博士論文

Liouville type theorems for the Navier-Stokes equations and applications

（ナヴィエ・ストークス方程式に対するリウヴィル型定理とその応用）

許　本源

Liouville type theorems for the Navier-Stokes equations and applications

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Contents

Abstract

In this thesis we study the Liouville type results for solutions to the Navier-Stokes equations, that is, the nonexistence of nontrivial bounded global (or entire) solutions to the Navier-Stokes equations.

We mainly consider the stationary Navier-Stokes equations in three-dimensional whole space and the non-stationary Navier-Stokes equations in half plane.

In Chapter 1, we give a brief introduction to this thesis.

In Chapter 2, we consider stationary solutions to the three-dimensional Navier-Stokes equations for viscous incompressible flows in the presence of a linear strain. For certain class of strains we prove a Liouville type theorem under suitable decay conditions on vorticity fields.

In Chapter 3, we establish a Liouville type result for a backward global solution to the Navier-Stokes equations in the half plane with the no-slip boundary condition. No assumptions on spatial decay for the vorticity nor the velocity field are imposed. We study the vorticity equations instead of the original Navier-Stokes equations. As an application, we extend the geometric regularity criterion for the Navier-Stokes equations in the three-dimensional half space under the no-slip boundary condition.

Chapter 2 is essentially based on [1]. And Chapter 3 is essentially based on [2].

All sections, formulas and theorems, etc., are cited only in the chapter where they appear.

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Acknowledgments

The author is grateful very much to his supervisor, Professor Yoshikazu Giga for suggesting the problems discussed in this thesis and for kind encouragement and valuable advice. He also wishes to express his great gratitude to Professor Yasunori Maekawa for many helpful discussions. He would like to thank Professor Jong-Shenq Guo for giving him several important references on Liouville type theorems for the Euler equations and the Navier-Stokes equations. The author is grateful to Professor Tai-Peng Tsai for helpful discussions especially on maximum principle and for useful comments on his works. He would next send his gratitude to Professor Tai-Chia Lin and Professor Okihiro Sawada for their warm encouragement and useful comments on the work discussed in Chapter 2

Chapter 1

Introduction

1 The subject of the thesis

The subject of this thesis is to study the Liouville type results for solutions to the Navier-Stokes equations, that is, the nonexistence of nontrivial bounded global (or entire) solutions to the Navier-Stokes equations. We proved Liouville type theorems for solutions to the Navier-Stokes equations in the following scenes.

(1) Entire solutions to the stationary Navier-Stokes equations with a liner strain in three-dimensional space.

(2) Backward global solutions to the Navier-Stokes equations in the half plane subject to the Dirichlet boundary condition.

We state the results in Chapter 2 and Chapter 3 respectively. We also extend the geometric regularity criterion for the Navier-Stokes equations in the three-dimensional half space under the no-slip boundary condition as an application of our Liouville type theorem in Chapter 3.

2 Introduction to Chapter 2

In Chapter 2 we consider stationary solutions to the three-dimensional Navier-Stokes equations for viscous incompressible flows with a linear strain:

$$\begin{cases} -\Delta U + Mx\cdot\nabla U + MU + U\cdot\nabla U + \nabla P &= 0 \qquad x\in\mathbb{R}^3, \\ \nabla\cdot U &= 0 \qquad x\in\mathbb{R}^3, \end{cases} \tag{NS$_\mathrm{M}$}$$

$$M = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad \lambda_i\in\mathbb{R}. \tag{2.1}$$

Here $U(x) = (U_1(x), U_2(x), U_3(x))$ represents the velocity field, $P(x)$ is the pressure field, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the space variable, and each λ_i is a given real number.

The system (NS$_\mathrm{M}$) is closely related with the original Navier-Stokes equations. For example, the first equation of the system (NS$_\mathrm{M}$) is formally obtained by considering the stationary solution to the Navier-Stokes equations of the form $U(x) + Mx$. If the trace of M, denoted by $\mathrm{Tr}(M)$ in the sequel, is equal to zero then the second equation of the system (NS$_\mathrm{M}$) is also recovered. This represents stationary solutions to the Navier-Stokes equations with a background flow Mx (which we called it linear strain). Even in the case $\mathrm{Tr}(M) \neq 0$, the system (NS$_\mathrm{M}$) is derived from the Navier-Stokes equations through self-similar solutions. For precise transform formula please see the Section 1 of Chapter 2. This observation shows that the system (NS$_\mathrm{M}$) describes three important classes of solutions to Navier-Stokes equations depending on the eigenvalues λ_i of M. And the sign of eigenvalues is closely related to the existence (or nonexistence) for nontrivial entire solutions. Our goal is to clarify this relation.

One important problem in three-dimensional Navier-Stokes equations is that: does blow-up phenomenon occur in finite time? When $\lambda_1 = \lambda_2 = \lambda_3 > 0$, the system ($\mathrm{NS_M}$) is called "Leray's equation", for it was suggested in [6] to prove the existence of blow-up solutions to Navier-Stokes equations by constructing backward self-similar solutions. For this particular case it was proved in [8] that any weak solution to Leray's equation in $L^3(\mathbb{R}^3)$ must be trivial. This result declared that Leray's idea does not give the construction of blow-up solutions to the Navier-Stokes equations. Although the eigenvalues λ_i in literature, such as [7, 8, 10] are assumed to be positive and identical, one can apply the method especially in [10] for proving the nonexistence of nontrivial solutions to the system ($\mathrm{NS_M}$) even when the eigenvalues are all positive but does not coincide with each other. On the other hand, when $\lambda_1 = \lambda_2 = \lambda_3 < 0$, the system ($\mathrm{NS_M}$) describes the forward self-similar solutions to Navier-Stokes equations, and their existence is already well known. For example, see [2, 4, 5, 9]. Furthermore, when $\lambda_1 < 0, \lambda_2 < 0, \displaystyle\sum_{i=1}^{3} \lambda_i = 0$, the system ($\mathrm{NS_M}$) has an explicit two-dimensional solution, called the Burgers vortex in [1].

In Chapter 2 we study the case when one of λ_i is negative and the other two are positive, for this case is essentially open in the literature. If λ_i is positive then the transport term $Mx \cdot \nabla$ possesses an expanding effect in x_i direction, which tends to trivialize solutions. Conversely, if λ_i is negative then the term $Mx\cdot\nabla$ induces a localization in x_i direction, bringing an effect to keep solutions nontrivial. Because of the expanding effect in two directions, one naturally expects that nontrivial (stationary) solutions tend to be absent in our case. However, the precise relation between the nonexistence and the size of the eigenvalues was not clarified. We contribute to this question by finding sufficient conditions for the nonexistence of nontrivial solutions in terms of the eigenvalues.

3 Introduction to Chapter 3

In Chapter 3 we establish a Liouville type result for a backward global solution to the Navier-Stokes equations in the half plane with the no-slip boundary condition. When we study evolution equations the Liouville problem for bounded *backward* solutions plays an important role in obtaining an a priori bound of *forward* solutions through a suitable scaling argument called a blow-up argument. Indeed, if one imposes a uniform continuity on the alignment of the vorticity direction, the blow-up limit of the three-dimensional (Navier-Stokes) flow must be a nontrivial bounded two-dimensional flow, and the problem is essentially reduced to the analysis of two-dimensional Liouville problem. If we assume that the possible blow-up is type I, then the limit flow is not allowed to be a constant. Thus the resolution of the Liouville problem is a crucial step to reach a contradiction. From this systematic argument we can exclude the possibility of type I blow-up for the original three-dimensional flows under a regularity condition on the vorticity direction.

Recently the paper [3] successfully completes the above argument when the velocity field satisfies the *perfect slip* boundary condition, but the problem was remained open for the case of the *no-slip* boundary condition. In Chapter 3 we prove a Liouville type theorem under *no-slip* boundary condition.

When we consider two-dimensional Liouville problem, one effective approach is to investigate vorticity equation. Different from the case of the whole space (or whole plane) or of the perfect slip boundary condition, maximum principle is no longer a useful tool to obtain an a priori bound of the vorticity field. We overcome this difficulty by using the boundary condition on the vorticity field. We also apply our Liouville type theorem to settle the problem left open in [3].

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Chapter 2

On nonexistence for stationary solutions to the Navier-Stokes equations with a linear strain

1 Introduction

In this chapter we consider stationary solutions to the three-dimensional Navier-Stokes equations for viscous incompressible flows with a linear strain:

$$\begin{cases} -\Delta U + Mx\cdot\nabla U + MU + U\cdot\nabla U + \nabla P &= 0 \qquad x\in\mathbb{R}^3, \\ \nabla\cdot U &= 0 \qquad x\in\mathbb{R}^3, \end{cases} \tag{$\mathrm{NS_M}$}$$

$$M = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad \lambda_i \in \mathbb{R}. \tag{1.1}$$

Here $U(x) = (U_1(x), U_2(x), U_3(x))$ represents the velocity field, $P(x)$ is the pressure field, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the space variable, and each λ_i is a given real number.

The system $(\mathrm{NS_M})$ is closely related with the original Navier-Stokes equations. For example, the first equation of $(\mathrm{NS_M})$ is formally obtained by considering the stationary solution to the Navier-Stokes equations of the form $U(x) + Mx$. If the trace of M, denoted by $\mathrm{Tr}(M)$ in the sequel, is equal to zero then the second equation of $(\mathrm{NS_M})$ is also recovered. Even in the case $\mathrm{Tr}(M) \neq 0$, $(\mathrm{NS_M})$ is derived from the Navier-Stokes equations through self-similar solutions. To formulate this relation in a more precise way, let us recall the three-dimensional Navier-Stokes equations for viscous incompressible flows:

$$\begin{cases} v_t - \Delta v + v\cdot\nabla v + \nabla p &= 0 \qquad t>0, \quad x\in\mathbb{R}^3, \\ \nabla\cdot v &= 0 \qquad t>0, \quad x\in\mathbb{R}^3, \end{cases} \tag{NS}$$

where $v = v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$ and $p = p(x,t)$. As stated above, when $\mathrm{Tr}(M) = 0$ the system $(\mathrm{NS_M})$ describes the stationary solutions to (NS) of the form $v(x) = U(x) + Mx$ and $p(x) = P(x) - \frac{1}{2}|Mx|^2$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^3$. The reader is referred to [8] for the analysis of the nonstationary problem (NS) with a linear strain, where more general matrices M are treated. If $\mathrm{Tr}(M) < 0$ then $(\mathrm{NS_M})$ is related with the forward self-similar solutions to (NS) with a linear strain, i.e., the solutions to (NS) of the form

$$v(x,t) = \frac{1}{\sqrt{2\alpha t}}(U+S_1)(\frac{x}{\sqrt{2\alpha t}}), \qquad p(x,t) = \frac{1}{2\alpha t}(P+S_2)(\frac{x}{\sqrt{2\alpha t}}), \tag{1.2}$$

where $\alpha = |\mathrm{Tr}(M)|/3$, $S_1(x) = (M - \frac{\mathrm{Tr}(M)}{3}I)x$, $S_2(x) = (\alpha^2|x|^2 - |Mx|^2)/2$. Finally, if $\mathrm{Tr}(M) > 0$ then $(\mathrm{NS_M})$ describes the backward self-similar solutions to (NS) with a linear strain,

$$
\begin{aligned}
v(x,t) &= \frac{1}{\sqrt{2\alpha(T-t)}}(U+S_1)\Big(\frac{x}{\sqrt{2\alpha(T-t)}}\Big),\\
p(x,t) &= \frac{1}{2\alpha(T-t)}(P+S_2)\Big(\frac{x}{\sqrt{2\alpha(T-t)}}\Big),
\end{aligned}
\tag{1.3}
$$

where $T > 0$, and S_1, S_2, and α are the same as above.

Despite of the simple structure of the matrix M in (1.1), the above observation shows that ($\mathrm{NS_M}$) describes three important classes of solutions to (NS) depending on the eigenvalues λ_i of M. However, it is still not clear whether ($\mathrm{NS_M}$) admits nontrivial solutions or not, except for the following cases:

$$
\text{(i) } \lambda_i > 0,\ \ i = 1,2,3 \quad \text{(ii) } \lambda_1 < 0,\ \lambda_2 < 0,\ \sum_{i=1}^{3} \lambda_i = 0, \quad \text{(iii) } \lambda_1 = \lambda_2 = \lambda_3 < 0.
$$

We note that the sign of the eigenvalues λ_i plays a critical role for the existence of nontrivial solutions to ($\mathrm{NS_M}$). Indeed, if λ_i is positive then the transport term $Mx \cdot \nabla$ possesses an expanding effect in x_i direction, which tends to trivialize solutions. Conversely, if λ_i is negative then the term $Mx \cdot \nabla$ induces a localization in x_i direction, bringing an effect to keep solutions nontrivial.

In this chapter we study the case when one of λ_i is negative and the other two are positive, for this case is essentially open in the literature and is also important as an intermediate case between (i) and (ii). By suitable scaling and coordinate transformation we may assume without loss of generality that

$$
\lambda_1 = -\lambda < 0, \qquad \lambda_2 = 1, \qquad \lambda_3 = \mu \geq 1. \tag{1.4}
$$

Our interest particularly lies in the situation $\sum_{i=1}^{3} \lambda_i = 0$ with (1.4), for it is contrasting with the case (ii), where the nontrivial solutions are known to exist; see the statement below. Because of the expanding effect in two directions, one naturally expects that nontrivial (stationary) solutions tend to be absent in our case. However, the precise relation between the nonexistence and the size of the eigenvalues is still not clarified. This chapter will contribute to this question by finding sufficient conditions for the nonexistence of nontrivial solutions in terms of the eigenvalues in (1.4), which will lead to a further understanding of the roles of the straining flows in fluid dynamics. Before stating our results, we briefly recall the known results on the cases (i)-(iii).

(i) $\lambda_i > 0$, $i = 1,2,3$: The most important example is $\lambda_1 = \lambda_2 = \lambda_3 > 0$. In this case ($\mathrm{NS_M}$) is called "Leray's equation", for it was suggested by [10] to prove the existence of blow-up solutions to (NS) by constructing backward self-similar solutions. For this particular case it was proved by [15] that any weak solution to Leray's equation in $L^3(\mathbb{R}^3)$ must be trivial. This result declared that Leray's idea does not give the construction of blow-up solutions to (NS). A simpler proof of the same conclusion was obtained by [14] under a slightly stronger assumption. The result of [14, 15] was extended by [17], where the condition of the spatial decay on U was completely removed. The expanding effect of $Mx \cdot \nabla$ in all directions was essentially used in [17]. Although the eigenvalues λ_i in [14, 15, 17] are assumed to be positive and identical, one can apply the method especially in [17] for proving the nonexistence of nontrivial solutions to ($\mathrm{NS_M}$) even when the eigenvalues are all positive but does not coincide with each other. We also refer to [3] for a related problem on the Euler equations.

(ii) $\lambda_1 < 0, \lambda_2 < 0, \sum_{i=1}^{3} \lambda_i = 0$: When $\lambda_1 = \lambda_2$ ($\mathrm{NS_M}$) has an explicit two-dimensional solution, called the Burgers vortex [1]. Even in the case $\lambda_1 \neq \lambda_2$ the analog of the Burgers vortex is known to exist; see [4, 5, 12, 13]. For stability of the Burgers vortex the reader is referred to a recent book [6, Chapter 2] and references cited there.

(iii) $\lambda_1 = \lambda_2 = \lambda_3 < 0$: In this case ($\mathrm{NS_M}$) describes the forward self-similar solutions to (NS), and their existence is already well known. For example, see [2, 7, 9, 16].

For more references about forward and backward self-similar solutions to (NS) the reader is referred to [6].

Now let us go back to the case (1.4) treated in the present chapter. In this case the solutions are more likely to be trivial due to the expanding effect of $Mx \cdot \nabla$ in two directions. However, the presence of the negative eigenvalue λ_1 gives rise to the interaction of the localization and the expansion through the diffusion and the nonlinearity, which makes the problem rather complicated. The aim of this chapter is to give sufficient conditions for (U, P) so that U must be a constant vector, by overcoming this difficulty. The key idea is to focus on the vorticity field $\Omega = \nabla \times U$. The assumptions and the main result of this chapter are stated as follows.

(C0) $|U(x)| + \dfrac{|P(x)|}{1+|x|} \in L^\infty(\mathbb{R}^3)$;

(C1) either (i) there is $\{x^{(n)}\} \subset \mathbb{R}^3$ such that

$$\lim_{n\to\infty} |x_1^{(n)}| = \infty, \quad \sup_n (|x_2^{(n)}| + |x_3^{(n)}|) < \infty, \quad \lim_{n\to\infty} \frac{P(x^{(n)})}{x_1^{(n)}} = 0$$

or

(ii) there is $\{x^{(n)}\} \subset \mathbb{R}^3$ such that $\lim_{n\to\infty} |x^{(n)}| = \infty, \quad \lim_{n\to\infty} U_1(x^{(n)}) = 0$;

(C2) $(1+|x|)|\Omega(x)| \in L^{p_0}(\mathbb{R}^3) \qquad$ for some $p_0 \in (1,3)$;

(C3) there is $\theta_0 > \lambda$ such that
either (i) $(1+|x_2|)^{\theta_0+1}|\Omega(x)| \in L^\infty(\mathbb{R}^3)$
or (ii) $(1+|x_3|)^{\frac{\theta_0}{\mu}+1}|\Omega(x)| \in L^\infty(\mathbb{R}^3)$ holds.

Theorem 1.1. *Let* $(U, P) \in (C^2(\mathbb{R}^3))^3 \times C^1(\mathbb{R}^3)$ *be a solution to* $(\mathrm{NS_M})$. *Assume that* **(C0)**-**(C3)** *hold. Then* $U \equiv \mathrm{const}$.

Remark 1.2. Under the conditions **(C0)** and **(C2)** it is not difficult to deduce $\nabla^k U \in L^\infty(\mathbb{R}^3)$ for each $k \in \mathbb{N}$. We will freely use this fact in the rest of the chapter.

This theorem implies that when the vorticity field decays sufficiently fast there are only trivial solutions to $(\mathrm{NS_M})$. We note that the absolute value of each eigenvalue represents the intensity of its straining effect, and it crucially acts on the structure of $(\mathrm{NS_M})$. In particular, the ratios of $|\lambda_1| = \lambda$ (localizing effect) and $\lambda_2 = 1, \lambda_3 = \mu$ (expanding effect) are important and they appear in the condition **(C3)**.

As in the previous papers [14, 15, 17], the key of our proof is to estimate the generalized pressure

$$\Pi(x) = \frac{1}{2}|U(x)|^2 + Mx \cdot U(x) + P(x). \tag{1.5}$$

However, the arguments in [14, 15, 17] rely on the positivity of each λ_i in the core part of the proof. So another new idea is needed to deal with the negative eigenvalue in our case. Under the conditions **(C0)** and **(C2)** the generalized pressure Π is written as $\Pi = a + \Pi_0$, where a is a constant and Π_0 decays uniformly at $|x| \to \infty$. The basic strategy is to investigate the spatial decay of Π_0 in details. In particular, we establish the pointwise estimates of $|\Pi_0(x)|$ from above and below that cannot be compatible to hold at the same time when Π_0 is not trivial. Theorem 1.1 is an immediate consequence of this result. As for the lower bound, we observe that Π_0 satisfies the inequality $\Delta\Pi_0 - Mx \cdot \nabla\Pi_0 - U \cdot \nabla\Pi_0 \geq 0$ and then apply the argument in [11] to get

$$|\Pi_0(x)| \geq C_{x_1}(1 + x_2^2 + (1 + x_3^2)^{\frac{1}{\mu}})^{-\frac{\theta}{2}} \qquad \text{if} \quad \Pi_0(x) \not\equiv 0, \tag{1.6}$$

where C_{x_1} is a positive constant independent of x_2 and x_3; see Proposition 3.5. In fact, when Π_0 decays at spatial infinity the estimate (1.6) is proved only under the conditions **(C0)** and **(C1')**: $\lim_{|x|\to\infty} |U_1(x)| = 0$. Especially, it is possible to derive the conclusion in Theorem 1.1 by alternatively assuming **(C0)**, **(C1')**, and suitable decay conditions on Π (or on Π_0) so as to contradict with (1.6). Although we do not need to pay much attention on vorticity fields in this alternative result, instead, there we are forced to assume strong spatial decay conditions on Π if $|\lambda|$ is large. But these are not so "realistic" assumptions because Π includes the pressure term P for which we cannot expect fast spatial decay in general even if U decays rapidly. On the other hand, the flows with localized vorticity fields are considered to be natural objects, and Theorem 1.1 excludes the possibility of the realization of such flows.

From mathematical point of view it is essential that Π_0 solves the Poisson equation with the inhomogeneous terms which are written in terms of the vorticity field Ω. Then under the assumptions in Theorem 1.1 the lower bound (1.6) is improved by

$$|\Pi_0(0,x_2,0)| \geq C_l(1+x_2^2)^{-l} \quad \text{or} \quad |\Pi_0(0,0,x_3)| \geq C_l(1+x_3^2)^{-l} \qquad \text{if} \quad \Pi_0(x) \not\equiv 0, \tag{1.7}$$

for all $l > 0$; see Proposition 3.8. Since $l > 0$ in (1.7) is arbitrary it is not difficult to obtain the upper bound of $|\Pi_0(x)|$ such that a contradiction arises. Indeed, after establishing several estimates of Ω by using the vorticity equations, we can deduce some polynomial decay of Π_0 from the analysis of the Poisson equation.

The plan of this chapter is as follows. In Section 2.1 we recall some equations which Π or Ω satisfies. In Section 2.2 we prove some estimates of Ω by using the vorticity equations. In this step we use the weighted estimates of the Ornstein-Uhlenbeck semigroup which are given in the appendix. In Section 2.3 we give the estimates of the velocity field from the Biot-Savart law. Section 3 is devoted to establish the pointwise estimates of Π_0. Then Theorem 1.1 is proved in Section 4.

2 Preliminaries

2.1 Fundamental equality

In this section we state several equalities which are fundamental in this chapter. Set

$$\Pi(x) = \frac{1}{2}|U(x)|^2 + P(x) + Mx \cdot U(x). \tag{2.1}$$

Let $\mathcal{L}$ be the differential operator defined by

$$\mathcal{L}f = \Delta f - Mx \cdot \nabla f. \tag{2.2}$$

Proposition 2.1. *Let* (U,P) *be a smooth solution to* $(\mathrm{NS_M})$. *Then the following equalities hold.*

$$\mathcal{L}\Pi - U \cdot \nabla \Pi = |\Omega|^2, \tag{2.3}$$

$$-\Delta U_j - (U \times \Omega)_j + \partial_j \Pi = -Mx \cdot (\nabla U_j - \partial_j U), \tag{2.4}$$

$$\mathcal{L}\Omega + (M - \mathrm{Tr}(M)I)\Omega = U \cdot \nabla \Omega - \Omega \cdot \nabla U. \tag{2.5}$$

Proof. Each equality is derived from a direct computation. we can easily check it without difficulty.

2.2 Estimates for vorticity

In this section we prove some estimates of Ω from the vorticity equations (2.5).

Proposition 2.2. *Assume that* **(C0)** *and* **(C2)** *hold. Let* $k = 0, 1, 2$. *Then*

$$(1+|x|)|\nabla^k \Omega(x)| \in L^p(\mathbb{R}^3) \qquad \text{for all} \quad p \in [p_0, \infty]. \tag{2.6}$$

Moreover, we have

$$(1+|x_2|)^{\theta_0+1}|\nabla^k \Omega(x)| \in L^\infty(\mathbb{R}^3) \qquad \text{if (i) of } (\mathbf{C3}) \text{ holds,} \tag{2.7}$$

$$(1+|x_3|)^{\frac{\theta_0}{\mu}+1}|\nabla^k \Omega(x)| \in L^\infty(\mathbb{R}^3) \qquad \text{if (ii) of } (\mathbf{C3}) \text{ holds.} \tag{2.8}$$

To prove Proposition 2.2 we introduce the semigroup $e^{t\mathcal{L}}f$ associated with the operator $\mathcal{L}$ defined by

$$\begin{aligned}(e^{t\mathcal{L}}f)(x) = {} & (2\pi)^{-\frac{3}{2}}(\det Q_t)^{-\frac{1}{2}} e^{-t\mathrm{Tr}(M)} \\ & \cdot \int_{\mathbb{R}^3} e^{-\frac{1}{2}\left\{\frac{\lambda e^{2\lambda t}}{e^{2\lambda t}-1}y_1^2 + \frac{1}{e^{2t}-1}y_2^2 + \frac{\mu}{e^{2\mu t}-1}y_3^2\right\}} f(e^{-tM}(x-y))\,\mathrm{d}y. \end{aligned} \tag{2.9}$$

Here

$$\det Q_t = \lambda^{-1}\mu^{-1}(e^{2t\lambda}-1)(1-e^{-2t})(1-e^{-2\mu t}). \tag{2.10}$$

The operator like $\mathcal{L}$ is well known as the Ornstein-Uhlenbeck operator. The representation (2.9) is easily obtained through the Fourier transform, so we proceed by admitting (2.9).

Lemma 2.3. *Let* $\theta_1, \theta_2, \theta_3 \geq 0$ *and* $1 \leq q \leq p \leq \infty$. *Set*

$$b(x) = (1+x_1^2)^{\theta_1} + (1+x_2^2)^{\theta_2} + (1+x_3^2)^{\theta_3}. \tag{2.11}$$

Then for each $k \in \mathbb{N} \cup \{0\}$ *there are positive constants* C *and* c *such that*

$$\|b\nabla^k e^{t\mathcal{L}} f\|_{L^p} \leq C t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} e^{ct}\|bf\|_{L^q}. \tag{2.12}$$

The proof of Lemma 2.3 will be stated in the appendix. The L^p-L^q estimates for $e^{t\mathcal{L}}$ without weight functions are obtained by [8] for a general class of M.

Proof of Proposition 2.2. We give the proof only for (2.6), since (2.7) and (2.8) are obtained in the similar manner. By taking (2.5) and the Laplace transform into account we set

$$\Phi(F) = \int_0^\infty e^{t\mathcal{L}} e^{t\left(M-(\mathrm{Tr}(M)+c')I\right)}\left(c'\Omega - U\cdot\nabla F + F\cdot\nabla U\right)\mathrm{d}t. \tag{2.13}$$

Here F satisfies $bF \in (L^{p_0}(\mathbb{R}^3) \cap L^{p_1}(\mathbb{R}^3))^3$ and $b\partial_j F \in (L^{p_0}(\mathbb{R}^3) \cap L^{p_2}(\mathbb{R}^3))^3$ for some $p_1, p_2 \in (p_0, \infty]$ satisfying $1/p_1 > 1/p_0 - 2/3$ and $1/p_2 > 1/p_0 - 1/3$, and $c' > 0$ is taken sufficiently large. Then by Lemma 2.3 and by using the L^∞ bound of U and ∇U, it is not difficult to see

$$
\begin{aligned}
&\|b\Phi(F)\|_{L^{p_0}\cap L^{p_1}} \leq C\|b\Omega\|_{L^{p_0}} + \delta(c')\big(\|bF\|_{L^{p_0}} + \|b\nabla F\|_{L^{p_0}}\big),\\
&\|b\nabla\Phi(F)\|_{L^{p_0}\cap L^{p_2}} \leq C\|b\Omega\|_{L^{p_0}} + \delta(c')\big(\|bF\|_{L^{p_0}} + \|b\nabla F\|_{L^{p_0}}\big),\\
&\|b\Phi(F_1) - b\Phi(F_2)\|_{L^{p_0}\cap L^{p_1}} \leq \delta(c')\big(\|bF_1 - bF_2\|_{L^{p_0}} + \|b\nabla F_1 - b\nabla F_2\|_{L^{p_0}}\big),\\
&\|b\nabla\Phi(F_1) - b\nabla\Phi(F_2)\|_{L^{p_0}\cap L^{p_2}} \leq \delta(c')\big(\|bF_1 - bF_2\|_{L^{p_0}} + \|b\nabla F_1 - b\nabla F_2\|_{L^{p_0}}\big).
\end{aligned}
$$

Here the constant $\delta(c')$ satisfies $\delta(c') \to 0$ as $c' \to \infty$. Hence by taking c' large enough we find a fixed point F_* of Φ from the contraction mapping theorem in the natural weighted Sobolev space. Since $\nabla^k U$ is bounded we can also show that F_* is smooth and bounded, and satisfies the equation

$$
\mathcal{L}F_* + (M - (\mathrm{Tr}(M) + c')I)F_* = -c'\Omega + U\cdot\nabla F_* - F_*\cdot\nabla U. \tag{2.14}
$$

Moreover, solving the adjoint equation of (2.14), we can show the uniqueness of solutions to (2.14) in $(L^{p_0}(\mathbb{R}^3))^3$ by standard argument. Thus we have $\Omega = F_*$, i.e., $b\Omega \in (L^{p_1}(\mathbb{R}^3))^3$ and $b\partial_j\Omega \in (L^{p_0}(\mathbb{R}^3)\cap L^{p_2}(\mathbb{R}^3))^3$. Repeating this argument at most finite times, we conclude that $b\Omega \in (L^\infty(\mathbb{R}^3))^3$ and $b\partial_j\Omega \in (L^\infty(\mathbb{R}^3))^3$. The property $b\partial^2_{ij}\Omega \in (L^p(\mathbb{R}^3))^3$ for $p \in [p_0,\infty]$ is then proved by the same argument as above, if one uses the equality $\nabla e^{t\mathcal{L}}f = e^{t\mathcal{L}}e^{-tM}\nabla f$. This completes the proof of Proposition 2.2.

2.3 Estimates for velocity

Let V be the velocity field recovered from Ω via the Biot-Savart law, i.e.,

$$
V(x) = (-\Delta)^{-1}\nabla\times\Omega = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(x-y)}{|x-y|^3}\times\Omega(y)\,\mathrm{d}y. \tag{2.15}
$$

Then by **(C0)** we have

$$
U = u_c + V \qquad\qquad u_c:\ \text{a constant vector.} \tag{2.16}
$$

Proposition 2.4. *Assume that* **(C0)** *and* **(C2)** *hold. Then*

$$
|V(x)| \leq C(1+|x|)^{-1}. \tag{2.17}
$$

Proof. We first note the inequality

$$
(1+|x|)|V(x)| \leq C\Big(\int_{\mathbb{R}^3}\frac{|\Omega(y)|}{|x-y|}\,\mathrm{d}y + \int_{\mathbb{R}^3}\frac{(1+|y|)|\Omega(y)|}{|x-y|^2}\,\mathrm{d}y\Big) =: C(I_1 + I_2).
$$

Then for $1/p_0' + 1/p_0 = 1$, the term I_1 is estimated as

$$
\begin{aligned}
I_1 &\leq \int_{|x-y|\leq 1}\frac{|\Omega(y)|}{|x-y|}\,\mathrm{d}y + \int_{|x-y|\geq 1}\frac{|\Omega(y)|}{|x-y|}\,\mathrm{d}y\\
&\leq C\|\Omega\|_{L^\infty} + \Big(\int_{|x-y|\geq 1}|x-y|^{-p_0'}(1+|y|)^{-p_0'}\,\mathrm{d}y\Big)^{\frac{1}{p_0'}}\|(1+|\cdot|)\Omega\|_{L^{p_0}} < \infty,
\end{aligned}
$$

since $p_0 \in (1,3)$. By Proposition 2.2 we have $(1+|x|)|\Omega(x)| \in L^{p_0}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Then by applying the Hardy-Littlewood-Sobolev inequality and the Calderón-Zygmund inequality, we get $I_2 \in L^\infty(\mathbb{R}^3)$. This completes the proof.

3 Estimates for Π

In this section we establish the estimates for Π, which is the core of the proof of Theorem 1.1. From (2.4) we have

$$-\Delta\Pi = -\nabla\cdot(U\times\Omega) + \sum_j \partial_j\big(Mx\cdot(\nabla U_j - \partial_j U)\big). \tag{3.1}$$

Taking (3.1) into account, we set

$$\begin{aligned}\Pi_0(x) &:= -(-\Delta)^{-1}\nabla\cdot(U\times\Omega) + \sum_j (-\Delta)^{-1}\partial_j\big(M(\cdot)\cdot(\nabla U_j - \partial_j U)\big)\\ &= C\sum_j \int_{\mathbb{R}^3} \frac{x_j-y_j}{|x-y|^3}\Big((U(y)\times\Omega(y))_j - My\cdot(\nabla U_j(y) - \partial_j U(y))\Big)\,\mathrm{d}y.\end{aligned} \tag{3.2}$$

3.1 Upper bound of $-\Pi_0$

Proposition 3.1. *Assume that* **(C0)** *and* **(C2)** *hold. Set $\langle x\rangle = (1+|x|^2)^{1/2}$. Then*

$$\|\Pi_0\|_{L^{q_0}} \le C(1+\|U\|_{L^\infty})\|\langle\cdot\rangle\Omega\|_{L^{p_0}}, \tag{3.3}$$
$$\|\nabla\Pi_0\|_{L^p} \le C(1+\|U\|_{L^\infty})\|\langle\cdot\rangle\Omega\|_{L^p}, \tag{3.4}$$
$$\|\nabla^2\Pi_0\|_{L^p} \le C\big((1+\|\nabla U\|_{L^\infty})\|\langle\cdot\rangle\Omega\|_{L^p} + (1+\|U\|_{L^\infty})\|\langle\cdot\rangle\nabla\Omega\|_{L^p}\big), \tag{3.5}$$

for $1/q_0 = 1/p_0 - 1/3$ and for all $p \in [p_0,\infty)$. In particular, $\Pi_0, \nabla\Pi_0 \in L^\infty(\mathbb{R}^3)$ and

$$\lim_{R\to\infty}\sup_{|x|\ge R}(|\Pi_0(x)| + |\nabla\Pi_0(x)|) = 0. \tag{3.6}$$

Moreover, if **(C3)** *holds in addition, then there is $\delta > 0$ such that*

$$|\Pi_0(0,x_2,0)| \le C(1+|x_2|)^{-\delta} \qquad \text{if (i) of } (\mathbf{C3}) \text{ holds,} \tag{3.7}$$
$$|\Pi_0(0,0,x_3)| \le C(1+|x_3|)^{-\delta} \qquad \text{if (ii) of } (\mathbf{C3}) \text{ holds.} \tag{3.8}$$

Proof. It is easy to see that

$$|\Pi_0(x)| \le C(1+\|U\|_{L^\infty})\int_{\mathbb{R}^3}\frac{1}{|x-y|^2}\langle y\rangle|\Omega(y)|\,\mathrm{d}y. \tag{3.9}$$

Hence by the Hardy-Littlewood-Sobolev inequality we have

$$\|\Pi_0\|_{L^{q_0}} \le C(1+\|U\|_{L^\infty})\|\langle\cdot\rangle\Omega\|_{L^{p_0}} \qquad \text{for } \frac{1}{q_0} = \frac{1}{p_0} - \frac{1}{3}. \tag{3.10}$$

Moreover, the Calderón-Zygmund inequality implies

$$\|\nabla\Pi_0\|_{L^p} \le C(1+\|U\|_{L^\infty})\|\langle\cdot\rangle\Omega\|_{L^p} < \infty \qquad \text{for all } p \in [p_0,\infty). \tag{3.11}$$

by Proposition 2.2. The estimate for $\|\nabla^2\Pi_0\|_{L^p}$ is obtained in the similar manner. To prove (3.7) we use the inequality (3.9) and observe that

$$\begin{aligned}
& (1+|x_2|)^\delta|\Pi_0(x)| \\
\le\ & C\int_{\mathbb{R}^3} \frac{1}{|x-y|^{2-\delta}}(1+|y|)|\Omega(y)| + \frac{1}{|x-y|^2}(1+|y|)(1+|y_2|)^\delta|\Omega(y)|\,\mathrm{d}y \\
=\ & C(I_1(x)+I_2(x)).
\end{aligned} \tag{3.12}$$

Since $(1+|x|)|\Omega(x)| \in L^{p_0}(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$ and $p_0 \in (1,3)$, if $\delta \in (0,\theta_0)$ is small enough, then it is not difficult to see $I_1 \in L^\infty(\mathbb{R}^3)$ by dividing the integral into $\int_{|x-y|\le 1}$ and $\int_{|x-y|\ge 1}$. As for I_2, we observe that

$$\begin{aligned}
I_2(0,x_2,0) &= \int_{\mathbb{R}^3} \frac{1}{(x_2-y_2)^2+y_1^2+y_3^2}(1+|y|)(1+|y_2|)^\delta|\Omega(y)|\,\mathrm{d}y \\
&\le C\int_{|y_1|+|y_3|\le 1} \frac{1}{(x_2-y_2)^2+y_1^2+y_3^2}(1+|y_2|)^{1+\delta}|\Omega(y)|\,\mathrm{d}y \\
&\quad + C\int_{|y_1|+|y_3|\ge 1} \frac{1}{(x_2-y_2)^2+y_1^2+y_3^2}(1+|y_2|)^{1+\delta}|\Omega(y)|\,\mathrm{d}y \\
&\quad + C\int_{|y_1|+|y_3|\ge 1} \frac{1}{|x_2-y_2|+|y_1|+|y_3|}(1+|y_2|)^{\delta}|\Omega(y)|\,\mathrm{d}y \\
&= I_{2,1}(x_2)+I_{2,2}(x_2)+I_{2,3}(x_2).
\end{aligned}$$

Then $I_{2,1} \in L^\infty(\mathbb{R})$ if $\delta \in (0,\theta_0)$. As for $I_{2,2}$, we note that for any $\epsilon > 0$ if $\delta < \epsilon\theta_0$ then $(1+|y_2|)^{1+\delta}|\Omega(y)| \le C\{(1+|y_2|)|\Omega(y)|\}^{1-\epsilon}$ by (i) of **(C3)**. Since $\{(1+|y|)|\Omega(y)|\}^{1-\epsilon} \in L^p(\mathbb{R}^3)$ for some $p \in (1,3)$ if $\epsilon > 0$ is sufficiently small due to **(C2)**, we have $I_{2,2} \in L^\infty(\mathbb{R}^3)$ by the Hölder inequality. Similarly, from $(1+|y_2|)^\delta|\Omega(y)| \le C|\Omega(y)|^{1-\epsilon}$ for any $\epsilon \in (0,1)$ with $\delta < \epsilon(1+\theta_0)$, we have

$$\begin{aligned}
& |I_{2,3}(x_2)| \\
& \le C\Big(\int_{|y_1|+|y_3|\ge 1} \frac{1}{(|x_2-y_2|+|y_1|+|y_3|)^{q'}(1+|y|)^{(1-\epsilon)q'}}\,\mathrm{d}y\Big)^{\frac{1}{q'}}\|\langle\cdot\rangle\Omega\|_{L^{(1-\epsilon)q}}^{1-\epsilon},
\end{aligned}$$

where $1/q' + 1/q = 1$. We choose $\epsilon > 0$ sufficiently small so that both $p_0 \le (1-\epsilon)q$ and $q < 3/(1+2\epsilon)$ are satisfied. Then the right-hand side of the above inequality is uniformly bounded with respect to x_2, since $(1-\epsilon)q' > 3/2$ in such case. The estimate (3.8) is proved in the same way. This completes the proof.

The condition **(C0)** implies $|\Pi(x)| \le C(1+|x|)$, and hence, we have from (3.1) and the definition of Π_0,

$$\Pi(x) = \sum_i a_i x_i + a_0 + \Pi_0(x), \tag{3.13}$$

for some $a_i \in \mathbb{R}$, $i = 0, 1, 2, 3$. Then (2.3) yields

$$(U + Mx)\cdot a = -|\Omega|^2 + \Delta\Pi_0 - (U + Mx)\cdot\nabla\Pi_0, \qquad a = (a_1, a_2, a_3). \tag{3.14}$$

By Proposition 3.1 the right-hand side of (3.14) has the order $o(|x|)$ at $|x| \to \infty$, so a must be the zero vector. Hence we have $\Pi = a_0 + \Pi_0$ and

$$\mathcal{L}\Pi_0 - U\cdot\nabla\Pi_0 = |\Omega|^2. \tag{3.15}$$

Since $|\Pi_0(x)| \to 0$ as $|x| \to \infty$ by Proposition 3.1, the strong maximum principle implies

Corollary 3.2. *Assume that* **(C0)** *and* **(C2)** *hold. Then either* $\Pi_0 \equiv 0$ *or* $\Pi_0(x) < 0$ *for all* $x \in \mathbb{R}^3$.

By using (2.4) we can derive the estimates for the derivatives of Π_0, which are different from the ones in Proposition 3.1.

Proposition 3.3. *Assume that* **(C0)**, **(C2)**, **(C3)** *hold. Let* $k = 1, 2$. *Then it follows that*

$$|\nabla^k\Pi_0(x)| \le C(1+|x_1|+|x_3|)(1+|x_2|)^{-\theta_0} \qquad \text{if (i) of } (\mathbf{C3}) \text{ holds,} \tag{3.16}$$

$$|\nabla^k\Pi_0(x)| \le C(1+|x_1|+|x_2|)(1+|x_3|)^{-\frac{\theta_0}{\mu}} \qquad \text{if (ii) of } (\mathbf{C3}) \text{ holds.} \tag{3.17}$$

Proof. It suffices to consider the case when (i) of **(C3)** holds. By (2.4) and $\Pi = a_0 + \Pi_0$ we have

$$\begin{aligned}\partial_j\Pi_0 = \partial_j\Pi &= \Delta U_j + (U\times\Omega)_j - Mx\cdot(\nabla U_j - \partial_j U)\\ &= -(\nabla\times\Omega)_j + (U\times\Omega)_j - Mx\cdot(\nabla U_j - \partial_j U)\\ &= I_1 + I_2 + I_3.\end{aligned} \tag{3.18}$$

Here we have used $\Delta U = -\nabla\times\Omega$. From Propositions 2.2, 2.4 we have

$$|I_1(x)| + |I_2(x)| \le C(1+|x_2|)^{-\theta_0-1}. \tag{3.19}$$

As for I_3, we have from **(C3)**,

$$|I_3(x)| \le C|x||\Omega(x)| \le C(1+|x_1|+|x_3|)(1+|x_2|)^{-\theta_0}. \tag{3.20}$$

The estimate for $\nabla^2\Pi_0$ is proved in the same way, due to Proposition 2.2. This completes the proof.

3.2 Lower bound of $-\Pi_0$

For the moment we consider a smooth nontrivial function f which satisfies

$$\mathcal{L}f - B \cdot \nabla f \geq 0, \qquad \lim_{R\to\infty} \sup_{|x|\geq R} |f(x)| = 0. \tag{3.21}$$

In this section B is always assumed to be a smooth vector function satisfying $\nabla \cdot B = 0$. The strong maximum principle implies that $f(x) < 0$ for all $x \in \mathbb{R}^3$. The aim of this section is to derive a lower bound on the spatial decay of $-f$. We start from the "rough" lower bound.

Proposition 3.4. *Let $f \in BC^2(\mathbb{R}^3)$ be a nontrivial solution to* (3.21). *Assume that*

$$\lim_{R\to\infty} \sup_{|x|\geq R} \frac{|B(x)|}{|x|} = 0. \tag{3.22}$$

Then for all $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$-f(x) \geq C_\epsilon e^{-\frac{\lambda(1+\epsilon)}{2}x_1^2 - \frac{\epsilon}{2}(x_2^2+\mu x_3^2)}, \qquad x \in \mathbb{R}^3. \tag{3.23}$$

Proof. We set

$$\tilde{f}(x) = -f(x)e^{-\frac{1}{2}({x_2}^2+\mu {x_3}^2)} = -f(x)e^{-\frac{1}{2}x^t M_0 x}, \tag{3.24}$$

where

$$M_\gamma = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{pmatrix} \qquad \text{for} \quad \gamma \in \mathbb{R}. \tag{3.25}$$

Then the direct calculations yield

$$\begin{aligned} \Delta\tilde{f} &= e^{-\frac{1}{2}x^t M_0 x}\big(- \Delta f + 2M_0 x \cdot \nabla f - f|M_0 x|^2 + f\mathrm{Tr}(M_0)\big), \\ (M_\lambda x - B) \cdot \nabla \tilde{f} &= e^{-\frac{1}{2}x^t M_0 x}\big(B \cdot \nabla f - M_\lambda x \cdot \nabla f + f M_0 x \cdot (M_\lambda x - B)\big). \end{aligned}$$

Thus we see

$$\begin{aligned} \tilde{\mathcal{L}}\tilde{f} &:= \Delta\tilde{f} + (-B + M_\lambda x) \cdot \nabla\tilde{f} + (\mathrm{Tr}(M_0) - M_0 x \cdot B)\tilde{f} \\ &= e^{-\frac{1}{2}x^t M_0 x}\big(- \Delta f + B \cdot \nabla f + Mx \cdot \nabla f\big) \\ &= e^{-\frac{1}{2}x^t M_0 x}(-\mathcal{L}f + B \cdot \nabla f) \leq 0. \end{aligned} \tag{3.26}$$

Now we set $N = 2\|\tilde{f}\|_{L^\infty} > 0$, and let $\delta \in (0, 1/4)$ and $K > 1$. Then we define the function F_δ by

$$F_\delta(x) = \frac{1}{w(x)} \log(\frac{\tilde{f}(x)}{N} + \delta) < 0,$$

where

$$w(x) = K + \frac{1}{2}(\lambda {x_1}^2 + {x_2}^2 + \mu {x_3}^2) = K + \frac{1}{2}x^t M_\lambda x.$$

Since

$$\nabla F_\delta = \frac{\nabla \tilde{f}}{w(\tilde{f} + N\delta)} - \frac{\nabla w}{w} F_\delta,$$

and

$$\begin{aligned}\Delta F_\delta &= \frac{\Delta \tilde{f}}{w(\tilde{f} + N\delta)} - 2\frac{\nabla w \cdot \nabla F_\delta}{w} - \frac{\Delta w}{w} F_\delta - \frac{|\nabla \tilde{f}|^2}{w(\tilde{f} + N\delta)^2} - \frac{|\nabla w|^2}{w} {F_\delta}^2 \\ &= \frac{\Delta \tilde{f}}{w(\tilde{f} + N\delta)} - 2\frac{\nabla w \cdot \nabla F_\delta}{w} - \frac{\Delta w}{w} F_\delta - w|\nabla F_\delta|^2 - 2F_\delta \nabla w \cdot \nabla F_\delta,\end{aligned}$$

we get from (3.26) the equation for F_δ such as

$$\begin{aligned}-\Delta F_\delta \geq \big(- B + M_\lambda x + 2\frac{\nabla w}{w} + 2F_\delta \nabla w\big) \cdot \nabla F_\delta + \frac{(\mathrm{Tr}(M_0) - M_0 x \cdot B)\tilde{f}}{w(\tilde{f} + N\delta)} \\ + \big((-B + M_\lambda x) \cdot \frac{\nabla w}{w} + \frac{\Delta w}{w} + \frac{|\nabla w|^2}{w} F_\delta\big) F_\delta + w|\nabla F_\delta|^2.\end{aligned}$$

Since $F_\delta < 0$, we have for large $p \in \mathbb{N}$,

$$\begin{aligned}&(2p-1)\int_{\mathbb{R}^3} |\nabla F_\delta|^2 F_\delta^{2(p-1)}\,\mathrm{d}x = \int_{\mathbb{R}^3} -\Delta F_\delta {F_\delta}^{2p-1}\,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^3} \big(- B + M_\lambda x + 2\frac{\nabla w}{w} + 2F_\delta \nabla w\big) \cdot \nabla F_\delta {F_\delta}^{2p-1}\,\mathrm{d}x \\ &+ \int_{\mathbb{R}^3} \{(-B + M_\lambda x) \cdot \frac{\nabla w}{w} + \frac{\Delta w}{w} + \frac{|\nabla w|^2}{w} F_\delta\} {F_\delta}^{2p}\,\mathrm{d}x \\ &+ \int_{\mathbb{R}^3} w|\nabla F_\delta|^2 {F_\delta}^{2p-1}\,\mathrm{d}x + \int_{\mathbb{R}^3} \frac{(\mathrm{Tr}(M_0) - M_0 x \cdot B)\tilde{f}}{w(\tilde{f} + N\delta)} {F_\delta}^{2p-1}\,\mathrm{d}x. \quad (3.27)\end{aligned}$$

By the integration by parts and $\nabla \cdot B = 0$ the first term of right hand side of (3.27) equals

$$\frac{1}{2p}\int_{\mathbb{R}^3} \nabla \cdot \big(- M_\lambda x - 2\frac{\nabla w}{w} - 2F_\delta \nabla w\big) {F_\delta}^{2p}\,\mathrm{d}x.$$

Since the third term of the right hand sider of (3.27) is nonpositive and

$\mathrm{Tr}(M_0) > 0$, we get

$$\begin{aligned}
&(2p-1)\int_{\mathbb{R}^3} |\nabla F_\delta|^2 {F_\delta}^{2(p-1)}\,\mathrm{d}x \\
\le\ & \frac{1}{p}\int_{\mathbb{R}^3} \big(-\frac{1}{2}\mathrm{Tr}(M_\lambda) - \nabla\cdot\frac{\nabla w}{w} - \nabla\cdot(F_\delta\nabla w)\big){F_\delta}^{2p}\,\mathrm{d}x \\
&+ \int_{\mathbb{R}^3} \big((-B+M_\lambda x)\cdot\nabla w + \Delta w + |\nabla w|^2 F_\delta\big)\frac{{F_\delta}^{2p}}{w}\,\mathrm{d}x \\
&+ \int_{\mathbb{R}^3} \frac{|M_0 x\cdot B|}{w|F_\delta|}{F_\delta}^{2p}\,\mathrm{d}x.
\end{aligned}$$

By the integration by parts we have

$$\int_{\mathbb{R}^3} \nabla\cdot(F_\delta\nabla w){F_\delta}^{2p}dx = \frac{2p}{2p+1}\int_{\mathbb{R}^3}\Delta w {F_\delta}^{2p+1}\,\mathrm{d}x,$$

and observe that $\nabla w = M_\lambda x$ and $\Delta w = \mathrm{Tr}(M_\lambda) > 0$. Thus we obtain

$$\begin{aligned}
&(2p-1)\int_{\mathbb{R}^3} |\nabla F_\delta|^2 {F_\delta}^{2(p-1)}\,\mathrm{d}x \\
\le\ & \int_{\mathbb{R}^3}\Bigg((-B+M_\lambda x)\cdot\nabla w - \frac{\mathrm{Tr}(M_\lambda)w}{2p} + \big(1-\frac{1}{p}-\frac{2wF_\delta}{2p+1}\big)\Delta w \\
&+ (F_\delta + \frac{1}{pw})|\nabla w|^2 + \frac{|M_0x\cdot B|}{|F_\delta|}\Bigg)\frac{{F_\delta}^{2p}}{w}\,\mathrm{d}x \\
=\ & \int_{\mathbb{R}^3}\Bigg((-B+M_\lambda x)\cdot M_\lambda x + \big(1-\frac{2wF_\delta}{2p+1}\big)\mathrm{Tr}(M_\lambda) \\
&+ (F_\delta + \frac{1}{pw})|M_\lambda x|^2 + \frac{|M_0x\cdot B|}{|F_\delta|}\Bigg)\frac{{F_\delta}^{2p}}{w}\,\mathrm{d}x \\
=\ & I_1 + I_2.
\end{aligned} \tag{3.28}$$

Here

$$\begin{aligned}
I_1 =\ & \int_{F_\delta > -1-\epsilon}\Bigg((-B+M_\lambda x)\cdot M_\lambda x + \big(1-\frac{2wF_\delta}{2p+1}\big)\mathrm{Tr}(M_\lambda) \\
&\qquad + (F_\delta + \frac{1}{pw})|M_\lambda x|^2 + \frac{|M_0x\cdot B|}{|F_\delta|}\Bigg)\frac{{F_\delta}^{2p}}{w}\,\mathrm{d}x \\
I_2 =\ & \int_{F_\delta \le -1-\epsilon}\Bigg((-B+M_\lambda x)\cdot M_\lambda x + \big(1-\frac{2wF_\delta}{2p+1}\big)\mathrm{Tr}(M_\lambda) \\
&\qquad + (F_\delta + \frac{1}{pw})|M_\lambda x|^2 + \frac{|M_0x\cdot B|}{|F_\delta|}\Bigg)\frac{{F_\delta}^{2p}}{w}\,\mathrm{d}x.
\end{aligned} \tag{3.29}$$

We claim that if $p \gg (\|F_\delta\|_{L^\infty}+1)(K+1)$ then there are positive constants C' and R' which are independent of p and δ such that

$$I_1 \le C'\|F_\delta\chi_{\{F_\delta > -1-\epsilon\}}\|_{L^{2p-1}}^{2p-1}, \qquad I_2 \le C'\|F_\delta\chi_{\{|x|\le R'\}}\|_{L^{2p}}^{2p}.$$

Indeed, we have

$$
\begin{aligned}
I_1 \leq \int_{F_\delta > -1-\epsilon} \Big(& \frac{|B\cdot M_\lambda x|}{w} + \frac{M_\lambda x \cdot M_\lambda x}{w} + \frac{\mathrm{Tr}(M_\lambda)}{w} - 2\mathrm{Tr}(M_\lambda)\frac{F_\delta}{2p+1} \\
& + \frac{|M_\lambda x|^2}{pw^2} + \frac{|M_0 x \cdot B|}{w|F_\delta|}\Big) {F_\delta}^{2p}\,\mathrm{d}x \\
\leq C\big(1 + \| & \frac{B\cdot M_\lambda x}{w}\|_{L^\infty} + \|\frac{B\cdot M_0 x}{w}\|_{L^\infty}\big)\|F_\delta \chi_{\{F_\delta > -1-\epsilon\}}\|_{L^{2p-1}}^{2p-1}.
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 \ \leq & \int_{F_\delta \leq -1-\epsilon} \Big(|B\cdot M_\lambda x| + M_\lambda x\cdot M_\lambda x + \mathrm{Tr}(M_\lambda)\big(1 - \frac{2wF_\delta}{2p+1}\big)\Big)\frac{{F_\delta}^{2p}}{w}\,\mathrm{d}x \\
& + \int_{F_\delta \leq -1-\epsilon} \big(-(1+\epsilon)|M_\lambda x|^2 + \frac{|M_\lambda x|^2}{pw} + |M_0 x\cdot B|\big)\frac{{F_\delta}^{2p}}{w}\,\mathrm{d}x \\
\leq & \int_{F_\delta \leq -1-\epsilon} \Big(\frac{|B\cdot M_\lambda x|}{w} + \frac{|B\cdot M_0 x|}{w} + \frac{\mathrm{Tr}(M_\lambda)}{w} - \mathrm{Tr}(M_\lambda)\frac{2F_\delta}{2p+1} \\
& + \frac{|M_\lambda x|^2}{pw^2} - \epsilon\frac{|M_\lambda x|^2}{w}\Big){F_\delta}^{2p}\,\mathrm{d}x.
\end{aligned}
$$

We observe that if R' and p are sufficiently large and $|x| \geq R'$ then

$$
\frac{|B\cdot M_\lambda x| + |B\cdot M_0 x| + \mathrm{Tr}(M_\lambda)}{w} - \mathrm{Tr}(M_\lambda)\frac{2F_\delta}{2p+1} + \frac{|M_\lambda x|^2}{pw^2} - \epsilon\frac{|M_\lambda x|^2}{w} \leq 0.
$$

Therefore

$$
I_2 \leq C\big(1 + \|\frac{B\cdot M_\lambda x}{w}\|_{L^\infty} + \|\frac{B\cdot M_0 x}{w}\|_{L^\infty}\big)\|F_\delta \chi_{\{|x|\leq R'\}}\|_{L^{2p}}^{2p}.
$$

So the claim holds by taking

$$
C' = C\big(1 + \|\frac{B\cdot M_\lambda x}{w}\|_{L^\infty} + \|\frac{B\cdot M_0 x}{w}\|_{L^\infty}\big).
$$

We have from the Sobolev inequality

$$
\|F_\delta\|_{L^{6p}}^{2p} = \|{F_\delta}^p\|_{L^6}^2 \leq C\|\nabla({F_\delta}^p)\|_{L^2}^2 = C\int_{\mathbb{R}^3} p^2 {F_\delta}^{2(p-1)}|\nabla F_\delta|^2\,\mathrm{d}x.
$$

Then by the claim and (3.28) we get

$$
\|F_\delta\|_{L^{6p}}^{2p} \leq C\frac{p^2}{2p-1}\big(\|F_\delta\chi_{\{F_\delta > -1-\epsilon\}}\|_{L^{2p-1}}^{2p-1} + \|F_\delta\chi_{\{|x|\leq R'\}}\|_{L^{2p}}^{2p}\big).
$$

Hence by letting $p \to \infty$ we have

$$
\|F_\delta\|_{L^\infty} \leq \|F_\delta\chi_{\{F_\delta > -1-\epsilon\}}\|_{L^\infty} + \|F_\delta\chi_{\{|x|\leq R'\}}\|_{L^\infty} \leq 1 + \epsilon + \|F_\delta\chi_{\{|x|\leq R'\}}\|_{L^\infty}.
$$

Since R' does not depend on δ and K, we have for $|x| \le R'$,

$$|F_\delta(x)| \le \frac{-1}{K + \frac{x^t M_\lambda x}{2}} \log(\frac{\inf_{|x|\le R'} \tilde{f}(x)}{N} + \delta) \le \frac{-1}{K} \log(\frac{\inf_{|x|\le R'} \tilde{f}(x)}{N}) \le \epsilon$$

if K is sufficiently large but independent of δ. So we have $\|F_\delta\|_{L^\infty} \le 1+2\epsilon$, that is, $\log(\frac{\tilde{f}(x)}{N} + \delta) \ge -(1+2\epsilon)\big(K + \frac{1}{2}(x^t M_\lambda x)\big)$, which implies

$$\frac{\tilde{f}(x)}{N} + \delta \ge e^{-(1+2\epsilon)K} e^{-\frac{(1+2\epsilon)}{2} x^t M_\lambda x}.$$

Hence the proof is complete by letting $\delta \to 0$ and from the definition of $\tilde{f}(x)$.

Next we show a more precise lower bound of $-f$ under the additional condition on B.

Proposition 3.5. *Let $f \in BC^2(\mathbb{R}^3)$ be a nontrivial solution to* (3.21)*. Assume that $B \in (L^\infty(\mathbb{R}^3))^3$ and*

$$\lim_{R\to\infty} \sup_{|x_1|\le R_0, |x_2|+|x_3|\ge R} |B_1(x)| = 0 \qquad \text{for all} \quad R_0 > 0. \tag{3.30}$$

Then for all $\theta > \lambda$ and $\epsilon > 0$ there is $C_{\theta,\epsilon} > 0$ such that

$$-f(x) \ge C_{\theta,\epsilon}(1 + x_2^2 + (1+x_3^2)^{\frac{1}{\mu}})^{-\frac{\theta}{2}} e^{-\frac{1+\epsilon}{2}\lambda x_1^2}. \qquad x \in \mathbb{R}^3. \tag{3.31}$$

Proof. For $\epsilon, \epsilon' > 0$ we set

$$W_{\epsilon,\epsilon'}(x) := (1 + x_2^2 + (1+x_3^2)^{\frac{1}{\mu}})^{-\frac{\theta}{2}} e^{\frac{-\epsilon'}{2}({x_2}^2 + \mu {x_3}^2) - \frac{1+\epsilon}{2}\lambda {x_1}^2},$$
$$H_{\epsilon,\epsilon'}(x) := \frac{W_{\epsilon,\epsilon'}(x)}{-f(x)} \ge 0.$$

Note that by Proposition 3.4 the function $H_{\epsilon,\epsilon'}(x)$ rapidly decays at spatial infinity for each $\epsilon, \epsilon' > 0$. The direct calculation shows

$$\begin{aligned}
\nabla H_{\epsilon,\epsilon'} &= -H_{\epsilon,\epsilon'}\frac{\nabla f}{f} - \frac{\nabla W_{\epsilon,\epsilon'}}{f} = -H_{\epsilon,\epsilon'}\frac{\nabla f}{f} + \frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}} H_{\epsilon,\epsilon'}, \\
\Delta H_{\epsilon,\epsilon'} &= -\frac{\Delta f}{f} H_{\epsilon,\epsilon'} - 2\frac{\nabla f}{f}\cdot\nabla H_{\epsilon,\epsilon'} - \frac{\Delta W_{\epsilon,\epsilon'}}{f} \\
&= -\frac{\Delta f}{f} H_{\epsilon,\epsilon'} + 2(\frac{\nabla H_{\epsilon,\epsilon'}}{H_{\epsilon,\epsilon'}} - \frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}})\cdot\nabla H_{\epsilon,\epsilon'} + \frac{\Delta W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}} H_{\epsilon,\epsilon'}.
\end{aligned}$$

Thus by (3.21) we have

$$\begin{aligned}
& -\Delta H_{\epsilon,\epsilon'} \\
\le\ & (B + Mx)\cdot\frac{\nabla f}{f} H_{\epsilon,\epsilon'} - 2(\frac{\nabla H_{\epsilon,\epsilon'}}{H_{\epsilon,\epsilon'}} - \frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}})\cdot\nabla H_{\epsilon,\epsilon'} - \frac{\Delta W_{\epsilon,\epsilon'}}{W} H_{\epsilon,\epsilon'} \\
\le\ & \big(\frac{2\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}} - B - Mx\big)\cdot\nabla H_{\epsilon,\epsilon'} - \big((-B - Mx)\cdot\frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}} + \frac{\Delta W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}}\big) H_{\epsilon,\epsilon'}.
\end{aligned}$$

Then the integration by parts yields

$$
\begin{aligned}
&(2p-1)\int_{\mathbb{R}^3}|\nabla H_{\epsilon,\epsilon'}|^2 H_{\epsilon,\epsilon'}^{2(p-1)}\,\mathrm{d}x\\
&\le -\frac{1}{2p}\int_{\mathbb{R}^3}\nabla\cdot\big(-B-Mx+2\frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}}\big)H_{\epsilon,\epsilon'}^{2p}\,\mathrm{d}x\\
&\qquad\qquad -\int_{\mathbb{R}^3}\big((-B-Mx)\cdot\frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}}+\frac{\Delta W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}}\big)H_{\epsilon,\epsilon'}^{2p}\,\mathrm{d}x\\
&= -\int_{\mathbb{R}^3}\{-\frac{\mathrm{Tr}(M)}{2p}+\frac{1}{p}\nabla\cdot\frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}}+\frac{\Delta W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}}-(B+Mx)\cdot\frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}}\}H_{\epsilon,\epsilon'}^{2p}dx
\end{aligned}
\tag{3.32}
$$

We observe that

$$
\begin{aligned}
\frac{\Delta W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}}-Mx\cdot\frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}} &= (\epsilon')^2x^tM_0x-\epsilon\lambda-\epsilon'\mathrm{Tr}(M_0)+O(\frac{1}{1+x_2^2+x_3^2})\\
&+(\theta+2\epsilon'\theta)\frac{{x_2}^2+(1+{x_3}^2)^{\frac{1}{\mu}-1}{x_3}^2}{1+{x_2}^2+(1+{x_3}^2)^{\frac{1}{\mu}}}-\lambda+\lambda^2\epsilon(1+\epsilon){x_1}^2+\epsilon'x^tM_0x,
\end{aligned}
\tag{3.33}
$$

$$
-B\cdot\frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}}=\lambda(1+\epsilon)B_1x_1+\epsilon'B\cdot M_0x+\frac{\theta}{\mu}\frac{\mu B_2x_2+B_3x_3(1+{x_3}^2)^{\frac{1}{\mu}-1}}{1+{x_2}^2+(1+{x_3}^2)^{\frac{1}{\mu}}},
\tag{3.34}
$$

and

$$
\nabla\cdot\frac{\nabla W_{\epsilon,\epsilon'}}{W_{\epsilon,\epsilon'}}=-(\lambda+\epsilon\lambda+\epsilon'+\epsilon'\mu)+O(\frac{1}{1+x_2^2+x_3^2}).
\tag{3.35}
$$

From the assumption on B and the condition $\theta>\lambda$, if ϵ and ϵ' are small enough and p is sufficiently large then there exists $R>0$ independent of ϵ' (but depending on ϵ) such that the integrand of the right hand side of (3.32) is nonnegative when $|x|\ge R$. Indeed, it suffices to consider each case of (i) $|x_1|\ge R/2$ and (ii) $|x_1|\le R/2$ and $(x_2^2+x_3^2)^{1/2}\ge R/2$; when $|x_1|\ge R/2$ the term $\lambda^2\epsilon(1+\epsilon){x_1}^2+\epsilon'x^tM_0x$ is dominant, and when $|x_1|\le R/2$ and $(x_2^2+x_3^2)^{1/2}\ge R/2$ the term

$$
(\theta+2\epsilon'\theta)\frac{{x_2}^2+(1+{x_3}^2)^{\frac{1}{\mu}-1}{x_3}^2}{1+{x_2}^2+(1+{x_3}^2)^{\frac{1}{\mu}}}+\epsilon'x^tM_0x
$$

becomes dominant by the assumptions. Therefore we have

$$
(2p-1)\int_{\mathbb{R}^3}|\nabla H_{\epsilon,\epsilon'}|^2H_{\epsilon,\epsilon'}^{2(p-1)}\,\mathrm{d}x\le C\|H_{\epsilon,\epsilon'}\chi_{\{|x|\le R\}}\|_{L^{2p}}^{2p},
$$

and then $\|H_{\epsilon,\epsilon'}\|_{L^{6p}}^{2p}\le Cp^2(2p-1)^{-1}\|H_{\epsilon,\epsilon'}\chi_{\{|x|\le R\}}\|_{L^{2p}}^{2p}$. By taking $p\to\infty$, we have $\|H_{\epsilon,\epsilon'}\|_{L^\infty}\le\|H_{\epsilon,\epsilon'}\chi_{\{|x|\le R\}}\|_{L^\infty}$ for all small $\epsilon'>0$, and thus

$\|H_{\epsilon,0}\|_{L^\infty} \leq \|H_{\epsilon,0}\chi_{\{|x|\leq R\}}\|_{L^\infty}$. Since $\inf_{|x|\leq R}(-f(x)) \neq 0$ for each $R > 0$, we have

$$0 < H_{\epsilon,0}(x) = \frac{(1 + x_2^2 + (1 + x_3^2)^{\frac{1}{\mu}})^{-\frac{\theta}{2}} e^{-\frac{1+\epsilon}{2}\lambda {x_1}^2}}{-f(x)} \leq C_{\theta,\epsilon} \qquad \text{if } \ |x| \leq R.$$

So we conclude that $|H_{\epsilon,0}(x)| \leq \|H_{\epsilon,0}\chi_{\{|x|\leq R\}}\|_{L^\infty} \leq C_{\theta,\epsilon}$, which gives

$$-f(x) \geq C_{\theta,\epsilon}(1 + x_2^2 + (1 + x_3^2)^{\frac{1}{\mu}})^{-\frac{\theta}{2}} e^{-\frac{1+\epsilon}{2}\lambda {x_1}^2}.$$

This completes the proof of Proposition 3.5.

Remark 3.6. The function $f(x) = -(1 + (x_2^2 + x_3^2)/2)^{-1} e^{-x_1^2}$ satisfies (3.21) with $B = 0$, $\lambda = 2$, and $\mu = 1$. Hence (3.31) is considered to be rather optimal under the conditions in Proposition 3.5.

Corollary 3.7. *Assume that* **(C0)**-**(C2)** *hold and that* $\Pi_0 \not\equiv 0$. *Then for all* $\theta > \lambda$ *and* $\epsilon > 0$ *there is* $C_{\theta,\epsilon} > 0$ *such that*

$$-\Pi_0(x) \geq C_{\theta,\epsilon}(1 + x_2^2 + (1 + x_3^2)^{\frac{1}{\mu}})^{-\frac{\theta}{2}} e^{-\frac{1+\epsilon}{2}\lambda x_1^2}. \qquad x \in \mathbb{R}^3. \tag{3.36}$$

Proof. From (2.16) and Proposition 2.4 it suffices to show $U_1 = V_1$; then the assumptions in Proposition 3.5 are satisfied. Assume that (i) of **(C1)** holds. Then by the relation $\Pi(x) = |U(x)|^2/2 + P(x) + Mx\cdot(u_c + V(x))$ we must have $u_c = (0, u_{c,2}, u_{c,3})$ since $\Pi(x) = a_0 + \Pi_0(x)$ is bounded function. Thus $U_1 = V_1$ follows. When (ii) of **(C1)** holds $u_c = (0, u_{c,2}, u_{c,3})$ is trivial due to Proposition 2.4. This completes the proof.

3.3 Lower bound of $-\Pi_0$ in (x_2, x_3) direction

Proposition 3.8. *Assume that* **(C0)**-**(C3)** *hold and that* $\Pi_0 \not\equiv 0$. *Then for any* $l > 0$ *there is* $C > 0$ *such that*

$$-\Pi_0(0, x_2, 0) \geq C(1 + |x_2|)^{-l} \qquad \text{if (i) of } (\mathbf{C3}) \text{ holds,} \tag{3.37}$$
$$-\Pi_0(0, 0, x_3) \geq C(1 + |x_3|)^{-l} \qquad \text{if (ii) of } (\mathbf{C3}) \text{ holds.} \tag{3.38}$$

Proof. We give the proof only for the case when (i) of **(C3)** holds, since the other case is proved in the same way. Set $g(x_2) = -\Pi_0(0, x_2, 0) > 0$. From (2.3), g satisfies

$$\begin{aligned}\partial_2^2 g - x_2\partial_2 g = {}& (\partial_1^2\Pi_0)(0, x_2, 0) + (\partial_3^2\Pi_0)(0, x_2, 0)\\ & - U(0, x_2, 0)\cdot(\nabla\Pi_0)(0, x_2, 0) - |\Omega(0, x_2, 0)|^2,\end{aligned}$$

and hence, by Proposition 3.3 and **(C0)**,

$$\partial_2^2 g - x_2\partial_2 g \leq C(1 + |x_2|)^{-\theta_0}. \tag{3.39}$$

Now we use the same argument as in Proposition 3.5 to establish the lower bound of g. Set

$$h_{l,\epsilon}(x_2) = \frac{w_{l,\epsilon}(x_2)}{g(x_2)}, \qquad w_{l,\epsilon}(x_2) = (1+x_2^2)^{-l} e^{-\epsilon x_2^2}, \qquad l, \epsilon > 0. \quad (3.40)$$

Then $h \in W^{2,p}(\mathbb{R}^3)$ for all $p \gg 1$, and we have the inequality

$$\begin{aligned}
&(2p-1) \int_{\mathbb{R}} |\partial_2 h_{l,\epsilon}(x_2)|^2 |h_{l,\epsilon}(x_2)|^{2(p-1)} \, \mathrm{d}x_2 \\
\leq\ & \frac{1}{2p} \int_{\mathbb{R}} \big(1 - 2\partial_2 (\frac{\partial_2 w_{l,\epsilon}}{w_{l,\epsilon}})\big) |h_{l,\epsilon}(x_2)|^{2p} \, \mathrm{d}x_2 \\
&\qquad - \int_{\mathbb{R}} \big(- \frac{x_2 \partial_2 w_{l,\epsilon}}{w_{l,\epsilon}} + \frac{\partial_2^2 w_{l,\epsilon}}{w_{l,\epsilon}} - C\frac{(1+|x_2|)^{-\theta_0}}{g} \big) |h_{l,\epsilon}(x_2)|^{2p} \, \mathrm{d}x_2.
\end{aligned} \quad (3.41)$$

Since $l > 0$, $\theta_0 > \lambda$, and $g(x_2) \geq C(1+|x_2|)^{-\theta}$ for all $\theta > \lambda$ by Corollary 3.7, there is $R \geq 1$ independent of $\epsilon > 0$ such that

$$(2p-1) \int_{\mathbb{R}} |\partial_2 h_{l,\epsilon}(x_2)|^2 |h_{l,\epsilon}(x_2)|^{2(p-1)} \, \mathrm{d}x_2 \leq C \|h_{l,\epsilon}\|_{L^{2p}(B_R)}^{2p}.$$

Then the Gagliardo-Nirenberg inequality yields

$$\|h_{l,\epsilon}^p\|_{L^\infty} \leq C \|h_{l,\epsilon}^p\|_{L^2}^{\frac{1}{2}} \|\partial_2 (h^p)\|_{L^2}^{\frac{1}{2}} \leq C p^{\frac{1}{4}} \|h^p\|_{L^2}^{\frac{1}{2}} \|h\|_{L^{2p}(B_R)}^{\frac{p}{2}},$$

that is, $\|h_{l,\epsilon}\|_{L^\infty} \leq (Cp)^{1/(4p)} \|h_{l,\epsilon}\|_{L^{2p}}^{\frac{1}{2}} \|h_{l,\epsilon}\|_{L^{2p}(B_R)}^{\frac{1}{2}}$. Tending $p \to \infty$, we get $\|h_{l,\epsilon}\|_{L^\infty} \leq \|h_{l,\epsilon}\|_{L^\infty(B_R)} < \infty$. Since R is independent of $\epsilon > 0$, we have $g(x_2) \geq C(1+|x_2|)^{-l}$ for all $l > 0$. This completes the proof.

4 Proof of Theorem 1.1

Proof of Theorem 1.1. If $\Pi_0 \not\equiv 0$ then the lower bound for Π_0 in Proposition 3.8 contradicts with the decay estimate of Π_0 in (3.7) or (3.8). Hence $\Pi_0 \equiv 0$, i.e., $\Pi \equiv$ const. Thus we have $\Omega \equiv 0$ from (2.3), which implies $U = u_c =$ const.

5 Appendix

Proof of Lemma 2.3. We first give the proof for $k = 0$. For simplicity of notations we set

$$\begin{aligned}
&h(t,x) = e^{-\frac{1}{2}\left(\frac{\lambda e^{2\lambda t}}{e^{2\lambda t}-1} x_1^2 + \frac{1}{e^{2t}-1} x_2^2 + \frac{\mu}{e^{2\mu t}-1} x_3^2\right)}, \qquad F(t,x) = f(e^{-tM} x), \\
&G(t) = (2\pi)^{-\frac{3}{2}} (\det Q_t)^{-\frac{1}{2}} e^{-t\mathrm{Tr}(M)}.
\end{aligned}$$

Then we have

$$
\begin{aligned}
b(x)(e^{t\mathcal{L}}f)(x) = & \; G(t)b(x)\int_{\mathbb{R}^3} h(t,y)F(t,x-y)\,\mathrm{d}y \\
= & \; G(t)\int_{\mathbb{R}^3} b(x)h(t,x-y)F(t,y)\,\mathrm{d}y,
\end{aligned}
$$

and by the definition of $b(x)$ we obtain

$$
|b(x)(e^{t\mathcal{L}}f)(x)| \leq CG(t)\int_{\mathbb{R}^3}\big(b(x-y)h(t,x-y)+h(t,x-y)b(y)\big)|F(t,y)|\,\mathrm{d}y. \tag{5.1}
$$

For $1 \leq q \leq p \leq \infty$ and $1 \leq r < \infty$ satisfying $1/p = 1/r + 1/q - 1$ we get by the Young inequality

$$
\|be^{t\mathcal{L}}f\|_{L^p} \leq CG(t)\big(\|bh(t)\|_{L^r}\|F(t)\|_{L^q} + \|h(t)\|_{L^r}\|bF(t)\|_{L^q}\big). \tag{5.2}
$$

We observe that

$$
\begin{aligned}
\|F(t)\|_{L^q}^q = & \; e^{t\mathrm{Tr}(M)}\int_{\mathbb{R}^3}|f(z)|^q\,\mathrm{d}z \\
\leq & \; e^{t\mathrm{Tr}(M)}\int_{\mathbb{R}^3}|b(z)|^q|f(z)|^q\,\mathrm{d}z = e^{t\mathrm{Tr}(M)}\|bf\|_{L^q}^q,
\end{aligned}
$$

and

$$
\begin{aligned}
\|bF(t)\|_{L^q}^q = & \; \int_{\mathbb{R}^3}|b(y)|^q|f(e^{-tM}y)|^q\,\mathrm{d}y \\
\leq & \; Ce^{ct}e^{t\mathrm{Tr}(M)}\int_{\mathbb{R}^3}|b(z)|^q|f(z)|^q\,\mathrm{d}z \leq Ce^{ct}\|bf\|_{L^q}^q,
\end{aligned}
$$

where C and c depend on θ_i and λ_i. So we have

$$
\|be^{t\mathcal{L}}f\|_{L^p} \leq C(\det\, Q_t)^{-\frac{1}{2}}e^{ct}\|bf\|_{L^q}\big(\|bh(t)\|_{L^r} + \|h(t)\|_{L^r}\big). \tag{5.3}
$$

The direct calculation implies

$$
\begin{aligned}
\|h(t)\|_{L^r} = & \; \Big(\int_{\mathbb{R}^3} e^{-\frac{r}{2}\left\{\frac{\lambda e^{2\lambda t}}{e^{2\lambda t}-1}y_1^2+\frac{1}{e^{2t}-1}y_2^2+\frac{\mu}{e^{2\mu t}-1}y_3^2\right\}}\,\mathrm{d}y\Big)^{\frac{1}{r}} \\
= & \; \Big(\int_{\mathbb{R}^3} e^{-z^2}\,\mathrm{d}z\Big)^{\frac{1}{r}} G_r(t) \leq CG_r(t),
\end{aligned}
$$

where

$$
G_r(t) = \Big(\frac{2}{r}\Big)^{\frac{3}{2r}}\Big(\frac{\lambda e^{2\lambda t}}{e^{2\lambda t}-1}\Big)^{\frac{-1}{2r}}\Big(\frac{1}{e^{2t}-1}\Big)^{\frac{-1}{2r}}\Big(\frac{\mu}{e^{2\mu t}-1}\Big)^{\frac{-1}{2r}}.
$$

Next we compute

$$\|bh(t)\|_{L^r} = \Big(\int_{\mathbb{R}^3} |b(y)|^r e^{-\frac{r}{2}\left(\frac{\lambda e^{2\lambda t}}{e^{2\lambda t}-1}y_1^2+\frac{1}{e^{2t}-1}y_2^2+\frac{\mu}{e^{2\mu t}-1}y_3^2\right)}\,\mathrm{d}y\Big)^{\frac{1}{r}}$$
$$\le C\bigg(\int_{\mathbb{R}^3}\big(1+\frac{2}{r}\frac{e^{2\lambda t}-1}{\lambda e^{2\lambda t}}{z_1}^2\big)^{\theta_1 r}$$
$$+\big(1+\frac{2}{r}(e^{2t}-1){z_2}^2\big)^{\theta_2 r}+\big(1+\frac{2}{r}\frac{e^{2\mu t}-1}{\mu}{z_3}^2\big)^{\theta_3 r}\,\mathrm{d}y\bigg)^{\frac{1}{r}} G_r(t).$$

Since $\displaystyle\int_{\mathbb{R}} |z_j|^{2\theta_j r} e^{-{z_j}^2}\,\mathrm{d}z_j < C$ for $1 \le r < \infty$ we have

$$\|bh(t)\|_{L^r} \le C\big(1+(\frac{2}{r}\frac{e^{2\lambda t}-1}{\lambda e^{2\lambda t}})^{\theta_1}+(\frac{2}{r}(e^{2t}-1))^{\theta_2}+(\frac{2}{r}\frac{e^{2\mu t}-1}{\mu})^{\theta_3}\big)G_r(t).$$

Then by combining the estimates of $\|h(t)\|_{L^r}$ and $\|bh(t)\|_{L^r}$ with (5.3) we obtain

$$\|be^{t\mathcal{L}}f\|_{L^p}$$
$$\le\ C\frac{e^{ct}\|bf\|_{L^q}G_r(t)\big(1+(\frac{2}{r}\frac{e^{2\lambda t}-1}{\lambda e^{2\lambda t}})^{\theta_1}+(\frac{2}{r}(e^{2t}-1))^{\theta_2}+(\frac{2}{r}\frac{e^{2\mu t}-1}{\mu})^{\theta_3}\big)}{(\det\, Q_t)^{\frac{1}{2}}}.$$

Observing that

$$(\det\, Q_t)^{-\frac{1}{2}}G_r(t) \le Ce^{(\frac{1+\mu}{r}-\lambda)t}\Big\{\frac{1}{(1-e^{-2t\lambda})(1-e^{-2t})(1-e^{-2t\mu})}\Big\}^{\frac{1}{2}(1-\frac{1}{r})},$$

we finally obtain

$$\|be^{t\mathcal{L}}f\|_{L^p} \le Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}e^{ct}\|bf\|_{L^q},$$

where the constants C and c depend only on θ_i, λ_i, p, and q. As for the case $r=\infty$, the only possibility is $p=\infty$ and $q=1$. Then the similar argument shows

$$\|be^{t\mathcal{L}}f\|_{L^\infty} \le C(\det\, Q_t)^{-\frac{1}{2}}e^{ct}\|bf\|_{L^1}\big(\|bh(t)\|_{L^\infty}+\|h(t)\|_{L^\infty}\big).$$

Since h and bh are bounded functions in time and space we complete the proof for $k=0$. For $k=1$ it will be sufficient to show that

$$\|b\partial_1 e^{t\mathcal{L}}f\|_{L^p} \le Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}e^{ct}\|bf\|_{L^q}.$$

But as in the case of $k=0$ it is not difficult to derive the inequality

$$\|b\partial_1 e^{t\mathcal{L}}f\|_{L^p}$$
$$\le\ Ce^{ct}\|bf\|_{L^q}\Big\{\frac{1}{(1-e^{-2t\lambda})(1-e^{-2t})(1-e^{-2t\mu})}\Big\}^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\Big(\frac{1}{1-e^{-2t\lambda}}\Big)^{\frac{1}{2}}$$
$$\le\ Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}e^{ct}\|bf\|_{L^q}.$$

The estimates (2.12) for higher order derivatives are proved in the same manner. This completes the proof.

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Chapter 3

A Liouville theorem for the planer Navier-Stokes equations with the no-slip boundary condition and its application to a geometric regularity criterion

1 Introduction

In this chapter we study a backward solution to the Navier-Stokes equations in the half plane

$$\partial_t u + \operatorname{div}(u \otimes u) - \Delta u + \nabla p = 0, \qquad \operatorname{div} u = 0 \qquad \text{in } (-\infty, 0) \times \mathbb{R}^2_+ \tag{1.1}$$

subject to the no-slip boundary condition

$$u = 0 \qquad \text{on } (-\infty, 0) \times \partial\mathbb{R}^2_+. \tag{1.2}$$

Here $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$, and $u = u(t,x) = (u_1(t,x), u_2(t,x))$, $p = p(t,x)$ denote the velocity field, the pressure field, respectively. We use the standard notation for derivatives; $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $\Delta = \sum_{j=1}^2 \partial_j^2$, $\operatorname{div} u = \sum_{j=1}^2 \partial_j u_j$, and $(u \otimes u)_{1\le i,j\le 2} = (u_i u_j)_{1\le i,j\le 2}$.

We are interested in the Liouville problem for (1.1) - (1.2), that is, the nonexistence of nontrivial bounded global solutions to (1.1) - (1.2). As is well known, in the study of evolution equations the Liouville problem for bounded *backward* solutions plays an important role in obtaining an a priori bound of *forward* solutions through a suitable scaling argument called a blow-up argument. For example, the reader is referred to [11] for semilinear parabolic equations, to [21, 29] for the axisymmetric Navier-Stokes equations (see also [7, 8] for a different approach), to [16, 12] for a geometric regularity criterion to the three-dimensional Navier-Stokes equations, and to a recent result [1] for the Stokes semigroup in L^∞ spaces.

This work is particularly motivated by [16, 12], where (1.1) - (1.2) is naturally derived from a blow-up argument for the three-dimensional Navier-Stokes equations in the half space. Indeed, if one imposes a uniform continuity on the alignment of the vorticity direction, the blow-up limit of the three-dimensional (Navier-Stokes) flow must be a nontrivial bounded two-dimensional flow, and the problem is essentially reduced to the analysis of (1.1) - (1.2). If, in addition, one assumes that the possible blow-up is type I, then the limit flow is not allowed to be a constant in time. Thus the resolution of the Liouville problem is a crucial step to reach a contradiction. From this systematic argument we can exclude the possibility of type I blow-up for the original three-dimensional flows under a regularity condition on the vorticity direction.

Recently the paper [16] successfully completes the above argument when the velocity field satisfies the *perfect slip* boundary condition, but the problem was remained open for the case of the *no-slip* boundary condition, which is physically more relevant. In this chapter we prove a Liouville type theorem for (1.1) - (1.2) under some conditions on the velocity field u, the pressure field p, and the vorticity field $\omega = \partial_1 u_2 - \partial_2 u_1$. Our result is useful enough to settle the problem left open in [16]; see Theorem 1.2 below. The details on this geometric regularity criterion will be discussed in Section 4.

When one discusses the Liouville problem the choice of function spaces is of course a crucial issue. Indeed, if u solves (1.1) - (1.2) and decays fast enough in time and space then it is easy to conclude that u is identically zero by a standard energy inequality. However, in view of application to the geometric regularity criterion, it is important to establish a Liouville type result within the framework of spatially nondecaying solutions. We should recall here that there are nontrivial shear flows whose velocity fields are bounded and decaying in time as $t \to -\infty$, while the pressure fields grow linearly at spatial infinity; see [30, 12], and see also (4.3) below. The appearance of the time-decaying shear flows is due to both the presence of the nontrivial boundary and the no-slip boundary condition in (1.1) - (1.2). Indeed, if we consider the whole space case or if we replace (1.2) by the perfect slip boundary condition, $\partial_2 u_1 = u_2 = 0$ on $\partial \mathbb{R}^2$, then such kind of flows does not exist. We note that these shear flows also solve the Stokes equations (i.e. nonlinear term is absent). Thus, even for the linearized problem, we need to impose some assumptions on the spatial growth of the pressure field to obtain a Liouville theorem. In fact, for the Stokes equations it is recently shown in [19] that any nontrivial bounded backward solution has to be a shear flow. Especially, the result of [19] gives a complete characterization of bounded backward solutions for the linear problem.

On the other hand, for the full Navier-Stokes equations there seems to be still few results on the Liouville type problem even in the case of the half plane. The crucial difficulty is that, though the vorticity field satisfies the heat-transport equations, maximum principle is no longer a useful tool to obtain an a priori bound of the vorticity field. Indeed, the no-slip boundary condition on the velocity field is in general a source of vorticity on the boundary, and maximum principle does not provide useful information about this vorticity production on the boundary. This is contrasting with the case of the whole plane or of the perfect slip boundary condition, where there is no vorticity production near the boundary and maximum principle is directly applied to derive an a priori bound of the vorticity field. Although the analysis of the vorticity equations is a core part also in the proof of our Liouville theorem, the key idea to overcome the difficulty is to use the *boundary condition* on the vorticity field, rather than maximum principle.

Roughly speaking, our Liouville theorem requires four kinds of assumptions. The first one is a uniform bound on the velocity field including their derivatives. The second one is on a structure of the pressure field, which is essential to exclude the shear flows in [30, 12] but is a natural requirement in order to restrict our solutions to mild solutions, i.e., solutions to the integral equations associated with (1.1) - (1.2). The third one is the type I temporal decay of the velocity field as $t \to -\infty$. The last one is the nonnegativity of the vorticity field. Precisely, the main result of this chapter is stated as follows.

Theorem 1.1. *Let* (u,p) *be a solution to* (1.1)-(1.2) *satisfying the following conditions.*

(C1) $\sup_{-\infty<t<0} \big(\|u(t)\|_{C^{2+\mu}} + \|\partial_t u(t)\|_{C^\mu}\big) < \infty$ *for some* $\mu \in (0,1)$.
(C2) $p = p_F + p_H$, *where* $p_F(t)$ *is the solution to* (2.3) *in Proposition 2.1 with* $F = -u(t)\otimes u(t)$ *and* $p_H(t)$ *is the solution to* (2.6) *in Proposition 2.2 with* $g = \omega(t)|_{x_2=0}$, *respectively.*
(C3) $\sup_{-\infty<t<0} (-t)^{1/2}\|u(t)\|_\infty < \infty$.
(C4) $\omega \geq 0$ *in* $(-\infty,0)\times\mathbb{R}^2_+$, *where* $\omega = \partial_1 u_2 - \partial_2 u_1$ *is the vorticity field.*

Then u *is identically zero.*

Here $\|\cdot\|_{C^{2+\mu}}$ and $\|\cdot\|_{C^\mu}$ denote the norms of the Hölder spaces (the definitions are stated in the end of this section), and $\|\cdot\|_\infty$ stands for the usual sup norm in the x variables.

The condition (C3) in Theorem 1.1 is compatible with the type I blow-up assumption for forward solutions. The sign condition (C4) on the vorticity field is a rather strong requirement at least in the class of spatially decaying solutions. Indeed, if there is a time t such that $\sup_{x_1}|u_1(t,x_1,x_2)| \to 0$ as $x_2 \to \infty$ then it is not difficult to see $u = 0$ even when (C3) is absent; see [12, Theorem 3.3]. However, in the framework of nondecaying solutions the situation is different and becomes complicated. We note that, as is observed in [12], there is a shear flow satisfying all of (C1), (C3), and (C4).

The key idea of the proof of Theorem 1.1 is to focus on the velocity field *formally* defined by the Biot-Savart law:

$$v(t,x) := \frac{1}{2\pi}\int_{\mathbb{R}^2_+} \Big(\frac{(x-y)^\perp}{|x-y|^2} - \frac{(x-y^*)^\perp}{|x-y^*|^2}\Big)\omega(t,y)\,\mathrm{d}y,\ x^\perp = (-x_2,x_1),\ y^* = (y_1,-y_2). \tag{1.3}$$

We note that v coincides with u when u and ω decay fast enough at spatial infinity. By formally taking the boundary trace of v_1 we observe that

$$v_1(t,x_1,0) = \frac{1}{\pi}\int_{\mathbb{R}^2+} \frac{y_2}{(x_1-y_1)^2+y_2^2}\omega(t,y)\,\mathrm{d}y. \tag{1.4}$$

Hence, if v_1 satisfies the no-slip boundary condition then the assumption (C4) implies $\omega = 0$, which leads to $u = 0$ by the Liouville theorem for bounded harmonic functions.

In order to justify the above formal argument we need to prove the following two claims:
Claim 1: The integral representation of the right-hand side of (1.3) is well-defined. In other words, the vorticity field has an enough spatial decay so that the integral in (1.3) converges.

Claim 2: The tangential component v_1 satisfies the no-slip boundary condition.

Both of two claims are far from trivial, for we have to start from the spatially nondecaying data, and the right-hand side of (1.4) is highly nonlocal. To show Claim 1 we make use of the type I temporal decay of u assumed in (C3). In fact, since (C3) is a scaling invariant bound, by applying the result of [6] or [27] we can establish the Gaussian pointwise bound of the Green function for the heat-transport operator $\partial_t - \Delta + u \cdot \nabla$ with the Neumann boundary condition. This pointwise estimate of the Green function leads to a polynomial decay of the vorticity field as $x_2 \to \infty$, which makes the integral of (1.3) well-defined. The key ingredient of the proof of Claim 2 is the boundary condition on the vorticity field. Indeed, combined with a calculation based on the integration by parts, the vorticity boundary condition yields $\partial_t v_1(t, x_1, 0) = 0$ for $-\infty < t < 0$ and $x_1 \in \mathbb{R}$, as is already observed in [24] in the setting of spatially decaying solutions. Then the no-slip boundary condition for v_1 is a consequence of the convergence $\lim_{t\to-\infty} v_1(t, x_1, 0) = 0$, which can be verified from the time decay condition (C3) and the polynomial decay of the vorticity field established in Claim 1.

As an application of Theorem 1.1, we can extend the geometric regularity criterion in [16] for the three-dimensional Navier-Stokes equations in the half space to the case of the no-slip boundary condition.

Theorem 1.2. *Let (u, p) be a spatially bounded mild solution to the Navier-Stokes equations* (4.1)-(4.2) *in* $(0, T) \times \mathbb{R}^3_+$*. Assume that the possible blow-up of u is type I, i.e.*

$$\sup_{0<t<T} (T - t)^{\frac{1}{2}} \|u(t)\|_\infty < \infty.$$

Let d be a positive number and let η be a nondecreasing continuous function on $[0, \infty)$ *satisfying* $\eta(0) = 0$*. Assume that η is a modulus of continuity in the x variables for the vorticity direction $\xi = \omega/|\omega|$, in the sense that*

$$|\xi(t, x) - \xi(t, y)| \leq \eta(|x - y|) \quad \text{for} \quad (t, x), (t, y) \in \Omega_d, \tag{CA}$$

where $\Omega_d = \{(t, x) \in (0, T) \times \mathbb{R}^3_+ \mid |\omega(x, t)| > d\}$*. Then u is bounded up to $t = T$.*

The condition (CA) is called a "continuous alignment" condition. This kind of geometric condition on the vorticity direction was firstly given in [9] for a finite energy solution in $\mathbb{R}^3$ with H^1 initial data. In [9] the modulus η is taken as $\eta(\sigma) = A\sigma$ with some constant $A > 0$, while the type I condition is not needed there. The condition in [9] was relaxed in [5], where η is allowed to be $\eta(\sigma) = A\sigma^{1/2}$; see [16] for further references on the related results. A corresponding result to [5] for slip boundary conditions is established in [3], where $\eta(\sigma) = A\sigma^{1/2}$ in (CA). However, under the no-slip boundary condition the regularity criterion, so far obtained in [4], needs an extra assumption that the boundary integral of the normal derivative of the square of the vorticity is sufficiently small. As far as we know, the present chapter gives the first contribution to the case of the no-slip boundary condition under the same assumption to the whole space. This is rather surprising since the geometric regularity criterion is still valid even if the vorticity is created from the boundary because of the no-slip boundary condition. As in [16], the proof of Theorem 1.2 is based on a blow-up argument.

Before concluding this section, we introduce Banach spaces with nondecaying functions. Let Ω be a domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Then, for $k \in \mathbb{N} \cup \{0\}$ and $\mu \in (0,1)$ the spaces $BC(\overline{\Omega})$, $C^k(\overline{\Omega})$, and $C^{k+\mu}(\overline{\Omega})$ are respectively defined by

$$
\begin{aligned}
&BC(\overline{\Omega}) = \big\{\, f \in C(\overline{\Omega}) \mid \|f\|_\infty = \sup_{x\in\overline{\Omega}} |f(x)| < \infty \,\big\},\\
&C^k(\overline{\Omega}) = \big\{ f \in BC(\overline{\Omega}) \mid \nabla^\alpha f \in BC(\overline{\Omega}),\, |\alpha| \le k, \quad \|f\|_{C^k} = \sum_{|\alpha|\le k} \|\nabla^\alpha f\|_\infty < \infty \,\big\},\\
&C^{k+\mu}(\overline{\Omega}) = \big\{ f \in C^k(\overline{\Omega}) \mid \\
&\qquad \|f\|_{C^{k+\mu}} = \|f\|_{C^k} + \sum_{|\alpha|=k} \sup_{x,y\in\overline{\Omega},\, x\ne y} \frac{|\nabla^\alpha f(x) - \nabla^\alpha f(y)|}{|x-y|^\mu} < \infty \,\big\}.
\end{aligned}
$$

Let us also introduce the BMO spaces as follows.

$$
\begin{aligned}
&BMO(\mathbb{R}^n) = \big\{ f \in L^1_{loc}(\mathbb{R}^n) \mid \|f\|_{BMO} = \sup_B \frac{1}{|B|} \int_B |f - \mathrm{Avg}_B f| \,\mathrm{d}x < \infty \,\big\},\\
&BMO(\Omega) = \big\{ f \in L^1_{loc}(\Omega) \mid \text{there is } g \in BMO(\mathbb{R}^n) \text{ such that } f = g \ \text{ a.e. in } \Omega,\\
&\qquad \|f\|_{BMO} = \inf\{\|g\|_{BMO} \mid g \in BMO(\mathbb{R}^n),\ f = g \ \text{ a.e. in } \Omega\} \,\big\}.
\end{aligned}
$$

In the definition of $\|\cdot\|_{BMO}$ the supremum is taken over all ball B in $\mathbb{R}^n$, $|B|$ is the volume of B, and $\mathrm{Avg}_B f = |B|^{-1} \int_B f \,\mathrm{d}x$.

This chapter is organized as follows. In Section 2 we consider the Stokes equations with a inhomogeneous term and derive the boundary condition on the vorticity field. We also obtain the integral equations for the vorticity field, which is useful to estimate the vorticity field directly. Section 3 is the core part of this chapter, and we study (1.1) - (1.2) under the conditions of Theorem 1.1. To this end we establish a temporal decay estimate in Section 3.1 and a spatial decay estimate in Section 3.2. Claim 1 and Claim 2 in this section are respectively stated as Lemma 3.1 and Lemma 3.2. These are proved in Section 3.3, which completes the proof of Theorem 1.1. Finally, as an application of Theorem 1.1, we prove Theorem 1.2 in Section 4.

After this work was completed, Professor Gregory Seregin kindly pointed out that a Liouville type result can be proved without using the vorticity equation [28]. However, his results need an assumption that the (kinetic) energy is bounded in time. This assumption imposes a decay at the spatial infinity and it is not enough to apply for proving a geometric regularity criterion such as Theorem 1.2.

2 Vorticity boundary condition for the Stokes flows

In this section we consider the Stokes equations

$$\partial_t u - \Delta u + \nabla p = \operatorname{div} F, \qquad \operatorname{div} u = 0 \qquad \text{in } \ (-L, 0) \times \mathbb{R}^2_+ \tag{2.1}$$

subject to the no-slip boundary condition

$$u = 0 \qquad \text{on } \ (-L, 0) \times \partial\mathbb{R}^2_+. \tag{2.2}$$

The aim of this section is to derive the boundary condition on the vorticity field

$$\omega = -\nabla^\perp \cdot u, \qquad \nabla^\perp = (\partial_2, -\partial_1)^\top.$$

If the flow possesses enough spatial decay then the vorticity boundary condition can be derived from the Biot-Savart law (e.g. [24]). Here we give an alternative derivation of the vorticity boundary condition in order to deal with nondecaying flows. The derivation is closely related with the structure of the pressure field. As is well-known, by acting the div operator in (2.1) the pressure field is recovered as a solution to the Poisson equations with the inhomogeneous Neumann boundary condition. With this in mind we start from

Proposition 2.1. *Assume that $F = (F_{ij})_{1 \le i,j \le 2} \in (C^2(\overline{\mathbb{R}^2_+}))^{2\times 2}$, $F_{ij} = \partial_2 F_{ij} = 0$ on $\partial\mathbb{R}^2_+$ for each i, j. Then there is a unique (up to a constant) solution $p_F \in BMO(\mathbb{R}^2_+)$ to*

$$\begin{cases} \Delta p_F = \operatorname{div}\operatorname{div} F & \text{in } \ \mathbb{R}^2_+, \\ \partial_2 p_F = 0 & \text{on } \ \partial\mathbb{R}^2_+, \end{cases} \tag{2.3}$$

such that

$$\|p_F\|_{BMO} \le C\|F\|_\infty, \qquad \|\nabla p_F\|_{C^\mu} \le C\|F\|_{C^{1+\mu}}, \qquad 0 < \mu < 1. \tag{2.4}$$

Proof. As usual, let us introduce the even extension: $\tilde{p}_F(x) = p_F(x)$ for $x_2 \ge 0$ and $\tilde{p}_F(x) = p_F(x^*)$ for $x_2 < 0$. The same extension is introduced also for F_{11} and F_{12}, while the odd extension is applied for F_{12} and F_{21}. We denote by $\tilde{F}$ the tensor extended in this manner. Then (2.3) is reduced to the Poisson equation $\Delta\tilde{p}_F = \operatorname{div}\operatorname{div}\tilde{F}$ in $\mathbb{R}^2$ by the assumption $F_{ij} =$

$\partial_2 F_{ij} = 0$ on $\partial\mathbb{R}^2_+$. Its solution is written as $\tilde{p}_F = -\mathrm{div}\,\mathrm{div}\,(-\Delta_{\mathbb{R}^2})^{-1}\tilde{F}$, where the operator $(-\Delta_{\mathbb{R}^2})^{-1}$ is defined as the convolution with the Newton potential in $\mathbb{R}^2$. It is well known that $\mathrm{div}\,\mathrm{div}\,(-\Delta_{\mathbb{R}^2})^{-1}$ defines a singular integral operator, and hence it is bounded in $BMO(\mathbb{R}^2)$, and $\nabla\mathrm{div}\,\mathrm{div}\,(-\Delta_{\mathbb{R}^2})^{-1}$ is bounded from $C^{1+\mu}(\mathbb{R}^2)$ to $C^{\mu}(\mathbb{R}^2)$, $0 < \mu < 1$. Thus (2.4) holds. The uniqueness is a consequence of the classical Liouville theorem for harmonic functions in $\mathbb{R}^2$. The proof is complete.

In order to recover the no-slip boundary condition on the velocity field we need to introduce the harmonic pressure field. As a preliminary, let us recall some results on the fractional power of the Laplace operator $-\partial_1^2$. As is well known, $-\partial_1^2$ is realized as a sectorial operator in $BC(\mathbb{R})$ (cf. [23]), and hence its fractional power $(-\partial_1^2)^{1/2}$ is also sectorial in $BC(\mathbb{R})$. The characterization of the interpolation spaces as in [23, Section 3.1.3] implies that

$$C^{1+\mu}(\mathbb{R}) \hookrightarrow D((-\partial_1^2)^{\frac{1}{2}}) \qquad \text{for all} \ \ \mu \in (0,1), \tag{2.5}$$

where $D((-\partial_1^2)^{1/2})$ is the domain of $(-\partial_1^2)^{1/2}$ in $BC(\mathbb{R})$. Note that the semigroup generated by $(-\partial_1^2)^{1/2}$ is nothing but the Poisson semigroup whose kernel is explicitly described.

Proposition 2.2. *Assume that* $g \in C^{1+\mu}(\mathbb{R})$ *for some* $\mu \in (0,1)$. *Then there is a unique (up to a constant) solution* $p_H \in L^1_{loc}(\mathbb{R}^2_+)$ *to*

$$\begin{cases} \Delta p_H = 0 & \text{in} \ \ \mathbb{R}^2_+, \\ \partial_2 p_H = \partial_1 g & \text{on} \ \ \partial\mathbb{R}^2_+, \end{cases} \tag{2.6}$$

such that

$$\sup_{x\in\mathbb{R}^2_+} x_2|\nabla p_H(x)| \le C\|g\|_\infty, \qquad \|\nabla p_H\|_{C^{\mu'}} \le C\|g\|_{C^{1+\mu}}, \quad 0 < \mu' < \mu. \tag{2.7}$$

Moreover, it follows that

$$\lim_{x_2\downarrow 0} \partial_1 p_H(x) = (-\partial_1^2)^{\frac{1}{2}} g(x_1) \qquad \text{in} \ \ BC(\mathbb{R}). \tag{2.8}$$

Remark 2.3. In Proposition 2.2 the weight estimate in (2.7) is essential in view of the uniqueness of solutions. In particular, if one tries to avoid the Poiseuille type flows as in [12] it is important to impose suitable conditions on the behavior of the harmonic pressure at spatial infinity.

Proof of Proposition 2.2. The solution p_H is constructed so as to satisfy the representation

$$\nabla p_H(x) = -\int_0^\infty \big(\nabla\partial_1 e^{-(x_2+y_2)(-\partial_1^2)^{\frac{1}{2}}} g\big)(x_1)\,\mathrm{d}y_2. \tag{2.9}$$

Indeed, if g is compactly supported then p_H is given by

$$p_H = -\int_0^\infty \partial_1 e^{-(x_2+y_2)(-\partial_1^2)^{1/2}} g \, \mathrm{d}y_2$$

, where the integral converges absolutely. Then we modify p_H by adding a constant so that the condition $p_H(0) = 0$ holds and both (2.6) and (2.9) are satisfied. We denote this modified solution by $p_H(g)$. The straightforward calculation of the Poisson semigroup yields that

$$\|\nabla^k e^{-x_2(-\partial_1^2)^{\frac{1}{2}}} g\|_{L^\infty_{x_1}} \le C x_2^{-k} \|g\|_\infty, \qquad k = 0, 1, 2, \tag{2.10}$$

and

$$\|\nabla p_H(g)\|_{C^{\mu'}} \le C \|g\|_{C^{1+\mu}}, \qquad 0 < \mu' < \mu < 1. \tag{2.11}$$

Then for general $g \in C^{1+\mu}(\mathbb{R})$ we approximate g by $g\chi_R$ with a smooth cut-off χ_R and take the limit of $p_H(g_R)$ at $R \to \infty$. Since $g_R \to g$ in $C^1(K)$ for each compact set $K \subset \mathbb{R}$ and $\sup_{R>0} \|g_R\|_{C^{1+\mu}} < \infty$, it is not difficult to show that there is a subsequence of $\{p_H(g_R)\}_{R>0}$ which converges to some p_H in $C^1(K')$ for each compact set $K' \subset \overline{\mathbb{R}^2_+}$. It is easy to see that p_H solves (2.6) and also satisfies (2.9) by the Lebesgue convergence theorem. The estimate (2.7) is a consequence of (2.10) and (2.11). To show (2.8) we observe from (2.9) that

$$\begin{aligned}
\partial_1 p_H(x) &= \int_0^\infty (-\partial_1^2) e^{-(x_2+y_2)(-\partial_1^2)^{\frac{1}{2}}} g \, \mathrm{d}y_2 \\
&= -\int_0^\infty (-\partial_1^2)^{\frac{1}{2}} \partial_{y_2} \big(e^{-(x_2+y_2)(-\partial_1^2)^{\frac{1}{2}}} g \big) \, \mathrm{d}y_2 \\
&= (-\partial_1^2)^{\frac{1}{2}} e^{-x_2(-\partial_1^2)^{\frac{1}{2}}} g, \qquad \text{for} \quad x_2 > 0.
\end{aligned}$$

Hence (2.8) follows from (2.5). The uniqueness of solutions to (2.6) is again reduced to the classical Liouville theorem for harmonic functions in $\mathbb{R}^2$ by a suitable reflection argument. The details are omitted here. The proof is now complete.

We are now in position to derive the vorticity boundary condition for nondecaying flows.

Lemma 2.4. *Assume that* $F = (F_{ij})_{1 \le i,j \le 2} \in C((-L, 0) \times (C^2(\overline{\mathbb{R}^2_+}))^{2\times 2})$, $F_{ij}(t) = \partial_2 F_{ij}(t) = 0$ *on* $(-L, 0) \times \partial\mathbb{R}^2_+$ *for each* i, j. *Let* (u, p) *be the solution to* (2.1)-(2.2) *such that*

(C1) $\sup_{-L<t<0} \big(\|u(t)\|_{C^{2+\mu}} + \|\partial_t u(t)\|_{C^\mu} \big) < \infty$ *for some* $\mu \in (0, 1)$, (C2) $p = p_F + p_H$, *where* $p_F(t)$ *is the solution to* (2.3) *in Proposition 2.1 with* $F = F(t)$ *and* $p_H(t)$ *is the solution to* (2.6) *in Proposition 2.2 with* $g = \omega(t)|_{x_2=0}$, *respectively.*

Then ω *satisfies*

$$\partial_t \omega - \Delta \omega = -\nabla^\perp \cdot \operatorname{div} F \qquad \text{in} \quad (-L, 0) \times \mathbb{R}^2_+ \tag{2.12}$$

with

$$\partial_2 \omega + (-\partial_1^2)^{\frac{1}{2}} \omega = -\partial_1 p_F \qquad \text{on} \quad (-L, 0) \times \partial\mathbb{R}^2_+. \tag{2.13}$$

Proof. It is straightforward to see (2.12). To show (2.13) we first recall the equality $-\Delta u = \nabla^\perp \omega$ and then (2.1) yields $\partial_2 \omega = -\partial_t u_1 - \partial_1 p + \tau \cdot \operatorname{div} F$ for $x_2 > 0$, where $\tau = (1, 0)^\top$. Thus we have

$$\begin{aligned} \lim_{x_2 \downarrow 0} \partial_2 \omega &= -\lim_{x_2 \downarrow 0} \partial_t u_1 - \lim_{x_2 \downarrow 0} \partial_1 p_F - \lim_{x_2 \downarrow 0} \partial_1 p_H + \lim_{x_2 \downarrow 0} \tau \cdot \operatorname{div} F \\ &= -\partial_1 p_F|_{x_2=0} - (-\partial_1^2)^{\frac{1}{2}} \omega|_{x_2=0} \qquad \text{by} \quad (2.8). \end{aligned}$$

The proof is now complete.

Lemma 2.4 leads to the integral equation for the vorticity field, which is useful to estimate the vorticity directly including near the boundary.

Let $G(t, x) = (4\pi t)^{-1} \exp\big(-|x|^2/(4t)\big)$ be the two-dimensional Gaussian. Then for each $t > 0$ we introduce the operator e^{tB} defined by

$$e^{tB} f = G(t) * f + G(t) \star f + \Gamma(t) \star f, \tag{2.14}$$

where

$$\Gamma(t) = 2 \int_0^\infty \big(\partial_1^2 + (-\partial_1^2)^{\frac{1}{2}} \partial_2\big) G(t + \tau) \, \mathrm{d}\tau \tag{2.15}$$

with the notations

$$f \ * \ h(x) = \int_{\mathbb{R}^2_+} f(x - y) h(y) \, \mathrm{d}y,$$

$$f \ \star \ h(x) = \int_{\mathbb{R}^2_+} f(x - y^*) h(y) \, \mathrm{d}y, \qquad y^* = (y_1, -y_2).$$

For $g \in C_0^\infty(\mathbb{R})$ we set

$$e^{tB}(g \delta_{\partial\mathbb{R}^2_+}) = \int_{\mathbb{R}} K(t, x, y)|_{y_2=0} g(y_1) \, \mathrm{d}y_1,$$

where $K(t, x, y)$ is the kernel of e^{tB}. Due to the pointwise estimate of $K(t, x, y)$ in (3.10), the term $e^{tB}(g \delta_{\partial\mathbb{R}^2_+})$ makes sense also for $g \in L^\infty(\mathbb{R})$. The operator e^{tB} naturally arises in the vorticity equations. Indeed, if

$f \in C_0^\infty(\mathbb{R}^2_+)$ then $e^{tB}f$ satisfies the (homogeneous) vorticity equations (2.12)-(2.13), i.e., $w(t) = e^{(t+L)B}f$ solves

$$\partial_t w - \Delta w = 0 \quad \text{in } (-L,0)\times\mathbb{R}^2_+, \qquad \partial_2 w + (-\partial_1^2)^{\frac{1}{2}} w = 0 \quad \text{on } (-L,0)\times\partial\mathbb{R}^2_+, \tag{2.16}$$

but with the initial data $w(-L) = \lim_{t\to -L} w(t) = f + \Gamma(0)\star f$ in $L^p(\mathbb{R}^2_+)$ for all $1 < p < \infty$; see [24, Sections 3,5] for details. Note that $\Gamma(0)\star$ is a singular integral operator. In particular, we have $\|\Gamma(0)\star f\|_{L^p} \le C\|f\|_{L^p}$ for all $f \in L^p(\mathbb{R}^2_+)$. If $f = -\nabla^\perp \cdot u$ with $u \in C^\infty_{0,\sigma}(\mathbb{R}^2_+)$ then $\Gamma(0)\star f = 0$ (see [24, Proposition 3.2]), hence in this case we recover the initial condition $w(-L) = f$, as desired. For each $t > 0$ let us introduce the operator $T(t) : (L^\infty(\mathbb{R}^2_+))^2 \to L^\infty(\mathbb{R}^2_+)$ as follows:

$$\langle T(t)v, f\rangle_{L^2} = \langle v_1, \partial_2 e^{tB} f\rangle_{L^2} - \langle v_2, \partial_1 e^{tB} f\rangle_{L^2} \qquad \text{for all } \ f \in L^1(\mathbb{R}^2_+). \tag{2.17}$$

Here $\langle , \rangle_{L^2}$ denote the inner product of $L^2(\mathbb{R}^2_+)$. The operator $T(t)$ is well-defined due to the estimate $\|\nabla e^{tB} f\|_{L^1} \le Ct^{-1/2}\|f\|_{L^1}$ by [24, Lemma 3.4] and the duality $L^1(\mathbb{R}^2_+)^* = L^\infty(\mathbb{R}^2_+)$. In particular, we have

$$\|T(t)v\|_\infty \le Ct^{-\frac{1}{2}}\|v\|_\infty, \qquad t > 0. \tag{2.18}$$

Lemma 2.5. *Assume that the conditions in Lemma 2.4 hold and* $\operatorname{div} F \in (L^\infty(\mathbb{R}^2_+))^2$. *Then ω satisfies the integral equation*

$$\omega(t) = T(t-s)u(s) + \int_s^t T(t-\tau)\operatorname{div} F(\tau)\,\mathrm{d}\tau + \int_s^t e^{(t-\tau)B}(\partial_1 p_{F(\tau)}\delta_{\partial\mathbb{R}^2_+})\,\mathrm{d}\tau \tag{2.19}$$

for $-L < s < t < 0$.

Remark 2.6. There are several solution formulas for the velocity field of the Stokes flows in the half space with the Dirichlet condition for example in [31, 35]. Ours differs from those in the literature since it is a convenient form to represent the vorticity field.

Proof of Lemma 2.5. Take any $\phi(\tau, x) \in C_0^\infty([s,t]\times\overline{\mathbb{R}^2_+})$. Multiplying (2.12) by ϕ and using the integration by parts, we observe that ω satisfies

$$\begin{aligned}\langle \omega(t), \phi(t)\rangle_{L^2} &= \int_{\mathbb{R}^2_+} u(s)\cdot\nabla^\perp\phi(s)\,\mathrm{d}x + \int_s^t\int_{\mathbb{R}^2_+} \operatorname{div} F\cdot\nabla^\perp\phi(\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad - \int_s^t\int_{\partial\mathbb{R}^2_+} (\phi\partial_2\omega - \omega\partial_2\phi)(\tau)\,\mathrm{d}x_1\,\mathrm{d}\tau \\ &\quad + \int_s^t\int_{\mathbb{R}^2_+} \omega(\partial_\tau\phi + \Delta\phi)(\tau)\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

Fix $R \gg 1$ and set $\phi_R(\tau,x) := (\chi_R e^{(t-\tau)B}\psi)(x)$, where $\psi \in C_0^\infty(\mathbb{R}^2_+)$ and $\chi_R = \chi_R(x)$ is a nonnegative smooth cut-off function in $\mathbb{R}^2$ such that $\chi_R(x) = 1$ if $|x| \le R$ and $\chi_R(x) = 0$ if $|x| \ge 2R$. We may assume that $\|\nabla^k \chi_R\|_\infty \le CR^{-k}$ for $k = 0,1,2$. Then we set

$$
\begin{aligned}
\langle \omega(t), \phi_R(t)\rangle_{L^2} &= \int_{\mathbb{R}^2_+} u(s)\cdot \nabla^\perp \phi_R(s)\,\mathrm{d}x + \int_s^t \int_{\mathbb{R}^2_+} \operatorname{div} F \cdot \nabla^\perp \phi_R(\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad - \int_s^t \int_{\partial\mathbb{R}^2_+} (\phi_R \partial_2 \omega - \omega \partial_2 \phi_R)(\tau)\,\mathrm{d}x_1\,\mathrm{d}\tau \\
&\quad + \int_s^t \int_{\mathbb{R}^2_+} \omega(\partial_\tau \phi_R + \Delta \phi_R)(\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\
&:= I_1 + I_2 - I_3 + I_4.
\end{aligned}
$$

As for I_1, we have

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^2_+} u(s)\cdot \nabla^\perp(\chi_R e^{(t-s)B}\psi)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^2_+} \big(u_1(s)(\partial_2\chi_R)e^{(t-s)B}\psi - u_2(s)(\partial_1\chi_R)e^{(t-s)B}\psi\big)\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^2_+} \chi_R\big(u_1(s)\partial_2 e^{(t-s)B}\psi - u_2(s)\partial_1 e^{(t-s)B}\psi\big)\,\mathrm{d}x. \qquad (2.20)
\end{aligned}
$$

Thanks to [24, Lemma 3.4] and (C1) we have $e^{(t-s)B}\psi \in L^p(\mathbb{R}^2_+)$ for any $1 < p \le \infty$, and the first term of right-hand side of (2.20) converges to zero in the limit $R \to \infty$. As for the second term of (2.20), we observe from [24, Lemma 3.4] that $\|\nabla e^{(t-s)B}\psi\|_{L^1} \le C(t-s)^{-1/2}\|\psi\|_{L^1}$. Hence the Hölder inequality implies that $u_1(s)\partial_2 e^{(t-s)B}\psi - u_2(s)\partial_1 e^{(t-s)B}\psi$ belongs to $L^1(\mathbb{R}^2_+)$ for $t > s$. Thus we have $\lim_{R\to\infty} I_1 = \langle T(t-s)u(s), \psi\rangle_{L^2}$ by the definition of $T(t-s)$. Similarly, by the assumption $\operatorname{div} F \in (L^\infty(\mathbb{R}^2_+))^2$ and by the Fubini theorem, we have $\lim_{R\to\infty} I_2 = \langle \int_s^t T(t-\tau)\operatorname{div} F(\tau)\,\mathrm{d}\tau, \psi\rangle_{L^2}$.

As for I_3, we recall the vorticity boundary condition (2.13). Then it follows that

$$
\begin{aligned}
I_3 &= \int_s^t \int_{\partial\mathbb{R}^2_+} (\phi_R\partial_2\omega - \omega\partial_2\phi_R)(\tau)\,\mathrm{d}x_1\,\mathrm{d}\tau \\
&= \int_s^t \int_{\partial\mathbb{R}^2_+} \big\{ -\chi_R\big(\partial_1 p_{F(\tau)} + (-\partial_1^2)^{\frac12}\omega\big)e^{(t-\tau)B}\psi \\
&\qquad\qquad - \omega\big(\chi_R\partial_2 e^{(t-\tau)B}\psi + (\partial_2\chi_R)e^{(t-\tau)B}\psi\big)\big\}\,\mathrm{d}x_1\,\mathrm{d}\tau \\
&= -\int_s^t \int_{\partial\mathbb{R}^2_+} \chi_R\big(\partial_1 p_{F(\tau)} e^{(t-\tau)B}\psi + \omega(-\partial_1^2)^{\frac12} e^{(t-\tau)B}\psi + \omega\partial_2 e^{(t-\tau)B}\psi\big)\,\mathrm{d}x_1\,\mathrm{d}\tau \\
&\quad - \int_s^t \int_{\partial\mathbb{R}^2_+} (\partial_2\chi_R)\omega e^{(t-\tau)B}\psi\,\mathrm{d}x_1\,\mathrm{d}\tau.
\end{aligned}
$$

Since $(-\partial_1^2)^{\frac{1}{2}} e^{(t-\tau)B}\psi + \partial_2 e^{(t-\tau)B}\psi = 0$ on $\partial\mathbb{R}^2_+$, we obtain

$$\begin{aligned}\lim_{R\to\infty} I_3 &= -\lim_{R\to\infty} \int_s^t \int_{\partial\mathbb{R}^2_+} \chi_R \partial_1 p_{F(\tau)} e^{(t-\tau)B}\psi \,\mathrm{d}x_1 \,\mathrm{d}\tau \\ &= -\langle \int_s^t e^{(t-\tau)B} \partial_1 p_{F(\tau)} \delta_{\partial\mathbb{R}^2_+}, \psi\rangle_{L^2}.\end{aligned}$$

Finally, we consider I_4. It is easy to check that

$$\partial_\tau \phi_R + \Delta\phi_R = (\Delta\chi_R) e^{(t-\tau)B}\psi + 2\nabla\chi_R \cdot \nabla e^{(t-\tau)B}\psi.$$

Since ω is bounded in space and time, by using $\|\nabla^k \chi_R\|_\infty \le CR^{-k}$ for $k = 1, 2$ and the estimate of $e^{(t-\tau)B}\psi$ the term I_4 is shown to converge to zero as $R \to \infty$. Combining the above calculations, we have

$$\begin{aligned}&\lim_{R\to\infty} \langle \omega(t), \phi_R(t)\rangle_{L^2} = \lim_{R\to\infty} [I_1 + I_2 - I_3 + I_4] \\ &= \langle T(t-s)u(s) + \int_s^t T(t-\tau) \mathrm{div}\, F(\tau)\,\mathrm{d}\tau + \int_s^t e^{(t-\tau)B}\big(\partial_1 p_{F(\tau)} \delta_{\partial\mathbb{R}^2_+}\big)\,\mathrm{d}\tau, \psi\rangle_{L^2}\end{aligned}$$

Note that $\phi_R(t) = \lim_{\tau\to t} \chi_R e^{(t-\tau)B}\psi = \chi_R(\psi + \Gamma(0) \star \psi)$. By the definition of $\Gamma(0)$ we have $(\partial_2 + (-\partial_1^2)^{1/2})\Gamma(0) \star \psi = 0$ in $\mathbb{R}^2_+$. Then, together with the divergence free property of u and $u = 0$ on $\partial\mathbb{R}^2_+$, we observe from the integration by parts and $\|\Gamma(0) \star \psi\|_{L^p} \le C\|\psi\|_{L^p}$ that

$$\lim_{R\to\infty} \langle u_1, \chi_R \partial_2(\Gamma(0) \star \psi)\rangle_{L^2} = \lim_{R\to\infty} \langle u_2, \chi_R \partial_1(\Gamma(0) \star \psi)\rangle_{L^2},$$

that is, $\lim_{R\to\infty} \langle \omega(t), \chi_R \Gamma(0) \star \psi\rangle_{L^2} = 0$ again from the integration by parts for $\omega = \partial_1 u_2 - \partial_2 u_1$. Thus it follows that

$$\begin{aligned}\lim_{R\to\infty} \langle \omega(t), \phi_R(t)\rangle_{L^2} &= \lim_{R\to\infty} \langle \omega(t), \chi_R\psi\rangle_{L^2} + \lim_{R\to\infty} \langle \omega(t), \chi_R \Gamma(0) \star \psi\rangle_{L^2} \\ &= \lim_{R\to\infty} \langle \omega(t), \chi_R\psi\rangle_{L^2} = \langle \omega(t), \psi\rangle_{L^2}.\end{aligned}$$

Since ψ is arbitrary the proof is now complete.

As an immediate consequence of Lemmas 2.4 and 2.5, we obtain the vorticity equation for the full nonlinear problem (1.1)-(1.2).

Proposition 2.7. *Let* (u, p) *be the solution to* (1.1)-(1.2) *such that*

(C1) $\sup_{-\infty<t<0} \big(\|u(t)\|_{C^{2+\mu}} + \|\partial_t u(t)\|_{C^\mu}\big) < \infty$ *for some* $\mu \in (0, 1)$,

(C2) $p = p_F + p_H$, *where* $p_F(t)$ *is the solution to* (2.3) *in Proposition 2.1 with* $F = -u(t) \otimes u(t)$ *and* $p_H(t)$ *is the solution to* (2.6) *in Proposition 2.2 with* $g = \omega(t)|_{x_2=0}$, *respectively.*

Then ω *satisfies*

$$\partial_t \omega - \Delta\omega = \nabla^\perp \cdot \mathrm{div}\,(u \otimes u) \qquad \text{in} \quad (-\infty, 0) \times \mathbb{R}^2_+ \tag{2.21}$$

with

$$\partial_2\omega + (-\partial_1^2)^{\frac{1}{2}}\omega = -\partial_1 p_F \qquad \text{on} \ \ (-\infty,0)\times\partial\mathbb{R}^2_+. \tag{2.22}$$

Moreover, ω *satisfies the integral equation* (2.19) *for* $-\infty < s < t < 0$*.*

3 Liouville type result

In this section we prove Theorem 1.1. As stated in the introduction, the key idea of the proof is to derive the spatial decay of vorticity fields in the vertical direction and to verify the relation of the Biot-Savart law between the velocity and the vorticity. More precisely, the core parts of the proof are the following two lemmas.

Lemma 3.1. *Under the conditions* (C1)*,* (C2)*, and* (C3) *of Theorem 1.1 the vorticity* ω *satisfies*

$$\sup_{(t,x)\in(-\infty,0)\times\mathbb{R}^2_+} x_2^{1+\theta}|\omega(t,x)| < \infty \qquad \text{for all} \ \ \theta\in(0,1). \tag{3.1}$$

Lemma 3.2. *Under the conditions* (C1)*,* (C2)*, and* (C3) *of Theorem 1.1 the velocity* u *is represented as*

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2_+}\Big(\frac{(x-y)^\perp}{|x-y|^2} - \frac{(x-y^*)^\perp}{|x-y^*|^2}\Big)\omega(t,y)\,\mathrm{d}y. \tag{3.2}$$

Here $x^\perp = (-x_2, x_1)^\top$ *and* $y^* = (y_1, -y_2)^\top$*.*

Proof of Theorem 1.1. We give a proof of Theorem 1.1 by admitting Lemmas 3.1 and 3.2. The proofs of these lemmas will be postponed to the latter sections. From (3.1) and (3.2) we observe that

$$0 = \lim_{x_2\downarrow 0} u_1(t,x) = \frac{1}{\pi}\int_{\mathbb{R}^2+}\frac{y_2}{(x_1-y_1)^2+y_2^2}\omega(t,y)\,\mathrm{d}y, \tag{3.3}$$

by the Lebesgue convergence theorem. Then (C4) implies that the integrand of the right-hand side of (3.3) has to be zero, that is, $\omega(t,x) = 0$ in $(-\infty,0)\times\mathbb{R}^2_+$. Then for all t the velocity $u(t)$ is harmonic and bounded in $\mathbb{R}^2_+$ and vanishes on $\partial\mathbb{R}^2_+$. Hence u must be zero by the classical Liouville theorem for harmonic functions. The proof is now complete.

3.1 Temporal decay of vorticity

The main result of this section is the following lemma.

Lemma 3.3. *Under the conditions* (C1), (C2), *and* (C3) *of Theorem 1.1 the vorticity* ω *satisfies*

$$\|\omega(t)\|_\infty \le C(-t)^{-1}|\log(-t)|^2, \qquad -\infty < t < -2, \tag{3.4}$$

$$\|(-\partial_1^2)^{\frac12}\omega(t)\|_\infty \le C(-t)^{-\frac32}|\log(-t)|^4, \qquad -\infty < t < -2. \tag{3.5}$$

Lemma 3.3 is proved by estimating the integral equations for the vorticity field in Proposition 2.7. To this end we first establish the $L^\infty - L^\infty$ estimates for the operators in (2.19).

Lemma 3.4. *Assume that* $v \in (C(\overline{\mathbb{R}^2_+}))^2$ *with* $v = 0$ *on* $\partial\mathbb{R}^2_+$ *and* $g \in BC(\mathbb{R})$. *Then*

$$\|(-\partial_1^2)^{\frac12}T(t)v\|_\infty \le Ct^{-1}\|v\|_\infty, \tag{3.6}$$

$$\|(-\partial_1^2)^{\frac{k}{2}}T(t)\partial_i v\|_\infty \le Ct^{-1-\frac{k}{2}}\|v\|_\infty, \qquad k = 0,1,\ i = 1,2, \tag{3.7}$$

$$\|(-\partial_1^2)^{\frac{k}{2}}e^{tB}(\partial_1^l g\delta_{\partial\mathbb{R}^2_+})\|_\infty \le Ct^{-\frac{1+k+l}{2}}\|g\|_\infty, \qquad k,l = 0,1. \tag{3.8}$$

Moreover, if $F = u\otimes u$ *with* $u \in (C^2(\overline{\mathbb{R}^2_+}))^2$ *satisfying* $\operatorname{div} u = 0$ *in* $\mathbb{R}^2_+$ *and* $u = 0$ *on* $\partial\mathbb{R}^2_+$ *then*

$$\|(-\partial_1^2)^{\frac12}T(t)\mathrm{div}F\|_\infty \le C\|u\|_\infty \min\{t^{-1}\|\omega\|_\infty,\ t^{-\frac12}\|\nabla\omega\|_\infty\}. \tag{3.9}$$

Proof. As in [24, Proposition 5.1], using the Fourier transform, we can derive the pointwise estimate for the kernel $K(t,x,y)$ of e^{tB} such as

$$\begin{aligned}&|(-\partial_1^2)^{\frac{k}{2}}\partial_1^l\partial_2^j K(t,x,y)|\\ &\le Ct^{-\frac{k+l+2}{2}}\Big(1+\frac{|(x_1-y_1)/\sqrt{t}|^{2+k+l}}{\log(e+|(x_1-y_1)/\sqrt{t}|^2)}+|(x_2-y_2)/\sqrt{t}|^{2+k+l+j}\Big)^{-1}.\end{aligned} \tag{3.10}$$

Since e^{tB} commutes with ∂_1, the estimates (3.6) and (3.8) are immediate from (3.10). As for (3.7), we give a proof only for the case $k = 1$ and $i = 2$. The other cases are proved in the same manner. By the definition of $T(t)$ in (2.17) we have

$$\begin{aligned}\langle(-\partial_1^2)^{\frac12}T(t)\partial_2 v, f\rangle_{L^2} &= \langle\partial_2 v_1, \partial_2(-\partial_1^2)^{\frac12}e^{tB}f\rangle_{L^2} - \langle\partial_2 v_2, \partial_1(-\partial_1^2)^{\frac12}e^{tB}f\rangle_{L^2}\\ &= -\langle v_1, \partial_2^2(-\partial_1^2)^{\frac12}e^{tB}f\rangle_{L^2} + \langle v_2, \partial_1\partial_2(-\partial_1^2)^{\frac12}e^{tB}f\rangle_{L^2}.\end{aligned}$$

Here we have used the integration by parts and the boundary condition $v = 0$ on $\partial\mathbb{R}^2_+$. Since (3.10) implies $\|\partial_2^2(-\partial_1^2)^{\frac12}e^{tB}f\|_1 + \|\partial_2^2(-\partial_1^2)^{\frac12}e^{tB}f\|_{L^1} \le Ct^{-3/2}\|f\|_{L^1}$, we obtain (3.8) by the duality argument. Finally we show

(3.9). Set $v = \operatorname{div} F$. Note that v vanishes on the boundary by the assumption. Then again by the definition of $T(t)$ we have

$$\begin{aligned}
\langle(-\partial_1^2)^{\frac12}T(t)\operatorname{div}F, f\rangle_{L^2} &= \langle v_1, \partial_2(-\partial_1^2)^{\frac12}e^{tB}f\rangle_{L^2} - \langle v_2, \partial_1(-\partial_1^2)^{\frac12}e^{tB}f\rangle_{L^2} \\
&= \langle -\nabla^\perp\cdot v, (-\partial_1^2)^{\frac12}e^{tB}f\rangle_{L^2} \\
&= \langle u\cdot\nabla\omega, (-\partial_1^2)^{\frac12}e^{tB}f\rangle_{L^2} = -\langle u\omega, \nabla(-\partial_1^2)^{\frac12}e^{tB}f\rangle_{L^2}.
\end{aligned}$$

Here we have used the equality $-\nabla^\perp\cdot\operatorname{div}(u\otimes u) = u\cdot\nabla\omega = \nabla\cdot(u\omega)$. By using the estimates $\|\nabla^k(-\partial_1^2)^{\frac12}e^{tB}f\|_{L^1} \le Ct^{-(1+k)/2}\|f\|_{L^1}$ for $k=0,1$, we obtain (3.9) by the duality argument. The proof is now complete.

Proof of Lemma 3.3. By Proposition 2.7 the vorticity ω satisfies the integral equation (2.19) for $-\infty < s < 2t < t < 0$ with $F(t) = -u(t)\otimes u(t)$. We set

$$I(t,s) = T(t-s)u(s), \qquad II(t,s) = \int_s^t T(t-\tau)\operatorname{div}F(\tau)\,\mathrm{d}\tau,$$

$$III(t,s) = \int_s^t e^{(t-\tau)B}(\partial_1 p_{F(\tau)}\delta_{\partial\mathbb{R}^2_+})\,\mathrm{d}\tau.$$

For $I(t,s)$ we have from (2.18), (3.6), and (C3) that

$$\|(-\partial_1^2)^{\frac{k}{2}}I(t,s)\|_\infty \le C(t-s)^{-\frac{1+k}{2}}\|u(s)\|_\infty \le C(t-s)^{-\frac{1+k}{2}}(-s)^{-\frac12} \to 0 \quad \text{as} \quad s\to-\infty. \tag{3.11}$$

Next we consider the term $II(t,s)$. When $\tau < t - 1/t^2$ we apply (3.7) and get

$$\|T(t-\tau)\operatorname{div}F(\tau)\|_\infty \le C(t-\tau)^{-1}\|u(\tau)\|_\infty^2 \le C(t-\tau)^{-1}(-\tau)^{-1}$$

by (C3), and we also have from (3.9) and (C3) that

$$\|(-\partial_1^2)^{\frac12}T(t-\tau)\operatorname{div}F(\tau)\|_\infty \le C(t-\tau)^{-1}(-\tau)^{-\frac12}\|\omega(\tau)\|_\infty.$$

When $t - 1/t^2 \le \tau < t$ we use (2.18), (3.6), and (3.9) to get

$$\|T(t-\tau)\operatorname{div}F(\tau)\|_\infty \le C(t-\tau)^{-\frac12}\|u(\tau)\cdot\nabla u(\tau)\|_\infty \le C(t-\tau)^{-\frac12}(-\tau)^{-\frac12}$$

and

$$\|(-\partial_1^2)^{\frac12}T(t-\tau)\operatorname{div}F(\tau)\|_\infty \le C(t-\tau)^{-\frac12}\|u(\tau)\|_\infty\|\nabla\omega(\tau)\|_\infty \le C(t-\tau)^{-\frac12}(-\tau)^{-\frac12}.$$

Collecting these, for $t < -2$ we have arrived at

$$\begin{aligned}
\lim_{s\to-\infty}\|II(t,s)\|_\infty &\le C\int_{-\infty}^{t-\frac{1}{t^2}}(t-\tau)^{-1}(-\tau)^{-1}\,\mathrm{d}\tau + C\int_{t-\frac{1}{t^2}}^t (t-\tau)^{-\frac12}(-\tau)^{-\frac12}\,\mathrm{d}\tau \\
&\le C(-t)^{-1}\log(-t),
\end{aligned} \tag{3.12}$$

and

$$\lim_{s\to-\infty}\|(-\partial_1^2)^{\frac12}II(t,s)\|_\infty \le C\int_{-\infty}^{t-\frac{1}{t^2}}(t-\tau)^{-1}(-\tau)^{-\frac12}\|\omega(\tau)\|_\infty\,\mathrm{d}\tau + C(-t)^{-\frac32}. \tag{3.13}$$

Finally we estimate $III(t,s)$. To this end we recall that p_F is the restriction of the function $-\mathrm{div}\,\mathrm{div}\,(-\Delta_{\mathbb{R}^2})^{-1}\tilde{F}$ on $\mathbb{R}^2_+$; see the proof of Proposition 2.1 for details and the definition of $\tilde{F}$. Then we decompose $\partial_1 p_{F(\tau)}$ as

$$\partial_1 p_{F(\tau)} = \sum_{j=1}^{3}\partial_1 p_{F(\tau),j} = -\Big(\int_0^{\frac{1}{\tau^4}} + \int_{\frac{1}{\tau^4}}^{\tau^4} + \int_{\tau^4}^{\infty}\Big)\partial_1\mathrm{div}\,\mathrm{div}\,G(\theta)*\tilde{F}(\tau)\,\mathrm{d}\theta.$$

Here $G(\theta,x)$ is the two-dimensional Gaussian. Firstly we observe that

$$\begin{aligned}\|(-\partial_1^2)^{\frac{k}{2}}e^{(t-\tau)B}(\partial_1 p_{F(\tau),1}\delta_{\partial\mathbb{R}^2_+})\|_\infty &\le C(t-\tau)^{-\frac12-\frac{k\kappa}{2}}\int_0^{\frac{1}{\tau^4}}\theta^{-\frac12-\frac{k(1-\kappa)}{2}}\|\mathrm{div}\,\mathrm{div}\,\tilde{F}(\tau)\|_\infty\,\mathrm{d}\tau\\ &\le C(t-\tau)^{-\frac12-\frac{k\kappa}{2}}(-\tau)^{-2+2k(1-\kappa)}\end{aligned}$$

for $k=0,1$ and $\kappa\in(0,1)$, where we have applied (3.8) and the interpolation argument using $(-\partial_1^2)^{1/2}=(-\partial_1^2)^{\kappa/2}(-\partial_1^2)^{(1-\kappa)/2}$ when $k=1$. By taking κ close to 1 we thus obtain

$$\|\int_{-\infty}^{t}(-\partial_1^2)^{\frac{k}{2}}e^{(t-\tau)B}(\partial_1 p_{F(\tau),1}\delta_{\partial\mathbb{R}^2_+})\,\mathrm{d}\tau\|_\infty \le C(-t)^{-\frac32},\quad k=0,1,\quad -\infty<t<-2. \tag{3.14}$$

The estimate of $\partial_1 p_{F(\tau),3}$ is easily calculated as

$$\|(-\partial_1^2)^{\frac{k}{2}}\partial_1 p_{F(\tau),3}\|_\infty \le C\int_{\tau^4}^{\infty}\theta^{-\frac{3+k}{2}}\,\mathrm{d}\theta\|\tilde{F}(\tau)\|_\infty \le C(-\tau)^{-3},\quad k=0,1.$$

Hence we have from (3.8),

$$\|\int_{-\infty}^{t}(-\partial_1^2)^{\frac{k}{2}}e^{(t-\tau)B}(\partial_1 p_{F(\tau),3}\delta_{\partial\mathbb{R}^2_+})\,\mathrm{d}\tau\|_\infty \le C(-t)^{-\frac32},\quad k=0,1,\quad -\infty<t<-2. \tag{3.15}$$

Now we consider the term related with $\partial_1 p_{F(\tau),2}$. By the definition of $\tilde{F}$ in Proposition 2.1 we take the even extension for u_1 and the odd extension for u_2. Each extension is denoted by $\tilde{u}_i$. This extension leads to the odd extension $\tilde{\omega}$ of the vorticity ω. Then it is straightforward to see $\mathrm{div}\,\tilde{F} = -\tilde{u}^\perp\tilde{\omega}-\nabla|\tilde{u}|^2/2$ with $\tilde{u}^\perp=(-\tilde{u}_2,\tilde{u}_1)^\top$, and thus, $\mathrm{div}\,\mathrm{div}\,\tilde{F}=-\mathrm{div}\,(\tilde{u}^\perp\tilde{\omega})-\Delta|\tilde{u}|^2/2$. Hence we have

$$\begin{aligned}\partial_1 p_{F(\tau),2} &= \int_{\frac{1}{\tau^4}}^{\tau^4}\partial_1\mathrm{div}\,G(\theta)\,\mathrm{d}\theta*(\tilde{u}^\perp\tilde{\omega})(\tau)+\frac12\int_{\frac{1}{\tau^4}}^{\tau^4}\partial_1\Delta G(\theta)\,\mathrm{d}\theta*|\tilde{u}|^2(\tau)\\ &= \int_{\frac{1}{\tau^4}}^{\tau^4}\partial_1\mathrm{div}\,G(\theta)\,\mathrm{d}\theta*(\tilde{u}^\perp\tilde{\omega})(\tau)+\frac12\partial_1 G(\tau^4)*|\tilde{u}|^2-\frac12 G(\tau^{-4})*\partial_1|\tilde{u}|^2(\tau).\end{aligned}$$

Since $\partial_1|\tilde{u}|^2 = 0$ on $\partial\mathbb{R}^2_+$ we have $\|G(\tau^{-4})*\partial_1|\tilde{u}|^2\|_{L^\infty(\partial\mathbb{R}^2_+)} \le C(-\tau)^{-2}\|\partial_1|\tilde{u}|^2\|_{C^1} \le C(-\tau)^{-2}$. Hence it follows that

$$\begin{aligned}\|\partial_1 p_{F(\tau),2}\|_{L^\infty(\partial\mathbb{R}^2_+)} &\le C\|\tilde{u}(\tau)\|_\infty\|\tilde{\omega}(\tau)\|_\infty \log(-\tau) + C(-\tau)^{-2}\\ &\le C\|\omega(\tau)\|_\infty(-\tau)^{-\frac{1}{2}}\log(-\tau) + C(-\tau)^{-2}.\end{aligned} \tag{3.16}$$

When $\tau < t - 1/t^4$ we have from (3.8) that

$$\begin{aligned}&\|e^{(t-\tau)B}(\partial_1 p_{F(\tau),2}\delta_{\partial\mathbb{R}^2_+})\|_\infty\\ &\le C(t-\tau)^{-1}\int_{\frac{1}{\tau^4}}^{\tau^4}\theta^{-1}\|\tilde{F}(\tau)\|_\infty \,\mathrm{d}\theta \le C(t-\tau)^{-1}(-\tau)^{-1}\log(-\tau),\end{aligned}$$

while (3.16) implies

$$\|(-\partial_1^2)^{\frac{1}{2}}e^{(t-\tau)B}(\partial_1 p_{F(\tau),2}\delta_{\partial\mathbb{R}^2_+})\|_\infty \le C(t-\tau)^{-1}\big(\|\omega(\tau)\|_\infty(-\tau)^{-\frac{1}{2}}\log(-\tau) + (-\tau)^{-2}\big).$$

As for the case $t - 1/t^4 \le \tau < t$, we have for $k = 0, 1$,

$$\begin{aligned}\|(-\partial_1^2)^{\frac{k}{2}}e^{(t-\tau)B}(\partial_1 p_{F(\tau),2}\delta_{\partial\mathbb{R}^2_+})\|_\infty &\le C(t-\tau)^{-\frac{1}{2}}\int_{\frac{1}{\tau^4}}^{\tau^4}\theta^{-1}\|\tilde{F}(\tau)\|_{C^2}\,\mathrm{d}\theta\\ &\le C(t-\tau)^{-\frac{1}{2}}\log(-\tau).\end{aligned}$$

Combining the above three yields

$$\Big\|\int_{-\infty}^{t} e^{(t-\tau)B}(\partial_1 p_{F(\tau),2}\delta_{\partial\mathbb{R}^2_+})\,\mathrm{d}\tau\Big\|_\infty \le C(-t)^{-1}|\log(-t)|^2, \quad -\infty < t < -2, \tag{3.17}$$

and

$$\begin{aligned}&\Big\|(-\partial_1^2)^{\frac{1}{2}}\int_{-\infty}^{t} e^{(t-\tau)B}(\partial_1 p_{F(\tau),2}\delta_{\partial\mathbb{R}^2_+})\,\mathrm{d}\tau\Big\|_\infty\\ &\le C\int_{-\infty}^{t-1/t^4}(t-\tau)^{-1}\|\omega(\tau)\|_\infty(-\tau)^{-\frac{1}{2}}\log(-\tau)\,\mathrm{d}\tau + C(-t)^{-\frac{3}{2}}, \quad -\infty < t < -2.\end{aligned} \tag{3.18}$$

The estimates (3.11), (3.12), (3.14), (3.15), (3.17) imply (3.4), and the estimates (3.11), (3.13), (3.14), (3.15), (3.18) together with (3.4) give (3.5). The proof is complete.

Remark 3.5. The proof of Lemma 3.3 implies that, from (3.4) and (3.16),

$$\|\partial_1 p_{F(\tau),2}\|_{L^\infty(\partial\mathbb{R}^2_+)} \le C(-\tau)^{-\frac{3}{2}}|\log(-\tau)|^3, \qquad -\infty < \tau < -2.$$

Since it is easy to see $\|\partial_1 p_{F(\tau),i}\|_{L^\infty(\partial\mathbb{R}^2_+)} \le (-\tau)^{-3/2}$ for $i = 1, 3$, we have

$$\|\partial_1 p_{F(\tau)}\|_{L^\infty(\partial\mathbb{R}^2_+)} \le C(-\tau)^{-\frac{3}{2}}|\log(-\tau)|^3, \qquad -\infty < \tau < -2. \tag{3.19}$$

This estimate will be used later.

3.2 Spatial decay of vorticity - proof of Lemma 3.1

In this section we derive spatial decay of the vorticity field and complete the proof of Lemma 3.1. The key idea is to regard (2.22) as the Neumann boundary condition $\partial_2\omega = g$ with the inhomogeneous term $g = -(-\partial_1^2)^{1/2}\omega|_{x_2=0} - \partial_1 p_F$. Then we use a representation formula of the vorticity in terms of the fundamental solution for the heat-transport operator $\partial_t - \Delta + \tilde{u}\cdot\nabla$ in $\mathbb{R}^2$, whose precise pointwise estimate has already been established by [6, 27]. Here $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)^\top$ is the extension of u to $\mathbb{R}^2$, where $\tilde{u}_1,\ \tilde{u}_2$ are the even, odd extensions of $u_1,\ u_2$, respectively. Note that this extension preserves the divergence-free condition when $u_2 = 0$ on $\partial\mathbb{R}^2_+$. The scaling invariant assumption (C3) is essential in establishing the spatial decay of the vorticity, for it leads to the global Gaussian estimate for the fundamental solution. We start from the following lemma.

Lemma 3.6. *Under the conditions* (C1)*,* (C2)*, and* (C3) *of Theorem 1.1 the vorticity ω is expressed as*

$$\omega(t) = \Gamma_u(t,s)\omega(s) - \int_s^t \Gamma_u(t,\tau)(g(\tau)\delta_{\partial\mathbb{R}^2_+})\,\mathrm{d}\tau, \qquad -\infty < s < t < 0, \tag{3.20}$$

with $g(\tau) = -(-\partial_1^2)^{1/2}\omega(\tau)|_{x_2=0} - \partial_1 p_{F(\tau)}$. Here $\Gamma_u(t,s)$ is the evolution operator defined by

$$\Gamma_u(t,s)f = \int_{\mathbb{R}^2_+} \big(\Gamma_{\tilde{u}}(t,x;s,y) + \Gamma_{\tilde{u}}(t,x;s,y^*)\big) f(y)\,\mathrm{d}y,$$

where $\Gamma_{\tilde{u}}(t,x;s,y)$ is the fundamental solution to the heat-transport equations

$$\partial_t w - \Delta w + \tilde{u}\cdot\nabla w = 0 \qquad \text{in } \ (-\infty,0)\times\mathbb{R}^2. \tag{3.21}$$

Moreover, it follows that

$$\|\Gamma_u(t,s)f\|_p \le C(t-s)^{-\frac{1}{q}+\frac{1}{p}}\|f\|_q, \qquad -\infty < s < t < 0, \quad 1 \le q \le p \le \infty, \tag{3.22}$$

$$0 < \Gamma_{\tilde{u}}(t,x;s,y) \le C_1(t-s)^{-1}\exp\Big(-C_2\frac{|x-y|^2}{t-s}\Big). \tag{3.23}$$

Here C_1 and C_2 depend only on $M = \sup_{-\infty<t<0}(-t)^{1/2}\|u(t)\|_\infty$.

Remark 3.7. In (3.20) the term $\Gamma_u(t,\tau)(g(\tau)\delta_{\partial\mathbb{R}^2_+})$ is defined as

$$\Gamma_u(t,\tau)(g(\tau)\delta_{\partial\mathbb{R}^2_+})(x) = 2\int_{\mathbb{R}} \Gamma_{\tilde{u}}(t,x;\tau,y_1,0)g(\tau,y_1)\,\mathrm{d}y_1. \tag{3.24}$$

Proof of Lemma 3.6. The existence of fundamental solutions to (3.21) is classical under the assumption of (C1); cf. [13]. The estimate (3.22) is a consequence of [6, Theorem 1] and the definition of $\Gamma_u(t,s)$. As for (3.23), we have from [6, Theorem 3] that

$$\Gamma_{\tilde{u}}(t,x;s,y) \le \frac{1}{4\pi(t-s)} \exp\Big(-\frac{1}{4(t-s)} \big(|x-y| - \int_s^t \|u(\tau)\|_\infty \,\mathrm{d}\tau\big)_+^2 \Big). \tag{3.25}$$

Here $(\alpha)_+ = \max\{0,\alpha\}$ for $\alpha \in \mathbb{R}$. The condition (C3) yields

$$\int_s^t \|u(\tau)\|_\infty \,\mathrm{d}\tau \le M \int_s^t (-\tau)^{-\frac{1}{2}} \,\mathrm{d}\tau \le 2M|t-s|^{\frac{1}{2}}, \qquad M = \sup_{-\infty<t<0} (-t)^{\frac{1}{2}} \|u(t)\|_\infty.$$

Hence if $|x-y| \ge 4(t-s)^{1/2}$ then (3.25) implies (3.23). On the other hand, if $|x-y| \le 4M(t-s)^{1/2}$ then again from (3.25) we have

$$\Gamma_{\tilde{u}}(t,x;s,y) \le \frac{1}{4\pi(t-s)} = \frac{1}{4\pi(t-s)} e^{\frac{|x-y|^2}{t-s}} e^{-\frac{|x-y|^2}{t-s}} \le \frac{e^{16M^2}}{4\pi(t-s)} e^{-\frac{|x-y|^2}{t-s}},$$

which is the desired estimate. The positivity of $\Gamma_{\tilde{u}}(t,x;s,y)$ is a consequence of the strong maximal principle and the details are omitted here. The representation (3.20) is derived from the fact that the equation $\partial_t\omega - \Delta\omega + u\cdot\nabla\omega = 0$ in $(-\infty,0)\times\mathbb{R}^2_+$ with the Neumann boundary condition $\partial_2\omega = g$ on $\partial\mathbb{R}^2_+$ is equivalent with the equation

$$\partial_t\tilde{w} - \Delta\tilde{w} + \tilde{u}\cdot\nabla\tilde{w} = -2g\delta_{\partial\mathbb{R}^2_+} \quad \text{in } (-\infty,0)\times\mathbb{R}^2, \tag{3.26}$$

where $\tilde{w}$ is the even extension of ω to $\mathbb{R}^2$. The proof is complete.

Proof of Lemma 3.1. By (C1), (3.5), and (3.19) the function $g(t) = -(-\partial_1^2)^{1/2}\omega(t)|_{x_2=0} - \partial_1 p_{F(t)}$ is estimated as

$$\|g(t)\|_{L^\infty(\partial\mathbb{R}^2_+)} \le C(-t)^{-\frac{3-\epsilon}{2}} \qquad -\infty < t < 0,\ \epsilon\in(0,1). \tag{3.27}$$

The estimate (3.23) and the representation (3.24) lead to

$$\|\Gamma_u(t,\tau)(g(\tau)\delta_{\partial\mathbb{R}^2_+})\|_\infty \le C(t-\tau)^{-\frac{1}{2}}(-\tau)^{-\frac{3-\epsilon}{2}}$$

for $\tau<0$ and $\epsilon\in(0,1)$. On the other hand, we have from (3.4) and (3.22) that $\|\Gamma_u(t,s)\omega(s)\|_\infty \le C(-s)^{-1}|\log(-s)|^2$ for $s \ll -1$. Thus by taking the limit $s\to-\infty$ in (3.20) we arrive at the expression

$$\omega(t,x) = -2\int_{-\infty}^t \int_{\mathbb{R}} \Gamma_u(t,x;\tau,y_1,0) g(\tau,y_1) \,\mathrm{d}y_1 \,\mathrm{d}\tau, \qquad t<0,\ x\in\mathbb{R}^2_+. \tag{3.28}$$

Let $\theta \in (0, 1-\epsilon)$. Then from (3.23) and (3.27) we have

$$x_2^{1+\theta}|\omega(t,x)| \le C \int_{-\infty}^{t} \int_{\mathbb{R}} (t-\tau)^{-1+\frac{1+\theta}{2}} e^{-c\frac{(x_1-y_1)^2}{t-\tau}} |g(\tau, y_1)| \,\mathrm{d}y_1$$
$$\le C \int_{-\infty}^{t} (t-\tau)^{\frac{\theta}{2}} (-\tau)^{-\frac{3-\epsilon}{2}} \,\mathrm{d}\tau \le C(-t)^{-\frac{1-\theta-\epsilon}{2}}. \qquad (3.29)$$

It is easy to see that the same argument with (C1) also yields $\sup_{-1<t<0, x\in\mathbb{R}^2_+} x_2^{1+\theta}|\omega(t,x)| < \infty$. The proof is complete.

3.3 Representation of solutions by the Biot-Savart law

In this section we give a proof of Lemma 3.2. To this end we denote by $v(t,x)$ the right-hand side of (3.2), which is well-defined by (3.1) and the estimate

$$\int_{\mathbb{R}^2_+} \Big| \frac{(x-y)^\perp}{|x-y|^2} - \frac{(x-y^*)^\perp}{|x-y^*|^2} \Big| (1+y_2)^{-1-\theta} \,\mathrm{d}y \le C \int_{\mathbb{R}^2_+} \frac{y_2}{|x-y||x-y^*|} (1+y_2)^{-1-\theta} \,\mathrm{d}y < \infty.$$

In particular, v is uniformly bounded in $(-\infty, 0) \times \mathbb{R}^2_+$. The goal is thus to show $u = v$. Since both u and v satisfy the divergence-free condition and their vorticity fields are given by the same ω, the difference $w = u - v$ is harmonic in $\mathbb{R}^2_+$. Moreover, u and v_2 vanishes on the boundary by the no-slip boundary condition and the definition of v. Hence, due to the Liouville theorem for harmonic functions we only need to prove the fact $v_1 = 0$ on $\partial\mathbb{R}^2_+$. We first note that v_1 is written as

$$\begin{aligned} v_1(t,\cdot,x_2) &= \partial_2 \int_0^{x_2} e^{-(x_2-y_2)(-\partial_1^2)^{\frac{1}{2}}} \int_{y_2}^{\infty} e^{-(z_2-y_2)(-\partial_1^2)^{\frac{1}{2}}} \omega(t,\cdot,z_2) \,\mathrm{d}z_2 \,\mathrm{d}y_2 \\ &= \int_{x_2}^{\infty} e^{-(y_2-x_2)(-\partial_1^2)^{\frac{1}{2}}} \omega(t,\cdot,y_2) \,\mathrm{d}y_2 \\ &\quad - \int_0^{x_2} \int_{y_2}^{\infty} (-\partial_1^2)^{\frac{1}{2}} e^{-(x_2-2y_2+z_2)(-\partial_1^2)^{\frac{1}{2}}} \omega(t,\cdot,z_2) \,\mathrm{d}z_2 \,\mathrm{d}y_2. \end{aligned} \qquad (3.30)$$

The last term of the right-hand side of (3.30) vanishes on $\partial\mathbb{R}^2_+$, so we focus on the first term which we will denote by $v_{1,1}(t,x)$. Fix any $\delta > 0$ and let $-t > 2\delta$ and $x_2 > \delta$. For sufficiently small $\epsilon \in (0, \delta/4)$ we denote by $\omega_\epsilon(t,x) = \int_{-\infty}^{\delta} \int_{\mathbb{R}^2_+} \eta_\epsilon(t-s, x-y)\omega(s,y) \,\mathrm{d}y \,\mathrm{d}s$ the mollification of ω. The mollifier η_ϵ is taken so that $\operatorname{supp} \eta_\epsilon \subset \{(t,x) \in \mathbb{R}^3 \mid |t|^2 + |x|^2 < \epsilon^2\}$. Then ω_ϵ satisfies

$$\partial_t \omega_\epsilon(t,x) = \Delta\omega_\epsilon(t,x) - \nabla \cdot (u\omega)_\epsilon(t,x) + F_\epsilon(t,x), \qquad (3.31)$$

where

$$
\begin{aligned}
(u\omega)_\epsilon(t,x) &= \int_{-\infty}^{\delta}\int_{\mathbb{R}^2_+} \eta_\epsilon(t-s,x-y)u\omega(s,y)\,\mathrm{d}y\,\mathrm{d}s,\\
F_\epsilon(t,x) &= -\eta_\epsilon(t-\delta)*\omega(\delta)(x)\\
&\quad -\int_{-\infty}^{\delta}\int_{\partial\mathbb{R}^2_+}\big(\eta_\epsilon(t-s,x-y)\partial_2\omega(s,y)+\partial_2\eta_\epsilon(t-s,x-y)\omega(s,y)\big)\,\mathrm{d}y_1\,\mathrm{d}s.
\end{aligned}
$$

By (3.1) and the definition of η_ϵ each term in (3.31) has the same spatial decay as ω. Set $v_{1,1,\epsilon}(t,\cdot,x_2) = \int_{x_2}^{\infty} e^{-(y_2-x_2)(-\partial_1^2)^{\frac{1}{2}}}\omega_\epsilon(t,\cdot,y_2)\,\mathrm{d}y_2$. Then we verify the calculation

$$
\partial_t v_{1,1,\epsilon}(t,\cdot,x_2) = \int_{x_2}^{\infty} e^{-(y_2-x_2)(-\partial_1^2)^{\frac{1}{2}}}\big(\Delta\omega_\epsilon - \nabla\cdot(u\omega)_\epsilon + F_\epsilon\big)(t,\cdot,y_2)\,\mathrm{d}y_2,
$$

and the integration by parts yields

$$
\begin{aligned}
\partial_t v_{1,1,\epsilon}(t,\cdot,x_2) &= -\partial_2\omega_\epsilon(t,\cdot,x_2) - (-\partial_1^2)^{\frac{1}{2}}\omega_\epsilon(t,\cdot,x_2)\\
&\quad - \int_{x_2}^{\infty}\nabla_x\cdot e^{-(y_2-x_2)(-\partial_1^2)^{\frac{1}{2}}}(u\omega)_\epsilon(t,\cdot,y_2)\,\mathrm{d}y_2\\
&\quad + (u_2\omega)_\epsilon(t,\cdot,x_2) + \int_{x_2}^{\infty} e^{-(y_2-x_2)(-\partial_1^2)^{\frac{1}{2}}}F_\epsilon(t,\cdot y_2)\,\mathrm{d}y_2.
\end{aligned}
\tag{3.32}
$$

From (C1) and (3.1) it is easy to see that the following convergence holds in the limit $\epsilon\to 0$ uniformly on each compact set of $\{(t,x)\mid t<-2\delta, x_2>\delta\}$:

$$
\begin{aligned}
-\partial_2\omega_\epsilon(t) - (-\partial_1^2)^{\frac{1}{2}}\omega_\epsilon(t) + (u_2\omega)_\epsilon(t) &\to -\partial_2\omega(t) - (-\partial_1^2)^{\frac{1}{2}}\omega(t) + u_2\omega(t),\\
\int_{x_2}^{\infty}\nabla\cdot e^{-(y_2-x_2)(-\partial_1^2)^{\frac{1}{2}}}(u\omega)_\epsilon(t,\cdot,y_2)\,\mathrm{d}y_2 &\to \int_{x_2}^{\infty}\nabla_x\cdot e^{-(y_2-x_2)(-\partial_1^2)^{\frac{1}{2}}}(u\omega)(t,\cdot,y_2)\,\mathrm{d}y_2,\\
\int_{x_2}^{\infty} e^{-(y_2-x_2)(-\partial_1^2)^{\frac{1}{2}}}F_\epsilon(t,\cdot,y_2)\,\mathrm{d}y_2 &\to 0.
\end{aligned}
$$

Thus we have for $s<t<-2\delta$ and $x_2>\delta$,

$$
\begin{aligned}
v_{1,1}(t)-v_{1,1}(s) &= \int_s^t \big(-\partial_2\omega(\tau) - (-\partial_1^2)^{\frac{1}{2}}\omega(\tau) + u_2\omega(\tau)\big)\,\mathrm{d}\tau\\
&\quad - \int_s^t\int_{x_2}^{\infty}\nabla_x\cdot e^{-(y_2-x_2)(-\partial_1^2)^{\frac{1}{2}}}(u\omega)(\tau,\cdot,y_2)\,\mathrm{d}y_2\,\mathrm{d}\tau.
\end{aligned}
\tag{3.33}
$$

Since $\delta>0$ is arbitrary we may take $x_2\to 0$ in (3.33). Then, recalling the definition of $v_{1,1}$ and (3.30), we take the trace $x_2\to 0$ and obtain from

(2.22) that

$$v_1(t) = v_1(s) + \int_s^t \Big(\partial_1 p_{F(\tau)} - \int_0^\infty \partial_1 e^{-y_2(-\partial_1^2)^{\frac{1}{2}}} (u_1\omega)(\tau,\cdot,y_2)\, \mathrm{d}y_2 \\ - \int_0^\infty (-\partial_1^2)^{\frac{1}{2}} e^{-y_2(-\partial_1^2)^{\frac{1}{2}}} (u_2\omega)(\tau,\cdot,y_2)\, \mathrm{d}y_2 \Big)\, \mathrm{d}\tau \tag{3.34}$$

on $\partial\mathbb{R}^2_+$. Since $p_{F(\tau)}$ is the solution given by Proposition 2.1 with $F(\tau) = -u(\tau) \otimes u(\tau)$, by using $\operatorname{div}\operatorname{div} F = -\operatorname{div}(u^\perp\omega) - \Delta|u|^2/2$ we have the representation

$$\partial_1 p_{F(\tau)} = -\partial_1 \frac{|u(\tau)|^2}{2} + \partial_1 \pi_{F(\tau)}, \tag{3.35}$$

where

$$\partial_1 \pi_{F(\tau)} = e^{-x_2(-\partial_1^2)^{\frac{1}{2}}} \int_0^\infty \big(\partial_1 e^{-y_2(-\partial_1^2)^{\frac{1}{2}}} (u_1\omega)(\tau,\cdot,y_2) + (-\partial_1^2)^{\frac{1}{2}} e^{-y_2(-\partial_1^2)^{\frac{1}{2}}} (u_2\omega)(\tau,\cdot,y_2) \big)\, \mathrm{d}y_2 \\ + \int_0^{x_2} \int_{y_2}^\infty \partial_1 (-\partial_1^2)^{\frac{1}{2}} e^{-(x_2-2y_2+z_2)(-\partial_1^2)^{\frac{1}{2}}} (u_1\omega)(\tau,\cdot,z_2)\, \mathrm{d}z_2\, \mathrm{d}y_2 \\ - \int_0^{x_2} \int_{y_2}^\infty \partial_1^2 e^{-(x_2-2y_2+z_2)(-\partial_1^2)^{\frac{1}{2}}} (u_2\omega)(\tau,\cdot,z_2)\, \mathrm{d}z_2\, \mathrm{d}y_2 \\ - \int_0^{x_2} e^{-(x_2-y_2)(-\partial_1^2)^{\frac{1}{2}}} (u_1\omega)(\tau,\cdot,y_2)\, \mathrm{d}y_2. \tag{3.36}$$

Thus (3.34)-(3.36) leads to $v_1(t) = v_1(s)$ on $\partial\mathbb{R}^2_+$ for all $-\infty < s < t < 0$. Then (3.1) and (3.4) imply that

$$v_1(t) = \lim_{s\to-\infty} v_1(s) = \lim_{s\to-\infty} \int_0^\infty e^{-y_2(-\partial_1^2)^{1/2}} \omega(s,\cdot,y_2)\, \mathrm{d}y_2 = 0$$

on $\partial\mathbb{R}^2_+$ by the Lebesgue convergence theorem. The proof is now complete.

4 Application to geometric regularity criterion

We shall extend a geometric regularity criterion [16] of solutions to the Navier-Stokes equations in $\mathbb{R}^3$ to the case when the domain is the half space $\mathbb{R}^3_+$ with the Dirichlet condition as an application of the Liouville type result (Theorem 1.1). As already discussed in [16] when one imposes the Neumann boundary problem (or the slip boundary condition), the extension is rather straightforward. This is because the rescaled two-dimensional vorticity equations still enjoy the maximum principle since there is no vorticity production from the boundary. We shall state our geometric regularity criterion for the Dirichlet problem in a rigorous way.

We consider the Navier-Stokes equations in the half space $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$

$$\partial_t u - \Delta u + \nabla \cdot (u \otimes u) + \nabla p = 0, \qquad \text{div } u = 0 \qquad \text{in } (0,T) \times \mathbb{R}^3_+ \tag{4.1}$$

with the Dirichlet boundary condition:

$$u = 0 \qquad \text{on } (0,T) \times \partial\mathbb{R}^3_+. \tag{4.2}$$

As mentioned in the introduction, we need to consider a spatially non-decaying solution to carry out what is called a blow-up argument. However, if one allows non-decaying solutions, the uniqueness of the initial-boundary value problem for (4.1)-(4.2) fails. Indeed, the Poiseuille type flow of the form

$$u = (u_1(t, x_3), 0, 0), \quad p(t, x_1) = -x_1 f(t), \tag{4.3}$$

solves (4.1)-(4.2) provided that u_1 solves the heat equation

$$\begin{aligned} \partial_t u_1 - \partial_3^2 u_1 &= f(t) \quad \text{in } (0,T) \times \{x_3 > 0\}, \\ u_1 &= 0 \quad \text{on } (0,T) \times \{x_3 = 0\}. \end{aligned}$$

with some f depending only on time. Since f can be chosen arbitrary, one is able to construct various solutions (u, p) to (4.1)-(4.2) of the form (4.3) with the same initial data. If one assumes that f is bounded and smooth, all such (u, p) is smooth and bounded. Hence this yields the non-uniqueness of the initial-boundary value problem for (4.1)-(4.2) when one allows non-decaying solutions.

A simple way to avoid non-uniqueness is to improve a relation between the pressure and the velocity. Taking the divergence of (4.1), we see

$$-\Delta p = \sum_{i,j=1}^{3} \partial_i \partial_j (u_i u_j) \quad \text{in } \mathbb{R}^3_+, \tag{4.4}$$

since $\operatorname{div} u = 0$. Next, taking the inner product of (4.1) with normal $n = (0, 0, -1)$, we have

$$\frac{\partial p}{\partial n} = -\Delta u \cdot n \quad \text{on } \partial\mathbb{R}^3_+. \tag{4.5}$$

It is convenient to decompose p into the sum $p_H + p_F$ as we did in earlier sections. Namely, for the harmonic pressure term p_H we require

$$-\Delta p_H = 0 \quad \text{in } \mathbb{R}^3_+, \tag{4.6}$$

$$\frac{\partial p_H}{\partial n} = -\Delta u \cdot n \quad \text{on } \partial\mathbb{R}^3_+. \tag{4.7}$$

and the pressure p_F coming from transport term we require

$$-\Delta p_F = \sum_{i,j=1}^{3} \partial_i \partial_j (u_i u_j), \quad F = (u_i u_j) \quad \text{in} \quad \mathbb{R}^3_+, \tag{4.8}$$

$$\frac{\partial p_F}{\partial n} = 0 \quad \text{on} \quad \partial \mathbb{R}^3_+. \tag{4.9}$$

Evidently, (4.6)-(4.9) implies (4.4)-(4.5) for $p = p_H + p_F$. Note that the $\Delta u \cdot n = -\mathrm{div}_{\partial \mathbb{R}^3_+}(\omega \times n)$ as noted in [1]. If one imposes smoothness and boundedness for u up to second derivatives, one can get the uniqueness of ∇p (determined from u) provided that p is restricted to avoid the linear growth at spatial infinity; see Proposition 2.1 and Proposition 2.2 and also [1]. The unique solution is formally written by using the Helmholtz projection $\mathbb{P}$ to the solenoidal space:

$$\nabla p = (I - \mathbb{P})(\Delta u - \nabla \cdot (u \otimes u)) \tag{4.10}$$

and the solution having this form is called a mild solution. It is not difficult to prove the uniqueness of the mild solution; see [14] for the whole space and [2] for the half space.

There is a large literature giving a growth condition for pressure so that the solution is a mild solution which is unique. Such type of result goes back to [10] and has been developed in the case of the whole space [15] and the half space [25]. A typical criterion for the whole space case is $p \in L^1((0,T); BMO(\mathbb{R}^3))$ [20]. There are references on this issue [25], [26] for further relaxation of growth assumptions for the pressure.

In this section we consider the mild solution. We know there is a unique local-in-time mild solution for the initial-boundary value problem for (4.1)-(4.2) for any bounded continuous initial velocity u_0 i.e., $u_0 \in BC(\overline{\mathbb{R}^3_+})$ which is solenoidal in the sense that $\mathrm{div}\, u_0 = 0$ in $\mathbb{R}^3_+$ and $u_0 \cdot n = 0$ on $\partial \mathbb{R}^3_+$ [32]; see also [2].

We are now in position to prove Theorem 1.2, which is a natural extension of the geometric regularity criterion of [16]. We shall prove this result by a blow-up argument. The basic strategy is the same as in [16]. However, to assert uniqueness of the limit we invoke our Liouville type result (Theorem 1.1). Of course, in some steps it is more involved because of the presence of the boundary.

Proof of Theorem 1.2. Step 1 (Construction of blow-up sequence). Assume that u blows up at $t = T$. Then there exists a sequence $\{(t_k, x_k)\}_{k=1}^{\infty} \subset [0, T) \times \mathbb{R}^3_+$ with $t_{k+1} > t_k$ such that

(i) $|u(t,x)| \le M_k$ for $t \le t_k,\ \ x \in \mathbb{R}^3_+$

(ii) $M_k = \|u(t_k)\|_\infty \to \infty,\ \ t_k \uparrow T$ as $k \to \infty$

(iii) $|u(t_k, x_k)| \ge M_k/2$

We rescale u, ω with respect to (t_k, x_k) i.e.

$$u_k(t,x) = \lambda_k u(t_k + {\lambda_k}^2 t, x_k + \lambda_k x)$$
$$\omega_k(t,x) = {\lambda_k}^2 \omega(t_k + {\lambda_k}^2 t, x_k + \lambda_k x), \quad T - t_k > {\lambda_k}^2 t > -t_k$$

with $\lambda_k = 1/M_k$. Since (4.1)-(4.2) is scaling invariant under the above rescaling, we see that u_k is a mild solution of (4.1)-(4.2) in $(-t_k {M_k}^2, 0] \times \mathbb{R}^3_{+,-c_k}$ with $c_k = x_{k,3} M_k$, where $x_k = (x_{k,1}, x_{k,2}, x_{k,3})$ and $\mathbb{R}^3_{+,-c} = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > -c\}$.
Step 2 (Compactness). By assumption (i) we have $|u_k| \leq 1$ in $(-t_k {M_k}^2, 0] \times \mathbb{R}^3_{+,-c_k}$. Since u_k is a mild solution, we know that ∇u_k is also bounded in $(-t_k {M_k}^2 + 1, 0] \times \mathbb{R}^3_{+,-c_k}$ by a result of [2]. Thus $(u_k, \omega_k) \rightharpoonup (\bar{u}, \bar{\omega})$ as $k \to \infty$ *-weakly in L^∞ with some $(\bar{u}, \bar{\omega})$ such that $|\bar{u}| \leq 1$ and $|\bar{\omega}| \leq c$ in $(-\infty, 0] \times \mathbb{R}^3_{+,-c}$ ($c = \lim_{k\to\infty} c_k$) by taking a subsequence. Moreover, $\bar{u}$ is a bounded global mild solution in $(-\infty, 0] \times \mathbb{R}^3_{+,-c}$. Note that there are two cases depending upon whether $\lim c_k = \infty$ or $\lim c_k < \infty$. In the first case $\mathbb{R}^3_{+,-c} = \mathbb{R}^3$ and the limit $\bar{u}$ solves the Navier-Stokes equations in the whole space. In the second case $\bar{u}$ solves the Navier-Stokes equations in the half space $\mathbb{R}^3_{+,-c}$ with the Dirichlet condition (cf. [11]).

We need some compactness to guarantee that u_k converges to u at least locally uniformly in $(-\infty, 0] \times \mathbb{R}^3_{+,-c}$ to guarantee that $u_k(0,0) \to \bar{u}(0,0)$.

In the whole space this can be guaranteed by the estimates of higher-order derivatives so that all space-time derivatives of u_k are bounded in $(-t_k {M_k}^2 + 1, 0]$ uniformly in k (e.g. [17]). In the case of the Dirichlet problem it seems to be unknown since it is nontrivial to handle normal derivatives. However, what we need here are local estimates, rather than global estimates.

We first note that the pressure defined by (4.10) is estimated as

$$\|p\|_{L^r(B_R(x_0)\cap\mathbb{R}^3_+)} \leq C(\|\omega\|_{L^\infty(\partial\mathbb{R}^3_+)} + \|u\|^2_{L^\infty(\mathbb{R}^3_+)}) \tag{4.11}$$

with C depending on R and $r \in (1, \infty)$ and independent of u and ω, where $B_R(x_0)$ is a closed ball of radius R centered at $x_0 \in \mathbb{R}^3_+$. Here we normalize p such that $p(x_0) = 0$. Decompose p into $p_H + p_F$. For p_F we have a BMO estimate $\|p\|_{BMO} \leq C\|u\|^2_\infty$. For the harmonic pressure term, as observed in [1], we have

$$\|x_3 \nabla p_H\|_{L^\infty(\mathbb{R}^3_+)} \leq C\|\omega\|_{L^\infty(\partial\mathbb{R}^3_+)}.$$

From these two estimates (4.11) easily follows. The estimate (4.11) enables us to localize the problem. We cut off u in $B_R(x_0) \cap \mathbb{R}^3_+$ with Bogovski type adjustment to apply the L^r maximal regularity of the Stokes equation problem in a smoothly bounded domain with the zero boundary condition, e.g. [18]. By (4.11) we observe that the external force has a local space-time L^r bound depending on u only through the space time sup norm of u

and ∇u. Thus we are able to control all $W_r^{2,1}(I \times (B_R(x_0) \cap \mathbb{R}^3_+))$ norm of u, where I is a bounded time interval $\subset (-\infty, 0]$. By the Sobolev embedding theorem we have a Hölder bound on ∇u in $Q = I \times (B_R(x_0) \cap \mathbb{R}^3_+)$. This is of course enough to ensure that u_k converges to $\bar{u}$ locally uniformly in $(-\infty, 0] \times \bar{\mathbb{R}}^3_{+,-c}$. By a bootstrap argument we improve the regularity of the pressure and observe that $u_k \to u$ locally uniformly for its all derivatives. Note that without a bound for the pressure one cannot localize the problem. Since (t_k, x_k) is taken so that $|u_k(0,0)| \geq 1/2$ by Step 1 (iii), we conclude that $|\bar{u}(0,0)| \geq 1/2$.
Step 3 (Characterization of the limit). We now apply the continuous alignment condition (CA) and our Liouville type result (Theorem 1.1) to conclude that $\bar{u}$ must be zero, which contradict with $|\bar{u}(0,0)| \geq 1/2$. Here is a sketch of the proof. We set the vorticity direction $\xi_k = \omega_k/|\omega_k|$. Then (CA) implies

$$|\xi_k(t,x) - \xi_k(t,y)| \leq \eta(\frac{|x-y|}{M_k}) \to 0,$$

so that $\bar{\xi} = \bar{\omega}/|\bar{\omega}|$ is independent of x. By the unique existence theory [2] of the mild solution $\bar{\xi}$ must be also constant in time. Thus $(\bar{u}, \bar{\omega})$ is a two-dimensional flow in $(-\infty, 0) \times \mathbb{R}^3_{+,-c}$. When $c = \infty$ the problem is reduced to the whole space case, and it is already proved in [16] that $\bar{u} = 0$, which leads to a desired contradiction. Hence it suffices to consider the case $c < \infty$. By a suitable change of coordinates we may assume that $\bar{u} = (\bar{u}_1(x_1,x_2), \bar{u}_2(x_1,x_2), 0)$ with $\bar{\omega} = (0,0,\bar{\omega}_3)$, $\bar{\omega}_3 \geq 0$ and $\bar{u}_1 = \bar{u}_2 = 0$ on $(-\infty, 0) \times \partial\mathbb{R}^2_+$ where $\mathbb{R}^2_+ = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$.

Now we shall apply Theorem 1.1 for $(\bar{u}, \bar{\omega})$. The condition (C4) is trivially fulfilled because $\bar{\omega_3} \geq 0$. The condition (C3) is inherited from the type I assumption. It remains to prove (C1) and (C2) for our mild solution $\bar{u}$, but thanks to Proposition 2.1 and 2.2, it is enough to prove (C1). By the construction of the blow-up sequence we know

$$\sup_{-\infty<t<0} \|\bar{u}(t)\|_\infty < \infty.$$

Applying a result of [2], we also know $\|\nabla \bar{u}(t)\|_\infty$ is bounded for all $t < 0$. Then we have to estimate the higher-order derivatives to prove (C1), which will be established in Lemma 4.1 below. This is sufficient to derive (C1) so we apply Theorem 1.1 to conclude $\bar{u} \equiv 0$, and reach a contradiction.

Lemma 4.1. *Let u be a mild solution of (4.1)-(4.2) in $(0,T) \times \mathbb{R}^2_+$ with initial data $u_0 = (u_{0,1}, u_{0,2})$ in $BC(\overline{\mathbb{R}^2_+})$. Assume that u_0 is solenoidal, i.e. $\operatorname{div} u_0 = 0$ in $\mathbb{R}^2_+$ and $u_{0,2} = 0$ on $\partial\mathbb{R}^2_+$. Assume that there exists $T > 0$ such that*

$$K = \sup_{0<t<T} \|u(t)\|_\infty < \infty.$$

Then there exists a constant C depending only on K and T such that

$$\sup_{0<t<T} (t^{\frac{m}{2}} \|\nabla^m u(t)\|_\infty + t^{1+\frac{l}{2}} \|\nabla^l \partial_t u(t)\|_\infty) \leq C$$

with $m = 1, 2, 3$ and $l = 0, 1$, where ∇^0 is interpreters as an identity operator.

Remark 4.2. We need the Hölder norm estimates in (C1), but these are obtained from a simple interpolation of L^∞ bounds in Lemma 4.1; see e.g. [22, Theorem 3.2.1], [34, Section 3.2].

The idea of the proof of Lemma 4.1 is to estimate the tangential derivatives with up to one normal derivative as in [13] or [17]. We also need to estimate the time derivative. In the meanwhile we estimate p_H and p_F, which enable us to estimate the normal derivatives. Except for the estimates of the pressure term the argument is rather conventional, so we give a sketch of the proof instead of giving a full detail. In the argument below we just use L^∞ norm so we simply write $\|f\|$ instead of $\|f\|_\infty$.

Sketch of the proof of Lemma 4.1. Step 1 (Tangential derivatives and time derivatives). We first note that the mild solution solves the integral equation

$$u(t) = S(t)u_0 + w, \qquad w = -\int_s^t S(t-s)\mathbb{P}\nabla \cdot (u \otimes u)(s)\,\mathrm{d}s, \tag{4.12}$$

where $S(t)$ is the Stokes semigroup. According to [2], we know

$$\|\nabla S(t)\mathbb{P}f\| \leq C_1 t^{-\frac{1}{2}} \|f\|, \qquad \|S(t)\mathbb{P}\nabla \cdot f\| \leq C_2 t^{-\frac{1}{2}} \|f\|. \tag{4.13}$$

Taking the derivatives ∇ in (4.12), we obtain, for $0 < \epsilon < 1$,

$$\begin{aligned} \|\nabla u\|(t) \leq{}& C_0 t^{-\frac{1}{2}} \|u_0\| + \int_{t(1-\epsilon)}^t \|\nabla S(t-s)\mathbb{P}\nabla \cdot (u \otimes u)(s)\|\,\mathrm{d}s \\ &+ \int_0^{t(1-\epsilon)} \|\nabla S(\frac{t-s}{2}) \cdot S(\frac{t-s}{2})\mathbb{P}\nabla \cdot (u \otimes u)(s)\|\,\mathrm{d}s \\ \leq{}& C_0 t^{-\frac{1}{2}} \|u_0\| + 2KC_1 \int_{t(1-\epsilon)}^t (t-s)^{-\frac{1}{2}} \|\nabla u(s)\|\,\mathrm{d}s + K^2 C_1 C_2 \int_0^{t(1-\epsilon)} (t-s)^{-1}\,\mathrm{d}s. \end{aligned}$$

This yields the estimate $\|\nabla u(t)\| \leq C_K t^{-1/2}$, $t \in (0, T)$ by using [13, Lemma 2.4]. Since the tangential derivative ∂_1 commutes with the Stokes semigroup, a similar argument yields $\|\partial_1^{m-1} \nabla u(t)\| \leq C_{Km} t^{-m/2}$, $t \in (0, T)$ for all $m = 1, 2, \cdots$. Note that the proof makes the sense if we know in advance that $\|\partial_1^{m-1} \nabla u(t)\|$ is finite and locally bounded in time in $(0, T)$; however we are able to justify this process by approximating u_0 by L^p_σ vector field, and the details are omitted here.

As for the time derivative, we differentiate (4.12) in t, which gives

$$\begin{aligned}\partial_t u(t) = \frac{\mathrm{d}S(t)}{\mathrm{d}t}u_0 - \int_{t(1-\epsilon)}^{t} S(t-s)\mathbb{P}\partial_s\nabla\cdot(u\otimes u)(s)\,\mathrm{d}s - S(\epsilon t)\mathbb{P}\nabla\cdot(u\otimes u)(t-\epsilon t)\\ - \int_0^{t(1-\epsilon)} \frac{\mathrm{d}}{\mathrm{d}t}S(t-s)\mathbb{P}\nabla\cdot(u\otimes u)(s)\,\mathrm{d}s.\end{aligned}$$

By using the estimate (4.13) and $\|\frac{\mathrm{d}S(t)}{\mathrm{d}t}f\| \leq Ct^{-1}\|f\|$ that is obtained from the explicit formula of the Stokes semigroup in [32, 35], we have

$$\begin{aligned}\|\partial_t u(t)\| &\leq C_K t^{-1} + \int_{t(1-\epsilon)}^{t} \frac{C_K}{(t-s)^{\frac{1}{2}}}\|\partial_s u(s)\|\,\mathrm{d}s + C_K(\epsilon t)^{-\frac{1}{2}} + \int_0^{t(1-\epsilon)} \frac{C_K}{(t-s)^{\frac{3}{2}}}\,\mathrm{d}s\\ &\leq C_{K,T,\epsilon}t^{-1} + \int_{t(1-\epsilon)}^{t} \frac{C_K}{(t-s)^{\frac{1}{2}}}\|\partial_s u(s)\|\,\mathrm{d}s.\end{aligned}$$

Then by using [13, Lemma 2.4] we have the estimate $\|\partial_t u(t)\| \leq C_{K,T}t^{-1}$. Similarly, we can also obtain the following estimate.

$$\begin{aligned}\|\nabla\partial_t u(t)\| \leq \|\nabla\frac{\mathrm{d}S(t)}{\mathrm{d}t}u_0\| + \int_{t(1-\epsilon)}^{t} \|\nabla S(t-s)\mathbb{P}\partial_s\nabla\cdot(u\otimes u)(s)\|\,\mathrm{d}s\\ + \|\nabla S(\epsilon t)\mathbb{P}\nabla\cdot(u\otimes u)(t-\epsilon t)\| + \int_0^{t(1-\epsilon)} \|\nabla\frac{\mathrm{d}}{\mathrm{d}t}S(t-s)\mathbb{P}\nabla\cdot(u\otimes u)(s)\|\,\mathrm{d}s.\end{aligned}$$

By using (4.13) again, we have

$$\|\nabla\partial_t u(t)\| \leq C_{K,T,\epsilon}t^{-\frac{3}{2}} + \int_{t(1-\epsilon)}^{t} \frac{C_K}{(t-s)^{\frac{1}{2}}}\|\nabla\partial_s u(s)\|\,\mathrm{d}s.$$

Hence, by [13, Lemma 2.4] we arrive at $\|\nabla\partial_t u(t)\| \leq C_{K,T}t^{-3/2}$.
Step 2 (Pressure estimates). In order to estimate the normal derivatives of the solution, we first consider p_F and p_H. Recall that p_F is expressed as $p_F = \int_{\mathbb{R}^2}\nabla\nabla\cdot E(x-y)(\tilde{u}\otimes\tilde{u})\,\mathrm{d}y$; see the proof of Proposition 2.1. Hence $\partial_1 p_F$ can be decomposed into $\int_{R^2}\nabla^3 E(x-y)(\tilde{u}\otimes\tilde{u})(1-\chi_R)\,\mathrm{d}y + \int_{\mathbb{R}^2}\nabla E(x-y)\partial_1\nabla\cdot\big(\tilde{u}\otimes\tilde{u}\chi_R\big)\,\mathrm{d}y$, where $E(x)$ is the newton potential and $\chi_R = \chi_R(x-y)$ is a smooth cut-off such that $\chi_R(x-y) = 1$ for $|x-y| \leq R$ and $\chi_R(x-y) = 0$ for $|x-y| \geq 2R$. Then we have

$$\|\partial_1 p_F(t)\| \leq R^{-1}\|u(t)\|^2 + R\big(\|\partial_1\nabla u(t)\|\|u(t)\| + \|\nabla u(t)\|^2\big),$$

which yields $\|\partial_1 p_F\| \leq Ct^{-1/2}$ by taking $R = t^{1/2}$, where C depends only on K and T. As for p_H, we see

$$\|\partial_1 p_H\|_{L^\infty_{x_1}}(t,x_2) \leq \int_{x_2}^{L} \|\partial_1\partial_2 p_H\|_{L^\infty_{x_1}}(t,y_2)\,\mathrm{d}y_2 + \|\partial_1 p_H\|_{L^\infty_{x_1}}(t,L).$$

Since $\partial_1\partial_2 p_H(w) = \partial_1^2 e^{-x_2(-\partial_1^2)^{1/2}}(\omega|_{x_2=0}) = e^{-x_2(-\partial_1^2)^{1/2}}(\partial_1^2\omega|_{x_2=0})$, we have

$$\|\partial_1\partial_2 p_H\|_{L^\infty_{x_1}}(t, y_2) \le C\|\partial_1^2\nabla u(t)\|. \tag{4.14}$$

Furthermore, it follows from (2.7) that $\|\partial_1 p_H\|_{L^\infty_{x_1}}(t, L) \le CL^{-1}\|\omega(t)\|$. Hence

$$\|\partial_1 p_H\|_{L^\infty_{x_1}}(t, x_2) \le C\big(L\|\partial_1^2\nabla u(t)\| + \frac{1}{L}\|\omega(t)\|\big), \quad 0 \le x_2 \le L.$$

By taking $L = t^{1/2}$, we have $\sup_{0\le x_2\le L}\|\partial_1 p_H\|_{L^\infty_{x_1}}(t, x_2) \le Ct^{-1}$. Then by maximum principle we obtain $\|\partial_1 p_H(t)\| \le Ct^{-1}$. The constant C depends only on K and T. From the similar argument we can extend the estimates to the higher-order tangential derivatives.
Step 3 (Normal derivatives). By combining the above estimates with the following equation

$$\partial_t u_1 - \Delta u_1 + (u, \nabla)u_1 + \partial_1 p = 0, \tag{4.15}$$

it is easy to check that $\|\partial_2^2 u(t)\| \le Ct^{-1}$. Finally, by differentiating (4.15) in the normal direction and by using (4.14), the estimate $\|\partial_2^3 u_1(t)\| \le Ct^{-3/2}$ follows. With the aid of the estimate $\|\partial_1^2 p(t)\| \le Ct^{-3/2}$ and the divergence free property of the solutions, we finally obtain $\|\partial_2^3 u_2(t)\| \le Ct^{-3/2}$ by differentiating (4.15) in the tangential direction. The proof is now complete.

5 A remark on L^∞ estimates for the higher order derivatives

In this section we consider L^∞ estimates for the higher order derivatives of solutions to the Navier-Stokes equations in the half plane.

There are several related results on this issue. For all $q \in [n, \infty]$, L^q estimates for the higher order derivatives of solutions to the Navier-Stokes equations is obtained by [17] in the whole space $\mathbb{R}^n$ for $n \ge 2$. For the Keller-Segel equation of parabolic-parabolic type, the spatial analyticity of solutions is proved by [33]. And [36] showed the spatial analyticity of the solution to the drift-diffusion equation. However, they considered this problem in the whole space for the most part.

In [17], they proved the L^q estimates for the lower order derivatives of solutions by using the Gronwall type inequality. And then they showed the L^q estimates for the higher order derivatives of solutions by induction. However, when we apply this method directly in the half plane case, it is difficult to obtain the integral estimates unless we have suitable commutativity between the semigroup and differential. In the previous section, we overcame this difficult by combining Gronwall type inequality used in

[13, 17] and the estimates in [2] to show the L^∞ estimates for the lower order derivatives. In this section, we will give some remarks and conjectures on the estimates for the higher order derivatives.

Conjecture 1. *Assume that the conditions in Lemma 4.1 hold,* $\|\nabla^k u\| \le Ct^{\frac{-k}{2}}$ *for* $k = 0, 1, 2, ..., m$ *and* $\|\partial^\alpha \partial_1 p\| \le Ct^{\frac{-|\alpha|-1}{2}}$ *for* $|\alpha| = m-1 \ge 0$. *Then* $\|\nabla^{m+1} u\| \le C_T t^{\frac{-m-1}{2}}$

A possible strategy to prove conjecture 1 is the following.

By Lemma 4.1, $\|\partial_1^\beta \nabla u\| \le Ct^{\frac{-\beta-1}{2}}$ for any $\beta \in \mathbb{N} \cup \{0\}$, so we only need to prove the terms involve normal derivatives higher than second order. Since the divergence free property of the solutions, $\partial_1^{m+1-i}\partial_2^i u_2 = -\partial_1^{m+2-i}\partial_2^{i-1} u_1$, we only need to observe the derivatives of u_1; i.e. we only need to investigate $\partial_1^{m-1-i}\partial_2^{i+2} u_1$ for $i = 0, 1, 2, ..., m-1$. (Since the estimates for $\partial_1^{m+1} u_1$ and $\partial_1^m \partial_2 u_1$ are already known.)

If we can obtain the estimates for $\partial_t u$ such as $\|\nabla^k \partial_t u\| \le Ct^{-k/2-1}$ then we can get the conclusion by induction as follows.

For $i = 0$, we take ∂_1^{m-1} to (4.15) to obtain the estimate for $\partial_1^{m-1}\partial_2^2 u_1$.

Similarly for $i = 1$, we take $\partial_1^{m-2}\partial_2$ to (4.15) to obtain the estimate for $\partial_1^{m-2}\partial_2^3 u_1$.

For $i = 2$, in order to obtain the estimate for $\partial_1^{m-3}\partial_2^4 u_1$, we need the estimate for $\partial_1^{m-1}\partial_2^2 u_1$ we just done in the case $i = 0$.

Hence, by similar argument, the proof is completed by induction.

Remark 5.1. As for the estimates for higher order derivatives of $\partial_t u$, when we apply the same argument as in the proof of Lemma 4.1, we have to deal with the integral term $\int_{t(1-\epsilon)}^t \|\nabla^k S(t-s)\mathbb{P}\partial_s \nabla \cdot (u \otimes u)(s)\| \, \mathrm{d}s$. Since we do not have the commutativity between the semigroup and differential in the normal direction, it is difficult to estimate this term. If we can extend the estimates in [2] to higher order; i.e. if we can prove the following estimates

$$\|\nabla^k S(t)\mathbb{P} f\| \le Ct^{-\frac{k}{2}}\|f\|, \tag{5.1}$$

it is hopeful to obtain the estimates for higher order space derivatives of $\partial_t u$.

Remark 5.2. If $\|\partial_1^m p\| \le C_1 t^{-m/2}$, we may obtain $\|\partial^\alpha \partial_1 p\| \le Ct^{\frac{-|\alpha|-1}{2}}$ by differentiating the Navier-Stokes equations in both direction alternatively and by induction.

According to this remark, if Conjecture 1 is proved, by obtaining the estimates of tangential derivatives of p, we can obtain the estimates for higher order derivatives of solutions to the Navier-Stokes equations in half plane.

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