a point $(a_1, \ldots, a_{n+1}) \in C^0_{n+1}(\mathbb{C})$ and consider the collection of non-crossing paths $\gamma_1, \ldots, \gamma_n$ in $\mathbb{C}$ such that γ_i joins a_i and a_{i+1}. We call it the A_n-chain of paths. Let τ_i be a path in $C_{n+1}(\mathbb{C})$ which corresponds to the half-twist along γ_i in counterclockwise. Through the projection $C_{n+1}(\mathbb{C}) \to M_n$, τ_i gives a closed path in M_n.

It is well-known that there is an isomorphism of groups $\pi_1(M_n, *) \cong B_{n+1}$ given by

$$[\tau_i] \mapsto \sigma_i$$

where the base point $*$ is a polynomial $\prod_{i=1}^{n+1}(z - a_i)$.

6.3 Quadratic differentials associated with polynomials

Let $\mathbb{P}^1$ be a Riemann sphere, and we fix the coordinate $z \in \mathbb{C} \cup \{\infty\} \cong \mathbb{P}^1$. For a polynomial $p_n \in M_n$, define the meromorphic quadratic differential ϕ on $\mathbb{P}^1$ by

$$\phi(z) := p_n(z)^{N-2} dz^{\otimes 2}.$$

ϕ has zeros of order $(N-2)$ at distinct $n+1$ points on $\mathbb{C} \subset \mathbb{P}^1$, and a unique pole of order $(d_{N,n}+2)$ at $\infty \in \mathbb{P}^1$.

We denote by $Q(N,n)$ the space of such quadratic differentials associated with $p_n \in M_n$:

$$Q(N,n) := \{\, p_n(z)^{N-2} dz^{\otimes 2} \,|\, p_n \in M_n \,\}.$$

Clearly, independent of N, the space $Q(N,n)$ is isomorphic to M_n as a complex manifold, but later we give this space various structures depending on N.

We introduce the action of $\mathbb{C}^*$ on $Q(N,n)$. For $\phi(z) = p_n(z)^{N-2} dz^{\otimes 2}$, through the isomorphism $Q(N,n) \cong M_n$, the action of $\mathbb{C}^*$ is defined by

$$(k \cdot \phi)(z) := \{(k \cdot p_n)(z)\}^{N-2} dz^{\otimes 2} \quad (k \in \mathbb{C}^*).$$

By using the equation $(k \cdot p_n)(kz) = k^{n+1} p_n(z)$, we have the following formula.

Lemma 6.2 *For $\phi \in Q(N,n)$, the equation*

$$(k \cdot \phi)(kz) = k^{d_{N,n}} \phi(z) \quad (k \in \mathbb{C}^*)$$

holds.

For $\phi \in Q(N,n)$, denote by $\mathrm{Zero}(\phi) \subset \mathbb{P}^1$ the set of zeros of ϕ and by $\mathrm{Crit}(\phi) \subset \mathbb{P}^1$ the set of critical points of ϕ. Note that $\mathrm{Crit}(\phi) = \mathrm{Zero}(\phi) \cup \{\infty\}$.

Consider the action $z \mapsto kz$ for $k \in \mathbb{C}^*$ and $z \in \mathbb{P}^1$. This action maps the critical points of ϕ to that of $k \cdot \phi$:

$$k\,(\mathrm{Crit}(\phi)) = \mathrm{Crit}(k \cdot \phi).$$

6.4 Homology group

Definition 6.3 *Depending on the parity of N, we introduce the homology groups $H_+(\phi)$ and $H_-(\phi)$ by the following.*

- *If N is even, $H_+(\phi)$ is defined by the relative homology group*

  $$H_+(\phi) := H_1(\mathbb{C}, \mathrm{Zero}(\phi); \mathbb{Z}).$$

- *If N is odd, $H_-(\phi)$ is given as in the following. Let S be a hyperelliptic curve defined by the equation $y^2 = p_n(z)$ and $\pi \colon S \to \mathbb{P}^1$ be an associated branched covering map of degree 2. Then, the homology group $H_-(\phi)$ is defined by*

  $$H_-(\phi) := H_1(S \setminus \pi^{-1}(\infty); \mathbb{Z}).$$

Note that if $N = 3$, the homology group $H_-(\phi)$ is the hat-homology group in [BS] which is written by $\hat{H}(\phi)$.

Lemma 6.4 *Both $H_+(\phi)$ and $H_-(\phi)$ are free abelian groups of rank n.*

Proof. If N is even, the result is clear. If N is odd, there are two cases, n is even or odd. When n is even, the genus of the hyperelliptic curve S is $n/2$ and the fiber $\pi^{-1}(\infty)$ consists of one point, hence the result follows. When n is odd, the genus of S is $(n-1)/2$, but the fiber $\pi^{-1}(\infty)$ consists of two points, hence the result also follows in this case. $\square$

Let $\psi := \sqrt{\phi} = p_n(z)^{\frac{N-2}{2}}\,dz$ be a square root of ϕ. (If N is odd, ψ is determined up to the sign.) If N even ψ is a holomorphic 1-form on $\mathbb{P}^1 \setminus \{\infty\}$, and if N is odd ψ is a holomorphic 1-form on $S \setminus \pi^{-1}(\infty)$.

Therefore for any cycle $\alpha \in H_\pm(\phi)$, the integration of 1-form ψ via the cycle α

$$Z_\phi(\alpha) := \int_\alpha \psi$$

is well-defined and gives the complex number $Z_\phi(\alpha) \in \mathbb{C}$.

Thus, we have a group homorphism $Z_\phi \colon H_\pm(\phi) \to \mathbb{C}$, called the period of ϕ.

6.5 Framing

Following Section 2.6 in [BS], we introduce the framings on $Q(N,n)$. Let Γ be a free abelian group of rank n. A Γ-framing of $\phi \in Q(N,n)$ is an isomorphism of abelian groups

$$\theta \colon \Gamma \longrightarrow H_\pm(\phi).$$

Denote by $Q(N,n)^\Gamma$ the space of pairs (ϕ,θ) consists of a quadratic differential $\phi \in Q(N,n)$ and a Γ-framing θ. The projection

$$Q(N,n)^\Gamma \longrightarrow Q(N,n)\,, \quad (\phi,\theta) \mapsto \phi$$

defines a principal $\mathrm{GL}(n,\mathbb{Z})$-bundle on $Q(N,n)$.

For $(\phi,\theta) \in Q(N,n)^\Gamma$, the composition of a framing θ and a period Z_ϕ gives a group homomorphism $Z_\phi \circ \theta \colon \Gamma \to \mathbb{C}$. Thus, we have the map

$$\mathcal{W}_N \colon Q(N,n)^\Gamma \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{C})$$

defined by $\mathcal{W}_N(\phi,\theta) := Z_\phi \circ \theta \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{C})$. We call $\mathcal{W}_N$ a period map.

Proposition 6.5 *The period map*

$$\mathcal{W}_N \colon Q(N,n)^\Gamma \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{C})$$

is a local isomorphism of complex manifolds.

As a consequence, the period map $\mathcal{W}_N$ determines local coordinates on $Q(N,n)^\Gamma$.

6.6 Intersection form

In this section, we introduce the intersection forms on $H_{\pm}(\phi)$.

First, we consider the case N is even. The exact sequence of relative homology groups gives the isomorphism of homology groups

$$H_+(\phi) = H_1(\mathbb{C}, \mathrm{Zero}(\phi); \mathbb{Z}) \cong \widetilde{H}_0(\mathrm{Zero}(\phi); \mathbb{Z})$$

where $\widetilde{H}_0(\mathrm{Zero}(\phi); \mathbb{Z})$ is the reduced homology group of $H_0(\mathrm{Zero}(\phi); \mathbb{Z})$, and this isomorphism is induced by the boundary map. There is a trivial intersection form on $H_0(\mathrm{Zero}(\phi); \mathbb{Z})$, and it induces the intersection form on $\widetilde{H}_0(\mathrm{Zero}(\phi); \mathbb{Z})$. Through the above isomorphism of homology groups, we have the intersection form

$$I_+ \colon H_+(\phi) \times H_+(\phi) \longrightarrow \mathbb{Z}.$$

Note that I_+ is symmetric and non-degenerate.

Next, consider the case N is odd. Let $\pi \colon S \to \mathbb{P}^1$ be the hyperelliptic curve introduced in Section 6.4, and $\iota \colon S \setminus \pi^{-1}(\infty) \to S$ be the inclusion. This inclusion induces the sujective group homomorphism

$$\iota_* \colon H_-(\phi) = H_1(S \setminus \pi^{-1}(\infty); \mathbb{Z}) \longrightarrow H_1(S; \mathbb{Z}).$$

Since there is a skew-symmetric non-degenerate intersection form on $H_1(S; \mathbb{Z})$, through the inclusion ι_*, we have the intersection form

$$I_- \colon H_-(\phi) \times H_-(\phi) \longrightarrow \mathbb{Z}.$$

ι_* is an isomorphism if n is even, and has the rank one kernel if n is odd. Therefore, I_- is skew-symmetric and non-degenerate if n is even, and skew-symmetric and has the rank one kernel if n is odd.

For $\phi \in Q(N, n)$, picking the order of $\mathrm{Zero}(\phi)$ and consider the A_n-chain of path $\gamma_1, \ldots, \gamma_n$. If N is even, the path γ_i determines the homology class $\hat{\gamma}_i \in H_+(\phi)$. If N is odd, the lift of γ_i on the hyperelliptic curve S also determines the homology class $\hat{\gamma}_i \in H_-(\phi)$.

The pair of the homology group $H_{\pm}(\phi)$ and the intersection form $I_{\pm}$ gives the geometric realization of the lattice L and the bilinear form $<, >_{\pm}$ introduced in Section 6.1.

Lemma 6.6 *There is an isomorphism $H_{\pm}(\phi) \cong L$ given by $\hat{\gamma}_i \mapsto \alpha_i$.*

Further, by choosing suitable orientations for $\hat{\gamma}_1, \ldots, \hat{\gamma}_n$, this isomorphism takes the intersection form $I_{\pm}$ to the bilinear form $<, >_{\pm}$.

Proof. Easy computations gives the result. □

6.7 Local system

For $\phi \in Q(N, n)$, by corresponding to the homology group $H_{\pm}(\phi)$, we have a local system

$$\bigcup_{\phi \in Q(N,n)} H_{\pm}(\phi) \longrightarrow Q(N, n).$$

For a continuous path $c\colon [0,1] \to Q(N,n)$ with $\phi_0 = c(0)$, $\phi_1 = c(1)$, the parallel transportation along c gives the identification

$$\mathrm{GM}(c)\colon H_\pm(\phi_0) \xrightarrow{\sim} H_\pm(\phi_1),$$

and determines a flat connection on this local system. We call it the Gauss-Manin connection.

Fix a point $(\phi_0, \theta_0) \in Q(N,n)^\Gamma$ and denote by $Q(N,n)^\Gamma_*$ the connected component containing (ϕ_0, θ_0). Any point $(\phi, \theta) \in Q(N,n)^\Gamma$ is contained in $Q(N,n)^\Gamma_*$ if and only if there is a continuous path $c\colon [0,1] \to Q(N,n)$ connecting ϕ_0 and ϕ such that the diagram

$$\begin{array}{ccc} & \Gamma & \\ {\scriptstyle\theta_0}\swarrow & & \searrow{\scriptstyle\theta} \\ H_\pm(\phi_0) & \xrightarrow{\mathrm{GM}(c)} & H_\pm(\phi) \end{array}$$

commutes.

Note that different connected components of $Q(N,n)^\Gamma$ are all isomorphic since these are related by the action of $\mathrm{GL}(n,\mathbb{Z})$.

We introduce the $\mathbb{C}$-action on $Q(N,n)^\Gamma_*$, which is compatible with the $\mathbb{C}$-action on the space of stability conditions given in Section 2.6.

Definition 6.7 *For a framed differential $(\phi, \theta) \in Q(N,n)^\Gamma_*$, the new framed differential $(\phi', \theta') := t \cdot (\phi, \theta)$ is defined by*

$$\phi' := e^{-(2\pi i/d_{N,n})\,t} \cdot \phi, \quad \theta' := \mathrm{GM}(e^{-(2\pi i/d_{N,n})\,t\,s}) \circ \theta$$

where $\mathrm{GM}(e^{-(2\pi i/d_{N,n})\,t\,s})$ is the Gauss-Manin connection of the path $e^{-(2\pi i/d_{N,n})\,t\,s} \cdot \phi$ for $s \in [0,1]$ which connects ϕ and ϕ'.

6.8 Monodromy representation

By using the intersection forms $I_\pm$ on $H_\pm(\phi)$ defined in Section 6.6, we compute how does the cycle change along the closed path in $Q(N,n)$, which are known as the Picard-Lefschetz formula.

Fix a base point $* = \prod_{i=1}^{n+1}(z - a_i)$ and let $\gamma_1, \dots, \gamma_n$ be the A_n-chain of paths. Let $\phi_0 := \left[\prod_{i=1}^{n+1}(z - a_i)\right]^{N-2} dz^{\otimes 2}$ be the differential corresponding to the base point $*$. Consider the half-twists $\tau_1, \dots, \tau_n$ as in Section 6.2 and take the orientations of $\hat{\gamma}_1, \dots, \hat{\gamma}_n \in H_\pm(\phi_0)$ as in Lemma 6.6.

Proposition 6.8 *The monodromy representation of $[\tau_i] \in \pi_1(M_n, *)$ on $H_\pm(\phi_0)$ is given by*

$$\mathrm{GM}(\tau_i)(\beta) = \beta - I_\pm(\hat{\gamma}_i, \beta)\,\hat{\gamma}_i.$$

Lemma 6.6 and Proposition 6.8 imply that the monodromy representation of the braid group $\pi_1(M_n, *) \cong B_{n+1}$ on $H_\pm(\phi_0)$ gives the geometric realization of two type representations $\rho_\pm$ introduced in Section 6.1. Further recall from Section 6.1 that the subgroup $P_\pm \subset B_{n+1}$ is the kernel of $\rho_\pm$ and the group $W_\pm$ is the image of $\rho_\pm$ on $L \cong H_\pm(\phi_0)$.

Let $\widetilde{M_n}$ be a universal covering of M_n. Note that the braid group B_{n+1} acts on $\widetilde{M_n}$ as the group of deck transformations.

Corollary 6.9 *The connected component $Q(N,n)_*^\Gamma$ is isomorphic to the space $\widetilde{M_n}/P_\pm$ and the projection map $Q(N,n)_*^\Gamma \to Q(N,n)$ gives a principal $W_\pm$-bundle.*

Proof. It follows from the above remarks. □

7 Trajectories on $\mathbb{P}^1$

Here, in order to fix notations, we collect basic facts and definitions for trajectories on $\mathbb{P}^1$, mainly from Section 3 in [BS]. In spite of the treatment of higher order zeros, since our Riemann surface is only $\mathbb{P}^1$, many results are simplified.

7.1 Foliations

Let $\phi \in Q(N,n)$. For any neighborhood of a point in $\mathbb{P}^1 \setminus \mathrm{Crit}(\phi)$, the differential $\phi(z) = \varphi(z)dz^{\otimes 2}$ defines the distinguished local coordinate w by

$$w := \int \sqrt{\varphi(z)}\, dz,$$

up to the transformation $w \mapsto \pm w + c\, (c \in \mathbb{C})$. This coordinate also characterized by

$$\phi(w) = dw \otimes dw.$$

The coordinate w determines the foliation on $\mathbb{P}^1 \setminus \mathrm{Crit}(\phi)$ by

$$\mathrm{Im}\,(w/e^{i\pi\theta}) = \text{constant}.$$

A straight arc $\gamma \colon I \to \mathbb{P}^1 \setminus \mathrm{Crit}(\phi)$ of the phase θ is an integral curve along the foliation $\mathrm{Im}\,(w/e^{i\pi\theta}) = \text{constant}$, defined on an open interval $I \subset \mathbb{R}$. In other words, γ is a straight arc if $\mathrm{Im}\,(w(t)/e^{i\pi\theta}) = \text{constant}$ where

$$w(t) = \int^{\gamma(t)} \sqrt{\varphi(z)}\, dz.$$

Note that the set of maximal straight arcs of the phase θ are just leaves of the foliation $\mathrm{Im}\,(w/e^{i\pi\theta}) = \text{constant}$.

The foliation of the phase $\theta = 0$ is called a horizontal foliation, and a maximal straight arc of the phase $\theta = 0$ is called a trajectory.

Lemma 7.1 *Let $\phi \in Q(N,n)$. For $k = re^{i\pi\theta} \in \mathbb{C}^*$, set $\phi' := k \cdot \phi$. Then, the distinguished coordinates w and w' corresponding to ϕ and ϕ' have the relation*

$$w' = k^{\frac{d_{N,n}}{2}} w.$$

In addition, the transformation $z \mapsto kz$ on $\mathbb{P}^1$ maps the horizontal straight arcs for ϕ to the straight arcs of the phase $(d_{N,n}\theta)/2$ for ϕ'.

Proof. Let $\gamma \colon I \to \mathbb{P}^1 \setminus \mathrm{Crit}(\phi)$ be a horizontal straight arc of ϕ. Then, the following computation gives the result:

$$\begin{aligned}\int^{k\gamma(t)} \sqrt{\varphi'(z)}\, dz &= \int^{k\gamma(t)} \sqrt{(k \cdot \varphi)(z)}\, dz \\ &= \int^{\gamma(t)} \sqrt{(k \cdot \varphi)(kz')}\, d(kz') \quad (\text{put } z = kz'.) \\ &= k^{\frac{d_{N,n}}{2}} \int^{\gamma(t)} \varphi(z')\, dz' \quad (\text{by Lemma 6.2}).\end{aligned}$$

□

A trajectory γ defined on a finite open interval (a,b) is called saddle trajectory. Since $\mathbb{P}^1$ is compact, γ is extended to a continuous path $\gamma \colon [a,b] \to \mathbb{P}^1$ with $\gamma(a), \gamma(b) \in \mathrm{Zero}(\phi)$.

Similarly, a saddle connection of the phase θ is defined to be a maximal straight arc of the phase θ defined on a finite open interval.

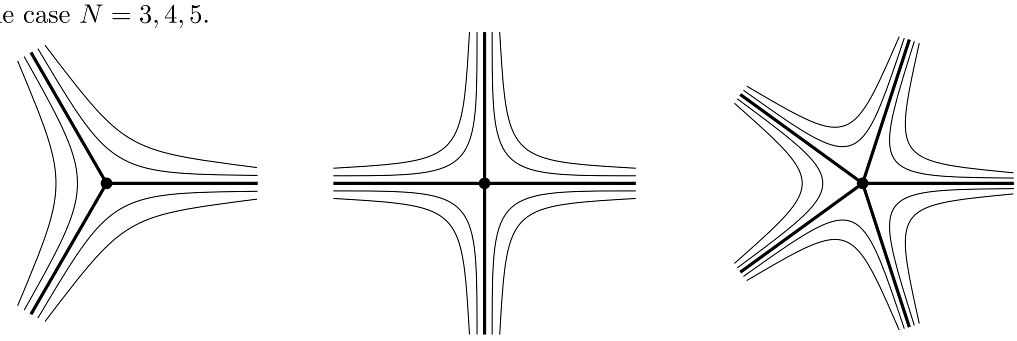
7.2 Foliation near a critical point

Here, we see the local behavior of the horizontal foliation near a critical point.

First, we consider the behavior near a zero. Let $\phi \in Q(N,n)$ and $p \in \mathrm{Zero}(\phi)$ be a zero of order $N-2$. Then, there is a local coordinate t of a neighborhood $p \in U \subset \mathbb{P}^1$ ($t = 0$ corresponds to p) such that

$$\phi = c^2\, t^{N-2} dt^{\otimes 2}, \quad c = \frac{1}{2} N.$$

Hence, on $U \setminus \{p\}$, the distinguished local coordinate w takes the form $w = t^{N/2}$. The following figures illustrate the local behaviors of the horizontal foliations in the case $N = 3, 4, 5$.



Note that there are N horizontal straight arcs departing from p. This fact is the reason why the differential ϕ determines the N-angulation.

Next, we consider the behavior near a pole. Since the order of a unique pole of $\phi \in Q(N,n)$ at $\infty \in \mathbb{P}^1$ is $(d_{N,n}+2)$, it is sufficient to treat a pole of order > 2. In this case, there is a neighborhood $\infty \in U \subset \mathbb{P}^1$ and a collection of $d_{N,n}$ distinguished tangent directions $v_i\,(i = 1,\dots,d_{N,n})$ at ∞, such that any trajectory entering U tends to ∞ and becomes asymptotic to one of the v_i. See Section 3.3 in [BS] for more detail.

7.3 Classification of trajectories

In this section, we summarize the global structure of the horizontal foliation on $\mathbb{P}^1$ by following Section 3.4 in [BS], and Section 9-11, 15 in [Str84].

Since $\mathbb{P}^1 \setminus \infty$ is contractible, the Theorem and Theorem in [Str84] implies that . Hence in our setting, every trajectory of ϕ is classified to exactly one of the following classes.

(1) A saddle trajectory approaches distinct zeros of ϕ at both ends.

(2) A separating trajectory approaches a zero of ϕ at one end and a pole $\infty \in \mathbb{P}^1$ at the other end.

(3) A generic trajectory approaches a pole $\infty \in \mathbb{P}^1$ at both ends.

As explained in the previous section, there are only finte horizontal straight arcs departing from zeros of ϕ. Therefore, the number of saddle trajectories and separating trajectories are finite. Actually, for ϕ, denote by s_ϕ the number of saddle trajectories, and by t_ϕ the number of separating trajectories. Then, the fact that there are N horizontal straight arcs departing from each zero of ϕ implies the equation

$$2s_\phi + t_\phi = N(n+1). \tag{7.1}$$

By removing these two type trajectories and $\mathrm{Crit}(\phi)$ from $\mathbb{P}^1$, the remaining open surface splits into a disjoint union of connected components. In our setting, each connected component coincides to one of the following two type surfaces.

(1) A half-plane is the upper half-plane

$$\{\, w \in \mathbb{C} \,|\, \mathrm{Im}\, w > 0 \,\}$$

equipped with the differential $dw^{\otimes 2}$. It is swept out by generic trajectories with the both end points at $\infty \in \mathbb{P}^1$. The boundary $\{\mathrm{Im}\, w = 0\}$ consists of saddle trajectories and two separating trajectories.

(2) A horizontal strip is the strip domain

$$\{\, w \in \mathbb{C} \,|\, a < \mathrm{Im}\, w < b \,\}$$

equipped with the differential $dw^{\otimes 2}$. it is also swept out by generic trajectories with the both end points at $\infty \in \mathbb{P}^1$. Each of two boundaries $\{\mathrm{Im}\, w = a\}$ and $\{\mathrm{Im}\, w = b\}$ consists of saddle trajectories and two separating trajectories.

Note that every saddle trajectory or separating trajectory appears just two times as a boundary element of these open surfaces.

Let k be the number of half planes and l be the number of horizontal strips appearing in such a decomposition of $\mathbb{P}^1$. Since every half plane has 2 separating trajectories on the boundary, and every horizontal strip has 4 separating trajectories on the two boundaries, we have the equation

$$2k + 4l = 2t_\phi. \tag{7.2}$$

We also note that near a pole of order $m > 2$, just $m - 2$ half planes appear. Since ϕ has a unique pole of order $(d_{N,n} + 2)$ at $\infty \in \mathbb{P}^1$, we have

$$k = d_{N,n} = (N - 2)(n + 1) + 2. \tag{7.3}$$

Lemma 7.2 *Let $\phi \in Q(N, n)$. For the number of saddle trajectories s_ϕ and the number of horizontal strips l, the equation*

$$s_\phi + l = n$$

holds.

Proof. This immediately follows by combining the equations (7.1), (7.2) and (7.3). □

Definition 7.3 *A differential $\phi \in Q(N, n)$ is called saddle-free if ϕ has no saddle trajectories ($s_\phi = 0$).*

For a saddle-free differential $\phi \in Q(N, n)$, Lemma 7.2 implies that exactly n horizontal strips appear in the decomposition of $\mathbb{P}^1$ by ϕ. In this situation, any boundary component of a half plane or a horizontal strip consists of only two separating trajectories.

7.4 Hat-homology classes and standard saddle connections

In this section, we collect some constructions of homology classes from Section 3.2 and Section 3.6 in [BS].

Let $\phi \in Q(N, n)$. Recall from Section 6.4 that the square root of ϕ defines a holomorphic 1-form $\psi = \sqrt{\phi}$ on $\mathbb{P}^1 \setminus \{\infty\}$ or $S \setminus \pi^{-1}(\infty)$ depending on the parity of N. Write by $\hat{S}$ the surface $\mathbb{P}^1 \setminus \{\infty\}$ if N is even, or $S \setminus \pi^{-1}(\infty)$ if N is odd.

We first note that the 1-form ψ defines the distinguished coordinate $\hat{w}$ on $\hat{S} \setminus \mathrm{Crit}(\psi)$ by $\psi = d\hat{w}$, up to the transformation $\hat{w} \mapsto \hat{w} + c \, (c \in \mathbb{C})$. As a result, ψ determines the horizontal foliation $\mathrm{Im}\, \hat{w} =$ constant on $\hat{S} \setminus \mathrm{Crit}(\psi)$. Since the sign of $d\hat{w}$ is uniquely determined, this foliation has the canonical orientation, which makes the tangent vector of a horizontal straight arc lies in $\mathbb{R}_{>0}$.

For a saddle trajectory $\gamma \colon [a, b] \to \mathbb{P}^1$, we can define a homology class $\hat{\gamma} \in H_\pm(\phi)$, called the hat-homology class of γ by the following.

If N is even, we orient γ to be compatible with the above orientation. Since $\gamma(a), \gamma(b) \in \mathrm{Zero}(\phi)$, the oriented curve γ defines a homology class $\hat{\gamma} := [\gamma] \in H_+(\phi) = H_1(\mathbb{C}, \mathrm{Zero}(\phi); \mathbb{Z})$. By definition, it satisfies $Z_\phi(\hat{\gamma}) \in \mathbb{R}_{>0}$.

If N is odd, since zeros of ϕ are branched points of the double cover $\pi \colon S \to \mathbb{P}^1$, the inverse image $\pi^{-1}(\gamma)$ becomes a closed curve in $S \setminus \pi^{-1}(\infty)$. We orient $\pi^{-1}(\gamma)$ to be compatible with the above orientation. Then, the oriented closed curve $\pi^{-1}(\gamma)$ defines a homology class homology class $\hat{\gamma} := [\pi^{-1}(\gamma)] \in H_-(\phi) = H_1(S \setminus \pi^{-1}(\infty); \mathbb{Z})$. As in the case N is even, it satisfies $Z_\phi(\hat{\gamma}) \in \mathbb{R}_{>0}$.

Similar construction works for a saddle connection $\gamma \colon [a, b] \to \mathbb{P}^1$ with the non-zero phase θ. In this case, the orientation is given to satisfy that $Z_\phi(\hat{\gamma}) \in \{\, re^{i\pi\theta} \mid r > 0 \,\} \subset \mathbb{H}$.

Let $h = \{a < \mathrm{Im}\, w < b\}$ be a horizontal strip given by the decomposition of $\mathbb{P}^1$ by a saddle-free differential ϕ. Then, each of two boundary components $\{\mathrm{Im}\, w = a\}$ and $\{\mathrm{Im}\, w = b\}$ contains exactly one zero of ϕ and consists of two separating trajectories intersecting at this zero. Hence, there is a unique saddle connection l_h connecting two zeros of two boundary components. We denote by $\gamma_h := \hat{l_h} \in H_\pm(\phi)$ the hat-homology class associated with the saddle connection l_h.

Since ϕ is saddle-free, Lemma 7.2 implies that there are precisely n horizontal strips in the decomposition of $\mathbb{P}^1$ by ϕ. Thus, we obtain n saddle connections $l_{h_1}, \dots, l_{h_n}$ and n hat-homology classes $\gamma_{h_1}, \dots, \gamma_{h_n}$. l_{h_i} is called a standard saddle connection, and γ_{h_i} is called a standard saddle class.

Lemma 7.4 ([BS], Lemma 3.2) *Standard saddle classes $\gamma_{h_1}, \dots, \gamma_{h_n}$ form a basis of $H_\pm(\phi)$.*

Proof. The argument of Lemma 3.2 in [BS] for $N = 3$ also works for general $N \geq 3$. □

7.5 Coloured quiver from saddle-free differential

Following Section 2 in [BT09], we introduce the notion of a coloured quiver. An $(N-2)$-coloured quiver $\mathcal{Q}$ consists of vertices $\{1, \dots, n\}$ and coloured arrows $i \xrightarrow{(c)} j$, where $c \in \{0, 1, \dots, N-2\}$. Write by $q_{ij}^{(c)}$ the number of arrows from i to j of colour (c).

In addition, we assume the following conditions:

(1) No loops, $q_{ii}^{(c)} = 0$ for all c.

(2) Monochromaticity, if $q_{ij}^{(c)} \neq 0$, then $q^{(c')} = 0$ for $c \neq c'$.

(3) Skew-symmetry, $q_{ij}^{(c)} = q_{ji}^{(N-2-c)}$.

Definition 7.5 *For a saddle-free differential $\phi \in Q(N, n)$, define a coloured quiver $\mathcal{Q}(\phi)$ by the following. Let $h_1, \dots, h_n$ be the horizontal strips in the decomposition of $\mathbb{P}^1$ by ϕ, and $l_{h_1}, \dots, l_{h_n}$ be the corresponding standard saddle* *connections. The vertices of $\mathcal{Q}(\phi)$ are standard saddle connections $\{l_{h_1}, \dots, l_{h_n}\}$, and the coloured arrow $l_{h_i} \xrightarrow{(c)} l_{h_j}$ for $i \neq j$ is given if l_{h_i} and l_{h_j} have the same zero p of ϕ as a boundary point and the number of separating trajectories appearing around p from l_{h_i} to l_{h_j} in counterclockwise is just $c+1$.*

We can easily check that such a coloured quiver $\mathcal{Q}(\phi)$ satisfies the additional three conditions.

For a coloured quiver $\mathcal{Q}$, the coloured quiver lattice $L_{\mathcal{Q}}$ is defined to be a free abelian group generated by vertices of $\mathcal{Q}$. Let $\alpha_1, \dots, \alpha_n$ be generators corresponding to vertices of $\mathcal{Q}$, then

$$L_{\mathcal{Q}} = \bigoplus_{i=1}^{n} \mathbb{Z}\,\alpha_i.$$

Define a bilinear form $<\,,\,>: L_{\mathcal{Q}} \times L_{\mathcal{Q}} \longrightarrow \mathbb{Z}$ by

$$< \alpha_i, \alpha_j >:= \delta_{ij} + (-1)^N \delta_{ij} - \sum_{c=0}^{N-2} (-1)^c q_{ij}^{(c)}.$$

Lemma 7.6 *The bilinear form $<\,,\,>: L_{\mathcal{Q}} \times L_{\mathcal{Q}} \to \mathbb{Z}$ is symmetric if N is even, and skew-symmetric if N is is odd. The diagonal part is computed that $< \alpha_i, \alpha_i >= 2$ if N is even, and $< \alpha_i, \alpha_i >= 0$ if N is odd for all i.*

Proof. The first part follows from the condition skew-symmetry. The second part follows from the condition no loops. □

Let us define a linear map

$$\mu: L_{\mathcal{Q}(\phi)} \xrightarrow{\sim} H_{\pm}(\phi)$$

by $\mu(\alpha_i) := \gamma_{h_i}$ where α_i is the basis corresponding to the vertex l_{h_i} of $\mathcal{Q}(\phi)$ and γ_{h_i} is the standard saddle class of l_{h_i}. By Lemma 7.4, the map μ is isomorphism.

Lemma 7.7 *The isomorphism $\mu: L_{\mathcal{Q}(\phi)} \xrightarrow{\sim} H_{\pm}(\phi)$ takes the bilinear form $<\,,\,>$ on $L_{\mathcal{Q}(\phi)}$ to the intersection form $I_{\pm}$ on $H_{\pm}(\phi)$.*

Proof. We can easily see that $I_+(\gamma_{h_i}, \gamma_{h_i}) = 2$ and $I_-(\gamma_{h_i}, \gamma_{h_i}) = 0$. Hence we need to prove that for $i \neq j$, if l_{h_i} and l_{h_j} have the same zero p of ϕ as a boundary point and the number of separating trajectories around p from l_{h_i} to l_{h_j} is $c+1$, then $I_{\pm}(\gamma_{h_i}, \gamma_{h_j}) = (-1)^{c+1}$.

First, consider the case N is even. Since the oriented foliation of $\psi = \sqrt{\phi}$ around p has the opposite orientations by crossing a separating trajectory, l_{h_i} and l_{h_j} have the same orientation relative to p if the number of separating trajectories around p from l_{h_i} to l_{h_j} is even, and have the different orientation relative to p if that is odd. Therefore the result follows in this case.

Next, consider the case N is odd. The oriented foliation of $\psi = \sqrt{\phi}$ on the hyperelliptic curve S around p also has the opposite orientations by crossing a separating trajectory. Therefore the lift of l_{h_i} and l_{h_j} on S is intersecting $+1$ if the number of separating trajectories around p from l_{h_i} to l_{h_j} is even, and -1 if that is odd. □

8 CY_N algebras of A_n-quivers

8.1 Ginzburg dga

Let $Q = (Q_0, Q_1)$ be a finite quiver with vertices Q_0 and arrows Q_1. The Ginzburg dga $\Gamma_N Q := (k\overline{Q}, d)$ is defined as follows. Define a graded quiver $\overline{Q}$ with vertices Q_0 and arrows:

- the original arrows Q_1 (degree 0);
- an opposite arrow $a^* \colon j \to i$ for an original arrow $a \colon i \to j \in Q_1$ (degree $N-2$);
- a loop t_i for each vertex $i \in Q_0$ (degree $N-1$).

Let $k\overline{Q}$ be a graded path algebra of $\overline{Q}$, and define a differential $d \colon k\overline{Q} \to k\overline{Q}$ of degree -1 by

- $da = da^* = 0$ for $a \in Q_1$
- $dt_i = e_i \left(\sum_{a \in Q_1} (aa^* - a^*a) \right) e_i$

where e_i is the idempotent at $i \in Q_0$.

Note that the 0-th homology is $\mathbf{H}_0(\Gamma_N Q) \cong kQ$.

Let $\mathcal{D}(\Gamma_N Q)$ be a derived category of right dg-modules over $\Gamma_N Q$, and $\mathcal{D}_{\mathrm{fd}}(\Gamma_N Q)$ be a full subcategory of $\mathcal{D}(\Gamma_N Q)$ consists of dg modules M whose homology is of finite total dimension:

$$\sum_{i \in \mathbb{Z}} \dim_k \mathbf{H}_i(M) < \infty.$$

Theorem 8.1 ([Kel11], Theorem 6.3) *The category $\mathcal{D}_{\mathrm{fd}}(\Gamma_N Q)$ is a Calabi-Yau N triangulated category.*

Proposition 8.2 ([Ami09], Lemma 2.2 and Proposition 2.3) *There is a canonical bounded t-structure $\mathcal{F} \subset \mathcal{D}_{\mathrm{fd}}(\Gamma_N Q)$ with the heart $\mathcal{H}$ such that the functor $\boldsymbol{H}_0 \colon \mathcal{D}_{\mathrm{fd}}(\Gamma_N Q) \to \text{mod-}\,\boldsymbol{H}_0(\Gamma_N Q)$ induces an equivalence of abelian categories $\mathcal{H}$ and $\text{mod-}\,\boldsymbol{H}_0(\Gamma_N Q)$.*

We write by $\mathcal{H}_\Gamma$ the heart determined by the above proposition, and call it the standard heart.

For a quiver $\overrightarrow{A_n}$, we set $\mathcal{D}_n^N := \mathcal{D}_{\mathrm{fd}}(\Gamma_N \overrightarrow{A_n})$.

8.2 Koszul duality

Here, we comment to the relationship between our category $\mathcal{D}_n^N$ and the category treated in [ST01, Tho06] which is defined as a perfect category of a graded path algebra of some graded quiver (see Section 4c in [ST01] or Definition 3.2 in [Tho06]). Throughout this section, we fix the straight forward orientation for a quiver $\overrightarrow{A_n}$, in which the arrows of $\overrightarrow{A_n}$ is given by $i \to i+1$ for $i = 1, \ldots, n-1$.

Let R_n^N be a graded quiver with vertices $\{1, \ldots, n\}$, degree 1 arrows $\alpha_i : i \to i+1\,(i = 1, \ldots, n-1)$ and degree $N-1$ arrows $\beta_i : i+1 \to i\,(i = 1, \ldots, n-1)$.

Write by kR_n^N a graded path algebra of R_n^N over k. The graded algebra A_n^N is defined to be the quotient of kR_n^N by the two-sided ideal generated by the elements:

- $\alpha_i \alpha_{i+1}\,(i = 1, \ldots, n-2)$
- $\beta_{i+1}\beta_i\,(i = 1, \ldots, n-2)$
- $\beta_i \alpha_i - \alpha_{i+1}\beta_{i+1}\,(i = 1, \ldots, n-2)$.

We give a trivial differential $d = 0 \colon A_n^N \to A_n^N$, and treat A_n^N as a dga. Let $P_i := e_i A_n^N$ be a projective right module corresponding to the vertex i where e_i is the idempotent at v_i.

Let $\mathcal{D}(A_n^N)$ be a derived category of right dg-modules over A_n^N, and $\operatorname{per}(A_n^N) \subset \mathcal{D}(A_n^N)$ be a smallest full triangulated subcategory containing $P_1, \ldots, P_n$. Then, by the Koszul duality, we have the following category equivalence.

Proposition 8.3 *The exact functor* $\operatorname{Hom}^\bullet(S, ?) \colon \mathcal{D}(\Gamma_N \overrightarrow{A_n}) \to \mathcal{D}(A_n^N)$ *induces the category equivalence of triangulated subcategories*

$$\operatorname{Hom}^\bullet(S, ?) \colon \mathcal{D}_n^N \xrightarrow{\sim} \operatorname{per}(A_n^N).$$

In particular, $\operatorname{Hom}^\bullet(S, S_i) = P_i$.

Proof. Set $S := \oplus_{i=1}^n S_i$. Then, the endomorphism dg-algebra $\operatorname{End}^\bullet S$ is isomorphic to A_n^N:

$$\operatorname{End}^\bullet S \cong A_n^N.$$

Therefore by the Koszul duality, the above functor is a category equivalence (see Section 9 in [KN13]). □

8.3 Artin braid groups and spherical twists

For a quiver $\overrightarrow{A_n}$ with vertices $\{1, \ldots, n\}$, we assume the vertex i is adjacent to vertices $i-1$ and $i+1$. Let S_i be the simple module corresponding to the vertex i. Such simple modules $S_1, \ldots, S_n$ of $\mathcal{H}_\Gamma$ are spherical in $\mathcal{D}_n^N$, hence they define spherical twists $\Phi_{S_1}, \ldots, \Phi_{S_n} \in \operatorname{Aut}(\mathcal{D}_n^N)$. The Seidel-Thomas braid group $\operatorname{Sph}(\mathcal{D}_n^N)$ is defined to be the subgroup of $\operatorname{Aut}(\mathcal{D}_n^N)$ generated by these spherical twists:

$$\operatorname{Sph}(\mathcal{D}_n^N) := \langle \Phi_1, \ldots, \Phi_n \rangle .$$

Theorem 8.4 ([ST01], Theorem 1.2 and Theorem 1.3) *For groups* B_{n+1} *and* $\operatorname{Sph}(\mathcal{D}_n^N)$*, the correspondence of generators* $\sigma_i \mapsto \Phi_{S_i}$ *is extended to the isomorphism of groups*

$$B_{n+1} \cong \operatorname{Sph}(\mathcal{D}_n^N).$$

Consider the K-group

$$K(\mathcal{D}_n^N) \cong \bigoplus_{i=1}^{n} \mathbb{Z}[S_i]$$

and the Euler form $\chi\colon K(\mathcal{D}_n^N) \times K(\mathcal{D}_n^N) \to \mathbb{Z}$. Recall from Section 6.1 that there is a free abelian group L and a bilinear form $<,>_\pm$ associated with a quiver $\overrightarrow{A_n}$.

Lemma 8.5 *Define the map $K(\mathcal{D}_n^N) \to L$ by $[S_i] \mapsto \alpha_i$. Then, this map is an isomorphism of abelian groups and takes the Euler form χ to the bilinear form $<,>_\pm$.*

At the K-group level, a spherical twist Φ_{S_i} induces the reflection

$$[\Phi_{S_i}]([E]) = [E] - \chi(S_i, E)[S_i]$$

on $K(\mathcal{D}_n^N)$. Therefore under the above isomorphism, spherical twists $\Phi_{S_1}, \dots, \Phi_{S_n}$ act as reflections $r_1^\pm, \dots, r_n^\pm$ in Section 6.1 on the K-group $K(\mathcal{D}_n^N)$:

$$[\Phi_{S_1}] = r_1^\pm, \dots, [\Phi_{S_n}] = r_n^\pm.$$

9 N-angulations of polygons and exchange graphs

9.1 N-angulations of polygons

A geometric construction of $(N-2)$-cluster category of type A_n is given in [BM08]. By using this construction, here we introduce the oriented graph associated with decompositions of a polygon into N-gons. This graph is the geometric realization of the cluster exchange graph of an $(N-2)$-cluster category for type A_n.

Let $\Pi_{d_{N,n}}$ be a $d_{N,n}$-gon. An $(N-2)$-diagonal of $\Pi_{d_{N,n}}$ is a diagonal which divides $\Pi_{d_{N,n}}$ into an $((N-2)i+2)$-gon and an $((N-2)(n+1-i)+2)$-gon $(i = 1, \dots, n)$. A collection of $(N-2)$-diagonals is called non-crossing if they intersect only the boundary of $\Pi_{d_{N,n}}$.

The maximal set of non-crossing $(N-2)$-diagonals of $\Pi_{d_{N,n}}$ is called the N-angulation. The number of $(N-2)$-diagonals in the N-angulation is n, and this divides $\Pi_{d_{N,n}}$ into $n+1$ pieces of N-gons.

For any $(N-2)$-diagonal δ, there are just two N-gons whose common edge is δ. By removing δ, we have a $(2N-2)$-gon. We call δ a diameter of this $(2N-2)$-gon.

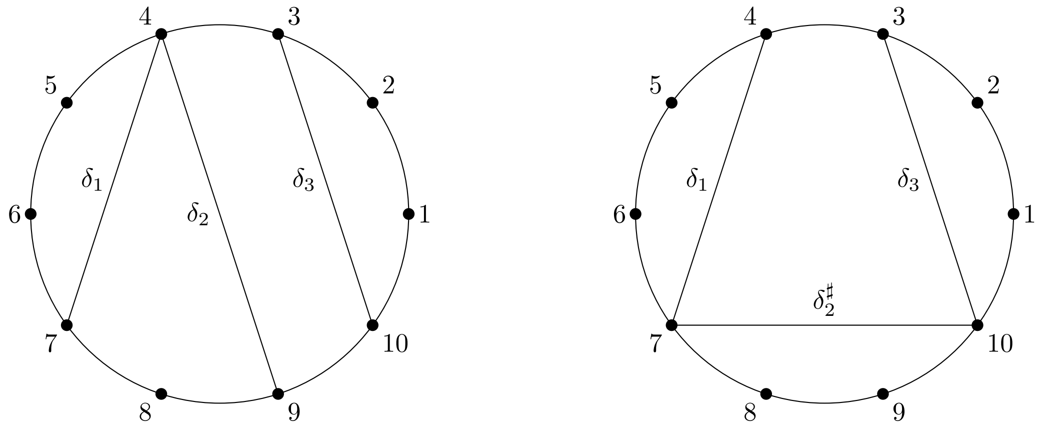
Let Δ be an N-angulation and $\delta \in \Delta$ be an $(N-2)$-diagonal. For Δ, we define the operation to make new N-angulations $\mu_\delta^\sharp(\Delta)$ and $\mu_\delta^\flat(\Delta)$ by the following.

For a diagonal $\delta \in \Delta$, there is a unique $(2N-2)$-gon which has δ as a diameter. By rotating two boundary points of δ one step counterclockwise

(clockwise), we have the new $(N-2)$-diagonal $\delta^\sharp$ ($\delta^\flat$). Thus, the new N-angulations is defined by

$$\mu_\delta^\sharp(\Delta) := (\Delta \backslash \{\delta\}) \cup \{\delta^\sharp\} \quad (\mu_\delta^\flat(\Delta) := (\Delta \backslash \{\delta\}) \cup \{\delta^\flat\}).$$

The following figure illustrates the example for $N = 4$ and $n = 3$.



Definition 9.1 *Define the graph* $\mathrm{A}(\Pi_{d_{N,n}}, N)$ *to be the oriented graph whose vertices are all* N*-angulations of* $\Pi_{d_{N,n}}$*, and for two* N*-angulations* Δ *and* Δ'*, the arrow* $\mathcal{H} \xrightarrow{\delta} \mathcal{H}'$ *is given if there is some diagonal* $\delta \in \Delta$ *such that* $\mu_\delta^\sharp(\Delta) = \Delta'$*.*

Lemma 9.2 ([BT09], Proposition 7.1) *The graph* $\mathrm{A}(\Pi_{d_{N,n}}, N)$ *is connected.*

9.2 Exchange graphs and braid groups

Definition 9.3 *For* $\mathcal{D}_n^N$ *, we define the exchange graph* $\mathrm{EG}(\mathcal{D}_n^N)$ *to be the oriented graph whose vertices are all finite hearts of* $\mathcal{D}_n^N$*, and for two hearts* $\mathcal{H}$ *and* $\mathcal{H}'$*, the arrow* $\mathcal{H} \xrightarrow{S} \mathcal{H}'$ *is given if these is some rigid simple object* $S \in \mathcal{H}$ *such that they relate by the simple forward tilt* $\mathcal{H}_S^\sharp = \mathcal{H}'$*.*

In particular, we denote by $\mathrm{EG}^\circ(\mathcal{D}_n^N) \subset \mathrm{EG}(\mathcal{D}_n^N)$ *the connected component containing the standard heart* $\mathcal{H}_\Gamma$*.*

Let $\mathcal{H} \in \mathrm{EG}(\mathcal{D}_n^N)$ be a finite heart and $S \in \mathcal{H}$ be a simple object. Since

$$\Phi(\mathcal{H}_S^\sharp) = (\Phi(\mathcal{H}))_{\Phi(S)}^\sharp,$$

holds for any autoequivalence $\Phi \in \mathrm{Aut}(\mathcal{D}_n^N)$, autoequivalences act on $\mathrm{EG}(\mathcal{D}_n^N)$ as automorphisms of the oriented graph. Denote by $\mathrm{Aut}^\circ(\mathcal{D}_n^N)$ the subgroup consisting of autoequivalences which preserve the connected component $\mathrm{EG}^\circ(\mathcal{D}_n^N)$.

For $\mathcal{H} \in \mathrm{EG}^\circ(\mathcal{D}_n^N, \mathcal{H}_0)$ and rigid simple object $S \in \mathcal{H}$, define inductively

$$\mathcal{H}_S^{m\sharp} := (\mathcal{H}_S^{(m-1)\sharp})_{S[m-1]}, \quad \mathcal{H}_S^{m\flat} := (\mathcal{H}_S^{(m-1)\flat})_{S[-m+1]}$$

for $m \geq 1$, and we set $\mathcal{H}^{m\sharp} := \mathcal{H}^{-m\flat}$ for $m < 0$.

Proposition 9.4 ([KQ], Corollary 8.3) *Let $\mathcal{H} \in \mathrm{EG}^\circ(\mathcal{D}_n^N)$ and consider simple objects $\operatorname{Sim}\mathcal{H} = \{T_1, \dots, T_n\}$. Then, spherical twists $\Phi_{T_1}, \dots, \Phi_{T_n}$ generate the Seidel-Thomas braid group $\mathrm{Sph}(\mathcal{D}_n^N)$ and give the equation*

$$\mathcal{H}_{T_i}^{(N-1)\sharp} = \Phi_{T_i}^{-1}(\mathcal{H}), \quad \mathcal{H}_{T_i}^{(N-1)\flat} = \Phi_{T_i}(\mathcal{H}).$$

This result implies that the action of $\mathrm{Sph}(\mathcal{D}_n^N)$ is free and preserves the connected component $\mathrm{EG}^\circ(\mathcal{D}_n^N)$.

For $N \geq 3$, we define the full subgraph of $\mathrm{EG}^\circ(\mathcal{D}_n^N)$ by

$$\mathrm{EG}_N^\circ(\mathcal{D}_n^N) := \{\, \mathcal{H} \in \mathrm{EG}^\circ(\mathcal{D}_n^N) \mid \mathcal{H}_\Gamma \leq \mathcal{H} \leq \mathcal{H}_\Gamma[N-2] \,\}.$$

In the following meaning, $\mathrm{EG}_N^\circ(\mathcal{D}_n^N)$ is the fundamental domain of the action of $\mathrm{Sph}(\mathcal{D}_n^N)$ on $\mathrm{EG}^\circ(\mathcal{D}_n^N)$.

Theorem 9.5 ([KQ], Theorem 8.5) *The map of graphs*

$$p_0 \colon \mathrm{EG}_N^\circ(\mathcal{D}_n^N) \longrightarrow \mathrm{EG}^\circ(\mathcal{D}_n^N)/\,\mathrm{Sph}(\mathcal{D}_n^N)$$

is embedding and bijection between vertices.

9.3 Correspondence between hearts and N-angulations

Combining the result for the geometric realization of $(N-2)$-cluster category of type A_n in [BM08] and the correspondence between cluster exchange graphs and heart exchange graphs in [KQ], we have the following result.

Theorem 9.6 ([BM08], Theorem 5.6 and [KQ], Theorem 8.5) *There is a canonical isomorphism of oriented graphs with labeled arrows*

$$\mathrm{A}(\Pi_{d_{N,n}}, N) \xrightarrow{\sim} \mathrm{EG}^\circ(\mathcal{D}_n^N)/\,\mathrm{Sph}(\mathcal{D}_n^N).$$

For the heart $\mathcal{H}(\Delta) \in \mathrm{EG}_N^\circ(\mathcal{D}_n^N)$ which corresponds to the N-angulation $\Delta \in \mathrm{A}(\Pi_{d_{N,n}}, N)$ ($\mathcal{H}(\Delta)$ is determined up to modulo $\mathrm{Sph}(\mathcal{D}_n^N)$), there is a bijection between $(N-2)$-diagonals of Δ and simple objects of $\mathcal{H}(\Delta)$ such that

$$\mathcal{H}(\mu_\delta^\sharp(\Delta)) \equiv \mathcal{H}(\Delta)_{S_\delta}^\sharp \qquad \mathrm{mod}\ \mathrm{Sph}(\mathcal{D}_n^N)$$

where δ is a $(N-2)$-diagonal in Δ and S_δ is the corresponding simple object in $\mathcal{H}(\Delta)$.

Proof. By Theorem 5.6 in [BM08], the graph $\mathrm{A}(\Pi_{d_{N,n}}, N)$ is isomorphic to the cluster exchange graph of $(N-2)$-cluster category for of type A_n, and $(N-2)$-diagonals of the N-angulations correspond to the indecomposable direct summands of $(N-2)$-cluster tilting objects which consists of the vertex set of a cluster exchange graph. (For the precise definition of a cluster exchange graph, see Definition 4.4 in [KQ]).

On the other hand, by Theorem 8.5 in [KQ], there is a canonical isomorphism between the cluster exchange graph of type A_n and $\mathrm{EG}^\circ(\mathcal{D}_n^N)/\,\mathrm{Sph}(\mathcal{D}_n^N)$ such that indecomposable direct summands of cluster tilting objects are mapped to the simple objects of the corresponding hearts. □

Remark 9.7 *By Theorem 9.5, for an N-angulation Δ, we can uniquely take the corresponding heart $\mathcal{H}(\Delta)$ in $\mathrm{EG}_N^{\circ}(\mathcal{D}_n^N)$. In the following, we always assume that $\mathcal{H}(\Delta)$ is taken in the fundamental domain $\mathrm{EG}_N^{\circ}(\mathcal{D}_n^N)$.*

Lemma 9.8 *Let $\Delta \in \mathrm{A}(\Pi_{d_{N,n}}, N)$ be an N-angulation and $\mathcal{H}(\Delta) \in \mathrm{EG}_N^{\circ}(\mathcal{D}_n^N)$ be the corresponding heart. Take a $(N-2)$-diagonal $\delta \in \Delta$ and consider a new N-angulation $\mu_\delta^\sharp(\Delta)$. Then, there is a unique spherical twist $\Phi \in \mathrm{Sph}(\mathcal{D}_n^N)$ such that*

$$\Phi(\mathcal{H}(\mu_\delta^\sharp(\Delta))) = \mathcal{H}(\Delta)_{S_\delta}^\sharp.$$

Proof. Since the action of $\mathrm{Sph}(\mathcal{D}_n^N)$ on $\mathrm{EG}^{\circ}(\mathcal{D}_n^N)$ is free, this follows from Theorem 9.6 and Remark 9.7. □

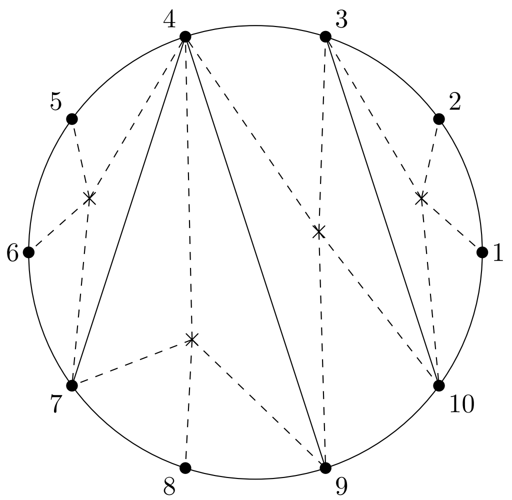
9.4 N-angulation from saddle-free differential

Let $\phi \in Q(N,n)$. Recall from Section 7.2 that ϕ defines $d_{N,n}$ distinguished tangent directions at $\infty \in \mathbb{P}^1$ since ϕ has a pole of order $(d_{N,n}+2)$ at that point. Consider the real oriented blow-up at $\infty \in \mathbb{P}^1$ (that is replacement of ∞ by S^1), then we have a disk Π. By adding $d_{N,n}$ points on the boundary of Π, which corresponds to $d_{N,n}$ distinguished tangent directions, we have a d-gon $\Pi_{d_{N,n}}$.

Lemma 9.9 *Let $\phi \in Q(N,n)$ be a saddle-free differential. Then, by taking one generic trajectory from each horizontal strip in the decomposition of $\mathbb{P}^1$ by ϕ, we have an N-angulation Δ_ϕ of $\Pi_{d_{N,n}}$.*

Proof. By Lemma 7.2, there are just n horizontal strips in the decomposition, and clearly generic trajectories taking from different horizontal strips are non-crossing each other. Hence, it is sufficient to prove that these generic trajectories are $(N-2)$-diagonals. Assume that a generic trajectory δ divides $\Pi_{d_{N,n}}$ into two disks Π_1 and Π_2, and Π_1 contains i zeros and Π_2 contains $(n+1-i)$ zeros. Then, Π_1 has $i-1$ horizontal strips and Ni separating trajectories. By the equation (7.2), the number of half planes in Π_1 is $((N-2)i+2)$, and this implies that Π_1 is $((N-2)i+2)$-gon. □

The following figure illustrates the example of the 4-angulation of 10-gon by a saddle-free differential. Generic trajectories are written by solid lines and separating trajectories are written by dotted lines.



We write by Δ_ϕ the N-angulation which is determined from a saddle-free differential $\phi \in Q(N, n)$ by using this lemma. We also write by $h(\delta)$ the horizontal strip which contains a generic trajectory δ.

9.5 Coloured quiver lattice and K-group

Definition 9.10 *For an N-angulation $\Delta \in \mathrm{A}(\Pi_{d_{N,n}}, N)$, define the $(N-2)$-coloured quiver $\mathcal{Q}(\Delta)$ whose vertices are all $(N-2)$-diagonals of Δ and the coloured arrow $\delta_i \xrightarrow{(c)} \delta_j$ is given if both δ_i and δ_j lie on the same N-gon of N-angulation and there is just c edges forming the segment of the boundary of the N-gon in counterclockwise.*

Definition 9.11 *For the heart $\mathcal{H} \in \mathrm{EG}^\circ(\mathcal{D}_n^N)$, define the coloured quiver $\mathcal{Q}(\mathcal{H})$ whose vertices are simple objects $S_1, \dots, S_n \in \mathcal{H}$ and the number of coloured arrows from S_i to S_j is $\dim_k \mathrm{Hom}^{c+1}_{\mathcal{D}_n^N}(S_i, S_j)$.*

Proposition 9.12 *Let $\Delta \in \mathrm{A}(\Pi_{d_{N,n}}, N)$ be an N-angulation and $\mathcal{H}(\Delta) \in \mathrm{EG}^\circ(\mathcal{D}_n^N)$ be the corresponding heart. Then, there is a canonical isomorphism of coloured quivers*

$$\mathcal{Q}(\Delta) = \mathcal{Q}(\mathcal{H}(\Delta)).$$

In particular, the correspondence of vertices is given by the correspondence between $(N-2)$-diagonals and simple objects in Theorem 9.6.

Proof. By Proposition 11.1 in [BT09], the coloured quiver $\mathcal{Q}(\Delta)$ is equal to the coloured quiver of the corresponding cluster tilting object. Theorem 8.6 in [KQ] implies that by the correspondence of cluster tilting objects and hearts in the proof of Theorem 9.6, the coloured quiver of the cluster tilting object is mapped to the coloured quiver of the corresponding heart. □

Recall Section 7.5 that for a coloured quiver $\mathcal{Q}$, we define a $\mathbb{Z}$-lattice $L_{\mathcal{Q}}$ and a bilinear form $<,>$ on $L_{\mathcal{Q}}$. For a coloured quiver $\mathcal{Q}(\Delta)$ with vertices $\{\delta_1, \ldots, \delta_n\}$, write by α_i the basis of $L_{\mathcal{Q}(\Delta)}$ corresponding to the vertex δ_i.

By using Proposition 9.12, we have the similar result of Lemma 9.10 in [BS].

Lemma 9.13 *Let Δ be an N-angulation and $\mathcal{H}(\Delta)$ be the corresponding heart. Then, there is an isomorphism of $\mathbb{Z}$-lattices*

$$\lambda \colon L_{\mathcal{Q}(\Delta)} \longrightarrow K(\mathcal{D}_n^N)$$

such that for each $(N-2)$-diagonal δ_i, the basis α_i is mapped to the class of the corresponding simple object S_{δ_i}. This isomorphism takes the bilinear form $<,>$ on $L_{\mathcal{Q}(\Delta)}$ to the Euler form χ on $K(\mathcal{D}_n^N)$.

Proof. First note that as stated in Remark 2.3, there is a canonical isomorphism of K-groups $K(\mathcal{H}(\Delta)) \cong K(\mathcal{D}_n^N)$. Since $K(\mathcal{H}(\Delta))$ is generated by the class of simple objects $[S_{\delta_1}], \ldots, [S_{\delta_n}]$, the map λ is an isomorphism.

Let $\mathcal{Q}(\Delta) = (q_{ij}^{(c)})$ be a coloured quiver of Δ. By Proposition 9.12, we have $q_{ij}^{(c)} = \dim_k \mathrm{Hom}^{c+1}(S_{\delta_i}, S_{\delta_j})$. The Euler form with respect to the basis $[S_{\delta_1}], \ldots, [S_{\delta_n}]$ is computed by

$$\begin{aligned}
\chi(S_{\delta_i}, S_{\delta_j}) &= \sum_{l \in \mathbb{Z}} (-1)^l \dim_k \mathrm{Hom}^l(S_{\delta_i}, S_{\delta_j}) \\
&= \dim_k \mathrm{Hom}^0(S_{\delta_i}, S_{\delta_j}) + (-1)^N \dim_k \mathrm{Hom}^N(S_{\delta_i}, S_{\delta_j}) - \sum_{c=0}^{N-2} (-1)^c \dim_k \mathrm{Hom}^{c+1}(S_{\delta_i}, S_{\delta_j}) \\
&= \delta_{ij} + (-1)^N \delta_{ij} - \sum_{c=0}^{N-2} (-1)^c q_{ij}^{(c)} \\
&= < \alpha_i, \alpha_j > .
\end{aligned}$$

Hence, the result follows. □

Note that for a saddle-free differential ϕ, there is a natural identification between the coloured quiver $\mathcal{Q}(\phi)$ defined in Section 7.5 and the coloured quiver $\mathcal{Q}(\Delta_\phi)$ associated with the N-angulation Δ_ϕ given by Definition 9.10. The correspondence of vertices is given by $l_{h(\delta)} \mapsto \delta$ where δ is a generic trajectory of ϕ, h is a horizontal strip containing δ and $l_{h(\delta)}$ is the standard saddle connection of $h(\delta)$. In the following, we identify $\mathcal{Q}(\phi)$ and $\mathcal{Q}(\Delta_\phi)$.

Corollary 9.14 *Let $\phi \in Q(N, n)$ be a saddle-free differential and Δ_ϕ be an N-angulation associated with ϕ. Then, there is an isomorphism of $\mathbb{Z}$-lattices*

$$\nu \colon K(\mathcal{D}_n^N) \xrightarrow{\sim} H_\pm(\phi)$$

such that for each $(N-2)$-diagonal δ, λ takes the class of simple object $[S_\delta]$ to the standard saddle class $\gamma_{h(\delta)}$ and the Euler form χ on $K(\mathcal{D}_n^N)$ to the intersection form $I_\pm$ on $H_\pm(\phi)$.

Proof. This immediately follows from Lemma 7.7 and Lemma 9.13. □

9.6 Coloured quiver mutation and K-group

Let Δ be an N-angulation and $\mu^{\sharp}_{\delta}(\Delta)$ be a new N-angulation given by the operation in Section 9.1 where δ is an $(N-2)$-diagonal of Δ. We call a coloured quiver $\mathcal{Q}(\mu^{\sharp}_{\delta}(\Delta))$ the coloured quiver mutation of $\mathcal{Q}(\Delta)$ at the vertex δ.

Let $\delta_1, \dots, \delta_n$ be $(N-2)$-diagonals of Δ and $\delta'_1, \dots, \delta'_n$ be $(N-2)$-diagonals of $\mu^{\sharp}_{\delta_i}(\Delta)$ ($\delta'_j = \delta_j$ if $j \neq i$ and $\delta'_i = \delta^{\sharp}_i$). Consider coloured quiver lattices

$$L_{\mathcal{Q}(\Delta)} = \bigoplus_{j=1}^{n} \mathbb{Z}\alpha_j, \quad L_{\mathcal{Q}(\mu^{\sharp}_{\delta_i}(\Delta))} = \bigoplus_{j=1}^{n} \mathbb{Z}\alpha'_j$$

where α_i (respectively α'_i) is the basis corresponding to δ_i (respectively δ'_i).

Define a linear map $F_i \colon L_{\mathcal{Q}(\Delta_i)} \to L_{\mathcal{Q}(\mu^{\sharp}_{\delta_i}(\Delta))}$ by

$$F_i(\alpha_j) := \begin{cases} -\alpha'_i & \text{if} \quad j = i \\ \alpha'_j + \tilde{q}^{(0)}_{ij}\alpha'_i & \text{if} \quad j \neq i. \end{cases}$$

where $\tilde{q}^{(0)}_{ij}$ is the number of arrows of colour (0) from δ'_i to δ'_j in the coloured quiver mutation $\mathcal{Q}(\mu^{\sharp}_{\delta}(\Delta))$. Then, the following result holds.

Lemma 9.15 *Let $\lambda \colon L_{\mathcal{Q}(\Delta)} \to K(\mathcal{D}^N_n)$ and $\lambda^{\sharp} \colon L_{(\mathcal{Q}(\mu^{\sharp}_{\delta_i}(\Delta))} \to K(\mathcal{D}^N_n)$ be the isomorphisms given by Lemma 9.13. Then, there is a unique spherical twist $\Phi \in \operatorname{Sph}(\mathcal{D}^N_n)$ such that the following* (1) *and* (2) *hold.*

(1) The diagram

$$\begin{array}{ccc} L_{\mathcal{Q}(\Delta)} & \xrightarrow{\;F_i\;} & L_{\mathcal{Q}(\mu^{\sharp}_{\delta_i}(\Delta))} \\ \Big\downarrow{\scriptstyle\lambda} & & \Big\downarrow{\scriptstyle\lambda^{\sharp}} \\ K(\mathcal{D}^N_n) & \xrightarrow{\;[\Phi]\;} & K(\mathcal{D}^N_n) \end{array}$$

commutes, where $[\Phi]$ is a linear automorphism on $K(\mathcal{D}^N_n)$ induced by Φ.

(2) Set $\mathcal{H} := \mathcal{H}(\Delta)$ and $\mathcal{H}' := \mathcal{H}(\mu^{\sharp}_{\delta_i}(\Delta))$. Then, $\Phi(\mathcal{H}) = (\mathcal{H}')^{\flat}_{S'_i}$ where $S'_i := S_{\delta^{\sharp}_i}$.

Proof. Part (2) follows from Lemma 9.7. Consider the relationship between simple objects in $\Phi(\mathcal{H})$ and $\mathcal{H}'$

$$\operatorname{Sim}\Phi(\mathcal{H}) = \{S_1, \dots, S_n\}, \quad \operatorname{Sim}\mathcal{H}' = \{S'_1, \dots, S'_n\}.$$

under the backward simple tilting. By Proposition 2.8, there are relations

$$S_i = S'_i[-1], \quad S_j = \operatorname{Cone}(S'_i[-1] \otimes \operatorname{Hom}^1(S'_i, S'_j) \to S'_j) \quad \text{for } j \neq i.$$

At the K-group level, we have

$$[S_i] = -[S'_i], \quad [S_j] = [S'_j] + \tilde{q}^{(0)}_{ij}[S'_i] \quad \text{for } j \neq i.$$

Let $\delta_1, \ldots, \delta_n$ and $\delta'_1, \ldots, \delta'_1$ be $(N-2)$-diagonals of Δ and $\mu^\sharp_{\delta_i}(\Delta)$ with the relations $\delta_j = \delta'_j$ if $j \neq i$ and $\delta^\sharp_i = \delta'_i$. Note that $\lambda(\alpha_j) = [S_{\delta_j}]$ and $\lambda^\sharp(\alpha'_j) = [S_{\delta'_j}]$. Since $S_j = \Phi(S_{\delta_j})$ and $S'_j = S_{\delta'_j}$, we conclude that

$$([\Phi] \circ \lambda)(\alpha_i) = \lambda^\sharp(-\alpha'_i), \quad ([\Phi] \circ \lambda)(\alpha_j) = \lambda^\sharp(\alpha'_j + \tilde{q}^{(0)}_{ij}\alpha'_i) \quad \text{for } j \neq i.$$

□

10 Proof of main theorem in Part II

10.1 Stratification

Following Section 5.2 in [BS], we introduce the stratification on $Q(N,n)$. Many results of [BS] for this stratification also hold in our cases. For $\phi \in Q(N,n)$, denote by s_ϕ the number of saddle trajectories. Define the subsets of $Q(N,n)$ by

$$D_p := \{\, \phi \in Q(N,n) \,|\, s_\phi \leq p \,\}.$$

(Note that our notation differs from that of [BS]. Our D_p is written B_{2p} in [BS].) Since Lemma 7.2 implies that $s_\phi \leq n$, we have $D_n = Q(N,n)$. We note that B_0 consists of all saddle-free differentials. These subsets have the following property.

Lemma 10.1 ([BS], Lemma 5.2) *The subsets $D_p \subset Q(N,n)$ form an increasing chain of dense open subsets*

$$D_0 \subset D_1 \subset \cdots \subset D_n = Q(N,n).$$

Let $F_0 := D_0$ and $F_p := D_p \setminus D_{p-1}$ for $p \geq 1$. Then we have a finite stratification

$$Q(N,n) = \bigcup_{p=0}^{n} F_p$$

by locally closed subset F_p.

Proposition 10.2 ([BS], Proposition 5.5 and Proposition 5.7) *Let $p \geq 1$ and assume that $\phi \in F_p$. Then, there is a neighborhood $\phi \in U \subset D_p$ a class $0 \neq \alpha \in H_\pm(\phi)$ such that*

$$\phi \in U \cap F_p \Longrightarrow Z_\phi(\alpha) \in \mathbb{R}.$$

Further, if $p \geq 2$, then we can take a neighborhood $\phi \in U \subset D_p$ to satisfy that $U \cap D_{p-1}$ is connected.

Following Section 5.2 in [BS], we define a generic differential. $\phi \in Q(N,n)$ is callled generic if for any $\gamma_1, \gamma_2 \in H_\pm(\phi)$,

$$\mathbb{R}Z_\phi(\gamma_1) = \mathbb{R}Z_\phi(\gamma_2) \Longrightarrow \mathbb{Z}\gamma_1 = \mathbb{Z}\gamma_2.$$

Lemma 10.3 *If $\phi \in Q(N,n)$ is generic, then $\phi \in D_1$.*

Proof. Assume that $\phi \in Q(N,n)$ is generic and $\phi \in Q(N,n) \setminus D_1$. Since at least ϕ has two different saddle trajectories, we write them by γ_1, γ_2, and by $\tilde{\gamma_1}, \tilde{\gamma_2} \in H_{\pm}(\phi)$ the corresponding saddle homology classes. Note that $\mathbb{Z}\hat{\gamma_1} \neq \mathbb{Z}\hat{\gamma_2}$. On the other hand, by the definition of the saddle trajectory, we have $\mathbb{Z}_\phi(\hat{\gamma_1}), \mathbb{Z}_\phi(\hat{\gamma_2}) \in \mathbb{R}$ and this contradicts to the definition of a generic differential. □

A differential $\phi \in Q(N,n)$ defines the length of a smooth path $\gamma \colon [0,1] \to \mathbb{P}^1 \setminus \{\infty\}$ by

$$l_\phi(\gamma) := \int_0^1 |\varphi(\gamma(t))|^{1/2} \left| \frac{d\gamma(t)}{dt} \right| dt$$

where $\phi(z) = \varphi(z) dz^{\otimes 2}$.

Proposition 6.8 in [BS] is easily extended in our cases.

Proposition 10.4 ([BS], Proposition 6.8) *Assume that for a sequence of framed differentials*

$$(\phi_k, \theta_k) \in Q(N,n)_*^\Gamma, \quad k \geq 1,$$

periods $Z_{\phi_k} \circ \theta_n \colon \Gamma \to \mathbb{C}$ converge as $k \to \infty$. Further, there is some constant $L > 0$, which is independent of k, such that any saddle connection γ of ϕ_k satisfies $l_{\phi_k}(\gamma) \geq L$. Then, there is some subsequence of $\{(\phi_k, \theta_k)\}_{k \geq 1}$ which converges in $Q(N,n)_*^\Gamma$.

10.2 Stability conditions from saddle-free differentials

Lemma 10.5 *$\phi \in B_0 \subset Q(N,n)$ be a saddle-free differential. Consider the corresponding heart $\mathcal{H}(\Delta_\phi) \in \mathrm{EG}_N^\circ(\mathcal{D}_n^N)$ and a linear map*

$$Z := Z_\phi \circ \nu \colon K(\mathcal{D}_n^N) \longrightarrow \mathbb{C}$$

where $Z_\phi \colon H_\pm(\phi) \to \mathbb{C}$ is a period integral of ϕ and $\nu \colon K(\mathcal{D}_n^N) \xrightarrow{\sim} H_\pm(\phi))$ is an isomorphism given by Corollary 9.14. Then, the pair $(\mathcal{H}(\Delta_\phi), Z)$ determines a unique stability condition $\sigma(\phi) = (\mathcal{H}(\Delta_\phi), Z) \in \mathrm{Stab}^\circ(\mathcal{D}_n^N)$.

10.3 Wall crossing

Let $\phi_1 \in F_1$ be a differential which has only one saddle trajectory. Write by Δ_{ϕ_1} the set of $(N-2)$-diagonals obtained by taking one generic trajectory from each horizontal strip of ϕ_1. By Lemma 7.2, $(n-1)$ horizontal strips appear in the decomposition of $\mathbb{P}^1$ by ϕ_1, so the number of $(N-2)$-diagonals in Δ_{ϕ_1} is $(n-1)$.

By Proposition 10.2, if we take $r > 0$ sufficiently small, then for any $0 < t \leq r$, the differentials

$$\phi(t) := e^{-i\pi t} \cdot \phi_1, \quad \phi^\sharp(t) := e^{+i\pi t} \cdot \phi_1$$

are saddle-free. Write $\phi := \phi(r)$ and $\phi^\sharp := \phi^\sharp(r)$.

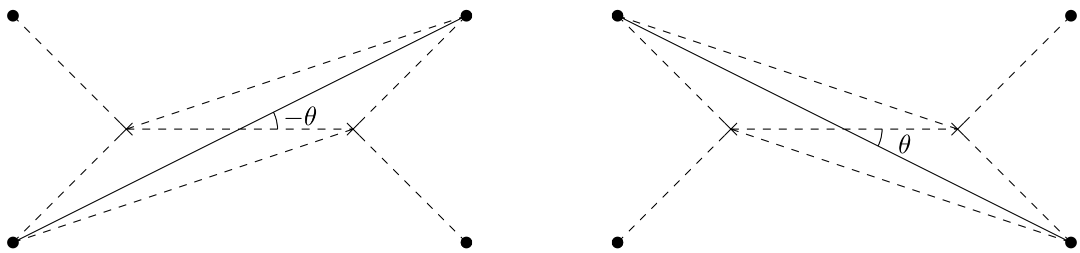
Let Δ_ϕ and $\Delta_{\phi^\sharp}$ be N-angulations determined by these saddle-free differentials. Write by δ a unique $(N-2)$-diagonal of Δ_ϕ which is not contained in Δ_{ϕ_1}:

$$\delta := \Delta_\phi \setminus \Delta_{\phi_1}.$$

Lemma 10.6 *Two N-angulations Δ_ϕ and $\Delta_{\phi^\sharp}$ are related by*

$$\mu_\delta^\sharp(\Delta_\phi) = \Delta_{\phi^\sharp}.$$

Proof. Let γ be a unique saddle trajectory of ϕ_1. By Lemma 7.1, the rotation $z \mapsto e^{\pm i\pi r}$ takes γ to the standard saddle connection which makes a constant angle $\pm\pi\theta$ (put $\theta := r\, d_{N,n}/2$) with the horizontal foliation of $e^{\pm i\pi r}\cdot\phi_1$. Therefore, the generic trajectories δ and $\delta^\sharp$ are given as in the following figure, and the result follows. □



As a result, we have an natural isomorphism of coloured quiver lattices $L_{\mathcal{Q}(\mu_\delta^\sharp(\Delta_\phi))} \cong L_{\mathcal{Q}(\Delta_{\phi^\sharp})}$.

Let

$$F_\delta \colon L_{\mathcal{Q}(\Delta)} \longrightarrow L_{\mathcal{Q}(\Delta_{\phi^\sharp})}$$

be the linear isomorphism defined in Section 9.6.

The relationship between F_δ and the Gauss-Manin connection

$$\mathrm{GM}(c)\colon H_\pm(\phi) \longrightarrow H_\pm(\phi^\sharp)$$

along the path $c(t) = e^{i\pi t}\cdot\phi_1\ (t \in [-r, r])$ is given by the next commutative diagram.

Lemma 10.7 *Let $\mu\colon L_{\mathcal{Q}(\Delta_\phi)} \to H_\pm(\phi)$ and $\mu^\sharp\colon L_{\mathcal{Q}(\Delta_{\phi^\sharp})} \to H_\pm(\phi^\sharp)$ be the isomorphisms given by Lemma 7.7. Then, the diagram*

$$\begin{array}{ccc} L_{\mathcal{Q}(\Delta_\phi)} & \xrightarrow{F_\delta} & L_{\mathcal{Q}(\Delta_{\phi^\sharp})} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu^\sharp} \\ H_\pm(\phi) & \xrightarrow{\mathrm{GM}(c)} & H_\pm(\phi^\sharp) \end{array}$$

commutes.

Proof. □

For $0 < t, r$, we denote by

$$\sigma(t) := \sigma(\phi(t)), \quad \sigma^\sharp(t) := \sigma(\phi^\sharp(t))$$

stability conditions in $\mathrm{Stab}^\circ(\mathcal{D}_n^N)$ constructed by Lemma 10.5 for saddle-free differentials $\phi(t)$ and $\phi^\sharp(t)$.

Proposition 10.8 ([BS], Proposition 10.7) *Let $\nu\colon K(\mathcal{D}_n^N) \to H_\pm(\phi)$ and $\nu^\sharp\colon K(\mathcal{D}_n^N) \to H_\pm(\phi^\sharp)$ be the isomorphisms given by Corollary 9.14. Then, there is a unique spherical twist $\Phi \in \mathrm{Sph}(\mathcal{D}_n^N)$ such that the following* (1) *and* (2) *hold.*

(1) The diagram

$$\begin{array}{ccc} K(\mathcal{D}_n^N) & \xrightarrow{[\Phi]} & K(\mathcal{D}_n^N) \\ {\scriptstyle\nu}\big\downarrow & & \big\downarrow{\scriptstyle\nu^\sharp} \\ H_\pm(\phi) & \xrightarrow{\mathrm{GM}(c)} & H_\pm(\phi^\sharp) \end{array}$$

commutes.

(2) The stability conditions $\Phi(\sigma(t))$ and $\sigma^\sharp(t)$ become arbitrarily close at $t \to 0$ in $\mathrm{Stab}^\circ(\mathcal{D}_n^N)$.

Proof. Part (1) is immediately follows from Lemma 9.15 and Lemma 10.7. For the proof of part (2), we first note that by Lemma 9.15 (2) and Lemma 10.6, the hearts $\mathcal{H}(\Delta_\phi)$ and $H(\Delta_{\phi^\sharp})$ are related by

$$\Phi(\mathcal{H}(\Delta_\phi)) = \mathcal{H}(\Delta_{\phi^\sharp})^\flat_{S_{\delta^\sharp}}.$$

By the definition of $\sigma(t)$ and $\sigma^\sharp(t)$ in Lemma 10.5, $\sigma(t)$ is contained in $\mathrm{Stab}(\mathcal{H}(\Delta_\phi))$ and $\sigma^\sharp(t)$ is contained in $\mathrm{Stab}(\mathcal{H}(\Delta_{\phi^\sharp}))$. Further, by Part (1), the central charges of $\sigma(t)$ and $\sigma^\sharp(t)$ approach the same point in $\mathrm{Hom}(K(\mathcal{D}_n^N), \mathbb{C})$ at $t \to 0$. Hence, by Lemma 2.18, we can take some open subset

$$U \subset \mathrm{Stab}(\mathcal{H}(\Delta_\phi)) \cup \mathrm{Stab}(\mathcal{H}(\Delta_{\phi^\sharp}))$$

such that U is mapped isomorphically on $\mathrm{Hom}(\mathcal{D}_n^N, \mathbb{C})$ and $\sigma(t), \sigma^\sharp(t) \in U$. Thus, the result follows. □

10.4 Construction of isomorphism

Definition 10.9 *Let $(\phi, \theta) \in Q(N, n)^\Gamma$ be a framed saddle-free differential and $\nu\colon K(\mathcal{D}_n^N) \to H_\pm(\phi)$ be an isomorphism in Lemma 9.14. We define the isomorphism κ by the composition of θ and ν^{-1}:*

$$\kappa := \nu^{-1} \circ \theta \colon \Gamma \xrightarrow{\sim} K(\mathcal{D}_n^N).$$

Fix a framed saddle-free differential $(\phi_0, \theta_0) \in Q(N, n)^\Gamma$ and let $Q(N, n)^\Gamma_*$ be the connected component which contains (ϕ_0, θ_0).

In the following, we identify $K(\mathcal{D}_n^N)$ with Γ by the isomorphism κ_0 for (ϕ_0, θ_0) given by Definition 10.9.

Lemma 10.10 *Let $(\phi, \theta) \in Q(N, n)^\Gamma_*$ be a framed saddle-free differential and $\kappa \colon \Gamma \to K(\mathcal{D}_n^N)$ be an isomorphism determined by Definition 10.9. Then, there is some spherical twist $\Phi \in \mathrm{Sph}(\mathcal{D}_n^N)$ such that the diagram*

$$\begin{array}{ccc} & \Gamma & \\ {\scriptstyle\kappa}\swarrow & & \searrow{\scriptstyle\kappa_0} \\ K(\mathcal{D}_n^N) & \xrightarrow{[\Phi]} & K(\mathcal{D}_n^N). \end{array}$$

commute. In addition, if there are two such spherical twists Φ_1 and Φ_2, then the induced linear isomorphisms on $K(\mathcal{D}_n^N)$ are the same:

$$[\Phi_1] = [\Phi_2].$$

Proof. $[\Phi_1] = [\Phi_2]$ immediately follows from

$$[\Phi_1] \circ \kappa = \kappa_0 = [\Phi_2] \circ \kappa.$$

Therefore, we show that such a spherical twist exsists. Take a path $c \colon [0, 1] \to Q(N, n)^\Gamma_*$ from (ϕ, θ) to (ϕ_0, θ_0). By Proposition 10.2, we can deform the path c to be intersecting only finitely many points of F_1, consisting of framed differentials having only one saddle trajectory.

By applying Proposition 10.8 to each of these points, we have a spherical twist $\Phi \in \mathrm{Sph}(\mathcal{D}_n^N)$ fitting into a diagram

$$\begin{array}{ccc} K(\mathcal{D}_n^N) & \xrightarrow{[\Phi]} & K(\mathcal{D}_n^N) \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\nu_0} \\ H_\pm(\phi) & \xrightarrow{\mathrm{GM}(c)} & H_\pm(\phi_0) \end{array}$$

where ν and ν_0 are isomorphisms given by Corollary 9.14. Further, recall the diagram

$$\begin{array}{ccc} & \Gamma & \\ {\scriptstyle\theta}\swarrow & & \searrow{\scriptstyle\theta_0} \\ H_\pm(\phi) & \xrightarrow{\mathrm{GM}(c)} & H_\pm(\phi_0) \end{array}$$

in Section 6.7. Combining the above two diagrams, we have the result. □

Let $\mathrm{Sph}^0(\mathcal{D}_n^N) \subset \mathrm{Sph}(\mathcal{D}_n^N)$ be the subgroup of spherical twists which act by the identity on $K(\mathcal{D}_n^N)$. Note that $\mathrm{Sph}(\mathcal{D}_n^N) \cong B_{n+1}$ by Theorem 8.4 and $\mathrm{Sph}(\mathcal{D}_n^N)$ act as the group $W_\pm$ on $K(\mathcal{D}_n^N)$ (see Section 8.3). Hence the group $\mathrm{Sph}^0(\mathcal{D}_n^N)$ is isomorphic to the group $P_\pm$.

Lemma 10.11 *The action of* $\mathrm{Sph}^0(\mathcal{D}_n^N)$ *on* $\mathrm{Stab}^\circ(\mathcal{D}_n^N)$ *is free and properly discontinuous.*

Proof. Let d be the metric on $\mathrm{Stab}^\circ(\mathcal{D}_n^N)$ (for more details, see Section 8 in [Bri07]). For any $\Phi \in \mathrm{Sph}^0(\mathcal{D}_n^N)$ and $\sigma \in \mathrm{Stab}^\circ(\mathcal{D}_n^N)$, the stability condition $\Phi(\sigma)$ and σ are not equal but have the same central charge. Hence, by Lemma 6.4 in [Bri07], we have $d(\sigma, \Phi(\sigma)) \geq 1$ and this implies the result. □

By completely the same argument of Proposition 11.3 in [BS], we can construct the following equivariant holomorphic map.

Proposition 10.12 *There is a* $W_\pm$*-equivariant and* $\mathbb{C}$*-equivariant holomorphic map of complex manifolds* K *such that the diagram*

$$\begin{array}{ccc} Q(N,n)_*^\Gamma & \xrightarrow{\quad K \quad} & \mathrm{Stab}^\circ(\mathcal{D}_n^N)/\,\mathrm{Sph}^0(\mathcal{D}_n^N) \\ & {\scriptstyle \mathcal{W}_N}\searrow \quad \swarrow{\scriptstyle \mathcal{Z}} & \\ & \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}) & \end{array}$$

commutes.

Proof. Let $t \in \mathbb{C}$ act by $Z(\gamma) \mapsto e^{i\pi t} \cdot Z(\gamma)$ for $Z \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C})$ and $\gamma \in \Gamma$. Then, by Definition 6.7 and Lemma 7.1, the map $\mathcal{W}_N$ is $\mathbb{C}$-equivariant. On the other hand, by the definition of $\mathbb{C}$-action on $\mathrm{Stab}^\circ(\mathcal{D}_n^N)$ given in Section 2.6, the map $\mathcal{Z}$ is also $\mathbb{C}$-equivariant. Since both $\mathcal{W}_N$ and $\mathcal{Z}$ are local isomorphisms, if K exists, then K is $\mathbb{C}$-equivariant.

Recall from Section 10.1 that there is the stratification

$$D_0 \subset D_1 \subset \cdots \subset D_n = Q(N,n).$$

We extend the stratification to $Q(N,n)_*^\Gamma$ by the obvious way (and use the same notation).

Here, we give the construction of K only on the stratum D_0 which consists of framed saddle-free differentials. By using Proposition 10.2 and Proposition 10.8, the argument of Proposition 11.3 in [BS] for the extension of K on larger strata D_p $(p \geq 1)$ also works in our case.

Let $(\phi, \theta) \in D_0$ be a framed saddle-free differential. By Lemma 10.5, we have the stability condition $\sigma(\phi) \in \mathrm{Stab}^\circ(\mathcal{D}_n^N)$. Let $\Phi \in \mathrm{Sph}(\mathcal{D}_n^N)$ be a spherical twist given by Lemma 10.10. Since Φ is determined up to the action of $P_\pm \subset \mathrm{Sph}(\mathcal{D}_n^N)$, the map $K \colon D_0 \to \mathrm{Stab}^\circ(\mathcal{D}_n^N)/P_\pm$ given by

$$K((\phi, \theta)) := \Phi(\sigma(\phi)) \in \mathrm{Stab}^\circ(\mathcal{D}_n^N)/P_\pm$$

is well-defined.

Finally, we see that K is $W_\pm$-equivariant. Note that the action of $W_\pm$ preserves the stratification and commutes with the action of $\mathbb{C}$. Hence if we show $W_\pm$-equivariance on D_0, then we can inductively extend it to larger strata. But, it is clear by the construction of K on D_0. □

Let $(\phi, \theta) \in Q(N,n)^{\Gamma}_{*}$ and we set $\sigma = K((\phi,\theta))$. The stability condition σ is determined up to the action of $P_{\pm} \subset \mathrm{Aut}(\mathcal{D}^N_n)$. For a class $0 \neq \alpha \in \Gamma$, let $\mathcal{M}_{\sigma}(\alpha)$ be the moduli space of σ-stable objects which have class α and phase in $(0,1]$ (here, we treat $\mathcal{M}_{\sigma}(\alpha)$ as a set).

Proposition 10.13 ([BS], Proposition 11.7) *Let ϕ be a generic differential. Then, the moduli space $\mathcal{M}_{\sigma}(\alpha)$ is empty or one point. In particular, $\mathcal{M}_{\sigma}(\alpha)$ is one point if and only if the class α corresponds to the class of a saddle connection of ϕ (up to sign).*

Proof. By Lemma 10.3, ϕ has at most one saddle trajectory. Hence, the result follows from the argument of Proposition 11.7 in [BS]. □

Proposition 10.14 *The map K in Proposition 10.12 is an isomorphism.*

Proof. By using Proposition 10.4 and Proposition 10.13, we can apply the same argument of Proposition 11.11 in [BS]. □

Theorem 10.15 *There is a B_{n+1}-equivariant and $\mathbb{C}$-equivariant isomorphism of complex manifolds $\widetilde{K}$ such that the diagram*

$$\begin{array}{ccc} \widetilde{M_n} & \xrightarrow{\ \widetilde{K}\ } & \mathrm{Stab}^{\circ}(\mathcal{D}^N_n) \\ & {\scriptstyle \mathcal{W}_N} \searrow \quad \swarrow {\scriptstyle \mathcal{Z}} & \\ & \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}) & \end{array}$$

commutes.

Proof. First, recall from Corollary 6.9 that $\widetilde{M_n}$ is a universal covering of the space $Q(N,n)^{\Gamma}_{*}$ and $\widetilde{M_n}/P_{\pm} \cong Q(N,n)^{\Gamma}_{*}$. Further, by Proposition 10.12 and Proposition 10.14, we have $\widetilde{M_n}/P_{\pm} \cong \mathrm{Stab}^{\circ}(\mathcal{D}^N_n)/P_{\pm}$ (note that $\mathrm{Sph}^0(\mathcal{D}^N_n) \cong P_{\pm}$).

Therefore, we can lift the map K to the B_{n+1}-equivariant covering map

$$\widetilde{K} \colon \widetilde{M_n} \to \mathrm{Stab}^{\circ}(\mathcal{D}^N_n),$$

and Lemma 10.11 implies that $\widetilde{K}$ is an isomorphism. □

Corollary 10.16 *The distinguished connected component* $\mathrm{Stab}^{\circ}(\mathcal{D}^N_n)$ *is contractible.*

Proof. Since $\widetilde{M_n}$ is contractible, it immediately follows from Theorem 10.15. □

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