博士論文

GOOD REDUCTION CRITERION FOR K3 SURFACES

（K3曲面の良い還元の判定法）

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GOOD REDUCTION CRITERION FOR K3 SURFACES

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Abstract. We prove a Néron–Ogg–Shafarevich type criterion for good reduction of K3 surfaces, which states that a K3 surface over a complete discrete valuation field has potential good reduction if its l-adic cohomology group is unramified. We also prove a p-adic version of the criterion. (These are analogues of the criteria for good reduction of abelian varieties.) The model of the surface will be in general not a scheme but an algebraic space. As a corollary of the criterion we obtain the surjectivity of the period map of K3 surfaces in positive characteristic.

1. Introduction

Let K be a complete discrete valuation field with perfect residue field of characteristic p. The Néron–Ogg–Shafarevich criterion states that an abelian variety A has good reduction if and only if $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_l)$ is an unramified representation[1] (or equivalently, the l-adic Tate module of A is unramified) for some prime $l \neq p$ (then it is so for all $l \neq p$). A p-adic counterpart of this result is that A has good reduction if and only if $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ is a crystalline representation.

Such criteria do not hold for general varieties (if a variety has good reduction then its cohomology groups are unramified/crystalline, but the converse is not true).

In this paper we prove criteria for K3 surfaces (in both l-adic and p-adic settings) similar to those for abelian varieties. We allow *algebraic spaces* as models and consider *potential* good reduction (that is, we allow finite extension of the base field). More precisely, our main theorem is the following.

Theorem 1.1. *Let X be a K3 surface over K which admits an ample line bundle L satisfying $p > L^2 + 4$. Assume that one of the following holds:*
(a) *For some prime $l \neq p$, $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified.*
(b) *(K is of characteristic 0 and) $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline.*
Then X has potential good reduction with an algebraic space model, that is, for some finite extension K'/K, there exists an algebraic space smooth proper over $\mathcal{O}_{K'}$ with generic fiber isomorphic to $X_{K'}$.

Remark 1.2. (1) We cannot replace "algebraic space" in the statement of the theorem by "scheme"; we present counterexamples in Section 5.2. Hence it is, in contrast to the case of abelian varieties, somewhat essential to allow algebraic spaces (not only schemes) when considering reduction of K3 surfaces.

Date: 2014/01/07.

[1]A representation of the absolute Galois group of a discrete valuation field is said to be unramified if the inertia subgroup acts trivially.

(2) We do not know whether the field extension is necessary. Under an additional assumption we give a bound for the extension degree (Corollary 4.4). See also the example in Section 5.3.

(3) If one prefers to stay in the category of schemes, we have the following: If X is as in the theorem, then for some finite extension K'/K, there exists a proper flat scheme $\mathcal{X}$ over $\mathcal{O}_{K'}$ with generic fiber isomorphic to $X_{K'}$ and special fiber having at worst rational double point singularities.

(4) The condition on the degree L^2 of the line bundle is satisfied for example in the following cases: (i) $p \geq 7$ and X is a double cover of $\mathbb{P}^2$ ramifying over a sextic; or (ii) $p \geq 11$ and X is a quartic surface in $\mathbb{P}^3$. On the other hand, for each p there exist (infinitely many) K3 surfaces not satisfying the condition, and for $2 \leq p \leq 5$ the condition is never satisfied.

(5) If we know in advance that X (or some surface birationally equivalent to X) has potential semistable reduction, the condition $p > L^2 + 4$ can be replaced by the (weaker) condition $p \geq 5$, since the inequality $p > L^2 + 4$ is used only in the first step of the proof (see the outline below). Potential semistable reduction of general surfaces in mixed characteristic is an open problem.

We review known results concerning this kind of criterion.

Serre–Tate [ST68, Theorem 1] proved the criterion for abelian varieties and gave the name "Néron–Ogg–Shafarevich criterion" after related works of Néron [Nér64], Ogg [Ogg67], and Shafarevich. The p-adic version is obtained by Coleman–Iovita [CI99, Theorem II.4.7] and Breuil [Bre00, Corollaire 1.6]. These criteria fail for general varieties, even for curves of genus ≥ 2 (but there are results of Oda [Oda95, Theorem 3.2] and Andreatta–Iovita–Kim [AIK13, Theorem 1.6] relating good reduction of curves with l-adic and p-adic fundamental groups respectively). Kulikov [Kul77a] essentially showed the (potential) criterion for complex K3 surfaces in the category of complex manifolds (not necessarily schemes). In mixed characteristic, Ito [Ito01, Corollary 4.3][2] and myself [Mat12, Theorem 0.1] proved analogues of the Néron–Ogg–Shafarevich criterion for some special kinds of K3 surfaces using the geometry of the surfaces (which are closely related to abelian surfaces).

The outline of the proof of Theorem 1.1 (given in Section 3) is as follows.

We follow a method of Maulik [Mau12] of studying reduction of K3 surfaces. Using the line bundle L and results of Saito [SaiT04] and Kawamata [Kaw94], we obtain a model of X with log terminal singularities which are well described. Using a result of Artin [Art74], we can resolve some of the singularities of that model in the category of algebraic spaces and we obtain a strictly semistable model (which is an algebraic space). The special fiber of that space is then an SNC (simple normal crossing) log K3 surface, which is classified by Nakkajima [Nakk00] (in a parallel way to the characteristic 0 case [Kul77a]). Using the unramified/crystalline assumption and a comparison result between the cohomology groups of the generic and the special fiber, we conclude that the special fiber is actually smooth.

[2]This paper of Ito is unpublished, but it is included in [Mat12, Theorem 1.18] as an appendix.

In Section 2 we prove (in both l-adic and p-adic settings) the comparison results used in the last step. If the strictly semistable model is a scheme (not merely an algebraic space), then everything we need is well-known. What we have to do is to generalize those results to the algebraic space case. In the l-adic case (Section 2.3), we prove the existence of weight spectral sequence by extending Saito's construction [SaiT03] of the (Steenbrink–Rapoport–Zink) weight spectral sequence in the scheme case to the algebraic space case. In the p-adic case (Section 2.2), we use Olsson's results [Ols07] on the Hyodo–Kato isomorphisms for algebraic spaces.

The remaining part of this thesis provides applications and examples.

In Section 4 we give an application to the period map of K3 surfaces. The period map $\iota_{2d,\mathbb{C}}$ for quasi-polarized complex K3 surfaces attaches the Hodge structure $\langle c_1(L)\rangle^{\perp} \subset H^2(X,\mathbb{Z})$ to each quasi-polarized complex K3 surface (X,L) (of fixed degree $2d$). This map is known to be surjective (Kulikov [Kul77b]). Recent results of Madapusi Pera [MP13a, MP13b] extends $\iota_{2d,\mathbb{C}}$ to a map $\iota_{2d,\mathbb{Z}_{(p)}}$ from the moduli stack of K3 surfaces over $\mathbb{Z}_{(p)}$ to an integral model over $\mathbb{Z}_{(p)}$ of a certain orthogonal Shimura variety, for arbitrary prime $p \neq 2$. Using our main theorem (and its proof) we prove that, under the condition $p > 18d+4$, that $\iota_{2d,\mathbb{Z}_{(p)}}$ is surjective (Theorem 4.1).

In Section 5 we give some (counter)examples related to the main theorem. In Section 5.2 we give a K3 surface having potential good reduction with an algebraic space model but not with a scheme model. In Section 5.3 we give a K3 surface with unramified H^2 which has good reduction with a scheme model only after replacing the base field by an (unramified) extension.

In Section 6 we give an application to K3 surfaces with complex multiplications. A complex K3 surface is said to have complex multiplication (CM) if its Hodge group is commutative, or equivalently if its Mumford–Tate group is commutative. It is known that a CM K3 surface is defined over some number field. Using our main theorem we can show that CM K3 surfaces have potential good reduction at any prime $\mathfrak{p}$, provided the residue characteristic p of $\mathfrak{p}$ satisfies $p > L^2 + 4$ (Theorem 6.3). This can be viewed as an analogue of the fact that abelian varieties with complex multiplications have good reduction [ST68, Theorem 6], although our result is restricted to large p.

Acknowledgments. The author expresses his sincere gratitude to his advisor Atsushi Shiho for supporting him in many ways. The author also thanks Takuma Hayashi, Tetsushi Ito, Teruhisa Koshikawa, Keerthi Madapusi Pera, Yukiyoshi Nakkajima, Takeshi Saito, Naoya Umezaki, and Kohei Yahiro for giving him helpful comments. This work was supported by Grant-in-Aid for JSPS Fellows Grant Number 12J08397.

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2. Comparison theorems for semistable algebraic spaces

In this section we prove comparison theorems of (l-adic and p-adic) cohomology groups of fibers of a semistable algebraic space, which we will use in the proof of the main theorem. For general properties of algebraic spaces, see [Knu71].

2.1. **Statement of the comparison theorems.** Let $\mathcal{O}_K$ be a complete discrete valuation ring with perfect residue field of characteristic $p > 0$. Denote by G_K the absolute Galois group of K. We first introduce the notion of semistable algebraic spaces over $\mathcal{O}_K$.

Recall that an algebraic space is said to be irreducible if the intersection of any two nonempty open subspaces (images of open immersions) is nonempty, and that an irreducible component of an algebraic space is a maximal irreducible closed subspace.

Definition 2.1. An algebraic space X over $\mathcal{O}_K$ is said to be *semistable* purely of dimension n if it is étale-locally isomorphic to $\mathcal{O}_K[x_1, \dots, x_{n+1}]/(x_1 \cdots x_r - \pi)$, where π is a uniformizer of $\mathcal{O}_K$. It is *strictly semistable* if moreover each irreducible component of the special fiber is smooth.

Although irreducibility is not an étale local property, an étale covering of a strictly semistable algebraic space is always strictly semistable. If X is a scheme, these definitions are of course equivalent to the usual ones.

We use the following notation for a strictly semistable algebraic space X over $\operatorname{Spec} \mathcal{O}_K$.

- X_K and X_k are the generic and special fibers of X.
- $X_{\overline{K}}$ and $X_{\overline{k}}$ are the corresponding geometric fibers.
- Z_i ($i = 1, \dots, m$) are the irreducible components of X_k. By assumption each Z_i is smooth and hence all connected components of each $(Z_i)_{\overline{k}}$ are smooth (in particular irreducible).
- $X_k^{(p)}$ is the disjoint union of the smooth $(n-p)$-dimensional (possibly empty) subspaces $Z_I = Z_{i_0} \cap \cdots \cap Z_{i_p}$ for sets $I = \{i_0, \dots, i_p\} \subset \{1, \dots, m\}$ of cardinality[3] $p+1$.
- $X_{\overline{k}}^{(p)}$ is constructed similarly from the irreducible components of $X_{\overline{k}}$. (Since each $(Z_i)_{\overline{k}}$ is smooth, this is naturally identified with $X_k^{(p)} \times_k \overline{k}$).

We now state the comparison theorems. Proofs will be given in the following subsections.

Proposition 2.2. *Assume K is of characteristic 0. Let X be a proper strictly semistable algebraic space over $\mathcal{O}_K$ whose fibers are 2-dimensional schemes. Let $W = W(k)$ and $K_0 = \operatorname{Frac} W$.*

(1) We have a p-adic spectral sequence

$$E_1^{p,q} = \bigoplus_{i \geq \max\{0,-p\}} H^{q-2i}_{\mathrm{crys}}(X_k^{(p+2i)}/W)(-i) \Rightarrow H^{p+q}_{\mathrm{logcrys}}(X_k/W).$$

[3]This numbering is the same to that of Saito [SaiT03]. Some authors (e.g. [RZ82], [Mok93], [Ill94]) write $-^{(p+1)}$ for this space.

Moreover this spectral sequence is compatible with the monodromy operator in the following sense: There is an endomorphism N on the Hyodo–Steenbrink complex $W_n A^\bullet$ defined by Mokrane [Mok93, Section 3.13] *which induces a map*

$$\begin{array}{ccc} E_1^{p,q} = \bigoplus_{i \geq \max\{0,-p\}} H^{q-2i}_{\mathrm{crys}}(X_k^{(p+2i)}/W)(-i) & \Longrightarrow & H^{p+q}_{\mathrm{logcrys}}(X_k/W) \\ \downarrow N & & \downarrow N \\ E_1^{p+2,q-2} = \bigoplus_{i-1 \geq \max\{0,-p-2\}} H^{q-2i}_{\mathrm{crys}}(X_k^{(p+2i)}/W)(-i+1) & \Longrightarrow & H^{p+q}_{\mathrm{logcrys}}(X_k/W) \end{array}$$

of spectral sequences.

(2) The spectral sequence degenerates at E_2 modulo torsion.

(3) The morphism N induces the following isomorphisms on E_2 terms modulo torsion

$$N \colon E_2^{-1,3}{}_{\mathbb{Q}} \xrightarrow{\sim} E_2^{1,1}(-1)_{\mathbb{Q}}, \quad N^2 \colon E_2^{-2,4}{}_{\mathbb{Q}} \xrightarrow{\sim} E_2^{2,0}(-2)_{\mathbb{Q}}.$$

(4) Assume moreover that X_k is an SNC log K3 surface (or more generally[4] that it is liftable to a semistable scheme over K_0). Then we have an isomorphism

$$H^2_{\mathrm{logcrys}}(X_k/W) \otimes_W K \cong (D_{\mathrm{pst}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \otimes_{K_0^{\mathrm{ur}}} \overline{K})^{G_K}$$

compatible with the operator N. In particular, if $H^2_{\text{ét}}$ is crystalline, then the operator N on the right hand side of the spectral sequence in (1) is zero modulo torsion.

Proposition 2.3. *Let X be as in the previous proposition (with no assumption on* $\operatorname{char} K$*). Let $l \neq p$ be a prime. Let Λ be $\mathbb{Z}/l^n\mathbb{Z}$, $\mathbb{Z}_l$, or $\mathbb{Q}_l$.*

(1) We have an l-adic spectral sequence

$$E_1^{p,q} = \bigoplus_{i \geq \max\{0,-p\}} H^{q-2i}_{\text{ét}}(X_{\overline{k}}^{(p+2i)}, \Lambda(-i)) \Rightarrow H^{p+q}_{\text{ét}}(X_{\overline{K}}, \Lambda)$$

compatible with the action of G_K. Moreover this spectral sequence is compatible with the monodromy operator in the following sense: Let T be an element of the inertia group I_K such that $t_l(T)$ is a generator of $\mathbb{Z}_l(1)$ (where $t_l \colon I_K \to \mathbb{Z}_l(1)$ is the canonical surjection). Then the endomorphism $N = T - 1$ of the complex $R\psi\Lambda$ of nearby cycles (defined later) induces a map

$$\begin{array}{ccc} E_1^{p,q} = \bigoplus_{i \geq \max\{0,-p\}} H^{q-2i}(X_{\overline{k}}^{(p+2i)}, \Lambda(-i)) & \Longrightarrow & H^{p+q}(X_{\overline{K}}, \Lambda) \\ \downarrow 1 \otimes t_l(T) & & \downarrow N \\ E_1^{p+2,q-2} = \bigoplus_{i-1 \geq \max\{0,-p-2\}} H^{q-2i}(X_{\overline{k}}^{(p+2i)}, \Lambda(-i+1)) & \Longrightarrow & H^{p+q}(X_{\overline{K}}, \Lambda) \end{array}$$

[4]This liftability assumption is satisfied for any projective SNC log K3 surface [Nakk00, Corollary 6.9].

of spectral sequences[5].

(2) The spectral sequence degenerates at E_2 *modulo torsion.*

(3) Let $\Lambda = \mathbb{Q}_l$. *The morphism* N *induces the following isomorphisms on* E_2 *terms*

$$N\colon E_2^{-1,3} \xrightarrow{\sim} E_2^{1,1}(-1), \quad N^2\colon E_2^{-2,4} \xrightarrow{\sim} E_2^{2,0}(-2).$$

2.2. **Proof of p-adic case.** We prove Proposition 2.2.

Note that (1)–(3) are statements for the special fiber, which is a (log) scheme.

(1),(2) These are proved (without restriction on the dimension) by Mokrane [Mok93, Section 3.23 and Proposition 3.18] and Nakkajima [Nakk05, Theorem 3.6] respectively.

(3) By [Nakk06, Remark 6.8 (1)] (cf. [Nakk, Theorem 8.3]) the weight monodromy conjecture is true for H^2 of 2-dimensional log schemes.

(4) If the algebraic space $\mathcal{X}$ is a scheme, then this follows from the isomorphism of C_{st}. We use results of Olsson [Ols07] to extend this argument to algebraic spaces (actually he treats a more general case of tame Deligne–Mumford stacks). A (ϕ, N, G_K)-structure (in the sense of Olsson [Ols07, Definition 0.1.1]) on a K-vector space T is a collection $(D, \phi, N, \{\rho_\pi\})$ of a finite dimensional K_0^{ur}-vector space D with a continuous semilinear G_K-action, a Frobenius-semilinear automorphism ϕ on D, a K_0^{ur}-linear nilpotent endomorphism N on D, and a family of isomorphisms $\rho_\pi\colon T \xrightarrow{\sim} (D \otimes_{K_0^{\mathrm{ur}}} \overline{K})^{G_K}$ indexed by the uniformizers π of K, subject to certain compatibilities.

Since the generic fiber X_K of our $\mathcal{X}$ is a scheme, $H^2_{\mathrm{dR}}(X_K/K)$ is equipped with a (ϕ, N, G_K)-structure with $D = D_{\mathrm{pst}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ through the C_{st} isomorphism of Tsuji (and de Jong's alteration). Olsson defined another (ϕ, N, G_K)-structure on $H^2_{\mathrm{dR}}(X_K/K)$ in the following way: Let $\mathcal{S}_H(\alpha)$ be the stack (over $\mathbb{Z}[t]$) defined in [Ols07, Sections 6.1–6.2] with parameters $\alpha = (\alpha_1, \ldots, \alpha_r) = (1, \ldots, 1)$ and $H = \mathfrak{S}_r$ for r large enough, and $\mathcal{S} = \mathcal{S}_H(\alpha)_{\mathcal{O}_K,\pi}$ be its base change by $\mathbb{Z}[t] \to \mathcal{O}_K\colon t \mapsto \pi$. Any semistable algebraic space over $\mathcal{O}_K$ with dimension at most r admits a canonical smooth morphism to $\mathcal{S}$ ([Ols07, Section 6.2]). Hence we have a smooth morphism $\mathcal{X} \to \mathcal{S}$. Then Olsson defines ([Ols07, Sections 6.4–6.5]) a projective system $(H^2_{\mathrm{crys}}(X_k/\mathcal{S}_{W_n}))_n$ equipped with ϕ and N, which depends only on the special fiber X_k, and he shows that $D = (\varprojlim_n H^2_{\mathrm{crys}}(X_k/\mathcal{S}_{W_n})) \otimes_W K_0^{\mathrm{ur}}$ gives a (ϕ, N, G_K)-structure on $H^2_{\mathrm{dR}}(X_K/K)$. Here $\mathcal{S}_{W_n} = \mathcal{S}_H(\alpha)_{W_n,0}$ is defined in the same way. He shows also that these two (ϕ, N, G_K)-structures are isomorphic ([Ols07, Theorem 9.6.9]). In particular, we have an isomorphism

$$(D_{\mathrm{pst}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \otimes_{K_0^{\mathrm{ur}}} \overline{K})^{G_K} \xrightarrow{\sim} (\varprojlim_n H^2_{\mathrm{crys}}(X_k/\mathcal{S}_{W_n})) \otimes_W K$$

compatible with N.

On the other hand, by assumption we have a semistable scheme Y over $\mathcal{O}_{K_0}$ with special fiber isomorphic to X_k. Then $H^2_{\mathrm{dR}}(Y_{K_0}/K_0)$ admits a (ϕ, N, G_{K_0})-structure with $D = (\varprojlim_n H^2_{\mathrm{crys}}(Y_k/\mathcal{S}_{W_n})) \otimes K_0^{\mathrm{ur}}$. There is also a (ϕ, N, G_{K_0})-structure with

[5] We fixed a small error in the corresponding formula in [SaiT03, Corollary 2.8]: his $i - 1 \geq \max\{0, -p\}$ should be $i - 1 \geq \max\{0, -p - 2\}$.

$D = H^2_{\mathrm{logcrys}}(Y_k/W) \otimes_W K_0^{\mathrm{ur}}$ defined by Hyodo–Kato [HK94, Section 3]. He shows that these two (ϕ, N, G_{K_0})-structures are isomorphic ([Ols07, Theorem 9.6.7]). In particular, we have an isomorphism

$$(\varprojlim_n H^2_{\mathrm{crys}}(Y_k/\mathcal{S}_{W_n})) \otimes_W K_0 \xrightarrow{\sim} H^2_{\mathrm{logcrys}}(Y_k/W) \otimes_W K_0$$

compatible with N.

Combining these with the obvious isomorphisms

$$(\varprojlim_n H^2_{\mathrm{crys}}(X_k/\mathcal{S}_{W_n})) \otimes_W K \cong (\varprojlim_n H^2_{\mathrm{crys}}(Y_k/\mathcal{S}_{W_n})) \otimes_W K \quad \text{and}$$
$$H^2_{\mathrm{logcrys}}(X_k/W) \otimes_W K \cong H^2_{\mathrm{logcrys}}(Y_k/W) \otimes_W K,$$

we obtain the desired isomorphism. (The compatibility condition of the (ϕ, N, G_K)-structure implies that this isomorphism is independent of the choice of π used in the constructions. We do not need this.)

2.3. **Proof of l-adic case.** In this subsection we prove Proposition 2.3.

We will follow Saito's construction [SaiT03, Sections 1–2] to formulate the weight spectral sequence of l-adic cohomology groups for semistable algebraic spaces. The point of his construction is to give a filtration of $R\psi\Lambda$ by (shifted) perverse sheaves (Lemma 2.5).

First, we will need the theory of étale sheaves, derived categories, six functors, and perverse sheaves on algebraic spaces. This is developed by Laszlo–Olsson in [LO08a, LO08b, LO09]. Since algebraic spaces "are étale-locally schemes", most properties of algebraic spaces (or objects on algebraic spaces) can be defined by taking (the pullback by) an étale covering by a scheme. For example, a perverse sheaf on an algebraic space Y of finite type over a field is defined to be an object of the derived category $D^b_c(Y, \Lambda)$ whose pullback by some (equivalently any) étale covering $U \to X$ by a scheme is a perverse sheaf on U.

Let $f\colon X \to S = \operatorname{Spec} \mathcal{O}_K$ be a strictly semistable algebraic space purely of relative dimension n. Define immersions i, $\bar{i}$, j, $\bar{j}$ by the diagram

$$\begin{array}{ccccc} X_{\overline{k}} & \xrightarrow{\bar{i}} & X_{S^{\mathrm{ur}}} & \xleftarrow{\bar{j}} & X_{\overline{K}} \\ \downarrow & & \downarrow & & \downarrow \\ X_k & \xrightarrow{i} & X & \xleftarrow{j} & X_K. \end{array}$$

Let $R\psi\Lambda = \bar{i}^* R\bar{j}_* \Lambda$ be the complex of nearby cycles, which we regard as an object of the derived category $D^+_c(X_k, \Lambda)$ with a continuous action of G_K.

We introduce some morphisms:

- $i_i\colon Z_i \hookrightarrow X$ and $j_i\colon X \setminus Z_i \hookrightarrow X$ are the immersions.
- $a_p\colon X_k^{(p)} \to X_k$ is the natural map (induced by the immersions).
- $i^{(p)} = i \circ a_p\colon X_k^{(p)} \to X$.
- $\theta\colon \Lambda \to i^* R^1 j_* \Lambda(1)$ is the map sending 1 to the boundary $\partial[\pi] \in H^1(K, \Lambda(1))$ of a prime element π of K with respect to the Kummer sequence.
- $\theta_i\colon \Lambda_{Z_i} \to i_i^* R^1 j_{i*} \Lambda(1)$ are defined similarly.
- $\theta'\colon a_{0*}\Lambda \to i^* R^1 j_* \Lambda(1)$ is the direct sum of the θ_i's.

These maps are all defined without problems in the algebraic space case.

Lemma 2.4 (cf. [SaiT03, Proposition 1.2, Corollary 1.3]). *(1) The map $\theta' \colon a_{0*}\Lambda \to i^*R^1j_*\Lambda(1)$ is an isomorphism and induces isomorphisms $\theta' \colon a_{p*}\Lambda \to i^*R^{p+1}j_*\Lambda(p+1)$ for $p \geq 0$ by cup-product.*

(2) Let $p \geq 0$. The canonical map $i^*R^pj_*\Lambda \to R^p\psi\Lambda$ is surjective. The map $\theta\cup \colon i^*R^pj_*\Lambda \to i^*R^{p+1}j_*\Lambda(1)$ induces a map $\bar{\theta} \colon R^p\psi\Lambda \to i^*R^{p+1}j_*\Lambda(1)$. The sequence

$$0 \to R^p\psi\Lambda \xrightarrow{\bar{\theta}} i^*R^{p+1}j_*\Lambda(1) \to R^{p+1}\psi\Lambda(1) \to 0$$

is exact.

(3) Let $\delta \colon \Lambda \to a_{0*}\Lambda$ be the canonical map. Then we have an isomorphism

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Lambda & \xrightarrow{\delta} & a_{0*}\Lambda & \xrightarrow{\delta\wedge} & \cdots & \xrightarrow{\delta\wedge} & a_{n*}\Lambda & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \theta'} & & & & \downarrow{\scriptstyle \theta'} & & \\ 0 & \longrightarrow & \Lambda & \xrightarrow{\theta} & i^*R^1j_*\Lambda(1) & \xrightarrow{\theta\cup} & \cdots & \xrightarrow{\theta\cup} & i^*R^{n+1}j_*\Lambda(n+1) & \longrightarrow & 0 \end{array}$$

of exact sequences.

(4) For $p \geq 0$, we have an exact sequence

$$0 \to R^p\psi\Lambda \xrightarrow{\bar{\theta}} i^*R^{p+1}j_*\Lambda(1) \xrightarrow{\theta\cup} \cdots \xrightarrow{\theta\cup} i^*R^{n+1}j_*\Lambda(n-p+1) \to 0.$$

Proof. (1), (2) Since the assertions are étale local, we can reduce to the scheme case [SaiT03].

(3), (4) These follow from (1) and (2). □

Lemma 2.5 (cf. [SaiT03, Lemma 2.5]). *(1) The object $R\psi\Lambda$ of $D^b_c(X_{\overline{k}}, \Lambda)$ is a $(-n)$-shifted perverse sheaf.*

(2) The canonical filtration $F'_pR\psi\Lambda = \tau_{\leq p}R\psi\Lambda$ is a filtration by sub $(-n)$-shifted perverse sheaves. Here $\tau_{\leq \bullet}$ is the truncation.

(3) For an integer $p \geq 0$, the map $\bar{\theta} \colon R^p\psi\Lambda \to i^*R^{p+1}j_*\Lambda(1)$ induces an isomorphism

$$\mathrm{Gr}^{F'}_p R\psi\Lambda = R^p\psi\Lambda[-p] \xrightarrow{\bar{\theta}} [i^*R^{p+1}j_*\Lambda(1) \xrightarrow{\theta\cup} \cdots \xrightarrow{\theta\cup} i^*R^{n+1}j_*\Lambda(n-p+1)]$$

where $i^*R^{n+1}j_*\Lambda(n-p+1)$ is put on degree n. The truncation

$$[i^*R^{p+q+1}j_*\Lambda(q+1) \xrightarrow{\theta\cup} \cdots \xrightarrow{\theta\cup} i^*R^{n+1}j_*\Lambda(n-p+1)]$$

defines a filtration $G'^q\,\mathrm{Gr}^{F'}_p R\psi\Lambda$ of $\mathrm{Gr}^{F'}_p R\psi\Lambda$ by sub $(-n)$-shifted perverse sheaves.

(4) Let T be an element of the inertia group I_K such that $t_l(T)$ is a generator of $\mathbb{Z}_l(1)$. For $p \geq 0$, the map $N = T - 1$ sends $F'_{p+1}R\psi\Lambda$ to $F'_pR\psi\Lambda$. The induced map

$$\bar{N} \colon \mathrm{Gr}^{F'}_{p+1} R\psi\Lambda = R^{p+1}\psi\Lambda[-(p+1)] \to \mathrm{Gr}^{F'}_p R\psi\Lambda = R^p\psi\Lambda[-p]$$

and the isomorphism in (3) make a commutative diagram

$$\begin{array}{ccc} R^{p+1}\psi\Lambda[-(p+1)] & \xrightarrow{\bar\theta} & [0 \to i^*R^{p+2}j_*\Lambda(1) \overset{\theta\cup}{\to} \cdots \overset{\theta\cup}{\to} i^*R^{n+1}j_*\Lambda(n-p)] \\ \downarrow{\scriptstyle \bar N} & & \downarrow{\scriptstyle \otimes t_l(T)} \\ R^p\psi\Lambda[-p] & \xrightarrow{\bar\theta} & [i^*R^{p+1}j_*\Lambda(1) \to i^*R^{p+2}j_*\Lambda(2) \overset{\theta\cup}{\to} \cdots \overset{\theta\cup}{\to} i^*R^{n+1}j_*\Lambda(n-p+1)] \end{array}$$

where, in the right column, the rightmost sheaves are put on degree n.

Proof. (1)–(3) We note that a smooth sheaf on a pure d-dimensional non-singular algebraic space over a field is a d-shifted perverse sheaf (as in the scheme case). Hence $a_{p*}\Lambda[-p]$ are $(-n)$-shifted perverse sheaves. The rest of the proof (using Lemma 2.4) is identical to the scheme case.

(4) It suffices to show the commutativity after taking an étale covering. Hence we can reduce to the scheme case. □

Lemma 2.6 (cf. [SaiT03, Corollary 2.6 (1),(2),(3),(5)]). *Let T and N be as above.*

(1) We have $N^{n+1} = 0$. The kernel filtration F_p defined by $F_pR\psi\Lambda = \mathrm{Ker}(N^{p+1}\colon R\psi\Lambda \to R\psi\Lambda)$ is equal to F'_p.

(2) The image filtration G^q defined by $G^q \mathrm{Gr}^F_p R\psi\Lambda = \mathrm{Im}(\bar N^q\colon \mathrm{Gr}^F_{p+q} R\psi\Lambda \to \mathrm{Gr}^F_p R\psi\Lambda)$ is equal to G'^q. The filtration $G^q \mathrm{Gr}^F_p R\psi\Lambda$ is induced by the image filtration $G^qR\psi\Lambda = \mathrm{Im}(N^q\colon R\psi\Lambda \to R\psi\Lambda)$.

(3) The isomorphism $\theta'\colon a_{(p+q)*}\Lambda \to i^*R^{p+q+1}j_*\Lambda(p+q+1)$ in Lemma 2.4 (1) induces an isomorphism

$$a_{(p+q)*}\Lambda(-p)[-(p+q)] \to \mathrm{Gr}^q_{G'} \mathrm{Gr}^{F'}_p R\psi\Lambda$$

of $(-n)$-shifted perverse sheaves.

(4) Let $p, q \geq 0$. The diagram

$$\begin{array}{ccc} a_{(p+q)*}\Lambda(-p)[-(p+q)] & \longrightarrow & \mathrm{Gr}^q_{G'} \mathrm{Gr}^{F'}_p R\psi\Lambda \\ \downarrow{\scriptstyle 1\otimes t_l(T)} & & \downarrow{\scriptstyle \bar N} \\ a_{(p+q)*}\Lambda(-(p-1))[-(p+q)] & \longrightarrow & \mathrm{Gr}^{q+1}_{G'} \mathrm{Gr}^{F'}_{p-1} R\psi\Lambda \end{array}$$

is commutative, where the horizontal maps are the isomorphisms in (3).

Proof. (1), (2) Same to the scheme case: by using Lemmas 2.4 and 2.5, we check the conditions which characterize the kernel and image filtrations.

(3) Same to the scheme case.

(4) Clear from Lemma 2.5 (4). □

Lemma 2.7 (cf. [SaiT03, Proposition 2.7]). *Let X be a strictly semistable algebraic space purely of relative dimension n over $\mathrm{Spec}\,\mathcal{O}_K$. Let T and N be as above. Let $M_\bullet$ be the monodromy filtration on $R\psi\Lambda$ defined by the (nilpotent) operator N. Then the isomorphism in Lemma 2.6 (3) induces an isomorphism*

$$\bigoplus_{p-q=r} a_{(p+q)*}\Lambda(-p)[-(p+q)] \to \mathrm{Gr}^M_r R\psi\Lambda$$

compatible with the action of G_k. *The filtration* $M_\bullet$ *and the canonical isomorphism are independent of the choice of* T.

Here the increasing filtration $M_\bullet$ *is defined by* $M_r A = \sum_{p-q=r} F_p A \cap G^q A$.

Proof. We combine the canonical isomorphism

$$\bigoplus_{p-q=r} \operatorname{Gr}_G^q \operatorname{Gr}_p^F R\psi\Lambda \to \operatorname{Gr}_r^M R\psi\Lambda$$

([SaiT03, Corollary 2.4]) with the isomorphisms in Lemma 2.6 (3). □

Proof of Proposition 2.3. (1) The filtration $M_\bullet$ induces (M. Saito [SaiM88, Lemme 5.2.18]) a spectral sequence

$$E_1^{p,q} = H^{p+q}(X_{\overline{k}}, \operatorname{Gr}_{-p}^M R\psi\Lambda) \Rightarrow H^{p+q}(X_{\overline{k}}, R\psi\Lambda).$$

The canonical isomorphism of Lemma 2.7 shows that the left-hand side is isomorphic to the left-hand side of the sequence in Proposition 2.3. Therefore it remains to show $H^{p+q}(X_{\overline{k}}, R\psi\Lambda) \cong H^{p+q}(X_{\overline{K}}, \Lambda)$, which is immediate if we can apply the proper base change theorem. That theorem for algebraic spaces (more generally for stacks) is proved by Liu–Zheng [LZ12, Theorem 0.1.1].

(Alternatively, if $\operatorname{char} K = 0$, we can use Artin's proof [Art73, Chapitre VII] of the theorem for algebraic spaces: although his proof is stated only for those of finite type over a base which is an algebraically closed field, it is valid when the base is an excellent Dedekind scheme (e.g. discrete valuation rings of characteristic zero).)

The compatibility with N follows from Lemma 2.6 (4).

(2) First we review the following proof of Nakayama in the case X is a scheme. Equip $\operatorname{Spec} O_K$ with a log structure by the homomorphism $\mathbb{N} \to \mathcal{O}_K : 1 \mapsto \pi$, (we fix a chart $\operatorname{Spec} O_K \to \operatorname{Spec} \mathbb{Z}[\mathbb{N}]$), and $\operatorname{Spec} k$ with the restriction. The special fiber X_k is naturally a semistable log scheme over $\operatorname{Spec} k$. Let $(X_k)_{1/l^n}$ be the log scheme $X_k \times_{\operatorname{Spec} \mathbb{Z}[\mathbb{N}]} \operatorname{Spec} \mathbb{Z}[l^{-n}\mathbb{N}]$. Let $(X_k)_{\mathrm{tl}}$ be the 2-limit $\varprojlim_n ((X_k)_{1/l^n})^{\log}_{\text{ét}}$ of log étale topoi and $\pi : (X_k)_{\mathrm{tl}} \to (X_k)^{\log}_{\text{ét}}$ be the projection. Denote by $\varepsilon : (X_k)^{\log}_{\text{ét}} \to (X_k)_{\text{ét}}$ the forgetting log morphism. Let $(X_k)^L$, L a finite extension of K, be the log scheme obtained from X_k by the base change $\operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathcal{O}_K$. Let $(X_k)^{\mathrm{tame}}$ be the 2-limit $\varprojlim_L ((X_k)^L)^{\log}_{\text{ét}}$, where L runs over the set of tame extensions L of K, and $\pi^{\mathrm{tame}} : (X_k)^{\mathrm{tame}} \to (X_k)^{\log}_{\text{ét}}$ be the projection. We have a natural morphism $(X_k)^{\mathrm{tame}} \to (X_k)_{\mathrm{tl}}$. Then we have isomorphisms

$$(*) \qquad R\varepsilon_*\pi_*\Lambda \xrightarrow{\sim} R\varepsilon_*\pi^{\mathrm{tame}}_*\Lambda = R\varepsilon'_*\Lambda \xrightarrow{\sim} R\varepsilon'_* R\Psi\Lambda = R\psi R\varepsilon_*\Lambda = R\psi\Lambda.$$

Here the first morphism is induced from the adjoint property and is proved to be isomorphism in [Naka00, proof of Proposition 1.9], the second isomorphism follows from the equality $\varepsilon' = \varepsilon \circ \pi^{\mathrm{tame}}$ of functors (note the identification $X_{\overline{k}(\log)} \xrightarrow{\sim} X_k^{\mathrm{tame}}$ in [Naka98, Proposition 3.1.3]), the third and the fourth isomorphism are respectively Theorem 3.2 and Section 3.1.6 of [Naka98], and the fifth isomorphism (where ε is the forgetting log morphism on the generic fiber) is trivial since the generic fiber has trivial log structure. The isomorphisms are known to be compatible with the actions of I_K. Hence we have an isomorphism between the corresponding spectral sequences. The rightmost side gives our spectral sequence and the log spectral sequence obtained from the leftmost side degenerates at E_2 ([Naka00, Theorem 2.1]).

The E_2-degeneration of the spectral sequence associated with the complex of log nearby cycles on the special fiber is true in the algebraic space case (since we assumed that X_k is a scheme). Therefore it suffices to show the isomorphism $(*)$ when X is an algebraic space. Take an étale covering $Y \to X$ by a scheme. We have canonical isomorphisms $(*)$ on Y_k, on $Y_k \times_{X_k} Y_k$, and on $Y_k \times_{X_k} Y_k \times_{X_k} Y_k$, which are compatible with pullbacks. Then the isomorphism on Y_k descends to an isomorphism of perverse sheaves on X_k, since giving a perverse sheaf on X_k is equivalent to giving a perverse sheaf on Y_k equipped with an isomorphism between its pullbacks to $Y_k \times_{X_k} Y_k$ satisfying a certain cocycle condition.

(3) The proof of [RZ82, Satz 2.13] applies once we have the same description of the morphism N (given in (1)) and the boundary maps in our case. So it suffices to show the following lemma.

(If we restrict to the case when X_k is liftable to a semistable scheme over a discrete valuation ring, which is enough for our application on K3 surfaces, then we can reduce directly to the scheme version ([SaiT03, Proposition 2.10]) of the next lemma since (from the above isomorphism) the spectral sequence depends only on the log special fiber .) □

Lemma 2.8 (cf. [SaiT03, Proposition 2.10]). *The boundary map*

$$d_1^{p,q} \colon E_1^{p,q} = \bigoplus_{i \geq \max\{0,-p\}} H^{q-2i}(X_{\overline{k}}^{(p+2i)}, \Lambda(-i)) \to E_1^{p+1,q}$$

of the weight spectral sequence is given by $\sum_{i \geq \max\{0,-p\}} \delta_{(p+2i)*} + \delta^*_{p+2i}$.

Here $\delta_{\bullet,*}$ and $\delta^*_\bullet$ are defined as follows. Recall that Z_I is the subscheme $Z_{i_0} \cap \cdots \cap Z_{i_p}$ for each subset $I = \{i_0, \ldots, i_p\}$ of the index set $\{1, \ldots, m\}$. For each pair of subsets $J \subset I$ with $|J| = |I| - 1$, let $i_{JI} \colon Z_I \to Z_J$ be the closed immersion and, writing $J = \{\ldots, \hat{i}_j, \ldots\} \subset I = \{i_0, \ldots, i_p\}$ with $i_0 < \cdots < i_p$, let $\varepsilon_{JI} = (-1)^j$. We define

$$\delta^*_p = \sum_{I,J} \varepsilon_{IJ} i^*_{IJ} \colon H^q(X_{\overline{s}}^{(p)}, \Lambda) \to H^q(X_{\overline{s}}^{(p+1)}, \Lambda)$$

where the sum runs pairs $I \subset J$ with $|I| = p$ and $|J| = p + 1$, and

$$\delta_{p*} = \sum_{I,J} \varepsilon_{JI} i_{JI,*} \colon H^q(X_{\overline{s}}^{(p)}, \Lambda) \to H^{q+2}(X_{\overline{s}}^{(p-1)}, \Lambda)$$

where the sum runs pairs $J \subset I$ with $|I| = p$ and $|J| = p - 1$.

Proof of Lemma 2.8. By the definition of the spectral sequence, the map $d_1^{p,q} \colon E_1^{p,q} \to E_1^{p+1,q}$ is the boundary map

$$H^{p+q}(X_{\overline{s}}, \operatorname{Gr}^M_{-p} R\psi\Lambda) \to H^{p+q+1}(X_{\overline{s}}, \operatorname{Gr}^M_{-p-1} R\psi\Lambda)$$

of the short exact sequence

$$0 \to \operatorname{Gr}^M_{-p-1} R\psi\Lambda \to M_{[-p-1,-p]} R\psi\Lambda \to \operatorname{Gr}^M_{-p} R\psi\Lambda \to 0$$

of $(-n)$-shifted perverse sheaves, where $M_{[a,b]}$ is the subquotient M_b/M_{a-1} (of which $\mathrm{Gr}^M_a = M_{[a,a]}$ is a special case). For an integer q, let (K^j_i, d'^j_i, d''^j_i), where $K^j_i = H^{q-i+j}(X_{\bar{s}}, \mathrm{Gr}^F_i \mathrm{Gr}^j_G R\psi\Lambda)$, be the double complex where the (anti-commuting) differentials d'^j_i and d''^j_i are respectively the boundary maps of the short exact sequences

$$0 \to \mathrm{Gr}^F_{i-1} \mathrm{Gr}^j_G R\psi\Lambda \to F_{[i-1,i]} \mathrm{Gr}^j_G R\psi\Lambda \to \mathrm{Gr}^F_i \mathrm{Gr}^j_G R\psi\Lambda \to 0$$

and

$$0 \to \mathrm{Gr}^{j+1}_G \mathrm{Gr}^F_i R\psi\Lambda \to G^{[j,j+1]} \mathrm{Gr}^F_i R\psi\Lambda \to \mathrm{Gr}^j_G \mathrm{Gr}^F_i R\psi\Lambda \to 0.$$

Then the complex $E_1^{\bullet,q}$ is the simple complex associated to the double complex (K^j_i, d'^j_i, d''^j_i).

Hence it suffices to show that the diagrams

(1)
$$\begin{array}{ccc} H^{q-2i}(X^{(i+j)}_{\bar{s}}, \Lambda(-i)) & \xrightarrow{\delta_{(i+j)*}} & H^{q-2i+2}(X^{(i+j-1)}_{\bar{s}}, \Lambda(-i+1)) \\ \downarrow & & \downarrow \\ H^{q-i+j}(X_{\bar{s}}, \mathrm{Gr}^F_i \mathrm{Gr}^j_G R\psi\Lambda) & \xrightarrow{d'} & H^{q-i+j+1}(X_{\bar{s}}, \mathrm{Gr}^F_{i-1} \mathrm{Gr}^j_G R\psi\Lambda) \end{array}$$

and

(2)
$$\begin{array}{ccc} H^{q-2i}(X^{(i+j)}_{\bar{s}}, \Lambda(-i)) & \xrightarrow{\delta^*_{i+j}} & H^{q-2i}(X^{(i+j+1)}_{\bar{s}}, \Lambda(-i)) \\ \downarrow & & \downarrow \\ H^{q-i+j}(X_{\bar{s}}, \mathrm{Gr}^j_G \mathrm{Gr}^F_i R\psi\Lambda) & \xrightarrow{d''} & H^{q-i+j+1}(X_{\bar{s}}, \mathrm{Gr}^{j+1}_G \mathrm{Gr}^F_i R\psi\Lambda) \end{array}$$

commute, where the vertical morphisms are induced from the isomorphisms in Lemma 2.6 (3).

First we consider (1). Since the commutativity for (i,j) is equivalent to that of $(i-1, j+1)$ by Lemma 2.6 (4), we can reduce to the case $j = 0$. In that case, it suffices to show the commutativity of the diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ a_{(i-1)*} R i^{(i-1)!} \Lambda(1)[i+1] & \longrightarrow & \mathrm{Gr}^F_{i-1} \mathrm{Gr}^0_G R\psi\Lambda \\ \downarrow & & \downarrow \\ R\mathcal{H}om([\Lambda_{X^{(i-1)}_{\bar{s}}} \xrightarrow{\delta\wedge} \Lambda_{X^{(i)}_{\bar{s}}}], \Lambda)(1)[2] & \longrightarrow & F_{[i-1,i]} \mathrm{Gr}^0_G R\psi\Lambda \\ \downarrow & & \downarrow \\ a_{i*} R i^{(i)!} \Lambda(1)[i+2] & \longrightarrow & \mathrm{Gr}^F_i \mathrm{Gr}^0_G R\psi\Lambda \\ \downarrow & & \downarrow \\ 0 & & 0. \end{array}$$

We easily reduce to the scheme case, which is shown in [SaiT03, Proposition 2.10]. (It would be also possible to extend his proof directly to the algebraic space case.)

The commutativity of (2) follows from the commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
a_{(i+j+1)*}\Lambda(-i)[-(i+j+1)] & \longrightarrow & \mathrm{Gr}_G^{j+1}\,\mathrm{Gr}_i^F\,R\psi\Lambda \\
\downarrow & & \downarrow \\
[a_{(i+j)*}\Lambda(-i) \xrightarrow{\delta\wedge} a_{(i+j+1)*}\Lambda(-i)] & \longrightarrow & G^{[j,j+1]}\,\mathrm{Gr}_i^F\,R\psi\Lambda \\
\downarrow & & \downarrow \\
a_{(i+j)*}\Lambda(-i)[-(i+j)] & \longrightarrow & \mathrm{Gr}_G^{j}\,\mathrm{Gr}_i^F\,R\psi\Lambda \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

which follows from Lemmas 2.5 (3) and 2.4 (3). □

3. Proof of the main theorem

In this section we use the following notation: we denote objects (schemes, line bundles on schemes, ...) over $\mathcal{O}_K$ by calligraphic letters like $\mathcal{X}$, and objects over fields by normal letters like X. For example, the generic (resp. special) fiber of an object $\mathcal{Y}$ over $\mathcal{O}_K$ is denoted by Y_η or Y_K (resp. Y_s or Y_k).

We follow a method of Maulik [Mau12, Section 4] of studying reduction of K3 surfaces. Let X and L be as in the statement of the theorem (we do not assume at this moment that H^2 is unramified). Put $L^2 = 2d$ (this value is always even).

First we construct (after field extension) a projective strictly semistable scheme $\mathcal{X}'$ whose generic fiber is birational to X. The assumption on L is used only in this step. (Hence, as we mentioned in the introduction, this assumption can be dropped if we admit the semistable reduction conjecture.)

Lemma 3.1. *Let X be a K3 surface of characteristic $\neq 2$ and L an ample line bundle on X. Then one of the following hold: (a) L is very ample, (b) $L = \mathcal{O}_X(kB)$ with $k = 1$ or 2 and B is an irreducible curve of arithmetic genus 2, or (c) X admits an irreducible curve E of arithmetic genus 1 and an irreducible curve Γ with $E \cdot \Gamma = r$ with $r = 1$ or 2. (The "curves" may be singular.)*

Proof. This follows from results of Saint-Donat [SD74] as follows. Throughout this proof all references refer that paper. By Theorem 8.1, either X admits an irreducible curve E with $p_a(E) = 1$ and a rational irreducible curve Γ with $E\cdot\Gamma = 1$, or $L = \mathcal{O}_X(C)$ with C an irreducible curve with $p_a(C) > 1$. If the former holds then we have (c). Assume that the latter holds. Note that since X is a K3 surface we have $C^2 = C \cdot (C + K) = 2p_a(C) - 2$. By Theorem 3.1, L is base-point free and hence we have a morphism $\phi_L \colon X \to \mathbb{P}^N$. Since $L^2 > 0$ the image of ϕ_L is

2-dimensional. By Section 4.1, the degree of ϕ_L is either 1 or 2, and if it is of degree 2 then (b) or (c) holds by Section 5.1 (if $C^2 = 2$) and Theorem 5.2 (if $C^2 > 2$). If ϕ_L is of degree 1 then it is an embedding since it contracts no curve (since L is ample), and this means that L is very ample. □

We first consider case (a). We embed X into a projective space $\mathbb{P}^N$ by $|L|$, and then take the composite with a projection $\mathbb{P}^N \to \mathbb{P}^1$ (which is a rational map) in a general position. We can resolve the points of indeterminacy of the rational map $X \to \mathbb{P}^1$ and obtain a morphism $X' \to \mathbb{P}^1$. By [Mau12, Lemma 4.2], all the fibers of this morphism are nodal, and general fibers are irreducible of genus $g = d+1$ (here we need $p > d+4$). By [Mau12, Remark 4.3] we can assume that general fibers are smooth. Then by [SaiT04, Corollary 1.9], after replacing K by a finite extension, we obtain a projective strictly semistable model $\mathcal{X}'$ of X' over $\mathcal{O}_K$. (Here we need $p > 2g + 2 = 2d + 4$.)

In case (b), the construction is similar[6] to the very ample case, in which we use the morphism $|B|\colon X \to \mathbb{P}^2$ in place of $X \hookrightarrow \mathbb{P}^N$. (We need the inequality $p > 2 + 4$, which is satisfied since $p > 2d + 4 \geq 2 + 4$.)

In case (c), it follows [PSS71, Theorem 1 of Section 3] (this requires $p \geq 5$) that the linear system $|E|$ induces an elliptic fibration $X \to \mathbb{P}^1$ with smooth general fibers. Then $X \setminus \Gamma \to \mathbb{P}^1$ is a hyperbolic fibration (whose general fibers are r-punctured elliptic curves). Hence we can apply [SaiT04, Corollary 1.9] similarly. (We need the inequality $p > 2g + 2 = 4$ and $p > r$, which is satisfied since $p > 2d + 4 \geq 6$.)

Applying the minimal model program ([Kaw94]; this requires $p \geq 5$) to this semistable scheme $\mathcal{X}'$, we obtain a "minimal model", a projective flat scheme $\mathcal{X}''$ over $\mathcal{O}_K$ satisfying the following properties: the generic fiber is smooth and birational to X' (hence to X), the irreducible components of the special fiber are geometrically normal, the relative canonical divisor $K_{\mathcal{X}''/\mathcal{O}_K}$ is nef and $\mathbb{Q}$-Cartier, and $(\mathcal{X}'', X''_s)$ has (at worst) log terminal singularities. (Here $K_{\mathcal{X}''/\mathcal{O}_K}$ is by definition the Weil divisor, defined up to linear equivalence, which agrees with $\bigwedge^2 \Omega^1_{(\mathcal{X}'')^{\mathrm{sm}}/\mathcal{O}_K}$ on the smooth part $(\mathcal{X}'')^{\mathrm{sm}}$. This is well-defined since $\mathcal{X}'' \setminus (\mathcal{X}'')^{\mathrm{sm}}$ is of codimension at least 2.)

Then it follows (since X is K3, see [Mau12, Section 4.3]) that X''_η is isomorphic to X and $K_{\mathcal{X}''/\mathcal{O}_K} = 0$.

We apply Kawamata's classification [Kaw94, Theorem 4.4] of log terminal singularities (of index 1): every non-smooth point of $\mathcal{X}''$ is one of the following types. (1) Semistable singularity (i.e., étale locally of the form $\mathcal{O}_K[x, y, z]/(xy - \pi)$ or $\mathcal{O}_K[x, y, z]/(xyz - \pi)$). (2) An isolated non-smooth point which is a rational double point in the special fiber X''_s.

[6]In detail: Let $C = (f = 0) \subset \mathbb{P}^2$ be the ramification divisor of $X \to \mathbb{P}^2$. Since $p > B^2 + 4 = 6 = \deg f$ (since B is the pullback of $\mathcal{O}(1)$) and C is smooth, C has only finitely many inflection points. Take a point on $\mathbb{P}^2$ which is not on the union of C and the tangent lines at the inflection points, and take the projection from that point. Then all the fibers of the resulting fibration $X' \to \mathbb{P}^1$ are nodal and general fibers are smooth irreducible. Now use Saito's result similarly.

Moreover, the irreducible components of the special fiber are normal (by the construction of the minimal model program) and hence regular in codimension one. Hence it follows that $\mathcal{X}''$ is *strictly* semistable away from points of type (2).

We note that, if $\mathcal{X}''$ is such a model over $\mathcal{O}_K$, then for any extension K' of K we can construct a model over $\mathcal{O}_{K'}$ satisfying the same properties. This follows from the result of Saito [SaiT04, Theorem 2.9.2] that there exists a log blow-up $\mathcal{Y} \to \mathcal{X}'' \otimes^{\log}_{\mathcal{O}_K} \mathcal{O}_{K'}$ such that $\mathcal{Y}$ is strictly semistable over $\mathcal{O}_{K'}$ away from points of type (2). Then since both log base change and log blow-up preserve the sheaf $\bigwedge^2 \Omega^1(\log)$ of top log differentials, and since the canonical divisor on $\mathcal{X}''$ and $\mathcal{Y}$ corresponds respectively to the line bundles $\bigwedge^2 \Omega^1_{(\mathcal{X}'')^{\mathrm{sm}}/O_K}(\log)$ and $\bigwedge^2 \Omega^1_{\mathcal{Y}^{\mathrm{sm}}/O_{K'}}(\log)$, it follows that $K_{\mathcal{Y}/\mathcal{O}_{K'}} = 0$. The other properties are immediate.

By [Art74, Theorem 2], singularities of type (2) can be resolved potentially *in the category of algebraic spaces*. That is, after we replace K by a finite extension (and replace $\mathcal{X}''$ as above), there exists an algebraic space $\mathcal{X}'''$ and a morphism $\phi\colon \mathcal{X}''' \to \mathcal{X}''$ satisfying the following conditions:

- ϕ is an isomorphism outside the singularities of type (2), and
- For each irreducible component Z of X''_s (note that Z is smooth outside points of type (2)), $\phi|_{\phi^{-1}(Z)}\colon \phi^{-1}(Z) \to Z$ is the minimal desingularization.

$\mathcal{X}'''$ is an algebraic space over $\mathcal{O}_K$ having (at worst) strictly semistable singularity. Then the special fiber X'''_s is a scheme, since it is covered by two open subschemes: $(X'''_s)^{\mathrm{sm}}$ (which is a scheme since smooth 2-dimensional) and the complement of rational double points in X''_s. Therefore X'''_s is an SNC surface. It is projective since it is a blow-up of the projective scheme X''_s. We want to show that this is an SNC log K3 surface (in the sense of Nakkajima [Nakk00]), that is, $\bigwedge^2 \Omega^1_{X'''_s/s}(\log)$ is trivial and $H^1(X'''_s, \mathcal{O}_{X'''_s}) = 0$.

Since $\phi_s\colon X'''_s \to X''_s$ is the blow-up of the rational double points, we have $\phi_s^*\colon H^1(X'''_s, \mathcal{O}_{X'''_s}) \cong H^1(X''_s, \mathcal{O}_{X''_s})$. Since X''_s is Gorenstein, the dualizing complex is represented by an invertible sheaf L. Let $\mathcal{U}$ be the complement of the points of type (2) in $\mathcal{X}''$. Then, since $\mathcal{U}$ is log smooth over $\mathcal{O}_K$, we have $\bigwedge^2 \Omega^1_{U_s/s}(\log) = \bigwedge^2 \Omega^1_{\mathcal{U}/\mathcal{O}_K}(\log)|_{U_s}$, which is trivial. Since $L|_{U_s}$ is isomorphic to this log canonical sheaf ([Tsu99, Theorem 2.21]), and since $X''_s \setminus U_s$ is of codimension at least 2, L itself is trivial. Then by duality we have $\dim H^2(X''_s, \mathcal{O}_{X''_s}) = \dim H^0(X''_s, \mathcal{O}_{X''_s})^\vee = 1$. Since the Euler–Poincaré characteristic of X''_s (which is equal to that of the generic fiber) is 2, we obtain $H^1(X''_s, \mathcal{O}_{X''_s}) = 0$, hence $H^1(X'''_s, \mathcal{O}_{X'''_s}) = 0$. Since $\bigwedge^2 \Omega^1_{U_s/s}(\log)$ is trivial, and since the resolution of rational double points (which are canonical singularities) does not change the canonical divisor, $\bigwedge^2 \Omega^1_{X'''_s/s}(\log)$ is also trivial. Thus X'''_s is an SNC log K3 surface.

Nakkajima [Nakk00, Proposition 3.4] gave a classification of SNC log K3 surfaces (which is parallel to Kulikov's classification in the complex case [Kul77a, Theorem II]) in arbitrary characteristic. Using that, we obtain the following list of possibilities for the shape of X'''_s:

Type I: A smooth K3 surface.

Type II: A union of surfaces $Z_1, \dots, Z_m$ with Z_1 and Z_m rational and others elliptic ruled. Double curves $Z_i \cap Z_j$ ($i \neq j$) are rulings (elliptic curves) if $|i-j| = 1$ and empty otherwise. (There are no triple points.)

Type III: A union of rational surfaces, whose dual graph of the configuration is a triangulation of S^2 (the sphere).

Now we use the unramified/crystalline hypothesis. Applying the comparison theorems (Propositions 2.2, 2.3) on $\mathcal{X}'''$, we observe that $E_2^{1,1}$ and $E_2^{2,0}$ are zero if H^2 of the generic fiber is unramified/crystalline. Therefore it suffices to show that if the special fiber is of Type II (resp. III) then $E_2^{1,1}$ (resp. $E_2^{2,0}$) is nonzero.

We can deduce this from the above description and the description of the map d_1 (given in Lemma 2.8 in the l-adic case and in [Mok93, Corollaire 4.14] in the p-adic case). We write the proof in the l-adic notation (the proof in the p-adic case is identical). Type II: observe that $E_1^{0,1} = H^1(X_s''^{(0)}, \Lambda)$ and $E_1^{1,1} = H^1(X_s''^{(1)}, \Lambda)$ are the direct sums of $\Lambda^{\oplus 2}$ respectively for each non-rational component and each double curve. Hence $E_2^{1,1} = \mathrm{Coker}(\mathrm{Res}\colon E_1^{0,1} \to E_1^{1,1}) \neq 0$. Type III: $E_1^{1,0} = H^0(X_s''^{(1)}, \Lambda)$ and $E_1^{2,0} = H^0(X_s''^{(2)}, \Lambda)$ are the direct sums of Λ respectively for each double curve and each triple point. Then $E_2^{2,0} = \mathrm{Coker}(\mathrm{Res}\colon E_1^{1,0} \to E_1^{2,0})$ is isomorphic to $H^2(S^2, \Lambda)$ (singular cohomology), which is nonzero.

Thus Theorem 1.1 is proved. For Remark 1.2 (3), $\mathcal{X}''$ is a (scheme) model of X with only rational double point singularities in the special fiber.

For the later application we need the following refinement. Recall that at each step of the minimal model program we have an opportunity of choosing which extremal ray to contract.

Proposition 3.2 ([Mau12, Theorem 4.1])**.** *Assume that L is very ample (and that we applied the construction of case (a)). By a suitable choice of the extremal ray at each step of the minimal model program, we can assume that the resulting model $\mathcal{X}'''$ admits a quasi-polarization which extends the polarization on X defined by L.*

By definition a quasi-polarization of $\mathcal{X}'''$ is an element of $\mathrm{Pic}(\mathcal{X}''')$ whose restriction to each geometric fiber is the class of a nef big line bundle in the Picard group.

Proof. It suffices to extend L to quasi-polarization on $\mathcal{X}''$. This is achieved by applying "minimal model program with scaling", as explained in [Mau12, Section 4.3]. Although his theorem is stated for the case X is supersingular, his argument can be applied to our more general case (provided X has potential good reduction). □

This proposition fails in the case L is not very ample. The problem is that, in case (c), the fibration we used (which is induced by $|E|$) is different from the one induced by $|L^{\otimes m}|$ (m large enough), and Maulik's argument applied to this fibration extends only E, not L, to a nef divisor on $\mathcal{X}''$.

4. Moduli spaces and period maps

The (potential) good reduction criterion is deeply related to the surjectivity of the period maps of K3 surfaces (I thank Tetsushi Ito and Keerthi Madapusi Pera for informing this concept to me). In this section we prove this surjectivity and, as a by-product, give a bound for the extension degree in Theorem 1.1.

First we shall introduce the moduli stacks of K3 surfaces, orthogonal Shimura varieties, period maps, and those with level structures, and their integral models. For precise definitions and proofs see [Riz06], [Mau12], [MP13b] and [MP13a][7].

Let d be a positive integer (which we fix throughout this section). A K3 surface over a scheme S is a smooth proper *algebraic space* over S whose fibers are K3 surfaces (over fields, in the usual sense). A primitive quasi-polarization (resp. a primitive polarization) of a K3 surface X over S is a section $\xi \in \Gamma(S, \underline{\mathrm{Pic}}(X/S))$ whose restriction $\xi_{\overline{s}}$ on each geometric fiber is the class of a nef big line bundle (resp. an ample line bundle) on $X_{\overline{s}}$ which is primitive (i.e., $\xi_{\overline{s}}$ is not a nontrivial multiple of an element of $\mathrm{Pic}(X_{\overline{s}})$). The degree of ξ is the self-intersection ξ^2, which is a locally constant integer on S. We denote by M_{2d} the Deligne–Mumford stack over $\mathbb{Z}[1/2]$ parametrizing K3 surfaces equipped with primitive quasi-polarizations of degree $2d$, and by M_{2d}° its substack where the quasi-polarization is a polarization. By definition, for a scheme S, $M_{2d}(S)$ (resp. $M_{2d}^{\circ}(S)$) is the groupoid[8] whose objects are the K3 surfaces over S equipped with a primitive quasi-polarization (resp. a primitive polarization) of degree $2d$. Then M_{2d} and M_{2d}° are of finite type over $\mathbb{Z}[1/2]$.

Let $U = \langle e, f\rangle$ be the quadratic lattice (= $\mathbb{Z}$-module equipped with a quadratic form) of rank 2 with $e \cdot e = f \cdot f = 0$ and $e \cdot f = 1$, and E_8 the E_8 lattice. Let $\Lambda = U^{\oplus 3} \oplus E_8^{\oplus 2}$ and $\Lambda_d = \langle e + df\rangle \oplus U^{\oplus 2} \oplus E_8^{\oplus 2} = \langle e - df\rangle^{\perp} \subset \Lambda$. Then for any complex K3 surface X, there exists a (non-canonical) isometry from $H^2(X, \mathbb{Z})$ to Λ. Here the quadratic form on $H^2(X, \mathbb{Z})$ is defined to be the canonical pairing $H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \xrightarrow{\cup} H^4(X, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$ multiplied by -1. Moreover, for any primitive quasi-polarization L of X of degree $2d$, we can choose such an isometry to take $c_1(L)$ to $e - df \in \Lambda$ (so that we obtain an isometry $PH^2((X, L), \mathbb{Z}) \xrightarrow{\sim} \Lambda_d$, where $PH^2((X, L)) = \langle c_1(L)\rangle^{\perp} \subset H^2(X)$). Let $\mathrm{Sh}(\Lambda_d)$ be the (canonical model defined over $\mathbb{Q}$ of the) orthogonal Shimura variety attached to the group $\mathrm{SO}(\Lambda_d \otimes \mathbb{Q})$, so that $\mathrm{Sh}(\Lambda_d)_{\mathbb{C}}$ parametrizes Hodge structures of a certain type on Λ_d. Then the period map[9] $\iota_{\mathbb{C}} \colon M_{2d,\mathbb{C}} \to \mathrm{Sh}(\Lambda_d)_{\mathbb{C}}$, attaching the Hodge structure $PH^2((X, L), \mathbb{Z})$ to each (X, L), descends to $\iota_{\mathbb{Q}} \colon M_{2d,\mathbb{Q}} \to \mathrm{Sh}(\Lambda_d)$.

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[7]One should be careful since the notation differs in these papers. We mainly follow that of [MP13b], but in order to avoid collision of notation we use $\mathbb{K}$ instead of his K and Λ_d instead of his L_d.

[8]A groupoid is a category such that all morphisms are isomorphisms. A set can be naturally regarded as a groupoid, but the groupoids $M_{2d}(S)$ and $M_{2d}^{\circ}(S)$ are not of that kind.

[9]To be precise, the period map is defined only on a suitable double covering $\tilde{M}_{2d}$ of M_{2d}. However, if $\mathbb{K}$ is neat, then $\tilde{M}_{2d,\mathbb{K}} \to M_{2d,\mathbb{K}}$ admits a (non-canonical) section, and the level structured period map $\iota_{\mathbb{K}}$ is indeed defined on $M_{2d,\mathbb{K}}$ via that section. Since we actually use only $\iota_{\mathbb{K}}$ for such $\mathbb{K}$'s, we omit this tilde for simplicity.

Let $\mathbb{K}_{\Lambda_d} \subset \mathrm{SO}(\Lambda_d)(\mathbb{A}_f)$ be the subgroup of the elements which preserve $\Lambda_d \otimes \hat{\mathbb{Z}}$ and act trivially on the discriminant group $\operatorname{disc}\Lambda_d = \Lambda_d^\vee/\Lambda_d$, where $\Lambda_d^\vee = \mathrm{Hom}(\Lambda_d, \mathbb{Z})$ is the dual lattice. An admissible subgroup of $\mathrm{SO}(\Lambda_d)(\mathbb{A}_f)$ is a compact open subgroup of $\mathbb{K}_{\Lambda_d}$. Let $\mathbb{K}$ be an admissible subgroup. We say that $\mathbb{K}$ is neat if, for every $g \in \mathrm{SO}(\Lambda_d)(\mathbb{A}_f)$, the discrete group $\mathrm{SO}(\Lambda_d)(\mathbb{Q}) \cap g\mathbb{K}g^{-1}$ is torsion-free. We say that $\mathbb{K}$ is prime to p if $\mathbb{K}$ is of the form $\mathbb{K}^p\mathbb{K}_p$ with $\mathbb{K}_p = \mathbb{K}_{\Lambda_d,p}$. (Here, as in the standard notation, $-_p$ and $-^p$ stands for the p-part and the prime-to-p part respectively.)

Let p be an odd prime, and $\mathbb{K}$ an admissible subgroup prime to p. For a morphism $S \to M_{2d,\mathbb{Z}_{(p)}}$ corresponding to $(f\colon X \to S, \xi)$ with S a scheme, let $I^p(S)$ be the set of isometries $\underline{\Lambda \otimes \hat{\mathbb{Z}}^p} \to R^2 f_* \hat{\mathbb{Z}}^p(1)$ taking $e - df$ to $c_1(\xi)$ (where $\hat{\mathbb{Z}}^p$ is the prime-to-p part of $\hat{\mathbb{Z}}$). This defines a sheaf I^p on $M_{2d,\mathbb{Z}_{(p)}}$ on which $\mathbb{K}^p_{\Lambda_d}$ acts naturally. We define a $\mathbb{K}^p$-level structure of a K3 surface $(f\colon X \to S, \xi)$ over $\mathbb{Z}_{(p)}$ to be a section of the sheaf $I^p/\mathbb{K}^p$ over S. Then there is a moduli stack $M_{2d,\mathbb{K},\mathbb{Z}_{(p)}}$ parametrizing objects equipped with $\mathbb{K}^p$-level structures, and there is a finite étale map $M_{2d,\mathbb{K},\mathbb{Z}_{(p)}} \to M_{2d,\mathbb{Z}_{(p)}}$ of degree $[\mathbb{K}_{\Lambda_d} : \mathbb{K}]$. We also have the Shimura variety (with level structure) $\mathrm{Sh}_{\mathbb{K}}(\Lambda_d)$ over $\mathbb{Q}$, which is a finite étale cover of $\mathrm{Sh}(\Lambda_d)$. The period map $\iota_{\mathbb{Q}}\colon M_{2d,\mathbb{Q}} \to \mathrm{Sh}(\Lambda_d)$ lifts to the period map (with level structure) $\iota_{\mathbb{K},\mathbb{Q}}\colon M_{2d,\mathbb{K},\mathbb{Q}} \to \mathrm{Sh}_{\mathbb{K}}(\Lambda_d)$. If $\mathbb{K}$ is neat, then $\mathrm{Sh}_{\mathbb{K}}(\Lambda_d)$ is a scheme (from general theory). If $\mathbb{K}$ is small enough (so that there are no nontrivial automorphisms of quasi-polarized K3 surfaces with $\mathbb{K}^p$-level structures) then $M_{2d,\mathbb{K},\mathbb{Z}_{(p)}}$ is an algebraic space.

Madapusi Pera ([MP13a]) recently constructed integral canonical models $\mathcal{S}(\Lambda_d)$ of $\mathrm{Sh}(\Lambda_d)$ over $\mathbb{Z}[1/2]$ and, for each $p > 2$ and for $\mathbb{K}$ prime to p, $\mathcal{S}_{\mathbb{K}}(\Lambda_d)_{(p)}$ of $\mathrm{Sh}_{\mathbb{K}}(\Lambda_d)$ over $\mathbb{Z}_{(p)}$. If $\mathbb{K}^p$ is small enough then $\mathcal{S}_{\mathbb{K}}(\Lambda_d)_{(p)}$ is a scheme. He extended the period maps to $\iota_{\mathbb{Z}[1/2]}\colon M_{2d,\mathbb{Z}[1/2]} \to \mathcal{S}(\Lambda_d)$ and $\iota_{\mathbb{K},\mathbb{Z}_{(p)}}\colon M_{2d,\mathbb{K},\mathbb{Z}_{(p)}} \to \mathcal{S}_{\mathbb{K}}(\Lambda_d)_{(p)}$, and showed that these maps are étale ([MP13b, Proposition 4.7 and its proof]).

It is known that $\iota_{\mathbb{C}}$ (and hence $\iota_{\mathbb{Q}}$) is surjective (Kulikov [Kul77b]: this follows from his result on degenerations from arguments similar to below). We show (under an assumption) that this is true also in characteristic p.

Theorem 4.1. *Assume $p > 18d + 4$. Take an admissible compact open prime-to-p subgroup $\mathbb{K} \subset \mathrm{SO}(\Lambda_d)(\mathbb{A}_f)$ small enough so that $M_{2d,\mathbb{K},\mathbb{Z}_{(p)}}$ is an algebraic space and $\mathcal{S}_{\mathbb{K}}(\Lambda_d)_{(p)}$ is a scheme. Then $\iota_{\mathbb{K},\mathbb{Z}_{(p)}}\colon M_{2d,\mathbb{K},\mathbb{Z}_{(p)}} \to \mathcal{S}_{\mathbb{K}}(\Lambda_d)_{(p)}$ is surjective.*

This follows from the following property of $\iota_{\mathbb{K},\mathbb{Z}_{(p)}}$.

Proposition 4.2. *Let p, d, and $\mathbb{K}$ be as in the above theorem. Then $\iota_{\mathbb{K},\mathbb{Z}_{(p)}}$ satisfies the following extension property: given any commutative diagram*

$$\begin{array}{ccccc} M^\circ_{2d,\mathbb{K},\mathbb{Z}_{(p)}} & \longrightarrow & M_{2d,\mathbb{K},\mathbb{Z}_{(p)}} & \xrightarrow{\iota_{\mathbb{K},\mathbb{Z}_{(p)}}} & \mathcal{S}_{\mathbb{K}}(\Lambda_d)_{(p)} \\ \uparrow & & & & \uparrow \\ \operatorname{Spec} K & & \longrightarrow & & \operatorname{Spec} \mathcal{O}_K \end{array}$$

with K a complete discrete valuation field with perfect residue field, there exists a finite unramified extension K' of K and a (not necessarily unique) morphism

$\operatorname{Spec} \mathcal{O}_{K'} \to M_{2d,\mathbb{K},\mathbb{Z}_{(p)}}$ *making the following diagram commutative:*

$$\begin{array}{ccccc}
M^{\circ}_{2d,\mathbb{K},\mathbb{Z}_{(p)}} & \longrightarrow & M_{2d,\mathbb{K},\mathbb{Z}_{(p)}} & \xrightarrow{\iota_{\mathbb{K},\mathbb{Z}_{(p)}}} & \mathcal{S}_{\mathbb{K}}(\Lambda_d)_{(p)} \\
\uparrow & & \nwarrow & & \uparrow \\
\operatorname{Spec} K & \longrightarrow & \longrightarrow & \longrightarrow & \operatorname{Spec} \mathcal{O}_K \\
\uparrow & & & \exists \; \nwarrow & \uparrow \\
\exists \operatorname{Spec} K' & \longrightarrow & \longrightarrow & \longrightarrow & \operatorname{Spec} \mathcal{O}_{K'}.
\end{array}$$

For simplicity, we write the maps by $M^{\circ}_{\mathbb{K}} \to M_{\mathbb{K}} \xrightarrow{\iota_{\mathbb{K}}} \mathcal{S}_{\mathbb{K}}$ or by $M^{\circ} \to M \xrightarrow{\iota} \mathcal{S}$.

Proof of Proposition 4.2. The morphism $\operatorname{Spec} K \to M$ corresponds to a primitively quasi-polarized K3 surface (X, ξ) over K with a $\mathbb{K}^p$-level structure. There is a "Kuga–Satake" abelian variety A of (X, ξ) defined over a finite extension K' of K (whose construction we recall in the next paragraph) satisfying an isomorphism

$$H^1_{\text{ét}}(A_{K'^{\text{sep}}}, \mathbb{Q}_l) \cong C(PH^2_{\text{ét}}(X_{K'^{\text{sep}}}, \mathbb{Q}_l)(1))$$

of l-adic Galois representations for each $l \neq p$ and a parallel isomorphism of p-adic representations (if $\operatorname{char} K = 0$), where C denotes the Clifford algebra of a quadratic space and $PH^2(-)(1)$ denotes the orthogonal complement of $\langle c_1(\xi) \rangle$ in $H^2(-)(1)$.

We shall recall the construction of A (for details see [MP13a, Section 3]). There are homomorphisms $\operatorname{GSpin}(\Lambda_d) \to \operatorname{SO}(\Lambda_d)$ and $\operatorname{GSpin}(\Lambda_d) \to \operatorname{GSp}(C(\Lambda_d))$, where $C(\Lambda_d)$ is equipped with a certain symplectic form. These homomorphisms induce finite morphisms between the corresponding Shimura varieties and finite morphisms between their integral models. Moreover, denoting by $\tilde{\operatorname{Sh}} = \tilde{\operatorname{Sh}}_{\tilde{\mathbb{K}}}(\Lambda_d)$ and $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}_{\tilde{\mathbb{K}}}(\Lambda_d)$ the GSpin Shimura variety and its integral model (where $\tilde{\mathbb{K}}$ is the inverse image of $\mathbb{K}$), we have finite étale morphisms $\tilde{\operatorname{Sh}}_{\tilde{\mathbb{K}}}(\Lambda_d) \to \operatorname{Sh}_{\mathbb{K}}(\Lambda_d)$ and $\tilde{\mathcal{S}}_{\tilde{\mathbb{K}}}(\Lambda_d)_{(p)} \to \mathcal{S}_{\mathbb{K}}(\Lambda_d)_{(p)}$. Replacing K by some finite extension K' such that $\operatorname{Spec} K' \to \mathcal{S}$ lifts to a morphism $\operatorname{Spec} K' \to \tilde{\mathcal{S}}$, we define the Kuga–Satake abelian variety A to be the restriction of the pullback of the universal abelian variety on the GSp Shimura variety to that K'-valued point of $\tilde{\mathcal{S}}$. This A is known to be independent up to isomorphism of the choice of the lift.

Since $\tilde{\mathcal{S}} \to \mathcal{S}$ is proper, $\operatorname{Spec} K' \to \tilde{\mathcal{S}}$ extends to a morphism $\operatorname{Spec} \mathcal{O}_{K'} \to \tilde{\mathcal{S}}$. This means that A extends to an abelian scheme over $\operatorname{Spec} \mathcal{O}_{K'}$ (again the pullback of the universal abelian variety). Hence $H^1(A)$ is unramified (as a representation of $G_{K'}$). Then $H^2(X)$ is also unramified: $PH^2(X)$ is unramified since $PH^2(X) \subset C(PH^2(X)) = H^1(A)(-1)$, and $c_1(L)$ is Galois-invariant (since L is defined over K).

We have $p > 18d+4 = (L^{\otimes 3})^2+4$, and by [SD74, Theorem 8.3] $L^{\otimes 3}$ is very ample. Therefore, applying Theorem 1.1 and Proposition 3.2 to the pair $(X, L^{\otimes 3})$, after replacing K' by a further finite extension K'' we obtain a proper smooth model $\mathcal{X}$ of X equipped with a quasi-polarization $\mathcal{L}_3$ extending $L^{\otimes 3}$. Since the closure $\mathcal{L}$ of L in $\mathcal{X}$ satisfies $\mathcal{L}^{\otimes 3} = \mathcal{L}_3$, it follows that $\mathcal{L}$ is itself a quasi-polarization and extends L. Also the level structure extends naturally, and hence we obtain a desired morphism $\operatorname{Spec} \mathcal{O}_{K''} \to M$. The commutativity follows easily.

(In this proof we used Theorem 1.1 for one $l \neq p$. If $\operatorname{char} K = 0$ we also could have used the p-adic criterion.) $\square$

Proof of Theorem 4.1. Since the image of ι is a dense open subset of $\mathcal{S}$ (since $\iota|_{M^\circ}$ is an open immersion and ι is étale), it suffices to show that for every closed point $s \in \mathcal{S}$ there exists a morphism $\operatorname{Spec} \mathcal{O}_K \to \mathcal{S}$ from a discrete valuation ring taking the closed point to s and the generic point into $\operatorname{Im} \iota|_{M^\circ}$. Considering (Zariski-)locally, we may assume $\mathcal{S} = \operatorname{Spec} A$, A a Noetherian integral domain, $\operatorname{Im} \iota|_{M^\circ} \supset \operatorname{Spec} A[1/f]$, $f \neq 0$, and s corresponds to a prime $\mathfrak{p} \subset A$, and what we want to show is that A admits a prime $\mathfrak{q} \subset \mathfrak{p}$ with $f \notin \mathfrak{q}$ and $\operatorname{ht} \mathfrak{q} = \operatorname{ht} \mathfrak{p} - 1$. If $\operatorname{ht} \mathfrak{p} = 1$ then we can take $\mathfrak{q} = (0)$. If $\operatorname{ht} \mathfrak{p} \geq 2$, then there exists infinitely many prime ideals of height 1 and only finitely many can contain f, hence we can take $\mathfrak{q} \subset \mathfrak{p}$ with $f \notin \mathfrak{q}$ and $\operatorname{ht} \mathfrak{q} = 1$. Now consider $A/\mathfrak{q}$ and use induction on the dimension. $\square$

Remark 4.3. We have just seen that Theorem 1.1 for single prime l implies Proposition 4.2. However, in turn, the surjectivity as in Proposition 4.2 implies Theorem 1.1 (under $p > 9L^2 + 4$) for any prime l (possibly $= p$), as shown in the proof of the next corollary. Therefore, at least for the K3 surfaces satisfying $p > 9L^2 + 4$, the l-adic criteria for primes $l \neq p$ are all equivalent to each other, and if $\operatorname{char} K = 0$ also to the p-adic one.

Corollary 4.4. *For each d there exists a (non-explicit) constant C satisfying the following property: For any (X, L) as in Theorem 1.1 with $L^2 = 2d$, if an additional condition $p > 18d+4 = 9L^2+4$ is satisfied, then the extension K'/K in the theorem can be taken to be of degree dividing C.*

Proof. Fix d. Let $\mathbb{K} \subset \mathrm{SO}(\Lambda_d)(\hat{\mathbb{Z}})$ be the group

$$\mathbb{K} = \mathbb{K}(3) = \{g \in \mathrm{SO}(\Lambda_d)(\hat{\mathbb{Z}}) : g \equiv 1 \pmod 3\} \cap \mathbb{K}_{\Lambda_d}.$$

Then $\mathbb{K}$ is admissible, prime to p if $p \neq 3$, and neat (since $1 + 3M(N, \mathbb{Z}_3)$ is torsion-free), and we can show as in [Mau12, Proposition 2.8] that $M_{2d,\mathbb{K},\mathbb{Z}[1/6d]}$ is an algebraic space. We fix an étale covering $M'_{\mathbb{Z}[1/6d]} \to M_{2d,\mathbb{K},\mathbb{Z}[1/6d]}$ by a scheme. Let $n_1, \ldots, n_k$ be the degrees of the connected components of $M'_{\mathbb{Z}[1/6d]}$ over their images, and let $C_1 = \operatorname{lcm}\{1, 2, \ldots, \max\{n_1, \ldots, n_k\}\}$. Let $C_2 = |\mathrm{GL}(21, \mathbb{F}_3)|$. We prove that, given (X, L) over K as in Theorem 1.1 with $L^2 = 2d$ satisfying an additional assumption $p > 18d + 4$ (hence in particular p does not divide $6d$), X has good reduction over an extension of K of degree dividing $C = C_1 C_2$ (note that this C depends only on d).

Take an extension K_1/K such that X_{K_1} admits a $\mathbb{K}^p$-level structure (this extension can be taken to be of degree dividing $[\mathbb{K}_{\Lambda_d} : \mathbb{K}]$, hence dividing C_2), so that we obtain a morphism $x\colon \operatorname{Spec} K_1 \to M_{\mathbb{K}} = M_{2d,\mathbb{K},\mathbb{Z}_{(p)}}$.

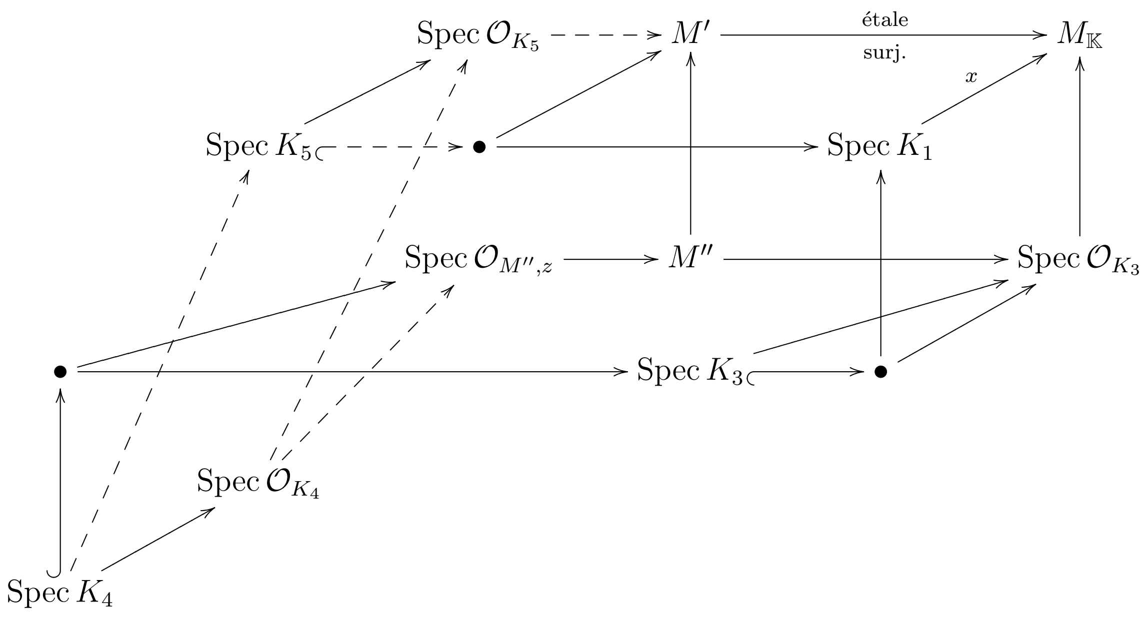
Lifting $\iota(x)\colon \operatorname{Spec} K_1 \to \mathcal{S}_{\mathbb{K}}(\Lambda_d)_{(p)}$ to a point of $\tilde{\mathcal{S}}_{\tilde{\mathbb{K}}}(\Lambda_d)_{(p)}$ (after replacing K_1 by a finite extension K_2), we obtain the Kuga–Satake abelian variety A of X. Since $H^1(A_{\overline{K}}, \mathbb{Q}_l) = C(PH^2(X_{\overline{K}}, \mathbb{Q}_l))$ is unramified/crystalline (since $H^2(X_{\overline{K}}, \mathbb{Q}_l)$

is so), A has good reduction, and the model of A over $\mathcal{O}_{K_2}$ is the pullback of the universal abelian scheme. Since $\tilde{\mathcal{S}}_{\tilde{\mathbb{K}}}(\Lambda_d)_{(p)}$ is finite over the integral model of the GSp Shimura variety, this gives a $\mathcal{O}_{K_2}$-valued point of $\tilde{\mathcal{S}}_{\tilde{\mathbb{K}}}(\Lambda_d)_{(p)}$ and hence a $\mathcal{O}_{K_2}$-valued point of $\mathcal{S}_{\mathbb{K}}(\Lambda_d)_{(p)}$. Applying Proposition 4.2, for some finite extension K_3/K_2 we obtain a $\mathcal{O}_{K_3}$-valued point of $M_{\mathbb{K}}$ compatible with the K_1-valued point x.

(This already proves that Proposition 4.2 implies Theorem 1.1 under $p > 9L^2+4$, as mentioned in the previous remark. Giving a bound requires a bit more work.)

Put $M' = M'_{\mathbb{Z}[1/6d]} \otimes \mathbb{Z}_{(p)}$. Put $M'' = M' \times_{M_{\mathbb{K}}} \operatorname{Spec} \mathcal{O}_{K_3}$ (this is a scheme). Note that $M'' \to \operatorname{Spec} \mathcal{O}_{K_3}$ is étale and surjective. Take a point $z \in M''$ mapping to the closed point of $\operatorname{Spec} \mathcal{O}_{K_3}$. Then the morphism $\operatorname{Spec} \mathcal{O}_{M'',z} \to \operatorname{Spec} \mathcal{O}_{K_3}$ is étale and surjective. The tensor product $\mathcal{O}_{M'',z} \otimes_{\mathcal{O}_{K_3}} K_3$ is a finite product of finite separable extensions of K_3. Take one of them, say K_4. Then the morphism $\operatorname{Spec} K_4 \to \operatorname{Spec} \mathcal{O}_{M'',z}$ factors through $\operatorname{Spec} \mathcal{O}_{K_4} \to \operatorname{Spec} \mathcal{O}_{M'',z}$ (by the assumption on z). Also, the morphism $\operatorname{Spec} K_4 \to M'$ factors through $M' \times_{M_{\mathbb{K}},x} \operatorname{Spec} K_1$ and hence through a component $\operatorname{Spec} K_5$ of $M' \times_{M_{\mathbb{K}},x} \operatorname{Spec} K_1$. Since the morphism $\operatorname{Spec} K_4 \to M'$ *of schemes* factors through both $\operatorname{Spec} \mathcal{O}_{K_4}$ and $\operatorname{Spec} K_5$, it follows (from an elementary set-theoretic argument on the corresponding rings) that the morphism factors through $\operatorname{Spec} \mathcal{O}_{K_5}$. This gives a $\mathcal{O}_{K_5}$-valued point of $M_{\mathbb{K}}$ compatible with x, and the degree of K_5/K_1 divides C_1.

(In the diagram below $\bullet$'s are the fibered products.)



$\square$

Remark 4.5. If $M_{\mathbb{K}} = M_{2d,\mathbb{K}(3),\mathbb{Z}_{(p)}}$ is already a scheme (not merely an algebraic space), then it follows immediately that $M_{\mathbb{K}}$ admits a $\operatorname{Spec} \mathcal{O}_{K_1}$-valued point, we obtain an explicit bound $C = C_2 = |\mathrm{GL}(21, \mathbb{F}_3)|$ independent of d, and moreover we can take the field extension to be unramified. But we do not know whether $M_{\mathbb{K}}$ is a scheme.

5. Some counterexamples

In this section we construct two explicit examples.

In Section 5.2 we construct a K3 surface which has good reduction with an algebraic space model but not with a scheme model. In Section 5.3 we construct a K3 surface which has good reduction with a scheme model only after a base field extension.

5.1. A sufficient condition for bad reduction.

We introduce a sufficient condition for a K3 surface not to have good reduction with a scheme model.

Proposition 5.1. *Let $\mathcal{X}$ be an irreducible projective scheme over $\mathcal{O}_K$. Assume that the generic fiber X_K is a K3 surface and that the special fiber has at least one rational double point and has no other singularities. If X_K has Picard number one, then X_K does not have good reduction with a scheme model. If furthermore X_K has geometric Picard number one (i.e., $X_{\overline{K}}$ has Picard number one), then X_K does not have potential good reduction with a scheme model.*

Proof. We prove the former assertion (then the latter follows immediately).

Assume to the contrary that there exists a *scheme* $\mathcal{X}'$ which is a proper smooth model of X_K (thus achieving a good reduction).

Let D be an effective divisor which generates $\mathrm{Pic}(X_K)$, $\mathcal{D}'$ its closure (as Weil divisor) in $\mathcal{X}'$, and D'_s the restriction of $\mathcal{D}'$ to X'_s. Then either D'_s is ample or not. We show that each assumption leads to a contradiction.

First we assume that D'_s is not ample. By the Nakai–Moishezon criterion, this means either $D'^2_s \leq 0$ or $D'_s \cdot C \leq 0$ for some curve $C \subset X'_s$. Since $D'^2_s = D'^2 > 0$, the former cannot occur. So take an irreducible curve $C \subset X'_s$ with $D'_s \cdot C \leq 0$. (The following argument is essentially given in [Art74, first page].) Take an affine open subset $U \subset \mathcal{X}'$ with $C \cap U \neq \emptyset$. Since $\mathcal{X}'$ is regular, the complement $\mathcal{X}' \setminus U$ is a divisor, and its components $\mathcal{Z}_i$ are all linearly equivalent to some positive multiple of $\mathcal{D}'$ (since X has Picard number one). We have $\mathcal{Z}_i \cdot C \leq 0$ by linear equivalence, and $\mathcal{Z}_i \cdot C \geq 0$ since $\mathcal{Z}_i$ does not contain C. Hence we have $(\mathcal{X}' \setminus U) \cdot C = 0$. Then the affine scheme U contains a complete curve C, which is absurd.

Now we assume that D'_s is ample. By [EGA3, Théorème 4.7.1] $\mathcal{D}'$ is a relatively ample divisor of $\mathcal{X}$ over $\mathcal{O}_K$. Then $(\mathcal{X}', \mathcal{D}')$ and $(\mathcal{X}, \mathcal{D})$ (where $\mathcal{D}$ is the closure of D in $\mathcal{X}$) have isomorphic generic fibers (as polarized varieties) but non-isomorphic special fibers (as varieties), as X'_s is smooth and X_s is singular. The following theorem shows that this is impossible. □

Theorem 5.2 (cf. Matsusaka–Mumford [MM64, Theorem 2])**.** *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be irreducible schemes proper over $\mathcal{O}_K$, $\mathcal{L}_1$ and $\mathcal{L}_2$ ample invertible sheaves on $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively, and $\mathcal{T} \subset \mathcal{X}_1 \times_{\mathcal{O}_K} \mathcal{X}_2$ an irreducible closed subscheme, flat over $\mathcal{O}_K$, such that its generic fiber T_η gives an isomorphism of polarized varieties $(X_{1\eta}, L_{1\eta})$ and $(X_{2\eta}, L_{2\eta})$. Assume $\mathcal{X}_1$ is smooth over $\mathcal{O}_K$, and $\mathcal{X}_2$ has smooth generic fiber and normal irreducible special fiber. Assume further that X_{2s} is not ruled. Then an irreducible component of T_s gives an isomorphism between X_{1s} and X_{2s}.*

If we further assume that $\mathcal{X}_2$ is smooth over $\mathcal{O}_K$, this is [MM64, Theorem 2]. However their proof works under this weaker assumption, as we check later in Section 5.4.

We also need the following proposition (this is merely a restatement of the argument on Artin's resolution in Section 3).

Proposition 5.3. *Let $\mathcal{X}$ be an irreducible proper scheme over $\mathcal{O}_K$. Assume that the generic fiber X_K is a smooth surface and that the special fiber is smooth except for (finitely many) rational double points. Then the surface X_K has potential good reduction with algebraic space model.*

5.2. **Example: good reduction only with an algebraic space model.** We construct an explicit example of a K3 surface which has potential good reduction with algebraic space models but not with scheme models. As we mentioned in the introduction, this suggests that allowing algebraic spaces is essential when considering reduction of K3 surfaces, in contrast to the case of abelian varieties.

Let $p \geq 7$ be a prime and $K = \mathbb{Q}_p$ (hence $\mathcal{O}_K = \mathbb{Z}_p$). Let $f \in \mathbb{Z}[x, y, z]$ be a homogeneous sextic polynomial satisfying the congruences

$$\begin{aligned} f &\equiv 2x^6 + x^4y^2 + 2x^3y^2z + x^2y^2z^2 + x^2yz^3 + 2x^2z^4 \\ &\qquad + xy^4z + xy^3z^2 + xy^2z^3 + 2xz^5 + 2y^6 + y^4z^2 + y^3z^3 \pmod 3, \\ f &\equiv y^6 + x^4y^2 + 3x^2y^4 + 2x^5z + 3xz^5 + z^6 \pmod 5, \text{ and} \\ f &\equiv xyz^4 + x^6 + y^6 \pmod p. \end{aligned}$$

(Such an f clearly exists from the Chinese remainder theorem.) Let $\mathcal{X}$ be the double covering of $\mathbb{P}^2_{\mathbb{Z}[1/2]}$ defined by the equation $w^2 = f(x, y, z)$ (so that it ramifies at the sextic defined by $f = 0$). As Elsenhans–Jahnel [EJ11, Example 5.1.1] showed, the congruences modulo 3 and 5 imply that $X_{\mathbb{Q}}$ (and hence X_K) are smooth K3 surfaces and have geometric Picard number one. On the other hand, from the congruence modulo p, the special fiber $X_{\mathbb{F}_p}$ of $\mathcal{X}_{\mathcal{O}_K}$ has exactly one rational double point at $x = y = 0$.

It follows from Propositions 5.1 and 5.3 that X_K is a desired example.

5.3. **Example: good reduction only after unramified field extension.** We construct a K3 surface over K which has good reduction with a scheme model over the integer ring of some unramified extension of K but not of K itself. (But there remains the possibility that it has good reduction with an algebraic space model over $\mathcal{O}_K$ without extension.)

This shows another difference between K3 surfaces and abelian varieties, as this situation does not occur in the case of abelian variety: whether an abelian variety has good reduction or not can be completely determined by the action of the inertia group, and an unramified extension does not change the inertia group.

The idea of our example is as follows: the model $\mathcal{X}$ over $\mathcal{O}_K$ has isolated singularities in the special fiber, which can be resolved by blowing up exactly one of the two curves $\mathcal{C}_+$ and $\mathcal{C}_-$, but these curves (and the corresponding resolutions and models) are defined only after base change by K'/K.

Let $\phi, \Phi \in \mathbb{Z}[x,y,z,w]$ be the (homogeneous) polynomials

$$\begin{aligned}\phi &= x^3 - x^2y - x^2z + x^2w - xy^2 - xyz + 2xyw + xz^2 + 2xzw \\ &\quad + y^3 + y^2z - y^2w + yz^2 + yzw - yw^2 + z^2w + zw^2 + 2w^3, \text{ and} \\ \Phi &= w\phi + (z^2 + xy + yz)(z^2 + xy).\end{aligned}$$

Van Luijk [vL07, proof of Theorem 3.1] proved that the K3 surface $Y_2 = (\Phi = 0) \subset \mathbb{P}^3_{\mathbb{F}_2}$ has geometric Picard number two (by computing its zeta function), and that any quartic surface defined over $\mathbb{Q}$ by a polynomial congruent to Φ modulo 2 is a smooth K3 surface and has geometric Picard number at most two (by using a reduction argument).

Now let $p \geq 5$ be a prime, a an integer with $a \not\equiv 0, \frac{27}{16} \pmod{p}$, and $b = cp^2$ where c is an integer which is a quadratic nonresidue modulo p and satisfies $c \equiv 1 \pmod{8}$. Let $f \in \mathbb{Z}[x,y,z,w]$ be a homogeneous cubic polynomial satisfying the congruences

$$\begin{aligned}f &\equiv \phi \pmod{2}, \text{ and} \\ f &\equiv x^3 + y^3 + z^3 + aw^3 \pmod{p}.\end{aligned}$$

Let $\mathcal{X} \subset \mathbb{P}^3_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[x,y,z,w]$ be the "quartic surface" defined by the equation $F = wf + g^2 - bh^2 = 0$, where

$$g = pz^2 + xy + \frac{p}{2}yz, \quad \text{and} \quad h = \frac{1}{2}yz$$

(note that $g^2 - bh^2$ has integral coefficients although g and h do not).

Let $K = \mathbb{Q}_p$. We show that X_K is a desired example.

Since $F \equiv \Phi \pmod{2}$, X_K is a smooth K3 surface and has geometric Picard number at most two from the above argument.

Over the (unramified) quadratic extension $K' = K(\sqrt{b})$ of K, we have two distinct irreducible divisors C_+ and C_-, where $C_\pm$ are respectively defined by $w = g \pm \sqrt{b}\,h = 0$. Since $C_+^2 < 0$ (since C_+ is a smooth rational curve) but $C_+ \cdot C_- \geq 0$, the classes of C_+ and C_- in the Picard group are linearly independent. Hence $X_{K'}$ has Picard number two. Since the action of $\operatorname{Gal}(K'/K)$ on $\operatorname{Pic}(X_{K'})$ is nontrivial (the nontrivial element takes C_+ to C_-), X_K has Picard number strictly less than two, hence one.

One easily checks that the singular points of $X_{\mathbb{F}_p} = (wf + g^2 = 0) \subset \mathbb{P}^3_{\mathbb{F}_p}$ are exactly the six points where $w = f = g = 0$ (recall that $f \equiv x^3 + y^3 + z^3 + aw^3$ and $g \equiv xy \pmod{p}$), and that these points are rational double points (of type A_1). We conclude by using Proposition 5.1(2) that X_K does not have good reduction.

On the other hand $X_{K'}$ has good reduction. We construct two smooth proper models $\mathcal{X}'_+$ and $\mathcal{X}'_-$. Let $\mathcal{C}_+$ and $\mathcal{C}_-$ be the subschemes of $\mathcal{X}_{\mathcal{O}_{K'}}$ defined by $w = g + \sqrt{b}\,h = 0$ and $w = g - \sqrt{b}\,h = 0$ respectively. Let $\psi_+ \colon \mathcal{X}'_+ \to \mathcal{X}_{\mathcal{O}_{K'}}$ be the blow-up at $\mathcal{C}_+ \subset \mathcal{X}_{\mathcal{O}_{K'}}$. Since the divisor $\mathcal{C}_+$ is Cartier at all the points where at least one of w, f, $g + \sqrt{b}\,h$, and $g - \sqrt{b}\,h$ is nonzero, the morphism ψ_+ is an isomorphism outside the subscheme $(w = f = g = bh = 0)$. One can easily check that this locus $(w = f = g = bh = 0)$ is the set of the six singular points of the special fiber. Some computation on local equations shows that ψ_+ is the minimal desingularization on the special fiber (and is an isomorphism on the generic fiber). Therefore $\mathcal{X}'_+$ is a smooth proper model of $X_{K'}$. By blowing up $\mathcal{C}_-$ we similarly obtain $\mathcal{X}'_-$, which is another smooth proper model.

5.4. **The Matsusaka–Mumford theorem.** We prove Theorem 5.2. This is a small refinement of the original theorem of Matsusaka–Mumford.

First we show the following theorem.

Theorem 5.4 (cf. Matsusaka–Mumford [MM64, Theorem 1])**.** *Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{T} \subset \mathcal{X} \times_{\mathcal{O}_K} \mathcal{Y}$ be irreducible schemes flat over $\mathcal{O}_K$ (of the same relative dimension n). Assume that the fibers of $\mathcal{X}$ and $\mathcal{Y}$ are integral, that $\mathcal{X}$ is proper smooth over $\mathcal{O}_K$, $\mathcal{Y}$ is normal, and that T_η gives a birational correspondence from X_η to Y_η. Assume moreover that Y_s is not ruled. Then the n-cycle T_s admits a (unique) decomposition $T_s = T'' + T^*$ to effective n-cycles, with T'' a birational correspondence from X_s to Y_s and with T^* satisfying $\mathrm{pr}_{1*}(T^*) = 0$ and $\mathrm{pr}_{2*}(T^*) = 0$ (where pr_i are the projections to X_s and Y_s).*

Proof. By the compatibility of specialization and push-forward we have $\mathrm{pr}_{2*}(T_s) = Y_s$, hence some (unique) component T'' of T_s with multiplicity one satisfies $\mathrm{pr}_2(T'') = Y_s$. We shall show $\mathrm{pr}_1(T'') = X_s$.

Let $A = \mathrm{pr}_1(T'') \subset X_s$. It follows from [Abh56, Proposition 3] that T'' is birationally equivalent to $A \times \mathbb{P}^n$ for some $n \geq 0$. (This follows from properties of the morphism $\mathcal{O}_{\mathcal{X},\alpha} \to \mathcal{O}_{\mathcal{T},\tau''}$ of local rings, where α and τ'' are respectively the generic points of A and T''. We need both local rings to be regular, hence we assumed $\mathcal{X}$ smooth and $\mathcal{Y}$ normal.) Since T'' is birationally equivalent to Y_s and Y_s is not ruled, we have $n = 0$, hence $\mathrm{pr}_1(T'') = X_s$. □

Proof of Theorem 5.2. Take the decomposition $T_s = T'' + T^*$ of Theorem 5.4. Matsusaka–Mumford shows (under assumption that $\mathcal{X}$ and $\mathcal{Y}$ are smooth) that the birational map induced by T'' is in fact an isomorphism. Kollár pointed out [Kol85, proof of Proposition 3.1.2], that their argument works for normal (not necessarily smooth) varieties. □

6. Application: potential good reduction of K3 surfaces with complex multiplications

As mentioned in [Riz05, Section 3.10 (A)], the Néron–Ogg–Shafarevich type (potential) good reduction criterion implies the potential good reduction of K3 surfaces with complex multiplications (CM).

We recall the Hodge group and the Hodge endomorphism algebra of a complex K3 surface (see [Zar83] for details). Let X be a complex K3 surface and $V = \mathrm{NS}(X)^\perp \subset H^2(X, \mathbb{Q})$ the transcendental lattice. The Hodge group Hdg of X is the minimal algebraic subgroup of $\mathrm{GL}(V)$ defined over $\mathbb{Q}$ such that $h(U^1) \subset \mathrm{Hdg}_\mathbb{R}$, where $h \colon \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to \mathrm{GL}(H^2(X, \mathbb{Z}) \otimes \mathbb{R})$ is the morphism attached to the Hodge structure and $U^1 \subset \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$ is the set of complex numbers of absolute value 1. It is known that the Hodge endomorphism algebra $E = \mathrm{End}_{\mathrm{Hdg}}(V)$ is either a totally-real field or a CM field, and that $\mathrm{Hdg} = \mathrm{SO}(V, \psi)$ or $\mathrm{Hdg} = \mathrm{U}(V, \psi)$ respectively, where $\psi : V \times V \to E$ is defined by

$$\langle ex, y \rangle = \mathrm{tr}_{E/\mathbb{Q}}(e\psi(x, y)),$$

where $\langle , \rangle$ is the pairing on the middle degree cohomology.

Definition 6.1. A complex K3 surface X is said to have complex multiplication (CM) if Hdg is commutative. Equivalently, X has complex multiplication if E is a CM field and $\mathrm{rank}_E V = 1$.

Remark 6.2. The definition of Rizov [Riz05, Definition 1.7] is that E is a CM field (without any condition on $\mathrm{rank}_E V$). He shows [Riz05, Corollary 3.19] that if X has complex multiplication (in his sense) with Hodge endomorphism algebra E then X is defined over an abelian extension of E.

Theorem 6.3. *Let X be a CM K3 surface (in the sense of Definition 6.1), which is defined over some number field, say K. Let $\mathfrak{p}$ be any prime of K such that its residue characteristic p satisfies $p > L^2 + 4$ for some ample line bundle L on X. Then X has potential good reduction at $\mathfrak{p}$ (with an algebraic space model).*

Remark 6.4. As in our main theorem, the assumption $p > L^2 + 4$ can be weakened to $p \geq 5$ if we admit the semistable reduction conjecture.

This can be viewed as an analogue of the theorem that abelian varieties with complex multiplication have potential good reduction [ST68, Theorem 6], although our result is restricted to large p.

This is also a generalization (for large p) of a previous result of ours on exceptional K3 surfaces. Exceptional K3 surfaces are the K3 surfaces in characteristic 0 of Picard number 20 (which is the maximum possible value in characteristic 0) and they are examples of CM K3 surfaces. We showed ([Mat12, Corollary 0.5]) that exceptional K3 surfaces have potential good reduction for $p \neq 2, 3$.

Proof of Theorem 6.3. We may assume that K contains E. Take a prime $l \neq p$. Since the Galois action preserves ψ, the image of G_K in $\mathrm{GL}(V \otimes \mathbb{Q}_l)$ is contained in $\mathrm{U}(V_{\mathbb{Q}_l}, \psi)(\mathbb{Q}_l)$. Since $\mathrm{rank}_E V = 1$, we have $\mathrm{U}(V_{\mathbb{Q}_l}, \psi)(\mathbb{Q}_l) \cong \mathrm{Ker}(\mathrm{Nm}\colon (E \otimes \mathbb{Q}_l)^* \to (E_0 \otimes \mathbb{Q}_l)^*) \subset (E \otimes \mathbb{Q}_l)^*$. This group is abelian and contains a pro-l group of finite index.

Since $\mathrm{Hdg}_{\mathbb{Q}_l}$ is abelian, the action of $G_{K_\mathfrak{p}}$ factors through $G_{K_\mathfrak{p}}^{\mathrm{ab}}$. Under the reciprocity map $K_\mathfrak{p}^* \to G_{K_\mathfrak{p}}^{\mathrm{ab}}$ of the local class field theory, the image of the inertia group $I_{K_\mathfrak{p}}$ in $G_{K_\mathfrak{p}}^{\mathrm{ab}}$ is isomorphic to $\mathcal{O}_{K_\mathfrak{p}}^*$, a group which contains a pro-p-group of finite index.

Since $l \neq p$, it follows that the image of $I_{K_\mathfrak{p}}$ in $\mathrm{U}(V_{\mathbb{Q}_l}, \psi)(\mathbb{Q}_l)$ is finite. Therefore, over a finite extension of K, $H^2_{\text{ét}}(X_{\overline{K_\mathfrak{p}}}, \mathbb{Q}_l)$ is unramified, and by Theorem 1.1 we conclude that X has potential good reduction at $\mathfrak{p}$. ☐

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