A construction of a universal finite type invariant of homology 3-spheres (ホモロジー3球面の普遍有限型不変量のひとつの構成)

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1 Introduction.

In this thesis, we give a construction of a topological invariant of rational homology 3-spheres via vector fields, which we denote by $\widetilde{z}$. The construction of $\widetilde{z}$ is a generalization of both that of z^{KKT} due to G. Kuperberg and D. Thurston ([14]) [1] and that of z^{FW}due to T. Watanabe ([32]). These two invariants z^{KKT} and z^{FW} are related to the Chern-Simons perturbation theory. More precisely we show that the construction of z^{KKT} is a special case of that of $\widetilde{z}$ when vector fields are given by a framing on a given rational homology 3-sphere, and the construction of z^{FW} is a special case of that of $\widetilde{z}$ when vector fields are gradient vector fields of Morse functions on a given rational homology 3-sphere. As a corollary, we have $z^{\mathrm{FW}} = z^{\mathrm{KKT}}$.

In this introduction, we first review two invariants z^{KKT} and z^{FW}. We next explain an outline of the construction of $\widetilde{z}$. We also give several remarks on the results of the thesis.

1 Background.

1.1 The Chern-Simons perturbation theory.

1.1.1 Witten's proposal and the Kontsevich invariant.

Around 1984, V. Jones defined an invariant of knots, which is now called the Jones polynomial, using the study of operator algebras in [12]. After that, quantum invariants associated with representations of Lie algebras were discovered for many representations of Lie algebras in late 1980s. The Jones polynomial is understood as the quantum sl_2 invariant. In 1989, E. Witten proposed that the partition function of the Chern-Simons field theory gives a topological invariant of links in 3-manifolds in [33]. Quantum invariants including the Jones polynomial are understood as the Witten invariant for links in S^3.

Around 1991, M. Kontsevich proposed a topological invariant of knots taking values in $\mathcal{A}(S^1)$. Here $\mathcal{A}(S^1)$ is the quotient space of the vector space generated by oriented Jacobi diagrams on S^1 divided by some relations. For each irreducible representation V of a simple Lie algebra g, there is a linear map from $\mathcal{A}(S^1)$ to $\mathcal{C}$ called a weight system. Then the quantum (g, V) invariant is recoverd from the Kontsevich invariant via the weight system. In this sense, the Kontsevich invariant is universal for quantum invariants.

1.1.2 Finite type invariants of knots.

Around 1989, V. A Vassiliev ([28]) defined the notion of finite type invariants of knots. Birman and Lin gave a combinatorial definition of the notion of finite type invariants in [3] and then they established a relation between the Jones polynomial and finite type invariants. Quantum invariants give examples of

[1]The notation z^{KKT} and z^{FW} differ from the original notations.

finite type invariants. Finite type invariants are characterized by the behavior under crossing changes. It is known that the Kontsevich invariant is universal for finite type invariants. R. Bott and C. Taubes constructed finite type invariants of knots via the configuration space integral in [4].

1.1.3 Quantum invariants of 3-manifolds.

The study of quantum invariants of 3-manifolds started from Witten's work. There are several ways of constructing mathematical invariants based on Witten's proposal. For example, an appropriate weighted sum of quantum invariants of a framed link corresponding to a given 3-manifold via Dehn surgery is a quantum invariant of such a 3-manifold. N. Reshetikhin and V. G. Turaev constructed the quantum $SU(2)$ invariant in this way in [26]. Many quantum invariants were discovered. T. Ohtsuki and T.T.Q. Le constructed the perturbative G invariant by an arithmetic expansion of the quantum G invariant.

1.1.4 Finite type invariants of integral homology 3-sphere.

T. Ohtsuki ([25]) defined the notion of finite type invariants of integral homology 3-spheres using integral surgeries instead of crossing changes in the definition of finite type invariants of knots. Garoufalidis-Levine also defined the notion of finite type invariants using Torelli surgeries and show that this definition coincides with Ohtsuki's definition for integral homology 3-spheres in [10]. S. Garoufalidis and Ohtsuki gave a relation between Jacobi diagrams and the theory of finite type invariants. Habiro gave a reformulation of this relation using claspers in [11].

T. T. Q. Le and J. Murakami and Ohtsuki ([15]) constructed a topological invariant of 3-manifolds from the Kontsevich invariant, which is called the LMO invariant. The LMO invariant takes values in $\mathcal{A}(\emptyset)$. The space $\mathcal{A}(\emptyset)$ is the quotient space of the vector space generated by oriented Jacobi diagrams divided by some relations. The degree of a Jacobi diagram is the half of the number of its vertexes. The space $A_n(\emptyset)$ is a vector subspace of $\mathcal{A}(\emptyset)$ spanned by Jacobi diagrams of degree n. The LMO invariant is universal for both finite type invariants of integral homology 3-spheres and perturbative quantum invariants.

The notion of finite type invariants was extended to 3-manifolds by T. D. Cochran and P. Melvin ([6]).

1.1.5 The Chern-Simons perturbation theory.

S. Axelrod and I. M. Singer ([1]) and Kontsevich ([13]) proposed topological invariants of 3-manifolds via the perturbative expansion of the Chern-Simons path integral. Their invariants are written by the configuration space integral. Axelrod and Singer, and Kontsevich gave a propagator to construct of their invariants. A propagator is a differential 2-form on the two point configuration space of a given 3-manifold. A propagator plays an important role in the configuration space integral.

Axelrod and Singer's propagator and Kontsevich's propagator are slightly different. Axelrod and Singer use Green functions and Riemannian metrics. Kontsevich assume only that a propagator is a closed form.

1.2 The Kontsevich-Kuperberg-Thurston invariant

G. Kuperberg and D. Thurston gave a topological invariant z^{KKT} of rational homology 3-spheres taking values in $\mathcal{A}(\emptyset)$ in [14] based on Kontsevich's work. Kuperberg and Thurston proved that z^{KKT} is universal for finite type invariants of homology 3-spheres. The invariant z_1^{KKT} which is the degree one part of z^{KKT} gives an alternative description of the Casson-Walker invariant. z^{KKT} is the sum of the principal term defined by using a framing and the correction term to the framings. The principal term of z^{KKT} is given by the configuration space integral. So this invariant is suitable for the study of surgery formulas. In fact Kuperberg and Thurston in [14] proved that z^{KKT} are of finite type by using surgery formulas. Furthermore, C. Lescop studied other type of surgery of rational homology 3-spheres, and then she gave several surgery formulas for z^{KKT} in [17], [18].

1.2.1 Related work on the Kontsevich-Kuperberg-Thurston invariant.

- A framing is a triple of linearly independent non-vanishing vector fields. Then we can regard the principal term of z^{KKT} as an invariant of a triple of linearly independent non-vanishing vector fields. Lescop gave an invariant of a non-vanishing vector field on a rational homology 3-sphere in [21]. Lescop's invariant is an extension of z_1^{KKT} for integral homology 3-spheres.
- Lescop constructed an invariant corresponding to z_1^{KKT} for 3-manifolds with first Betti number one in [19].
- K. Sakai studied the space of (higher dimensional) long knots using the configuration space integral in [27]. He constructed a map from some graph complex to the de Rham complex of the space of long knots and then he obtained a cohomology class which is an extension of the Haefliger invariant.
- I. Volic studied the space of pure braids in $\mathbb{R}^n$ via the configuration space integral in [29]. In particular, he obtained finite type invariants of braids in the case of $n = 3$.
- T. Moriyama constructed an invariant of an embedded 3-manifold in a 6-manifold satisfying some properties in [22]. His construction was inspired by the configuration space integral used in the construction of z_1^{KKT}. Moriyama's invariant recovers the Haefliger invariant and Milnor's triple linking number of algebraically split 3-component links in $\mathbb{R}^3$ and the Casson-Walker invariant. He gave a direct proof of a vanishing property of the Rokhlin invariant as an application ([23]).

- T. Watanabe studied families of higher dimensional homology spheres by using the configuration space integral in [30], [31]. Then he gave non-trivial homotopy classes of the diffeomorphism group of a sphere.

1.3 Fukaya's invariant.

In the 1990s, K. Fukaya constructed an invariant of a pair of two local systems on a 3-manifold via Morse functions in [7]. A broken graph is a graph given by cutting some edges. For a local system on a 3-manifold, we compute the number by counting flow graphs of several labeled broken graphs. Here a flow graph is a map from the given broken graph to the given 3-manifold such that the image of each edge is a trajectory of the Morse function corresponding to the label of such an edge. Each flow graph has a weight given by using holonomy of the given local system. Fukaya's invariant is defined as the difference of such numbers corresponding to given two local systems. He proposed that this difference is independent of the choice of Morse functions (so this is an invariant of the pair of two local systems on a 3-manifold).

M. Futaki pointed out that Fukaya's invariant sometimes depends on the choice of Morse functions in [9]. Then Fukaya's invariant is not a topological invariant of a local system on a 3-manifold.

Fukaya's construction is similar to the construction of the degree one part of the Chern-Simons perturbation theory in many points. Fukaya conjectured that there are relationships between Fukaya's invariant and the Chern-Simons Perturbation theory by Axelrod and Singer.

1.3.1 Related work on flow graphs.

M. Betz and R. L. Cohen ([2]) constructed cohomology operations to recover the cup products and the Poincaré duality and the Steenrod squares using flow graphs. Fukaya also recover the cup product and the Massey product.

1.4 Watanabe's invariant.

As stated above, the number of flow graphs in Fukaya's construction depends on the choice of both local systems and 3-manifolds. Fukaya considered a "relative invariant" of two local systems.

Watanabe constructed an invariant of a rational homology 3-sphere (with trivial local system). The principal term of Watanabe's invariant is defined by counting the moduli spaces of flow graphs as Fukaya's construction. In the case of trivial local system, the problem pointed out by Futaki does not occur. But a difference of the numbers corresponding to two local systems are trivial. Watanabe constructed the correction term to Morse functions, which is called the anomaly term. Then he defined a topological invariant of rational homology 3-spheres to be the sum of the principal term and the anomaly term. Watanabe also constructed an invariant z_n^{FW} taking values in $\mathcal{A}_n(\emptyset)$ by using flow graphs of higher loop graphs in [32].

Watanabe conjectured that his invariants related to the Chern-Simons perturbation theory by Kontsevich.

2 The main result.

In this thesis, we give a construction of a topological invariant of rational homology 3-spheres via vector fields, which we denote by $\widetilde{z}$ (Theorem 2). The construction of $\widetilde{z}$ is a generalization of both that of z^{KKT} and that of z^{FW}. $\widetilde{z}$ is the sum of the principal term and the anomaly term.

We show that the construction of z^{KKT} is a special case of that of $\widetilde{z}$ when vector fields are given by a framing on a given rational homology 3-sphere (Theorem 8), and the construction of z^{FW} is a special case of that of $\widetilde{z}$ when vector fields are gradient vector fields of Morse functions on a given rational homology 3-sphere (Theorem 9). As a corollary, we have $z^{\mathrm{KKT}} = z^{\mathrm{FW}}$ for any rational homology 3-sphere (Corollary 10).

We determine the constant μ_n which appeared in the construction of the anomaly term of $\widetilde{z}$ (Theorem 6). The constant μ_n is equivalent to the constant used in [32]. This constant was not determined.

2.1 Construction of $\widetilde{z}$.

Let Y be a rational homology 3-sphere with a base point $\infty \in Y$. Let $N(\infty; Y)$ be a regular neighborhood (that is diffeomorphic to an open ball) of ∞ in Y and let $N(\infty; S^3)$ be a regular neighborhood of ∞ in $S^3 = \mathbb{R}^3 \cup \{\infty\}$. We fix an orientation preserving diffeomorphism $\varphi^\infty : (N(\infty; Y), \infty) \overset{\cong}{\to} (N(\infty; S^3), \infty)$.We identify $N(\infty; Y)$ with $N(\infty; S^3)$ under φ^∞. Take $a \in S^2 \subset \mathbb{R}^3$. We often regard $a \in \mathbb{R}^3$ as the section of a trivial $\mathbb{R}^3$ bundle. The map $q_a : \mathbb{R}^3 \to \mathbb{R}$ is defined by $q_a(x) = \langle x, a \rangle$ where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^3$.

Definition 1 (admissible vector field)**.** A vector field $\gamma \in \Gamma T(Y \setminus \infty)$ is an *admissible vector field (with respect to a)* if the following conditions hold.

- $\gamma|_{N(\infty;Y)\setminus\infty} = -\mathrm{grad}\ q_a|_{N(\infty;S^3)\setminus\infty}$,
- γ is transverse to the zero section in $T(Y \setminus \infty)$.

Take $a_1, \cdots, a_{3n} \in S^2$ are enough "generic" points. Let γ_i be an admissible vector field with respect to a_i. We construct the principal term $\widetilde{z}_n(Y; \vec{\gamma}) \in \mathcal{A}_n(\emptyset)$ by using the configuration space integral and the correction term (called anomaly term) $\widetilde{z}^{\mathrm{anomaly}}(\vec{\gamma})$ to remove the ambiguity in the choice of $\vec{\gamma}$. Then we prove the following main theorem.

Theorem 2 (Theorem 4.14.)**.**

$$\widetilde{z}_n(Y) = \widetilde{z}_n(Y; \vec{\gamma}) - \widetilde{z}_n^{\mathrm{anomaly}}(\vec{\gamma}) \in \mathcal{A}_n(\emptyset)$$

does not depend on the choice of $\vec{\gamma}$. Thus $\widetilde{z}_n(Y)$ is a topological invariant of Y.

2.1.1 The principal term.

The 6-manifold with corner $C_2(Y)$ is a compactification of $(Y \setminus \infty)^2 \setminus \Delta$ similar to the Fulton-MacPherson compactification ([8]). Here Δ is the diagonal. We define an 3-submanifold $c(\gamma_i)$ of $\partial C_2(Y)$ using γ_i for $i = 1, \cdots, 3n$. We use the same symbol $c(\gamma_i)$ for its homology class. Since Y is a rational homology 3-sphere, there is a closed 2-form $\omega(\gamma_i) \in \Omega^2(C_2(Y))$ satisfying the following conditions:

- $\omega(\gamma_i)|_{\partial C_2(Y)}$ represents the Poincaré dual of $c(\gamma_i)$. [2]
- The support of $\omega(\gamma_i)|_{\partial C_2(Y)}$ is concentrated in near $c(\gamma_i)$.

We call $\omega(\gamma_i)$ a propagator with respect to γ_i.

Let $\omega(\gamma_i)$ be a propagator with respect to γ_i for each $i \in \{1, \cdots, 3n\}$. To simplify notation, we write $\vec{\gamma}$ instead of $(\gamma_1, \cdots, \gamma_{3n})$. Set

$$\mathcal{E}_n = \{\text{edge oriented, connected labeled Jacobi diagram of degree } n\},$$

$$\mathcal{A}_n(\emptyset) = \{\text{oriented Jacobi diagram of degree } n\}_{\mathbb{R}}/\mathrm{IHX}, \mathrm{AS},$$

where $\{\text{oriented Jacobi diagram of degree } n\}_{\mathbb{R}}$ is the real vector space generated by the set $\{\text{oriented Jacobi diagram of degree } n\}$. Here a Jacobi diagram of degree n is a trivalent graph with $2n$ vertexes without simple loop. There is a natural map $\mathcal{E}_n \to \mathcal{A}_n(\emptyset)$. We denote $[\Gamma]$ the image of Γ under this natural map. For each $\Gamma \in \mathcal{E}_n$ and each $i \in \{1, \cdots, 3n\}$, there is a map $P_i(\Gamma) : (Y \setminus \infty)^{2n} \setminus \Delta \to C_2(Y)$. Here $\Delta = \{(x_1, \cdots, x_{2n}) \in Y^{2n} \mid \sharp\{x_1, \cdots, x_{2n}\} < 2n\}$.

Definition 3.

$$\widetilde{z}_n(Y; \vec{\gamma}) = \sum_{\Gamma \in \mathcal{E}_n} \left(\int_{(Y \setminus \infty)^{2n} \setminus \Delta} \bigwedge_i P_i(\Gamma)^* \omega(\gamma_i) \right) [\Gamma] \in \mathcal{A}_n(\emptyset).$$

When $\vec{\gamma}$ is enough generic, $\widetilde{z}_n(Y; \vec{\gamma})$ depends only on the choice of $\vec{\gamma}$.

2.1.2 The anomaly term.

Let Y be an oriented closed 3-manifold (possibly not rational homology sphere) and let $\vec{\gamma} = (\gamma_1, \cdots, \gamma_{3n})$ be a generic family of vector fields on Y transverse to the zero section in TY. Let X be a connected oriented 4-manifold with $\partial X = Y$ and $\chi(X) = 0$.

Take a unit vector filed η_X on X such that $\eta_X|_{\partial X}$ coincides with the outward unit vector field of $TY = T(\partial X) \subset TX|_Y$. Let $T^v X$ be the normal bundle of η_X in TX. We remark that $T^v X|_Y = TY$.

Let $\beta_i \in \Gamma T^v X$ be a vector field of $T^v X$ transverse to the zero section in $T^v X$ satisfying $\beta_i|_Y = \gamma_i$. By using β_i we have a 4-cycle $c(\beta_i)$ of $(C_2(Y), \partial C_2(Y))$.

[2] There is a smooth manifold structure on $\partial C_2(Y)$.

For a real vector space V, we define

$$\breve{S}_{2n}(V) = \{\{1, \cdots, 2n\} \hookrightarrow V\}/\text{dilations and translations}.$$

For an $\mathbb{R}^3$ vector bundle E on M, we denote by $\breve{S}_{2n}(E)$ the fiber bundle over M where the fiber over $x \in M$ is $\breve{S}_{2n}(E_x)$. For each $\Gamma \in \mathcal{E}_n$ and each $i \in \{1, \cdots, 3n\}$, there is a map $\phi_i(\Gamma) : \breve{S}_{2n}(T^v X) \to ST^v X$.

Definition 4.

$$I(X; \vec{\gamma}) = \sum_{\Gamma} \sharp(\bigcap_i \Phi_i(\Gamma)^{-1} c_0(\beta_i))[\Gamma] \in \mathcal{A}_n(\emptyset).$$

We prove the following Lemma by a cobordism argument similar to Watanabe's argument in [32].

Lemma 5. *There exists $\mu_n \in \mathcal{A}_n(\emptyset)$ such that*

$$\widetilde{I}(\vec{\gamma}) = I(X; \vec{\gamma}) - \mu_n \mathrm{Sign} X$$

does not depend on the choice of X, β_i.

We determine the constant μ_n by computing a framed cobordism group. The constant μ_n is equivalent to the constant μ_n introduced by Watanabe in [32]. μ_n were not given explicitly in [32].

Theorem 6 (Lemma 7.10.)**.** $\mu_n = \frac{3}{4}\delta_n$. *Here $\delta_n \in \mathcal{A}_n(\emptyset)$ is the constant due to C. Lescop ([14]). Lescop gave an explicit formula of δ_n.*

In this setting, we describe the definition of $\widetilde{z}_n^{\text{anomaly}}(\vec{\gamma})$. Let Y be a rational homology 3-sphere and let $\gamma_1, \cdots, \gamma_{3n}$ be admissible vector fields with respect to $a_1, \cdots, a_{3n}$ respectively. Take a framing τ_{S^3} of S^3 satisfying $\tau_{S^3}|_{S^3 \setminus N'(\infty; S^3)} = \tau_{\mathbb{R}^3}$. Here $\tau_{\mathbb{R}^3}$ is the standard framing of $\mathbb{R}^3$ and $N'(\infty; S^3)$ is a neighborhood of ∞ smaller than $N(\infty; S^3)$, i.e., $\infty \in N'(\infty; S^3) \subset N(\infty; S^3)$. By the definition of admissible vector fields, $\gamma_i' = \gamma_i|_{Y \setminus N(\infty; Y)} \cup \tau_{S^3}^* a_i|_{N'(\infty; S^3)}$ is a vector filed on Y. We denote by $\vec{\gamma}_i' = (\gamma_1', \cdots, \gamma_{3n}')$.

Definition 7.

$$\widetilde{z}_n^{\text{anomaly}}(\vec{\gamma}) = \widetilde{I}(\vec{\gamma}') - \frac{1}{3}\sigma_{S^3}(\tau_{S^3})\mu_n.$$

Here $\sigma_{S^3}(\tau_{S^3}) \in \mathbb{Z}$ is the signature defect of τ_{S^3}.

We remark that the above definition is well-defined i.e. $\widetilde{z}_n^{\text{anomaly}}(\vec{\gamma})$ is independent of the choice of τ_{S^3}.

2.2 Proof of $z^{\text{KKT}} = \widetilde{z} = z^{\text{FW}}$.

We describe the outline of the proof of the following two Theorems.

Theorem 8 (Theorem 7.1.)**.** $z_n^{\text{KKT}}(Y) = \widetilde{z}_n(Y)$ *for any rational homology 3-sphere Y, for any $n \in \mathbb{N}$.*

Theorem 9 (Theorem 7.2.). *$z_n^{\mathrm{FW}}(Y) = \widetilde{z}_n(Y)$ for any rational homology 3-sphere Y, for any $n \in \mathbb{N}$.*

Corollary 10 (Corollary 7.3.). *$z_n^{\mathrm{FW}}(Y) = z_n^{\mathrm{KKT}}(Y)$ for any rational homology 3-sphere Y, for any $n \in \mathbb{N}$.*

2.2.1 Proof of $z^{\mathrm{KKT}} = \widetilde{z}$.

Let $\tau : T(Y \setminus \infty) \to \underline{\mathbb{R}^3}$ be a framing of $Y \setminus \infty$ satisfying $\tau|_{N(\infty;Y)\setminus\infty} = \tau_{\mathbb{R}^3}$ where $\tau_{\mathbb{R}^3}$ is the standard framing of $\mathbb{R}^3 \subset S^3$ and $\underline{\mathbb{R}^3}$ is the trivial vector bundle over an appropriate base space. $z^{\mathrm{KKT}}(Y)$ is defined as sum of the principal term $z^{\mathrm{KKT}}(Y;\tau) \in \mathcal{A}_n(\emptyset)$ and the correction term $-\delta_n \sigma_{Y\setminus\infty}(\tau) \in \mathcal{A}_n(\emptyset)$. Here $\delta_n \in \mathcal{A}_n(\emptyset)$ is the constant independent of both Y and τ. $\sigma_{Y\setminus\infty}(\tau)$ is defined by $\sigma_{Y\setminus\infty}(\tau) = \sigma_Y(\tau|_{Y\setminus N(\infty;Y)} \cup \tau_{S^3}|_{N'(\infty;S^3)}) - \sigma_{S^3}(\tau_{S^3})$. The admissible vector fields $\tau^*\vec{a} = (\tau^* a_1, \cdots, \tau^* a_{3n})$ satisfy $\tau^* a_i(x) \neq \tau^* a_j(x)$ for any $x \in Y \setminus \infty$ and for any $i \neq j$. By the definition of $z^{\mathrm{KKT}}(Y;\tau)$, we have $z^{\mathrm{KKT}}(Y;\tau) = \widetilde{z}(Y;\tau^*\vec{a})$.

Then it is sufficient to show that $\frac{1}{4}\sigma_{Y\setminus\infty}(\tau)\delta_n = \widetilde{z}(\tau^*\vec{a})$. We prove this equality by using a cobordism argument.

2.2.2 Proof of $z^{\mathrm{FW}} = \widetilde{z}$.

Let$f_1, \cdots, f_{3n} : Y \setminus \infty \to \mathbb{R}$ be generic Morse functions on $Y \setminus \infty$ such that the restriction of Morse function f_i to $N(\infty;Y) \setminus \infty$ coincides with the orthogonal projection $q_{a_i} : \mathbb{R}^3 \to \mathbb{R}$. $z^{\mathrm{FW}}(Y)$ is defined as sum of the principal term $z^{\mathrm{FW}}(Y;\vec{f}) \in \mathcal{A}_n(\emptyset)$ and the anomaly term $-z^{\mathrm{anomaly}}(\vec{f}) \in \mathcal{A}_n(\emptyset)$.

By the assumption of $f_1, \cdots, f_{3n}$, $\mathrm{grad}\vec{f} = (\mathrm{grad} f_1, \cdots, \mathrm{grad} f_{3n})$ is a family of admissible vector fields. By the definition of anomaly terms, we have $z^{\mathrm{anomaly}}(\vec{f}) = \widetilde{z}^{\mathrm{anomaly}}(\mathrm{grad}\vec{f})$.

It is sufficient to show that $z^{\mathrm{FW}}(Y;\vec{f}) = \widetilde{z}(Y;\mathrm{grad}\vec{f})$. Watanabe's original definition of $z_n^{\mathrm{FW}}(Y;\vec{f})$ is by counting flow graphs of broken graphs. We use an alternative description of the definition of $z_n^{\mathrm{FW}}(Y;\vec{f})$ in terms of intersections of manifolds. For each f_i, $M(\pm f_i)$ is a weighted sum of 4-dimensional submanifolds of Y^2. Then,

$$z_n^{\mathrm{FW}}(Y;\vec{f}) = \sum_{\Gamma\in\mathcal{E}_n} \sharp \bigcap P_i(\Gamma)^{-1}(\mathcal{M}(\pm f_i)).$$

We make a 4-cycle of $(C_2(Y), \partial C_2(Y))$ by modifying $\mathcal{M}(\pm f_i)$ and compare such 4-cycles with propagators used in the construction of $\widetilde{z}_n$. The equality $z^{\mathrm{FW}}(Y;\vec{f}) = \widetilde{z}(Y;\mathrm{grad}\vec{f})$ is followed by this comparison with the intersection theory and the Poincaré duality.

3 Remarks.

(1) Theorem 8 and Theorem 9 imply that $\widetilde{z}$ gives a geometric description of a universal finite type invariant. In particular, $\widetilde{z}_1$ and $z_1^{\mathrm{FW}}(Y)$ give a description of the Casson-Walker invariant using vector fields. Lescop gave

a combinatorial description of the principal term of z_1^{KKT} via Heegaard diagram in [20]. Lescop did not describe the anomaly term in such a combinatorial way. One of the difficulties is that framings are less flexible than vector fields. It is expected that a combinatorial description of the Casson-Walker invariant is obtained via our description of the anomaly term of $\widetilde{z}_1$.

(2) Our construction of $\widetilde{z}$ is expected to be useful for explicit computations. It is possible to regard a framing used in the construction of z^{KKT} as non-vanishing vector fields. In this thesis, we give an extension of this construction for (possibly vanishing) vector fields. Since to take vector fields is easier than to take a framing, our construction is expected to be useful for explicit computations. Surgery formula is a tool to investigate an invariant. To find surgery formulas of z^{KKT}, we have to deal with surgery of a manifold equipped with a framing. Sometimes a framing does not admit a local replacement according to surgery on the manifold, but it is always possible to replace a vector field locally. We can consider another type surgery that changes only vector fields.

(3) The anomaly term $\widetilde{z}^{\mathrm{anomaly}}(\vec{\gamma})$ is equivalent to the anomaly term $z^{\mathrm{anomaly}}(\vec{f})$ of z^{FW}. But we reformulate the construction. Some cobordisms are used in both our construction and Watanabe's construction of the anomaly term. But the conditions of such cobordisms are different. This difference plays an important role in the proof of Theorem 9 and Theorem 6.

(4) Fukaya conjectured that Fukaya's invariant related to the Chern-Simons perturbation theory by Axelrod and Singer. Fukaya remaked in [7] that his conjecture may be regarded as the higher genus analogue of the coincidence of the Reidemeister torsion and the analytic torsion established by J. Cheeger [5] and W. Müller [24]. On the other hand Watanabe conjectured that his invariant is related to invariants from the Chern-Simons perturbation theory by Kontsevich. Corollary 10 gives an answer to this conjecture.

4 Organization of the thesis.

The organization of this thesis is as follows. In Section 2 we prepare some notations. In Section 3 we review notions and facts about configuration spaces and graphs discussed by Lescop [16] and Watanabe [32]. In Section 4 we define the invariants $\widetilde{z}_n$ using vector fields and prove the independence of the choice of vector fields. In Section 5 we review the construction of z^{KKT} according to [16] by Lescop. In Section 6 we review Watanabe's construction of z^{FW} in [32] with a little modification. In Section 7 we prove that the construction of $\widetilde{z}$ is a generalization of both that of z^{KKT} and that of z^{FW}. In Section 8 we prove some lemmas, which are used in Section 6,7 for a compactification of the moduli space of flow graphs used in Sections 6 and 7. In Appendix we give an alternative and more direct proof of $\widetilde{z}_1 = z_1^{\mathrm{KKT}}$.

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Contents

2 Notation and some remarks.

In this thesis, all manifolds are smooth and oriented. Homology and cohomology are with rational coefficients. Let c be a $\mathbb{Q}$-linear sum of finitely many maps from compact k-dimensional manifolds with corners to a topological space X. We consider c as a k-chain of X via appropriate (not unique) triangulations of each k-manifold. Let Y be a submanifold of a manifold X. Let $c = \sum_i a_i(f_i : \Sigma_i \to X)$ be a chain of X, where $f_i : \Sigma_i \to X$ are smooth maps from compact manifolds with corners and a_i are rational numbers. If f_i is transverse to Y for each i, then we say that c is transverse to Y.

When B is a submanifold of a manifold A, We denote by $A(B)$ the manifold given by real blowing up of A along B. Namely $A(B) = (A \setminus B) \cup S\nu_B$ where ν_B is the normal bundle of $B \subset A$ and $S\nu_B$ is the sphere bundle of ν_B (see [16] for more details of real blow up). Note that if a submanifold $C \subset A$ is transverse to B, then $C(A \cap B)$ is a proper embedded submanifold of $A(B)$.

Let us denote by $\Delta \subset A^n$ the fat diagonal of the n-times direct product of a manifold A: $\Delta = \{(x_1, \ldots, x_n) \in A^n \mid \sharp\{x_1, \ldots, x_n\} < n\}$.

Let us denote by $\underline{\mathbb{R}^k}$ the trivial vector bundle over an appropriate base space with rank $k \in \mathbb{N}$. For a real vector space X, we denote by SX or $S(X)$ the unit sphere of X and for a real vector bundle $E \to B$ over a manifold B, we denote by SE or $S(E)$ the unit sphere bundle of E.

2.1 Notations about 3-manifolds and Morse functions.

Let $f : Y \to \mathbb{R}$ be a Morse function on a 3-dimensional manifold Y with a metric satisfying the Morse-Smale condition. Let $\mathrm{grad} f$ be the gradient vector field of f with respect to the metric of Y. Let us denote by $\mathrm{Crit}(f)$ the set of all critical points of f. Let $\{\Phi_f^t\}_{t \in \mathbb{R}} : Y \to Y$ be the 1-parameter group of diffeomorphisms associated to $-\mathrm{grad}\, f$. We denote by

$$\mathcal{A}_p = \{x \in Y \mid \lim_{t \to \infty} \Phi_f^t(x) = p\} \text{ and}$$

$$\mathcal{D}_p = \{x \in Y \mid \lim_{t \to -\infty} \Phi_f^t(x) = p\}$$

the ascending manifold and descending manifold at $p \in \mathrm{Crit}(f)$ respectively.

2.2 Conventions on orientations.

Boundaries are oriented by the outward normal first convention. Products are oriented by the order of the factors. Let $y \in B$ be a regular point of a smooth map $f : A \to B$ between smooth manifolds A and B. Let us orient $f^{-1}(y)$ by the following rules: $T_x f^{-1}(y) \oplus f^* T_y B = T_x A$, for any $x \in f^{-1}(y)$ where $f^* : T_y B \to T_x A$ is a linear map satisfying $f_* \circ f^* = \mathrm{id}_{T_y B}$. We denote by $-X$ the orientation reversed manifold of an oriented manifold X.

Suppose that Y, f and $\mathrm{grad} f$ are given as above. Let us orient ascending manifolds and descending manifolds by imposing the condition: $T_p \mathcal{A}_p \oplus T_p \mathcal{D}_p \cong T_p Y$ for any $p \in \mathrm{Crit}(f)$. Let $p, q \in \mathrm{Crit}(f)$ be the critical points of index 2 and 1 respectively. By the Morse-Smale condition, $\mathcal{D}_p \cap \mathcal{A}_q$ is a 1-manifold. Let us orient $\mathcal{D}_p \cap \mathcal{A}_q$ by the following rule:

$$T_{q'}(\mathcal{D}_p \cap \mathcal{A}_q) \oplus T_{q'} \mathcal{D}_q \cong T_{q'} \mathcal{D}_p,$$

where $q' \in \mathcal{D}_p \cap \mathcal{A}_q$ is a point near q.

3 Configuration space and Jacobi diagrams.

In this section, we introduce some notations about configuration spaces and Jacobi diagrams. Most of this section follows [18].

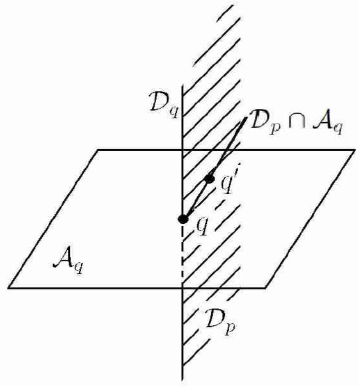


Figure 1: The orientation of $\mathcal{D}_p \cap \mathcal{A}_q$.

3.1 The configuration space $C_{2n}(Y)$.

Let Y be a rational homology 3-sphere with a basepoint ∞. Let $N(\infty; Y)$ be a regular neighborhood (that is diffeomorphic to an open ball) of ∞ in Y and let $N(\infty; S^3)$ be a regular neighborhood of ∞ in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ [1]. We fix a diffeomorphism $\varphi^\infty : (N(\infty; Y), \infty) \overset{\cong}{\to} (N(\infty; S^3), \infty)$ between $N(\infty; Y)$ and $N(\infty; S^3)$. We identify $N(\infty; Y)$ with $N(\infty; S^3)$ under φ^∞.

Let $\breve{C}_{2n}(Y) = (Y \setminus \infty)^{2n} \setminus \Delta = \{\{1, \ldots, 2n\} \hookrightarrow Y \setminus \infty\}$ and let $C_{2n}(Y)$ the compactification of $\breve{C}_{2n}(Y)$ given by Lescop [18, §3]. (This compactification is similar to Fulton-MacPherson compactification [9]). Roughly speaking, $C_{2n}(Y)$ is obtained from Y^{2n} by real blowing up along all diagonals and $\{(x_1, \ldots, x_{2n}) \mid \exists i \text{ such that } x_i = \infty\}$. See §3 in [18] for the complete definition. Note that $C_2(Y)$ is given by real blowing up Y^2 along (∞, ∞), $\infty \times (Y \setminus \infty), (Y \setminus \infty) \times \infty$ and Δ in turn. Let us denote by $q : C_2(Y) \to (Y \setminus \infty)^2$ the composition of the blow down maps. Then $\partial C_2(Y) = ST_\infty Y \times (Y \setminus \infty) \cup (Y \setminus \infty) \times ST_\infty Y \cup S\nu_{\Delta(Y \setminus \infty)} \cup q^{-1}(\infty^2)$. We identify $S\nu_{\Delta(Y\setminus\infty)}$ with $STY|_{Y\setminus\infty}$ by the canonical isomorphism $S\nu_{\Delta Y} \cong STY$. The involution $Y^2 \to Y^2, (x, y) \mapsto (y, x)$ induces an involution of $C_2(Y)$. We denote by $\iota : C_2(Y) \to C_2(Y)$ this involution.

Let $p_1 : (\partial C_2(Y) \supset) ST_\infty Y \times (Y \setminus \infty) \to ST_\infty Y \overset{\varphi^\infty}{=} ST_\infty S^3 = S^2$ and $p_2 : (\partial C_2(Y) \supset)(Y \setminus \infty) \times ST_\infty Y \to ST_\infty Y = ST_\infty S^3 = S^2$ be the projections. We denote by $\iota_{S^2} : S^2 \to S^2$ the involution induced by $\times(-1) : \mathbb{R}^3 \to \mathbb{R}^3$.

Let $p_c : C_2(S^3) \to S^2$ be the extension of the map $\mathrm{int} C_2(S^3) = (\mathbb{R}^3 \times \mathbb{R}^3) \setminus \Delta \to S^2$, $(x, y) \mapsto (y - x)/\|y - x\|$. Since it is possible to identify $q^{-1}(N(\infty; Y)^2) \subset \partial C_2(Y)$ with $q^{-1}(N(\infty; S^3)^2) \subset \partial C_2(S^3)$ by φ^∞, we get a map $\partial C_2(Y) \supset q^{-1}((N(\infty; Y) \setminus \infty)^2) \overset{p_c}{\to} S^2$. Since $p_1, \iota_{S^2} \circ p_2$ and p_c are compatible on boundary, these maps define the map

$$p_Y : \partial C_2(Y) \setminus S\nu_{\Delta(Y \setminus N(\infty; Y))} \to S^2.$$

(Here we note that $\partial C_2(Y) \setminus S\nu_{\Delta(Y\setminus N(\infty;Y))} = ST_\infty Y \times (Y \setminus \infty) \cup (Y \setminus \infty) \times ST_\infty Y \cup q^{-1}(N(\infty; Y)^2)$.)

[1] We sometimes denote $\{\infty\}$ briefly by ∞.

3.2 More on the boundary $\partial C_{2n}(Y)$.

For $B \subset \{1, \ldots, 2n\}$, we set

$$F(\infty; B) = q^{-1}(\{(x_1, \ldots, x_{2n}) \mid x_i = \infty \text{ iff } i \in B,\ \text{if } i, j \notin B \text{ then } x_i \neq x_j\}),$$

and for $B \subset \{1, \ldots, 2n\}(\sharp B \geq 2)$, we set

$$F(B) = q^{-1}(\{(x_1, \ldots, x_{2n}) \in (Y \backslash \infty)^{2n} \mid \exists y, x_i = y \,\text{iff}\, i \in B,\ \text{if}\, i, j \notin B \,\text{then}\, x_i \neq x_j\}).$$

Under these notations, $\partial C_{2n}(Y) = \bigcup_B F(\infty; B) \cup \bigcup_{\sharp B \geq 2} F(B)$. We remark that $\partial C_{2n}(Y)$ has smooth structure (See [18, § 3]).

Let X be a 3-dimensional real vector space. Let V be a finite set. we define $\breve{S}_V(X)$ to be the set of injective maps from V to X up to translations and dilations. Set $\underline{k} = \{1, \ldots, k\}$. We denote $S_{\underline{k}}(X)$ by $S_k(X)$. Note that $\breve{S}_2(X) = S(X)$. For an $\mathbb{R}^3$ vector bundle $E \to M$, we denote by $\breve{S}_V(E) \to M$ the fiber bundle where the fiber over $x \in M$ is $\breve{S}_V(E_x)$. Under these notations, $F(2n) = \breve{S}_{2n}(T(Y \setminus \infty))$.

We remark that $F(B)$ has a fiber bundle structure where the typical fiber is $\breve{S}_B(\mathbb{R}^3)$.

Lescop gave a compactification $S_V(X), S_V(E)$ of $\breve{S}_V(X), \breve{S}_V(E)$ respectively in [18]. Let $f(B)(X) = \breve{S}_B(X) \times \breve{S}_{\{b\} \cup B}(X)$, for $B \subset V$ with $B \neq V$ and $\sharp B \geq 2$. Let $f(B)(E) \to M$ be the fiber bundle where the fiber over $x \in M$ is $f(B)(E_x)$. Under this notation,

$$\partial S_V(X) = \bigcup_{\sharp B \geq 2} f(B)(X), \partial S_V(E) = \bigcup_{\sharp B \geq 2} f(B)(E)$$

(See Proposition 2.8 in [18]). We remark that $f(B)(E)$ has a fiber bundle structure where the typical fiber is $\breve{S}_B(\mathbb{R}^3)$.

3.3 Jacobi diagrams.

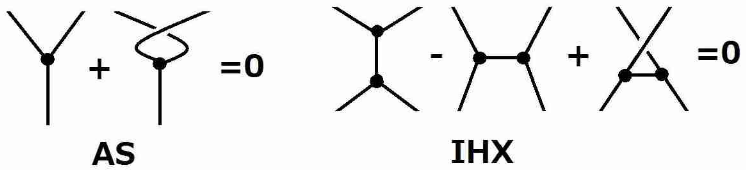
A *Jacobi diagram* of degree n is defined to be a trivalent graph with $2n$ vertexes and $3n$ edges without simple loops. For a Jacobi diagram $\overline{\Gamma}$, we denote by $H(\overline{\Gamma}), E(\overline{\Gamma})$ and $V(\overline{\Gamma})$ the set of half edges, the set of edges and the set of vertexes respectively. An *orientation* of a vertex of $\overline{\Gamma}$ is a cyclic order of the three half-edges that meet at the vertex. A Jacobi diagram is *oriented* if all its vertexes are oriented. Let

$$\mathcal{A}_n(\emptyset) = \{\text{degree } n \text{ oriented Jacobi diagrams}\}_{\mathbb{R}} / \mathrm{AS}, \mathrm{IHX},$$

where $\{\text{degree } n \text{ oriented Jacobi diagrams}\}_{\mathbb{R}}$ is the real vector space generated by the set $\{\text{degree } n \text{ oriented Jacobi diagrams}\}$. Here the relations AS and IHX are locally represented by the following pictures. Let

$$\mathcal{E}_n = \{\Gamma = (\overline{\Gamma}, \varphi_E, \varphi_V, \mathrm{ori}_E)\}$$

Here $\overline{\Gamma}$ is a connected Jacobi diagram of degree n, $\varphi_E : E(\overline{\Gamma}) \cong \{1, 2, \ldots, 3n\}$ and $\varphi_V : V(\overline{\Gamma}) \cong \{1, 2, \ldots, 2n\}$ are labels of edges and vertexes respectively, and ori_E is



Here the orientation of each vertex is given by counterclockwise order of the half edges.

a collection of orientations of each edge. These data and an orientation of $\overline{\Gamma}$ induce two orientations of $H(\overline{\Gamma})$. The first one is the edge-orientation induced by φ_E and ori_E. The second one is the vertex-orientation induced by φ_V and orientation of $\overline{\Gamma}$. We choose the orientation of $\overline{\Gamma}$ so that the edge-orientation coincides with the vertex-orientation. Let us denote by $[\Gamma] \in \mathcal{A}_n(\emptyset)$ the oriented Jacobi diagram given by Γ in such a way.

Remark 3.1. The notation $\mathcal{A}_{2n,3n}$ used by Watanabe [34] coincides with the notation $\mathcal{A}_n(\emptyset)$ used by Lescop [18] as $\mathbb{R}$-vector spaces.

4 Construction of an invariant of rational homology 3-spheres via vector fields.

Let n be a natural number. In this section, we construct an invariant $\widetilde{z}_n$ using vector fields. The idea of construction of $\widetilde{z}_n$ is based on the principal term of Kontsevich-Kuperberg-Thurston invariant ([16], [18]) and the construction of the anomaly part of Watanabe's invariant ([34]).

Let Y be a rational homology 3-sphere with a basepoint ∞. Let $N(\infty; Y)$ be a regular neighborhood (that is diffeomorphic to an open ball) of ∞ in Y and let $N(\infty; S^3)$ be a regular neighborhood of ∞ in $S^3 = \mathbb{R}^3 \cup \infty$. We fix a diffeomorphism $\varphi^\infty : (N(\infty; Y), \infty) \overset{\cong}{\to} (N(\infty; S^3), \infty)$ between $N(\infty; Y)$ and $N(\infty; S^3)$. We identify $N(\infty; Y)$ with $N(\infty; S^3)$ under φ^∞.

In Subsection 4.1, we will define the notion of admissible vector fields on $T(Y\setminus\infty)$. In Subsections 4.2, 4.4, we will define $\widetilde{z}_n(Y; \vec{\gamma})$ and $\widetilde{z}_n^{\mathrm{anomaly}}(\vec{\gamma})$ using a family of admissible vector fields $\vec{\gamma}$. Thus we obtain a topological invariant $\widetilde{z}_n(Y) = \widetilde{z}_n(Y; \vec{\gamma}) - \widetilde{z}_n^{\mathrm{anomaly}}(\vec{\gamma})$ of Y in Subsection 4.5. We will prove well-definedness of $\widetilde{z}_n$ in Subsection 4.6.

4.1 Admissible vector fields on $T(Y \setminus \infty)$.

For $a \in S^2 \subset \mathbb{R}^3$, the map $q_a : \mathbb{R}^3 \to \mathbb{R}$ is defined by $q_a(x) = \langle x, a\rangle$ where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^3$. Write $\pm a = \{a, -a\}$.

Definition 4.1. A vector field $\gamma \in \Gamma T(Y \setminus \infty)$ is an *admissible vector field (with respect to a)* if the following conditions hold.

- $\gamma|_{N(\infty;Y)\setminus\infty} = -\mathrm{grad}\; q_a|_{N(\infty;S^3)\setminus\infty}$,

- γ is transverse to the zero section in $T(Y \setminus \infty)$.

Example 4.2. We give two important examples of admissible vector fields with respect to a.

(1) Let $\tau_{\mathbb{R}^3} : T\mathbb{R}^3 \xrightarrow{\cong} \underline{\mathbb{R}^3}$ be the standard framing of $T\mathbb{R}^3$. We regard $a \in \mathbb{R}^3$ as a constant section of the trivial bundle $\underline{\mathbb{R}^3}$. For a framing $\tau : T(Y\setminus\infty) \xrightarrow{\cong} \underline{\mathbb{R}^3}$ such that $\tau|_{N(\infty;Y)\setminus\infty} = \tau_{\mathbb{R}^3}|_{N(\infty;S^3)\setminus\infty}$, the pull-back vector field $\tau^* a$ is an admissible vector field with respect to $-a$.

(2) For a Morse function $f : Y \setminus \infty \to \mathbb{R}$ such that $f|_{N(\infty;Y)\setminus\infty} = q_a|_{N(\infty;S)\setminus\infty}$, $\mathrm{grad} f$ is an admissible vector field with respect to a.

The following lemma plays an important role in the next subsection. For an admissible vector field γ, let

$$\bar{c}_\gamma = \overline{\left\{ \frac{\gamma(x)}{\|\gamma(x)\|} \in ST_xY \;\middle|\; x \in Y \setminus (\infty \cup \gamma^{-1}(0)) \right\}}^{\text{closure}} \subset ST(Y \setminus \infty).$$

Here we choose the orientation of $\bar{c}_\gamma$ such that the restriction of the projection $STY \to Y$ to $\bar{c}_\gamma$ is orientation preserving.

Lemma 4.3.
$$c_0(\gamma) = \bar{c}_\gamma \cup \bar{c}_{-\gamma}$$
is a submanifold of $ST(Y \setminus \infty)$ without boundary.

To prove this lemma, we first remark the following lemma. Let $n, k \geq 0$ be integers. Let $s : (\mathbb{R}^{n+k}, 0) \to (\mathbb{R}^n, 0)$ be a C^∞ map which is transverse to the origin $0 \in \mathbb{R}^n$.

Lemma 4.4. *There is a diffeomorphism $\varphi : (\mathbb{R}^{n+k}, 0) \to (\mathbb{R}^{n+k}, 0)$ such that $s \circ \varphi$ coincides with $p_{\mathbb{R}^n}$ as germs at $0 \in \mathbb{R}^{n+k}$. Here $p_{\mathbb{R}^n} : \mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is the orthogonal projection.*

Proof. This is a consequence of the implicit function theorem. $\square$

Proof of Lemma 4.3. It is sufficient to check this claim near $\gamma^{-1}(0)$. Let $x \in \gamma^{-1}(0)$. We fix a trivialization $\psi : T(Y \setminus \infty)|_{U_0} \xrightarrow{\cong} U_0 \times \mathbb{R}^3$ on a neighborhood U_0 of x in Y. By the above Lemma 4.4, there is a neighborhood $U \subset U_0$ of x and local coordinates $\varphi : \mathbb{R}^3 \xrightarrow{\cong} U$(which is independent of ψ) such that $(\varphi^{-1} \times \mathrm{id}) \circ \psi \circ \gamma \circ \varphi : \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ is represented by $(\varphi^{-1} \times \mathrm{id}) \circ \psi \circ \gamma \circ \varphi(x) = (x, x)$. We fix these local trivialization and coordinates and we write γ instead of $(\varphi^{-1} \times \mathrm{id}) \circ \psi \circ \gamma \circ \varphi$.

We first show that $\partial \bar{c}_\gamma \cap STU = -(\partial \bar{c}_{-\gamma} \cap STU)$ as oriented manifolds. Let $D_+ = \bar{c}_\gamma \cap STU$ and $D_- = \bar{c}_{-\gamma} \cap STU$. Under the above local coordinates, $D_+ = \{(tx, x/\|x\|) \mid x \in S^2, t \in [0, \infty)\} \subset (S^2 \times [0, \infty)/(S^2 \times \{0\})) \times S^2 = \mathbb{R}^3 \times S^2$ and $D_- = \{(tx, -x/\|x\|) \mid x \in S^2, t \in [0, \infty)\}$. Both the projection $\pi : D_+ \to \mathbb{R}^3$ and the projection $\pi : D_- \to \mathbb{R}^3$ are orientation preserving (or reversing). Let $g : \mathbb{R}^3 \times S^2 \to \mathbb{R}^3 \times S^2$ be the bundle map defined by $(x, v) \mapsto (x, -v)$. Since both the orientation of ∂D_+ and that of ∂D_- are induced by that of the base space $\mathbb{R}^3$, the map $g : \partial D_+ \xrightarrow{\cong} \partial D_-$ is orientation preserving. On the other hand, $g|_{\{0\}\times S^2} : \{0\} \times S^2 \to \{0\} \times S^2$ is orientation reversing. Hence, the identity map $\mathrm{id} : \{0\}\times S^2 \to \{0\}\times S^2$ is an orientation reversing map from ∂D_+ to ∂D_-. Therefore $\partial \bar{c}_\gamma = \partial D_+ = -\partial D_- = -\partial \bar{c}_{-\gamma}$.

We next prove that $c_0(\gamma) \cap STU$ is a submanifold of $STU \cong \mathbb{R}^3 \times S^2$. Let $p_2 : \mathbb{R}^3 \times S^2 \to S^2$ be the projection. For each $v \in S^2$, we have $(p_2|_{c_0(\gamma)})^{-1}(v) = \mathbb{R}v \times \{v\} \subset \mathbb{R}^3 \times S^2$. The set $\bigcup_{v\in S^2} \mathbb{R}v \times \{v\}$ is a submanifold of $\mathbb{R}^3 \times S^2$. In fact, for any $v_0 \in S^2$ and for any sufficiently small neighborhood $B_{v_0} \subset S^2$ of v_0 we can take a diffeomorphism

$$\Phi_{v_0} : (\mathbb{R}^3 \times B_{v_0}, \bigcup_{v\in B_{v_0}} \mathbb{R}v \times \{v\}) \xrightarrow{\cong} (\mathbb{R}^3 \times B_{v_0}, \mathbb{R}w_0 \times B_{v_0})$$

as follows[2]. Here $w_0 \in S^2 \subset \mathbb{R}^3$ is a point orthogonal to v_0 in $\mathbb{R}^3$ and $\mathbb{R}w_0$ is the 1-dimensional vector subspace of $\mathbb{R}^3$ spanned by w_0. For each $v \in B_{v_0}$, let $m(v, w_0) \in S^2$ be the middle point of the geodesic segment from v to w_0. Let $\rho(v, w_0) \in SO(3)$ be the rotation with axis directed by $m(v, w_0)$ and with angle π. So $\rho(v, w_0)$ exchanges v and w_0. Then we can define $\Phi_{v_0} : \mathbb{R}^3 \times B_{v_0} \to \mathbb{R}^3 \times B_{v_0}$ by $\Phi_{v_0}(x, v) = (\rho(v, w_0)(x), v)$ for each $(x, v) \in \mathbb{R}^3 \times B_{v_0}$.

Therefore $c_0(\gamma) \cap (\mathbb{R}^3 \times S^2) = \bigcup_{v\in S^2} \mathbb{R}v \times \{v\}$ is a submanifold of $\mathbb{R}^3 \times S^2$. $\square$

4.2 The principal term $\widetilde{z}(Y; \vec{\gamma})$.

In this subsection, we define the principal term $\widetilde{z}(Y; \vec{\gamma})$ of the invariant $\widetilde{z}(Y)$. We define

$$c(\gamma) = p_Y^{-1}(\pm a) \cup c_0(\gamma) \subset \partial C_2(Y).$$

By the definition of γ and Lemma 4.3, $c(\gamma)$ is a closed 3-manifold. Therefore $[c(\gamma)] \in H_3(\partial C_2(Y); \mathbb{R})$.

Let $\omega^a_{S^2}$ be an anti-symmetric closed 2-form on S^2 such that $\omega^a_{S^2}$ represents the Poincaré dual of $[\pm a]$ and the support of $\omega^a_{S^2}$ is concentrated in a small neighborhood of $\pm a$. Let $\omega_\partial(\gamma)$ be a closed 2-form on $\partial C_2(Y)$ satisfying the following conditions.

- $2\omega_\partial(\gamma)$ represents the Poincaré dual of $[c(\gamma)]$,
- The support of $\omega_\partial(\gamma)$ is concentrated in a small neighborhood of $c(\gamma)$,
- $\iota^*\omega_\partial(\gamma) = -\omega_\partial(\gamma)$ and
- $\omega_\partial(\gamma)|_{\partial C_2(Y)\setminus S\nu_{\Delta(Y\setminus N(\infty;Y))}} = \frac{1}{2} p_Y^* \omega^a_{S^2}$.

Since Y is a rational homology 3-sphere, the restriction $H^2(C_2(Y); \mathbb{R}) \to H^2(\partial C_2(Y); \mathbb{R})$ is an isomorphism. Thus there is a closed 2-form $\omega(\gamma)$ on $C_2(Y)$ satisfying the following conditions.

- $\omega(\gamma)|_{\partial C_2(Y)} = \omega_\partial(\gamma)$ and
- $\iota^*\omega(\gamma) = -\omega(\gamma)$.

[2]The author is indebted to Professor Christine Lescop for this construction.

Definition 4.5 (propagator)**.** We call $\omega(\gamma)$ a *propagator with respect to* γ.

Take $a_1, \ldots, a_{3n} \in S^2$ (we may take, for example, $a_1 = \ldots = a_{3n}$). Let γ_i be an admissible vector field with respect to a_i and let $\omega(\gamma_i)$ be a propagator with respect to γ_i for each $i \in \{1, \ldots, 3n\}$. To simplify notation, we write $\vec{\gamma}$ instead of $(\gamma_1, \ldots, \gamma_{3n})$.

For each $\Gamma = (\overline{\Gamma}, \varphi_E, \varphi_V, \mathrm{ori}_E) \in \mathcal{E}_n$ and for each $\varphi_E^{-1}(i) \in E(\overline{\Gamma})$, let $s(i), t(i) \in \{1, \ldots, 2n\}$ denote the labels of the initial vertex and the terminal vertex of $\varphi_E^{-1}(i)$ respectively. The embedding $\{1,2\} \cong \{s(i), t(i)\} \hookrightarrow \{1, \ldots, 2n\}$ induces the projection $\pi_{\breve{C}_{2n}(Y)} : \breve{C}_{2n}(Y) \to \breve{C}_2(Y)$. Furthermore it is possible to extend $\pi_{\breve{C}_{2n}(Y)}$ to $C_{2n}(Y)$ by the definition of $C_{2n}(Y)$. We denote by $P_i(\Gamma) : C_{2n}(Y) \to C_2(Y)$ such the extended map (see [18] §2.3 for more details).

Definition 4.6.

$$\widetilde{z}_n(Y;\vec{\gamma}) = \sum_{\Gamma \in \mathcal{E}_n} \left(\int_{C_{2n}(Y)} \bigwedge_i P_i(\Gamma)^* \omega(\gamma_i) \right) [\Gamma] \in \mathcal{A}_n(\emptyset).$$

Remark 4.7. By the above definition, the value $\widetilde{z}_n(Y;\vec{\gamma})$ often depends on the choices of $\omega(\gamma_i)$ even if we fix $\vec{\gamma}$. We will prove in Subsection 4.6 that $\widetilde{z}_n(Y;\vec{\gamma})$, however, depends only on the choice of $\vec{\gamma}$ for generic $\vec{\gamma}$.

4.3 Alternative description of $\widetilde{z}_n(Y;\vec{\gamma})$.

In this subsection, we give an alternative description of $\widetilde{z}_n(Y;\vec{\gamma})$ using cohomologies of simplicial complexes with coefficients in $\mathbb{R}$. This description will be needed in Section 7. The admissible vector field γ_i with respect to a_i and the 3-cycle $c(\gamma_i) \subset \partial C_2(Y)$ are as above. Let $T_{C_2(Y)}$ be the simplicial decomposition of $C_2(Y)$ given by pulling back a simplicial decomposition of $C_2(Y)/\iota$. So the simplicial decomposition $T_{C_2(Y)}$ is compatible with the action of ι. By replacing such a simplicial decomposition if necessary, we may assume that each simplex of $T_{C_2(Y)}$ is transverse to $c(\gamma_i)$. Let $\omega^s_\partial(\gamma_i) \in S^2(\partial C_2(Y))$ be the 2-cocycle defined by $\omega^s_\partial(\gamma_i)(\sigma) = \frac{1}{2}\sharp(\sigma \cap c(\gamma_i))$ for each 2-chain σ in $T_{C_2(Y)}|_{\partial C_2(Y)}$. Thus $\omega^s_\partial(\gamma_i)$ is anti-symmetric under the involution ι. Let $\omega^s(\gamma_i)$ be an extension of $\omega^s_\partial(\gamma_i)$ to $C_2(Y) = |T_{C_2(Y)}|$ satisfying the following conditions.

- $\omega^s(\gamma_i)|_{\partial C_2(Y)} = \omega^s_\partial(\gamma_i)$ and
- $\iota^* \omega^s(\gamma_i) = -\omega^s(\gamma_i)$.

We call it a *simplicial propagator*. Take an appropriate simplicial decomposition of $C_{2n}(Y)$. Then we have the 2-cocycle $P_i(\Gamma)^*\omega^s(\gamma_i) \in S^2(C_{2n}(Y))$. By the construction, $\bigwedge_i P_i(\Gamma)^*\omega^s(\gamma_i)$ is a cocycle in $(C_{2n}(Y), \partial C_{2n}(Y))$. If necessary we replace the simplicial decompositions with a smaller one, we have the following lemma via the intersection theory.

Lemma 4.8 (Alternative description of $\tilde{z}_n(Y;\gamma)$)**.** *If* $(\bigcap_i P_i(\Gamma)^{-1}\mathrm{support}(\omega^s(\gamma_i)))$ $\cap\, \partial C_{2n}(Y) = \emptyset$ *for any* Γ,

$$\tilde{z}_n(Y;\vec{\gamma}) = \sum_{\Gamma\in\mathcal{E}_n} \langle \bigwedge_i P_i(\Gamma)^*\omega^s(\gamma_i), [C_{2n}(Y), \partial C_{2n}(Y)]\rangle[\Gamma] \in \mathcal{A}_n(\emptyset).$$

Here $[C_{2n}(Y), \partial C_{2n}(Y)]$ *denotes the fundamental homology class and* $\langle,\rangle$ *denotes the Kronecker product.*

4.4 The anomaly term $\tilde{z}_n^{\mathrm{anomaly}}(\vec{\gamma})$.

In this subsection, we define the anomaly term $\tilde{z}_n^{\mathrm{anomaly}}(Y;\vec{\gamma})$ of the invariant $\tilde{z}_n(Y)$. The idea of the construction of this anomaly term is based on the construction of the anomaly term of Watanabe's invariant [34]. Let Y, ∞, $a_1,\ldots,a_{3n}\in S^2$, $\gamma_1,\ldots,\gamma_{3n}$ (admissible vector fields with respect to $a_1,\ldots,a_{3n}$ respectively) and $\omega(\gamma_1),\ldots,\omega(\gamma_{3n})$ be the same as above. Let X be a connected oriented 4-manifold with $\partial X = Y$ and $\chi(X) = 0$. For example, we can take $X = (T^4\sharp\mathbb{C}P^2)\setminus B^4$ when $Y = S^3$. For a framing τ' of TY or $\mathbb{R}\oplus TY$, we denote by $\sigma_Y(\tau')\in\mathbb{Z}$ the signature defect of τ'. Let τ_{S^3} be a framing[3] of TS^3 satisfying the following two conditions:

- $\sigma_{S^3}(\tau_{S^3}) = 2$,
- $\tau_{S^3}|_{S^3\setminus N'(\infty;S^3)} = \tau_{\mathbb{R}^3}|_{S^3\setminus N'(\infty;S^3)}$.

Here $N'(\infty;S^3)$ is a neighborhood of ∞ smaller than $N(\infty;S^3)$, i.e., $\infty\in N'(\infty;S^3)\subset N(\infty;S^3)$.

Remark 4.9. There is no special meaning in the number "2" in the condition $\sigma_{S^3}(\tau_{S^3}) = 2$. The anomaly term $\tilde{z}_n^{\mathrm{anomaly}}(\vec{\gamma})$ is independent of the choice of τ_{S^3} even if $\sigma_{S^3}(\tau_{S^3}) \neq 2$. We remark that there is no framing τ on S^3 such that $\sigma_{S^3}(\tau) = 0$.

Let η_Y be the outward unit vector field of $TY = T(\partial X)\subset TX|_Y$ in TX. Since $\chi(X) = 0$, it is possible to extend η_Y to a unit vector field of TX. We denote by $\eta_X\in\Gamma TX$ such an extended vector field. Let T^vX be the normal bundle of η_X. We remark that $T^vX|_Y = TY$.

The vector field $\tau_{S^3}^*a_i$ of $TY|_{N(\infty;Y)}$ is the pull-back of $a_i\in S^2\subset\mathbb{R}^3$ along $\tau_{S^3}|_{N(\infty;Y)}$ [4]. Since $\gamma_i|_{Y\setminus N(\infty;Y)}\in\Gamma T(Y\setminus N(\infty;Y))$ and $\tau_{S^3}^*a_i|_{N(\infty;Y)}\in\Gamma TY|_{N(\infty;Y)}$ are compatible, these vector fields define the vector field $\gamma_i'\in\Gamma TY$. Let $\beta_i\in\Gamma T^vX$ be a vector field of T^vX transverse to the zero section in T^vX satisfying $\beta_i|_Y = \gamma_i'$. By a similar argument of Lemma 4.3,

$$c_0(\beta_i) = \overline{\left\{ \frac{\beta(x)}{\|\beta(x)\|}, \frac{-\beta(x)}{\|\beta(x)\|}\in S(T^vX)_x \;\middle|\; x\in X\setminus\beta^{-1}(0)\right\}}^{\mathrm{closure}} \subset ST^vX$$

is a submanifold of ST^vX satisfying $\partial c_0(\beta_i)\subset STY$. Hence $c_0(\beta_i)$ is a cycle of $(ST^vX, \partial ST^vX)$. Here we choose the orientation of $c_0(\beta_i)$ such that the restriction of the projection $ST^vX\to X$ to $c_0(\beta_i)$ is orientation preserving.

[3] There is such a framing. For example, the Lie framing $\tau_{SU(2)}$ of $S^3 = SU(2)$ satisfies $\sigma_{S^3}(\tau_{SU(2)}) = 2$. See R. Kirby and P. Melvin [14] for more details. We can get τ_{S^3} by modifying $\tau_{SU(2)}$.

[4] We sometimes regard a framing as a bundle map to the trivial bundle over a point.

We note that $c_0(\beta_i)$ satisfies $c_0(\beta_i) \cap S\nu_{\Delta(Y \setminus N(\infty;Y))} = c_0(\gamma_i)$. Let $W(\gamma_i)$ be a closed 2-form on ST^vX satisfying the following conditions.

- $2W(\gamma_i)$ represents the Poincaré dual of $[c_0(\beta_i), \partial c_0(\beta_i)]$,
- The support of $W(\gamma_i)$ is concentrated in a small neighborhood of $c_0(\beta_i)$,
- $W(\gamma_i)|_{ST(Y \setminus N(\infty;Y))} = \omega_\partial(\gamma_i)|_{S\nu_{\Delta(Y\setminus N(\infty;Y))}}$ and
- $W(\gamma_i)|_{STN(\infty;Y)} = \frac{1}{2}\tau_{S^3}^*\omega_{S^2}^{a_i}$.

For $i \in \{1,2,\ldots,3n\}$, let $\phi_i^0(\Gamma) : \breve{S}_{2n}(T^vX) \to S_2(T^vX)$ be the map induced by $\{1,2\} \cong \{s(i),t(i)\} \hookrightarrow \{1,\ldots,2n\}$. It is possible to extend $\phi_i^0(\Gamma)$ to $S_{2n}(T^vX)$. We denote by $\phi_i(\Gamma) : S_{2n}(T^vX) \to S(T^vX)$ such the extended map. By an argument similar to Proposition 4.17 in [34], the following lemma holds.

Lemma 4.10. *There exists $\mu_n \in \mathcal{A}_n(\emptyset)$ such that*

$$-\mu_n \mathrm{Sign}X + \sum_{\Gamma \in \mathcal{E}_n} \int_{S_{2n}(T^vX)} \bigwedge_i \phi_i(\Gamma)^* W(\gamma_i)[\Gamma] \in \mathcal{A}_n(\emptyset)$$

does not depend on the choice of X, β_i, and $W(\gamma_i)$.

Proof of Lemma 4.10. Let X be a closed 4-manifold with $\mathrm{Sign}X = 0$ and $\chi(X) = 0$. When X is not connected, we assume that the Euler number of each component of X is zero. Let η_X be a unit vector field of TX and let T^vX be the normal bundle of η_X in TX. Let $\beta_1, \ldots, \beta_{3n}$ be a family of sections of T^vX that are transverse to the zero section in T^vX. Let W_i be a closed 2-form that represents the Poincaré dual of $c_0(\beta_i)$ in ST^vX, for $i = 1, \ldots, 3n$. By a cobordism argument, it is sufficient to show that $\sum_{\Gamma \in \mathcal{E}_n} \int_{S_{2n}(T^vX)} \bigwedge_i \phi_i(\Gamma)^* W_i[\Gamma] = 0$.

We first prove that there exist an oriented compact 5-manifold Z and unit vector fields $\eta_Z^1, \eta_Z^2 \in \Gamma TZ$ such that:

- $\partial Z = X \sqcup X$,
- η_Z^1, η_Z^2 are linearly independent at any point in Z, i.e., (η_Z^1, η_Z^2) is a 2-framing of TZ,
- $\eta_Z^1|_{\partial Z}$ is the outward unit vector field of $X = \partial Z$, and
- $\eta_Z^2|_{\partial Z} = \eta_X \sqcup \eta_X$.

Since $\mathrm{Sign}X = 0$, there exists a connected compact oriented 5-manifold Z_0 such that $\partial Z_0 = X$. Let $\eta_{Z_0} \in \Gamma TZ_0|_X$ be the outward unit vector field of $X = \partial Z_0$. By attaching 2-handles along the knots generating $H_1(Z_0; \mathbb{Z}/2)$ if necessary, we may assume that $H_1(Z_0;\mathbb{Z}/2) \cong H^4(Z_0; \partial Z_0; \mathbb{Z}/2) = 0$. Thus the primary obstruction o_{Z_0} to extend the 2-framing (η_{Z_0}, η_X) of $TZ_0|_X$ into Z_0 is in $H^5(Z_0, \partial Z_0; \pi_4(V_{5,2})) = H^5(Z_0, \partial Z_0; \mathbb{Z}/2)$. Let $Z = Z_0 \sharp Z_0$. Then the obstruction to extend the 2-framing $(\eta_{Z_0} \sqcup \eta_{Z_0}, \eta_X \sqcup \eta_X)$ of $TZ|_{X \sqcup X}$ into Z is $o_{Z_0} + o_{Z_0} = 0 \in H^5(Z, \partial Z; \mathbb{Z}/2)$. So we can take η_Z^1, η_Z^2 satisfying the above conditions.

Let T^vZ be the normal bundle of $\langle \eta_Z^1, \eta_Z^2\rangle$ in TZ. Then T^vZ is a rank 3 sub-bundle of TZ satisfying $T^vZ|_X = T^vX$. Let $\widetilde{\beta}_i \in \Gamma T^vZ$ be a vector field transverse to the zero section in T^vZ satisfying $T^vZ|_X = \beta_i$. Then $c_0(\widetilde{\beta}_i) = \overline{\left\{ \frac{\widetilde{\beta}_i(x)}{\|\widetilde{\beta}_i(x)\|}, \frac{-\widetilde{\beta}_i(x)}{\|\widetilde{\beta}_i(x)\|} \in S(T^vZ)_x \;\middle|\; x \in Z \setminus \widetilde{\beta}_i^{-1}(0) \right\}}^{\text{closure}} \subset ST^vZ$ is a submanifold of ST^vZ satisfying $\partial c_0(\widetilde{\beta}_i) = c_0(\beta_i)$. Let $W(\widetilde{\beta}_i)$ be a closed 2-form on ST^vZ that represents the Poincaré dual of $[c_0(\widetilde{\beta}_i), \partial c_0(\widetilde{\beta}_i)]$ and satisfying $W(\widetilde{\beta}_i)|_{ST^vX} = W_i$. By Stokes' theorem, we have

$$
\begin{aligned}
0 &= \sum_{\Gamma \in \mathcal{E}_n} \int_{S_{2n}(T^vZ)} d\left(\bigwedge_i \phi_i(\Gamma)^* W(\widetilde{\beta}_i) \right) [\Gamma] \\
&= 2 \sum_{\Gamma \in \mathcal{E}_n} \int_{S_{2n}(T^vX)} \bigwedge_i \phi_i(\Gamma)^* W_i [\Gamma] + \sum_{\Gamma \in \mathcal{E}_n} \int_{\partial S_{2n}(T^vZ)} \bigwedge_i \phi_i(\Gamma)^* W(\widetilde{\beta}_i)[\Gamma] \\
&= 2 \sum_{\Gamma \in \mathcal{E}_n} \int_{S_{2n}(T^vX)} \bigwedge_i \phi_i(\Gamma)^* W_i [\Gamma] + \sum_{\Gamma \in \mathcal{E}_n} \sum_{2 \le \sharp B < 2n} \int_{f(B)(T^vZ)} \bigwedge_i \phi_i(\Gamma)^* W(\widetilde{\beta}_i)[\Gamma] \\
&= 2 \sum_{\Gamma \in \mathcal{E}_n} \int_{S_{2n}(T^vX)} \bigwedge_i \phi_i(\Gamma)^* W_i [\Gamma].
\end{aligned}
$$

The last equality is given by Lemma 4.20. □

Let τ_Y be a framing of $T(Y \setminus \infty)$ satisfying $\tau_Y|_{N(\infty;Y)\setminus\infty} = \tau_{\mathbb{R}^3}|_{N(\infty;S^3)\setminus\infty}$. Then $\tau_Y^*\vec{a} = (\tau_Y^* a_1, \ldots, \tau_Y^* a_{3n})$ is a family of admissible vector fields. Let $\tau_Y' = \tau_Y|_{Y \setminus N(\infty;Y)} \cup \tau_{S^3}|_{N(\infty;S^3)}$. So τ_Y' is a framing of TY. Take $W(\tau_Y^* a_i)|_{STY} = \frac{1}{2}(\tau_Y')^* \omega_{S^2}^{a_i}$.

Lemma 4.11. *$\int_{S_{2n}(T^vX)} \bigwedge_i \phi_i(\Gamma)^* W(\tau_Y^* a_i)$ is independent of the choice of $a_1, \ldots, a_{3n}$.*

Proof. Let a_i' be an alternative choice of a_i for any i. Let $\widetilde{\omega}_{S^2}^i$ be a closed 2-form on $S^2 \times [0,1]$ satisfying $\widetilde{\omega}_{S^2}^i|_{S^2 \times \{0\}} = \omega_{S^2}^{a_i}$ and $\widetilde{\omega}_{S^2}^i|_{S^2 \times \{1\}} = \omega_{S^2}^{a_i'}$. Let $STY \times [0,1] \subset ST^vX$ be the collar of STY such that $STY \times \{0\} = \partial ST^vX$. We take $W(\tau_Y^* a_i)|_{STY \times [0,1]} = \frac{1}{2}(\tau_Y')^* \widetilde{\omega}_{S^2}^i$. Thus $W(\tau_Y^* a_i)|_{STY \times \{1\}} = W(\tau_Y^* a_i')$. Since Lemma 4.20 (1), we have

$$
\begin{aligned}
&\int_{S_{2n}(T^vX)} \bigwedge_i \phi_i(\Gamma)^* W(\tau_Y^* a_i) - \int_{S_{2n}(T^vX)} \bigwedge_i \phi_i(\Gamma)^* W(\tau_Y^* a_i') \\
&= \int_{S_{2n}(TY) \times [0,1]} \bigwedge_i \phi_i(\Gamma)^* W(\tau_Y^* a_i) \\
&= \frac{1}{2^{3n}} \int_{S_{2n}(TY) \times [0,1]} \bigwedge_i \phi_i(\Gamma)^* (\tau_Y' \times \mathrm{id})^* \widetilde{\omega}_{S^2}^i
\end{aligned}
$$

The map $S_{2n}(TY) \times [0,1] \xrightarrow{\prod_i \phi_i(\Gamma)} (STY \times [0,1])^{3n} \xrightarrow{(\tau_Y' \times \mathrm{id})^{3n}} (S^2 \times [0,1])^{3n}$ factors through $S_{2n}(\mathbb{R}^3) \times [0,1]$. Hence we have $((\tau_Y' \times \mathrm{id})^{3n} \circ \prod_i \phi_i(\Gamma))^* (\bigwedge_i \widetilde{\omega}_{S^2}^i) \in \mathrm{Im}(\Omega^{6n}(S_{2n}(\mathbb{R}^3) \times [0,1]) \to \Omega^{6n}(S_{2n}(TY) \times [0,1]))$. Since $\dim S_{2n}(\mathbb{R}^3) \times [0,1] = 6n - 3 < 6n = \dim \bigwedge_i \phi_i(\Gamma)^* (\tau_Y' \times \mathrm{id})^* \widetilde{\omega}_{S^2}^i$, we have $\int_{S_{2n}(TY) \times [0,1]} \bigwedge_i \phi_i(\Gamma)^* (\tau_Y')^* \widetilde{\omega}_{S^2}^i = 0$. □

Because of the above two lemmas, $-\mu_n \mathrm{Sign} X + \sum_{\Gamma \in \mathcal{E}_n} \int_{S_{2n}(T^v X)} \bigwedge_i \phi_i(\Gamma)^* W(\tau_{\mathbb{R}^3}^* a_i)[\Gamma]$ is independent of the choice of a 4-manifold X bounded by S^3 and a family $a_1, \ldots, a_{3n}$. We define

$$c_n = -\mu_n \mathrm{Sign} X + \sum_{\Gamma \in \mathcal{E}_n} \int_{S_{2n}(T^v X)} \bigwedge_i \phi_i(\Gamma)^* W(\tau_{\mathbb{R}^3}^* a_i)[\Gamma] \in \mathcal{A}_n(\emptyset).$$

Definition 4.12.

$$\widetilde{z}_n^{\text{anomaly}}(\vec{\gamma}) = -\mu_n \mathrm{Sign} X + \sum_{\Gamma \in \mathcal{E}_n} \int_{S_{2n}(T^v X)} \bigwedge_i \phi_i(\Gamma)^* W(\gamma_i)[\Gamma] - c_n \in \mathcal{A}_n(\emptyset).$$

Remark 4.13. We will show that $\mu_n = \frac{3}{2} c_n$ in Lemma 7.10 and Lemma 7.11. We can show that $\mu_1 = 72[\theta] \in \mathbb{Q}[\theta] = \mathcal{A}_1(\emptyset)$ by explicit computation (cf. the proof of Proposition A.1).

4.5 Definition of the invariant.

Theorem 4.14.

$$\widetilde{z}_n(Y) = \widetilde{z}_n(Y; \vec{\gamma}) - \widetilde{z}_n^{\text{anomaly}}(\vec{\gamma}) \in \mathcal{A}_n(\emptyset)$$

does not depend on the choice of $\vec{\gamma}$. Thus $\widetilde{z}_n(Y)$ is a topological invariant of Y.

Definition 4.15.

$$\widetilde{z}_n(Y) = \widetilde{z}_n(Y; \vec{\gamma}) - \widetilde{z}_n^{\text{anomaly}}(\vec{\gamma}) \in \mathcal{A}_n(\emptyset).$$

4.6 Well-definedness of $\widetilde{z}_n(Y)$ (proof of Theorem 4.14).

In this section we give the proof of well-definedness of $\widetilde{z}_n(Y)$, i.e., Theorem 4.14. The proof of well-definedness of $\widetilde{z}_n$ is almost parallel to that of z_n^{KKT} by Lescop [18].

Fix $i \in \{1, \ldots, 3n\}$. For any $j \in \{1, \ldots, 3n\}$, let a'_j, γ'_j, β'_j, $\omega(\gamma'_j)$ and $W(\gamma'_j)$ be alternative choices of a_j, γ_j, β_j, $\omega(\gamma_j)$ and $W(\gamma_j)$ respectively. Here $a'_j = a_j, \gamma'_j = \gamma_j$, $\omega(\gamma'_j) = \omega(\gamma_j)$, $\beta'_j = \beta_j$ and $W(\omega'_j) = W(\omega_j)$ for $j \neq i$. By the same argument as that of Proposition 2.15 in [18], we have the following lemma.

Lemma 4.16. *There exist a one-form $\eta_{S^2} \in \Omega^1(S^2)$ such that $d\eta_{S^2} = \omega_{S^2}^{a'_j} - \omega_{S^2}^{a_j}$, and a one-form $\eta \in \Omega^1(C_2(Y))$ such that*

- $d\eta = \omega(\gamma'_i) - \omega(\gamma_i)$,
- $\eta|_{\partial C_2(Y) \setminus S\nu_{\Delta(Y \setminus N(\infty; Y))}} = p_Y^* \eta_{S^2}$.

Similarly, the following lemma holds.

Lemma 4.17. *There exists a one-form $\eta_X \in \Omega^1(ST^v X)$ such that*

- $d\eta_X = W(\gamma'_i) - W(\gamma_i)$,
- $\eta_X|_{ST(Y \setminus N(\infty; Y))} = \eta|_{S\nu_{\Delta(Y \setminus N(\infty; Y))}}$,

- $\eta_X|_{ST^vX|_{N(\infty;Y)}} = \tau_{S^3}^*\eta_{S^2}$.

Proof. Set $\eta_X^0 = \eta|_{ST(Y\setminus N(\infty;Y))} \cup \tau_{S^3}^*\eta_{S^2}$. By the construction of $c_0(\beta_i), c_0(\beta_i')$, we have $[W(\gamma_i)] = [W(\gamma_i')] \in H^2(ST^vX)$ (cf. Lemma A.2). Thus there is a one-form $\eta_X^1 \in \Omega^1(ST^vX)$ such that $d\eta_X^1 = W(\gamma_i) - W(\gamma_i')$. Since $H^1(STY) = 0$, there is a function $\mu_X \in \Omega^0(STY)$ such that $d\mu_X = \eta_X^1|_{STY} - \eta_X^0$. Let $h : ST^vX \to \mathbb{R}$ be a C^∞ function such that $h \equiv 1$ near $STY(= \partial ST^vX)$ and $h \equiv 0$ far from STY. We can take $\eta_X = \eta_X^1 - d(h\mu_X)$ using collar of STY in ST^vX. $\square$

Set

$$\widetilde{\omega}_j = \begin{cases} \omega(\gamma_j)(= \omega(\gamma_j')) & j \neq i, \\ \eta & j = i. \end{cases}$$

Set

$$\widetilde{W}_j = \begin{cases} W(\gamma_j)(= W(\gamma_j')) & j \neq i, \\ \eta_X & j = i. \end{cases}$$

By Stokes' theorem,

$$\begin{aligned} &\int_{C_{2n}(Y)} \bigwedge_j P_j(\Gamma)^*\omega(\gamma_j) - \int_{C_{2n}(Y)} \bigwedge_j P_j(\Gamma)^*\omega(\gamma_j') \\ =\ & \int_{\partial C_{2n}(Y)} \bigwedge_j P_j(\Gamma)^*\widetilde{\omega}_j \\ =\ & \sum_{F\subset \partial C_{2n}(Y):\text{face}} \int_F \bigwedge_j P_j(\Gamma)^*\widetilde{\omega}_j. \end{aligned}$$

Lemma 4.18 (Lescop [18, Lemma 2.17])**.** *For any non-empty subset B of $2n = \{1,\dots,2n\}$, for any $\Gamma \in \mathcal{E}_n$,*

$$\int_{F(\infty;B)} \bigwedge_j P_j(\Gamma)^*\widetilde{\omega}_j = 0.$$

Lemma 4.19 (Lescop [18, Lemmas 2.18, 2.19, 2.20, and 2.21])**.** *For any $B \subset \{1,\dots,2n\}$ with $\sharp B \geq 2$ and $B \neq \{1,\dots 2n\}$*

$$\sum_{\Gamma\in\mathcal{E}_n} \left(\int_{F(B)} \bigwedge_j P_j(\Gamma)^*\widetilde{\omega}_j \right) [\Gamma] = 0.$$

The following lemma is proved as Lemma 2.18, 2.19, 2.20, and 2.21 in [18] (See also the proof of Proposition 2.10 in [18]).

Lemma 4.20. *For any $B \subset \{1,\dots,2n\}$ with $2 \leq \sharp B < 2n$,*

(1) $\sum_{\Gamma\in\mathcal{E}_n} \int_{f(B)(T^vX)} \bigwedge_j \phi_j(\Gamma)^*\widetilde{W}_j[\Gamma] = 0$,

(2) $\sum_{\Gamma\in\mathcal{E}_n} \int_{f(B)(T^vZ)} \bigwedge_j \phi_j(\Gamma)^*W(\widetilde{\beta}_j)[\Gamma] = 0$ *(See the proof of Lemma 4.10 for the notation $Z, W(\widetilde{\beta}_j)$).*

By Lemma 4.18 and Lemma 4.19,

$$\begin{aligned}
&\widetilde{z}_n(Y;\vec{\gamma})-\widetilde{z}_n(Y;\vec{\gamma}')\\
=&\sum_{\Gamma\in\mathcal{E}_n}\left(\int_{C_{2n}(Y)}\bigwedge_j P_j(\Gamma)^*\omega(\gamma_j)\right)[\Gamma]-\sum_{\Gamma\in\mathcal{E}_n}\left(\int_{C_{2n}(Y)}\bigwedge_j P_j(\Gamma)^*\omega(\gamma_j')\right)[\Gamma]\\
=&\sum_{\Gamma\in\mathcal{E}_n}\left(\int_{F(2n)}\bigwedge_j P_j(\Gamma)^*\widetilde{\omega}_j\right)[\Gamma].
\end{aligned}$$

Since $F(2n)=\breve{S}(T(Y\setminus\infty))$, the restriction of $P_j(\Gamma)$ to $F(2n)$ coincides with $\phi_j^0(\Gamma):\breve{S}_{2n}(T(Y\setminus\infty))\to S\nu_{\Delta(Y\setminus\infty)}\subset\partial C_2(Y)$. Therefore

$$\begin{aligned}
&\sum_{\Gamma\in\mathcal{E}_n}\int_{F(2n)}\bigwedge_j P_j(\Gamma)^*\widetilde{\omega}_j[\Gamma]\\
=&\sum_{\Gamma\in\mathcal{E}_n}\int_{\breve{S}_{2n}(T(Y\setminus\infty))}\bigwedge_j \phi_j^0(\Gamma)^*\widetilde{\omega}_j[\Gamma]\\
=&\sum_{\Gamma\in\mathcal{E}_n}\int_{\breve{S}_{2n}(T(Y\setminus N(\infty;Y)))}\bigwedge_j \phi_j^0(\Gamma)^*\widetilde{\omega}_j[\Gamma]+\sum_{\Gamma\in\mathcal{E}_n}\int_{\breve{S}_{2n}(T(N(\infty;Y)\setminus\infty))}\bigwedge_j \phi_j^0(\Gamma)^*\widetilde{\omega}_j[\Gamma]\\
=&\sum_{\Gamma\in\mathcal{E}_n}\int_{\breve{S}_{2n}(T(Y\setminus N(\infty;Y)))}\bigwedge_j \phi_j^0(\Gamma)^*\widetilde{\omega}_j[\Gamma].
\end{aligned}$$

The last equality comes from the following lemma.

Lemma 4.21. $\sum_{\Gamma\in\mathcal{E}_n}\int_{\breve{S}_{2n}(T(N(\infty;Y)\setminus\infty))}\bigwedge_j\phi_j^0(\Gamma)^*\widetilde{\omega}_j[\Gamma]=0.$

Proof. Since $\breve{S}_{2n}(T(N(\infty;Y)\setminus\infty))=(N(\infty;Y)\setminus\infty)\times\breve{S}_{2n}(\mathbb{R}^3)$ and $\widetilde{\omega}_j|_{ST(N(\infty;Y)\setminus\infty)}=\tau_{S^3}^*\omega_{S^2}$ (or $\tau_{S^3}^*\eta_{S^2}$), the form $\bigwedge_j\phi_j^0(\Gamma)^*\widetilde{\omega}_j|_{\breve{S}_{2n}(T(N(\infty;Y)\setminus\infty))}$ is in the image of the map $(\tau_{S^3})^{3n}\circ\prod_j\phi_j^0(\Gamma)$. The map $(\tau_{S^3})^{3n}\circ\prod_j\phi_j^0(\Gamma)|_{\breve{S}_{2n}(T(N(\infty;Y)\setminus\infty))}:\breve{S}_{2n}(T(N(\infty;Y)\setminus\infty))\to(ST(N(\infty;Y)\setminus\infty))^{3n}\to(S^2)^{3n}$ factors through $\breve{S}_{2n}(\mathbb{R}^3)$. Since $\dim\breve{S}_{2n}(\mathbb{R}^3)=6n-4<6n-1=\dim\bigwedge_j\phi_j^0(\Gamma)^*\widetilde{\omega}_j$, we have $\sum_{\Gamma\in\mathcal{E}_n}\int_{\breve{S}_{2n}(T^vY|_{N(\infty;Y)})}\bigwedge_j\phi_j^0(\Gamma)^*\widetilde{\omega}_j[\Gamma]=0$. ∎

On the other hand, by Stokes' theorem,

$$\begin{aligned}
&\widetilde{z}_n^{\text{anomaly}}(\vec{\gamma})-\widetilde{z}_n^{\text{anomaly}}(\vec{\gamma}')\\
=&\sum_{\Gamma\in\mathcal{E}_n}\int_{S_{2n}(T^vY)}\bigwedge_j\phi_j(\Gamma)^*\widetilde{W}_j[\Gamma]+\sum_{\Gamma\in\mathcal{E}_n}\int_{\partial S_{2n}(T^vX)}\bigwedge_j\phi_j(\Gamma)^*\widetilde{W}_j[\Gamma]\\
\overset{(*)}{=}&\sum_{\Gamma\in\mathcal{E}_n}\int_{S_{2n}(T^vY)}\bigwedge_j\phi_j(\Gamma)^*\widetilde{W}_j[\Gamma]\\
=&\sum_{\Gamma\in\mathcal{E}_n}\int_{S_{2n}(T(Y\setminus N(\infty;Y))}\bigwedge_j\phi_j(\Gamma)^*\widetilde{W}_j[\Gamma]+\sum_{\Gamma\in\mathcal{E}_n}\int_{S_{2n}(T^vY|_{N(\infty;Y)})}\bigwedge_j\phi_j(\Gamma)^*\widetilde{W}_j[\Gamma]\\
=&\sum_{\Gamma\in\mathcal{E}_n}\int_{S_{2n}(T(Y\setminus N(\infty;Y))}\bigwedge_j\phi_j(\Gamma)^*\widetilde{W}_j[\Gamma].
\end{aligned}$$

The equality $(*)$ is given by Lemma 4.20(1) and the last equality comes from the following lemma.

Lemma 4.22. $\sum_{\Gamma\in\mathcal{E}_n}\int_{S_{2n}(T^vY|_{N(\infty;Y)})}\bigwedge_j \phi_j(\Gamma)^*\widetilde{W}(\gamma'_j)[\Gamma]=0.$

The proof of this lemma is parallel to the proof of Lemma 4.21.

Since $\widetilde{W}_j|_{S\nu_{\Delta(Y\setminus N(\infty;Y))}}=\widetilde{\omega}_j|_{S\nu_{\Delta(Y\setminus N(\infty;Y))}}$ for any j, we have

$$\begin{aligned}\widetilde{z}_n(Y;\vec{\gamma})-\widetilde{z}_n(Y;\vec{\gamma}')&=\sum_{\Gamma\in\mathcal{E}_n}\int_{\check{S}_{2n}(T(Y\setminus N(\infty;Y)))}\bigwedge_j \phi^0_j(\Gamma)^*\widetilde{\omega}_j[\Gamma]\\&=\sum_{\Gamma\in\mathcal{E}_n}\int_{S_{2n}(T(Y\setminus N(\infty;Y)))}\bigwedge_j \phi_j(\Gamma)^*\widetilde{W}_j[\Gamma]=\widetilde{z}_n^{\mathrm{anomaly}}(\vec{\gamma})-\widetilde{z}_n^{\mathrm{anomaly}}(\vec{\gamma}').\end{aligned}$$

Now we finish the proof of Theorem 4.14.

5 Review of the Kontsevich-Kuperberg-Thurston invariant z_n^{KKT}.

In this section, we review the construction of z_n^{KKT} for rational homology 3-spheres. This section is based on [18].

Let $\tau_Y: T(Y\setminus\infty)\xrightarrow{\cong}\underline{\mathbb{R}^3}$ be a framing satisfying $\tau_Y|_{N(\infty;Y)\setminus\infty}=\tau_{\mathbb{R}^3}$. $\tau_Y|_{Y\setminus N(\infty;Y)}\cup\tau_{S^3}|_{N(\infty;S^3)}$ is a framing of TY by the assumption of τ_Y. We define

$$\begin{aligned}\sigma_{Y\setminus\infty}(\tau_Y)&=\sigma_Y(\tau_Y|_{Y\setminus N(\infty;Y)}\cup\tau_{S^3}|_{N(\infty;S^3)})-\sigma_{S^3}(\tau_{S^3})\\&=\sigma_Y(\tau_Y|_{Y\setminus N(\infty;Y)}\cup\tau_{S^3}|_{N(\infty;S^3)})-2\end{aligned}$$

and call it the *signature defect* of τ_Y of a framing of $Y\setminus\infty$. For example $\sigma_{\mathbb{R}^3}(\tau_{\mathbb{R}^3})=0$.

The canonical isomorphism $S\nu_{\Delta(Y\setminus\infty)}\cong ST(Y\setminus\infty)$ and the framing τ_Y induce the map $p_\Delta(\tau_Y): S\nu_{\Delta(Y\setminus\infty)}\to S^2$. By the assumption of τ_Y, maps $p_\Delta(\tau_Y)$ and $p_Y:\partial C_2(Y)\setminus S\nu\to S^2$ are compatible. So we get the map $p(\tau_Y)=p_Y\cup p_\Delta(\tau_Y):\partial C_2(Y)\to S^2$. Let $\omega_{S^2}\in\Omega^2(S^2)$ be an anti-symmetric 2-form satisfying $\int_{S^2}\omega_{S^2}=1$. Let $\omega(\tau_Y)$ be an anti-symmetric closed 2-from on $C_2(Y)$ satisfying $\omega(\tau_Y)|_{\partial C_2(Y)}=p(\tau_Y)^*\omega_{S^2}\in\Omega^2(\partial C_2(Y))$.

Proposition 5.1 (Lescop [18, Theorem 1.9 and Proposition 2.11])**.** *There exists a constant $\delta_n\in\mathcal{A}_n(\emptyset)$ such that*

$$\sum_{\Gamma\in\mathcal{E}_n}\int_{C_{2n}(Y)}\left(\bigwedge_i P_i(\Gamma)^*\omega(\tau_Y)\right)[\Gamma]-\frac{\sigma_{Y\setminus\infty}(\tau_Y)}{4}\delta_n\in\mathcal{A}_n(\emptyset)$$

does not depend on the choice of τ_Y.

Definition 5.2 (Kuperberg and Thurston [16], Lescop [18])**.**

$$z_n^{\mathrm{KKT}}(Y;\tau_Y)=\sum_{\Gamma\in\mathcal{E}_n}\int_{C_{2n}(Y)}\left(\bigwedge_i P_i(\Gamma)^*\omega(\tau_Y)\right)[\Gamma],$$

$$z_n^{\mathrm{KKT}}(Y) = z_n^{\mathrm{KKT}}(Y;\tau_Y) - \frac{\sigma_{Y\setminus\infty}(\tau_Y)}{4}\delta_n \in \mathcal{A}_n(\emptyset).$$

We remark that δ_n is given by the explicit formula in Proposition 2.10 in [18].

Remark 5.3. The universal finite type invariant Z_n^{KKT} described in [18] equals to the degree n part of $\exp(\sum_n \frac{1}{2^{3n}(3n)!(2n)!} z_n^{\mathrm{KKT}})$. See before Lemma 2.12 in [18] for more details.

Remark 5.4. We will show that $\delta_n = \frac{4}{3}\mu_n$ in Lemma 7.10.

6 Review of Watanabe's Morse homotopy invariant z_n^{FW}.

In this section we give a modified construction of Watanabe's Morse homotopy invariant $z_{2n,3n}^{\mathrm{FW}}$ [34] for rational homology 3-spheres. We will remark the differences between our modified construction and Watanabe's original construction after the definition of $z_{2n,3n}^{\mathrm{FW}}(Y)$. The invariant $z_{2n,3n}^{\mathrm{FW}}(Y)$ is a sum of the principal term $z_{2n,3n}^{\mathrm{FW}}(Y;\vec{f})$ and the anomaly term $z_{2n,3n}^{\mathrm{anomaly}}(\vec{f})$ of $\vec{f}$ where $\vec{f} = (f_1, f_2, \dots, f_{3n})$ is a family of Morse functions on $Y \setminus \infty$.

Fix a point $a \in S^2$.

Definition 6.1. A Morse function $f : Y \setminus \infty \to \mathbb{R}$ is an *admissible Morse function with respect to a* if it satisfies the following conditions.

- $f|_{N(\infty;Y)\setminus\infty} = q_a|_{N(\infty;S^3)\setminus\infty}$ and
- f has no critical point of index 0 or 3.

Let $\mathrm{Crit}(f) = \{p_1, \dots, p_k, q_1, \dots, q_k\}$ be the set of critical points of f where $\mathrm{ind}(p_i) = 2, \mathrm{ind}(q_i) = 1$. Let

$$0 \to C_2(Y \setminus \infty; f) \xrightarrow{\partial} C_1(Y \setminus \infty; f) \to 0$$

be the Morse complex of f with rational coefficients. Let $g : C_1(Y \setminus \infty; f) \to C_2(Y\setminus\infty; f), g([q_i]) = \sum_j g_{ij}[p_j]$ be the inverse map of the boundary map $\partial : C_2(Y\setminus\infty; f) \to C_1(Y \setminus \infty; f), \partial[p_i] = \sum_j \partial_{ij}[q_j]$. ($g$ is called a combinatorial propagator in [34].)

We now construct $\mathcal{M}(f)$ which is the weighted sum of (non-compact) 4-manifolds in $Y^2 \setminus \Delta$. Let $M_{\to}(f) = \mathrm{pr}(\varphi^{-1}(\Delta))$ where $\varphi : Y \times Y \times (0,\infty) \to Y \times Y$ is the map defined by $(x, y, t) \mapsto (y, \Phi_f^t(x))$ and $\mathrm{pr} : Y \times Y \times (0,\infty) \to Y \times Y$ is the projection. We choose the orientation of $M_{\to}(f)$ such that the inclusion $Y \times (0,\varepsilon) \hookrightarrow M_{\to}(f), (x,t) \mapsto (x, \Phi_f^t(x))$ preserves orientations. We define

$$\mathcal{M}(f) = M_{\to}(f) - \sum_{i,j} g_{ij}(\mathcal{A}_{q_i} \times \mathcal{D}_{p_j}) \setminus \Delta.$$

We remark that the orientation of $\mathcal{M}(f)$ does not depend on the choice of orientations of $\mathcal{A}_{q_i}, \mathcal{D}_{p_j}$.

Let $a_1, \ldots, a_{3n} \in S^2 \subset \mathbb{R}^3$ be points such that any distinct three points of them are linearly independent in $\mathbb{R}^3$. Let $f_i : Y \backslash \infty \to \mathbb{R}$ be a sufficiently generic admissible Morse function with respect to a_i for each $i = 1, \ldots, 3n$. We write $\vec{f} = (f_1, \ldots, f_{3n})$ to simplify notation. We replace a metric of Y such that the Morse-Smale condition holds for each f_i if necessary.

Set $\mathcal{M}(\pm f_i) = \mathcal{M}(f_i) + \mathcal{M}(-f_i)$.

Definition 6.2. For a generic $\vec{f}$,

$$z^{\mathrm{FW}}_{2n,3n}(Y; \vec{f}) = \sum_{\Gamma \in \mathcal{E}_n} \frac{1}{2^{3n}} \sharp \left(\bigcap_{i=1}^{3n} P_i(\Gamma)|^{-1}_{(Y \backslash \infty)^{2n} \backslash \Delta}(\mathcal{M}(\pm f_i)) \right) [\Gamma] \in \mathcal{A}_n(\emptyset).$$

We next define the anomaly part. Set grad $\vec{f} = (\mathrm{grad}\, f_1, \ldots, \mathrm{grad}\, f_{3n})$.

Definition 6.3.

$$z^{\mathrm{anomaly}}_{2n,3n}(\vec{f}) = z^{\mathrm{anomaly}}_n(\mathrm{grad}\, \vec{f}).$$

Definition 6.4 (Watanabe [34])**.**

$$z^{\mathrm{FW}}_{2n,3n}(Y) = z^{\mathrm{FW}}_{2n,3n}(Y; \vec{f}) - z^{\mathrm{anomaly}}_{2n,3n}(\vec{f}).$$

Remark 6.5. A difference between our modified construction of $z^{\mathrm{FW}}_{2n,3n}$ and Watanabe's original construction in [34] is the conditions for Morse functions. Our Morse function is on $Y \setminus \infty$ and explicitly written on $N(\infty; Y) \setminus \infty$. On the other hand, Watanabe uses any Morse functions on Y. We note that $Y \setminus \infty \subset Y \sharp S^3$ where $Y \sharp S^3$ is the connected sum of Y and S^3 at $\infty \in Y$ and $0 \in S^3$. Then it is possible to extend $f : Y \setminus \infty \to \mathbb{R}$ to $Y \sharp S^3 \cong Y$ in a standard way. Then we can show that

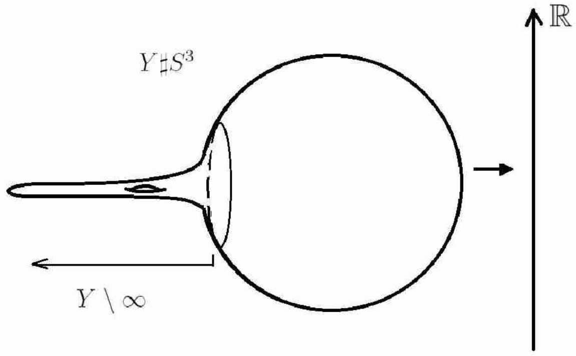


Figure 2: The extension of f to $Y \sharp S^3$

the difference between the value $z^{\mathrm{FW}}_{2n,3n}(Y)$ described in this section and the value of Watanabe's original invariant of Y is a constant which is independent of Y.

We must prove that $\sharp \left(\bigcap_i P_i(\Gamma)|^{-1}_{(Y \backslash \infty)^{2n} \backslash \Delta}(\mathcal{M}(\pm f_i)) \right)$ is well-defined for generic $\vec{f}$, because Morse functions used in the above definition differ from Morse functions used in the original definition in [34] near $N(\infty;Y)\setminus\infty$ (See Remark 6.5 for more details).

Lemma 6.6. *$P_1(\Gamma)|^{-1}_{(Y\setminus\infty)^{2n}\setminus\Delta}(\mathcal{M}(\pm f_i))$, $\dots, P_{3n}(\Gamma)|^{-1}_{(Y\setminus\infty)^{2n}\setminus\Delta}(\mathcal{M}(\pm f_i))$ transversally intersect at finitely many points, for generic $f_1,\dots,f_{3n}$ and $a_1,\dots,a_{3n}$, for any $\Gamma\in\mathcal{E}_n$.*

Proof. Let $x=(x_1,\dots,x_{2n})\in\bigcap_i P_i(\Gamma)|^{-1}_{(Y\setminus\infty)^{2n}\setminus\Delta}(\mathcal{M}(\pm f_i))\subset(Y\setminus\infty)^{2n}\setminus\Delta$.

<u>The case of $x\in(Y\setminus N(\infty;Y))^{2n}$.</u>

Thanks to §2.4 of [34], the transversality at x is given by generic $\vec{f}$.

<u>The case of $x\notin(Y\setminus N(\infty;Y))^{2n}$.</u>

We show that for generic $a_1,\dots,a_{3n}$, there are no such x. (Then, in particular, $\bigcap_i P_i(\Gamma)|^{-1}_{(Y\setminus\infty)^{2n}\setminus\Delta}(\mathcal{M}(\pm f_i))$ is a 0-dimensional compact manifold). Let $B=\{i\in\{1,\dots,2n\}\mid x_i\in Y\setminus N(\infty;Y)\}$. Let

$$E_B=\{i\in\{1,\dots,3n\}\cong E(\Gamma)\mid\{s(i),t(i)\}\subset B\},$$

$$E_B^\partial=\{i\in\{1,\dots,3n\}\cong E(\Gamma)\mid\{s(i),t(i)\}\cap B\neq\emptyset\}\setminus E_B.$$

Let Γ/B be the labeled graph obtained from Γ by collapsing B to a point b_0 and removing all edges in E_B. Here the labels of edges and vertexes of Γ/B are taken from $\{1,\dots,3n\}\setminus E_B$, $\{0,1,\dots,2n\}\setminus B$ respectively (the label of b_0 is 0). Note that $\sharp(V(\Gamma/B)-\{b_0\})=2n-\sharp B$ and $\sharp E(\Gamma/B)\geq 3n-\frac{3\sharp B}{2}$.

Let $\pi:Y\setminus\infty\to Y/(Y\setminus N(\infty;Y))\overset{\tau_\infty}{\cong}\mathbb{R}^3$ be the map obtained by collapsing $Y\setminus N(\infty;Y)$ to the point $0\in\mathbb{R}^3$. Let $\pi'_i:\mathbb{R}\to\mathbb{R}$ be the map obtained by collapsing $\mathrm{Im}(f_i:Y\setminus N(\infty;Y)\to\mathbb{R})$ to $0\in\mathbb{R}$. Then $\pi'_i\circ f_i=q_{a_i}\circ\pi:Y\setminus\infty\to\mathbb{R}$. Let $x':V(\Gamma/B)-\{b_0\}\hookrightarrow\mathbb{R}^3$ be the restriction of $\pi\circ x:V(\Gamma)\hookrightarrow\mathbb{R}^3$ to $V(\Gamma/B)-\{b_0\}\subset V(\Gamma)$. Let $a'\in(S^2)^{E(\Gamma/B)}$ be the points obtained from $a=(a_1,\dots,a_{3n})$ removing all a_i, $i\in E_B$. We define the map

$$\varphi:(\mathbb{R}^3)^{V(\Gamma/B)-\{b_0\}}\setminus\Delta\to(S^2)^{E(\Gamma/B)}$$

as

$$\varphi(y)=\left(\frac{y_{s(i)}-y_{t(i)}}{\|y_{s(i)}-y_{t(i)}\|}\right)_{i\in E(\Gamma/B)}.$$

Here if $i\in E_B^\partial$ then either $s(i)$ or $t(i)$ is 0. Then $x'\in\varphi^{-1}(a')$. By the following lemma, there is no x' for a generic a'. Therefore there is no x for a generic a.

Lemma 6.7. *For a generic a' we have $\varphi^{-1}(a')=\emptyset$.*

Proof. For any $y\in\varphi^{-1}(a')$ and for any $t\in(0,\infty)$, we have $ty\in\varphi^{-1}(a')$. Thus if $\varphi^{-1}(a')\neq\emptyset$, we have $\dim\varphi^{-1}(a')\geq 1$. On the other hand, $\dim((\mathbb{R}^3)^{V(\Gamma/B)-\{b_0\}})=6n-3\sharp B\leq 2\sharp E(\Gamma/B)=\dim((S^2)^{E(\Gamma/B)})$. Hence we have $\dim\varphi^{-1}(a')\leq 0$ for a generic a'. This is a contradiction. □

□

7 Proof of Theorem 8 and Theorem 9.

In this section we show that the construction of $\widetilde{z}$ is a generalization of that of z^{KKT} and that of z^{FW}. (Theorem 8 (Theorem 7.1) and Theorem 9 (Theorem 7.2)). As a corollary, we have $z_n^{\mathrm{KKT}} = z_n^{\mathrm{FW}}$.

Theorem 7.1 (Theorem 8 in the introduction). *$z_n^{\mathrm{KKT}}(Y) = \widetilde{z}_n(Y)$ for any rational homology 3-sphere Y, for any $n \in \mathbb{N}$.*

Theorem 7.2 (Theorem 9 in the introduction). *$z_n^{\mathrm{FW}}(Y) = \widetilde{z}_n(Y)$ for any rational homology 3-sphere Y, for any $n \in \mathbb{N}$.*

Corollary 7.3 (Corollary10 in the introduction). *$z_n^{\mathrm{FW}}(Y) = z_n^{\mathrm{KKT}}(Y)$ for any rational homology 3-sphere Y, for any $n \in \mathbb{N}$.*

A cobordism argument essentially used in the proof of Theorem 2 (see Remark 7.13) gives us the equalities $\mu_n = \frac{3}{4}\delta_n$ (Lemma 7.10) , $c_n = \frac{1}{2}\delta_n$ (Lemma 7.11).

7.1 Proof of $\widetilde{z}_n(Y) = z_n^{\mathrm{KKT}}(Y)$.

We follow the notations used in Section 5. For example, Y is a rational homology 3-sphere and $\infty \in Y$ is a basepoint, and so on. Let $\tau_Y : T(Y \setminus \infty) \overset{\cong}{\to} \underline{\mathbb{R}^3}$ be a framing of $Y \setminus \infty$ satisfying $\tau_Y|_{N(\infty;Y)\setminus\infty} = \tau_{\mathbb{R}^3}|_{N(\infty;S^3)\setminus\infty}$. We denote $\tau_Y^*\vec{a} = (\tau_Y^* a, \ldots, \tau_Y^* a)$ for $a \in S^2$. We take $\omega_{S^2} = \frac{1}{2}\omega_{S^2}^a$ in the definition of $z_n^{\mathrm{KKT}}(Y;\tau_Y)$, and we take $\omega(\tau_Y^* a) = \omega(\tau_Y)$ in the definition of $\widetilde{z}_n(Y;\tau_Y^*\vec{a})$. Thus

$$\widetilde{z}_n(Y;\tau_Y^*\vec{a}) = \sum_{\Gamma\in\mathcal{E}_n}\int_{C_{2n}(Y)}\bigwedge_i P_i(\Gamma)^*\omega(\tau_Y)[\Gamma] = z_n^{\mathrm{KKT}}(Y;\tau_Y).$$

Then we only need to show that

$$\widetilde{z}_n^{\mathrm{anomaly}}(\tau_Y^*\vec{a}) = \frac{1}{4}\sigma_{Y\setminus\infty}(\tau_Y)\delta_n$$

in this condition.

The idea of the proof of $\widetilde{z}_n^{\mathrm{anomaly}}(\tau_Y^*\vec{a}) = \frac{1}{4}\sigma_{Y\setminus\infty}(\tau_Y)\delta_n$ is as follows. We first prove this equality in the case of $Y = S^3$. The well-definedness of $\widetilde{z}_n^{\mathrm{anomaly}}(Y)$ implies that $\widetilde{z}_n^{\mathrm{anomaly}}(\tau^*\vec{a}) = \frac{1}{4}\sigma_{\mathbb{R}^3}(\tau)\delta_n$ for any framing τ of $S^3 \setminus \infty$. The general case is reduced to the case of $Y = S^3$ by a cobordism argument.

We introduce notation. For a compact 4-manifold X such that $\partial X = Y$ and $\chi(X) = 0$, we denote $\widetilde{z}^{\mathrm{anomaly}}(\vec{\gamma};X) = \sum_{\Gamma\in\mathcal{E}_n}\int_{S_{2n}(T^vX)}\bigwedge_i \phi_i(\Gamma)^*W(\gamma_i)[\Gamma] = \widetilde{z}_n^{\mathrm{anomaly}}(\vec{\gamma}) + \mu_n\mathrm{Sign}X + c_n$. Then $\widetilde{z}^{\mathrm{anomaly}}(\vec{\gamma}) = \widetilde{z}^{\mathrm{anomaly}}(\vec{\gamma};X) - \mu_n\mathrm{Sign}X - c_n$ by the definition.

Lemma 7.4. $\widetilde{z}_n(S^3) = z^{\mathrm{KKT}}(S^3)$.

Proof. Let X be a compact 4-manifold with $\partial X = S^3$ and $\chi(X) = 0$.

$$\begin{aligned}\widetilde{z}_n(S^3) &= \widetilde{z}_n(S^3;\tau_{\mathbb{R}^3}^*\vec{a}) - \widetilde{z}_n^{\mathrm{anomaly}}(\tau_{\mathbb{R}^3}^*\vec{a};X) + \mu_n\mathrm{Sign}X + c_n\\ &= \widetilde{z}_n(S^3;\tau_{\mathbb{R}^3}^*\vec{a})\\ &= z_n^{\mathrm{KKT}}(S^3;\tau_{\mathbb{R}^3}).\end{aligned}$$

Since $\sigma_{\mathbb{R}^3}(\tau_{\mathbb{R}^3}) = 0$, we have $z_n^{\mathrm{KKT}}(S^3; \tau_{\mathbb{R}^3}) = z_n^{\mathrm{KKT}}(S^3)$.

Therefore $\widetilde{z}_n(S^3) = z_n^{\mathrm{KKT}}(S^3; \tau_{\mathbb{R}^3}) = z_n^{\mathrm{KKT}}(S^3)$. $\square$

Since $\widetilde{z}_n^{\mathrm{anomaly}}(S^3)$ is independent of the choice of framing on $\mathbb{R}^3 = S^3 \setminus \infty$, we have the following corollary.

Corollary 7.5. *For any framing τ on $\mathbb{R}^3 = S^3 \setminus \infty$ such that $\tau|_{N(\infty;S^3)\setminus\infty} = \tau_{\mathbb{R}^3}|_{N(\infty;S^3)\setminus\infty}$, the equality $\widetilde{z}_n^{\mathrm{anomaly}}(\tau^*\vec{a}) = \frac{1}{4}\sigma_{\mathbb{R}^3}(\tau)\delta_n$ holds.*

Recall that the framing τ_Y of $T(Y\setminus\infty)$ gives the framing $\tau_Y \cup \tau_{S^3} = \tau_Y|_{Y\setminus N(\infty;Y)} \cup \tau_{S^3}|_{N(\infty;S^3)}$ of TY and $\sigma_{Y\setminus\infty}(\tau_Y) = \sigma_Y(\tau_Y \cup \tau_{S^3}) - \sigma(\tau_{S^3}) = \sigma_Y(\tau_Y \cup \tau_{S^3}) - 2$. We give a spin structure on Y using $\tau_Y \cup \tau_{S^3}$.

Lemma 7.6. *There exists a positive integer k and a spin 4-manifold X_0 such that $\chi(X_0) = 0$ and $\partial X_0 = Y \sqcup k(-S^3)$ as spin manifolds. Here $-S^3$ is S^3 with the opposite orientation.*

Proof. Since the 3-dimensional spin cobordism group equals to zero, there exists a spin 4-manifold $\widetilde{X}$ such that $\partial\widetilde{X} = Y$. Let $k = \chi(\widetilde{X})$. We may assume that $k \geq 0$, by replacing $\widetilde{X}$ by $\widetilde{X}\sharp nK3$ for sufficiently large integer n if necessary. Let X_0 be the spin 4-manifold obtained by removing k disjoint 4-balls, i.e., $X_0 = \widetilde{X} \setminus kB^4$. Then $\chi(X_0) = 0$ and $\partial X_0 = Y \sqcup k(-S^3)$. $\square$

Remark 7.7. Since $\chi(X_0\sharp T^4) = \chi(X_0) - 2, \chi(X_0\sharp K3) = \chi(X_0) + 22$ and T^4, $K3$ are spin, it is possible to choose $k + 2n$ instead of k for any $n \in \mathbb{Z}$.

Remark 7.8. Since the Euler number of a closed spin 4-manifold is even, the number $k(Y) = k \mod 2 \in \mathbb{Z}/2$ is an invariant of a spin 3-manifold Y. It is known that $k(Y) = \mathrm{rk} H_1(Y;\mathbb{Z}/2) + 1$ (See Theorem 2.6 in [14]). We also remark that $k(Y) \equiv \sigma_{Y\setminus\infty}(\tau_Y) + 1 \mod 2$.

Let X_0 be a spin 4-manifold such that $\chi(X_0) = 0$ and $\partial X_0 = Y \sqcup k(-S^3)$ for some $k \geq 1$. We denote by S_i^3 the i-th S^3-boundary of X_0. Then $\partial X_0 = Y \sqcup -S_1^3 \sqcup \ldots \sqcup -S_k^3$. By the obstruction theory, it is possible to extend the framing $\eta_Y \oplus (\tau_Y \cup \tau_{S^3})$ of $TX_0|_Y$ to X_0 where η_Y is the outward unit vector field on $Y \subset \partial X_0$ (see [14] for more details). We choose such an extended framing $\widetilde{\tau}_X$. We may assume that $\widetilde{\tau}_X^{*\,t}(1,0,0,0)|_{k(-S^3)}$ is the inward unit vector field on $k(-S^3) \subset \partial X_0 \subset X_0$. If necessary we modify $\widetilde{\tau}_X$ by using homotopy, we may assume that there exist a framing τ_i of $S_i^3 \setminus \infty$ such that $\tau_i|_{N(\infty;S_i^3)\setminus\infty} = \tau_{\mathbb{R}^3}|_{N(\infty;S^3)\setminus\infty}$ and $-\eta_i \oplus (\tau_i \cup \tau_{S^3}) = \widetilde{\tau}_X|_{-S_i^3}$. Here $-\eta_i$ is the inward unit vector field on $-S_i^3 \subset X_0$.

Let X' be a compact oriented 4-manifold with $\chi(X') = 0$ and $\partial X' = S^3$. Then $X_0 \cup kX'$ is a compact 4-manifold with $\chi(X_0 \cup kX') = 0$ and $\partial(X_0 \cup kX') = Y$.

Lemma 7.9. *The following three equalities hold.*

(1) $\widetilde{z}_n^{\mathrm{anomaly}}(\tau_Y^*\vec{a}; X_0 \cup kX') = \sum_{i=1}^k \widetilde{z}_n^{\mathrm{anomaly}}(\tau_i^*\vec{a}; X')$.

(2) $\sigma_{Y\setminus\infty}(\tau_Y) = \sum_{i=1}^k \sigma_{\mathbb{R}^3}(\tau_i) + 2(k-1) - 3\mathrm{Sign}X_0$.

(3) $\widetilde{z}_n^{\mathrm{anomaly}}(\tau_Y^*\vec{a}) = \frac{1}{4}\sigma_{Y\setminus\infty}(\tau_Y)\delta_n + \left(\frac{3}{4}\delta_n - \mu_n\right)\mathrm{Sign}X_0 + \frac{k-1}{2}\delta_n + (k-1)c_n$.

Proof. (1) We take a 3-bundle $T^v(X_0 \sqcup kX') \subset T(X_0 \sqcup kX')$ over $X_0 \cup kX'$ such that $T^v(X_0 \sqcup kX')|_{X_0}$ is the normal bundle of $\widetilde{\tau}_X^{*\,t}(1,0,0,0)$. We denote $T^vX_0 = T^v(X_0 \sqcup kX')|_{X_0}$, $T^v(kX') = T^v(X_0 \sqcup kX')|_{kX'}$. Let β be a section of $T^v(X_0 \sqcup kX')$ such that $\beta|_{X_0} = \widetilde{\tau}_X^* a$ and β is transverse to the zero section in $T^v(X_0 \sqcup kX')$. In this setting, we can take $W(\tau_Y^* a)|_{ST^vX_0} = \widetilde{\tau}_X^*\omega_{S^2}$. Then $\tilde{z}_n^{\mathrm{anomaly}}(\tau_Y^*\vec{a}; X_0 \sqcup kX') = \sum_{\Gamma\in\mathcal{E}_n}\int_{S_{2n}(T^v(X_0\sqcup kX'))}\bigwedge_i \phi_i(\Gamma)^* W(\tau_Y^* a)[\Gamma] = \sum_\Gamma \int_{S_{2n}(T^vX_0)}\bigwedge_i \phi_i(\Gamma)^*\widetilde{\tau}_X^*\omega_{S^2}[\Gamma]$
$+ \sum_\Gamma \int_{S_{2n}(T^v(kX'))}\bigwedge_i \phi_i(\Gamma)^* W(\tau_Y^* a)[\Gamma]$.

We show that $\int_{S_{2n}(T^vX_0)}\bigwedge_i \phi_i(\Gamma)^*\widetilde{\tau}_X^*\omega_{S^2} = 0$ for any $\Gamma \in \mathcal{E}_n$. The map $(\widetilde{\tau}_X)^{3n} \circ (\prod_i \phi_i(\Gamma)) : S_{2n}(T^vX_0) \to (S^2)^{3n}$ factors through $S_{2n}(\mathbb{R}^3)$:

$$\begin{tikzcd}
S_{2n}(T^vX_0) \arrow[r, "\prod_i \phi_i(\Gamma)"] \arrow[d, "\widetilde{\tau}_X"'] \arrow[dr, phantom, "\circlearrowleft"] & (ST^vX_0)^{3n} \arrow[d, "(\widetilde{\tau}_X)^{3n}"] \\
S_{2n}(\mathbb{R}^3) \arrow[r] & (S^2)^{3n}.
\end{tikzcd}$$

Hence we have $\bigwedge_i \phi_i(\Gamma)^*\widetilde{\tau}_X^*\omega_{S^2}|_{ST^vX_0} = ((\prod \widetilde{\tau}_X)^{3n} \circ \bigwedge_i \phi_i(\Gamma))^*(\omega_{S^2})^{3n}$
$\in \mathrm{Im}(\Omega^{6n}(S_{2n}(\mathbb{R}^3)) \to \Omega^{6n}(S_{2n}(T^vX_0)))$. Since $\dim \breve{S}_{2n}(\mathbb{R}^3) = 6n-4 < 6n = \dim \bigwedge_i \phi_i(\Gamma)^*\widetilde{\tau}_X^*\omega_{S^2}$, we have $\bigwedge_i \phi_i(\Gamma)^*\widetilde{\tau}_X^*\omega_{S^2} = 0$.

Therefore

$$\begin{aligned}
\tilde{z}_n^{\mathrm{anomaly}}(\tau_Y^*\vec{a}; X_0 \sqcup kX') &= \sum_{\Gamma\in\mathcal{E}_n}\int_{S_{2n}(T^vkX')}\bigwedge_i \phi_i(\Gamma)^* W(\tau_Y^* a)[\Gamma] \\
&= \sum_{i=1}^k \tilde{z}_n^{\mathrm{anomaly}}(\tau_i^*\vec{a}; X').
\end{aligned}$$

(2) By the obstruction theory and the definition of the signature defect, we have $\sigma_Y(\tau_Y \cup \tau_{S^3}) + 3\mathrm{Sign}X_0 = \sum_{i=1}^k \sigma_{S^3}(\tau_i \cup \tau_{S^3})$. Since $\sigma_{Y\setminus\infty}(\tau_Y) = \sigma_Y(\tau_Y \cup \tau_{S^3}) - 2$ and $\sigma_{\mathbb{R}^3}(\tau_i) = \sigma_{S^3}(\tau_i \cup \tau_{S^3}) - 2$, the equality (2) holds.

(3)

$$\begin{aligned}
\tilde{z}_n^{\mathrm{anomaly}}(\tau_Y^*\vec{a}) &= \tilde{z}_n^{\mathrm{anomaly}}(\tau_Y^*\vec{a}; X_0 \sqcup kX') - \mu_n\mathrm{Sign}(X_0 \sqcup kX') - c_n \\
&\overset{(1)}{=} \tilde{z}_n^{\mathrm{anomaly}}(\tau_1^*\vec{a}; X') - \mu_n\mathrm{Sign}X' - c_n \\
&+ \ldots + \tilde{z}_n^{\mathrm{anomaly}}(\tau_k^*\vec{a}; X') - \mu_n\mathrm{Sign}X' - c_n \\
&-\mu_n\mathrm{Sign}X_0 + (k-1)c_n \\
&= \sum_i \tilde{z}_n^{\mathrm{anomaly}}(\tau_i^*\vec{a}) - \mu_n\mathrm{Sign}X_0 + (k-1)c_n \\
&\overset{\text{Corollary 7.5}}{=} \sum_i \frac{1}{4}\sigma_{\mathbb{R}^3}(\tau_i)\delta_n - \mu_n\mathrm{Sign}X_0 + (k-1)c_n \\
&\overset{(2)}{=} \frac{1}{4}(\sigma_{Y\setminus\infty}(\tau_Y) - 2(k-1) + 3\mathrm{Sign}X_0)\delta_n - \mu_n\mathrm{Sign}X_0 + (k-1)c_n.
\end{aligned}$$

□

We next compute μ_n, c_n and prove that $\tilde{z}_n^{\mathrm{anomaly}}(\tau_Y^*\vec{a}) = \frac{1}{4}\sigma_{Y\setminus\infty}(\tau_Y)\delta_n$ by using the above lemma.

Lemma 7.10. $\mu_n = \frac{3}{4}\delta_n$.

Proof. Let $X_0 = K3\sharp 11T^4 \setminus (B^4 \sqcup B^4)$. Then X_0 is a spin 4-manifold satisfying $\chi(X_0) = 0$, $\mathrm{Sign}X_0 = 16$ and $\partial X_0 = S^3 \sqcup -S^3$. By Lemma 7.9 (3), we have $0 = \widetilde{z}_n^{\mathrm{anomaly}}(\tau_{\mathbb{R}^3}^*\vec{a}) = (\frac{3}{4}\delta_n - \mu_n)\mathrm{Sign}X_0$. Since $\mathrm{Sign}X_0 = 16 \neq 0$, we have $\mu_n = \frac{3}{4}\delta_n$. $\square$

Lemma 7.11. $c_n = \frac{1}{2}\delta_n$.

Proof. Let $X_0 = K3\sharp 10T^4 \setminus (B^4 \sqcup 3B^4)$. Then X_0 is a spin 4-manifold satisfying $\chi(X_0) = 0$, $\mathrm{Sign}X_0 = 16$ and $\partial X_0 = S^3 \sqcup 3(-S^3)$. By Lemma 7.9 (3) and Lemma 7.10, we have $0 = \widetilde{z}_n^{\mathrm{anomaly}}(\tau_{\mathbb{R}^3}^*\vec{a}) = -\delta_n + 2c_n$. Then $c_n = \frac{1}{2}\delta_n$. $\square$

Proposition 7.12. $\widetilde{z}_n^{\mathrm{anomaly}}(\tau_Y^*\vec{a}) = \frac{1}{4}\sigma_{Y\setminus\infty}(\tau_Y)\delta_n$.

Proof. Take $X_0, k, \widetilde{\tau}_X$ as in Lemma 7.9. By Lemma 7.9 (3), Lemma 7.10 and Lemma 7.11, we have $\widetilde{z}_n^{\mathrm{anomaly}}(\tau_Y^*\vec{a}) = \frac{1}{4}\sigma_{Y\setminus\infty}(\tau_Y)\delta_n - \frac{k-1}{2}\delta_n + (k-1)c_n = \frac{1}{4}\sigma_{Y\setminus\infty}(\tau_Y)\delta_n$. $\square$

Remark 7.13. We can rewrite the above proof of $\widetilde{z}^{\mathrm{anomaly}}(Y;\tau_Y^*\vec{a}) = \frac{1}{4}\sigma_{Y\setminus\infty}(\tau_Y)\delta_n$ as follows. We first remark that we can extend the definition of $\widetilde{z}_n^{\mathrm{anomaly}}(\tau_Y^*\vec{a})$ to any framing τ of a closed oriented 3-manifold Y by using τ instead of $\tau_Y \cup \tau_{S^3}$ in the construction. Let $\mathcal{M}$ be the set of all framed three manifolds: $\mathcal{M} = \{(Y,\tau)\}$. We define two maps $\widetilde{I}$ and σ as

$$\widetilde{I} : \mathcal{M} \to \mathcal{A}_n(\emptyset), \quad \widetilde{I}(Y,\tau) = \widetilde{z}_n^{\mathrm{anomaly}}(\tau^*\vec{a}),$$

$$\sigma : \mathcal{M} \to \mathcal{A}_n(\emptyset), \quad \sigma(Y,\tau) = \frac{\delta_n}{4}(\sigma_Y(\tau) - \sigma_{S^3}(\tau_{S^3})).$$

Then it is sufficient to show that

$$\widetilde{I}(Y,\tau) = \sigma(Y,\tau).$$

We now introduce a cobordism group Ω_3^{triv} defined as follows.

$$\Omega_3^{\mathrm{triv}} = \mathcal{M}/\sim$$

where $(Y_0,\tau_0) \sim (Y_1,\tau_1)$ if and only if there exists a oriented compact 4-manifold X and framing $\widetilde{\tau} : TX \xrightarrow{\cong} \underline{\mathbb{R}^4}$ such that:

- $\partial X = Y_0 \sqcup -Y_1$,
- $\widetilde{\tau}|_{Y_0} = \eta_0 \oplus \tau_0$, when η_0 is the outward unit vector field of Y_0,
- $\widetilde{\tau}|_{-Y_1} = -\eta_1 \oplus -\tau_1$, when $-\eta_1$ is the inward unit vector field of $-Y_1$, and
- $\mathrm{Sign}X = 0$.

There is a natural map $\pi : \mathcal{M} \to \Omega_3^{\mathrm{triv}}$. The proof of Lemma 7.9 (1) implies that the map $\widetilde{I}$ factors through Ω_3^{triv}: there is a homomorphism $\widetilde{I}^c : \Omega_3^{\mathrm{triv}} \to \mathcal{A}_n(\emptyset)$ such that $\widetilde{I} = \widetilde{I}^c \circ \pi$. The proof of Lemma 7.9 (2) implies that the map σ factors through Ω_3^{triv}: there is a homomorphism $\sigma^c : \Omega_3^{\mathrm{triv}} \to \mathcal{A}_n(\emptyset)$ such that $\sigma = \sigma^c \circ \pi$.

Lemma 7.6 implies that $\Omega_3^{\rm triv}\otimes\mathbb{Q}\cong\mathbb{Q}$. Corollary 7.5 and the study of framings on S^3 say that $\widetilde{I}^c(S^3,\tau)=\sigma^c(S^3,\tau)$ for some non-trivial element (S^3,τ) of $\Omega_3^{\rm triv}$.

Then we have

$$\widetilde{I}(Y,\tau)=\widetilde{I}^c\circ\pi(Y,\tau)=\sigma^c\circ\pi(Y,\tau)=\sigma(Y,\tau)$$

for any (Y,τ). In particular, $\widetilde{z}^{\rm anomaly}(\tau_Y^*\vec{a})=\frac{1}{4}\sigma_{Y\setminus\infty}(\tau_Y)\delta_n$ for any framing τ_Y on a punctured rational homology 3-sphere $Y\setminus\infty$.

7.2 Proof of $\widetilde{z}_n(Y)=z^{\rm FW}_{2n,3n}(Y)$.

Let f be an admissible Morse function with respect to $a\in S^2$. The weighted sum $\mathcal{M}(f)+\mathcal{M}(-f)$ consists of weighted pairs of two distinct points on a gradient trajectory. There is a compactification $\mathcal{M}_S(\pm f)$ of $\mathcal{M}(f)+\mathcal{M}(-f)$ by adding pairs of points on broken trajectories as the Morse theory. Then $\mathcal{M}_S(\pm f)$ becomes a 4-cycle in $(C_2(Y),\partial C_2(Y))$ (Lemma 8.4). See Section 8 for the detail of the above argument.

Lemma 7.14. *$\partial\mathcal{M}_S(\pm f)=c(\mathrm{grad}f)$ for any admissible Morse function f.*

Proof. Since $\mathrm{grad}f|_{N(\infty;Y)}=\mathrm{grad}q_a$, if $(x,u)\in\partial\mathcal{M}_S(\pm f)\cap((Y\setminus\infty)\times ST_\infty Y)$ then $u=\pm a$. On the other hand, $\partial\mathcal{M}_S(\pm f)\cap(\{x\}\times ST_\infty Y)=\{(x,a),(x,-a)\}$ for any $x\notin\mathrm{Crit}(f)$. Since $\partial\mathcal{M}_S(\pm f)$ is a 3-cycle, we have $\partial\mathcal{M}_S(f)\cap((Y\setminus\infty)\times ST_\infty Y)=(Y\setminus\infty)\times(\pm a)$. With a similar argument, we have $\partial C_2(Y)\setminus S\nu_{\Delta(Y\setminus\infty)}=p_Y^{-1}(\pm a)$. By this fact and Lemma 8.5 we conclude the proof. □

We follow the notations $a_1,\dots,a_{3n}$, $f_1,\dots,f_{3n}$ as in Section 6. In the following proposition, the notion "generic $\vec{f}$" means that $\partial C_{2n}(Y)\cap(\bigcap_i P_i(\Gamma)^{-1}\mathcal{M}_S(\pm f_i))=\emptyset$ for any $\Gamma\in\mathcal{E}_n$. We remark that there exists such a $\vec{f}$ (See Remark 7.16).

Proposition 7.15. *For generic $\vec{f}$, $z^{\rm FW}_{2n,3n}(Y;\vec{f})=\widetilde{z}_n(Y;\mathrm{grad}\vec{f})$.*

Proof. We define the 2-cocycle $\omega^s_i(\mathrm{grad}\ f_i)\in S^2(|T_{C_2(Y)}|)$ by $\omega^s(\mathrm{grad}\ f_i)(\sigma)=\frac{1}{2}\sharp(\sigma\cap\mathcal{M}_S(f_i))$ for each 2-chain σ of $T_{C_2(Y)}$.

By the construction, $\omega^s(\mathrm{grad}\ f_i)$ is a simplicial propagator for each i. By the intersection theory and Lemma 4.8, we have

$$z^{\rm FW}_{2n,3n}(Y;\vec{f})=\langle\bigwedge_i P_i(\Gamma)^*\omega^s(\mathrm{grad}\ f_i),[C_{2n}(Y),\partial C_{2n}(Y)]\rangle=\frac{1}{2^{3n}}\sharp\left(\bigcap_i P_i(\Gamma)^{-1}\mathcal{M}_S(f_i)\right)$$

for any $\Gamma\in\mathcal{E}_n$. □

Remark 7.16. We can show that $\partial C_{2n}(Y)\cap(\bigcap_i P_i(\Gamma)^{-1}\mathcal{M}_S(\pm f_i))=\emptyset$ for generic $\vec{f}$ by an argument similar to Lemma 2.7 in Watanabe [34]. For example, we take the following Φ'_Γ instead of Φ in Lemma 2.7 in [34] when we prove $F(\{1,2,4\})\cap$

$(\bigcap_{i=1}^6 P_i(\mathrm{Smooth}(\Gamma))^{-1}\mathcal{M}_S(\pm f_i)) = \emptyset$ for the graph Γ in the picture (2.2) in [34] (See Example 2.6 in [34] and see §3.4 of [34] for the definition of the operator Smooth).

$$\phi'_\Gamma : F(\{1,2,4\}) \times \left(\bigcup_{f_1 \in \mathcal{U}_1} \mathcal{A}_p(f_1) \cap \mathcal{D}_q(f_1) \right) \times (\mathbb{R}_{>0})^3 \times \prod_{i=2}^4 \mathcal{U}_i$$

$$\to Y^3 \times (TY)^2 \times (TY)^2 \times Y^3,$$

$$\begin{aligned} &\Phi'_\Gamma(((x_1,[w_1,w_2,w_4]),x_3),u,t_2,t_3,t_4,f_2,f_3,f_4) \\ &= ((x_1,u,\Phi_{f_6}^{t_6}(x_3)),(\mathrm{grad}_{x_1} f_2, \frac{w_2-w_1}{\|w_2-w_1\|}), \\ &\quad (\mathrm{grad}_{x_1} f_3, \frac{w_4-w_2}{\|w_4-w_2\|}),(x_3,\Phi_{f_4}^{t_4}(x_1),\Phi_{f_5}^{t_5}(x_1))). \end{aligned}$$

Here $x_1 \in Y \setminus \infty$, $[w_1,w_2,w_3] \in \breve{S}_{\{1,2,4\}} T_{x_1} Y$, $x_3 \in Y \setminus \{x_1,\infty\}$. Let

$$\begin{aligned} \Delta'_\Gamma = \{ &((y_1,y_1,y_1),((y_2,s_2v_2),(y_2,t_2v_2)),((y_3,s_3v_3),(y_3,t_3v_3)),(y_4,y_4,y_4)) \\ &\mid (y_1,y_2,y_3,y_4) \in (Y\setminus\infty)^4, t_i, s_i \geq 0, v_i \in T_{y_i}Y\}. \end{aligned}$$

Then Φ'_Γ is transverse to Δ'_Γ as Lemma 2.7 in [34].

It is obvious that $z_{2n,3n}^{\mathrm{anomaly}}(Y;\vec{f}) = \mathfrak{z}_n^{\mathrm{anomaly}}(Y;\mathrm{grad}\vec{f})$ by the definitions of the anomaly parts.

8 Compactification of moduli space $\mathcal{M}(f)$.

In this section we give a compactification $\mathcal{M}_S(\pm f)$ of $\mathcal{M}(f) \cup \mathcal{M}(-f)$ and then show that $\mathcal{M}_S(\pm f)$ is a 4-cycle in $(C_2(Y), \partial C_2(Y))$. Let $M_\to(f) = \varphi^{-1}|_{Y^2\times(0,\infty)}(\Delta)$ where $\varphi : Y^2 \times (-\infty,\infty) \to Y^2, (x,y,t) \mapsto (y, \Phi_f^t(x))$.

Lemma 8.1 (Watanabe [34, Proposition 2.12] (cf. [5])). *There is a manifold with corners $\overline{M}_\to(f)$ satisfying the following conditions.*

(1) $\overline{M}_\to(f) = \{g : I \to Y \mid I \subset -\mathbb{R}$,
g is a piecewise smooth map, $f(g(t)) = t$, $\frac{dg(t)}{dt} = \frac{\mathrm{grad}_{g(t)} f}{\|\mathrm{grad}_{g(t)} f\|^2}$ for any $t\}$ *as sets,*

(2) $\mathrm{int}\overline{M}_\to(f) = M_\to(f)$, *and*

(3) $\partial \overline{M}_\to(f) = \sum_i \mathcal{A}_{p_i} \times \mathcal{D}_{p_i} + \sum_j \mathcal{A}_{q_j} \times \mathcal{D}_{q_j}$.

Note that $\mathrm{int}(\overline{M}_\to(f) + \overline{M}_\to(-f)) = \varphi^{-1}(\Delta)$. We denote by $\overline{M}_\to(f) \to (Y \setminus \infty)^2$ the continuous map that is the extension of the embedding $M_\to(f) \to (Y\setminus\infty)^2$ to $\overline{M}_\to(f)$. For simplicity of notation, we write $\overline{M}_\to(f)$ instead of $\overline{M}_\to(f) \to (Y\setminus\infty)^2$.

Similarly we denote by $\overline{\mathcal{A}_{p_i}} \to Y$ the extension of $B^1(1) \cong \mathcal{A}_{p_i} \to Y$ to $\overline{B^1(1)}$ and we write $\overline{\mathcal{A}_{p_i}}$ instead of $\overline{\mathcal{A}_{p_i}} \to Y$ (We remark that $\mathcal{A}_{p_i}$ is diffeomorphic to $B^1(1)$ the interior of unit disk in $\mathbb{R}^1$). We also define $\overline{\mathcal{D}_{p_i}}, \overline{\mathcal{A}_{q_j}}$, and so on.

Lemma 8.2. (1) *$\overline{M}_{\to}(f) + \overline{M}_{\to}(-f)$ is transverse to Δ.*

(2) *$\overline{\mathcal{A}_{q_j}} \times \overline{\mathcal{D}_{p_i}}$ is transverse to Δ.*

Proof. (1) $\mathrm{grad}f$(which is the section of $\nu_{\Delta(Y\setminus\infty)}$) is transverse to the zero section in $\nu_{\Delta(Y\setminus\infty)}$. $\mathcal{A}_p \times \mathcal{D}_p \subset Y^2$ is transverse to Δ for any critical point $p \in \mathrm{Crit}(f) = \mathrm{Crit}(-f)$. Thanks to Lemma 8.1 (2),(3), this finishes the proof of (1).

(2) is immediate from the Morse-Smale condition. $\square$

By this Lemma, $(\overline{M}_{\to}(f) + \overline{M}_{\to}(-f))(\Delta)$ and $(\overline{\mathcal{A}_{q_j}} \times \overline{\mathcal{D}_{p_i}})(\Delta)$ are well-defined. It is clear that $(\overline{M}_{\to}(f) + \overline{M}_{\to}(-f))(\Delta) =$ $(\overline{M}_{\to}(f)+\overline{M}_{\to}(-f))\setminus\Delta\cup\{(x, \frac{\pm\mathrm{grad}_x f}{\|\mathrm{grad}_x f\|}) \mid x \in Y\setminus(\infty\cup\mathrm{Crit}(f))\}$ by the construction.

Definition 8.3. $\mathcal{M}^0_S(\pm f) = (\overline{M}_{\to}(f) + \overline{M}_{\to}(-f))(\Delta) + \sum_{i,j} g_{ij}(\overline{\mathcal{A}_{q_i}} \times \overline{\mathcal{D}_{p_j}})(\Delta) + \sum_{i,j}(-g_{ij})(\overline{\mathcal{D}_{p_j}} \times \overline{\mathcal{A}_{q_i}})(\Delta)$.

Let $\mathcal{M}_S(\pm f)$ be the extension of $\mathcal{M}^0_S(\pm f)$ to $C_2(Y)$.

Lemma 8.4. *$\mathcal{M}_S(\pm f)$ is a 4-cycle in $(C_2(Y), \partial C_2(Y))$.*

Proof. Since $\mathrm{Im}(\partial(\overline{\mathcal{A}_{q_i}} \times \overline{\mathcal{D}_{p_j}}) \to Y^2) = \sum_k \partial_{ki}\overline{\mathcal{A}_{p_k}} \times \overline{\mathcal{D}_{p_j}} + \sum_k \partial_{jk}\overline{\mathcal{A}_{q_i}} \times \overline{\mathcal{D}_{q_k}}$,

$$\begin{aligned}
&\mathrm{Im}(\sum_{i,j} g_{ij}\partial(\overline{\mathcal{A}_{q_i}} \times \overline{\mathcal{D}_{p_j}} \to Y^2)) \\
=& \sum_{i,j,k} g_{ij}\partial_{ki}\overline{\mathcal{A}_{p_k}} \times \overline{\mathcal{D}_{p_j}} + \sum_{i,j,k} g_{ij}\partial_{jk}\overline{\mathcal{A}_{q_i}} \times \overline{\mathcal{D}_{q_k}} \\
=& \sum_{i,j,k} \delta_{kj}\overline{\mathcal{A}_{p_k}} \times \overline{\mathcal{D}_{p_j}} + \sum_{i,j,k} \delta_{ik}\overline{\mathcal{A}_{q_i}} \times \overline{\mathcal{D}_{q_k}} \\
=& \sum_{j} \overline{\mathcal{A}_{p_j}} \times \overline{\mathcal{D}_{p_j}} + \sum_{j} \overline{\mathcal{A}_{q_j}} \times \overline{\mathcal{D}_{q_j}} \\
=& \partial\overline{M}_{\to}(f) \setminus \Delta.
\end{aligned}$$

Therefore $\partial\mathcal{M}_S(\pm f) \setminus \partial C_2(Y) = \emptyset$. $\square$

Under the identification $S\nu_{\Delta(Y\setminus\infty)} \cong ST(Y\setminus\infty)$, we have the following description.

Lemma 8.5. $\partial\mathcal{M}_S(\pm f) \cap ST(Y\setminus\infty) = \overline{\{(x, \frac{\pm\mathrm{grad}_x f}{\|\mathrm{grad}_x f\|}) \mid x \in Y\setminus(\infty\cup\mathrm{Crit}(f)\}}$.

Proof. Note that $(\overline{\mathcal{A}_{q_i}} \times \overline{\mathcal{D}_{p_j}}) \cap \Delta = \overline{\mathcal{A}_{q_i} \cap \mathcal{D}_{p_j}}$. By the definition of blow up, we have $\partial\mathcal{M}_S(\pm f) \cap S\nu_{\Delta(Y\setminus\infty)}$

$$= \overline{\left\{\left(x, \frac{\pm\mathrm{grad}_x f}{\|\mathrm{grad}_x f\|}\right)\right\}} + \sum_{i,j} g_{ij}\pi^{-1}(\overline{\mathcal{A}_{q_i} \cap \mathcal{D}_{p_j}}) + \sum_{i,j}(-g_{ij})\pi^{-1}(\overline{\mathcal{D}_{p_j} \cap \mathcal{A}_{q_i}})$$

where $\pi : STY \to Y$ is the projection.

Since $\sum_{i,j} g_{ij}\pi^{-1}(\overline{\mathcal{A}_{q_i} \cap \mathcal{D}_{p_j}}) + \sum_{i,j}(-g_{ij})\pi^{-1}(\overline{\mathcal{D}_{p_j} \cap \mathcal{A}_{q_i}}) = 0$ as chains, we conclude the proof. $\square$

A Another proof of $\widetilde{z}_1(Y) = z_1^{\mathrm{KKT}}(Y)$.

In this section we give a more direct proof of Proposition 7.12 in the case of $n = 1$. Remark that $\mathcal{A}_1(\emptyset) = \mathbb{Q}[\theta]$ and $\sharp\mathcal{E}_1 = 96$.

Proposition A.1 (Proposition 7.12 in the case of $n = 1$). $\widetilde{z}_1^{\mathrm{anomaly}}(\tau_Y^* \vec{a}) = \frac{1}{4}\sigma_{Y\setminus\infty}(\tau_Y)\delta_1$.

To show this proposition we first prepare some notations and a lemma. Let $\pi_1 : F_X \to ST^vX$ be the tangent bundle along the fiber of $\pi_2 : ST^vX \to X$. Let T^vX/TY be the real vector bundle over X/Y obtained by collapsing Y to a point using the framing $\tau_Y \cup \tau_{S^3} = \tau_Y|_{Y\setminus N(\infty;Y)} \cup \tau_{S^3}|_{N(\infty;Y)}$. We define $F_{X/Y}, ST^vX/STY$ in same way as above.

Let $e(F_X;\tau_Y) \in H^2(ST^vX/STY) = H^2(ST^vX, STY)$ be the Euler class of $F_{X/Y}$ and let $p_1(F_X;\tau_Y) \in H^4(ST^vX/STY) = H^4(ST^vX, STY)$ be the first Pontrjagin class of $F_{X/Y}$. By a standard argument, for example the Chern-Weil theory, we have $p_1(F_X;\tau_Y) = e_1(F_X;\tau_Y)^2$.

Lemma A.2. $2[W(\tau_Y^* a)] = e(F_X;\tau_Y) \in H^2(ST^vX/STY)$.

Proof. Let β be a section of T^vX such that $\beta|_{\partial X} = (\tau_Y \cup \tau_{S^3})^* a$ as Subsection 4.4. We define the map $f : ST^vX \to \mathbb{R}$ by

$$f(u) = \langle u, \beta(\pi_2(u))\rangle_{(T^vX)_{\pi_2(u)}}$$

where $\langle,\rangle_{(T^vX)_x}$ is the standard inner product on $(T^vX)_x (\cong \mathbb{R}^3)$. We define the vector field $V \in \Gamma F_X$ by $V|_{(ST^vX)_x} = \mathrm{grad}(f|_{(ST^vX)_x})$ for any $x \in X$. Thus V is transverse to the zero section in F_X and $V^{-1}(0) = c_0(\beta)$. Thus the Poincaré dual of $(c_0(\beta), \partial c_0(\beta))$ represents $e(F_X;\tau_Y)$. Since the closed 2-form $2W(\tau_Y^* a)$ represents the Poincaré dual of $(c_0(\beta), \partial c_0(\beta))$ and $W(\tau_Y^* a)|_{STY} = (\tau_Y \cup \tau_{S^3})^* \omega_{S^2}^a$, we conclude the proof. ☐

Proof of Proposition A.1. By the Lemma A.2, we have

$$\begin{aligned}
\int_{S_2(T^vX)} W(\tau_Y^* a)^3 &= \frac{1}{8}\int_{S_2(T^vX)} e(F_X;\tau_Y)^3 \\
&= \frac{1}{8}\int_{S_2(T^vX)} e(F_X;\tau_Y)p_1(F_X;\tau_Y) \\
&\overset{(*)}{=} \frac{1}{8}\int_{S_2(T^vX)} e(F_X;\tau_Y)\pi_2^* p_1(TX;\tau_Y) \\
&= \frac{1}{4}\int_X p_1(TX;\tau_Y) \\
&= \frac{1}{4}\sigma_Y(\tau_Y \cup \tau_{S^3}) + \frac{3}{4}\mathrm{Sign}X \\
&= \frac{1}{4}\sigma_{Y\setminus\infty}(\tau_Y) + \frac{3}{4}\mathrm{Sign}X + \frac{1}{2}.
\end{aligned}$$

The equality $(*)$ is given by the following two relations: $\underline{\mathbb{R}} \oplus F_X = \pi^* T^v X$ and $\underline{\mathbb{R}} \oplus T^v X = TX$. Then we have

$$\tilde{z}^{\text{anomaly}}(\tau_Y^* \vec{a}) = 96 \int_{S_2(T^v X)} W(\tau_Y^* a)^3[\theta] - \mu_1 \mathrm{Sign} X - c_1$$

$$= \frac{96}{4}[\theta]\sigma_{Y \setminus \infty}(\tau_Y) + (72[\theta] - \mu_1)\mathrm{Sign} X - (c_1 - 48[\theta]).$$

Since this equality holds for any τ_Y and X, then we have $\mu_1 = 72[\theta]$, $c_1 = 48[\theta]$, $\delta_1 = 96[\theta]$. Thus $\tilde{z}_1^{\text{anomaly}}(\tau_Y^* \vec{a}) = \frac{1}{4}\sigma_{Y \setminus \infty}(\tau_Y)\delta_1$. $\square$

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