博士論文

論文題目　On contact submanifolds of
the odd dimensional Euclidean spaces
(奇数次元ユークリッド空間の
接触部分多様体について)

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全体の序文

Title On contact submanifolds of the odd dimensional Euclidean spaces

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A contact structure is a maximally non-integrable hyperplane field η on an odd dimensional manifold N. If the normal bundle of η is orientable, we say that the contact structure η is co-oriented. This is equivalent to that there is a global defining 1-form α of η. Then, for a global contact form α, $d\alpha$ induces a symplectic vector bundle structure on η. The conformal class of the symplectic vector bundle structure does not depend on the choice of α. If an embedded odd dimensional submanifold M of the contact manifold (N, η) is transverse to η and the intersection $\xi = TM \cap \eta|_M$ is a symplectic subbundle of η, then (M, ξ) is called a contact submanifold of (N, η). The embedding $M \to N$ is called a contact embedding of (M, ξ) in (N, η). Similarly, an immersed contact submanifold (M, ξ) and a contact immersion $M \to N$ are defined.

Gromov formulated a very general way to construct a solution of partial differential relations ([3]). It is called the homotopy principle (h-principle). The Smale-Hirsch theory is a typical example of the h-principle. If the codimesion is positive, the classification problem of immersions is reduced to that of formal immersions, where formal immersions are bundle monomorphisms between the tangent bundles. When a problem of the differential topology can be reduced to that of the homotopy theory in such a way, we say that the h-principle holds. Gromov proved that the h-principle holds in various situations in contact geometry. An almost contact structure on an odd dimensional oriented manifold N^{2n+1} is a pair (β_1, β_2) of a global 1-form β_1 and a global 2-form β_2 satisfying the condition $\beta_1 \wedge \beta_2^n \neq 0$. Given an odd dimensional manifold N^{2n+1}, let S^+_{cont} and $\mathbb{S}^+_{cont}$ be the space of co-oriented almost contact structures on N^{2n+1} and the space of co-oriented contact structures on N^{2n+1}, respectively. Gromov showed the following theorem.

Theorem 1 (Gromov [2]) The inclusion $\mathbb{S}^+_{cont} \to S^+_{cont}$ is a homotopy equivalence for an odd dimensional open manifold N^{2n+1}.

For contact immersions and contact embeddings, he proved the following h-principles.

Theorem 2 (Gromov [3]) If the codimension is positive, the h-principle holds for contact immersions. If the codimension is greater than two, the h-principle holds for contact embeddings.

Here a formal contact immersion $F : TM \to TN$ covering $f : M \to N$ is a monomorphism which is transversal to η and sends the contact hyperplane ξ_p to a symplectic subspace of $\eta_{f(p)}$ for each $p \in M$. A formal contact immersion $F : TM \to TN$ is called a formal contact embedding if the base map $f : M \to N$ is an embedding and it is homotopic via homotopy of monomorphisms to the differential map df.

The aim of this thesis is to study contact submanifolds in the odd dimensional Euclidean spaces using the above h-principles. Concretely, we consider the following three problems.
(1) Is there any non-trivial obstruction for the existence of codimension two contact embeddings?
(2) Determine codimension two contact submanifolds in $\mathbb{R}^{2n+1}$ for some contact structure on $\mathbb{R}^{2n+1}$.
(3) Which closed co-oriented contact $(2n+1)$-manifold can be a contact submanifold of the standard contact structure on $\mathbb{R}^{4n+1}$?

The research of contact submanifolds has not been well developed. Recently, however, people began looking at contact submanifolds in relation with higher dimensional generalization of the Lutz-twist and the contact homology of higher dimensional contact manifolds. The precedent results about the above three problems are the following. For the problem (1), there have been no known obstructions other than that for the existence of an embedding as a manifold. For the problem (2), the link of a complex hypersurface singularity in $\mathbb{C}^{n+1}$ gives an example of codimension two contact submanifold of the standard contact structure η_0 on $\mathbb{R}^{2n+1}$. However, there are few examples of contact manifolds and the research is restricted to contact embeddings of typical contact manifolds. For the problem (3), the following theorem is known.

Theorem 3 (Gromov [3], see also [9], [8]) Any closed co-orientable contact $(2m+1)$-manifold can be an immersed contact submanifold of $(\mathbb{R}^{4m+1}, \eta_0)$, and it can be a contact submanifold of $(\mathbb{R}^{4m+3}, \eta_0)$.

Hence, we would like to know about the existence of a contact embedding of a co-oriented contact $(2n+1)$-manifold in $(\mathbb{R}^{4n+1}, \eta_0)$.

In this thesis, we show the following results. Theorems 4 and 5 are partial answers to the problem (1). For a co-oriented contact structure ξ, the Chern class of the contact structure ξ is defined as the Chern class of a complex vector bundle structure compatible with the symplectic structure on ξ.

Theorem 4 If a closed contact manifold (M^{2n-1}, ξ) is a contact submanifold of a co-oriented contact manifold (N^{2n+1}, η) such that $H^2(N^{2n+1}; \mathbb{Z}) = 0$, then the first Chern class $c_1(\xi)$ is trivial. Moreover, if the $2j$-th integral cohomology group $H^{2j}(N^{2n+1}; \mathbb{Z})$ is also trivial, then the j-th Chern class $c_j(\xi)$ is trivial.

By Theorem 4, the total Chern class of a codimension two contact submanifold of $\mathbb{R}^{2n+1}$ is trivial. In particular, a closed co-oriented contact 3-manifold with non-trivial first Chern class cannot be embedded in $(\mathbb{R}^5, \eta_0)$ as a contact submanifold. There are infinitely many such contact manifolds. For, every closed orientable 3-manifold admits a contact structure in each homotopy class of tangent 2-plane fields by the theorem of Lutz [6] and Martinet [7].

Theorem 5 There are infinitely many contact structures on S^7 which do not admit contact embeddings into $\mathbb{R}^9$ for any contact structure.

Theorem 5 gives examples of contact manifolds with trivial total Chern class which cannot be codimension two contact submanifolds of $\mathbb{R}^{2n+1}$. The proof of Theorem 5 relies on the following theorem of Ding and Geiges.

Theorem 6 (Ding-Geiges [1]) Any almost contact structure on S^7 can be realized as a contact structure.

For the problem (2), we show the following theorem.

Theorem 7 For any closed co-oriented contact 3-manifold (M^3, ξ) with $c_1(\xi) = 0$, there is a contact structure η on $\mathbb{R}^5$ such that we can embed (M^3, ξ) in $(\mathbb{R}^5, \eta)$ as a contact submanifold. For any closed, co-oriented, simply-connected contact 5-manifold (M^5, ξ) with $c_1(\xi) = 0$, there is a contact structure η on $\mathbb{R}^7$ such that we can embed (M^5, ξ) in $(\mathbb{R}^7, \eta)$ as a contact submanifold.

In the proof of Theorem 7, we use the relative version of Theorem 1. We prove that the induced contact structure on a tubular neighborhood of $M^{2n-1} \subset \mathbb{R}^{2n+1}$ can be extended over $\mathbb{R}^{2n+1}$ as an almost contact structure

and apply the h-principle to the extended almost contact structure.

For the problem (3), we prove the following theorem.

Theorem 8 Let (M^{2m+1}, ξ) be a closed co-oriented $(2m+1)$-contact manifold which satisfies either of the following conditions:
(i) m is odd, $m \geq 3$ and $H_1(M^{2m+1} : \mathbb{Z}) = 0$,
(ii) m is even, $m \geq 4$ and M^{2m+1} is 2-connected,
(iii) $m = 2$ and M^5 is simply-connected.
Then, there exists a contact embedding of (M^{2m+1}, ξ) in $(\mathbb{R}^{4m+1}, \eta_0)$.

For the proof of Theorem 8, it is enough to show the existence of formal contact embeddings by Theorem 2. We show it by using Li's classification of immersions of k-manifolds in $(2k-1)$-manifolds ([5]) and Kervaire's result about the unstable homotopy groups of $SO(2k)$ and $U(k)$ ([4]) for appropriate k.

Moreover, we prove that the existence problem of a formal contact embedding in the standard contact structure $(\mathbb{R}^{2n+1}, \eta_0)$ and that of a contact embedding in $(\mathbb{R}^{2n+1}, \eta)$ for some contact structure η are equivalent.

Theorem 9 Let (M^{2m+1}, ξ) be a contact manifold and $f : M^{2m+1} \to \mathbb{R}^{2n+1}$ be an embedding. The following three statements are equivalent.
(i) There is a formal contact embedding of (M^{2m+1}, ξ) in $(\mathbb{R}^{2n+1}, \eta_0)$ which covers the embedding f.
(ii) There is a contact immersion of (M^{2m+1}, ξ) in $(\mathbb{R}^{2n+1}, \eta_0)$ which is regularly homotopic to the embedding f.
(iii) There is a contact structure η on $\mathbb{R}^{2n+1}$ such that f is a contact embedding of (M^{2m+1}, ξ) in $(\mathbb{R}^{2n+1}, \eta)$.

By Theorems 7 and 9, for any closed co-oriented contact 3-manifold with trivial first Chern class, there is a formal contact embedding in $(\mathbb{R}^5, \eta_0)$. This fact can be also obtained by determining the regular homotopy classes of immersions which contain embeddings and those which contain contact immersions. First, we determine $\mathrm{CI}[(M^3, \xi), (\mathbb{R}^5, \eta_0)]$ the set of regular homotopy classes of immersions $M^3 \to \mathbb{R}^5$ which contain contact immersions of (M^3, ξ) into $(\mathbb{R}^5, \eta_0)$.

Theorem 10 The normal Euler class for a contact immersion of (M^3, ξ) into $(\mathbb{R}^5, \eta_0)$ is equal to $-c_1(\xi)$. Furthermore, there is a bijection between $\mathrm{CI}[(M^3, \xi), (\mathbb{R}^5, \eta_0)]$ and $H^3(M^3; \mathbb{Z})/(-2c_1(\xi) \smile H^1(M^3; \mathbb{Z}))$.

Note that the right-hand side of the formula in Theorem 10 is written in the terminology of the theorem of Wu ([11]). On the other hand, Saeki and Takase characterized $\mathrm{Emb}[M^3, \mathbb{R}^5]$, the set of the regular homotopy classes of immersions of M^3 in $\mathbb{R}^5$ which contains an embedding ([10]). Their result and Theorem 10 show that the condition $c_1(\xi) = 0$ is equivalent to that the intersection $\mathrm{Emb}[M^3, \mathbb{R}^5] \cap \mathrm{CI}[(M^3, \xi), (\mathbb{R}^5, \eta_0)]$ is not empty.

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On contact submanifolds of the odd dimensional Euclidean spaces

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Contents

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1 Introduction

A contact structure is a maximally non-integrable hyperplane field η on an odd dimensional manifold N. If the normal bundle of η is orientable, we say that the contact structure η is co-oriented. This is equivalent to that there is a global defining 1-form α of η. Then, for a global contact form α, $d\alpha$ induces a symplectic vector bundle structure on η. The conformal class of the symplectic vector bundle structure does not depend on the choice of α. If an embedded odd dimensional submanifold M of the contact manifold (N, η) is transverse to η and the intersection $\xi = TM \cap \eta|_M$ is a symplectic subbundle of η, then (M, ξ) is called a contact submanifold of (N, η). The embedding $M \to N$ is called a contact embedding of (M, ξ) in (N, η). Similarly, an immersed contact submanifold (M, ξ) and a contact immersion $M \to N$ are defined.

Gromov formulated a very general way to construct a solution of partial differential relations ([16]). It is called the homotopy principle (h-principle). The Smale-Hirsch theory ([46], [20]) is a typical example of the h-principle (see §4). If the codimesion is positive, the classification problem of immersions is reduced to that of formal immersions, where formal immersions are bundle monomorphisms between the tangent bundles. When a problem of the differential topology can be reduced to that of the homotopy theory in such a way, we say that the h-principle holds. Gromov proved that the h-principle holds in various situations in contact geometry. An almost contact structure on an odd dimensional oriented manifold N^{2n+1} is a pair (β_1, β_2) of a global 1-form β_1 and a global 2-form β_2 satisfying the condition $\beta_1 \wedge \beta_2^n \neq 0$. Given an odd dimensional manifold N^{2n+1}, let S^+_{cont} and $\mathbb{S}^+_{cont}$ be the space of co-oriented almost contact structures on N^{2n+1} and the space of co-oriented contact structures on N^{2n+1}, respectively. Gromov showed that the inclusion

$$\mathbb{S}^+_{cont} \to S^+_{cont}$$

is a homotopy equivalence for an odd dimensional open manifold N^{2n+1} (Theorem 5.5).

For contact immersions and contact embeddings, he proved the following h-principles.

h-principle for contact immersions. Let (M, ξ) and (N, η) be contact manifolds of dimensions $2m + 1$ and $2n + 1$, respectively, where $m < n$. A contact homomorphism $F_0 : TM \to TN$ covering $f_0 : M \to N$ is homotopic to the differential map $F_1 = df_1$ of a contact immersion $f_1 : M \to N$ (Theorem 5.1).

Here a contact homomorphism $F_0 : TM \to TN$ covering $f_0 : M \to N$ is a

monomorphism which is transversal to η and sends the contact hyperplane ξ_p to a symplectic subspace of $\eta_{f_0(p)}$ for each $p \in M$. It is also called a formal contact immersion of (M, ξ) in (N, η).

h-principle for contact embeddings. Let (M, ξ) and (N, η) be contact manifolds of dimensions $2m+1$ and $2n+1$, respectively, where $m+1 < n$. Suppose that the differential map $F_0 = df_0$ of an embedding $f_0 : M \to N$ is homotopic (via a homotopy of monomorphisms $F_t : TM \to TN$, $t \in [0,1]$, covering f_0) to a contact homomorphism $F_1 : TM \to TN$. Then there exists an isotopy $f_t : M \to N$, $t \in [0,1]$, such that the embedding $f_1 : M \to N$ is contact and the differential df_1 is homotopic to F_1 through contact homomorphisms. Moreover, one can choose the isotopy f_t to be arbitrarily C^0-close to f_0 (Theorem 5.2).

The contact homomorphism F_1 of Theorem 5.2 is called a formal contact embedding of (M, ξ) in (N, η). In other words, if the codimension is positive, the h-principle holds for contact immersions, and if the codimension is greater than two, the h-principle holds for contact embeddings.

The aim of this thesis is to study contact submanifolds in the odd dimensional Euclidean spaces using the above h-principles. Concretely, we consider the following three problems.
(1) Is there any non-trivial obstruction for the existence of codimension two contact embeddings?
(2) Determine codimension two contact submanifolds in $\mathbb{R}^{2n+1}$ for some contact structure on $\mathbb{R}^{2n+1}$.
(3) Which closed co-oriented contact $(2n+1)$-manifold can be a contact submanifold of the standard contact structure on $\mathbb{R}^{4n+1}$?

The research of contact submanifolds has not been well developed. Recently, however, people began looking at contact submanifolds in relation with higher dimensional generalization of the Lutz-twist and the contact homology of higher dimensional contact manifolds ([27], [37], [48]). The precedent results about the above three problems are the following. For the problem (1), there have been no known obstructions other than that for the existence of an embedding as a manifold. For the problem (2), the link of a complex hypersurface singularity in $\mathbb{C}^{n+1}$ gives an example of codimension two contact submanifold of the standard contact structure η_0 on $\mathbb{R}^{2n+1}$. However, there are few examples of contact manifolds and the research is restricted to contact embeddings of typical contact manifolds. For the problem (3), it is known that for any closed co-oriented contact $(2n+1)$-manifold (M^{2n+1}, ξ), there is a contact embedding of (M^{2n+1}, ξ) in the standard con-

tact structure on $\mathbb{R}^{4n+3}$ (Theorem 7.2). We would like to know about the existence of a contact embedding of a co-oriented contact $(2n+1)$-manifold in $(\mathbb{R}^{4n+1}, \eta_0)$.

1.1 Our results

In this thesis, we show the following results. Theorems 1.1 and 1.2 are partial answers to the problem (1). For a co-oriented contact structure ξ, the Chern class of the contact structure ξ is defined as the Chern class of a complex vector bundle structure compatible with the conformal symplectic structure on ξ.

Theorem 1.1 *If a closed contact manifold (M^{2n-1}, ξ) is a contact submanifold of a co-oriented contact manifold (N^{2n+1}, η) such that $H^2(N^{2n+1}; \mathbb{Z}) = 0$, then the first Chern class $c_1(\xi)$ is trivial. Moreover, if the $2j$-th integral cohomology group $H^{2j}(N^{2n+1}; \mathbb{Z})$ is also trivial, then the j-th Chern class $c_j(\xi)$ is trivial.*

By Theorem 1.1, the total Chern class of a codimension two contact submanifold of $\mathbb{R}^{2n+1}$ is trivial. In particular, a closed co-oriented contact 3-manifold with non-trivial first Chern class cannot be embedded in $(\mathbb{R}^5, \eta_0)$ as a contact submanifold. There are infinitely many such contact manifolds. For, every closed orientable 3-manifold admits a contact structure in each homotopy class of tangent 2-plane fields by the theorem of Lutz and Martinet ([30], [31]).

Theorem 1.2 *There are infinitely many contact structures on S^7 which do not admit contact embeddings into $\mathbb{R}^9$ for any contact structure.*

Theorem 1.2 gives examples of contact manifolds with trivial total Chern class which cannot be codimension two contact submanifolds of $\mathbb{R}^{2n+1}$. The proof of Theorem 1.2 relies on the theorem of Ding and Geiges which says that any almost contact structure on S^7 can be realized as a contact structure (Theorem 6.2).

For the problem (2), we show the following theorems. Theorem 1.3 is the converse of Theorem 1.1 when $N^{2n+1} = \mathbb{R}^5$.

Theorem 1.3 *Let (M^3, ξ) be a closed co-oriented contact 3-manifold with $c_1(\xi) = 0$. Then there is a contact structure η on $\mathbb{R}^5$ such that we can embed (M^3, ξ) in $(\mathbb{R}^5, \eta)$ as a contact submanifold.*

Theorem 1.4 *Let M^5 be a closed, oriented, simply-connected 5-manifold and ξ be a co-oriented contact structure on M^5 with $c_1(\xi) = 0$. Then there is a contact structure η on $\mathbb{R}^7$ such that we can embed (M^5, ξ) in $(\mathbb{R}^7, \eta)$ as a contact submanifold.*

In the proofs of Theorems 1.3 and 1.4, we use the relative version of Theorem 5.5 (Theorem 5.6). We prove that the induced contact structure on a tubular neighborhood of $M^{2n-1} \subset \mathbb{R}^{2n+1}$ can be extended over $\mathbb{R}^{2n+1}$ as an almost contact structure and apply the h-principle to the extended almost contact structure.

Moreover, we prove that the existence problem of a formal contact embedding in the standard contact structure $(\mathbb{R}^{2n+1}, \eta_0)$ and that of a contact embedding in $(\mathbb{R}^{2n+1}, \eta)$ for some contact structure η are equivalent.

Theorem 1.5 *Let (M^{2m+1}, ξ) be a contact manifold and $f : M^{2m+1} \to \mathbb{R}^{2n+1}$ be an embedding. The following three statements are equivalent.*

1. *There is a formal contact embedding of (M^{2m+1}, ξ) in $(\mathbb{R}^{2n+1}, \eta_0)$ which covers the embedding f.*
2. *There is a contact immersion of (M^{2m+1}, ξ) in $(\mathbb{R}^{2n+1}, \eta_0)$ which is regularly homotopic to the embedding f.*
3. *There is a contact structure η on $\mathbb{R}^{2n+1}$ such that f is a contact embedding of (M^{2m+1}, ξ) in $(\mathbb{R}^{2n+1}, \eta)$.*

By Theorems 1.3 and 1.5, for any closed co-oriented contact 3-manifold with trivial first Chern class, there is a formal contact embedding in $(\mathbb{R}^5, \eta_0)$. This fact can be also obtained by determining the regular homotopy classes of immersions which contain embeddings and those which contain contact immersions. First, we determine the regular homotopy classes of immersions of M^3 in $\mathbb{R}^5$ which contains a contact immersion.

Theorem 1.6 *The normal Euler class for a contact immersion of (M^3, ξ) into $(\mathbb{R}^5, \eta_0)$ is equal to $-c_1(\xi)$. Furthermore, there is a bijection*

$$\mathrm{CI}[(M^3, \xi), (\mathbb{R}^5, \eta_0)] \approx H^3(M^3; \mathbb{Z})/(-2c_1(\xi) \smile H^1(M^3; \mathbb{Z})),$$

where $\mathrm{CI}[(M^3, \xi), (\mathbb{R}^5, \eta_0)]$ is the set of regular homotopy classes of immersions $M^3 \to \mathbb{R}^5$ which contain contact immersions of (M^3, ξ) into $(\mathbb{R}^5, \eta_0)$ and $\smile$ denotes the cup product.

Note that the right-hand side of the formula in Theorem 1.6 is written in the terminology of Theorem 4.2.

On the other hand, Saeki, Szűcs and Takase characterized $\mathrm{Emb}[M^3, \mathbb{R}^5]$, the set of the regular homotopy classes of immersions of M^3 in $\mathbb{R}^5$ which contains an embedding (Theorem 4.5). Their result and Theorem 1.6 show that the condition $c_1(\xi) = 0$ is equivalent to that the intersection

$$\mathrm{Emb}[M^3, \mathbb{R}^5] \cap \mathrm{CI}[(M^3, \xi), (\mathbb{R}^5, \eta_0)]$$

is not empty.

For the problem (3), Gromov's h-principle for contact embeddings is applied. If $n \geq 2$, then the h-principle holds for contact embeddings of (M^{2n+1}, ξ) in $(\mathbb{R}^{4n+1}, \eta_0)$. That is, the existence of a formal contact embedding implies that of a contact embedding. Then we show the following theorem.

Theorem 1.7 *Let (M^{2m+1}, ξ) be a closed co-oriented $(2m+1)$-contact manifold which satisfies either of the following conditions:*

1. *m is odd, $m \geq 3$ and $H_1(M^{2m+1} : \mathbb{Z}) = 0$,*
2. *m is even, $m \geq 4$ and M^{2m+1} is 2-connected,*
3. *$m = 2$ and M^5 is simply-connected.*

Then, there exists a contact embedding of (M^{2m+1}, ξ) in $(\mathbb{R}^{4m+1}, \eta_0)$.

For the proof of Theorem 1.7, it is enough to prove that the intersection

$$\mathrm{Emb}[M^{2m+1}, \mathbb{R}^{4m+1}] \cap \mathrm{CI}[(M^{2m+1}, \xi), (\mathbb{R}^{4m+1}, \eta_0)]$$

is not empty. We show it by using Li's classification of immersions of k-manifolds in $(2k-1)$-manifolds (Theorem 7.6) and Kervaire's result about the unstable homotopy groups of $SO(2k)$ and $U(k)$ ([25]) for appropriate k.

1.2 Plan of this thesis

In §2, we review several basic facts on contact geometry and the conformal symplectic normal bundle of a contact submanifold which we use to prove Theorem 1.1 in §3. We summarize the Smale-Hirsch theory and the Saeki-Takase theory in §4, Gromov's h-principle for contact immersions, contact embeddings and contact structures on an open manifold in §5. Then we prove Theorems 1.2, 1.3, 1.4 and 1.5 in §6 and Theorems 1.6 and 1.7 in §7. In §8, we list closed contact 3-manifolds which are known to admit contact embeddings into the standard contact structure on $\mathbb{R}^5$.

2 Preliminary

We give necessary definitions and theorems following Geiges [12].

Definition 2.1 (Contact structures) *Let M^{2n+1} be a $(2n+1)$ dimensional manifold. A contact structure is a maximally non-integrable hyperplane field ξ. If ξ is locally defined by a 1-form as $\xi = \ker \alpha$, then $\alpha \wedge (d\alpha)^n \neq 0$. The pair (M^{2n+1}, ξ) is called a contact manifold. If the normal bundle of ξ is orientable, we say that ξ is co-oriented. This is equivalent to that ξ is globally defined by a 1-form α. The 1-form α is called a contact form.*

Definition 2.2 (Reeb vector fields) *Let α be a contact form on M^{2n+1}. Then the vector field R_α such that $\iota_{R_\alpha} d\alpha = 0$ and $\alpha(R_\alpha) = 1$ is uniquely determined. The vector field R_α is called the Reeb vector field of α.*

We introduce fundamental examples of contact structures.

Example 2.3 *Let $(x_1, y_1, \cdots, x_n, y_n, z)$ be the coordinates on $\mathbb{R}^{2n+1}$. Then the 1-form*

$$\alpha_1 = dz + \sum_{j=1}^{n} x_j dy_j$$

is a contact 1-form. The contact structure $\xi_1 = \ker \alpha_1$ is called the standard contact structure on $\mathbb{R}^{2n+1}$.

Example 2.4 *Let $(x_1, y_1, \cdots, x_{n+1}, y_{n+1})$ be the coordinates on $\mathbb{R}^{2n+2}$. Then the standard contact structure ξ_0 on the unit sphere S^{2n+1} in $\mathbb{R}^{2n+2}$ is given by the contact form*

$$\alpha_0 = \sum_{j=1}^{n+1} x_j dy_j - y_j dx_j.$$

The contact manifold (S^{2n+1}, ξ_0) is called the standard contact $(2n+1)$-sphere.

Remark 2.5 *The contact structure $\xi_0 = \ker \alpha_0$ can be represented as the complex tangency of $S^{2n+1} \subset \mathbb{C}^{n+1}$, that is,*

$$\xi_0 = TS^{2n+1} \cap J(TS^{2n+1}),$$

where J is the standard complex structure on $\mathbb{C}^{n+1}$.

Definition 2.6 (Contactomorphisms) *Two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) are said to be contactomorphic if there is a diffeomorphism $f : M_1 \to M_2$ such that $f_*(\xi_1) = \xi_2$. If $\xi_i = \ker \alpha_i$, $i = 1, 2$, this is equivalent to that $f^*\alpha_2$ is equal to α_1 multiplied by a non-vanishing function.*

Proposition 2.7 (Proposition 2.1.8 in [12]) *The two contact manifolds* $(\mathbb{R}^{2n+1}, \xi_1)$ *and* $(S^{2n+1} \setminus \{p\}, \xi_0)$ *are contactomorphic for any point* $p \in S^{2n+1}$.

Theorem 2.8 (Gray stability) *Let* α_t, $t \in [0,1]$, *be a smooth family of contact forms on a closed manifold* M^{2n+1}. *Then there is an isotopy* $\{\phi_t\}_{t\in[0,1]}$ *of* M^{2n+1} *such that* $(\phi_t)_*(\ker \alpha_0) = \ker \alpha_t$ *for each* $t \in [0,1]$.

Theorem 2.9 (Darboux's theorem) *Let* α *be a contact form on* M^{2n+1} *and* p *a point of* M^{2n+1}. *Then there are coordinates*

$$(x_1, \cdots, x_n, y_1, \cdots, y_n, z)$$

on a neighborhood $U \subset M^{2n+1}$ *of* p *such that* $p = (0, \cdots, 0)$ *and*

$$\alpha|_U = dz + \sum_{j=1}^{n} x_j dy_j.$$

Definition 2.10 (Symplectic vector bundles) *A symplectic vector bundle* (E, ω) *over a manifold* B *is a smooth vector bundle* $\pi : E \to B$ *together with a symplectic linear form* ω_b *on each fiber* $E_b = \pi^{-1}(b)$, $b \in B$, *with* ω_b *varying smoothly in* b.

Definition 2.11 (Complex vector bundle structures) *A complex structure on a vector bundle* $E \to B$ *is a family* J_b *of complex structures on the fibers* E_b, *with* J_b *varying smoothly in* b.

A complex structure on a symplectic vector bundle (E, ω) is called ω-compatible if J_b is ω_b-compatible on E_b for each $b \in B$, i.e., if $(u, v) \mapsto \omega_b(u, J_b v)$ is a positive definite symmetric bilinear form. Let (E, ω) be a sympletic vector bundle over the manifold B. Since $Sp(2n)/U(n)$ is contractible, the space $J(\omega)$ of ω-compatible complex vector bundle structures on E is non-empty and contractible. This fact enables us to define the Chern classes of a symplectic vector bundle (E, ω) to be the Chern classes of the complex vector bundle (E, J), where J is any ω-compatible complex bundle structure on E.

Definition 2.12 (The Chern classes of a contact structure) *Let* α *be a global defining* 1*-form of a co-oriented contact structure* (M^{2n-1}, ξ). *Since the* 2*-form* $d\alpha$ *induces a symplectic structure on* ξ, $(\xi, d\alpha|_\xi)$ *is a symplectic vector bundle over* M^{2n-1}. *Since the conformal class of the symplectic bundle structure does not depend on the choice of* α, *we define the Chern classes of* ξ *to be the Chern classes of this symplectic vector bundle.*

Definition 2.13 (Contact submanifolds) *Let $(N^{2n+1}, \eta = \ker \beta)$ be a co-oriented contact manifold. An odd dimensional submanifold M is a contact submaifold if $\eta_M = TM \cap \eta|_M$ is a contact structure on M. It is equivalent to saying that the 1-form $i^*\beta$ is a contact form on M, where $i : M \to N$ is the inclusion.*

Definition 2.14 (Conformal symplectic normal bundles) *Let (M, η_M) be a contact submanifold of a co-oriented contact manifold $(N, \eta = \ker \beta)$. The vector bundle η splits along M into the Whitney sum of the two subbundles*

$$\eta|_M = \eta_M \oplus (\eta_M)^\perp,$$

where η_M is the contact plane bundle on M given by $\eta_M = TM \cap \eta|_M$ and $(\eta_M)^\perp$ is the symplectic orthogonal of η_M in $\eta|_M$ with respect to the form $d\beta$. We can identify $(\eta_M)^\perp$ with the normal bundle νM. Moreover, $d\beta$ induces a symplectic structure on $(\eta_M)^\perp$. The conformal class of the symplectic structure does not depend on the choice of β. We call $(\eta_M)^\perp$ the conformal symplectic normal bundle of M in N.

The conformal symplectic normal bundle determines the structure on a tubular neighborhood of a contact submanifold. The next theorem is called the tubular neighborhood theorem for a contact submanifold.

Theorem 2.15 (Theorem 2.5.15 in [12]) *Let (N_i, η_i), $i = 1, 2$, be co-oriented contact manifolds with compact contact submanifolds (M_i, ξ_i). Suppose that there is an isomorphism of conformal symplectic normal bundles $\Phi\colon (\eta_{1M_1})^\perp \to (\eta_{2M_2})^\perp$ covering a contactomorphism $\phi\colon (M_1, \xi_1) \to (M_2, \xi_2)$. Then there exists a neighborhood of M_1 in N_1 that is contactomorphic to a neighborhood of M_2 in N_2.*

Example 2.16 *Transverse loops in a co-oriented contact manifold (N^{2n+1}, η) are contact submanifolds. Since the symplectic group $Sp(2n)$ is connected, there is only one conformal symplectic $\mathbb{R}^{2n}$-bundle over S^1. A model for the neighborhood of a transverse loop is given by*

$$(S^1 \times \mathbb{R}^{2n}, \eta = \ker (dz + \sum_{j=1}^{n} (x_j dy_j - y_j dx_j))),$$

where z denotes the S^1-coordinate. Theorem 2.15 says that in suitable local coordinates a neighborhood of any transverse loop is described by this model.

Example 2.17 *We can also give a model of a neighborhood of a codimension two contact submanifold with trivial normal bundle. Since the conformal*

symplectic structure on 2*-dimensional trivial vector bundle is unique, a model for the neighborhood of the contact submanifold is given by*

$$(M^{2n-1} \times \mathbb{R}^2, \ker(\alpha + xdy - ydx) = \ker(\alpha + r^2 d\theta)),$$

where α *is a defining form of the contact submanifold* (M^{2n-1}, ξ)*.*

3 Proof of Theorem 1.1

We need the following proposition for the proof of Theorem 1.1.

Proposition 3.1 (Theorem 11.3 in [34]) *Let* K^k *be a closed orientable* k*-manifold,* L^l *an orientable* l*-manifold with* $H^{l-k}(L^l; \mathbb{Z}) = 0$ *and* $f\colon K^k \to L^l$ *an embedding. Then the Euler class of the normal bundle is trivial.*

Proof of Theorem 1.1. Let $f : M^{2n-1} \to N^{2n+1}$ be an embedding such that

$$f_*(TM^{2n-1}) \cap \eta|_{f(M^{2n-1})} = f_*\xi.$$

Then, the vector bundle η splits along M^{2n-1} such that

$$\eta|_M = \eta_M \oplus (\eta_M)^\perp,$$

where $\eta_M = f_*\xi$ and $(\eta_M)^\perp$ is the conformal symplectic normal bundle. By the assumption $H^2(N^{2n+1}; \mathbb{Z}) = 0$ and Proposition 3.1, the Euler class of the normal bundle of f is zero. Since the normal bundle of f is 2-dimensional, it is topologically trivial. Since the conformal symplectic structure on 2-dimensional trivial vector bundle is unique, the normal bundle of M^{2n-1} is also trivial as a conformal symplectic vector bundle. That is, $(\eta_M)^\perp$ is 2-dimensional trivial symplectic vector bundle. Hence,

$$c((\eta_M)^\perp) = 1 + c_1((\eta_M)^\perp) = 1 \ \text{ and } \ c(\eta|_M) = c(\eta_M)c((\eta_M)^\perp) = c(\eta_M).$$

By the naturality of the first Chern class and the condition $H^2(N^{2n+1}; \mathbb{Z}) = 0$, it follows that $c_1(\eta|_M) = f^*c_1(\eta) = 0$. Thus, $c_1(\xi) = c_1(\eta_M) = c_1(\eta|_M) = 0$. If $H^{2j}(N^{2n+1}; \mathbb{Z}) = 0$, then $c_j(\xi) = c_j(\eta_M) = c_j(\eta|_M) = f^*c_j(\eta) = 0$. □

Remark 3.2 *Martinet [31] and Lutz [30] proved that any closed orientable* 3*-manifold admits a contact structure in each homotopy class of tangent* 2*-plane fields. Thus there are infinitely many contact* 3*-manifolds which cannot be contact submanifolds of* $\mathbb{R}^5$*.*

4 Classification of immersions by regular homotopy

For given smooth manifolds M and V, the classification of immersions of M in V by regular homotopy is a fundamental problem. The following theorem by Smale and Hirsch completely answered to this in terms of homotopy theory.

Theorem 4.1 (Smale [46], Hirsch [20]) *Let M and V be smooth manifolds with the dimensions n and q, respectively, where $n < q$. Let* $\mathrm{Imm}(M, V)$ *and* $\mathrm{Mon}(TM, TV)$ *be the spaces of immersions of M into V and fiberwise injective homomorphism from TM to TV with C^∞ topology, respectively. Then, the map*

$$d : \mathrm{Imm}(M, V) \to \mathrm{Mon}(TM, TV); f \mapsto df$$

is a weak homotopy equivalence.

Let M^3 be a closed connected oriented 3-manifold and $\mathrm{Imm}[M^3, \mathbb{R}^5]$ be the set of regular homotopy classes of immersions of M^3 into $\mathbb{R}^5$. We fix a trivialization τ of TM^3 once and for all. Then $\pi_0(\mathrm{Mon}(TM, TV))$ is identified with the set $[M^3, V_{5,3}]$ of homotopy classes of continuous mapping from M^3 to the Stiefel manifold $V_{5,3} = SO(5)/SO(2)$. The Smale-Hirsch theory provides a bijection

$$\mathrm{Imm}[M^3, \mathbb{R}^5] \approx [M^3, V_{5,3}].$$

Based on this bijection, Wu [51] showed the following.

Theorem 4.2 (Wu [51], see also [29], [43]) *The normal Euler class χ_f for an immersion $f : M^3 \to \mathbb{R}^5$ is of the form $2C$ for some $C \in H^2(M^3; \mathbb{Z})$, and for any $C \in H^2(M^3; \mathbb{Z})$, there is an immersion f such that $\chi_f = 2C$. Furthermore, there is a bijection*

$$\mathrm{Imm}[M^3, \mathbb{R}^5]_\chi \approx \coprod_{C \in H^2(M^3;\mathbb{Z}) \text{ with } 2C=\chi} H^3(M^3; \mathbb{Z})/(4C \smile H^1(M^3; \mathbb{Z})),$$

where $\mathrm{Imm}[M^3, \mathbb{R}^5]_\chi$ *is the set of regular homotopy classes of immersions with normal Euler class $\chi \in H^2(M^3; \mathbb{Z})$ and $\smile$ denotes the cup product.*

Saeki, Szűcs and Takase examined the set $\mathrm{Emb}[M^3, \mathbb{R}^5]$ of the regular homotopy classes which contains an embedding. We note that the normal bundle of an embedding of M^3 in $\mathbb{R}^5$ is trivial by Proposition 3.1. Since the regular homotopy does not change the isomorphism type of the normal bundle, it follows that

$$\mathrm{Emb}[M^3, \mathbb{R}^5] \subset \mathrm{Imm}[M^3, \mathbb{R}^5]_0.$$

Let $\Gamma_2(M^3)$ be the finite set $\{C \in H^2(M^3;\mathbb{Z}) \mid 2C = 0\}$. By Theorem 4.2, the set $\mathrm{Imm}[M^3,\mathbb{R}^5]_0$ can be identified with $\Gamma_2(M^3) \times \mathbb{Z}$.

Definition 4.3 (Wu invariant) *The projection*

$$c : \mathrm{Imm}[M^3,\mathbb{R}^5]_0 \to \Gamma_2(M^3)$$

is called the Wu invariant of the immersion of the parallelized 3*-manifold with trivial normal bundle.*

The following explanation due to [43] gives a geometrical description of the Wu invariant. A normal trivialization ν of an element $f \in \mathrm{Imm}[M^3,\mathbb{R}^5]_0$ and the trivialization of TM^3 define a map $M^3 \to \mathrm{SO}(5)$ and it induces a homomorphism

$$\pi_1(M^3) \to \pi_1(\mathrm{SO}(5)),$$

namely, an element $\tilde{c}_\nu$ in $H^1(M^3;\mathbb{Z}_2)$. If we change ν by an element

$$z \in [M^3, \mathrm{SO}(2)] = H^1(M^3;\mathbb{Z}),$$

then $\tilde{c}_{\nu+z} = \tilde{c}_\nu + \rho(z)$, where ρ is the mod 2 reduction map in the Bockstein exact sequence:

$$H^1(M^3;\mathbb{Z}) \xrightarrow{\rho} H^1(M^3;\mathbb{Z}_2) \longrightarrow H^2(M^3;\mathbb{Z}) \xrightarrow{\times 2} H^2(M^3;\mathbb{Z}).$$

Hence the coset of $\tilde{c}_\nu$ in

$$H^1(M^3;\mathbb{Z}_2)/\rho(H^1(M^3;\mathbb{Z})) \cong \Gamma_2(M^3) = \ker\left\{\times 2 : H^2(M^3;\mathbb{Z}) \to H^2(M^3;\mathbb{Z})\right\}$$

does not depend on ν, and the coset of $\tilde{c}_\nu$ corresponds to the Wu invariant $c(f) \in \Gamma_2(M^3)$.

Theorem 4.4 (Saeki-Szűcs-Takase [43]) *For any element $C \in \Gamma_2(M^3)$, there exists an embedding $f : M^3 \to \mathbb{R}^5$ with the Wu invariant $c(f) = C$.*

Saeki and Takase obtained a more detailed structure of the subset

$$\mathrm{Emb}[M^3,\mathbb{R}^5] \subset \mathrm{Imm}[M^3,\mathbb{R}^5]_0.$$

Theorem 4.5 (Saeki-Takase [44]) *In each $\mathbb{Z}$-component of*

$$\mathrm{Imm}[M^3,\mathbb{R}^5]_0 \approx \mathbb{Z} \amalg \cdots \amalg \mathbb{Z},$$

$\mathrm{Emb}[M^3,\mathbb{R}^5]$ *is a subgroup isomorphic either to* $24\mathbb{Z}$ *or to* $12\mathbb{Z}$.

This theorem and Theorem 1.6 proves that if $c_1(\xi) = 0$, the intersection

$$\mathrm{Emb}[M^3,\mathbb{R}^5] \cap \mathrm{CI}[(M^3,\xi),(\mathbb{R}^5,\eta_0)]$$

is isomorphic either to $24\mathbb{Z}$ or to $12\mathbb{Z}$.

5 Gromov's h-principle

5.1 Contact immersions and contact embeddings

Gromov proved the h-principle for contact immersions and contact embeddings. For contact immersions, the h-principle holds when the codimension is positive. For contact embeddings, it holds when the codimension is greater than two.

Theorem 5.1 (Gromov [16]; see also [10]) *Let (M, ξ) and (N, η) be contact manifolds of dimensions $2m+1$ and $2n+1$, respectively, where $m < n$. A contact homomorphism $F_0 : TM \to TN$ covering $f_0 : M \to N$ is homotopic to the differential map $F_1 = df_1$ of a contact immersion $f_1 : M \to N$.*

Theorem 5.2 (Gromov [16]; see also [10]) *Let (M, ξ) and (N, η) be contact manifolds of dimensions $2m+1$ and $2n+1$, respectively, where $m+1 < n$. Suppose that the differential map $F_0 = df_0$ of an embedding $f_0 : M \to N$ is homotopic (via a homotopy of monomorphisms $F_t : TM \to TN$, $t \in [0, 1]$, covering f_0) to a contact homomorphism $F_1 : TM \to TN$. Then there exists an isotopy $f_t : M \to N$ such that the embedding $f_1 : M \to N$ is contact and the differential df_1 is homotopic to F_1 through contact homomorphisms. Moreover, one can choose the isotopy f_t to be arbitrarily C^0-close to f_0.*

5.2 Almost contact structures on an open manifold

Definition 5.3 (Almost contact structures) *Let N^{2n+1} be an odd dimensional oriented manifold. An almost contact structure on N^{2n+1} is a pair (β_1, β_2) consisting of a global 1-form β_1 and a global 2-form β_2 satisfying the condition $\beta_1 \wedge \beta_2^n \neq 0$.*

Remark 5.4 *We can define an almost contact structure on N^{2n+1} as a reduction of the structure group of TN^{2n+1} from $SO(2n+1)$ to $U(n)$. Since a pair (β_1, β_2) satisfying $\beta_1 \wedge \beta_2^n \neq 0$ can be seen as the cooriented hyperplane field $\ker \beta_1$ with an almost complex structure compatible with the symplectic structure $\beta_2|_{\ker \beta_1}$, the two definitions are equivalent up to homotopy.*

As we stated in §1, Gromov proved the following theorem.

Theorem 5.5 (Gromov [15]) *Let N^{2n+1} be an odd dimensional open manifold. Then the inclusion*

$$\mathbb{S}^+_{cont} \to S^+_{cont}$$

is a homotopy equivalence.

He also proved the relative version of the above theorem. In particular, the following is true. It is useful for extending a contact structure from a neighborhood of a subcomplex of codimension greater than one.

Theorem 5.6 (Gromov [15]) *Let N^{2n+1} be a $(2n+1)$-dimensional open manifold and M^m a compact submanifold with the dimension $m < 2n$. Any almost contact structure on N^{2n+1} which is already a contact structure on a neighborhood of M^m is homotopic to a contact structure which coincides with the original one on a small neighborhood of M^m.*

Let $(M^{2n-1}, \xi = \ker \alpha)$ be a closed cooriented contact manifold and M^{2n-1} be embedded in $\mathbb{R}^{2n+1}$. By Proposition 3.1, there exists an embedding

$$F\colon M^{2n-1} \times D^2 \to \mathbb{R}^{2n+1}.$$

The form $\alpha + r^2 d\theta$ induces a contact form β on $U = F(M^{2n-1} \times D^2)$. By Theorem 5.6, in order to extend the given contact structure, it is enough to extend it as an almost contact structure. Almost contact structures on N^{2n+1} correspond to sections of the principal $SO(2n+1)/U(n)$ bundle associated with the tangent bundle TN^{2n+1}. Since the tangent bundle of the manifold U in $\mathbb{R}^{2n+1}$ is trivialized, we can identify the almost contact structure on U with a map

$$\tilde{g} : M^{2n-1} \times D^2 \to SO(2n+1)/U(n).$$

Since the extendability of the map $\tilde{g}$ over $\mathbb{R}^{2n+1}$ is equivalent to the null-homotopy of $\tilde{g}$, we obtain the following proposition. This is the key proposition for the proofs of Theorem 1.3, Theorem 1.4, Theorem 1.2 and Theorem 1.5.

Proposition 5.7 *We can embed (M^{2n-1}, ξ) in $\mathbb{R}^{2n+1}$ as a contact submanifold for some contact structure, if and only if there exists an embedding*

$$F\colon M^{2n-1} \times D^2 \to \mathbb{R}^{2n+1}$$

such that the map $g : M^{2n-1} \to SO(2n+1)/U(n)$ induced by the underlying almost contact structure of $(M^{2n-1} \times D^2, \ker(\alpha + r^2 d\theta))$ is null-homotopic.

We note that the homogeneous space $SO(2n+1)/U(n)$ is diffeomorphic to the symmetric space $SO(2n+2)/U(n+1)$ (see [14], see also [12], [26]). In the following, let us denote the symmetric space $SO(2n)/U(n)$ by Γ_n. If $k \leq 2n-2$, the homotopy groups $\pi_k(\Gamma_n)$ are said to be stable. The stable homotopy groups $\pi_k(\Gamma_n)$ are computed by Bott.

Proposition 5.8 (Bott [3]) *If $k \leq 2n-2$, then*

$$\pi_k(\Gamma_n) = \pi_{k+1}(SO) = \begin{cases} 0 & (k \equiv 1,3,4,5 \pmod 8), \\ \mathbb{Z} & (k \equiv 2,6 \pmod 8), \\ \mathbb{Z}_2 & (k \equiv 0,7 \pmod 8). \end{cases}$$

The unstable homotopy groups $\pi_{2m+r}(\Gamma_m)$ are computed for $-1 \leq r \leq 1$ by Massey [32], Harris [19] and Ōshima [41], and for $2 \leq r \leq 4$ by Kachi [23].

6 Proofs of Theorems 1.2, 1.3, 1.4 and 1.5

6.1 Proof of Theorem 1.3

Proof of Theorem 1.3. By Proposition 5.7, it is enough to show the existence of an embedding

$$F\colon M^3 \times D^2 \to \mathbb{R}^5$$

such that the map $g\colon M^3 \to SO(5)/U(2)$ induced by F is null-homotopic. Let us take a triangulation of M^3 and $M^{(l)}$ be its l dimensional skeleton, i.e.,

$$M^{(0)} \subset M^{(1)} \subset M^{(2)} \subset M^{(3)} = M^3.$$

The condition $c_1(\xi) = 0$ is equivalent to that ξ is a trivial plane bundle over M^3. A trivialization τ of ξ and the Reeb vector field R of α give a trivialization of TM^3. Let us fix this trivialization. By Theorem 4.4, there exists an embedding $f : M^3 \to \mathbb{R}^5$ such that $c(f) = 0$, i.e., for a normal trivialization ν of f,

$$\tilde{c}_\nu \in \rho(H^1(M^3;\mathbb{Z})) \subset H^1(M^3;\mathbb{Z}_2).$$

By changing ν by an element in $\rho^{-1}(\tilde{c}_\nu)$, we obtain a normal trivialization ν such that

$$\tilde{c}_\nu = 0 \in H^1(M^3;\mathbb{Z}_2).$$

This means the trivialization ν and the trivialization τ of TM^3 define a map $h\colon M^3 \to SO(5)$ which induces trivial map in π_1. That is, $h|_{M^{(1)}}$ is null-homotopic. Since $\pi_2(SO(5)) = 0$, $h|_{M^{(2)}}$ is null-homotopic. Then for the projection

$$\pi\colon SO(5) \to SO(5)/U(2),$$

$\pi \circ h|_{M^{(2)}}$ is null-homotopic. Since $\pi_3(SO(5)/U(2)) = 0$ by the diffeomorphism $SO(5)/U(2) \cong \mathbb{C}P^3$ (see [12]), $\pi \circ h$ is null-homotopic. As a tubular neighborhood of $f(M^3)$ in $\mathbb{R}^5$, we can take an embedding $F : M^3 \times D^2 \to \mathbb{R}^5$ satisfying the desired condition. □

6.2 Proof of Theorem 1.4

We prove Theorem 1.4 in a way similar to that of Theorem 1.3 using Proposition 5.7.

Proposition 6.1 ([6]) *Every closed, orientable, simply-connected, spin 5-manifold M^5 can be embedded in $\mathbb{R}^6$.*

Note that a manifold M is said to be spin if $w_2(TM) = 0$. Let M^5 be a closed, oriented, simply-connected 5-manifold and ξ be a co-oriented contact structure on M^5 with $c_1(\xi) = 0$. Since $c_1(\xi)$ is an integral lift of $w_2(\xi) = w_2(TM^5)$, M^5 is a spin manifold. By the above proposition, we can embed M^5 in $\mathbb{R}^6$, thus in $\mathbb{R}^7$.

Proof of Theorem 1.4. We embed $M^5 \times D^2$ in $\mathbb{R}^7$ as a tubular neighborhood of $M^5 \subset \mathbb{R}^7$. The form $\alpha + r^2 d\theta$ defines a contact structure on $M^5 \times D^2$. The underlying almost contact structure determines a map

$$g : M^5 \to SO(7)/U(3).$$

By Proposition 5.7, it is enough to prove that g is null-homotopic. By Proposition 5.8,

$$\pi_k(SO(7)/U(3)) = \begin{cases} 0 & (k = 1, 3, 4, 5), \\ \mathbb{Z} & (k = 2, 6). \end{cases}$$

Hence the only obstruction is in $H^2(M^5; \pi_2(SO(7)/U(3)))$. By Lemma 8.19 in [18], the twice of this obstruction is equal to the first Chern class $c_1(\xi)$, which is zero by assumption. On the other hand, the second integral cohomology group $H^2(M^5; \mathbb{Z})$ is torsion free, since the manifold M^5 is simply connected. Therefore, the obstruction vanishes. Thus g is always null-homotopic and this completes the proof of Theorem 1.4. □

6.3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Since the 7-dimensional sphere S^7 is parallelizable, the homotopy type of almost contact structures on S^7 corresponds to the following unstable homotopy group computed in [19]

$$\pi_7(SO(7)/U(3)) = \mathbb{Z} \oplus \mathbb{Z}_2.$$

Theorem 6.2 (Ding-Geiges [7]) *Any almost contact structure on S^7 can be realized by a contact structure.*

Geiges in [11] constructed a contact structure corresponding to $(0,1) \in \mathbb{Z} \oplus \mathbb{Z}_2$ using the projective space $\mathbb{H}P^2$ over the quaternions. There is no almost complex structure on $\mathbb{H}P^2$. However, on $\mathbb{H}P^2 \backslash D^8$, we have a unique almost complex structure. The obstruction for extending the almost complex structure over D^8 is in the cohomology group

$$H^8(\mathbb{H}P^2; \pi_7(SO(8)/U(4))) = \pi_7(SO(8)/U(4)) = \mathbb{Z} \oplus \mathbb{Z}_2.$$

Geiges proved that the obstruction class corresponds to the element $(0,1) \in \mathbb{Z} \oplus \mathbb{Z}_2$. Due to the following theorem, we can realize the almost complex structure by a Stein structure on $\mathbb{H}P^2 \backslash D^8$ with a convex boundary.

Theorem 6.3 (Eliashberg [9]) *Let M^{2n+1} be the boundary of a handlebody W^{2n+2} $(n \geq 2)$ that contains only handles of index $\leq n+1$. Then M admits a contact structure in every homotopy class of almost contact structures that is induced from an almost complex structure on W.*

Then the almost contact structure of the obtained contact structure on the boundary $\partial D^8 = S^7$ corresponds to $(0,1) \in \mathbb{Z} \oplus \mathbb{Z}_2$. Now we prove Theorem 1.2.

Proof of Theorem 1.2. We prove that the contact structure $\xi = \ker \alpha$ on S^7 corresponding to $(m,1)$ cannot be a contact submanifold of $\mathbb{R}^9$ for any contact structure. For any embedding $i : S^7 \to \mathbb{R}^9$, the contact structure $\ker(\alpha + r^2 d\theta)$ on the tubular neighborhood $S^7 \times D^2$ of $S^7 = i(S^7) \subset \mathbb{R}^9$ defines a map

$$g : S^7 \to SO(9)/U(4).$$

Since the Smale invariant of the embedding $i : S^7 \to \mathbb{R}^9$ is an even element of $\pi_7(V_{9,7}) \cong \pi_7(SO(9)) = \mathbb{Z}$ (see [21]), the homotopy class of the map $g : S^7 \to SO(9)/U(4)$ does not depend on the choice of an embedding i. Since the natural map $\pi_7(SO(7)/U(3)) \to \pi_7(SO(9)/U(4))$ maps $(m,1) \in \mathbb{Z} \oplus \mathbb{Z}_2$ to $1 \in \mathbb{Z}_2$, the map g corresponds to the element

$$1 \in \pi_7(SO(9)/U(4)) = \mathbb{Z}_2.$$

Thus the map g is not null-homotopic. By Proposition 5.7, (S^7, ξ) cannot be a contact submanifold of $\mathbb{R}^9$. This completes the proof of Theorem 1.2. $\square$

6.4 Proof of Theorem 1.5

Proof of Theorem 1.5. Since it is easily proved that (1) and (2) are equivalent, we prove the equivalence of (2) and (3).

(3) $\Rightarrow$ (2). The differential map

$$df : TM^{2m+1} \to T\mathbb{R}^{2n+1}$$

is a formal contact embedding of (M^{2m+1}, ξ) in $(\mathbb{R}^{2n+1}, \eta)$ covering the map

$$f : M^{2m+1} \to \mathbb{R}^{2n+1}.$$

By the contractibility of $\mathbb{R}^{2n+1}$, the bundle map df is homotopic via bundle monomorphisms to a formal contact immersion of (M^{2m+1}, ξ) in $(\mathbb{R}^{2n+1}, \eta_0)$ which covers the constant map to the origin $\mathbf{0} \in \mathbb{R}^{2n+1}$. Thus, Gromov's h-principle for contact immersions implies the existence of a contact immersion of (M^{2m+1}, ξ) in $(\mathbb{R}^{2n+1}, \eta_0)$. Moreover, it is regularly homotopic to the embedding f since the differential map of such a contact immersion is homotopic to df via bundle monomorphisms.
(2) $\Rightarrow$ (3). Let

$$g : (M^{2m+1}, \xi) \to (\mathbb{R}^{2n+1}, \eta_0)$$

be a contact immersion which is regularly homotopic to an embedding f. We can take a small immersed tubular neighborhood U of $g(M^{2m+1})$ so that U is regularly homotopic to some tubular neighborhood V of $f(M^{2m+1})$. Thus we can give a contact structure η_V on V by pulling back the contact structure on U. Since the contact structure on U is the restriction of the standard contact structure η_0, the map

$$M^{2m+1} \to SO(2n+1)/U(n)$$

induced by the underlying almost contact structure is homotopic to the constant map. Since U and V are regularly homotopic, the map

$$h : M^{2m+1} \to SO(2n+1)/U(n)$$

induced by the structure η_V on $V \subset \mathbb{R}^{2n+1}$ is null-homotopic. By the same argument as Proposition 5.7, there exists a contact structure η on $\mathbb{R}^{2n+1}$ for which f is a contact embedding. $\square$

7 Proofs of Theorems 1.6 and 1.7

In this section, we prove Theorems 1.6 and 1.7 by using the h-principle for contact immersions and contact embeddings. Before the proof, we review the precedent results about contact immersions and contact embeddings (Theorems 7.1 and 7.2).

7.1 Precedent results

Asymptotically holomorphic geometry is useful to construct contact immersions and contact embeddings in the standard contact sphere. It was first established by Donaldson [8] and developed by Auroux [1] in symplectic geometry. Ibort, Martínez and Presas [22] applied their method to contact geometry. They proved the existence of asymptotically holomorphic functions on a closed contact manifold. Based on their result, Giroux and Mohsen showed the existence of supporting open book decompositions for a closed contact manifold ([13]).

The standard contact sphere $S^{2n+1} \subset \mathbb{C}^{2n+1}$ carries the trivial open book decomposition defined by $\arg z_1$ with the binding $B = S^{2n+1} \cap \{z_1 = 0\}$. Based on the result of Ibort, Martínez and Presas, Mori and Martínez showed the existence of a contact immersion and a contact embedding with a supporting open book decomposition.

Theorem 7.1 (Mori [36], Martínez [33]) *Let (M^{2n+1}, ξ) be a closed co-oriented contact $(2n+1)$-manifold. There exists an immersion*

$$I : (M^{2n+1}, \xi) \to (S^{4n+1}, \xi_0)$$

such that $I^*\alpha_0 = e^f\alpha$ and $e^f\alpha$ is adapted to the open book decomposition on M^{2n+1} defined by $I^*(\arg z_1)$.

Theorem 7.2 (Martínez [33]) *Let (M^{2n+1}, ξ) be a closed co-oriented contact $(2n+1)$-manifold. There exists an embedding*

$$I : (M^{2n+1}, \xi) \to (S^{4n+3}, \xi_0)$$

such that $I^*\alpha_0 = e^f\alpha$ and $e^f\alpha$ is adapted to the open book decomposition on M^{2n+1} defined by $I^*(\arg z_1)$.

For the above pairs of the dimensions, the existence part has already been proved by Gromov's h-principle. To prove Theorem 1.6, we need the proof of the following theorem by the h-principle.

Theorem 7.3 *Let (M^3, ξ) be a closed cooriented contact 3-manifold. Then, there exists an contact immersion of (M^3, ξ) into the standard contact structure $(\mathbb{R}^5, \eta_0)$.*

Proof. Let α be a global defining 1-form of ξ and R_α be the Reeb vector field of α. By Gromov's h-principle for contact immersions, it is enough to prove the existence of a formal contact immersion, namely, the monomorphism

$$\bar{F} : TM^3 \to T\mathbb{R}^5$$

satisfying that

1. the underlying map $F : M^3 \to \mathbb{R}^5$ is the constant map to the origin $\mathbf{0} \in \mathbb{R}^5$,
2. $\bar{F}(\xi_x)$ is a complex line in the contact hyperplane $\eta_0(\mathbf{0}) \cong \mathbb{C}^2$, and
3. $\bar{F}(R_\alpha(x)) = (\dfrac{\partial}{\partial z})(\mathbf{0})$.

The tangent bundle of M^3 and the tangent space of $\mathbb{R}^5$ at the origin split into the Whitney sums

$$TM^3 = \mathbb{R}(R_\alpha) \oplus \xi$$

and

$$T_{\mathbf{0}}\mathbb{R}^5 = \mathbb{R}(\frac{\partial}{\partial z}) \oplus \eta_0(\mathbf{0}) \cong \mathbb{R} \oplus \mathbb{C}^2.$$

Thus it is enough to prove the existence of the map

$$g : M^3 \to \mathbb{C}P^1$$

such that the pull-back g^*L is isomorphic to ξ, where L is the tautological complex line bundle over $\mathbb{C}P^1$. Such a map exists by the following argument. Let $\tilde{\gamma}$ be the universal $U(1)$ bundle over $\mathbb{C}P^\infty$ and $\tilde{L}$ be the associated complex line bundle. Since $BU(1) = \mathbb{C}P^\infty$, the map

$$h : M^3 \to \mathbb{C}P^\infty$$

satisfying $h^*\tilde{L} \cong \xi$ is uniquely determined up to homotopy. By the cellular approximation theorem, we can take a homotopy $\{h_t\}_{t\in[0,1]}$ such that

$$h_0 = h \ \text{ and } \ h_1(M^3) \subset \mathbb{C}P^1.$$

Therefore, if we put $g = h_1$, then $g^*L \cong \xi$. This completes the proof. □

By the same argument, we can prove the existence of a contact immersion of (M^{2n+1}, ξ) in $(\mathbb{R}^{4n+1}, \eta_0)$. In this case, we use the inclusions of complex Grassmann manifolds

$$\mathrm{Gr}^{\mathbb{C}}_{n+1,n} \subset \mathrm{Gr}^{\mathbb{C}}_{n+2,n} \subset \cdots \subset BU(n),$$

and the associated cellular decomposition.

Since there is only one regular homotopy class of immersions of M^{2n+1} in $\mathbb{R}^{4n+3}$, the existence of a contact immersion and that of an embedding imply the existence of a formal contact embedding (see Theorem 1.5). By Theorem 5.2, there exists a contact embedding of (M^{2n+1}, ξ) in $(\mathbb{R}^{4n+3}, \eta_0)$.

7.2 Proof of Theorem 1.6

Since every closed orientable 3-manifold M^3 is parallelizable, we have a one-to-one correspondence between the following sets.

1. Homotopy classes of unit vector fields X on M^3.
2. Homotopy classes of cooriented 2-plane fields ξ in TM^3.
3. Homotopy classes of maps $f : M^3 \to S^2$.

Definition 7.4 (d^2-invariant, d^3-invariant) *Let η be the 2-plane field corresponding to the constant map $M^3 \to S^2$. For any cooriented 2-plane filed ξ on M^3, there is an obstruction*

$$d^2(\xi, \eta) \in H^2(M^3; \pi_2(S^2)) = H^2(M^3; \mathbb{Z})$$

for ξ to be homotopic to η over the 2-skeleton of M^3 and, if $d^2(\xi, \eta) = 0$ and after homotoping ξ to η over the 2-skeleton, an obstruction

$$d^3(\xi, \eta) \in H^3(M^3; \pi_3(S^2)) = H^3(M^3; \mathbb{Z})$$

for ξ to be homotopic to η over M^3.

By the Pontryagin-Thom map, homotopy classes of maps $f : M^3 \to S^2$ are in one-to-one correspondence with framed cobordism classes of framed links in M^3. Studying the framed cobordism classes of framed links, Pontryagin described the set $[M^3, S^2]$ of homotopy classes of maps $f : M^3 \to S^2$ in the following way.

Theorem 7.5 (Pontryagin [42], see also [5] and [47]) *There is a bijection*

$$[M^3, S^2] \approx \coprod_{D \in H_1(M^3;\mathbb{Z})} \mathbb{Z}/(2D \frown H_2(M^3; \mathbb{Z})),$$

where $\frown$ denotes the cap product.

Note that the element $D \in H_1(M^3; \mathbb{Z})$ and the element in $\mathbb{Z}$ correspond to the Poincaré duals of d^2-invariant and d^3-invariant of a 2-plane field, respectively. Theorem 7.5 is a key to the proof of Theorem 1.6.

Proof of Theorem 1.6. In the proof of Theorem 7.3, once we take $g : M^3 \to \mathbb{C}P^1$, an element of $[M^3, V_{5,3}]$ is determined by the monomorphism $\bar{F} : TM^3 \to T\mathbb{R}^5$. We denote it by $s(g)$. For any $x \in M^3$, the orthogonal

$g(x)^\perp$ in $\mathbb{C}^2$ determines the normal plane of $\bar{F}(T_xM^3)$. Hence, we see that $\chi_{F'} = -c_1(\xi)$ for the contact immersion F'. Let us take a triangulation of M^3 and $M^{(l)}$ be its l dimensional skeleton, i.e.,

$$M^{(0)} \subset M^{(1)} \subset M^{(2)} \subset M^{(3)} = M^3.$$

We may assume that $M^{(3)} \backslash M^{(2)}$ is one 3-cell c_3. The restriction $g^{(2)}$ of g to the 2-skeleton $M^{(2)}$ is uniquely determined by $c_1(\xi)$ up to homotopy. However, there is ambiguity of extending $g^{(2)}$ over M^3. Thus g can differ by the connected sum of an element of $\pi_3(\mathbb{C}P^1) \cong \mathbb{Z}$. Let $i : S^3 \to V_{4,2} \cong S^3 \times S^2$ and $j : V_{4,2} \to V_{5,3}$ be the natural inclusions. Then the induced homomorphism

$$(j \circ i)_* : \pi_3(S^3) \to \pi_3(V_{5,3})$$

is an isomorphism. Therefore, $s(g)$ can differ by the connected sum of an element of $\pi_3(V_{5,3}) \cong \mathbb{Z}$. On the other hand, Pontryagin classified the set $[M^3, S^2]$ (Theorem 7.5). We note that the map $g : M^3 \to S^2$ satisfies that

$$PD(D) = c_1(g^*L) = c_1(\xi) \in 2H^2(M^3; \mathbb{Z})$$

for the corresponding D. Therefore,

$$\mathbb{Z}/(2D \frown H_2(M^3; \mathbb{Z})) \approx H^3(M^3; \mathbb{Z})/(-2c_1(\xi) \smile H^1(M^3; \mathbb{Z})).$$

This bijection corresponds the set $\{[g] \in [M^3, S^2] \mid c_1(g^*L) = c_1(\xi)\}$ to the set $\mathrm{CI}[(M^3, \xi), (\mathbb{R}^5, \eta)]$. This completes the proof of Theorem 1.6. □

7.3 Proof of Theorem 1.7

Li classified the regular homotopy classes of immersions of an n-manifold in the $(2n-1)$-dimensional Euclidean space.

Theorem 7.6 (Li [29]) *Let M^n be a connected manifold with $n > 3$. Then, there is a bijection*

$$\mathrm{Imm}[M^n, \mathbb{R}^{2n-1}] \approx H^{n-1}(M^n, \pi_{n-1}(V_{2n-1,n})) \times H^n(M^n, \pi_n(V_{2n-1,n})).$$

This theorem enables us to prove Theorem 1.7.

Proof of Theorem 1.7. By Theorem 1.5 and Theorem 5.2, it is enough to prove that the intersection

$$\mathrm{Emb}[M^{2m+1}, \mathbb{R}^{4m+1}] \cap \mathrm{CI}[(M^{2m+1}, \xi), (\mathbb{R}^{4m+1}, \eta_0)]$$

is not empty. By Theorem 7.6 and $H_1(M^{2m+1};\mathbb{Z})=0$,

$$\mathrm{Imm}[M^{2m+1},\mathbb{R}^{4m+1}]\approx\pi_{2m+1}(V_{4m+1,2m+1}).$$

By Whitehead's theorem in [50],

$$\pi_{2m+1}(V_{4m+1,2m+1})=\begin{cases}\mathbb{Z}_4 & (m \text{ is odd}),\\ \mathbb{Z}_2\oplus\mathbb{Z}_2 & (m \text{ is even}).\end{cases}$$

By looking at the complex vector bundle structures compatible with the conformal symplectic structures of ξ and η_0, a contact immersion $(M^{2m+1},\xi)\to(\mathbb{R}^{4m+1},\eta_0)$ defines an $U(m)$-equivariant map from the complex m-frame bundle of ξ to the complex Stiefel manifold $V^{\mathbb{C}}_{2m,m}$, or a section of a $V^{\mathbb{C}}_{2m,m}$ bundle over M^{2m+1}. By Theorem 7.1, there are such sections.

Since we are interested in the regular homotopy classes of immersions $M^{2m+1}\to\mathbb{R}^{4m+1}$, we look at the homotopy classes of the sections of the $V_{4m+1,2m+1}$ bundle over M^{2m+1} induced from the sections of the $V^{\mathbb{C}}_{2m,m}$ bundle.

Since $V^{\mathbb{C}}_{2m,m}$ is $2m$-connected, two sections of the $V^{\mathbb{C}}_{2m,m}$ bundle over M^{2m+1} are homotopic on $2m$-dimensional skeleton of M^{2m+1}. Hence the difference of sections of the $V^{\mathbb{C}}_{2m,m}$ bundle is counted by an element of

$$\pi_{2m+1}(V^{\mathbb{C}}_{2m,m})=\pi_{2m+1}(U(2m)/U(m))=\mathbb{Z}.$$

Let

$$i:U(2m)/U(m)\to SO(4m)/SO(2m)=V_{4m,2m}$$

and

$$j:SO(4m)/SO(2m)\to SO(4m+1)/SO(2m)=V_{4m+1,2m+1}$$

be the natural inclusions. The induced homomorphism

$$(j\circ i)_*:\pi_{2m+1}(V^{\mathbb{C}}_{2m,m})\to\pi_{2m+1}(V_{4m+1,2m+1})$$

describes the difference of $\mathrm{Imm}[M^{2m+1},\mathbb{R}^{4m+1}]$. By the homotopy exact sequence of the fibration

$$SO(4m)/SO(2m)\to SO(4m+1)/SO(2m)\to S^{4m},$$

$j_*:\pi_{2m+1}(V_{4m,2m})\to\pi_{2m+1}(V_{4m+1,2m+1})$ is an isomorphism. Hence, we examine the image of $i_*:\pi_{2m+1}(V^{\mathbb{C}}_{2m,m})\to\pi_{2m+1}(V_{4m,2m})$.

(**1**) The case where m is odd ($m\geq 3$) and $H_1(M^{2m+1};\mathbb{Z})=0$. The surjectivity of i_* is showed in the proof of Lemma 2.1 in [25]. The reason is as follows. The generator of $\pi_{2m+1}(U(2m)/U(m))=\mathbb{Z}$ is induced by the generator z of

$$\pi_{2m+1}(U(m+1)/U(m))=\pi_{2m+1}(S^{2m+1})=\mathbb{Z}.$$

On the other hand,

$$\pi_{2m+1}(V_{4m+1,2m+1}) = \pi_{2m+1}(V_{4m,2m}) = \cdots = \pi_{2m+1}(V_{2m+3,3}) = \mathbb{Z}_4$$

is generated by x and y with the relation $2y = 0, 2x = y$, where x and y are the images of the generators of $\pi_{2m+1}(V_{2m+2,2}) = \mathbb{Z} \oplus \mathbb{Z}_2$ ([50]). Since z induces x, i_* is surjective. This means that

$$\mathrm{Imm}[M^{2m+1}, \mathbb{R}^{4m+1}] = \mathrm{CI}[(M^{2m+1}, \xi), (\mathbb{R}^{4m+1}, \eta_0)].$$

On the other hand, there exists an embedding of M^{2m+1} in $\mathbb{R}^{4m+1}$ by a result of Haefliger and Hirsch ([17]). Thus the intersection

$$\mathrm{Emb}[M^{2m+1}, \mathbb{R}^{4m+1}] \cap \mathrm{CI}[(M^{2m+1}, \xi), (\mathbb{R}^{4m+1}, \eta_0)]$$

is not empty.
(**2**) The case where m is even ($m \geq 4$) and M^{2m+1} is 2-connected. In this case, i_* is not surjective. $\pi_{2m+1}(V_{4m+1,2m+1}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is generated by x and y, where x and y are the images of the generators of $\pi_{2m+1}(V_{2m+2,2}) = \mathbb{Z} \oplus \mathbb{Z}_2$ ([50]). Since z induces x, i_* is not surjective and

$$\mathrm{CI}[(M^{2m+1}, \xi), (\mathbb{R}^{4m+1}, \eta_0)] = \mathbb{Z}_2 \subset \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \mathrm{Imm}[M^{2m+1}, \mathbb{R}^{4m+1}].$$

The subset $\mathrm{CI}[(M^{2m+1}, \xi), (\mathbb{R}^{4m+1}, \eta_0)]$ of $\mathrm{Imm}[M^{2m+1}, \mathbb{R}^{4m+1}]$ is characterized by the fact that the normal bundle ν of a contact immersion of M^{2m+1} in $\mathbb{R}^{4m+1}$ carries a complex vector bundle structure. It is necessary because ν is isomorphic to the quotient η_0/ξ which is a complex vector bundle. To show that it is sufficient, we consider the obstruction for the normal bundle of an immersion M^{2m+1} in $\mathbb{R}^{4m+1}$ to have a complex structure and show that it takes a nontrivial value for some immersion.

Let ν be the normal bundle of an immersion $M^{2m+1} \to \mathbb{R}^{4m+1}$. The obstructions for the normal bundle ν to have a complex vector bundle structure lie in the groups $H^i(M^{2m+1}; \pi_{i-1}(\Gamma_m))$. The regular homotopy classes of immersions correspond to the homotopy classes of sections of the $V_{4m+1,2m+1}$ bundle over M^{2m+1}. Since $\pi_{2m+1}(V_{4m+1,2m+1}) \approx \mathrm{Imm}[M^{2m+1}, \mathbb{R}^{4m+1}]$, we may assume that the sections restricted to the $2m$-skeleton are induced from a section of a $V^{\mathbb{C}}_{2m,m}$ bundle associated with a contact immersion. Hence ν restricted to the $2m$-skeleton has a complex vector bundle structure. Since all the lower obstructions vanish, the only obstruction Ω lies in

$$H^{2m+1}(M^{2m+1}; \pi_{2m}(\Gamma_m)) = \pi_{2m}(\Gamma_m),$$

where

$$\pi_{2m}(\Gamma_m) = \begin{cases} \mathbb{Z}_2 & (m \equiv 2 \pmod 4), \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & (m \equiv 0 \pmod 4), \end{cases}$$

by Harris' calculation [19]. Since the image of the composition of the two homomorphisms

$$\pi_{2m+1}(V_{4m+1,2m+1}) \to \pi_{2m}(\mathrm{SO}(2m))$$

and

$$\pi_{2m}(\mathrm{SO}(2m)) \to \pi_{2m}(\Gamma_m)$$

is isomorphic to $\mathbb{Z}_2$ ([25]), the obstruction Ω takes a nontrivial value for some immersion. Therefore, a regular homotopy class of immersions of M^{2m+1} in $\mathbb{R}^{4m+1}$ contains a contact immersion if and only if its normal bundle ν admits a complex structure.

Since $\mathrm{Emb}[M^{2m+1}, \mathbb{R}^{4m+1}]$ is a one point set by a theorem of Haefliger and Hirsch [17], in order to show that

$$\mathrm{Emb}[M^{2m+1}, \mathbb{R}^{4m+1}] \subset \mathrm{CI}[(M^{2m+1}, \xi), (\mathbb{R}^{4m+1}, \eta_0)],$$

it is enough to show that the normal bundle of an embedding admits a complex vector bundle structure.

Since M^{2m+1} is 2-connected and $2 < \frac{1}{2}(2m+1-4)$, it also embeds in $\mathbb{R}^{4m-1}$ by a theorem of Haefliger and Hirsch [17]. We denote the normal bundle of an embedding in $\mathbb{R}^{4m-1}$ by ν_{2m-2}. Then $\nu_{2m-2} \oplus \varepsilon^{\mathbb{C}} = \nu$. Since the restriction of ν_{2m-2} to $M \setminus \{x_0\}$ carries a stable complex structure and $M \setminus \{x_0\}$ is isotopic to a neighborhood of the $(2m-2)$-skeleton, it admits a complex structure. Thus the only obstruction ω for ν_{2m-2} to carry a complex structure lies in

$$H^{2m+1}(M^{2m+1}; \pi_{2m}(\Gamma_{m-1})) = \pi_{2m}(\Gamma_{m-1}),$$

which is

$$\pi_{2m}(\Gamma_{m-1}) = \begin{cases} \mathbb{Z}_{(24,m-2)/2} & (m \equiv 2 \pmod 4), \\ \mathbb{Z}_{(24,m-2)} & (m \equiv 0 \pmod 4), \end{cases}$$

by Kachi's calculation [23]. By the natural homomorphism

$$\pi_{2m}(\Gamma_{m-1}) \to \pi_{2m}(\Gamma_m),$$

ω is mapped to Ω. In the following, we show that $\Omega = 0$.
(**i**)$m \equiv 2 \pmod 4$. We consider the homotopy exact sequence

$$\pi_{2m+1}(S^{2m-2}) \to \pi_{2m}(\Gamma_{m-1}) \to \pi_{2m}(\Gamma_m),$$

namely,

$$\mathbb{Z}_{24} \to \mathbb{Z}_{(24,m-2)/2} \to \mathbb{Z}_2.$$

To show the triviality of $\pi_{2m}(\Gamma_{m-1}) \to \pi_{2m}(\Gamma_m)$, we show the surjectivity of $\pi_{2m+1}(S^{2m-2}) \to \pi_{2m}(\Gamma_{m-1})$. Let θ be the generator of the stable group $\pi_{2m+1}(S^{2m-2}) = \mathbb{Z}_{24}$. Kervaire [25] computed the boundary map

$$\partial : \pi_{2m+1}(S^{2m-2}) \to \pi_{2m}(SO(2m-2))$$

of the homotopy exact sequence of the fibration $SO(2m-1) \to S^{2m-2}$. The element $\partial\theta$ is the generator of $\pi_{2m}(SO(2m-2)) = \mathbb{Z}_{12}$. Thus ∂ is surjective. The homomorphism

$$p_* : \pi_{2m}(SO(2m-2)) \to \pi_{2m}(\Gamma_{m-1})$$

is also surjective by the homotopy exact sequence

$$\pi_{2m}(SO(2m-2)) \to \pi_{2m}(\Gamma_{m-1}) \to \pi_{2m-1}(U(m-1)) = 0$$

for the fibration $SO(2m-2) \to \Gamma_{m-1}$. Therefore, the composition

$$p_* \circ \partial : \pi_{2m+1}(S^{2m-2}) \to \pi_{2m}(\Gamma_{m-1})$$

is surjective. Hence $\pi_{2m}(\Gamma_{m-1}) \to \pi_{2m}(\Gamma_m)$ is the trivial map and $\Omega = 0$.
(**ii**)$m \equiv 0 \pmod 4$. We consider the composition of the natural homomorphisms

$$\pi_{2m}(\Gamma_{m-1}) \to \pi_{2m}(\Gamma_m) \to \pi_{2m}(\Gamma_{m+1}),$$

namely,

$$\mathbb{Z}_{(24,m-2)} \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_2,$$

where $\pi_{2m}(\Gamma_{m+1})$ is given by Proposition 5.8. We show that the composition is surjective. By Kervaire's computation [25], the natural homomorphism $\pi_{2m}(SO(2m-2)) \to \pi_{2m}(SO(2m+2))$ is surjective. Since

$$p_* : \pi_{2m}(SO(2m+2)) \to \pi_{2m}(\Gamma_{m+1})$$

is an isomorphism,

$$\pi_{2m}(\Gamma_{m-1}) \to \pi_{2m}(\Gamma_{m+1})$$

is also a surjection. In other words, the generator of $\pi_{2m}(\Gamma_{m-1}) = \mathbb{Z}_{(24,m-2)}$ is mapped to the generator of $\pi_{2m}(\Gamma_{m+1}) = \mathbb{Z}_2$. Since there is a contact embedding of (M^{2m+1}, ξ) in $(\mathbb{R}^{4m+3}, \eta_0)$, the image of Ω in the stable homotopy group $\pi_{2m}(\Gamma_{m+1})$ is trivial. Hence, ω is an even element in $\pi_{2m}(\Gamma_{m-1}) = \mathbb{Z}_{(24,m-2)}$. Therefore Ω is zero in $\pi_{2m}(\Gamma_m) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
(**3**) The case where $m = 2$ and M^5 is simply-connected. We show that the normal bundle of an embedding $M^5 \subset \mathbb{R}^9$ admits a complex vector bundle

structure. Since M^5 is simply-connected, $M^5 \setminus \{x_0\}$ is isotopic to the regular neighborhood of the 3-skeleton $M^{(3)}$. Hence we can embed $M^5 \setminus \{x_0\}$ in $\mathbb{R}^7$. We denote the normal bundle of $M^5 \setminus D^5 \subset \mathbb{R}^7$ by ν_1. Since ν_1 is orientable, it carries a complex structure and its restriction to the boundary 4-sphere is the trivial complex line bundle. Since the regular homotopy class of an embedding $f : S^4 \to \mathbb{R}^7$ is trivial, the boundary of $M^5 \setminus D^5$ can be capped by an embedded disk D^5 in $\mathbb{R}^9$. Indeed, we can construct an embedded disk D^5 as follows. We treat the normal coordinates (s, t) of $\mathbb{R}^7 \subset \mathbb{R}^9$ as parameters. Let f_t, $t \in [0, 1]$, be a regular homotopy of immersions between the embedding $f_0 = f : S^4 \to \mathbb{R}^7$ and the standard embedding $f_1 : S^4 \to \mathbb{R}^7$. The map $(f_t, 0, t) : S^4 \times [0, 1] \to \mathbb{R}^9$ is an immersion and we can make it an embedding by perturbing the s-coordinate in a neighborhood of the self-intersection. Now the embedded boundary component $(f_1, 0, 1)(S^4 \times \{1\}) \subset \mathbb{R}^7 \times \{0\} \times \{1\} \subset \mathbb{R}^9$ is capped by the standard embedded 5-hemisphere. In such a way, we obtain the embedding of M^5 in $\mathbb{R}^9$ such that $M^5 \setminus D^5$ is contained in $\mathbb{R}^7 \times \{0\} \times \{0\}$. Let us denote the normal bundle of M^5 in $\mathbb{R}^9$ by ν. Then $\nu|_{M^5 \setminus D^5} = \nu_1 \oplus \varepsilon^{\mathbb{C}}$. By construction, the complex vector bundle structure on $\nu|_{M^5 \setminus D^5} = \nu_1 \oplus \varepsilon^{\mathbb{C}}$ extends to ν. Therefore, ν admits a complex vector bundle structure and

$$\mathrm{Emb}[M^{2m+1}, \mathbb{R}^{4m+1}] \cap \mathrm{CI}[(M^{2m+1}, \xi), (\mathbb{R}^{4m+1}, \eta_0)]$$

is not empty. □

Remark 7.7 *Let $f : (M^{2m+1}, \xi) \to (\mathbb{R}^{4m+1}, \eta_0)$ be a contact embedding and ν be the normal bundle of f. Then the total Chern classes of ξ and ν satisfy*

$$c(\xi)c(\nu) = 1.$$

On the other hand, the normal Euler class of an embedding f is zero by Proposition 3.1. Hence $c_m(\nu) = 0$. Thus, we obtain a necessary condition for (M^{2m+1}, ξ) to be a contact submanifold of $(\mathbb{R}^{4m+1}, \eta_0)$. If $m = 1, 2$ and 3, then

$$c_1(\xi) = 0,\ c_2(\xi) - c_1(\xi)^2 = 0 \ \text{ and } \ c_3(\xi) - 2c_1(\xi)c_2(\xi) + c_1(\xi)^3 = 0,$$

respectively.

8 A list of contact 3-manifolds in the standard contact 5-space

We list known examples of closed contact 3-manifolds which can be contact submanifolds in $(\mathbb{R}^5, \eta_0)$ (or the standard contact 5-sphere). The links of following isolated surface singularities are typical examples.

1. The quasi-homogeneous singularities.

2. The cusp singularities.

Let (z_1, z_2, z_3) be the coordinates on $\mathbb{C}^3$. The algebraic variety defined by a quasi-homogeneous equation carries a $\mathbb{C}^*$-action. Thus the link carries a S^1-action and is diffeomorphic to a Seifert manifold, in general. The restriction of the canonical contact form on the 5-sphere to the singularity link induces a contact structure. Its Reeb vector field is tangent to the fiber circles.

Example 8.1 ([35]) *The link of the A_p-singularity $z_1^2 + z_2^2 + z_3^{p+1} = 0$ is the unique tight contact structure on the lens space $L(p+1, p)$.*

Example 8.2 ([35]) *The link of the singularity $z_1^2 + z_2^3 + z_3^5 = 0$ is the unique tight contact structure on the Poincaré homology sphere $\Sigma(2, 3, 5)$.*

Example 8.3 ([45]; see also [2], [37], [39]) *The simple elliptic singularities,*

$$\begin{aligned}
\tilde{E}_6 &: z_1^3 + z_2^3 + z_3^3 + \lambda_1 z_1 z_2 z_3 = 0 \ \ ({\lambda_1}^3 + 27 \neq 0),\\
\tilde{E}_7 &: z_1^2 + z_2^4 + z_3^4 + \lambda_2 z_1 z_2 z_3 = 0 \ \ ({\lambda_2}^4 - 64 \neq 0),\\
\tilde{E}_8 &: z_1^2 + z_2^3 + z_3^6 + \lambda_3 z_1 z_2 z_3 = 0 \ ({\lambda_3}^6 - 432 \neq 0).
\end{aligned}$$

The link of each singularity is diffeomorphic to the S^1 bundle over T^2 with the Euler class -3, -2 and -1, respectively. It can be seen as a parabolic T^2-bundle over S^1 and the contact structure is the unique universally tight and minimally twisting contact structure on it.

Example 8.4 ([28]; see also [2], [24], [37], [39]) *The cusp singularity,*

$$T_{pqr} : \ z_1^p + z_2^q + z_3^r + \lambda z_1 z_2 z_3 = 0 \ (\lambda \neq 0, p^{-1} + q^{-1} + r^{-1} < 1).$$

The link is diffeomorphic to a hyperbolic T^2 bundle over S^1 with monodromy

$$\begin{pmatrix} r-1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q-1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p-1 & -1 \\ 1 & 0 \end{pmatrix}.$$

The canonical contact structure is the positive contact structure associated to the suspension Anosov flow.

An oriented 3-manifold admits at most one Milnor fillable contact structure ([4]). Hence, we obtain only restricted range of contact submanifolds by surface singularity links. We also have some examples of contact submanifolds which are not surface singularity links.

Example 8.5 (Mori [38], Niederkrüger-Presas [40]) *An overtwisted contact structure on S^3 associated with the negative Hopf band.*

The following three examples can be obtained by the toric method found by Mori [38].

Example 8.6 *All the tight contact structures on the* 3*-torus.*

Example 8.7 (Furukawa) *A tight contact structure and an overtwisted contact structure on the lens space $L(p, 1)$.*

Example 8.8 (Furukawa) *Some universally tight contact structures on a T^2 bundle over S^1 with monodromy*

$$\begin{pmatrix} r-1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q-1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p-1 & -1 \\ 1 & 0 \end{pmatrix}.$$

For an embedded manifold $M^n \subset \mathbb{R}^{n+1}$, the unit cotangent bundle ST^*M^n is a contact submanifold of $ST^*\mathbb{R}^{n+1}$. A compact $(2n+1)$-dimensional submanifold of $ST^*\mathbb{R}^{n+1}$ can be a contact submanifold of the standard contact $(2n+1)$-sphere. Thus we obtain the following example.

Example 8.9 *The unit cotangent bundle of a closed orientable surface.*

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