博士論文

Congruences of Hilbert modular forms over real quadratic fields and the special values of L-functions

（実２次体上の Hilbert 保型形式の合同式と L 関数の特殊値）

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CONGRUENCES OF HILBERT MODULAR FORMS OVER REAL QUADRATIC FIELDS AND THE SPECIAL VALUES OF L-FUNCTIONS

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Abstract. The purpose of this article is to carry out the first step towards a generalization of the method of Greenberg–Vatsal in order to provide evidence for the Iwasawa main conjecture for Hilbert modular forms in the residually reducible case. In the case of a real quadratic field, we show how a congruence between a Hilbert cusp form and a Hilbert Eisenstein series of the same parallel weight 2 give rise to congruences between the algebraic parts of the critical values of the associated L-functions.

0. Introduction

0.1. **Introduction.** The motivation of this work is to investigate the Iwasawa main conjecture for a Hilbert modular form whose associated Galois representation is residually reducible. By the ingenious method of Ribet and Wiles, residually reducible representations provide a powerful means of the proof of the Iwasawa main conjecture for GL_1 over a totally real number field. However, the advanced recent work of Skinner and Urban [Ski–Ur] for the Iwasawa main conjecture for GL_2 over $\mathbb{Q}$ has not treated this case. For this reason, we are interested in providing evidence for the Iwasawa main conjecture in the residually reducible case following the work of Greenberg and Vatsal [Gre–Vat].

The purpose of this paper is to show how congruences between the Fourier coefficients of Hilbert Hecke eigenforms give rise to congruences between the special values of the associated L-functions. The study of this topic for elliptic modular forms was initiated by Mazur [M] using the arithmetic of modular curves in order to investigate a weak analogue of the Birch and Swinnerton–Dyer conjecture. Mazur's congruence formula was generalized by Stevens ([Ste1], [Ste2]). Using this tool, Vatsal [Vat] has proved congruences between special values of the L-functions of an elliptic cusp form and those of the L-functions of an elliptic Eisenstein series of the same weight 2. Based on this congruences, Greenberg and Vatsal [Gre–Vat] have studied the Iwasawa invariant of elliptic curves in towers of cyclotomic fields. In particular, they proved the Iwasawa main conjecture for certain elliptic curves. Their work is motivated by Kato's result [Kato] on the Iwasawa main conjecture for elliptic modular forms.

In this paper, we present a way to obtain congruences of the special values of the L-functions from congruences between a Hilbert cusp form and a Hilbert Eisenstein series of the same parallel weight 2 under some conditions. This is a generalization of the works explained above by Mazur [M], Stevens [Ste2], and Vatsal [Vat].

Let F be a totally real number field with narrow class number 1 and degree $n = [F : \mathbb{Q}]$ and $\triangle_F$ the discriminant of F. Let $\mathfrak{n}$ be an integral ideal of F such that $(\mathfrak{n}, 6\triangle_F) = 1$. Let $p \geq n + 2$ be a prime number such that $(p, 6\mathfrak{n}\triangle_F) = 1$. Let $\mathcal{O}$ be the ring of integers of a finite extension K over $\mathbb{Q}_p$ and $\varpi \in \mathcal{O}$ a uniformizer. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and an embedding $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$.

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Theorem 0.1 (=Theorem 3.1)**.** *Let φ and ψ be totally even (resp. totally odd) $\mathcal{O}$-valued narrow ray class characters of conductor $\mathfrak{m}_\varphi$ and $\mathfrak{m}_\psi$ such that $\mathfrak{m}_\varphi\mathfrak{m}_\psi = \mathfrak{n}$ and $\epsilon = -\mathbf{1}$ (resp. $\epsilon = \mathbf{1}$) the character on the Weyl group W_G. Put $\chi = \varphi\psi$, which is a totally even character. Assume that $\varphi \neq \mathbf{1}$ and the algebraic Iwasawa μ-invariants of the splitting fields of φ and ψ are equal to* 0*. Let $\boldsymbol{f} \in S_2(\mathfrak{n}, \mathcal{O})$ be a normalized Hecke eigenform for every Hecke operator $T(\mathfrak{m})$ and $U(\mathfrak{m})$ with character χ. We assume the following four conditions, where $Y(\mathfrak{n})$ denotes the Shimura variety defined by (1.2):*

(*a*) *$H_c^{n+1}(Y(\mathfrak{n}), \mathcal{O})$ is torsion-free;*
(*b*) *$H^n(\partial\left(Y(\mathfrak{n})^{BS}\right), \mathcal{O})$ is torsion-free;*
(*c*) *the Hilbert Eisenstein series $\boldsymbol{E} = \boldsymbol{E}_2(\varphi, \psi) \in M_2(\mathfrak{n}, \mathcal{O})$ with character χ satisfies $\boldsymbol{f} \equiv \boldsymbol{E} \pmod{\varpi}$ (for the definition, see just before Theorem 3.1);*
(*d*) *$C(\mathfrak{q}, \boldsymbol{E}) \not\equiv N(\mathfrak{q}) (\mathrm{mod} \varpi)$ for some prime ideal $\mathfrak{q}$ dividing $\mathfrak{n}$, where $C(\mathfrak{q}, \boldsymbol{E})$ is the $U(\mathfrak{q})$-eigenvalue of $\boldsymbol{E}$.*

Then there exist a complex number $\Omega_{\boldsymbol{f}}^{\epsilon} \in \mathbb{C}^\times$ and a p-adic unit $u \in \mathcal{O}^\times$ such that, for every primitive narrow ray class character $\eta : \mathrm{Cl}_F^+(\mathfrak{m}_\eta) \to \overline{\mathbb{Q}}^\times$ of conductor $\mathfrak{m}_\eta$ such that $\mathfrak{n}|\mathfrak{m}_\eta$ and $\eta = \epsilon$ on $W_G \simeq \mathbb{A}_{F,\infty}^\times/\mathbb{A}_{F,\infty,+}^\times$, the both values $\tau(\eta^{-1})D(1, \boldsymbol{f}, \eta)/(2\pi\sqrt{-1})^n\Omega_{\boldsymbol{f}}^{\epsilon}$ and $\tau(\eta^{-1})D(1, \boldsymbol{E}, \eta)/(2\pi\sqrt{-1})^n$ belong to $\mathcal{O}(\eta)$ and the following congruence holds:

$$\tau(\eta^{-1})\frac{D(1, \boldsymbol{f}, \eta)}{(2\pi\sqrt{-1})^n\Omega_{\boldsymbol{f}}^{\epsilon}} \equiv u\tau(\eta^{-1})\frac{D(1, \boldsymbol{E}, \eta)}{(2\pi\sqrt{-1})^n} \pmod{\varpi}.$$

Here $\tau(\eta^{-1})$ is the Gauss sum attached to η^{-1}, $D(1, *, \eta)$ is given by the Dirichlet series in the sense of Shimura (for the definition, see (1.12)), $\mathcal{O}(\eta)$ is the ring of integers of $K(\eta)$, and $K(\eta)$ is the field generated by elements of $\mathrm{im}(\eta)$ *over K.*

Remark 0.2. The assumption that the algebraic Iwasawa μ-invariants of the splitting fields of φ and ψ are equal to 0 is satisfied if the splitting fields of φ and ψ are abelian extensions over $\mathbb{Q}$ by the Ferrero–Washington theorem.

This result can be regarded as an analogue of Vatsal's result [Vat] in the case $F = \mathbb{Q}$ and weight $k = 2$. However, our methods to prove the main theorem have some limitations, such as the need for the torsion-freeness of the compact support cohomology and the boundary cohomology. In the case F is a real quadratic field with narrow class number 1, the assumption (a) is equivalent to the p-torsion-freeness of the maximal abelian quotient of the fundamental group of the Shimura variety $Y(\mathfrak{n})$. This has been studied by M. Kuga in [Kuga]. By using his method and the theorem of Serre (congruence subgroup property), we will prove the p-torsion-freeness under some assumptions (Proposition 2.26). Moreover, if $\mathfrak{n}$ is a prime ideal, then the assumption (b) is satisfied under some assumptions (Proposition 2.27). We will also give an example of a congruence between a Hilbert cusp form and a Hilbert Eisenstein series of the same parallel weight 2 satisfying the all assumptions of Theorem 0.1 (Example 2.28).

The organization of this paper is as follows.

In §1, we summarize results on the Hilbert modular varieties and Hilbert modular forms in the analytic and algebraic settings. Moreover, we state basic properties of Hilbert Eisenstein series, which are of great utility in the following sections.

In §2, we give an analogue of Stevens's results [Ste2]. We will construct a desired n-cocycle $\pi_{\mathbf{h}}$ associated to a Hilbert modular form $\mathbf{h}$ of a general multiple weight $k \geq 2t$ (Definition 2.4), which is based on the method of Yoshida ([Yo], [Yo2]). This provides the following three results:

(i) Mellin transform for a more general Hilbert modular form (§2.7, §2.8);
(ii) Integrality of the cohomology class of a Hilbert Eisenstein series (Corollary 2.24);
(iii) Construction of an example of a congruence between a Hilbert cusp form and a Hilbert Eisenstein series (Example 2.28).

The result (i) can be regarded as an analogue of results of Stevens ([Ste1], [Ste2]). He expected that his methods would be generalized to Hilbert modular forms [Ste1].

This cocycle allows us to determine the structure of the congruence module attached to a Hilbert Eisenstein series (Theorem 2.22), based on Berger [Be] and Emerton [Eme] by using cohomological congruence. This method and result can be regarded as cohomological treatment of the arguments of Ribet [Ri] and Wiles [Wil]. As an application, we prove (ii) and (iii) under some assumptions.

In §3, we generalize Vatsal's results [Vat]. For a normalized Hecke eigenform $\mathbf{f}$ and a Hilbert Eisenstein series $\mathbf{E}$ of the same parallel weight 2 related by congruences of the Hecke eigenvalues $C(\mathfrak{q}, \mathbf{f}) \equiv C(\mathfrak{q}, \mathbf{E}) (\mathrm{mod}\ \varpi)$ for all prime ideal $\mathfrak{q}$, we derive congruences between the special values of the associated L-functions (Theorem 0.1=Theorem 3.1). One of the key ingredients in our proof is to describe the special values of the L-functions attached to Hilbert modular forms using the evaluation maps (Proposition 2.19 and Proposition 2.20). This description allows us to prove congruences between the special values by using the cohomological congruence obtained by §4.

In §4, we present a way to show how congruences between the Fourier coefficients of Hilbert Hecke eigenforms give rise to congruences between the cocycles (Theorem 4.1) by using integral p-adic Hodge theory for open varieties with constant coefficients. Theorem 4.1 is crucial to prove congruences of integral cohomology classes between $[\pi_{\mathbf{f}}]/\Omega_{\mathbf{f}}$ and $[\pi_{\mathbf{E}}]$ modulo ϖ and the main theorem (Theorem 0.1=Theorem 3.1). It may be regarded as an analogue of multiplicity one theorem for modulo p parabolic cohomology in the case where the residual Galois representations $\bar{\rho}_{\mathbf{f}}$ $(= \rho_{\mathbf{f}} \bmod \varpi)$ associated to a Hilbert cusp form $\mathbf{f}$ is reducible. In the case $\bar{\rho}_{\mathbf{f}}$ is irreducible, under some assumptions, a multiplicity one theorem is known to be true by [Dim2] for a general totally real number field.

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0.2. **Notation.** In this paper, p and l always denote distinct prime numbers. We denote by $\mathbb{N}$ the set of natural numbers (that is, positive integers), denote by $\mathbb{Z}$ (resp. $\mathbb{Z}_p$) the ring of rational integers (resp. p-adic integers), and also denote by $\mathbb{Q}$ (resp. $\mathbb{Q}_p$) the rational number field (resp. the p-adic number field). Let $\widehat{\mathbb{Z}} = \prod_{l<\infty} \mathbb{Z}_l$, where l runs over all rational primes. We fix algebraic closures $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and fix embeddings

$$\overline{\mathbb{Q}} \xrightarrow{\iota_p} \overline{\mathbb{Q}}_p \to \mathbb{C},$$

where $\mathbb{C}$ denotes the complex number field.

We assume that every ring is commutative with identity. For a ring R and $n \in \mathbb{N}$, we use the following notation:

$$\begin{aligned} \mathrm{M}_n(R) &= \{(n \times n)\text{-matrices with entries in } R\}; \\ \mathrm{GL}_n(R) &= \{M \in \mathrm{M}_n(R) \mid M \text{ is an invertible matrix}\}; \\ \mathrm{SL}_n(R) &= \{M \in \mathrm{GL}_n(R) \mid \det(M) = 1\}. \end{aligned}$$

Moreover, if R is a subring of $\mathbb{R}$, we put

$$\mathrm{GL}_n(R)_+ = \{M \in \mathrm{GL}_n(R) \mid \det(M) > 0\}.$$

Let F be a totally real number field of degree $n = [F\colon \mathbb{Q}]$, $\mathfrak{o}_F$ the ring of integers of F, and $\mathbb{A}_F$ the adéle of F. We abbreviate $\mathbb{A}_{\mathbb{Q}}$ to $\mathbb{A}$. We have the usual decomposition $\mathbb{A}_F = \mathbb{A}_{F,f} \times \mathbb{A}_{F,\infty}$ into finite and infinite adéle parts and denote adélic variables by $x = (x_0, x_\infty)$. For any $x \in \mathbb{A}_F$ and any place v of F, x_v denotes the v-component of x. For any element $x \in \mathbb{A}_F$, any subset X of $\mathbb{A}_F$, and any ideal $\mathfrak{n}$ of $\mathfrak{o}_F$, we write $x_{\mathfrak{n}}$ and $X_{\mathfrak{n}}$ for the projection of x and X to $\prod_{\mathfrak{q}|\mathfrak{n}} F_{\mathfrak{q}}$, where $F_{\mathfrak{q}}$ denotes the $\mathfrak{q}$-adic completion of F. Let $N = \mathrm{Nr}_{F/\mathbb{Q}}$ be the norm map of $F/\mathbb{Q}$, $\mathfrak{d}_F \subset \mathfrak{o}_F$ the different of F, and $\triangle_F = N(\mathfrak{d}_F)$ the discriminant of F. A narrow ray class character modulo an integral ideal $\mathfrak{b}$ of F is a homomorphism

$$\chi : \mathrm{Cl}_F^+(\mathfrak{b}) \to \mathbb{C}^\times.$$

Let $r \in (\mathbb{Z}/2\mathbb{Z})^n$ be the sign of χ:

$$\chi((\alpha)) = \mathrm{sgn}(\alpha)^r \text{ for } \alpha \equiv 1 \pmod{\mathfrak{b}}.$$

The character χ is called as totally even (resp. totally odd) if the sing $r = (0, \cdots, 0)$ (resp. $r = (1, \cdots, 1)$).

For an algebraic group $H_{/\mathbb{Q}}$, we shall abbreviate $H(\mathbb{R})$ to H_∞ and denote by $H_{\infty,+}$ the connected component of H_∞ with the identity. We define the reductive algebraic group $G_{/\mathbb{Q}}$ to be $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2/F}$, where $\mathrm{Res}_{F/\mathbb{Q}}$ denotes the Weil restriction of scalars. We shall denote by $B_{/\mathbb{Q}} = B^+_{/\mathbb{Q}}$ (resp. $B^-_{/\mathbb{Q}}$) the standard Borel subgroup of upper (resp. lower) triangular matrices and $U_{/\mathbb{Q}} = U^+_{/\mathbb{Q}}$ (resp. $U^-_{/\mathbb{Q}}$) its unipotent radical of $G_{/\mathbb{Q}}$. Let J_F be the set of all real embeddings of F into $\mathbb{R}$. We have $G_\infty = \mathrm{GL}_2(\mathbb{R})^{J_F} = \mathrm{GL}_2(\mathbb{R})^n$, $G_{\infty,+} = \mathrm{GL}_2(\mathbb{R})_+^{J_F} = \mathrm{GL}_2(\mathbb{R})_+^n$, and $G(\mathbb{A}) = \mathrm{GL}_2(\mathbb{A}_F)$.

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1. Hilbert modular variety and Hilbert modular form

1.1. **Analytic Hilbert modular forms.** We recall the definitions of classical Hilbert modular forms. For more detail, refer to [Shi], [Hida88], [Hida91], [Hida94], [Ge–Go].

Let $\mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper half plane. Then $\mathrm{GL}_2(\mathbb{R})_+$ acts on $\mathfrak{H}$ by

$$\alpha z = \frac{az + b}{cz + d}$$

for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})_+$ and $z \in \mathfrak{H}$. We consider the left action of $\mathrm{GL}_2(\mathbb{R})_+^{J_F}$ on $\mathfrak{H}^{J_F}$ defined by

$$\alpha z = \left(\frac{a_\sigma z_\sigma + b_\sigma}{c_\sigma z_\sigma + d_\sigma}\right)_{\sigma \in J_F}$$

for $z = (z_\sigma)_{\sigma \in J_F} \in \mathfrak{H}^{J_F}$ and $\alpha = \left(\begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}\right)_{\sigma \in J_F} \in \mathrm{GL}_2(\mathbb{R})_+^{J_F}$. We define an action of the element $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ on $\mathfrak{H}$ by $z \mapsto -\bar{z}$. Then the action of $\mathrm{GL}_2(\mathbb{R})_+^{J_F}$ extends to that of G_∞ on $\mathfrak{H}^{J_F}$. Let $\underline{\mathbf{i}} = (\sqrt{-1}, \cdots, \sqrt{-1}) \in \mathfrak{H}^{J_F}$. Let $K_\infty = \mathrm{Stab}_{G(\mathbb{R})}(\underline{\mathbf{i}})$ and $K_{\infty,+} = \mathrm{Stab}_{G(\mathbb{R})_+}(\underline{\mathbf{i}})$ be the stabilizers of $\underline{\mathbf{i}}$. For each subset $J \subset J_F$ and $\alpha \in G_\infty$, we put

$$J^\alpha = \{\sigma \in J_F \mid \sigma \in J \ \text{ if} \det(\alpha_\sigma) > 0,\ \sigma \in J_F - J \ \text{ if} \det(\alpha_\sigma) < 0\}.$$

For each subset $J \subset J_F$, we define an automorphic factor $j_J(\alpha, z) \in \mathbb{C}^{J_F}$ as follows: for $\alpha = \left(\begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}\right)_{\sigma \in J_F} \in \mathrm{GL}_2(\mathbb{R})^{J_F}$ and $z \in \mathfrak{H}^{J_F}$,

$$j_J(\alpha, z) = \left(c_\sigma z_\sigma^J + d_\sigma\right)_\sigma,$$

where

$$z_\sigma^J = \begin{cases} z_\sigma & \text{if } \sigma \in J, \\ \overline{z}_\sigma & \text{if} \, \sigma \in J_F - J. \end{cases} \tag{1.1}$$

It satisfies the cocycle condition: for each $\alpha, \beta \in G_\infty$,

$$j_J(\alpha\beta, z) = j_{J^\beta}(\alpha, \beta z) j_J(\beta, z).$$

For an ideal $\mathfrak{n}$ of $\mathfrak{o}_F$, we define open compact subgroups of $G(\mathbb{A}_f)$ to be

$$K_1(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\widehat{\mathbb{Z}}) \middle| c \in \mathfrak{n}, d - 1 \in \mathfrak{n} \right\}.$$

The adélic Hilbert modular variety of level $K_1(\mathfrak{n})$ is defined as

$$\begin{aligned} Y(\mathfrak{n}) &= G(\mathbb{Q})\backslash G(\mathbb{A})/K_1(\mathfrak{n})K_{\infty,+} \\ &= G(\mathbb{Q})_+\backslash G(\mathbb{A})_+/K_1(\mathfrak{n})K_{\infty,+}, \end{aligned} \tag{1.2}$$

where $G(\mathbb{A})_+ = G(\mathbb{A}_f)G_{\infty,+}$ and $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G_{\infty,+}$. We recall that $Y(\mathfrak{n})$ is a disjoint union of finitely many arithmetic quotients Y_i as follows. Let $T = \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$. The determinant map $\det : G \to T$ induces

$$\det : G(\mathbb{Q})\backslash G(\mathbb{A})/K_1(\mathfrak{n})K_{\infty,+} \to T(\mathbb{Q})\backslash T(\mathbb{A})/\det(K_1(\mathfrak{n})K_{\infty,+}).$$

Moreover, we have

$$T(\mathbb{Q})\backslash T(\mathbb{A})/\det(K_1(\mathfrak{n})K_{\infty,+}) \simeq F^\times\backslash \mathbb{A}_F^\times/\widehat{\mathfrak{o}}_F^\times \mathbb{A}_{F,\infty,+}^\times,$$

where $\mathbb{A}_{F,\infty,+}^\times = \mathbb{R}_+^{\times J_F}$. It is isomorphic to the narrow ideal class group Cl_F^+ of F via $x \mapsto [x] = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(x_{\mathfrak{p}})}$, where $\mathfrak{p}$ runs over through the set of all prime ideals of $\mathfrak{o}_F$. Let $h_F^+ = \sharp \mathrm{Cl}_F^+$ be the narrow class number of F and $t_1, \cdots, t_{h_F^+} \in \mathbb{A}_F^\times$ such that $t_{i,\infty} = 1$ and the corresponding fractional ideals $[t_1], \cdots, [t_{h_F^+}]$ form a complete set of representatives for Cl_F^+. Throughout the paper, we assume that

$$\text{for each } i, \text{ both } \mathfrak{d}_F \text{ and } [t_i] \text{ are prime to } p. \tag{1.3}$$

Let $D \in \mathbb{A}_F^\times$ be such that $[D] = \mathfrak{d}_F$ and $D_\infty = 1$. We put

$$x_i = \begin{pmatrix} D^{-1}t_i^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

By the strong approximation theorem, we have

$$G(\mathbb{A}) = \coprod_{i=1}^{h_F^+} G(\mathbb{Q})x_i G_{\infty,+}K_1(\mathfrak{n}).$$

It implies the canonical decomposition

$$Y(\mathfrak{n}) \simeq \coprod_{i=1}^{h_F^+} Y_i, \tag{1.4}$$

where

$$Y_i = \Gamma_i(K_1(\mathfrak{n}))\backslash\mathfrak{H}^{J_F},$$
$$\Gamma_i(K_1(\mathfrak{n})) = G(\mathbb{Q})_+ \cap x_i K_1(\mathfrak{n}) x_i^{-1} G(\mathbb{R})_+.$$

We will be mostly interested in the following special congruence subgroups of $G(\mathbb{Q})$:

$$\begin{aligned}
\Gamma_{0,i}(\mathfrak{n}) &= \Gamma_0(\mathfrak{d}_F[t_i], \mathfrak{n}) \\
&= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F) \middle| a, d \in \mathfrak{o}_F, b \in \mathfrak{d}_F^{-1}[t_i]^{-1}, c \in \mathfrak{n}\mathfrak{d}_F[t_i], ad - bc \in \mathfrak{o}_{F,+}^\times \right\}; \\
\Gamma_{1,i}(\mathfrak{n}) &= \Gamma_i(K_1(\mathfrak{n})) = \Gamma_1(\mathfrak{d}_F[t_i], \mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{0,i}(\mathfrak{n}) \middle| d \equiv 1 \bmod \mathfrak{n} \right\}; \\
\Gamma_{1,i}^1(\mathfrak{n}) &= \Gamma_1^1(\mathfrak{d}_F[t_i], \mathfrak{n}) = \Gamma_{1,i}(\mathfrak{n}) \cap \mathrm{SL}_2(F),
\end{aligned} \tag{1.5}$$

where $\mathfrak{o}_{F,+}^\times \subset \mathfrak{o}_F^\times$ denotes the subgroup of totally positive units. Then we have

$$Y_i^1 = \Gamma_{1,i}^1(\mathfrak{n})\backslash\mathfrak{H}^{J_F}$$

and the $\mathfrak{o}_{F,+}^\times/\mathfrak{o}_{F,\mathfrak{n}}^{\times 2}$-covering map

$$\tau_i : Y_i^1 \to Y_i,$$

where $\mathfrak{o}_{F,\mathfrak{n}}^\times \subset \mathfrak{o}_F^\times$ denotes the subgroup consisting of elements congruent to 1 modulo $\mathfrak{n}$. We put

$$Y^1(\mathfrak{n}) = \coprod_{i=1}^{h_F^+} Y_i^1. \tag{1.6}$$

We define the subset of weights $\mathfrak{X}(T) \subset \mathbb{Z}[J_F] \times \frac{1}{2}\mathbb{Z}[J_F]$ by

$$\mathfrak{X}(T) = \{\kappa = (k - 2t, m) \mid k - 2t + 2m \in \mathbb{Z} \cdot t\},$$

where $t = \sum_{\sigma \in J_F} \sigma$.

We fix a subset $J \subset J_F$ and $\kappa = (k - 2t, m) \in \mathfrak{X}(T)$ such that $k - 2t + 2m = 0$ as [Shi]. For any $\alpha \in G(\mathbb{A})$ and $\mathbb{C}$-valued function $\mathbf{f}$ on $G(\mathbb{A})$, we define the function $\mathbf{f}|_{\kappa,J}\alpha$ on $G(\mathbb{A})$ by

$$(\mathbf{f}|_{\kappa,J}\alpha)(x) = \det(\alpha_\infty)^{k-t+m} j_{J\alpha_\infty}(\alpha_\infty, \underline{\mathbf{i}})^{-k}\mathbf{f}(x\alpha^{-1}).$$

Here we used the convention that, for $z \in (F \otimes \mathbb{R})_+^\times$ and $\nu \in \mathbb{Q}[J_F]$, $z^\nu = \prod_\sigma z_\sigma^{\nu_\sigma}$ and, for $z \in \mathfrak{H}^{J_F}$ and $\nu \in \mathbb{Z}[J_F]$, $z^\nu = \prod_\sigma z_\sigma^{\nu_\sigma}$. We abbreviate $\mathbf{f}|_{\kappa,J_F}$ to $\mathbf{f}|_\kappa$.

First we recall the adélic definition of the Hilbert modular forms, following [Shi] and [Hida88]. The space

$$S_{\kappa,J}(K_1(\mathfrak{n}),\mathbb{C})$$

of Hilbert cusp forms of weight κ with respect to level $K_1(\mathfrak{n})$ and type J is the $\mathbb{C}$-vector space of functions $\mathbf{f} : G(\mathbb{A}) \to \mathbb{C}$ satisfying the following four conditions (a),(b),(c), and (d):
(a) $\mathbf{f}|_{\kappa,J}u = \mathbf{f}$ for all $u \in K_1(\mathfrak{n})K_{\infty,+}$;
(b) $\mathbf{f}(\gamma x) = \mathbf{f}(x)$ for $\gamma \in G(\mathbb{Q})$;
For each $z \in \mathfrak{H}^{J_F}$, we can choose $u_\infty \in G_{\infty,+}$ such that $z = u_\infty \underline{\mathbf{i}}$. We define a function by $f_{x_j} : \mathfrak{H}^{J_F} \to \mathbb{C} : z \mapsto \det(u_\infty)^{-k+t-m} j_J(u_\infty, \underline{\mathbf{i}})^k \mathbf{f}(x_j u_\infty)$. Then it is well-defined, that is, it is independent of the choice of $u_\infty \in G_{\infty,+}$ by (a).
(c) f_{x_j} is holomorphic at z_σ for $\sigma \in J$ and anti-holomorphic at z_σ for $\sigma \in J_F - J$;
(d) $\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \mathbf{f}(ux)du = 0$ for all $x \in G(\mathbb{A})$ for each additive Haar measure du on $U(\mathbb{Q})\backslash U(\mathbb{A})$.

Also, the space

$$M_{\kappa,J_F}(K_1(\mathfrak{n}),\mathbb{C})$$

of holomorphic Hilbert modular forms of weight κ with respect to level $K_1(\mathfrak{n})$ and type J_F is the $\mathbb{C}$-vector space of functions $\mathbf{f} : G(\mathbb{A}) \to \mathbb{C}$ satisfying the condition (a), (b), and (c) as above in the case $J = J_F$.

We remark that this adélic definition of [Shi] is related to that of [Hida88], which is explicitly given by the proof of [Hida88, Proposition 4.1].

We fix a narrow ray class character $\chi : \mathrm{Cl}_F^+(\mathfrak{m}) \to \mathbb{C}^\times$ whose conductor $\mathfrak{m}$ dividing $\mathfrak{n}$ of infinite type $-k+2t-2m = 0$. We define the space $M_{\kappa,J_F}(K_1(\mathfrak{n}),\chi,\mathbb{C})$ (resp. $S_{\kappa,J}(K_1(\mathfrak{n}),\chi,\mathbb{C})$) to be the subspace of $M_{\kappa,J_F}(K_1(\mathfrak{n}),\mathbb{C})$ (resp. $S_{\kappa,J}(K_1(\mathfrak{n}),\mathbb{C})$) satisfying $\mathbf{f}(xb) = \chi^{-1}(b)\mathbf{f}(x)$ for any $b \in \mathbb{A}_F^\times$. We note that

$$M_{\kappa,J_F}(K_1(\mathfrak{n}),\mathbb{C}) \simeq \bigoplus_{\chi} M_{\kappa,J_F}(K_1(\mathfrak{n}),\chi,\mathbb{C}), \quad S_{\kappa,J}(K_1(\mathfrak{n}),\mathbb{C}) \simeq \bigoplus_{\chi} S_{\kappa,J}(K_1(\mathfrak{n}),\chi,\mathbb{C}),$$

where χ runs over all narrow ray class characters whose conductor $\mathfrak{m}$ dividing $\mathfrak{n}$ of infinite type 0.

Next we recall the definition of the Hilbert modular forms over the Hilbert upper half plane $\mathfrak{H}^{J_F}$. The space

$$S_{\kappa,J}(\Gamma_{1,i}(\mathfrak{n}),\mathbb{C})$$

of Hilbert cusp forms of weight κ with respect to level $\Gamma_{1,i}(\mathfrak{n})$ and type J is the $\mathbb{C}$-vector space of functions $f : \mathfrak{H}^{J_F} \to \mathbb{C}$ which is holomorphic at z_σ for $\sigma \in J$ and anti-holomorphic at z_σ for $\sigma \in J_F - J$ satisfying $f|_{\kappa,J}\gamma = f$ for all $\gamma \in \Gamma_{1,i}(\mathfrak{n})$ and vanishing at all cusps, where $(f|_{\kappa,J}\gamma)(z) = \det(\gamma)^{k-t+m} j_J(\gamma,z)^{-k} f(\gamma z)$.

The space

$$M_{\kappa,J_F}(\Gamma_{1,i}(\mathfrak{n}),\mathbb{C})$$

of holomorphic Hilbert modular forms of weight κ with respect to level $\Gamma_{1,i}(\mathfrak{n})$ and type J_F is the $\mathbb{C}$-vector space of holomorphic functions $f : \mathfrak{H}^{J_F} \to \mathbb{C}$ satisfying $f|_{\kappa,J_F}\gamma = f$ for all $\gamma \in \Gamma_{1,i}(\mathfrak{n})$.

Then the map $\mathbf{f} \mapsto (f_{x_i})_i$ induces

$$M_{\kappa,J_F}(K_1(\mathfrak{n}),\mathbb{C}) \simeq \bigoplus_{i=1}^{h_F^+} M_{\kappa,J_F}(\Gamma_{1,i}(\mathfrak{n}),\mathbb{C}), \qquad S_{\kappa,J}(K_1(\mathfrak{n}),\mathbb{C}) \simeq \bigoplus_{i=1}^{h_F^+} S_{\kappa,J}(\Gamma_{1,i}(\mathfrak{n}),\mathbb{C})$$

(cf. [Hida91, p.323] and [Hida88, (2.6a)]).

We define the Hecke operator acting on $M_{\kappa,J_F}(K_1(\mathfrak{n}),\mathbb{C})$ and $S_{\kappa,J}(K_1(\mathfrak{n}),\mathbb{C})$ as follows. We define the semigroups $\widehat{R}(\mathfrak{n})$ and $R_{ij}(\mathfrak{n})$ as

$$\widehat{R}(\mathfrak{n}) = G(\mathbb{A}_f) \cap \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\widehat{\mathfrak{o}}_F) \middle| c \in \mathfrak{n}\widehat{\mathfrak{o}}_F,\ d_v \in \mathcal{O}_v^\times \text{ whenever } \mathfrak{p}_v|\mathfrak{n} \right\},$$
$$R_{ij}(\mathfrak{n}) = G(\mathbb{Q}) \cap x_j \widehat{R}(\mathfrak{n}) x_i^{-1}.$$

Then the Hecke character χ defines a character on $\widehat{R}(\mathfrak{n})$ and $R_{ij}(\mathfrak{n})$ by

$$\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \chi(d_\mathfrak{n}).$$

For $y \in \widehat{R}(\mathfrak{n})$ and the double coset decomposition

$$(K_1(\mathfrak{n})K_{\infty,+})\, y\, (K_1(\mathfrak{n})K_{\infty,+}) = \coprod_i (K_1(\mathfrak{n})K_{\infty,+})\, y_i,$$

we define

$$\mathbf{f}|[(K_1(\mathfrak{n})K_{\infty,+})\, y\, (K_1(\mathfrak{n})K_{\infty,+})](x) = \sum_i \mathbf{f}(xy_i^{-1}). \tag{1.7}$$

In particular, the Hecke operator acting on $M_{\kappa,J_F}(K_1(\mathfrak{n}),\chi,\mathbb{C})$ and $S_{\kappa,J}(K_1(\mathfrak{n}),\chi,\mathbb{C})$ is given by

$$\mathbf{f}|[(K_1(\mathfrak{n})K_{\infty,+})\, y\, (K_1(\mathfrak{n})K_{\infty,+})](x) = \sum_i \chi(y_i)^{-1}\mathbf{f}(xy_i^\iota),$$

where $y^\iota = \det(y)y^{-1}$.

The definition of the Hecke operator acting on the Hilbert modular forms over the Hilbert upper half plane and the relation between this Hecke operator and adélic one is explicitly given by [Shi, §2].

1.2. **Dirichlet series associated to a Hilbert modular form.**

The aim of this subsection is to describe the definition and properties of Dirichlet series attached to Hilbert modular forms, following [Shi].

Let $\mathbf{h} = (h_i)_i \in M_{\kappa,J_F}(K_1(\mathfrak{n}),\mathbb{C})$. Assume that

$$\kappa = ((k-2)t, m) \text{ satisfies } (k-2)t + 2m = 0 \text{ for } 2 \le k \in \mathbb{Z}.$$

Then $\mathbf{h}$ has the Fourier expansion of the form

$$\begin{aligned} \mathbf{h}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) =& c_\infty([y]\mathfrak{d}_F,\mathbf{h})N([y]\mathfrak{d}_F)^{-k/2}y_\infty^{k/2} \\ &+ \sum_{0 \ll \xi \in F} c(\xi[y]\mathfrak{d}_F,\mathbf{h})N(\xi)^{k/2}y_\infty^{k/2}e_F(\sqrt{-1}\xi y_\infty)e_F(\xi x) \end{aligned} \tag{1.8}$$

given by [Shi, (2.18)] and [Hida88, Proposition 4.1] for any $x \in \mathbb{A}_F$ and $y \in \mathbb{A}_F^\times$ with $0 \ll y_\infty$. Here $\mathfrak{m} \mapsto c(\mathfrak{m},\mathbf{h})$ is a function on fractional ideals of F vanishing outside integral ideals and e_F is the additive character of $F\backslash\mathbb{A}_F$ characterized by $e_F(x_\infty) = \exp(2\pi\sqrt{-1}x_\infty)$ for $x_\infty \in \mathbb{A}_{F,\infty}$ (for the definition, see, for example, [Ge–Go, Appendix C.2]). Here we used the convention that $y_\infty^{k/2} = \prod_\sigma y_{\infty,\sigma}^{k_\sigma/2}$. In particular, for $z = x_\infty + \sqrt{-1}y_\infty \in \mathfrak{H}^{J_F}$, we have

$$h_i(z) = y_\infty^{-k/2}\mathbf{h}\left(x_i \begin{pmatrix} y_\infty & x_\infty \\ 0 & 1 \end{pmatrix}\right) = y_\infty^{-k/2}\mathbf{h}\left(\begin{pmatrix} t_i^{-1}D^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_\infty & x_\infty \\ 0 & 1 \end{pmatrix}\right)$$
$$= c_\infty([t_i]^{-1}, \mathbf{h})N([t_i])^{k/2} + \sum_{0 \ll \xi \in [t_i]} c(\xi[t_i]^{-1}, \mathbf{h})N(\xi)^{k/2}e_F(\xi z). \tag{1.9}$$

We simply denote by

$$a_\infty(0, h_i) = c_\infty([t_i]^{-1}, \mathbf{h})N([t_i])^{k/2} \text{ and } a_\infty(\xi, h_i) = c(\xi[t_i]^{-1}, \mathbf{h})N(\xi)^{k/2}$$

for any $0 \ll \xi \in [t_i]$. For $\mathbf{h} = (h_i)_i \in M_{\kappa,J_F}(K_1(\mathfrak{n}), \mathbb{C})$, we denote by

$$C_{\infty,i}(0, \mathbf{h}) = N([t_i])^{-k/2}a_\infty(0, h_i), \tag{1.10}$$

$$C(\mathfrak{m}, \mathbf{h}) = N(\mathfrak{m})^{k/2}c(\mathfrak{m}, \mathbf{h}) \tag{1.11}$$

for all non-zero integral ideals $\mathfrak{m}$ of F.

Let η be a character of the narrow ray class group $\mathrm{Cl}_F^+(\mathfrak{m}_\eta)$. The Dirichlet series in the sense of Shimura [Shi, (2.25)] is defined by

$$\sum_{\mathfrak{m}} C(\mathfrak{m}, \mathbf{h})\eta(\mathfrak{m})\mathrm{N}(\mathfrak{m})^{-s}, \tag{1.12}$$

where $\mathfrak{m}$ runs over all integral ideals of F. It converges absolutely for sufficiently large $\mathrm{Re}(s) \gg 0$ and extends to a meromorphic function on the complex plane (see, for example, §2.7 in this paper). For each $\mathbf{h} \in M_{\kappa,J_F}(K_1(\mathfrak{n}), \mathbb{C})$, let $D(s, \mathbf{h}, \eta)$ denote this analytic continuation. If η is the trivial character, we simply write $D(s, \mathbf{h})$.

1.3. **Hilbert Eisenstein series.** We recall the definition and properties of the Hilbert Eisenstein series. For more detail, refer to [Shi, §3].

We fix integral ideals $\mathfrak{a}, \mathfrak{b}$ of F. Let φ (resp. ψ) be a character of $\mathrm{Cl}_F^+(\mathfrak{a})$ (resp. $\mathrm{Cl}_F^+(\mathfrak{b})$) with sign q (resp. r) $\in (\mathbb{Z}/2\mathbb{Z})^n$. We may regard φ (resp. ψ) as a function of all integral ideals of F by defining $\varphi(\mathfrak{m}) = 0$ (resp. $\psi(\mathfrak{m}) = 0$) if $\mathfrak{m}$ is not prime to $\mathfrak{a}$ (resp. $\mathfrak{b}$). Then a function $\mathrm{sgn}(x)^r\psi(x\mathfrak{h}^{-1})$ of $x \in \mathfrak{h}$ depends only on x modulo $\mathfrak{a}\mathfrak{h}$ for a fractional ideal $\mathfrak{h}$ of F. If ψ is primitive, that is, the conductor $\mathfrak{m}_\psi$ is exactly $\mathfrak{b}$, then, by [Shi, (3.11)], we have

$$\sum_{b \in [t_i]\mathfrak{h}^{-1}/\mathfrak{b}[t_i]\mathfrak{h}^{-1}} \mathrm{sgn}(b)^r\psi(b[t_i]^{-1}\mathfrak{h})e_F(tb) = \mathrm{sgn}(t)^r\psi^{-1}(t\mathfrak{b}[t_i]\mathfrak{d}_F\mathfrak{h}^{-1})\tau(\psi) \tag{1.13}$$

for a fractional ideal $\mathfrak{h}$ of F and $t \in \mathfrak{b}^{-1}\mathfrak{d}_F^{-1}[t_i]^{-1}\mathfrak{h}$, where $\tau(\psi)$ is the Gauss sum attached to ψ defined by

$$\tau(\psi) = \sum_{x \in \mathfrak{b}^{-1}\mathfrak{d}_F^{-1}/\mathfrak{d}_F^{-1}} \mathrm{sgn}(x)^r\psi(x\mathfrak{b}\mathfrak{d}_F)e_F(x).$$

The following proposition is obtained by [Shi, Proposition 3.4] and [Da–Da–Po, Proposition 2.1].

Proposition 1.1. *Let $k \geq 2$ be an integer such that $(k, \cdots, k) \equiv q + r \pmod 2$. Assume that both φ and ψ are primitive. Then there exists an Eisenstein series $\boldsymbol{E}_k(\varphi, \psi) = (E_k(\varphi, \psi)_i)_i \in M_{\kappa,J_F}(K_1(\mathfrak{a}\mathfrak{b}), \varphi\psi, \mathbb{C})$ satisfying the following properties.*

(1) $D(s, \boldsymbol{E}_k(\varphi, \psi)) = L(s, \varphi)L(s - k + 1, \psi)$.

(2) $C(\mathfrak{m}, \boldsymbol{E}_k(\varphi, \psi)) = \sum_{\mathfrak{c}|\mathfrak{m}} \varphi\left(\frac{\mathfrak{m}}{\mathfrak{c}}\right)\psi(\mathfrak{c})N(\mathfrak{c})^{k-1}$ *for each integral ideal* $\mathfrak{m}$ *of* F.

(3)

$$C_{\infty,i}(0, E_k(\varphi,\psi)) = \begin{cases} 2^{-n}\varphi^{-1}([t_i])L(1-k,\varphi^{-1}\psi) & \text{if } \mathfrak{a} = 1, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1.2. *Assume that $[F:\mathbb{Q}] > 1$, $h_F^+ = 1$, and $\mathfrak{d}_F[t_1] = \mathfrak{o}_F$. Under the same notation and assumptions of Proposition 1.1, the constant term $a_{x/y}(0, E_k(\varphi,\psi)_1)$ of $\boldsymbol{E}_k(\varphi,\psi) = E_k(\varphi,\psi)_1$ at the cusp $x/y \in \mathbb{P}^1(F)$ is given by the followings: fix $\alpha = \begin{pmatrix} x & \beta \\ y & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}_F)$ such that $\alpha(\infty) = x/y$. If $y \notin \mathfrak{m}_\psi$ and $\psi \neq \boldsymbol{1}$, then $a_{x/y}(0, E_k(\varphi,\psi)_1) = 0$. If $y \in \mathfrak{m}_\psi$ or $\psi = \boldsymbol{1}$, then*

$$\begin{aligned} a_{x/y}(0, E_k(\varphi,\psi)_1) = &\frac{N([t_1])^{k/2}}{2^n}\frac{\tau(\varphi\psi^{-1})}{\tau(\psi^{-1})}\left(\frac{N(\mathfrak{m}_\psi)}{N(\mathfrak{m}_{\varphi\psi^{-1}})}\right)^k \mathrm{sgn}(-y)^q\varphi(-y\mathfrak{m}_\psi^{-1})\mathrm{sgn}(-x)^r\psi^{-1}(-x) \\ &\times\left(\prod_{\mathfrak{q}|\mathfrak{m}_\varphi\mathfrak{m}_\psi, \mathfrak{q}\nmid\mathfrak{m}_{\varphi\psi^{-1}}} (1-\varphi\psi^{-1}(\mathfrak{q})N(\mathfrak{q})^{-k})\right) L(1-k,\varphi^{-1}\psi). \end{aligned}$$

Proof. We follow the arguments in the proof of [Da–Da–Po, Proposition 2.1] and [Fre, Chapter III, Theorem 4.9]. We simply write $\mathfrak{a} = \mathfrak{m}_\varphi$ and $\mathfrak{b} = \mathfrak{m}_\psi$. In order to prove it, we recall the construction of the Eisenstein series $\mathbf{E}_k(\varphi,\psi)$ from [Shi, §3] and [Da–Da–Po, Proposition 2.1]. Let

$$U = \{u \in \mathfrak{o}_F^\times \mid N(u)^k = 1, u \equiv 1 \bmod \mathfrak{ab}\}$$

be a subgroup of $\mathfrak{o}_F^\times$ with finite index. For $z \in \mathfrak{H}^n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(2s+k) > 2$, we define

(1.14)

$$\begin{aligned} E_k(\varphi,\psi)_1(z,s) = &N([t_1])^{1-k/2}[\mathfrak{o}_F^\times : U]^{-1}\Gamma(k)^n N(\mathfrak{b})^{-1}\tau(\psi)\sum_{\mathfrak{h}\in\mathrm{Cl}_F}\sum_{a\in\mathfrak{h}/\mathfrak{ah}}\sum_{t\in\mathfrak{b}^{-1}\mathfrak{d}_F^{-1}[t_1]^{-1}\mathfrak{h}/\mathfrak{d}_F^{-1}[t_1]^{-1}\mathfrak{h}} \\ &\times \mathrm{sgn}(a)^q\varphi(a\mathfrak{h}^{-1})\mathrm{sgn}(-t)^r\psi(-t\mathfrak{b}\mathfrak{d}_F[t_1]\mathfrak{h}^{-1})N(\mathfrak{h})^{k-1} \\ &\times E_{k,U}(z,s;a,t;\mathfrak{ah},\mathfrak{d}_F^{-1}[t_1]^{-1}\mathfrak{h}), \end{aligned}$$

where Cl_F is the ideal class group of F and

$$\begin{aligned} &E_{k,U}(z,s;a,t;\mathfrak{ah},\mathfrak{d}_F^{-1}[t_1]^{-1}\mathfrak{h}) \\ &\quad = \triangle_F^{1/2}N(\mathfrak{d}_F^{-1}[t_1]^{-1}\mathfrak{h})(-1)^{kn}(2\pi\sqrt{-1})^{-kn}\sum_{(a',b')U}(a'z+b')^{-k}|a'z+b'|^{-2s}. \end{aligned}$$

Here the sum runs over representatives $(a',b') \neq (0,0)$ modulo U which acts by the diagonal multiplication, such that $a' - a \in \mathfrak{ah}$ and $b' - t \in \mathfrak{d}_F^{-1}[t_1]^{-1}\mathfrak{h}$. This series converges for $\mathrm{Re}(2s+k) > 2$ and can be continued to a holomorphic function in the whole plane if $n = [F:\mathbb{Q}] > 1$ ([Shi, p.656]). Then $E_k(\varphi,\psi)_1(z) = \lim_{s\to 0} E_k(\varphi,\psi)_1(z,s)$ is holomorphic in z if $n = [F:\mathbb{Q}] > 1$ ([Shi, p.656]).

We put $C = \triangle_F^{1/2}\Gamma(k)^n[\mathfrak{o}_F^\times : U]^{-1}N(\mathfrak{d}_F)^{-1}(-2\pi\sqrt{-1})^{-kn}$. For $z \in \mathfrak{H}^n$,

$$\begin{aligned}
&E_{k,U}(z,s;a,t;\mathfrak{a}\mathfrak{h},\mathfrak{d}_F^{-1}[t_1]^{-1}\mathfrak{h})|\alpha \\
&= E_{k,U}(\alpha z,s;a,t;\mathfrak{a}\mathfrak{h},\mathfrak{d}_F^{-1}[t_1]^{-1}\mathfrak{h})(yz+\delta)^{-k} \\
&= \triangle_F^{1/2}(-2\pi\sqrt{-1})^{-kn}\sum_{(a',b')U}(a'\alpha z+b')^{-k}(yz+\delta)^{-k}|a'\alpha z+b'|^{-2s} \\
&= \triangle_F^{1/2}(-2\pi\sqrt{-1})^{-kn}\sum_{(a',b')U}((a'x+b'y)z+(a'\beta+b'\delta))^{-k}|a'\alpha z+b'|^{-2s}
\end{aligned}$$

Then this series contributes to the constant term of $E_k(\varphi,\psi)_1|\alpha$ only when $a'x+b'y=0$.

(1) First suppose that $y \notin \mathfrak{b}$. Since $\mathfrak{d}_F[t_1]=\mathfrak{o}_F$ and $b'y=-a'x \in (y)\mathfrak{b}^{-1}\mathfrak{h}\cap\mathfrak{h}$, we see that $b' \in \mathfrak{h}$ and and hence $\mathrm{sgn}(-b')^r\psi^{-1}(-b'\mathfrak{b}\mathfrak{d}_F[t_1]\mathfrak{h}^{-1})=0$ if $\mathfrak{b}\neq 1$. Thus, the constant term $a_{x/y}(0,E_k(\varphi,\psi)_1)=0$ if $\mathfrak{b}\neq 1$.

In the case $\mathfrak{b}=1$, since $\mathfrak{d}_F[t_1]=\mathfrak{o}_F$, the constant term of $E_k(\varphi,\psi)_1|\alpha$ is equal to

(1.15)
$$C\cdot N([t_1])^{-k/2}\sum_{\mathfrak{h}\in \mathrm{Cl}_F}\sum_{a\in\mathfrak{h}/\mathfrak{a}\mathfrak{h}}\mathrm{sgn}(a)^q\varphi(a\mathfrak{h}^{-1})N(\mathfrak{h})^k\sum_{\substack{(a',b')U,(a',b')\neq(0,0)\\ a'-a\in\mathfrak{a}\mathfrak{h},b'\in\mathfrak{h},a'x+b'y=0}}(a'\beta+b'\delta)^{-k-2s}$$

at $s=0$. Suppose that $x\neq 0$. Using $a'x+b'y=0$ and $x\delta-\beta y=1$, we have $a'+b'y/x\in\mathfrak{a}\mathfrak{h}$ and $a'\beta+b'y=b'/x$. Thus the constant term of $E_k(\varphi,\psi)_1|\alpha$ is equal to

$$C\cdot N([t_1])^{-k/2}\sum_{\mathfrak{h}\in\mathrm{Cl}_F}\sum_{\substack{b'U\\ b'\in\mathfrak{h},b'\neq 0}}\mathrm{sgn}\left(-\frac{b'y}{x}\right)^q\varphi\left(-\frac{b'y}{x}\mathfrak{h}^{-1}\right)N(\mathfrak{h})^kN\left(\frac{b'}{x}\right)^{-k-2s} \tag{1.16}$$

at $s=0$. Since the map $(x^{-1}\mathfrak{h},b')\mapsto(b'/x)(x^{-1}\mathfrak{h})^{-1}\subset\mathfrak{o}_F$ from the set $\{(x^{-1}\mathfrak{h},b')\mid(\mathfrak{h},b')\text{ in }(1.16)\}$ to the set of non-zero integral ideals of F is a surjective $[\mathfrak{o}_F^\times:U]$-to-1 map, the value (1.16) is equal to

$$C\cdot N([t_1])^{-k/2}\mathrm{sgn}(-y)^q\varphi(-y)[\mathfrak{o}_F^\times:U]L(k,\varphi).$$

Therefore, using the functional equation for the Hecke L-functions (see, for example, [Mi, Theorem 3.3.1]), the constant term $a_{x/y}(0,E_k(\varphi,\psi)_1)$ is equal to

$$a_{x/y}(0,E_k(\varphi,\psi)_1)=\frac{N([t_1])^{k/2}}{2^n}\tau(\varphi)N(\mathfrak{m}_\varphi)^{-k}\mathrm{sgn}(-y)^q\varphi(-y)L(1-k,\varphi^{-1})$$

as desired.

Next suppose that $x=0$. Then $\beta y=1$ and (a',b') in (1.15) satisfies $b'=0$ and $a'\beta=-a'/y$ and hence the constant term of $E_k(\varphi,\psi)_1|\alpha$ is equal to

$$C\cdot N([t_1])^{-k/2}\sum_{\mathfrak{h}\in\mathrm{Cl}_F}\sum_{\substack{a'U\\ a'\in\mathfrak{h},a'\neq 0}}\mathrm{sgn}\left(a'\right)^q\varphi\left(a'\mathfrak{h}^{-1}\right)N(\mathfrak{h})^kN\left(-\frac{a'}{y}\right)^{-k-2s} \tag{1.17}$$

at $s=0$. Therefore, in the same way as above, our assertion follows from the map $(y^{-1}\mathfrak{h},a')\mapsto(a'/y)(y^{-1}\mathfrak{h})^{-1}\subset\mathfrak{o}_F$ from the set $\{(y^{-1}\mathfrak{h},a')\mid(\mathfrak{h},a')\text{ in }(1.17)\}$ to the set of non-zero integral ideals of F is a surjective $[\mathfrak{o}_F^\times:U]$-to-1 map and the functional equation for the Hecke L-functions:

$$a_{x/y}(0,E_k(\varphi,\psi)_1)=\frac{N([t_1])^{k/2}}{2^n}\tau(\varphi)N(\mathfrak{m}_\varphi)^{-k}\mathrm{sgn}(-y)^q\varphi(-y)L(1-k,\varphi^{-1}).$$

(2) Next suppose that $y \in \mathfrak{b}$. The constant term of $E_k(\varphi,\psi)_1|\alpha$ is equal to

$$C \cdot N([t_1])^{-k/2} N(\mathfrak{b})^{-1} \tau(\psi) \sum_{\mathfrak{h}\in \mathrm{Cl}_F} N(\mathfrak{h})^k \sum_{\substack{(a',b')U,(a',b')\neq(0,0)\\ a'\in\mathfrak{h},b'\in\mathfrak{b}^{-1}\mathfrak{h},a'x+b'y=0}} \times \mathrm{sgn}(a')^q \varphi(a'\mathfrak{h}^{-1}) \mathrm{sgn}(-b')^r \psi^{-1}(-b'\mathfrak{b}\mathfrak{h}^{-1})(a'\beta+b'\delta)^{-k-2s} \tag{1.18}$$

at $s=0$. We note that the map $(a',b') \mapsto a'\beta+b'\delta$ from the set $\{(a',b')$ in (1.18)$\}$ to $\mathfrak{b}^{-1}\mathfrak{h} - \{0\}$ is bijective. Indeed, for $(a'\beta+b'\delta) \in \mathfrak{b}^{-1}\mathfrak{h}$, we have $(a'\beta+b'\delta)x = b'$ and $-(a'\beta+b'\delta)y = a'$ since $a'x+b'y=0$ and $x\delta-\beta y=1$. Thus the constant term of $E_k(\varphi,\psi)_1|\alpha$ is equal to

$$C \cdot N([t_1])^{-k/2} N(\mathfrak{b})^{-1} \tau(\psi) \sum_{\mathfrak{h}\in \mathrm{Cl}_F} \sum_{\substack{dU\\ d\neq 0, d\in\mathfrak{b}^{-1}\mathfrak{h}}} \times \mathrm{sgn}(-dy)^q \varphi(-dy\mathfrak{h}^{-1}) \mathrm{sgn}(-dx)^r \psi^{-1}(-dx\mathfrak{b}\mathfrak{h}^{-1}) N(\mathfrak{h})^k N(d)^{-k-2s} \tag{1.19}$$

at $s=0$. Since the map $(\mathfrak{h},d) \mapsto d\mathfrak{b}\mathfrak{h}^{-1} \subset \mathfrak{o}_F$ from the set $\{(\mathfrak{h},d)$ in (1.19)$\}$ to the set of non-zero integral ideals of F is a surjective $[\mathfrak{o}_F^\times : U]$-to-1 map, the constant term (1.19) is equal to

$$\begin{aligned} &C \cdot N([t_1])^{-k/2} N(\mathfrak{b})^{-1} \mathrm{sgn}(-y)^q \varphi(-y) \mathrm{sgn}(-x)^r \psi^{-1}(-x) \\ &\times \varphi(\mathfrak{b}^{-1}) N(\mathfrak{b})^k [\mathfrak{o}_F^\times : U] L(k,\varphi\psi^{-1}) \prod_{\mathfrak{q}|\mathfrak{m}_\varphi\mathfrak{m}_\psi, \mathfrak{q}\nmid\mathfrak{m}_{\varphi\psi^{-1}}} (1-\varphi\psi^{-1}(\mathfrak{q})N(\mathfrak{q})^{-k}). \end{aligned}$$

Therefore, using the functional equation for the Hecke L-functions, we obtain that the constant term $a_{x/y}(0,E_k(\varphi,\psi)_1)$ is equal to

$$\begin{aligned} a_{x/y}(0,E_k(\varphi,\psi)_1) = &\frac{N([t_1])^{k/2}}{2^n} \frac{\tau(\varphi\psi^{-1})}{\tau(\psi^{-1})} \left(\frac{N(\mathfrak{m}_\psi)}{N(\mathfrak{m}_{\varphi\psi^{-1}})}\right)^k \mathrm{sgn}(-y)^q \varphi(-y\mathfrak{m}_\psi^{-1}) \mathrm{sgn}(-x)^r \psi^{-1}(-x) \\ &\times \left(\prod_{\mathfrak{q}|\mathfrak{m}_\varphi\mathfrak{m}_\psi, \mathfrak{q}\nmid\mathfrak{m}_{\varphi\psi^{-1}}} (1-\varphi\psi^{-1}(\mathfrak{q})N(\mathfrak{q})^{-k})\right) L(1-k,\varphi^{-1}\psi). \end{aligned}$$

as desired. □

1.4. **Geometric Hilbert modular variety.** We recall the algebraic Hilbert modular varieties and its toroidal compactifications. For more detail, refer to [Dim2], [Dim–Ti], and [Ti–Xi].

A Hilbert-Blumenthal abelian variety (HBAV for short) over a scheme S with respect to $\mathfrak{o}_F$ is a pair (A,ι) consisting of an abelian scheme $\pi : A \to S$ together with an embedding of algebras $\iota : \mathfrak{o}_F \hookrightarrow \mathrm{End}(A_{/S})$ such that $\pi_*(\Omega^1_{A/S}) \simeq \mathfrak{d}_F^{-1} \otimes \mathcal{O}_S$, that is, $\mathrm{Lie}(A)$ is locally free $(\mathfrak{o}_F \otimes \mathcal{O}_S)$-module of rank 1. We remark that if $A_{/S}$ is a HBAV, then its dual $A^\vee{}_{/S}$ has a natural structure of HBAV. We fix an ideal $\mathfrak{n}$ of $\mathfrak{o}_F$ and put $\triangle = N_{F/\mathbb{Q}}(\mathfrak{n}\mathfrak{d}_F)$. Let $\mu_\mathfrak{n}$ be the closed subscheme of $\mathbb{G}_m \otimes_\mathbb{Z} \mathfrak{d}_F^{-1}$ defined by $\mu_\mathfrak{n}(R) = \{x \in \mathbb{G}_m(R) \otimes_\mathbb{Z} \mathfrak{d}_F^{-1} \mid \mathfrak{n}x = 0\}$. Let $\mathfrak{c}$ be a fractional ideal of F and $\mathfrak{c}_+ = \mathfrak{c} \cap (F\otimes\mathbb{R})_+^\times$ the cone of totally positive elements in $\mathfrak{c}$. If $A_{/S}$ is a HBAV, the functor from the category of S-schemes to the category of sets $X \mapsto A(X) \otimes_{\mathfrak{o}_F} \mathfrak{c}$ is represented by an HBAV, denoted by $A \otimes_{\mathfrak{o}_F} \mathfrak{c}$. A $\mathfrak{c}$-polarization on a HBAV $A_{/S}$ is an $\mathfrak{o}_F$-linear isomorphism $\lambda : A \otimes_{\mathfrak{o}_F} \mathfrak{c} \xrightarrow{\simeq} A^\vee$ such that, under the isomorphism $\mathrm{Hom}_{\mathfrak{o}_F}(A, A^\vee) \simeq \mathrm{Hom}_{\mathfrak{o}_F}(A, A\otimes_{\mathfrak{o}_F}\mathfrak{c})$ given by $f \mapsto \lambda\circ f$, the symmetric elements of $\mathrm{Hom}_{\mathfrak{o}_F}(A, A^\vee)$ correspond precisely to $\mathfrak{c} \subset \mathrm{Hom}_{\mathfrak{o}_F}(A, A\otimes_{\mathfrak{o}_F}\mathfrak{c})$, and the symmetric polarizations correspond precisely to $\mathfrak{c}_+$. A $\mu_{\mathfrak{n}}$-level structure on a HBAV $A_{/S}$ is an $\mathfrak{o}_F$-linear closed immersion $\alpha : \mu_{\mathfrak{n}} \hookrightarrow A$ of group schemes over S.

We consider the contravariant functor $\mathcal{F}_{1,\mathfrak{c}}$ from the category of $\mathbb{Z}[1/\triangle]$-schemes to the category of sets:

$$\mathcal{F}_{1,\mathfrak{c}} : S \mapsto \{(A, \iota, \lambda, \alpha)\}_{/\simeq}, \tag{1.20}$$

where (A, ι) is a HBAV over S endowed with a $\mathfrak{c}$-polarizations λ and a $\mu_{\mathfrak{n}}$-level structure α and $\{*\}_{/\simeq}$ indicates the set of isomorphism classes of $*$.

Throughout the paper, we assume that

$$(\mathfrak{n}, 6\triangle_F) = 1. \tag{1.21}$$

Then $\Gamma_1(\mathfrak{c}, \mathfrak{n})$ as in (1.5) is torsion-free ([Dim–Ti, Lemma 1.4]) and the functor $\mathcal{F}_{1,\mathfrak{c}}$ is representable by a quasi-projective, smooth, geometrically connected $\mathbb{Z}[1/\triangle]$-scheme $M_{1,\mathfrak{c}} = M(\Gamma_1^1(\mathfrak{c}, \mathfrak{n}))$ of relative dimension $n = [F : \mathbb{Q}]$ ([Dim–Ti, Theorem 4.1]).

Let $\mathfrak{o}_{F,+}^\times \subset \mathfrak{o}_F^\times$ be the subgroup of totally positive units and $\mathfrak{o}_{F,\mathfrak{n}}^\times \subset \mathfrak{o}_F^\times$ the subgroup consisting of elements congruent to 1 modulo $\mathfrak{n}$. The finite group $\mathfrak{o}_{F,+}^\times/\mathfrak{o}_{F,\mathfrak{n}}^{\times 2}$ acts on $M_{1,\mathfrak{c}}$ by $[\varepsilon] \cdot (A, \iota, \lambda, \alpha) = (A, \iota, \iota(\varepsilon) \circ \lambda, \alpha)$ for $[\varepsilon] \in \mathfrak{o}_{F,+}^\times/\mathfrak{o}_{F,\mathfrak{n}}^{\times 2}$. We denoted by $M_{\mathfrak{c}} = M(\Gamma_1(\mathfrak{c}, \mathfrak{n}))$ the quotient of $M_{\mathfrak{c}}^1$ by $\mathfrak{o}_{F,+}^\times/\mathfrak{o}_{F,\mathfrak{n}}^{\times 2}$. It is a coarse moduli scheme of the contravariant functor $\mathcal{F}_{\mathfrak{c}}$ from the category of $\mathbb{Z}[1/\triangle]$-schemes to the category of sets:

$$\mathcal{F}_{\mathfrak{c}} : S \mapsto \{(A, \iota, [\lambda], \alpha)\}_{/\simeq}, \tag{1.22}$$

where (A, ι) is a HBAV over S endowed with an $\mathfrak{o}_{F,+}^\times$-orbit of $\mathfrak{c}$-polarizations $[\lambda]$ and a $\mu_{\mathfrak{n}}$-level structure α ([Dim–Ti, Corollary 4.2]). Also, $M_{\mathfrak{c}}$ is a quasi-projective, smooth, geometrically connected $\mathbb{Z}[1/\triangle]$-scheme of relative dimension $n = [F : \mathbb{Q}]$. We put

$$M_1 = \coprod_{i=1}^{h_F^+} M_{1,[t_i]}, \qquad M = \coprod_{i=1}^{h_F^+} M_{[t_i]},$$

where $\{[t_i]\}_{i=1}^{h_F^+}$ is a set of representatives of Cl_F^+ as (1.3).

Toroidai compactifications $M_{1,\mathfrak{c}}^{\mathrm{tor}}$ and $M_{\mathfrak{c}}^{\mathrm{tor}}$ of $M_{1,\mathfrak{c}}$ and $M_{\mathfrak{c}}$ are smooth and proper over $\mathbb{Z}[1/\triangle]$ and the boundaries $M_{1,\mathfrak{c}}^{\mathrm{tor}} - M_{\mathfrak{c}}^1$ and $M_{\mathfrak{c}}^{\mathrm{tor}} - M_{\mathfrak{c}}$ are relative simple normal crossing divisors of $M_{1,\mathfrak{c}}^{\mathrm{tor}}$ and $M_{\mathfrak{c}}^{\mathrm{tor}}$, respectively ([Dim, Theorem 7.2]). We put

$$M_1^{\mathrm{tor}} = \coprod_{i=1}^{h_F^+} M_{1,[t_i]}^{\mathrm{tor}}, \qquad M^{\mathrm{tor}} = \coprod_{i=1}^{h_F^+} M_{[t_i]}^{\mathrm{tor}}.$$

Let $\pi : \mathcal{A} \to M_{1,\mathfrak{c}}$ be the universal HBAV. There exists a semi-abelian scheme $\overline{\pi} : \mathcal{G} \to M_{1,\mathfrak{c}}^{\mathrm{tor}}$ extending $\pi : \mathcal{A} \to M_{1,\mathfrak{c}}$ such that a neighbourhood of the boundary corresponding to a cusp is the Tate semi-abelian scheme ([Dim–Ti, Theorem 6.4]). We have a vector bundle $\underline{\omega}_{\mathfrak{c}} = \overline{\pi}_* \Omega_{\mathcal{G}/M_{1,\mathfrak{c}}^{\mathrm{tor}}}$, which is a locally free $\mathcal{O}_{M_{1,\mathfrak{c}}^{\mathrm{tor}}} \otimes \mathfrak{o}_F$-module of rank 1.

Let $\widetilde{F}$ be the Galois closure of F in $\overline{\mathbb{Q}}$ and $\mathfrak{o}_{F'}$ the ring of integers of the number field $F' = \widetilde{F}(\varepsilon^{t/2}; \varepsilon \in \mathfrak{o}_{F,+}^\times)$. For a $\mathbb{Z}[1/\triangle]$-scheme S, we denote by $S_{\mathfrak{o}_{F'}} = S \times_{\mathbb{Z}[1/\triangle]} \mathfrak{o}_{F'}[1/\triangle]$ its base change to $\mathrm{Spec}(\mathfrak{o}_{F'}[1/\triangle])$. The finite group $\mathfrak{o}_{F,+}^\times/\mathfrak{o}_{F,\mathfrak{n}}^{\times 2}$ acts on $\underline{\omega}_{\mathfrak{c}}$ over $M_{1,\mathfrak{c},\mathfrak{o}_{F'}}^{\mathrm{tor}}$ via $[\varepsilon] \cdot s = \varepsilon^{-1/2}[\varepsilon]^* s$, where s denotes a local section of $\underline{\omega}_{\mathfrak{c}}$. Then $\underline{\omega}_{\mathfrak{c}}$ descends to a locally free $\mathcal{O}_{M_{\mathfrak{c},\mathfrak{o}_{F'}}^{\mathrm{tor}}} \otimes \mathfrak{o}_F$-module of rank 1.

We remark that if B is an $\mathfrak{o}_{F'}[1/\triangle_F]$-algebra, then we can decompose

$$B \otimes \mathfrak{o}_F = \bigoplus_{\sigma \in J_F} B$$

by the map $b \otimes a \mapsto (ba^\sigma)_{\sigma \in J_F}$. In particular, $\underline{\omega}$ decomposes into a direct sum of line bundles

$$\underline{\omega}_{\mathfrak{c}} = \bigoplus_{\sigma \in J_F} \underline{\omega}_{\mathfrak{c},\sigma}.$$

1.5. **Geometric Hilbert modular form and log de Rham cohomology.** We use the terminology of logarithmic structures in Kato [Kato2]. Let $\mathcal{Y}$ be a regular scheme and $\mathcal{D}$ a reduced divisor with normal crossings on $\mathcal{Y}$. Then the subsheaf L of monoids on $\mathcal{Y}_{\text{ét}}$ defind by

$$L(\mathcal{U})=\{g \in \mathcal{O}_{\mathcal{Y}}(\mathcal{U}) \mid g \text{ is invertible outside } \mathcal{D} \times_{\mathcal{Y}} \mathcal{U}\} \tag{1.23}$$

for each étale $\mathcal{Y}$-scheme $\mathcal{U}$ is a fine log structure ([Kato, (2.5)]).

Let $D = M^{\text{tor}}_{1,[t_i]} - M_{1,[t_i]}$ be the boundary. Then we define a log scheme $(M^{\text{tor}}_{1,[t_i]}, L)$ to be the scheme $M^{\text{tor}}_{1,[t_i]}$ endowed with the log structure $L=\{g \in \mathcal{O}_{M^{\text{tor}}_{1,[t_i]}} \mid g$ is invertible outside $D\}$. By [Dim–Ti, Theorem6.4], there is a toroidal compactification $\mathcal{A}^{\text{tor}}$ of the semi-abelian scheme $\mathcal{G}$ on $M_{1,[t_i]}$ such that $\mathcal{A}^{\text{tor}}$ is smooth and proper over $\mathbb{Z}[1/\triangle]$, $\overline{\pi} : \mathcal{A}^{\text{tor}} \to M^{\text{tor}}_{1,[t_i]}$ extending $\pi : \mathcal{A} \to M_{1,[t_i]}$ is semi-stable, and $\mathcal{A}^{\text{tor}} - \mathcal{A}$ is a relative normal crossing divisor above D. We define a log scheme $(\mathcal{A}^{\text{tor}}, L')$ to be the scheme $\mathcal{A}^{\text{tor}}$ endowed with the log structure $L'=\{g \in \mathcal{O}_{\mathcal{A}^{\text{tor}}} \mid g$ is invertible outside $\overline{\pi}^{-1}(D)\}$. Then the morphisms of log schemes $(\mathcal{A}^{\text{tor}}, L') \to (M^{\text{tor}}_{1,[t_i]}, L)$ and $(M^{\text{tor}}_{1,[t_i]}, L) \to (\mathrm{Spec}(\mathbb{Z}[1/\triangle]), \mathrm{triv})$ are log smooth ([Kato2, Theorem 3.5]) and hence both $\Omega^j_{\mathcal{A}^{\text{tor}}/M^{\text{tor}}_{1,[t_i]}}(\log(D)) = \Omega^j_{\mathcal{A}^{\text{tor}}/M^{\text{tor}}_{1,[t_i]}}(\log(L'/L))$ and $\Omega^j_{M^{\text{tor}}_{1,[t_i]}}(\log(D)) = \Omega^j_{M^{\text{tor}}_{1,[t_i]}}(\log(L))$ are locally free of finite type ([Kato2, Theorem 3.10]).

We fix an algebra $R_0 = \mathfrak{o}_{F'}[1/\triangle]$. For any $\mathbb{Z}[1/\triangle]$-algebra R and $\mathbb{Z}[1/\triangle]$-scheme $\mathcal{Y}$, we denote by $\mathcal{Y}_R$ its base change to $\mathrm{Spec}(R)$. Moreover, for any $\mathbb{Z}[1/\triangle]$-algebra R and $\mathbb{Z}[1/\triangle]$-log scheme $(\mathcal{Y}, L)$, we denote by $(\mathcal{Y}, L)_R$ its base change to $(\mathrm{Spec}(R), \mathrm{triv})$ with the trivial log structure. Let $\Omega^j_{\mathcal{A}^{\text{tor}}_R/M^{\text{tor}}_{1,[t_i],R}}(\log(D))$ (resp. $\Omega^j_{M^{\text{tor}}_{1,[t_i],R}/R}(\log(D))$) denote the differential module defined by the log smooth morphism $(\mathcal{A}^{\text{tor}}, L')_R \to (M^{\text{tor}}_{1,[t_i]}, L)_R$ (resp. $(M^{\text{tor}}_{1,[t_i]}, L)_R \to (\mathrm{Spec}(\mathbb{Z}[1/\triangle]), \mathrm{triv})_R$).

We define the de Rham cohomology sheaf on $M^{\text{tor}}_{1,[t_i],R}$ as

$$\mathcal{H}^1_{[t_i]} = R^1\overline{\pi}_*\Omega^\bullet_{\mathcal{A}^{\text{tor}}_R/M^{\text{tor}}_{1,[t_i],R}}(\log(D)).$$

Then, under the assumption (1.3), we have an exact sequence

$$0 \to \underline{\omega}_{[t_i]} \to \mathcal{H}^1_{[t_i]} \to \underline{\omega}^{-1}_{[t_i]} \otimes \mathfrak{d}^{-1}_F[t_i] \to 0 \tag{1.24}$$

([Dim2, §1.9]). This sequence (1.24) defines the Hodge filtration

$$\mathcal{H}^1 = F^0(\mathcal{H}^1) \supset F^1(\mathcal{H}^1) = \underline{\omega}_{[t_i]} \supset F^2(\mathcal{H}^1) = 0.$$

We have the canonical integrable connection

$$\nabla\colon \mathcal{H}^1_{[t_i]} \to \mathcal{H}^1_{[t_i]} \otimes_{\mathcal{O}_{M^{\text{tor}}_{1,[t_i],R}}} \Omega^1_{M^{\text{tor}}_{1,[t_i],R}/R}(\log(D)).$$

Then $\underline{\omega}_{[t_i]}$, $\mathcal{H}^1_{[t_i]}$, and ∇ descend to $M^{\mathrm{tor}}_{[t_i],R}$ ([Dim2, §1.9]) and hence we use the same notation. We define a complex of sheaves $\Omega^\bullet(\mathcal{H}^1_{[t_i]})$ as follows:

$$\Omega^\bullet(\mathcal{H}^1_{[t_i]}) := \mathcal{H}^1_{[t_i]} \otimes_{\mathcal{O}_{M^{\mathrm{tor}}_{1,[t_i],R}}} \Omega^\bullet_{M^{\mathrm{tor}}_{1,[t_i],R}/R}(\log(D)).$$

We define

$$H^m(M^{\mathrm{tor}}_{1,[t_i],R}, \mathcal{H}^1_{[t_i]}, \nabla) = H^m(M^{\mathrm{tor}}_{1,[t_i],R}, \Omega^\bullet(\mathcal{H}^1_{[t_i]}))$$

by the hyper cohomology of this complex. The Kodaira–Spencer map

$$\theta\colon \underline{\omega}_{[t_i]} \hookrightarrow \mathcal{H}^1_{[t_i]} \xrightarrow{\nabla} \mathcal{H}^1_{[t_i]} \otimes_{\mathcal{O}_{M^{\mathrm{tor}}_{1,[t_i],R}}} \Omega^1_{M^{\mathrm{tor}}_{1,[t_i],R}/R}(\log(D)) \to \underline{\omega}^{-1}_{[t_i]} \otimes_{\mathcal{O}_{M^{\mathrm{tor}}_{1,[t_i],R}}} \Omega^1_{M^{\mathrm{tor}}_{1,[t_i],R}/R}(\log(D)),$$

which is $\mathcal{O}_{M^{\mathrm{tor}}_{1,[t_i],R}}$-linear, induces an isomorphism

$$\Omega^1_{M^{\mathrm{tor}}_{1,[t_i],R}/R}(\log(D)) \simeq \underline{\omega}^2_{[t_i]} \simeq \bigoplus_{\sigma\in J_F} \underline{\omega}^{\otimes 2}_{[t_i],\sigma}. \tag{1.25}$$

For a weight $\kappa = (k-2t, m) \in \mathfrak{X}(T)$, we put

$$\underline{\omega}^\kappa_{[t_i]} = \bigotimes_{\sigma\in J_F} \left(\underline{\omega}^{k_\sigma-2}_{[t_i],\sigma} \otimes (\wedge\mathcal{H}^1_{[t_i],\sigma})^{m_\sigma}\right).$$

The coherent sheaves on $M_{1,[t_i]}$ above descend to $M_{[t_i]}$ and then we shall use the same convention as above on $M_{[t_i]}$.

Definition 1.3. ([Dim2, §1.5] and [Ti–Xi, §1.5]). Let R be an $\mathfrak{o}_{F'}[1/\triangle]$-algebra and $\kappa = (k-2t, m) \in \mathfrak{X}(T)$. We define the space of Hilbert modular forms of weight κ and level $\Gamma^1_1(\mathfrak{d}_F[t_i], \mathfrak{n})$ and $\Gamma_1(\mathfrak{d}_F[t_i], \mathfrak{n})$ with coefficients in R to be

$$M_\kappa(\Gamma^1_1(\mathfrak{d}_F[t_i], \mathfrak{n}), R) = H^0(M_{1,[t_i],R}, \underline{\omega}^\kappa_{[t_i]} \otimes \underline{\omega}^{2t}_{[t_i]}),$$
$$M_\kappa(\Gamma_1(\mathfrak{d}_F[t_i], \mathfrak{n}), R) = H^0(M_{[t_i],R}, \underline{\omega}^\kappa_{[t_i]} \otimes \underline{\omega}^{2t}_{[t_i]}),$$

respectively. If $F \neq \mathbb{Q}$, then, by the Koecher's principle, we have $M_\kappa(\Gamma_1(\mathfrak{d}_F[t_i], \mathfrak{n}), R) = H^0(M^{\mathrm{tor}}_{[t_i],R}, \underline{\omega}^\kappa_{[t_i]} \otimes \underline{\omega}^{2t}_{[t_i]})$. We define the subspace of Hilbert cusp forms as

$$S_\kappa(\Gamma^1_1(\mathfrak{d}_F[t_i], \mathfrak{n}), R) = H^0(M^{\mathrm{tor}}_{1,[t_i],R}, \underline{\omega}^\kappa_{[t_i]} \otimes \underline{\omega}^{2t}_{[t_i]}(-D)),$$
$$S_\kappa(\Gamma_1(\mathfrak{d}_F[t_i], \mathfrak{n}), R) = H^0(M^{\mathrm{tor}}_{[t_i],R}, \underline{\omega}^\kappa_{[t_i]} \otimes \underline{\omega}^{2t}_{[t_i]}(-D)),$$

respectively. We denote by

$$M_\kappa(M_1, R) = \bigoplus_{i=1}^{h_F^+} M_\kappa(\Gamma^1_1(\mathfrak{d}_F[t_i], \mathfrak{n}), R), \qquad M_\kappa(M, R) = \bigoplus_{i=1}^{h_F^+} M_\kappa(\Gamma_1(\mathfrak{d}_F[t_i], \mathfrak{n}), R),$$
$$S_\kappa(M_1, R) = \bigoplus_{i=1}^{h_F^+} S_\kappa(\Gamma^1_1(\mathfrak{d}_F[t_i], \mathfrak{n}), R), \qquad S_\kappa(M, R) = \bigoplus_{i=1}^{h_F^+} S_\kappa(\Gamma_1(\mathfrak{d}_F[t_i], \mathfrak{n}), R).$$

1.6. **Hecke operator on geometric modular variety and geometric modular form.** First we define the Hecke correspondence $T(\mathfrak{a})$ (if $\mathfrak{a}$ is prime to $\mathfrak{n}$) and $U(\mathfrak{a})$ (if $\mathfrak{a}$ is not prime to $\mathfrak{n}$) on the $M_1 = \prod_{i=1}^{h_F^+} M_{1,[t_i]}$ and $M_1^{\mathrm{tor}} = \prod_{i=1}^{h_F^+} M^{\mathrm{tor}}_{1,[t_i]}$.

Let $\mathfrak{a}$ be an integral ideal of F and fix a pair (i, j) such that $[t_i]\mathfrak{a} = [t_j]$ in Cl^+_F. We consider the functor $\mathcal{F}_{1,\mathfrak{a},i,j}$ from the category of $\mathbb{Z}[1/\triangle]$-schemes to the category of sets:

$$\mathcal{F}_{1,\mathfrak{a},i,j} : S \mapsto \{(A, \iota, \lambda, \alpha, C, \beta)\}_{/\simeq}, \tag{1.26}$$

where $(A, \iota, \lambda, \alpha)_{/S}$ is a $[t_i]$-polarized HBAV over S with $\mu_{\mathfrak{n}}$-level structure, $C \subset A[\mathfrak{a}]$ is an $\mathfrak{o}_F$-stable closed subscheme, which is disjoint from $\alpha(\mu_{\mathfrak{n}})$ and étale locally isomorphic to the constant group scheme $\mathfrak{o}_F/\mathfrak{a}$ over $\mathfrak{o}_F$, and β is an $\mathfrak{o}_{F,\mathfrak{n}}^{\times,2}$-orbit of isomorphisms $([t_i]\mathfrak{a}, ([t_i]\mathfrak{a})_+) \simeq ([t_j], [t_j]_+)$, where $\mathfrak{c}_+ = \mathfrak{c} \cap (F \otimes \mathbb{R})_+^\times$ is the totally positive cone for a fractional ideal $\mathfrak{c}$ of F.

Then we have a projection

$$\pi_1 \colon \mathcal{F}_{1,\mathfrak{a},i,j} \to \mathcal{F}_{1,[t_i]} : (A, \iota, \lambda, \alpha, C, \beta) \longmapsto (A, \iota, \lambda, \alpha).$$

Also we have a projection

$$\pi_2 \colon \mathcal{F}_{1,\mathfrak{a},i,j} \to \mathcal{F}_{1,[t_i]} : (A, \iota, \lambda, \alpha, C, \beta) \longmapsto (A/C, \iota', \lambda', \alpha'),$$

where ι' is the embbeging $\mathfrak{o}_F \hookrightarrow \mathrm{End}(A/C)$ naturally induced by ι and the projection $A \to A/C$, α is the composition of $\alpha : \mu_{\mathfrak{n}} \hookrightarrow A$ and the projection $A \to A/C$, and λ' is the $[t_j]$-polarization of A/C explicitly given by [Ki–La, §1.9].

The functor $\mathcal{F}_{1,\mathfrak{a},i,j}$ is representable by $M_{1,\mathfrak{a},i,j}$ explicitly constructed by [Ki–La, §1.9]. We put $M_{1,\mathfrak{a}} = \prod_{i=1}^{h_F^+} M_{1,\mathfrak{a},i,j}$. Then the two projections

$$M_{1,\mathfrak{a}} \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} M_1$$

induce algebraic correspondences $T(\mathfrak{a})$ and $U(\mathfrak{a})$ on M_1. We now define the Hecke correspondence $T(\mathfrak{a})$ and $U(\mathfrak{a})$ on M_1^{tor} as the closure of $T(\mathfrak{a})$ and $U(\mathfrak{a})$ in $M_1^{\mathrm{tor}} \times M_1^{\mathrm{tor}}$, repsectively.

According to [Dim2, §2.4] and [Ki–La, §1.11], we get $\pi_{1,*}\pi_2^*\underline{\omega}^\kappa \to \pi_{1,*}\pi_1^*\underline{\omega}^\kappa \to \underline{\omega}^\kappa$ and an action of $T(\mathfrak{a})$ and $U(\mathfrak{a})$ on the space of geometric modular forms $M_\kappa(M_1, R)$ and $S_\kappa(M_1, R)$. Moreover, we get an action of $T(\mathfrak{m})$ and $U(\mathfrak{m})$ on $M_\kappa(M, R)$ and $S_\kappa(M, R)$ by using the projection

$$\sum_{[\varepsilon] \in \mathfrak{o}_{F,+}^\times / \mathfrak{o}_{F,\mathfrak{n}}^{\times,2}} [\varepsilon] : M_\kappa(M_1, R) \to M_\kappa(M, R). \tag{1.27}$$

According to [Dim2, §2.4] and [Ki–La, §1.11.8], this Hecke action over $\mathbb{C}$ coincides with the usual Hecke operator as (1.7).

2. Integrality of n-cocycles

2.1. **Group cohomology.** To state our theorem, we need to recall some properties about group cohomology. Let Γ be a congruence subgroup of $G(\mathbb{Q}) = \mathrm{GL}_2(F)$ and $\overline{\Gamma} = \Gamma/(\Gamma \cap F^\times)$.

Definition 2.1. (The standard $R[\overline{\Gamma}]$-free resolution of R). Let R be a commutative ring and M a left $R[\overline{\Gamma}]$-module. We define $F_q = R[\overline{\Gamma}]^{\otimes(q+1)}$ and regard it as an $R[\overline{\Gamma}]$-module via the multiplication of $R[\overline{\Gamma}]$ on the first factor. Then F_q is a free $R[\overline{\Gamma}]$-module with a basis $\{[\overline{\gamma_1}, \cdots, \overline{\gamma_q}] = 1 \otimes \overline{\gamma_1} \otimes \cdots \otimes \overline{\gamma_q} \mid \overline{\gamma_i} \in \overline{\Gamma}\}$. We define the $R[\overline{\Gamma}]$-linear boundary map $\partial_q \colon F_q \to F_{q-1}$ by $\partial_1[\overline{\gamma}] = \overline{\gamma} - 1$ and

$$\partial_q[\overline{\gamma_1}, \cdots, \overline{\gamma_q}] = \overline{\gamma_1}[\overline{\gamma_2}, \cdots, \overline{\gamma_q}] + \sum_{j=1}^{q-1} (-1)^j [\overline{\gamma_1}, \cdots, \overline{\gamma_j\gamma_{j+1}}, \cdots, \overline{\gamma_q}] + (-1)^q [\overline{\gamma_1}, \cdots, \overline{\gamma_{q-1}}]$$

for $q > 1$. It is well known that (F_*, ∂_*) is a $R[\overline{\Gamma}]$-free resolution of R.

Let $C^q = C^q(\overline{\Gamma}, M)$ be the space of functions on $\overline{\Gamma}^q$ with values in M for $q \geq 1$ and M for $q = 0$. Note that $\mathrm{Hom}_{R[\overline{\Gamma}]}(F_q, M) \cong C^q$. Then the differential map $d^q \colon C^q \to C^{q+1}$ induced by ∂_* on F_* is given by $d^0 u(\overline{\gamma}) = (\overline{\gamma} - 1)u$ for $u \in M$ if $q = 0$, and if $q > 0$,

$$\begin{aligned} d^q u(\overline{\gamma_1}, \cdots, \overline{\gamma_{q+1}}) &= \overline{\gamma_1} u(\overline{\gamma_2}, \cdots, \overline{\gamma_{q+1}}) \\ &\quad + \sum_{j=1}^{q} (-1)^j u(\overline{\gamma_1}, \cdots, \overline{\gamma_j \gamma_{j+1}}, \cdots, \overline{\gamma_q}) + (-1)^{i+1} u(\overline{\gamma_1}, \cdots, \overline{\gamma_q}). \end{aligned}$$

The associated q-th cohomology group of $\overline{\Gamma}$ with coefficients in M is given by

$$H^q(\overline{\Gamma}, M) = Z^q(\overline{\Gamma}, M)/B^q(\overline{\Gamma}, M),$$

where

$$Z^q(\overline{\Gamma}, M) = \ker(d^q \colon C^q \to C^{q+1}) \qquad \text{and} \qquad B^q(\overline{\Gamma}, M) = \mathrm{im}(d^{q-1} \colon C^{q-1} \to C^q).$$

2.2. **Construction of n-cocycle.** In this subsection, we will construct an n-cocycle associated to a Hilbert modular form, which is a generalization of the Eichler–Shimura cocycles. This work explicitly gives the isomorphism between de Rham cohomology group and group cohomology (cf. [Be Ph.D., Proposition 2.5]). In order to do it, we strictly follow the arguments in the method of Yoshida in [Yo]. We put $J_F = \{\sigma_1, \cdots, \sigma_n\}$. For each subset $J \subset J_F$, we put

$$dz_{J} = \bigwedge_{i=1}^{n} dz_{\sigma_i}^J, \tag{2.1}$$

where z_σ^J is defined by (1.1).

Hereafter, we assume that $k_i \geq 2$ and $k - 2t + 2m \in 2\mathbb{Z} \cdot t$ for $\kappa = (k - 2t, m) \in \mathfrak{X}(T)$ and $k = \sum_{i=1}^n k_i \sigma_i \in \mathbb{Z}[J_F]$. For any $\mathbb{Z}$-algebra A, a non-negative integer $\ell \in \mathbb{Z}_{\geq 0}$, and $\begin{pmatrix} u \\ v \end{pmatrix} \in A^2$, we put

$$\begin{bmatrix} u \\ v \end{bmatrix}^\ell = {}^t(u^\ell, u^{\ell-1}v, \cdots, uv^{\ell-1}, v^\ell).$$

We consider the column vector space $L_\ell(A) \simeq A^{\ell+1} \simeq \mathrm{Sym}^\ell(A^2)$. For any $\mathbb{Z}$-algebra A, we define the ℓ-th symmetric tensor representation ρ_ℓ of $\mathrm{GL}_2(A)$ on $L_\ell(A) \simeq \mathrm{Sym}^\ell(A^2)$ by

$$\rho_\ell(g) \begin{bmatrix} u \\ v \end{bmatrix}^\ell = \left[g \begin{pmatrix} u \\ v \end{pmatrix} \right]^\ell.$$

Let $L_{k-2}(A) = \otimes_{i=1}^n L_{k_i - 2}(A)$ on which $\mathrm{GL}_2(A)$ acts via the representation $\rho = \rho_{k_1-2} \otimes \cdots \otimes \rho_{k_n-2}$.

Recall that $\widetilde{F}$ is the Galois closure of F in $\overline{\mathbb{Q}}$ and $F' = \widetilde{F}(\varepsilon^{t/2} : \varepsilon \in \mathfrak{o}_{F,+}^\times)$. Put $m_\kappa = k - t + m$. For an $\mathfrak{o}_{F'}$-algebra A containing the values of u^{m_κ} for all $u \in \mathfrak{o}_{F'} \cap (F')^\times$, we define the $A[(\mathrm{M}_2(\mathfrak{o}_{F'}) \cap \mathrm{GL}_2(F'))^{J_F}]$-module $L_\kappa(A)$ as follows: let $L_\kappa(A)$ be the A-module $L_{k-2}(A)$ with a left action by

$$g \bullet P = \det(g)^{-m_\kappa + t} \rho(g) P$$

for $g \in (\mathrm{M}_2(\mathfrak{o}_{F'}) \cap \mathrm{GL}_2(F'))^{J_F}$ and $P \in L_{k-2}(A)$. In particular, $G(\mathbb{A}_f)$ naturally acts on $L_\kappa(A \otimes_{\mathfrak{o}_{F'}} \mathbb{A}_{F'})$. For $\widehat{\mathfrak{o}}_{F'} = \mathfrak{o}_{F'} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and each i with $1 \leq i \leq h_F^+$, we consider the i-part $L_{\kappa,i}(A)$ of $L_\kappa(A)$ defined by

$$L_{\kappa,i}(A) = L_\kappa(A \otimes_{\mathfrak{o}_{F'}} F') \cap x_i \bullet L_\kappa(A \otimes_{\mathfrak{o}_{F'}} \widehat{\mathfrak{o}}_{F'}).$$

Then the semigroup $R_{ii}(\mathfrak{n})$ as §1.1 acts on $L_{\kappa,i}(A)$.

From now on, in this subsection, we fix i with $1 \leq i \leq h_F^+$ and abbreviate $\Gamma_{1,i}(\mathfrak{n})$ to Γ and $L_{\kappa,i}(A)$ to $L_\kappa(A)$.

We define a $L_\kappa(\mathbb{C})$-valued holomorphic n-form $\omega(h)$ on $\mathfrak{H}^n$ attached to a holomorphic function h on $\mathfrak{H}^n$ by

$$\omega(h) = h(z) \begin{bmatrix} z_1 \\ 1 \end{bmatrix}^{k_1-2} \otimes \cdots \otimes \begin{bmatrix} z_n \\ 1 \end{bmatrix}^{k_n-2} dz_{J_F}. \tag{2.2}$$

If $h \in M_\kappa(\Gamma, \mathbb{C})$, then, by definition of the slash operator, for $g \in \mathrm{GL}_2(\mathbb{R})_+^n$, we have

$$(h|g)(z) = \det(g)^{m_\kappa} j(g,z)^{-k} h(gz).$$

We remark that

$$\begin{bmatrix} g_i z_i \\ 1 \end{bmatrix}^{k_i-2} = j(g_i, z_i)^{-k_i+2} \rho_{k_i-2}(g_i) \begin{bmatrix} z_i \\ 1 \end{bmatrix}^{k_i-2}.$$

Then we get

$$\begin{aligned} g^*\omega(h) &= h(gz) \begin{bmatrix} g_1 z_1 \\ 1 \end{bmatrix}^{k_1-2} \otimes \cdots \otimes \begin{bmatrix} g_n z_n \\ 1 \end{bmatrix}^{k_n-2} dg_1 z_1 \wedge \cdots \wedge dg_n z_n \\ &= \det(g)^{-m_\kappa+t} \rho(g)(h|g)(z) \begin{bmatrix} z_1 \\ 1 \end{bmatrix}^{k_1-2} \otimes \cdots \otimes \begin{bmatrix} z_n \\ 1 \end{bmatrix}^{k_n-2} dz_{J_F}. \end{aligned}$$

Under the condition $k - 2t + 2m \in 2\mathbb{Z} \cdot t$, the center $\Gamma \cap F^\times$ of Γ acts trivially on $L_\kappa(\mathbb{C})$. Then we obtain the pull-back formula

$$\gamma^*\omega(h) = \gamma \bullet \omega(h) \tag{2.3}$$

for any $\overline{\gamma} \in \overline{\Gamma} = \Gamma/(\Gamma \cap F^\times)$ and a lift $\gamma \in \Gamma$ of $\overline{\gamma}$.

Fix a base point $w = (w_1, \cdots, w_n) \in \mathfrak{H}^{J_F}$. We define a $L_\kappa(\mathbb{C})$-valued holomorphic function as

$$F(z) = \int_{\omega_1}^{z_1} \cdots \int_{\omega_n}^{z_n} \omega(h). \tag{2.4}$$

Moreover, we put $\overline{\gamma} * F(z) = \gamma \bullet F(\gamma^{-1}z)$ for each $\overline{\gamma} \in \overline{\Gamma}$. We remark that

$$\frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_n} (\overline{\gamma} * F - F)(z) = 0.$$

Lemma 2.2. ([Yo, Chapter V, Lemma 5.1]). *Let $D \subset \mathbb{C}^n$ be an open domain and contractible. Let f be a holomorphic function on D.*

(1) *Assume that*

$$\frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_n} f(z) = 0.$$

Then there exist holomorphic functions $g_i(z)$ on D such that $g_i(z)$ is independent of z_i and f is decomposed into

$$f(z) = \sum_{i=1}^n g_i(z).$$

(2) *Moreover, assume that* $n \geq 2$ *and* $f(z) = \sum_{i=1}^{n} g_i(z)$ *is a decomposition as* (1). *If* $f(z)$ *is independent of* z_1, *then there exist holomorphic functions* $h_i(z)$ *on* D *such that* $h_i(z)$ *is independent of* z_1 *and* z_i *and* f *is decomposed into*

$$f(z) = \sum_{i=2}^{n} h_i(z).$$

Remark 2.3. This decomposition is not unique in general.

Then, by applying Lemma 2.2 (1) to $(-1)(\overline{\gamma} * F - F)$, we obtain a decomposition

$$(-1)(\overline{\gamma} * F - F)(z) = \sum_{i=1}^{n} g_i^{(1)}(\overline{\gamma})(z), \tag{2.5}$$

where, for each i, $g_i^{(1)}(\overline{\gamma})(z)$ is a holomorphic function on $\mathfrak{H}^n$ and independent of z_i. We can explicitly describe $g_n^{(1)}(\overline{\gamma})(z)$ as follows. We have

$$\begin{aligned}
&(\overline{\gamma} * F(z) - F)(z) \\
&= \int_{\gamma\omega_1}^{z_1} \cdots \int_{\gamma\omega_{n-1}}^{z_{n-1}} \left(\int_{\omega_n}^{z_n} + \int_{\gamma\omega_n}^{\omega_n} \right) \omega(h) - \int_{\omega_1}^{z_1} \cdots \int_{\omega_n}^{z_n} \omega(h) \\
&= \int_{\gamma\omega_1}^{z_1} \cdots \int_{\gamma\omega_{n-1}}^{z_{n-1}} \int_{\gamma\omega_n}^{\omega_n} \omega(h) + \left(\int_{\gamma\omega_1}^{z_1} \cdots \int_{\gamma\omega_{n-1}}^{z_{n-1}} - \int_{\omega_1}^{z_1} \cdots \int_{\omega_{n-1}}^{z_{n-1}} \right) \int_{\omega_n}^{z_n} \omega(h).
\end{aligned} \tag{2.6}$$

By applying Lemma 2.2 (1) to the second term of (2.6), we can choose $-g_n^{(1)}(\overline{\gamma})(z)$ as the first term of (2.6):

$$g_n^{(1)}(\overline{\gamma})(z) = \int_{\gamma\omega_1}^{z_1} \cdots \int_{\gamma\omega_{n-1}}^{z_{n-1}} \int_{\omega_n}^{\gamma\omega_n} \omega(h). \tag{2.7}$$

By regarding (2.5) as a 1-cochain in $C^1(\overline{\Gamma}, L_\kappa(\mathbb{C}))$, we obtain

$$dg_n^{(1)}(\overline{\gamma_1}, \overline{\gamma_2})(z) = -\sum_{i=1}^{n-1} dg_i^{(1)}(\overline{\gamma_1}, \overline{\gamma_2})(z)$$

for $\overline{\gamma_1}, \overline{\gamma_2} \in \overline{\Gamma}$, where d is the boundary map in group cohomology. The left hand side is independent of z_n and each $dg_i^{(1)}(\overline{\gamma_1}, \overline{\gamma_2})(z)$ with $1 \leq i \leq n-1$ is independent of z_i. Thus, by Lemma 2.2 (2), we can decompose

$$(-1)^2 dg_n^{(1)}(\overline{\gamma_1}, \overline{\gamma_2})(z) = \sum_{i=1}^{n-1} g_i^{(2)}(\overline{\gamma_1}, \overline{\gamma_2})(z), \tag{2.8}$$

where, for each i, $g_i^{(2)}(\overline{\gamma_1}, \overline{\gamma_2})(z)$ is a holomorphic function and independent of z_i. Similar to (2.6), we explicitly give $g_{n-1}^{(2)}(\overline{\gamma_1}, \overline{\gamma_2})(z)$ as follows. A direct calculation shows

$$
\begin{aligned}
(2.9)\qquad & dg_n^{(1)}(\overline{\gamma_1}, \overline{\gamma_2})(z) = \overline{\gamma_1} * g_n^{(1)}(\overline{\gamma_2})(z) - g_n^{(1)}(\overline{\gamma_1\gamma_2})(z) + g_n^{(1)}(\overline{\gamma_1})(z) \\
&= \int_{\gamma_1\gamma_2\omega_1}^{z_1} \cdots \int_{\gamma_1\gamma_2\omega_{n-1}}^{z_{n-1}} \int_{\gamma_1\omega_n}^{\gamma_1\gamma_2\omega_n} \omega(h) \\
&\quad - \int_{\gamma_1\gamma_2\omega_1}^{z_1} \cdots \int_{\gamma_1\gamma_2\omega_{n-1}}^{z_{n-1}} \int_{\omega_n}^{\gamma_1\gamma_2\omega_n} \omega(h) + \int_{\gamma_1\omega_1}^{z_1} \cdots \int_{\gamma_1\omega_{n-1}}^{z_{n-1}} \int_{\omega_n}^{\gamma_1\omega_n} \omega(h) \\
&= \int_{\gamma_1\gamma_2\omega_1}^{z_1} \cdots \int_{\gamma_1\gamma_2\omega_{n-1}}^{z_{n-1}} \int_{\gamma_1\omega_n}^{\omega_n} \omega(h) + \int_{\gamma_1\omega_1}^{z_1} \cdots \int_{\gamma_1\omega_{n-1}}^{z_{n-1}} \int_{\omega_n}^{\gamma_1\omega_n} \omega(h) \\
&= \int_{\gamma_1\gamma_2\omega_1}^{z_1} \cdots \int_{\gamma_1\gamma_2\omega_{n-2}}^{z_{n-2}} \int_{\gamma_1\gamma_2\omega_{n-1}}^{\gamma_1\omega_{n-1}} \int_{\gamma_1\omega_n}^{\omega_n} \omega(h) \\
&\quad + \left(\int_{\gamma_1\gamma_2\omega_1}^{z_1} \cdots \int_{\gamma_1\gamma_2\omega_{n-2}}^{z_{n-2}} - \int_{\gamma_1\omega_1}^{z_1} \cdots \int_{\gamma_1\omega_{n-2}}^{z_{n-2}} \right) \times \left(\int_{\gamma_1\omega_{n-1}}^{z_{n-1}} \int_{\gamma_1\omega_n}^{\omega_n} \right) \omega(h).
\end{aligned}
$$

Similar to (2.7), by using Lemma 2.2 (1), we can choose as

$$
(2.10)\qquad g_{n-1}^{(2)}(\overline{\gamma_1}, \overline{\gamma_2})(z) = \int_{\gamma_1\gamma_2\omega_1}^{z_1} \cdots \int_{\gamma_1\gamma_2\omega_{n-2}}^{z_{n-2}} \int_{\gamma_1\omega_{n-1}}^{\gamma_1\gamma_2\omega_{n-1}} \int_{\omega_n}^{\gamma_1\omega_n} \omega(h).
$$

By repeating this arguments, for $1 \leq m \leq n-1$ and $\overline{\gamma_1}, \cdots \overline{\gamma_m} \in \overline{\Gamma}$, we get

$$
(2.11)\qquad (-1)^m dg_{n-m+2}^{(m-1)}(\overline{\gamma_1}, \cdots, \overline{\gamma_m})(z) = \sum_{i=1}^{n-m+1} g_i^{(m)}(\overline{\gamma_1}, \cdots, \overline{\gamma_m})(z),
$$

with

$$
\begin{aligned}
& g_{n-m+1}^{(m)}(\overline{\gamma_1}, \cdots, \overline{\gamma_m})(z) \\
&= \int_{\gamma_1\cdots\gamma_m\omega_1}^{z_1} \cdots \int_{\gamma_1\cdots\gamma_m\omega_{n-m}}^{z_{n-m}} \int_{\gamma_1\cdots\gamma_{m-1}\omega_{n-m+1}}^{\gamma_1\cdots\gamma_m\omega_{n-m+1}} \cdots \int_{\gamma_1\omega_{n-1}}^{\gamma_1\gamma_2\omega_{n-1}} \int_{\omega_n}^{\gamma_1\omega_n} \omega(h).
\end{aligned}
$$

Thus, we obtain a n-cocycle $dg_2^{(n-1)}(\overline{\gamma_1}, \cdots, \overline{\gamma_n})(z)$ because it is a constant function. We have

$$
dg_2^{(n-1)}(\overline{\gamma_1}, \cdots, \overline{\gamma_n})(z) = \int_{\gamma_1\cdots\gamma_{n-1}\omega_1}^{\gamma_1\cdots\gamma_n\omega_1} \cdots \int_{\gamma_1\omega_{n-1}}^{\gamma_1\gamma_2\omega_{n-1}} \int_{\omega_n}^{\gamma_1\omega_n} \omega(h).
$$

Therefore we obtain (1) of the following theorem.

Proposition-Definition 2.4. *Let $h \in M_\kappa(\Gamma, \mathbb{C})$ and $w = (w_1, \cdots, w_n) \in \mathfrak{H}^n$ a base point. Assume that $k - 2t + 2m \in 2\mathbb{Z} \cdot t$ and $k_\sigma \geq 2$ for each $\sigma \in J_F$.*

(1) *For $\overline{\gamma_i} \in \overline{\Gamma}$ and a lift $\gamma_i \in \Gamma$ of $\overline{\gamma_i}$ with $1 \leq i \leq n$, a map*

$$
\pi_{h,\omega} \colon \overline{\Gamma}^n \longrightarrow L_\kappa(\mathbb{C})
$$

defined by

$$
\pi_{h,\omega}(\overline{\gamma_1}, \cdots, \overline{\gamma_n}) = \int_{\gamma_1\cdots\gamma_{n-1}\omega_1}^{\gamma_1\cdots\gamma_n\omega_1} \cdots \int_{\gamma_1\omega_{n-1}}^{\gamma_1\gamma_2\omega_{n-1}} \int_{\omega_n}^{\gamma_1\omega_n} \omega(h)
$$

is an n-cocycle.

(2) *The cohomology class $[\pi_h] = [\pi_{h,\omega}] \in H^n(\overline{\Gamma}, L_\kappa(\mathbb{C}))$ does not depend on the choice of the base point $\omega = (\omega_i)_i \in \mathfrak{H}^{J_F}$.*

Proof. The assertion (2) is proved by [Yo, Theorem 5.2]. □

Remark 2.5. If $n = [F : \mathbb{Q}] = 1$ and h is a cusp form, then $\pi_{h,\omega}$ is the as usual Eichler–Shimura cocycle.

2.3. **Hecke operators on group cohomology.** In this subsection, we will prove that the map from the space of modular forms to the group cohomology

$$M_{\kappa,J_F}(\Gamma,\mathbb{C}) \to H^n(\overline{\Gamma}, L_\kappa(\mathbb{C})) : h \mapsto [\pi_h]$$

is compatible with the action of Hecke operators.

In this subsection, we fix i with $1 \leq i \leq h_F^+$ and abbreviate $\Gamma_{1,i}(\mathfrak{n})$ to Γ and $L_{\kappa,i}(A)$ to $L_\kappa(A)$. We recall the definitions of the Hecke operators on the space of modular forms and the group cohomology. Assume that, for $\alpha \in G(\mathbb{Q})$, we have the decomposition $\Gamma\alpha\Gamma = \coprod_{i\in I} \Gamma\alpha_i$ as a finite disjoint union. For each $h \in M_\kappa(\Gamma,\mathbb{C})$, we define the Hecke operator

$$h|[\Gamma\alpha\Gamma] = \sum_{i\in I} h|_\kappa \alpha_i.$$

For each cusp $s \in \mathbb{P}^1(F)$, we write $\overline{\Gamma_s}$ for the stabilizer of s in $\overline{\Gamma}$. Let $C(\Gamma)$ be a set of representatives for Γ-equivalence classes of cusps, which is a finite set. Then we note that for each cusp s, we can find $\gamma \in \Gamma$ and $s_0 \in C(\Gamma)$ such that $\gamma s = s_0$. The q-th parabolic cohomology group $H^q_{\mathrm{par}}(\overline{\Gamma}, M)$ of $\overline{\Gamma}$ with coefficients in a $\overline{\Gamma}$-module M is defined by the exact sequence

$$0 \to H^q_{\mathrm{par}}(\overline{\Gamma}, M) \to H^q(\overline{\Gamma}, M) \to \bigoplus_{s\in C(\Gamma)} H^q(\overline{\Gamma_s}, M). \tag{2.12}$$

Fix a cusp $s \in C(\Gamma)$. We decompose

$$\Gamma\alpha\Gamma = \coprod_{i\in I^s} \Gamma\beta_i^s\Gamma_s \text{ and } \Gamma\beta_i^s\Gamma_s = \coprod_{j\in J_i^s} \Gamma\beta_i^s\delta_{i,j}^s \text{ with } \delta_{i,j}^s \in \Gamma_s$$

as a finite disjoint union. By [Hida93, Lemma 3.3], we have a decomposition

$$\Gamma_{\beta_i^s(s)}\beta_i^s\Gamma_s = \coprod_{j\in J_i^s} \Gamma_{\beta_i^s(s)}\beta_i^s\delta_{i,j}^s.$$

First we define the Hecke operator $[\Gamma\alpha\Gamma]$ on $H^q(\overline{\Gamma}, M)$ as follows (cf. [Hida93, p.288, 289] or [Yo2, §1]). For each $\overline{\gamma} \in \overline{\Gamma}$, fix a lift $\gamma \in \Gamma$ of $\overline{\gamma}$. For each i, j, let $\alpha_{i,j} = \beta_i^s\delta_{i,j}^s$, $\gamma^{(i,j)} \in \Gamma$, and $\gamma(i,j) = (\gamma(i), \gamma(j)') \in \mathbb{Z} \times \mathbb{Z}$ such that

$$\alpha_{i,j}\gamma = \gamma^{(i,j)}\alpha_{\gamma(i,j)}.$$

For each cocycle u, we define

$$(u|\Gamma\alpha\Gamma])(\overline{\gamma}_1, \cdots, \overline{\gamma}_q) = \sum_{i\in I^s, j\in J_i^s} (\beta_i^s\delta_{i,j}^s)^{-1} u\left(\overline{\gamma_1^{(i,j)}}, \cdots, \overline{\gamma_q^{\gamma_{q-1}\circ\cdots\circ\gamma_1(i,j)}}\right).$$

Since $\partial(u|\Gamma\alpha\Gamma]) = \partial(u)|\Gamma\alpha\Gamma]$, it is well-defined.

Next we define an action of the Hecke algebra on the boundary cohomology (cf. [Hida93, p.288, 289]). For $c = (c_t)_{t\in C(\Gamma)} \in \bigoplus_{t\in C(\Gamma)} H^i(\overline{\Gamma_t}, M)$, we define

$$(c|[\Gamma\alpha\Gamma])_s = \sum_{i\in I^s} c_{\beta_i^s(s)}|[\Gamma_{\beta_i^s(s)}\beta_i^s\Gamma_s].$$

As in the proof of [Hida86, Proposition 4.2], its definition is independent of the choice of β_i^s and, via this action, the boundary cohomology $\bigoplus_{t\in C(\Gamma)} H^i(\overline{\Gamma_t}, M)$ becomes a Hecke module.

Proposition 2.6. *The sequence (2.12) is an exact sequence of Hecke modules.*

Proof. For each cocycle u, it suffices to check that

$$\mathrm{res}(u|[\Gamma\alpha\Gamma])_s = (\mathrm{res}(u)|[\Gamma\alpha\Gamma])_s$$

for each cusp $s \in C(\Gamma)$. Suppose that $\overline{\gamma}_k \in \overline{\Gamma_s}$ for all k. Then $\gamma_1^{(i,j)}\beta^s_{\gamma_1(i)}\delta^s_{\gamma_1(i),\gamma_1(j)'} = \beta^s_i\delta^s_{i,j}\gamma_1 \in \Gamma\beta^s_i\Gamma_s$ and hence $\beta^s_{\gamma_1(i)} = \beta^s_i$, $\gamma_1(i) = i$, and $\gamma_1^{(i,j)} = \beta^s_i\delta^s_{i,j}\gamma_1{\delta^s_{i,\gamma_1(j)'}}^{-1}{\beta^s_i}^{-1} \in \Gamma_{\beta^s_i(s)}$. Moreover, we have $\gamma_2^{\gamma_1(i,j)}\beta^s_{\gamma_2(i)}\delta^s_{\gamma_2(i),\gamma_2\circ\gamma_1(j)'} = \beta^s_i\delta^s_{i,\gamma_1(j)'}\gamma_2 \in \Gamma\beta^s_i\Gamma_s$ and hence $\beta^s_{\gamma_2(i)} = \beta^s_i$, $\gamma_2(i) = i$, and $\gamma_2^{\gamma_1(i,j)} \in \Gamma_{\beta^s_i(s)}$. Repeating this arguments, we get

$$\begin{aligned}
\mathrm{res}(u|[\Gamma\alpha\Gamma])_s(\overline{\gamma}_1,\cdots,\overline{\gamma}_q) &= \sum_{i,j}(\beta^s_i\delta^s_j)^{-1}\mathrm{res}(u)\left(\overline{\gamma_1^{(i,j)}},\cdots,\overline{\gamma_q^{\gamma_{q-1}\circ\cdots\circ\gamma_1(i,j)}}\right)\\
&= \sum_i \mathrm{res}(u)_{\beta^s_i(s)}[\Gamma_{\beta^s_i(s)}\beta^s_i\Gamma_s]\\
&= (\mathrm{res}(u)|[\Gamma\alpha\Gamma])_s(\overline{\gamma}_1,\cdots,\overline{\gamma}_n)
\end{aligned}$$

as desired. □

The following proposition is the main result in this subsection.

Proposition 2.7. *Assume that $\alpha \in G(\mathbb{Q})$. We fix $\Gamma\alpha\Gamma = \coprod_{i\in I}\Gamma\alpha_i$. For each α_j and $\overline{\gamma} \in \overline{\Gamma}$ with a lift $\gamma \in \Gamma$ of $\overline{\gamma}$, let $\overline{\gamma^{(j)}} \in \overline{\Gamma}$ and $\gamma(j) \in \mathbb{Z}$ such that*

$$\alpha_j\gamma = \gamma^{(j)}\alpha_{\gamma(j)}.$$

Let $h \in M_\kappa(\Gamma,\mathbb{C})$ and $\omega = (\omega_1,\cdots,\omega_n) \in \mathfrak{H}^n$ a base point. Then, for $\overline{\gamma}_1,\cdots,\overline{\gamma}_n \in \overline{\Gamma}$, we have

$$\pi_{h|[\Gamma\alpha\Gamma],\omega}(\overline{\gamma}_1,\cdots,\overline{\gamma}_n) = \sum_{i\in I}\alpha_i^{-1}\bullet\pi_{h,\omega}\left(\overline{\gamma_1^{(i)}},\cdots,\overline{\gamma_n^{(\gamma_{n-1}\circ\cdots\circ\gamma_1(i))}}\right).$$

In particular,

$$\pi_{h|[\Gamma\alpha\Gamma],\omega} = \pi_{h,\omega}|[\Gamma\alpha\Gamma].$$

Proof. By using the pull-back formula, we have

$$\begin{aligned}
\omega(h|[\Gamma\alpha\Gamma]) &= \sum_{i\in I}\omega(h|_\kappa\alpha_i)\\
&= \sum_{i\in I}\alpha_i^{-1}\bullet\alpha_i^*\omega(h).
\end{aligned}$$

Then we have

$$\begin{aligned}
F_{h|[\Gamma\alpha\Gamma]}(z) &= \int_{\omega_1}^{z_1}\cdots\int_{\omega_n}^{z_n}\omega(h|[\Gamma\alpha\Gamma])\\
&= \sum_{i\in I}\alpha_i^{-1}\bullet\int_{\alpha_i\omega_1}^{\alpha_i z_1}\cdots\int_{\alpha_i\omega_n}^{\alpha_i z_n}\omega(h).
\end{aligned}$$

For $\overline{\gamma} \in \overline{\Gamma}$, similar to (2.7), we shall explicitly give a decomposition

$$(-1)(\overline{\gamma} * F_{h|[\Gamma\alpha\Gamma]} - F_{h|[\Gamma\alpha\Gamma]})(z) = \sum_{j=1}^n g^{(1)}_{j,\alpha}(\overline{\gamma})(z),$$

where, for each j, $g^{(1)}_{j,\alpha}(\overline{\gamma})(z)$ is independent of z_j. We put

$$F_i(z) = \int_{\alpha_i\omega_1}^{z_1} \cdots \int_{\alpha_i\omega_n}^{z_n} \omega(h).$$

Then we have

$$(\overline{\gamma} * F_{h|[\Gamma\alpha\Gamma]} - F_{h|[\Gamma\alpha\Gamma]})(z) = \sum_{i\in I} \left\{\gamma\alpha_i^{-1} \bullet F_i(\alpha_i\gamma^{-1}z) - \alpha_i^{-1} \bullet F_i(\alpha_i z)\right\}.$$

For the moment we admit the following decomposition:

$$\sum_{i\in I} \left\{\gamma\alpha_i^{-1} \bullet F(\alpha_i\gamma^{-1}z) - \alpha_i^{-1} \bullet F(\alpha_i z)\right\} = \sum_{j=1}^{n} g^{(1),*}_{j,\alpha}(\overline{\gamma})(z), \tag{2.13}$$

where the holomorphic function $F(z)$ is defined by (2.4) and, for each j, $g^{(1),*}_{j,\alpha}(\gamma)(z)$ is independent of z_j. We remark that there is the canonical decomposition

$$F_{h|[\Gamma\alpha\Gamma]}(z) = \sum_{i\in I} \alpha_i^{-1} \bullet F(\alpha_i z) + \sum_{i\in I}\sum_{j=1}^{n} \alpha_i^{-1} \bullet F^{(1)}_{j,\alpha}(z),$$

where, for each j, $F^{(1)}_{j,\alpha}(z)$ is independent of z_j. Thus, by combining these, we can choose $g^{(1)}_{n,\alpha}(\overline{\gamma})(z)$ as

$$g^{(1)}_{n,\alpha}(\overline{\gamma})(z) = g^{(1),*}_{n,\alpha}(\overline{\gamma})(z) + \sum_{i\in I} \left\{\gamma\alpha_i^{-1} \bullet F^{(1)}_{n,\alpha}(\alpha_i\gamma^{-1}z) - \alpha_i^{-1} \bullet F^{(1)}_{n,\alpha}(\alpha_i z)\right\} + x(\overline{\gamma}) \tag{2.14}$$

for some 1-cocycle $x(\overline{\gamma}) \in L_\kappa(\mathbb{C})$. We shall explicitly give $g^{(1),*}_{n,\alpha}(\overline{\gamma})(z)$ as follows. By regarding (2.14) as an equation of $\overline{\Gamma}$, we obtain

$$dg^{(1)}_{n,\alpha}(\overline{\gamma_1}, \overline{\gamma_2})(z) = dg^{(1),*}_{n,\alpha}(\overline{\gamma_1}, \overline{\gamma_2})(z)$$

up to 1-coboundary, where d is the boundary map in group cohomology. By substituting i by $\gamma(i)$ in the first term, we get

$$\begin{aligned}
\sum_{i\in I} \left\{\gamma\alpha_i^{-1} \bullet F(\alpha_i\gamma^{-1}z) - \alpha_i^{-1} \bullet F(\alpha_i z)\right\} &= \sum_{i\in I} \left\{\gamma\alpha_{\gamma(i)}^{-1} \bullet F(\alpha_{\gamma(i)}\gamma^{-1}z) - \alpha_i^{-1} \bullet F(\alpha_i z)\right\} \\
&= \sum_{i\in I} \left\{\alpha_i^{-1}\gamma^{(i)} \bullet F({\gamma^{(i)}}^{-1}\alpha_i z) - \alpha_i^{-1} \bullet F(\alpha_i z)\right\} \\
&= \sum_{i\in I} \alpha_i^{-1} \bullet \left\{\gamma^{(i)} * F(\alpha_i z) - F(\alpha_i z)\right\}.
\end{aligned}$$

Thus we obtain

$$g^{(1),*}_{n,\alpha}(\overline{\gamma})(z) = \sum_{i\in I} \alpha_i^{-1} \bullet g^{(1)}_n\left(\overline{\gamma^{(i)}}\right)(\alpha_i z),$$

where $g^{(1)}_n$ is given by (2.7).

Moreover, by substituting i by $\gamma(i)$ in the first term, we get

$$
\begin{aligned}
&dg^{(1),*}_{n,\alpha}(\overline{\gamma_1},\overline{\gamma_2})(z)\\
&=\sum_{i\in I}\gamma_1\alpha_i^{-1}\bullet g_n^{(1)}(\overline{\gamma_2^{(i)}})(\alpha_i\gamma_1^{-1}z)-\sum_{i\in I}\alpha_i^{-1}\bullet\left\{g_n^{(1)}(\overline{(\gamma_1\gamma_2)^{(i)}})(\alpha_i z)-g_n^{(1)}(\overline{\gamma_1^{(i)}})(\alpha_i z)\right\}\\
&=\sum_{i\in I}\gamma_1\alpha_{\gamma_1(i)}^{-1}\bullet g_n^{(1)}(\overline{\gamma_2^{\gamma_1(i)}})(\alpha_{\gamma_1(i)}\gamma_1^{-1}z)\\
&\quad-\sum_{i\in I}\alpha_i^{-1}\bullet\left\{g_n^{(1)}(\overline{(\gamma_1\gamma_2)^{(i)}})(\alpha_i z)-g_n^{(1)}(\overline{\gamma_1^{(i)}})(\alpha_i z)\right\}\\
&=\sum_{i\in I}\alpha_i^{-1}\gamma_1^{(i)}\bullet g_n^{(1)}(\overline{\gamma_2^{\gamma_1(i)}})({\gamma_1^{(i)}}^{-1}\alpha_i z)\\
&\quad-\sum_{i\in I}\alpha_i^{-1}\bullet\left\{g_n^{(1)}(\overline{\gamma_1^{(i)}\gamma_2^{(\gamma_1(i))}})(\alpha_i z)-g_n^{(1)}(\overline{\gamma_1^{(i)}})(\alpha_i z)\right\}\\
&=\sum_{i\in I}\alpha_i^{-1}\bullet\left\{\gamma_1^{(i)}*g_n^{(1)}(\overline{\gamma_2^{\gamma_1(i)}})(\alpha_i z)-g_n^{(1)}(\overline{\gamma_1^{(i)}\gamma_2^{(\gamma_1(i))}})(\alpha_i z)+g_n^{(1)}(\overline{\gamma_1^{(i)}})(\alpha_i z)\right\}.
\end{aligned}
$$

Thus, similar to above, we obtain

$$g^{(2),*}_{n-1,\alpha}(\overline{\gamma_1},\overline{\gamma_2})(z)=\sum_{i\in I}\alpha_i^{-1}\bullet g^{(2)}_{n-1}(\overline{\gamma_1^{(i)}},\overline{\gamma_2^{(\gamma_1(i))}})(\alpha_i z),$$

where $g^{(2)}_{n-1}$ is given by (2.10).

Repeating this computations proves the theorem. □

2.4. **Constant term of n-cocycle.** In this subsection, for $E\in M_\kappa(\Gamma,\mathbb{C})$, we describe the image of the n-cocycle $[\pi_E]$ under the restriction map in group cohomology. It is important for us to determine the structure of congruence modules attached to an Eisenstein series E and prove the integrality the cocycle $[\pi_E]$ in §2.10.

We fix i with $1\le i\le n$. For $z,z'\in\mathfrak{H}$, let $\{z,z'\}$ denote the oriented geodesic path joining z to z'. We define a new n-cocycle $\pi^{(i)}_{E,\omega}$ as

$$
\begin{aligned}
\pi^{(i)}_{E,\omega}(\overline{\gamma_1},\cdots,\overline{\gamma_n})=&\gamma_1\cdots\gamma_{n-i}\bullet\int_{I_1}\cdots\int_{I_{i-1}}\int_{\omega_i}^{\gamma_{n-i+1}\omega_i}\int_{I_{i+1}}\cdots\int_{I_n}\omega(E)\\
&+b_1^{(i)}(\overline{\gamma_1},\cdots,\overline{\gamma_n})-b_2^{(i)}(\overline{\gamma_1},\cdots,\overline{\gamma_n}),
\end{aligned}
$$

where $\widetilde{E}(z)=E(z)-a_\infty(0,E)$,

$$
\begin{aligned}
b_1^{(i)}(\overline{\gamma_1},\cdots,\overline{\gamma_n})&=\gamma_1\cdots\gamma_{n-i}\gamma_{n-i+1}\bullet\int_{I'_1}\cdots\int_{I'_{i-1}}\int_{\omega_i}^{i\infty}\int_{I'_{i+1}}\cdots\int_{I'_n}\omega(\widetilde{E})\\
&\quad-\gamma_1\cdots\gamma_{n-i}\gamma_{n-i+1}\bullet\int_{I'_1}\cdots\int_{I'_{i-1}}\int_{0}^{\omega_i}\int_{I'_{i+1}}\cdots\int_{I'_n}\omega(a_\infty(0,E))\\
b_2^{(i)}(\overline{\gamma_1},\cdots,\overline{\gamma_n})&=\gamma_1\cdots\gamma_{n-i}\bullet\int_{I_1}\cdots\int_{I_{i-1}}\int_{\omega_i}^{i\infty}\int_{I_{i+1}}\cdots\int_{I_n}\omega(\widetilde{E})\\
&\quad-\gamma_1\cdots\gamma_{n-i}\bullet\int_{I_1}\cdots\int_{I_{i-1}}\int_{0}^{\omega_i}\int_{I_{i+1}}\cdots\int_{I_n}\omega(a_\infty(0,E)),
\end{aligned}
$$

$$I_j = (\gamma_1 \cdots \gamma_{n-i})^{-1}\{\gamma_1 \cdots \gamma_{n-j}\omega_j, \gamma_1 \cdots \gamma_{n-j+1}\omega_j\},$$
$$I'_j = \gamma_{n-i+1}^{-1} I_j.$$

We remark that $b_1^{(i)}(\overline{\gamma_1}, \cdots, \overline{\gamma_n})$ and $b_2^{(i)}(\overline{\gamma_1}, \cdots, \overline{\gamma_n})$ converge absolutely by the same way as in the proof of Proposition 2.12.

Proposition 2.8. *For $E \in M_\kappa(\Gamma, \mathbb{C})$, a cocycle $\pi_{E,\omega}^{(i)}$ satisfies the following properties.*

(1) *the value $\pi_{E,\omega}^{(i)}(\gamma_1, \cdots, \gamma_n)$ is independent on ω_i.*

(2) *$\pi_{E,\omega}^{(i)}$ is cohomologous to $\pi_{E,\omega}$.*

Proof. (1) follows from a direct calculation.

To prove (2), we put

$$\begin{aligned}
&v^{(i)}(\overline{\gamma_1}, \cdots, \overline{\gamma_{n-1}}) \\
&= \gamma_1 \cdots \gamma_{n-i} \bullet \int_{\gamma_{n-i+1}\cdots\gamma_{n-2}\omega_1}^{\gamma_{n-i+1}\cdots\gamma_{n-1}\omega_1} \cdots \int_{\omega_{i-1}}^{\gamma_{n-i+1}\omega_{i-1}} \int_{\omega_i}^{i\infty} \\
&\quad \times \int_{\gamma_{n-i}^{-1}\omega_{i+1}}^{\omega_{i+1}} \cdots \int_{(\gamma_2\cdots\gamma_{n-i})^{-1}\omega_{n-1}}^{(\gamma_3\cdots\gamma_{n-i})^{-1}\omega_{n-1}} \int_{(\gamma_1\cdots\gamma_{n-i})^{-1}\omega_n}^{(\gamma_2\cdots\gamma_{n-i})^{-1}\omega_{n-1}} \omega(\widetilde{E}) \\
&\quad - \gamma_1 \cdots \gamma_{n-i} \bullet \int_{\gamma_{n-i+1}\cdots\gamma_{n-2}\omega_1}^{\gamma_{n-i+1}\cdots\gamma_{n-1}\omega_1} \cdots \int_{\omega_{i-1}}^{\gamma_{n-i+1}\omega_{i-1}} \int_0^{\omega_i} \\
&\quad \times \int_{\gamma_{n-i}^{-1}\omega_{i+1}}^{\omega_{i+1}} \cdots \int_{(\gamma_2\cdots\gamma_{n-i})^{-1}\omega_{n-1}}^{(\gamma_3\cdots\gamma_{n-i})^{-1}\omega_{n-1}} \int_{(\gamma_1\cdots\gamma_{n-i})^{-1}\omega_n}^{(\gamma_2\cdots\gamma_{n-i})^{-1}\omega_n} \omega(a_\infty(0, E)).
\end{aligned}$$

We claim that

$$dv^{(i)}(\overline{\gamma_1}, \cdots, \overline{\gamma_n}) = (-1)^{n-i}\left\{\pi_{E,\omega}^{(i)}(\overline{\gamma_1}, \cdots, \overline{\gamma_n}) - \pi_{E,\omega}(\overline{\gamma_1}, \cdots, \overline{\gamma_n})\right\}. \tag{2.15}$$

The proof will now proceed in two steps.

Step1:

$$\begin{aligned}
&\overline{\gamma_1} \bullet v^{(i)}(\overline{\gamma_2}, \cdots, \overline{\gamma_n}) + \sum_{1 \le j \le n-i} (-1)^j v^{(i)}(\overline{\gamma_1}, \cdots, \overline{\gamma_j\gamma_{j+1}}, \cdots, \overline{\gamma_n}) \\
&= (-1)^{n-i} b_1^{(i)}(\overline{\gamma_1} \cdots \overline{\gamma_n}).
\end{aligned}$$

Proof. For each $1 \le k \le n-i$, we prove $(*)_k$ by induction on k:

$$\begin{aligned}
(*)_k \quad &\overline{\gamma_1} \bullet v^{(i)}(\overline{\gamma_2}, \cdots, \overline{\gamma_n}) + \sum_{1 \le j \le k} (-1)^k v^{(i)}(\overline{\gamma_1}, \cdots, \overline{\gamma_j\gamma_{j+1}}, \cdots, \overline{\gamma_n}) \\
&= (-1)^k\Bigg\{\gamma_1 \cdots \gamma_{n-i+1} \bullet \int_{\gamma_{n-i+2}\cdots\gamma_{n-1}\omega_1}^{\gamma_{n-i+2}\cdots\gamma_n\omega_1} \cdots \int_{\omega_i}^{i\infty} \cdots \\
&\quad \times \int_{(\gamma_{k+2}\cdots\gamma_{n-i+1})^{-1}\omega_{n-k}}^{(\gamma_{k+3}\cdots\gamma_{n-i+1})^{-1}\omega_{n-k}} \int_{(\gamma_k\cdots\gamma_{n-i+1})^{-1}\omega_{n-k+1}}^{(\gamma_{k+1}\cdots\gamma_{n-i+1})^{-1}\omega_{n-k+1}} \cdots \int_{(\gamma_1\cdots\gamma_{n-i+1})^{-1}\omega_n}^{(\gamma_2\cdots\gamma_{n-i+1})^{-1}\omega_n} \omega(\widetilde{E}) \\
&\quad - \gamma_1 \cdots \gamma_{n-i+1} \bullet \int_{\gamma_{n-i+2}\cdots\gamma_{n-1}\omega_1}^{\gamma_{n-i+2}\cdots\gamma_n\omega_1} \cdots \int_0^{\omega_i} \cdots \\
&\quad \times \int_{(\gamma_{k+2}\cdots\gamma_{n-i+1})^{-1}\omega_{n-k}}^{(\gamma_{k+3}\cdots\gamma_{n-i+1})^{-1}\omega_{n-k}} \int_{(\gamma_k\cdots\gamma_{n-i+1})^{-1}\omega_{n-k+1}}^{(\gamma_{k+1}\cdots\gamma_{n-i+1})^{-1}\omega_{n-k+1}} \cdots \int_{(\gamma_1\cdots\gamma_{n-i+1})^{-1}\omega_n}^{(\gamma_2\cdots\gamma_{n-i+1})^{-1}\omega_n} \omega(a_\infty(0, E))\Bigg\}.
\end{aligned}$$

The statement is true when $k=1$. Indeed, we have

$$\begin{aligned}
&\overline{\gamma_1}\bullet v^{(i)}(\overline{\gamma_2},\cdots,\overline{\gamma_n})-v^{(i)}(\overline{\gamma_1\gamma_2},\cdots,\overline{\gamma_n})\\
&=\gamma_1\cdots\gamma_{n-i+1}\bullet\int\cdots\int_{\omega_i}^{i\infty}\cdots\int_{(\gamma_3\cdots\gamma_{n-i+1})^{-1}\omega_{n-1}}^{(\gamma_4\cdots\gamma_{n-i+1})^{-1}\omega_{n-1}}\int_{(\gamma_2\cdots\gamma_{n-i+1})^{-1}\omega_n}^{(\gamma_3\cdots\gamma_{n-i+1})^{-1}\omega_n}\omega(\widetilde{E})\\
&\quad-\gamma_1\cdots\gamma_{n-i+1}\bullet\int\cdots\int_{\omega_i}^{i\infty}\cdots\int_{(\gamma_3\cdots\gamma_{n-i+1})^{-1}\omega_{n-1}}^{(\gamma_4\cdots\gamma_{n-i+1})^{-1}\omega_{n-1}}\int_{(\gamma_1\gamma_2\cdots\gamma_{n-i+1})^{-1}\omega_n}^{(\gamma_3\cdots\gamma_{n-i+1})^{-1}\omega_n}\omega(\widetilde{E})\\
&\quad-\gamma_1\cdots\gamma_{n-i+1}\bullet\int\cdots\int_{0}^{\omega_i}\cdots\int_{(\gamma_3\cdots\gamma_{n-i+1})^{-1}\omega_{n-1}}^{(\gamma_4\cdots\gamma_{n-i+1})^{-1}\omega_{n-1}}\int_{(\gamma_2\cdots\gamma_{n-i+1})^{-1}\omega_n}^{(\gamma_3\cdots\gamma_{n-i+1})^{-1}\omega_n}\omega(a_\infty(0,E))\\
&\quad+\gamma_1\cdots\gamma_{n-i+1}\bullet\int\cdots\int_{0}^{\omega_i}\cdots\int_{(\gamma_3\cdots\gamma_{n-i+1})^{-1}\omega_{n-1}}^{(\gamma_4\cdots\gamma_{n-i+1})^{-1}\omega_{n-1}}\int_{(\gamma_1\gamma_2\cdots\gamma_{n-i+1})^{-1}\omega_n}^{(\gamma_3\cdots\gamma_{n-i+1})^{-1}\omega_n}\omega(a_\infty(0,E))\\
&=(-1)\Big\{\gamma_1\cdots\gamma_{n-i+1}\bullet\int\cdots\int_{0}^{\omega_i}\cdots\int_{(\gamma_3\cdots\gamma_{n-i+1})^{-1}\omega_{n-1}}^{(\gamma_4\cdots\gamma_{n-i+1})^{-1}\omega_{n-1}}\int_{(\gamma_1\cdots\gamma_{n-i+1})^{-1}\omega_n}^{(\gamma_2\cdots\gamma_{n-i+1})^{-1}\omega_n}\omega(\widetilde{E})\\
&\quad+\gamma_1\cdots\gamma_{n-i+1}\bullet\int\cdots\int_{0}^{\omega_i}\cdots\int_{(\gamma_3\cdots\gamma_{n-i+1})^{-1}\omega_{n-1}}^{(\gamma_4\cdots\gamma_{n-i+1})^{-1}\omega_{n-1}}\int_{(\gamma_1\gamma_2\cdots\gamma_{n-i+1})^{-1}\omega_n}^{(\gamma_2\cdots\gamma_{n-i+1})^{-1}\omega_n}\omega(a_\infty(0,E))\Big\}.
\end{aligned}$$

Suppose that $(*)_k$. By adding $(-1)^{k+1}v^{(i)}(\overline{\gamma_1},\cdots,\overline{\gamma_{k+1}\gamma_{k+2}},\cdots,\overline{\gamma_n})$ to $(*)_k$, we have

$$\begin{aligned}
&\overline{\gamma_1}\bullet v^{(i)}(\overline{\gamma_2},\cdots,\overline{\gamma_n})+\sum_{1\le j\le k+1}(-1)^j v^{(i)}(\overline{\gamma_1},\cdots,\overline{\gamma_j\gamma_{j+1}},\cdots,\overline{\gamma_n})\\
&=(-1)^{k+1}\Big\{\gamma_1\cdots\gamma_{n-i+1}\bullet\int_{\gamma_{n-i+2}\cdots\gamma_{n-1}\omega_1}^{\gamma_{n-i+2}\cdots\gamma_n\omega_1}\cdots\int_{\omega_i}^{i\infty}\cdots\\
&\quad\times\left(\int_{(\gamma_{k+3}\cdots\gamma_{n-i+1})^{-1}\omega_{n-k}}^{(\gamma_{k+2}\cdots\gamma_{n-i+1})^{-1}\omega_{n-k}}+\int_{(\gamma_{k+1}\cdots\gamma_{n-i+1})^{-1}\omega_{n-k}}^{(\gamma_{k+3}\cdots\gamma_{n-i+1})^{-1}\omega_{n-k}}\right)\cdots\int_{(\gamma_1\cdots\gamma_{n-i+1})^{-1}\omega_n}^{(\gamma_2\cdots\gamma_{n-i+1})^{-1}\omega_n}\omega(\widetilde{E})\\
&\quad-\gamma_1\cdots\gamma_{n-i+1}\bullet\int_{\gamma_{n-i+2}\cdots\gamma_{n-1}\omega_1}^{\gamma_{n-i+2}\cdots\gamma_n\omega_1}\cdots\int_{0}^{\omega_i}\cdots\\
&\quad\times\left(\int_{(\gamma_{k+3}\cdots\gamma_{n-i+1})^{-1}\omega_{n-k}}^{(\gamma_{k+2}\cdots\gamma_{n-i+1})^{-1}\omega_{n-k}}+\int_{(\gamma_{k+1}\cdots\gamma_{n-i+1})^{-1}\omega_{n-k}}^{(\gamma_{k+3}\cdots\gamma_{n-i+1})^{-1}\omega_{n-k}}\right)\cdots\int_{(\gamma_1\cdots\gamma_{n-i+1})^{-1}\omega_n}^{(\gamma_2\cdots\gamma_{n-i+1})^{-1}\omega_n}\omega(a_\infty(0,E))\Big\}\\
&=(-1)^{k+1}\Big\{\gamma_1\cdots\gamma_{n-i+1}\bullet\int_{\gamma_{n-i+2}\cdots\gamma_{n-1}\omega_1}^{\gamma_{n-i+2}\cdots\gamma_n\omega_1}\cdots\int_{\omega_i}^{i\infty}\cdots\\
&\quad\times\int_{(\gamma_{k+1}\cdots\gamma_{n-i+1})^{-1}\omega_{n-k}}^{(\gamma_{k+2}\cdots\gamma_{n-i+1})^{-1}\omega_{n-k}}\cdots\int_{(\gamma_1\cdots\gamma_{n-i+1})^{-1}\omega_n}^{(\gamma_2\cdots\gamma_{n-i+1})^{-1}\omega_n}\omega(\widetilde{E})\\
&\quad-\gamma_1\cdots\gamma_{n-i+1}\bullet\int_{\gamma_{n-i+2}\cdots\gamma_{n-1}\omega_1}^{\gamma_{n-i+2}\cdots\gamma_n\omega_1}\cdots\int_{0}^{\omega_i}\cdots\\
&\quad\times\int_{(\gamma_{k+1}\cdots\gamma_{n-i+1})^{-1}\omega_{n-k}}^{(\gamma_{k+2}\cdots\gamma_{n-i+1})^{-1}\omega_{n-k}}\cdots\int_{(\gamma_1\cdots\gamma_{n-i+1})^{-1}\omega_n}^{(\gamma_2\cdots\gamma_{n-i+1})^{-1}\omega_n}\omega(a_\infty(0,E))\Big\}
\end{aligned}$$

as desired. $\square$

Step2:

$$\sum_{n-i+1\le n-j\le n-1}(-1)^{n-j}v^{(i)}(\overline{\gamma_1},\cdots,\overline{\gamma_{n-j}\gamma_{n-j+1}},\cdots,\overline{\gamma_n})+(-1)^n v^{(i)}(\overline{\gamma_1},\cdots,\overline{\gamma_{n-1}})$$
$$=(-1)^{n-i+1}b_2^{(i)}(\overline{\gamma_1}\cdots\overline{\gamma_n}).$$

Proof. We prove $(*)'_k$ by induction on $n-i+1\le n-k\le n-1$:

$(*)'_k$
$$\sum_{n-k\le n-j\le n-1}(-1)^{n-j}v^{(i)}(\overline{\gamma_1},\cdots,\overline{\gamma_{n-j}\gamma_{n-j+1}},\cdots,\overline{\gamma_n})+(-1)^n v^{(i)}(\overline{\gamma_1},\cdots,\overline{\gamma_{n-1}})$$
$$=(-1)^{n-k}\Big\{\gamma_1\cdots\gamma_{n-i}\bullet\int\cdots\int_{\gamma_{n-i+1}\cdots\gamma_{n-k}\omega_k}^{\gamma_{n-i+1}\cdots\gamma_{n-k+1}\omega_k}\int_{\gamma_{n-i+1}\cdots\gamma_{n-k-2}\omega_{k+1}}^{\gamma_{n-i+1}\cdots\gamma_{n-k-1}\omega_{k+1}}\cdots\int_{\omega_i}^{i\infty}\cdots\omega(\widetilde{E})$$
$$-\gamma_1\cdots\gamma_{n-i}\bullet\int\cdots\int_{\gamma_{n-i+1}\cdots\gamma_{n-k}\omega_k}^{\gamma_{n-i+1}\cdots\gamma_{n-k+1}\omega_k}\int_{\gamma_{n-i+1}\cdots\gamma_{n-k-2}\omega_{k+1}}^{\gamma_{n-i+1}\cdots\gamma_{n-k-1}\omega_{k+1}}\cdots\int_0^{\omega_i}\cdots\omega(a_\infty(0,E))\Big\}.$$

First suppose $k=1$. We have

$$(-1)^{n-1}v^{(i)}(\overline{\gamma_1},\cdots,\overline{\gamma_{n-1}\gamma_n})+(-1)^n v^{(i)}(\overline{\gamma_1},\cdots,\overline{\gamma_{n-1}})$$
$$=(-1)^{n-1}\Big\{\gamma_1\cdots\gamma_{n-i}\bullet\int_{\gamma_{n-i+1}\cdots\gamma_{n-2}\omega_1}^{\gamma_{n-i+1}\cdots\gamma_{n-1}\gamma_n\omega_1}\int_{\gamma_{n-i+1}\cdots\gamma_{n-3}\omega_2}^{\gamma_{n-i+1}\cdots\gamma_{n-2}\omega_2}\cdots\int_{\omega_i}^{i\infty}\cdots\omega(\widetilde{E})$$
$$-\gamma_1\cdots\gamma_{n-i}\bullet\int_{\gamma_{n-i+1}\cdots\gamma_{n-2}\omega_1}^{\gamma_{n-i+1}\cdots\gamma_{n-1}\omega_1}\int_{\gamma_{n-i+1}\cdots\gamma_{n-3}\omega_2}^{\gamma_{n-i+1}\cdots\gamma_{n-2}\omega_2}\cdots\int_{\omega_i}^{i\infty}\cdots\omega(\widetilde{E})$$
$$-\gamma_1\cdots\gamma_{n-i}\bullet\int_{\gamma_{n-i+1}\cdots\gamma_{n-2}\omega_1}^{\gamma_{n-i+1}\cdots\gamma_{n-1}\gamma_n\omega_1}\int_{\gamma_{n-i+1}\cdots\gamma_{n-3}\omega_2}^{\gamma_{n-i+1}\cdots\gamma_{n-2}\omega_2}\cdots\int_0^{\omega_i}\cdots\omega(a_\infty(0,E))$$
$$+\gamma_1\cdots\gamma_{n-i}\bullet\int_{\gamma_{n-i+1}\cdots\gamma_{n-2}\omega_2}^{\gamma_{n-i+1}\cdots\gamma_{n-1}\omega_1}\int_{\gamma_{n-i+1}\cdots\gamma_{n-3}\omega_2}^{\gamma_{n-i+1}\cdots\gamma_{n-2}\omega_2}\cdots\int_0^{\omega_i}\cdots\omega(a_\infty(0,E))\Big\}$$
$$=(-1)^{n-1}\Big\{\gamma_1\cdots\gamma_{n-i}\bullet\int_{\gamma_{n-i+1}\cdots\gamma_{n-1}\omega_1}^{\gamma_{n-i+1}\cdots\gamma_n\omega_1}\int_{\gamma_{n-i+1}\cdots\gamma_{n-3}\omega_2}^{\gamma_{n-i+1}\cdots\gamma_{n-2}\omega_2}\cdots\int_{\omega_i}^{i\infty}\cdots\omega(\widetilde{E})$$
$$-\gamma_1\cdots\gamma_{n-i}\bullet\int_{\gamma_{n-i+1}\cdots\gamma_{n-1}\omega_1}^{\gamma_{n-i+1}\cdots\gamma_n\omega_1}\int_{\gamma_{n-i+1}\cdots\gamma_{n-3}\omega_2}^{\gamma_{n-i+1}\cdots\gamma_{n-2}\omega_2}\cdots\int_0^{\omega_i}\cdots\omega(a_\infty(0,E))\Big\}$$

as desired.

Next suppose that $(*)'_k$. By adding $(-1)^{n-k-1}v^{(i)}(\overline{\gamma_1},\cdots,\overline{\gamma_{n-k-1}\gamma_{n-k}},\cdots,\overline{\gamma_n})$ to $(*)'_k$, we get

$$\begin{aligned}
&\sum_{n-k-1\le n-j\le n-1}(-1)^{n-j}v^{(i)}(\overline{\gamma_1},\cdots,\overline{\gamma_{n-j}\gamma_{n-j+1}},\cdots,\overline{\gamma_n})+(-1)^n v^{(i)}(\overline{\gamma_1},\cdots,\overline{\gamma_{n-1}})\\
&=(-1)^{n-k-1}\Big\{\gamma_1\cdots\gamma_{n-i}\bullet\int_{\gamma_{n-i+1}\cdots\gamma_{n-1}\omega_1}^{\gamma_{n-i+1}\cdots\gamma_n\omega_1}\cdots\\
&\quad\times\left(\int_{\gamma_{n-i+1}\cdots\gamma_{n-k-1}\omega_{k+1}}^{\gamma_{n-i+1}\cdots\gamma_{n-k-2}\omega_{k+1}}+\int_{\gamma_{n-i+1}\cdots\gamma_{n-k-2}\omega_{k+1}}^{\gamma_{n-i+1}\cdots\gamma_{n-k-1}\gamma_{n-k}\omega_{k+1}}\right)\cdots\int_{\omega_i}^{i\infty}\cdots\omega(\widetilde{E})\\
&\quad-\gamma_1\cdots\gamma_{n-i}\bullet\int_{\gamma_{n-i+1}\cdots\gamma_{n-1}\omega_1}^{\gamma_{n-i+1}\cdots\gamma_n\omega_1}\cdots\\
&\quad\times\left(\int_{\gamma_{n-i+1}\cdots\gamma_{n-k-1}\omega_{k+1}}^{\gamma_{n-i+1}\cdots\gamma_{n-k-2}\omega_{k+1}}+\int_{\gamma_{n-i+1}\cdots\gamma_{n-k-2}\omega_{k+1}}^{\gamma_{n-i+1}\cdots\gamma_{n-k-1}\gamma_{n-k}\omega_{k+1}}\right)\cdots\int_{0}^{\omega_i}\cdots\omega(a_\infty(0,E))\Big\}\\
&=(-1)^{n-k-1}\Big\{\gamma_1\cdots\gamma_{n-i}\bullet\int_{\gamma_{n-i+1}\cdots\gamma_{n-1}\omega_1}^{\gamma_{n-i+1}\cdots\gamma_n\omega_1}\cdots\int_{\gamma_{n-i+1}\cdots\gamma_{n-k-1}\omega_{k+1}}^{\gamma_{n-i+1}\cdots\gamma_{n-k-1}\gamma_{n-k}\omega_{k+1}}\cdots\int_{\omega_i}^{i\infty}\cdots\omega(\widetilde{E})\\
&\quad-\gamma_1\cdots\gamma_{n-i}\bullet\int_{\gamma_{n-i+1}\cdots\gamma_{n-1}\omega_1}^{\gamma_{n-i+1}\cdots\gamma_n\omega_1}\cdots\int_{\gamma_{n-i+1}\cdots\gamma_{n-k-1}\omega_{k+1}}^{\gamma_{n-i+1}\cdots\gamma_{n-k-1}\gamma_{n-k}\omega_{k+1}}\cdots\int_{0}^{\omega_i}\cdots\omega(a_\infty(0,E))\Big\}
\end{aligned}$$

as desired. □

Therefore, by **Step1** and **Step2**, we obtain

$$dv^{(i)}(\overline{\gamma_1},\cdots,\overline{\gamma_n})=(-1)^{n-i}\{\pi^{(i)}_{E,\omega}(\overline{\gamma_1},\cdots,\overline{\gamma_n})-\pi_{E,\omega}(\overline{\gamma_1},\cdots,\overline{\gamma_n})\}.$$

□

Now we describe the image of $[\pi_E]$ under the restriction map.

Proposition 2.9. *Fix i with $1\le i\le h_F^+$ and let $\Gamma=\Gamma_{1,i}(\mathfrak{n})$. Let Φ_p be the composite field of $\iota_p(F^\sigma(\sqrt{-1}))$ in $\overline{\mathbb{Q}}_p$ for all $\sigma\in J_F$ and $\mathcal{O}$ the ring of integers of a finite extension K over Φ_p containing the values of u^{m_κ} for all $u\in\mathfrak{o}^\times_{F',+}$ as §2.2. Here $\iota_p:\overline{\mathbb{Q}}\to\overline{\mathbb{Q}}_p$ is the fixed embedding. Assume that $E\in M_\kappa(\Gamma,\mathcal{O})$ with $\kappa=(k-2t,m)\in\mathfrak{X}(T)$ and $k_\sigma-1<p$ for all $\sigma\in J_F$. Then we have the following properties:*

(1)
$$\mathrm{res}([\pi_E])\in\bigoplus_{s\in C(\Gamma)}\widetilde{H}^n(\overline{\Gamma_s},L_{\kappa,i}(\mathcal{O})),$$
where $\widetilde{H}^n(\overline{\Gamma_s},L_{\kappa,i}(\mathcal{O}))=\mathrm{im}\left(H^n(\overline{\Gamma_s},L_{\kappa,i}(\mathcal{O}))\to H^n(\overline{\Gamma_s},L_{\kappa,i}(K))\right)$ is the torsion-free part of $H^n(\overline{\Gamma_s},L_{\kappa,i}(\mathcal{O}))$.

(2) *Suppose that E vanishes at a cusp $s\in C(\Gamma)$. Then*
$$\mathrm{res}([\pi_E])=0\ \textit{in}\ \widetilde{H}^n(\overline{\Gamma_s},L_{\kappa,i}(\mathcal{O})).$$

Proof. We treat the case $s=\infty$ (the case $s\neq\infty$ is similar). By the previous proposition,

$$\pi^{(n)}_{E,\omega}(\overline{\gamma_1},\cdots,\overline{\gamma_n})=\pi_{E,\omega}(\overline{\gamma_1},\cdots,\overline{\gamma_n})+dv^{(n)}(\overline{\gamma_1},\cdots,\overline{\gamma_n})$$

is independent on ω_n. With the help of Proposition 2.12, the first term of $b_1^{(n)}(\overline{\gamma_1},\cdots,\overline{\gamma_n})$ and $b_2^{(n)}(\overline{\gamma_1},\cdots,\overline{\gamma_n})$ converge to 0 when ω_n tends to $\sqrt{-1}\infty$. For any $\overline{\gamma_1}\in\overline{\Gamma}_\infty$ and a lift

$\gamma_1 \in \Gamma_\infty$ of $\overline{\gamma_1}$, when ω_n tends to $\sqrt{-1}\infty$, so does $\gamma_1\omega_n$. Thus we obtain

$$\begin{aligned}
&\lim_{\omega_n \to \sqrt{-1}\infty} \pi^{(n)}_{E,(\sqrt{-1},\cdots,\sqrt{-1},\omega_n)}(\overline{\gamma}_1, \cdots, \overline{\gamma}_n) \\
&\quad = \lim_{\omega_n \to \sqrt{-1}\infty} \int_{I_1} \int_{I_{n-1}} \left(\int_{\omega_n}^{\gamma_1\omega_n} - \int_{\gamma_1 0}^{\gamma_1\omega_n} + \int_0^{\omega_n} \right) \omega(a_\infty(0, E)) \\
&\quad = \int_{I_1} \cdots \int_{I_{n-1}} \int_0^{\gamma_1 0} \omega(a_\infty(0, E))
\end{aligned}$$

as desired. □

Proposition 2.10. *Assume that $h_F^+ = 1$ and both φ and ψ are totally even or totally odd primitive characters. Let $\boldsymbol{E} = E_2(\varphi, \psi)_1$ as Proposition 1.1. Under the same notation and assumptions of Proposition 2.9 with $h_F^+ = 1$, $[\pi_{\boldsymbol{E}}]$ is rational, that is,*

$$[\pi_{\boldsymbol{E}}] \in H^n(Y(\mathfrak{n}), K).$$

Proof. We use the same notation of Proposition 2.9 and §2.10. Let $\mathfrak{p}_{\mathbf{E}}$ denote the maximal ideal of $\mathbb{H}_2(\mathfrak{n}, \mathcal{O}) \otimes K$ generated by $T(\mathfrak{m}) - C(\mathfrak{m}, \mathbf{E}), S(\mathfrak{m}) - \chi^{-1}(\mathfrak{m}), U(\mathfrak{m}) - C(\mathfrak{m}, \mathbf{E})$ for all integral ideals $\mathfrak{m}$ of F. By Proposition 2.9, $\mathrm{res}([\pi_{\mathbf{E}}])$ is rational. Moreover, as mentioned in Remark 2.23, we will see that $[\pi_{\mathbf{E}}] = [\pi_{\mathbf{E}}]^{\epsilon_{\mathbf{E}}}$. Let $[\omega] \in H^n(Y(\mathfrak{n}), K)_{\mathfrak{p}_{\mathbf{E}}}[\epsilon_{\mathbf{E}}]$ mapping to $\mathrm{res}([\pi_{\mathbf{E}}])$. Then we have $[\pi_{\mathbf{E}}] - [\omega] \in H^n_{\mathrm{par}}(Y(\mathfrak{n}), \mathbb{C})_{\mathfrak{p}_{\mathbf{E}}}[\epsilon_{\mathbf{E}}]$. The partial Eichler–Shimura–Harder isomorphism (2.27) and the q-expansion principle over $\mathbb{C}$ imply that $H^n_{\mathrm{par}}(Y(\mathfrak{n}), \mathbb{C})_{\mathfrak{p}_{\mathbf{E}}}[\epsilon_{\mathbf{E}}] = 0$ and hence $[\pi_{\mathbf{E}}]$ is rational. □

2.5. **Borel–Serre compactification.** In this subsection, we recall the Borel–Serre compactification. For more detail, refer to [Bo–Se], [Bo–Ji], [Ha], [Hida93], [Gha].

We fix i with $1 \le i \le h_F^+$ and abbreviate $\Gamma_{1,i}(\mathfrak{n})$ to $\Gamma_{1,i}$ and $\overline{\Gamma_{1,i}} \backslash \mathfrak{H}^{J_F}$ to Y_i.

The Borel–Serre compactification $(\mathfrak{H}^{J_F})^{\mathrm{BS}}$ of $\mathfrak{H}^{J_F}$ is a locally compact manifold on which $\mathrm{GL}_2(F)$ acts. We describe the boundary of $(\mathfrak{H}^{J_F})^{\mathrm{BS}}$ at the cusp ∞ as follows. Let $X = \{(y, x) \in (F \otimes \mathbb{R})_+^\times \times (F \otimes \mathbb{R}) \mid y_1 \cdots y_n = 1\}$. We have

$$\mathfrak{H}^{J_F} \xrightarrow{\simeq} \mathbb{R}_+^\times \times X : (x_{\sigma_i} + \sqrt{-1} y_{\sigma_i})_i \mapsto \left(\prod_{i=1}^n y_{\sigma_i}, \left(N(y_{\sigma_i})^{-\frac{1}{n}} y_{\sigma_i}, x_i \right)_i \right),$$

which is compatible with the action of $\Gamma_{1,i,\infty}$. Here $\Gamma_{1,i,\infty}$ acts trivially on the first factor of the right hand side. Then the boundary of $(\mathfrak{H}^{J_F})^{\mathrm{BS}}$ at the cusp ∞ is given by $(\mathbb{R}_+^\times \cup \{\infty\}) \times X$ (see, for example, [Ha, §2.1] and [Hida93, p.273]).

The Borel–Serre compactification $Y_i^{\mathrm{BS}} = \overline{\Gamma_{1,i}} \backslash (\mathfrak{H}^{J_F})^{\mathrm{BS}}$ of Y_i is a compact manifold with boundary $D_s^i = (\mathbb{R}_+^\times \cup \{\infty\}) \times \overline{\Gamma_{1,i,s}} \backslash \alpha(X)$ at each cusp $s = \alpha(\infty)$ for $\alpha \in \mathrm{SL}_2(F)$, where $\alpha(X) = \{\alpha(x + \sqrt{-1}y) \mid (y, x) \in X\}$ (see, for example, [Ha, §2.1] and [Hida93, p.273]).

Let $\mathcal{O}$ be the ring of integers of a finite extension over $\mathbb{Q}_p$. We assume that $\overline{\Gamma_{1,i}}$ is p-torsion-free. Then the cohomology of Y_i^{BS} has the following property:

$$H^m(Y_i^{\mathrm{BS}}, \underline{M}) \simeq H^m(Y_i, \underline{M}) \simeq H^m(\overline{\Gamma_{1,i}}, M)$$

for any $\mathcal{O}[\overline{\Gamma_{1,i}}]$-module M, where $\underline{M}$ is the sheaf associated to M. Moreover,

$$H^m(\partial(Y_i^{\mathrm{BS}}), \underline{M}) \simeq \bigoplus_{s \in C(\Gamma_{1,i})} H^m(\overline{\Gamma_{1,i,s}}, M)$$

for any such module M.

2.6. **Fundamental domain.** In this subsection, we will construct a relative homology class which is related to the special values of L-function attached to Hilbert modular forms.

We fix i with $1 \leq i \leq h_F^+$ and abbreviate $\overline{\Gamma_{1,i}(\mathfrak{n})}\backslash\mathfrak{H}^{J_F}$ to Y_i. Let E be a subgroup of $\mathfrak{o}_{F,+}^\times$ with finite index.

First, we remark that a fundamental domain of $\mathbb{R}_+^{J_F}/E$ is given by

$$\Omega_E = \prod_{j=1}^{n-1}\{\varepsilon_j^{r_j} \mid r_j \in [0,1)\} \times \mathbb{R}_+ \hookrightarrow X \times \mathbb{R}_+ \simeq \mathfrak{H}^{J_F} :$$

$$(\varepsilon_1^{r_1}, \cdots, \varepsilon_{n-1}^{r_{n-1}}, -\log(r_n)) \mapsto (((\varepsilon^{r(i)})_i, 0), -\log(r_n)) \mapsto \sqrt{-1}y\varepsilon^r,$$

where $\varepsilon^r = (\varepsilon^{r(i)})_i$ with $\varepsilon^{r(i)} = (\prod_{j=1}^{n-1}\varepsilon_j^{r_j})^{\sigma_i}$ and $y = -\log(r_n)$. We put

$$\overline{\Omega}_E = \prod_{j=1}^{n-1}\{\varepsilon_j^{r_j} \mid r_j \in [0,1]\} \times (\mathbb{R}_{\geq 0} \cup \{\infty\}).$$

For a closed unit interval $I = [0,1]$, we define a singular n-cube ℓ_i associated to $\overline{\Omega}_E$ as a C^∞-map

$$\ell_i : I^n \to \overline{\Omega}_E \to (\mathfrak{H}^{J_F})^{\mathrm{BS}}$$

given by

$$(r_1, \cdots, r_n) \mapsto (\varepsilon_1^{r_1}, \cdots, \varepsilon_{n-1}^{r_{n-1}}, -\log(r_n)) \mapsto \sqrt{-1}y\varepsilon^r.$$

Let $c_{E,i} = \mathrm{proj} \circ \ell_i$ be the composition of ℓ_i and the canonical projection $(\mathfrak{H}^{J_F})^{\mathrm{BS}} \to Y_i^{\mathrm{BS}}$. Let

$$D_{0,\infty}^i = D_\infty^i \sqcup D_0^i$$

be a subspace of the boundary $\partial(Y_i^{\mathrm{BS}})$ of Y_i^{BS}. Then we have the partial n-cycle $[c_{E,i}]$:

Definition 2.11.

$$[c_{E,i}] \in H_n(Y_i^{\mathrm{BS}}, D_{0,\infty}^i; \mathbb{Z}),$$

$$[c_E] = ([c_{E,i}])_i \in \bigoplus_{i=1}^{h_F^+} H_n(Y_i^{\mathrm{BS}}, D_{0,\infty}^i; \mathbb{Z}).$$

2.7. **Twisted Mellin transform.** The aim of this subsection is to give a Mellin transform of a Hilbert modular form. In order to do it, we must need the following analytic properties. We use the same notation as §1.2 and §2.6.

Proposition 2.12. *Let $h \in M_\kappa(\Gamma_{1,i}(\mathfrak{n}), \mathbb{C})$.*

(1) *Under the same notation as §1.2 and §2.6, the integral*

$$\int_{\text{image of } c_{E,i}} y^{(s-1)n} w(\widetilde{h}) = \int_{[0,1]^{n-1}} \int_{\sqrt{-1}\mathbb{R}_+} y^{(s-1)n} w(\widetilde{h})$$

converges absolutely for $\mathrm{Re}(s) \gg 0$ and extends to a meromorphic function on the complex plane which is holomorphic at $s = 1$. Here $w(h)$ is defined by (2.2) and $\widetilde{h}(z) = h(z) - a_\infty(0, h)$.

(2) *Moreover, if h vanishes at the cusps 0 and ∞, then the integral above converges absolutely for all $s \in \mathbb{C}$.*

Proof. (1) For $\mathrm{Re}(s) \gg 0$, we have

$$\int_{[0,1]^{n-1}} \int_{\sqrt{-1}\mathbb{R}_+} y^{(s-1)n} w(\widetilde{h}) = \int_{[0,1]^{n-1}} \left(\int_{\sqrt{-1}}^{\sqrt{-1}\infty} + \int_0^{\sqrt{-1}} \right) y^{(s-1)n} w(\widetilde{h}). \tag{2.16}$$

Now we calculate the second term. We put $\sigma = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \in G_{\infty,+}$. Then by the pull-back formula, we have

(2.17)

$$\begin{aligned}
&\int_{[0,1]^{n-1}} \int_0^{\sqrt{-1}} y^{(s-1)n} w(\widetilde{h}) \\
&= - \int_{[-1,0]^{n-1}} \int_{\sqrt{-1}}^{\sqrt{-1}\infty} y^{(1-s)n} \sigma \bullet w(\widetilde{h|\sigma}) \\
&\quad - \int_{[-1,0]^{n-1}} \int_{\sqrt{-1}}^{\sqrt{-1}\infty} y^{(1-s)n} \sigma \bullet w(a_\infty(0, h|\sigma)) - \int_{[0,1]^{n-1}} \int_0^{\sqrt{-1}} y^{(s-1)n} w(a_\infty(0,h)).
\end{aligned}$$

The second (resp. third) term of (2.17) converges for $\mathrm{Re}(s) \geq k$ (resp. $\mathrm{Re}(s) \geq 1$). For each non-negative integers m, m', since

$$\int_1^\infty y^{(1-s)n+m} dy = \frac{-1}{(1-s)n+m+1} \quad \text{and} \quad \int_0^1 y^{(s-1)n+m'} dy = \frac{1}{(s-1)n+m'+1},$$

the second and third terms of (2.17) are holomorphic at $s = 1$. In order to prove that the first term of (2.16) and (2.17) converge absolutely and entire at $s = 1$, it is enough to show that

$$\int_{[a,b]^{n-1}} \int_1^\infty y^{(s-1)n} \widetilde{h}(\sqrt{-1} y \varepsilon^r) y^m dr dy \tag{2.18}$$

is absolutely convergent and entire at $s = 1$ for any $a, b \in \mathbb{R}$ with $a \leq b$.

Our proof of this claim is based on [Ga, §1.7 and §1.9]. Recall that the absolutely convergent function $\widetilde{h}(z)$ has the Fourier expansion of the form:

$$\widetilde{h}(z) = \sum_{0 \ll \xi \in [t_i]} a_\infty(\xi, h) e_F(\xi z).$$

There is a positive constant $M > 0$ such that $N(\xi) > M$ for each $0 \ll \xi \in [t_i]$. Then there is $\varepsilon > 0$ such that $N(\xi) > M + \varepsilon$ for any such ξ. Thus, by the argument in [Ga, p.29], we have an estimate

$$\exp\left(n M^{\frac{1}{n}} y\right) \left| \widetilde{h}(\sqrt{-1} y \varepsilon^r) \right| \leq \sum_{0 \ll \xi \in [t_i]} |a_\infty(\xi, h)| \exp\left(-\pi \left(2 - \left(\frac{M}{M+\varepsilon} \right)^{\frac{1}{n}} \right) \mathrm{Tr}(\xi y \varepsilon^r) \right).$$

Since $\widetilde{h}(z)$ is absolutely convergent, so is the latter series. Thus, there are positive constants $C, C' > 0$ such that

$$\left| \widetilde{h}(\sqrt{-1} y \varepsilon^r) \right| \leq C \exp\left(-C' y\right)$$

for $y \geq 1$ and each $r \in [a,b]^{n-1}$. Therefore, the integral (2.18) is dominated by

$$\int_{[a,b]^{n-1}} \int_1^\infty \exp(-C' y) y^{\mathrm{Re}(s) n - n + m} dr dy$$

and hence is absolutely convergent and entire function of $s \in \mathbb{C}$.

The assertion (2) follows from the argument in the proof of (1) and the vanishing of the second and third terms in (2.17). $\square$

We assume that $h_F^+ = 1$ and fix a Hilbert cusp form $\mathbf{f}$ and a Hilbert Eisenstein series $\mathbf{E}_2(\varphi, \psi)$ as Proposition 1.1 satisfying the following conditions:

$$\begin{aligned} &\mathbf{f} \in S_\kappa(K_1(\mathfrak{n}), \chi, \mathbb{C}) \text{ and} \\ &\mathbf{E}_2(\varphi, \psi) = E_2(\varphi, \psi)_1 \in M_\kappa(K_1(\mathfrak{n}), \chi, \mathbb{C}) \text{ vanishes at the cusp } \infty. \end{aligned} \tag{2.19}$$

Hereafter we write $\mathbf{h} = \mathbf{f}$ or $\mathbf{E}_2(\varphi, \psi)$. We express the special values of Dirichlet series $D(1, \mathbf{h}, \eta)$ as a Mellin transform for a more general modular form $\mathbf{h}$ (cf. [Oda, §16], [Hida94, §7, §8], and [Ochi, §3]).

Let $\eta : \mathrm{Cl}_F^+(\mathfrak{m}_\eta) \to \overline{\mathbb{Q}}^\times$ be a primitive character whose conductor $\mathfrak{m}_\eta$ is prime to $\mathfrak{d}_F[t_1]$, and $\mathfrak{n}|\mathfrak{m}_\eta$. Let $(\mathfrak{m}_\eta^{-1}/\mathfrak{o}_F)^\times$ (resp. $(\mathfrak{m}_\eta^{-1}\mathfrak{d}_F^{-1}[t_1]^{-1}/\mathfrak{d}_F^{-1}[t_1]^{-1})^\times$) be the subset of $\mathfrak{m}_\eta^{-1}/\mathfrak{o}_F$ (resp. $\mathfrak{m}_\eta^{-1}\mathfrak{d}_F^{-1}[t_1]^{-1}/\mathfrak{d}_F^{-1}[t_1]^{-1}$) consisting of elements whose annihilator is $\mathfrak{m}_\eta$.

Hereafter we fix a non-canonical isomorphism of $\mathfrak{o}_F$-modules $\mathfrak{m}_\eta^{-1}\mathfrak{d}_F^{-1}[t_1]^{-1}/\mathfrak{d}_F^{-1}[t_1]^{-1} \simeq \mathfrak{m}_\eta^{-1}/\mathfrak{o}_F \simeq \mathfrak{o}_F/\mathfrak{m}_\eta$ and a non-canonical bijection induced from it $(\mathfrak{m}_\eta^{-1}\mathfrak{d}_F^{-1}[t_1]^{-1}/\mathfrak{d}_F^{-1}[t_1]^{-1})^\times \simeq (\mathfrak{m}_\eta^{-1}/\mathfrak{o}_F)^\times \simeq (\mathfrak{o}_F/\mathfrak{m}_\eta)^\times$. Hence we may canonically identify $(\mathfrak{m}_\eta^{-1}\mathfrak{d}_F^{-1}[t_1]^{-1}/\mathfrak{d}_F^{-1}[t_1]^{-1})^\times/\mathfrak{o}_{F,+}^\times$ with a subgroup of $\mathrm{Cl}_F^+(\mathfrak{m}_\eta)$ under the canonical extension

$$1 \to (\mathfrak{o}_F/\mathfrak{m}_\eta)^\times/\mathfrak{o}_{F,+}^\times \to \mathrm{Cl}_F^+(\mathfrak{m}_\eta) \to \mathrm{Cl}_F^+ \to 1. \tag{2.20}$$

Let η_1 denote the function on $(\mathfrak{m}_\eta^{-1}\mathfrak{d}_F^{-1}[t_1]^{-1}/\mathfrak{d}_F^{-1}[t_1]^{-1})^\times/\mathfrak{o}_{F,+}^\times$ defined by $\eta_1(\bar{b}) = \eta(\bar{b}\mathfrak{b}\mathfrak{d}_F[t_1])$. We note that $\eta_1(\xi\bar{b}) = \eta(\xi)\eta_1(\bar{b})$ for any $\bar{b} \in (\mathfrak{m}_\eta^{-1}\mathfrak{d}_F^{-1}[t_1]^{-1}/\mathfrak{d}_F^{-1}[t_1]^{-1})^\times/\mathfrak{o}_{F,+}^\times$ and $0 \ll \xi \in [t_1]$ prime to $\mathfrak{m}_\eta$.

Recall that the Gauss sum $\tau(\eta)$ of η is defined by

$$\tau(\eta) = \sum_{b \in (\mathfrak{m}_\eta^{-1}\mathfrak{d}_F^{-1}/\mathfrak{d}_F^{-1})^\times} \eta(b) e_F(b),$$

where b runs over a set of representatives of $(\mathfrak{m}_\eta^{-1}\mathfrak{d}_F^{-1}/\mathfrak{d}_F^{-1})^\times$.

Let $E = \mathfrak{o}_{F,\mathfrak{m}_\eta,+}^\times = \{e \in \mathfrak{o}_{F,+}^\times \mid e \equiv 1 \pmod{\mathfrak{m}_\eta}\}$.

Hereafter we fix a set S (resp. T) of representatives of $(\mathfrak{m}_\eta^{-1}\mathfrak{d}_F^{-1}[t_1]^{-1}/\mathfrak{d}_F^{-1}[t_1]^{-1})^\times/\mathfrak{o}_{F,+}^\times$ in $\mathfrak{m}_\eta^{-1}\mathfrak{d}_F^{-1}[t_1]^{-1}$ (resp. $\mathfrak{o}_{F,+}^\times/E$ in $\mathfrak{o}_{F,+}^\times$) satisfying the condition that

$$\text{each cusp } b \in S \text{ is } \Gamma_{0,1}(\mathfrak{n})\text{-equivalent to the cusp } \infty. \tag{2.21}$$

Here we note that the existence of such set follows from the assumption $\mathfrak{n}|\mathfrak{m}_\eta$. Indeed, fix a generator m (resp. c) of $\mathfrak{m}_\eta$ (resp. $\mathfrak{d}_F[t_1]$) and a set S' of representatives of $(\mathfrak{o}_F/\mathfrak{m}_\eta)^\times$ satisfying that each $x \in S'$ is prime to mc. Then $\{x/mc \mid x \in S'\}$ is a set of representatives for $(\mathfrak{m}_\eta^{-1}\mathfrak{d}_F^{-1}[t_1]^{-1}/\mathfrak{d}_F^{-1}[t_1]^{-1})^\times/\mathfrak{o}_{F,+}^\times$. The assumption $\mathfrak{n}|\mathfrak{m}_\eta$ implies that $mc \in \mathfrak{n}\mathfrak{d}_F[t_1]$ and hence there is $\begin{pmatrix} x & * \\ mc & * \end{pmatrix} \in \Gamma_{0,1}^1(\mathfrak{n})$ as desired.

Let $\bar{b}$ denote the image of $b \in \mathfrak{m}_\eta^{-1}\mathfrak{d}_F^{-1}[t_1]^{-1}$ in $(\mathfrak{m}_\eta^{-1}\mathfrak{d}_F^{-1}[t_1]^{-1}/\mathfrak{d}_F^{-1}[t_1]^{-1})^\times/\mathfrak{o}_{F,+}^\times$ under the canonical map. Then

$$\begin{aligned}
(2.22)\qquad & N([t_1])^{s-k/2}\sum_{b\in S}\sum_{u\in T}\eta_1(\bar{b})^{-1}h_1(z+bu)\\
&= N([t_1])^{s-k/2}\sum_{0\ll\xi\in[t_1]}a_\infty(\xi,h_1)\sum_{b\in S}\sum_{u\in T}\eta_1(\bar{b})^{-1}e_F(\xi bu)e_F(\xi z)\\
&= N([t_1])^{s-k/2}\tau(\eta^{-1})\sum_{0\ll\xi\in[t_1]}a_\infty(\xi,h_1)\eta(\xi[t_1]^{-1})e_F(\xi z).
\end{aligned}$$

Here the last equality follows from [Shi, (3.11)] (or (1.13) in this paper).

By taking $\Omega_E = \coprod_{u\in T}u^{-1}\Omega_{\mathfrak{o}_{F,+}^\times}$, we have

$$\begin{aligned}
& N([t_1])^{s-k/2}\sum_{b\in S}\eta_1(\bar{b})^{-1}\int_{\Omega_E}h_1(z+b)y^{(s-1)t}dz_{J_F}\\
&= N([t_1])^{s-k/2}\sum_{b\in S}\eta_1(\bar{b})^{-1}\sum_{u\in T}\int_{u^{-1}\Omega_{\mathfrak{o}_{F,+}^\times}}h_1(z+b)y^{(s-1)t}dz_{J_F}\\
&= N([t_1])^{s-k/2}\sum_{b\in S}\eta_1(\bar{b})^{-1}\sum_{u\in T}\int_{\Omega_{\mathfrak{o}_{F,+}^\times}}h_1(z+bu)y^{(s-1)t}dz_{J_F}\\
&= \int_{\Omega_{\mathfrak{o}_{F,+}^\times}}N([t_1])^{s-k/2}\sum_{b\in S}\sum_{u\in T}\eta_1(\bar{b})^{-1}h_1(z+bu)y^{(s-1)t}dz_{J_F}.
\end{aligned}$$

Here we note that each integral is well-defined by using Proposition 2.12 (2), our assumption (2.19), and the condition (2.21). By using the Fourier expansion of (2.22), for $\mathrm{Re}(s)\gg 0$, we have

$$\begin{aligned}
& N([t_1])^{s-k/2}\sum_{b\in S}\eta_1(\bar{b})^{-1}\int_{\sqrt{-1}(F\otimes\mathbb{R})_+^\times/E}h_1(z+b)y^{(s-1)t}dz_{J_F}\\
&= \tau(\eta^{-1})N([t_1])^{s-k/2}\sum_{0\ll\xi\in[t_1]}a_\infty(\xi,h_1)\eta(\xi[t_1]^{-1})\int_{\Omega_{\mathfrak{o}_{F,+}^\times}}e_F(\xi z)y^{(s-1)t}dz_{J_F}\\
&= \tau(\eta^{-1})\sum_{0\ll\xi\in[t_1]}\frac{a_\infty(\xi,h_1)\eta(\xi[t_1]^{-1})N([t_1])^{-k/2}}{N(\xi[t_1]^{-1})^s}\int_{\Omega_{\mathfrak{o}_{F,+}^\times}}e_F(\xi z)(\xi y)^{(s-1)t}\bigwedge_{j=1}^n d\xi^{\sigma_j}z_{\sigma_j}\\
&= \tau(\eta^{-1})\sum_{\xi\mathfrak{o}_{F,+}^\times}\frac{a_\infty(\xi,h_1)\eta(\xi[t_1]^{-1})N([t_1])^{-k/2}}{N(\xi[t_1]^{-1})^s}\int_{\sqrt{-1}(F\otimes\mathbb{R})_+^\times}e_F(\xi z)(\xi y)^{(s-1)t}\bigwedge_{j=1}^n d\xi^{\sigma_j}z_{\sigma_j}\\
&= \tau(\eta^{-1})L(s,\mathbf{h},\eta)(2\pi)^{-sn}\sqrt{-1}^n\Gamma(s)^n.
\end{aligned}$$

Here we note that each integral is well-defined by using Proposition 2.12 (1), and we may regard $h_1(z+b)$ as a function on $\sqrt{-1}(F\otimes\mathbb{R})_+^\times/E$ since $h_1(uz+b)=h_1(z+b)$ for any $u\in E$. Furthermore, the integrals in the first term of this equation are independent of the choice of a lift b of $\bar{b}$. Indeed, $\int_{\sqrt{-1}(F\otimes\mathbb{R})_+^\times/E}h_1(z+bu)y^{(s-1)t}dz_{J_F} = \int_{\sqrt{-1}(F\otimes\mathbb{R})_+^\times/E}h_1(z+b)y^{(s-1)t}dz_{J_F}$ for any $u\in\mathfrak{o}_{F,+}^\times$ by substituting z by zu^{-1}, and $h_1(z+b)=h_1(z+b+a)$ for any $a\in\mathfrak{d}_F^{-1}[t_1]^{-1}$ since $\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\in\Gamma_{1,1}^1(\mathfrak{n})$. Hence the integral depends only on the image $\bar{b}$ of b in

$(\mathfrak{m}_\eta^{-1}\mathfrak{d}_F^{-1}[t_1]^{-1}/\mathfrak{d}_F^{-1}[t_1]^{-1})^\times/\mathfrak{o}_{F,+}^\times$ and we will denote it by

$$\int_{\sqrt{-1}(F\otimes\mathbb{R})_+^\times/E} h_1(z+\bar{b})y^{(s-1)t}dz_{J_F}.$$

Thus we obtain the following Mellin transform.

Proposition 2.13. *Assume that* $h_F^+ = 1$. *Let* $\kappa = (0,0)$, $\mathbf{h} = h_1 \in M_{(0,0)}(K_1(\mathfrak{n}), \chi, \mathbb{C})$ *satisfying (2.19). Let* $\eta : \mathrm{Cl}_F^+(\mathfrak{m}_\eta) \to \overline{\mathbb{Q}}^\times$ *be a primitive character whose conductor* $\mathfrak{m}_\eta$ *is prime to* $\mathfrak{d}_F[t_1]$, *and* $\mathfrak{n}|\mathfrak{m}_\eta$. *Then*

$$\sum_{b\in S}\eta_1(\bar{b})^{-1}\int_{\sqrt{-1}(F\otimes\mathbb{R})_+^\times/\mathfrak{o}_{F,\mathfrak{m},+}^\times} h_1(z+\bar{b})dz_{J_F}$$
$$= \tau(\eta^{-1})L(1,\mathbf{h},\eta)(-2\pi\sqrt{-1})^{-n}.$$

Remark 2.14. As mentioned above, the assumption $\mathfrak{n}|\mathfrak{m}_\eta$ and the conditions (2.19) and (2.21) imply that each integral is well-defined.

Remark 2.15. If $\mathbf{h}$ is a Hilbert cusp form, then the Mellin transform as Proposition 2.13 is satisfied without the assumption $\mathfrak{n}|\mathfrak{m}_\eta$.

We consider a Mellin transform in the anti-holomorphic case. Let $W_G = K_\infty/K_{\infty,+} = \{w_J \mid J \subset J_F\}$ be the Weyl group, where $w_J \in K_\infty$ such that $w_{J,\sigma} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ if $\sigma \in J$ and $w_{J,\sigma} = \begin{pmatrix}-1&0\\0&1\end{pmatrix}$ if $\sigma \in J_F - J$. Recall that the Weyl group acts on the space of Hilbert modular forms via $\mathbf{h} \mapsto \mathbf{h}|[K_\infty w_J K_\infty]$ for each subset $J \subset J_F$.

Proposition 2.16. *Under the same notation and assumptions of Proposition 2.13,*

$$\sum_{b\in S}\eta_1(\bar{b})^{-1}\int_{\sqrt{-1}(F\otimes\mathbb{R})_+^\times/\mathfrak{o}_{F,\mathfrak{m},+}^\times} h_{J,1}(z+\bar{b})dz_J$$
$$= \tau(\eta^{-1})L(1,\boldsymbol{h},\eta)\eta_\infty(\iota_J)(-2\pi\sqrt{-1})^{-n},$$

where dz_J *is defined by (2.1) and* $\iota_J \in \mathbb{A}_{F,\infty}$ *such that* $\iota_{J,\sigma} = 1$ *if* $\sigma \in J$ *and* $\iota_{J,\sigma} = -1$ *if* $\sigma \in J_F - J$.

Proof. Since $h_F^+ = 1$, we can take $a \in \mathfrak{o}_F^\times$ such that $\sigma(a) > 0$ if $\sigma \in J$ and $\sigma(a) < 0$ if $\sigma \in J_F - J$. By putting $\gamma = \begin{pmatrix}a&0\\0&1\end{pmatrix}$, the action of $[K_\infty w_J K_\infty]$ on $Y(\mathfrak{n}) = Y_1$ is given by $z \mapsto \gamma^{-1}z$. Then, by the definition, we have

$$h_{J,1}(z)dz_J = h_1(\gamma^{-1}z)(-1)^{\sharp(J_F-J)}dz_J$$

and

$$h_1(\gamma^{-1}z) = \sum_{0\ll\xi\in[t_1]} c(\xi[t_1]^{-1},\mathbf{h})N(\xi)e_F(\xi\gamma^{-1}z)$$
$$= \sum_{\mu\in[t_1],\{\mu\}=J} c(\mu[t_1]^{-1},\mathbf{h})|N(\mu)|e_F(\sqrt{-1}\mu y_\infty^{w_J})e_F(\mu x_\infty).$$

Here $\{\mu\} = \{\sigma \in J_F | \mu^\sigma > 0\}$, $y_{\infty,\sigma}^{w_J} = y_{\infty,\sigma}$ if $\sigma \in J$, $y_{\infty,\sigma}^{w_J} = -y_{\infty,\sigma}$ if $\sigma \in J_F - J$, the first equality follows from (1.9), and the last equality follows from the substitution $\mu = a^{-1}\xi$.

By the similar way as in the proof of Proposition 2.13, we obtain

$$\begin{aligned} &N([t_1])^{s-1}\sum_{b\in S}\eta_1(\bar{b})^{-1}\int_{\sqrt{-1}(F\otimes\mathbb{R})_+^\times/\mathfrak{o}_{F,\mathfrak{m},+}^\times} h_{J,1}(z+\bar{b})y^{(s-1)t}dz_J \\ &= \tau(\eta^{-1})L(s,\mathbf{h},\eta)\eta_\infty(\iota_J)(2\pi)^{-sn}(\sqrt{-1})^n\Gamma(s)^n. \end{aligned}$$

□

Remark 2.17. If $\mathbf{h}$ is a Hilbert cusp form, then the Mellin transform as Proposition 2.16 is satisfied without the assumption $\mathfrak{n}|\mathfrak{m}_\eta$.

2.8. **Relation between cocycle and Dirichlet series.** In this subsection, we give a cohomological treatment of Dirichlet series (1.12).

We consider the adélic Hilbert modular varieties $Y(\mathfrak{n}) = Y_1$ as (1.4). Let C_∞^1 be the subset of $C(\Gamma_{1,1}(\mathfrak{n}))$ consisting of s equivalent to the cusp ∞ over $\Gamma_{0,1}(\mathfrak{n})$. As the previous subsection, we assume that $h_F^+ = 1$ and fix a primitive character η whose conductor is $\mathfrak{m}_\eta$ and a lift $b\in S$ of $\bar{b}\in(\mathfrak{m}_\eta^{-1}\mathfrak{d}_F^{-1}[t_1]^{-1}/\mathfrak{d}_F^{-1}[t_1]^{-1})^\times/\mathfrak{o}_{F,+}^\times$. We consider the following subset H_b of $\mathfrak{H}^{J_F}$:

$$H_b = b+\sqrt{-1}(F\otimes\mathbb{R})_+^\times = \{b+\sqrt{-1}y \mid y\in(F\otimes\mathbb{R})_+^\times\}.$$

We define an action of $\mathfrak{o}_{F,\mathfrak{m}_\eta,+}^\times$ on H_b by

$$\varepsilon*(z_\sigma)_{\sigma\in J_F} = (\varepsilon^\sigma z_\sigma-(\varepsilon^\sigma-1)b)_{\sigma\in J_F}.$$

Since $(\varepsilon-1)b\in\mathfrak{d}_F^{-1}[t_1]^{-1}$ for any $\varepsilon\in\mathfrak{o}_{F,\mathfrak{m}_\eta,+}^\times$, we see that $\varepsilon*(z_\sigma)_\sigma$ is $\Gamma_{1,1}^1(\mathfrak{n})$-equivalent to $\varepsilon(z_\sigma)_\sigma$. Then we have $H_b/\mathfrak{o}_{F,\mathfrak{m}_\eta,+}^\times\to Y(\mathfrak{n})$ and it induces

$$H_c^n(Y(\mathfrak{n}),A)\to H_c^n(H_b/\mathfrak{o}_{F,\mathfrak{m}_\eta,+}^\times,A)$$

for $A=\mathcal{O}$, K, or $\mathbb{C}$. Since the cusp b is $\Gamma_{0,1}(\mathfrak{n})$-equivalent to the cusp ∞, it factors through the relative singular cohomology:

$$H^n(Y(\mathfrak{n})^{\mathrm{BS}},D_{C_\infty}(\mathfrak{n});A)\to H^n(Y(\mathfrak{n})^{\mathrm{BS}},D_{b,\infty}(\mathfrak{n});A)\to H_c^n(H_b/\mathfrak{o}_{F,\mathfrak{m}_\eta,+}^\times,A). \tag{2.23}$$

Here D_s^1 is the boundary of the Borel–Serre compactification Y_1^{BS} of Y_1 at each cusp s as §2.5, $D_{C_\infty}^1=\coprod_{s\in C_\infty^1}D_s^1$, $D_{C_\infty}(\mathfrak{n})=D_{C_\infty}^1$, $D_{b,\infty}^1=D_b^1\sqcup D_\infty^1$, and $D_{b,\infty}(\mathfrak{n})=D_{b,\infty}^1$.

Then we define the evaluation map

$$\mathrm{ev}_{b,1,A}:\widetilde{H}^n(Y(\mathfrak{n})^{\mathrm{BS}},D_{C_\infty}(\mathfrak{n});A)\to A \tag{2.24}$$

by the composition of (2.23) and the trace map $H_c^n(H_b/\mathfrak{o}_{F,\mathfrak{m}_\eta,+}^\times,A)\to A$, where

$$\widetilde{H}^n(Y(\mathfrak{n})^{\mathrm{BS}},D_{C_\infty}(\mathfrak{n});A)=H^n(Y(\mathfrak{n})^{\mathrm{BS}},D_{C_\infty}(\mathfrak{n});A)/A\text{-torsion}.$$

In order to relate our cohomology class $[\pi_*]$ and the special values of the L-functions, we recall the relative de Rham theory, which is proved by Borel [Bo, Theorem 5.2] for general locally symmetric spaces.

Let $\Omega^\bullet(Y_1,\mathbb{C})$ denote the complex of $\mathbb{C}$-valued C^∞-differential $\Gamma_{1,1}(\mathfrak{n})$-invariant forms in $\mathfrak{H}^{J_F}$. Moreover, let $\Omega_{\mathrm{fd}}^\bullet(Y_1,D_{C_\infty}^1;\mathbb{C})$ denote the complex of forms in $\Omega^\bullet(Y_1,\mathbb{C})$ which, together with their exterior differentials, are fast decreasing at each cusp $s\in C_\infty^1$. By the argument in the proof of [Bo, Theorem 5.2] on the stalks at the boundary, we have

$$H_{\mathrm{dR}}^n(Y_1,\Omega_{\mathrm{fd}}^\bullet(Y_1,D_{C_\infty}^1;\mathbb{C}))\simeq H^n(Y_1^{\mathrm{BS}},D_{C_\infty}^1;\mathbb{C}).$$

Let's fix $\mathbf{h}=h_1\in M_{(0,0)}(K_1(\mathfrak{n}),\chi,\mathbb{C})$ satisfying the assumptions of Proposition 2.9 and (2.19). Then, under the same notation of Proposition 2.9, by Proposition 2.10, $[\pi_{\mathbf{h}}]=$

$[\pi_{h_1}] \in H^n(Y(\mathfrak{n}), K)$, that is, $[\pi_{\mathbf{h}}]$ is rational. Moreover, by Proposition 2.9 (2), it is zero in the partial boundary cohomology $H^n(D_{C_\infty}(\mathfrak{n}), \mathbb{C})$. Let $[\pi_{\mathbf{h}}]_{\mathrm{rel}}$ denote the relative cohomology class in $H^n(Y(\mathfrak{n})^{\mathrm{BS}}, D_{C_\infty}(\mathfrak{n}); \mathbb{C})$ mapping to $[\pi_{\mathbf{h}}]$.

Proposition 2.18. *Assume that $h_F^+ = 1$. Let $\boldsymbol{h} = h_1 \in M_{(0,0)}(K_1(\mathfrak{n}), \chi, \mathbb{C})$ be a Hecke eigenform for all $T(\mathfrak{m})$ and $U(\mathfrak{m})$ satisfying the assumptions of Proposition 2.9, (2.19), and $C(\mathfrak{q}, \boldsymbol{h}) \not\equiv N(\mathfrak{q}) (\mathrm{mod}\ \varpi)$ for at least one prime ideal $\mathfrak{q}$ of $\mathfrak{o}_F$ dividing $\mathfrak{n}$. Then, under the same notation of Proposition 2.9, $[\pi_{\boldsymbol{h}}]_{\mathrm{rel}}$ is rational:*

$$[\pi_{\boldsymbol{h}}]_{\mathrm{rel}} \in H^n(Y(\mathfrak{n})^{BS}, D_{C_\infty}(\mathfrak{n}); K).$$

Proof. We use the same notation of the proof of (3.5). Let $\mathfrak{m}'_{\mathbf{h}}$ be the maximal ideal of $\mathbb{H}_2(\mathfrak{n}, \mathcal{O})'$ generated by ϖ and $U(\mathfrak{q}) - C(\mathfrak{q}, \mathbf{h})$ for all ideals $\mathfrak{q}$ of $\mathfrak{o}_F$ dividing $\mathfrak{n}$, which acts on the relative singular cohomology $H^n(Y(\mathfrak{n})^{\mathrm{BS}}, D_{C_\infty}(\mathfrak{n}); K)$. Let $[c]_{\mathrm{rel}} \in H^n(Y(\mathfrak{n})^{\mathrm{BS}}, D_{C_\infty}(\mathfrak{n}); K)_{\mathfrak{m}'_{\mathbf{h}}}$ mapping to $[\pi_{\mathbf{h}}] \in H^n(Y(\mathfrak{n}), K)_{\mathfrak{m}'_{\mathbf{h}}}$. Then $[c]_{\mathrm{rel}} - [\pi_{\mathbf{h}}]_{\mathrm{rel}}$ is in the image of $H^{n-1}(D_{C_\infty}(\mathfrak{n}), \mathbb{C})_{\mathfrak{m}'_{\mathbf{h}}}$. As we will mention just after (3.5), we have $H^{n-1}(D_{C_\infty}(\mathfrak{n}), \mathbb{C})_{\mathfrak{m}'_{\mathbf{h}}} = 0$ under the assumptions that $h_F^+ = 1$ and $C(\mathfrak{q}, \mathbf{h}) \not\equiv N(\mathfrak{q}) (\mathrm{mod}\ \varpi)$ for at least one prime ideal $\mathfrak{q}$ of $\mathfrak{o}_F$ dividing $\mathfrak{n}$. Thus $[\pi_{\mathbf{h}}]_{\mathrm{rel}} = [c]_{\mathrm{rel}}$ is rational. □

We note that, if $[\pi_{\mathbf{h}}]_{\mathrm{rel}} \in \widetilde{H}^n(Y(\mathfrak{n})^{\mathrm{BS}}, D_{C_\infty}(\mathfrak{n}); A)$, then, as mentioned just before Proposition 2.13, the value $\mathrm{ev}_{b,1,A}([\pi_{\mathbf{h}}]_{\mathrm{rel}})$ depends only on $\bar{b}$ and hence we will denote it by $\mathrm{ev}_{\bar{b},1,A}([\pi_{\mathbf{h}}]_{\mathrm{rel}})$. Then, by combining these observations and Proposition 2.13, we obtain the following description.

Proposition 2.19. *Assume that $h_F^+ = 1$, $\kappa = (0, 0)$, and $\mathbf{h} = h_1 \in M_{(0,0)}(K_1(\mathfrak{n}), \chi, \mathbb{C})$ satisfying (2.19) and $[\pi_{\mathbf{h}}]_{\mathrm{rel}} \in \widetilde{H}^n(Y(\mathfrak{n})^{BS}, D_{C_\infty}(\mathfrak{n}); A)$. Let $\eta : \mathrm{Cl}_F^+(\mathfrak{m}_\eta) \to \overline{\mathbb{Q}}^\times$ be a primitive character whose conductor $\mathfrak{m}_\eta$ is prime to $\mathfrak{d}_F[t_1]$, and $\mathfrak{n}|\mathfrak{m}_\eta$. Then*

$$\begin{aligned}&\sum_{b \in S} \eta_i(\bar{b})^{-1} \mathrm{ev}_{\bar{b},1,A}([\pi_{\boldsymbol{h}}]_{\mathrm{rel}}) \\ &= \tau(\eta^{-1}) L(1, \boldsymbol{h}, \eta)(-2\pi\sqrt{-1})^{-n} \in A(\eta).\end{aligned}$$

We also treat in the anti-holomorphic case under the assumption $h_F^+ = 1$. Since $h_F^+ = 1$, the action of $[K_\infty w_J K_\infty]$ on $Y(\mathfrak{n})$ is given by $z \to \gamma^{-1} z$, where $\gamma = \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}$ for some $\xi \in \mathfrak{o}_F^\times$ such that $\xi^\sigma > 0$ if $\sigma \in J$ and $\xi^\sigma < 0$ if $\sigma \in J_F - J$. By this description, we see that $[K_\infty w_J K_\infty]$ preserves the component $D_{C_\infty}(\mathfrak{n})$ and hence $[K_\infty w_J K_\infty]$ acts on $\widetilde{H}^n(Y(\mathfrak{n})^{\mathrm{BS}}, D_{b,\infty}(\mathfrak{n}); A)$.

We note that the group cohomology class $[\pi_{h_1}]_{\mathrm{rel}}|[K_\infty w_J K_\infty]$ corresponds to the de Rham cohomology class $h_{1,J}(z) dz_{{}_J}$ via the de Rham theorem. By using Proposition 2.16, the similar calculation shows the following proposition in the anti-holomorphic case.

Proposition 2.20. *Under the same notation and assumptions of Proposition 2.19, we have*

$$\begin{aligned}&\sum_{b \in S} \eta_1(\bar{b})^{-1} \mathrm{ev}_{\bar{b},1,A}\left([\pi_{\boldsymbol{h}}]_{\mathrm{rel}}|[K_\infty w_J K_\infty]\right) \\ &= \tau(\eta^{-1}) L(1, \boldsymbol{h}, \eta) \eta_\infty(\iota_J)(-2\pi\sqrt{-1})^{-n} \in A(\eta),\end{aligned}$$

where $\iota_J \in \mathbb{A}_{F,\infty}$ such that $\iota_{J,\sigma} = 1$ if $\sigma \in J$ and $\iota_{J,\sigma} = -1$ if $\sigma \in J_F - J$.

2.9. **Duality theorem between Hecke algebra and Hilbert modular form.** Hereafter, we simply write

$$M_2(\mathfrak{n},\mathbb{C}) = M_{(0,0),J_F}(K_1(\mathfrak{n}),\mathbb{C}),\ S_2(\mathfrak{n},\mathbb{C}) = S_{(0,0),J_F}(K_1(\mathfrak{n}),\mathbb{C})$$

and

$$M_2(\Gamma_{1,i}(\mathfrak{n}),\mathbb{C}) = M_{(0,0),J_F}(\Gamma_{1,i}(\mathfrak{n}),\mathbb{C}),\ S_2(\Gamma_{1,i}(\mathfrak{n}),\mathbb{C}) = S_{(0,0),J_F}(\Gamma_{1,i}(\mathfrak{n}),\mathbb{C}).$$

Recall that $\mathbf{h} = (h_i)_i \in M_2(\mathfrak{n},\mathbb{C})$ has the Fourier expansion of the form (1.9). For a subring A of $\mathbb{C}$, we put

$$M_2(\Gamma_{1,i}(\mathfrak{n}),A) = M_2(\Gamma_{1,i}(\mathfrak{n}),\mathbb{C}) \cap A[[e_F(\xi z) : \xi = 0 \text{ or } 0 \ll \xi \in F]],$$
$$S_2(\Gamma_{1,i}(\mathfrak{n}),A) = S_2(\Gamma_{1,i}(\mathfrak{n}),\mathbb{C}) \cap A[[e_F(\xi z) : \xi = 0 \text{ or } 0 \ll \xi \in F]],$$

and

$$M_2(\mathfrak{n},A) = \bigoplus_{i=1}^{h_F^+} M_2(\Gamma_{1,i}(\mathfrak{n}),A), \qquad S_2(\mathfrak{n},A) = \bigoplus_{i=1}^{h_F^+} S_2(\Gamma_{1,i}(\mathfrak{n}),A). \tag{2.25}$$

Let Φ_p be the field as Proposition 2.9. We fix a finite extension K of Φ_p. Let $\mathcal{O}$ be the ring of integers of K, ϖ a uniformizer, and κ the residue field. We shall write $A = K$ or $\mathcal{O}$ and use the same notation as §1.2.

We define the Hecke operators $T(\varpi_{\mathfrak{q}}^e)$ for a prime ideal $\mathfrak{q}$ of $\mathfrak{o}_F$ and a uniformizer $\varpi_{\mathfrak{q}}$ of $\mathfrak{o}_{F_{\mathfrak{q}}}$ and $S(\varpi_{\mathfrak{q}}^e)$ for a prime ideal $\mathfrak{q}$ of $\mathfrak{o}_F$ such that $\mathfrak{q} \nmid \mathfrak{n}$ by the following double coset:

$$T(\varpi_{\mathfrak{q}}^e) = K_1(\mathfrak{n}) \begin{pmatrix} \varpi_{\mathfrak{q}}^e & 0 \\ 0 & 1 \end{pmatrix} K_1(\mathfrak{n}) \text{ and } S(\varpi_{\mathfrak{q}}^e) = K_1(\mathfrak{n}) \begin{pmatrix} \varpi_{\mathfrak{q}}^e & 0 \\ 0 & \varpi_{\mathfrak{q}}^e \end{pmatrix} K_1(\mathfrak{n}).$$

We put $T(\mathfrak{q}^e) = T(\varpi_{\mathfrak{q}}^e)$ and $S(\mathfrak{q}^e) = S(\varpi_{\mathfrak{q}}^e)$ for a prime ideal $\mathfrak{q}$ prime to $\mathfrak{n}$, and $U(\mathfrak{q}^e) = T(\varpi_{\mathfrak{q}}^e)$ for a prime ideal $\mathfrak{q}$ dividing $\mathfrak{n}$. Then we define

$$T(\mathfrak{m}) = \prod_{\mathfrak{q}\nmid\mathfrak{n}} T(\mathfrak{q}^{e(\mathfrak{q})}) \text{ and } S(\mathfrak{m}) = \prod_{\mathfrak{q}\nmid\mathfrak{n}} S(\mathfrak{q}^{e(\mathfrak{q})})$$

for any integral ideal $\mathfrak{m} = \prod_{\mathfrak{q}\nmid\mathfrak{n}} \mathfrak{q}^{e(\mathfrak{q})}$ of F prime to $\mathfrak{n}$ and

$$U(\mathfrak{m}) = \prod_{\mathfrak{q}\mid\mathfrak{n}} U(\mathfrak{q}^{e(\mathfrak{q})})$$

for any integral ideal $\mathfrak{m} = \prod_{\mathfrak{q}\mid\mathfrak{n}} \mathfrak{q}^{e(\mathfrak{q})}$ of F dividing $\mathfrak{n}$.

Let $\mathbb{H}_2(\mathfrak{n},A)$ (resp. $\mathcal{H}_2(\mathfrak{n},A)$) be the commutative Hecke A-subalgebra of $\mathrm{End}_{\mathbb{C}}(M_2(\mathfrak{n},\mathbb{C}))$ (resp. $\mathrm{End}_{\mathbb{C}}(S_2(\mathfrak{n},\mathbb{C}))$) generated by $T(\mathfrak{m})$, $S(\mathfrak{m})$ for all ideals $\mathfrak{m}$ of $\mathfrak{o}_F$ prime to $\mathfrak{n}$, and $U(\mathfrak{m})$ for all ideals $\mathfrak{m}$ of $\mathfrak{o}_F$ dividing $\mathfrak{n}$ as (1.7).

Then, by [Shi, (2.23)], there is a relation between the Hecke operators and the Fourier expansion of the following form: for $V(\mathfrak{m}') = T(\mathfrak{m}')$ or $U(\mathfrak{m}')$, we have

$$C(\mathfrak{m},\mathbf{f}|V(\mathfrak{m}')) = \sum_{\mathfrak{m}+\mathfrak{m}'\subset\mathfrak{c}} N(\mathfrak{c})C(\mathfrak{c}^{-2}\mathfrak{m}\mathfrak{m}',\mathbf{f}|S(\mathfrak{c})). \tag{2.26}$$

According to [Hida88, Theorem 4.11] and [Hida91, Theorem 2.2 (ii)], the space $S_2(\mathfrak{n},A)$ (resp. $M_2(\mathfrak{n},A)$) is stable under $\mathcal{H}_2(\mathfrak{n},A)$ (resp. $\mathbb{H}_2(\mathfrak{n},A)$).

Theorem 2.21 (Duality theorem)**.** *Assume that $p > 3$ is prime to the discriminant $\triangle_F$ of F. Let K be any finite extension of Φ_p and $\mathcal{O}$ its ring of integers . Then, for $A = K$ or $\mathcal{O}$,*

$$\langle\ ,\ \rangle : \mathbb{H}_2(\mathfrak{n},A) \times M_2(\mathfrak{n},A) \to A : (t,\boldsymbol{f}) \mapsto C(\mathfrak{o}_F,\boldsymbol{f}|t)$$

is a perfect pairing.

Proof. We follow the arguments in the proof of [Hida88, Theorem 5.1] and [Hida91, Theorem 2.2 (iii)].

First we assume that $A = K$. According to the proof of [Hida91, Theorem 2.2 (iii)], $M_2(\mathfrak{n}, K)$ is of finite dimension over K. Thus, it is enough to prove the non-degeneracy of the pairing. Suppose $\langle t, \mathbf{f}\rangle = 0$ for all t. By the relation (2.26), we have

$$C(\mathfrak{m}, \mathbf{f}) = C(\mathfrak{o}_F, \mathbf{f}|V(\mathfrak{m})) = \langle V(\mathfrak{m}), \mathbf{f}\rangle = 0$$

for $V(\mathfrak{m}) = T(\mathfrak{m})$ or $U(\mathfrak{m})$ and all integral ideals $\mathfrak{m}$ of F. Thus, $\mathbf{f}$ is a constant function and hence $\mathbf{f} = 0$ because the weight of $\mathbf{f}$ is positive. Conversely, if $\langle t, \mathbf{f}\rangle = 0$ for all $\mathbf{f}$, then for $V(\mathfrak{m}) = T(\mathfrak{m})$ or $U(\mathfrak{m})$ and all integral ideals $\mathfrak{m}$ of F, we have

$$C(\mathfrak{m}, \mathbf{f}|t) = C(\mathfrak{o}_F, \mathbf{f}|tV(\mathfrak{m})) = C(\mathfrak{o}_F, \mathbf{f}|V(\mathfrak{m})t) = \langle t, \mathbf{f}|V(\mathfrak{m})\rangle = 0.$$

Thus, $\mathbf{f}|t = 0$ and hence $h = 0$ as an operator. This proves the assertion for $A = K$.

Next suppose that $A = \mathcal{O}$. It suffices to prove that

$$M_2(\mathfrak{n}, \mathcal{O}) \simeq \mathrm{Hom}_{\mathcal{O}}\left(\mathbb{H}_2(\mathfrak{n}, \mathcal{O}), \mathcal{O}\right).$$

If $\phi : \mathbb{H}_2(\mathfrak{n}, \mathcal{O}) \to \mathcal{O}$ is an $\mathcal{O}$-linear map, then we can extend to a K-linear map $\phi : \mathbb{H}_2(\mathfrak{n}, K) \to K$. Thus, by the duality theorem for a field K, we get $\mathbf{f} \in M_2(\mathfrak{n}, K)$ such that $\langle t, \mathbf{f}\rangle = \phi(h)$ for all $t \in \mathbb{H}_2(\mathfrak{n}, \mathcal{O})$. Then for $V(\mathfrak{m}) = T(\mathfrak{m})$ or $U(\mathfrak{m})$ and every ideal $\mathfrak{m}$ of $\mathfrak{o}_F$, we have

$$C(\mathfrak{m}, \mathbf{f}) = C(\mathfrak{o}_F, \mathbf{f}|V(\mathfrak{m})) = \langle V(\mathfrak{m}), \mathbf{f}\rangle = \phi(V(\mathfrak{m})) \in \mathcal{O}.$$

Suppose that the constant term of $\mathbf{f}$ does not belong to $\mathcal{O}$, that is, $a_\infty(0, f_i) \notin \mathcal{O}$ for some i. Let $r \in \mathbb{Z}$ be the positive integer such that $\varpi^r a_\infty(0, f_i) \in \mathcal{O}^\times$. Then the q-expansion of $\varpi^r f_i$ is equal to $\varpi^r a_\infty(0, f_i)$ modulo ϖ. By [An–Go], the kernel of q-expansion map on the space of Hilbert modular forms of all parallel weight is generated by $H_{p-1}-1$, where H_{p-1} is the Hasse invariant of level 1 and parallel weight $p-1$. Then we have $\varpi^r f_i - \varpi^r a_\infty(0, f_i) = \alpha(H_{p-1}-1)$ for some $\alpha \in \kappa$. Since the weight of H_{p-1} is $p-1 > 2$, this contradictions. Thus $\mathbf{f} \in M_2(\mathfrak{n}, \mathcal{O})$ as desired. □

2.10. **Congruence modules and Integrality of cocycles.** In this subsection, we will determine the structure of a congruence module associated to an Eisenstein series. As applications, we will prove the integrality of Eisenstein cocycles based on [Be, §4] and [Eme] and construct an example of a congruence between a Hilbert cusp form and an Eisenstein series based on [Ri] and [Wil].

We use the same notation as §1.2, §2.4, and §2.9. We simply write $\Gamma_{1,i} = \Gamma_{1,i}(\mathfrak{n})$, $Y_i = \overline{\Gamma_{1,i}}\backslash\mathfrak{H}^{J_F}$. For $? = \phi$ or par and $X = Y(\mathfrak{n})$, Y_i, or $\partial(Y_i^{\mathrm{BS}})$ as (1.4) and §2.5, we write

$$\widetilde{H}^m_?(X, \mathcal{O}) = \mathrm{im}\left(H^m_?(X, \mathcal{O}) \to H^m_?(X, K)\right)$$

for the torsion-free part of $H^m_?(X, \mathcal{O})$, where, for $A = \mathcal{O}$ or K, $H^m_c(X, A)$ is the compact support cohomology of X with coefficients in A and $H^m_{\mathrm{par}}(X, A) = \mathrm{im}\left(H^m_c(X, A) \to H^m(X, A)\right)$ is the parabolic cohomology of X with coefficients in A. For $[\pi] \in H^n(X, K)$, let

$$\delta([\pi]) = \left\{a \in \mathcal{O} \middle| a[\pi] \in \widetilde{H}^n(X, \mathcal{O})\right\}$$

be the denominator ideal in the sense of Berger ([Be, §4.1]). We fix an Eisenstein series $\mathbf{E} = \mathbf{E}_2(\varphi, \psi)$ as Proposition 1.1 such that primitive narrow ray class characters φ and ψ

satisfy $\mathfrak{m}_\varphi \mathfrak{m}_\psi = \mathfrak{n}$ and

(Eis condition) φ and ψ are $\mathcal{O}$-valued totally even (resp. totally odd) such that $\varphi \neq \mathbf{1}$ and the algebraic Iwasawa μ-invariants of the splitting fields of φ and ψ are equal to 0 (see Remark 0.2).

Then, by Proposition 1.1 (3), the Eisenstein series $\mathbf{E}$ satisfies (2.19). We put the character $\epsilon_{\mathbf{E}} = -\mathbf{1}$ (resp. $\epsilon_{\mathbf{E}} = \mathbf{1}$) on the Weyl group W_G if both φ and ψ are totally even (resp. totally odd). Put $\chi = \varphi\psi$. We denote by $\mathbb{I}$ an ideal of $\mathbb{H}_2(\mathfrak{n}, \mathcal{O})$ generated by $T(\mathfrak{m}) - C(\mathfrak{m}, \mathbf{E}), S(\mathfrak{m}) - \chi^{-1}(\mathfrak{m}), U(\mathfrak{m}) - C(\mathfrak{m}, \mathbf{E})$ for all integral ideals $\mathfrak{m}$ of F. Let $\mathcal{I}$ be the image of $\mathbb{I}$ under the canonical surjection $\mathbb{H}_2(\mathfrak{n}, \mathcal{O}) \twoheadrightarrow \mathcal{H}_2(\mathfrak{n}, \mathcal{O})$. The module $\mathcal{H}_2(\mathfrak{n}, \mathcal{O})/\mathcal{I}$ is the congruence module associated to the Eisenstein series $\mathbf{E}$ in the sense of Hida. By [Hida88, p.329–333], the spaces of classical modular forms $S_2(\Gamma_{1,i}, \mathcal{O})$ and $M_2(\Gamma_{1,i}, \mathcal{O})$ can be embedded into the space of geometric modular forms $M_{(0,0)}(M, \mathcal{O})$. For this reason, if $f \in M_2(\Gamma_{1,i}, \mathcal{O})$, then the constant term of f at each cusp point belongs to $\mathcal{O}$ by the q-expansion principle. Thus, by Theorem 2.21 (Duality theorem), for each i with $1 \leq i \leq h_F^+$ and each cusp $s \in C(\Gamma_{1,i})$, we can take $A_{i,s} \in \mathbb{H}_2(\mathfrak{n}, \mathcal{O})$ corresponding a map

$$M_2(\mathfrak{n}, \mathcal{O}) \to \mathcal{O} : \mathbf{f} \mapsto a_s(0, f_i),$$

where $a_s(0, f_i)$ is the constant term of f_i at s. Let (i_0, s_0) be a pair such that $v_p(a_{s_0}(0, E_{i_0})) \leq v_p(a_s(0, E_i))$ for each (i, s) under the p-adic valuation v_p. We put

$$C = a_{s_0}(0, E_{i_0}).$$

In order to state the main theorem of this subsection, we recall the Eichler–Shimura–Harder isomorphism. The theorem [Hida93, Theorem 1.1] says that the $\mathbb{C}$-vector space $H^n_{\mathrm{par}}(Y(\mathfrak{n}), \mathbb{C})/H^n_{\mathrm{cusp}}(Y(\mathfrak{n}), \mathbb{C})$ is spanned by the cohomology classes of the invariant forms $\omega_{J'} = \bigwedge_{\sigma \in J'} y_\sigma^{-2} dx_\sigma \wedge dy_\sigma$ with $\sharp J' = n/2$ if $n = [F : \mathbb{Q}]$ is even. Moreover, by [Hida88, §7], both $H^n_{\mathrm{par}}(Y(\mathfrak{n}), \mathbb{C})$ and $H^n_{\mathrm{cusp}}(Y(\mathfrak{n}), \mathbb{C})$ are W_G-modules. Since $h_F^+ = 1$, as mentioned just after Proposition 2.19, for each subset $J \subset J_F$, the Weyl action of $((1_\sigma)_{\sigma \in J}, (-1_\sigma)_{\sigma \in J_F - J}) \in W_G$ on $Y(\mathfrak{n}) = Y_1$ is given by

$$\begin{aligned}((x_\sigma + \sqrt{-1}y_\sigma)_{\sigma \in J}, &(x_\sigma + \sqrt{-1}y_\sigma)_{\sigma \in J_F - J}) \\ &\mapsto \left(\xi^\sigma (x_\sigma + \sqrt{-1}y_\sigma)_{\sigma \in J}, (-\xi)^\sigma(-x_\sigma + \sqrt{-1}y_\sigma)_{\sigma \in J_F - J}\right)\end{aligned}$$

for some $\xi \in \mathfrak{o}_F^\times$. In the case $n = [F : \mathbb{Q}]$ is even, if a character ϵ on W_G satisfying $\sharp\{\sigma \in J_F \mid \epsilon(-1_\sigma) = -1\} \neq n/2$, then $H^n_{\mathrm{par}}(Y(\mathfrak{n}), \mathbb{C})[\epsilon] = H^n_{\mathrm{cusp}}(Y(\mathfrak{n}), \mathbb{C})[\epsilon]$, where $V[\epsilon] = \{v \in V \mid w \cdot v = \epsilon(w)v \text{ for } w \in W_G\}$ is the ϵ-isotypic part of this action for any W_G-module V. In particular, we obtain

$$H^n_{\mathrm{par}}(Y(\mathfrak{n}), \mathbb{C})[\epsilon_{\mathbf{E}}] \simeq H^n_{\mathrm{cusp}}(Y_1(\mathfrak{n}), \mathbb{C})[\epsilon_{\mathbf{E}}] \simeq S_2(\mathfrak{n}, \mathbb{C}) \tag{2.27}$$

as Hecke modules (cf. [Hida94, §2,§3]). Thus we will use that the Hecke algbra $\mathcal{H}_2(\mathfrak{n}, \mathcal{O})$ is isomorphic to the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}\left(\widetilde{H}^n_{\mathrm{par}}(Y(\mathfrak{n}), \mathcal{O})[\epsilon_{\mathbf{E}}]\right)$. Moreover, we can decompose

$$H^n(Y(\mathfrak{n}), \mathbb{C})[\epsilon_{\mathbf{E}}] \simeq H^n_{\mathrm{par}}(Y(\mathfrak{n}), \mathbb{C})[\epsilon_{\mathbf{E}}] \oplus H^n_{\mathrm{Eis}}(Y(\mathfrak{n}), \mathbb{C})[\epsilon_{\mathbf{E}}]$$

and the Hodge-Tate weight of $H^n_{\mathrm{Eis}}(Y(\mathfrak{n}), \mathbb{C})$ is n by Proposition 4.6. Here $H^n_{\mathrm{Eis}}(Y(\mathfrak{n}), \mathbb{C})$ is the Eisenstein cohomology (for the definition, see **Step2** and **Step3** in the proof of Proposition 4.6). Then we have $\mathbb{H}_2(\mathfrak{n}, \mathcal{O}) \to \mathrm{End}_{\mathcal{O}}\left(\widetilde{H}^n(Y(\mathfrak{n}), \mathcal{O})[\epsilon_{\mathbf{E}}]\right)$. Let $\mathbb{H}_2(\mathfrak{n}, \mathcal{O})[\epsilon_{\mathbf{E}}]$ (resp. $I[\epsilon_{\mathbf{E}}]$) denote the image of $\mathbb{H}_2(\mathfrak{n}, \mathcal{O})$ (resp. each ideal I of $\mathbb{H}_2(\mathfrak{n}, \mathcal{O})$) under this map.

Theorem 2.22. *Let $F/\mathbb{Q}$ be a totally real number field with $h_F^+ = 1$ and $p > 3$ a prime number such that p is prime to $\mathfrak{n}$ and $\triangle_F$. We assume the following two conditions:*
(a) both $H^n(\partial(Y(\mathfrak{n})^{BS}), \mathcal{O})$ and $H_c^{n+1}(Y(\mathfrak{n}), \mathcal{O})$ are torsion-free;
(b) $C(\mathfrak{q}, \boldsymbol{E}) \not\equiv N(\mathfrak{q})(\mathrm{mod}\varpi)$ for some prime ideal $\mathfrak{q}$ dividing $\mathfrak{n}$, where $C(\mathfrak{q}, \boldsymbol{E})$ is the $U(\mathfrak{q})$-eigenvalue of $\boldsymbol{E}$.

Then there is an isomorphism of $\mathcal{O}$-modules

$$\mathbb{H}_2(\mathfrak{n}, \mathcal{O})[\epsilon_{\boldsymbol{E}}]/(\mathbb{I} + \sum_{s \in C(\Gamma_{1,1})} \mathcal{O}A_{1,s})[\epsilon_{\boldsymbol{E}}] \simeq \mathcal{H}_2(\mathfrak{n}, \mathcal{O})/\mathcal{I} \simeq \mathcal{O}/C.$$

Proof. By the definition, we have the canonical surjection

$$\mathbb{H}_2(\mathfrak{n}, \mathcal{O})[\epsilon_{\mathbf{E}}]/(\mathbb{I} + \sum_{s \in C(\Gamma_{1,1})} \mathcal{O}A_{1,s})[\epsilon_{\mathbf{E}}] \twoheadrightarrow \mathcal{H}_2(\mathfrak{n}, \mathcal{O})/\mathcal{I}. \tag{2.28}$$

Let $\mathbf{G} = \mathbf{E}/C$ and $[\pi_{\mathbf{G}}]^{\epsilon_{\mathbf{E}}} = [\pi_{G_1}]^{\epsilon_{\mathbf{E}}} \in H^n(Y(\mathfrak{n}), \mathbb{C})[\epsilon_{\mathbf{E}}]$. Here $[\pi_{G_1}]^{\epsilon_{\mathbf{E}}}$ stands for the projection to the $\epsilon_{\mathbf{E}}$-part of $[\pi_{G_1}]$. We have $[\pi_{\mathbf{G}}] \in H^n(Y(\mathfrak{n}), K)$ by Proposition 2.10. Let $\delta_{\mathbf{G}} = \delta([\pi_{\mathbf{G}}]^{\epsilon_{\mathbf{E}}})$. Next, we construct a surjection

$$\mathcal{H}_2(\mathfrak{n}, \mathcal{O})/\mathcal{I} \twoheadrightarrow \mathcal{O}/\delta_{\mathbf{G}}. \tag{2.29}$$

By the calculation of the constant term of an n-cocycle (Proposition 2.9), we have

$$\mathrm{res}([\pi_{\mathbf{G}}]) = \mathrm{res}[\pi_{G_1}] \in \widetilde{H}^n(\partial(Y_1^{\mathrm{BS}}), \mathcal{O}) \simeq \bigoplus_{s \in C(\Gamma_{1,1})} \widetilde{H}^n(\overline{\Gamma_{1,1}(\mathfrak{n})}_s, \mathcal{O}).$$

The torsion-free assumption implies $\widetilde{H}_c^{n+1}(Y(\mathfrak{n}), \mathcal{O}) = H_c^{n+1}(Y(\mathfrak{n}), \mathcal{O})$. Moreover, by the definition, the image of $\mathrm{res}([\pi_{\mathbf{G}}])$ under the connecting homomorphism $H^n(\partial(Y(\mathfrak{n})^{\mathrm{BS}}), K) \to H_c^{n+1}(Y(\mathfrak{n}), K)$ is equal to 0. Thus, there is $[c] = [c_1]$ in $\widetilde{H}^n(Y(\mathfrak{n}), \mathcal{O})[\epsilon_{\mathbf{E}}]$ such that

$$\mathrm{res}([c]) = \mathrm{res}([\pi_{\mathbf{G}}]^{\epsilon_{\mathbf{E}}}).$$

Thus we get

$$[c] - [\pi_{\mathbf{G}}]^{\epsilon_{\mathbf{E}}} \in H^n_{\mathrm{par}}(Y(\mathfrak{n}), K)[\epsilon_{\mathbf{E}}].$$

We fix $d \in \delta_{\mathbf{G}}$. We put $[e_0] = d([c] - [\pi_{\mathbf{G}}]^{\epsilon_{\mathbf{E}}}) \in \widetilde{H}^n_{\mathrm{par}}(Y(\mathfrak{n}), \mathcal{O})[\epsilon_{\mathbf{E}}]$. Then we may assume $[e_0] \neq 0$. Indeed, if $[e_0] = 0$, then $[c] = [\pi_{\mathbf{G}}]^{\epsilon_{\mathbf{E}}}$ and hence $\delta_{\mathbf{G}} = 1$. Let $[e_0], \cdots, [e_v]$ be an $\mathcal{O}$-basis of $\widetilde{H}^n_{\mathrm{par}}(Y(\mathfrak{n}), \mathcal{O})[\epsilon_{\mathbf{E}}]$. For each $t \in \mathcal{H}_2(\mathfrak{n}, \mathcal{O})$, we write

$$t([e_0]) = \sum_{i=0}^{v} \lambda_i(t)[e_i]$$

with $\lambda_i(t) \in \mathcal{O}$. Thus we define a surjection

$$\mathcal{H}_2(\mathfrak{n}, \mathcal{O}) \twoheadrightarrow \mathcal{O}/\delta_{\mathbf{G}} : t \mapsto \lambda_0(t).$$

This $\mathcal{O}$-homomorphism factors through the congruence module $\mathcal{H}_2(\mathfrak{n}, \mathcal{O})/\mathcal{I}$. Indeed, for $t \in \mathcal{I}$ and its lift $\widetilde{t} \in \mathbb{I}$, we have

$$t([e_0]) = d \cdot \widetilde{t}([c] - [\pi_{\mathbf{G}}]^{\epsilon_{\mathbf{E}}}) = d \cdot \widetilde{t}[c] \equiv 0 \pmod{d},$$

because the map $\mathbf{G} \mapsto [\pi_{\mathbf{G}}]$ is compatible with the Hecke operators (Proposition 2.7). Thus we get

$$\mathcal{H}_2(\mathfrak{n}, \mathcal{O})/\mathcal{I} \twoheadrightarrow \mathcal{O}/\delta_{\mathbf{G}}.$$

Next we construct

$$\mathcal{O}/\delta_{\mathbf{G}} \twoheadrightarrow \mathcal{O}/C. \tag{2.30}$$

Let η_p be a non-trivial primitive narrow ray class character corresponding to a character of $\mathrm{Gal}(F(\zeta_{p^\infty})/F)$ of finite order with $\eta_p = \epsilon_{\mathbf{E}}$ on $W_G \simeq \mathbb{A}_{F,\infty}^\times/\mathbb{A}_{F,\infty,+}^\times$. We put $\eta = \eta_p\varphi^{-1}\psi^{-1}$. Note that $\mathfrak{n}|\mathfrak{m}_\eta$. We fix $d \in \delta_{\mathbf{G}}$ and then $d[\pi_{\mathbf{G}}] = d/C \cdot [\pi_{\mathbf{E}}] \in \widetilde{H}^n(Y(\mathfrak{n}), \mathcal{O})$. Moreover, $d[\pi_{\mathbf{G}}]_{\mathrm{rel}} \in H^n(Y(\mathfrak{n})^{\mathrm{BS}}, D_{C_\infty}(\mathfrak{n}); K)$ by Proposition 2.18. We claim that $d[\pi_{\mathbf{G}}]_{\mathrm{rel}}$ is integral:

$$d[\pi_{\mathbf{G}}]_{\mathrm{rel}} \in \widetilde{H}^n(Y(\mathfrak{n})^{\mathrm{BS}}, D_{C_\infty}(\mathfrak{n}); \mathcal{O}). \tag{2.31}$$

For the moment we admit the claim. Since $\mathfrak{n}|\mathfrak{m}_\eta$, by using Proposition 2.19 and Proposition 2.20,

$$\begin{aligned}
&\sum_{b\in S} \eta_1(\bar{b})^{-1}\mathrm{ev}_{b,1,\mathcal{O}}(d[\pi_{\mathbf{G}}]_{\mathrm{rel}}^{\epsilon_{\mathbf{E}}}) \\
&= \frac{d}{C}\cdot\tau(\eta^{-1})\cdot\frac{\sqrt{-1}^n}{(2\pi)^n}\cdot D(1,\mathbf{E},\eta) \\
&= \frac{d}{C}\cdot\tau(\eta^{-1})\cdot\frac{\sqrt{-1}^n}{2^n}\cdot\frac{L(1,\eta\varphi)}{\pi^n}\cdot L(0,\eta\psi) \\
&= \frac{d}{C}\cdot\frac{(-1)^n}{2^n\triangle_F^{1/2}}\cdot\frac{\tau(\varphi\psi)\varphi\psi(\mathfrak{m}_{\eta_p})\eta_p(\mathfrak{m}_\psi)}{\tau(\psi)\psi(\mathfrak{m}_{\eta_p})\eta_p(\mathfrak{m}_{\varphi\psi})}\cdot L(0,\eta_p^{-1}\psi)\cdot L(0,\eta_p\varphi^{-1}) \in \mathcal{O}(\eta).
\end{aligned}\tag{2.32}$$

Here the first equality follows from Proposition 2.19 and Proposition 2.20, the second equality follows from Proposition 1.1 (1), and the last equality follows from the functional equation for Hecke L-functions (see, for example, [Mi, Theorem 3.3.1]) using that $\eta\varphi = \eta_p\psi^{-1}$ is totally odd and [Mi, (3.3.11)]. Since both $\eta_p\psi^{-1}$ and $\eta_p\varphi^{-1}$ are totally odd, the left hand side is non-zero by using the functional equation for Hecke L-functions (see, for example, [Da–Da–Po, Lemma 1.1]). We remark that the second and third terms in (2.32) are prime to p. Moreover, by (Eis condition) with the help of the Iwasawa main conjecture for totally real number fields proved by Wiles [Wil], the p-adic valuation of $L(0, \eta_p^{-1}\psi)$ and $L(0, \eta_p\varphi^{-1})$ are smaller than that of ϖ for all but finitely many narrow ray class character η_p with $\eta_p = \epsilon_{\mathbf{E}}$ on W_G. Therefore we obtain that $C \mid d$ as required.

Thus it remains to prove the claim (2.31). We use the same notation as the proof of (3.5). Let $\mathfrak{m}'_{\mathbf{E}}$ be the maximal ideal of $\mathbb{H}_2(\mathfrak{n}, \mathcal{O})'$ generated by ϖ and $U(\mathfrak{q}) - C(\mathfrak{q}, \mathbf{E})$ for all ideals $\mathfrak{q}$ of $\mathfrak{o}_F$ dividing $\mathfrak{n}$, which acts on the torsion-free part of the relative singular cohomology $\widetilde{H}^n(Y(\mathfrak{n})^{\mathrm{BS}}, D_{C_\infty}(\mathfrak{n}); \mathcal{O})$. By Proposition 2.9 and (2.19), $d[\pi_{\mathbf{G}}]$ is zero in the torsion-free part of the partial boundary cohomology $\widetilde{H}^n(D_{C_\infty}(\mathfrak{n}), \mathcal{O})_{\mathfrak{m}'_{\mathbf{E}}}$. Moreover, the torsion-free assumption implies $\widetilde{H}^n(D_{C_\infty}(\mathfrak{n}), \mathcal{O})_{\mathfrak{m}'_{\mathbf{E}}} = H^n(D_{C_\infty}(\mathfrak{n}), \mathcal{O})_{\mathfrak{m}'_{\mathbf{E}}}$. If we fix $[\omega]' \in H^n(Y(\mathfrak{n}), \mathcal{O})_{\mathfrak{m}'_{\mathbf{E}}}$ mapping to $d[\pi_{\mathbf{G}}] \in \widetilde{H}^n(Y(\mathfrak{n}), \mathcal{O})_{\mathfrak{m}'_{\mathbf{E}}}$, then $[\omega]'$ is zero in $H^n(D_{C_\infty}(\mathfrak{n}), \mathcal{O})_{\mathfrak{m}'_{\mathbf{E}}}$. Let $[\omega]'_{\mathrm{rel}} \in H^n(Y(\mathfrak{n})^{\mathrm{BS}}, D_{C_\infty}(\mathfrak{n}); \mathcal{O})_{\mathfrak{m}'_{\mathbf{E}}}$ mapping to $[\omega]' \in H^n(Y(\mathfrak{n}), \mathcal{O})_{\mathfrak{m}'_{\mathbf{E}}}$ and let $[\omega]_{\mathrm{rel}}$ denote the image of $[\omega]'_{\mathrm{rel}}$ in $\widetilde{H}^n(Y(\mathfrak{n})^{\mathrm{BS}}, D_{C_\infty}(\mathfrak{n}); \mathcal{O})_{\mathfrak{m}'_{\mathbf{E}}}$. Then $[\omega]_{\mathrm{rel}} - d[\pi_{\mathbf{G}}]_{\mathrm{rel}}$ is in the image of $H^{n-1}(D_{C_\infty}(\mathfrak{n}), K)_{\mathfrak{m}'_{\mathbf{E}}}$. As we will mention just after (3.5), we have $H^{n-1}(D_{C_\infty}(\mathfrak{n}), \mathbb{C})_{\mathfrak{m}'_{\mathbf{E}}} = 0$ under the assumptions that $h_F^+ = 1$ and $C(\mathfrak{q}, \mathbf{E}) \not\equiv N(\mathfrak{q}) (\mathrm{mod}\ \varpi)$ for at least one prime ideal $\mathfrak{q}$ of $\mathfrak{o}_F$ dividing $\mathfrak{n}$. Thus $d[\pi_{\mathbf{G}}]_{\mathrm{rel}} = [\omega]_{\mathrm{rel}}$ is integral as desired.

Furthermore, by the definition, $A_{1,s} = a_s(0, E_1)$ in $\mathbb{H}_2(\mathfrak{n}, \mathcal{O})/\mathbb{I}$ and hence we have

$$\mathcal{O}/C \twoheadrightarrow \mathbb{H}_2(\mathfrak{n}, \mathcal{O})[\epsilon_{\mathbf{E}}]/(\mathbb{I} + \sum_{s\in C(\Gamma_{1,1})} \mathcal{O}A_{1,s})[\epsilon_{\mathbf{E}}]. \tag{2.33}$$

Then (2.28), (2.29), (2.30), and (2.33) prove the theorem. □

Remark 2.23. Since $\sum_{b\in S}\eta_1(\bar{b})^{-1}\mathrm{ev}_{b,1,\mathbb{C}}(d[\pi_{\mathbf{G}}]^{\epsilon_{\mathbf{E}}}_{\mathrm{rel}})\neq 0$ by the proof of (2.30) in Theorem 2.22, we see that $[\pi_{\mathbf{G}}]^{\epsilon_{\mathbf{E}}}_{\mathrm{rel}}\neq 0$ in $H^n(Y(\mathfrak{n}),\mathbb{C})$. Thus, by Proposition 4.6, we can verify that

$$[\pi_{\mathbf{E}}]^{\epsilon_{\mathbf{E}}}=[\pi_{\mathbf{E}}].$$

By the proof of Theorem 2.22, we obtain $\delta_{\mathbf{G}}=(C)$ and the following corollary, which we will use in §3 and §4.5.

Corollary 2.24. *Under the same assumptions of Theorem 2.22,*

$$[\pi_{\boldsymbol{E}}]\in\widetilde{H}^n(Y(\mathfrak{n}),\mathcal{O}),$$

that is, $[\pi_{\boldsymbol{E}}]$ *is an integral cocycle. Moreover, the modulo* ϖ *cohomology class of* $[\pi_{\boldsymbol{E}}]$ *is non-zero:*

$$[\pi_{\boldsymbol{E}}]\neq 0 \ \textit{in}\ \widetilde{H}^n(Y(\mathfrak{n}),\kappa),$$

where

$$\widetilde{H}^n(Y(\mathfrak{n}),\kappa)=H^n(Y(\mathfrak{n}),\kappa)/(\textit{image of }\mathcal{O}\textit{-torsion of }H^n(Y(\mathfrak{n}),\mathcal{O})).$$

2.11. **Real quadratic field case.** We give an example of a congruence between a Hilbert cusp form and a Hilbert Eisenstein series.

We use the same notation as the proof of Theorem 2.22 and simply write $\Gamma_1=\Gamma_{1,1}(\mathfrak{n})$ and $\Gamma_1^1=\Gamma_{1,1}^1(\mathfrak{n})$. Hereafter, in this subsection, we assume that F is a real quadratic field with $h_F^+=1$. First we show the following lemma.

Lemma 2.25. *Assume the following four conditions* (1)*,* (2)*,* (3)*, and* (4)*:*

(1) $H^3_c(Y(\mathfrak{n}),\mathcal{O})$ *is torsion-free;*
(2) $H^2(\partial\left(Y(\mathfrak{n})^{BS}\right),\mathcal{O})$ *is torsion-free;*
(3) $C(\mathfrak{q},\boldsymbol{E})\not\equiv N(\mathfrak{q})(\mathrm{mod}\varpi)$ *for some prime ideal* $\mathfrak{q}$ *dividing* $\mathfrak{n}$*, where* $C(\mathfrak{q},\boldsymbol{E})$ *is the* $U(\mathfrak{q})$*-eigenvalue of* $\boldsymbol{E}$*;*
(4) *the ideal* $(C)\neq 0,\mathcal{O}$*.*

Then there exist a finite extension K' *of* K *with the ring of integer* $\mathcal{O}\hookrightarrow\mathcal{O}'$ *and a uniformizer* ϖ' *such that* $(\varpi')\cap\mathcal{O}=(\varpi)$ *and a Hecke eigenform* $\boldsymbol{f}\in S_2(\mathfrak{n},\mathcal{O}')$ *for all* $T(\mathfrak{m})$ *and* $U(\mathfrak{m})$ *with character* χ *such that* $\boldsymbol{f}\equiv\boldsymbol{E}\ (\mathrm{mod}\varpi')$*, that is,*

$$C(\mathfrak{m},\boldsymbol{f})\equiv C(\mathfrak{m},\boldsymbol{E})(\mathrm{mod}\varpi')$$

for any integral ideal $\mathfrak{m}$ *of* F*.*

Proof. By the proof of Theorem 2.22, if $(C)\neq 0,\mathcal{O}$, then $[e_0]\neq 0\in\widetilde{H}^2_{\mathrm{par}}(Y(\mathfrak{n}),\mathcal{O})[\epsilon_{\mathbf{E}}]$ is cohomologous to $-[\pi_{\mathbf{E}}]$ modulo ϖ and the Hecke eigenvalues of $[e_0]$ are the same as those of $-[\pi_{\mathbf{E}}]$ modulo ϖ for all $t\in\mathbb{H}_2(\mathfrak{n},\mathcal{O})$. The Deligne–Serre lifting lemma ([Del–Se, Lemma 6.11]) in the case $R=\mathcal{O}$, $M=\widetilde{H}^2_{\mathrm{par}}(Y(\mathfrak{n}),\mathcal{O})[\epsilon_{\mathbf{E}}]$, and $\mathbb{T}=\mathcal{H}_2(\mathfrak{n},\mathcal{O})$ says that there exist a finite extension K' of K with the ring of integer $\mathcal{O}\hookrightarrow\mathcal{O}'$ and a uniformizer ϖ' such that $(\varpi')\cap\mathcal{O}=(\varpi)$ and a non-zero eigenvector $[\pi]\in\widetilde{H}^2_{\mathrm{par}}(Y(\mathfrak{n}),\mathcal{O})[\epsilon_{\mathbf{E}}]\otimes\mathcal{O}'$ for all $t\in\mathbb{T}$ with eigenvalues $\lambda(V(\mathfrak{m}))$ satisfying

$$\lambda(V(\mathfrak{m}))\equiv C(\mathfrak{m},\mathbf{E})\ (\mathrm{mod}\ \varpi'),$$

where $V(\mathfrak{m})=T(\mathfrak{m})$ or $U(\mathfrak{m})$. Then, by the partial Eichler–Shimura–Harder isomorphism (2.27), we may regard $[\pi]\in S_2(\mathfrak{n},\mathbb{C})$ and hence we get a Hecke eigenform $\mathbf{f}$ for all $T(\mathfrak{m})$ and $U(\mathfrak{m})$ such that $[\pi]=[\pi_{\mathbf{f}}]$. By using the relation between Hecke eigenvalues and Fourier coefficients, we may assume that $\mathbf{f}\in S_2(\mathfrak{n},\mathcal{O}')$ with character χ. Therefore we obtain the congruence between a Hecke eigenform and our Eisenstein series

$$\mathbf{f}\equiv\mathbf{E}\ (\mathrm{mod}\varpi').$$

□

In order to construct an example of the congruence between a Hilbert cusp form and a Hilbert Eisenstein series, we shall prove (1) and (2) of Lemma 2.25 in certain case and give a Hilbert Eisenstein series satisfying (3) and (4) of Lemma 2.25 based on a numerical table in [Oka].

The first question we have to ask is torsion-freeness of (1). By the Poincaré–Lefschetz duality theorem, we obtain

$$H^3_c(Y(\mathfrak{n}),\mathcal{O}) \simeq H_1(Y(\mathfrak{n}),\mathcal{O}).$$

Proposition 2.26. *Assume that* $\mathfrak{n}$ *is prime to* $6\triangle_F$. *If* p *is prime to* $6\mathfrak{n}$ *and* $\sharp(\mathfrak{o}_{F,+}^{\times}/\mathfrak{o}_{F,\mathfrak{n}}^{\times 2})$, *then the assumption (1) of Lemma 2.25 is satisfied.*

Proof. Since $\mathfrak{n}$ is prime to 2, we have $\overline{\Gamma_1^1} = \Gamma_1^1$ and hence $\overline{\Gamma_1}/\Gamma_1^1 \simeq \mathfrak{o}_{F,+}^{\times}/\mathfrak{o}_{F,\mathfrak{n}}^{\times 2}$. Thus, by the Poincaré–Lefschetz duality theorem, it suffices to show that $\Gamma_1^{1,\mathrm{ab}}$ is p-torsion-free if p is prime to $6\mathfrak{n}$. This torsion-free problem will be solved by the method of Kuga [Kuga] and the theorem of Serre [Se] as follows. Since $\alpha^{-1}\Gamma_1^1\alpha = \Gamma_1(\mathfrak{o}_F,\mathfrak{n}) \cap \mathrm{SL}_2(\mathfrak{o}_F)$ for some $\alpha \in \mathrm{GL}_2(F)$, we may assume $\Gamma_1^1 = \Gamma_1(\mathfrak{o}_F,\mathfrak{n}) \cap \mathrm{SL}_2(\mathfrak{o}_F)$. Thus the theorem [Se, Theorem 3] shows that $\Gamma_1^{1,\mathrm{ab}}$ is torsion group. By the congruence subgroup property [Se, Corollary 3 of Theorem 2], there is an integral ideal $\mathfrak{m}$ of F such that the principal congruence subgroup $\Gamma(\mathfrak{m})$ satisfies $\Gamma(\mathfrak{m}) \subset [\Gamma_1^1 : \Gamma_1^1] \subset \Gamma_1^1$. In particular, we have

$$\Gamma_1^{1,\mathrm{ab}} \simeq \left(\Gamma_1^1/\Gamma(\mathfrak{m})\right)^{\mathrm{ab}}.$$

We estimate the order of right hand side as follows. Let $H = \Gamma_1^1/\Gamma(\mathfrak{m})$. We decompose $\mathrm{SL}_2(\mathfrak{o}_F)/\Gamma(\mathfrak{m}) = \prod_i \mathrm{SL}_2(\mathfrak{o}_F/\mathfrak{q}_i^{r_i})$ and $H = \prod_i H_{\mathfrak{q}_i}$. We define $\widehat{H}_{\mathfrak{q}_i}$ by the following cartesian diagram:

$$\begin{tikzcd}
H_{\mathfrak{q}_i} \arrow[r, hook] & \mathrm{SL}_2(\mathfrak{o}_F/\mathfrak{q}_i^{r_i}) \\
\widehat{H}_{\mathfrak{q}_i} \arrow[r, hook] \arrow[u, two heads] \arrow[ur, phantom, "\square"] & \mathrm{SL}_2(\mathfrak{o}_{F_{\mathfrak{q}_i}}). \arrow[u, two heads]
\end{tikzcd}$$

Here we note that, since SL_2 is connected semi-simple, for each positive integer r and prime ideal $\mathfrak{q}$ of $\mathfrak{o}_F$, the canonical map $\mathrm{SL}_2(\mathfrak{o}_{F_\mathfrak{q}}) \twoheadrightarrow \mathrm{SL}_2(\mathfrak{o}_F/\mathfrak{q}^r)$ is surjective.

Then our assertion follows from the following claim: We fix a positive integer $r = r_i$ and a prime ideal $\mathfrak{q} = \mathfrak{q}_i$ of $\mathfrak{o}_F$. Let l be the prime number such that $(l) = \mathfrak{q} \cap \mathbb{Z}$.

Claim (a) $\widehat{H}_\mathfrak{q}^{\mathrm{ab}} = 1$ in the case $\widehat{H}_\mathfrak{q} = \mathrm{SL}_2(\mathfrak{o}_{F_\mathfrak{q}})$ and $(\mathfrak{q}, 6) = 1$.

(b) $\widehat{H}_\mathfrak{q}^{\mathrm{ab}}$ is an l-group in the case $\widehat{H}_\mathfrak{q} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{q}^r \right\}$.

The assertion (a) is obtained by ([Fe–Si, Proposition 4.8]).

The assertion (b) follows from the arguments in [Fe–Si] as follows. For each non-negative integer m, let $\widehat{\Gamma}(\mathfrak{q}^m) = \ker\left(\mathrm{SL}_2(\mathfrak{o}_{F_\mathfrak{q}}) \twoheadrightarrow \mathrm{SL}_2(\mathfrak{o}_F/\mathfrak{q}^m)\right)$. The direct calculation with the help of the proof of [Fe–Si, Lemma 4.4] shows that $\widehat{H}_\mathfrak{q}$ is generated by all elementary unipotents in $\widehat{H}_\mathfrak{q}$. Then the image of $\widehat{H}_\mathfrak{q}/(\widehat{H}_\mathfrak{q} \cap \widehat{\Gamma}(\mathfrak{q}))$ in $\mathrm{SL}_2(\mathfrak{o}_{F_\mathfrak{q}}/\mathfrak{q})$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and hence it is an l-group. Moreover, by using the proof of [Fe–Si, Propositon 4.8], we have $\mathrm{EL}_2(\mathfrak{q}^{4m}) \subset [\widehat{H}_\mathfrak{q} : \widehat{H}_\mathfrak{q}] \subset \widehat{H}_\mathfrak{q}$. Here $\mathrm{EL}_2(\mathfrak{o}_{F_\mathfrak{q}})$ is the subgroup of $\mathrm{SL}_2(\mathfrak{o}_{F_\mathfrak{q}})$ generated by all elementary unipotents and $\mathrm{EL}_2(\mathfrak{q}^{4m}) = \mathrm{EL}_2(\mathfrak{o}_{F_\mathfrak{q}}) \cap \widehat{\Gamma}(\mathfrak{q}^{4m})$. As mentioned in the proof of [Fe–Si, Lemma

4.5], $\mathrm{EL}_2(\mathfrak{q}^{4m})$ is a subgroup of $\widehat{\Gamma}(\mathfrak{q}^{4m})$ with index a power of l. Since $\widehat{\Gamma}(\mathfrak{q})/\widehat{\Gamma}(\mathfrak{q}^{4m})$ is an l-group, so is $\widehat{H}_{\mathfrak{q}}/\widehat{\Gamma}(\mathfrak{q}^{4m})$. In particular, $\widehat{H}_{\mathfrak{q}}^{\mathrm{ab}}$ is an l-group as desired. □

The second point to be discussed is (2). Let ε_0 be the fundamental unit of F and ε_+ be a generator of $\mathfrak{o}_{F,+}^{\times}$:

$$\varepsilon_+ = \begin{cases} \varepsilon_0 & \text{if } N(\varepsilon_0) = 1, \\ \varepsilon_0^2 & \text{if } N(\varepsilon_0) = -1. \end{cases} \tag{2.34}$$

Proposition 2.27. *If $p \nmid N(\varepsilon_+ - 1)$ and $\mathfrak{n}$ is a prime ideal $\mathfrak{q}$ of $\mathfrak{o}_F$ prime to $6\triangle_F$, then the assumption (2) of Lemma 2.25 is satisfied.*

Proof. We simply write $\Gamma = \Gamma_{1,1}$ and may assume $\Gamma = \Gamma_1(\mathfrak{o}_F, \mathfrak{n})$ by taking conjugation. We recall the arguments in [Gha, §3]. Using the description of the boundary cohomology as §2.5, it suffices to show that $H^2(\overline{\Gamma_s}, \mathcal{O}) = H^2(\overline{\alpha^{-1}\Gamma\alpha \cap B_\infty}, \mathcal{O})$ is torsion-free for each cusp $s \in C(\Gamma)$, where $s \in \mathbb{P}^1(F)$ and $\alpha \in \mathrm{SL}_2(\mathfrak{o}_F)$ such that $\alpha(\infty) = s$, B_∞ is the standard Borel subgroup of upper triangular matrices, and the bar $^-$ means image in $\mathrm{GL}_2(F)/(\mathrm{GL}_2(F) \cap F^\times)$. Moreover, as mentioned in [Gha, p. 260], $H^2(\overline{\alpha^{-1}\Gamma\alpha \cap B_\infty}, \mathcal{O})$ is torsion-free if and only if $H^1(\overline{\alpha^{-1}\Gamma\alpha \cap B_\infty}, K/\mathcal{O})$ is divisible. A main tool for our proof is the Hochschild–Serre spectral sequence

$$E_2^{i,j} = H^i(\overline{\alpha^{-1}\Gamma\alpha \cap B_\infty}/\overline{\alpha^{-1}\Gamma\alpha \cap U_\infty}, H^j(\overline{\alpha^{-1}\Gamma\alpha \cap U_\infty}, K/\mathcal{O})) \Rightarrow H^{i+j}(\overline{\alpha^{-1}\Gamma\alpha \cap B_\infty}, K/\mathcal{O}),$$

where U_∞ is the unipotent radical of B_∞. As a similar calculation in [Gha, §3.4.2], our assertion follows from the following (2.35), (2.36), and the exact sequence (2.37). Indeed, it degenerates at E_2 and hence it is enough to prove that each E_2^{i+j} with $i + j = 1$ is divisible. Using (2.36) and (2.37), we have $E_2^{1,0} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, K/\mathcal{O}) = K/\mathcal{O}$. Moreover, for each $\varepsilon \in \mathfrak{o}_{F,+}^{\times}$, $b \in \mathfrak{q}^{1-e}$, and $f \in E_2^{0,1}$, we have $f(\varepsilon b) = f(b)$ under the isomorphisms (2.35) and (2.36). Then $N(\varepsilon_+ - 1)f = 0$ and hence $f = 0$ if $p \nmid N(\varepsilon_+ - 1)$ as desired.

It remains to prove (2.35), (2.36), and (2.37). Fix $s = x/y \in \mathbb{P}^1(F)$ with $x, y \in \mathfrak{o}_F$ and $(x, y) = 1$ and $\alpha = \begin{pmatrix} x & \beta \\ y & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}_F)$ such that $\alpha(\infty) = s$. We may assume that if $(y, \mathfrak{q}) = 1$, then $(\delta, \mathfrak{q}) = \mathfrak{q}$. Indeed, since $(x\mathfrak{q}, y) = 1$, there is $\begin{pmatrix} x & \beta \\ y & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}_F)$ with $(\delta, \mathfrak{q}) = 1$. We prove the following claims:

$$\alpha^{-1}\Gamma\alpha \cap U_\infty \simeq \mathfrak{q}^{1-e} \ \text{ if } (y, \mathfrak{q}) = \mathfrak{q}^e; \tag{2.35}$$

$$\overline{\alpha^{-1}\Gamma\alpha \cap T_\infty} \simeq \mathfrak{o}_{F,+}^{\times}; \tag{2.36}$$

$$1 \to \overline{\alpha^{-1}\Gamma\alpha \cap U_\infty} \to \overline{\alpha^{-1}\Gamma\alpha \cap B_\infty} \to \overline{\alpha^{-1}\Gamma\alpha \cap T_\infty} \to 1, \tag{2.37}$$

where T_∞ is the standard torus. For $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \alpha^{-1}\Gamma\alpha \cap B_\infty$, the direct calculation shows

$$\alpha \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \alpha^{-1} = \begin{pmatrix} x & \beta \\ y & \delta \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} \delta & -\beta \\ -y & x \end{pmatrix} = \begin{pmatrix} a\delta x - bxy - \beta dy & -a\beta x + bx^2 + \beta dx \\ a\delta y - by^2 - \delta dy & -a\beta y + bxy + \delta dx \end{pmatrix}.$$

First we prove (2.35). Suppose that $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \alpha^{-1}\Gamma\alpha \cap U_\infty$. The condition $\alpha \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \alpha^{-1} \in \Gamma$ is equivalent to $bx^2 \in \mathfrak{o}_F$, $by^2 \in \mathfrak{q}$, and $bxy \in \mathfrak{q}$. Since $(x, y) = 1$, we have $b \in \mathfrak{o}_F$. If $(y, \mathfrak{q}) = \mathfrak{q}^e$, then $b \in \mathfrak{q}^{1-e}$ as desired.

Next we prove (2.36). Suppose that $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \alpha^{-1}\Gamma\alpha \cap T_\infty$. The condition $\alpha \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \alpha^{-1} \in \Gamma$ is equivalent to $ad \in \mathfrak{o}_{F,+}^\times$, $(a-d)\delta y \in \mathfrak{q}$, and $-a\beta y + d\delta x \equiv 1 (\mathrm{mod}\ \mathfrak{q})$.

Suppose $(y, \mathfrak{q}) = 1$. Since $x\delta - \beta y = 1$, we have $a \equiv 1(\mathrm{mod}\ \mathfrak{q})$ and hence $\overline{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}} = \overline{\begin{pmatrix} 1 & 0 \\ 0 & a^{-1}d \end{pmatrix}}$. Moreover, for each $\varepsilon \in \mathfrak{o}_{F,+}^\times$, $\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \in \alpha^{-1}\Gamma\alpha \cap T_\infty$ as desired.

By the same argument, if $(y, \mathfrak{q}) = \mathfrak{q}$, then $d \equiv 1(\mathrm{mod}\ \mathfrak{q})$ and hence $\overline{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}} = \overline{\begin{pmatrix} ad^{-1} & 0 \\ 0 & 1 \end{pmatrix}}$ and, for each $\varepsilon \in \mathfrak{o}_{F,+}^\times$, $\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} \in \alpha^{-1}\Gamma\alpha \cap T_\infty$ as desired.

Finally we prove (2.37). For any $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \alpha^{-1}\Gamma\alpha \cap B_\infty$, it suffices to show that $\begin{pmatrix} 1 & \varepsilon b \\ 0 & 1 \end{pmatrix} \in \alpha^{-1}\Gamma\alpha \cap U_\infty$ for each $\varepsilon \in \mathfrak{o}_F^\times$, which is equivalent to $bx^2 \in \mathfrak{o}_F$, $by^2 \in \mathfrak{q}$, and $bxy \in \mathfrak{q}$ by the proof of (2.35). The condition $\alpha \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \alpha^{-1} \in \Gamma$ implies that $bxy \in \mathfrak{o}_F$, $bx^2 \in \mathfrak{o}_F$, and $by^2 \in \mathfrak{q}$. Then it suffices to check that $bxy \in \mathfrak{q}$. Since $(x, y) = 1$, we have $b \in \mathfrak{o}_F$. Since $by^2 \in \mathfrak{q}$, if $(y, \mathfrak{q}) = 1$, then $b \in \mathfrak{q}$ as desired. $\square$

Example 2.28. We give an example satisfying the assumptions of Lemma 2.25 in the case $F = \mathbb{Q}(\sqrt{2})$ with $\mathfrak{o}_F = \mathbb{Z}[\sqrt{2}]$, $h_F^+ = 1$, $\triangle_F = 8$, $\varepsilon_0 = 1 + \sqrt{2}$, and $\varepsilon_+ = 3 + 2\sqrt{2}$. According to [Oka, §4, p.1137], for the non-trivial character $\chi : \mathrm{Gal}(F(\sqrt{5})/F) \to \overline{\mathbb{Q}}^\times$ whose infinite type is the identity and conductor is a prime ideal $\mathfrak{n} = (5)$ of $\mathfrak{o}_F$, we have

$$L(-1, \chi) = \frac{28}{5}. \tag{2.38}$$

A pair of characters $\varphi = \chi^{-1}$ and the trivial character $\psi = \mathbf{1}$ satisfies (Eis condition). We see that $p = 7$ with $(p, 6\triangle_F) = 1$ and the Eisenstein series $\mathbf{E}_2(\varphi, \psi)$ with respect to level $\Gamma_{1,1}((5))$ satisfy the assumptions (1), (2), (3), and (4) of Lemma 2.25. Indeed, (1) (resp. (2)) follows from Proposition 2.26 (resp. Proposition 2.27) since $(7, 2 \cdot \sharp(\mathfrak{o}_F/5)^\times) = 1$ (resp. $(7, 2+2\sqrt{2}) = 1$). Moreover, (3) (resp. (4)) can be confirmed by $C((5), \mathbf{E}_2(\varphi, \psi)) = 0$ (resp. (2.38) and Proposition 1.2). Thus we can lift the Eisenstein series $\mathbf{E}_2(\varphi, \psi)$ to a Hecke eigenform modulo 7.

3. Congruences for L-functions

The purpose of this section is to prove the main theorem (Theorem 0.1=Theorem 3.1) of this paper. In this section we use the same notation as §2.10.

3.1. **Canonical periods.** Let $\mathbf{f} \in S_2(\mathfrak{n}, \mathcal{O})$ be a normalized Hecke eigenform for all $T(\mathfrak{m})$ and $U(\mathfrak{m})$ with character χ. Let ϵ denote $\epsilon_{\mathbf{E}}$. We would like to define the canonical period $\Omega_{\mathbf{f}}^\epsilon$ in the sense of Vatsal [Vat]. We denote by $\mathfrak{p}_{\mathbf{f}}$ the prime ideal of Hecke algebra $\mathcal{H}_2(\mathfrak{n}, \mathcal{O})$ over $\mathcal{O}$ generated by $T(\mathfrak{q}) - C(\mathfrak{q}, \mathbf{f})$ and $S(\mathfrak{q}) - \chi^{-1}(\mathfrak{q})$ for all ideals $\mathfrak{q}$ of $\mathfrak{o}_F$ outside $\mathfrak{n}$ and $U(\mathfrak{q}) - C(\mathfrak{q}, \mathbf{f})$ for $\mathfrak{q}$ dividing $\mathfrak{n}$. We identify the Weyl group $W_G = K_\infty/K_{\infty,+}$ with $\{\pm 1\}^{J_F}$ via the determinant map. By [Hida88, §2, §7], the Weyl group W_G acts on the space of Hilbert cusp forms and $H_{\mathrm{par}}^n(Y(\mathfrak{n}), \mathcal{O})$. Moreover, this action commutes with the Hecke operators $T(\mathfrak{m})$, $U(\mathfrak{m})$, and $S(\mathfrak{m})$ for all ideals $\mathfrak{m}$ of $\mathfrak{o}_F$.

The partial Eichler–Shimura–Harder isomorphism (2.27) and the q-expansion principle over $\mathbb{C}$ imply that ϵ-part of the eigenspace of the Weyl action is free of rank 1:

$$H^n_{\mathrm{par}}(Y(\mathfrak{n}), \mathbb{C})[\mathfrak{p}_{\mathbf{f}}, \epsilon] \simeq \mathbb{C},$$
$$\widetilde{H}^n_{\mathrm{par}}(Y(\mathfrak{n}), \mathcal{O})[\mathfrak{p}_{\mathbf{f}}, \epsilon] \simeq \mathcal{O},$$

where $\widetilde{H}^n_{\mathrm{par}}(Y(\mathfrak{n}), \mathcal{O})$ is the torsion-free part of $H^n_{\mathrm{par}}(Y(\mathfrak{n}), \mathcal{O})$ as §2.10. We choose a generator $[\delta_{\mathbf{f}}]^\epsilon$ of $\widetilde{H}^n_{\mathrm{par}}(Y(\mathfrak{n}), \mathcal{O})[\mathfrak{p}_{\mathbf{f}}, \epsilon]$. We write $[\pi_{\mathbf{f}}]^\epsilon$ for the projection of $[\pi_{\mathbf{f}}]$ to the ϵ-part. Since $[\delta_{\mathbf{f}}]^\epsilon, [\pi_{\mathbf{f}}]^\epsilon \in H^n_{\mathrm{par}}(Y(\mathfrak{n}), \mathbb{C})[\mathfrak{p}_{\mathbf{f}}, \epsilon]$, there exists a complex number $\Omega^\epsilon_{\mathbf{f}} \in \mathbb{C}^\times$ such that

$$[\pi_{\mathbf{f}}]^\epsilon = \Omega^\epsilon_{\mathbf{f}}[\delta_{\mathbf{f}}]^\epsilon. \tag{3.1}$$

The complex number $\Omega^\epsilon_{\mathbf{f}}$ is called the canonical period in the sense of Vatsal.

3.2. **Congruences of special values.** For modular forms $\mathbf{f}, \mathbf{g} \in M_2(\mathfrak{n}, \mathcal{O})$, we define the congruence of modular forms $\mathbf{f} \equiv \mathbf{g} (\mathrm{mod} \varpi)$ by $C(\mathfrak{m}, \mathbf{f}) \equiv C(\mathfrak{m}, \mathbf{g}) (\mathrm{mod} \varpi)$ for any integral ideal $\mathfrak{m}$ of F.

Theorem 3.1. *Let $p \geq [F : \mathbb{Q}] + 2$ be a prime number such that p is prime to $\mathfrak{n}$ and $6\triangle_F$. Assume that $h^+_F = 1$. Let φ and ψ be narrow ray class characters satisfying (Eis condition) as §2.10 and $\epsilon = \epsilon_{\boldsymbol{E}}$ the character on the Weyl group W_G defined just after (Eis condition). Put $\chi = \varphi\psi$. Let $\boldsymbol{f} \in S_2(\mathfrak{n}, \mathcal{O})$ a normalized Hecke eigenform for all $T(\mathfrak{m})$ and $U(\mathfrak{m})$ with character χ. We assume the following three conditions:*

(a) both $H^n(\partial (Y(\mathfrak{n})^{BS}), \mathcal{O})$ and $H^{n+1}_c(Y(\mathfrak{n}), \mathcal{O})$ are torsion-free;
(b) the Hilbert Eisenstein series $\boldsymbol{E} = \boldsymbol{E}_2(\varphi, \psi) \in M_2(\mathfrak{n}, \mathcal{O})$ with character χ satisfies $\boldsymbol{f} \equiv \boldsymbol{E} (\mathrm{mod} \varpi)$;
(c) $C(\mathfrak{q}, \boldsymbol{E}) \not\equiv N(\mathfrak{q}) (\mathrm{mod} \varpi)$ for some prime ideal $\mathfrak{q}$ dividing $\mathfrak{n}$, where $C(\mathfrak{q}, \boldsymbol{E})$ is the $U(\mathfrak{q})$-eigenvalue of $\boldsymbol{E}$.

Then there exist a complex number $\Omega^\epsilon_{\boldsymbol{f}} \in \mathbb{C}^\times$ and a p-adic unit $u \in \mathcal{O}^\times$ such that, for every primitive narrow ray class character $\eta : \mathrm{Cl}^+_F(\mathfrak{m}_\eta) \to \overline{\mathbb{Q}}^\times$ of conductor $\mathfrak{m}_\eta$ such that $\mathfrak{n}|\mathfrak{m}_\eta$ and $\eta = \epsilon$ on $W_G \simeq \mathbb{A}^\times_{F,\infty}/\mathbb{A}^\times_{F,\infty,+}$, the both values $\tau(\eta^{-1})D(1, \boldsymbol{f}, \eta)/(2\pi\sqrt{-1})^n\Omega^\epsilon_{\boldsymbol{f}}$ and $\tau(\eta^{-1})D(1, \boldsymbol{E}, \eta)/(2\pi\sqrt{-1})^n$ belong to $\mathcal{O}(\eta)$ and the following congruence holds:

$$\tau(\eta^{-1})\frac{D(1, \boldsymbol{f}, \eta)}{(2\pi\sqrt{-1})^n\Omega^\epsilon_{\boldsymbol{f}}} \equiv u\tau(\eta^{-1})\frac{D(1, \boldsymbol{E}, \eta)}{(2\pi\sqrt{-1})^n} \ (\mathrm{mod}\ \varpi).$$

Here $\tau(\eta^{-1})$ is the Gauss sum attached to η^{-1}, $D(1, *, \eta)$ is given by the Dirichlet series in the sense of Shimura (1.12), $\mathcal{O}(\eta)$ is the ring of integers of $K(\eta)$, and $K(\eta)$ is the field generated by elements of $\mathrm{im}(\eta)$ *over K.*

Remark 3.2. In the case $[F : \mathbb{Q}] = 2$, if $\mathfrak{n}$ is a prime ideal $\mathfrak{q}$ of $\mathfrak{o}_F$ prime to $6\triangle_F$, then the condition (a) is satisfied under the assumptions of Proposition 2.26 and Proposition 2.27.

Proof. We may take $[t_1]$ such that $\mathfrak{d}_F[t_1] = \mathfrak{o}_F$. We abbreviate $Y(\mathfrak{n}) = Y_1$ to Y and $\Gamma_{1,1}(\mathfrak{n})$ to Γ. The assumptions $\mathbf{f} \equiv \mathbf{E} (\mathrm{mod}\ \varpi)$ and Theorem 4.1 imply the following congruence of cocycles : for some p-adic unit $u \in \mathcal{O}^\times$,

$$[\delta_{\mathbf{f}}]^\epsilon = u[\pi_{\mathbf{E}}]^\epsilon \text{ in } \widetilde{H}^n_{\mathrm{par}}(Y, \kappa)[\epsilon].$$

Here we note that $[\pi_{\mathbf{E}}]^\epsilon = [\pi_{\mathbf{E}}] \neq 0$ in $\widetilde{H}^n_{\mathrm{par}}(Y, \kappa)$ by Remark 2.23 and Corollary 2.24.

Let $C_\infty = \{c \in C(\Gamma) \mid c \text{ is } \Gamma_{0,1}(\mathfrak{n})\text{-equivalent to the cusp } \infty\}$ and

$$D_{C_\infty} = \coprod_{s \in C_\infty} D_s^1 \hookrightarrow Y^{\mathrm{BS}}.$$

For $A = \mathcal{O}$ or κ, we define the partial parabolic cohomology $H^n_{\mathrm{par}}(Y, D_{C_\infty}; A)$ by

$$H^n_{\mathrm{par}}(Y, D_{C_\infty}; A) = \mathrm{im}\left(H^n(Y^{\mathrm{BS}}, D_{C_\infty}; A) \to H^n(Y, A)\right).$$

For an $\mathcal{O}$-module M, M_{torsion} stands for the torsion part of M. For $A = \mathcal{O}$ or κ, we define $\widetilde{H}^n(Y^{\mathrm{BS}}, D_{C_\infty}; A)$, $\widetilde{H}^n_{\mathrm{par}}(Y, D_{C_\infty}; A)$, and $\widetilde{H}^m(D_{C_\infty}, A)$ as follows:

$$\widetilde{H}^m(Y^{\mathrm{BS}}, D_{C_\infty}; A) = H^m(Y^{\mathrm{BS}}, D_{C_\infty}; A)/\left(\text{image of } H^m(Y^{\mathrm{BS}}, D_{C_\infty}; \mathcal{O})_{\mathrm{torsion}}\right),$$
$$\widetilde{H}^m_{\mathrm{par}}(Y, D_{C_\infty}; A) = H^m_{\mathrm{par}}(Y, D_{C_\infty}; A)/\left(\text{image of } H^m_{\mathrm{par}}(Y, D_{C_\infty}; \mathcal{O})_{\mathrm{torsion}}\right),$$
$$\widetilde{H}^m(D_{C_\infty}, A) = H^m(D_{C_\infty}, A)/\left(\text{image of } H^m(D_{C_\infty}, \mathcal{O})_{\mathrm{torsion}}\right).$$

By the definition, we have

$$[\delta_{\mathbf{f}}]^\epsilon = u[\pi_{\mathbf{E}}]^\epsilon \text{ in } \widetilde{H}^n_{\mathrm{par}}(Y, D_{C_\infty}; \kappa)[\epsilon].$$

We must show the following congruence of cocycles:

$$[\delta_{\mathbf{f}}]^\epsilon_{\mathrm{rel}} = u[\pi_{\mathbf{E}}]^\epsilon_{\mathrm{rel}} \text{ in } \widetilde{H}^n(Y^{\mathrm{BS}}, D_{C_\infty}; \kappa). \tag{3.2}$$

Let $\mathbb{H}_2(\mathfrak{n}, \mathcal{O})' = \langle U(\mathfrak{m})\rangle$ be the sub-algebra of the Hecke algebra $\mathbb{H}_2(\mathfrak{n}, \mathcal{O})$ generated by $U(\mathfrak{m})$ for all ideals $\mathfrak{m}$ of $\mathfrak{o}_F$ dividing $\mathfrak{n}$ and $\mathfrak{m}'_{\mathbf{f}}$ a maximal ideal of $\mathbb{H}_2(\mathfrak{n}, \mathcal{O})'$ generated by ϖ and $U(\mathfrak{q}) - C(\mathfrak{q}, \mathbf{f})$ for all ideals $\mathfrak{q}$ of $\mathfrak{o}_F$ dividing $\mathfrak{n}$. Since each Hecke correspondence $U(\mathfrak{q})$ preserves the component D_{C_∞}, $\mathbb{H}_2(\mathfrak{n}, \mathcal{O})'$ acts on $\widetilde{H}^{n-1}(D_{C_\infty}, A)$ and $\widetilde{H}^n(Y^{\mathrm{BS}}, D_{C_\infty}; A)$ for $A = \mathcal{O}$, κ, or $\mathbb{C}$.

Since $h_F^+ = 1$, for any prime ideal $\mathfrak{q}$ of $\mathfrak{o}_F$ dividing $\mathfrak{n}$, we fix a totally positive generator $g_{\mathfrak{q}}$ of $\mathfrak{q}$. By **Step1** in the proof of Theorem 4.6, for each cusp $t \in C(\Gamma)$, we know that a basis of $H^{n-1}(D_t, \mathbb{C})$ is given by ω_t.

Claim: the $U(\mathfrak{q})$-eigenvalue of ω_t is equal to $N(\mathfrak{q})$ for each $t \in C_\infty$.

Proof. We wrtie $t = \gamma(\infty)$ for some $\gamma \in \Gamma_{0,1}(\mathfrak{n})$. The canonical map $\gamma : D_{C_\infty} \to D_{C_\infty}$ induces $\gamma^* : H^{n-1}(D_{C_\infty}, \mathbb{C}) \to H^{n-1}(D_{C_\infty}, \mathbb{C})$. By the definition of γ, we have $\gamma^*\omega_t \in H^{n-1}(D_\infty, \mathbb{C})$. In order to prove **Claim**, we first compute the $U(\mathfrak{q})$-eigenvalue of ω_∞. We decompose as §2.3:

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & g_{\mathfrak{q}} \end{pmatrix} \Gamma = \coprod_{i \in I^\infty} \Gamma\beta_i^\infty\Gamma_\infty, \; \Gamma\beta_i^\infty\Gamma_\infty = \coprod_{j \in J_i^\infty} \Gamma\beta_i^\infty\delta_{i,j}^\infty \text{ with } \delta_{i,j}^\infty \in \Gamma_\infty, \text{ and}$$

$$\Gamma_{\beta_i^\infty(\infty)}\beta_i^\infty\Gamma_\infty = \coprod_{j \in J_i^\infty} \Gamma_{\beta_i^\infty(\infty)}\beta_i^\infty\delta_{i,j}^\infty.$$

Remark that, by the definition of the Hecke action on the boundary cohomology (see [Hida93, (3.1c)] or §2.3 in this paper),

$$\left(\omega_\infty | [\Gamma \begin{pmatrix} 1 & 0 \\ 0 & g_{\mathfrak{q}} \end{pmatrix} \Gamma]\right)_\infty = \sum_{i \in I^\infty} \omega_{\beta_i^\infty(\infty)} | [\Gamma_{\beta_i^\infty(\infty)}\beta_i^\infty\Gamma_\infty] \tag{3.3}$$

with $\beta_i^\infty(\infty)$ equivariant to the cusp ∞ over Γ: $\beta_i^\infty(\infty) = \beta_i^\infty\delta_j^\infty(\infty) \sim_\Gamma \infty$.

We use the following decomposition:

$$\Gamma \begin{pmatrix} g_{\mathfrak{q}} & 0 \\ 0 & 1 \end{pmatrix} \Gamma = \coprod_{b \in \mathfrak{o}_F/\mathfrak{q}} \begin{pmatrix} g_{\mathfrak{q}} & b \\ 0 & 1 \end{pmatrix} \Gamma, \tag{3.4}$$

where b runs over a set of representative of $\mathfrak{o}_F/\mathfrak{q}$.

In order to check it, note that, for any $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \begin{pmatrix} g_{\mathfrak{q}} & 0 \\ 0 & 1 \end{pmatrix} \Gamma$, we have $c \equiv 0 (\mathrm{mod}\ \mathfrak{n})$, $d \equiv 1 (\mathrm{mod}\ \mathfrak{n})$, and $\det(\beta) = g_{\mathfrak{q}} u$ for some $u \in \mathfrak{o}_{F,+}^{\times}$. Since $\mathfrak{q}$ divides $\mathfrak{n}$, we have $(c, d) = 1$ and hence there is $\gamma_1 = \begin{pmatrix} d & * \\ -c & * \end{pmatrix} \in \Gamma$ with $\det(\gamma_1) = 1$ such that

$$\begin{aligned} \beta\gamma_1 \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & * \\ -c & * \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \det(\beta) & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} g_{\mathfrak{q}} & b' \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

This proves (3.4) as desired.

For our calculation, we explicitly decompose

$$\Gamma_{\beta_i^{\infty}(\infty)} \beta_i^{\infty} \Gamma_{\infty} = \coprod_{j \in J_i^{\infty}} \Gamma_{\beta_i^{\infty}(\infty)} \beta_i^{\infty} \delta_{i,j}^{\infty}.$$

Remark that $\gamma_\delta \beta_i^{\infty} \delta = \begin{pmatrix} 1 & b_i \\ 0 & g_{\mathfrak{q}} \end{pmatrix}$ for some $\gamma_\delta \in \Gamma$ and $\delta \in \Gamma_\infty$. Since $\beta_i^{\infty}\delta(\infty) \sim_{\Gamma} \infty$, we have $\gamma' \gamma_\delta \beta_i^{\infty} \delta(\infty) = \infty$ for some $\gamma' \in \Gamma$ and hence $\gamma'\gamma_\delta\beta_i^{\infty}\delta$ belongs to the standard Borel subgroup B_∞^+ of upper triangular matrices. Moreover,

$$\Gamma_{\beta_i^{\infty}(\infty)} \beta_i^{\infty} \Gamma_{\infty} = (\gamma'\gamma_\delta)^{-1} \Gamma_\infty \gamma' \gamma_\delta \beta_i^{\infty} \delta \Gamma_\infty$$

and

$$g_{\mathfrak{q}} (\gamma' \gamma_\delta \beta_i^{\infty} \delta)^{-1} = \begin{pmatrix} g_{\mathfrak{q}} u_a & * \\ 0 & u_d \end{pmatrix}$$

for some $u_a u_d \in \mathfrak{o}_{F,+}^{\times}$ with $u_d \equiv 1 (\mathrm{mod} \mathfrak{n})$. Since $\begin{pmatrix} u_a^{-1} & 0 \\ 0 & u_d^{-1} \end{pmatrix} \in \Gamma_\infty$, we have $\sharp J_i^{\infty} = N(\mathfrak{q})$. Thus, by the same way as above, if we write $\gamma_j' \gamma_j \beta_i^{\infty} \delta_{i,j}^{\infty}(\infty) = \infty$ for some $\gamma_j', \gamma_j \in \Gamma$, then

$$\begin{aligned} \left(\omega_\infty | [\Gamma \begin{pmatrix} 1 & 0 \\ 0 & g_{\mathfrak{q}} \end{pmatrix} \Gamma] \right)_\infty &= \sum_{j=1}^{N(\mathfrak{q})} (\beta_i^{\infty} \delta_{i,j}^{\infty})^* \omega_{\beta_i^{\infty}(\infty)} \\ &= \sum_{j=1}^{N(\mathfrak{q})} (\beta_i^{\infty} \delta_{i,j}^{\infty})^* (\gamma_j' \gamma_j)^* \omega_\infty \\ &= \sum_{j=1}^{N(\mathfrak{q})} (\gamma_j' \gamma_j \beta_i^{\infty} \delta_{i,j}^{\infty})^* \omega_\infty \\ &= N(\mathfrak{q}) \omega_\infty. \end{aligned}$$

Here we note that ω_∞ is invariant under the element of $B_\infty^+ \cap G_{\infty,+}$. Thus we get $(\gamma^*\omega_t)|U(\mathfrak{q}) = N(\mathfrak{q})\gamma^*\omega_t$ and hence

$$(\gamma^{-1})^* \left((\gamma^*\omega_t)|U(\mathfrak{q})\right) = N(\mathfrak{q})\omega_t.$$

For the proof of **Claim**, we finally show that

$$\Gamma\gamma\Gamma \cdot \Gamma \begin{pmatrix} 1 & 0 \\ 0 & g_\mathfrak{q} \end{pmatrix} \Gamma \cdot \Gamma\gamma^{-1}\Gamma = \Gamma \begin{pmatrix} 1 & 0 \\ 0 & g_\mathfrak{q} \end{pmatrix} \Gamma.$$

Since Γ is a normal subgroup of $\Gamma_{0,1}(\mathfrak{n})$, we have $\gamma\Gamma = \Gamma\gamma$ and hence it is enough to show that

$$\Gamma\gamma \begin{pmatrix} 1 & 0 \\ 0 & g_\mathfrak{q} \end{pmatrix} \gamma^{-1}\Gamma = \Gamma \begin{pmatrix} 1 & 0 \\ 0 & g_\mathfrak{q} \end{pmatrix} \Gamma.$$

This follows from the same arguments as in the proof of (3.4) because if we write $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \begin{pmatrix} g_\mathfrak{q} & 0 \\ 0 & 1 \end{pmatrix} \gamma^{-1}$, then we have $c \equiv 0 (\mathrm{mod}\ \mathfrak{n})$, $d \equiv 1 (\mathrm{mod}\ \mathfrak{n})$, $\det(\beta) = g_\mathfrak{q}$, and $\mathfrak{q}$ divides $\mathfrak{n}$. $\square$

Moreover, a direct calculation shows that

$$\begin{aligned} E_1|[\Gamma \begin{pmatrix} 1 & 0 \\ 0 & g_\mathfrak{q} \end{pmatrix} \Gamma](z) &= \sum_b E_1| \begin{pmatrix} 1 & b \\ 0 & g_\mathfrak{q} \end{pmatrix} (z) \\ &= \sum_b N(g_\mathfrak{q})^{-1} \sum_{0 \ll \xi \in [t_1]} a_\infty(\xi, E_1) e_F \left(\xi \frac{b}{g_\mathfrak{q}}\right) e_F \left(\xi \frac{z}{g_\mathfrak{q}}\right) \\ &= \sum_{0 \ll \xi \in g_\mathfrak{q} \cdot [t_1]} a_\infty(\xi, E_1) e_F \left(\xi \frac{z}{g_\mathfrak{q}}\right). \end{aligned}$$

Then the eigenvalue of this series is equal to $C([g_\mathfrak{q}], \mathbf{E}) = C(\mathfrak{q}, \mathbf{E})$. Then, by our assumption that $C(\mathfrak{q}, \mathbf{E}) \not\equiv N(\mathfrak{q}) (\mathrm{mod} \varpi)$ for some prime ideal $\mathfrak{q}$ dividing $\mathfrak{n}$, we have

$$\widetilde{H}^{n-1}(D_{C_\infty}, \mathcal{O})_{\mathfrak{m}'_\mathfrak{f}} = 0 \tag{3.5}$$

since $\widetilde{H}^{n-1}(D_{C_\infty}, \mathcal{O})_{\mathfrak{m}'_\mathfrak{f}} \hookrightarrow H^{n-1}(D_{C_\infty}, \mathbb{C})_{\mathfrak{m}'_\mathfrak{f}} = 0$.

We consider the following diagram:

(3.6)

$$\begin{array}{ccccc} \left(\text{image of } H^n(Y^{\mathrm{BS}}, D_{C_\infty}; \mathcal{O})_{\text{torsion}}\right)_{\mathfrak{m}'_\mathfrak{f}} & \hookrightarrow & H^n(Y^{\mathrm{BS}}, D_{C_\infty}; \kappa)_{\mathfrak{m}'_\mathfrak{f}} & \twoheadrightarrow & \widetilde{H}^n(Y^{\mathrm{BS}}, D_{C_\infty}; \kappa)_{\mathfrak{m}'_\mathfrak{f}} \\ \downarrow{\bigstar\bigstar} & & \downarrow & & \downarrow{\bigstar} \\ \left(\text{image of } H^n_{\mathrm{par}}(Y, D_{C_\infty}; \mathcal{O})_{\text{torsion}}\right)_{\mathfrak{m}'_\mathfrak{f}} & \hookrightarrow & H^n_{\mathrm{par}}(Y, D_{C_\infty}; \kappa)_{\mathfrak{m}'_\mathfrak{f}} & \twoheadrightarrow & \widetilde{H}^n_{\mathrm{par}}(Y, D_{C_\infty}; \kappa)_{\mathfrak{m}'_\mathfrak{f}}. \end{array}$$

Thus, by (3.5), $\bigstar\bigstar$ is surjective. Since $\left(\text{image of } H^{n-1}(D_{C_\infty}, \mathcal{O})_{\text{torsion}}\right)_{\mathfrak{m}'_\mathfrak{f}} \subset \ker(\bigstar\bigstar)$, the snake lemma for (3.6) implies that $\widetilde{H}^{n-1}(D_{C_\infty}, \kappa)_{\mathfrak{m}'_\mathfrak{f}} \twoheadrightarrow \ker(\bigstar)$.

Claim: $\ker(\bigstar) = 0$.

Proof. It is enough to show that $\widetilde{H}^{n-1}(D_{C_\infty}, \kappa)_{\mathfrak{m}'_\mathfrak{f}} = 0$. By our assumption that the boundary cohomology of Y^{BS} is torsion-free, the exact sequence $0 \to \mathcal{O} \xrightarrow{\times\varpi} \mathcal{O} \to \kappa \to 0$ implies $\mathrm{mod}\, \varpi : H^{n-1}(D_{C_\infty}, \mathcal{O})_{\mathfrak{m}'_\mathfrak{f}} \twoheadrightarrow H^{n-1}(D_{C_\infty}, \kappa)_{\mathfrak{m}'_\mathfrak{f}}$ and hence we get

$$\mathrm{mod}\, \varpi : \widetilde{H}^{n-1}(D_{C_\infty}, \mathcal{O})_{\mathfrak{m}'_\mathfrak{f}} \twoheadrightarrow \widetilde{H}^{n-1}(D_{C_\infty}, \kappa)_{\mathfrak{m}'_\mathfrak{f}}.$$

Then our claim follows from this and (3.5). □

Therefore we obtain the congruence of cocycles (3.2).

Using the functoriality of the trace map for $\mathcal{O} \to \kappa$ and the vanishing of the image of $H^n(Y^{\mathrm{BS}}, D_{C_\infty}; \mathcal{O})_{\mathrm{torsion}}$ under the evaluation map $\mathrm{ev}_{b,1,\mathcal{O}}$ as (2.24), $\mathrm{ev}_{b,1,\mathcal{O}}$ induces

$$\mathrm{ev}_{b,1,\kappa} : \widetilde{H}^n(Y^{\mathrm{BS}}, D_{C_\infty}; \kappa) \to \kappa.$$

Then our assertion follows from this, (3.2), Proposition 2.19, and Proposition 2.20. □

4. On torsion cohomology in the Hilbert modular case

4.1. **Comparison theorem for torsion cohomology.** In this subsection, we will briefly review the fully faithful functor from the category of finitely generated filtered φ-module to the category of $\mathcal{O}$-representations of $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of finite length, and state the comparison theorem between the parabolic étale cohomology and the parabolic log-crystalline cohomology, which we will use in the following subsections.

Let $\mathcal{O}$ be the ring of integers of a finite extension K over $\mathbb{Q}_p$, ϖ a uniformizer, and κ the residue field. For a non-negative integer $r \in \mathbb{Z}$, we denote by $\mathbf{MF}^r_{\mathcal{O},\mathrm{tor}}$ the category of the following triples $(M, \{\mathrm{Fil}^i M\}_i, \{\varphi^i\}_i)$:

(1) M is a finitely generated $\mathcal{O}$-module;
(2) $\{\mathrm{Fil}^i M\}_{i\in\mathbb{Z}}$ is a decreasing filtration on M by sub-$\mathcal{O}$-modules such that $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{r+1} M = 0$;
(3) $\varphi^i \colon \mathrm{Fil}^i M \to M$ is an $\mathcal{O}$-linear homomorphism;
(4) $\varphi^i\,|_{\mathrm{Fil}^{i+1}M} = p\varphi^{i+1}$;
(5) $\sum_{i=0}^r \varphi^i(\mathrm{Fil}^i M) = M$.

A morphisms in $\mathbf{MF}^r_{\mathcal{O},\mathrm{tor}}$ is a homomorphism of filtered $\mathcal{O}$-modules compatible with $\varphi^\bullet$. We say that a morphism $\eta : M \to M'$ in $\mathbf{MF}^r_{\mathcal{O},\mathrm{tor}}$ is strict if $\eta(\mathrm{Fil}^i M) = \mathrm{Fil}^i M' \cap \eta(M)$ for each $i \in \mathbb{Z}$. It is known that any morphism in $\mathbf{MF}^r_{\mathcal{O},\mathrm{tor}}$ is strict and hence $\mathbf{MF}^r_{\mathcal{O},\mathrm{tor}}$ is an abelian category ([Fo–La, Proposition 1.8]).

The kernel and cokernel of η in $\mathbf{MF}^r_{\mathcal{O},\mathrm{tor}}$ are explicitly given as follows. For an object $(M, \{\mathrm{Fil}^i M\}_i, \{\varphi^i\}_i) \in \mathbf{MF}^r_{\mathcal{O},\mathrm{tor}}$ and the sub-$\mathcal{O}$-module $N = \ker(\eta) \subset M$, we define a filtration $\mathrm{Fil}^i N$ and an $\mathcal{O}$-linear homomorphism φ^i_N by $\mathrm{Fil}^i N = N \cap \mathrm{Fil}^i M$ and $\varphi^i_N = \varphi^i|_N$, respectively. For $N' = \mathrm{coker}(\eta)$, we define a filtration $\mathrm{Fil}^i N'$ and an $\mathcal{O}$-linear homomorphism φ'^i_N by $\mathrm{Fil}^i N' = \mathrm{Fil}^i M'/\eta(\mathrm{Fil}^i M) \hookrightarrow N'$ and the morphism induced by φ^i_M and φ'^i_M, respectively. In particular, for a morphism $\eta : M \to M'$ in $\mathbf{MF}^r_{\mathcal{O},\mathrm{tor}}$, we have $\mathrm{im}(\eta) = \mathrm{coim}(\eta) \in \mathbf{MF}^r_{\mathcal{O},\mathrm{tor}}$ and $\mathrm{Fil}^i \mathrm{im}(\eta) \simeq \left(\mathrm{Fil}^i M + \ker(\eta)\right)/\ker(\eta)$.

Let $\mathbf{MF}^r_{\kappa,\mathrm{tor}}$ be the full subcategory of $\mathbf{MF}^r_{\mathcal{O},\mathrm{tor}}$ consisting of objects M satisfying $\varpi M = 0$. We denote by $\mathbf{Rep}_{\mathcal{O}}(G_{\mathbb{Q}_p})$ the category of representations of $G_{\mathbb{Q}_p}$ on $\mathcal{O}$-modules of finite length. For $0 \le r \le p-2$, there exists a fully faithful functor

$$T_{\mathrm{cris}} \colon \mathbf{MF}^r_{\mathcal{O},\mathrm{tor}} \to \mathbf{Rep}_{\mathcal{O}}(G_{\mathbb{Q}_p}).$$

given by Fontaine–Laffaille ([Fo–La], [Br–Me], [Wach]). We denote by $\mathbf{Rep}^r_{\mathcal{O},\mathrm{cris}}(G_{\mathbb{Q}_p})$ the essential image of $\mathbf{MF}^r_{\mathcal{O},\mathrm{tor}}$ by T_{cris}. We say that the Hodge–Tate weight of $T \in \mathbf{Rep}^r_{\mathcal{O},\mathrm{cris}}(G_{\mathbb{Q}_p})$ is the $s \in \mathbb{Z}$ for which $\mathrm{Gr}^s M \neq 0$, where $M \in \mathbf{MF}^r_{\mathcal{O},\mathrm{tor}}$ is the corresponding module such that $T_{\mathrm{cris}}(M) \simeq T$.

The comparison theorem for log-smooth varieties with constant coefficients (proved by Faltings ([Fa, Theorem 5.3]) and improved by Breuil–Tsuji ([Br, Theorem 3.2.4.6]=[Tsu, Theorem 5.1] and [Br, Theorem 3.2.4.7])) shows that, for $(X^{\rm tor}, X) = (M_1^{\rm tor}, M_1)$ or $(M^{\rm tor}, M)$ and $n \le p-2$, there are canonical $G_{\mathbb{Q}_p}$-equivariant $\mathcal{O}$-linear isomorphisms

$$\begin{aligned} H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathcal{O}) &\simeq T_{\rm cris}\left(H^n_{\text{log-cris}}(X^{\rm tor}_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} \mathcal{O}\right), \\ H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \kappa &\simeq T_{\rm cris}\left(H^n_{\text{log-cris}}(X^{\rm tor}_{\mathbb{F}_p}) \otimes_{\mathbb{F}_p} \kappa\right). \end{aligned} \tag{4.1}$$

Here the filtration on $H^n_{\text{log-cris}}(X^{\rm tor}_{\mathbb{F}_p}) \otimes_{\mathbb{F}_p} \kappa$ is given by the Hodge to de Rham spectral sequence

$$E_1^{i,j} = H^j(X^{\rm tor}_{\mathbb{F}_p}, \Omega^i_{X^{\rm tor}_{\mathbb{F}_p}/\mathbb{F}_p}(\log(D))) \Rightarrow H^{i+j}(X^{\rm tor}_{\mathbb{F}_p}, \Omega^\bullet_{X^{\rm tor}_{\mathbb{F}_p}/\mathbb{F}_p}(\log(D))), \tag{4.2}$$

where $D = X^{\rm tor} - X$. This spectral sequence degenerates at E_1 by [Ill, Corollary 4.13].

The comparison theorem for cohomology with compact support (proved by Faltings ([Fa, Theorem 5.3])) says that, for $(X^{\rm tor}, X) = (M_1^{\rm tor}, M_1)$ or $(M^{\rm tor}, M)$ and $n \le p-2$, there are canonical $G_{\mathbb{Q}_p}$-equivariant $\mathcal{O}$-linear isomorphisms

$$\begin{aligned} H^n_{\text{ét},c}(X_{\overline{\mathbb{Q}}_p}, \mathcal{O}) &\simeq T_{\rm cris}\left(H^n_{\text{log-cris},!}(X^{\rm tor}_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} \mathcal{O}\right), \\ H^n_{\text{ét},c}(X_{\overline{\mathbb{Q}}_p}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \kappa &\simeq T_{\rm cris}\left(H^n_{\text{log-cris},!}(X^{\rm tor}_{\mathbb{F}_p}) \otimes_{\mathbb{F}_p} \kappa\right). \end{aligned} \tag{4.3}$$

Here the filtration on $H^n_{\text{log-cris},!}(X^{\rm tor}_{\mathbb{F}_p}) \otimes_{\mathbb{F}_p} \kappa$ is given by the Hodge to de Rham spectral sequence

$$E_1^{i,j} = H^j(X^{\rm tor}_{\mathbb{F}_p}, \Omega^i_{X^{\rm tor}_{\mathbb{F}_p}/\mathbb{F}_p}(\log(D))(-D)) \Rightarrow H^{i+j}(X^{\rm tor}_{\mathbb{F}_p}, \Omega^\bullet_{X^{\rm tor}_{\mathbb{F}_p}/\mathbb{F}_p}(\log(D))(-D)). \tag{4.4}$$

This spectral sequence is degenerate at E_1 by [Fa, p.59, Theorem 4.1].

For $? = \phi$ or $!$, we simply write

$$\begin{aligned} H^n_{\text{log-cris},?}(X^{\rm tor})_{\mathcal{O}} &= H^n_{\text{log-cris},?}(X^{\rm tor}_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} \mathcal{O}, \\ H^n_{\text{log-cris},?}(X^{\rm tor})_{\kappa} &= H^n_{\text{log-cris},?}(X^{\rm tor}_{\mathbb{F}_p}) \otimes_{\mathbb{F}_p} \kappa. \end{aligned}$$

For $A = \mathcal{O}$ or κ, we define the parabolic étale cohomology $H^n_{\text{ét,par}}(X_{\overline{\mathbb{Q}}_p}, A)$ and parabolic log-crystalline cohomology $H^n_{\text{log-cris,par}}(X^{\rm tor})_A$ in $\mathbf{MF}^r_{\mathcal{O},\rm tor}$ by

$$\begin{aligned} H^n_{\text{ét,par}}(X_{\overline{\mathbb{Q}}_p}, A) &= \operatorname{im}\left(H^n_{\text{ét},c}(X_{\overline{\mathbb{Q}}_p}, A) \to H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, A)\right), \\ H^n_{\text{log-cris,par}}(X^{\rm tor})_A &= \operatorname{im}\left(H^n_{\text{log-cris},!}(X^{\rm tor})_A \to H^n_{\text{log-cris}}(X^{\rm tor})_A\right). \end{aligned}$$

By the comparison theorem (4.1) and (4.3), we obtain $G_{\mathbb{Q}_p}$-equivariant $\mathcal{O}$-linear isomorphisms

$$\begin{aligned} H^n_{\text{ét,par}}(X_{\overline{\mathbb{Q}}_p}, \mathcal{O}) &\simeq T_{\rm cris}\left(H^n_{\text{log-cris,par}}(X^{\rm tor})_{\mathcal{O}}\right), \\ H^n_{\text{ét,par}}(X_{\overline{\mathbb{Q}}_p}, \kappa) &\simeq T_{\rm cris}\left(H^n_{\text{log-cris,par}}(X^{\rm tor})_{\kappa}\right). \end{aligned}$$

Moreover, by the definition of the Hodge filtration on $H^n_{\text{log-cris,par}}(X^{\text{tor}})_\kappa$, we have the following commutative diagram:

$$\begin{tikzcd} H^0(X^{\text{tor}}_\kappa, \Omega^n_{X^{\text{tor}}_\kappa/\kappa}(\log(D))(-D)) \arrow[r, two heads] \arrow[dd, hook] & \mathrm{Fil}^n H^n_{\text{log-cris},!}(X^{\text{tor}})_\kappa \arrow[d] \\ & \mathrm{Fil}^n H^n_{\text{log-cris,par}}(X^{\text{tor}})_\kappa \arrow[d, hook] \\ H^0(X^{\text{tor}}_\kappa, \Omega^n_{X^{\text{tor}}_\kappa/\kappa}(\log(D))) \arrow[r, "\simeq"] & \mathrm{Fil}^n H^n_{\text{log-cris}}(X^{\text{tor}})_\kappa. \end{tikzcd}$$

Here the isomorphism on the bottom of the diagram follows from the degeneration of the Hodge to de Rham spectral sequence (4.2) and hence we get

$$\mathrm{Fil}^n H^n_{\text{log-cris,par}}(X^{\text{tor}})_\kappa \simeq H^0(X^{\text{tor}}_\kappa, \Omega^n_{X^{\text{tor}}_\kappa/\kappa}(\log(D))(-D)) = S_2(\mathfrak{n}, \kappa).$$

4.2. **Analogue of a multiplicity-one theorem.** In this subsection, we prove the following main theorem of §4 which will be proved in §4.6.

Hereafter, we assume that $n = [F : \mathbb{Q}] \leq p - 2$ and $\mathcal{O}$ is the ring of integers of a finite extension K of the composite field of $\iota_p(F')$ and Φ_p. Here $\iota_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}_p}$ is the fixed embedding and F' (resp. Φ_p) is defined in §1.4 (resp. Proposition 2.9). Let ϖ be a uniformizer and κ the residue field.

Theorem 4.1. *Let $p \geq n + 2$ be a prime number such that p is prime to $\mathfrak{n}$ and $6\triangle_F$. Assume that $h^+_F = 1$. Let φ and ψ be narrow ray class characters satisfying (Eis condition) as §2.10 and $\epsilon = \epsilon_{\boldsymbol{E}}$ the character on the Weyl group W_G defined just after (Eis condition). Put $\chi = \varphi\psi$. Let $\boldsymbol{f} \in S_2(\mathfrak{n}, \mathcal{O})$ a normalized Hecke eigenform for all $T(\mathfrak{m})$ and $U(\mathfrak{m})$ with character χ. We assume the following three conditions:*

(*a*) *both $H^n(\partial\left(Y(\mathfrak{n})^{BS}\right), \mathcal{O})$ and $H^{n+1}_c(Y(\mathfrak{n}), \mathcal{O})$ are torsion-free;*
(*b*) *the Hilbert Eisenstein series $\boldsymbol{E} = \boldsymbol{E}_2(\varphi, \psi) \in M_2(\mathfrak{n}, \mathcal{O})$ with character χ satisfies $\boldsymbol{f} \equiv \boldsymbol{E} (\mathrm{mod}\ \varpi)$;*
(*c*) *$C(\mathfrak{q}, \boldsymbol{E}) \not\equiv N(\mathfrak{q}) (\mathrm{mod} \varpi)$ for some prime ideal $\mathfrak{q}$ dividing $\mathfrak{n}$, where $C(\mathfrak{q}, \boldsymbol{E})$ is the $U(\mathfrak{q})$-eigenvalue of $\boldsymbol{E}$.*

Then there exists a p-adic unit $u \in \mathcal{O}^\times$ such that

$$[\delta_{\boldsymbol{f}}]^\epsilon = u[\pi_{\boldsymbol{E}}] \quad \textit{in } \widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \kappa),$$

where

$$\widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \kappa) = H^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \kappa) / \left(\textit{image of } H^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})_{\text{torsion}}\right)$$

and M_{torsion} stands for the torsion part of M for an $\mathcal{O}$-module M.

Remark 4.2. Dimitrov [Dim2, Theorem 6.7] proved that a multiplicity-one theorem holds for the $\mathbf{f}$-parts of $H^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \kappa)$ and $H^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})$ if the residual Galois representation $\bar{\rho}_{\mathbf{f}}$ is irreducible under some assumptions.

Hereafter we assume the condition (Eis condition) and the congruence of all Hecke eigenvalues between a Hilbert cusp form $\mathbf{f} \in S_2(\mathfrak{n}, \mathcal{O})$ and a Hilbert Eisenstein series $\mathbf{E} = \mathbf{E}(\varphi, \psi) \in M_2(\mathfrak{n}, \mathcal{O})$ with character $\chi = \varphi\psi$, that is, $\mathbf{f} \equiv \mathbf{E}$ (mod ϖ).

Let $\mathfrak{p}_{\mathbf{E}}$ (resp. $\mathfrak{p}_{\mathbf{f}}$) be the prime ideal of the Hecke algebra $\mathbb{H}_2(\mathfrak{n},\mathcal{O})$ (resp. $\mathcal{H}_2(\mathfrak{n},\mathcal{O})$) assosiated $\mathbf{E}$ (resp. $\mathbf{f}$). In order to prove the main theorem, we consider three p-adic Galois representation $\widetilde{\widetilde{V}}$, $\widetilde{\widetilde{V}}_{\mathbf{f}}$, and $\widetilde{V}_{\mathbf{E}}$ defined as follows.

For ?$=\phi$ or par, we write the torsion-free part of cohomologies as

$$\widetilde{H}^n_{\text{ét},?}(M_{\overline{\mathbb{Q}}},\mathcal{O}) = \operatorname{im}\left(H^n_{\text{ét},?}(M_{\overline{\mathbb{Q}}},\mathcal{O}) \to H^n_{\text{ét},?}(M_{\overline{\mathbb{Q}}},K)\right),$$
$$\widetilde{H}^n_{\text{log-cris},?}(M^{\text{tor}})_{\mathcal{O}} = \operatorname{im}\left(H^n_{\text{log-cris},?}(M^{\text{tor}})_{\mathcal{O}} \to H^n_{\text{log-cris},?}(M^{\text{tor}})_K\right).$$

We define $\widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}},\kappa)$ in $\mathbf{Rep}^{p-2}_{\mathcal{O},\text{cris}}(G_{\mathbb{Q}_p})$ and $\widetilde{H}^n_{\text{log-cris,par}}(M^{\text{tor}})_\kappa$ in $\mathbf{MF}^{p-2}_{\kappa,\text{tor}}$ by the followings:

$$\widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}},\kappa) = H^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}},\kappa)/\left(\text{image of } H^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}},\mathcal{O})_{\text{torsion}}\right),$$
$$\widetilde{H}^n_{\text{log-cris,par}}(M^{\text{tor}})_\kappa = H^n_{\text{log-cris,par}}(M^{\text{tor}})_\kappa/\left(\text{image of } H^n_{\text{log-cris,par}}(M^{\text{tor}})_{\mathcal{O},\text{torsion}}\right).$$

By the comparison theorem (4.1) and (4.3), we have

$$\widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}},\kappa) \simeq T_{\text{cris}}(\widetilde{H}^n_{\text{log-cris,par}}(M^{\text{tor}})_\kappa).$$

In §4.3 and §4.4, we will consider the $\mathbf{f}$-parts of $\widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}},\mathcal{O})$ and $\widetilde{H}^n_{\log-\text{cris,par}}(M^{\text{tor}})_{\mathcal{O}}$ etc. defined by

$$\widetilde{\widetilde{V}} = \widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}},\kappa)[\mathfrak{p}_{\mathbf{f}}] \quad \text{and} \quad \widetilde{\widetilde{M}} = \widetilde{H}^n_{\log-\text{cris,par}}(M^{\text{tor}})_\kappa[\mathfrak{p}_{\mathbf{f}}],$$
$$\widetilde{V}_{\mathbf{f}} = \widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}},\mathcal{O})[\mathfrak{p}_{\mathbf{f}}] \quad \text{and} \quad \widetilde{M}_{\mathbf{f}} = \widetilde{H}^n_{\text{log-cris,par}}(M^{\text{tor}})_{\mathcal{O}}[\mathfrak{p}_{\mathbf{f}}],$$
$$\widetilde{\widetilde{V}}_{\mathbf{f}} = \widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}},\mathcal{O})[\mathfrak{p}_{\mathbf{f}}]/\varpi \quad \text{and} \quad \widetilde{\widetilde{M}}_{\mathbf{f}} = \widetilde{H}^n_{\text{log-cris,par}}(M^{\text{tor}})_{\mathcal{O}}[\mathfrak{p}_{\mathbf{f}}]/\varpi.$$

By applying the comparison theorem (4.1) and (4.3), we get

$$\widetilde{\widetilde{V}} \simeq T_{\text{cris}}(\widetilde{\widetilde{M}}), \qquad \widetilde{V}_{\mathbf{f}} \simeq T_{\text{cris}}(\widetilde{M}_{\mathbf{f}}), \qquad \widetilde{\widetilde{V}}_{\mathbf{f}} \simeq T_{\text{cris}}(\widetilde{\widetilde{M}}_{\mathbf{f}}).$$

A main tool for our proof is the torsion-free Eisenstein part

$$\widetilde{V}_{\mathbf{E}} = \widetilde{H}^n_{\text{ét}}(M_{\overline{\mathbb{Q}}},\mathcal{O})[\mathfrak{p}_{\mathbf{E}}] \quad \text{and} \quad \widetilde{M}_{\mathbf{E}} = \widetilde{H}^n_{\text{log-cris}}(M^{\text{tor}})_{\mathcal{O}}[\mathfrak{p}_{\mathbf{E}}].$$

By the comparison theorem (4.1), we have

$$\widetilde{V}_{\mathbf{E}} \simeq T_{\text{cris}}(\widetilde{M}_{\mathbf{E}}).$$

We will show that the Hodge–Tate weight of $\widetilde{V}_{\mathbf{E}}$ is $n=[F:\mathbb{Q}]$ by Proposition 4.6.

4.3. Rank of $\mathrm{Fil}^n(\widetilde{\widetilde{M}})$.

Let us begin our analysis by computing the rank of $\mathrm{Fil}^n(\widetilde{\widetilde{M}})$.

Proposition 4.3. *Let $\triangle = N(\mathfrak{n}\mathfrak{d}_F)$. Assume $(p,\triangle)=1$. Then $\mathrm{Fil}^n(\widetilde{\widetilde{M}})$ is free of rank 1 over κ.*

Proof. By the definition,

$$\begin{aligned}&\mathrm{Fil}^n(\widetilde{H}^n_{\log-\text{cris,par}}(M^{\text{tor}})_\kappa)\\ &= H^0(M^{\text{tor}}_\kappa,\Omega^n_{M^{\text{tor}}_\kappa/\kappa}(\log(D))(-D))/\left(\text{image of } H^0(M^{\text{tor}}_{\mathcal{O}},\Omega^n_{M^{\text{tor}}_{\mathcal{O}}/\mathcal{O}}(\log(D))(-D))_{\text{torsion}}\right)\\ &= H^0(M^{\text{tor}}_\kappa,\Omega^n_{M^{\text{tor}}_\kappa/\kappa}(\log(D))(-D)).\end{aligned}$$

Then we have

$$\mathrm{Fil}^n(\widetilde{\widetilde{M}}) = H^0(M^{\text{tor}}_\kappa,\Omega^n_{M^{\text{tor}}_\kappa/\kappa}(\log(D))(-D))[\mathfrak{p}_{\mathbf{f}}].$$

Our assertion follows from

$$H^0(M_\kappa^{\mathrm{tor}}, \Omega^n_{M_\kappa^{\mathrm{tor}}/\kappa}(\log(D))(-D))[\mathfrak{p}_{\mathbf{f}}] \simeq \kappa,$$

which is proved by the q-expansion principle [Dim2, Proposition 1.10] and Hecke relations between Fourier coefficients and Hecke eigenvalues. ☐

4.4. **Rank of $\mathrm{Fil}^n(\widetilde{\overline{M_{\mathbf{f}}}})$.** The second point to be discussed is $\mathrm{Fil}^n(\widetilde{\overline{M_{\mathbf{f}}}})$.

Lemma 4.4. *Assume that* $\operatorname{coker}\Big(H^n_{\text{ét,c}}(M_{\overline{\mathbb{Q}}}, \mathcal{O}) \to H^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})\Big)$ *is torsion-free. Then the canonical morphism*

$$\widetilde{\overline{V}}_{\mathbf{f}} \to \widetilde{\overline{V}} \quad \textit{is injective.}$$

Proof. First, we claim that

$$\widetilde{\overline{V}}_{\mathbf{f}} = \widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})[\mathfrak{p}_{\mathbf{f}}]/\varpi \to \widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})/\varpi \quad \text{is injective.} \tag{4.5}$$

Since $\widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})/\widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})[\mathfrak{p}_{\mathbf{f}}]$ is torsion-free, the snake lemma for

$$\begin{tikzcd}
\widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})[\mathfrak{p}_{\mathbf{f}}] \arrow[r, hook, "\times\varpi"] \arrow[d, hook] & \widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})[\mathfrak{p}_{\mathbf{f}}] \arrow[r, two heads, "\bmod \varpi"] \arrow[d, hook] & \widetilde{\overline{V}}_{\mathbf{f}} \arrow[d, "(4.5)"] \\
\widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O}) \arrow[r, hook, "\times\varpi"] & \widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O}) \arrow[r, two heads, "\bmod \varpi"] & \widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})/\varpi
\end{tikzcd}$$

implies the injectivity of (4.5).

Next, we claim that

$$\widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})/\varpi \to \widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \kappa) \quad \text{is injective.} \tag{4.6}$$

If the map ★ in the diagram (4.7) is injective, our claim follows from the snake lemma for

$$\begin{tikzcd}
H^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})_{\text{torsion}} \arrow[r] \arrow[dd] & H^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O}) \arrow[r, two heads] \arrow[d, "\bmod \varpi"] & \widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O}) \arrow[d, "\bmod \varpi"] \\
 & H^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})/\varpi \arrow[r, two heads] \arrow[d, "\bigstar"] & \widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})/\varpi \arrow[d, "(4.6)"] \\
\text{image of } H^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})_{\text{torsion}} \arrow[r] & H^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \kappa) \arrow[r, two heads] & \widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \kappa).
\end{tikzcd} \tag{4.7}$$

The injectivity of the map ♢ in the diagram (4.8) follows from the snake lemma and the assumption that the cokernel of $H^n_{\text{ét,c}}(M_{\overline{\mathbb{Q}}}, \mathcal{O}) \to H^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})$ is torsion-free. Thus the injectivity of the map ★ follows from the following commutative diagram:

$$\begin{tikzcd}
H^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O}) \arrow[r, hook] \arrow[d, "\bmod \varpi"] & H^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathcal{O}) \arrow[r, two heads] \arrow[d, "\bmod \varpi"] & H^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})/H^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O}) \\
H^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})/\varpi \arrow[r, hook, "\diamondsuit"] \arrow[d, "\bigstar"] & H^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})/\varpi \arrow[d, hook] & \\
H^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \kappa) \arrow[r, hook] & H^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \kappa). &
\end{tikzcd} \tag{4.8}$$

□

Proposition 4.5. $\mathrm{Fil}^n(\widetilde{\widetilde{M}}_{\boldsymbol{f}}) \neq 0$.

Proof. Since $\widetilde{\widetilde{V}}_{\mathbf{f}} \hookrightarrow \widetilde{\widetilde{V}}$, we have $\widetilde{\widetilde{M}}_{\mathbf{f}} \hookrightarrow \widetilde{\widetilde{M}}$ and hence

$$\mathrm{Fil}^n(\widetilde{\widetilde{M}}_{\mathbf{f}}) \hookrightarrow \mathrm{Fil}^n(\widetilde{\widetilde{M}}).$$

Then our assertion follows from Proposition 4.3 and $\mathbf{f} \equiv \mathbf{E} \not\equiv 0 \pmod{\varpi}$. □

4.5. **The Hodge–Tate weight and rank of $\widetilde{V}_{\mathbf{E}}$.** Finally, we consider the torsion-free Eisenstein part

$$\widetilde{V}_{\mathbf{E}} = \widetilde{H}^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})[\mathfrak{p}_{\mathbf{E}}].$$

We abbreviate $\Gamma_{1,1}(\mathfrak{n})$ to Γ. The following proposition is a key to prove our theorem.

Proposition 4.6. *Assume that $F \neq \mathbb{Q}$, $\overline{\Gamma} = \Gamma/(\Gamma \cap F^\times)$ is p-torsion-free, and $C(\mathfrak{q}, \boldsymbol{E}) \neq N(\mathfrak{q})$ for some prime ideal $\mathfrak{q}$ dividing $\mathfrak{n}$, where $C(\mathfrak{q}, \boldsymbol{E})$ is the $U(\mathfrak{q})$-eigenvalue of $\boldsymbol{E}$. Then $\widetilde{V}_{\boldsymbol{E}}$ is free of rank 1 over $\mathcal{O}$ and the Hodge-Tate weight is n.*

Proof. We denote by X the complex manifolds $Y = Y(\mathfrak{n})$ or $Y^1 = Y^1(\mathfrak{n})$ defined in §1.1. We shall decompose

$$H^n(X, \mathbb{C}) = H^n_{\text{par}}(X, \mathbb{C}) \oplus H^n_{\text{Eis}}(X, \mathbb{C}),$$

where $H^n_{\text{Eis}}(X, \mathbb{C})$ is the Eisenstein cohomology (for the definition, see **Step3**).

By the comparison theorem between étale cohomology, Betti cohomology, and de Rham cohomology, it suffices to prove the following two claims:

(1) $H^n_{\text{Eis}}(Y, \mathbb{C}) = F^n H^n_{\text{Eis}}(Y, \mathbb{C})$

(2) $H^n_{\text{Eis}}(Y, \mathbb{C})$ is stable under the Hecke correspondences and

$$H^n(Y, \mathbb{C})[\mathfrak{p}_{\mathbf{E}}] = H^n_{\text{Eis}}(Y, \mathbb{C})[\mathfrak{p}_{\mathbf{E}}] \simeq \mathbb{C}.$$

First, we prove (1). In the case $X = Y^1$, Freitag shows that the Hodge number of the Eisenstein cohomology is equal to n ([Fre, Chapter III, Proposition 3.5 and Theorem 4.9]). In the case $X = Y$, we follow the arguments in the Freitag's proof.

Step1: To give a basis of $H^{n-1}(\overline{\Gamma_t}, \mathbb{C})$ and $H^n(\overline{\Gamma_t}, \mathbb{C})$ for each cusp t.

Let $\alpha \in G(\mathbb{Q})$ be such that $\alpha^{-1}(t) = \infty$. We may assume that $t = \infty$ by the pull-back by α. We shall prove that a basis of $H^{n-1}(\overline{\Gamma_\infty}, \mathbb{C})$ (resp. $H^n(\overline{\Gamma_\infty}, \mathbb{C})$) over $\mathbb{C}$ is given by

$$\omega_\infty^{n-1} = \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{n-1}}{y_{n-1}} \text{ (resp. } \omega_\infty^n = dx_1 \wedge \cdots \wedge dx_n).$$

We remark that these forms are closed and $\overline{\Gamma_\infty}$-invariant.

Let $D = \{z \in \mathfrak{H}^n \mid N(y) = y_1 \cdots y_n = 1\}$ be the boundary of the Borel–Serre compactification Y^{BS} of Y at the cusp ∞ as §2.5. The group Γ_∞ which consists of transformations of the form

$$z \mapsto uz + b, \ N(u) = 1$$

acts on D. We may identify D with $\mathbb{R}^{2n-1}$ by

$$D \simeq \mathbb{R}^{2n-1} : z \mapsto (x_1, \cdots, x_n, u_1, \cdots, u_{n-1})$$

with coordinates $\{x_i\}_{i=1}^n$ and $\{u_i = \log(y_i)\}_{i=1}^{n-1}$. Since

$$\overline{\Gamma_\infty} \backslash \mathfrak{H}^n \simeq \mathbb{R} \times (\overline{\Gamma_\infty} \backslash D) : z \mapsto (\log(N(y)), N(y)^{-1/n} z),$$

$\overline{\Gamma_\infty} \backslash D \hookrightarrow \overline{\Gamma_\infty} \backslash \mathfrak{H}^n$ is a homotopy equivalence and hence it suffices to compute $H^*(\overline{\Gamma_\infty} \backslash D, \mathbb{C})$.

For subsets $b, c \subset \{1, \cdots, n\}$, we consider a $\overline{\Gamma_\infty}$-invariant harmonic differential m-form $\omega = \sum f_{b,c}(x,u) dx_b \wedge du_c$. By the same argument of [Fre, p.145, 146], the functions $f_{b,c}(x,u)$ are independent of x and if $f_{b,c}(x,u) \neq 0$, then $b = \phi$ or $\{1, \cdots, n\}$.

In the case $b = \phi$, $H^{n-1}(\overline{\Gamma_\infty} \backslash D, \mathbb{C})$ is isomorphic to the de Rham cohomology of a lattice $\log(\mathfrak{o}_{F,+}^\times) \subset \mathbb{R}^{n-1}$. In the same way as [Fre, p.146], one shows that ω_∞^{n-1} is a basis as desired. In the case $b = \{1, \cdots, n\}$, $H^n(\overline{\Gamma_\infty} \backslash D, \mathbb{C})$ is isomorphic to the de Rham cohomology of a lattice and hence this case is similar.

Step2: To construct the Eisenstein operator

$$E : \bigoplus_{t \in C(\Gamma)} H^n(\overline{\Gamma_t}, \mathbb{C}) \to H^n(\overline{\Gamma}, \mathbb{C}).$$

We may assume $t = \infty$. As in [Fre, Chapter III, Remark 3.1], $\omega_\infty^n = dx_1 \wedge \cdots \wedge dx_n$ is cohomologous to $dz_1 \wedge \cdots \wedge dz_n$ up to scalar. We put

$$\omega_\infty = dz_1 \wedge \cdots \wedge dz_n.$$

As in the proof of [Fre, Chapter III, Proposition 3.5], in order to construct $\overline{\Gamma}$-invariant forms from $\overline{\Gamma_\infty}$-invariant forms, the Eisenstein operator E is defined by symmetrization:

$$E(\omega_\infty) = \text{“} \sum_{M \in \overline{\Gamma_\infty} \backslash \overline{\Gamma}} M^* \omega_\infty \text{”}.$$

Here “ ” means that it can be defined by using analytic continuation of Eisenstein series. Note that, for $M = \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \in \overline{\Gamma}$, $M^* \omega_\infty = N(cz+d)^{-2} \omega_\infty$. If there exists the limit $E_{2,0}^\Gamma(z) = \lim_{s \to 0} E_{2,0}^\Gamma(z,s)$, then the Eisenstein operator E is well-defined:

$$E(\omega_\infty) = \sum_{M \in \overline{\Gamma_\infty} \backslash \overline{\Gamma}} M^* \omega_\infty := \lim_{s \to 0} \sum_{M \in \overline{\Gamma_\infty} \backslash \overline{\Gamma}} |N(cz+d)|^{-2s} M^* \omega_\infty = E_{2,0}^\Gamma(z) \omega_\infty,$$

where $E_{2,0}^\Gamma(z,s)$ is an Eisenstein series of the following type:

$$E_{2,0}^\Gamma(z,s) = \sum_{M \in \overline{\Gamma_\infty} \backslash \overline{\Gamma}} N(cz+d)^{-2} |N(cz+d)|^{-2s}.$$

Analytic continuation of the Eisenstein series follows from [Shi, Proposition 3.2] as follows. We use the same notation $H(z,s,\mathbf{1})$ as [Shi, (3.14)] for $\mathfrak{b} = \mathfrak{d}_F[t_1]$, $\mathfrak{c} = \mathfrak{n}$. Since $\overline{\Gamma_\infty} \backslash \overline{\Gamma} \simeq \Gamma_\infty \backslash \Gamma$, we have $E_{2,0}^\Gamma(z,s) = H(z,s,\mathbf{1})$. Thus, by [Shi, Proposition 3.2], if $n = [F : \mathbb{Q}] > 1$, then $E_{2,0}^\Gamma(z,s)$ can be continued to a meromorphic function on the whole s-plane and holomorphic at $s = 0$ as desired.

Step3: To show that Eisenstein operator E is a section of the restriction map $H^n(\overline{\Gamma}, \mathbb{C}) \to H^n(\overline{\Gamma_t}, \mathbb{C})$ for each cusp t.

As in the proof of [Fre, Chapter III, Proposition 3.3], it suffices to compute the constant term of $E_{2,0}^\Gamma(z)$ at the cusp t is equal to 1 (resp. 0) if $t \sim_\Gamma \infty$ (resp. $t \nsim_\Gamma \infty$). As in the same way ([Fre, Chapter I, §5]), the constant term can be computed by using the formula

$$\lim_{N(y) \to \infty} \lim_{s \to 0} E_{2,0}^\Gamma(z,s)|M = \lim_{s \to 0} \lim_{N(y) \to \infty} E_{2,0}^\Gamma(z,s)|M.$$

For example, at the cusp $t = \infty$, we have

$$\lim_{N(y) \to \infty} N(cz+d)^{-2} |N(cz+d)|^{-2s} = \begin{cases} 1 & \text{if } c = 0, \\ 0 & \text{if } c \neq 0. \end{cases}$$

We define the Eisenstein cohomology $H^n_{\mathrm{Eis}}(Y,\mathbb{C})$ by

$$H^n_{\mathrm{Eis}}(Y,\mathbb{C}) = \mathrm{im}(E).$$

Therefore, since $E^{\Gamma}_{2,0}(z)$ is holomorphic, the Hodge number of the Eisenstein cohomology is n, that is, $H^n_{\mathrm{Eis}}(Y,\mathbb{C}) = F^n H^n_{\mathrm{Eis}}(Y,\mathbb{C})$.

Next, we prove (2). Since the $U(\mathfrak{q})$-eigenvalue of each invariant form $\omega_{J'}$ as in the proof of Theorem 2.22 is $N(\mathfrak{q})$ by the decomposition (3.4), the assumption $C(\mathfrak{q},\mathbf{E}) \neq N(\mathfrak{q})$ and the q-expansion principle over $\mathbb{C}$ imply that $H^n_{\mathrm{par}}(Y,\mathbb{C})[\mathfrak{p}_{\mathbf{E}}] = 0$. Thus, if the Eisenstein cohomology $H^n_{\mathrm{Eis}}(Y,\mathbb{C})$ is stable under the Hecke correspondences, we get (2):

$$H^n(Y,\mathbb{C})[\mathfrak{p}_{\mathbf{E}}] = H^n_{\mathrm{Eis}}(Y,\mathbb{C})[\mathfrak{p}_{\mathbf{E}}] \simeq \mathbb{C}.$$

We prove this stability of the Hecke correspondence. We use the same notation as §2.3. Let $c = (c_t)_{t\in C(\Gamma)} \in \bigoplus_{t\in C(\Gamma)} H^n(\overline{\Gamma}_t,\mathbb{C})$ such that $c_t = 0$ if $t \neq \infty$ and $c_\infty = [\omega_\infty]$. Let's fix $\alpha \in \mathrm{GL}_2(F)$ such that $\Gamma\alpha\Gamma = \coprod_{i\in I}\Gamma\alpha_i$ as a finite disjoint union. It suffices to show that

$$E(c)|[\Gamma\alpha\Gamma] = E(c|[\Gamma\alpha\Gamma]). \tag{4.9}$$

By the definition of the Eisenstein operator E, the left hand side is equal to

$$\begin{aligned} E(c)|[\Gamma\alpha\Gamma] &= \sum_{i\in I}\alpha_i^*\left(\lim_{s\to 0}\sum_{M\in\Gamma_\infty\backslash\Gamma}|N(j(M,z))|^{-2s}M^*\omega_\infty\right) \\ &= \sum_{i\in I}\lim_{s\to 0}\sum_{M\in\Gamma_\infty\backslash\Gamma}|N(j(M,\alpha_i(z)))|^{-2s}\alpha_i^*M^*\omega_\infty \\ &= \sum_{i\in I}\lim_{s\to 0}|N(j(\alpha_i,z))|^{2s}\sum_{M\in\Gamma_\infty\backslash\Gamma}|N(j(M\alpha_i,z))|^{-2s}(M\alpha_i)^*\omega_\infty \\ &= \lim_{s\to 0}\sum_{i\in I}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma\alpha_i}|N(j(\gamma,z))|^{-2s}\gamma^*\omega_\infty \\ &= \lim_{s\to 0}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma\alpha\Gamma}|N(j(\gamma,z))|^{-2s}\gamma^*\omega_\infty. \end{aligned} \tag{4.10}$$

We consider the right hand side of (4.9). For each $s \in \mathbb{P}^1(F)$, we put

$$\mathscr{S}_s = \{\gamma \in \Gamma_\infty\backslash\Gamma\alpha\Gamma \mid \gamma(s) = \infty\}.$$

Note that

$$\Gamma_\infty\backslash\Gamma\alpha\Gamma = \coprod_{s\in\mathbb{P}^1(F)}\mathscr{S}_s.$$

For each $s \in \mathbb{P}^1(F)$, there exist a unique $t \in C(\Gamma)$ and a unique $M \in \Gamma_t\backslash\Gamma$ such that $M(s) = t$ and hence

$$\coprod_{t\in C(\Gamma)}\coprod_{M\in\Gamma_t\backslash\Gamma}\mathscr{S}_{M^{-1}(t)} \to \Gamma_\infty\backslash\Gamma\alpha\Gamma \quad \text{is bijective.} \tag{4.11}$$

We put $c|[\Gamma\alpha\Gamma] = ([\omega'_t])_{t\in C(\Gamma)}$. We claim that

$$\omega'_t = \sum_{\gamma\in\mathscr{S}_t}\gamma^*\omega_\infty. \tag{4.12}$$

By the definition of $c|[\Gamma\alpha\Gamma]$ as §2.3,

$$\begin{aligned}\omega'_t &= \sum_{i\in I^t}\sum_{j\in J_i^t}(\beta_i^t\delta_{i,j}^t)^*\omega_{\beta_i^t(t)}\\ &= \sum_{i\in I^t_\infty}\sum_{j\in J_i^t}(\beta_i^t\delta_{i,j}^t)^*\omega_{\beta_i^t(t)},\end{aligned}$$

where $I^t_\infty = \{i \in I^t \mid \beta_i^t(t) \sim_\Gamma \infty\}$. For each $i \in I^t_\infty$, we may assume that $\beta_i^t(t) = \infty$ by replacing β_i^t by $\gamma_i^t\beta_i^t$ with $\gamma_i^t \in \Gamma$ and $\gamma_i^t\beta_i^t(t) = \infty$. Then, in order to prove (4.12), it suffices to show the following decomposition:

$$\mathscr{S}_t = \coprod_{i\in I^t_\infty}\coprod_{j\in J_i^t}\Gamma_\infty\beta_i^t\delta_{i,j}^t.$$

Proof. $(\supset)$: It follows from $\beta_i^t\delta_{i,j}^t(\infty) = \infty$.

$(\subset)$: Using the decomposition of $\Gamma\alpha\Gamma$ as §2.3, we have

$$\Gamma_\infty\backslash\Gamma\alpha\Gamma = \coprod_{i\in I^t}\coprod_{j\in J_i^t}\Gamma_\infty\backslash\Gamma\beta_i^t\delta_{i,j}^t.$$

For each $\Gamma_\infty\gamma\beta_i^t\delta_{i,j}^t \in \mathscr{S}_t$ with $\gamma \in \Gamma$, we have $\gamma\beta_i^t(t) = \infty$ and hence $i \in I^t_\infty$ and $\gamma \in \Gamma_\infty$. In particular, $\Gamma_\infty\gamma\beta_i^t\delta_{i,j}^t = \Gamma_\infty\beta_i^t\delta_{i,j}^t$ as desired. □

Thus we obtain

$$\begin{aligned}E(c|[\Gamma\alpha\Gamma]) &= \sum_{t\in C(\Gamma)}\sum_{\gamma\in\mathscr{S}_t}E(\gamma^*\omega_\infty)\\ &= \sum_{t\in C(\Gamma)}\sum_{\gamma\in\mathscr{S}_t}\lim_{s_{t,\gamma}\to 0}\sum_{M\in\Gamma_t\backslash\Gamma}|N(j(\gamma M, z))|^{-2s_{t,\gamma}}(\gamma M)^*\omega_\infty.\end{aligned}$$

Here the first equality follows from (4.12) and the second equality follows from the definition of the Eisenstein operator E. Thus we get

$$E(c|[\Gamma\alpha\Gamma]) = E(c)|[\Gamma\alpha\Gamma]$$

as desired. Here the equality follows from $\mathscr{S}_t \cdot M = \mathscr{S}_{M^{-1}(t)}$, (4.11), and (4.10). □

Under the same assumptions of main theorem 4.1, we show that $\bmod\varpi : H^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathcal{O}) \twoheadrightarrow H^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})/\varpi$ induces

$$\bmod\varpi : \widetilde{V}_{\mathbf{E}} \to \widetilde{\widetilde{V}}.$$

Let $[c] \in H^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})$ mapping to $[\pi_{\mathbf{E}}] \in \widetilde{H}^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})$ and let $\overline{[c]}$ denote the image of $[c]$ in $H^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})/\varpi$. Our assumptions that $\operatorname{coker}\left(H^n_{\text{ét,c}}(M_{\overline{\mathbb{Q}}}, \mathcal{O}) \to H^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})\right)$ is torsion-free and $\mathbf{f} \equiv \mathbf{E} (\bmod\ \varpi)$ imply that $[\pi_{\mathbf{E}}]$ is zero in $\operatorname{coker}\left(H^n_{\text{ét,c}}(M_{\overline{\mathbb{Q}}}, \mathcal{O}) \to H^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})\right)/\varpi$ by Proposition 2.9. Thus $\overline{[c]}$ is zero in $\operatorname{coker}\left(H^n_{\text{ét,c}}(M_{\overline{\mathbb{Q}}}, \mathcal{O}) \to H^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})\right)/\varpi$. With the help of the injectivity of $\diamondsuit$ in the diagram (4.8), we see that $\overline{[c]}$ belongs to $H^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})/\varpi$. Then our claim follows from the injectivity of $\widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})/\varpi \xrightarrow{\spadesuit} \widetilde{H}^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})/\varpi$ and

(4.6). One see that the injectivity of $\spadesuit$ is obtained by the following diagram:

$$
\begin{tikzcd}
H^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})_{\text{torsion}}/\varpi \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & H^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})/\varpi \arrow[r, two heads] \arrow[d, hook, "\diamondsuit"] & \widetilde{H}^n_{\text{ét,par}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})/\varpi \arrow[d, "\spadesuit"] \\
H^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})_{\text{torsion}}/\varpi \arrow[r, hook] & H^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})/\varpi \arrow[r, two heads] & \widetilde{H}^n_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathcal{O})/\varpi.
\end{tikzcd}
\tag{4.13}
$$

We define

$$L = \operatorname{im}\left(\operatorname{mod}\varpi : \widetilde{V}_{\mathbf{E}} \to \widetilde{\widetilde{V}}\right).$$

With the help of Corollary 2.24, we obtain the following proposition:

Proposition 4.7. *Under the same assumptions of Theorem 4.1 and Proposition 4.6, L is a free of rank 1 over κ with Hodge-Tate weight n.*

4.6. **Proof of Theorem 4.1.** We consider the following diagram :

$$
\begin{tikzcd}
\widetilde{V}_{\mathbf{E}} \arrow[dr] \arrow[drr, "\operatorname{mod}\varpi"] & & \\
& L \arrow[r, hook] & \widetilde{\widetilde{V}} \\
\widetilde{\widetilde{V}}_{\mathbf{f}} \arrow[urr, hook, "(4.5)"'] & &
\end{tikzcd}
$$

By the comparison theorem between étale cohomology and Betti cohomology, we may regard this diagram as W_G-equivariant. We put $N = \operatorname{im}(\operatorname{mod}\varpi : \widetilde{M}_{\mathbf{E}} \to \widetilde{\widetilde{M}})$. By combining Proposition 4.3, Proposition 4.7, and Remark 2.23, we have $L = L[\epsilon]$ and $\operatorname{Fil}^n(\widetilde{\widetilde{M}}) = N$. Thus, by Lemma 4.4 and Proposition 4.5, there exists a subrepresentation L' of $\widetilde{\widetilde{V}}_{\mathbf{f}}$ such that $L \simeq L'$. By the diagram as W_G-modules, L' is stable under the action of W_G and $L' = L'[\epsilon]$. The partial Eichler–Shimura–Harder isomorphism (2.27) over $\mathbb{C}$ says that $\widetilde{\widetilde{V}}_{\mathbf{f}}[\epsilon]$ is free of rank 1 over κ. Therefore, we obtain the main theorem 4.1.

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