# A cellular approach to the Hecke–Clifford superalgebra

(セルラー代数の手法による Hecke–Clifford スーパー代数の研究)

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# Contents

# Introduction

The purpose of this paper is to classify the simple modules over the Hecke–Clifford superalgebra by use of an extended theory of cellular algebras. The original theory of cellular algebras is developed by Graham and Lehrer [**GL96**] as an axiomatization of various algebras arising as endomorphism algebra on natural representation of classical groups and quantum groups: the symmetric group algebra, the Brauer algebra, the partition algebra, the Iwahori–Hecke algebra, the Birman–Murakami–Wenzl algebra and so on so forth. First recall the notion of cellular algebra with more general one introduced by Du and Rui [**DR98**]. The definition below is based on that given by König and Xi [**KX98**], which is equivalent but slightly ring-theoretic than the original one. Let $(\Lambda, \leq)$ be a partially ordered set. In this introduction we assume that the set $\Lambda$ is finite for simplicity.

Definition 0.1. Let $A$ be an algebra over a commutative ring $\Bbbk$. $A$ is called a *standardly based algebra* on $\Lambda$ if it is equipped with a particular basis over $\Bbbk$

$$\{a_{ij}^{\lambda} \in A \mid \lambda \in \Lambda, i \in I(\lambda), j \in J(\lambda)\}$$

parametrized by families of finite sets $I(\lambda)$ and $J(\lambda)$ for each $\lambda$ which satisfies the following properties.

(1) For each $\lambda \in \Lambda$, the $\Bbbk$-submodule $A^{<\lambda} \subset A$ spanned by

$$\{a_{ij}^{\mu} \in A \mid \mu < \lambda, i \in I(\mu), j \in J(\mu)\}$$

is a 2-sided ideal of $A$.

(2) For each $\lambda \in \Lambda$, there exist a left $A$-module $M_\lambda = \Bbbk\{m_i^\lambda \mid i \in I(\lambda)\}$ and a right $A$-module $N_\lambda = \Bbbk\{n_j^\lambda \mid j \in J(\lambda)\}$, which also have parametrized bases, such that

$$M_\lambda \otimes_{\Bbbk} N_\lambda \to A/A^{<\lambda},$$
$$m_i^\lambda \otimes n_j^\lambda \mapsto a_{ij}^\lambda$$

is a homomorphism between $(A, A)$-bimodules.

$A$ is also called a *cellular algebra* if $I(\lambda) = J(\lambda)$ for all $\lambda$ and the map $a_{ij}^\lambda \mapsto a_{ji}^\lambda$ defines an anti-involution on the algebra $A$.

We here do not pay much attention to anti-involutions, so standardly based algebras are fundamental for us. Intuitively the cell $\Bbbk\{a_{ij}^\lambda \mid i \in I(\lambda), j \in J(\lambda)\}$ for each $\lambda \in \Lambda$ is made to imitate the structure of matrix algebra, so that the modules $M_\lambda$ and $N_\lambda$ respectively correspond to column and row vector spaces. As a semisimple algebra decompose into a direct sum of matrix algebras, a cellular algebra has a filtration whose successive quotients are such cells.

One of the most striking result of the theory is the classification of simple modules performed as follows. First we can show that there is a canonical $A$-bilinear form

$$(\,,\,)\colon N_\lambda \times M_\lambda \to \Bbbk$$

between $M_\lambda$ and $N_\lambda$ for each $\lambda$. Now suppose $\Bbbk$ is a field and let

$$L_\lambda \coloneqq M_\lambda \big/ \{x \in M_\lambda \mid (y, x) = 0 \text{ for all } y \in N_\lambda\}$$

for each $\lambda$. Graham and Lehrer [**GL96**, Theorem 3.4] prove that an $A$-module $L_\lambda$ is either zero or simple, and the set $\{L_\lambda \mid \lambda \in \Lambda, L_\lambda \neq 0\}$ consists of pairwise distinct all simple $A$-modules. This is an analogue of the fact that each matrix component of a semisimple algebra produces its simple module.

However, this strategy does not work well in representation theory of superalgebras; there are no known non-trivial cellular superalgebras in the original definition. This is essentially because there is another kind of simple superalgebras in addition to matrix algebras, namely matrix algebras over the Clifford superalgebra. The key idea is that we allow a generalized cellular algebra to have such a new kind of cells.

The construction above of simple modules, though the developers of the theory might have not noticed, implicitly use the notion of Morita context which connect the two algebras $A/A^{<\lambda}$ and $\Bbbk$.

Definition 0.2. A *Morita context* between the algebras $A$ and $B$ is a pair of an $(A,B)$-bimodule $M$ and a $(B,A)$-bimodule $N$ equipped with bimodule homomorphisms $\eta\colon M \otimes_B N \to A$ and $\rho\colon N \otimes_A M \to B$ which satisfy the associativity laws

$$\eta(x \otimes y) \cdot x' = x \cdot \rho(y \otimes x'), \qquad \rho(y \otimes x) \cdot y' = y \cdot \eta(x \otimes y')$$

for each $x, x' \in M$ and $y, y' \in N$.

This is a weaker version of the Morita equivalence [**Mor58**] and studied in detail by Nicholson and Watters [**NW88**]. Morita's original theorem says that if both $\eta$ and $\rho$ are surjective then we have a category equivalence between the module categories of these algebras, and every category equivalence is obtained in this form. For such data, we can prove the following statement

Theorem 0.3. *Let $I \subset A$, $J \subset B$ be the images of $\eta$ and $\rho$ respectively. Let $\operatorname{Irr}(A)$ be the set of isomorphism class of simple $A$-modules, and let $\operatorname{Irr}^I(A)$ be its subset consisting of simple modules $V$ such that $IV = V$. We similarly define $\operatorname{Irr}^J(B)$. For a $B$-module $W$, let $DW$ be the image of the $A$-homomorphism*

$$\begin{aligned} M \otimes_B W &\to \operatorname{Hom}_B(N, W) \\ m \otimes w &\mapsto (n \mapsto \rho(n \otimes m)w). \end{aligned}$$

*Then $W \mapsto DW$ induces a one-to-one correspondence $\operatorname{Irr}^I(A) \xleftrightarrow{1:1} \operatorname{Irr}^J(B)$.*

We will prove it in Theorem 3.15 in a more general setting: we also treat a Morita context between two abelian categories instead of that between two algebras $A$ and $B$, so that it is redefined as that between their module categories $A\text{-}\mathcal{M}od$ and $B\text{-}\mathcal{M}od$. We do this process for two reasons. First since our purpose is a classification of simple objects in the module category, it is more essential to deal directly with the module category $A\text{-}\mathcal{M}od$ rather than the algebra $A$ itself. Second we expect that our strategy works in more general settings outside representation theory of algebras.

Anyway, note that for a standardly based algebra, for each $\lambda \in \Lambda$ the embedding $M_\lambda \otimes N_\lambda \hookrightarrow A/A^{<\lambda}$ and the bilinear form $N_\lambda \otimes_A M_\lambda \to \Bbbk$ make pair $(M_\lambda, N_\lambda)$ into a Morita context between the algebras $A/A^{<\lambda}$ and $\Bbbk$, and $L_\lambda$ above is just $D\Bbbk$ where $\Bbbk$ is viewed as a trivial $\Bbbk$-module. The classification of simple modules of a cellular algebra is a consequence of this theorem.

For a general Morita context we do not need that one algebra is a base ring $\Bbbk$. Hence by replacing $\Bbbk$ with a more general one, such as the Clifford superalgebra, we can define generalized cellular algebras in order to obtain a similar method of classification which we can apply to more various things. In this paper we introduce the notion of *standardly filtered algebra* over a family of algebras $\{B_\lambda\}_{\lambda\in\Lambda}$; see

Definition 4.1. A standardly filtered algebra $A$ also consists of a Morita context $(M_\lambda, N_\lambda)$ for each $\lambda \in \Lambda$, between quotient algebras $A/A^{<\lambda}$ of $A$ and $B_\lambda/B'_\lambda$ of $B_\lambda$. Let $B''_\lambda/B'_\lambda \subset B_\lambda/B'_\lambda$ the image of the Morita context map $N_\lambda \otimes_A M_\lambda \to B_\lambda/B'_\lambda$, and write $\mathrm{Irr}^{B''_\lambda}_{B'_\lambda}(B_\lambda) := \mathrm{Irr}^{B''_\lambda/B'_\lambda}(B_\lambda/B'_\lambda)$. These data induce the following classification which generalizes [**GL96**, Theorem 3.4].

THEOREM 0.4. *The Morita contexts induce a one-to-one correspondence*

$$\mathrm{Irr}(A) \xleftrightarrow{1:1} \bigsqcup_{\lambda \in \Lambda} \mathrm{Irr}^{B''_\lambda}_{B'_\lambda}(B_\lambda).$$

In the classical case, each $B_\lambda$ is taken to be a base field $\Bbbk$ so that $\mathrm{Irr}^{B''_\lambda}_{B'_\lambda}(B_\lambda)$ is either $\{\Bbbk\}$ or $\varnothing$. Thus in this case $\mathrm{Irr}(A)$ is in bijection with some subset of $\Lambda$. If so, we simply say that $A$ is a standardly filtered algebra over $\Bbbk$ on the set $\Lambda$, similarly as before. In any case the classification of simple modules of $A$ can be reduced to those of $B_\lambda$'s via this correspondence. In this paper we also introduce the notion of generalized standardly based algebra and that of generalized cellular algebra over the family $\{B_\lambda\}$, not over the single base ring $\Bbbk$. It seems better to list several examples rather than to introduce its detailed definition. In many cases a standardly filtered algebra is produced from a category as follows.

LEMMA 0.5. *Let $\mathcal{A}$ be a $\Bbbk$-linear (super)category. For each $\lambda \in \Lambda$, let us take an object $X_\lambda \in \mathcal{A}$ and a subalgebra $B_\lambda \subset \mathrm{End}_{\mathcal{A}}(X_\lambda)$. Let $\mathcal{A}^\lambda \subset \mathcal{A}$ be a 2-sided ideal of $\mathcal{A}$ generated by $X_\lambda$; that is,*

$$\mathcal{A}^\lambda(X,Y) := \mathrm{Hom}_{\mathcal{A}}(X_\lambda, Y) \circ \mathrm{Hom}_{\mathcal{A}}(X, X_\lambda) \subset \mathrm{Hom}_{\mathcal{A}}(X,Y).$$

*Now suppose that*

$$\mathrm{Hom}_{\mathcal{A}}(X_\mu, X_\lambda) = \sum_{\nu \le \lambda,\mu} \mathcal{A}^\nu(X_\mu, X_\lambda)$$

*for each pair of $\lambda, \mu \in \Lambda$ and*

$$\mathrm{End}_{\mathcal{A}}(X_\lambda) = B_\lambda + \sum_{\mu < \lambda} \mathcal{A}^\mu(X_\lambda, X_\lambda)$$

*for each $\lambda \in \Lambda$. Then for every $\omega \in \Lambda$, $\mathrm{End}_{\mathcal{A}}(X_\omega)$ is a standardly filtered algebra over $\{B_\lambda\}$. Here the Morita contexts for $A$ above is given by*

$$A/A^{<\lambda} := \mathrm{End}_{\mathcal{A}/\mathcal{A}^{<\lambda}}(X_\omega), \qquad M_\lambda := \mathrm{Hom}_{\mathcal{A}/\mathcal{A}^{<\lambda}}(X_\lambda, X_\omega),$$
$$B_\lambda/B'_\lambda := \mathrm{End}_{\mathcal{A}/\mathcal{A}^{<\lambda}}(X_\lambda), \qquad N_\lambda := \mathrm{Hom}_{\mathcal{A}/\mathcal{A}^{<\lambda}}(X_\omega, X_\lambda)$$

*where $\mathcal{A}^{<\lambda} := \sum_{\mu<\lambda} \mathcal{A}^\mu$. The homomorphisms equipped on this Morita context is just the composition of morphisms.*

EXAMPLE 0.6. The Iwahori–Hecke algebra $H_n(q)$ of type $\mathsf{A}_{n-1}$ for $q \in \Bbbk$ is standardly filtered over $\Bbbk$ on the set of compositions of $n$. If moreover $q$ is an invertible element, it is also standardly based and the index set can be restricted to partitions of $n$. For the proof we take the module category of $H_n$ as $\mathcal{A}$ and for a composition $\lambda$ we pick up the corresponding parabolic module as $X_\lambda$. Then the Morita contexts above is given by the Specht modules. See Part II in this paper. In particular, for $q = 1$ the symmetric group algebra $\Bbbk\mathfrak{S}_n$ is also standardly based.

EXAMPLE 0.7. The Hecke–Clifford superalgebra $H^c_n(a;q)$, which is our main target, is also standardly filtered on the same sets over the Clifford superalgebras with different quadratic form for each composition. The key point is that there is a right action of this Clifford superalgebra on the super-analogue of the Specht module. It is also standardly based in a generalized sense if $2aq \in \Bbbk^\times$ and the $q$-characteristic of $\Bbbk$ is greater than $n/2$. The strategy of the proof is same as that for the Iwahori–Hecke algebra. See Part III. For the special case $q = 1$, we obtain that the Sergeev superalgebra $W_n(a) = C_n(a) \rtimes \mathfrak{S}_n$ is also standardly filtered over the Clifford superalgebras.

EXAMPLE 0.8. The Temperley–Lieb algebra $\mathrm{TL}_n(t)$ is standardly based over $\Bbbk$ on the set of natural numbers $\{0, 1, \ldots, n\}$. We can simply take the Temperley–Lieb category as $\mathcal{A}$, and for $k \leq n$ the "$k$-points" object as $X_k$.

EXAMPLE 0.9. The partition algebra $P_n(t)$ is standardly based over $\{\Bbbk\mathfrak{S}_k\}_{0\leq k\leq n}$, the symmetric group algebras. Note the natural inclusion $\Bbbk\mathfrak{S}_n \subset P_n(t)$. We take Deligne's "representation category $\mathcal{R}ep(\mathfrak{S}_t)$ of the symmetric group $\mathfrak{S}_t$ for $t \in \Bbbk$" (see [**Del07**]) as $\mathcal{A}$ and similarly "$k$-points" object $[1]^{\otimes k}$ as $X_k$. In addition, by using that $\Bbbk\mathfrak{S}_k$ is standardly based over $\Bbbk$ on the set of partitions of $k$, we obtain that $P_n(t)$ is also standardly based over $\Bbbk$ on the set of all partitions of $k \leq n$. It gives a simple alternative proof of [**Xi99**]. We can prove similar results for the Brauer algebra and the walled Brauer algebra using the Deligne's category for "the orthogonal group $O_t$" and "the general linear group $GL_t$" respectively.

As we listed in the examples above, the *Iwahori–Hecke algebra* $H_n = H_n(q)$ is one of the most important example of a cellular algebra whose cellular basis is given by Kazhdan and Lusztig's canonical basis [**KL79**] or Murphy's basis [**Mur92, Mur95**]. It first comes from a study of flag varieties over the finite fields, and also appears as an endomorphism algebra of a certain representation of the quantum general linear group via an analogue of the Schur–Weyl duality, then considered as a $q$-analogue of the symmetric group algebra. Now suppose $\Bbbk$ is a field and $q \in \Bbbk$ be a non-zero element. When $q$ is not a root of unity, its representations are very similar to those of the symmetric group in characteristic zero, and a concrete construction of the simple modules called Young's seminormal form is given by Hoefsmit [**Hoe74**]. For modular representations $q = \sqrt[e]{1}$, its simple modules are studied by Dipper and James [**DJ86, DJ87**] in a cellular way. In addition to cellular representation theory there is a beautiful approach on the classification of simple modules made by Lascoux, Leclerc, Thibon [**LLT96**], Ariki [**Ari96**], Grojnowski [**Gro**], Brundan [**Bru98**], Kleshchev [**Kle95**] and others, called the *categorification*. Based on their works it is proven that the union set $\bigsqcup_{n\in\mathbb{N}} \mathrm{Irr}(H_n)$ of simple modules of $H_n$ for all $n \in \mathbb{N}$ has a structure of Kashiwara crystal [**Kas02**] over the affine quantum enveloping algebra $U_v(\widehat{\mathfrak{sl}}_e)$ of type $\mathsf{A}^{(1)}_{e-1}$, and is isomorphic to the crystal basis $B(\Lambda_0)$ of the irreducible representation $V(\Lambda_0)$ whose highest weight is the fundamental weight $\Lambda_0$. One can obtain each simple module by applying Kashiwara operators $\tilde{f}_i$ on the trivial module of $H_0 = \Bbbk$. However this construction is too abstract and hard to compute in practical use. Compared with this Lie theory, the cellular theory has advantages that we can construct simple modules in a concrete way, and that we can apply it even when $\Bbbk$ is a more general commutative ring: we only requires that $q \in \Bbbk$ is invertible.

THEOREM 0.10. *Suppose $q \in \Bbbk$ is invertible. Then there is a one-to-one correspondence*

$$\mathrm{Irr}(H_n) \stackrel{1:1}{\longleftrightarrow} \bigsqcup_{\lambda\colon \text{partition}} \mathrm{Irr}^{\Bbbk f_\lambda}(\Bbbk).$$

*Here $f_\lambda := [\lambda_1 - \lambda_2]![\lambda_2 - \lambda_3]! \cdots [\lambda_r]! \in \Bbbk$ for each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ where $[k]!$ denotes the $q$-factorial.*

Our main target in this paper is the *Hecke–Clifford superalgebra* $H^c_n = H^c_n(a; q)$ for $a, q \in \Bbbk$, which is a super version of the Iwahori–Hecke algebra. It is introduced by Olshanski [**Ols92**] as a partner of the quantum Queer superalgebra via the Schur–Weyl duality and is considered as a $q$-analogue of the wreath product

$W_n = C_n \rtimes \mathfrak{S}_n$ of the Clifford superalgebra, which is called the *Sergeev superalgebra.* It is known that the spin representation theory of the symmetric group $\mathfrak{S}_n$ is controlled by $W_n$; see [**BK02**] or [**Kle05**]. Young's seminormal form of $H_n^c$ for characteristic zero case is independently founded by Hill, Kujawa and Sussan [**HKS11**] and Wan [**Wan10**], and the categorification method for $q = \sqrt[e]{1}$ is developed by Brundan and Kleshchev [**BK01**] for odd $e$ and by Tsuchioka [**Tsu10**] for even $e$ using the affine quantum enveloping algebra of type $\mathsf{A}_{e-1}^{(2)}$ and of type $\mathsf{D}_{e/2}^{(2)}$ respectively. Hence this paper fills the missing one: the cellular representation theory of $H_n^c$. In our cellular method the classification can be done in a very weak assumption same as the case of the Iwahori–Hecke algebra above. This is our main theorem.

Theorem 0.11. *Suppose $q \in \Bbbk$ is invertible. Then there is a one-to-one correspondence*

$$\operatorname{Irr}(H_n^c) \xleftrightarrow{1:1} \bigsqcup_{\lambda:\ \text{partition}} \operatorname{Irr}_{\Theta_\lambda}^{\Delta_\lambda + \Theta_\lambda}(\Gamma_\lambda).$$

*Here $\Gamma_\lambda$ is the Clifford superalgebra defined on the quadratic form with respect to the scalars $a[\![\lambda_1]\!], a[\![\lambda_2]\!], \ldots, a[\![\lambda_r]\!]$ where $[\![k]\!]$ denotes the $q^2$-integer, and $\Delta_\lambda, \Theta_\lambda \subset \Gamma_\lambda$ are its 2-sided ideals defined in Chapter 11.*

By the way, this paper also contains a quite different topic which we use as a tool to classify simple modules in a cellular method: the representation theory in non-integral rank. This field is pioneered by Deligne [**Del07**] who invented the representation category of the "symmetric group $\mathfrak{S}_t$" for $t$ which is not necessarily a natural number. This category can be considered as a realization of a stable structure of the ordinary representation category of $\mathfrak{S}_n$ for large $n$ behaves polynomially in $n$. There are several variations of this category: ones for linear algebraic groups $GL_t$, $O_t$ and $Sp_t$ we used above are also obtained by Deligne [**Del07**], and for finite general linear group $GL_t(\mathbb{F}_q)$ for the wreath product $G^t \rtimes \mathfrak{S}_t$ by Knop [**Kno06, Kno07**], and the symmetric tensor product of the category $\operatorname{Sym}^t(\mathcal{C}) = (\mathcal{C} \boxtimes \cdots \boxtimes \mathcal{C})^{\mathfrak{S}_t}$ by the author [**Mor12**] as a generalization of the module category of wreath product. The examples above are made from the structures of tensor category of ordinary representation categories. We here introduce another kind of its variations: the module category over the Iwahori–Hecke algebra $H_t$ and the Hecke–Clifford superalgebra $H_t^c$. Since its module categories are not tensor categories, we construct them by a slightly different method. The module category of $H_t^c$ for $t \notin \mathbb{N}$ is used to complete the proof of the main theorem above.

This paper consists of three parts. The purpose of Part I is to extend the theory of cellular algebras so that we can apply it to our target, the Hecke–Clifford superalgebra. We start by reviewing the enriched category theory in Chapter 1 in order to treat representation category of superalgebra. Next in Chapter 2 and 3 we define Morita contexts between two categories, and develop the theory of classification of simple objects. We then in Chapter 4 introduce the notion of standardly filtered algebra by use of Morita contexts.

Part II treats the representation theory of the Iwahori–Hecke algebra. In Chapter 5 using results in Part I we reconstruct the cellular representation theory from a generalized viewpoint to make it suitable for our purpose. The rest ones Chapter 6, 7 and 8 are prepared for introducing the representation category of the Iwahori–Hecke algebra in non-integral rank. We see that this category also has a cellular structure.

Part III deals with the representation theory of the Hecke–Clifford superalgebra, which is our main part. It starts with Chapter 9 which develops its cellular representation theory parallel to the previous part. Then Chapter 10 interrupts and introduce the representation category of the Hecke–Clifford superalgebra in non-integral rank. Finally in Chapter 11 we continue the study of the cellular representation theory and using representations in non-integral rank we completes the classification of its simple modules.

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# Part I

# Cellular Algebras Arising from Morita Contexts

CHAPTER 1

# Basics on enriched categories

Throughout in this paper, we fix a commutative ring $\Bbbk$. Tensor products over $\Bbbk$ are simply denoted by $\otimes$. We denote by $\Bbbk^\times$ the set of invertible elements in $\Bbbk$. We here recall the basic notions of enriched categories in a special case. For details we refer the reader to the textbook [**Kel82**].

## 1. Enriched categories

Let us denote by $\mathcal{M}$ the symmetric tensor category of $\Bbbk$-modules, by $\mathcal{S}$ that of $\Bbbk$-supermodules and by $\mathcal{G}$ that of graded $\Bbbk$-modules. So they consist of $\Bbbk$-modules $V = \bigoplus_{i\in I} V_i$ graded by the abelian group $I = \{1\}$, $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}$ respectively, and $\Bbbk$-homomorphisms which respect these gradings. When we take an element $x \in V$, we always assume that $x$ is a homogeneous element. For such $x$, we denote by $|x| \in I$ the degree of $x$. The symmetries on them are defined as

$$\begin{aligned} V \otimes W &\to W \otimes V \\ x \otimes y &\mapsto (-1)^{|x||y|} y \otimes x \end{aligned}$$

using the Koszul sign $(-1)^{|x||y|}$. If you want to use the naïve symmetry in the graded case, you should concentrate on evenly graded spaces for convention (actually we can take an arbitrary abelian group $I$ with a homomorphism $I \to \{\pm 1\}$).

Now let $\mathcal{V}$ be one of $\mathcal{M}$, $\mathcal{S}$ or $\mathcal{G}$. In each case, a $\mathcal{V}$-category is called a $\Bbbk$-*linear category*, a $\Bbbk$-*linear supercategory* or a $\Bbbk$-*linear graded category*. Shortly, a $\mathcal{V}$-category $\mathcal{C}$ consists of a hom object $\mathrm{Hom}_\mathcal{C}(X, Y) \in \mathcal{V}$ for each pair of objects $X, Y \in \mathcal{C}$ instead of a hom set. By taking the degree-zero part $\mathrm{Hom}_{\mathcal{C}_0}(X, Y) \coloneqq \mathrm{Hom}_\mathcal{C}(X, Y)_0$ we obtain the underlying category $\mathcal{C}_0$. When we write $f\colon X \to Y$ we mean that $f$ is a homogeneous element of $\mathrm{Hom}_\mathcal{C}(X, Y)$. For $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$, the *tensor product* $\mathcal{V}$-*category* $\mathcal{C} \boxtimes \mathcal{D}$ and the *opposite* $\mathcal{V}$-*category* $\mathcal{C}^{\mathrm{op}}$ are defined through the symmetry on $\mathcal{V}$. Their morphisms are in the form $f \boxtimes g$ and $f^{\mathrm{op}}$ respectively and the compositions are given by

$$\begin{aligned} (f_1 \boxtimes g_1) \circ (f_2 \boxtimes g_2) &\coloneqq (-1)^{|g_1||f_2|} (f_1 \circ f_2) \boxtimes (g_1 \circ g_2), \\ f_1^{\mathrm{op}} \circ f_2^{\mathrm{op}} &\coloneqq (-1)^{|f_1||f_2|} (f_2 \circ f_1)^{\mathrm{op}}. \end{aligned}$$

Similarly a $\mathcal{V}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ is a collection of a degree preserving homomorphism $\mathrm{Hom}_\mathcal{C}(X, Y) \to \mathrm{Hom}_\mathcal{D}(FX, FY)$ for each pair of $X, Y \in \mathcal{C}$. By taking its degree-zero part we obtain its underlying usual functor $F_0\colon \mathcal{C}_0 \to \mathcal{D}_0$. A $\mathcal{V}$-natural transformation $F \to G$ is defined as a usual natural transformation between the underlying functors $F_0 \to G_0$ which satisfies an additional condition.

We denote by $\mathcal{H}om(\mathcal{C}, \mathcal{D})_0$ the usual category consisting of $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations between them. The set (or the class) of natural transformations between $\mathcal{V}$-functors $F, G\colon \mathcal{C} \to \mathcal{D}$ is denoted by $\mathrm{Hom}_{\mathcal{C},\mathcal{D}}(F, G)_0$ for short. We can also complete this category to a $\mathcal{V}$-category $\mathcal{H}om(\mathcal{C}, \mathcal{D})$ (except that it may not be locally small) by letting its hom object $\mathrm{Hom}_{\mathcal{C},\mathcal{D}}(F, G)$ as a equalizer of the parallel morphisms

$$\prod_{X\in\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(FX,GX) \rightrightarrows \prod_{Y,Z\in\mathcal{C}} \mathrm{Hom}_{\mathcal{V}}\bigl(\mathrm{Hom}_{\mathcal{C}}(Y,Z),\mathrm{Hom}_{\mathcal{D}}(FY,GZ)\bigr).$$

We represent its homogeneous element as $h\colon F\to G$. The reader should pay attention to that its naturality means

$$(hY)\circ(Ff) = (-1)^{|h||f|}(Gf)\circ(hX)$$

for every $f\colon X\to Y$. Note that a $\mathcal{V}$-algebra $A$ (a monoid object in $\mathcal{V}$) is nothing but a $\mathcal{V}$-category $\mathcal{C}$ with a single object $*\in\mathcal{C}$ such that $A\simeq \mathrm{End}_{\mathcal{C}}(*)$. Then the category of left $A$-modules, which we denote by $A$-$\mathcal{M}od$, is equivalent to the functor category $\mathcal{H}om(\mathcal{C},\mathcal{V})$. The *tensor product $\mathcal{V}$-algebra* $A\otimes B$ and the *opposite $\mathcal{V}$-algebra* $A^{\mathrm{op}}$ are special cases of the operations for $\mathcal{V}$-categories above. We also denote by $\mathcal{M}od$-$A\simeq A^{\mathrm{op}}$-$\mathcal{M}od$ the category of right $A$-modules.

## 2. Alternative definitions

Now for a while consider the super case $\mathcal{V}=\mathcal{S}$. $\mathcal{S}$ has the *parity change functor* $\Pi\colon\mathcal{S}\to\mathcal{S}$ which exchanges the grading

$$(\Pi V)_0 \coloneqq V_1,\quad (\Pi V)_1\coloneqq V_0$$

for supermodule $V=V_0\oplus V_1$. In a general $\mathcal{S}$-category $\mathcal{C}$, for an object $Y\in\mathcal{C}$ its parity change $\Pi Y\in\mathcal{C}$, if exists, is defined as a representation of the $\mathcal{S}$-functor

$$\mathrm{Hom}_{\mathcal{C}}(X,\Pi Y)\simeq \Pi\,\mathrm{Hom}_{\mathcal{C}}(X,Y).$$

If every object in $\mathcal{C}$ has its parity change, $\Pi$ can be defined as an $\mathcal{S}$-functor $\Pi\colon\mathcal{C}\to\mathcal{C}$ and we also have an $\mathcal{S}$-natural isomorphism

$$\Pi\,\mathrm{Hom}_{\mathcal{C}}(X,Y)\simeq\mathrm{Hom}_{\mathcal{C}}(\Pi X,Y).$$

We say that such $\mathcal{C}$ is $\Pi$-*closed*. Instead of to treat the theory of enriched categories directly, we can view a $\Pi$-closed $\mathcal{S}$-category as a usual category with additional informations as follows.

LEMMA 1.1. *Giving a $\Pi$-closed $\mathcal{S}$-category $\mathcal{C}$ is equivalent to giving an $\mathcal{M}$-category $\mathcal{C}_0$ with an $\mathcal{M}$-functor $\Pi\colon\mathcal{C}_0\to\mathcal{C}_0$ and an isomorphism $\xi\colon\Pi^2\simeq\mathrm{Id}_{\mathcal{C}_0}$ such that $\xi\Pi=\Pi\xi$ as $\mathcal{S}$-natural isomorphisms $\Pi^3\to\Pi$.*

PROOF. Clearly a $\Pi$-closed $\mathcal{S}$-category induces such a datum. Conversely let $\mathcal{C}_0$ be an $\mathcal{M}$-category equipped with $\Pi\colon\mathcal{C}_0\to\mathcal{C}_0$ and $\xi\colon\Pi^2\simeq\mathrm{Id}_{\mathcal{C}_0}$. For each $X,Y\in\mathcal{C}_0$, let $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ be a supermodule defined by

$$\mathrm{Hom}_{\mathcal{C}}(X,Y)_0\coloneqq\mathrm{Hom}_{\mathcal{C}_0}(X,Y),\quad \mathrm{Hom}_{\mathcal{C}}(X,Y)_1\coloneqq\mathrm{Hom}_{\mathcal{C}_0}(X,\Pi Y).$$

Then we can define the composition $\mathrm{Hom}_{\mathcal{C}}(Y,Z)\otimes\mathrm{Hom}_{\mathcal{C}}(X,Y)\to\mathrm{Hom}_{\mathcal{C}}(X,Z)$ by

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}_0}(Y,Z)\otimes\mathrm{Hom}_{\mathcal{C}_0}(X,Y) &\to \mathrm{Hom}_{\mathcal{C}_0}(X,Z),\\
\mathrm{Hom}_{\mathcal{C}_0}(Y,\Pi Z)\otimes\mathrm{Hom}_{\mathcal{C}_0}(X,Y) &\to \mathrm{Hom}_{\mathcal{C}_0}(X,\Pi Z),\\
\mathrm{Hom}_{\mathcal{C}_0}(Y,Z)\otimes\mathrm{Hom}_{\mathcal{C}_0}(X,\Pi Y) &\simeq \mathrm{Hom}_{\mathcal{C}_0}(\Pi Y,\Pi Z)\otimes\mathrm{Hom}_{\mathcal{C}_0}(X,\Pi Y)\\
&\to \mathrm{Hom}_{\mathcal{C}_0}(X,\Pi Z),\\
\mathrm{Hom}_{\mathcal{C}_0}(Y,\Pi Z)\otimes\mathrm{Hom}_{\mathcal{C}_0}(X,\Pi Y) &\simeq \mathrm{Hom}_{\mathcal{C}_0}(\Pi Y,\Pi^2 Z)\otimes\mathrm{Hom}_{\mathcal{C}_0}(X,\Pi Y)\\
&\simeq \mathrm{Hom}_{\mathcal{C}_0}(\Pi Y,Z)\otimes\mathrm{Hom}_{\mathcal{C}_0}(X,\Pi Y)\\
&\to \mathrm{Hom}_{\mathcal{C}_0}(X,Z).
\end{aligned}$$

The condition $\xi\Pi=\Pi\xi$ is needed for that composition of three odd morphisms is associative. $\square$

LEMMA 1.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be $\Pi$-closed $\mathcal{S}$-categories. Then giving an $\mathcal{S}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ is equivalent to giving an $\mathcal{M}$-functor $F_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ between their underlying $\mathcal{M}$-categories equipped with an isomorphism $\alpha\colon F_0\Pi \simeq \Pi F_0$ which makes the diagram below commutes:*

$$\begin{tikzcd} F_0\Pi^2 \arrow[r, "\alpha\Pi"] \arrow[dr, "F_0\xi"'] & \Pi F_0 \Pi \arrow[r, "\Pi\alpha"] & \Pi^2 F_0 \arrow[dl, "\xi F_0"] \\ & F_0. & \end{tikzcd}$$

PROOF. The one direction is clear. So let $F_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ be an $\mathcal{M}$-functor equipped with an isomorphism $\alpha\colon F_0\Pi \simeq \Pi F_0$. On objects simply let $FX \coloneqq F_0X$. Then we can define the degree preserving map $F\colon \operatorname{Hom}_{\mathcal{C}}(X, Y) \to \operatorname{Hom}_{\mathcal{C}}(FX, FY)$ by

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}_0}(X, Y) &\to \operatorname{Hom}_{\mathcal{C}_0}(FX, FY), \\ \operatorname{Hom}_{\mathcal{C}_0}(X, \Pi Y) &\to \operatorname{Hom}_{\mathcal{C}_0}(FX, F\Pi Y) \simeq \operatorname{Hom}_{\mathcal{C}_0}(FX, \Pi FY). \end{aligned}$$

The commutativity of the diagram above is used to ensure that $F$ preserves composition of two odd morphisms. ☐

LEMMA 1.3. *Let $F$ and $G$ be $\mathcal{S}$-functors $\mathcal{C} \to \mathcal{D}$ between $\Pi$-closed $\mathcal{S}$-categories. Then a natural transformation $h\colon F_0 \to G_0$ is $\mathcal{S}$-natural if and only if the square*

$$\begin{tikzcd} F_0\Pi \arrow[r, "h\Pi"] \arrow[d, "\alpha"'] & G_0\Pi \arrow[d, "\alpha"] \\ \Pi F_0 \arrow[r, "\Pi h"'] & \Pi G_0 \end{tikzcd}$$

*commutes.*

PROOF. The $\mathcal{S}$-naturality just says that the two parallel maps, which are induced by $h$, $\operatorname{Hom}_{\mathcal{C}}(X, Y) \rightrightarrows \operatorname{Hom}_{\mathcal{C}}(FX, GY)$ coincide. By the usual naturality it is satisfied for the even parts. The condition above is equivalent to that it is also holds for the odd parts. ☐

Next consider the graded case $\mathcal{V} = \mathcal{G}$. Analogously we have the *$k$-th degree shift functor* $\Sigma^k$ defined by $(\Sigma^k V)_i \coloneqq V_{i+k}$ for $V \in \mathcal{G}$. We say that a $\mathcal{G}$-category is $\Sigma$-*closed* if each $Y \in \mathcal{C}$ has its degree shift $\Sigma^k Y$ defined by

$$\Sigma^k \operatorname{Hom}_{\mathcal{C}}(X, Y) \simeq \operatorname{Hom}_{\mathcal{C}}(X, \Sigma^k Y).$$

$\Sigma$-closed $\mathcal{G}$-category can be also characterized as follows. The proofs are similar as before so we omit them.

LEMMA 1.4. *Giving a $\Sigma$-closed $\mathcal{G}$-category $\mathcal{C}$ is equivalent to giving a $\mathcal{M}$-category $\mathcal{C}_0$ with a $\mathcal{M}$-functor $\Sigma\colon \mathcal{C}_0 \to \mathcal{C}_0$ which is an equivalence.*

LEMMA 1.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be $\Sigma$-closed $\mathcal{G}$-categories. Then giving a $\mathcal{G}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ is equivalent to giving a $\mathcal{M}$-functor $F_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ equipped with an isomorphism $\alpha\colon F_0\Sigma \simeq \Sigma F_0$.*

LEMMA 1.6. *Let $F$ and $G$ be $\mathcal{G}$-functors $\mathcal{C} \to \mathcal{D}$ between $\Sigma$-closed $\mathcal{G}$-categories. Then a natural transformation $f\colon F_0 \to G_0$ is $\mathcal{G}$-natural if and only if the square*

$$\begin{tikzcd} F_0\Sigma \arrow[r, "f\Sigma"] \arrow[d, "\alpha"'] & G_0\Sigma \arrow[d, "\alpha"] \\ \Sigma F_0 \arrow[r, "\Sigma f"'] & \Sigma G_0 \end{tikzcd}$$

*commutes.*

## 3. Limits in enriched category

Let $\mathcal{C}$ be a $\mathcal{V}$-category. We say a usual functor $\mathcal{I} \to \mathcal{C}_0$ from a small category $\mathcal{I}$ to the underlying category of $\mathcal{C}$ a *diagram* in $\mathcal{C}$. So a diagram consists of $Y_i \in \mathcal{C}$ for each $i \in \mathcal{I}$ and a degree-zero morphism $Y_i \to Y_j$ for each arrow $i \to j$ in $\mathcal{I}$. For a diagram $\{Y_i\}_{i\in\mathcal{I}}$, its *(conical) $\mathcal{V}$-limit* is an object $\varprojlim_i Y_i \in \mathcal{C}$ with a family of degree-zero morphisms $\varprojlim_i Y_i \to Y_i$ for each $i$ which satisfies the $\mathcal{V}$-natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X, \varprojlim_i Y_i) \simeq \varprojlim_i \mathrm{Hom}_{\mathcal{C}}(X, Y_i).$$

Note that the usual limit only implies

$$\mathrm{Hom}_{\mathcal{C}_0}(X, \varprojlim_i Y_i) \simeq \varprojlim_i \mathrm{Hom}_{\mathcal{C}_0}(X, Y_i).$$

Since taking the degree-zero part $V \mapsto V_0$ is continuous, the $\mathcal{V}$-limit of an diagram is also its usual limit. The converse does not hold in general but there are no differences between them in a suitable condition.

LEMMA 1.7. *Suppose $\mathcal{C}$ is an $\mathcal{M}$-category (resp. a $\Pi$-closed $\mathcal{S}$-category, a $\Sigma$-closed $\mathcal{G}$-category). Then for any diagram its limit in $\mathcal{C}_0$ is automatically its $\mathcal{V}$-limit in $\mathcal{C}$.*

PROOF. It is trivial for the $\mathcal{V} = \mathcal{M}$ case. When $\mathcal{V} = \mathcal{S}$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(X, \varprojlim_i Y_i) &\simeq \mathrm{Hom}_{\mathcal{C}_0}(X, \varprojlim_i Y_i) \oplus \mathrm{Hom}_{\mathcal{C}_0}(\Pi X, \varprojlim_i Y_i)\\ &\simeq \varprojlim_i \mathrm{Hom}_{\mathcal{C}_0}(X, Y_i) \oplus \varprojlim_i \mathrm{Hom}_{\mathcal{C}_0}(\Pi X, Y_i)\\ &\simeq \varprojlim_i \mathrm{Hom}_{\mathcal{C}}(X, Y_i).\end{aligned}$$

The similar proof works for $\mathcal{V} = \mathcal{G}$ since a limit commutes with direct sums in a Grothendieck category $\mathcal{G}$. □

The dual notion *(conical) $\mathcal{V}$-colimit* of a diagram is introduced similarly. Next we introduce the notion of abelian $\mathcal{V}$-category.

DEFINITION 1.8. We say that a $\mathcal{S}$-category (resp. a $\mathcal{G}$-category) $\mathcal{C}$ is *abelian* if it is $\Pi$-closed (resp. $\Sigma$-closed) and its underlying category $\mathcal{C}_0$ is abelian.

As usual, an $\mathcal{M}$-closed category $\mathcal{C}$ is called abelian when $\mathcal{C}_0$ is abelian. By the lemma above, in an abelian $\mathcal{V}$-category we can use several categorical notions such as zero object, direct sum, kernel, cokernel, monomorphism, epimorphism and exactness defined as via enriched Hom functors without any modifications. We are also allowed to operate homological computations as follows.

LEMMA 1.9. *Let $\mathcal{C}$ be an abelian $\mathcal{V}$-category. Then $P \in \mathcal{C}$ is projective in $\mathcal{C}_0$ if and only if the functor $\mathrm{Hom}_{\mathcal{C}}(P, \bullet)\colon \mathcal{C} \to \mathcal{V}$ is exact.*

PROOF. The "if" part follows from that taking the degree-zero part $\mathcal{V} \to \mathcal{M}; V \mapsto V_0$ is exact. The "only if" part can be proven similarly as the lemma above. □

We here study limits and colimits in a functor category. Let $\{F_i\colon \mathcal{C} \to \mathcal{D}\}$ be a diagram in $\mathcal{H}om(\mathcal{C}, \mathcal{D})$. If the $\mathcal{V}$-limit $\varprojlim_i F_i X$ exists for each $X \in \mathcal{C}$, then the $\mathcal{V}$-limit $\varprojlim_i F_i$ also exists and is defined as

$$(\varprojlim_i F_i)X \coloneqq \varprojlim_i F_i X.$$

Dually $\mathcal{V}$-colimits of this diagram is also computed value-wise. We remark that the composition of $\mathcal{V}$-functors

$$\begin{aligned}\mathcal{H}om(\mathcal{D},\mathcal{E}) \boxtimes \mathcal{H}om(\mathcal{C},\mathcal{D}) &\to \mathcal{H}om(\mathcal{C},\mathcal{E}),\\ F \boxtimes G &\mapsto FG\end{aligned}$$

is also defined as a $\mathcal{V}$-functor. By definition, the right multiplication

$$\bullet G\colon \mathcal{H}om(\mathcal{D},\mathcal{E}) \to \mathcal{H}om(\mathcal{C},\mathcal{E})$$

is both $\mathcal{V}$-continuous (i.e. preserves $\mathcal{V}$-limits) and $\mathcal{V}$-cocontinuous (i.e. does $\mathcal{V}$-colimits). In contrast, the left multiplication

$$F\bullet\colon \mathcal{H}om(\mathcal{C},\mathcal{D}) \to \mathcal{H}om(\mathcal{C},\mathcal{E})$$

preserves certain $\mathcal{V}$-limits or $\mathcal{V}$-colimits for all $\mathcal{C}$ if and only if $F$ does.

Suppose that $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$ are both abelian, and $\mathcal{C}$ has enough projectives. Then for each $F\colon \mathcal{C}\to\mathcal{D}$, its $i$-th left derived $\mathcal{V}$-functor $L_iF\colon \mathcal{C}\to\mathcal{D}$ makes sense as usual. $L_i$ can be viewed as a $\mathcal{V}$-endofunctor on $\mathcal{H}om(\mathcal{C},\mathcal{D})$, which is also abelian, and from a short exact sequence $0\to F\to G\to H\to 0$ of $\mathcal{V}$-functors they yield a long exact sequence

$$\cdots \to L_2H \to L_1F \to L_1G \to L_1H \to L_0F \to L_0G \to L_0H \to 0.$$

Dually, when $\mathcal{C}$ has enough injectives we define the $i$-th right derivation $R^i$.

We are most interested in the zeroth derivation $L_0$. By definition there is a canonical $\mathcal{V}$-natural transformation $L_0F\to F$. $L_0F\colon \mathcal{C}\to\mathcal{D}$ is right exact and of course $L_0F\simeq F$ when $F$ is already right exact. For another right exact $\mathcal{V}$-functor $G\colon\mathcal{C}\to\mathcal{D}$, the map

$$\mathrm{Hom}_{\mathcal{C},\mathcal{D}}(G, L_0F) \to \mathrm{Hom}_{\mathcal{C},\mathcal{D}}(G,F)$$

is an isomorphism since its inverse map is given by

$$\mathrm{Hom}_{\mathcal{C},\mathcal{D}}(G,F) \to \mathrm{Hom}_{\mathcal{C},\mathcal{D}}(L_0G, L_0F) \simeq \mathrm{Hom}_{\mathcal{C},\mathcal{D}}(G, L_0F).$$

Thus $L_0F$ is considered as the most applicative right exact approximation of $F$.

## 4. Adjunctions

An adjunction from $\mathcal{C}$ to $\mathcal{D}$ is a pair of adjoint $\mathcal{V}$-functors $F\colon\mathcal{C}\to\mathcal{D}$ and $F^\vee\colon\mathcal{D}\to\mathcal{C}$ where $F$ is left adjoint to $F^\vee$. That is, it is called so if there is a $\mathcal{V}$-natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(FX,Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, F^\vee Y).$$

Then $F$ is $\mathcal{V}$-cocontinuous and $F^\vee$ is $\mathcal{V}$-continuous. It is also characterized by degree-zero natural transformations $\delta\colon \mathrm{Id}_{\mathcal{C}}\to F^\vee F$ and $\epsilon\colon FF^\vee\to\mathrm{Id}_{\mathcal{D}}$ which satisfy the zig-zag identities

$$\mathrm{id}_F = (F \xrightarrow{F\delta} FF^\vee F \xrightarrow{\epsilon F} F), \qquad \mathrm{id}_{F^\vee} = (F^\vee \xrightarrow{\delta F^\vee} F^\vee F F^\vee \xrightarrow{F^\vee\epsilon} F^\vee).$$

$\delta$ and $\epsilon$ are called the unit and the counit of the adjunction respectively. For a $\mathcal{V}$-functor $F\colon\mathcal{C}\to\mathcal{D}$, the rest datum $(F^\vee,\delta,\varepsilon)$ which makes an adjunction is uniquely determined up to unique isomorphism if it exists. In order to keep notations simple we say that "$F$ is an adjunction from $\mathcal{C}$ to $\mathcal{D}$" when $F\colon\mathcal{C}\to\mathcal{D}$ has a fixed right adjoint $\mathcal{V}$-functor $F^\vee$. We denote by $\mathcal{A}dj(\mathcal{C},\mathcal{D})$ the full subcategory of $\mathcal{H}om(\mathcal{C},\mathcal{D})$ consisting of adjunctions.

For an adjunction $F\colon\mathcal{C}\to\mathcal{D}$, the left multiplication $F\bullet$ is left adjoint to $F^\vee\bullet$ while the right multiplication $\bullet F$ is right adjoint to $\bullet F^\vee$. Thus for two parallel adjunction $F,G\colon\mathcal{C}\to\mathcal{D}$, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{C},\mathcal{D}}(F,G) \simeq \mathrm{Hom}_{\mathcal{D},\mathcal{C}}(G^\vee, F^\vee).$$

We here list few examples of adjunction category.

EXAMPLE 1.10. Let $A$ and $B$ be $\mathcal{V}$-algebras. For an $(A, B)$-bimodule $M$, the $\mathcal{V}$-functors between their module categories

$$F\colon B\text{-}\mathcal{M}od \to A\text{-}\mathcal{M}od, \quad W \mapsto M \otimes_B W, \qquad F^\vee\colon A\text{-}\mathcal{M}od \to B\text{-}\mathcal{M}od, \quad V \mapsto \mathrm{Hom}_A(M, V)$$

form an adjunction from $B$-$\mathcal{M}od$ to $A$-$\mathcal{M}od$. Conversely let $F\colon B\text{-}\mathcal{M}od \to A\text{-}\mathcal{M}od$ be an arbitrary adjunction. When we put $M := FB$ the multiplication on $B$ from right makes $M$ a right $B$-module. We have

$$F^\vee V \simeq \mathrm{Hom}_B(B, F^\vee V) \simeq \mathrm{Hom}_A(FB, V) = \mathrm{Hom}_A(M, V)$$

so every adjunction between module categories can be obtained in this way. In addition, $\mathcal{A}dj(B\text{-}\mathcal{M}od, A\text{-}\mathcal{M}od)$ is equivalent to $A$-$\mathcal{M}od$-$B$, the category of $(A, B)$-bimodules. In particular it is abelian and the embedding to $\mathcal{H}om(B\text{-}\mathcal{M}od, A\text{-}\mathcal{M}od)$ is right exact, but not left exact in general.

EXAMPLE 1.11. Suppose $\Bbbk$ is a field, and for $A$ and $B$ above let $A\text{-}\mathcal{M}od^f$, $B\text{-}\mathcal{M}od^f$ be the categories of their finite dimensional left modules. Suppose that $N$ is a $(B, A)$-bimodule which satisfies these finiteness conditions:

(1) if a right $A$-module $V$ is finite dimensional then so is $\mathrm{Hom}_{A^{\mathrm{op}}}(V, N)$,
(2) if a left $B$-module $W$ is finite dimensional then so is $\mathrm{Hom}_B(W, N)$,
(3) $N$ is locally finite dimensional, that is, it is the union of its finite dimensional $(B, A)$-submodules.

We denote by $V^\vee := \mathrm{Hom}_{\mathcal{V}}(V, \Bbbk)$ the dual space of a finite dimensional vector space. Then the functors

$$F\colon B\text{-}\mathcal{M}od^f \to A\text{-}\mathcal{M}od^f, \quad W \mapsto \mathrm{Hom}_B(W, N)^\vee, \qquad F^\vee\colon A\text{-}\mathcal{M}od^f \to B\text{-}\mathcal{M}od^f, \quad V \mapsto \mathrm{Hom}_{A^{\mathrm{op}}}(V^\vee, N)$$

form an adjunction via the natural isomorphism

$$\begin{aligned}\mathrm{Hom}_A(\mathrm{Hom}_B(W, N)^\vee, V) &\simeq \mathrm{Hom}_{A^{\mathrm{op}}}(V^\vee, \mathrm{Hom}_B(W, N))\\ &\simeq \mathrm{Hom}_B(W, \mathrm{Hom}_{A^{\mathrm{op}}}(V^\vee, N)).\end{aligned}$$

In this case $N$ is recovered from $F$ by the formula $N \simeq \varinjlim_{B \twoheadrightarrow B'} (FB')^\vee$ where $B'$ runs over all finite dimensional quotient algebras of $B$. Every adjunctions are obtained in this way and $\mathcal{A}dj(B\text{-}\mathcal{M}od^f, A\text{-}\mathcal{M}od^f)$ is equivalent to the opposite of the category of such $(B, A)$-bimodules. This category does not necessarily have kernels.

EXAMPLE 1.12. For a small $\mathcal{V}$-category $\mathcal{A}$, we call a $\mathcal{V}$-functor $\mathcal{A} \to \mathcal{V}$ "a left $\mathcal{A}$-module", and denote by $\mathcal{A}\text{-}\mathcal{M}od := \mathcal{H}om(\mathcal{A}, \mathcal{V})$. Similarly, for another $\mathcal{V}$-category $\mathcal{B}$, a right $\mathcal{B}$-module (resp. an $(\mathcal{A}, \mathcal{B})$-bimodule) is just a $\mathcal{V}$-functor $\mathcal{B}^{\mathrm{op}} \to \mathcal{V}$ (resp. $\mathcal{B}^{\mathrm{op}} \boxtimes \mathcal{A} \to \mathcal{V}$). Since a $\mathbb{Z}$-linear category is a "ring with several objects" as Mitchell [**Mit72**] noticed, this terminology is a generalization for usual algebras.

When $M$ is an $(\mathcal{A}, \mathcal{B})$-bimodule and $N$ is an $(\mathcal{A}, \mathcal{C})$-bimodule, we can form an $(\mathcal{B}, \mathcal{C})$-bimodule $\mathrm{Hom}_{\mathcal{A}}(M, N)$ defined as

$$\begin{aligned}&\mathcal{C}^{\mathrm{op}} \boxtimes \mathcal{B} \to \mathcal{V},\\ &Z \boxtimes Y \mapsto \mathrm{Hom}_{\mathcal{A},\mathcal{V}}(M(Y, \bullet), N(Z, \bullet)).\end{aligned}$$

On the other hand, if $P$ is an $(\mathcal{B}, \mathcal{C})$-bimodule, there is an $(\mathcal{A}, \mathcal{C})$-bimodule denoted by $M \otimes_{\mathcal{B}} P$, which sends $Z \boxtimes X \in \mathcal{C}^{\mathrm{op}} \boxtimes \mathcal{A}$ to the coequalizer of the parallel maps

$$\bigoplus_{Y', Y'' \in \mathcal{B}} M(Y', X) \otimes \mathrm{Hom}_{\mathcal{B}}(Y'', Y') \otimes P(Z, Y'') \rightrightarrows \bigoplus_{Y \in \mathcal{B}} M(Y, X) \otimes P(Z, Y).$$

Similarly as above, an adjunction $F\colon \mathcal{B}\text{-}\mathcal{M}od \to \mathcal{A}\text{-}\mathcal{M}od$ is represented by some $(\mathcal{A},\mathcal{B})$-bimodule $M$ using $\otimes$ and Hom. The identity functor on $\mathcal{A}\text{-}\mathcal{M}od$ corresponds to the $(\mathcal{A},\mathcal{A})$-bimodule $\mathrm{Hom}_{\mathcal{A}}(\bullet,\bullet)$.

## 5. The category of adjunctions

First of all, we make sure a well-known fact that adjunctions are closed under colimits, especially cokernels. Let $\{F_i\}$ be a diagram in $\mathcal{A}dj(\mathcal{C},\mathcal{D})$. For each arrow $i \to j$ between indices, the $\mathcal{V}$-natural transformation $F_i \to F_j$ induces the corresponding $F_j^\vee \to F_i^\vee$. Thus they form the diagram $\{F_i^\vee\}$ in $\mathcal{H}om(\mathcal{D},\mathcal{C})$ whose arrows are reversed.

Proposition 1.13. *Let $\{F_i\}$ be as above, and suppose that the $\mathcal{V}$-colimit $F := \varinjlim_i F_i$ and the $\mathcal{V}$-limit $F^\vee := \varprojlim_i F_i^\vee$ are both exist. Then $F$ is left adjoint to $F^\vee$. Moreover the canonical morphisms $F_i \to F$ and $F^\vee \to F_i^\vee$ are mapped to each other by the isomorphism*

$$\mathrm{Hom}_{\mathcal{C},\mathcal{D}}(F_i, F) \simeq \mathrm{Hom}_{\mathcal{D},\mathcal{C}}(F^\vee, F_i^\vee).$$

We here give two proofs for this easy but important result.

First proof. First we prove the lemma by studying the functors value-wise. The statements are obvious by the $\mathcal{V}$-natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(FX,Y) \simeq \varprojlim_i \mathrm{Hom}_{\mathcal{D}}(F_iX,Y) \simeq \varprojlim_i \mathrm{Hom}_{\mathcal{C}}(X,F_i^\vee Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X,F^\vee Y).$$

□

Second proof. The second is a "2-categorical" proof. For each arrow $i \to j$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Id}_{\mathcal{C}} \arrow[r] \arrow[d] & F_j^\vee F_j \arrow[r] \arrow[d] & F_j^\vee F \arrow[d] \\
F_i^\vee F_i \arrow[r] & F_i^\vee F_j \arrow[r] & F_i^\vee F
\end{tikzcd}$$

so it induces the unit $\mathrm{Id}_{\mathcal{C}} \to \varprojlim_i F_i^\vee F \simeq F^\vee F$. The counit $FF^\vee \to \mathrm{Id}_{\mathcal{D}}$ is defined analogously. To prove that $F \to FF^\vee F \to F$ is equal to $\mathrm{id}_F$, it suffices to show that its pullback $F_i \to F \to FF^\vee F \to F$ is equal to $F_i \to F$ for each $i$. It follows from the commutativity of the diagram below:

$$\begin{tikzcd}
F \arrow[d] & F_i \arrow[l] \arrow[r] \arrow[d] \arrow[rr, bend left, "\mathrm{id}"] & F_iF_i^\vee F_i \arrow[r] \arrow[d] & F_i \arrow[d] \\
FF^\vee F \arrow[rrr, bend right] & F_iF^\vee F \arrow[l] \arrow[r] & F_iF_i^\vee F \arrow[r] & F.
\end{tikzcd}$$

Similarly $F^\vee \to F^\vee FF^\vee \to F^\vee$ is also equal to the identity so $F$ and $F^\vee$ are adjoint to each other. The diagram above also shows us that $F_i \to F$ also coincides with the composite $F_i \to F_iF^\vee F \to F_iF_i^\vee F \to F$ which is induced by $F^\vee \to F_i^\vee$. □

Remark 1.14. Although we are now studying the 2-category of $\mathcal{V}$-categories, the second proof also works for an arbitrary 2-category such that the right multiplication of a 1-cell is both $\mathcal{V}$-continuous and $\mathcal{V}$-cocontinuous.

We will soon use the next corollary.

COROLLARY 1.15. *Suppose that both $\mathcal{C}$ and $\mathcal{D}$ are abelian. A sequence of adjunctions $F \to G \to H \to 0$ is exact in $\mathcal{H}om(\mathcal{C},\mathcal{D})$ if and only if so is the corresponding sequence $0 \to H^\vee \to G^\vee \to F^\vee$ in $\mathcal{H}om(\mathcal{D},\mathcal{C})$. In particular, $F \to G$ is epic if and only if the corresponding $G^\vee \to F^\vee$ is monic.*

As we have seen in the examples, the $\mathcal{V}$-category $\mathcal{A}dj(\mathcal{C},\mathcal{D})$ may have limits which differ from those taken in $\mathcal{H}om(\mathcal{C},\mathcal{D})$. We here give a simple sufficient condition for $\mathcal{A}dj(\mathcal{C},\mathcal{D})$ to be abelian.

LEMMA 1.16. *Let $\mathcal{C}$ and $\mathcal{D}$ be abelian $\mathcal{V}$-categories, and suppose that $\mathcal{C}$ has enough projectives and $\mathcal{D}$ has enough injectives. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be parallel adjunctions with a degree-zero natural transformation $F \to G$. Let $K := \operatorname{Ker}(F \to G)$ and $C := \operatorname{Coker}(G^\vee \to F^\vee)$. Then $L_0K$ is left adjoint to $R^0C$, so it is an adjunction. Moreover, $L_0K$ is the kernel of $F \to G$ in $\mathcal{A}dj(\mathcal{C},\mathcal{D})$.*

PROOF. Let $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and take a projective resolution $P' \to P \to X \to 0$ and an injective resolution $0 \to Y \to Q \to Q'$ respectively. By definition $\operatorname{Hom}_{\mathcal{D}}((L_0K)X, Y)$ is the kernel of the map

$$\operatorname{Hom}_{\mathcal{D}}(KP, Q) \to \operatorname{Hom}_{\mathcal{D}}(KP', Q) \oplus \operatorname{Hom}_{\mathcal{D}}(KP, Q').$$

Since $P$ is projective and $Q$ is injective, each term can be represented as

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}}(KP, Q) &\simeq \operatorname{Coker}(\operatorname{Hom}_{\mathcal{D}}(GP, Q) \to \operatorname{Hom}_{\mathcal{D}}(FP, Q))\\ &\simeq \operatorname{Coker}(\operatorname{Hom}_{\mathcal{C}}(P, G^\vee Q) \to \operatorname{Hom}_{\mathcal{C}}(P, F^\vee Q))\\ &\simeq \operatorname{Hom}_{\mathcal{C}}(P, CQ).\end{aligned}$$

Hence we have a natural isomorphism $\operatorname{Hom}_{\mathcal{D}}((L_0K)X, Y) \simeq \operatorname{Hom}_{\mathcal{C}}(X, (R^0C)Y)$ since its right-hand side also has a similar representation. For any $H \in \mathcal{A}dj(\mathcal{C},\mathcal{D})$ there is a natural isomorphism

$$\operatorname{Hom}_{\mathcal{C},\mathcal{D}}(H, L_0K) \simeq \operatorname{Hom}_{\mathcal{C},\mathcal{D}}(H, K) \simeq \operatorname{Ker}(\operatorname{Hom}_{\mathcal{C},\mathcal{D}}(H, F) \to \operatorname{Hom}_{\mathcal{C},\mathcal{D}}(H, G)).$$

Thus $L_0K$ is the kernel of $F \to G$ taken in $\mathcal{A}dj(\mathcal{C},\mathcal{D})$. $\square$

PROPOSITION 1.17. *Let $\mathcal{C}$ and $\mathcal{D}$ be as above. Then $\mathcal{A}dj(\mathcal{C},\mathcal{D})$ is abelian and the embedding $\mathcal{A}dj(\mathcal{C},\mathcal{D}) \hookrightarrow \mathcal{H}om(\mathcal{C},\mathcal{D})$ is right exact.*

PROOF. Clearly it is closed under $\Pi$ or $\Sigma$. By Proposition 1.13 and Lemma 1.16 it also has finite direct sums, kernels and cokernels. In $\mathcal{A}dj(\mathcal{C},\mathcal{D})$ the image of the morphism $F \to G$ is isomorphic to its coimage, since they give same value $\operatorname{Image}(FP \to GP)$ on enough projectives $P \in \mathcal{C}$. The cokernel of a morphism in $\mathcal{A}dj(\mathcal{C},\mathcal{D})$ is equal to that taken in $\mathcal{H}om(\mathcal{C},\mathcal{D})$ so the embedding is right exact. $\square$

CHAPTER 2

# Ideal functors in abelian categories

From now on, we omit all prefixes "$\mathcal{V}$-" in order to avoid redundant descriptions, so "a category" means a $\mathcal{V}$-category, "a functor" a $\mathcal{V}$-functor, etc.

We here study some kind of endofunctors which we call *ideal functors*. These are analogues of 2-sided ideals in a ring. Later it is used to divide the category into two parts by a Morita context.

## 1. Ideal functors

Let $\mathcal{C}$ be an abelian category (i.e. an abelian $\mathcal{V}$-category) and consider the category $\mathcal{E}nd(\mathcal{C}) \coloneqq \mathcal{H}om(\mathcal{C}, \mathcal{C})$ which is also abelian and has the specific object $\mathrm{Id}_{\mathcal{C}}$, the identity functor.

Definition 2.1. A subfunctor $I \subset \mathrm{Id}_{\mathcal{C}}$ is called an *ideal functor* on $\mathcal{C}$ if its cokernel $T_I \coloneqq \mathrm{Coker}(I \hookrightarrow \mathrm{Id}_{\mathcal{C}})$ is an adjunction. The cokernel of the corresponding morphism $T_I^\vee \hookrightarrow \mathrm{Id}_{\mathcal{C}}$ (monic by Corollary 1.15) is denoted by $I^\circ$.

Example 2.2. Consider the case that $\mathcal{C}$ is a module category $A$-$\mathcal{M}od$. Then the set of quotient adjunctions of $\mathrm{Id}_{A\text{-}\mathcal{M}od}$ is in bijection with the set of quotient $(A, A)$-bimodules of $A$, that is, $A/I$ for a 2-sided ideal $I \subset A$. Thus the corresponding ideal functor maps an $A$-module $V$ to the kernel of the map

$$V \twoheadrightarrow T_I V \coloneqq A/I \otimes_A V \simeq V/IV,$$

namely $IV$. This is why we call such kind of functor an "ideal functor". In this case the right adjoint $T_I^\vee$ can be represented as

$$T_I^\vee V \coloneqq \mathrm{Hom}_A(A/I, V) \simeq \{v \in V \mid Iv = 0\}.$$

Example 2.3. When $\mathcal{C} = \mathcal{A}$-$\mathcal{M}od$ is a module category of a category $\mathcal{A}$, an ideal functor on $\mathcal{C}$ is also corresponds to a 2-sided ideal $\mathcal{I} \subset \mathcal{A}$. Here a 2-sided ideal in a category is a collection of spaces of morphisms

$$\mathcal{I}(V, W) \subset \mathrm{Hom}_{\mathcal{A}}(V, W)$$

for each pair of $V, W \in \mathcal{A}$, which is closed under compositions with all morphisms in $\mathcal{A}$. From such an ideal we can form a quotient category $\mathcal{A}/\mathcal{I}$, whose hom sets are defined by pairwise quotient.

Example 2.4. The socle $\mathrm{Soc}(X)$ of an object $X \in \mathcal{C}$ is the sum of all simple subobjects of $X$. Dually, its top $\mathrm{Top}(X)$ is defined as $X/\mathrm{Rad}(X)$ where the radical $\mathrm{Rad}(X)$ is the intersection of all its maximal subobjects of $X$. If these objects exist for every $X \in \mathcal{C}$, then Soc, Top and Rad can be defined as endofunctors on $\mathcal{C}$.

Suppose that for all object $X \in \mathcal{C}$, $\mathrm{Top}(X)$ and $\mathrm{Soc}(X)$ are both semisimple. Then the functor Top is left adjoint to Soc, thus $\mathrm{Rad} = \mathrm{Ker}(\mathrm{Id}_{\mathcal{C}} \twoheadrightarrow \mathrm{Top})$ is an ideal functor.

Example 2.5. If $I$ is an ideal functor on $\mathcal{C}$, $(I^\circ)^{\mathrm{op}}$ is an ideal functor on $\mathcal{C}^{\mathrm{op}}$.

On a general abelian category, a typical example of ideal functor is the image of an adjunction.

PROPOSITION 2.6. *Let $F\colon \mathcal{C} \to \mathcal{C}$ be an adjunction with a degree-zero natural transformation $F \to \mathrm{Id}_{\mathcal{C}}$. Then $I := \mathrm{Image}(F \to \mathrm{Id}_{\mathcal{C}})$ is an ideal functor. $I^\circ$ is also the image of the corresponding natural transformation $\mathrm{Id}_{\mathcal{C}} \to F^\vee$.*

PROOF. By Proposition 1.13, the cokernel $T_I = \mathrm{Coker}(F \to \mathrm{Id}_{\mathcal{C}})$ has the right adjoint functor $T_I^\vee := \mathrm{Ker}(\mathrm{Id}_{\mathcal{C}} \to F^\vee)$ so it is an adjunction. Thus by definition $I$ is an ideal functor. Since $0 \to T_I^\vee \to \mathrm{Id}_{\mathcal{C}} \to F^\vee$ is exact by Corollary 1.15, $F^\vee$ contains $I^\circ = \mathrm{Coker}(T_I^\vee \hookrightarrow \mathrm{Id}_{\mathcal{C}})$ as a subfunctor. □

REMARK 2.7. Suppose $\mathcal{C}$ has enough projectives and injectives. Then for an ideal functor $I$ on $\mathcal{C}$, $L_0 I$ is an adjunction and $L_0 I \twoheadrightarrow I$ is epic by Proposition 1.17. Thus in this case every ideal functor is obtained as the image of an adjunction. When $\mathcal{C} = A\text{-}\mathcal{M}od$, $L_0 I$ for a 2-sided ideal $I \subset A$ is just the tensor functor $I \otimes_A \bullet$ where $I$ is viewed as an $(A, A)$-bimodule.

On the other hand, let $A$ be a polynomial algebra $\Bbbk[x_1, x_2, \ldots]$ in infinitely many variables over a field $\Bbbk$ and consider the case $\mathcal{C} = A\text{-}\mathcal{M}od^f$, the category of its finite dimensional modules. Then $I\colon V \mapsto \sum_i x_i V$ is clearly an ideal functor on $A\text{-}\mathcal{M}od^f$ but there are no adjunctions which cover $I$.

We here list basic properties of ideal functors. For simplicity if there is a canonical isomorphism $F \to G$ between objects which is clear from the context, we write $F = G$ for short.

LEMMA 2.8. *Let $I \subset \mathrm{Id}_{\mathcal{C}}$ be an ideal functor. Then*
(1) *the morphisms $T_I = T_I \cdot \mathrm{Id}_{\mathcal{C}} \to T_I^2$ and $T_I = \mathrm{Id}_{\mathcal{C}} \cdot T_I \to T_I^2$ coincide,*
(2) *$T_I^\vee T_I = T_I = T_I^2$,*
(3) *$I T_I = 0 = I^\circ T_I$.*

PROOF. (1) follows from that the epimorphism $\mathrm{Id}_{\mathcal{C}} \twoheadrightarrow T_I$ equalize these two morphisms. Since $\mathrm{Id}_{\mathcal{C}} \twoheadrightarrow T_I$ factors through $\mathrm{Id}_{\mathcal{C}} \to T_I^\vee T_I \hookrightarrow T_I$, the monomorphism $T_I^\vee T_I \hookrightarrow T_I$ is also epic. So it must be an isomorphism since the functor category is abelian. Similarly $T_I^\vee = T_I T_I^\vee$ holds and we obtain

$$T_I = T_I^\vee T_I = T_I T_I^\vee T_I = T_I^2$$

so (2) holds. (3) is just a rephrasing of (2). □

In general an ideal functor is neither left exact nor right exact. Still, we can prove the following useful properties.

PROPOSITION 2.9. *An ideal functor preserves all images.*

PROOF. Let $I \subset \mathrm{Id}_{\mathcal{C}}$ be an ideal functor. A functor is called *mono* (resp. *epi*) if it preserves all monomorphisms (resp. epimorphisms). Since $\mathrm{Id}_{\mathcal{C}}$ is clearly a mono functor, so is its subfunctor $I$. $I$ is also epi because $\mathrm{Id}_{\mathcal{C}}$ is epi and the cokernel $T_I$ is right exact; apply the nine lemma. Thus $I$ preserves all epi-mono factorizations. □

Recall that for a possibly infinite family of subobjects $\{X_i \subset X\}$, their sum, if exists, is the minimum subobject $\sum_i X_i \subset X$ which contains all $X_i$.

LEMMA 2.10. *Let $X \in \mathcal{C}$ be an object and $Y, Z \subset X$ its subobjects. Then $IZ \subset Y$ if and only if $Z \subset \mathrm{Ker}(X \twoheadrightarrow X/Y \twoheadrightarrow I^\circ(X/Y))$ (in other words, $X/Z$ is a quotient of $I^\circ(X/Y)$).*

PROOF. Suppose $IZ \subset Y$, or equivalently, the composite $IZ \hookrightarrow Z \to X/Y$ is zero. Then $Z \to X/Y$ factors through $Z \twoheadrightarrow T_I Z$. Hence $Z \to X/Y \twoheadrightarrow I^\circ(X/Y)$ factors through $I^\circ T_I Z = 0$ so it must be zero. Taking the opposite category we can dually prove the other implication. □

PROPOSITION 2.11. *An ideal functor commutes with summation.*

PROOF. Let $I \subset \mathrm{Id}_{\mathcal{C}}$ be an ideal functor. Take an object $X \in \mathcal{C}$ and a family of subobjects $\{X_i \subset X\}$ whose sum $\sum_i X_i$ exists. Note that $X_i \hookrightarrow \sum_i X_i$ induces $IX_i \hookrightarrow I \sum_i X_i$. Thus if $\sum_i IX_i$ exists, it is contained in $I \sum_i X_i$.

Conversely, let $Y \subset X$ be a subobject which contains every $IX_i$. Let

$$Y' := \mathrm{Ker}(X \twoheadrightarrow X/Y \twoheadrightarrow I^\circ(X/Y)).$$

Then $X_i \subset Y'$ for all $i$ by the "only if " part of Lemma 2.10, so $\sum_i X_i \subset Y'$. On the other hand, its "if" part says that $IY' \subset Y$. Thus $I \sum_i X_i \subset IY' \subset Y$, so $\sum_i IX_i$ exists and actually

$$\sum_i IX_i = I \sum_i X_i.$$

□

## 2. Subcategories defined by ideal functors

In this section we fix an ideal functor $I$ on $\mathcal{C}$. Using an ideal functor, we define two full subcategory of $\mathcal{C}$ in the following manner.

LEMMA 2.12. *For an object $X \in \mathcal{C}$, the following conditions are equivalent.*

(1) $IX = 0$ *(*$\iff X = T_I X$*),*
(2) $I^\circ X = 0$ *(*$\iff T_I^\vee X = X$*).*

PROOF. Similar to the proof of $T_I = T_I^2$ in Lemma 2.8. □

DEFINITION 2.13. An object $X \in \mathcal{C}$ is called *$I$-annihilated* if it satisfies the conditions above. We denote by $\mathcal{C}_I$ the full subcategory of $\mathcal{C}$ consisting of $I$-annihilated objects.

The other subcategory is defined analogously.

DEFINITION 2.14. An object $X \in \mathcal{C}$ is called

(1) *$I$-accessible* if $IX = X$ ($\iff T_I X = 0$),
(2) *$I$-torsion-free* if $X = I^\circ X$ ($\iff T_I^\vee X = 0$).

We denote by $\mathcal{C}^I$ the full subcategory of $\mathcal{C}$ consisting of $I$-accessible $I$-torsion-free objects.

By definition it is clear that these subcategories are closed under the parity change $\Pi$ or the degree shift $\Sigma$.

EXAMPLE 2.15. Let $A$ be a ring and $I \subset A$ a 2-sided ideal. Then an $A$-module $V$ is $I$-annihilated if and only if it can be defined over the quotient ring $A/I$. Thus there is a canonical category equivalence $(A\text{-}\mathcal{M}od)_I \simeq (A/I)\text{-}\mathcal{M}od$.

EXAMPLE 2.16. Suppose that every object in $\mathcal{C}$ is of finite length. Then the assumption in Example 2.4 is satisfied. For an ideal functor $I = \mathrm{Rad}$, we have that $\mathcal{C}^{\mathrm{Rad}} = \{0\}$ (Nakayama's lemma) and $\mathcal{C}_{\mathrm{Rad}}$ consists of all semisimple objects in $\mathcal{C}$.

Clearly the intersection of $\mathcal{C}^I$ and $\mathcal{C}_I$ is $\{0\}$. In addition, there are no non-zero morphisms between objects in these categories.

LEMMA 2.17. *Let $X, Y, Z \in \mathcal{C}$ and suppose that $X$ is $I$-accessible, $Y$ is $I$-torsion-free and $Z$ is $I$-annihilated. Then* $\mathrm{Hom}_{\mathcal{C}}(X, Z) = 0$ *and* $\mathrm{Hom}_{\mathcal{C}}(Z, Y) = 0$.

PROOF. Follows from

$$\mathrm{Hom}_{\mathcal{C}}(X, Z) = \mathrm{Hom}_{\mathcal{C}}(X, T_I^\vee Z) \simeq \mathrm{Hom}_{\mathcal{C}}(T_I X, Z) = 0,$$
$$\mathrm{Hom}_{\mathcal{C}}(Z, Y) = \mathrm{Hom}_{\mathcal{C}}(T_I Z, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(Z, T_I^\vee Y) = 0.$$

□

The important property is that simple objects in $\mathcal{C}$ are divided into these subcategories. The proof of the lemma below is obvious.

LEMMA 2.18. *When $X \in \mathcal{C}$ is simple, these three conditions are all equivalent.*

(1) *$X$ is $I$-accessible,*
(2) *$X$ is $I$-torsion-free,*
(3) *$X$ is not $I$-annihilated.* □

NOTATION 2.19. We denote by $\operatorname{Irr}\mathcal{C}$ the isomorphism class of simple objects in $\mathcal{C}$. For an ideal functor $I$ on $\mathcal{C}$, we also denote by $\operatorname{Irr}\mathcal{C}^I$ and $\operatorname{Irr}\mathcal{C}_I$ the subsets of $\operatorname{Irr}\mathcal{C}$ whose members are simple objects contained in respective subcategories.

By the lemma, we have a decomposition $\operatorname{Irr}\mathcal{C} = \operatorname{Irr}\mathcal{C}^I \sqcup \operatorname{Irr}\mathcal{C}_I$. Note that the definitions of $\operatorname{Irr}\mathcal{C}^I$ and $\operatorname{Irr}\mathcal{C}_I$ need both the category $\mathcal{C}$ and the ideal functor $I$, not only the subcategories themselves.

PROPOSITION 2.20.

(1) *$I$-accessible objects are closed under quotients, extensions and direct sums,*
(2) *$I$-torsion-free objects are closed under subobjects, extensions and direct products,*
(3) *$I$-annihilated objects are closed under subobjects, quotients, direct products and direct sums.*

*In particular, $\mathcal{C}^I$ is an exact subcategory of $\mathcal{C}$ in Quillen's sense, and $\mathcal{C}_I$ is an abelian subcategory.*

PROOF. Follow from that $T_I$ is cocontinuous and that $T_I^\vee$ is continuous. □

Obviously $\operatorname{Irr}\mathcal{C}_I$ is equal to the isomorphism class of simple objects in an abelian category $\mathcal{C}_I$, so this notation makes no confusions. Let us denote by the embedding $\mathcal{C}_I \hookrightarrow \mathcal{C}$ of abelian category by $\Phi_I$. Namely, for $X \in \mathcal{C}_I$, we explicitly write $\Phi_I X \in \mathcal{C}$ when we need to emphasis the categories in which these objects belong.

LEMMA 2.21. *$\Phi_I$ has both the left adjoint functor $\Phi_I^\wedge$ and the right adjoint functor $\Phi_I^\vee$ such that $\Phi_I\Phi_I^\wedge = T_I$, $\Phi_I\Phi_I^\vee = T_I^\vee$ and $\Phi_I^\wedge\Phi_I = \Phi_I^\vee\Phi_I = \mathrm{Id}_{\mathcal{C}_I}$.*

PROOF. Since $T_I X$ and $T_I^\vee X$ for any $X \in \mathcal{C}$ are $I$-annihilated, the functors $\Phi_I^\wedge$ and $\Phi_I^\vee$ can be defined as the restriction of $T_I$ and $T_I^\vee$ respectively. For any $Y \in \mathcal{C}_I$, we have naturally

$$\operatorname{Hom}_{\mathcal{C}}(X, \Phi_I Y) = \operatorname{Hom}_{\mathcal{C}}(X, T_I^\vee \Phi_I Y) \simeq \operatorname{Hom}_{\mathcal{C}}(T_I X, \Phi_I Y) = \operatorname{Hom}_{\mathcal{C}_I}(\Phi_I^\wedge X, Y),$$
$$\operatorname{Hom}_{\mathcal{C}}(\Phi_I Y, X) = \operatorname{Hom}_{\mathcal{C}}(T_I \Phi_I Y, X) \simeq \operatorname{Hom}_{\mathcal{C}}(\Phi_I Y, T_I^\vee X) = \operatorname{Hom}_{\mathcal{C}_I}(Y, \Phi_I^\vee X).$$

Thus these two functors are respective adjoints of the embedding. □

COROLLARY 2.22. *For an ideal functor $J$ on another abelian category $\mathcal{D}$, the functor*

$$\mathcal{H}om(\mathcal{C}_I, \mathcal{D}_J) \to \mathcal{H}om(\mathcal{C}, \mathcal{D}),$$
$$F \mapsto \Phi_J F \Phi_I^\wedge$$

*is fully faithful. Its image is equivalent to the full subcategory*

$$\{G \colon \mathcal{C} \to \mathcal{D} \mid G = T_J G T_I\} \subset \mathcal{H}om(\mathcal{C}, \mathcal{D})$$

*and the inverse is induced by*

$$\mathcal{H}om(\mathcal{C}, \mathcal{D}) \to \mathcal{H}om(\mathcal{C}_I, \mathcal{D}_J),$$
$$G \mapsto \Phi_J^\wedge G \Phi_I.$$

*In particular, $F$ is an adjunction if and only if so is $\Phi_J F \Phi_I^\wedge$.* □

Remark 2.23. $\mathcal{C}_I$ is characterized up to equivalence by these data: let $\mathcal{E}$ be an abelian category with an adjoint $\Phi\colon \mathcal{E} \to \mathcal{C}$ which also has a left adjoint functor $\Phi^\wedge$, and suppose that the counit $\Phi^\wedge\Phi \to \mathrm{Id}_\mathcal{E}$ is an isomorphism and the unit $\mathrm{Id}_\mathcal{C} \to \Phi\Phi^\wedge$ is epic. Then $\mathcal{E}$ is canonically equivalent to $\mathcal{C}_I$ where $I := \mathrm{Ker}(\mathrm{Id}_\mathcal{C} \twoheadrightarrow \Phi\Phi^\wedge)$.

## 3. Ideal operations

In this section we consider operations against ideal functors: summation, product and quotient. These are analogues of those against usual 2-sided ideals in rings. Firstly we introduce summation of ideal functors.

Proposition 2.24. *Let $\{I_i\}$ be a family of ideal functors on $\mathcal{C}$, and suppose that their sum $\sum_i I_i \subset \mathrm{Id}_\mathcal{C}$ and the intersection $\bigcap_i T_{I_i}^\vee \subset \mathrm{Id}_\mathcal{C}$ exist. Then $\sum_i I_i$ is again an ideal functor.*

Proof. The cokernel of $\sum_i I_i \hookrightarrow \mathrm{Id}_\mathcal{C}$ is the pushout of adjunctions $\mathrm{Id}_\mathcal{C} \twoheadrightarrow T_{I_i}$ under $\mathrm{Id}_\mathcal{C}$, which is left adjoint to the pullback $\bigcap_i T_{I_i}^\vee$ by Proposition 1.13. □

Remark 2.25. In contrast, the intersection of ideal functors is not an ideal functor. This is because even in the module category of a ring, the equation $(I \cap J)V = IV \cap JV$ does not hold in general.

In particular, a finite sum always exists. We can represent finite sum in another way as follows.

Proposition 2.26. *Let $I$ and $J$ be ideal functors on $\mathcal{C}$. Then $T_I T_J = T_{I+J}$.*

Proof. We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, two heads] & \mathrm{Id}_\mathcal{C} \arrow[r] \arrow[d, two heads] & T_I \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & IT_J \arrow[r] & T_J \arrow[r] & T_I T_J \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact since the right multiplication of $T_J$ is exact, and $I \to IT_J$ is epic since $I$ preserves images. These properties imply that the right square is cocartesian. In other words, $T_I T_J$ is the pushout of $T_I$ and $T_J$ under $\mathrm{Id}_\mathcal{C}$. □

Secondly we consider the product of two ideal functors defined by composition.

Lemma 2.27. *For ideal functors $I$ and $J$ on $\mathcal{C}$, we have $T_I J = J T_{IJ}$.*

Here we let $T_{IJ} := \mathrm{Coker}(IJ \hookrightarrow \mathrm{Id}_\mathcal{C})$ though we have not yet prove that $IJ$ is an ideal functor.

Proof. Let us consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & IJ \arrow[r] \arrow[d, equal] & J \arrow[r] \arrow[d, hook] & T_I J \arrow[r] \arrow[d, hook, dashed] & 0 \\ 0 \arrow[r] & IJ \arrow[r] & \mathrm{Id}_\mathcal{C} \arrow[r] & T_{IJ} \arrow[r] & 0 \end{tikzcd}$$

whose rows are exact. The commutative square induces $T_I J \to T_{IJ}$ which is monic by the four lemma. On the other hand, we have another commutative diagram

$$\begin{tikzcd} J \arrow[r, two heads] \arrow[d, hook] & JT_{IJ} \arrow[d, hook] \\ \mathrm{Id}_\mathcal{C} \arrow[r, two heads] & T_{IJ}. \end{tikzcd}$$

where $J \to JT_{IJ}$ is epic and $JT_{IJ} \to T_{IJ}$ is monic. Thus we have two epi-mono factorization of the diagonal morphism so that its images must be equal. □

Lemma 2.28. *For $I$ and $J$ as above, we have $I^\circ J = JI^\circ$.*

Proof. Follows in a similar way from that both are the image of $J \hookrightarrow \mathrm{Id}_{\mathcal{C}} \twoheadrightarrow I^\circ$. ☐

Proposition 2.29. *Let $I$ and $J$ be ideal functors on $\mathcal{C}$. Then $IJ$ is also an ideal functor such that $(IJ)^\circ = J^\circ I^\circ$.*

Proof. Let $T_{IJ} \coloneqq \mathrm{Coker}(IJ \hookrightarrow \mathrm{Id}_{\mathcal{C}})$ and $T_{IJ}^\vee \coloneqq \mathrm{Ker}(\mathrm{Id}_{\mathcal{C}} \twoheadrightarrow J^\circ I^\circ)$. It suffices to prove that these functors actually form an adjunction. By the lemma above, we have $IJT_{IJ} = IT_IJ = 0$. So $JT_{IJ}$ is $I$-annihilated and this implies $JI^\circ T_{IJ} = I^\circ JT_{IJ} = 0$. Thus $I^\circ T_{IJ}$ is $J$-annihilated so $J^\circ I^\circ T_{IJ} = 0$. Equivalently, we have $T_{IJ}^\vee T_{IJ} = T_{IJ}$. We can prove $T_{IJ}T_{IJ}^\vee = T_{IJ}^\vee$ in a similar way. Using these isomorphisms, we can define the unit $\mathrm{Id}_{\mathcal{C}} \twoheadrightarrow T_{IJ} = T_{IJ}^\vee T_{IJ}$ and the counit $T_{IJ}T_{IJ}^\vee = T_{IJ}^\vee \hookrightarrow \mathrm{Id}_{\mathcal{C}}$. Now it is obvious that these morphisms satisfy the zig-zag identities. ☐

Lastly we study about quotient of ideal functors.

Definition 2.30. Let $I$ and $J$ be two ideal functors on $\mathcal{C}$ such that $J \subset I \subset \mathrm{Id}_{\mathcal{C}}$. Since $\mathcal{C}_J$ is closed under subobjects, $X \in \mathcal{C}_J$ implies $IX \in \mathcal{C}_J$. We denote this restricted functor of $I$ by $I_J \colon \mathcal{C}_J \to \mathcal{C}_J$.

Recall that we denote by $\Phi_J \colon \mathcal{C}_J \to \mathcal{C}$ the embedding of abelian category, and the endofunctor category can be also exactly embedded as

$$\mathcal{E}nd(\mathcal{C}_J) \to \mathcal{E}nd(\mathcal{C}),$$
$$F \mapsto \Phi_J F \Phi_J^\wedge.$$

Lemma 2.31. *Let $I$ and $J$ be as above. Then*

$$\Phi_J I_J \Phi_J^\wedge = IT_J = \mathrm{Ker}(T_J \twoheadrightarrow T_I) \simeq I/J.$$

Proof. By definition, $\Phi_J I_J = I\Phi_J$ so $\Phi_J I_J \Phi_J^\wedge = IT_J$. Since $T_IT_J = T_{I+J} = T_I$, it is equal to

$$IT_J = \mathrm{Ker}(T_J \twoheadrightarrow T_IT_J) = \mathrm{Ker}(T_J \twoheadrightarrow T_I).$$

The last isomorphism is obvious. ☐

Proposition 2.32. *For $I$ and $J$ as above, $I_J$ is an ideal functor on $\mathcal{C}_J$. Moreover every ideal functors on $\mathcal{C}_J$ is obtained in this way, and ideal functors on $\mathcal{C}_J$ are in one-to-one correspondence with those on $\mathcal{C}$ which contain $J$.*

Proof. Via the above embedding, $\mathrm{Coker}(I_J \hookrightarrow \mathrm{Id}_{\mathcal{C}_J})$ is mapped to an adjunction $\mathrm{Coker}(IT_J \hookrightarrow T_J) = T_I$. Thus $I_J$ is an ideal functor and $I$ is recovered from $I_J$ in this way.

Conversely, suppose that $K$ is an ideal functor on $\mathcal{C}_J$. Then $T_{\tilde{K}} \coloneqq \Phi_J T_K \Phi_J^\wedge$ is a quotient adjunction of $T_J$ so $\tilde{K} \coloneqq \mathrm{Ker}(\mathrm{Id}_{\mathcal{C}} \twoheadrightarrow T_J \twoheadrightarrow T_{\tilde{K}})$ is an ideal functor which contains $J$. Moreover we have

$$\tilde{K}\Phi_J = \mathrm{Ker}(\Phi_J \twoheadrightarrow T_{\tilde{K}}\Phi_J) = \mathrm{Ker}(\Phi_J \twoheadrightarrow \Phi_J T_K) = \Phi_J K,$$

which means that $\tilde{K}_J \simeq K$. Thus these operations gives a one-to-one correspondence up to unique isomorphism. ☐

Now the next statements are clear.

Proposition 2.33. *Let $I, J \subset \mathrm{Id}_{\mathcal{C}}$ be ideal functors. Then*

1. $\mathcal{C}_{I+J} = \mathcal{C}_I \cap \mathcal{C}_J$,
2. $\mathcal{C}^{IJ} = \mathcal{C}^I \cap \mathcal{C}^J$,
3. *if $J \subset I$ then $(\mathcal{C}_J)_{I_J} = \mathcal{C}_I$ and $(\mathcal{C}_J)^{I_J} = \mathcal{C}^I \cap \mathcal{C}_J$.* ☐

## 4. Compatibility with extension

In this section we fix an abelian category $\mathcal{C}$ and an ideal functor $I \subset \mathrm{Id}_{\mathcal{C}}$. For each pair of $X, Y \in \mathcal{C}$, let us denote $\mathrm{Ext}^i_{\mathcal{C}}(X,Y)_0 \coloneqq \mathrm{Ext}^i_{\mathcal{C}_0}(X,Y)$ the Ext group taken in $\mathcal{C}_0$. It is defined as the set of equivalence classes of exact sequences

$$0 \to Y \to E_i \to \cdots \to E_1 \to A \to 0.$$

We also define the graded Ext group $\mathrm{Ext}^i_{\mathcal{C}}(X,Y)$ as

$$\mathrm{Ext}^i_{\mathcal{C}}(X,Y) \coloneqq \mathrm{Ext}^i_{\mathcal{C}}(X,Y)_0 \oplus \mathrm{Ext}^i_{\mathcal{C}}(X,\Pi Y)_0$$

for the super case $\mathcal{V} = \mathcal{S}$ and

$$\mathrm{Ext}^i_{\mathcal{C}}(X,Y) \coloneqq \bigoplus_k \mathrm{Ext}^i_{\mathcal{C}}(X,\Sigma^k Y)_0$$

for the graded case $\mathcal{V} = \mathcal{G}$.

Every exact sequences in $\mathcal{C}_I$ are still exact in $\mathcal{C}$, so we have a canonical map

$$\mathrm{Ext}^i_{\mathcal{C}_I}(X,Y) \to \mathrm{Ext}^i_{\mathcal{C}}(\Phi_I X, \Phi_I Y).$$

However this map is rarely an isomorphism because when we take an exact sequence in $\mathcal{C}$ whose both ends are in $\mathcal{C}_I$, the rest terms do not need to belong in $\mathcal{C}_I$. In this section we give some characterizations of that $\Phi_I$ preserves Ext functors. These results are a reformulation of those in [**APT92**].

In a module category case, the condition for $\mathrm{Ext}^1$ is well-known: it is equivalent to that the ideal is idempotent. We can easily generalize this fact as follows.

Proposition 2.34. *The followings are equivalent.*

(1) $\mathrm{Ext}^1_{\mathcal{C}_I}(X,Y) \simeq \mathrm{Ext}^1_{\mathcal{C}}(X,Y)$ *for every* $X, Y \in \mathcal{C}_I$,
(2) $\mathcal{C}_I$ *is closed under extensions,*
(3) $I^2 = I$.

Proof. (1) $\Leftrightarrow$ (2) is immediate by definition. Now let $0 \to X \to Y \to Z \to 0$ be a short exact sequence in $\mathcal{C}$ and suppose $X, Z \in \mathcal{C}_I$. Then clearly $I^2 Y = 0$ so (3) implies (2). Conversely assume that (3) fails so that $I^2 \subsetneq I$. Then there exists $X \in \mathcal{C}$ such that $I^2 X \subsetneq IX$. By $T_I I = I T_{I^2}$ the sequence

$$0 \to T_I I X \to T_{I^2} X \to T_I X \to 0$$

is exact. Both the left and the right term is in $\mathcal{C}_I$ but the middle is not since $I^2 X \subsetneq IX$ implies $I T_{I^2} X = T_I I X \neq 0$. Hence (2) does not hold, so we have (2) $\Leftrightarrow$ (3). $\square$

Lemma 2.35. *Suppose that $\mathcal{C}$ has enough projectives.*

(1) *If $P \in \mathcal{C}$ is projective, then so is $\Phi_I^\wedge P \in \mathcal{C}_I$.*
(2) *$\mathcal{C}_I$ also has enough projectives.*
(3) *Each projective object in $\mathcal{C}_I$ is a direct summand of $\Phi_I^\wedge P$ for some projective object $P \in \mathcal{C}$.*

Proof. (1) follows from that its right adjoint $\Phi_I$ is exact. For any $X \in \mathcal{C}_I$, there is a projective object $P \in \mathcal{C}$ and an epimorphism $P \twoheadrightarrow \Phi_I X$ which induces $\Phi_I^\wedge P \twoheadrightarrow X$ so (2) and (3) follow from (1). $\square$

In the rest of this section, we require that $\mathcal{C}$ has enough projectives and injectives in order to use its homological properties. Then we have that $\mathrm{Ext}^i_{\mathcal{C}}(X, \bullet)$ and $\mathrm{Ext}^i_{\mathcal{C}}(\bullet, Y)$ are the $i$-th left derived functors of $\mathrm{Hom}_{\mathcal{C}}(X, \bullet)$ and $\mathrm{Hom}_{\mathcal{C}}(\bullet, Y)$ respectively.

Lemma 2.36. *Let $2 \le k \le \infty$. For $X \in \mathcal{C}$, the followings are equivalent.*

(1) $\mathrm{Ext}^i_{\mathcal{C}_I}(\Phi_I^\wedge X, Y) \simeq \mathrm{Ext}^i_{\mathcal{C}}(X, \Phi_I Y)$ *for any* $Y \in \mathcal{C}_I$ *and* $0 \le i < k$,

(2) $\mathrm{Ext}^i_{\mathcal{C}}(X, \Phi_I Q) = 0$ *for any injective* $Q \in \mathcal{C}_I$ *and* $1 \leq i < k$,
(3) $(L_i \Phi_I^\wedge)X = 0$ *for any* $1 \leq i < k$,
(4) $(L_0 I)X \simeq IX$ *and* $(L_i I)X = 0$ *for any* $1 \leq i < k-1$.

Proof. Let $P_k \to \cdots \to P_1 \to P_0 \to X \to 0$ be a projective resolution of $X$. First (1) $\Rightarrow$ (2) is trivial. Assume (2) then it implies

$$0 \to \mathrm{Hom}_{\mathcal{C}_I}(\Phi_I^\wedge X, Q) \to \mathrm{Hom}_{\mathcal{C}_I}(\Phi_I^\wedge P_0, Q) \to \cdots \to \mathrm{Hom}_{\mathcal{C}_I}(\Phi_I^\wedge P_k, Q)$$

is exact for any injective $Q \in \mathcal{C}_I$. Since $\mathcal{C}_I$ has enough injectives by the dual of Lemma 2.35, the sequence

$$\Phi_I^\wedge P_k \to \cdots \to \Phi_I^\wedge P_1 \to \Phi_I^\wedge P_0 \to \Phi_I^\wedge X \to 0$$

must be exact so (3) holds. Conversely, if (3) is satisfied, the sequence above is a projective resolution of $\Phi_I^\wedge X$. Thus

$$\begin{aligned}\mathrm{Ext}^i_{\mathcal{C}_I}(\Phi_I^\wedge X, Y) &\simeq H_i(\mathrm{Hom}_{\mathcal{C}_I}(\Phi_I^\wedge P_i, Y))\\ &\simeq H_i(\mathrm{Hom}_{\mathcal{C}}(P_i, \Phi_I Y))\\ &\simeq \mathrm{Ext}^i_{\mathcal{C}}(X, \Phi_I Y)\end{aligned}$$

so (1) holds. Finally we have $\Phi_I(L_i \Phi_I^\wedge) = L_i T_I$ since $\Phi_I$ is exact. Hence (3) and (4) are equivalent by that the sequence

$$0 \to L_1 T_I \to L_0 I \to \mathrm{Id}_{\mathcal{C}} \to T_I \to 0$$

is exact and that $L_i T_I \simeq L_{i-1} I$. □

One can easily check that if every $X \in \mathcal{C}$ satisfies the above conditions for $k = 2$, then it is also true for $k = \infty$. In this situation, we can rewrite the conditions as follows.

Corollary 2.37. *The followings are equivalent.*

(1) $\mathrm{Ext}^i_{\mathcal{C}_I}(\Phi_I^\wedge X, Y) \simeq \mathrm{Ext}^i_{\mathcal{C}}(X, \Phi_I Y)$ *for any* $X \in \mathcal{C}$, $Y \in \mathcal{C}_I$ *and* $i \geq 0$,
(2) $\Phi_I$ *sends injectives to injectives,*
(3) $\Phi_I^\wedge$ *is exact,*
(4) $I$ *is an adjunction.* □

However this condition is too strong for our purpose because we only need objects of the form $\Phi_I X \in \mathcal{C}$ for $X \in \mathcal{C}_I$. Now we state a criteria of Ext preserving property for ideal functors.

Proposition 2.38. *Let* $2 \leq k \leq \infty$. *Then the followings are equivalent.*

(1) $\mathrm{Ext}^i_{\mathcal{C}_I}(X, Y) \simeq \mathrm{Ext}^i_{\mathcal{C}}(\Phi_I X, \Phi_I Y)$ *for any* $X, Y \in \mathcal{C}_I$ *and* $0 \leq i < k$,
(2) $\mathrm{Ext}^i_{\mathcal{C}}(\Phi_I P, \Phi_I Q) = 0$ *for any projective* $P \in \mathcal{C}_I$, *injective* $Q \in \mathcal{C}_I$ *and* $1 \leq i < k$,
(3) $(L_i \Phi_I^\wedge)\Phi_I = 0$ *for any* $1 \leq i < k$,
(4) $(L_i \Phi_I^\wedge)\Phi_I P = 0$ *for any projective* $P \in \mathcal{C}_I$ *and* $1 \leq i < k$.
(5) $(L_i I)\Phi_I = 0$ *for any* $0 \leq i < k-1$,
(6) $(L_i I)\Phi_I P = 0$ *for any projective* $P \in \mathcal{C}_I$ *and* $0 \leq i < k-1$.

*When* $k \geq 3$, *it is also equivalent to that:*

(7) $(L_0 I)^2 \simeq L_0 I$, *and* $(L_i I)IP = 0$ *for any projective* $P \in \mathcal{C}$ *and* $1 \leq i < k-2$.

*Since the conditions (1) and (2) are self-dual, we can replace the rest conditions by their dual statements.*

PROOF. (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) and (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (6) follow from the previous lemma. (3) $\Rightarrow$ (4) is obvious. Now suppose (4). Let $X \in \mathcal{C}$ and take an exact sequence $0 \to K \to P \to X \to 0$ with $P$ projective. By applying $(L_i\Phi_I^\wedge)\Phi_I$ it yields the long exact sequence

$$\cdots \to (L_1\Phi_I^\wedge)\Phi_I K \to (L_1\Phi_I^\wedge)\Phi_I P \to (L_1\Phi_I^\wedge)\Phi_I X \to K \to P \to X \to 0.$$

Then the assumption implies $(L_1\Phi_I^\wedge)\Phi_I X = 0$. Moreover we have $(L_i\Phi_I^\wedge)\Phi_I X \simeq (L_{i-1}\Phi_I^\wedge)\Phi_I K$ for each $2 \leq i < k$ so by induction all of them must be zero. Hence (4) implies (3).

Now suppose $k \geq 3$ and we prove that (6) is also equivalent to (7). First by Lemma 2.35, (6) can be replaced by

(6') $(L_iI)T_IP = 0$ for any projective $P \in \mathcal{C}$ and $0 \leq i < k-1$.

Applying $L_iI$'s to the exact sequence $0 \to IP \to P \to T_IP \to 0$, we obtain that it is equivalent to that $(L_0I)IP \simeq IP$ and $(L_iI)IP = 0$ for $1 \leq i < k-2$. Moreover, the first condition is equivalent to that $(L_0I)^2 \simeq L_0I$ since a right exact functor is determined by the values on projectives which generate whole $\mathcal{C}$. $\square$

Hence it also yields a well-known statement as a special case: for an algebra $A$ and its 2-sided ideal $I \subset A$, $\mathrm{Ext}^2_{A/I} = \mathrm{Ext}^2_A$ if and only if the multiplication $I \otimes_A I \to I$ is an isomorphism.

EXAMPLE 2.39. Suppose $I^2 = I$ and $I$ sends projectives to projectives. Then for $k = \infty$ the condition (7) above is easily verified.

## 5. Ideal filters

In this section we consider a family of ideal functors indexed by a partially ordered set $(\Lambda, \leq)$. Such situation occurs mainly in the study of cellular algebras or quasi-hereditary algebras. In this section we fix an abelian category $\mathcal{C}$ which is closed under sums and intersections of subobjects with cardinality $\#\Lambda$.

DEFINITION 2.40. An *ideal filter* on $\mathcal{C}$ indexed by $(\Lambda, \leq)$ is a family of ideal functors $\{I^{\leq\lambda}\}_{\lambda\in\Lambda}$ which satisfies these three conditions:

$$I^{\leq\lambda} \subset I^{\leq\mu} \quad \text{if } \lambda \leq \mu, \qquad \mathrm{Id}_{\mathcal{C}} = \sum_{\lambda} I^{\leq\lambda}, \qquad I^{\leq\lambda}I^{\leq\mu} \subset \sum_{\nu\leq\lambda,\mu} I^{\leq\nu}.$$

From now on $\{I^{\leq\lambda}\}$ denotes an ideal filter on $\mathcal{C}$ indexed by $(\Lambda, \leq)$.

NOTATION 2.41. For each $\lambda \in \Lambda$, we define

$$I^{<\lambda} := \sum_{\mu<\lambda} I^{\leq\mu}.$$

When an ideal filter on $\mathcal{C}$ is fixed, we write $\mathcal{C}^{\leq\lambda} := \mathcal{C}^{I^{\leq\lambda}}$ and $\mathcal{C}_{<\lambda} := \mathcal{C}_{I^{<\lambda}}$ for short. We also denote $\mathcal{C}[\lambda] := \mathcal{C}^{\leq\lambda} \cap \mathcal{C}_{<\lambda}$ and $\mathrm{Irr}\,\mathcal{C}[\lambda] := \{V \in \mathrm{Irr}\,\mathcal{C} \mid V \in \mathcal{C}[\lambda]\}$.

The purpose of introducing an ideal filter is to divide the category into a small subcategories as follows.

LEMMA 2.42. *If $\lambda \neq \mu$ we have $\mathcal{C}[\lambda] \cap \mathcal{C}[\mu] = \{0\}$, so that $\mathrm{Irr}\,\mathcal{C}[\lambda] \cap \mathrm{Irr}\,\mathcal{C}[\mu] = \varnothing$.*

PROOF. Suppose that $X \in \mathcal{C}[\lambda] \cap \mathcal{C}[\mu]$. If $\lambda < \mu$, we have

$$X = I^{\leq\lambda}X \subset I^{<\mu}X = 0.$$

Otherwise $\lambda \not\leq \mu$. Then

$$X = I^{\leq\lambda}I^{\leq\mu}X \subset \sum_{\nu\leq\lambda,\mu} I^{\leq\nu}X \subset I^{<\lambda}X = 0.$$

In either case, we have that $X = 0$. $\square$

A partially ordered set $(\Lambda, \leq)$ is said to be *well-founded* if its every non-empty subset has a minimal element. Under the axiom of choice, this property is equivalent to the conditions below:

(1) there are no infinite descending chains $\lambda_1 > \lambda_2 > \cdots$,
(2) there is a well-ordering extension of $\leq$.

PROPOSITION 2.43. *Suppose that $\Lambda$ is well-founded. Then*

$$\operatorname{Irr}\mathcal{C} = \bigsqcup_{\lambda\in\Lambda} \operatorname{Irr}\mathcal{C}[\lambda].$$

PROOF. Suppose that $X \in \mathcal{C}$ is simple. Since we have $0 \neq X = \sum_\lambda I^{\leq\lambda}X$, the set $\{\lambda \in \Lambda \mid I^{\leq\lambda}X \neq 0\}$ is non-empty. By the assumption it has a minimal element $\lambda \in \Lambda$, so that $X \in \mathcal{C}[\lambda]$. □

In practice we can choose a partially ordered set from various choices to obtain a same result. First observe that we can remove redundant indices from $\Lambda$.

LEMMA 2.44. *Suppose that there is a subset $\Lambda_0 \subset \Lambda$ such that $\mathrm{Id}_{\mathcal{C}} = \sum_{\lambda\in\Lambda_0} I^{\leq\lambda}$. Let $\Lambda' \coloneqq \{\mu \in \Lambda \mid \exists\lambda \in \Lambda_0$ s.t. $\mu \leq \lambda\}$ be the order ideal generated by $\Lambda_0$. Then*

(1) *$\{I^{\leq\lambda}\}_{\lambda\in\Lambda'}$ is also an ideal filter indexed by $(\Lambda', \leq)$,*
(2) *$I^{<\lambda} = I^{\leq\lambda}$ unless $\lambda \in \Lambda'$, so that $\mathcal{C}[\lambda] = \{0\}$.*

PROOF. (1) is obvious. Suppose $\lambda \notin \Lambda'$. Then

$$I^{\leq\lambda} \subset \sum_{\mu\in\Lambda'} I^{\leq\mu}I^{\leq\lambda} \subset \sum_{\mu\in\Lambda'}\sum_{\nu\leq\lambda,\mu} I^{\leq\nu} \subset I^{<\lambda}$$

so (2) holds. □

Next we consider how these subcategories will be affected when we strengthen the order on $\Lambda$.

LEMMA 2.45. *Let $\trianglelefteq$ be an extension of $\leq$, that is, another partial ordering on $\Lambda$ such that $\lambda \leq \mu$ implies $\lambda \trianglelefteq \mu$. For each $\lambda \in \Lambda$, define*

$$I^{\trianglelefteq\lambda} \coloneqq \sum_{\mu\trianglelefteq\lambda} I^{\leq\mu}.$$

*Then*

(1) *$\{I^{\trianglelefteq\lambda}\}$ is also an ideal filter indexed by $(\Lambda, \trianglelefteq)$,*
(2) *$\mathcal{C}^{\leq\lambda} \cap \mathcal{C}_{<\lambda} = \mathcal{C}^{\trianglelefteq\lambda} \cap \mathcal{C}_{\triangleleft\lambda}$,*
(3) *there is a surjective morphism $I^{\leq\lambda}/I^{<\lambda} \twoheadrightarrow I^{\trianglelefteq\lambda}/I^{\triangleleft\lambda}$.*

PROOF. Let us check the conditions in the definition for $\{I^{\trianglelefteq\lambda}\}$. The first two clearly hold. Since ideal functors commute with summation, we also have the third one

$$I^{\trianglelefteq\lambda}I^{\trianglelefteq\mu} = \sum_{\nu\trianglelefteq\lambda,\,\pi\trianglelefteq\mu} I^{\leq\nu}I^{\leq\pi} \subset \sum_{\nu\trianglelefteq\lambda,\,\pi\trianglelefteq\mu}\sum_{\rho\leq\nu,\pi} I^{\leq\rho} \subset \sum_{\nu\trianglelefteq\lambda,\,\pi\trianglelefteq\mu}\sum_{\rho\trianglelefteq\nu,\pi} I^{\trianglelefteq\rho} = \sum_{\rho\trianglelefteq\lambda,\mu} I^{\trianglelefteq\rho}.$$

Thus $\{I^{\trianglelefteq\lambda}\}$ is an ideal filter.

In order to prove (2), first note that

$$I^{\triangleleft\lambda} = \sum_{\mu\triangleleft\lambda} I^{\trianglelefteq\mu} = \sum_{\mu\triangleleft\lambda}\sum_{\nu\trianglelefteq\mu} I^{\leq\mu} = \sum_{\nu\triangleleft\lambda} I^{\leq\nu}.$$

Suppose that $X \in \mathcal{C}^{\leq\lambda} \cap \mathcal{C}_{<\lambda}$. Since $I^{\leq\lambda} \subset I^{\trianglelefteq\lambda}$, clearly $X \in \mathcal{C}^{\trianglelefteq\lambda}$. Moreover,

$$I^{\triangleleft\lambda}X = I^{\triangleleft\lambda}I^{\leq\lambda}X = \sum_{\mu\triangleleft\lambda} I^{\leq\mu}I^{\leq\lambda}X \subset \sum_{\mu\triangleleft\lambda}\sum_{\nu\leq\lambda,\mu} I^{\leq\nu}X \subset I^{<\lambda}X = 0$$

so $X \in \mathcal{C}_{\lhd\lambda}$. Conversely, suppose that $X \in \mathcal{C}^{\unlhd\lambda} \cap \mathcal{C}_{\lhd\lambda}$. Then $I^{<\lambda} \subset I^{\lhd\lambda}$ immediately implies that $X \in \mathcal{C}_{<\lambda}$. We also have

$$X = I^{\unlhd\lambda}X = I^{\leq\lambda}X + I^{\lhd\lambda}X = I^{\leq\lambda}X,$$

that is, $X$ is $I^{\leq\lambda}$-accessible. We can prove that $X$ is $I^{\leq\lambda}$-torsion-free in a similar manner so $X \in \mathcal{C}^{\leq\lambda}$. Thus $\mathcal{C}^{\leq\lambda} \cap \mathcal{C}_{<\lambda} = \mathcal{C}^{\unlhd\lambda} \cap \mathcal{C}_{\lhd\lambda}$.

Finally (3) follows from $I^{\leq\lambda} \subset I^{\unlhd\lambda}$, $I^{<\lambda} \subset I^{\lhd\lambda}$ and $I^{\unlhd\lambda} = I^{\leq\lambda} + I^{\lhd\lambda}$. □

Hence the notation $\mathcal{C}[\lambda] := \mathcal{C}^{\leq\lambda} \cap \mathcal{C}_{<\lambda}$ does not depend on taking extension of ordering. Unfortunately, the functor $I^{\leq\lambda}/I^{<\lambda}$ *does* change by extension of ordering. The condition for stability of this functor is described as follows.

Definition 2.46. For each $\lambda$, let

$$I^{\ngeq\lambda} := \sum_{\mu\ngeq\lambda} I^{\leq\mu}.$$

An ideal filter $\{I^{\leq\lambda}\}$ is said to be *rigid* if it satisfies $I^{\leq\lambda} \cap I^{\ngeq\lambda} = I^{<\lambda}$ for every $\lambda$.

Note that the condition $I^{\leq\lambda} \cap I^{\ngeq\lambda} \supset I^{<\lambda}$ is always satisfied. Clearly if $\leq$ is a total order then every ideal filter is rigid.

Proposition 2.47. *Suppose $\{I^{\leq\lambda}\}$ is rigid. Then for any extension $\unlhd$ of $\leq$,*

(1) *$\{I^{\unlhd\lambda}\}$ is also rigid,*
(2) *the canonical morphism $I^{\leq\lambda}/I^{<\lambda} \twoheadrightarrow I^{\unlhd\lambda}/I^{\lhd\lambda}$ is an isomorphism.*

Proof. Note that $I^{\lhd\lambda} \subset I^{\ntrianglerighteq\lambda} \subset I^{\ngeq\lambda}$. These inclusions imply

$$I^{\unlhd\lambda} \cap I^{\ntrianglerighteq\lambda} = (I^{\leq\lambda} + I^{\lhd\lambda}) \cap I^{\ntrianglerighteq\lambda} = (I^{\leq\lambda} \cap I^{\ntrianglerighteq\lambda}) + I^{\lhd\lambda} \subset (I^{\leq\lambda} \cap I^{\ngeq\lambda}) + I^{\lhd\lambda}.$$

By the assumption, the right hand side is equal to $I^{<\lambda} + I^{\lhd\lambda} = I^{\lhd\lambda}$, so $\{I^{\unlhd\lambda}\}$ is rigid. Moreover

$$I^{\leq\lambda} \cap I^{\lhd\lambda} \subset I^{\leq\lambda} \cap I^{\ngeq\lambda} = I^{<\lambda}$$

so $I^{\leq\lambda} \cap I^{\lhd\lambda} = I^{<\lambda}$. Thus the morphism $I^{\leq\lambda}/I^{<\lambda} \twoheadrightarrow I^{\unlhd\lambda}/I^{\lhd\lambda}$ is an isomorphism since its kernel is $(I^{\leq\lambda} \cap I^{\lhd\lambda})/I^{<\lambda} = 0$. □

When an ideal filter is rigid we have an additional result on simple objects.

Proposition 2.48. *Suppose $\{I^{\leq\lambda}\}$ is rigid and $\Lambda$ is well-founded. Then in the Grothendieck group of finite length objects in $\mathcal{C}$, we have the equation*

$$[X] = \sum_{\lambda\in\Lambda} [I^{\leq\lambda}X/I^{<\lambda}X].$$

Proof. By the proposition above, by taking its extension we may assume that $\leq$ is a well-ordering. Suppose $X \in \mathcal{C}$ is of finite length. Let us denote by $l(Y)$ the length of $Y \in \mathcal{C}$. Since $X = \sum_\lambda I^{\leq\lambda}X$, the set $\{\lambda \in \Lambda \,|\, l(I^{\leq\lambda}X) \geq k\}$ is not empty for each $0 \leq k \leq l(X)$. Let $\lambda_k$ be its minimum element. Then we have

$$I^{<\lambda_k}X = \begin{cases} 0 & \text{if } k = 0, \\ I^{\leq\lambda_{k-1}}X & \text{if } \lambda_k \neq \lambda_{k-1}. \end{cases}$$

Hence by taking the composition series

$$0 \subset I^{\leq\lambda_0}X \subset I^{\leq\lambda_1}X \subset \cdots \subset I^{\leq\lambda_{l(X)}}X = X$$

we have

$$[X] = [I^{\leq\lambda_0}X] + \sum_{1\leq k\leq l(X)} [I^{\leq\lambda_k}X/I^{\leq\lambda_{k-1}}X] = \sum_{\lambda\in\Lambda_0} [I^{\leq\lambda}X/I^{<\lambda}X]$$

where $\Lambda_0 = \{\lambda_0, \lambda_1, \ldots, \lambda_{l(X)}\}$ (overlapping elements are excluded). It is also clear that $I^{<\lambda}X = I^{\leq\lambda}X$ when $\lambda \notin \Lambda_0$, so the statement holds. □

CHAPTER 3

# Morita context between abelian categories

The classical Morita theory [**Mor58**] treats a category equivalence between respective module categories of two rings $A$ and $B$. It is performed as a tensor functor $P \otimes_B \bullet \colon B\text{-}\mathcal{M}od \to A\text{-}\mathcal{M}od$ and a hom functor $\mathrm{Hom}_A(P, \bullet) \colon A\text{-}\mathcal{M}od \to B\text{-}\mathcal{M}od$ by use of a progenerator $P$, which is an $(A, B)$-bimodule such that finitely generated and projective as both left and right modules. To make this correspondence symmetric, we can take a $(B, A)$-bimodule $P' := \mathrm{Hom}_A(P, A)$ and rewrite $\mathrm{Hom}_A(P, \bullet) \simeq P' \otimes_A \bullet$. A Morita context between rings is a weaker notion of Morita equivalence consists of such pair $(P, P')$, which still provides an equivalence between certain full subcategories of the module categories. We here introduce a more generalized notion, a Morita context between two abelian categories.

## 1. Morita context and its trace ideals

Definition 3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories. A *Morita context* between $\mathcal{C}$ and $\mathcal{D}$ is a pair of adjunctions $F \colon \mathcal{D} \to \mathcal{C}$ and $G \colon \mathcal{C} \to \mathcal{D}$ equipped with two degree-zero natural transformations $\eta \colon FG \to \mathrm{Id}_{\mathcal{C}}$ and $\rho \colon GF \to \mathrm{Id}_{\mathcal{D}}$ such that $F\rho = \eta F$ as morphisms $FGF \rightrightarrows F$ and $\rho G = G\eta$ as $GFG \rightrightarrows G$. These equations are called the associativity laws.

When $\mathcal{C} = A\text{-}\mathcal{M}od$ and $\mathcal{D} = B\text{-}\mathcal{M}od$ are respectively the module categories of algebras $A$ and $B$, the above definition of Morita context between $\mathcal{C}$ and $\mathcal{D}$ coincides with Definition 0.2 of that between $A$ and $B$ we introduced in the introduction.

Remark 3.2. Iglesias and Torrecillas [**IT95, IT98**] has defined a more general notion called *wide (right) Morita context*. They only required that $F$ and $G$ are right exact.

For a while we fix a Morita context $(F, G)$ between $\mathcal{C}$ and $\mathcal{D}$ as above.

Notation 3.3. We denote by

$$\begin{aligned} &\bar{\eta} \colon G \to F^\vee, && \bar{\rho} \colon F \to G^\vee, \\ &\eta^\vee \colon \mathrm{Id}_{\mathcal{C}} \to G^\vee F^\vee, && \rho^\vee \colon \mathrm{Id}_{\mathcal{D}} \to F^\vee G^\vee \end{aligned}$$

the morphisms induced by adjunctions. Let $D$ and $D'$ be the images of $\bar{\eta} \colon G \to F^\vee$ and $\bar{\rho} \colon F \to G^\vee$ respectively.

The functors $D \colon \mathcal{C} \to \mathcal{D}$ and $D' \colon \mathcal{D} \to \mathcal{C}$ are called *Morita context functors*. Similarly as ideal functors, a Morita context functor is not left nor right exact. However it has a following property again.

Lemma 3.4. *Morita context functors preserve all images.*

Proof. $D$ is both mono and epi since it is a subobject of a left exact functor $F^\vee$ as well as a quotient of a right exact functor $G$. $\square$

Let $I \subset \mathrm{Id}_{\mathcal{C}}$ and $J \subset \mathrm{Id}_{\mathcal{D}}$ be the images of $\eta \colon FG \to \mathrm{Id}_{\mathcal{C}}$ and $\rho \colon GF \to \mathrm{Id}_{\mathcal{D}}$ respectively. These are ideal functors on the respective categories by Proposition 2.6, which we call *trace ideals*. First we study how these functors act on the subcategories defined by these ideal functors $I$ and $J$.

LEMMA 3.5. *Suppose $X \in \mathcal{C}$ and consider three morphisms $\eta X \colon FGX \to X$, $\bar{\eta} X \colon GX \to F^\vee X$ and $\eta^\vee X \colon X \to G^\vee F^\vee X$.*

(1) *$X$ is $I$-accessible if and only if $\eta X$ is epic.*
(2) *$X$ is $I$-torsion-free if and only if $\eta^\vee X$ is monic.*
(3) *$X$ is $I$-annihilated if and only if $\eta X = 0$ (equivalently, $\bar{\eta} X = 0$ or $\eta^\vee X = 0$). In particular, it is also equivalent to that $DX = \mathrm{Image}(\bar{\eta} X) = 0$.*

PROOF. Obvious by definition. □

LEMMA 3.6. *Suppose $X \in \mathcal{C}$.*

(1) *If $X$ is $I$-accessible then $GX$ is $J$-accessible.*
(2) *If $X$ is $I$-torsion-free then $F^\vee X$ is $J$-torsion-free.*
(3) *If $X$ is $I$-annihilated then both $GX$ and $F^\vee X$ are $J$-annihilated.*

*In particular, so is $DX$ in each cases.*

PROOF. Since $\rho GX \colon GFGX \to GX$ is equal to $G\eta X$ and $G$ is right exact, if $\eta X$ is epic then so is $\rho GX$. This means that if $X$ is $I$-accessible then $GX$ is $J$-accessible by the lemma above. (2) and (3) can be proven in a similar manner. The last statement follows from that these properties are inherited to subobjects or quotients. □

LEMMA 3.7. $\mathrm{Coker}(DX \hookrightarrow F^\vee X)$ *is $J$-annihilated for any $X \in \mathcal{C}$.*

PROOF. Let $C \coloneqq \mathrm{Coker}(D \hookrightarrow F^\vee) = \mathrm{Coker}(G \to F^\vee)$ and consider the commutative diagram

$$\begin{tikzcd}
GFG \arrow[r] \arrow[d] & GFF^\vee \arrow[r] \arrow[d, "\rho F^\vee"] \arrow[dl] & GFC \arrow[r] \arrow[d, "\rho C"] & 0 \\
G \arrow[r, "\bar{\eta}"] & F^\vee \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

Its rows are exact since $GF$ is right exact. Since $\rho F^\vee \colon GFF^\vee \to F^\vee$ factors through $G$ by the associativity on $FGF$, the induced morphism $\rho C \colon GFC \to C$ is zero. In other words, $JC = 0$. □

PROPOSITION 3.8. *If $X \in \mathcal{C}$ is $I$-accessible, $DX$ is the unique largest $J$-accessible subobject of $F^\vee X$.*

PROOF. Let $Y \subset F^\vee X$ be $J$-accessible. Then by Proposition 2.20, $Y' \coloneqq DX + Y$ and $Y'/DX$ are also $J$-accessible. On the other hand, by the lemma above $Y'/DX \subset F^\vee X/DX$ must be $J$-annihilated too. Hence we conclude that $Y' = DX$, that is, $Y \subset DX$. □

## 2. Category equivalence

The first remarkable result which Morita context brings is the equivalence of categories between respective subcategories defined by ideal functors.

THEOREM 3.9. *$D$ and $D'$ induce a category equivalence $\mathcal{C}^I \simeq \mathcal{D}^J$.*

To prove this theorem, first we list several endofunctors on $\mathcal{C}$ into a diagram.

LEMMA 3.10. *Consider the following epi-mono factorizations*

$$\eta \colon FG \twoheadrightarrow I \hookrightarrow \mathrm{Id}_{\mathcal{C}}, \qquad \bar{\rho} G \colon FG \twoheadrightarrow D'G \hookrightarrow G^\vee G, \qquad \bar{\rho} D \colon FD \twoheadrightarrow D'D \hookrightarrow G^\vee D.$$

*These epimorphisms factor through*

$$FG \twoheadrightarrow FD \twoheadrightarrow I \twoheadrightarrow D'G \twoheadrightarrow D'D.$$

*Dually, monomorphisms* $\mathrm{Id}_{\mathcal{C}} \hookrightarrow G^\vee F^\vee$, $D'F \hookrightarrow G^\vee F^\vee$ *and* $D'D \hookrightarrow G^\vee D$ *factor through*

$$D'D \hookrightarrow D'F^\vee \hookrightarrow I^\circ \hookrightarrow G^\vee D \hookrightarrow G^\vee F^\vee.$$

*These chains of morphisms fit into the commutative diagram*

$$\begin{tikzcd}
FG \arrow[r, two heads] & FD \arrow[r, two heads] \arrow[d] \arrow[ddl, two heads] & I \arrow[r, two heads] \arrow[d, hook] & D'G \arrow[r, two heads] \arrow[d, hook] & D'D \arrow[ddl, hook] \\
& FF^\vee \arrow[r] \arrow[d, two heads] & \mathrm{Id}_{\mathcal{C}} \arrow[r] \arrow[d, two heads] & G^\vee G \arrow[d] & \\
D'D \arrow[r, hook] & D'F^\vee \arrow[r, hook] & I^\circ \arrow[r, hook] & G^\vee D \arrow[r, hook] & G^\vee F^\vee.
\end{tikzcd}$$

Proof. First the morphisms $FG \twoheadrightarrow FD$ and $D'G \twoheadrightarrow D'D$ at both ends are induced by $\bar{\eta}\colon G \twoheadrightarrow D \hookrightarrow F^\vee$. Since the functors $F$ and $D'$ are both epi, these morphisms are epic. Now consider the diagram

$$\begin{tikzcd}
& FG \arrow[dl, two heads] \arrow[d, two heads] \arrow[dr, two heads] & \\
FD \arrow[d] & I \arrow[d, hook] & D'G \arrow[d, hook] \\
FF^\vee \arrow[r] & \mathrm{Id}_{\mathcal{C}} \arrow[r] & G^\vee G.
\end{tikzcd}$$

The right pentagon is commutative by the help of the associativity on $GFG$ while the commutativity of the left is trivial. Since $I$ and $D'G$ are the images of the respective pentagons, there exist the unique morphisms $FD \twoheadrightarrow I \twoheadrightarrow D'G$ which make the diagram commutes. Now it is left us to check the commutativity for $\bar{\rho}D$. We have the diagram

$$\begin{tikzcd}
FG \arrow[r, two heads] \arrow[d, two heads] & D'G \arrow[d] \\
FD \arrow[r, two heads] \arrow[ur, two heads] & D'D
\end{tikzcd}$$

where the outer square trivially commutes. Since $FG \twoheadrightarrow FD$ is epic and the upper triangle commutes, the lower also does. The dual statement goes similarly and the last commutativity has been already proven. $\square$

Corollary 3.11. *$D'D$ is equal to the image of the composite $I \hookrightarrow \mathrm{Id}_{\mathcal{C}} \twoheadrightarrow I^\circ$. In particular, if $X \in \mathcal{C}^I$ then canonically $X \simeq D'DX$.* $\square$

Putting this corollary and Lemma 3.6 together, we obtain Theorem 3.9.

Remark 3.12. Though the categories $\mathcal{C}^I$ and $\mathcal{D}^J$ are equivalent, their exact structures may differ. For example, let $A$ and $B$ be the upper triangle matrix algebras

$$A \coloneqq \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}, \qquad B \coloneqq \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$$

over $\Bbbk$. Let $M$ and $N$ be the bimodules

$$M \coloneqq \left\{ \begin{pmatrix} * & * \\ 0 & * \\ 0 & * \end{pmatrix} \right\}, \qquad N \coloneqq \left\{ \begin{pmatrix} * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$$

and define $\eta\colon M \otimes_B N \to A$ and $\rho\colon N \otimes_A M \to B$ by matrix multiplication. These data define a Morita context between $A$-$\mathcal{M}od$ and $B$-$\mathcal{M}od$. $\rho$ is surjective and the image of $\eta$ is

$$I := \left\{ \begin{pmatrix} * & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix} \right\} \subset A,$$

so the Morita context functors induce a category equivalence $(A\text{-}\mathcal{M}od)^I \simeq B\text{-}\mathcal{M}od$. However it sends a short exact sequence

$$0 \longrightarrow \left\{ \begin{pmatrix} * \\ 0 \end{pmatrix} \right\} \longrightarrow \left\{ \begin{pmatrix} * \\ * \end{pmatrix} \right\} \longrightarrow \left\{ \begin{pmatrix} - \\ * \end{pmatrix} \right\} \longrightarrow 0$$

in $B$-$\mathcal{M}od$ to the sequence

$$0 \longrightarrow \left\{ \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix} \right\} \longrightarrow \left\{ \begin{pmatrix} * \\ * \\ * \end{pmatrix} \right\} \longrightarrow \left\{ \begin{pmatrix} - \\ - \\ * \end{pmatrix} \right\} \longrightarrow 0$$

in $A$-$\mathcal{M}od$, which is obviously not exact at the middle term.

As we have seen in this remark, the category equivalence does not preserve extensions in general. However, it is true if one of the categories is semisimple.

Lemma 3.13. *If $\mathcal{D}$ is semisimple, then* $\operatorname{Ext}^1_{\mathcal{C}}(X, Y) = 0$ *for any* $X, Y \in \mathcal{C}^I$.

Proof. Let $0 \to Y \to E \to X \to 0$ be a short exact sequence in $\mathcal{C}$. Since $\mathcal{C}^I$ is closed under extensions, $E$ is also in $\mathcal{C}^I$. Now $\mathcal{C}^I \simeq \mathcal{D}^J$ is semisimple, so that this sequence splits. $\square$

On the other hand, the Ext preserving property for the other category is induced from the following condition.

Lemma 3.14. *Suppose that $\rho\colon GF \to \mathrm{Id}_{\mathcal{D}}$ is surjective. Then* $\operatorname{Ext}^1_{\mathcal{C}}(X, Y) \simeq \operatorname{Ext}^1_{\mathcal{C}_I}(X, Y)$ *for any* $X, Y \in \mathcal{C}_I$.

Proof. By the assumption $FGFG \to FG$ is also surjective. This implies $I^2 = I$, so that we can use Proposition 2.34. $\square$

## 3. Correspondence on simple objects

Next we prove the correspondence between the simple objects in the respective subcategories.

Theorem 3.15. *$D$ and $D'$ induce a one-to-one correspondence* $\operatorname{Irr} \mathcal{C}^I \overset{1:1}{\longleftrightarrow} \operatorname{Irr} \mathcal{D}^J$.

Though we have proven the category equivalence $\mathcal{C}^I \simeq \mathcal{D}^J$, we have to prove this theorem independently since we do not know how to characterize the set $\operatorname{Irr} \mathcal{C}^I$ from the category $\mathcal{C}^I$ itself. Actually, using Lemma 3.6 again, this theorem is obtained as an immediate corollary of the next theorem.

Theorem 3.16. *Let $X \in \operatorname{Irr} \mathcal{C}^I$. Then $DX$ is the simple socle of $F^\vee X$ as well as the simple top of $GX$.*

Proof. By the assumption $X \notin \mathcal{C}_I$, we have $DX \neq 0$. Take any non-zero subobject $Y \hookrightarrow F^\vee X$. Then the corresponding morphism $FY \to X$ is also non-zero, so it must be epic since $X$ is simple. Now consider the commutative diagram

$$\begin{tikzcd}
GFY \arrow[r] \arrow[d, two heads] \arrow[rr, bend left, two heads] & GFF^\vee X \arrow[r] \arrow[d, two heads] \arrow[dr, "\rho F^\vee X"] & GX \arrow[d, two heads, "\bar{\eta} X"] \\
JY \arrow[r, hook] & JF^\vee X \arrow[r, hook] & F^\vee X.
\end{tikzcd}$$

Since $G$ is right exact $GFY \to GX$ is also epic. So we have

$$JY = \mathrm{Image}(GFY \to F^\vee X) = \mathrm{Image}(GX \to F^\vee X) = DX.$$

This implies that $DX$ is contained in an arbitrary non-zero subobject $Y \hookrightarrow F^\vee X$, so it must be a simple socle of $F^\vee X$. Dually it is also a simple top of $GX$. $\square$

Putting it together with Lemma 3.6 and Lemma 3.7 we obtain the next corollary.

Corollary 3.17. *Let $X \in \operatorname{Irr} \mathcal{C}$ and $Y \in \operatorname{Irr} \mathcal{D}^J$. Then*

$$[GX : Y] = [F^\vee X : Y] = \begin{cases} 1 & Y \simeq DX, \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

Here $[M : S]$ is the multiplicity of a simple object $S$ in the composition factors of $M$. If $M$ is not of finite length this symbol does not make sense in general, but the formula above can be always read in an appropriate manner.

When we replace $X$ above to its injective hull or its projective cover, we obtain similar statements.

Proposition 3.18. *Let $X \in \operatorname{Irr} \mathcal{C}^I$ and suppose that it has a projective cover $P \twoheadrightarrow X$. Then $GP \twoheadrightarrow GX \twoheadrightarrow DX$ is the top of $GP$.*

Proof. Take $Y \in \operatorname{Irr} \mathcal{D}$ and let $C \coloneqq \operatorname{Coker}(D'Y \hookrightarrow G^\vee Y)$. By the projectiveness of $P$, the sequence

$$0 \to \operatorname{Hom}_{\mathcal{C}}(P, D'Y) \to \operatorname{Hom}_{\mathcal{C}}(P, G^\vee Y) \to \operatorname{Hom}_{\mathcal{C}}(P, C) \to 0$$

is exact. By Lemma 3.7, $C$ is $I$-annihilated. Hence it has no subquotients isomorphic to $X$, so $\operatorname{Hom}_{\mathcal{C}}(P, C) = 0$ by a property of projective cover. Thus

$$\operatorname{Hom}_{\mathcal{D}}(GP, Y) \simeq \operatorname{Hom}_{\mathcal{C}}(P, G^\vee Y) \simeq \operatorname{Hom}_{\mathcal{C}}(P, D'Y).$$

Now $D'Y$ is simple or zero, so there is a non-zero morphism $GP \twoheadrightarrow Y$ if and only if $D'Y \simeq X$, or equivalently, $Y \simeq DX$. Moreover

$$\operatorname{Hom}_{\mathcal{D}}(GP, DX) \simeq \operatorname{Hom}_{\mathcal{C}}(P, X) \simeq \operatorname{End}_{\mathcal{C}}(X) \simeq \operatorname{End}_{\mathcal{D}}(DX).$$

Thus $GP \twoheadrightarrow DX$ is the unique its simple quotient. $\square$

Remark that $\operatorname{Ker}(GP \twoheadrightarrow DP)$ is $J$-annihilated by the dual of Lemma 3.7 but $\operatorname{Ker}(DP \twoheadrightarrow DX)$ is not in general, so may contains a composition factor in $\mathcal{C}^J$.

## 4. Morita context among multiple categories

We here generalize the notion of Morita context, from that between two categories to that among more than two categories. Let us take an index set $\Lambda$ which is not necessarily finite. We assume that every category appears in this section is closed under sums and intersections with cardinality $\#\Lambda$.

Definition 3.19. Let $\{\mathcal{C}_\lambda\}_{\lambda \in \Lambda}$ be a family of abelian categories indexed by a set $\Lambda$. A *Morita context* among $\{\mathcal{C}_\lambda\}$ is a family of adjunctions $F_{\lambda\mu} \colon \mathcal{C}_\mu \to \mathcal{C}_\lambda$ indexed by a pair of $\lambda, \mu \in \Lambda$, equipped with a family of degree-zero natural transformations $\eta_{\lambda\mu\nu} \colon F_{\lambda\mu} F_{\mu\nu} \to F_{\lambda\nu}$ indexed by a triple of $\lambda, \mu, \nu \in \Lambda$ which satisfies the following conditions.

(1) (The associativity law) For each $\lambda, \mu, \nu, \pi \in \Lambda$, the square

$$\begin{tikzcd} F_{\lambda\mu}F_{\mu\nu}F_{\nu\pi} \arrow[r, "\eta_{\lambda\mu\nu}F_{\nu\pi}"] \arrow[d, "F_{\lambda\mu}\eta_{\mu\nu\pi}"'] & F_{\lambda\nu}F_{\nu\pi} \arrow[d, "\eta_{\lambda\nu\pi}"] \\ F_{\lambda\mu}F_{\mu\pi} \arrow[r, "\eta_{\lambda\mu\pi}"'] & F_{\lambda\pi} \end{tikzcd}$$

commutes.

(2) (The unit law) For each $\lambda$, there is a fixed isomorphism $F_{\lambda\lambda} \simeq \mathrm{Id}_{\mathcal{C}_\lambda}$ such that $\eta_{\lambda\lambda\mu}$ and $\eta_{\lambda\mu\mu}$ are respectively equal to

$$F_{\lambda\lambda}F_{\lambda\mu} \simeq \mathrm{Id}_{\mathcal{C}_\lambda}F_{\lambda\mu} \simeq F_{\lambda\mu}, \qquad F_{\lambda\mu}F_{\mu\mu} \simeq F_{\lambda\mu}\mathrm{Id}_{\mathcal{C}_\mu} \simeq F_{\lambda\mu}.$$

One can easily verify that when $\#\Lambda = 2$ this definition is equivalent to the previous one.

Remark 3.20. Let $\mathbf{A}$ be a 2-category which consists of abelian categories as 0-cells, adjunctions as 1-cells and natural transformation as 2-cells. Consider $\Lambda$ as a codiscrete category, that is, we regard that there exists unique morphism $\mu \to \lambda$ for each $\lambda, \mu \in \Lambda$. Then a Morita context is just a lax functor $\mathcal{F}\colon \Lambda \to \mathbf{A}$ (where $\mathcal{C}_\lambda = \mathcal{F}(\lambda)$, $F_{\lambda\mu} = \mathcal{F}(\mu \to \lambda)$) such that the unit $\mathrm{Id}_{\mathcal{F}(\lambda)} \to \mathcal{F}(\lambda \to \lambda)$ is an isomorphism for every $\lambda \in \Lambda$.

Example 3.21. Let $\mathcal{A}$ be a category, and take an object $X_\lambda \in \mathcal{A}$ for each $\lambda$. Let $A_\lambda \coloneqq \mathrm{End}_{\mathcal{A}}(X_\lambda)$ be its endomorphism algebra. Then for each pair of $\lambda, \mu$, $\mathrm{Hom}_{\mathcal{A}}(X_\mu, X_\lambda)$ is a $(A_\lambda, A_\mu)$-bimodule so it induces an adjunction $A_\mu\text{-}\mathcal{M}od \to A_\lambda\text{-}\mathcal{M}od$. Moreover the composition of morphisms

$$\mathrm{Hom}_{\mathcal{A}}(X_\mu, X_\lambda) \otimes \mathrm{Hom}_{\mathcal{A}}(X_\nu, X_\mu) \to \mathrm{Hom}_{\mathcal{A}}(X_\nu, X_\lambda)$$

gives a natural transformations between these adjunctions which is associative and unital. Hence these define a Morita context among the categories $\{A_\lambda\text{-}\mathcal{M}od\}$. Conversely all Morita contexts among module categories are obtained in this way.

Example 3.22. More generally, take a small full subcategory $\mathcal{A}_\lambda \subset \mathcal{A}$ for each $\lambda$. Then a collection of a $(\mathcal{A}_\lambda, \mathcal{A}_\mu)$-module

$$\begin{aligned} \mathcal{A}_\mu^{\mathrm{op}} \boxtimes \mathcal{A}_\lambda &\to \mathcal{V} \\ X \boxtimes Y &\mapsto \mathrm{Hom}_{\mathcal{A}}(X, Y) \end{aligned}$$

also defines a Morita context among $\{\mathcal{A}_\lambda\text{-}\mathcal{M}od\}$.

Suppose that $\{F_{\lambda\mu}\}_{\lambda,\mu\in\Lambda}$ is a Morita context among categories $\{\mathcal{C}_\lambda\}_{\lambda\in\Lambda}$. For each triple of $\alpha, \lambda, \mu \in \Lambda$, let $I^\alpha_{\lambda\mu}$ be a subfunctor of $F_{\lambda\mu}$ defined by

$$I^\alpha_{\lambda\mu} \coloneqq \mathrm{Image}(\eta_{\lambda\alpha\mu}\colon F_{\lambda\alpha}F_{\alpha\mu} \to F_{\lambda\mu}).$$

In particular, $I^\alpha_{\lambda\lambda} \subset F_{\lambda\lambda} \simeq \mathrm{Id}_{\mathcal{C}_\lambda}$ is an ideal functor on $\mathcal{C}_\lambda$. The unit law implies that $I^\lambda_{\lambda\mu} = I^\mu_{\lambda\mu} = F_{\lambda\mu}$. By the associativity law there are natural transformations

$$I^\alpha_{\lambda\mu}F_{\mu\nu} \to I^\alpha_{\lambda\nu} \quad \text{and} \quad F_{\lambda\mu}I^\beta_{\mu\nu} \to I^\beta_{\lambda\nu}$$

induced by $\eta_{\lambda\mu\nu}$. Since $F_{\lambda\mu}$ is right exact, they induce

$$(F_{\lambda\mu}/I^\alpha_{\lambda\mu})(F_{\mu\nu}/I^\beta_{\mu\nu}) \to F_{\lambda\nu}/(I^\alpha_{\lambda\nu} + I^\beta_{\lambda\nu}).$$

Now take a subset $\Lambda' \subset \Lambda$. Then clearly the restriction $\{F_{\lambda\mu}\}_{\lambda,\mu\in\Lambda'}$ gives a Morita context among the subcollection $\{\mathcal{C}_\lambda\}_{\lambda\in\Lambda'}$. In contrast, we can also take a "quotient" of this Morita context with respect to $\Lambda'$ as follows.

PROPOSITION 3.23. *For each $\lambda, \mu \in \Lambda$, let*

$$I'_{\lambda\mu} := \sum_{\alpha \in \Lambda'} I^{\alpha}_{\lambda\mu}$$

*and $\mathcal{C}'_\lambda := (\mathcal{C}_\lambda)_{I'_{\lambda\lambda}}$. Then there exists a Morita context $\{F'_{\lambda\mu}\}$ among the abelian categories $\{\mathcal{C}'_\lambda\}$ defined by*

$$F'_{\lambda\mu} := \Phi^{\wedge}_{I'_{\lambda\lambda}}(F_{\lambda\mu}/I'_{\lambda\mu})\Phi_{I'_{\mu\mu}}.$$

PROOF. Since $F_{\lambda\mu}$ is cocontinuous, by taking colimits of natural transformation above we obtain

$$(F_{\lambda\mu}/I'_{\lambda\mu})(F_{\mu\nu}/I'_{\mu\nu}) \to F_{\lambda\nu}/I'_{\lambda\nu}.$$

Moreover, by the unit law we have $T_{I'_{\lambda\lambda}}(F_{\lambda\mu}/I'_{\lambda\mu})T_{I'_{\mu\mu}} = F_{\lambda\mu}/I'_{\lambda\mu}$. Thus there are a natural transformation

$$\eta'_{\lambda\mu\nu} \colon F'_{\lambda\mu}F'_{\mu\nu} = \Phi^{\wedge}_{I'_{\lambda\lambda}}(F_{\lambda\mu}/I'_{\lambda\mu})(F_{\mu\nu}/I'_{\mu\nu})\Phi_{I'_{\nu\nu}} \to \Phi^{\wedge}_{I'_{\lambda\lambda}}(F_{\lambda\mu}/I'_{\lambda\nu})\Phi_{I'_{\nu\nu}} = F'_{\lambda\nu}$$

and an isomorphism

$$F'_{\lambda\lambda} \simeq \Phi^{\wedge}_{I'_{\lambda\lambda}}T_{I'_{\lambda\lambda}}\Phi_{I'_{\lambda\lambda}} = \mathrm{Id}_{\mathcal{C}_\lambda}$$

which form a Morita context. □

Note that $\mathcal{C}'_\alpha = \{0\}$ for every $\alpha \in \Lambda'$, so the quotient Morita context above should be considered as parameterized by the complement set $\Lambda \setminus \Lambda'$. When $\Lambda'$ has a decomposition $\Lambda' = \Lambda'_1 \sqcup \Lambda'_2$, taking the quotient by $\Lambda'$ is equal to first taking by $\Lambda'_1$, then by $\Lambda'_2$.

## 5. Morita context with a partial order

As a special case of quotient, let us consider the case that $\Lambda'$ in the previous section consists of a single element $\alpha$. For each $\lambda \in \Lambda \setminus \{\alpha\}$, the pair $(F_{\lambda\alpha}, F_{\alpha\lambda})$ is a Morita context between two categories $\mathcal{C}_\lambda$ and $\mathcal{C}_\alpha$. Hence we have a category equivalence

$$(\mathcal{C}_\lambda)^{I^{\alpha}_{\lambda\lambda}} \simeq (\mathcal{C}_\alpha)^{I^{\lambda}_{\alpha\alpha}}$$

and a one-to-one correspondence

$$\mathrm{Irr}(\mathcal{C}_\lambda)^{I^{\alpha}_{\lambda\lambda}} \xleftrightarrow{1:1} \mathrm{Irr}(\mathcal{C}_\alpha)^{I^{\lambda}_{\alpha\alpha}}.$$

In practice we should choose $\alpha$ such that the structure of $\mathcal{C}_\alpha$ is very simple so that we can describe a part of $\mathcal{C}_\lambda$, which may be hard to study, by terms of $\mathcal{C}_\alpha$. Now on the collection of the rest part $\mathcal{C}'_\lambda = (\mathcal{C}_\lambda)_{I^{\alpha}_{\lambda\lambda}}$ we have a new Morita context, so we can recursively continue this process for $\mathcal{C}'_\lambda$ by choosing another $\beta \in \Lambda$ to decompose $\mathcal{C}_\lambda$ into small parts. In order to perform this strategy at one time, we introduce a partial order on the set $\Lambda$ as we do before in the previous chapter. Intuitively it indicates the order of $\alpha, \beta, \ldots$ we pick up from $\Lambda$.

DEFINITION 3.24. Let $\{F_{\lambda\mu}\}$ be a Morita context among the categories $\{\mathcal{C}_\lambda\}$. A partial order $\leq$ on the set $\Lambda$ is said to be *compatible* with $\{F_{\lambda\mu}\}$ if it satisfies

$$F_{\lambda\mu} = \sum_{\nu \leq \lambda, \mu} I^{\nu}_{\lambda\mu}, \quad \text{where} \quad I^{\nu}_{\lambda\mu} := \mathrm{Image}(F_{\lambda\nu}F_{\nu\mu} \to F_{\lambda\mu})$$

for each pair of $\lambda, \mu \in \Lambda$.

When $\lambda$ and $\mu$ are comparable then the condition above is trivially satisfied. Hence every total order on $\Lambda$ is compatible.

LEMMA 3.25. *If $\Lambda$ is well-founded, then the condition above is equivalent to that*

$$F_{\lambda\mu} = \sum_{\nu<\lambda} I^{\nu}_{\lambda\mu}$$

*is satisfied for each pair of $\lambda, \mu \in \Lambda$ such that $\lambda \not\leq \mu$.*

PROOF. Clearly the first condition implies the second. Suppose the second one. We prove the first condition for a fixed $\mu$ by transfinite induction on $\lambda$. So assume that for every $\nu < \lambda$ we have $F_{\nu\mu} = \sum_{\pi\leq\nu,\mu} I^{\pi}_{\nu\mu}$. If $\lambda \leq \mu$ then the condition is trivially satisfied so assume $\lambda \not\leq \mu$. Then by the assumption we have $F_{\lambda\mu} = \sum_{\nu<\lambda} I^{\nu}_{\lambda\mu}$. Each $I^{\nu}_{\lambda\mu}$ is contained in

$$\sum_{\pi\leq\nu,\mu} \mathrm{Image}(F_{\lambda\nu}F_{\nu\pi}F_{\pi\mu} \to F_{\lambda\mu}) \subset \sum_{\pi\leq\lambda,\mu} I^{\pi}_{\lambda\mu}$$

since $F_{\lambda\mu}$ is cocontinuous. Thus the condition is also satisfied for $\lambda$. □

Now assume that a partial order $\leq$ is compatible with $\{F_{\lambda\mu}\}$. Let us denote

$$I^{\leq\lambda}_{\alpha\beta} := \sum_{\mu\leq\lambda} I^{\lambda}_{\alpha\beta} \quad \text{and} \quad I^{<\lambda}_{\alpha\beta} := \sum_{\mu<\lambda} I^{\lambda}_{\alpha\beta}.$$

PROPOSITION 3.26. *For each $\omega \in \Lambda$ the family $\{I^{\leq\lambda}_{\omega\omega}\}_{\lambda\in\Lambda}$ is an ideal filter on $\mathcal{C}_\omega$.*

PROOF. For simplicity let us write $I^{\lambda} := I^{\lambda}_{\omega\omega}$ and $I^{\leq\lambda} := I^{\leq\lambda}_{\omega\omega}$. The first two conditions in Definition 2.40 are obvious. So we prove $I^{\leq\lambda}I^{\leq\mu} \subset \sum_{\rho\leq\lambda,\mu} I^{\leq\rho}$ for each $\lambda, \mu \in \Lambda$. Let us take $\nu \leq \lambda$ and $\pi \leq \mu$. Since $F_{\omega\nu}$ is cocontinuous, we have

$$I^{\nu}I^{\pi} \subset \mathrm{Image}(F_{\omega\nu}F_{\nu\pi}F_{\pi\omega} \to \mathrm{Id}_{\mathcal{C}_\omega}) \subset \sum_{\rho\leq\nu,\pi} I^{\rho}.$$

Hence by taking sum we obtain the inclusion as desired. □

Using this ideal filter the category $\mathcal{C}_\omega$ is divided into $\mathcal{C}_\omega[\lambda] = (\mathcal{C}_\omega)^{\leq\lambda} \cap (\mathcal{C}_\omega)_{<\lambda}$. For each $\lambda$, by taking the quotient with respect to the subset $\Lambda' = \{\mu \in \Lambda \mid \mu < \lambda\}$ we have a Morita context between $(\mathcal{C}_\omega)_{<\lambda}$ and $(\mathcal{C}_\lambda)_{<\lambda}$ whose trace ideal in $(\mathcal{C}_\omega)_{<\lambda}$ is just $(I^{\leq\lambda}_{\omega\omega})_{I^{<\lambda}_{\omega\omega}}$. The corresponding trace ideal in $(\mathcal{C}_\lambda)_{<\lambda}$ is $(I^{\omega}_{\lambda\lambda} + I^{<\lambda}_{\lambda\lambda})_{I^{<\lambda}_{\lambda\lambda}}$. Thus by letting

$$\mathcal{C}_\lambda\langle\omega\rangle := (\mathcal{C}_\lambda)^{I^{\omega}_{\lambda\lambda}} \cap (\mathcal{C}_\lambda)_{<\lambda}$$

and $\mathrm{Irr}\,\mathcal{C}_\lambda\langle\omega\rangle := \{V \in \mathrm{Irr}\,\mathcal{C} \mid V \in \mathcal{C}_\lambda\langle\omega\rangle\}$ we obtain the following theorem.

THEOREM 3.27. *For each $\lambda \leq \omega$, there is a Morita context between $(\mathcal{C}_\omega)_{<\lambda}$ and $(\mathcal{C}_\lambda)_{<\lambda}$ which induces a category equivalence $\mathcal{C}_\omega[\lambda] \simeq \mathcal{C}_\lambda\langle\omega\rangle$ and a one-to-one correspondence $\mathrm{Irr}\,\mathcal{C}_\omega[\lambda] \xleftrightarrow{1:1} \mathrm{Irr}\,\mathcal{C}_\lambda\langle\omega\rangle$. If $\lambda \not\leq \omega$, then $\mathcal{C}_\omega[\lambda] = 0$.* □

COROLLARY 3.28. *If $\Lambda$ is well-founded, we have*

$$\mathrm{Irr}\,\mathcal{C}_\omega = \bigsqcup_{\lambda\leq\omega} \mathrm{Irr}\,\mathcal{C}_\omega[\lambda] \xleftrightarrow{1:1} \bigsqcup_{\lambda\leq\omega} \mathrm{Irr}\,\mathcal{C}_\lambda\langle\omega\rangle.$$

□

CHAPTER 4

# Generalized cellular algebras

Now we concentrate on representation theory of algebras. Here continuously the term "an algebra" means a $\mathcal{V}$-algebra. With the help of the category equivalence $\mathcal{A}dj(B\text{-}\mathcal{M}od, A\text{-}\mathcal{M}od) \simeq A\text{-}\mathcal{M}od\text{-}B$, we can interpret all the notions we have introduced in the previous chapters into the language of modules. For example, ideal functors on the category are replaced by 2-sided ideals in an algebra. So an ideal filter is just a collection of 2-sided ideals which satisfies the similar conditions.

In this chapter we fix a partially ordered set $(\Lambda, \leq)$ and an indexed family $\{B_\lambda\}_{\lambda\in\Lambda}$ of algebras. We introduce a generalized notion of standardly based algebra and that of cellular algebra over the family $\{B_\lambda\}$, not over the single base algebra $\Bbbk$. We also study its Morita invariance motivated by the work of König and Xi [**KX99**].

## 1. Standard filter

We start from a very general setting. In the last of previous chapter we decompose a category into small parts in order to study them one by one. A standardly filtered algebra is defined so that we can perform similar strategy for its module category.

Definition 4.1. Let $A$ be an algebra. A *prestandard filter* of $A$ over $\{B_\lambda\}$ is a datum consisting of:

- an ideal filter $\{A^{\leq\lambda}\}_{\lambda\in\Lambda}$ on $A$,
- for each $\lambda \in \Lambda$, a 2-sided ideal $B'_\lambda \subset B_\lambda$,
- for each $\lambda \in \Lambda$, a Morita context $(M_\lambda, N_\lambda)$ between $A/A^{<\lambda}$ and $B_\lambda/B'_\lambda$ whose trace ideal in $A$ is $A^{\leq\lambda}/A^{<\lambda}$.

Moreover if it satisfies $A^{\leq\mu}M_\lambda = 0$ and $N_\lambda A^{\leq\mu} = 0$ for each pair of $\lambda, \mu$ such that $\lambda \not\leq \mu$, we call it a *standard filter*. An algebra equipped with a standard filter is called a *standardly filtered algebra*.

Now Lemma 0.5 in the introduction is just a reformulation of Theorem 3.27. By Lemma 3.25, if $\Lambda$ is well-founded the first assumption of the lemma can be weakened to

$$\mathrm{Hom}_{\mathcal{A}}(X_\mu, X_\lambda) = \mathcal{A}^{<\lambda}(X_\mu, X_\lambda).$$

Note that in the settings of the lemma, for $\omega_1, \dots, \omega_n \in \Lambda$, the algebra

$$\mathrm{End}_{\mathcal{A}}\Big(\bigoplus_i X_{\omega_i}\Big) = \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{A}}(X_{\omega_i}, X_{\omega_j})$$

is also standardly filtered. It can be proven by adding a new index $\infty$ which is greater than any element of $\Lambda$ so that $X_\infty = \bigoplus_i X_{\omega_i}$, then remove it since it is needless by that $\mathcal{A}^{\leq\infty} = \mathcal{A}^{<\infty}$.

If each $B_\lambda$ is just the base ring $\Bbbk$, we simply say it is a standard filter over $\Bbbk$ instead of over the family $\{\Bbbk\}$. Actually the condition for being a standardly filtered algebra can be weakened as follows.

LEMMA 4.2. *Suppose that $(M, N)$ is a Morita context between algebras $A$ and $B$, and let us write its equipped maps as $\eta\colon M \otimes_B N \to A$ and $\rho\colon N \otimes_A M \to B$. Let*

$$\begin{aligned} B' &:= \{b \in B \mid \eta(mb \otimes n) = 0 \text{ for all } m \in M, n \in N\}, \\ M' &:= \{m \in M \mid \eta(m \otimes n) = 0 \text{ for all } n \in N\}, \\ N' &:= \{n \in N \mid \eta(m \otimes n) = 0 \text{ for all } m \in M\}. \end{aligned}$$

*Then $(M/M', N/N')$ is a Morita context between algebras $A$ and $B/B'$ with the same trace ideal in $A$.*

PROOF. First by definition $\eta\colon M/M' \otimes_B N/N' \to A$ is well-defined. In addition we have $MB' \subset M'$ and $B'N \subset N'$ so that $M/M'$ and $N/N'$ can be considered as modules over $B/B'$. Moreover $\rho(M' \otimes_A N), \rho(M \otimes_A N') \subset B'$ by the associativity, so that $\rho\colon N/N' \otimes_A M/M' \to B/B'$ is also well-defined. Now it is clear that these data form a Morita context between $A$ and $B/B'$. $\square$

Note that $B'$ above is the common annihilator of $M/M'$ and $N/N'$, so that these are faithful modules over $B/B'$.

PROPOSITION 4.3. *If an algebra $A$ has a prestandard filter, it also has a standard filter.*

PROOF. Take a prestandard filter of $A$ as above. For each $\lambda$, let $A^{\not\geq\lambda} := \sum_{\mu\not\geq\lambda} A^{\leq\mu}$. Then

$$\begin{aligned} \eta(A^{\not\geq\lambda} M_\lambda \otimes_{B_\lambda} N_\lambda) &= A^{\not\geq\lambda} A^{\leq\lambda} + A^{<\lambda} = A^{<\lambda}, \\ \eta(M_\lambda \otimes_{B_\lambda} N_\lambda A^{\not\geq\lambda}) &= A^{\leq\lambda} A^{\not\geq\lambda} + A^{<\lambda} = A^{<\lambda}. \end{aligned}$$

Hence taking $M'_\lambda \subset M_\lambda$ and $N'_\lambda \subset N_\lambda$ as in the lemma above, we have inclusions $A^{\not\geq\lambda} M_\lambda \subset M'_\lambda$ and $N_\lambda A^{\not\geq\lambda} \subset N'_\lambda$. Thus by replacing $(M_\lambda, N_\lambda)$ with the quotients $(M_\lambda/M'_\lambda, N_\lambda/N'_\lambda)$ we obtain a standard filter. $\square$

The notion of standardly filtered algebra is a Morita invariant and inherited by Peirce decomposition.

PROPOSITION 4.4. *Let $A$ be a standardly filtered algebra over $\{B_\lambda\}$.*

(1) *For any idempotent $e \in A$, the algebra $eAe$ is also standardly filtered.*
(2) *If an algebra $A'$ is Morita equivalent to $A$ (i.e. $A$-$\mathcal{M}od \simeq A'$-$\mathcal{M}od$), $A'$ is also standardly filtered.*

PROOF. (1) follows from that the pair $(eM_\lambda, N_\lambda e)$ forms a Morita context between $eAe/eA^{<\lambda}e$ and $B_\lambda/B'_\lambda$. (2) is a consequence of that the definition of standard filter on an algebra can be translated into the language of its module category. $\square$

For an algebra $A$ and a 2-sided ideal $I \subset A$, let us write

$$\mathrm{Irr}(A) := \mathrm{Irr}(A\text{-}\mathcal{M}od) \quad \text{and} \quad \mathrm{Irr}^I(A) := \mathrm{Irr}(A\text{-}\mathcal{M}od^I) = \mathrm{Irr}(A) \setminus \mathrm{Irr}(A/I)$$

for short. More generally, for $J \subset I \subset A$ let

$$\mathrm{Irr}^I_J(A) := \mathrm{Irr}^{I/J}(A/J) = \mathrm{Irr}(A/J) \setminus \mathrm{Irr}(A/I).$$

Then Proposition 2.43 and Theorem 3.15 immediately bring us the following classification of simple $A$-modules. This is a generalization of [**GL96**, Theorem 3.4].

THEOREM 4.5. *Suppose that $\Lambda$ is well-founded. Let $A$ be a standardly filtered algebra over $\{B_\lambda\}$ and take its prestandard filter as above. For each $\lambda$, let $B''_\lambda/B'_\lambda \subset B_\lambda/B'_\lambda$ be the trace ideal of the Morita context. Then there is a one-to-one correspondence*

$$\operatorname{Irr}(A) = \bigsqcup_{\lambda\in\Lambda} \operatorname{Irr}^{A^{\le\lambda}}_{A^{<\lambda}}(A) \xleftrightarrow{1:1} \bigsqcup_{\lambda\in\Lambda} \operatorname{Irr}^{B''_\lambda}_{B'_\lambda}(B_\lambda)$$

*induced by Morita contexts.* □

In the classical case, each $B_\lambda$ is taken to be a base field so that $\#\operatorname{Irr}(B_\lambda) = 1$. Thus in this case $\operatorname{Irr}(A)$ is in bijection with some subset of $\Lambda$.

Let us write $[M : S]$ the multiplicity of a simple module $S$ in the composition factors of $M$. The analogue of the decomposition matrix of cellular algebra can be defined as follows. It also satisfies the unitriangular property.

LEMMA 4.6. *Take a standard filter of $A$ as above. Let $\lambda, \mu \in \Lambda$ and take $S \in \operatorname{Irr}^{A^{\le\mu}}_{A^{<\mu}}(A)$, $T \in \operatorname{Irr}(B_\lambda)$. Then unless $\lambda \le \mu$*

$$[M_\lambda \otimes_{B_\lambda} T : S] = [\operatorname{Hom}_{B_\lambda}(N_\lambda, T) : S] = 0.$$

*Moreover, if $\lambda = \mu$,*

$$[M_\lambda \otimes_{B_\lambda} T : S] = [\operatorname{Hom}_{B_\lambda}(N_\lambda, T) : S] = \begin{cases} 1 & \text{if } T \simeq DS, \\ 0 & \text{otherwise.} \end{cases}$$

*Here $D$ is the Morita context functor which induces $\operatorname{Irr}^{A^{\le\lambda}}_{A^{<\lambda}}(A) \xleftrightarrow{1:1} \operatorname{Irr}^{B''_\lambda}_{B'_\lambda}(B_\lambda)$.*

PROOF. The first equation follows from that $M_\lambda \otimes_{B_\lambda} T$ and $\operatorname{Hom}_{B_\lambda}(N_\lambda, T)$ are $A^{\not\ge\lambda}$-annihilated. The second follows from Corollary 3.17 if $T \in \operatorname{Irr}(B/B'_\lambda)$; otherwise $M_\lambda \otimes_{B_\lambda} T = \operatorname{Hom}_{B_\lambda}(N_\lambda, T) = 0$ so the formula also holds trivially. □

## 2. Well-based standard filter

Graham and Lehrer [**GL96**, Theorem 3.7] also proved that for a cellular algebra we can compute its Cartan matrix by its decomposition matrix. A general standardly filtered algebra does not have this property, so we strengthen its conditions to prove an analogue of the theorem.

DEFINITION 4.7. Let $(M, N)$ be a Morita context between algebras $A$ and $B$. We say that $(M, N)$ is *well-based* over $B$ if $M$ and $N$ are both finitely generated and projective over $B$ and the map $M \otimes_B N \to A$ is injective.

DEFINITION 4.8. A prestandard filter of $A$ is said to be *well-based* if

(1) the ideal filter $\{A^{\le\lambda}\}$ is rigid,
(2) the Morita context $(M_\lambda, N_\lambda)$ is well-based over $B_\lambda/B'_\lambda$ for every $\lambda \in \Lambda$.

An algebra equipped with a well-based standard filter is called a *weakly standardly based algebra*.

We can prove a statement similar to Proposition 4.3 for weakly standardly based algebras. The proof is clear from the lemmas below.

LEMMA 4.9. *Let $B$ be an algebra. Let $M$ be a finitely generated projective right $B$-module and $N$ be a left $B$-module. Then $x \in M$ satisfies $0 = x \otimes y \in M \otimes_B N$ for all $y \in N$ if and only if $x \in M \cdot \operatorname{Ann}_B(N)$. Here $\operatorname{Ann}_B(N) \coloneqq \{b \in B \mid bN = 0\}$ denotes the (left) annihilator of $N$.*

PROOF. The "if" part is obvious so we prove the "only if" part. We may assume that there is an $m \times m$ idempotent matrix $e = (e_{ij})$ such that $M = eB^m$. Since $M \subset B^m$ is an direct summand, we can regard $M \otimes_B N \subset B^m \otimes_B N = N^m$. Suppose $x = {}^t(x_1, \dots, x_m) \in M$ satisfies $x \otimes N = 0$. This means that $0 = x \otimes n = {}^t(x_1 n, \dots, x_m n) \in N^m$ for all $n \in N$, that is, $x_1, \dots, x_m \in \mathrm{Ann}_B(N)$. Thus

$$x = ex = {}^t(e_{11}, \dots, e_{m1})x_1 + \cdots + {}^t(e_{1m}, \dots, e_{mm})x_m \in M \cdot \mathrm{Ann}_B(N).$$

□

LEMMA 4.10. *Let $A$, $B$, $M$ and $N$ as in Lemma 4.2. If $(M, N)$ is well-based over $B$, then so is $(M/M', N/N')$ over $B/B'$.*

PROOF. By the lemma above, we have $M' = M \cdot \mathrm{Ann}_B(N) \subset MB'$. We already has the other inclusion so $M' = MB'$, hence $M/M' \simeq M \otimes_B (B/B')$ is finitely generated and projective over $B/B'$. The same holds for $N/N'$. It is clear that $\eta\colon M/M' \otimes_B N/N' \to A$ is also injective. □

PROPOSITION 4.11. *If an algebra $A$ has a well-based prestandard filter, it also has a well-based standard filter.* □

We also prove the statements similar to Proposition 4.4.

PROPOSITION 4.12. *Let $A$ be a weakly standardly based algebra over $\{B_\lambda\}$.*

(1) *For any idempotent $e \in A$, the algebra $eAe$ is also weakly standardly based.*
(2) *If an algebra $A'$ is Morita equivalent to $A$, $A'$ is also weakly standardly based.*

PROOF. (1) follows from that $eM_\lambda$ and $N_\lambda e$ are also finitely generated and projective, and that $eM_\lambda \otimes_{B_\lambda} N_\lambda e \simeq e(M_\lambda \otimes_{B_\lambda} N_\lambda)e$. (2) follows from the following categorical characterization of being finitely generated and projective: a right (resp. left) $B$-module $M$ is finitely generated projective if and only if the functor $M \otimes_B \bullet$ (resp. $\mathrm{Hom}_B(M, \bullet)$) also have its left (resp. right) adjoint functor. □

Now [**GL96**, Theorem 3.7] can be generalized as follows.

THEOREM 4.13. *Suppose that $\Bbbk$ is a field, $\Lambda$ is well-founded and each $B_\lambda$ is finite dimensional and semisimple. Let $A$ be a weakly standardly based algebra over $\{B_\lambda\}$ and take its well-based prestandard filter. Let $S_1, S_2 \in \mathrm{Irr}(A)$ and suppose that they have projective covers $P_i \twoheadrightarrow S_i$. Then*

$$[P_2 : S_1] = \dim_\Bbbk \mathrm{End}_A(S_2) \sum_\lambda \sum_{T \in \mathrm{Irr}(B_\lambda)} \frac{[M_\lambda \otimes_{B_\lambda} T : S_1][\mathrm{Hom}_{B_\lambda}(N_\lambda, T) : S_2]}{\dim_\Bbbk \mathrm{End}_{B_\lambda}(T)}.$$

Note that we can take $\mu \in \Lambda$ such that $S_2 \in \mathrm{Irr}^{A^{\leq\mu}}_{A^{<\mu}}(A)$ then by the Morita context we have an isomorphism $\mathrm{End}_A(S_2) \simeq \mathrm{End}_{B_\mu}(DS_2)$, so that its dimension is also easy to compute.

PROOF. Since the ideal filter is rigid, we have $[P_2 : S_1] = \sum_\lambda [A^{\leq\lambda}P_2/A^{<\lambda}P_2 : S_1]$ by Proposition 2.48. Then for each $\lambda$, we have

$$[A^{\leq\lambda}P_2/A^{<\lambda}P_2 : S_1] = \dim_\Bbbk \mathrm{Hom}_A(P_1, A^{\leq\lambda}P_2/A^{<\lambda}P_2)/\dim_\Bbbk \mathrm{End}_A(S_1)$$

and by using $A^{\leq\lambda}/A^{<\lambda} \simeq M_\lambda \otimes_{B_\lambda} N_\lambda$ and that $P_2$ is flat,

$$\begin{aligned}\mathrm{Hom}_A(P_1, A^{\leq\lambda}P_2/A^{<\lambda}P_2) &\simeq \mathrm{Hom}_A(P_1, M_\lambda \otimes_{B_\lambda} N_\lambda \otimes_A P_2)\\ &\simeq \mathrm{Hom}_{B_\lambda}(M_\lambda^\vee \otimes_A P_1, N_\lambda \otimes_A P_2)\end{aligned}$$

where $M_\lambda^\vee := \mathrm{Hom}^{\mathrm{op}}_{B_\lambda}(M_\lambda, B_\lambda)$. Since $B_\lambda$ is semisimple,

$$\dim_{\Bbbk} \mathrm{Hom}_{B_\lambda}(M_\lambda^\vee \otimes_A P_1, N_\lambda \otimes_A P_2) \\ = \sum_{T \in \mathrm{Irr}(B_\lambda)} [M_\lambda^\vee \otimes_A P_1 : T][N_\lambda \otimes_A P_2 : T] \dim_{\Bbbk} \mathrm{End}_{B_\lambda}(T).$$

Moreover we have

$$\mathrm{Hom}_{B_\lambda}(M_\lambda^\vee \otimes_A P_1, T) \simeq \mathrm{Hom}_A(P_1, M_\lambda \otimes_{B_\lambda} T)$$

which implies

$$[M_\lambda^\vee \otimes_A P_1 : T] \dim_{\Bbbk} \mathrm{End}_{B_\lambda}(T) = [M_\lambda \otimes_{B_\lambda} T : S_1] \dim_{\Bbbk} \mathrm{End}_A(S_1).$$

Similarly we have

$$[N_\lambda \otimes_A P_2 : T] \dim_{\Bbbk} \mathrm{End}_{B_\lambda}(T) = [\mathrm{Hom}_{B_\lambda}(N_\lambda, T) : S_2] \dim_{\Bbbk} \mathrm{End}_A(S_2).$$

Putting them all together, we obtain the equation. □

Quasi-hereditary algebra is an important class of algebra introduced by Cline, Parshall and Scott [**CPS88**]. The condition for a non-generalized standardly based algebra to be quasi-hereditary is given by Graham and Lehrer [**GL96**], and Du and Rui [**DR98**]. We can prove an analogous partial result for our generalized standardly based algebra.

LEMMA 4.14. *Let $(M, N)$ be a well-based Morita context between $A$ and $B$, and suppose that the algebra $B$ is semisimple. If $\rho\colon N \otimes_A M \to B$ is surjective, then the trace ideal $I := \eta(M \otimes_B N) \subset A$ is generated by an idempotent, and finitely generated and projective as both a left and a right $A$-module.*

PROOF. By the Artin–Wedderburn theorem and the Morita equivalence, we may assume that $B$ is a product of finitely many division algebras:

$$B = D_1 \times D_2 \times \cdots \times D_l$$

(here we mean that every non-zero *homogeneous* element $x \in D_i$ is invertible). Let us write $1_i = (0, \dots, 1, \dots, 0) \in B$ the identity element of each $D_i$. Since $\rho$ is surjective, for each $i$ we can find $m_i \in M$ and $n_i \in N$ such that $\rho(n_i \otimes m_i)1_i \neq 0$. By multiplying elements in $D_i$ we may assume that $\rho(n_i \otimes m_i) = 1_i$, $1_i n_i = n_i$ and $m_i 1_i = m_i$. Thus $\rho(n_i \otimes m_j) = 0$ for $i \neq j$. Let $e := \sum_i \eta(m_i \otimes n_i) \in A$. Then by the associativity $e$ is an idempotent. Moreover the maps

$$\begin{aligned} M &\to Ae, & Ae &\to M,\\ m &\mapsto \sum_i \eta(m \otimes n_i), & a &\mapsto \sum_i a m_i \end{aligned}$$

are inverses of each other. Hence $M \simeq Ae$ is a finitely generated and projective left $A$-module. Similarly $N \simeq eA$ as right $A$-modules so that $I \simeq M \otimes_B N$ is finitely generated and projective from both sides. By these isomorphisms we also have $I = AeA$. □

Hence in this case we have $\mathrm{Ext}^i_{A/I} \simeq \mathrm{Ext}^i_A$ for any $i$ by Proposition 2.38. We also have $\mathrm{Ext}^1_A(V, W) = 0$ for $V, W \in \mathrm{Irr}^I(A)$ by Lemma 3.13.

## 3. Standard basis

We here give the definition of class of algebras which is more closely related to the original one of cellular algebra.

Definition 4.15. A *(generalized) standard basis* of an algebra $A$ is a direct sum decomposition

$$A = \bigoplus_{\lambda \in \Lambda} A^\lambda$$

as a $\Bbbk$-module (not as a left or right $A$-module) such that for each $\lambda$

$$A^{\leq \lambda} := \bigoplus_{\mu \leq \lambda} A^\mu \quad \text{and} \quad A^{< \lambda} := \bigoplus_{\mu < \lambda} A^\mu$$

are both 2-sided ideals of $A$, equipped with for each $\lambda$ an isomorphism of $(A, A)$-bimodules

$$M_\lambda \otimes_{B_\lambda} N_\lambda \simeq A^{\leq \lambda}/A^{<\lambda}$$

for a pair of an $(A, B_\lambda)$-bimodule $M_\lambda$ and a $(B_\lambda, A)$-bimodule $N_\lambda$ which are both finitely generated and free over $B_\lambda$. An algebra equipped with a standard basis is called a *(generalized) standardly based algebra.*

When every $B_\lambda$ is the base algebra $\Bbbk$, this definition coincides with the original we given at the beginning.

Proposition 4.16. *A standardly based algebra is a weakly standardly based algebra.*

Proof. Since $A^{\leq\lambda}A^{\leq\mu} \subset A^{\leq\lambda} \cap A^{\leq\mu} = \bigoplus_{\nu \leq \lambda, \mu} A^\nu$, the collection $\{A^{\leq\lambda}\}_{\lambda\in\Lambda}$ is an ideal filter on $A$. As proved in [**GL96**, Proposition 2.4], we can construct a suitable $(B_\lambda, B_\lambda)$-homomorphism $\rho\colon M_\lambda \otimes_A N_\lambda \to B_\lambda$ for each $\lambda$ which completes a Morita context between $A/A^{<\lambda}$ and $B_\lambda$. □

The converse is also holds when the following assumptions are satisfied.

Proposition 4.17. *Suppose that $\Lambda$ is well-founded and every $B_\lambda$ is projective over $\Bbbk$. Then a weakly standardly based algebra is a standardly based algebra if $M_\lambda$ and $N_\lambda$ are free over $B_\lambda$ for every $\lambda$.*

Proof. Let $A$ be a weakly standardly based algebra. Since its ideal filter $\{A^{\leq\lambda}\}$ is rigid, by taking a well-ordering extension, we obtain a well-ordered filtration of $A$ whose successive quotients are $A^{\leq\lambda}/A^{<\lambda}$. Each of them is isomorphic to $M_\lambda \otimes_{B_\lambda} N_\lambda$ which is projective over $\Bbbk$, so we can lift $M_\lambda \otimes_{B_\lambda} N_\lambda \hookrightarrow A/A^{<\lambda}$ to some $\Bbbk$-linear map $\iota_\lambda\colon M_\lambda \otimes_{B_\lambda} N_\lambda \hookrightarrow A$. Then as a $\Bbbk$-module $A$ decompose into a direct sum of $\Bbbk$-modules $A^\lambda := \iota_\lambda(M_\lambda \otimes_{B_\lambda} N_\lambda)$ as desired. □

It is a natural question to ask whether the freeness condition of the definition of standardly based algebra can be weakened to the projectiveness. So suppose that we are given an $(A, B)$-bimodule $M$ and a $(B, A)$-bimodule $N$ which are both finitely generated and projective over $B$, equipped with an injective $(A, A)$-homomorphism $\eta\colon M \otimes_B N \hookrightarrow A$. By replacing them with their quotients, we may assume that $B'$, $M'$ and $N'$ taken as in lemma 4.2 are all zero. The existence of $\rho\colon N \otimes_B M \to A$ fails in this general situation: consider the following counterexample that

$$A = \Bbbk, \qquad B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}, \qquad M = \left\{ \begin{pmatrix} * & * \end{pmatrix} \right\}, \qquad N = \left\{ \begin{pmatrix} * \\ * \end{pmatrix} \right\}$$

with natural isomorphism $\eta\colon M \otimes_B N \simeq A$. We state a sufficient condition for its existence as follows. This is a generalization of [**GL96**, Proposition 2.4].

LEMMA 4.18. *Let $A$, $B$, $M$ and $N$ as above. Let $\mathrm{Tr}_{B^{\mathrm{op}}}(M), \mathrm{Tr}_B(N) \subset B$ be the trace ideals of $M$ and $N$ in $B$, that is,*

$$\mathrm{Tr}_{B^{\mathrm{op}}}(M) := \mathrm{Image}(M^\vee \otimes_A M \to B), \qquad \mathrm{Tr}_B(N) := \mathrm{Image}(N \otimes_A N^\vee \to B)$$

*where $M^\vee := \mathrm{Hom}_{B^{\mathrm{op}}}(M, B)$ and $N^\vee := \mathrm{Hom}_B(N, B)$. If $M$ is $\mathrm{Tr}_B(N)$-accessible and $N$ is $\mathrm{Tr}_{B^{\mathrm{op}}}(M)$-accessible, then there exists a unique $(B, B)$-homomorphism $\rho\colon N \otimes_A M \to B$ which makes $(M, N)$ into a Morita context between $A$ and $B$.*

PROOF. The uniqueness of $\rho$ is clear from that $B' = 0$, so we prove its existence. First consider the sequence

$$\begin{tikzcd} M \otimes_B N \otimes_A M \otimes_B N \arrow[r, shift left, "\eta\otimes_A(M\otimes_B N)"] \arrow[r, shift right, "(M\otimes_B N)\otimes_A \eta"'] & M \otimes_B N \arrow[r, hook, "\eta"] & A. \end{tikzcd}$$

Since the two parallel homomorphisms above are equalized by $\eta$ which is injective, these are equal. This implies that the diagram below is commutative:

$$\begin{tikzcd} N \otimes_A M \otimes_B N \otimes_A N^\vee \arrow[r, "N\otimes_A\eta\otimes_A N^\vee"] \arrow[d] & N \otimes_A N^\vee \arrow[r] & B \arrow[d, hook] \\ N \otimes_A M \arrow[rr] & & \mathrm{End}_A(M)^{\mathrm{op}}. \end{tikzcd}$$

Here the map at the bottom is given by $n \otimes m \mapsto (m' \mapsto \eta(m' \otimes n)m)$. By the assumption that $M$ is $\mathrm{Tr}_B(N)$-accessible, the left vertical arrow is surjective. On the other hand, since $M$ is faithful over $B$, the right vertical arrow is injective. Hence the diagram induces a $(B, B)$-homomorphism $\rho\colon N \otimes_A M \to B$ which satisfies $\eta(m' \otimes n)m = m'\rho(n \otimes m)$ for all $m, m' \in M$ and $n \in N$. Dually we can prove the existence of another $\rho'\colon N \otimes_A M \to B$ such that $n\eta(m \otimes n') = \rho'(n \otimes m)n'$, but we have $\rho = \rho'$ by the uniqueness. $\square$

When $M$ and $N$ are free over $B$ the accessibility condition is trivially satisfied, so that this proof is essentially the same as the original one by Graham and Lehrer. Note that this condition is not necessary: consider the example above with replacing $A$ with $\Bbbk \oplus \Bbbk\epsilon$, $\epsilon^2 = 0$ so that $\epsilon M = 0$, $N\epsilon = 0$ and $\eta\colon M \otimes_B N \simeq A\epsilon$, which clearly has $\rho = 0$. One necessary condition is that $\eta(M \otimes_B N)M \subset M\,\mathrm{Tr}_B(N)$ and $N\eta(M \otimes_B N) \subset \mathrm{Tr}_{B^{\mathrm{op}}}(M)N$, but the author does not know it is sufficient for the existence of $\rho$ or not.

## 4. Involution on algebras

For an algebra $A$, we call an algebra homomorphism $A \to A^{\mathrm{op}}$ whose square is equal to the identity an *anti-involution* on $A$. That is, it is a degree-zero map $\bullet^*\colon A \to A$ satisfying

$$1^* = 1, \quad (ab)^* = (-1)^{|a||b|}b^*a^* \quad \text{and} \quad a^{**} = a$$

(beware the Koszul sign). If $A$ has an anti-involution, for each left $A$-module $M$ there is a corresponding right $A$-module $M^*$ whose underlying set is equal to $M$ and action is defined by $x^* \cdot a^* := (-1)^{|a||x|}(ax)^*$, where we write $x^* \in M^*$ the element corresponds to $x \in M$. Similarly for a right $A$-module $N$ we denote by $N^*$ the corresponding left $A$-module, so that $M^{**} \simeq M$.

DEFINITION 4.19. Let $A$ and $B$ be algebras with anti-involution. A Morita context $(M, N)$ between $A$ and $B$ is said to be *involutive* if there is an isomorphism $M \simeq N^*$ of $(A, B)$-bimodules (so $M^* \simeq N$) which satisfies

$$\eta(x \otimes y)^* = (-1)^{|x||y|}\eta(y^* \otimes x^*), \quad \rho(y \otimes x)^* = (-1)^{|x||y|}\rho(x^* \otimes y^*)$$

for every $x \in M$, $y \in N$.

Now we assume that each $B_\lambda$ has a fixed anti-involution.

DEFINITION 4.20. A standardly filter on an algebra $A$ with anti-involution is said to be *involutive* if for each $\lambda$, $A^{\leq\lambda}$ and $B'_\lambda$ are closed under anti-involution and the Morita context $(M_\lambda, N_\lambda)$ between $A/A^{<\lambda}$ and $B_\lambda/B'_\lambda$ is involutive. An algebra equipped with an involutive well-based standard filter is called a *weakly cellular algebra*.

Note that in the settings of Lemma 0.5, when $\mathcal{A}$ has an anti-involution $\mathcal{A} \to \mathcal{A}^{\mathrm{op}}$ which fixes all $X_\lambda$ and $B_\lambda$, it produces an involutive standard filter. The statements below are clear from the definition.

LEMMA 4.21. *Let $A$, $B$, $M$ and $N$ as in Lemma 4.2. If $A$ and $B$ have their anti-involutions and $(M, N)$ is involutive, then $(B')^* = B'$ and $(M/M', N/N')$ is also involutive.* □

PROPOSITION 4.22. *If an algebra $A$ with anti-involution has a involutive (well-based) prestandard filter, it also has an involutive (well-based) standard filter.* □

To deal its Morita invariant property we should be careful with the compatibility between Morita equivalence and anti-involution. See the hypotheses $(*)$ and $(\dagger)$ in [**KX99**]. We here say that algebras $A$ and $A'$ with anti-involution are *involutively Morita equivalent* if the category equivalence makes the diagram

$$\begin{tikzcd} A\text{-}\mathcal{M}od \arrow[r, "*"] \arrow[d, "\sim"'] & \mathcal{M}od\text{-}A \arrow[d, "\sim"] \\ A'\text{-}\mathcal{M}od \arrow[r, "*"'] & \mathcal{M}od\text{-}A' \end{tikzcd}$$

commutes up to natural isomorphism. Note that if $A'$ is equivalent to $A$ we can find an idempotent $m \times m$ matrix $e = (e_{ij})$ over $A$ such that $A' \simeq e \cdot \mathrm{Mat}_m(A) \cdot e$. The condition above is equivalent to that we can also take $e$ so that $e^*_{ij} = e_{ji}$.

The proofs of the statements below are obvious by Proposition 4.12.

PROPOSITION 4.23. *Let $A$ be a weakly cellular algebra over $\{B_\lambda\}$.*

(1) *For any idempotent $e \in A$ such that $e^* = e$, the algebra $eAe$ with the same anti-involution is also weakly cellular.*
(2) *If an algebra $A'$ with anti-involution is involutively Morita equivalent to $A$, $A'$ is also weakly standardly filtered.* □

In [**KX99**] it is also proved that even if we are not given a such anti-involution on $A'$ we can construct it from that on $A$. Thus their result is stronger than above.

Finally we give the definition of cellular algebra in terms of basis. Note the next lemma which follows by the uniqueness of $\rho$.

LEMMA 4.24. *Suppose that $M$, $N$ and $\eta$ in Lemma 4.18 satisfies $M \simeq N^*$ and $\eta(x \otimes y)^* = (-1)^{|x||y|}\eta(y^* \otimes x^*)$. Then the induced map $\rho\colon N \otimes_A M \to B$ also satisfies $\rho(y \otimes x)^* = (-1)^{|x||y|}\rho(x^* \otimes y^*)$ so that the Morita context $(M, N)$ between $A$ and $B$ is involutive.*

DEFINITION 4.25. A standardly based algebra $A$ over $\{\mathcal{B}_\lambda\}$ with anti-involution is called a *(generalized) cellular algebra* if each component $A^\lambda$ is closed under anti-involution and the isomorphism $M_\lambda \otimes_{B_\lambda} N_\lambda \simeq A^{\leq\lambda}/A^{<\lambda}$ satisfies the involutive property similar as above.

Again, this definition is same as the original one when every $B_\lambda$ is just the base ring $\Bbbk$. Then the next statement is obvious from the lemma above.

PROPOSITION 4.26. *A cellular algebra is a weakly cellular algebra.* □

We prove the converse in suitable conditions.

PROPOSITION 4.27. *Suppose that the assumptions in Proposition 4.17 are satisfied, in addition to that $2 \in \Bbbk$ is invertible. Then a weakly cellular algebra is a cellular algebra if $M_\lambda$ and $N_\lambda$ are free over $B_\lambda$ for every $\lambda$.*

PROOF. The problem is that $\iota_\lambda$ we chose in the proof of Proposition 4.17 does not preserve anti-involution. So we retake a new map $\iota'_\lambda \colon M_\lambda \otimes_{B_\lambda} N_\lambda \hookrightarrow A$ defined by

$$\iota'_\lambda(x \otimes y) \coloneqq \frac{\iota_\lambda(x \otimes y) + (-1)^{|x||y|}\iota_\lambda(y^* \otimes x^*)^*}{2}.$$

Then $\iota'_\lambda$ is also a lift of $M_\lambda \otimes_{B_\lambda} N_\lambda \hookrightarrow A/A^{<\lambda}$ which satisfies $\iota'_\lambda(x \otimes y)^* = (-1)^{|x||y|}\iota'_\lambda(y^* \otimes x^*)$. Thus by putting $A^\lambda \coloneqq \iota'_\lambda(M_\lambda \otimes_{B_\lambda} N_\lambda)$ we obtain a desired direct sum decomposition. $\square$

# Part II

# Representation Theory of the Iwahori–Hecke Algebra

CHAPTER 5

# Cellular structure on the Iwahori–Hecke algebra

In this chapter we review the definition and the representation theory of the Iwahori–Hecke algebra. We give a new proof for that the Iwahori–Hecke algebra and the associated $q$-Schur algebra are cellular with respect to Murphy's basis [**Mur92, Mur95**] in a more simple and sophisticated way than his original one or given in [**Mat99**]. We need this reconstruction in order to make this proof to be fit in non-integral rank case we discuss later. In addition we give a generalized theorem that classify its simple modules on a very few assumptions.

## 1. The symmetric groups

We here introduce standard notions on Young tableaux used in representation theory of the symmetric group and the Iwahori–Hecke algebra, and we briefly recall some of their basic facts. We refer the standard textbooks [**Hum90**], [**Ful97**] and [**Mat99**] for details.

We write $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of natural numbers. We denote by $\mathfrak{S}_n$ the symmetric group of rank $n \in \mathbb{N}$ acting on the set $\{1, 2, \dots, n\}$ from left. For $1 \le i \le n-1$, let $s_i$ be the basic transposition $(i, i+1)$. As a Coxeter group, $\mathfrak{S}_n$ is generated by the elements $s_1, s_2, \dots, s_{i-1}$. With respect to this generator set, the *length* of $w \in \mathfrak{S}_n$ is equal to its inversion number

$$\ell(w) = \#\{(i,j) \mid 1 \le i < j \le n \text{ and } w(j) < w(i)\}.$$

A *composition* of $n \in \mathbb{N}$ is an infinite sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of natural numbers whose sum, written as $|\lambda| := \sum_i \lambda_i$, is equal to $n$. Alternatively we often represent a composition $\lambda$ as a finite tuple $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ if it satisfies $\lambda_i = 0$ for all $r > i$. For such $\lambda$, the corresponding *parabolic subgroup* (also called the *Young subgroup*)

$$\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_r} \subset \mathfrak{S}_n$$

is defined. It is known that the quotient set $\mathfrak{S}_n/\mathfrak{S}_\lambda$ has the minimal length coset representatives

$$\mathfrak{D}_\lambda := \{w \in \mathfrak{S}_n \mid \ell(ws_i) > \ell(w) \text{ for every } s_i \in \mathfrak{S}_\lambda\}.$$

With respect to this set, every $w \in \mathfrak{S}_n$ is uniquely decomposed as $w = uv$ to a pair of $u \in \mathfrak{D}_\lambda$ and $v \in \mathfrak{S}_\lambda$ which satisfies $\ell(w) = \ell(u) + \ell(v)$. For another composition $\mu$, the sets

$$\mathfrak{D}_\mu^{-1} = \{w \in \mathfrak{S}_n \mid \ell(s_i w) > \ell(w) \text{ for every } s_i \in \mathfrak{S}_\mu\}$$

and $\mathfrak{D}_\lambda \cap \mathfrak{D}_\mu^{-1}$ are the minimal length representatives of the left cosets $\mathfrak{S}_\mu \backslash \mathfrak{S}_n$ and the double cosets $\mathfrak{S}_\mu \backslash \mathfrak{S}_n / \mathfrak{S}_\lambda$ respectively.

The *Poincaré polynomial* of a subset $S \subset \mathfrak{S}_n$ is defined by $P_S(q) := \sum_{w \in S} q^{\ell(w)} \in \mathbb{Z}[q]$. If $S$ has a decomposition $S = S_1 \cdot S_2$ which preserves lengths, it follows by definition that $P_S(q) = P_{S_1}(q) P_{S_2}(q)$. We have a $q$-factorial as the Poincaré polynomial of whole $\mathfrak{S}_n$,

$$P_{\mathfrak{S}_n}(q) = [n]! := [1][2] \cdots [n],$$

where $[k]$ is a $q$-integer $[k] = 1 + q + \cdots + q^{k-1}$, which follows inductively from the decomposition $\mathfrak{S}_n = \bigsqcup_{1 \le k \le n} s_k s_{k+1} \dots s_n \mathfrak{S}_{n-1}$. Then for a composition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of $n$, we obtain

$$P_{\mathfrak{S}_\lambda}(q) = P_{\mathfrak{S}_{\lambda_1}}(q) P_{\mathfrak{S}_{\lambda_2}}(q) \cdots P_{\mathfrak{S}_{\lambda_r}}(q) = [\lambda_1]![\lambda_2]! \cdots [\lambda_r]!$$

and

$$P_{\mathfrak{D}_\lambda}(q) = \frac{P_{\mathfrak{S}_n}(q)}{P_{\mathfrak{S}_\lambda}(q)} = \frac{[n]!}{[\lambda_1]![\lambda_2]! \cdots [\lambda_r]!}.$$

We write this polynomial as

$$\begin{bmatrix} n \\ \lambda \end{bmatrix} = \begin{bmatrix} n \\ \lambda_1, \lambda_2, \dots, \lambda_r \end{bmatrix} := P_{\mathfrak{D}_\lambda}(q)$$

and call it the *$q$-multinomial coefficient.* In particular, when $\lambda = (n-k, k)$ is of length 2, the Poincaré polynomial of $\mathfrak{D}_{(n-k,k)}$ is given by *a $q$-binomial coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{bmatrix} n \\ n-k, k \end{bmatrix} = \frac{[n][n-1] \cdots [n-k+1]}{[k]!}.$$

## 2. Combinatorics on tableaux

The *Young diagram* of a composition $\lambda$ is defined by

$$Y(\lambda) := \{(i,j) \mid 1 \le i,\ 1 \le j \le \lambda_i\}.$$

We represent it by boxes placed in the fourth quadrant arranged as matrix indices (the English notation):

$$(3,2) = \ydiagram{3,2}, \quad (2,4,1) = \ydiagram{2,4,1}, \quad (2,0,3) = \ydiagram{2,0,3}.$$

A *tableau* of shape $\lambda$ is a function $\mathsf{T} \colon Y(\lambda) \to \{1, 2, \dots\}$. The *weight* of a tableau $\mathsf{T}$ is a composition $\mu = (\mu_1, \mu_2, \dots)$ whose $i$-th component is $\mu_i := \#\mathsf{T}^{-1}(i)$. $\mathsf{T}$ is said to be *row-semistandard* if it satisfies $\mathsf{T}(i,j) \le \mathsf{T}(i,j+1)$ for each pair of adjacent boxes $(i,j), (i,j+1) \in Y(\lambda)$, that is, entries in each row of $\mathsf{T}$ are weakly increasing. We denote by $\mathrm{Tab}_{\lambda;\mu}$ the set of row-semistandard tableaux of shape $\lambda$ and weight $\mu$. For example,

$$\mathrm{Tab}_{(2,3);(3,1,0,1)} = \left\{ \ytableaushort{11,124}, \ytableaushort{12,114}, \ytableaushort{14,112}, \ytableaushort{24,111} \right\}.$$

A row-semistandard tableau is also called a *row-standard tableau* if its weight is $(1^n) = (1, 1, \dots, 1)$. We denote by $\mathrm{Tab}_\lambda := \mathrm{Tab}_{\lambda;(1^n)}$ the set of row-standard tableaux of shape $\lambda$.

The set $\mathfrak{D}_\lambda$ is in bijection with the set $\mathrm{Tab}_\lambda$ by the following correspondence: for a row-standard tableau $\mathsf{T}$, we obtain a permutation $d(\mathsf{T}) \in \mathfrak{D}_\lambda$ by reading its entries from left to right for each rows from top to bottom. For example,

$$\mathsf{T} = \ytableaushort{1245,378,6} \quad \text{corresponds to} \quad d(\mathsf{T}) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 4 & 5 & 3 & 7 & 8 & 6 \end{pmatrix} = s_3 s_4 s_6 s_7.$$

Actually any tableau of weight $(1^n)$ provides a permutation in this manner, and the increasing condition on the rows just say that this permutation is in $\mathfrak{D}_\lambda$. We denote by $\varpi_\lambda$ the longest element in $\mathfrak{D}_\lambda$. Its corresponding tableau is obtained by putting numbers on $Y(\lambda)$ from bottom to top, conversely as before. For each $\mathsf{T} \in \mathrm{Tab}_\lambda$, let us write $\ell(\mathsf{T}) := \ell(d(\mathsf{T}))$ for short which we also call the *length* of $\mathsf{T}$. $\ell(\mathsf{T})$ can be also expressed as the inversion number

$$\ell(\mathsf{T}) = \left\{ \big((i,j),(k,l)\big) \,\middle|\, i < k \text{ and } T(k,l) < T(i,j) \right\}.$$

Next let us take another composition $\mu$ and consider the action $\mathfrak{S}_\mu \curvearrowright \mathfrak{S}_n/\mathfrak{S}_\lambda$. For $\mathsf{S} \in \mathrm{Tab}_{\lambda;\mu}$, we denote by $\mathrm{Tab}_{\mathsf{S}}$ the set $\{\mathsf{T} \in \mathrm{Tab}_\lambda \mid \mathsf{T}|_\mu = \mathsf{S}\}$ where $\mathsf{T}|_\mu$ is a row-semistandard tableau of weight $\mu$ obtained from $\mathsf{T}$ by replacing its entries $1, 2, \ldots, \mu_1$ by 1, $\mu_1+1, \ldots, \mu_1+\mu_2$ by 2, and so forth. For example, for

$$\mathsf{S} = \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 1&4&4\\ \cline{1-3} 3\\ \cline{1-1}\end{array}\,,$$

we have

$$\mathrm{Tab}_{\mathsf{S}} = \left\{ \begin{array}{|c|c|c|c|}\hline 1&2&4&5\\ \hline 3&7&8\\ \cline{1-3} 6\\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|c|}\hline 1&3&4&5\\ \hline 2&7&8\\ \cline{1-3} 6\\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|c|}\hline 2&3&4&5\\ \hline 1&7&8\\ \cline{1-3} 6\\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|c|}\hline 1&2&4&6\\ \hline 3&7&8\\ \cline{1-3} 5\\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|c|}\hline 1&3&4&6\\ \hline 2&7&8\\ \cline{1-3} 5\\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|c|}\hline 2&3&4&6\\ \hline 1&7&8\\ \cline{1-3} 5\\ \cline{1-1}\end{array} \right\}.$$

Then via the one-to-one correspondence $\mathrm{Tab}_\lambda \overset{1:1}{\longleftrightarrow} \mathfrak{S}_n/\mathfrak{S}_\lambda$, each subset $\mathrm{Tab}_{\mathsf{S}} \subset \mathrm{Tab}_\lambda$ clearly corresponds to each orbit of the action above. Hence the set $\mathrm{Tab}_{\lambda;\mu}$ is in bijection with the set $\mathfrak{D}_\lambda \cap \mathfrak{D}_\mu^{-1}$. Namely, for each $\mathsf{S} \in \mathrm{Tab}_{\lambda;\mu}$, there is a unique tableau $\mathsf{S}_\downarrow \in \mathrm{Tab}_{\mathsf{S}}$ which has the minimal length, so that $d(\mathsf{S}_\downarrow) \in \mathfrak{D}_\lambda \cap \mathfrak{D}_\mu^{-1}$. We can construct $\mathsf{S}_\downarrow$ from $\mathsf{S}$ in the following manner: first we mark subscripts $1, 2, \ldots, \mu_k$ to all $k$'s in $\mathsf{S}$ for each number $k$ along with the above reading order. Then $\mathsf{S}_\downarrow$ is obtained by replacing the entries of $\mathsf{S}$ by $1, 2, \ldots, n$ with respect to the total order

$$1_1 < 1_2 < \cdots < 1_{\mu_1} < 2_1 < 2_2 < \cdots < 2_{\mu_2} < \cdots .$$

For example, the row-semistandard tableau $\mathsf{S}$ above is marked as

$$\begin{array}{|c|c|c|c|}\hline 1_1&1_2&2_1&3_1\\ \hline 1_3&4_1&4_2\\ \cline{1-3} 3_2\\ \cline{1-1}\end{array}$$

and gives the corresponding row-standard tableau $\mathsf{S}_\downarrow = \mathsf{T}$ in the previous example; so $d(\mathsf{S}_\downarrow) = d(\mathsf{T}) = s_3 s_4 s_6 s_7$. Other elements in $\mathrm{Tab}_{\mathsf{S}}$ can be constructed from $\mathsf{S}_\downarrow$ as follows: let $\#_{ij}(\mathsf{S})$ be the number of $j$'s in the $i$-th row of $\mathsf{S}$, and $\mathsf{S}[j]$ be the composition of $\mu_j$ defined by $\mathsf{S}[j]_i := \#_{ij}(\mathsf{S})$. We define $\mathfrak{D}_{\mathsf{S}} \subset \mathfrak{S}_n$ by

$$\mathfrak{D}_{\mathsf{S}} := \mathfrak{D}_{\mathsf{S}[1]} \times \mathfrak{D}_{\mathsf{S}[2]} \times \cdots \times \mathfrak{D}_{\mathsf{S}[r]} \subset \mathfrak{S}_\mu \subset \mathfrak{S}_n.$$

Then we have a one-to-one correspondence $\mathfrak{D}_{\mathsf{S}} \to \mathrm{Tab}_{\mathsf{S}}$; $w \mapsto w\mathsf{S}_\downarrow$ which preserves lengths, that is, $\ell(w\mathsf{S}_\downarrow) = \ell(w) + \ell(\mathsf{S}_\downarrow)$. Let $\varpi_{\mathsf{S}} \in \mathfrak{D}_{\mathsf{S}}$ be its longest element $\varpi_{\mathsf{S}} := (\varpi_{\mathsf{S}[1]}, \varpi_{\mathsf{S}[2]}, \ldots, \varpi_{\mathsf{S}[r]})$. The tableau $\mathsf{S}^\uparrow := \varpi_{\mathsf{S}} \mathsf{S}_\downarrow \in \mathrm{Tab}_{\mathsf{S}}$ which has maximal length is obtained by replacing the entries of $\mathsf{S}$ from bottom to top, contrary to $\mathsf{S}_\downarrow$.

The matrix $(\#_{ij}(\mathsf{S}))_{i,j\geq 1}$ uniquely determines a row-semistandard tableau $\mathsf{S}$, and its shape $\lambda$ and its weight $\mu$ are recovered from this matrix as

$$\lambda_i = \sum_j \#_{ij}(\mathsf{S}) \quad \text{and} \quad \mu_j = \sum_i \#_{ij}(\mathsf{S}).$$

So for each $\mathsf{S}$, there exists a unique tableau $\mathsf{S}^*$ of shape $\mu$ and of weight $\lambda$, which satisfies $\#_{ij}(\mathsf{S}^*) = \#_{ji}(\mathsf{S})$. We call it the *dual tableau* of $\mathsf{S}$. For example, for $\mathsf{S}$ above, its dual is

$$\mathsf{S}^* = \begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 1\\ \cline{1-2} 1&3\\ \cline{1-2} 2&2\\ \cline{1-2}\end{array}\,.$$

It easily follows that $d((\mathsf{S}^*)_\downarrow) = d(\mathsf{S}_\downarrow)^{-1}$. So taking dual $\mathrm{Tab}_{\lambda;\mu} \to \mathrm{Tab}_{\mu;\lambda}$; $\mathsf{S} \mapsto \mathsf{S}^*$ corresponds to the inversion $\mathfrak{D}_\lambda \cap \mathfrak{D}_\mu^{-1} \to \mathfrak{D}_\mu \cap \mathfrak{D}_\lambda^{-1}$; $w \mapsto w^{-1}$ via the bijection $d$.

## 3. The Iwahori–Hecke algebra

Hereafter we fix a parameter $q \in \Bbbk$. For each $n \in \mathbb{N}$, the *Iwahori–Hecke algebra* $H_n = H_n(q)$ of rank $n$ (or of type $\mathsf{A}_{n-1}$) is an algebra generated by elements $T_1, T_2, \dots, T_{n-1}$ with defining relations

$$T_iT_j = T_jT_i \quad \text{if } |i-j| \geq 2, \qquad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad (T_i - q)(T_i + 1) = 0.$$

Here for $n = 0$ or $1$, it is defined as $H_0 = H_1 = \Bbbk$. For each $w \in \mathfrak{S}_n$, we take a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ and define an element $T_w \coloneqq T_{i_1}T_{i_2}\cdots T_{i_r}$ of $H_n$. Then it is known that it does not depend on choice of expression, and that $H_n$ is a free $\Bbbk$-module with basis $\{T_w \mid w \in \mathfrak{S}_n\}$. Thus $H_n$ can be considered as a $q$-deformation of $\Bbbk\mathfrak{S}_n$, the group ring of the symmetric group. The element $T_w$ is invertible if and only if $q \in \Bbbk^\times$; in such a case, we have $T_i^{-1} = q^{-1}(T_i - q + 1)$ and $T_w^{-1} = T_{i_r}^{-1} \cdots T_{i_2}^{-1}T_{i_1}^{-1}$. By definition, if $u, v \in \mathfrak{S}_n$ satisfy $\ell(uv) = \ell(u) + \ell(v)$ then $T_{uv} = T_uT_v$. The algebra $H_n$ has an anti-involution defined by $(T_w)^* \coloneqq T_{w^{-1}}$. Thus the category of left $H_n$-modules is equivalent to that of right modules.

For a composition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of $n$, let $H_\lambda$ be a subalgebra of $H_n$ spanned by $\{T_w \mid w \in \mathfrak{S}_\lambda\}$. Then $H_n$ is free as a right $H_\lambda$-module with basis $\{T_w \mid w \in \mathfrak{D}_\lambda\}$ by the decomposition $\mathfrak{S}_n = \mathfrak{D}_\lambda\mathfrak{S}_\lambda$. As an abstract algebra, we have an isomorphism

$$H_\lambda \simeq H_{\lambda_1} \otimes H_{\lambda_2} \otimes \cdots \otimes H_{\lambda_r}.$$

It is called a *parabolic subalgebra* of $H_n$.

In representation theory of the symmetric groups and the Iwahori–Hecke algebra, it is important to treat modules over these algebras for all ranks at once. So it is better to consider the direct sum of all their module categories. Convolution product of modules is defined as a binary operation on this category.

Definition 5.1. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a composition of $n$. For each $i = 1, 2, \dots, r$, let $V_i$ be an $H_{\lambda_i}$-module. We define the $H_n$-module

$$V_1 * V_2 * \cdots * V_r \coloneqq H_n \otimes_{H_\lambda} (V_1 \boxtimes V_2 \boxtimes \cdots \boxtimes V_r)$$

where $\boxtimes$ denotes the outer tensor product of modules. It is called the *convolution product* of $V_1, V_2, \dots, V_r$.

Obviously this product is associative up to natural isomorphism. By the basis theorem, we have a direct sum decomposition

$$V_1 * V_2 * \cdots * V_r = \bigoplus_{w \in \mathfrak{D}_\lambda} T_w(V_1 \boxtimes V_2 \boxtimes \cdots \boxtimes V_r)$$

as a $\Bbbk$-module. The convolution product $*$ defines a structure of tensor category on the direct sum of the module categories $\bigoplus_n (H_n\text{-}\mathcal{M}od)$. This tensor category also admits a braiding

$$\begin{aligned} \sigma(V, W) \colon V * W &\to W * V \\ x \boxtimes y &\mapsto T_{\varpi_{(n,m)}}(y \boxtimes x) \end{aligned}$$

in a weak sense; it satisfies the hexagon axioms of braiding but is not invertible unless $q \in \Bbbk^\times$. Here $\varpi_{(n,m)}$ is the longest element in $\mathfrak{D}_{(n,m)}$ defined by

$$\varpi_{(n,m)}(i) \coloneqq \begin{cases} i + m & \text{if } 1 \leq i \leq n, \\ i - n & \text{if } n + 1 \leq i \leq m + n. \end{cases}$$

The hexagon axioms follow from the decompositions

$$\varpi_{(n+p,m)} = (\varpi_{(n,m)}, 1_p) \cdot (1_n, \varpi_{(p,m)}), \quad \varpi_{(p,m+n)} = (1_m, \varpi_{(p,n)}) \cdot (\varpi_{(p,m)}, 1_n)$$

which preserve lengths. Here we denote by $1_n$ the unit element of $\mathfrak{S}_n$.

## 4. Parabolic modules and the $q$-Schur algebra

Let $\lambda$ be a composition. We define an element $m_\lambda \in H_\lambda$ by

$$m_\lambda := \sum_{w \in \mathfrak{S}_\lambda} T_w.$$

Note that $T_i(1+T_i) = (1+T_i)T_i = q(1+T_i)$. Hence $m_\lambda$ satisfies $T_w m_\lambda = m_\lambda T_w = q^{\ell(w)} m_\lambda$ for all $w \in \mathfrak{S}_\lambda$ since it can be also written as

$$m_\lambda = \sum_{\substack{w \in \mathfrak{S}_\lambda, \\ \ell(s_i w) > \ell(w)}} (1+T_i)T_w = \sum_{\substack{w \in \mathfrak{S}_\lambda, \\ \ell(w s_i) > \ell(w)}} T_w(1+T_i)$$

for each $s_i \in \mathfrak{S}_\lambda$. In particular, $\Bbbk m_\lambda$ is a 2-sided ideal of $H_\lambda$.

Let $M_\lambda := H_n m_\lambda$ be a left ideal of $H_n$ generated by $m_\lambda$, which we call a *parabolic module*. In particular, the *trivial module* $\mathbb{1}_n := M_{(n)}$ is a free $\Bbbk$-module of rank one spanned by $m_n := m_{(n)}$, on which every $T_w$ acts by a scalar $q^{\ell(w)}$. Since the action $H_n \curvearrowleft H_\lambda$ is free, $M_\lambda$ is isomorphic to $H_n \otimes_{H_\lambda} \Bbbk m_\lambda$ as an $H_n$-module; so it has a basis $\{T_w m_\lambda \mid w \in \mathfrak{D}_\lambda\}$ over $\Bbbk$. Or equivalently, by using convolution product, we can also represent it as $M_\lambda \simeq \mathbb{1}_{\lambda_1} * \mathbb{1}_{\lambda_2} * \cdots * \mathbb{1}_{\lambda_r}$. Elements of $M_\lambda \subset H_n$ are characterized as

$$M_\lambda = \{x \in H_n \mid x T_w = q^{\ell(w)} x \text{ for all } w \in \mathfrak{S}_\lambda\}$$

because for $x = \sum_{w \in \mathfrak{S}_n} x_w T_w$ ($x_w \in \Bbbk$), $xT_i = qx$ is equivalent to that $x_w = x_{ws_i}$ for all $w \in \mathfrak{S}_n$. For each $w \in \mathfrak{D}_\lambda$, we take the corresponding row-standard tableau $\mathsf{T}$ such that $w = d(\mathsf{T})$ and write $m_{\mathsf{T}} := T_w m_\lambda$. The action of $H_n$ on it is described as follows: suppose each number $i$ is contained in the $r(i)$-th row of $\mathsf{T}$. Then

$$T_i \cdot m_{\mathsf{T}} = \begin{cases} q m_{\mathsf{T}} & \text{if } r(i) = r(i+1), \\ m_{s_i \mathsf{T}} & \text{if } r(i) < r(i+1), \\ q m_{\mathsf{T}} + (q-1) m_{s_i \mathsf{T}} & \text{if } r(i) > r(i+1). \end{cases}$$

We similarly define right ideals $M^*_\lambda := m_\lambda H_n$ and $\mathbb{1}^*_n := M^*_{(n)}$. Then we have

$$M^*_\lambda = \{x \in H_n \mid T_w x = q^{\ell(w)} x \text{ for all } w \in \mathfrak{S}_\lambda\}.$$

Now take two compositions $\lambda, \mu$ of $n$. Since $M_\mu$ is a cyclic module generated by $m_\mu$ with the relations $T_w m_\mu = q^{\ell(w)} m_\mu$ for every $w \in \mathfrak{S}_\mu$, by taking the image of the generator $m_\mu$ we have an isomorphism

$$\mathrm{Hom}_{H_n}(M_\mu, M_\lambda) \simeq \{x \in M_\lambda \mid T_w x = q^{\ell(w)} x \text{ for all } w \in \mathfrak{S}_\mu\} = M_\lambda \cap M^*_\mu.$$

Let us write $M_{\lambda;\mu} := M_\lambda \cap M^*_\mu$. The collection of these $\Bbbk$-modules has a natural product

$$\begin{aligned} \circ_\mu \colon M_{\mu;\nu} \otimes M_{\lambda;\mu} &\to M_{\lambda;\nu}, \\ x m_\mu \otimes m_\mu y &\mapsto x m_\mu y. \end{aligned}$$

According to the isomorphism above, this product corresponds to the opposite of the composition of homomorphisms. Note that $M_{\lambda;\mu}$ naturally acts on the parabolic module $M_\mu$ from right, so that the composition is given by the reversed product. On the other hand, $M_{\lambda;\mu}$ is also isomorphic to $\mathrm{Hom}_{H_n^{\mathrm{op}}}(M^*_\lambda, M^*_\mu)$, the set of homomorphisms between right modules. In this view, the product is just same as the composition of such homomorphisms. Anyway, the algebra with this product

$$\mathscr{S}_{r,n} := \bigoplus_{\lambda,\mu} M_{\lambda;\mu} \simeq \mathrm{End}_{H_n}\Big(\bigoplus_\lambda M_\lambda\Big)^{\mathrm{op}} \simeq \mathrm{End}_{H_n^{\mathrm{op}}}\Big(\bigoplus_\lambda M^*_\lambda\Big)$$

is called the *$q$-Schur algebra* which is introduced by Dipper and James [**DJ89**]. Here $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_r)$ runs over all compositions of $n$ whose components are zero except for the first $r$ ones. Note that the Iwahori–Hecke algebra itself can be obtained similarly:

$$H_n = M_{(1^n);(1^n)} \simeq \mathrm{End}_{H_n}(M_{(1^n)})^{\mathrm{op}} \simeq \mathrm{End}_{H_n^{\mathrm{op}}}(M^*_{(1^n)}).$$

Since we can write

$$M_{\lambda;\mu} = \{x \in H_n \mid T_v x T_w = q^{\ell(v)+\ell(w)} x \text{ for all } v \in \mathfrak{S}_\mu, w \in \mathfrak{S}_\lambda\},$$

it has a basis $\{\sum_{v\in\mathfrak{S}_\mu w\mathfrak{S}_\lambda} T_v \mid w \in \mathfrak{D}_\lambda \cap \mathfrak{D}_\mu^{-1}\}$ which corresponds to the double cosets $\mathfrak{S}_\mu\backslash\mathfrak{S}_n/\mathfrak{S}_\lambda$. Similarly as before, for $w \in \mathfrak{D}_\lambda \cap \mathfrak{D}_\mu^{-1}$ we take the corresponding row-semistandard tableau $\mathsf{S} \in \mathrm{Tab}_{\lambda;\mu}$ such that $w = d(\mathsf{S}_\downarrow)$ and write $m_\mathsf{S} \coloneqq \sum_{v\in\mathfrak{S}_\mu w\mathfrak{S}_\lambda} T_v$. As an element of $M_\lambda$, we can decompose it as $m_\mathsf{S} = \sum_{\mathsf{T}\in\mathrm{Tab}_\mathsf{S}} m_\mathsf{T}$. The anti-involution on $H_n$ induces a map

$$\bullet^* \colon M_{\lambda;\mu} \to M_{\mu;\lambda}$$

which induces that on $\mathscr{S}_{r,n}$. By definition we have $(m_\mathsf{S})^* = m_{\mathsf{S}^*}$.

## 5. Decomposing a tableau

In this section we observe that for each $\mathsf{S} \in \mathrm{Tab}_{\lambda;\mu}$, $m_\mathsf{S} \in M_{\lambda;\mu}$ has a canonical decomposition

$$m_\mathsf{S} = m_\mu \circ_\nu m_{P_{w,\nu}} \circ_{w\nu} m_\lambda$$

into three tableaux. We first explain each of these terms.

Let $\mu$ and $\nu$ be compositions of $n$. We say that $\nu$ is a *refinement* of $\mu$ when there is an increasing sequence of indices $1 \le a_1 \le a_2 \le \cdots$ such that $\mu_i = \sum_{a_i \le j < a_{i+1}} \nu_j$. Clearly it is equivalent to that $\mathfrak{S}_\nu \subset \mathfrak{S}_\mu$. Hence $m_\mu$ is contained in both $M_{\mu;\nu}$ and $M_{\nu;\mu}$. As elements of these sets, $m_\mu$ is respectively represented by tableaux $\mathsf{S}$ and its dual $\mathsf{S}^*$ defined by $\mathsf{S}^*(j,k) \coloneqq i$ for $a_i \le j < a_{i+1}$, such as

$$\mathsf{S} = \begin{array}{|c|c|c|c|c}\hline 1&2&2&3& \\ \hline 4&4&4&5&\multicolumn{1}{c|}{5} \\ \hline\end{array} \qquad \text{and} \qquad \mathsf{S}^* = \begin{array}{|c|c|c|}\hline 1 & \multicolumn{2}{c}{} \\ \hline 1&1& \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{2}{c}{} \\ \hline 2&2&2 \\ \hline 2&2& \multicolumn{1}{c}{} \\ \hline\end{array}$$

for $\mu = (4,5)$ and $\nu = (1,2,1,3,2)$. For $\mathsf{T} \in \mathrm{Tab}_{\lambda;\nu}$ of weight $\nu$, let $\mathsf{T}|_\mu \in \mathrm{Tab}_{\lambda;\mu}$ be a row-standard tableau of weight $\mu$ obtained by replacing each entry $j$ in $\mathsf{T}$ such that $a_i \le j < a_{i+1}$ with $i$, similarly as before. Since $(1^n)$ is a refinement of every composition, this notation coincides with the previous one.

Lemma 5.2. *Let $\nu$ be a refinement of $\mu$, and take $a_1 \le a_2 \cdots$ as above.*

(1) *For $\mathsf{S} \in \mathrm{Tab}_{\lambda;\mu}$, we have*

$$m_\mu \circ_\mu m_\mathsf{S} = \sum_{\mathsf{T}\in\mathrm{Tab}_{\lambda;\nu}, \mathsf{T}|_\mu = \mathsf{S}} m_\mathsf{T} \in M_{\lambda;\nu}$$

*where $m_\mu$ is regarded as an element of $M_{\mu;\nu}$.*

(2) *For $\mathsf{T} \in \mathrm{Tab}_{\lambda;\nu}$, we have*

$$m_\mu \circ_\nu m_\mathsf{T} = \left(\prod_i q^{\ell_i} \prod_k \begin{bmatrix} \#_{ki}(\mathsf{T}|_\mu) \\ \#_{ka_i}(\mathsf{T}), \#_{k,a_i+1}(\mathsf{T}) \ldots, \#_{k,a_{i+1}-1}(\mathsf{T}) \end{bmatrix}\right) m_{\mathsf{T}|_\mu}.$$

*Here $\ell_i \coloneqq \#\{((k,l),(k',l')) \mid k < k',\, a_i \le \mathsf{T}(k',l') < \mathsf{T}(k,l) < a_{i+1}\}$ is the inversion number of $\mathsf{T}$ for entries $j$ such that $a_i \le j < a_{i+1}$.*

Proof. (1). By definition, as an element of $M_\lambda$, $m_\mu \circ_\mu m_\mathsf{S}$ is just $m_\mathsf{S} = \sum_{\mathsf{R}\in\mathrm{Tab}_\mathsf{S}} m_\mathsf{R}$. Hence the formula is clear from that $(\mathsf{R}|_\nu)|_\mu = \mathsf{R}|_\mu$.

(2). Let us write $\mathsf{S} \coloneqq \mathsf{T}|_\mu$. It suffices to prove for the universal case $\Bbbk = \mathbb{Z}[q]$ where $q$ is an indeterminate. First we compute an ordinal product $m_\mu \cdot m_\mathsf{T}$ in

$M_\lambda$. We can take $w \in \mathfrak{S}_\mu$ such that $\mathsf{T}_\downarrow = w \cdot \mathsf{S}_\downarrow$, then $\ell(w) = \sum_i \ell_i$. Since $m_\mathsf{T} = \sum_{v \in \mathfrak{D}_\mathsf{T}} m_{v \cdot \mathsf{T}_\downarrow}$ and $\mathfrak{D}_\mathsf{T} \subset \mathfrak{S}_\mu$,

$$m_\mu \cdot m_\mathsf{T} = q^{\ell(w)} P_{\mathfrak{D}_\mathsf{T}}(q) m_\mu \cdot m_{\mathsf{S}_\downarrow} = q^{\ell(w)} P_{\mathfrak{D}_\mathsf{T}}(q) \Big( \prod_{k,i} [\#_{ki}(\mathsf{S})]! \Big) m_\mathsf{S}.$$

On the other hand, we have $m_\mu \cdot m_\mathsf{T} = (\prod_j [\nu_j]!) m_\mu \circ_\nu m_\mathsf{T}$. Since $M_\lambda$ is a free module over an integral domain $\mathbb{Z}[q]$, we can cancel this coefficient. Thus the formula follows from

$$\frac{P_{\mathfrak{D}_\mathsf{T}}(q) \prod_{k,i} [\#_{ki}(\mathsf{S})]!}{\prod_j [\nu_j]!} = \frac{\prod_{k,i} [\#_{ki}(\mathsf{S})]!}{\prod_{k,j} [\#_{kj}(\mathsf{T})]!} = \prod_k \begin{bmatrix} \#_{ki}(\mathsf{S}) \\ \#_{ka_i}(\mathsf{T}), \dots, \#_{k,a_{i+1}-1}(\mathsf{T}) \end{bmatrix}.$$

□

Next we introduce the middle term of the decomposition.

Definition 5.3. Let $\nu = (\nu_1, \nu_2, \dots, \nu_r)$ be a composition of $n$ and $w \in \mathfrak{S}_r$. Let us write $w\nu := (\nu_{w(1)}, \nu_{w(2)}, \dots, \nu_{w(r)})$. We define $\mathsf{P}_{w,\nu} \in \mathrm{Tab}_{w\nu;\nu}$ by

$$\mathsf{P}_{w,\nu}(i,j) := w(i)$$

and call it the *permutation tableau* with respect to $w$.

The composition with a permutation tableau is complicated in general, so we prove a multiplication formula only for a special case.

Lemma 5.4. *Let $\nu$ and $w$ as above. Suppose $\mathsf{T} \in \mathrm{Tab}_{\lambda;w\nu}$ satisfies that for each pair of boxes $(i,j),(k,l) \in Y(\lambda)$, $i \le k$ and $\mathsf{T}(k,l) < \mathsf{T}(i,j)$ implies $w(\mathsf{T}(k,l)) < w(\mathsf{T}(i,j))$. Then we have $m_{\mathsf{P}_{w,\nu}} \circ_{w\nu} m_\mathsf{T} = m_{w\mathsf{T}}$.*

Proof. The tableau $w\mathsf{T}$ is also row-standard by the assumption. By the definition of permutation tableau, there is a permutation $v \in \mathfrak{S}_n$ such that $m_{\mathsf{P}_{w,\nu}} = T_v m_{w\nu}$. The formula follows from that every $\mathsf{R} \in \mathrm{Tab}_\mathsf{T}$ satisfies $v\mathsf{R} \in \mathrm{Tab}_{w\mathsf{T}}$ and $\ell(v\mathsf{R}) = \ell(v) + \ell(\mathsf{R})$. □

Proposition 5.5. *For each $\mathsf{S} \in \mathrm{Tab}_{\lambda;\mu}$, there exists a unique pair $(\nu, w)$ of a composition $\nu = (\nu_1, \nu_2, \dots, \nu_r)$ with $\nu_1, \nu_2, \dots, \nu_r > 0$ and a permutation $w \in \mathfrak{S}_r$ such that $w\nu$ and $\nu$ are respectively refinements of $\lambda$ and $\mu$, and*

$$m_\mathsf{S} = m_\mu \circ_\nu m_{P_{w,\nu}} \circ_{w\nu} m_\lambda.$$

Proof. For such $\mathsf{S}$, it suffices to put

$$\nu := (\#_{11}(\mathsf{S}), \#_{21}(\mathsf{S}), \dots, \#_{12}(\mathsf{S}), \#_{22}(\mathsf{S}), \dots, \#_{13}(\mathsf{S}), \#_{23}(\mathsf{S}), \dots),$$
$$w\nu := (\#_{11}(\mathsf{S}), \#_{12}(\mathsf{S}), \dots, \#_{21}(\mathsf{S}), \#_{22}(\mathsf{S}), \dots, \#_{31}(\mathsf{S}), \#_{32}(\mathsf{S}), \dots)$$

with removing zero entries $\#_{ij}(\mathsf{S}) = 0$, and take the corresponding permutation $w$. Then by the two lemmas above we have a desired decomposition. For example,

$$\begin{array}{|c|c|c|c|c|} \hline 2&2&3&3&3 \\ \hline 1&1&1&1 \\ \cline{1-4} 1&3&3 \\ \cline{1-3} \end{array} = \begin{array}{|c|c|c|c|} \hline 1&1&1&1 \\ \hline 1 \\ \cline{1-1} 2&2 \\ \cline{1-2} 3&3&3 \\ \cline{1-3} 3&3 \\ \cline{1-2} \end{array} \circ_{(4,1,2,3,2)} \begin{array}{|c|c|c|c|} \hline 3&3 \\ \cline{1-2} 4&4&4 \\ \hline 1&1&1&1 \\ \hline 2 \\ \cline{1-1} 5&5 \\ \cline{1-2} \end{array} \circ_{(2,3,4,1,2)} \begin{array}{|c|c|c|c|c|} \hline 1&1&2&2&2 \\ \hline 3&3&3&3 \\ \cline{1-4} 4&5&5 \\ \cline{1-3} \end{array}$$

where we represent an element $m_\mathsf{T}$ by the tableau $\mathsf{T}$ itself for short. The uniqueness is obvious from this construction. □

## 6. Good tableaux

We introduce a partial order $\leq$ on the set of compositions of $n \in \mathbb{N}$ called the *dominance order*. Here for two compositions $\lambda$ and $\mu$, they are defined to be $\lambda \leq \mu$ if and only if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \leq \mu_1 + \mu_2 + \cdots + \mu_k$$

is satisfied for each $k \in \mathbb{N}$. It is not a total order; for example, the compositions $(3,3)$ and $(4,1,1)$ are incomparable. According to the *reversed* dominance order, we make a filtration on the module category as we did in the previous part. For each composition $\lambda$, the set $\{\mu \,|\, \mu > \lambda\}$ is finite. Hence the set of all compositions with the reversed dominance order is a well-founded partially ordered set.

Notation 5.6. Let $X, Y \in H_n\text{-}\mathcal{M}od$. For a composition $\lambda$, let

$$\mathcal{H}^\lambda(X,Y) \coloneqq \operatorname{Hom}_{H_n}(M_\lambda, Y) \circ \operatorname{Hom}_{H_n}(X, M_\lambda)$$

be the set of homomorphisms which factor through $M_\lambda$. In other words, $\mathcal{H}^\lambda$ is a 2-sided ideal of $H_n\text{-}\mathcal{M}od$ generated by $M_\lambda$. By using the dominance order we define

$$\mathcal{H}^{\geq\lambda}(X,Y) \coloneqq \sum_{\mu\geq\lambda} \mathcal{H}^\mu(X,Y), \quad \mathcal{H}^{>\lambda}(X,Y) \coloneqq \sum_{\mu>\lambda} \mathcal{H}^\mu(X,Y)$$

and

$$\operatorname{Hom}^{(\lambda)}_{H_n}(X,Y) \coloneqq \operatorname{Hom}_{H_n}(X,Y)/\mathcal{H}^{>\lambda}(X,Y).$$

The last one is a hom set in the quotient category $(H_n\text{-}\mathcal{M}od)/\mathcal{H}^{>\lambda}$.

When $X$ and $Y$ above are parabolic modules, we write corresponding submodules or quotient modules of $M_{\lambda;\mu}$ as $M^\nu_{\lambda;\mu}$, $M^{\geq\nu}_{\lambda;\mu}$, $M^{>\nu}_{\lambda;\mu}$ and $M^{(\nu)}_{\lambda;\mu}$ respectively. In particular,

$$M^{(\nu)}_{\lambda;\mu} \simeq \operatorname{Hom}^{(\nu)}_{H_n}(M_\mu, M_\lambda)$$

is the $\Bbbk$-module equipped with the reversed composition as product. As its special case we let $S_{\lambda;\mu} \coloneqq M^{(\lambda)}_{\lambda;\mu}$. Then $S_{\lambda;\lambda}$ is a quotient algebra of $M_{\lambda;\lambda}$ and $S_{\lambda;\mu}$ is a *right* module over this algebra. When $\mu = (1^n)$ we simply write $S_\lambda \coloneqq S_{\lambda;(1^n)}$. $S_\lambda$ is also a *left* module over $H_n \simeq M_{(1^n);(1^n)}$ and called the *Specht module*. We denote equalities in the quotient set $S_{\lambda;\mu}$ by the symbol $\equiv$.

Note that if a composition $\lambda = (\lambda_1, \lambda_2, \dots)$ has $\lambda_i = 0$ such that $\lambda_{i+1} \neq 0$, letting $\tilde{\lambda} \coloneqq (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots)$ we have $\lambda < \tilde{\lambda}$ and $M_\lambda \simeq M_{\tilde{\lambda}}$. Hence for such $\lambda$, $M_\lambda$ is zero in the quotient category $(H_n\text{-}\mathcal{M}od)/\mathcal{H}^{>\lambda}$; in particular we have $S_{\lambda;\mu} = 0$ for all $\mu$. We can remove such needless compositions from the index set. Then the rest is now a finite set.

For a while we fix $n \in \mathbb{N}$ and $\lambda, \mu$ denote compositions of $n$. In order to study this quotient category, we introduce a combinatorial notion on tableaux as follows.

Definition 5.7. Let $\mathsf{T} \in \mathrm{Tab}_{\lambda;\mu}$ be a row-semistandard tableau. We say that a box $(i,j) \in Y(\lambda)$ in the Young diagram is *good* if it satisfies $\mathsf{T}(i,j) \geq i$, and $\mathsf{T}$ is said to be *good* if all boxes in $Y(\lambda)$ are good.

Lemma 5.8. *$S_{\lambda;\mu}$ is spanned by* $\{m_\mathsf{T} \,|\, \mathsf{T} \in \mathrm{Tab}_{\lambda;\mu}$ which is good$\}$.

Proof. Suppose that $\mathsf{T}$ is not good. For such $\mathsf{T}$, let us define a tableau $\mathsf{T}_1$ of shape $\lambda$ by

$$\mathsf{T}_1(i,j) \coloneqq \min\{i, \mathsf{T}(i,j)\}.$$

Next let $\mathsf{T}_2$ be a tableau obtained by moving up all ungood boxes $\boxed{i}$ of $\mathsf{T}$ to its $i$-th row, so that $\mathsf{T}_2$ is good. For example, when

$$\mathsf{T} = \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&2&3\\\hline 2&2&3&3\\\cline{1-4} 1&2&3\\\cline{1-3}\end{array}$$

which has ungood 1 and 2 in the third row, we let

$$\mathsf{T}_1 = \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&1\\\hline 2&2&2&2\\\cline{1-4} 1&2&3\\\cline{1-3}\end{array} \quad \text{and} \quad \mathsf{T}_2 = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&2&3\\\hline 2&2&2&3&3\\\cline{1-5} 3\\\cline{1-1}\end{array}.$$

Let $\nu$ be the weight of $\mathsf{T}_1$, which is equal to the shape of $\mathsf{T}_2$. For each $k$ we have

$$\nu_1 + \nu_2 + \cdots + \nu_k = \lambda_1 + \lambda_2 + \cdots + \lambda_k + \#\{(i,j) \in Y(\lambda) \mid i > k,\, T(i,j) \leq k\}.$$

Since $\mathsf{T}$ is not good, we have $\nu > \lambda$ so that $m_{\mathsf{T}_2} \circ_\nu m_{\mathsf{T}_1} \equiv 0$ in $S_{\lambda;\mu}$.

On the other hand, observe that the $i$-th row of $\mathsf{T}_2$ is obtained by reading entries of $\mathsf{T}$ at boxes $(k,l)$ such that $\mathsf{T}_1(k,l) = i$ from bottom to top. So taking $w := d(\mathsf{T}_2^\uparrow) \in \mathfrak{D}_\nu$ we have $\mathsf{T}^\uparrow = w\mathsf{T}_1^\uparrow$ and $\ell(\mathsf{T}^\uparrow) = \ell(w) + \ell(\mathsf{T}_1^\uparrow)$. This induces the following decomposition in $M_{\lambda;\mu}$:

$$m_{\mathsf{T}_2} \circ_\nu m_{\mathsf{T}_1} = m_{\mathsf{T}} + \sum_{\mathsf{S} \in \mathrm{Tab}_{\lambda;\mu},\, \ell(\mathsf{S}^\uparrow) < \ell(\mathsf{T}^\uparrow)} c_{\mathsf{S}} m_{\mathsf{S}} \qquad (c_{\mathsf{S}} \in \Bbbk).$$

Hence in $S_{\lambda;\mu}$ we can replace ungood $m_{\mathsf{T}}$ by a linear combination of elements $m_{\mathsf{S}}$ which has smaller lengths. Consequently it inductively follows that any tableau can be written as a linear combination of good ones. □

LEMMA 5.9. (1) $\mathrm{Tab}_{\lambda;\lambda}$ *has only one good tableau.*
(2) *There are no good tableau in* $\mathrm{Tab}_{\lambda;\mu}$ *unless* $\lambda \geq \mu$.

PROOF. If $\mathsf{T} \in \mathrm{Tab}_{\lambda;\mu}$ is good, then for each $k$, all $i$'s in $\mathsf{T}$ less than or equal to $k$ are placed in its $k$-th row or upper. The number of such numbers ($= \mu_1 + \cdots + \mu_k$) must be equal to or less than that of such boxes ($= \lambda_1 + \cdots + \lambda_k$) so we have $\lambda \geq \mu$. Moreover if $\lambda = \mu$, all $i$'s in $\mathsf{T}$ must be in its $i$-th row. □

By these two lemmas, the statements below are obvious.

COROLLARY 5.10. (1) $S_{\lambda;\lambda}$ *is spanned by* $m_\lambda$. *Hence it is isomorphic to a quotient ring of* $\Bbbk$.
(2) $S_{\lambda;\mu} = 0$ *unless* $\lambda \geq \mu$. □

Hence it satisfies the assumptions in Lemma 0.5, so it produces several standardly filtered algebras.

THEOREM 5.11. *The Iwahori–Hecke algebra* $H_n = M_{(1^n;1^n)}$ *and the* $q$*-Schur algebra* $\mathscr{S}_{r,n} = \bigoplus_{\lambda,\mu} M_{\lambda;\mu}$ *are standardly filtered algebras over* $\Bbbk$ *on the set of compositions. Here for each composition* $\nu$*, their ideal filter and attached Morita contexts is given by*

$$H_n^{\geq\nu} := M_{(1^n;1^n)}^{\geq\nu} \quad \textit{with} \quad (S_\nu, S_\nu^*)$$

*and*

$$\mathscr{S}_{r,n}^{\geq\nu} := \bigoplus_{\lambda,\mu} M_{\lambda;\mu}^{\geq\nu} \quad \textit{with} \quad \Big(\bigoplus_\lambda S_{\nu;\lambda}, \bigoplus_\lambda S_{\nu;\lambda}^*\Big)$$

*where* $S_{\nu;\lambda}^* := M_{\lambda;\nu}^{(\nu)}$ *and* $S_\nu^* := S_{\nu;(1^n)}^*$. *These standard filters are involutive.* □

It seems to be an interesting problem to determine the $\Bbbk$-module structure of $S_{\lambda;\mu}$ (or more general $M_{\lambda;\mu}^{(\nu)}$) in detail. For the case that $q$ is invertible we can completely determine its structure by taking its free basis as we will study in later sections. In the other case the situation is more complicated so that these modules even need not to be free. The author conjectures that $S_{\lambda;\lambda}$ is isomorphic to $\Bbbk$ or $\Bbbk/q^{h(\lambda)}\Bbbk$ for some $h(\lambda) \in \mathbb{N}$ determined by the shape of $\lambda$, but a general one is still unable to describe.

## 7. Local transformations in Specht modules

In this section we prove useful formulas for computation on Specht modules.

LEMMA 5.12. *Suppose we have an equation $\sum_{\mathsf{T}} c_{\mathsf{T}} m_{\mathsf{T}} \equiv 0$ in $S_{\lambda;\mu}$ for some $c_{\mathsf{T}} \in \Bbbk$. Take an arbitrary sequence $a_1 \leq a_2 \leq \cdots \leq a_k$. For each $\mathsf{T} \in \mathrm{Tab}_{\lambda;\mu}$ let $\mathsf{T}^+$ be the tableau obtained by adding a new row $\boxed{a_1}\boxed{a_2}\boxed{\cdots}\boxed{a_k}$ at the top of $\mathsf{T}$. Then we have an equation $\sum_{\mathsf{T}} c_{\mathsf{T}} m_{\mathsf{T}^+} \equiv 0$ in $S_{(k,\lambda);\mu^+}$, where $(k,\lambda) := (k, \lambda_1, \lambda_2, \dots)$ and $\mu^+_j = \mu_j + \#\{i \mid a_i = j\}$.*

PROOF. First note that the convolution functor with trivial module

$$\mathbb{1}_k * \bullet \colon (H_n\text{-}\mathcal{M}od)/\mathcal{H}^{>\lambda} \to (H_{k+n}\text{-}\mathcal{M}od)/\mathcal{H}^{>(k,\lambda)}$$

is still well-defined, because for any $V \to W$ which factors through some $M_\nu$ for $\nu > \lambda$, corresponding $\mathbb{1}_k * V \to \mathbb{1}_k * W$ factors through $M_{(k,\nu)}$ with $(k,\nu) > (k,\lambda)$. For each $\mathsf{T}$, let us define $\mathsf{T}^\# \in \mathrm{Tab}_{(k,\lambda);(k,\mu)}$ by

$$\mathsf{T}^\#(i,j) = \begin{cases} 1 & \text{if } i = 1, \\ \mathsf{T}(i-1,j) + 1 & \text{otherwise,} \end{cases}$$

so that $m_{\mathsf{T}^\#} = \mathbb{1}_k * m_{\mathsf{T}}$. On the other hand, let $\mathsf{R} \in \mathrm{Tab}_{(k,\mu);\mu^+}$ be the tableau defined by

$$\mathsf{R}(i,j) = \begin{cases} a_j & \text{if } i = 1, \\ i-1 & \text{otherwise.} \end{cases}$$

Then we have $m_{\mathsf{T}^+} = m_{\mathsf{R}} \circ_{(k,\mu)} m_{\mathsf{T}^\#}$ by the decomposition of $m_{\mathsf{R}}$ according to Proposition 5.5 and the formulas in Lemma 5.2 and Lemma 5.4. Hence

$$\sum_{\mathsf{T}} c_{\mathsf{T}} m_{\mathsf{T}^+} = m_{\mathsf{R}} \circ_{(k,\mu)} \sum_{\mathsf{T}} c_{\mathsf{T}} m_{\mathsf{T}^\#} = m_{\mathsf{R}} \circ_{(k,\mu)} \Big( \mathbb{1}_k * \sum_{\mathsf{T}} c_{\mathsf{T}} m_{\mathsf{T}} \Big) \equiv 0.$$

□

By the same argument, we can also add a new row to the bottom of tableaux. For the bottom row of a tableau we have another kind of formula.

LEMMA 5.13. *Let $\sum_{\mathsf{T}} c_{\mathsf{T}} m_{\mathsf{T}} \equiv 0 \in S_{\lambda;\mu}$ as above. Take a number $a$ which is greater than or equal to any entries of $\mathsf{T}$ (so $\mu_i = 0$ for $i > a$). For each $\mathsf{T} \in \mathrm{Tab}_{\lambda;\mu}$, let $\mathsf{T}^+$ be the tableau obtained by joining a bar $\boxed{a}\boxed{a}\boxed{\cdots}\boxed{a}$ of length $l$ at the right of the bottom row of $\mathsf{T}$. Then we also have $\sum_{\mathsf{T}} c_{\mathsf{T}} \left[\begin{smallmatrix} \#_{ra}(\mathsf{T})+l \\ l \end{smallmatrix}\right] m_{\mathsf{T}^+} \equiv 0$ in $S_{\lambda^+;\mu^+}$ where $\lambda^+ := (\lambda_1, \dots, \lambda_{r-1}, \lambda_r + l)$ and $\mu^+ := (\mu_1, \dots, \mu_{a-1}, \mu_a + l)$.*

PROOF. For a composition $\nu = (\nu_1, \dots, \nu_r)$, we write $(\nu, l) := (\nu_1, \dots, \nu_r, l)$. We define $\mathsf{T}^\# \in \mathrm{Tab}_{(\lambda,l);\mu^+}$ for each $\mathsf{T} \in \mathrm{Tab}_{\lambda;\mu}$ by

$$\mathsf{T}^\#(i,j) := \begin{cases} \mathsf{T}(i,j) & \text{if } i \leq r, \\ a & \text{if } i = r+1 \end{cases}$$

and $\mathsf{R} \in \mathrm{Tab}_{\lambda^+;(\lambda,l)}$ by

$$\mathsf{R}(i,j) := \begin{cases} i & \text{if } i < r \text{ or } (i = r,\ j \leq \lambda_r), \\ r+1 & \text{if } i = r,\ j > \lambda_r, \end{cases}$$

so that $\left[\begin{smallmatrix} \#_{ra}(\mathsf{T})+l \\ l \end{smallmatrix}\right] m_{\mathsf{T}^+} = m_{\mathsf{T}^\#} \circ_{(\lambda,l)} m_{\mathsf{R}}$ similarly to the previous proof. By the similar argument we can prove $\sum_{\mathsf{T}} c_{\mathsf{T}} m_{\mathsf{T}^\#} \equiv 0$, and more strongly, this element can be written as a linear combination of elements which factor through $M_{(\nu,l)}$ for $\nu > \lambda$. This implies $(\nu, l) \not\leq \lambda^+$; thus by Corollary 5.10, in $S_{\lambda^+;\mu^+}$ we have

$$\sum_{\mathsf{T}} c_{\mathsf{T}} \begin{bmatrix} \#_{ra}(\mathsf{T}) + l \\ l \end{bmatrix} m_{\mathsf{T}^+} = \sum_{\mathsf{T}} c_{\mathsf{T}} m_{\mathsf{T}^\#} \circ_\nu m_{\mathsf{R}} \equiv 0.$$

□

The formula below will be needed for a later computation.

Lemma 5.14. *Let $k, l, n \in \mathbb{N}$ such that $k \le l \le n$ and let $\lambda := (n-k, k)$ and $\mu := (n-l, l)$. For each $i$, let $\mathsf{T}_i \in \mathrm{Tab}_{\lambda;\mu}$ be the tableau determined by $\#_{21}(\mathsf{T}_i) = i$, that is, it is in the form*

$$\mathsf{T}_i = \begin{array}{l}\boxed{1\,|\,1\,|\cdots|\,1\,|\,2\,|\,2\,|\cdots|\,2}\\ \underbrace{\boxed{1\,|\cdots|\,1}}_{i}\boxed{2\,|\cdots|\,2}\end{array}.$$

*Then we have $m_{\mathsf{T}_i} \equiv (-1)^i q^{\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix} m_{\mathsf{T}_0}$ in $S_{\lambda;\mu}$.*

Proof. We prove it by an induction on $k$. The case $i = 0$ is trivial so assume that $0 < i \le k$. For $i < k$, using the assumption of induction, the formula is implied by the lemma above. On the other hand, by Lemma 5.2 (1) we have

$$0 \equiv \boxed{1\,|\,1\,|\,1\,|\cdots|\,1\,|\,2\,|\cdots|\,2} \circ_{(n)} \begin{array}{l}\boxed{1\,|\,1\,|\,1\,|\cdots|\cdots|\cdots|\cdots|\,1}\\ \boxed{1\,|\,1\,|\cdots|\cdots|\cdots|\,1}\end{array} = \sum_{0 \le i \le k} m_{\mathsf{T}_i},$$

so that the statement also holds for $i = k$ by the formula

$$\sum_{0 \le i \le k} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix} = 0 \qquad \text{implied by} \qquad \prod_{0 \le i < k} (1 + q^k t) = \sum_{0 \le i \le k} q^{\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix} t^i.$$

□

Multiplying an element to the both-hand sides of this formula for $i = k = l$, we obtain the following corollary by Lemma 5.2 (1).

Corollary 5.15. *Let $\lambda = (n-k, k)$ as above. For arbitrary entries $a_1 \le \cdots \le a_k$, we have*

$$\begin{array}{l}\boxed{1\,|\,1\,|\cdots|\,1\,|\,a_1\,|\cdots|\,a_k}\\ \boxed{1\,|\cdots|\,1}\end{array} \equiv (-1)^k q^{\binom{k}{2}} \begin{array}{l}\boxed{1\,|\,1\,|\cdots|\cdots|\cdots|\cdots|\,1}\\ \boxed{a_1\,|\cdots|\,a_k}\end{array}.$$

□

## 8. Semistandard tableaux

Hereafter in this chapter we assume $q \in \Bbbk^\times$. Then the braiding $\sigma$ of the convolution $*$ is now invertible so we have $M_\lambda \simeq M_{w\lambda}$ for any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ and $w \in \mathfrak{S}_r$. Recall that a composition $\lambda$ is called a *partition* if it is a descending sequence: $\lambda_1 \ge \lambda_2 \ge \ldots$. So in this case, unless $\lambda$ is a partition, we can take some $w$ such that $\lambda < w\lambda$, so that $M_\lambda$ is zero in the quotient category $(H_n\text{-}\mathcal{M}od)/\mathcal{H}^{>\lambda}$ again.

A row-semistandard tableau $\mathsf{T} \in \mathrm{Tab}_{\lambda;\mu}$ is called a *semistandard tableau* if its shape $\lambda$ is a partition and for all vertically adjacent boxes $(i,j), (i+1,j) \in Y(\lambda)$ it satisfies $T(i,j) < T(i+1,j)$; or equivalently, all its columns are strictly increasing. We denote by $\mathrm{STab}_{\lambda;\mu}$ the set of all semistandard tableaux of shape $\lambda$ of weight $\mu$. Note that the strictly increasing condition clearly implies that every semistandard tableau is good. Now we can improve a lemma in the previous section.

Lemma 5.16. *$S_{\lambda;\mu}$ is spanned by $\{m_{\mathsf{T}} \mid \mathsf{T} \in \mathrm{STab}_{\lambda;\mu}\}$.*

Proof. The statement is clear if $\lambda$ is not a partition, so we may assume so. Suppose $\mathsf{T}$ is not semistandard and take its box $(k,l) \in Y(\lambda)$ such that $\mathsf{T}(k,l) \ge \mathsf{T}(k+1,l)$. Let $\nu$ be a composition

$$\nu := (\lambda_1, \ldots, \lambda_{k-1}, l-1, \lambda_k + 1, \lambda_{k+1} - l, \lambda_{k+2}, \lambda_{k+3}, \ldots).$$

We define tableaux $\mathsf{T}_1 \in \mathrm{Tab}_{\lambda;\nu}$ and $\mathsf{T}_2 \in \mathrm{Tab}_{\nu;\mu}$ by

$$\mathsf{T}_1(i,j) = \begin{cases} i & \text{if } i < k \text{ or } (i = k,\ j < l) \text{ or } (i = k+1,\ j \leq l), \\ i+1 & \text{otherwise}, \end{cases}$$

$$\mathsf{T}_2(i,j) = \begin{cases} \mathsf{T}(i,j) & \text{if } i \leq k \text{ or } (i = k+1,\ j \leq l), \\ \mathsf{T}(k, j-1) & \text{if } i = k+1,\ j > l, \\ \mathsf{T}(k+1, j+l) & \text{if } i = k+2, \\ \mathsf{T}(i-1, j) & \text{if } i > k+2. \end{cases}$$

For example, when

$$\mathsf{T} = \begin{array}{lllllll} 1 & 1 & 1 & 2 & 2 & 3 & 4 \\ 1 & 2 & 2 & 2 & 3 & & \\ 2 & 5 & & & & & \end{array}$$

and $(k,l) = (1,4)$, the corresponding tableaux are

$$\mathsf{T}_1 = \begin{array}{lllllll} 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 3 & & \\ 4 & 4 & & & & & \end{array} \quad \text{and} \quad \mathsf{T}_2 = \begin{array}{llllllll} 1 & 1 & 1 & & & & & \\ 1 & 2 & 2 & 2 & 2 & 2 & 3 & 4 \\ 3 & & & & & & & \\ 2 & 5 & & & & & & \end{array}.$$

So intuitively $\mathsf{T}_2$ is obtained by picking up entries of $\mathsf{T}$ in the polygonal chain

$$\begin{array}{lllllll} \square & \square & \square & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \square & & \\ \square & \square & & & & & \end{array}$$

which turns at $(k,l)$ and $(k+1,l)$ as a new row. Then by the same argument in the proof of Lemma 5.8, $m_{\mathsf{T}_2} \circ_\nu m_{\mathsf{T}_1} \in M_{\lambda;\mu}$ has the leading term $m_{\mathsf{T}}$. Now let $\nu^+$ be another composition

$$\nu^+ := (\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + 1, l-1, \lambda_{k+1} - l, \lambda_{k+2}, \ldots, \lambda_r)$$

which is obtained by swapping middle two entries of $\nu$. By the assumption $q \in \Bbbk^\times$, we have $M_\nu \simeq M_{\nu^+}$. Hence $m_{\mathsf{T}_2} \circ_\nu m_{\mathsf{T}_1}$ also factors through $M_{\nu^+}$. On the other hand, we have clearly $\nu^+ \not\trianglelefteq \lambda$. Hence by Corollary 5.10 (2), $m_{\mathsf{T}_2} \circ_\nu m_{\mathsf{T}_1} \equiv 0$ in $S_{\lambda;\mu}$. By induction on length as before we obtain the statement. $\square$

Theorem 5.17. *Recall the assumption $q \in \Bbbk^\times$. Then $M_{\lambda;\mu}$ has a basis*

$$\bigsqcup_{\nu:\ \text{partition}} \{ m_{\mathsf{S}} \circ_\nu m_{\mathsf{T}^*} \mid \mathsf{S} \in \mathrm{STab}_{\nu;\mu}, \mathsf{T} \in \mathrm{STab}_{\nu;\lambda} \}.$$

Proof. First we prove that the set above spans the hom space. Take an appropriate total order on the set of all compositions $\{\nu_1, \nu_2, \ldots, \nu_p = \lambda, \ldots\}$ which is stronger than the reversed dominance order, so that $i \leq j$ whenever $\nu_i \geq \nu_j$. We take a filtration on $M_{\lambda;\mu}$ by letting $M^{\leq k}_{\lambda;\mu} := \sum_{i \leq k} M^{\nu_i}_{\lambda;\mu}$ for each $k$ so that $M_{\lambda;\mu} = M^{\leq p}_{\lambda;\mu}$. Then, on each composition factor, by inclusion $M^{>\nu_k}_{\lambda;\mu} \subset M^{\leq k-1}_{\lambda;\mu}$ there is a natural surjective map

$$\circ_{\nu_k} : S_{\nu_k;\mu} \otimes S^*_{\nu_k;\lambda} \twoheadrightarrow M^{\geq \nu_k}_{\lambda;\mu} / M^{>\nu_k}_{\lambda;\mu} \twoheadrightarrow M^{\leq k}_{\lambda;\mu} / M^{\leq k-1}_{\lambda;\mu}$$

here recall that $S_{\nu;\mu} = M^{(\nu)}_{\nu;\mu}$ and we define $S^*_{\nu;\lambda} := M^{(\nu)}_{\lambda;\nu}$. Hence by Lemma 5.16, the right-hand side is spanned by $\{m_{\mathsf{S}} \circ_{\nu_k} m_{\mathsf{T}^*}\}$ above.

Now remember the Robinson–Schensted–Knuth correspondence [**Knu70**]

$$\mathrm{Tab}_{\lambda;\mu} \overset{1:1}{\longleftrightarrow} \bigsqcup_{\nu:\ \text{partition}} \mathrm{STab}_{\nu;\lambda} \times \mathrm{STab}_{\nu;\mu}.$$

Hence the rank of the free $\Bbbk$-module $M_{\lambda;\mu}$ is equal to the number of elements in the generating set above. Consequently this set is also linearly independent, so that it forms a basis. $\square$

COROLLARY 5.18. (1) *$S_{\lambda;\mu}$ has a basis $\{m_{\mathsf{T}} \mid \mathsf{T} \in \mathrm{STab}_{\lambda;\mu}\}$. In particular,*

$$S_{\lambda;\lambda} \simeq \begin{cases} \Bbbk & \text{if } \lambda \text{ is a partition,} \\ 0 & \text{otherwise.} \end{cases}$$

(2) *The product*

$$\circ_\nu \colon S_{\nu;\mu} \otimes S^*_{\nu;\lambda} \to M^{(\nu)}_{\lambda;\mu}$$

*is injective.*

(3) *$H_n$ and $\mathscr{S}_{r,n}$ are cellular algebras.* □

Now for the $q$-Schur algebra $\mathscr{S}_{r,n} = \bigoplus_{\lambda,\mu} M_{\lambda;\mu}$ its simple modules are easily classified. For each parition $\nu$, if $\nu$ is at most of length $r$, then the trace ideal of the Morita context $(\bigoplus_\lambda S_{\nu;\lambda}, \bigoplus_\lambda S^*_{\nu;\lambda})$ in $S_{\nu;\nu} \simeq \Bbbk$ is clearly $\Bbbk$. Otherwise the Morita context is zero since $\lambda, \mu \not\trianglelefteq \nu$ for all such $\lambda, \mu$. Hence we obtain a following classification.

THEOREM 5.19. *When $q \in \Bbbk^\times$, there is a one-to-one correspondence*

$$\mathrm{Irr}(\mathscr{S}_{r,n}) \xleftrightarrow{1:1} \{\nu = (\nu_1, \dots, \nu_r); \text{partition}\} \times \mathrm{Irr}(\Bbbk)$$

*induced by the Morita context functors. Here for a pair of $\nu$ and $V \in \mathrm{Irr}(\Bbbk)$, the corresponding simple module is given by*

$$\mathrm{Image}\Bigl(\bigoplus_\lambda S_{\nu;\lambda} \otimes V \to \mathrm{Hom}_\Bbbk\Bigl(\bigoplus_\lambda S^*_{\nu;\lambda}, V\Bigr)\Bigr).$$

## 9. Identification of the ideals

Recall the assumption $q \in \Bbbk^\times$. We then proceed to the classification of simple modules of the Iwahori–Hecke algebra $H_n \simeq M_{(1^n);(1^n)}$. For each partition $\lambda$, let $J_\lambda = S^*_\lambda \cdot S_\lambda$ be the trace ideal of the Morita context $(S_\lambda, S^*_\lambda)$ in $S_{\lambda;\lambda} \simeq \Bbbk$; here note that the product $\circ_{(1^n)}$ is just the ordinal multiplication. Since $S^*_\lambda$ is generated by $m_\lambda$, we have $J_\lambda = m_\lambda \cdot S_\lambda$. In order to classify simple modules, we have to determine it.

LEMMA 5.20. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a partition. For such $\lambda$, let*

$$f_\lambda \coloneqq [\lambda_1 - \lambda_2]![\lambda_2 - \lambda_3]! \dots [\lambda_r]!.$$

*Then we have inclusions $\Bbbk f^r_\lambda \subset J_\lambda \subset \Bbbk f_\lambda$. In particular, $\mathrm{Irr}^{J_\lambda}(\Bbbk) = \mathrm{Irr}^{\Bbbk f_\lambda}(\Bbbk)$.*

PROOF. First we prove $J_\lambda \subset \Bbbk f_\lambda$. So it suffices to prove that for an arbitrary $\mathsf{T} \in \mathrm{Tab}_\lambda$ we have $m_\lambda \cdot m_{\mathsf{T}} \in \Bbbk f_\lambda m_\lambda$ as an element of $S_{\lambda;\lambda}$. Note that taking a refinement $\mu \coloneqq (\lambda_1, 1^{n-\lambda_1})$ of $\lambda$ we can decompose $m_\lambda \in M^*_\lambda$ as $m_\lambda \circ_\mu m_\mu$. So let $\mathsf{S} \coloneqq \mathsf{T}|_\mu$. Explicitly, $\mathsf{S}$ is a row-semistandard tableau of shape $\lambda$ of weight $\mu$ defined by

$$\mathsf{S}(i,j) \coloneqq \begin{cases} 1 & \text{if } 1 \le \mathsf{T}(i,j) \le \lambda_1, \\ \mathsf{T}(i,j) - \lambda_1 + 1 & \text{otherwise.} \end{cases}$$

Let $\nu \coloneqq \mathsf{S}[1]$ be the composition of $\lambda_1$ where $\nu_i$ is the number of entries $1, 2, \dots, \lambda_1$ in the $i$-th row of $\mathsf{T}$. Then by Lemma 5.2 (2) we obtain that

$$m_\mu \cdot m_{\mathsf{T}} = q^\ell [\nu_1]![\nu_2]! \cdots [\nu_r]!\, m_{\mathsf{S}}$$

for some $\ell \in \mathbb{N}$. In particular, the coefficient can be divided by $[\nu_1]!$. Let $\lambda \setminus \nu$ be the composition of $n - \lambda_1$ defined by $(\lambda \setminus \nu)_i \coloneqq \lambda_i - \nu_i$. Since $m_{\mathsf{S}}$ factors through $M_{(\lambda_1, \lambda \setminus \nu)}$ as before, if $\nu_1 < \lambda_1 - \lambda_2$ then $\lambda \not\trianglerighteq (\lambda_1, \lambda \setminus \nu)$, which implies $m_{\mathsf{S}} \equiv 0$ in $S_{\lambda;\mu}$. Thus the statement trivially holds in this case. Otherwise $[\nu_1]!$ can be divided by $[\lambda_1 - \lambda_2]!$. By induction, for $\lambda' = (\lambda_2, \dots, \lambda_r)$ we may assume that $m_{\lambda'} \cdot S_{\lambda'} \subset \Bbbk f_{\lambda'} m_{\lambda'}$. Note that $S_{\lambda;\mu} = \mathbb{1}_{\lambda_1} * S_{\lambda'}$. Therefore

$$m_\lambda \cdot m_{\mathsf{T}} \in [\lambda_1 - \lambda_2]!\, m_\lambda \circ_\mu S_{\lambda;\mu} \subset [\lambda_1 - \lambda_2]!(\mathbb{1}_{\lambda_1} * \Bbbk f_{\lambda'} m_{\lambda'}) = \Bbbk f_\lambda m_\lambda.$$

Next we prove the other inclusion $\Bbbk f_\lambda^r \subset J_\lambda$. Let $\mathsf{R} \in \mathrm{Tab}_{\lambda;\lambda}$ be the row-semistandard tableau determined by $\#_{ij}(\mathsf{R}) = \lambda_{i+j-1} - \lambda_{i+j}$. For example, when $\lambda = (6, 4, 1)$,

$$\mathsf{R} = \begin{ytableau} 1 & 1 & 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 1 \end{ytableau}\,.$$

Then by taking its underlying row-standard tableau $\mathsf{R}_\downarrow \in \mathrm{Tab}_\lambda$, by Lemma 5.2 (2) again we obtain

$$m_\lambda \cdot m_{\mathsf{R}_\downarrow} = [\lambda_1 - \lambda_2]![\lambda_2 - \lambda_3]!^2 \cdots [\lambda_r]!^r m_{\mathsf{R}}.$$

On the other hand, by using Corollary 5.15 repeatedly, we also obtain that $m_{\mathsf{R}} \in \Bbbk^\times m_\lambda$ in $S_{\lambda;\lambda}$. For example,

$$\begin{ytableau} 1 & 1 & 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 1 \end{ytableau} \equiv - \begin{ytableau} 1 & 1 & 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 2 \end{ytableau} \equiv -q^6 \begin{ytableau} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 \\ 2 \end{ytableau} \equiv q^6 \begin{ytableau} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 \end{ytableau}\,.$$

This implies $J_\lambda m_\lambda \supset \Bbbk m_\lambda \cdot m_{\mathsf{R}_\downarrow} \supset \Bbbk f_\lambda^r m_\lambda$ as desired. □

This completes the classification we noted in the introduction.

THEOREM 5.21. *When $q \in \Bbbk^\times$, there is a one-to-one correspondence*

$$\mathrm{Irr}(H_n) \overset{1:1}{\longleftrightarrow} \bigsqcup_{\lambda\colon \text{partition}} \mathrm{Irr}^{\Bbbk f_\lambda}(\Bbbk)$$

*induced by the Morita context functors. For a partition $\lambda$ and $V \in \mathrm{Irr}^{\Bbbk f_\lambda}(\Bbbk)$, the corresponding simple module is*

$$\mathrm{Image}(S_\lambda \otimes V \to \mathrm{Hom}_\Bbbk(S_\lambda^*, V)).$$

□

Finally let us consider the case that $\Bbbk$ is a field. Let $e \in \mathbb{N} \cup \{\infty\}$ be the *$q$-characteristic* of $\Bbbk$, namely $e := \min\{k \mid [k] = 0\}$ (the case $e = \infty$ is usually written as $e = 0$, but we use this definition for simplicity). A partition $\lambda$ is called $e$-restricted if $\lambda_i - \lambda_{i+1} < e$ holds for every $i$. Then clearly we have that $\lambda$ is $e$-restricted if and only if $f_\lambda = 0$. Thus as a corollary of the theorem we obtain the well-known classification.

COROLLARY 5.22. *If $\Bbbk$ is a field whose $q$-characteristic is $e$ (we still assume that $q \in \Bbbk^\times$), there is a one-to-one correspondence*

$$\mathrm{Irr}(H_n) \overset{1:1}{\longleftrightarrow} \{e\text{-restricted partition}\}.$$

The right-hand side set is actually the crystal $B(\Lambda_0)$ of type $\mathsf{A}_{e-1}^{(1)}$ under the description of Misra and Miwa [**MM90**].

CHAPTER 6

# Stable structure of the module category

In this chapter we focus on the behavior of the module category of the Iwahori–Hecke algebra $H_d$ for large $d \gg 0$. Several things will be stable in the large rank which are easier to study than the unstable ones, especially in the super case we will treat in the next part.

## 1. Induction and restriction

Recall that the category $\bigoplus_n (H_n\text{-}\mathcal{M}od)$ has the convolution product $*$. We define the *induction functor* as taking convolution with trivial representation. It plays a central role in what follows.

Definition 6.1. Let $k, n \in \mathbb{N}$. For an $H_n$-module $V$, we denote by $\mathrm{Ind}_k V$ the $H_{k+n}$-module

$$\mathrm{Ind}_k V := \mathbb{1}_k * V.$$

This defines a functor $\mathrm{Ind}_k \colon H_n\text{-}\mathcal{M}od \to H_{k+n}\text{-}\mathcal{M}od$ between module categories.

It is clear from the direct sum decomposition that the functor $\mathrm{Ind}_k$ is exact. We prove that this functor has both left and right adjoint.

Definition 6.2. Let $k, n \in \mathbb{N}$. For an $H_{k+n}$-module $W$, we define $H_n$-modules

$$\begin{aligned}
\mathrm{Res}_k W &:= \mathrm{Hom}_{H_{(k,n)}}(\mathbb{1}_k \boxtimes H_n, W|_{(k,n)}) \\
&\simeq \{x \in W \mid T_i x = qx \text{ for } 1 \le i \le k\}, \\
\mathrm{Res}'_k W &:= (H_n \boxtimes \mathbb{1}_k^*) \otimes_{H_{(n,k)}} W|_{(n,k)} \\
&\simeq W/\!\sim, \text{ where } T_i x \sim qx \text{ for } n+1 \le i \le n+k
\end{aligned}$$

where we denote by $W|_\lambda$ the restricted $H_\lambda$-module. $\mathrm{Res}_k$ and $\mathrm{Res}'_k$ are functors $H_{k+n}\text{-}\mathcal{M}od \to H_n\text{-}\mathcal{M}od$, and we call them the *subrestriction* and the *quorestriction* functors.

Note that the definition of $\mathrm{Res}_k$ uses the composition $(k, n)$ while that of $\mathrm{Res}'_k$ does the reversed one $(n, k)$. In other words, to define the action of $H_n$ on the module, $\mathrm{Res}_k$ takes the last $n$ indices $\{k+1, k+2, \dots, k+n\}$ while $\mathrm{Res}'_k$ does the first $n$ ones $\{1, 2, \dots, n\}$ of them. Hence the two restriction functors naturally commute.

Proposition 6.3. $\mathrm{Res}_k$ *(resp.* $\mathrm{Res}'_k$*) is the right (resp. left) adjoint functor of* $\mathrm{Ind}_k$.

The adjointness for $\mathrm{Res}_k$ is obvious from the Frobenius reciprocity. To prove that for $\mathrm{Res}'_k$, we first prove the next lemma.

Lemma 6.4. *There is an isomorphism of* $(H_{(k,n)}, H_{k+n})$*-bimodules*

$$\mathrm{Hom}_{H^{\mathrm{op}}_{(k,n)}}(H_{k+n}, H_{(k,n)}) \simeq {}^\sigma H_{k+n}.$$

Here the left-hand side above is the set of homomorphisms between *right* $H_{(k,n)}$-modules. $\sigma$ denotes the canonical isomorphism $\sigma\colon H_{(k,n)} \simeq H_{(n,k)}$ and ${}^{\sigma}H_{k+n}$ is the $(H_{(k,n)}, H_{k+n})$-bimodule whose underlying set is just $H_{k+n}$ but the left $H_{(k,n)}$-action is twisted by $\sigma$.

Proof. Since $\{T_w \mid w \in \mathfrak{D}_{(k,n)}\}$ is a basis of the free right $H_{(k,n)}$-module $H_{k+n}$, its dual basis, which we denote by $\{\delta_w \mid w \in \mathfrak{D}_{(k,n)}\}$, is a basis of the free left $H_{(k,n)}$-module $\mathrm{Hom}^{\mathrm{op}}_{H_{(k,n)}}(H_{k+n}, H_{(k,n)})$. Explicitly, the function $\delta_w$ is defined by

$$\begin{aligned}\delta_w\colon H_{k+n} &\to H_{(k,n)}\\ T_{uv} &\mapsto \begin{cases} T_v & \text{if } u = w,\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

for a pair of $u \in \mathfrak{D}_{(k,n)}$ and $v \in \mathfrak{S}_{(k,n)}$. If $w, s_i w \in \mathfrak{D}_{(k,n)}$ and $\ell(s_i w) > \ell(w)$, we have

$$\delta_{s_i w} \cdot T_i = \delta_w + (q-1)\delta_{s_i w}$$

by definition. So $\mathrm{Hom}^{\mathrm{op}}_{H_{(k,n)}}(H_{k+n}, H_{(k,n)})$ is generated by $\delta_{\varpi_{(k,n)}}$ as a right $H_{k+n}$-module. Since the rank of this $\Bbbk$-module is $(k+n)!$, which coincides with the rank of the algebra, the action of $H_{k+n}$ on $\delta_{\varpi_{(k,n)}}$ is free. Thus there is an isomorphism of right $H_{k+n}$-modules

$$\begin{aligned}{}^{\sigma}H_{k+n} &\to \mathrm{Hom}^{\mathrm{op}}_{H_{(k,n)}}(H_{k+n}, H_{(k,n)}),\\ {}^{\sigma}x &\mapsto \delta_{\varpi_{(k,n)}} \cdot x,\end{aligned}$$

where we denote by ${}^{\sigma}x \in {}^{\sigma}H_{k+n}$ the element corresponds to $x \in H_{k+n}$. Moreover this isomorphism also respects left $H_{(k,n)}$-action. Actually let us take any generator $s_i \in \mathfrak{S}_{(k,n)}$ and let $j \coloneqq \varpi_{(k,n)}(i)$ so that $\sigma(T_i) = T_j$. Then we have

$$s_j\varpi_{(k,n)} = \varpi_{(k,n)}s_i \notin \mathfrak{D}_{(k,n)} \quad \text{and} \quad \ell(s_j\varpi_{(k,n)}) > \ell(\varpi_{(k,n)}).$$

Hence for each $u \in \mathfrak{D}_{(k,n)}$,

$$\delta_{\varpi_{(k,n)}}(T_jT_u) = \begin{cases} T_i & \text{if } u = \varpi_{(k,n)},\\ 0 & \text{otherwise.}\end{cases}$$

In other words, $\delta_{\varpi_{(k,n)}} \cdot T_j = T_i\delta_{\varpi_{(k,n)}}$. Thus it satisfies $T_v\delta_{\varpi_{(k,n)}} = \delta_{\varpi_{(k,n)}} \cdot \sigma(T_v)$ for each $v \in \mathfrak{S}_{(k,n)}$. $\square$

Proof of Proposition 6.3. As a consequence of the lemma, we have another definition of $\mathrm{Ind}_k$:

$$\begin{aligned}\mathrm{Ind}_k V &= (\mathbb{1}_k \boxtimes V) \otimes_{H_{(k,n)}} H_{k+n}\\ &\simeq \mathrm{Hom}_{H_{(k,n)}}(\mathrm{Hom}^{\mathrm{op}}_{H_{(k,n)}}(H_{k+n}, H_{(k,n)}), \mathbb{1}_k \boxtimes V)\\ &\simeq \mathrm{Hom}_{H_{(k,n)}}({}^{\sigma}H_{k+n}, \mathbb{1}_k \boxtimes V)\\ &\simeq \mathrm{Hom}_{H_{(n,k)}}(H_{k+n}, V \boxtimes \mathbb{1}_k).\end{aligned}$$

Here in the second line we take the double dual of $H_{k+n}$ with respect to the free right action of $H_{(k,n)}$. Now the adjointness is clear. $\square$

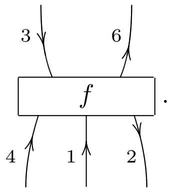
Notation 6.5. As a convention, for integers $k, n \in \mathbb{Z}$ which do not satisfy $k, n \geq 0$, we also define $H_n$-$\mathcal{M}od$ as the zero category and $\mathrm{Ind}_k$, $\mathrm{Res}_k$ and $\mathrm{Res}'_k$ as the zero functor.

## 2. Diagrammatic natural transformations

In this subsection we define important natural transformations between functors $\mathrm{Ind}_k$, $\mathrm{Res}_k$ and $\mathrm{Res}'_k$. Before we introduce them, let us explain *string diagrams*, which are useful for calculation in theory of 2-categories.

In a diagram we represent a functor by a colored string. The right (resp. left) region separated by a string stands for the domain (resp. codomain) category of the corresponding functor. A composite of these functors is represented by a sequence of strings arranged horizontally. In particular, the identity functor is represented by the "no strings" diagram. A natural transformation between such functors are represented by a figure connecting these sequences from top to bottom. Note that an object $X \in \mathcal{C}$ can be considered as a functor $X : \{*\} \to \mathcal{C}$, where $\{*\}$ denotes the category with single object, so we also represent it by a string.

In this paper, we represent the functor $\mathrm{Ind}_k$ by a down arrow $\downarrow$, and both $\mathrm{Res}_k$, $\mathrm{Res}'_k$ by up arrows $\uparrow$ which are labeled by $k$. For example, $f : \mathrm{Ind}_3\mathrm{Res}_6 \to \mathrm{Res}_4\mathrm{Res}_1\mathrm{Ind}_2$ is represented by a figure like

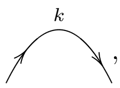
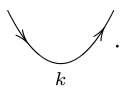



Note that the diagram above can not distinguish $\mathrm{Res}_k$ from $\mathrm{Res}'_k$, but we only use diagrams when it is clear from the context.

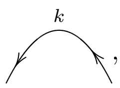
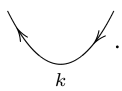
The adjointness between $\mathrm{Ind}_k$ and $\mathrm{Res}_k$ yields natural transformations

$$\delta_k : \mathrm{Id} \to \mathrm{Res}_k\mathrm{Ind}_k, \qquad \epsilon_k : \mathrm{Ind}_k\mathrm{Res}_k \to \mathrm{Id}$$

called the unit and the counit respectively. We represent these morphisms by the cap and the cup diagrams:

$\delta_k =$  , $\qquad \epsilon_k =$  .


We also have the the unit $\delta'_k : \mathrm{Id} \to \mathrm{Ind}_k\mathrm{Res}'_k$ counit $\epsilon'_k : \mathrm{Res}'_k\mathrm{Ind}_k \to \mathrm{Id}$ induced by the other adjunction. We represent them by the same diagrams as above but arrows are reversed:

$\delta'_k =$  , $\qquad \epsilon'_k =$  .


Now let $k, l \in \mathbb{N}$. We define three $H_{k+l}$-homomorphisms

$$\begin{aligned} \mu_{(k,l)} &: M_{(k,l)} \to \mathbb{1}_{k+l}, & \Delta_{(k,l)} &: \mathbb{1}_{k+l} \to M_{(k,l)} & \sigma_{(k,l)} &: M_{(l,k)} \to M_{(k,l)}, \\ m_{(k,l)} &\mapsto m_{k+l}, & m_{k+l} &\mapsto \sum_{w \in \mathfrak{D}_{(k,l)}} T_w m_{(k,l)}, & m_{(l,k)} &\mapsto T_{\varpi_{(k,l)}} m_{(k,l)} \end{aligned}$$

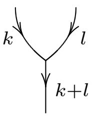
which correspond to the tableaux

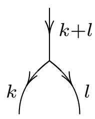
$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&1&1&2&2&\cdots&2\\\hline\end{array}, \qquad \begin{array}{|c|c|c|c|c|c|}\hline 1&1&\cdots&1&1&1\\\hline 1&1&\cdots&1\\\cline{1-4}\end{array}, \qquad \begin{array}{|c|c|c|c|c|c|}\hline 2&2&\cdots&2&2&2\\\hline 1&1&\cdots&1\\\cline{1-4}\end{array}$$

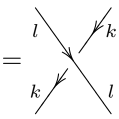
respectively. These homomorphisms induce natural transformations between functors $H_n$-$\mathcal{M}od \to H_{k+l+n}$-$\mathcal{M}od$,

$$\mu_{(k,l)} : \mathrm{Ind}_k\mathrm{Ind}_l \to \mathrm{Ind}_{k+l}, \ \Delta_{(k,l)} : \mathrm{Ind}_{k+l} \to \mathrm{Ind}_k\mathrm{Ind}_l, \ \sigma_{(k,l)} : \mathrm{Ind}_l\mathrm{Ind}_k \to \mathrm{Ind}_k\mathrm{Ind}_l$$

which we denote by the same symbols. Again, if $k$ and $l$ do not satisfy $k, l \geq 0$, then these morphisms are defined to be zero. We represent these natural transformations by the string diagrams

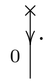
$\mu_{(k,l)} =$ 

, $\Delta_{(k,l)} =$ 

, $\sigma_{(k,l)} =$ 

,

that is, junction, branch, and crossing of strings respectively. Finally, an obvious isomorphism $\mathrm{Ind}_0 \simeq \mathrm{Id}$ and its inverse are represented by broken strings:

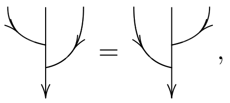
and 

.

An equation which hold between compositions of these morphisms can be expressed as a local transformation of string diagrams. In particular, we are allowed to move diagrams through regular homotopy by the naturality of homomorphisms. In addition, diagrams can pass through under or over another string since $\sigma$ satisfies the axioms of braiding. We here list other equations which we use later.

Proposition 6.6. *The following equations hold. Here in the diagrams labels which can be deduced from others are omitted from strings.*

(1) *The associativity and the coassociativity laws:*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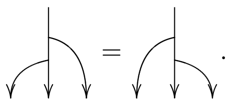
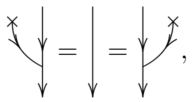
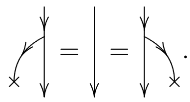
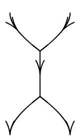
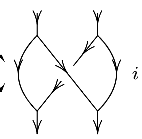

.

(2) *The unit and the counit laws:*

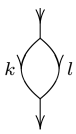

,

.

(3) *The graded bialgebra relation:*


$= \sum_i$ 


*where $i$ ranges over all integers and determines all labels on other strings. The summand is zero for all but finite $i$.*

(4) *The bubble elimination:*



$= \begin{bmatrix} k+l \\ k \end{bmatrix}$ 
.

*Here $\begin{bmatrix} k+l \\ k \end{bmatrix}$ is the $q$-binomial coefficient.*

Remark 6.7. The equations (1), (2) and (3) say that $\bigoplus_n \mathbb{1}_n$ has a structure of bialgebra in the braided tensor category $\bigoplus_n (H_n\text{-}\mathcal{M}od)$. It also admits the antipode of Hopf algebra by (4) defined by a scalar multiplication of $(-1)^n q^{\binom{n}{2}}$ on each $\mathbb{1}_n$. When $q = 1$, this algebra is nothing but the divided power algebra $\Bbbk[x, x^2/2, x^3/6, \ldots]$.

To prove the proposition, we first state a lemma.

LEMMA 6.8. *Let $d, m, n \in \mathbb{N}$ such that $m, n \leq d$. For each $i \in \mathbb{N}$ such that $m+n-d \leq i \leq m, n$, we define $g_i \in \mathfrak{S}_d$ by*

$$g_i := (1_{d-m-n+i}, \varpi_{(m-i,n-i)}, 1_i) \in \mathfrak{S}_{(d-m-n+i,m+n-2i,i)}.$$

*Then there is a decomposition*

$$\mathfrak{D}_{(d-n,n)} = \bigsqcup_i (\mathfrak{D}_{(d-m-n+i,n-i)} \times \mathfrak{D}_{(m-i,i)}) g_i$$

*which preserves lengths.*

PROOF. The set $\mathrm{Tab}_{(d-n,n);(d-m,m)}$ consisting of tableaux such as

$$\mathsf{S} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&2&2\\ \hline 1&2&2&2 \\ \cline{1-4}\end{array}$$

which is determined by $\#_{22}(\mathsf{S})$, the number of 2's in the second row. Let $\mathsf{S}_i$ be the tableau which has $i = \#_{22}(\mathsf{S}_i)$. Then the other matrix entries are given by

$$\#_{11}(\mathsf{S}_i) = d-m-n+i, \qquad \#_{12}(\mathsf{S}_i) = m-i, \qquad \#_{21}(\mathsf{S}_i) = n-i.$$

Thus we have a decomposition

$$\mathrm{Tab}_{(d-n,n)} = \bigsqcup_i \mathrm{Tab}_{\mathsf{S}_i} = \bigsqcup_i (\mathfrak{D}_{(d-m-n+i,n-i)} \times \mathfrak{D}_{(m-i,i)}) \cdot \mathsf{S}_{i\downarrow}$$

where $i$ ranges over the natural numbers such that these numbers are non-negative. The corresponding element $d(\mathsf{S}_{i\downarrow}) \in \mathfrak{D}_{(d-n,n)} \cap \mathfrak{D}^{-1}_{(d-m,m)}$ is the permutation which swaps the numbers $\{d-m-n+i+1, \ldots, d-m\}$ for $\{d-m+1, \ldots, d-i\}$, which is just $g_i$. By translating the decomposition above from the set of tableaux into that of permutations, we obtain the statement. $\square$

Now we back to the proof of the proposition.

PROOF OF PROPOSITION 6.6. Every morphisms represented by diagrams above are induced by some homomorphisms $M_\mu \to M_\lambda$ so it suffices to prove equations between such homomorphisms. It is enough to check that these homomorphisms send the generator $m_\mu \in M_\mu$ to a same element.

First clearly the both-hand sides of the left equation of (1) sends $m_{(k,l,m)}$ to $m_{k+l+m}$. The right one follows by taking its dual. (2) is also obvious.

Next the left-hand side of (3) is the homomorphism

$$\begin{aligned} m_{(d-m,m)} &\overset{\mu}{\mapsto} m_d \\ &\overset{\Delta}{\mapsto} \sum_{w \in \mathfrak{D}_{(d-n,n)}} T_w m_{(d-n,n)}. \end{aligned}$$

By the decomposition of $\mathfrak{D}_{(d-n,n)}$ in the Lemma 6.8, it can be decomposed as

$$\sum_{w \in \mathfrak{D}_{(d-n,n)}} T_w m_{(d-n,n)} = \sum_i \sum_{\substack{u \in \mathfrak{D}_{(d-m-n+i,n-i)}, \\ v \in \mathfrak{D}_{(m-i,i)}}} T_{(u,v)} T_{g_i} m_{(d-n,n)}.$$

On the other hand, the summand in the right-hand side is the homomorphism

$$\begin{aligned} m_{(d-m,m)} &\overset{\Delta}{\mapsto} \sum_{u,v} T_{(u,v)} m_{(d-m-n+i,n-i,m-i,i)} \\ &\overset{\sigma}{\mapsto} \sum_{u,v} T_{(u,v)} T_{g_i} m_{(d-m-n+i,m-i,n-i,i)} \\ &\overset{\mu}{\mapsto} \sum_{u,v} T_{(u,v)} T_{g_i} m_{(d-n,n)}. \end{aligned}$$

Thus the both-hand sides are equal.

The left-hand side of (4) is

$$m_{k+l} \mapsto \sum_{w\in\mathfrak{D}_{(k,l)}} T_w m_{(k,l)} \mapsto \sum_{w\in\mathfrak{D}_{(k,l)}} q^{\ell(w)} m_{k+l}.$$

So the equation is a consequence of $P_{\mathfrak{D}_{(k,l)}}(q) = \begin{bmatrix} k+l \\ k \end{bmatrix}$. □

## 3. Homomorphisms between induced modules

We here study the set of homomorphisms between two induced modules $\mathrm{Ind}_{d-m}V$ and $\mathrm{Ind}_{d-n}W$ in terms of string diagrams. The key observation is that this set stabilizes for sufficiently large rank $d$. In order to explain this, first we consider a parabolic restriction of an induced module.

LEMMA 6.9. *Let $d, m, n \in \mathbb{N}$ such that $m, n \le d$. For each $W \in H_n$-$\mathcal{M}od$ there is an isomorphism*

$$(\mathrm{Ind}_{d-n}W)|_{(d-m,m)} \simeq \bigoplus_i \mathrm{Ind}^{(1)}_{d-m-n+i}\mathrm{Ind}^{(2)}_{m-i}(W|_{(n-i,i)})$$

*of $H_{(d-m,m)}$-modules. Here $\mathrm{Ind}^{(1)}_k$ and $\mathrm{Ind}^{(2)}_k$ are the functors*

$$\mathrm{Ind}^{(1)}_k \colon H_{(p,q)}\text{-}\mathcal{M}od \to H_{(k+p,q)}\text{-}\mathcal{M}od, \quad \mathrm{Ind}^{(2)}_k \colon H_{(p,q)}\text{-}\mathcal{M}od \to H_{(p,k+q)}\text{-}\mathcal{M}od$$

*defined by $\mathrm{Ind}_k$ on each components.*

PROOF. The decomposition of $\mathfrak{D}_{(d-n,n)}$ in the Lemma 6.8 gives the direct sum decomposition

$$(\mathrm{Ind}_{d-n}W)|_{(d-m,m)} = \bigoplus_i \bigoplus_{\substack{u\in\mathfrak{D}_{(d-m-n+i,n-i)},\\ v\in\mathfrak{D}_{(m-i,i)}}} T_{(u,v)}T_{g_i}(\mathbb{1}_{d-n}\boxtimes W)$$

of left $H_{(d-m,m)}$-module. Let $\lambda$ be the composition $(d-m-n+i, n-i, m-i, i)$. Since $T_{g_i}$ swaps the middle two terms of the action, we have an isomorphism

$$T_{g_i}(\mathbb{1}_{d-n}\boxtimes W) \simeq \mathbb{1}_{d-m-n+i}\boxtimes W^{(1)}\boxtimes \mathbb{1}_{m-i}\boxtimes W^{(2)}$$

of $H_\lambda$-modules. Here we used *Sweedler's notation*; we wrote the isomorphism above as if $W|_{(n-i,i)}$ can be expressed as an outer tensor product $W^{(1)}\boxtimes W^{(2)}$. Though it does not hold in general, we can define an $H_\lambda$-action on the underlying set $W$ according to this notation. Then we have

$$\begin{aligned}\bigoplus_{u,v} T_{(u,v)}T_{g_i}(\mathbb{1}_{d-n}\boxtimes W) &\simeq H_{(d-m,m)}\otimes_{H_\lambda}(\mathbb{1}_{d-m-n+i}\boxtimes W^{(1)}\boxtimes \mathbb{1}_{m-i}\boxtimes W^{(2)})\\ &\simeq \mathrm{Ind}_{d-m-n+i}W^{(1)}\boxtimes \mathrm{Ind}_{m-i}W^{(2)}\\ &\simeq \mathrm{Ind}^{(1)}_{d-m-n+i}\mathrm{Ind}^{(2)}_{m-i}(W|_{(n-i,i)}).\end{aligned}$$

The statement follows by taking the direct sum of them. □

We are now ready to prove the following.

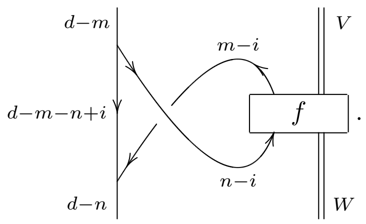
PROPOSITION 6.10. *Let $d, m, n \in \mathbb{N}$ such that $m, n \le d$. For each $V \in H_m$-$\mathcal{M}od$ and $W \in H_n$-$\mathcal{M}od$, we have an isomorphism of $\Bbbk$-modules*

$$\mathrm{Hom}_{H_d}(\mathrm{Ind}_{d-m}V, \mathrm{Ind}_{d-n}W) \simeq \bigoplus_{m+n-d\le i} \mathrm{Hom}_{H_i}(\mathrm{Res}'_{m-i}V, \mathrm{Res}_{n-i}W)$$

*natural in $V$ and $W$. Here for each $H_i$-homomorphism $f\colon \mathrm{Res}'_{m-i}V \to \mathrm{Res}_{n-i}W$, the corresponding $H_d$-homomorphism is defined as the composite*

$$\begin{aligned}\mathrm{Ind}_{d-m}V &\xrightarrow{\Delta\delta' V} \mathrm{Ind}_{d-m-n+i}\mathrm{Ind}_{n-i}\mathrm{Ind}_{m-i}\mathrm{Res}'_{m-i}V\\ &\xrightarrow{\mathrm{Ind}\sigma f} \mathrm{Ind}_{d-m-n+i}\mathrm{Ind}_{m-i}\mathrm{Ind}_{n-i}\mathrm{Res}_{n-i}W\\ &\xrightarrow{\mu\epsilon W} \mathrm{Ind}_{d-n}W\end{aligned}$$

*which can be illustrated as follows:*




Note that the summand in the right-hand side is zero unless $0 \le i \le m, n$. In particular, as we varies the rank $d \in \mathbb{N}$ larger, this set is stable for $d \ge m+n$. By the adjointness and the Yoneda lemma, this lemma is equivalent to either of the following statement.

Corollary 6.11. *Let $d, m, n \in \mathbb{N}$ be as above.*

(1) *For each $W \in H_n$-$\mathcal{M}od$, there is a natural isomorphism of $H_m$-modules*

$$\mathrm{Res}_{d-m}\mathrm{Ind}_{d-n}W \simeq \bigoplus_{m+n-d\le i} \mathrm{Ind}_{m-i}\mathrm{Res}_{n-i}W.$$

(2) *For each $V \in H_m$-$\mathcal{M}od$, there is a natural isomorphism of $H_n$-modules*

$$\mathrm{Res}'_{d-n}\mathrm{Ind}_{d-m}V \simeq \bigoplus_{m+n-d\le i} \mathrm{Ind}_{n-i}\mathrm{Res}'_{m-i}V.$$

We prove (1) of this corollary.

Proof. First we prove it for the case $m = 0$. It is easy to check that an element $x \in \mathrm{Ind}_{d-n}W$, written as

$$x = \sum_{w\in\mathfrak{D}_{(d-n,n)}} T_w(m_{d-n} \boxtimes x_w)$$

by elements $x_w \in W$, is in $\mathrm{Res}_d\mathrm{Ind}_{d-n}W$ if and only if these elements satisfy

$$x_w = x_1 \in \mathrm{Res}_n W$$

for every $w$. Hence we have an isomorphism

$$\begin{aligned}\mathrm{Res}_n W &\simeq \mathrm{Res}_d\mathrm{Ind}_{d-n}W\\ x &\mapsto \sum_{w\in\mathfrak{D}_{(d-n,n)}} T_w(m_{d-n}\boxtimes x)\end{aligned}$$

of $H_0 \simeq \Bbbk$-modules.

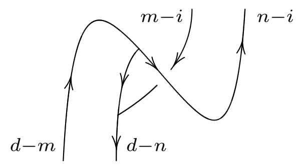
Now let $m$ be arbitrary. Under the isomorphism $H_{(0,n)} \simeq H_n$, we can rewrite $\mathrm{Res}_{d-m}\mathrm{Ind}_{d-n}W$ as $\mathrm{Res}^{(1)}_{d-m}((\mathrm{Ind}_{d-n}W)|_{(d-m,m)})$, where the functor $\mathrm{Res}^{(1)}_k$ is defined similarly as $\mathrm{Ind}^{(1)}_k$. So recall the direct sum decomposition in Lemma 6.9. By the case of $m = 0$ above, we have

$$\begin{aligned}\mathrm{Res}^{(1)}_{d-m}\mathrm{Ind}^{(1)}_{d-m-n+i}\mathrm{Ind}^{(2)}_{m-i}(W|_{(n-i,i)}) &\simeq \mathrm{Ind}^{(2)}_{m-i}\mathrm{Res}^{(1)}_{n-i}(W|_{(n-i,i)})\\ &\simeq \mathrm{Ind}_{m-i}\mathrm{Res}_{n-i}W.\end{aligned}$$

Taking the direct sum for all $i$, we obtain the desired isomorphism. The inverse of this isomorphism is given by

$$\begin{aligned}\mathrm{Ind}_{m-i}\mathrm{Res}_{n-i}W &\to \mathrm{Res}_{d-m}\mathrm{Ind}_{d-n}W,\\ m_{m-i}\boxtimes x &\to \sum_{u\in\mathfrak{D}_{(d-m-n+i,n-i)}} T_{(u,1)}T_{g_i}(m_{d-n}\boxtimes x),\end{aligned}$$

by the definitions of the isomorphisms, which is represented by the diagram




.

This induces the function defined as in Proposition 6.10. □

## 4. Tableaux and strings

Let $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_r)$ be a composition. We denote by $\lambda'=(\lambda_2,\dots,\lambda_r)$ one obtained from $\lambda$ by removing its first component. Conversely, for such $\lambda'$ we write $\lambda=(\lambda_1,\lambda')$ for short.

Recall that a parabolic module $M_\lambda$ can be expressed as an induced module: $M_\lambda\simeq\mathrm{Ind}_{\lambda_1}M_{\lambda'}$. Hence we have two tools to describe the set of homomorphisms between two parabolic modules; the first one is the tableaux basis introduced in Subsection 3 and the second is string diagrams according to Proposition 6.10. In this subsection we explain the connection between these two.

For two composition $\lambda$ and $\pi$, we write $\pi\subset\lambda$ when $\pi_i\le\lambda_i$ holds for every $i$, that is, there is an inclusion of Young diagrams $Y(\pi)\subset Y(\lambda)$. Then the next statement inductively follows from Corollary 6.11.

Lemma 6.12. *For a composition $\lambda$ and $k\in\mathbb{N}$, we have an isomorphism*

$$\mathrm{Res}_k M_\lambda\simeq\mathrm{Res}'_k M_\lambda\simeq\bigoplus_{\pi\subset\lambda,\,|\pi|=|\lambda|-k} M_\pi.$$

Hence for two compositions $\lambda$ and $\mu$ of $d\in\mathbb{N}$, we have an isomorphism

$$\begin{aligned}\mathrm{Hom}_{H_d}(M_\mu,M_\lambda)&\simeq\bigoplus_i\mathrm{Hom}_{H_i}(\mathrm{Res}'_{d-\mu_1-i}M_{\mu'},\mathrm{Res}_{d-\lambda_1-i}M_{\lambda'})\\ &\simeq\bigoplus_{\pi\subset\lambda',\,\rho\subset\mu',\,|\pi|=|\rho|}\mathrm{Hom}_{H_{|\pi|}}(M_\rho,M_\pi)\end{aligned}$$

by Proposition 6.10. Let us explain this isomorphism more precisely. We will see that for each tableau $\mathsf{S}\in\mathrm{Tab}_{\lambda;\mu}$ which gives the basis element $m_{\mathsf{S}}$ in the left-hand side, there is a tableau ${}^{\backprime}\mathsf{S}'\in\mathrm{Tab}_{\pi;\rho}$ which produces the corresponding element $m_{{}^{\backprime}\mathsf{S}'}$ in the right-hand side. Here, for a tableau $\mathsf{S}$ of shape $\lambda$, $\mathsf{S}'$ denotes the tableau obtained by cutting off the first row of $\mathsf{S}$ so that its shape is $\lambda'$. Dually ${}^{\backprime}\mathsf{S}$ is defined by ${}^{\backprime}\mathsf{S}:=((\mathsf{S}^*)')^*$, that is, it is obtained from $\mathsf{S}$ by removing all 1's and decreasing other entries. ${}^{\backprime}\mathsf{S}':={}^{\backprime}(\mathsf{S}')=({}^{\backprime}\mathsf{S})'$ is defined as the result of these two commuting operations.

First for two composition $\alpha\subset\lambda$, let $\mathsf{R}_{\lambda;\alpha}$ be the row-semistandard tableau of shape $\lambda$ defined by

$$\mathsf{R}_{\lambda;\alpha}(i,j)=\begin{cases}1 & \text{if } j\le\alpha_i,\\ i+1 & \text{otherwise.}\end{cases}$$

The weight of $\mathsf{R}_{\lambda;\alpha}$ is given by $(|\alpha|, \lambda \setminus \alpha)$, where $\lambda \setminus \alpha$ is the composition defined by $(\lambda \setminus \alpha)_i \coloneqq \lambda_i - \alpha_i$. For example, for $\lambda = (4,3,5,2)$ and $\alpha = (1,3,0,1)$, we have

$$\mathsf{R}_{\lambda;\alpha} = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline 4 & 4 & 4 & 4 & 4 \\ \hline 1 & 5 \\ \hline \end{array}\,.$$

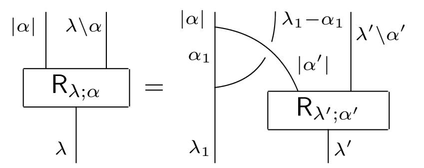
Then the element $m_{\mathsf{R}_{\lambda;\alpha}}$ satisfies

$$m_{\mathsf{R}_{\lambda;\alpha}} = \sum_{w \in \mathcal{D}_\alpha} T_w m_{\mathsf{R}_{\lambda;\alpha\downarrow}} = \sum_{w \in \mathcal{D}_{(\alpha_1, |\alpha'|)}} T_w T_g (m_{\lambda_1} \boxtimes m_{\mathsf{R}_{\lambda';\alpha'}})$$

where $g \in \mathfrak{S}_{|\lambda|}$ is the permutation

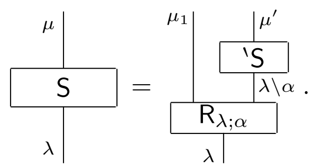
$$g = (1_{\alpha_1}, \varpi_{(\lambda_1 - \alpha_1, |\alpha'|)}, 1_{|\lambda'| - |\alpha'|}) \in \mathfrak{S}_{(\alpha_1, \lambda_1 - \alpha_1 + |\alpha'|, |\lambda'| - |\alpha'|)}.$$

Using string diagrams, this equation can be represented as

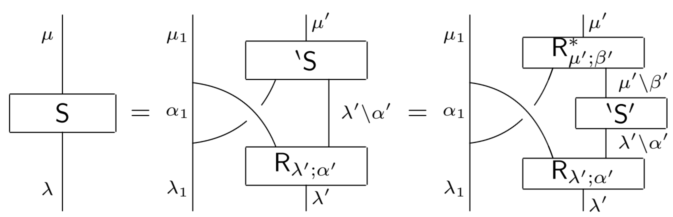



by regarding $m_{\mathsf{R}_{\lambda;\alpha}}$ as a homomorphism $M_{(|\alpha|, \lambda\setminus\alpha)} \to M_\lambda$. Here we represent the homomorphism $m_{\mathsf{S}}$ by a boxed $\mathsf{S}$ and the module $M_\lambda$ by a string with the label $\lambda$ for short. By comparing it to the form of the homomorphism obtained in Proposition 6.10, it inductively follows that this homomorphism, via adjunction, corresponds to the embedding $M_{\lambda\setminus\alpha} \hookrightarrow \operatorname{Res}_{|\alpha|} M_\lambda$ which sends the generator $m_{\lambda\setminus\alpha}$ to the element $m_{\mathsf{R}_{\lambda;\alpha}}$. This fact can be also directly verified by that the module $M_\lambda = \bigoplus_{\mathsf{T} \in \mathrm{Tab}_\lambda} \Bbbk m_{\mathsf{T}}$ is decomposed into summands by subrestriction according to the position of numbers $1, 2, \ldots, |\alpha|$ on each row-standard tableau $\mathsf{T}$. Dually, the homomorphism $M_\lambda \to M_{(|\alpha|, \lambda\setminus\alpha)}$ corresponds to the projection $\operatorname{Res}'_{|\alpha|} M_\lambda \twoheadrightarrow M_{(|\alpha|, \lambda\setminus\alpha)}$ is given by the dual element $m_{\mathsf{R}^*_{\lambda;\alpha}}$.

Now for a general $\mathsf{S} \in \mathrm{Tab}_{\lambda;\mu}$, let $\alpha = \mathsf{S}[1]$. Then we have a decomposition




Using the equation above and the decomposition of $({}^{\backprime}\mathsf{S})^*$ again, we obtain

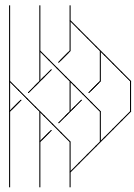



where $\beta = \mathsf{S}^*[1]$. Summarizing the above, we have proved that according to the isomorphism in Proposition 6.10, $m_{\mathsf{S}} \colon M_\mu \to M_\lambda$ corresponds to the composite

$$\operatorname{Res}'_{\lambda_1 - \alpha_1} M_{\mu'} \twoheadrightarrow M_{\mu'\setminus\beta'} \xrightarrow{m_{{}^{\backprime}\mathsf{S}'}} M_{\lambda'\setminus\alpha'} \hookrightarrow \operatorname{Res}_{|\alpha'|} M_{\lambda'}$$

as we noted above. Of course we need not to stop here; we can continue the decomposition of three smaller tableaux in the right-hand side so on and forth. At last of this process we obtain a "fish-scale diagram" like below:

$$m_{\mathsf{S}} =$$



.

In particular, every homomorphism between parabolic modules can be constructed from three parts of diagram: $\mu$, $\Delta$ and $\sigma$. The composition of these homomorphisms can be computed using local transformations listed in Proposition 6.6.

CHAPTER 7

# Fakemodules over the Iwahori–Hecke algebra

In this chapter, we introduce a category $\underline{H}_t$-$\mathcal{M}od$ which interpolates usual module categories $H_n$-$\mathcal{M}od$ in some sense. The index $t$, which is not necessarily a natural number, is considered as a rank of the "fake Hecke algebra $\underline{H}_t$", an imaginary object which does not really exist. We call an object and a morphism in $\underline{H}_t$-$\mathcal{M}od$ an *$\underline{H}_t$-fakemodule* and an *$\underline{H}_t$-fakemorphism* respectively, which are made to control hidden behaviors of their underlying usual module and homomorphism.

## 1. Binomial sequences

First we explain what actually the index $t$ is.

Definition 7.1. A *$q$-binomial sequence* in $\Bbbk$ is a function $t\colon \mathbb{N} \to \Bbbk$, whose values are written as $k \mapsto \left[\begin{smallmatrix} t \\ k \end{smallmatrix}\right]$, which satisfies

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = 1 \qquad \text{and} \qquad \begin{bmatrix} t \\ k \end{bmatrix}\begin{bmatrix} t \\ l \end{bmatrix} = \sum_{0\le i\le k,l} q^{(k-i)(l-i)} \begin{bmatrix} l \\ i \end{bmatrix}\begin{bmatrix} k+l-i \\ l \end{bmatrix}\begin{bmatrix} t \\ k+l-i \end{bmatrix}.$$

We denote by $B_q(\Bbbk)$ the set of all $q$-binomial sequences in $\Bbbk$.

This definition is an abstraction of properties of usual $q$-binomial coefficients as we can see in the following examples.

Lemma 7.2. *For each $n \in \mathbb{N}$, the function $k \mapsto \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ is a $q$-binomial sequence. This map $\mathbb{N} \to B_q(\Bbbk)$ is injective.*

Proof. When $l > n$ the multiplicative relation is trivial. Otherwise we can check it by the formula

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{0\le i\le k,l} q^{(k-i)(l-i)} \begin{bmatrix} l \\ i \end{bmatrix}\begin{bmatrix} n-l \\ k-i \end{bmatrix},$$

which follows inductively from

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]}{[k]}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = \frac{q^k[n-k]+[k]}{[k]}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = q^k\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix},$$

and

$$\begin{bmatrix} n-l \\ k-i \end{bmatrix}\begin{bmatrix} n \\ l \end{bmatrix} = \frac{[n]!}{[k-i]![l]![n-k-l+i]!} = \begin{bmatrix} k+l-i \\ l \end{bmatrix}\begin{bmatrix} n \\ k+l-i \end{bmatrix}.$$

We have $n = \max\left\{k \,\middle|\, \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] \ne 0\right\}$ so $n$ is recovered from the sequence $\left\{\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]\right\}_{k\in\mathbb{N}}$. □

Lemma 7.3. *Suppose that $q$ and the $q$-integers $[k]$ for all $k \ge 1$ are invertible. Then for any element $x \in \Bbbk$,*

$$\begin{bmatrix} t \\ k \end{bmatrix} := q^{-\binom{k}{2}} \frac{x(x-[1])\cdots(x-[k-1])}{[k]!}$$

*is a $q$-binomial sequence. Conversely every $q$-binomial sequence is determined by $x = \left[\begin{smallmatrix} t \\ 1 \end{smallmatrix}\right]$ in this way. Hence the set $B_q(\Bbbk)$ is in bijection with $\Bbbk$.*

Proof. Similar as above. □

NOTATION 7.4. Let $t$ be a $q$-binomial sequence. By convention, for $k < 0$ we put $\left[{t \atop k}\right] = 0$. We also write $[t] := \left[{t \atop 1}\right]$ and $q^t := 1 + (q-1)[t]$ for short. These notations are of course compatible with the usual ones for a $q$-binomial $n \in \mathbb{N}$.

We give other strange examples without proofs.

EXAMPLE 7.5. Suppose that $1 - q^k$ is invertible for all $k \geq 1$. Then there is a $q$-binomial sequence $\infty$ defined by

$$\begin{bmatrix} \infty \\ k \end{bmatrix} := \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

Under a suitable topology, it is actually the limit value $\lim_{n\to\infty} \left[{n \atop k}\right]$ according to the Taylor expansion at $q = 0$.

EXAMPLE 7.6. When $q = 0$, the multiplicative relation reduces to

$$\begin{bmatrix} t \\ k \end{bmatrix}\begin{bmatrix} t \\ l \end{bmatrix} = \begin{bmatrix} t \\ \max\{k, l\} \end{bmatrix}.$$

Hence a 0-binomial sequence is nothing but a descending sequence of idempotents. In particular, if $\Bbbk$ has no non-trivial idempotents, we have $B_0(\Bbbk) = \mathbb{N} \cup \{\infty\}$.

EXAMPLE 7.7. By the generalized Lucas' theorem, we have a congruence equation

$$\binom{n + e^k}{k} \equiv \binom{n}{k} \pmod e$$

holds for every $e, n, k \in \mathbb{N}$. So for each $e$-adic number $n \in \mathbb{Z}_e$, a 1-binomial sequence

$$\binom{n}{k} := \binom{n \bmod e^k}{k} \in \mathbb{Z}/e\mathbb{Z}$$

is defined. Here to denote a 1-binomial sequence we prefer the symbol $\binom{t}{k}$ to $\left[{t \atop k}\right]$. The map $\mathbb{Z}_e \to B_1(\mathbb{Z}/e\mathbb{Z})$ is also injective.

In order to study a global property of $q$-binomial sequences, we introduce a binary operation on the set $B_q(\Bbbk)$ by imitating formulas hold for usual $q$-binomial coefficients.

PROPOSITION 7.8. *Let $t$ be a $q$-binomial sequence in $\Bbbk$.*

(1) *For $n \in \mathbb{N}$, let*

$$\begin{bmatrix} t+n \\ k \end{bmatrix} = \sum_{0 \leq i \leq k, n} q^{(k-i)(n-i)} \begin{bmatrix} n \\ i \end{bmatrix}\begin{bmatrix} t \\ k-i \end{bmatrix}.$$

*Then $t + n$ is a $q$-binomial sequence.*

(2) *More generally, for another $q$-binomial sequence $u$, let*

$$\begin{bmatrix} t+u \\ k \end{bmatrix} = \sum_{0 \leq i \leq k} q^{\binom{i}{2}}(q-1)^i[i]! \sum_{0 \leq j \leq k-i} \begin{bmatrix} k-j \\ i \end{bmatrix}\begin{bmatrix} i+j \\ i \end{bmatrix}\begin{bmatrix} t \\ k-j \end{bmatrix}\begin{bmatrix} u \\ i+j \end{bmatrix}.$$

*Then $t + u$ is also a $q$-binomial sequence.*

(3) *$B_q(\Bbbk)$ forms a commutative monoid with respect to this addition. Its unit element is $\left[{0 \atop k}\right] = \delta_{k0}$.*

In particular we have $[t + u] = [t] + [u] + (q-1)[t][u]$. It implies

$$q^t q^u = (1 + (q-1)[t])(1 + (q-1)[u]) = 1 + (q-1)[t+u] = q^{t+u}$$

as desired. To see that (1) is a special case of (2), first we get

$$\begin{aligned}\begin{bmatrix} t+1 \\ k \end{bmatrix} &= \begin{bmatrix} t \\ k \end{bmatrix} + \begin{bmatrix} t \\ k-1 \end{bmatrix} + (q-1)[k]\begin{bmatrix} t \\ k \end{bmatrix} \\ &= q^k \begin{bmatrix} t \\ k \end{bmatrix} + \begin{bmatrix} t \\ k-1 \end{bmatrix}\end{aligned}$$

by letting $u = 1$. Then $t + n$ is obtained as $t + 1 + 1 + \cdots + 1$ inductively. We will prove the rest statements in the next subsection.

The shift operation $t \mapsto t + 1$ is fundamental in study. Unfortunately, this map is not invertible in general as follows.

LEMMA 7.9. *$1 \in B_q(\Bbbk)$ has the inverse element $-1$ if and only if $q \in \Bbbk^\times$. If so, the map $\mathbb{Z} \to B_q(\Bbbk)$ is also injective.*

PROOF. Follows from that $-1$ must and can be defined by $\left[\begin{smallmatrix} -1 \\ k \end{smallmatrix}\right] = (-1)^k q^{-\binom{k}{2}}$. □

However, an element $t - 1$ is unique if it exists.

LEMMA 7.10. *The map $B_q(\Bbbk) \to B_q(\Bbbk)$; $t \mapsto t + 1$ is injective.*

PROOF. Suppose that two $q$-binomial sequences $t$ and $u$ satisfy $t + 1 = u + 1$. Then by definition

$$q^k \left( \begin{bmatrix} t \\ k \end{bmatrix} - \begin{bmatrix} u \\ k \end{bmatrix} \right) + \left( \begin{bmatrix} t \\ k-1 \end{bmatrix} - \begin{bmatrix} u \\ k-1 \end{bmatrix} \right) = 0.$$

Hence there are elements $0 = \epsilon_0, \epsilon_1, \epsilon_2, \ldots \in \Bbbk$ which satisfy $q\epsilon_i = \epsilon_{i-1}$ for all $i \geq 1$, and using these elements we can write $\left[\begin{smallmatrix} t \\ k \end{smallmatrix}\right] - \left[\begin{smallmatrix} u \\ k \end{smallmatrix}\right] = (-1)^k \epsilon_{\binom{k+1}{2}}$. Note that $\epsilon_i \epsilon_j = q^j \epsilon_{i+j} \epsilon_j = 0$. We prove that all these elements must be zero.

By the multiplicative laws of $t$ and $u$ respectively, we have

$$\begin{aligned} 0 &= \left( [t] + q[2] \begin{bmatrix} t \\ 2 \end{bmatrix} - [t]^2 \right) - \left( [u] + q[2] \begin{bmatrix} u \\ 2 \end{bmatrix} - [u]^2 \right) \\ &= -\epsilon_1 + q[2]\epsilon_3 + 2[t]\epsilon_1 \\ &= \epsilon_2 + 2[t]\epsilon_1 \end{aligned}$$

so $\epsilon_2 = -2[t]\epsilon_1$. It implies that $\epsilon_1 = -2[t]\epsilon_0 = 0$ (and $\epsilon_2 = 0$) so $[t] = [u]$. Then for $k \geq 2$, similarly

$$\begin{aligned} 0 &= \left( [k] \begin{bmatrix} t \\ k \end{bmatrix} + q^k [k+1] \begin{bmatrix} t \\ k+1 \end{bmatrix} - [t] \begin{bmatrix} t \\ k \end{bmatrix} \right) - \left( [k] \begin{bmatrix} u \\ k \end{bmatrix} + q^k [k+1] \begin{bmatrix} u \\ k+1 \end{bmatrix} - [u] \begin{bmatrix} u \\ k \end{bmatrix} \right) \\ &= (-1)^k [k] \epsilon_{\binom{k+1}{2}} + (-1)^{k+1} q^k [k+1] \epsilon_{\binom{k+2}{2}} - (-1)^k [t] \epsilon_{\binom{k+1}{2}} \\ &= -(-1)^k \epsilon_{\binom{k+1}{2}+1} - (-1)^k [t] \epsilon_{\binom{k+1}{2}} \end{aligned}$$

so $\epsilon_{\binom{k+1}{2}+1} = -[t]\epsilon_{\binom{k+1}{2}}$. By the same argument we have $\epsilon_i = 0$ for all $i \leq \binom{k+1}{2} + 1$. Since we can take an arbitrary large $k$, the statement follows. □

We will use $q$-binomial sequences to specify the "rank" of the Iwahori–Hecke algebra $\underline{H}_t$. However, in the following construction of its fakemodule category, we will need to use values $\left[\begin{smallmatrix} t-m \\ k \end{smallmatrix}\right]$ for all $m \in \mathbb{N}$. Hence we have to use $q$-binomial sequences only which have following property:

DEFINITION 7.11. A *$q$-binomial sequence* $t$ in $\Bbbk$ is said to be *total* if $t - m$ exists for all $m \in \mathbb{N}$. We denote by $B_q^+(\Bbbk)$ the set of total $q$-binomial sequences; so

$$B_q^+(\Bbbk) := \bigcap_{m \in \mathbb{N}} (B_q(\Bbbk) + m).$$

The subset $B_q^+(\Bbbk)$ is an ideal of $B_q(\Bbbk)$ with respect to the addition. As we noted above, if $q \in \Bbbk^\times$ then $B_q^+(\Bbbk) = B_q(\Bbbk) \supset \mathbb{Z}$.

EXAMPLE 7.12. The $q$-binomial sequence $\infty$ defined in Example 7.5 satisfies $\infty = \infty + t$ for any $t \in B_q(\Bbbk)$; so in particular it is total. It is easy to see that when $q = 0$ there are no total $q$-binomial sequences other than $\infty$.

## 2. Universal binomial ring

Although Proposition 7.8 can be proved directly in principle, the proof will be too complicated. Instead of doing this, we introduce the universal ring $\Bbbk\{T\}$ which parametrizes all $q$-binomial sequences in $\Bbbk$ and makes the proof easier.

DEFINITION 7.13. We denote by $\Bbbk\{T\}$ a commutative algebra generated by elements $\begin{bmatrix} T \\ k \end{bmatrix}$ for all $k \in \mathbb{N}$ with relations similar as those for $q$-binomial sequences:

$$\begin{bmatrix} T \\ 0 \end{bmatrix} = 1 \qquad \text{and} \qquad \begin{bmatrix} T \\ k \end{bmatrix}\begin{bmatrix} T \\ l \end{bmatrix} = \sum_{0 \le i \le k,l} q^{(k-i)(l-i)} \begin{bmatrix} l \\ i \end{bmatrix}\begin{bmatrix} k+l-i \\ l \end{bmatrix}\begin{bmatrix} T \\ k+l-i \end{bmatrix}.$$

We call it the *$q$-binomial ring* over $\Bbbk$.

By definition, giving a $q$-binomial in $\Bbbk$ is equivalent to giving an algebra homomorphism $\Bbbk\{T\} \to \Bbbk$. In the language of algebraic geometry, the affine scheme $\operatorname{Spec}(\Bbbk\{T\})$ represents a functor which sends a commutative algebra $A$ to the set $B_q(A)$. When the assumptions in Lemma 7.3 hold, this functor is simply to take the underlying set of $A$. Hence we have:

COROLLARY 7.14. *When $q$ and $[k]$ for all $k \ge 1$ are invertible, $\Bbbk\{T\}$ is isomorphic to $\Bbbk\big[[T]\big]$, the polynomial ring over $\Bbbk$ in indeterminate $[T]$. Under this isomorphism,*

$$\begin{bmatrix} T \\ k \end{bmatrix} = q^{-\binom{k}{2}} \frac{[T]([T]-[1])\cdots([T]-[k-1])}{[k]!}.$$

The reader should not confuse it with $\Bbbk[\![T]\!]$, the ring of formal power series! Using this universal ring, we can rewrite Proposition 7.8 as follows:

PROPOSITION 7.15. *$\Bbbk\{T\}$ is a cocommutative bialgebra over $\Bbbk$ with the coproduct*

$$\begin{bmatrix} T \\ k \end{bmatrix} \mapsto \sum_{0 \le i \le k} q^{\binom{i}{2}}(q-1)^i [i]! \sum_{0 \le j \le k-i} \begin{bmatrix} k-j \\ i \end{bmatrix}\begin{bmatrix} i+j \\ i \end{bmatrix}\left(\begin{bmatrix} T \\ k-j \end{bmatrix} \otimes \begin{bmatrix} T \\ i+j \end{bmatrix}\right)$$

*and the counit*

$$\begin{bmatrix} T \\ k \end{bmatrix} \mapsto \delta_{k0}.$$

*In other words,* $\operatorname{Spec}(\Bbbk\{T\})$ *is an affine commutative monoid scheme.*

The cocommutativity and the counit law are obvious so the problems are the well-definedness and the coassociativity of the coproduct. Before we prove them, we remark a small observation. Let $\mathbb{Z}[q]$ be a polynomial ring over integers. Then there is a canonical ring homomorphism $\mathbb{Z}[q] \to \Bbbk$ which sends its indeterminate to $q \in \Bbbk$, and induces an isomorphism of algebras

$$\Bbbk\{T\} \simeq \Bbbk \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q]\{T\}.$$

Thus it is enough to prove this proposition for $\Bbbk = \mathbb{Z}[q]$. Since $\mathbb{Z}[q]$ is an integral domain, it can be embedded in its field of fractions $\mathbb{Q}(q)$, the rational function field. In this case, we can concretely construct the algebra $\mathbb{Z}[q]\{T\}$ as follows.

LEMMA 7.16. *$\mathbb{Q}(q)\{T\}$ is isomorphic to the polynomial ring $\mathbb{Q}(q)\big[[T]\big]$. Moreover, the $\mathbb{Z}[q]$-algebra homomorphism $\mathbb{Z}[q]\{T\} \to \mathbb{Q}(q)\{T\}$ induced by $\mathbb{Z}[q] \hookrightarrow \mathbb{Q}(q)$ is injective. As a $\mathbb{Z}[q]$-module, $\mathbb{Z}[q]\{T\}$ is free with basis $\{\left[\begin{smallmatrix}T\\k\end{smallmatrix}\right]\}_{k\in\mathbb{N}}$.*

PROOF. Clearly $q \in \mathbb{Q}(q)$ satisfies the assumption in Corollary 7.14 so the first statement follows. In this polynomial ring, each $\left[\begin{smallmatrix}T\\k\end{smallmatrix}\right]$ has degree $k$ with respect to the indeterminate $[T]$. Since $\mathbb{Z}[q]\{T\}$ is spanned by the set $\{\left[\begin{smallmatrix}T\\k\end{smallmatrix}\right]\}_{\Bbbk\in\mathbb{N}}$ and its image in $\mathbb{Q}(q)\big[[T]\big]$ is linearly independent, so it is in $\mathbb{Z}[q]\{T\}$. This also implies that the $\mathbb{Z}[q]$-linear map $\mathbb{Z}[q]\{T\} \to \mathbb{Q}(q)\{T\}$ is injective. $\square$

COROLLARY 7.17. *$\Bbbk\{T\}$ is also a free $\Bbbk$-module with basis $\{\left[\begin{smallmatrix}T\\k\end{smallmatrix}\right]\}_{k\in\mathbb{N}}$.*

Now define a coproduct on $\mathbb{Q}(q)\{T\} \simeq \mathbb{Q}(q)\big[[T]\big]$ by

$$[T] \mapsto [T] \otimes 1 + 1 \otimes [T] + (q-1)[T] \otimes [T].$$

Then by a direct computation it is clear that this coproduct is coassociative.

LEMMA 7.18. *This coproduct on $\mathbb{Q}(q)\{T\} \simeq \mathbb{Q}(q)\big[[T]\big]$ coincides with one given in Proposition 7.15. Hence this proposition holds for $\Bbbk = \mathbb{Q}(q)$.*

PROOF. Since we are working in the polynomial ring $\mathbb{Q}(q)\big[[T]\big] \otimes_{\mathbb{Q}(q)} \mathbb{Q}(q)\big[[T]\big]$, it suffices to prove that two coproducts coincide when they are composed with the substituting map

$$[T] \otimes 1 \mapsto [m], \quad 1 \otimes [T] \mapsto [n]$$

for all $m, n \in \mathbb{N}$. The second one sends $[T]$ to $[m+n]$ by definition, so $\left[\begin{smallmatrix}T\\k\end{smallmatrix}\right]$ is mapped to $\left[\begin{smallmatrix}m+n\\k\end{smallmatrix}\right]$. On the other hand, we can prove the formula

$$q^{kl} = \sum_{0\le i\le k,l} q^{\binom{i}{2}}(q-1)^i[i]!\begin{bmatrix}k\\i\end{bmatrix}\begin{bmatrix}l\\i\end{bmatrix}$$

which holds for each $k, l \in \mathbb{N}$ by induction. Hence

$$\begin{aligned}\begin{bmatrix}m+n\\k\end{bmatrix} &= \sum_j q^{(k-j)(n-j)}\begin{bmatrix}n\\j\end{bmatrix}\begin{bmatrix}m\\k-j\end{bmatrix}\\ &= \sum_{i,j} q^{\binom{i}{2}}(q-1)^i[i]!\begin{bmatrix}k-j\\i\end{bmatrix}\begin{bmatrix}n-j\\i\end{bmatrix}\begin{bmatrix}n\\j\end{bmatrix}\begin{bmatrix}m\\k-j\end{bmatrix}\\ &= \sum_{i,j} q^{\binom{i}{2}}(q-1)^i[i]!\begin{bmatrix}k-j\\i\end{bmatrix}\begin{bmatrix}i+j\\i\end{bmatrix}\begin{bmatrix}m\\k-j\end{bmatrix}\begin{bmatrix}n\\i+j\end{bmatrix}\end{aligned}$$

which is precisely the first one. $\square$

Now the proposition is obvious from this lemma.

LEMMA 7.19. *Proposition 7.15 holds for $\Bbbk = \mathbb{Z}[q]$. Consequently it also holds for arbitrary ring $\Bbbk$ with $q \in \Bbbk$.*

PROOF. Since $\mathbb{Z}[q]\{T\}$ is free, its tensor product spaces can be naturally embedded in those of $\mathbb{Q}(q)\{T\}$. By definition values of the coproduct on $\mathbb{Z}[q]\{T\}$ live in these spaces. Hence the well-definedness and the associativity on $\mathbb{Z}[q]\{T\}$ are induced from those on $\mathbb{Q}(q)\{T\}$. $\square$

In addition to this proof, the $q$-binomial ring helps us to prove several formulas for $q$-binomial sequences; in many cases, these equations can be stated over $\mathbb{Z}[q]$. Then by a similar argument it suffices to prove them for natural numbers instead of for general ones. For example, we can easily show the next statement when $t \in \mathbb{N}$. Thus it is also true for an arbitrary $t \in B_q(\Bbbk)$.

LEMMA 7.20. *Let $t$ be a $q$-binomial sequence and $k, l \in \mathbb{N}$. Then*

$$\begin{bmatrix} t \\ k \end{bmatrix}\begin{bmatrix} t+l \\ l \end{bmatrix} = \begin{bmatrix} k+l \\ k \end{bmatrix}\begin{bmatrix} t+l \\ k+l \end{bmatrix}.$$

At last of this subsection, we also define the universal ring $\Bbbk\{T\}^+$ for total $q$-binomial sequences. Again $B_q^+(\Bbbk)$ is in bijection with the set of algebra homomorphisms $\Bbbk\{T\}^+ \to \Bbbk$. Note that the map $t \mapsto t+1$ induces a $\Bbbk$-algebra homomorphism $\Bbbk\{T+1\} \to \Bbbk\{T\}$; here $\Bbbk\{T+1\}$ is the $q$-binomial ring in a new "indeterminate" $T+1$. Explicitly, the map is

$$\begin{bmatrix} T+1 \\ k \end{bmatrix} \mapsto q^k \begin{bmatrix} T \\ k \end{bmatrix} + \begin{bmatrix} T \\ k-1 \end{bmatrix}.$$

DEFINITION 7.21. A *total $q$-binomial ring* $\Bbbk\{T\}^+$ is defined as a direct limit

$$\Bbbk\{T\}^+ := \varinjlim(\cdots \to \Bbbk\{T+1\} \to \Bbbk\{T\} \to \Bbbk\{T-1\} \to \cdots).$$

By this definition, there is a canonical algebra homomorphism $\Bbbk\{T\} \to \Bbbk\{T\}^+$ which corresponds to the embedding $B_q^+(A) \hookrightarrow B_q(A)$. Its basic properties can be described as follows.

PROPOSITION 7.22. (1) *$\Bbbk\{T\} \to \Bbbk\{T\}^+$ is injective if and only if $q$ is not a zero divisor.*
(2) *$\Bbbk\{T\} \to \Bbbk\{T\}^+$ is an isomorphism if and only if $q \in \Bbbk^\times$.*

PROOF. It suffices to prove that so is $\Bbbk\{T+1\} \to \Bbbk\{T\}$ in each case. It follows from that this map can be represented by an upper triangular matrix whose diagonal is $(1, q, q^2, \dots)$, with respect to the bases $\{\left[\begin{smallmatrix} T \\ k \end{smallmatrix}\right]\}_{k\in\mathbb{N}}$ and $\{\left[\begin{smallmatrix} T+1 \\ k \end{smallmatrix}\right]\}_{k\in\mathbb{N}}$. $\square$

In particular, $\mathbb{Z}[q]\{T\}^+$ is also embedded in the polynomial ring $\mathbb{Q}(q)\big[[T]\big]$ as the union of all $\mathbb{Z}[q]\{T+m\}$. Unfortunately, the structure of $\Bbbk\{T\}^+$ is more complicated than that of $\Bbbk\{T\}$ when $q \notin \Bbbk^\times$; it is not even free in general.

## 3. The category of induced fakemodules

We here introduce the most basic objects in the category $\underline{H}_t$-$\mathcal{M}od$ written as $\underline{\mathrm{Ind}}_{t-m}V$, which we call an *induced fakemodule*. This fakemodule imitates the usual induced module $\mathrm{Ind}_{d-m}V$. In this subsection we define the full subcategory $\underline{H}_t$-$\mathcal{M}od_0$, which consists of fakemodules in this form, in terms of generators and relations.

DEFINITION 7.23. Let $t$ be a total $q$-binomial sequence in $\Bbbk$. We define a category $\underline{H}_t$-$\mathcal{M}od_0$ as follows. An object in the category $\underline{H}_t$-$\mathcal{M}od_0$ is an $H_m$-module $V$ for some $m \in \mathbb{N}$, represented by the symbol $\underline{\mathrm{Ind}}_{t-m}V$. Morphisms between these objects are generated by

$$\underline{\mathrm{Ind}}_{t-m}f \colon \underline{\mathrm{Ind}}_{t-m}V \to \underline{\mathrm{Ind}}_{t-m}W,$$

defined for each $H_m$-homomorphism $f \colon V \to W$, and

$$\underline{\mu}_{(t-m-k,k)}V \colon \underline{\mathrm{Ind}}_{t-m-k}\mathrm{Ind}_k V \to \underline{\mathrm{Ind}}_{t-m}V,$$
$$\underline{\Delta}_{(t-m-k,k)}V \colon \underline{\mathrm{Ind}}_{t-m}V \to \underline{\mathrm{Ind}}_{t-m-k}\mathrm{Ind}_k V$$

defined for each $H_m$-module $V$ and $k \in \mathbb{N}$, with relations listed below. The first two of them are:

(a) $\underline{\mathrm{Ind}}_{t-m}$ is a $\Bbbk$-linear functor $H_m$-$\mathcal{M}od \to \underline{H}_t$-$\mathcal{M}od$. That is,

$$\underline{\mathrm{Ind}}_{t-m}\mathrm{id}_V = \mathrm{id}_{\underline{\mathrm{Ind}}_{t-m}V}, \quad \underline{\mathrm{Ind}}_{t-m}(f \circ g) = \underline{\mathrm{Ind}}_{t-m}f \circ \underline{\mathrm{Ind}}_{t-m}g$$

and

$$\underline{\mathrm{Ind}}_{t-m}(af + bg) = a \cdot \underline{\mathrm{Ind}}_{t-m}f + b \cdot \underline{\mathrm{Ind}}_{t-m}g$$

for suitable $H_m$-homomorphisms $f, g$ and scalars $a, b \in \Bbbk$.

(b) $\underline{\mu}_{(t-m-k,k)}$ and $\underline{\Delta}_{(t-m-k,k)}$ are natural transformations between functors $H_m$-$\mathcal{M}od \to \underline{H}_t$-$\mathcal{M}od$, respectively $\underline{\mathrm{Ind}}_{t-m-k}\mathrm{Ind}_k \rightleftharpoons \underline{\mathrm{Ind}}_{t-m}$. That is, the square below and its dual commute for any $H_m$-homomorphism $f\colon V \to W$:

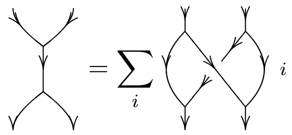
$$\begin{tikzcd}
\underline{\mathrm{Ind}}_{t-m-k}\mathrm{Ind}_k V \arrow[r, "\underline{\mu}_{(t-m-k,k)}V"] \arrow[d, "\underline{\mathrm{Ind}}_{t-m-k}\mathrm{Ind}_k f"'] & \underline{\mathrm{Ind}}_{t-m} V \arrow[d, "\underline{\mathrm{Ind}}_{t-m} f"] \\
\underline{\mathrm{Ind}}_{t-m-k}\mathrm{Ind}_k W \arrow[r, "\underline{\mu}_{(t-m-k,k)}W"'] & \underline{\mathrm{Ind}}_{t-m} W.
\end{tikzcd}$$

The rest relations are represented by diagrams as we do before. To represent the functor $\underline{\mathrm{Ind}}$ and the natural transformations $\underline{\mu}$ and $\underline{\Delta}$, we use same diagrams as Ind, $\mu$ and $\Delta$. Here arrows which represent $\underline{\mathrm{Ind}}$ always appear in leftmost of each diagram.

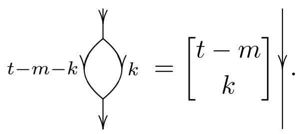
(1) The associativity and the coassociativity laws:

(2) The unit and the counit laws:

(3) The graded bialgebra relation:

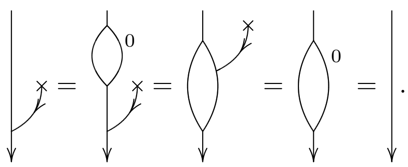


(4) The bubble elimination:



As we mentioned above, an object and a morphism in $\underline{H}_t$-$\mathcal{M}od$ is called an *$\underline{H}_t$-fakemodule* and an *$\underline{H}_t$-fakemorphism* respectively. We denote by $\mathrm{Hom}_{\underline{H}_t}$ the set of fakemorphisms between fakemodules instead of $\mathrm{Hom}_{\underline{H}_t\text{-}\mathcal{M}od_0}$ for simplicity.

Remark 7.24. The relation (2) above is needless since we can deduce it from (1) and (4) using the unit law on the ordinal $\mathrm{Ind}_0$:



We still list this relation here for convenience of later proofs.

Especially, when the rank $t$ is an usual integral rank $d \in \mathbb{N}$, we obtain a category $\underline{H}_d$-$\mathcal{M}od_0$ which is similar to but slightly different from original $H_d$-$\mathcal{M}od$. For example, if $m > d$ then a module $\mathrm{Ind}_{d-m}V$ is zero by definition while the corresponding fakemodule $\underline{\mathrm{Ind}}_{d-m}V$ is not. Note that in the definition we use values of $q$-binomial coefficients $\left[{d-m \atop k}\right]$ for negative integers. So to define $\underline{H}_d$-$\mathcal{M}od_0$ we must have that the $q$-binomial sequence $d$ is total, or equivalently, $q \in \Bbbk^\times$.

PROPOSITION 7.25. *Suppose that $q \in \Bbbk^\times$. Then for each $d \in \mathbb{N}$, there is a full and surjective functor $P\colon \underline{H}_d\text{-}\mathcal{M}od_0 \to H_d\text{-}\mathcal{M}od$ such that $P \circ \underline{\mathrm{Ind}}_{d-m} = \mathrm{Ind}_{d-m}$, $P(\underline{\mu}_{(d-m-k,k)}V) = \mu_{(d-m-k,k)}V$ and $P(\underline{\Delta}_{(d-m-k,k)}V) = \Delta_{(d-m-k,k)}V$.*

PROOF. This functor is well-defined since these relations are satisfied in $H_d$-$\mathcal{M}od$ by Proposition 6.6. For each $H_d$-module $V$, there is an $\underline{H}_d$-fakemodule $\underline{\mathrm{Ind}}_0 V$ which is mapped to it by $P$, so $P$ is surjective. Moreover, any $H_d$-homomorphism $f\colon \mathrm{Ind}_{d-m}V \to \mathrm{Ind}_{d-n}W$ comes from

$$\underline{\mathrm{Ind}}_{d-m}V \xrightarrow{\underline{\Delta}_{(0,d-m)}V} \underline{\mathrm{Ind}}_0\mathrm{Ind}_{d-m}V \xrightarrow{\underline{\mathrm{Ind}}_0 f} \underline{\mathrm{Ind}}_0\mathrm{Ind}_{d-n}W \xrightarrow{\underline{\mu}_{(0,d-m)}W} \underline{\mathrm{Ind}}_{d-n}W.$$

Thus $P$ is also full. □

We call this $P\colon \underline{H}_d\text{-}\mathcal{M}od_0 \to H_d\text{-}\mathcal{M}od$ a *realization functor*, which makes a fakemodule into an usual module.

REMARK 7.26. $\underline{\mu}_{(0,d-m)}$ and $\underline{\Delta}_{(0,d-m)}$ we used above induce natural transformations $\underline{\mathrm{Ind}}_0 \circ P \leftrightarrows \mathrm{Id}$ between endofunctors on $\underline{H}_d$-$\mathcal{M}od_0$. Each $\underline{\mathrm{Ind}}_{d-m}V \in \underline{H}_t$-$\mathcal{M}od_0$ is a direct summand of $\underline{\mathrm{Ind}}_0\mathrm{Ind}_{d-m}V$ via these morphisms if $m \leq d$, and otherwise $\underline{\mathrm{Ind}}_0\mathrm{Ind}_{d-m}V = 0$.

The next is our first main theorem; we can completely describe the set of morphisms in $\underline{H}_t$-$\mathcal{M}od_0$ as follows.

THEOREM 7.27 (Basis theorem). *For $V \in H_m$-$\mathcal{M}od$ and $W \in H_n$-$\mathcal{M}od$,*

$$\mathrm{Hom}_{\underline{H}_t}(\underline{\mathrm{Ind}}_{t-m}V, \underline{\mathrm{Ind}}_{t-n}W) \simeq \bigoplus_i \mathrm{Hom}_{H_i}(\mathrm{Res}'_{m-i}V, \mathrm{Res}_{n-i}W).$$

*This isomorphism is defined similarly as in Proposition 6.10 using $\underline{\mathrm{Ind}}$, $\underline{\Delta}$ and $\underline{\mu}$.*

Note that the right-hand side does not depend on $t$, but composition of these fakemorphisms are different for each $t$. We will prove this theorem in the next subsection. From this result immediately we obtain the statement below.

COROLLARY 7.28. *The functor $\underline{\mathrm{Ind}}_{t-m}\colon H_m\text{-}\mathcal{M}od \to \underline{H}_t\text{-}\mathcal{M}od_0$ has both the right adjoint $\mathrm{PRes}_{t-m}$ and the left adjoint $\mathrm{PRes}'_{t-m}$ defined by*

$$\mathrm{PRes}_{t-m}\underline{\mathrm{Ind}}_{t-n}W \coloneqq \bigoplus_i \mathrm{Ind}_{m-i}\mathrm{Res}_{n-i}W,$$

$$\mathrm{PRes}'_{t-n}\underline{\mathrm{Ind}}_{t-m}V \coloneqq \bigoplus_i \mathrm{Ind}_{n-i}\mathrm{Res}'_{m-i}V.$$

For the origins of the names of these adjoint functors, see Chapter 8. Comparing the theorem with Proposition 6.10, we also obtain the next corollary.

COROLLARY 7.29. *Suppose $q \in \Bbbk^\times$ and let $d \in \mathbb{N}$.*

(1) *For $V \in H_m$-$\mathcal{M}od$ and $W \in H_n$-$\mathcal{M}od$, the kernel of the map*

$$\mathrm{Hom}_{\underline{H}_d}(\underline{\mathrm{Ind}}_{d-m}V, \underline{\mathrm{Ind}}_{d-n}W) \twoheadrightarrow \mathrm{Hom}_{H_d}(\mathrm{Ind}_{d-m}V, \mathrm{Ind}_{d-n}W)$$

*induced by the realization functor $P\colon \underline{H}_d\text{-}\mathcal{M}od_0 \to H_d\text{-}\mathcal{M}od$ is*

$$\bigoplus_{i<m+n-d} \mathrm{Hom}_{H_i}(\mathrm{Res}'_{m-i}V, \mathrm{Res}_{n-i}W).$$

*In particular, this realization map is an isomorphism when $m + n \leq d$.*

(2) *The kernel of the realization functor $P\colon \underline{H}_d\text{-}\mathcal{M}od_0 \to H_d\text{-}\mathcal{M}od$ is generated by objects $\underline{\mathrm{Ind}}_{d-m}V$ for all $V \in H_m$-$\mathcal{M}od$ such that $m > d$. That is, every morphism between fakemodules which is annihilated by $P$ is a sum of morphisms which factor through some $\underline{\mathrm{Ind}}_{d-m}V$.*

## 4. Proof of the basis theorem

Instead of to prove the basis theorem of $\underline{H}_t$-$\mathcal{M}od_0$ directly, we first prove a similar theorem for another category which covers it. Let $\mathcal{C}_t$ be a category defined similarly as $\underline{H}_t$-$\mathcal{M}od_0$ but without the bubble elimination relation (4) in Definition 7.23 (note that this definition actually does not depend on $t$ at all). Then the basis theorem of $\mathcal{C}_t$ can be stated as follows.

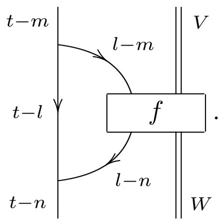
LEMMA 7.30. *The set of morphisms in $\mathcal{C}_t$ is given by*

$$\mathrm{Hom}_{\mathcal{C}_t}(\underline{\mathrm{Ind}}_{t-m}V, \underline{\mathrm{Ind}}_{t-n}W) \simeq \bigoplus_{m,n\le l} \mathrm{Hom}_{H_l}(\mathrm{Ind}_{l-m}V, \mathrm{Ind}_{l-n}W).$$

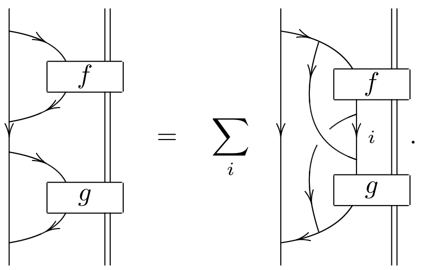
*Here for $f\colon \mathrm{Ind}_{l-m}V \to \mathrm{Ind}_{l-n}W$, the corresponding morphism is*

$$\underline{\mathrm{Ind}}_{t-m}V \xrightarrow{\Delta V} \underline{\mathrm{Ind}}_{t-l}\mathrm{Ind}_{l-m}V \xrightarrow{\underline{\mathrm{Ind}} f} \underline{\mathrm{Ind}}_{t-l}\mathrm{Ind}_{l-n}W \xrightarrow{\mu W} \underline{\mathrm{Ind}}_{t-n}W,$$

*represented as*




PROOF. Trivially we can rewrite the identities and the generators of $\mathcal{C}_t$ in this form. Moreover, by the defining relations, the composition of such morphisms can be also transformed into such form:

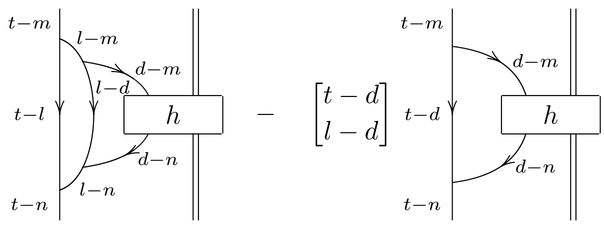



Hence the set of morphisms in $\mathcal{C}$ is spanned by this form.

On the other hand, according to the right-hand side diagram above, we can define the product on the collection of $\Bbbk$-modules $\bigoplus_l \mathrm{Hom}_{H_l}(\mathrm{Ind}_{l-m}V, \mathrm{Ind}_{l-n}W)$. The identity law of this product is clear and the reader can also verify its associativity. Thus this product defines a category, which coincides with $\mathcal{C}_t$ since it is the largest category which satisfies the defining relations. □

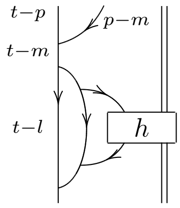
We can obtain the category $\underline{H}_t$-$\mathcal{M}od_0$ by taking quotient of $\mathcal{C}_t$ with respect to the relation (4). Let $\mathcal{I}_t$ be the kernel of the full and surjective functor $\mathcal{C}_t \to \underline{H}_t$-$\mathcal{M}od_0$, that is, the 2-sided ideal in $\mathcal{C}_t$ generated by the difference of the both-hand sides of this relation. By studying this kernel, we can prove the target basis theorem.

LEMMA 7.31. *$\mathcal{I}_t$ is spanned by morphisms of the form*

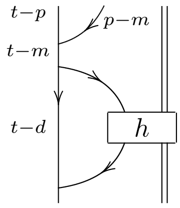


*for all* $d, l, m, n \in \mathbb{N}$, $V \in H_m\text{-}\mathcal{M}od$, $W \in H_n\text{-}\mathcal{M}od$ *and* $h\colon \mathrm{Ind}_{d-m}V \to \mathrm{Ind}_{d-n}W$.

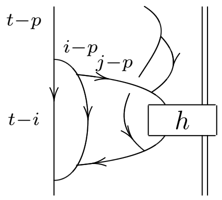
Proof. It is clear that the morphism above will be annihilated in $\underline{H}_t\text{-}\mathcal{M}od_0$, and that a generator of $\mathcal{I}_t$ itself can be written in such form. Hence it suffices to prove that the collection of spaces spanned by these morphisms forms a 2-sided ideal in $\mathcal{C}_t$, that is, it is closed under taking composition with generators. The only its non-trivial part is the case to compose $\underline{\mu}$ on the top (and dually, $\underline{\Delta}$ on the bottom). Actually we can compute



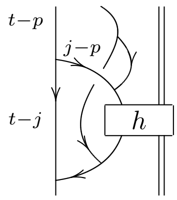
$$- \begin{bmatrix} t-d \\ l-d \end{bmatrix}$$



$$= \sum_{i,j} q^{(i-j)(i-l)} \begin{bmatrix} j-d \\ i-l \end{bmatrix}$$



$$- \sum_{j} \begin{bmatrix} t-d \\ l-d \end{bmatrix}$$

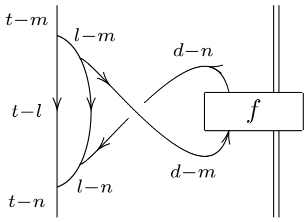


in $\mathcal{C}_t$ using local transformations listed in Definition 7.23 and Proposition 6.6. By the equation

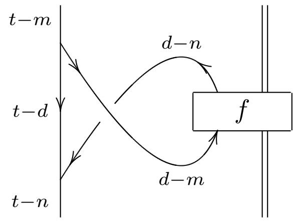
$$\sum_{i} q^{(i-j)(i-l)} \begin{bmatrix} j-d \\ i-l \end{bmatrix} \begin{bmatrix} t-j \\ i-j \end{bmatrix} = \begin{bmatrix} t-d \\ l-d \end{bmatrix},$$

this morphism is decomposed as a linear combination of above ones. □

Lemma 7.32. *$\mathcal{I}_t$ is spanned by morphisms of the form*



$$- \begin{bmatrix} t-d \\ l-d \end{bmatrix}$$



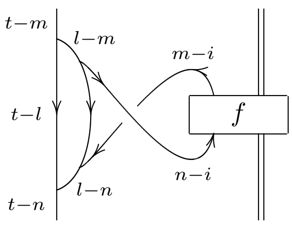
*for all* $d, l, m, n \in \mathbb{N}$, $V \in H_m\text{-}\mathcal{M}od$, $W \in H_n\text{-}\mathcal{M}od$ *and* $f\colon \mathrm{Res}'_{d-n}V \to \mathrm{Res}_{d-m}W$.

Proof. Recall the direct sum decomposition of Hom-space in Proposition 6.10:

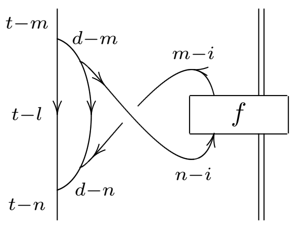
$$\mathrm{Hom}_{H_d}(\mathrm{Ind}_{d-m}V, \mathrm{Ind}_{d-n}W) \simeq \bigoplus_{m+n-d \leq i} \mathrm{Hom}_{H_i}(\mathrm{Res}'_{m-i}V, \mathrm{Res}_{n-i}W).$$

According to this isomorphism, replace $h\colon \mathrm{Ind}_{d-m}V \to \mathrm{Ind}_{d-n}W$ in Lemma 7.31 with the homomorphism corresponds to $f\colon \mathrm{Res}'_{m-i}V \to \mathrm{Res}_{n-i}W$. The result is

$$\begin{bmatrix} l-m-n+i \\ l-d \end{bmatrix}$$



$$- \begin{bmatrix} t-d \\ l-d \end{bmatrix}$$



.

Hence $\mathcal{I}_t$ is also spanned by these morphisms. In particular, by letting $i = m+n-d$ we obtain the morphism above. For a general $i$, since by Lemma 7.20 we have

$$\begin{bmatrix} t-m-n+i \\ l-m-n+i \end{bmatrix} \begin{bmatrix} l-m-n+i \\ l-d \end{bmatrix} = \begin{bmatrix} t-d \\ l-d \end{bmatrix} \begin{bmatrix} t-m-n+i \\ d-m-n+i \end{bmatrix},$$

it can be transformed into a linear combination of these morphisms. $\square$

We are now ready to finish the proof.

Proof of Theorem 7.27. By Lemma 7.30 and Proposition 6.10, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}_t}(\underline{\mathrm{Ind}}_{t-m}V, \underline{\mathrm{Ind}}_{t-n}W) &\simeq \bigoplus_l \mathrm{Hom}_{H_l}(\mathrm{Ind}_{l-m}V, \mathrm{Ind}_{l-n}W) \\ &\simeq \bigoplus_l \bigoplus_{m+n-l\leq i} \mathrm{Hom}_{H_k}(\mathrm{Res}'_{m-i}V, \mathrm{Res}_{n-i}W).\end{aligned}$$

Let us write $H(i,l)$ the summand in the right-hand side above. By Lemma 7.32, the kernel of the map

$$\mathrm{Hom}_{\mathcal{C}_t}(\underline{\mathrm{Ind}}_{t-m}V, \underline{\mathrm{Ind}}_{t-n}W) \to \mathrm{Hom}_{\underline{H}_t}(\underline{\mathrm{Ind}}_{t-m}V, \underline{\mathrm{Ind}}_{t-n}W)$$

is the direct sum of images of the maps

$$\begin{aligned}\mathrm{Hom}_{H_i}(\mathrm{Res}'_{m-i}V, \mathrm{Res}_{n-i}W) &\to H(i,l) \oplus H(i, m+n-i), \\ f &\mapsto (f, -\left[\begin{smallmatrix} t-m-n+i \\ l-m-n+i \end{smallmatrix}\right] f)\end{aligned}$$

for all $i \in \mathbb{N}$ and $l > 0$. Hence we have an isomorphism

$$\begin{aligned}\mathrm{Hom}_{\underline{H}_t}(\underline{\mathrm{Ind}}_{t-m}V, \underline{\mathrm{Ind}}_{t-n}W) &\simeq \bigoplus_i H(i, m+n-i) \\ &\simeq \bigoplus_i \mathrm{Hom}_{H_i}(\mathrm{Res}'_{m-i}V, \mathrm{Res}_{n-i}W).\end{aligned}$$

$\square$

## 5. Parabolic fakemodules

Recall that induction is taking convolution product with the trivial module. By the definition of the category, we can define convolution product of a fakemodule and a usual module as follows.

Definition 7.33. Let $t$ be a total $q$-binomial sequence and $n \in \mathbb{N}$. We define the convolution product $*\colon \underline{H}_t\text{-}\mathcal{M}od_0 \times H_n\text{-}\mathcal{M}od \to \underline{H}_{t+n}\text{-}\mathcal{M}od_0$ as follows. First on objects we put

$$(\underline{\mathrm{Ind}}_{t-m}V) * W := \underline{\mathrm{Ind}}_{t-m}(V * W)$$

for each $V \in H_m\text{-}\mathcal{M}od$ and $W \in H_n\text{-}\mathcal{M}od$. By the associativity of convolution, we can also define

$$\begin{aligned}(\underline{\mathrm{Ind}}_{t-m}f) * g &:= \underline{\mathrm{Ind}}_{t-m}(f * g), \\ (\underline{\mu}_{(t-k-m,k)}V) * W &:= \underline{\mu}_{(t-k-m,k)}(V * W), \\ (\underline{\Delta}_{(t-k-m,k)}V) * W &:= \underline{\Delta}_{(t-k-m,k)}(V * W)\end{aligned}$$

on morphisms. It is easy to check that these morphisms satisfy the defining relations.

We denote by $\underline{\mathbb{1}}_t$ the trivial fakemodule $\underline{\mathrm{Ind}}_t\mathbb{1}_0$. Then an induced fakemodule can be also written as $\underline{\mathrm{Ind}}_{t-m}V \simeq \underline{\mathbb{1}}_{t-m} * V$ using convolution. This product is also associative, so it provides a structure of right $\bigoplus_n(H_n\text{-}\mathcal{M}od)$-module for the category $\bigoplus_m(\underline{H}_{t+m}\text{-}\mathcal{M}od_0)$.

Recall again that a parabolic module $M_\lambda$ is a special case of an induced module. We here introduce parabolic fakemodules into our category $\underline{H}_t\text{-}\mathcal{M}od_0$ by imitating this construction.

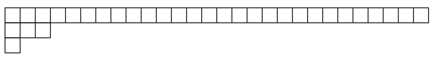
Definition 7.34. Let $t$ be a total $q$-binomial sequence. A *fakecomposition* $\lambda = (\lambda_1, \lambda')$ of $t$ is a pair of a total $q$-binomial sequence $\lambda_1$ and a composition $\lambda'$ such that $|\lambda| \coloneqq \lambda_1 + |\lambda'| = t$. For such $\lambda$, we write $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ where $\lambda_i \coloneqq \lambda'_{i-1}$ for $i \geq 2$. Let $\underline{M}_\lambda \in \underline{H}_t\text{-}\mathcal{M}od_0$ be a fakemodule defined by

$$\underline{M}_\lambda \coloneqq \underline{\mathrm{Ind}}_{\lambda_1} M_{\lambda'} \simeq \mathbb{1}_{\lambda_1} * \mathbb{1}_{\lambda_2} * \mathbb{1}_{\lambda_3} * \cdots * \mathbb{1}_{\lambda_l}.$$

Let $\lambda$ and $\mu$ be two fakecompositions of $t$. Similarly as before, we let $\underline{M}_{\lambda;\mu} \coloneqq \mathrm{Hom}_{\underline{H}_t}(\underline{M}_\mu, \underline{M}_\lambda)^{\mathrm{op}}$, that is, we equip these $\Bbbk$-modules with the reversed composition

$$\circ_\mu \colon \underline{M}_{\mu;\nu} \otimes \underline{M}_{\lambda;\mu} \to \underline{M}_{\lambda;\nu}.$$

Let $\lambda|_d$ and $\mu|_d$ be corresponding fakecompositions of $d \in \mathbb{N}$ obtained by replacing their first components. By Theorem 7.27 the set of $H_d$-homomorphisms $M_{\mu|d} \to M_{\lambda|d}$ stabilizes for sufficiently large $d$ into the set of $\underline{H}_t$-fakemorphisms $\underline{M}_\mu \to \underline{M}_\lambda$. So as a basis of $\underline{M}_{\lambda;\mu}$ we can take the set $\mathrm{Tab}_{\lambda|d;\mu|d}$ for $d \gg 0$ which converges to a finite set. Intuitively we think of Young diagrams whose first rows are very long:

.

Let $\underline{\mathrm{Tab}}_{\lambda;\mu}$ be the set consisting of such tableaux. Formally we define

$$\underline{\mathrm{Tab}}_{\lambda;\mu} \coloneqq \varinjlim_d \mathrm{Tab}_{\lambda|d;\mu|d}$$

where the map $\mathrm{Tab}_{\lambda|d;\mu|d} \hookrightarrow \mathrm{Tab}_{\lambda|d+1;\mu|d+1}$ is inserting $\boxed{1}$ on the first row of the tableau from left. For example, when $\lambda = (t-2, 2)$ and $\mu = (t-3, 2, 1)$, $\underline{\mathrm{Tab}}_{(t-2,2);(t-3,2,1)}$ consisting of the tableaux

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&1&1&2\\\hline 2&3\\\cline{1-2}\end{array}\,,\qquad \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&1&1&3\\\hline 2&2\\\cline{1-2}\end{array}\,,\qquad \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&1&2&2\\\hline 1&3\\\cline{1-2}\end{array}\,,$$

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&1&2&3\\\hline 1&2\\\cline{1-2}\end{array}\,,\qquad \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&2&2&3\\\hline 1&1\\\cline{1-2}\end{array}$$

regardless of $t$. Note that for such a tableau $\mathsf{S}$, a usual tableau '$\mathsf{S}$' is well-defined; this does not depend on how long the first row of $\mathsf{S}$ is. We denote by the symbol $\underline{m}_{\mathsf{S}} \in \underline{M}_{\lambda;\mu}$ the fakemorphism $\underline{M}_\mu \to \underline{M}_\lambda$ corresponding to $\mathsf{S}$. It is defined by the usual homomorphism $m_{`\mathsf{S}'}$ according to the basis theorem similarly as we did in Section 4. The number $\#_{ij}(\mathsf{S})$ is also well-defined for $(i,j) \neq (1,1)$, and we define $\#_{11}(\mathsf{S})$ as a $q$-binomial sequence

$$\#_{11}(\mathsf{S}) \coloneqq t - \sum_{(i,j)\neq(1,1)} \#_{ij}(\mathsf{S}).$$

When $q \in \Bbbk^\times$, for a fakecomposition $\lambda$ of $d \in \mathbb{N}$ the realization functor $P$ sends the fakemodule $\underline{M}_\lambda$ to $M_\lambda$ if $\lambda$ is a composition (that is, $\lambda_1 \geq 0$) and otherwise 0. For two compositions $\lambda$ and $\mu$, the realization of morphisms is given by

$$\begin{aligned} P \colon \underline{M}_{\lambda;\mu} &\twoheadrightarrow M_{\lambda;\mu} \\ \underline{m}_{\mathsf{S}} &\mapsto \begin{cases} m_{\mathsf{S}} & \text{if } \mathsf{S} \in \mathrm{Tab}_{\lambda;\mu}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

More precisely, to realize the $\underline{H}_d$-fakemorphism $\underline{m}_{\mathsf{S}}$ to an $H_d$-homomorphism $m_{\mathsf{S}}$, we should cut off superfluous 1's in the first row of $\mathsf{S}$. When there are not enough such 1's, i.e. $\#_{11}(\mathsf{S}) < 0$, it produces a zero homomorphism. If $t = 4$ in the example above, the realization map

$$P \colon \underline{M}_{(2,2);(1,2,1)} \to M_{(2,2);(1,2,1)}$$

is given by

$$\begin{ytableau} 1 & 1 & \cdots & 1 & 1 & 1 & 2 \\ 2 & 3 \end{ytableau} \mapsto \begin{ytableau} 1 & 2 \\ 2 & 3 \end{ytableau}, \qquad \begin{ytableau} 1 & 1 & \cdots & 1 & 1 & 1 & 3 \\ 2 & 2 \end{ytableau} \mapsto \begin{ytableau} 1 & 3 \\ 2 & 2 \end{ytableau},$$

$$\begin{ytableau} 1 & 1 & \cdots & 1 & 1 & 2 & 2 \\ 1 & 3 \end{ytableau} \mapsto \begin{ytableau} 2 & 2 \\ 1 & 3 \end{ytableau}, \qquad \begin{ytableau} 1 & 1 & \cdots & 1 & 1 & 2 & 3 \\ 1 & 2 \end{ytableau} \mapsto \begin{ytableau} 2 & 3 \\ 1 & 2 \end{ytableau},$$

$$\begin{ytableau} 1 & 1 & \cdots & 1 & 2 & 2 & 3 \\ 1 & 1 \end{ytableau} \mapsto 0.$$

We can also compute their product by regarding $t$ as a large number. For example,

$$\begin{aligned}
&\begin{ytableau} 1 & 1 & \cdots & 1 & 3 \\ 1 & 2 \end{ytableau} \circ_{(t-2,1,1)} \begin{ytableau} 1 & 1 & \cdots & 1 & 1 & 2 \\ 2 \end{ytableau} \\
&= (1 + T_{t-3} + \cdots + T_1 T_2 \cdots T_{t-3}) T_{t-1} T_{t-2} \cdot (1 + T_{t-1}) \underline{m}_{(t-1,1)} \\
&= (1 + T_{t-3} + \cdots + T_1 T_2 \cdots T_{t-3}) \cdot q(1 + T_{t-2}) T_{t-1} \underline{m}_{(t-1,1)} \\
&= q[t-2] T_{t-1} \underline{m}_{(t-1,1)} + q(1 + T_{t-3} + \cdots + T_1 T_2 \cdots T_{t-3}) T_{t-2} T_{t-1} \underline{m}_{(t-1,1)} \\
&= q[t-2] \begin{ytableau} 1 & 1 & \cdots & 1 & 1 & 3 \\ 2 \end{ytableau} + q \begin{ytableau} 1 & 1 & \cdots & 1 & 2 & 3 \\ 1 \end{ytableau}.
\end{aligned}$$

The correctness of this calculation is guaranteed by the following logic: the composite can be computed in a free module over $\mathbb{Z}[q]\{T\}$ embedded in $\mathbb{Q}(q)\big[[T]\big]$, and the equation holds when $[T]$ is replaced with $[d]$ for all $d \gg 0$; hence by nature the both-hand sides are equal with polynomial coefficients.

We define the dominance order on the set of fakecompositions so that $\lambda \leq \mu$ if and only if $\lambda|_d \leq \mu|_d$ for all $d \gg 0$, then the reversed dominance order is still well-founded. According to this dominance order we introduce a quotient module $\underline{M}^{(\nu)}_{\lambda;\mu}$ of $\underline{M}_{\lambda;\mu}$ and $\underline{S}_{\lambda;\mu} := \underline{M}^{(\lambda)}_{\lambda;\mu}$ similarly as before. By the utterly same proofs as before, we obtain an analogous theorems on these modules.

THEOREM 7.35.

(1) *$\underline{S}_{\lambda;\lambda}$ is spanned by $\underline{m}_\lambda$.*
(2) *$\underline{S}_{\lambda;\mu} = 0$ unless $\lambda \geq \mu$.* □

We also say that a fakecomposition $\lambda$ is a *fakepartition* if $\lambda'$ is a partition, and the set of semistandard tableaux

$$\underline{\mathrm{STab}}_{\lambda;\mu} := \varinjlim_{d} \mathrm{STab}_{\lambda|d;\mu|d}.$$

THEOREM 7.36. *Assume $q \in \Bbbk^\times$. Then*

(1) *$\underline{M}_{\lambda;\mu}$ has a basis*
$$\bigsqcup_{\nu \colon \text{fakepartition}} \{\underline{m}_{\mathsf{S}} \circ_\nu \underline{m}_{\mathsf{T}^*} \mid \mathsf{S} \in \underline{\mathrm{STab}}_{\nu;\mu}, \mathsf{T} \in \underline{\mathrm{STab}}_{\nu;\lambda}\}.$$
(2) *$\underline{S}_{\lambda;\mu}$ has a basis $\{\underline{m}_{\mathsf{T}} \mid \mathsf{T} \in \underline{\mathrm{STab}}_{\lambda;\mu}\}$ so*
$$\underline{S}_{\lambda;\lambda} = \begin{cases} \Bbbk & \text{if } \lambda \text{ is a fakepartition,} \\ 0 & \text{otherwise.} \end{cases}$$
(3) *The product*
$$\circ_\nu \colon \underline{S}_{\nu;\mu} \otimes \underline{S}^*_{\nu;\lambda} \to \underline{M}^{(\nu)}_{\lambda;\mu}$$
*is injective.* □

Now let us define $\underline{\mathscr{S}}_{r,t} \coloneqq \bigoplus_{\lambda,\mu} \underline{M}_{\lambda;\mu}$ with anti-involution $\bullet^*\colon \underline{M}_{\lambda;\mu} \to \underline{M}_{\mu;\lambda}$. Note that the index set is now an infinite set so that it does not have 1. It should be regarded as just a category rather than a non-unital ring, so by a $\underline{\mathscr{S}}_{r,t}$-module we mean a graded space $V = \bigoplus_\lambda V_\lambda$ such that each $1 \in \underline{M}_{\lambda;\lambda}$ acts on $V$ as a projection to $V_\lambda$. When $q \in \Bbbk^\times$, the $q$-Schur algebra $\mathscr{S}_{r,d}$ for $d \in \mathbb{N}$ is obtained from $\underline{\mathscr{S}}_{r,d}$ as a quotient. By the theorems above, $\underline{\mathscr{S}}_{r,t}$ is standardly filtered on the set of fakecompositions of $t$, and when $q \in \Bbbk^\times$ it is also cellular over the set of fakepartitions of $t$ in some sense. It provides the following classification.

THEOREM 7.37. *If $q \in \Bbbk^\times$, we have a one-to-one correspondence*

$$\mathrm{Irr}(\underline{\mathscr{S}}_{r,t}) \xleftrightarrow{1:1} \{\nu = (\nu_1, \dots, \nu_r); \text{fakepartition}\} \times \mathrm{Irr}(\Bbbk).$$

## 6. Completion of category

The category $\underline{H}_t\text{-}\mathcal{M}od_0$ we defined is sometimes inconvenient to study since it does not allow us to apply various categorical operations. At the last of this section we show that the category $\underline{H}_t\text{-}\mathcal{M}od_0$ can be naturally embedded to a larger category $\underline{H}_t\text{-}\mathcal{M}od$ which is closed under taking direct sums, direct summands and direct limits (i.e. filtered colimits). The category $\underline{H}_t\text{-}\mathcal{M}od$ is constructed from $\underline{H}_t\text{-}\mathcal{M}od_0$ using the process of several completions of category, namely *pseudo-abelian envelope* (see [**Del07**, 1]) and *indization* (see [**KS06**, 6]). Let us recall the general notions of them.

DEFINITION 7.38. A category $\mathcal{C}$ is called *idempotent complete* (or *Karoubian*) if every idempotent $e\colon X \to X$ in $\mathcal{C}$ splits, that is, there exists $Y \in \mathcal{C}$ and morphisms $p\colon X \to Y$, $i\colon Y \to X$ such that $ip = e$ and $pi = \mathrm{id}_Y$.

The *idempotent completion* (or *Karoubification*) of $\mathcal{C}$, denoted by $\mathcal{C}^{\mathrm{kar}}$, is the category consisting of all pairs of $X \in \mathcal{C}$ and an idempotent $e\colon X \to X$, written as $eX$, as objects. Its morphisms are defined by

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{kar}}}(eX, fY) \coloneqq f \cdot \mathrm{Hom}_{\mathcal{C}}(X, Y) \cdot e.$$

DEFINITION 7.39. A $\mathbb{Z}$-linear category $\mathcal{C}$ is called *additive* if it is closed under finite direct sum. The *additive envelope* $\mathcal{C}^{\mathrm{add}}$ of $\mathcal{C}$ consists of formal finite direct sums $\bigoplus_i X_i$ of objects in $\mathcal{C}$, with morphisms

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{add}}}(\bigoplus_i X_i, \bigoplus_j Y_j) \coloneqq \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j).$$

DEFINITION 7.40. A $\mathbb{Z}$-linear category $\mathcal{C}$ is called *pseudo-abelian* if it is additive and idempotent complete. The *pseudo-abelian envelope* of $\mathcal{C}$ is defined by $\mathcal{C}^{\mathrm{psab}} \coloneqq (\mathcal{C}^{\mathrm{add}})^{\mathrm{kar}}$.

Then it is easy to prove that $\mathcal{C}^{\mathrm{psab}}$ is a pseudo-abelian category contains $\mathcal{C}$ as a full subcategory, which is the smallest in the following sense: if $\mathcal{D}$ is another pseudo-abelian category, we have a category equivalence

$$\mathcal{H}om(\mathcal{C}^{\mathrm{psab}}, \mathcal{D}) \simeq \mathcal{H}om(\mathcal{C}, \mathcal{D})$$

induced by the canonical functor $\mathcal{C} \to \mathcal{C}^{\mathrm{psab}}$. Here we used the symbol $\mathcal{H}om$ to denote the category of $\mathbb{Z}$-linear functors. In other words, every $\mathbb{Z}$-linear category can be naturally extended to a pseudo-abelian category without loss of informations.

Recall that a module of an algebra $A$ is called *finitely presented* if it is isomorphic to the cokernel of some $A$-homomorphism $A^m \to A^n$ with $m, n \in \mathbb{N}$. Let us denote by $A$-*mod* the full subcategory of $A\text{-}\mathcal{M}od$ consisting of finitely presented modules. Note that $H_n$ is finitely presented over $\Bbbk$, so an $H_n$-module $V$ is finitely presented if and only if it is finitely presented as a $\Bbbk$-module.

LEMMA 7.41. *Let $V$ be a finitely presented $H_n$-module. Then* $\mathrm{Ind}_k V$ *and* $\mathrm{Res}'_k V$ *are also finitely presented.*

PROOF. Since these functors are right exact, it suffices to show that $\mathrm{Ind}_k H_n$ and $\mathrm{Res}'_k H_n$ are finitely presented. It follows from the definitions of these modules. □

REMARK 7.42. When $\Bbbk$ is not a coherent ring, $\mathrm{Res}_k$ need not to have this property. For example, let $\Bbbk$ be a commutative $\mathbb{F}_2$-algebra generated by $y, x_1, x_2, \ldots$ with relations $x_i y = 0$ for all $i$, and let $q = 1$. Consider the $\Bbbk\mathfrak{S}_2$-module $V = \Bbbk^2$ with action defined by the matrix

$$s_1 \mapsto \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}.$$

Then its fixed point set $\mathrm{Res}_2 V = V^{\mathfrak{S}_2}$ is $\Bbbk \oplus \mathrm{Ker}\, y$, which is not even finitely generated.

So a fakemodule $\underline{\mathrm{Ind}}_{t-m} V$ should be also regarded as "finitely presented" in some sense. Note that finitely presented modules are closed under taking direct sums and direct summands. With these facts in mind, we define the category $\underline{H}_t$-*mod*, which is again a full subcategory of $\underline{H}_t$-$\mathcal{M}od$, consisting of finitely presented $\underline{H}_t$-fakemodules.

DEFINITION 7.43. For a total $q$-binomial sequence $t$, let $\underline{H}_t$-$mod_0$ be the full subcategory of $\underline{H}_t$-$\mathcal{M}od_0$ consisting of objects $\underline{\mathrm{Ind}}_{t-m} V$ such that $V$ is finitely presented. Then we put

$$\underline{H}_t\text{-}mod := (\underline{H}_t\text{-}mod_0)^{\mathrm{psab}}.$$

To obtain a general fakemodule from these finitely presented ones, we use the process of indization. Recall that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is called *finitary* if $\mathcal{C}$ admits direct limits and $F$ preserves all direct limits. The next fact indicates what we should do. Though it is well-known (according to Breaz [**Bre13**] this result is due to Lenzing [**Len69**]), we here note a sketch of the proof for convenience of the reader.

LEMMA 7.44. *Let $A$ be an algebra and $V$ be an $A$-module. Then*

(1) *$V$ is isomorphic to a direct limit of finitely presented modules,*
(2) *$V$ is finitely presented if and only if the functor* $\mathrm{Hom}_A(V, \bullet)$ *is finitary.*

PROOF. First we can find a free resolution $A^{\oplus\Lambda} \xrightarrow{f} A^{\oplus\Pi} \to V \to 0$ whose ranks are not necessarily finite. Then

$$V \simeq \varinjlim_{(\Lambda_0,\Pi_0)} \mathrm{Coker}(A^{\oplus\Lambda_0} \xrightarrow{f} A^{\oplus\Pi_0})$$

where $\Lambda_0 \subset \Lambda$, $\Pi_0 \subset \Pi$ runs over all pairs of finite subsets which satisfy $f(A^{\oplus\Lambda_0}) \subset A^{\oplus\Pi_0}$. Hence we can assume that $V \simeq \varinjlim_i V_i$, which is a direct limit of finitely presented modules. If $\mathrm{Hom}_A(V, \bullet)$ is finitary, then $\mathrm{End}_A(V) \simeq \varinjlim_i \mathrm{Hom}_A(V, V_i)$ so $\mathrm{id}_V$ factors through some $V_i$. Thus $V$ is isomorphic to a direct summand of $V_i$, which is also finitely presented. The "only if" part follows from that direct limits commute with finite limits and that the functor $\mathrm{Hom}_A(A, \bullet)$ is clearly finitary. □

We call such an isomorphism $V \simeq \varinjlim_i V_i$ a *presentation* of $V$. By using this fact, we can extend the notion of finitely presented modules to a general category as follows.

Definition 7.45. Let $\mathcal{C}$ be a category which admits direct limits. An object $X \in \mathcal{C}$ is called *finitely presented* if the functor $\mathrm{Hom}_{\mathcal{C}}(X, \bullet)$ is finitary. We denote by $\mathcal{C}^{\mathrm{fp}}$ the subcategory of $\mathcal{C}$ consisting of finitely presented objects. $\mathcal{C}$ is called *locally finitely presented* if every its object is isomorphic to a direct limit of finitely presented ones.

Note that if there are presentations of objects $X \simeq \varinjlim_i X_i$ and $Y \simeq \varinjlim_j Y_j$, we can represent the set of morphisms $X \to Y$ by using them as

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \simeq \varprojlim_i (X_i, Y) \simeq \varprojlim_i \varinjlim_j (X_i, Y_j).$$

According to this isomorphism, we can define the category of so-called ind-objects.

Definition 7.46. Let $\mathcal{C}$ be a category. An *ind-object* in $\mathcal{C}$ is a formal direct limit $\varinjlim_i X_i$ of objects in $\mathcal{C}$. The *indization* $\mathcal{C}^{\mathrm{ind}}$ of $\mathcal{C}$ is a category consisting of ind-objects. The set of morphisms between ind-objects is defined by

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{ind}}}(\varinjlim_i X_i, \varinjlim_j Y_j) \coloneqq \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j).$$

Again $\mathcal{C}$ is contained in $\mathcal{C}^{\mathrm{ind}}$ as a full subcategory. $\mathcal{C}^{\mathrm{ind}}$ admits arbitrary direct limits, so in particular it is idempotent complete; here the image of an idempotent $e \colon X \to X$ is obtained as the direct limit

$$eX \simeq \varinjlim(\cdots \to X \xrightarrow{e} X \to \cdots).$$

When $\mathcal{C}$ is an additive category, $\mathcal{C}^{\mathrm{ind}}$ also admits arbitrary direct sums.

Every object in $\mathcal{C}$ is finitely presented in $\mathcal{C}^{\mathrm{ind}}$ by definition, so $\mathcal{C}^{\mathrm{ind}}$ is locally finitely presented. Conversely, a finitely presented object in $\mathcal{C}^{\mathrm{ind}}$ is isomorphic to a direct summand of some object in $\mathcal{C}$ by the same argument as above. Hence we have a category equivalence

$$(\mathcal{C}^{\mathrm{ind}})^{\mathrm{fp}} \simeq \mathcal{C}^{\mathrm{kar}}.$$

When $\mathcal{D}$ is another category which admits direct limits, a functor $F \colon \mathcal{C} \to \mathcal{D}$ can be extended to a finitary functor $F \colon \mathcal{C}^{\mathrm{ind}} \to \mathcal{D}$ defined by $F(\varinjlim_i X_i) \coloneqq \varinjlim_i FX_i$. The category of functors $\mathcal{C} \to \mathcal{D}$ is equivalent to that of finitary functors $\mathcal{C}^{\mathrm{ind}} \to \mathcal{D}$ via this correspondence.

Using these definitions, we can simply rewrite Lemma 7.44 as follows:

Corollary 7.47. *For an algebra $A$, we have*

$$A\text{-}\mathit{mod} = (A\text{-}\mathcal{M}\mathit{od})^{\mathrm{fp}} \qquad \text{and} \qquad A\text{-}\mathcal{M}\mathit{od} \simeq (A\text{-}\mathit{mod})^{\mathrm{ind}}.$$

With this fact in mind, the category $\underline{H}_t$-$\mathcal{M}od$ is defined as the category of ind-objects.

Definition 7.48. For a total $q$-binomial sequence $t$, let

$$\underline{H}_t\text{-}\mathcal{M}\mathit{od} \coloneqq (\underline{H}_t\text{-}\mathit{mod})^{\mathrm{ind}}.$$

We still use the notation $\mathrm{Hom}_{\underline{H}_t}$ to denote the set of morphisms in $\underline{H}_t$-$\mathcal{M}od$.

$\underline{H}_t$-$\mathcal{M}od$ is a locally finitely presented additive category. Since $\underline{H}_t$-$mod$ is idempotent complete, we have $(\underline{H}_t\text{-}\mathcal{M}\mathit{od})^{\mathrm{fp}} \simeq \underline{H}_t\text{-}\mathit{mod}$ as desired.

Example 7.49. Consider a fakemodule defined as a direct limit of parabolic ones

$$\Omega_t \coloneqq \varinjlim_k \underline{M}_{(t-k,1^k)}$$

where each fakemorphism $\underline{M}_{(t-k+1,1^{k-1})} \to \underline{M}_{(t-k,1^k)}$ is given by a tableau

$$\mathsf{S}_k \coloneqq \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&\cdots&1&1\\ \hline 1 \\ \cline{1-1} 2 \\ \cline{1-1} 3 \\ \cline{1-1} \cdots \\ \cline{1-1} k \\ \cline{1-1} \end{array}.$$

For each fakecomposition $\lambda$, there is a natural fakemorphism $\underline{M}_\lambda \to \underline{M}_{(\lambda_1,1^{|\lambda'|})} \to \Omega_t$; so it is considered as the union of all parabolic fakemodules. One has

$$\mathrm{Hom}_{\underline{H}_t}(\mathbb{1}_t, \Omega_t) \simeq \varinjlim_k \mathrm{Hom}_{\underline{H}_t}(\mathbb{1}_t, M_{(t-k,1^k)}) \simeq \Bbbk.$$

In particular, $\Omega_t$ is not zero.

## 7. Extension of functors

It is still left to us to study the relationship between the categories $\underline{H}_t$-$\mathcal{M}od_0$ and $\underline{H}_t$-$\mathcal{M}od$. The key is the next lemma.

LEMMA 7.50. *The functors* $\mathrm{Ind}_k$, $\mathrm{Res}_k$ *and* $\mathrm{Res}'_k$ *are finitary.*

PROOF. $\mathrm{Ind}_k$ and $\mathrm{Res}'_k$ have their right adjoint functors so are obviously finitary. Let $W$ be an $H_{k+n}$-module and take its presentation $W \simeq \varinjlim_j W_j$. Then for each finitely presented $H_n$-module $V$,

$$\begin{aligned}\mathrm{Hom}_{H_n}(V, \mathrm{Res}_k W) &\simeq \mathrm{Hom}_{H_{k+n}}(\mathrm{Ind}_k V, W)\\ &\simeq \varinjlim_j \mathrm{Hom}_{H_{k+n}}(\mathrm{Ind}_k V, W_j)\\ &\simeq \varinjlim_j \mathrm{Hom}_{H_n}(V, \mathrm{Res}_k W_j)\\ &\simeq \mathrm{Hom}_{H_n}(V, \varinjlim_j \mathrm{Res}_k W_j)\end{aligned}$$

because $\mathrm{Ind}_k V$ is also finitely presented. This also holds for arbitrary $V \in H_m$-$\mathcal{M}od$ since it is a direct limit of finitely presented modules. Consequently, we have an isomorphism $\mathrm{Res}_k W \simeq \varinjlim_j \mathrm{Res}_k W_j$ by the Yoneda lemma. □

We define the embedding functor $\underline{H}_t$-$\mathcal{M}od_0 \to \underline{H}_t$-$\mathcal{M}od$ as follows. Recall that an object in $\underline{H}_t$-$\mathcal{M}od_0$ is the induced fakemodule $\underline{\mathrm{Ind}}_{t-m} V$ of an $H_m$-module $V$. Then we can choose its presentation $V \simeq \varinjlim_i V_i$. In order to make the functor $\underline{\mathrm{Ind}}_{t-m}$ into finitary, we have to map $\underline{\mathrm{Ind}}_{t-m} V \in \underline{H}_t$-$\mathcal{M}od_0$ to the direct limit $\varinjlim_i \underline{\mathrm{Ind}}_{t-m} V_i \in \underline{H}_t$-$\mathcal{M}od$ of finitely presented fakemodules. The morphism generators $\underline{\mathrm{Ind}}_{t-m} f$, $\underline{\mu}_{(t-m-k,k)} V$ and $\underline{\mu}_{(t-m-k,k)} V$ are also naturally mapped to morphisms in $\underline{H}_t$-$\mathcal{M}od$ by direct limit.

PROPOSITION 7.51. *The functor* $\underline{H}_t$-$\mathcal{M}od_0 \to \underline{H}_t$-$\mathcal{M}od$ *is well-defined and fully faithful.*

PROOF. Since the images of the generators clearly satisfy the defining relations, the functor $W \mapsto \varinjlim_j \underline{\mathrm{Ind}}_{t-n} W_j$ is defined once a presentation $W \simeq \varinjlim_j W_j$ is fixed for each $W \in H_n$-$\mathcal{M}od$. To prove that this does not depend on choice of presentation of $W$, it suffices to show that so is the functor $X \mapsto \mathrm{Hom}_{\underline{H}_t}(X, \varinjlim_j \underline{\mathrm{Ind}}_{t-n} W_j)$ up to isomorphism. If $V \in H_m\text{-}mod_0$, by Theorem 7.27 actually we have

$$\begin{aligned}\mathrm{Hom}_{\underline{H}_t}(\underline{\mathrm{Ind}}_{t-m}V, \varinjlim_j \underline{\mathrm{Ind}}_{t-n}W_j) &\simeq \varinjlim_j \mathrm{Hom}_{\underline{H}_t}(\underline{\mathrm{Ind}}_{t-m}V, \underline{\mathrm{Ind}}_{t-n}W_j)\\ &\simeq \varinjlim_j \bigoplus_k \mathrm{Hom}_{H_k}(\mathrm{Res}'_{m-k}V, \mathrm{Res}_{n-k}W_j)\\ &\simeq \bigoplus_k \mathrm{Hom}_{H_k}(\mathrm{Res}'_{m-k}V, \varinjlim_j \mathrm{Res}_{n-k}W_j)\\ &\simeq \bigoplus_k \mathrm{Hom}_{H_k}(\mathrm{Res}'_{m-k}V, \mathrm{Res}_{n-k}W)\end{aligned}$$

which does not depend on presentation. Here we used that $\mathrm{Res}'_{m-k}V$ is finitely presented and that $\mathrm{Res}_{n-k}$ is finitary. So the independence holds for $X \in \underline{H}_t\text{-}mod$. Since an object in $\underline{H}_t\text{-}\mathcal{M}od$ is a direct limit of finitely presented fakemodules, it also holds for arbitrary $X \in \underline{H}_t\text{-}\mathcal{M}od$. Thus the functor is well-defined. Moreover, for another $V \in H_m\text{-}\mathcal{M}od$, by taking presentation $V \simeq \varinjlim_i V_i$ we also have

$$\begin{aligned}\mathrm{Hom}_{\underline{H}_t}(\varinjlim_i \underline{\mathrm{Ind}}_{t-m}V_i, \varinjlim_j \underline{\mathrm{Ind}}_{t-n}W_j) &\simeq \varprojlim_i \bigoplus_k \mathrm{Hom}_{H_k}(\mathrm{Res}'_{m-k}V_i, \mathrm{Res}_{n-k}W_j)\\ &\simeq \bigoplus_k \mathrm{Hom}_{H_k}(\varinjlim_i \mathrm{Res}'_{m-k}V_i, \mathrm{Res}_{n-k}W_j)\\ &\simeq \bigoplus_k \mathrm{Hom}_{H_k}(\mathrm{Res}'_{m-k}V, \mathrm{Res}_{n-k}W)\end{aligned}$$

because the direct sum is finite and $\mathrm{Res}'_{m-k}$ is also finitary. It is isomorphic to $\mathrm{Hom}_{\underline{H}_t}(\underline{\mathrm{Ind}}_{t-m}V, \underline{\mathrm{Ind}}_{t-n}W)$ computed in $\underline{H}_t\text{-}\mathcal{M}od_0$ so this functor is fully faithful. □

Consequently, we can think $\underline{H}_t\text{-}\mathcal{M}od_0$ as a full subcategory of $\underline{H}_t\text{-}\mathcal{M}od$. Under this embedding, the induction functor can be extended to

$$\underline{\mathrm{Ind}}_{t-m} \colon H_m\text{-}\mathcal{M}od \to \underline{H}_t\text{-}\mathcal{M}od_0 \to \underline{H}_t\text{-}\mathcal{M}od.$$

This functor coincides with the one induced from $\underline{\mathrm{Ind}}_{t-m} \colon H_m\text{-}mod_0 \to \underline{H}_t\text{-}mod$ by indization; so in particular it is finitary. To define a functor from $\underline{H}_t\text{-}\mathcal{M}od$, we can use the next lemma.

Lemma 7.52. *Let $\mathcal{C}$ be a pseudo-abelian category which admits direct limits, and let $F$ be an additive functor $\underline{H}_t\text{-}\mathcal{M}od_0 \to \mathcal{C}$. Suppose that for each $m \in \mathbb{N}$ the functor $F \circ \underline{\mathrm{Ind}}_{t-m} \colon H_m\text{-}\mathcal{M}od \to \mathcal{C}$ is finitary, and*

$$F(\underline{\mu}_{(t-m-k,k)}V) = \varinjlim_i F(\underline{\mu}_{(t-m-k,k)}V_i), \quad F(\underline{\Delta}_{(t-m-k,k)}V) = \varinjlim_i F(\underline{\Delta}_{(t-m-k,k)}V_i)$$

*hold for each $H_m$-module $V \simeq \varinjlim_i V_i$ and $k \in \mathbb{N}$. Then $F$ can be uniquely extended to a finitary additive functor $\underline{H}_t\text{-}\mathcal{M}od \to \mathcal{C}$ up to isomorphism.*

Proof. By the property of pseudo-abelian envelope and that of indization, the restriction $\underline{H}_t\text{-}mod_0 \to \mathcal{C}$ of $F$ can be extended to $\underline{H}_t\text{-}\mathcal{M}od \to \mathcal{C}$. By definition this functor is finitary and additive. Moreover it coincides with $F$ on $\underline{H}_t\text{-}\mathcal{M}od_0$ up to isomorphism by the assumptions. The uniqueness is obvious. □

By this lemma we can extend the functors we have defined listed in below:

(1) For $d \in \mathbb{N}$, we have the realization functor $P \colon \underline{H}_d\text{-}\mathcal{M}od \to H_d\text{-}\mathcal{M}od$ which makes a usual module from a fakemodule. This is also full and surjective by the same argument as before. Note that the fakemodule $\Omega_t$ defined in Example 7.49 is not zero but will be disappear by realization in every usual module category.

(2) The convolution product can be also defined as $*\colon \underline{H}_t\text{-}\mathcal{M}od \times H_n\text{-}\mathcal{M}od \to \underline{H}_{t+n}\text{-}\mathcal{M}od$ since $V \mapsto V * W$ is finitary.
(3) The extended functor $\mathrm{PRes}_{t-m}\colon \underline{H}_t\text{-}\mathcal{M}od \to H_m\text{-}\mathcal{M}od$ (resp. $\mathrm{PRes}'_{t-m}$) is still the right (resp. left) adjoint of $\underline{\mathrm{Ind}}_{t-m}\colon H_m\text{-}\mathcal{M}od \to \underline{H}_t\text{-}\mathcal{M}od$. Here the proof for $\mathrm{PRes}_{t-m}$ uses that $\underline{\mathrm{Ind}}_{t-m}V$ is finitely presented when so is $V$, and that for $\mathrm{PRes}'_{t-m}$ is obvious.

CHAPTER 8

# Operations on fakemodules

Though it is not important for the main purpose of this paper, the aim of this chapter is to define various operations acting on fakemodules. We also auxiliary introduce several variations of the fakemodule category $\underline{H}_t$-$\mathcal{M}od$.

## 1. Fakemodules over the parabolic subalgebra

First we define the analogue of the subrestriction functor $\mathrm{Res}_k\colon H_{k+n}\text{-}\mathcal{M}od \to H_n\text{-}\mathcal{M}od$, and we prove that $\mathrm{PRes}_{t-m}$, the right adjoint functor of $\underline{\mathrm{Ind}}_{t-m}$, factors through this functor. Before we define it directly, it is convenient to introduce the category of fakemodules over the parabolic subalgebra "$\underline{H}_{(t,u)} \subset \underline{H}_{t+u}$" in a similar way as before.

Definition 8.1. Let $t, u$ be total $q$-binomial sequences. We define a category $\underline{H}_{(t,u)}$-$\mathcal{M}od_0$ which consists of induced fakemodules in the form $\underline{\mathrm{Ind}}^{(1)}_{t-m}\underline{\mathrm{Ind}}^{(2)}_{u-n}V$ for every $V \in H_{(m,n)}$-$\mathcal{M}od$ as objects. Its morphisms are generated by

$$\underline{\mathrm{Ind}}^{(1)}_{t-m}\underline{\mathrm{Ind}}^{(2)}_{u-n}f\colon \underline{\mathrm{Ind}}^{(1)}_{t-m}\underline{\mathrm{Ind}}^{(2)}_{u-n}V \to \underline{\mathrm{Ind}}^{(1)}_{t-m}\underline{\mathrm{Ind}}^{(2)}_{u-n}W,$$

corresponds to each $H_{(m,n)}$-homomorphism $f\colon V \to W$, and

$$\begin{aligned}\underline{\mu}^{(1)}_{(t-m-k,k)}\underline{\mu}^{(2)}_{(u-n-l,l)}V\colon \underline{\mathrm{Ind}}^{(1)}_{t-m-k}\underline{\mathrm{Ind}}^{(2)}_{u-n-l}\mathrm{Ind}^{(1)}_k\mathrm{Ind}^{(2)}_l V &\to \underline{\mathrm{Ind}}^{(1)}_{t-m}\underline{\mathrm{Ind}}^{(2)}_{u-n}V,\\ \underline{\Delta}^{(1)}_{(t-m-k,k)}\underline{\Delta}^{(2)}_{(u-n-l,l)}V\colon \underline{\mathrm{Ind}}^{(1)}_{t-m}\underline{\mathrm{Ind}}^{(2)}_{u-n}V &\to \underline{\mathrm{Ind}}^{(1)}_{t-m-k}\underline{\mathrm{Ind}}^{(2)}_{u-n-l}\mathrm{Ind}^{(1)}_k\mathrm{Ind}^{(2)}_l V\end{aligned}$$

for each $V$ and $k, l \in \mathbb{N}$, with relations similar to $\underline{H}_t$-$\mathcal{M}od_0$. We complete it into a locally finitely presented additive category $\underline{H}_{(t,u)}$-$\mathcal{M}od$ similarly as before.

Analogously to Theorem 7.27 one can also prove the basis theorem for this category by a similar method. We left the details of the proof to the reader.

Theorem 8.2. *For $V \in H_{(m,n)}$-$\mathcal{M}od$ and $W \in H_{(p,q)}$-$\mathcal{M}od$, we have*

$$\begin{aligned}&\mathrm{Hom}_{\underline{H}_{(t,u)}}(\underline{\mathrm{Ind}}^{(1)}_{t-m}\underline{\mathrm{Ind}}^{(2)}_{u-n}V, \underline{\mathrm{Ind}}^{(1)}_{t-p}\underline{\mathrm{Ind}}^{(2)}_{u-q}W)\\ &\qquad\simeq \bigoplus_{i,j}\mathrm{Hom}_{H_{(i,j)}}(\mathrm{Res}'^{(1)}_{m-i}\mathrm{Res}'^{(2)}_{n-j}V, \mathrm{Res}^{(1)}_{p-i}\mathrm{Res}^{(2)}_{q-j}W).\end{aligned}$$

The most useful tool is the parabolic restriction functor $X \mapsto X|_{(t,u)}$ defined as follows.

Definition 8.3. First we define a functor $|_{(t,u)}\colon \underline{H}_{t+u}$-$\mathcal{M}od_0 \to \underline{H}_{(t,u)}$-$\mathcal{M}od$. We put

$$(\underline{\mathrm{Ind}}_{t+u-m}V)|_{(t,u)} \coloneqq \bigoplus_i \underline{\mathrm{Ind}}^{(1)}_{t-m+i}\underline{\mathrm{Ind}}^{(2)}_{u-i}(V|_{(m-i,i)})$$

on objects. For $f\colon V \to W$, we straightforwardly define

$$(\underline{\mathrm{Ind}}_{t+u-m}f)|_{(t,u)} \coloneqq \sum_i \underline{\mathrm{Ind}}^{(1)}_{t-m+i}\underline{\mathrm{Ind}}^{(2)}_{u-i}(f|_{(m-i,i)}).$$

To define the map on other generators, we use that

$$\begin{aligned}(\underline{\mathrm{Ind}}_{t+u-k-m}\mathrm{Ind}_k V)|_{(t,u)} &= \bigoplus_j \underline{\mathrm{Ind}}^{(1)}_{t-k-m+j}\underline{\mathrm{Ind}}^{(2)}_{u-j}((\mathrm{Ind}_k V)|_{(m-j,j)}) \\ &\simeq \bigoplus_{i,j} \underline{\mathrm{Ind}}^{(1)}_{t-k-m+j}\underline{\mathrm{Ind}}^{(2)}_{u-j}\mathrm{Ind}^{(1)}_{k+i-j}\mathrm{Ind}^{(2)}_{j-i}(V|_{(m-i,i)}).\end{aligned}$$

Under this isomorphism, we define the fakemorphism $(\underline{\Delta}_{(t+u-m-k,k)}V)|_{(t,u)}$ as

$$(\underline{\Delta}_{(t+u-m-k,k)}V)|_{(t,u)} \coloneqq \sum_{i,j} \underline{\Delta}^{(1)}_{(t-k-m+j,k+i-j)}\underline{\Delta}^{(2)}_{(u-j,j-i)}(V|_{(m-i,i)}).$$

In contrast, the fakemorphism $(\underline{\mu}_{(t+u-m-k,k)}V)|_{(t,u)}$ is defined by

$$(\underline{\mu}_{(t+u-m-k,k)}V)|_{(t,u)} \coloneqq \sum_{i,j} q^{(k+i-j)(u-j)}\underline{\mu}^{(1)}_{(t-k-m+j,k+i-j)}\underline{\mu}^{(2)}_{(u-j,j-i)}(V|_{(m-i,i)})$$

with an additional factor $q^{(k+i-j)(u-j)}$ which comes from the exchange of $\underline{\mathbb{1}}_{u-j}$ and $\mathbb{1}_{k+i-j}$. Then we extend it to a finitary functor $|_{(t,u)}\colon \underline{H}_{t+u}\text{-}\mathcal{M}od \to \underline{H}_{(t,u)}\text{-}\mathcal{M}od$.

We left it to the reader to check the relations. For example, the bubble elimination relation can be verified by the formula

$$\begin{bmatrix} t+u-m \\ k \end{bmatrix} = \sum_j q^{(k+i-j)(u-j)} \begin{bmatrix} u-i \\ j-i \end{bmatrix} \begin{bmatrix} t-m+i \\ k+i-j \end{bmatrix}.$$

Recall the definition of $\mathrm{Res}_k$. We define the subrestriction functor acting on fakemodules by imitating this definition, using an analogue of the functor

$$\mathrm{Hom}^{(1)}_{H_k}(V,W) \coloneqq \mathrm{Hom}_{H_{(k,n)}}(V \boxtimes H_n, W) \in H_n\text{-}\mathcal{M}od$$

for $V \in H_k\text{-}\mathcal{M}od$ and $W \in H_{(k,n)}\text{-}\mathcal{M}od$. Clearly we have the outer tensor product

$$\begin{aligned}\boxtimes\colon \underline{H}_t\text{-}\mathcal{M}od \times \underline{H}_u\text{-}\mathcal{M}od &\to \underline{H}_{(t,u)}\text{-}\mathcal{M}od, \\ (\underline{\mathrm{Ind}}_{t-m}V, \underline{\mathrm{Ind}}_{u-n}W) &\mapsto \underline{\mathrm{Ind}}^{(1)}_{t-m}\underline{\mathrm{Ind}}^{(2)}_{u-n}(V \boxtimes W).\end{aligned}$$

The functor $\mathrm{Hom}^{(1)}_{\underline{H}_t}$ is defined as the right adjoint of this outer tensor product.

Lemma 8.4. *For each $X \in \underline{H}_t$-mod, the functor $X \boxtimes \bullet\colon \underline{H}_u\text{-}\mathcal{M}od \to \underline{H}_{(t,u)}\text{-}\mathcal{M}od$ has the right adjoint functor $\mathrm{Hom}^{(1)}_{\underline{H}_t}(X,\bullet)\colon \underline{H}_{(t,u)}\text{-}\mathcal{M}od \to \underline{H}_u\text{-}\mathcal{M}od$.*

Proof. It suffices to prove the case $X = \underline{\mathrm{Ind}}_{t-m}V$ where $V \in H_m$-*mod*. For $Y = \underline{\mathrm{Ind}}^{(1)}_{t-p}\underline{\mathrm{Ind}}^{(2)}_{u-q}W$ where $W \in H_{(p,q)}\text{-}\mathcal{M}od$, we put

$$\mathrm{Hom}^{(1)}_{\underline{H}_t}(X,Y) \coloneqq \underline{\mathrm{Ind}}_{u-q}\Big(\bigoplus_i \mathrm{Hom}^{(1)}_{H_i}(\mathrm{Res}'_{m-i}V, \mathrm{Res}^{(1)}_{p-i}W)\Big)$$

then by the respective basis theorems Theorem 7.27 and Theorem 8.2, the natural isomorphism

$$\mathrm{Hom}_{\underline{H}_{(t,u)}}(X \boxtimes \bullet, Y) \simeq \mathrm{Hom}_{\underline{H}_u}(\bullet, \mathrm{Hom}^{(1)}_{\underline{H}_t}(X,Y))$$

holds as desired. Here we used that the functor $X \boxtimes \bullet$ is finitary. Since $X \boxtimes \bullet$ also preserves the finitely presented property, for a general $Y \simeq \varinjlim_j Y_j \in \underline{H}_{(t,u)}\text{-}\mathcal{M}od$ we can extend it by

$$\mathrm{Hom}^{(1)}_{\underline{H}_t}(X,Y) \coloneqq \varinjlim_j \mathrm{Hom}^{(1)}_{\underline{H}_t}(X,Y_j).$$

□

Definition 8.5. We put $\mathrm{Res}_t Y \coloneqq \mathrm{Hom}^{(1)}_{\underline{H}_t}(\underline{\mathbb{1}}_t, Y|_{(t,u)})$ for $Y \in \underline{H}_{t+u}\text{-}\mathcal{M}od$. Thus $\mathrm{Res}_t$ is a functor $\underline{H}_{t+u}\text{-}\mathcal{M}od \to \underline{H}_u\text{-}\mathcal{M}od$.

By definition, we have

$$\mathrm{Res}_t \underline{\mathrm{Ind}}_{t+u-n} W \simeq \bigoplus_i \underline{\mathrm{Ind}}_{u-i} \mathrm{Res}_{n-i} W$$

which implies

$$P(\mathrm{Res}_{t-m} \underline{\mathrm{Ind}}_{t-n} W) \simeq \bigoplus_i \mathrm{Ind}_{m-i} \mathrm{Res}_{n-i} W = \mathrm{PRes}_{t-m} \underline{\mathrm{Ind}}_{t-n} W$$

where $P\colon \underline{H}_m\text{-}\mathcal{M}od \to H_m\text{-}\mathcal{M}od$ is the realization functor. Since both functors are finitary we have $\mathrm{PRes}_{t-m} \simeq P \circ \mathrm{Res}_{t-m}$. Note that the definition of $\mathrm{Res}_{t-m}$ requires that the $q$-binomial $m$ is total, so that $q \in \Bbbk^\times$, while $\mathrm{PRes}_{t-m}$ can be defined for an arbitrary $q$.

## 2. Right fakemodules and tensor product

Next we define the analogue of the quorestriction functor $\mathrm{Res}'_k\colon H_{k+n}\text{-}\mathcal{M}od \to H_n\text{-}\mathcal{M}od$. Let us denote by $\mathcal{M}od\text{-}A$ the category of *right* $A$-modules for an algebra $A$. We first introduce the category of right fakemodules $\mathcal{M}od\text{-}\underline{H}_t$ whose objects imitate the following modules.

Definition 8.6. For a right $H_n$-module $V$, we define a right $H_{k+n}$-module $\mathrm{Ind}^\tau_k V$ by

$$\mathrm{Ind}^\tau_k V \coloneqq V * \mathbb{1}^*_k = (V \boxtimes \mathbb{1}^*_k) \otimes_{H_{(n,k)}} H_n.$$

Beware that the order of convolution product is exchanged, compared with the definition of $\mathrm{Ind}_k$. The functor $\mathrm{Ind}^\tau_k\colon \mathcal{M}od\text{-}H_n \to \mathcal{M}od\text{-}H_{k+n}$ has the right adjoint

$$\mathrm{Res}^\tau_k W \coloneqq \mathrm{Hom}^{\mathrm{op}}_{H_{(n,k)}}(H_n \boxtimes \mathbb{1}^*_k, W|_{(n,k)})$$

and the left adjoint

$$\mathrm{Res}'^\tau_k W \coloneqq W|_{(k,n)} \otimes_{H_{(k,n)}} (\mathbb{1}_k \boxtimes H_n)$$

similar to $\mathrm{Ind}_k$.

Definition 8.7. For a total $q$-binomial sequence $t$, we define a category $\mathcal{M}od_0\text{-}\underline{H}_t$ which contains objects in the form $\underline{\mathrm{Ind}}^\tau_{t-m} V$ for each $V \in \mathcal{M}od\text{-}H_m$, with morphisms generated by

$$\underline{\mathrm{Ind}}^\tau_{t-m} f\colon \underline{\mathrm{Ind}}^\tau_{t-m} V \to \underline{\mathrm{Ind}}^\tau_{t-m} W$$

for each $f\colon V \to W$ and

$$\underline{\mu}^\tau_{(t-m-k,k)}\colon \underline{\mathrm{Ind}}^\tau_{t-m-k} \mathrm{Ind}^\tau_k V \to \underline{\mathrm{Ind}}^\tau_{t-m} V,$$
$$\underline{\Delta}^\tau_{(t-m-k,k)}\colon \underline{\mathrm{Ind}}^\tau_{t-m} V \to \underline{\mathrm{Ind}}^\tau_{t-m-k} \mathrm{Ind}^\tau_k V.$$

We make its completion $\mathcal{M}od\text{-}\underline{H}_t$ similarly as before. We use the symbol $\mathrm{Hom}^{\mathrm{op}}_{\underline{H}_t}$ to denote the set of fakemorphisms between right fakemodules.

Now let $\tau$ be another anti-involution on the algebra $H_n$ defined by $\tau(T_i) \coloneqq T_{n-i}$. For a left $H_n$-module $V$, let $V^\tau$ be the right $H_n$-module obtained by twisting the action on $V$ via $\tau$. Then we have a category equivalence $\bullet^\tau\colon H_n\text{-}\mathcal{M}od \to \mathcal{M}od\text{-}H_n$. Clearly

$$(V * W)^\tau \simeq W^\tau * V^\tau \quad \text{and} \quad (\mathbb{1}_n)^\tau \simeq \mathbb{1}^*_n,$$

hence we have $(\mathrm{Ind}_k V)^\tau \simeq \mathrm{Ind}^\tau_k V^\tau$. It immediately induces a category equivalence $\bullet^\tau\colon \underline{H}_t\text{-}\mathcal{M}od \to \mathcal{M}od\text{-}\underline{H}_t$ defined by $(\underline{\mathrm{Ind}}_{t-m} V)^\tau \simeq \underline{\mathrm{Ind}}^\tau_{t-m} V^\tau$. So actually we have not defined anything new. By this equivalence, the basis theorem of this category is given as follows.

THEOREM 8.8. *For each $V \in \mathcal{M}od\text{-}H_m$ and $W \in \mathcal{M}od\text{-}H_n$,*

$$\mathrm{Hom}^{\mathrm{op}}_{\underline{H}_t}(\underline{\mathrm{Ind}}^{\tau}_{t-m}V, \underline{\mathrm{Ind}}^{\tau}_{t-n}W) \simeq \bigoplus_i \mathrm{Hom}^{\mathrm{op}}_{H_i}(\mathrm{Res}'^{\tau}_{m-i}V, \mathrm{Res}^{\tau}_{n-i}W).$$

Recall that for a left $A$-module $V$ and a $\Bbbk$-module $Z$, the set of $\Bbbk$-homomorphisms $\mathrm{Hom}_{\Bbbk}(A, Z)$ has a canonical structure of right $A$-module.

LEMMA 8.9. *For $V \in H_m\text{-}\mathcal{M}od$ and $Z \in \Bbbk\text{-}\mathcal{M}od$, we have a natural isomorphism*

$$\mathrm{Hom}_{\Bbbk}(\mathrm{Ind}_k V, Z) \simeq \mathrm{Ind}^{\tau}_k \mathrm{Hom}_{\Bbbk}(V, Z).$$

PROOF. By Lemma 6.4,

$$\begin{aligned}
\mathrm{Hom}_{\Bbbk}(\mathrm{Ind}_k V, Z) &\simeq \mathrm{Hom}^{\mathrm{op}}_{H_{(k,m)}}(H_{k+m}, \mathrm{Hom}_{\Bbbk}(\mathbb{1}_k \boxtimes V, Z)) \\
&\simeq \mathrm{Hom}_{\Bbbk}(\mathbb{1}_k \boxtimes V, Z) \otimes_{H_{(k,m)}} \mathrm{Hom}^{\mathrm{op}}_{H_{(k,m)}}(H_{k+m}, H_{(k,m)}) \\
&\simeq (\mathbb{1}_k \boxtimes \mathrm{Hom}_{\Bbbk}(V, Z)) \otimes_{H_{(k,m)}} {}^{\sigma}H_{k+m} \\
&\simeq (\mathrm{Hom}_{\Bbbk}(V, Z) \boxtimes \mathbb{1}_k) \otimes_{H_{(m,k)}} H_{k+m} \\
&\simeq \mathrm{Ind}^{\tau}_k \mathrm{Hom}_{\Bbbk}(V, Z).
\end{aligned}$$

□

With this fact in mind, we introduce the dual of a fakemodule defined as below.

DEFINITION 8.10. We define

$$\mathrm{Hom}_{\Bbbk}(\underline{\mathrm{Ind}}_{t-m}V, Z) := \underline{\mathrm{Ind}}^{\tau}_{t-m} \mathrm{Hom}_{\Bbbk}(V, Z)$$

for $V \in H_m\text{-}\mathcal{M}od$ and $Z \in \Bbbk\text{-}\mathcal{M}od$. On morphisms, we put

$$\mathrm{Hom}_{\Bbbk}(\underline{\mathrm{Ind}}_{t-m}f, g) := \mathrm{Ind}^{\tau}_{t-m} \mathrm{Hom}_{\Bbbk}(f, g)$$

and

$$\begin{aligned}
\mathrm{Hom}_{\Bbbk}(\underline{\mu}_{(t-m-k,k)}V, Z) &:= \underline{\mu}^{\tau}_{(t-m-k,k)} \mathrm{Hom}_{\Bbbk}(V, Z), \\
\mathrm{Hom}_{\Bbbk}(\underline{\Delta}_{(t-m-k,k)}V, Z) &:= \underline{\Delta}^{\tau}_{(t-m-k,k)} \mathrm{Hom}_{\Bbbk}(V, Z).
\end{aligned}$$

according to the isomorphism

$$\mathrm{Hom}_{\Bbbk}(\underline{\mathrm{Ind}}_{t-m-k}\mathrm{Ind}_k V, Z) \simeq \underline{\mathrm{Ind}}^{\tau}_{t-m-k}\mathrm{Ind}^{\tau}_k \mathrm{Hom}_{\Bbbk}(V, Z).$$

This correspondence defines a functor $\mathrm{Hom}_{\Bbbk}\colon (\underline{H}_t\text{-}\mathcal{M}od_0)^{\mathrm{op}} \times \Bbbk\text{-}\mathcal{M}od \to \mathcal{M}od_0\text{-}\underline{H}_t$.

The relations of generators on the map above are trivial. Unfortunately, we can not extend the domain of this functor to $(\underline{H}_t\text{-}\mathcal{M}od)^{\mathrm{op}}$ because we should have $\mathrm{Hom}_{\Bbbk}(\varinjlim_j Y_j, Z) \simeq \varprojlim_j \mathrm{Hom}_{\Bbbk}(Y_j, Z)$ but $\mathcal{M}od\text{-}\underline{H}_t$ does not admit limits. However we can naturally consider the trinity

$$\mathrm{Hom}^{\mathrm{op}}_{\underline{H}_t}(X, \mathrm{Hom}_{\Bbbk}(Y, Z))$$

of $X \in \mathcal{M}od\text{-}\underline{H}_t$, $Y \in \underline{H}_t\text{-}\mathcal{M}od$ and $Z \in \Bbbk\text{-}\mathcal{M}od$, so that

$$\mathrm{Hom}^{\mathrm{op}}_{\underline{H}_t}(\varinjlim_i X_i, \mathrm{Hom}_{\Bbbk}(\varinjlim_j Y_j, Z)) \simeq \varprojlim_i \varprojlim_j \mathrm{Hom}^{\mathrm{op}}_{\underline{H}_t}(X_i, \mathrm{Hom}_{\Bbbk}(Y_j, Z)).$$

Briefly $\mathrm{Hom}_{\Bbbk}(Y, Z)$ is a presheaf on the category $\mathcal{M}od\text{-}\underline{H}_t$.

LEMMA 8.11. *For each pair of $X \in \mathcal{M}od\text{-}\underline{H}_t$ and $Y \in \underline{H}_t\text{-}\mathcal{M}od$, the functor $\mathrm{Hom}^{\mathrm{op}}_{\underline{H}_t}(X, \mathrm{Hom}_{\Bbbk}(Y, \bullet))$ is representable by a $\Bbbk$-module which we denote by $X \otimes_{\underline{H}_t} Y$.*

PROOF. For $V \in \mathcal{M}od\text{-}H_m$ and $W \in H_n\text{-}\mathcal{M}od$, we put

$$\underline{\mathrm{Ind}}^{\tau}_{t-m} V \otimes_{\underline{H}_t} \underline{\mathrm{Ind}}_{t-n} W := \bigoplus_i \mathrm{Res}'^{\tau}_{m-i} V \otimes_{H_i} \mathrm{Res}'_{n-i} W.$$

Then by Theorem 8.8, we have actually

$$\begin{aligned}\mathrm{Hom}^{\mathrm{op}}_{\underline{H}_t}(\underline{\mathrm{Ind}}^{\tau}_{t-m} V, \mathrm{Hom}_{\Bbbk}(\underline{\mathrm{Ind}}_{t-n} W, Z)) &= \mathrm{Hom}^{\mathrm{op}}_{\underline{H}_t}(\underline{\mathrm{Ind}}^{\tau}_{t-m} V, \underline{\mathrm{Ind}}^{\tau}_{t-n} \mathrm{Hom}_{\Bbbk}(W, Z)) \\ &\simeq \bigoplus_i \mathrm{Hom}^{\mathrm{op}}_{H_i}(\mathrm{Res}'^{\tau}_{m-i} V, \mathrm{Res}^{\tau}_{n-i} \mathrm{Hom}_{\Bbbk}(W, Z))\end{aligned}$$

and for each $i$

$$\begin{aligned}\mathrm{Hom}^{\mathrm{op}}_{H_i}(\mathrm{Res}'^{\tau}_{m-i} V, \mathrm{Res}^{\tau}_{n-i} \mathrm{Hom}_{\Bbbk}(W, Z)) &\simeq \mathrm{Hom}^{\mathrm{op}}_{H_m}(\mathrm{Ind}^{\tau}_{n-i}\mathrm{Res}'^{\tau}_{m-i} V, \mathrm{Hom}_{\Bbbk}(W, Z)) \\ &\simeq \mathrm{Hom}^{\mathrm{op}}_{H_m}(\mathrm{Ind}^{\tau}_{n-i}\mathrm{Res}'^{\tau}_{m-i} V \otimes_{H_m} W, Z) \\ &\simeq \mathrm{Hom}^{\mathrm{op}}_{H_m}(\mathrm{Res}'^{\tau}_{m-i} V \otimes_{H_i} \mathrm{Res}'^{\tau}_{n-i} W, Z)\end{aligned}$$

so the natural isomorphism holds. For general $X \in \mathcal{M}od\text{-}\underline{H}_t$ and $Y \in \underline{H}_t\text{-}\mathcal{M}od$, it suffices to put

$$(\varinjlim_i X_i) \otimes_{\underline{H}_t} (\varinjlim_j Y_j) \simeq \varinjlim_i \varinjlim_j (X_i \otimes_{\underline{H}_t} Y_j).$$

$\square$

Hence we can define tensor product as a functor $\otimes_{\underline{H}_t} \colon \mathcal{M}od\text{-}\underline{H}_t \times \underline{H}_t\text{-}\mathcal{M}od \to \Bbbk\text{-}\mathcal{M}od$. By introducing the dual functor on another direction, we can also prove a natural isomorphism

$$\mathrm{Hom}_{\Bbbk}(X \otimes_{\underline{H}_t} Y, Z) \simeq \mathrm{Hom}_{\underline{H}_t}(Y, \mathrm{Hom}_{\Bbbk}(X, Z)).$$

In a similar method we define $\otimes^{(2)}_{\underline{H}_t} \colon \mathcal{M}od\text{-}\underline{H}_t \times \underline{H}_{(u,t)}\text{-}\mathcal{M}od \to \underline{H}_u\text{-}\mathcal{M}od$ which is an analogue of

$$V \otimes^{(2)}_{H_m} W := (H_n \boxtimes V) \otimes_{H_{(n,m)}} W$$

for $V \in \mathcal{M}od\text{-}H_m$ and $W \in H_{(n,m)}\text{-}\mathcal{M}od$, so that

$$\underline{\mathrm{Ind}}_{t-m} V \otimes^{(2)}_{\underline{H}_t} \underline{\mathrm{Ind}}^{(1)}_{u-n} \underline{\mathrm{Ind}}^{(2)}_{t-p} W = \underline{\mathrm{Ind}}_{u-n}\Big(\bigoplus_i \mathrm{Res}'^{\tau}_{m-i} V \otimes^{(2)}_{H_i} \mathrm{Res}'^{(2)}_{p-i} W\Big).$$

By introducing the enriched dual $\mathrm{Hom}_{\Bbbk} \colon (\mathcal{M}od_0\text{-}\underline{H}_t)^{\mathrm{op}} \times \underline{H}_u\text{-}\mathcal{M}od_0 \to \underline{H}_{(u,t)}\text{-}\mathcal{M}od_0$ defined by

$$\mathrm{Hom}_{\Bbbk}(\underline{\mathrm{Ind}}^{\tau}_{t-m} V, \underline{\mathrm{Ind}}_{u-n} W) := \underline{\mathrm{Ind}}^{(1)}_{u-n} \underline{\mathrm{Ind}}^{(2)}_{t-m} \mathrm{Hom}_{\Bbbk}(V, W),$$

it can be defined by a natural isomorphism

$$\mathrm{Hom}_{\underline{H}_u}(X \otimes^{(2)}_{\underline{H}_t} Y, Z) \simeq \mathrm{Hom}_{\underline{H}_{(u,t)}}(Y, \mathrm{Hom}_{\Bbbk}(X, Z)).$$

Finally let $\mathrm{Res}'_t \colon \underline{H}_{t+u}\text{-}\mathcal{M}od \to \underline{H}_u\text{-}\mathcal{M}od$ be the functor

$$\mathrm{Res}'_t X := \underline{\mathbb{1}}^*_t \otimes^{(2)}_{\underline{H}_t} X|_{(u,t)}.$$

Then we have

$$\mathrm{Res}'_t \underline{\mathrm{Ind}}_{t+u-m} V \simeq \bigoplus_i \underline{\mathrm{Ind}}_{u-i} \mathrm{Res}'_{m-i} V$$

which implies

$$P(\mathrm{Res}'_{t-n} \underline{\mathrm{Ind}}_{t-m} V) \simeq \bigoplus_i \mathrm{Ind}_{n-i} \mathrm{Res}'_{m-i} V \simeq \mathrm{PRes}'_{t-n} \underline{\mathrm{Ind}}_{t-m} V$$

when $q \in \Bbbk^{\times}$. Thus $P \circ \mathrm{Res}'_{t-n} \simeq \mathrm{PRes}'_{t-m}$ also holds in this case.

## 3. The Kronecker product over the symmetric group

In the rest of this chapter, we consider the case $q = 1$. Since $H_n$ is now isomorphic to the symmetric group algebra $\Bbbk\mathfrak{S}_n$, it is better to denote by $\Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od$ the category of fakemodules rather than $\underline{H}_t$-$\mathcal{M}od$. Note that the category $\Bbbk\mathfrak{S}_n$-$\mathcal{M}od$ has another structure of symmetric tensor category, namely the Kronecker tensor product $V \otimes W$ on which the action of $w \in \mathfrak{S}_n$ is defined by $w \cdot (x \otimes y) \coloneqq wx \otimes wy$. This tensor category is closed, that is, it has internal homs $[V, W]$ whose underlying set is $\mathrm{Hom}_\Bbbk(V, W)$, where for $f : V \to W$, $(w \cdot f)(x) \coloneqq wf(w^{-1}x)$. These constructions implicitly use the structure of Hopf algebra on the group algebra $\Bbbk\mathfrak{S}_n$ induced by the diagonal embedding $\mathfrak{S}_n \to \mathfrak{S}_{(n,n)}$ and the inversion $\mathfrak{S}_n \to \mathfrak{S}_n^{\mathrm{op}}$. We introduce a similar structure into our category $\Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od$.

First we remark the next observation. Let us denote by $\delta^* : \Bbbk\mathfrak{S}_{(n,n)}$-$\mathcal{M}od \to \Bbbk\mathfrak{S}_n$-$\mathcal{M}od$ the pullback functor of modules through $\delta : \mathfrak{S}_n \to \mathfrak{S}_{(n,n)}$.

Lemma 8.12. *Let $d, n \in \mathbb{N}$ such that $d \geq n$. For $X \in \Bbbk\mathfrak{S}_{(d,n)}$-$\mathcal{M}od$, we have*

$$\delta^*(\mathrm{Ind}^{(2)}_{d-n} X) \simeq X^{(1)} * \delta^*(X^{(2)} \boxtimes X^{(3)})$$

*under Sweedler's notation $X|_{(d-n,n,n)} \simeq X^{(1)} \boxtimes X^{(2)} \boxtimes X^{(3)}$. That is, it is obtained by inducing the $\Bbbk\mathfrak{S}_{(d-n,n)}$-module $X^{(1)} \boxtimes \delta^*(X^{(2)} \boxtimes X^{(3)})$ whose underlying set is just $X$.*

Proof. Since $w \in \mathfrak{D}_{(d-n,n)}$ acts on $X$ by an isomorphism, we have

$$\begin{aligned}\mathrm{Ind}^{(2)}_{d-n} X &= \bigoplus_{w \in \mathfrak{D}_{(d-n,n)}} X^{(1)} \boxtimes X^{(2)} \boxtimes w(\mathbb{1}_{d-n} \boxtimes X^{(3)}) \\ &= \bigoplus_{w \in \mathfrak{D}_{(d-n,n)}} \delta(w) \cdot (X^{(1)} \boxtimes X^{(2)} \boxtimes \mathbb{1}_{d-n} \boxtimes X^{(3)}).\end{aligned}$$

Then clearly $\delta^*(X^{(1)} \boxtimes X^{(2)} \boxtimes \mathbb{1}_{d-n} \boxtimes X^{(3)}) \simeq X^{(1)} \boxtimes \delta^*(X^{(2)} \boxtimes X^{(3)})$ as $\Bbbk\mathfrak{S}_{(d-n,n)}$-modules, so the statement holds. $\square$

Similarly, for $Y \in \Bbbk\mathfrak{S}_{(n,d)}$-$\mathcal{M}od$ such that $Y|_{(n,d-n,n)} \simeq Y^{(1)} \boxtimes Y^{(2)} \boxtimes Y^{(3)}$,

$$\delta^*(\mathrm{Ind}^{(1)}_{d-n} Y) \simeq Y^{(2)} * \delta^*(Y^{(1)} \boxtimes Y^{(3)}).$$

Hence more generally, for $V \in \Bbbk\mathfrak{S}_{(m,n)}$-$\mathcal{M}od$ by Lemma 6.9 we have

$$\begin{aligned}\delta^*(\mathrm{Ind}^{(1)}_{d-m}\mathrm{Ind}^{(2)}_{d-n} V) &\simeq \bigoplus_i \mathrm{Ind}_{d-m-n+i} V^{(1)} * \delta^*(\mathrm{Ind}_{n-i} V^{(2)} \boxtimes V'') \\ &\simeq \bigoplus_i \mathrm{Ind}_{d-m-n+i}(V^{(1)} * V^{(3)} * \delta^*(V^{(2)} \boxtimes V^{(4)}))\end{aligned}$$

where $V \simeq V' \boxtimes V''$, $V'|_{(m-i,i)} \simeq V^{(1)} \boxtimes V^{(2)}$ and $V''|_{(n-i,i)} \simeq V^{(3)} \boxtimes V^{(4)}$ for each $i$. We analogously introduce the diagonal pullback of $\Bbbk\underline{\mathfrak{S}}_{(t,t)}$-fakemodules.

Definition 8.13. For $V \in \Bbbk\mathfrak{S}_{(m,n)}$-$\mathcal{M}od$, we let

$$\delta^*(\underline{\mathrm{Ind}}^{(1)}_{t-m}\underline{\mathrm{Ind}}^{(2)}_{t-n} V) \coloneqq \bigoplus_i \underline{\mathrm{Ind}}_{t-m-n+i}(V^{(1)} * V^{(3)} * \delta^*(V^{(2)} \boxtimes V^{(4)}))$$

under Sweedler's notation we used above. This correspondence defines a functor $\delta^* : \Bbbk\underline{\mathfrak{S}}_{(t,t)}$-$\mathcal{M}od \to \Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od$. Here the map on morphisms are defined through the parabolic restriction similarly as in Definition 8.3.

By use of this diagonal pullback, we define the Kronecker tensor product on $\Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od$ as follows.

DEFINITION 8.14. For $X, Y \in \Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od$, let

$$X \otimes Y := \delta^*(X \boxtimes Y).$$

Thus $\otimes$ is a functor $\Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od \times \Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od \to \Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od$.

In particular, for $V \in \Bbbk\mathfrak{S}_m$-$\mathcal{M}od$ and $W \in \Bbbk\mathfrak{S}_n$-$\mathcal{M}od$, we have

$$\underline{\mathrm{Ind}}_{t-m}V \otimes \underline{\mathrm{Ind}}_{t-n}W = \bigoplus_i \underline{\mathrm{Ind}}_{t-m-n+i}(V^{(1)} * W^{(1)} * (V^{(2)} \otimes W^{(2)}))$$

where we write $V|_{(m-i,i)} \simeq V^{(1)} \boxtimes V^{(2)}$ and $W|_{(n-i,i)} \simeq W^{(1)} \boxtimes W^{(2)}$ for each $i$. We left it to reader to verify that the tensor product $\otimes$ endowed with the unit object $\underline{\mathbb{1}}_t$ actually satisfies the axioms of symmetric tensor category.

We similarly define internal homs on $\Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od$. For later convenience we use the anti-involution $\tau$ on $\mathfrak{S}_n$ defined by $s_i \mapsto s_{n-i}$ instead of the usual inversion. Since both gives modules isomorphic to each other, this choice do not matter.

DEFINITION 8.15. For $X \in \Bbbk\underline{\mathfrak{S}}_t$-*mod* and $Y \in \Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od$, we let

$$[X, Y] := \delta^* \operatorname{Hom}_{\Bbbk}(X^\tau, Y).$$

Then $[,]$ is a functor $(\Bbbk\underline{\mathfrak{S}}_t\text{-}mod)^{\mathrm{op}} \times \Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od \to \Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od$.

By the same reason as before, the left argument $X$ is restricted to finitely presented fakemodules, so strictly speaking it is not a true internal hom. For an arbitrary $X \in \Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od$, $[X, Y]$ can be defined as a presheaf on $\Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od$. The property of internal hom (in a weaker sense) is now verified as follows.

LEMMA 8.16. *For $X$ and $Y$ above, we have a natural isomorphism*

$$\mathrm{PRes}_t[X, Y] = \operatorname{Hom}_{\Bbbk\underline{\mathfrak{S}}_t}(\underline{\mathbb{1}}_t, [X, Y]) \simeq \operatorname{Hom}_{\Bbbk\underline{\mathfrak{S}}_t}(X, Y).$$

PROOF. For $V$ as in Definition 8.13, we have

$$\mathrm{PRes}_t\delta^*(\underline{\mathrm{Ind}}^{(1)}_{t-m}\underline{\mathrm{Ind}}^{(2)}_{t-n}V) \simeq \bigoplus_i \mathrm{Res}_{m-i}V^{(1)} \boxtimes \mathrm{Res}_{n-i}V^{(3)} \boxtimes \mathrm{Res}_i\delta^*(V^{(2)} \boxtimes V^{(4)}).$$

Explicitly, the summand is the subspace of $V$ consisting of elements which satisfy

$$(w_1, w_2, w_3, w_4) \cdot x = (1_{m-i}, 1_i, 1_{n-i}, w_4w_2^{-1}) \cdot x$$

for every $(w_1, w_2, w_3, w_4) \in \mathfrak{S}_{(m-i,i,n-i,i)}$. In particular, when $V$ above is in the form $\operatorname{Hom}_{\Bbbk}(V^\tau, W)$ for $V \in H_m$-$\mathcal{M}od$ and $W \in H_n$-$\mathcal{M}od$, the summand is naturally isomorphic to $\operatorname{Hom}_{\Bbbk\mathfrak{S}_i}(\mathrm{Res}'_{m-i}V, \mathrm{Res}_{n-i}W)$. Hence by Theorem 7.27,

$$\begin{aligned}\mathrm{PRes}_t[\underline{\mathrm{Ind}}_{t-m}V, \underline{\mathrm{Ind}}_{t-n}W] &\simeq \mathrm{PRes}_t\delta^*\big(\underline{\mathrm{Ind}}^{(1)}_{t-m}\underline{\mathrm{Ind}}^{(2)}_{t-n} \operatorname{Hom}_{\Bbbk}(V^\tau, W)\big) \\ &\simeq \bigoplus_i \operatorname{Hom}_{\Bbbk\mathfrak{S}_i}(\mathrm{Res}'_{m-i}V, \mathrm{Res}_{n-i}W) \\ &\simeq \operatorname{Hom}_{\Bbbk\underline{\mathfrak{S}}_t}(\underline{\mathrm{Ind}}_{t-m}V, \underline{\mathrm{Ind}}_{t-n}W).\end{aligned}$$

Therefore by taking direct limits the statement holds. □

PROPOSITION 8.17. *For $X \in \Bbbk\underline{\mathfrak{S}}_t$-mod, the functor $X \otimes \bullet$ is left adjoint to the functor $[X, \bullet]$.*

PROOF. We here note the sketch of the proof. For $Y \in \Bbbk\underline{\mathfrak{S}}_t$-*mod* and $Z \in \Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od$, as objects in the category $\Bbbk\underline{\mathfrak{S}}_{(t,t,t)}$-$\mathcal{M}od$ which we define similarly as before, a natural isomorphism

$$\operatorname{Hom}_{\Bbbk}(X^\tau \otimes Y^\tau, Z) \simeq \operatorname{Hom}_{\Bbbk}(Y^\tau, \operatorname{Hom}_{\Bbbk}(X^\tau, Z))$$

holds. Then we can apply the diagonal pullback twice, so that

$$[X \otimes Y, Z] \simeq \delta^* \operatorname{Hom}_{\Bbbk}(\delta^*(X \otimes Y)^\tau, Z) \simeq \delta^* \operatorname{Hom}_{\Bbbk}(Y^\tau, \delta^* \operatorname{Hom}_{\Bbbk}(X^\tau, Z)) \simeq [Y, [X, Z]].$$

Then by the previous lemma, applying $\mathrm{PRes}_t$ we obtain

$$\mathrm{Hom}_{\Bbbk\underline{\mathfrak{S}}_t}(X \otimes Y, Z) \simeq \mathrm{Hom}_{\Bbbk\underline{\mathfrak{S}}_t}(Y, [X, Z])$$

as desired. It also holds for an arbitrary $Y \in \Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od$. $\square$

We finish this chapter by describing the relation between the motivating Deligne's category [**Del07**] and our $\Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od$. Here we write $\mathcal{R}ep(\mathfrak{S}_t)$ the Deligne's category for rank $\binom{t}{1} \in \Bbbk$.

Proposition 8.18. *Deligne's category $\mathcal{R}ep(\mathfrak{S}_t)$ is a tensor full subcategory of $\Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od$.*

Proof. We define a tensor functor $\mathcal{R}ep(\mathfrak{S}_t) \to \Bbbk\underline{\mathfrak{S}}_t$-$\mathcal{M}od$ which is fully faithful as follows. For objects, we put for each $m \in \mathbb{N}$

$$\begin{aligned} \mathcal{R}ep(\mathfrak{S}_t) &\to \Bbbk\underline{\mathfrak{S}}_t\text{-}\mathcal{M}od \\ [m] &\mapsto \underline{M}_{(t-m,1^m)}. \end{aligned}$$

Recall that in the Deligne's category morphisms are represented by a *recollement*, that is, an equivalence relation on the union set $U \sqcup V$ with its quotient set $C = (U \sqcup V)/\sim$ such that $U \to C$ and $V \to C$ are both injective. We define a map on morphisms by putting for each recollement $C$ between the sets $\{1, 2, \ldots, m\}$ and $\{1', 2', \ldots, n'\}$,

$$\begin{aligned} \mathrm{Hom}_{\underline{\mathfrak{S}}_t}([n], [m]) &\to \underline{M}_{(t-m,1^m);(t-n,1^n)} \\ (C) &\mapsto \underline{m}_{\mathsf{S}(C)} \end{aligned}$$

where the row-semistandard tableau $\mathsf{S}(C)$ is determined by

$$\#_{i+1,j+1}(\mathsf{S}(C)) \coloneqq \begin{cases} 1 & i \sim j' \text{ in } C, \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, 2 \ldots, m$ and $j = 1, 2, \ldots, n$. We left it to the reader that this map preserves composition and tensor product of morphisms. Clearly the map $C \mapsto \mathsf{S}(C)$ is bijective, so that this functor is fully faithful. $\square$

We remark that for $d \in \mathbb{N}$ Deligne's category $\mathcal{R}ep(\mathfrak{S}_d)$ only depends on the scalar value $d \in \Bbbk$ while our $\Bbbk\underline{\mathfrak{S}}_d$-$\mathcal{M}od$ gives different categories for each $d$. So $\Bbbk\underline{\mathfrak{S}}_d$-$\mathcal{M}od$ is considered to be capturing more precise structures in the modular case.

# Part III

# Representation Theory of the Hecke–Clifford Superalgebra

CHAPTER 9

# Cellular structure on the Hecke–Clifford superalgebra, I

We are now ready to introduce the main topic of this paper, the Hecke–Clifford superalgebra. In this chapter we introduce analogues of the Murphy basis, the $q$-Schur algebra and the Specht modules for this superalgebra and develop the cellular representation theory parallel to the Iwahori–Hecke algebra.

## 1. The Clifford superalgebra

First we define the most basic superalgebra, the Clifford superalgebra.

DEFINITION 9.1. Let $n \in \mathbb{N}$ and take $a_1, \dots, a_n \in \Bbbk$. The *Clifford superalgebra* (or the *Clifford–Grassman superalgebra*) $C_n(a_1, \dots, a_n)$ is generated by the odd elements $c_1, \dots, c_n$ with relations

$$c_i^2 = a_i, \qquad c_i c_j = -c_j c_i \quad \text{for } i \neq j.$$

We have a canonical isomorphism $C_n(a_1, \dots, a_n) \simeq C_1(a_1) \otimes \cdots \otimes C_1(a_n)$ (note that by the help of Koszul sign $c_i$ and $c_j$ for $i \neq j$ (anti-)commutes). Clearly $C_1(a) = \Bbbk \oplus \Bbbk c_1$ so $\{c_1^{p_1} c_2^{p_2} \cdots c_n^{p_n} \mid p_k \in \{0,1\}\}$ is a basis of $C_n(a_1, \dots, a_n)$.

REMARK 9.2. More generally, for each free $\Bbbk$-module $V$ equipped with a quadratic form $Q\colon V \to \Bbbk$, we have the corresponding Clifford superalgebra $C_Q$ generated by $V$ with the relation $v^2 = Q(v)$. When $\Bbbk$ is a field whose characteristic is different from 2, we can always take an orthogonal basis with respect to $Q$, so that $C_Q$ is isomorphic to the above form.

The classification of simple modules of $C_n(a_1, \dots, a_n)$ is well-known for special cases (see [**Kle05**, 12]). We here state a more general result.

PROPOSITION 9.3. *Suppose $\Bbbk$ is a field. Then $C_n(a_1, \dots, a_n)$ has a unique maximal 2-sided ideal. In particular, it has a unique simple module up to isomorphism and parity change $\Pi$.*

PROOF. First we prove the case that $\Bbbk$ is algebraically closed. By replacing $c_i$ with $c_i/\sqrt{a_i}$ for $a_i \neq 0$ and permuting the generators, we may assume that it is in the form $C_n(1, \dots, 1, 0, \dots, 0)$. If the characteristic of $\Bbbk$ is 2, $C_2(1,1)$ is isomorphic to $C_2(1,0)$ since $c_1 + c_2$ (anti-)commutes with $c_1$ and its square is zero. Otherwise $C_2(1,1)$ is isomorphic to the matrix algebra $\mathrm{Mat}_{1|1}(\Bbbk) := \mathrm{End}_{\Bbbk}(\Bbbk \oplus \Pi\Bbbk)$ via the isomorphism using the Pauli matrices below:

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad c_2 \mapsto \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix},$$

$$c_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad c_1 c_2 \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$

Thus in both cases, $C_n(a_1, \dots, a_n)$ is isomorphic to $\mathrm{Mat}_{p|p}(\Bbbk) \otimes C_q(1) \otimes C_r(0, \dots, 0)$ for some $p, r \in \mathbb{N}$ and $q \in \{0,1\}$. Then central idempotent elements $c_1, \dots, c_r \in C_r(0, \dots, 0)$ are contained in its Jacobson radical, and the quotient superalgebra

$\mathrm{Mat}_{p|p}(\Bbbk) \otimes C_q(1)$ with respect to these elements is Morita equivalent to $C_q(1)$, which is clearly simple (note that $(1 \pm c_1)C_1(1)$ is not considered as an ideal because it is not homogeneous).

Now let $\Bbbk$ be an arbitrary field. Take a proper 2-sided ideal $I \subsetneq C_n$. Let $\bar{\Bbbk}$ be an algebraic closure of $\Bbbk$. Then $I \otimes \bar{\Bbbk} \subsetneq C_n \otimes \bar{\Bbbk}$ is also a proper 2-sided ideal so contained in the Jacobson radical of $C_n \otimes \bar{\Bbbk}$ by the previous case. Since the Jacobson radical is nilpotent, so is $I$. Hence $I$ is contained in the Jacobson radical of $C_n$. □

## 2. The Hecke–Clifford superalgebra

Henceforth we fix elements $a, q \in \Bbbk$. Let us write $C_n = C_n(a) \coloneqq C_n(a, \dots, a)$ for short.

DEFINITION 9.4. The *Hecke-Clifford superalgebra* $H_n^c = H_n^c(a;q)$ is generated by $C_n(a)$ and $H_n(q)$ with relations

$$T_i c_j = c_j T_i \quad \text{for } j \neq i, i+1, \qquad T_i c_i = c_{i+1} T_i, \qquad (T_i - q + 1)c_{i+1} = c_i(T_i - q + 1).$$

Note that if $q \in \Bbbk^\times$, the second relation implies the third.

Here in order to make it compatible with the notions in the previous part we slightly modified the original definition by Olshanski [**Ols92**]. When $q = 1$, $H_n^c$ is isomorphic to the wreath product of the Clifford superalgebra

$$W_n(a) \coloneqq C_1(a) \wr \mathfrak{S}_n = C_n(a) \rtimes \mathfrak{S}_n$$

which is called the *Sergeev superalgebra*. In this case there is a natural anti-homomorphism $*\colon W_n(a)^{\mathrm{op}} \to W_n(-a)$ between superalgebras defined by $s_i^* \coloneqq s_i$ and $c_i^* \coloneqq c_i$ (note that due to the Koszul sign we have $(c_i^{\mathrm{op}})^2 = -(c_i^2)^{\mathrm{op}} = -a$), but unfortunately this involution does not have its $q$-analogue. On the other hand, $H_n^c$ has another kind of involution $\tau\colon H_n^c(a)^{\mathrm{op}} \to H_n^c(-a)$ defined by $\tau(T_i) \coloneqq T_{n-i}$, $\tau(c_i) \coloneqq c_{n-i+1}$.

The next basis theorem is well-known, but we make its proof by ourself since we modified the definition.

PROPOSITION 9.5. *The multiplication maps $C_n \otimes H_n \to H_n^c$ and $H_n \otimes C_n \to H_n^c$ are isomorphisms of supermodules.*

PROOF. We prove the first isomorphism. By the defining relations this map is surjective. In order to show that it is also injective, we construct an action of $H_n^c$ on $C_n \otimes H_n$ by

$$\begin{aligned} T_i(x \otimes y) &\coloneqq s_i(x) \otimes T_i y + (q-1)t_i(x) \otimes y \\ c_i(x \otimes y) &\coloneqq c_i x \otimes y \end{aligned}$$

for $x \in C_n$ and $y \in H_n$. Here $s_i$ is the automorphism of superalgebra $C_n$ which exchanges $c_i$ and $c_{i+1}$, and $t_i$ is the $\Bbbk$-linear map $C_n \to C_n$ defined by

$$\begin{aligned} t_i(1) &\coloneqq 0, & t_i(c_{i+1}) &\coloneqq -c_i + c_{i+1}, \\ t_i(c_i) &\coloneqq 0, & t_i(c_i c_{i+1}) &\coloneqq a + c_i c_{i+1} \end{aligned}$$

and $t_i(zc_j) = t_i(z)c_j$ for $j \neq i, i+1$. It is a routine work to verify that the action is well-defined. This action satisfies $xy \cdot (1 \otimes 1) = x \otimes y$ for $x \in C_n$ and $y \in H_n$ so it defines the inverse map $H_n^c \to C_n \otimes H_n$.

Now we have $H_n^c \simeq C_n \otimes H_n$ so that $H_n^c$ is a free supermodule over $\Bbbk$ of rank $2^n n!$ with a basis $\{c_1^{p_1} \cdots c_n^{p_n} T_w\}$. By the commutation relation $\{T_w c_1^{p_1} \cdots c_n^{p_n}\}$ also forms a basis of $H_n^c$. This implies the second isomorphism. □

In particular, $C_n$ and $H_n$ can be identified with subsuperalgebras of $H_n^c$. For each left $H_n$-module $V$, $C_n \otimes V \simeq H_n^c \otimes_{H_n} V$ is naturally a left $H_n^c$-module.

REMARK 9.6. By the commutation relation, for $n \geq 2$, $I_n := \sum_{1 \leq i < j \leq n} (c_i - c_j) H_n^c$ is a 2-sided ideal of $H_n^c$ whose quotient superalgebra is

$$H_n^c / I_n \simeq C_1 \otimes H_n \otimes (\Bbbk / 2a\Bbbk).$$

Now suppose that $2a = 0$. Since $(c_i - c_j)^2 = 2a = 0$ and $(c_i - c_j)(c_i - c_k) = -(c_j - c_k)(c_i - c_j)$ for mutual different $i$, $j$ and $k$, the ideal $I_n$ is nilpotent. Thus it is contained in the Jacobson ideal of $H_n^c$, so that

$$\begin{aligned} \mathrm{Irr}(H_n^c) = \mathrm{Irr}(H_n^c/I_n) \simeq \mathrm{Irr}(C_1 \otimes H_n) = \{V, \Pi V \mid V \in \mathrm{Irr}(H_n), aV = 0\} \\ \sqcup \{V \oplus c_1 V \mid V \in \mathrm{Irr}(H_n), aV = V\}. \end{aligned}$$

Hence the classification of simple module of $H_n^c$ is reduced to that of $H_n$.

The next computation is a key of our theory. Recall that $m_n = \sum_{w \in \mathfrak{S}_n} T_w \in H_n$.

LEMMA 9.7. *Let $\gamma_n^L := c_1 + qc_2 + \cdots + q^{n-1} c_n$ and $\gamma_n^R := q^{n-1} c_1 + q^{n-2} c_2 + \cdots + c_n$. Then for $1 \leq i_1 < i_2 < \cdots < i_r \leq n$,*

$$m_n c_{i_1} c_{i_2} \cdots c_{i_r} m_n = \begin{cases} \left(\frac{a(q-1)}{[2]}\right)^s [n]!\, m_n & \text{if } r = 2s, \\ \left(\frac{a(q-1)}{[2]}\right)^s [n-1]!\, \gamma_n^L m_n & \text{if } r = 2s+1 \end{cases}$$

*(note that $[2], [4], [6], \ldots$ can be divided by $[2]$). Moreover we have $\gamma_n^L m_n = m_n \gamma_n^R$.*

PROOF. Since $H_n^c$ is free over $\Bbbk$, it suffices to prove for the field of rational functions $\Bbbk = \mathbb{Q}(a, q)$ in variables $a$ and $q$, which contains the universal ring $\mathbb{Z}[a, q]$. If $i_{j-1} < i_j - 1$ holds for some $j$, we have

$$\begin{aligned} m_n \cdots c_{i_j} \cdots m_n &= q^{-1} m_n \cdots c_{i_j} \cdots T_{i_j - 1} m_n \\ &= q^{-1} m_n T_{i_j - 1} \cdots c_{i_j - 1} \cdots m_n \\ &= m_n \cdots c_{i_j - 1} \cdots m_n. \end{aligned}$$

Hence we may assume $i_j = j$. Then for $r = 0$ or $1$, we have $m_n^2 = [n]!\, m_n$ and

$$m_n c_1 m_n = (c_1 + c_2 T_1 + \cdots + c_n T_{n-1} \cdots T_2 T_1) m'_{n-1} m_n = [n-1]!\, \gamma_n^L m_n$$

where $m'_{n-1} = m_{(1,n-1)}$. Moreover we have

$$\begin{aligned} q m_n c_1 c_2 c_3 \cdots m_n &= m_n c_1 c_2 c_3 \cdots T_1 m_n \\ &= m_n c_1 T_1 c_1 c_3 \cdots m_n \\ &= m_n (T_1 c_2 + (q-1)(c_1 - c_2)) c_1 c_3 \cdots m_n \\ &= a(q-1) m_n c_3 \cdots m_n - m_n c_1 c_2 c_3 \cdots m_n \end{aligned}$$

so that

$$m_n c_1 c_2 c_3 \cdots m_n = \frac{a(q-1)}{[2]} m_n c_3 \cdots m_n.$$

Hence inductively we obtain the equation. Similarly as above we have

$$m_n c_n m_n = m_n m_{n-1} (c_n + T_{n-1} c_{n-1} + \cdots + T_{n-1} \cdots T_2 T_1 c_1) = [n-1]!\, m_n \gamma_n^R$$

which implies $\gamma_n^L m_n = m_n \gamma_n^R$. $\square$

## 3. Parabolic supermodules

Analogously to the Iwahori–Hecke algebra, for each composition $\lambda$ we introduce the *parabolic subalgebra*

$$H^c_\lambda = \bigoplus_{w\in\mathfrak{S}_\lambda} C_n T_w = \bigoplus_{w\in\mathfrak{S}_\lambda} T_w C_n \simeq H^c_{\lambda_1}\otimes H^c_{\lambda_2}\otimes\cdots\otimes H^c_{\lambda_r}.$$

Then $H^c_n$ is again a free right $H^c_\lambda$-module with a basis $\{T_w \,|\, w\in\mathfrak{D}_\lambda\}$. For $m_\lambda = \sum_{w\in\mathfrak{S}_\lambda} T_w$, $C_n m_\lambda$ is a left (but not right) ideal of $H^c_\lambda$ and the *parabolic module* $M^c_\lambda \coloneqq H^c_n m_\lambda \simeq H^c_n \otimes_{H^c_\lambda} C_n m_\lambda$ is defined as its induced module. Then by the basis theorem $H^c_n \simeq C_n\otimes H_n$ we have $M^c_\lambda \simeq C_n\otimes M_\lambda$, and

$$M^c_\lambda = \{x\in H^c_n \,|\, xT_w = q^{\ell(w)}x \text{ for all } w\in\mathfrak{S}_\lambda\}.$$

In particular we define the *trivial module* $\mathbb{1}^c_n \coloneqq M^c_{(n)} \simeq C_n$. Similarly right modules $M^{c*}_\lambda \coloneqq m_\lambda H^c_n \simeq M^*_\lambda\otimes C_n$ and $\mathbb{1}^{c*}_n \coloneqq M^{c*}_{(n)}$ are defined. Then we have

$$\mathrm{Hom}_{H^c_n}(M^c_\mu, M^c_\lambda) \simeq M^c_\lambda \cap M^{c*}_\mu$$

equipped with the reversed product

$$\begin{aligned}\circ_\mu\colon (M^c_\mu\cap M^{c*}_\nu)\otimes(M^c_\lambda\cap M^{c*}_\mu) &\to M^c_\lambda\cap M^{c*}_\nu,\\ xm_\mu\otimes m_\mu y &\mapsto xm_\mu y.\end{aligned}$$

With Lemma 9.7 in mind, for each composition $\lambda$ we define elements

$$\begin{aligned}
\gamma^L_{\lambda;1} &\coloneqq c_{[1,2,\ldots,\lambda_1]}, & \gamma^R_{\lambda;1} &\coloneqq c_{[\lambda_1,\ldots,2,1]},\\
\gamma^L_{\lambda;2} &\coloneqq c_{[\lambda_1+1,\lambda_1+2,\ldots,\lambda_1+\lambda_2]}, & \gamma^R_{\lambda;2} &\coloneqq c_{[\lambda_1+\lambda_2,\ldots,\lambda_1+2,\lambda_1+1]},\\
&\cdots & &\cdots
\end{aligned}$$

where $c_{[i_1,i_2,\ldots,i_r]} \coloneqq c_{i_1} + qc_{i_2} + \cdots + q^{r-1}c_{i_r}$. Then we have $\gamma^L_{\lambda;i}m_\lambda = m_\lambda\gamma^R_{\lambda;i}$. So let us define the endomorphism $\gamma_{\lambda;i}$ acts on $M^c_\lambda$ and $M^{c*}_\lambda$ as

$$xm_\lambda\cdot\gamma_{\lambda;i} \coloneqq xm_\lambda\gamma^R_{\lambda;i} = x\gamma^L_{\lambda;i}m_\lambda, \qquad \gamma_{\lambda;i}\cdot m_\lambda y \coloneqq \gamma^L_{\lambda;i}m_\lambda y = m_\lambda\gamma^R_{\lambda;i}y.$$

Note that these endomorphisms anti-commute and we have

$$(\gamma^L_{\lambda;i})^2 = (\gamma^R_{\lambda;i})^2 = a[\![\lambda_i]\!]$$

where $[\![k]\!]$ is a $q^2$-integer $1+q^2+\cdots+q^{2(k-1)}$. We can abstractly define a superalgebra consisting of these actions as follows.

Definition 9.8. Let $\lambda$ be a composition. Let $\Gamma_\lambda$ be a superalgebra generated by odd elements $\gamma_{\lambda;1},\gamma_{\lambda;2},\ldots$ with relations

$$(\gamma_{\lambda;i})^2 = a[\![\lambda_i]\!], \qquad \gamma_{\lambda;i}\gamma_{\lambda;j} = -\gamma_{\lambda;j}\gamma_{\lambda;i} \quad \text{for } i\neq j, \qquad \gamma_{\lambda;i}=0 \quad \text{if } \lambda_i = 0.$$

Hence it is just isomorphic to the Clifford superalgebra $C_r(a[\![\lambda_{i_1}]\!],\ldots,a[\![\lambda_{i_r}]\!])$ where $\{i_1,\ldots,i_r\}$ are indices such that $\lambda_i\neq 0$. By the action above $M^c_\lambda$ (resp. $M^{c*}_\lambda$) is now an $(H^c_n,\Gamma_\lambda)$-bimodule (resp. a $(\Gamma_\lambda, H^c_n)$-bimodule). Since the set

$$\{(\gamma^L_{\lambda;i_1})^{p_1}(\gamma^L_{\lambda;i_2})^{p_2}\cdots(\gamma^L_{\lambda;i_r})^{p_r} \,|\, p_k\in\{0,1\}\}$$

is linearly independent in $C_n$, the map $\Gamma_\lambda\to\Gamma_\lambda m_\lambda\subset M^c_\lambda\cap M^{c*}_\lambda$ is an inclusion of superalgebra.

## 4. Circled tableaux

In order to denote elements of the parabolic module $M^{\mathrm{c}}_{\lambda}$ graphically we introduce the notion of *circled tableau* [**Sag87**]. Here a circled tableau of shape $\lambda$ is a map $Y(\lambda) \to \{1, 2, \dots, \} \sqcup \{\textcircled{1}, \textcircled{2}, \dots\}$. From a circled tableau $\mathsf{T}$ we obtain its underlying ordinal tableau $\mathsf{T}^{\times}$ by removing circles from numbers. The weight of a circled tableau is defined as that of underlying tableau. We say that a circled tableau is *row-standard* if its underlying tableau is row-standard. Let $\mathrm{Tab}^{\mathrm{c}}_{\lambda}$ be the set of row-standard circled tableau of shape $\lambda$. For $\mathsf{T} \in \mathrm{Tab}^{\mathrm{c}}_{\lambda}$ we define the corresponding element $m_{\mathsf{T}} \coloneqq T_w c_{i_1} \cdots c_{i_r} m_{\lambda}$ where $i_1, \dots, i_r$ are indices of positions of circled entries in $\mathsf{T}$ according to the top-to-bottom reading order and $w = d(\mathsf{T}^{\times})$. For example,

$$\text{for } \mathsf{T} = \begin{array}{|c|c|c|c|}\hline \textcircled{1} & 2 & 4 & \textcircled{5} \\ \hline 3 & \textcircled{7} & 8 \\ \cline{1-3} \textcircled{6} \\ \cline{1-1} \end{array}, \quad m_{\mathsf{T}} = T_3 T_4 T_6 T_7 c_1 c_4 c_6 c_8 m_{(4,3,1)}.$$

For such $\mathsf{T}$, we define its length as $\ell(\mathsf{T}) \coloneqq \ell(d(\mathsf{T}^{\times}))$. If we focus only on leading terms with respect to this length, we have

$$T_w c_{i_1} \cdots c_{i_r} = c_{w(i_1)} \cdots c_{w(i_r)} T_w + (\text{lower terms})$$

so by $M^{\mathrm{c}}_{\lambda} \simeq C_n \otimes M_{\lambda}$ the set $\{m_{\mathsf{T}} \mid \mathsf{T} \in \mathrm{Tab}^{\mathrm{c}}_{\lambda}\}$ forms a basis of $M^{\mathrm{c}}_{\lambda}$. The action of $T_i$ is described as

$$T_i \cdot m_{\mathsf{T}} = \begin{cases} m_{s_i \mathsf{T}} & \text{if } r(i) < r(i+1), \\ q m_{\mathsf{T}} + (q-1) m_{s_i \mathsf{T}} & \text{if } r(i) > r(i+1) \end{cases}$$

where $r(i)$ is the index of the row which contains $i$ or $\textcircled{$i$}$ similarly as before, and $s_i \mathsf{T}$ is the circled tableau whose underlying tableau is $(s_i \mathsf{T})^{\times} = s_i(\mathsf{T}^{\times})$ and which has circles at the same boxes as $\mathsf{T}$. If $r(i) = r(i+1)$, putting $j = i+1$ it acts by

$$T_i \cdot \begin{array}{|c|c|}\hline i & j \\ \hline \end{array} = q \begin{array}{|c|c|}\hline i & j \\ \hline \end{array}, \qquad T_i \cdot \begin{array}{|c|c|}\hline i & \textcircled{$j$} \\ \hline \end{array} = \begin{array}{|c|c|}\hline \textcircled{$i$} & j \\ \hline \end{array} + (q-1) \begin{array}{|c|c|}\hline i & \textcircled{$j$} \\ \hline \end{array},$$

$$T_i \cdot \begin{array}{|c|c|}\hline \textcircled{$i$} & j \\ \hline \end{array} = q \begin{array}{|c|c|}\hline i & \textcircled{$j$} \\ \hline \end{array}, \qquad T_i \cdot \begin{array}{|c|c|}\hline \textcircled{$i$} & \textcircled{$j$} \\ \hline \end{array} = a(q-1) \begin{array}{|c|c|}\hline i & j \\ \hline \end{array} - \begin{array}{|c|c|}\hline \textcircled{$i$} & \textcircled{$j$} \\ \hline \end{array}$$

In contrast the action of $c_i$ is hard to describe due to the commutation relation of $C_n$ and $H_n$, but on leading terms we have

$$c_i \cdot \boxed{i} = \pm \boxed{\textcircled{$i$}} + \cdots, \qquad c_i \cdot \boxed{\textcircled{$i$}} = \pm a \boxed{i} + \cdots$$

as desired. Here the signs above are taken to be $+$ if it has even number of circles before this box with respect to the reading order, and otherwise $-$. The right action of $\Gamma_{\lambda}$ is easy: for example,

$$\begin{array}{|c|c|c|c|}\hline \textcircled{1} & 2 & 4 & \textcircled{5} \\ \hline 3 & \textcircled{7} & 8 \\ \cline{1-3} \textcircled{6} \\ \cline{1-1} \end{array} \cdot \gamma_{(4,3,1);2} = \begin{array}{|c|c|c|c|}\hline \textcircled{1} & 2 & 4 & \textcircled{5} \\ \hline \textcircled{3} & \textcircled{7} & 8 \\ \cline{1-3} \textcircled{6} \\ \cline{1-1} \end{array} - aq \begin{array}{|c|c|c|c|}\hline \textcircled{1} & 2 & 4 & \textcircled{5} \\ \hline 3 & 7 & 8 \\ \cline{1-3} \textcircled{6} \\ \cline{1-1} \end{array} - q^2 \begin{array}{|c|c|c|c|}\hline \textcircled{1} & 2 & 4 & \textcircled{5} \\ \hline 3 & \textcircled{7} & \textcircled{8} \\ \cline{1-3} \textcircled{6} \\ \cline{1-1} \end{array}.$$

Beware the signs due to the exchange of $c_i$ and $c_j$.

REMARK 9.9. Usually the shifted form

$$\begin{array}{cccc} \cline{1-4} \multicolumn{1}{|c|}{\textcircled{1}} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{\textcircled{5}} \\ \cline{1-4} & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{\textcircled{7}} & \multicolumn{1}{c|}{8} \\ \cline{2-4} & & \multicolumn{1}{|c|}{\textcircled{6}} & \\ \cline{3-3} \end{array}$$

is used in literatures for circled tableaux. We continue to use the non-shifted form since it seems to be troublesome to change the notations from the previous part.

Furthermore we introduce the set of row-semistandard circled tableau $\mathrm{Tab}^{\mathrm{c}}_{\lambda;\mu}$ to denote elements of $M^{\mathrm{c}}_{\lambda} \cap M^{\mathrm{c}*}_{\mu}$. We call a circled tableau of shape $\lambda$ and of weight $\mu$ is *row-semistandard* if its underlying tableau is row-semistandard and it does not contain parts of the form $\begin{array}{|c|c|}\hline \textcircled{$i$} & i \\ \hline \end{array}$ or $\begin{array}{|c|c|}\hline \textcircled{$i$} & \textcircled{$i$} \\ \hline \end{array}$. In other words, circled numbers must be placed at the rightmost of a bar $\begin{array}{|c|c|c|c|}\hline i & i & \cdots & i \\ \hline\end{array}$ in a row. It is also equivalent to say that each row is weakly increasing with respect to the order

$$1 < \textcircled{1} < 2 < \textcircled{2} < 3 < \textcircled{3} < \cdots$$

and a circled number can not be adjacent to itself. For such $\mathsf{S} \in \mathrm{Tab}^{c}_{\lambda;\mu}$, we define an element $m_{\mathsf{S}} \in M^c_\lambda \cap M^{c*}_\mu$ as follows: first we make a formal linear combination of tableaux from $\mathsf{S}$ by distributing

$$\begin{array}{|c|c|c|c|}\hline i & i & \cdots & \textcircled{i} \\ \hline\end{array} \mapsto \begin{array}{|c|c|c|c|}\hline \textcircled{i} & i & \cdots & i \\ \hline\end{array} + q \begin{array}{|c|c|c|c|}\hline i & \textcircled{i} & \cdots & i \\ \hline\end{array} \cdots + q^{r-1} \begin{array}{|c|c|c|c|}\hline i & i & \cdots & \textcircled{i} \\ \hline\end{array}$$

for each circled bar of length $r$, then by replacing each term $q^l\mathsf{R}$ with the sum of $q^l m_{\mathsf{T}} \in M^c_\lambda$ for all $\mathsf{T}$ such that $\mathsf{T}^\times \in \mathrm{Tab}_{\mathsf{S}^\times}$ and its positions of circles are same as that of $\mathsf{R}$. For example, for

$$\mathsf{S} = \begin{array}{cccc} 1 & \textcircled{1} & 2 & \textcircled{3} \\ 1 & 5 & \textcircled{5} & \\ \textcircled{4} & & & \end{array}$$

we have

$$\begin{aligned} m_{\mathsf{S}} = {} & \begin{array}{cccc} \textcircled{1} & 2 & 4 & \textcircled{5} \\ 3 & \textcircled{7} & 8 & \\ \textcircled{6} & & & \end{array} + q \begin{array}{cccc} 1 & \textcircled{2} & 4 & \textcircled{5} \\ 3 & \textcircled{7} & 8 & \\ \textcircled{6} & & & \end{array} + q \begin{array}{cccc} \textcircled{1} & 2 & 4 & \textcircled{5} \\ 3 & 7 & \textcircled{8} & \\ \textcircled{6} & & & \end{array} + q^2 \begin{array}{cccc} 1 & \textcircled{2} & 4 & \textcircled{5} \\ 3 & 7 & \textcircled{8} & \\ \textcircled{6} & & & \end{array} \\ & + \begin{array}{cccc} \textcircled{1} & 3 & 4 & \textcircled{5} \\ 2 & \textcircled{7} & 8 & \\ \textcircled{6} & & & \end{array} + q \begin{array}{cccc} 1 & \textcircled{3} & 4 & \textcircled{5} \\ 2 & \textcircled{7} & 8 & \\ \textcircled{6} & & & \end{array} + q \begin{array}{cccc} \textcircled{1} & 3 & 4 & \textcircled{5} \\ 2 & 7 & \textcircled{8} & \\ \textcircled{6} & & & \end{array} + q^2 \begin{array}{cccc} 1 & \textcircled{3} & 4 & \textcircled{5} \\ 2 & 7 & \textcircled{8} & \\ \textcircled{6} & & & \end{array} \\ & + \begin{array}{cccc} \textcircled{2} & 3 & 4 & \textcircled{5} \\ 1 & \textcircled{7} & 8 & \\ \textcircled{6} & & & \end{array} + q \begin{array}{cccc} 2 & \textcircled{3} & 4 & \textcircled{5} \\ 1 & \textcircled{7} & 8 & \\ \textcircled{6} & & & \end{array} + q \begin{array}{cccc} \textcircled{2} & 3 & 4 & \textcircled{5} \\ 1 & 7 & \textcircled{8} & \\ \textcircled{6} & & & \end{array} + q^2 \begin{array}{cccc} 2 & \textcircled{3} & 4 & \textcircled{5} \\ 1 & 7 & \textcircled{8} & \\ \textcircled{6} & & & \end{array} . \end{aligned}$$

PROPOSITION 9.10. *The set $\{m_{\mathsf{S}} \mid \mathsf{S} \in \mathrm{Tab}^c_{\lambda;\mu}\}$ is linearly independent in $M^c_\lambda \cap M^{c*}_\mu$. Moreover if $[2] \in \Bbbk$ is not a zero-divisor, it is also a basis of $M^c_\lambda \cap M^{c*}_\mu$.*

PROOF. For each $\mathsf{S} \in \mathrm{Tab}^c_{\lambda;\mu}$, take $\mathsf{T} \in \mathrm{Tab}^c_\lambda$ so that $\mathsf{T}^\times = (\mathsf{S}^\times)_\downarrow$ and $\mathsf{T}$ has a circle at each box whose position is the leftmost of circled bars $\begin{array}{|c|c|c|c|}\hline i & i & \cdots & \textcircled{i} \\ \hline\end{array}$ in $\mathsf{S}$. Then the coefficient of $m_{\mathsf{S}}$ at the basis element $m_{\mathsf{T}}$ is 1. Since this map $\mathsf{S} \mapsto \mathsf{T}$ is injective, the set $\{m_{\mathsf{S}} \mid \mathsf{S} \in \mathrm{Tab}^c_{\lambda;\mu}\}$ is linearly independent in $M^c_\lambda$.

On the other hand, let us take $x \in M^c_\lambda$ and write $x = \sum_{\mathsf{T} \in \mathrm{Tab}^c_\lambda} c_{\mathsf{T}} m_{\mathsf{T}}$. Suppose that $x \in M^{c*}_\mu$ and let $s_i \in \mathfrak{S}_\mu$. Then by the above description of the action of $T_i$, for $\mathsf{T} \in \mathrm{Tab}^c_\lambda$ such that $r(i) \neq r(i+1)$, $T_i x = qx$ implies $c_{\mathsf{T}} = c_{s_i\mathsf{T}}$. On the other hand, suppose $r(i) = r(i+1)$. Let $\mathsf{T}_1$, $\mathsf{T}_2$, $\mathsf{T}_3$ and $\mathsf{T}_4$ be circled tableaux obtained by replacing $i$ and $j = i+1$ in $\mathsf{T}$ by $\begin{array}{|c|c|}\hline i & j \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline \textcircled{i} & j \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline \textcircled{i} & j \\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline \textcircled{i} & \textcircled{j} \\ \hline\end{array}$ respectively. Then by $(1+T_i)x = [2]x$, we have $[2](c_{\mathsf{T}_2} - qc_{\mathsf{T}_3}) = [2]c_{\mathsf{T}_4} = 0$. Using the assumption that $[2]$ is not a zero-divisor, we obtain $c_{\mathsf{T}_2} = qc_{\mathsf{T}_3}$ and $c_{\mathsf{T}_4} = 0$. Hence $x$ can be written as a linear combination of $m_{\mathsf{S}}$. It is also clear that this condition is sufficient for that $x \in M^{c*}_\mu$. $\square$

Unfortunately, if the assumption is not satisfied then this statement may fail. For example, when $q = 1$ and $2 = 0$ in $\Bbbk$, the element $c_1c_2m_2$ is incidentally contained in $\mathbb{1}^c_2 \cap \mathbb{1}^{c*}_2$. The set $M^c_\lambda \cap M^{c*}_\mu$ is not suitable for our use, so instead we use a well-behaved set

$$M^c_{\lambda;\mu} := \Bbbk\{m_{\mathsf{S}} \mid \mathsf{S} \in \mathrm{Tab}^c_{\lambda;\mu}\} \subset M^c_\lambda \cap M^{c*}_\mu .$$

This free $\Bbbk$-module is preserved by an extension of scalars. Since the universal ring $\Bbbk = \mathbb{Z}[a, q]$ satisfies the assumption, it is closed under product

$$\circ_\mu \colon M^c_{\mu;\nu} \otimes M^c_{\lambda;\mu} \to M^c_{\lambda;\nu}.$$

By definition we can represent $\gamma^L_{\lambda;i} m_\lambda = m_\lambda \gamma^R_{\lambda;i} \in M^c_\lambda \cap M^{c*}_\lambda$ by a circled tableau, so $\Gamma_\lambda m_\lambda$ is contained in $M^c_{\lambda;\lambda}$. Hence $\Gamma_\mu$ also acts on $M_{\lambda;\mu}$ from left (resp. $M_{\mu;\nu}$

from right) and the product $\circ_\mu$ above is $\Gamma_\mu$-bilinear, so that we can define it as

$$\circ_\mu \colon M^c_{\mu;\nu} \otimes_{\Gamma_\mu} M^c_{\lambda;\mu} \to M^c_{\lambda;\nu}.$$

Let $\mathscr{S}^c_{r,n} := \bigoplus_{\lambda,\mu} M^c_{\lambda;\mu}$ where $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_r)$ run over compositions of at most $r$ components as before. We call it the *queer $q$-Schur superalgebra*. When $q = 1$ and $2 \neq 0$, it is equal to the Schur superalgebra of type $\mathsf{Q}$ introduced in [**BK02**].

We finish this section with a remark on involution. For $\mathsf{S} \in \mathrm{Tab}^c_{\lambda;\mu}$, we can similarly define an element $m^*_{\mathsf{S}} \in M^c_\mu \cap M^{c*}_\lambda$ in the dual manner by multiplying elements on $m_\lambda$ from right. When $q = 1$, it is actually the dual element of $m_{\mathsf{S}} \in W(-a)$ mapped via the anti-involution $*\colon W_n(-a)^{\mathrm{op}} \to W_n(a)$. For such $\mathsf{S}$ we define its dual tableau $\mathsf{S}^* \in \mathrm{Tab}^c_{\mu;\lambda}$ so that $(\mathsf{S}^\times)^* = (\mathsf{S}^*)^\times$ and $\mathsf{S}$ has $\textcircled{j}$ in its $i$-th row if and only if $\mathsf{S}^*$ has $\textcircled{i}$ in its $j$-th row. Then by the commutation relation on Lemma 9.7 $m^*_{\mathsf{S}}$ has the leading term $m_{\mathsf{S}^*}$ but they are not equal unless $q = 1$. The map $m_{\mathsf{S}} \mapsto m_{\mathsf{S}^*}$ does not either preserve the reversed product in general.

## 5. Good circled tableaux

Analogously to the non-super case, we introduce a filtration into our subcategory of $H^c_n$-$\mathcal{M}od$. According to this filtration we decompose the set of simple modules of $H^c_n$ into small parts.

Definition 9.11. For each compositions $\lambda$, $\mu$, and $\nu$, let

$$M^{c\,\nu}_{\lambda;\mu} := M^c_{\nu;\mu} \circ_\nu M^c_{\lambda;\nu} \subset M^c_{\lambda;\mu}.$$

Then we define

$$M^{c\,\geq\nu}_{\lambda;\mu} := \sum_{\pi \geq \nu} M^{c\,\pi}_{\lambda;\mu}, \quad M^{c\,>\nu}_{\lambda;\mu} := \sum_{\pi > \nu} M^{c\,\pi}_{\lambda;\mu}$$

and finally

$$M^{c(\nu)}_{\lambda;\mu} := M^c_{\lambda;\mu} / M^{c\,\geq\nu}_{\lambda;\mu}.$$

In particular we let $S^c_{\lambda;\mu} := M^{c(\lambda)}_{\lambda;\mu}$ and $S^c_\lambda := S^c_{\lambda;(1^n)}$.

We say that a circled tableau $\mathsf{T} \in \mathrm{Tab}^c_{\lambda;\mu}$ is *good* if its underlying tableau $\mathsf{T}^\times$ is good.

Lemma 9.12. *$S^c_{\lambda;\mu}$ is spanned by* $\{m_{\mathsf{T}} \mid \mathsf{T} \in \mathrm{Tab}^c_{\lambda;\mu}$ which is good$\}$.

Proof. Similarly to the proof of Lemma 5.8, we prove it inductively by replacing each $m_{\mathsf{T}}$ for ungood $\mathsf{T} \in \mathrm{Tab}^c_{\lambda;\mu}$ with tableaux which have smaller lengths. However in this case we can not perform this method at a time, so we do for each number one by one. Suppose that $\mathsf{T}$ has ungood $\boxed{i}$ or $\boxed{\textcircled{i}}$, which we choose so that $i$ is minimum. In particular, $i$'s in its $i$-th row are at the leftmost of $\mathsf{T}$ if exist. If it has circled $\textcircled{i}$ in its $i$-th row, multiplying $\gamma_{\lambda;i}$ from right we can represent $m_{\mathsf{T}}$ by tableaux without this circle; for example,

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&\textcircled{2}&3\\\hline 2&2&3&\textcircled{3}\\\cline{1-4} \textcircled{1}&\textcircled{2}&3\\\cline{1-3}\end{array} \cdot \gamma_{\lambda;1} = -\begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&\textcircled{1}&\textcircled{2}&3\\\hline 2&2&3&\textcircled{3}\\\cline{1-4} \textcircled{1}&\textcircled{2}&3\\\cline{1-3}\end{array} - aq^4 \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&2&3\\\hline 2&2&3&\textcircled{3}\\\cline{1-4} \textcircled{1}&\textcircled{2}&3\\\cline{1-3}\end{array} - q^5 \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&\textcircled{2}&\textcircled{3}\\\hline 2&2&3&\textcircled{3}\\\cline{1-4} \textcircled{1}&\textcircled{2}&3\\\cline{1-3}\end{array}.$$

Hence we may assume that $i$ in the $i$-th row of $\mathsf{T}$ is not circled. We define tableaux $\mathsf{T}_1, \mathsf{T}_2$ from $\mathsf{T}$ by moving up ungood $\boxed{i}$ and $\boxed{\textcircled{i}}$ as we did in the proof of Lemma 5.8, and if such $\textcircled{i}$ is circled we remove this circle from $\mathsf{T}_2$ and put on the same box at $\mathsf{T}_1$. For example, for

$$\mathsf{T} = \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&\textcircled{2}&3\\\hline 2&2&3&\textcircled{3}\\\cline{1-4} \textcircled{1}&\textcircled{2}&3\\\cline{1-3}\end{array}$$

we have

$$\mathsf{T}_1 = \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&1\\\hline 2&2&2&2&\multicolumn{2}{c}{}\\\cline{1-4} \textcircled{1}&3&3&\multicolumn{3}{c}{}\\\cline{1-3}\end{array} \quad \text{and} \quad \mathsf{T}_2 = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&\textcircled{2}&3\\\hline 2&2&3&\textcircled{3}&\multicolumn{3}{c}{}\\\cline{1-4} \textcircled{2}&3&\multicolumn{5}{c}{}\\\cline{1-2}\end{array}.$$

By taking leading terms, we also have a decomposition $0 \equiv m_{\mathsf{T}_2} \circ_\nu m_{\mathsf{T}_1} = \pm m_{\mathsf{T}} +$ (lower terms). □

This leads us the following parallel results.

COROLLARY 9.13. (1) $S^c_{\lambda;\lambda}$ *is spanned by* $m_\lambda$ *over* $\Gamma_\lambda$.
(2) $S^c_{\lambda;\mu} = 0$ *unless* $\lambda \geq \mu$. □

THEOREM 9.14. $H^c_n$ *and* $\mathscr{S}^c_{r,n}$ *are standardly filtered algebras over* $\{\Gamma_\lambda\}$ *on compositions* $\lambda$. □

Note that we have a natural map

$$M^{c(\nu)}_{\lambda;\mu} \to \mathrm{Hom}^{(\nu)}_{H^c_n}(M^c_\mu, M^c_\lambda)$$

where the right-hand side is the set of homomorphisms in the quotient category naïvely defined by using the whole category $H^c_n$-$\mathcal{M}od$, but in general this map is not surjective nor injective when the assumption in Proposition 9.10 is not satisfied. Although we can also define a standard filter using the right-hand side, this filter is ill-behaved with extension of scalars. In contrast, we have

$$S^c_{\lambda;\mu} \simeq S^c_{\lambda;\mu}(\mathbb{Z}[a,q]) \otimes_{\mathbb{Z}[a,q]} \Bbbk$$

as desired since $M^c_{\lambda;\mu}$ has a free basis. So $S^c_{\lambda;\mu}$ is certainly the right definition.

In these modules we have the local transformation lemma by a similar proof as before.

LEMMA 9.15. *Suppose we have an equation* $\sum_{\mathsf{T}} c_{\mathsf{T}} m_{\mathsf{T}} \equiv 0$ *in* $S^c_{\lambda;\mu}$ *for some* $c_{\mathsf{T}} \in \Bbbk$. *For each* $\mathsf{T} \in \mathrm{Tab}_{\lambda;\mu}$ *let* $\mathsf{T}^+$ *be the tableau obtained by adding a new common row at the top (resp. the bottom) of* $\mathsf{T}$. *Then we have* $\sum_{\mathsf{T}} c_{\mathsf{T}} m_{\mathsf{T}^+} \equiv 0$. □

## 6. Shifted semistandard circled tableaux

In this section, we consider the case $\Bbbk = \mathbb{Q}(a,q)$. Recall that a composition $\lambda$ is called a *strict partition* if it is strictly decreasing: $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0 = \lambda_{r+1} = \lambda_{r+2} = \cdots$.

DEFINITION 9.16. A row-semistandard circled tableau $\mathsf{T} \in \mathrm{STab}^c_{\lambda;\mu}$ is called *shifted semistandard* if its shape $\lambda$ is a strict partition and it does not contain any of the patterns

$$\begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{i}\\\hline \multicolumn{1}{|c|}{i} & \\\cline{1-1}\end{array},\ \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{\textcircled{i}}\\\hline \multicolumn{1}{|c|}{i} & \\\cline{1-1}\end{array} \qquad \text{and} \qquad \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{j}\\\hline \multicolumn{1}{|c|}{i} & \\\cline{1-1}\end{array},\ \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{j}\\\hline \multicolumn{1}{|c|}{\textcircled{i}} & \\\cline{1-1}\end{array},\ \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{\textcircled{j}}\\\hline \multicolumn{1}{|c|}{i} & \\\cline{1-1}\end{array},\ \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{\textcircled{j}}\\\hline \multicolumn{1}{|c|}{\textcircled{i}} & \\\cline{1-1}\end{array} \quad \text{for } i < j$$

(in particular, its underlying tableau $\mathsf{T}^\times$ is semistandard). In other words, its entries are also weakly increasing along with each diagonal line so that a non-circled number does not continue. We denote by $\mathrm{STab}^c_{\lambda;\mu}$ the set of shifted semistandard circled tableaux of shape $\lambda$ of weight $\mu$.

A similar notion of *generalized shifted tableau* is introduced in [**Sag87**]. The only difference is that he use the order

$$\textcircled{1} < 1 < \textcircled{2} < 2 < \textcircled{3} < 3 < \cdots$$

instead of ours. The set of shifted semistandard circled tableaux is clearly is in bijection with that of his generalized shifted tableaux by the following circle moving:

$$\begin{array}{ccccccc} &&&&&\multicolumn{1}{|c|}{i}&\multicolumn{1}{c|}{i}\\\cline{3-7} &&\multicolumn{1}{|c|}{i}&\multicolumn{1}{c|}{i}&\multicolumn{1}{c|}{\textcircled{i}}&&\\\cline{1-5} \multicolumn{1}{|c|}{i}&\multicolumn{1}{c|}{\textcircled{i}}&&&&&\\\cline{1-2}\end{array} \mapsto \begin{array}{ccccccc} &&&&&\multicolumn{1}{|c|}{\textcircled{i}}&\multicolumn{1}{c|}{i}\\\cline{3-7} &&\multicolumn{1}{|c|}{\textcircled{i}}&\multicolumn{1}{c|}{i}&\multicolumn{1}{c|}{i}&&\\\cline{1-5} \multicolumn{1}{|c|}{i}&\multicolumn{1}{c|}{i}&&&&&\\\cline{1-2}\end{array}, \qquad \begin{array}{ccccccc} &&&&&\multicolumn{1}{|c|}{i}&\multicolumn{1}{c|}{\textcircled{i}}\\\cline{3-7} &&\multicolumn{1}{|c|}{i}&\multicolumn{1}{c|}{i}&\multicolumn{1}{c|}{\textcircled{i}}&&\\\cline{1-5} \multicolumn{1}{|c|}{i}&\multicolumn{1}{c|}{\textcircled{i}}&&&&&\\\cline{1-2}\end{array} \mapsto \begin{array}{ccccccc} &&&&&\multicolumn{1}{|c|}{\textcircled{i}}&\multicolumn{1}{c|}{i}\\\cline{3-7} &&\multicolumn{1}{|c|}{\textcircled{i}}&\multicolumn{1}{c|}{i}&\multicolumn{1}{c|}{i}&&\\\cline{1-5} \multicolumn{1}{|c|}{\textcircled{i}}&\multicolumn{1}{c|}{i}&&&&&\\\cline{1-2}\end{array}.$$

LEMMA 9.17. *Let $m, k \in \mathbb{N}$ such that $m \geq k$. Let $\lambda := (m, k)$ and $\mu := (k, m)$. Then in $S^{\mathrm{c}}_{\lambda;\mu}$ we have*

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&\textcircled{1}&2&\cdots&2\\\hline 2&2&\cdots&2\\\cline{1-4}\end{array} \equiv \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&2&\cdots&2\\\hline 2&2&\cdots&\textcircled{2}\\\cline{1-4}\end{array}.$$

PROOF. By Lemma 5.14 and the assumption $q \in \Bbbk^\times$, we have

$$\begin{aligned}\gamma_{\mu;2} \cdot \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&2&\cdots&2\\\hline 2&2&\cdots&2\\\cline{1-4}\end{array} &= (-1)^k q^{-\binom{k}{2}} \gamma_{\mu;2} \cdot \begin{array}{|c|c|c|c|c|c|c|}\hline 2&2&\cdots&\cdots&\cdots&\cdots&2\\\hline 1&1&\cdots&1\\\cline{1-4}\end{array}\\ &= (-1)^k q^{-\binom{k}{2}} \begin{array}{|c|c|c|c|c|c|c|}\hline 2&2&\cdots&\cdots&\cdots&\cdots&\textcircled{2}\\\hline 1&1&\cdots&1\\\cline{1-4}\end{array}\\ &= (-1)^k q^{-\binom{k}{2}} \begin{array}{|c|c|c|c|c|c|c|}\hline 2&2&\cdots&\cdots&\cdots&\cdots&2\\\hline 1&1&\cdots&1\\\cline{1-4}\end{array} \cdot \gamma_{\lambda;1}\\ &= \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&2&\cdots&2\\\hline 2&2&\cdots&2\\\cline{1-4}\end{array} \cdot \gamma_{\lambda;1}.\end{aligned}$$

If $m = k$, we have

$$\gamma_{\lambda;2} \cdot \begin{array}{|c|c|c|c|}\hline 1&1&\cdots&1\\\hline 2&2&\cdots&2\\\hline\end{array} = \begin{array}{|c|c|c|c|}\hline 1&1&\cdots&1\\\hline 2&2&\cdots&\textcircled{2}\\\hline\end{array}, \qquad \begin{array}{|c|c|c|c|}\hline 1&1&\cdots&1\\\hline 2&2&\cdots&2\\\hline\end{array} \cdot \gamma_{\lambda;1} = \begin{array}{|c|c|c|c|}\hline 1&1&\cdots&\textcircled{1}\\\hline 2&2&\cdots&2\\\hline\end{array}.$$

Otherwise both-hand sides can be computed by Lemma 9.7 as

$$\gamma_{\mu;2} \cdot \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&2&\cdots&2\\\hline 2&2&\cdots&2\\\cline{1-4}\end{array} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&2&\cdots&2\\\hline 2&2&\cdots&\textcircled{2}\\\cline{1-4}\end{array} + q^k \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&2&\cdots&\textcircled{2}\\\hline 2&2&\cdots&2\\\cline{1-4}\end{array}$$

and

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&2&\cdots&2\\\hline 2&2&\cdots&2\\\cline{1-4}\end{array} \cdot \gamma_{\lambda;1} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&\textcircled{1}&2&\cdots&2\\\hline 2&2&\cdots&2\\\cline{1-4}\end{array} + q^k \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&2&\cdots&\textcircled{2}\\\hline 2&2&\cdots&2\\\cline{1-4}\end{array}$$

so these equations also imply the statement. □

LEMMA 9.18. *$S^{\mathrm{c}}_{\lambda;\lambda} = 0$ unless $\lambda$ is a strict partition.*

PROOF. If $\lambda$ is not a partition it holds by the same reason as the non-super case. Otherwise if $\lambda$ is not a strict partition, it contains $\lambda_i = \lambda_{i+1} > 0$. So it suffices to prove for $\lambda = (k, k)$. By the lemma above we have

$$(\gamma_{\lambda;1} - \gamma_{\lambda;2}) \cdot \begin{array}{|c|c|c|c|}\hline 1&1&\cdots&1\\\hline 2&2&\cdots&2\\\hline\end{array} = \begin{array}{|c|c|c|c|}\hline 1&1&\cdots&\textcircled{1}\\\hline 2&2&\cdots&2\\\hline\end{array} - \begin{array}{|c|c|c|c|}\hline 1&1&\cdots&1\\\hline 2&2&\cdots&\textcircled{2}\\\hline\end{array} \equiv 0.$$

On the other hand, $(\gamma_{\lambda;1} - \gamma_{\lambda;2})^2 = 2a[\![k]\!] \in \Bbbk^\times$. Hence we have $m_\lambda \equiv 0$ in $S^{\mathrm{c}}_{\lambda;\lambda}$. □

LEMMA 9.19. *$S^{\mathrm{c}}_{\lambda;\mu}$ is spanned by $\{m_{\mathsf{T}} \mid \mathsf{T} \in \mathrm{STab}^{\mathrm{c}}_{\lambda;\mu}\}$.*

PROOF. If $\lambda$ is not a strict partition the statement is clear by the previous lemma, so we may assume so. First we prove the statement for two special cases.

Case 1: $\lambda = (m, k)$ and $\mu = (k, k, m - k)$. Then every good but non-shifted-semistandard circled tableaux have an underlying tableau

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&3&\cdots&3\\\hline 2&2&\cdots&2\\\cline{1-4}\end{array}$$

and can be made from this tableau by multiplying $\gamma_{\mu;i}$. By Lemma 9.17,

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&\textcircled{1}&3&\cdots&3\\\hline 2&2&\cdots&2\\\cline{1-4}\end{array} \equiv \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&3&\cdots&3\\\hline 2&2&\cdots&\textcircled{2}\\\cline{1-4}\end{array} + (\text{lower terms}).$$

Hence the statement holds since $(\gamma_{\mu;1} - \gamma_{\mu;2})^2 = 2a[\![k]\!]$ is invertible again and every good tableau which has smaller length is shifted semistandard.

Case 2: $\lambda = (m, k)$ and $\mu = (k, l, m - l - l', l')$ where $l < k$ and $l' < m - k$, so

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&\cdots&\cdots&\cdots&1&3&\cdots&4\\\hline 2&\cdots&2&3&\cdots&3\\\cline{1-6}\end{array} \quad\text{and}\quad \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&\cdots&\cdots&\cdots&1&3&\cdots&4\\\hline 2&\cdots&2&3&\cdots&\textcircled{3}\\\cline{1-6}\end{array}$$

are all the good but non-shifted-semistandard circled tableaux. Similar to above,

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&\cdots&\cdots&\cdots&\textcircled{1}&3&\cdots&4\\\hline 2&\cdots&2&3&\cdots&3\\\cline{1-6}\end{array} \equiv \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&\cdots&\cdots&\cdots&1&3&\cdots&4\\\hline 2&\cdots&\textcircled{2}&3&\cdots&3\\\cline{1-6}\end{array} + q^l \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&\cdots&\cdots&\cdots&1&3&\cdots&4\\\hline 2&\cdots&2&3&\cdots&\textcircled{3}\\\cline{1-6}\end{array} + \cdots.$$

Then by multiplying $\gamma_{\mu;1}$ and $\gamma_{\mu;2}$ from left respectively, we obtain

$$a[\![k]\!]\begin{ytableau} 1 & 1 & \cdots & \cdots & \cdots & 1 & 3 & \cdots & 4 \\ 2 & \cdots & 2 & 3 & \cdots & 3 \end{ytableau} \equiv \begin{ytableau} 1 & 1 & \cdots & \cdots & \cdots & \textcircled{1} & 3 & \cdots & 4 \\ 2 & \cdots & \textcircled{2} & 3 & \cdots & 3 \end{ytableau} + q^l \begin{ytableau} 1 & 1 & \cdots & \cdots & \cdots & \textcircled{1} & 3 & \cdots & 4 \\ 2 & \cdots & 2 & 3 & \cdots & \textcircled{3} \end{ytableau} + \cdots$$

and

$$-\begin{ytableau} 1 & 1 & \cdots & \cdots & \cdots & \textcircled{1} & 3 & \cdots & 4 \\ 2 & \cdots & \textcircled{2} & 3 & \cdots & 3 \end{ytableau} \equiv a[\![l]\!]\begin{ytableau} 1 & 1 & \cdots & \cdots & \cdots & 1 & 3 & \cdots & 4 \\ 2 & \cdots & 2 & 3 & \cdots & 3 \end{ytableau} + q^l \begin{ytableau} 1 & 1 & \cdots & \cdots & \cdots & 1 & 3 & \cdots & 4 \\ 2 & \cdots & \textcircled{2} & 3 & \cdots & \textcircled{3} \end{ytableau} + \cdots$$

so that

$$\begin{aligned} a([\![k]\!] + [\![l]\!]) & \begin{ytableau} 1 & 1 & \cdots & \cdots & \cdots & 1 & 3 & \cdots & 4 \\ 2 & \cdots & 2 & 3 & \cdots & 3 \end{ytableau} \\ & \equiv q^l \begin{ytableau} 1 & 1 & \cdots & \cdots & \cdots & 1 & 3 & \cdots & 4 \\ 2 & \cdots & \textcircled{2} & 3 & \cdots & \textcircled{3} \end{ytableau} - q^l \begin{ytableau} 1 & 1 & \cdots & \cdots & \cdots & 1 & 3 & \cdots & 4 \\ 2 & \cdots & \textcircled{2} & 3 & \cdots & \textcircled{3} \end{ytableau} + \cdots \end{aligned}$$

with $a([\![k]\!] + [\![l]\!]) \in \Bbbk^\times$. Multiplying $\gamma_{\lambda;1}$ from right we can similarly decompose the second tableau above.

Now we proceed to a general case. Let $\mathsf{T} \in \mathrm{Tab}^c_{\lambda;\mu}$ which is not shifted semistandard. Then there is a prohibited pattern at boxes $(k, l+1)$ and $(k+1, l)$. Choose $(k, l)$ so that $\mathsf{T}$ has no such patterns at the bottom right boxes of $(k, l)$ except for it. We prove that we can replace the element $m_\mathsf{T}$ by a linear combination of $m_\mathsf{R}$ where either $\mathsf{R}$ has no such patterns in this region or $\mathsf{R}$ satisfies $\ell(\mathsf{R}^\uparrow) < \ell(\mathsf{T}^\uparrow)$. Then by induction it can be written as a linear combination of shifted standard ones.

First consider the case $\mathsf{T}(k+1, l) = i, \textcircled{$i$}$ and $\mathsf{T}(k, l+1) = j, \textcircled{$j$}$ with $i < j$. Similar to the proof of Lemma 5.16, we define

$$\nu := (\lambda_1, \ldots, \lambda_{k-1}, l, l, \lambda_k - l, \lambda_{k+1} - l, \lambda_{k+2}, \lambda_{k+3}, \ldots)$$

and $\mathsf{T}_1 \in \mathrm{Tab}_{\lambda;\nu}$, $\mathsf{T}_2 \in \mathrm{Tab}^c_{\nu;\mu}$ by

$$\mathsf{T}_1(i,j) = \begin{cases} i & \text{if } i < k \text{ or } (i = k,\ j \leq l) \text{ or } (i = k+1,\ j \leq l), \\ i+2 & \text{otherwise,} \end{cases}$$

and

$$\mathsf{T}_2(i,j) = \begin{cases} \mathsf{T}(i,j) & \text{if } i \leq k+1, \\ \mathsf{T}(i-2, j+l) & \text{if } i = k+2 \text{ or } i = k+3, \\ \mathsf{T}(i-2, j) & \text{if } i > k+3. \end{cases}$$

Then the leading term of $m_{\mathsf{T}_2} \circ_\nu m_{\mathsf{T}_1}$ is $m_\mathsf{T}$. Applying the decomposition in Case 1 above to $m_{\mathsf{T}_1}$, we can replace $m_{\mathsf{T}_2} \circ_\nu m_{\mathsf{T}_1}$ by a linear combination of tableaux with smaller lengths so the induction goes forward.

Next consider the other case $\mathsf{T}(k+1, l) = i$ and $\mathsf{T}(k, l+1) = i$ or $\textcircled{$i$}$. Let $(k+1, l_1+1)$ and $(k, l_2)$ be the ends of the bars $\begin{ytableau} i & i & \cdots & i \end{ytableau}$ which start from $(k+1, l)$ and $(k, l+1)$ respectively. In this case we define

$$\nu := (\lambda_1, \ldots, \lambda_{k-1}, l, l_1, l_2 - l_1, \lambda_k - l_2, \lambda_{k+1} - l, \lambda_{k+2}, \lambda_{k+3}, \ldots)$$

and $\mathsf{T}_1 \in \mathrm{Tab}_{\lambda;\nu}$, $\mathsf{T}_2 \in \mathrm{Tab}^c_{\nu;\mu}$ by

$$\mathsf{T}_1(i,j) = \begin{cases} i & \text{if } i < k \text{ or } (i = k,\ j \leq l) \text{ or } (i = k+1,\ j \leq l_1), \\ k+2 & \text{if } (i = k,\ l < j \leq l_2) \text{ or } (i = k+1,\ l_1 < j \leq l), \\ i+3 & \text{otherwise,} \end{cases}$$

and

$$\mathsf{T}_2(i,j) = \begin{cases} \mathsf{T}(i,j) & \text{if } i \le k+1, \\ \mathsf{T}(k+1, j+l_1) & \text{if } i = k+2,\ j \le l - l_1 \\ \mathsf{T}(k, j+l_1) & \text{if } i = k+2,\ j > l - l_1 \\ \mathsf{T}(k, j+l_2) & \text{if } i = k+3, \\ \mathsf{T}(k+1, j+l) & \text{if } i = k+4, \\ \mathsf{T}(i-3, j) & \text{if } i > k+4. \end{cases}$$

In addition, if $\mathsf{T}(k, l_2) = \textcircled{i}$ is circled, we remove the corresponding circle from $\mathsf{T}_2$ and move it to $\mathsf{T}_1(k, l_2) = i$. Then the top term of $m_{\mathsf{T}_2} \circ_\nu m_{\mathsf{T}_1}$ is again $\pm m_{\mathsf{T}}$. For example, when

$$\mathsf{T} = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & \textcircled{2} & \textcircled{3} & 4 \\ \hline 1 & 2 & 2 & 3 & \textcircled{3} \\ \cline{1-5} \textcircled{2} & 5 \\ \cline{1-2} \end{array}$$

and $(k, l) = (1, 3)$, by picking up entries at

$$\begin{array}{|c|c|c|c|c|c|c|} \hline \phantom{0} & \phantom{0} & \phantom{0} & \bullet & \bullet & \phantom{0} & \phantom{0} \\ \hline \phantom{0} & \bullet & \bullet & \phantom{0} & \phantom{0} \\ \cline{1-5} \phantom{0} & \phantom{0} \\ \cline{1-2} \end{array}$$

then moving a circle from $\mathsf{T}_2$ to $\mathsf{T}_1$ we obtain

$$\mathsf{T}_1 = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 3 & \textcircled{3} & 4 & 4 \\ \hline 2 & 3 & 3 & 5 & 5 \\ \cline{1-5} 6 & 6 \\ \cline{1-2} \end{array} \quad \text{and} \quad \mathsf{T}_2 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 \\ \cline{1-3} 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline \textcircled{3} & 4 \\ \cline{1-2} 3 & \textcircled{3} \\ \cline{1-2} \textcircled{2} & 5 \\ \cline{1-2} \end{array}.$$

Now according to Case 2 above, up to lower terms, we can replace $\mathsf{T}_1$ by a linear combination of $m_{\mathsf{S}}$ such that $\mathsf{S}^\times = \mathsf{T}_1^\times$, $\mathsf{S}(k+1, l) = \textcircled{k+2}$ and $\mathsf{S}$ has circles only at the boxes $(k, 1), \dots, (k, \lambda_k)$ and $(k+1, 1), \dots, (k+1, l)$. Then the top term of $m_{\mathsf{T}_2} \circ_\nu m_{\mathsf{S}}$ is $\pm m_{\mathsf{R}}$, where $\mathsf{R}$ also satisfies that $\mathsf{R}^\times = \mathsf{T}^\times$, $\mathsf{R}(k+1, l) = \textcircled{i}$ and the positions of circles of $\mathsf{R}$ and $\mathsf{T}$ only differ at these boxes. Hence $\mathsf{R}$ also does not have bad patterns at the bottom right region of $(k, l)$. This completes the induction. $\square$

Let $\mathrm{STab}^{c\prime}_{\lambda;\mu}$ be the subset of $\mathrm{STab}^{c}_{\lambda;\mu}$ consisting of tableaux whose entries in the rightmost of each rows are not circled. Then clearly $\#\mathrm{STab}^{c}_{\lambda;\mu} = 2^{l(\lambda)} \cdot \#\mathrm{STab}^{c\prime}_{\lambda;\mu}$ where $l(\lambda)$ is the number of non-zero components of $\lambda$.

COROLLARY 9.20. *$S^c_{\lambda;\mu}$ is spanned by $\{m_{\mathsf{T}} \mid \mathsf{T} \in \mathrm{STab}^{c\prime}_{\lambda;\mu}\}$ over $\Gamma_\lambda$.*

By a similar proof we can prove that $S^{c*}_{\lambda;\mu} := M^{c(\lambda)}_{\mu;\lambda}$ is also spanned by $\{m_{\mathsf{T}^*} \mid \mathsf{T} \in \mathrm{STab}^{c}_{\lambda;\mu}\}$. Now parallel to Theorem 5.17 we obtain the following basis theorem.

THEOREM 9.21. *When $\Bbbk = \mathbb{Q}(a, q)$, $M^c_{\lambda;\mu}$ has a basis*

$$\bigsqcup_{\nu\colon \text{strict partition}} \{m_{\mathsf{S}} \circ_\nu m_{\mathsf{T}^*} \mid \mathsf{S} \in \mathrm{STab}^c_{\nu;\mu}, \mathsf{T} \in \mathrm{STab}^{c\prime}_{\nu;\lambda}\}.$$

PROOF. The proof of that this set spans $M^c_{\lambda;\mu}$ is same as that of Theorem 5.17. For that of linear independence we use the one-to-one correspondence

$$\mathrm{Tab}^c_{\lambda;\mu} \overset{1:1}{\longleftrightarrow} \bigsqcup_{\nu\colon \text{strict partition}} \mathrm{STab}^c_{\nu;\lambda} \times \mathrm{STab}^{c\prime}_{\nu;\mu}$$

induced by Sagan's shifted Knuth correspondence [**Sag87**, Theorem 8.1]. $\square$

Corollary 9.22. (1) *$S^c_{\lambda;\mu}$ has a basis $\{m_{\mathsf{T}} \,|\, \mathsf{T} \in \mathrm{STab}^c_{\lambda;\mu}\}$. In particular,*

$$S^c_{\lambda;\lambda} \simeq \begin{cases} \Gamma_\lambda & \text{if } \lambda \text{ is a strict partition,} \\ 0 & \text{otherwise.} \end{cases}$$

(2) *The product*

$$\circ_\nu : S^c_{\nu;\mu} \otimes_{\Gamma_\nu} S^{c*}_{\nu;\lambda} \to M^{c(\nu)}_{\lambda;\mu}$$

*is injective.*

(3) *$H^c_n$ and $\mathscr{S}^c_{r,n}$ are standardly based algebras.* □

Remark 9.23. The basis theorem above for $H^c_n$ (i.e. $\lambda = \mu = (1^n)$) also holds in the following more weaker conditions: $\Bbbk$ is an arbitrary commutative ring and $2aq \in \Bbbk^\times$, and the $q^2$-integers $[\![k]\!]$ are also invertible for $1 \leq k \leq n/2$. Note that we need not to use Case 2 in the proof of Lemma 9.19. This implies that $H^c_n$ is also standardly based over $\{\Gamma_\lambda\}$ in these conditions.

CHAPTER 10

# Fakemodules over the Hecke–Clifford superalgebra

Here we have a break on the classification of simple modules. In this chapter we introduce the module category of the Hecke–Clifford superalgebra $H_t^c$ for a non-integral rank $t \in B_q(\Bbbk)$ parallel as before. Its stable structure is used to determine the structure of the ordinary module category in the next chapter.

## 1. Stable structures

As in the previous part, we have a convolution product of modules

$$V_1 * V_2 * \cdots * V_r := H_n^c \otimes_{H_\lambda^c} (V_1 \boxtimes V_2 \boxtimes \cdots \boxtimes V_r)$$

for $V_i \in H_{\lambda_i}^c\text{-}\mathcal{M}od$. Since we have $H_n^c = \bigoplus_{w \in \mathfrak{D}_\lambda} T_w H_\lambda^c$, it has a similar decomposition

$$V_1 * V_2 * \cdots * V_r = \bigoplus_{w \in \mathfrak{D}_\lambda} T_w (V_1 \boxtimes V_2 \boxtimes \cdots \boxtimes V_r)$$

of supermodules. Then we introduce the induction functor

$$\begin{aligned} \mathrm{Ind}_k^c \colon H_n^c\text{-}\mathcal{M}od &\to H_{k+n}^c\text{-}\mathcal{M}od, \\ V &\mapsto \mathbb{1}_k^c * V \end{aligned}$$

and two restriction functors

$$\begin{aligned} \mathrm{Res}_k^c \colon H_{k+n}^c\text{-}\mathcal{M}od &\to H_n^c\text{-}\mathcal{M}od, \\ W &\mapsto \mathrm{Hom}_{H_{(k,n)}^c}(\mathbb{1}_k^c \boxtimes H_n^c, W|_{(k,n)}) \\ \mathrm{Res}_k^{c\prime} \colon H_{k+n}^c\text{-}\mathcal{M}od &\to H_n^c\text{-}\mathcal{M}od, \\ W &\mapsto (H_n^c \boxtimes \mathbb{1}_k^{c*}) \otimes_{H_{(n,k)}^c} W|_{(n,k)}. \end{aligned}$$

$\mathrm{Res}_k^c$ and $\mathrm{Res}_k^{c\prime}$ are respectively the right and the left adjoint functors of $\mathrm{Ind}_k^c$. Similar to the previous part it is proven by the following dual lemma.

Lemma 10.1. *There is an isomorphism of $(H_{(k,n)}^c, H_{k+n}^c)$-bimodules*

$$\mathrm{Hom}_{H_{(k,n)}^{c\,\mathrm{op}}}(H_{k+n}^c, H_{(k,n)}^c) \simeq {}^\sigma H_{k+n}^c$$

*where $\sigma$ denotes $H_{(k,n)}^c \simeq H_{(n,k)}^c$.*

Proof. Similarly as before we can prove that the left-hand side is a free right $H_{k+n}^c$-module generated by $\delta_{\varpi_{(k,n)}}$. For $w \in \mathfrak{D}_{(k,n)}$ we have

$$\delta_{\varpi_{(k,n)}}(c_{w(i)} T_w) = \delta_{\varpi_{(k,n)}}(T_w c_i + \cdots) = \begin{cases} c_i & \text{if } w = \varpi_{(k,n)}, \\ 0 & \text{otherwise} \end{cases}$$

so that $\delta_{\varpi_{(k,n)}} \cdot c_{w(i)} = c_i \delta_{\varpi_{(k,n)}}$. This implies that the map above is also a homomorphism of left $H_{k,n}^c$-modules. □

The parabolic restriction lemma can be stated completely the same as the non-super case.

Lemma 10.2. *Let $d, m, n \in \mathbb{N}$ such that $m, n \le d$. For each $W \in H_n^c$-$\mathcal{M}od$ there is an isomorphism*

$$(\mathrm{Ind}_{d-n}^c W)|_{(d-m,m)} \simeq \bigoplus_i \mathrm{Ind}_{d-m-n+i}^{c(1)} \mathrm{Ind}_{m-i}^{c(2)} (W|_{(n-i,i)})$$

*of $H_{(d-m,m)}^c$-modules.*

Proof. In this proof instead of $T_i$ we use $T_i' \coloneqq T_i - q + 1$ so that $T_i' c_{i+1} = c_i T_i'$. Since $T_i'$'s also satisfy the braid relations, we can define $T_w' \coloneqq T_{i_1}' T_{i_2}' \cdots T_{i_l}'$ for $w = s_{i_1} s_{i_2} \cdots s_{i_l}$. Then we have the decomposition

$$H_n^c = \bigoplus_{w \in \mathfrak{D}_{(d-n,n)}} T_w' H_{(d-n,n)}^c = \bigoplus_i \bigoplus_{\substack{u \in \mathfrak{D}_{(d-m-n+i,n-i)},\\ v \in \mathfrak{D}_{(m-i,i)}}} T_{(u,v)}' C_n T_{g_i}' H_{(d-n,n)}$$

where $g_i \coloneqq (1_{d-m-n+i}, \varpi_{(m-i,n-i)}, 1_i)$. Now $T_{g_i}'$ satisfies $c_j T_{g_i}' = T_{g_i}' c_{j+m-i}$ for $d-m-n+i+1 \le j \le d-m$ and $c_j T_{g_i}' = T_{g_i}' c_j$ for $j \ge d-i+1$, so that

$$C_n T_{g_i}' H_{(d-n,n)} = C_{d-n}' T_{g_i}' (H_{d-n} \otimes H_n^c)$$

where we write $C_{d-n}' \coloneqq C_{d-m-n+i} \otimes 1_{n-i} \otimes C_{m-i} \otimes 1_i$. Thus we have

$$(\mathrm{Ind}_{d-n}^c W)|_{(d-m,m)} = \bigoplus_i \bigoplus_{u,v} T_{(u,v)}' C_{d-n}' T_{g_i}' (\mathbb{1}_{d-n} \boxtimes W).$$

As $H_{(d-m-n+i,n-i,m-i,i)}^c$-modules, we have an isomorphism

$$\begin{gathered} \mathbb{1}_{d-m-n+i}^c \boxtimes W^{(1)} \boxtimes \mathbb{1}_{m-i}^c \boxtimes W^{(2)} \simeq C_{d-n}' T_{g_i}' (\mathbb{1}_{d-n} \boxtimes W) \\ m_{d-m-n+i} \boxtimes x \boxtimes m_{m-i} \boxtimes y \mapsto T_{g_i}' (m_{d-n} \boxtimes x \boxtimes y) \end{gathered}$$

under the Sweedler's notation $W|_{(n-i,i)} \simeq W^{(1)} \boxtimes W^{(2)}$, which implies the statement. □

Proposition 10.3. *Let $d, m, n \in \mathbb{N}$ such that $m, n \le d$. Let $V \in H_m^c$-$\mathcal{M}od$, $W \in H_n^c$-$\mathcal{M}od$ and suppose that the action of $[2] \in \Bbbk$ on $W$ is injective. Then we have a natural isomorphism of $\Bbbk$-supermodules*

$$\begin{aligned} \mathrm{Hom}_{H_d^c}(\mathrm{Ind}_{d-m}^c V, \mathrm{Ind}_{d-n}^c W) \simeq & \bigoplus_{m+n-d \le i} \mathrm{Hom}_{H_i^c}(\mathrm{Res}_{m-i}^{c\prime} V, \mathrm{Res}_{n-i}^c W) \\ & \oplus \bigoplus_{m+n-d < i} \Pi \, \mathrm{Hom}_{H_i^c}(\mathrm{Res}_{m-i}^{c\prime} V, \mathrm{Res}_{n-i}^c W). \end{aligned}$$

Proof. It suffices to prove

$$\mathrm{Res}_{d-m}^c \mathrm{Ind}_{d-n}^c W \simeq \bigoplus_{m+n-d \le i} \mathrm{Ind}_{m-i}^c \mathrm{Res}_{n-i}^c W \oplus \bigoplus_{m+n-d < i} \Pi \, \mathrm{Ind}_{m-i}^c \mathrm{Res}_{n-i}^c W.$$

By the lemma above we can reduce it to the case $m = 0$:

$$\mathrm{Res}_d^c \mathrm{Ind}_{d-n}^c W \simeq \begin{cases} \mathrm{Res}_n^c W \oplus \Pi \, \mathrm{Res}_n^c W & \text{if } d > n, \\ \mathrm{Res}_n^c W & \text{if } d = n. \end{cases}$$

The case $d = n$ is clear. For the other case $d > n$, using the decomposition $H_n^c = \bigoplus_{w \in \mathfrak{D}_{d-n,n}} T_w H_{(d-n,n)}^c$ it follows from the isomorphism

$$\mathrm{Res}_{d-n}^{c(1)} (\mathbb{1}_{d-n}^c \boxtimes W) = (\Bbbk m_{d-n} \boxtimes W) \oplus (\Bbbk \gamma_{d-n}^L m_{d-n} \boxtimes W) \simeq W \oplus \Pi W$$

which we can prove similarly to Proposition 9.10 under the assumption. □

## 2. String diagrams in the super case

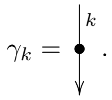
We introduce diagrammatic natural transformations

$\mu_{(k,l)}\colon \mathrm{Ind}^c_k\mathrm{Ind}^c_l \to \mathrm{Ind}^c_{k+l},\ \Delta_{(k,l)}\colon \mathrm{Ind}^c_{k+l} \to \mathrm{Ind}^c_k\mathrm{Ind}^c_l,\ \sigma_{(k,l)}\colon \mathrm{Ind}^c_l\mathrm{Ind}_k \to \mathrm{Ind}^c_k\mathrm{Ind}^c_l$

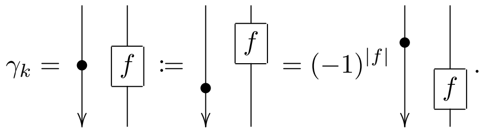
similar to before. In addition, we have another odd natural transformation

$$\gamma_k\colon \mathrm{Ind}^c_k \to \mathrm{Ind}^c_k$$

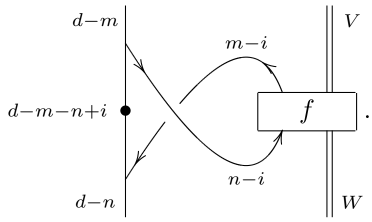
induced by $\gamma_k\colon \mathbb{1}_k \to \mathbb{1}_k;\ m_k \mapsto \gamma^L_k m_k$. By convention we put $\gamma_k = 0$ for $k \leq 0$. We represent this natural transformation by a dot on a string:




Beware that the odd naturality means that we can transform diagrams up to Koszul sign:



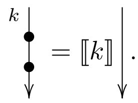

Then the homomorphism $\mathrm{Ind}^c_{d-m}V \to \mathrm{Ind}^c_{d-n}W$ corresponds to $f\colon \mathrm{Res}^{c\prime}_{m-i}V \to \mathrm{Res}^c_{n-i}W$ in the second summand of Proposition 10.3 is illustrated as



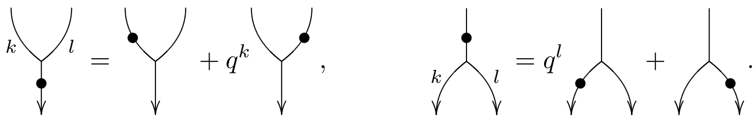

We can still apply the local transformations of diagrams listed in Proposition 6.6, in addition to the followings.

PROPOSITION 10.4. *The following equations hold.*

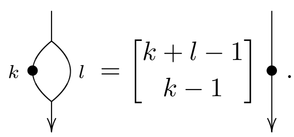
(5) *The square of dot:*




(6) *The distribution of dot:*




(7) *The bubble elimination with dot:*




PROOF. (5) just say that $\gamma_k^2 = [\![k]\!]$. (6) follows from

$$m_{k+l}\gamma^R_{k+l} = \gamma^L_{k+l}m_{k+l} = (\gamma^L_{(k,l);1} + q^k\gamma^L_{(k,l);2})m_{k+l}$$

and

$$\gamma^L_{k+l}m_{k+l} = m_{k+l}\gamma^R_{k+l} = \sum_{w\in\mathfrak{D}_{(k,l)}} T_w m_{(k,l)}(q^l\gamma^R_{(k,l);1} + \gamma^R_{(k,l);2}).$$

Finally it suffices to prove (7) for $\Bbbk = \mathbb{Q}(a,q)$. By Lemma 9.7,

$$m_{k+l}\gamma^R_{(k,l);1}m_{k+l} = [k][k+l-1]!\gamma^L_{k+l}m_{k+l}.$$

On the other hand,

$$m_{k+l}\gamma^R_{(k,l);1}m_{k+l} = \sum_{w\in\mathfrak{D}_{(k,l)}} T_w m_{(k,l)}\gamma^R_{(k,l);1}m_{k+l} = [k]![l]! \sum_{w\in\mathfrak{D}_{(k,l)}} T_w\gamma^L_{(k,l);1}m_{k+l}$$

so that

$$\sum_{w\in\mathfrak{D}_{(k,l)}} T_w\gamma^L_{(k,l);1}m_{k+l} = \frac{[k][k+l-1]!}{[k]![l]!}\gamma^L_{k+l}m_{k+l} = \begin{bmatrix} k+l-1 \\ k-1 \end{bmatrix}\gamma^L_{k+l}m_{k+l}.$$

□

## 3. The category of fakemodules

Now let $t$ be a total $q$-binomial sequence in $\Bbbk$. We here similarly define the fakemodule supercategory $\underline{H}^c_t$-$\mathcal{M}od$ of the superalgebra "$\underline{H}^c_t$" in terms of generators and relations. For such $t$, we write $[\![t]\!] \coloneqq \begin{bmatrix} t+1 \\ 2 \end{bmatrix} - \begin{bmatrix} t \\ 2 \end{bmatrix}$, the "$q^2$-integer" of $t$. Of course it coincides with the usual one for a natural number.

Definition 10.5. First we define the supercategory $\underline{H}^c_t$-$\mathcal{M}od_0$. An object in the supercategory $\underline{H}^c_t$-$\mathcal{M}od_0$ is an $H^c_m$-module $V$ for some $m \in \mathbb{N}$, written as $\underline{\mathrm{Ind}}^c_{t-m}V$ and called a *$\underline{H}^c_t$-fakemodule.* Morphisms between these fakemodules are generated by

$$\underline{\mathrm{Ind}}^c_{t-m}f\colon \underline{\mathrm{Ind}}^c_{t-m}V \to \underline{\mathrm{Ind}}^c_{t-m}W,$$

defined for each $H_m$-homomorphism $f\colon V \to W$,

$$\underline{\mu}_{(t-m-k,k)}V\colon \underline{\mathrm{Ind}}^c_{t-m-k}\mathrm{Ind}^c_k V \to \underline{\mathrm{Ind}}^c_{t-m}V,$$
$$\underline{\Delta}_{(t-m-k,k)}V\colon \underline{\mathrm{Ind}}^c_{t-m}V \to \underline{\mathrm{Ind}}^c_{t-m-k}\mathrm{Ind}^c_k V$$

defined for each $H_m$-module $V$ and $k \in \mathbb{N}$, and

$$\Upsilon_{t-m}V\colon \underline{\mathrm{Ind}}^c_{t-m}V \to \underline{\mathrm{Ind}}^c_{t-m}V$$

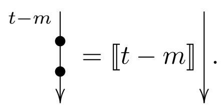
defined for each $H_m$-module $V$. The parity of $\underline{\mathrm{Ind}}^c_{t-m}f$ is defined to be same as $f$, those of $\underline{\mu}_{(t-m-k,k)}V$ and $\underline{\Delta}_{(t-m-k,k)}V$ are even, and that of $\Upsilon_{t-m}V$ is odd. The relations between them are similar ones listed in Definition 7.23 in addition to the followings: the naturality

(c) $\Upsilon_{t-m}$ is an odd natural transformation. That is, the square below commutes up to Koszul sign:

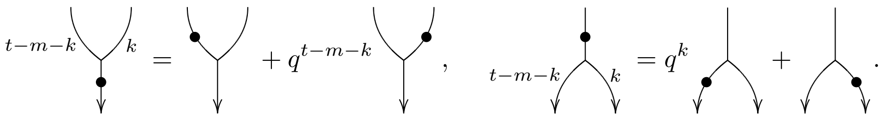
$$\begin{tikzcd} \underline{\mathrm{Ind}}_{t-m}V \arrow[r, "\Upsilon_{t-m}V"] \arrow[d, "\underline{\mathrm{Ind}}_{t-m}f"'] \arrow[dr, phantom, "(-1)^{|f|}"] & \underline{\mathrm{Ind}}_{t-m}V \arrow[d, "\underline{\mathrm{Ind}}_{t-m}f"] \\ \underline{\mathrm{Ind}}_{t-m}W \arrow[r, "\Upsilon_{t-m}W"'] & \underline{\mathrm{Ind}}_{t-m}W. \end{tikzcd}$$

and diagrammatic transformations.

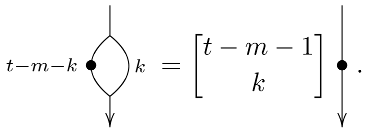
(5) The square of dot:



(6) The distribution of dot:




(7) The bubble elimination with dot:




Similarly as before, we can complete it to locally finitely presented supercategory $\underline{H}^c_t$-$\mathcal{M}od$. We denote by $\mathrm{Hom}_{\underline{H}^c_t}$ the set of fakemorphisms between fakemodules.

The relations listed above are satisfied in the ordinary module category by Proposition 10.4, so that when $q \in \Bbbk^\times$ we have the realization functor $P\colon \underline{H}^c_d$-$\mathcal{M}od \to H^c_d$-$\mathcal{M}od$ for each $d \in \mathbb{N}$, which is full and surjective. We can prove the basis theorem for this supercategory.

THEOREM 10.6 (Basis theorem). *For $V \in H^c_m$-$\mathcal{M}od$ and $W \in H^c_n$-$\mathcal{M}od$, if the action of $[2] \in \Bbbk$ on $W$ is injective, we have*

$$\begin{aligned}\mathrm{Hom}_{\underline{H}^c_t}(\underline{\mathrm{Ind}}^c_{t-m}V, \underline{\mathrm{Ind}}^c_{t-n}W) \simeq \bigoplus_i \mathrm{Hom}_{H^c_i}(\mathrm{Res}^{c\prime}_{m-i}V, \mathrm{Res}^c_{n-i}W)\\ \oplus \bigoplus_i \Pi\,\mathrm{Hom}_{H^c_i}(\mathrm{Res}^{c\prime}_{m-i}V, \mathrm{Res}^c_{n-i}W).\end{aligned}$$

PROOF. Similarly to the proof of Theorem 7.27, we first define the supercategory $\mathcal{C}^c_t$ which is defined by the relations above except for the bubble eliminations (4) and (7). Then for this supercategory we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}^c_t}(\underline{\mathrm{Ind}}^c_{t-m}V, \underline{\mathrm{Ind}}^c_{t-n}W) \simeq \bigoplus_{m,n\le l} \mathrm{Hom}_{H^c_l}(\mathrm{Ind}^c_{l-m}V, \mathrm{Ind}^c_{l-n}W)\\ \oplus \bigoplus_{m,n\le l} \Pi\,\mathrm{Hom}_{H^c_l}(\mathrm{Ind}^c_{l-m}V, \mathrm{Ind}^c_{l-n}W).\end{aligned}$$

Studying the kernel by use of Proposition 10.3 we obtain the desired isomorphism. We left the details to the reader. □

## 4. Parabolic fakemodules

Now for a fakecomposition $\lambda = (\lambda_1, \lambda')$ of $t$, we define the *parabolic fakemodule* $\underline{M}^c_\lambda := \underline{\mathrm{Ind}}^c_{\lambda_1} M^c_{\lambda'}$. Similarly as before, we define the set of row-standard circled tableaux

$$\underline{\mathrm{Tab}}^c_{\lambda;\mu} := \varinjlim_d \mathrm{Tab}^c_{\lambda|d;\mu|d}$$

and the fakemorphism $\underline{m}_{\mathsf{S}}\colon \underline{M}^c_\mu \to \underline{M}^c_\lambda$ for each $\mathsf{S} \in \underline{\mathrm{Tab}}^c_{\lambda;\mu}$. Let $\underline{M}^c_{\lambda;\mu}$ be a supermodule spanned by a linearly independent set $\{\underline{m}_{\mathsf{S}} \mid \mathsf{S} \in \underline{\mathrm{Tab}}^c_{\lambda;\mu}\}$. The superalgebra $\underline{\Gamma}_\lambda$ is defined similarly as $\Gamma_\lambda$ with generators $\underline{\gamma}_{\lambda;1}$ and $\gamma_{\lambda;2}, \gamma_{\lambda;3}, \dots$ by using $(\underline{\gamma}_{\lambda;1})^2 = a[\![\lambda_1]\!]$ instead of usual $q^2$-integers, but the relation $\underline{\gamma}_{\lambda;1} = 0$ is omitted even if $\lambda_1 = 0$. Hence as an abstract superalgebra, we have

$$\underline{\Gamma}_\lambda \simeq C_1(a[\![\lambda_1]\!]) \otimes \Gamma_{\lambda'}.$$

Then there is an inclusion of superalgebra $\underline{\Gamma}_\lambda \hookrightarrow \underline{M}^c_{\lambda;\lambda}$, so that the reversed product

$$\circ_\mu\colon \underline{M}^c_{\mu;\nu} \otimes_{\underline{\Gamma}_\mu} \underline{M}^c_{\lambda;\mu} \to \underline{M}^c_{\lambda;\nu}$$

is defined. When $q \in \Bbbk^\times$, for $d \in \mathbb{N}$ the realization functor $P$ sends $\underline{m}_{\mathsf{S}} \in \underline{M}^c_{\lambda;\mu}$ to the corresponding $m_{\mathsf{S}} \in M^c_{\lambda;\mu}$ if $\#(\mathsf{S}^\times) > 0$ or $\#(\mathsf{S}^\times) = 0$ and $\mathsf{S}$ has no $\textcircled{1}$ in its first row, and otherwise zero. For example, when $\lambda = \mu = (1,1)$, $P$ is given by

$$\begin{array}{|c|c|c|c|}\hline 1&\cdots&1&1\\\hline 2\\\cline{1-1}\end{array} \mapsto \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&\cdots&1&1\\\hline \textcircled{2}\\\cline{1-1}\end{array} \mapsto \begin{array}{|c|}\hline 1\\\hline \textcircled{2}\\\hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&\cdots&1&\textcircled{1}\\\hline 2\\\cline{1-1}\end{array} \mapsto \begin{array}{|c|}\hline \textcircled{1}\\\hline 2\\\hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&\cdots&1&\textcircled{1}\\\hline \textcircled{2}\\\cline{1-1}\end{array} \mapsto \begin{array}{|c|}\hline \textcircled{1}\\\hline \textcircled{2}\\\hline\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1&\cdots&1&2\\\hline 1\\\cline{1-1}\end{array} \mapsto \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&\cdots&1&2\\\hline \textcircled{1}\\\cline{1-1}\end{array} \mapsto \begin{array}{|c|}\hline 2\\\hline \textcircled{1}\\\hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&\cdots&1&\textcircled{2}\\\hline 1\\\cline{1-1}\end{array} \mapsto \begin{array}{|c|}\hline \textcircled{2}\\\hline 1\\\hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1&\cdots&1&\textcircled{2}\\\hline \textcircled{1}\\\cline{1-1}\end{array} \mapsto \begin{array}{|c|}\hline \textcircled{2}\\\hline \textcircled{1}\\\hline\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1&\cdots&\textcircled{1}&2\\\hline 1\\\cline{1-1}\end{array} \mapsto 0, \quad \begin{array}{|c|c|c|c|}\hline 1&\cdots&\textcircled{1}&2\\\hline \textcircled{1}\\\cline{1-1}\end{array} \mapsto 0, \quad \begin{array}{|c|c|c|c|}\hline 1&\cdots&\textcircled{1}&\textcircled{2}\\\hline 1\\\cline{1-1}\end{array} \mapsto 0, \quad \begin{array}{|c|c|c|c|}\hline 1&\cdots&\textcircled{1}&\textcircled{2}\\\hline \textcircled{1}\\\cline{1-1}\end{array} \mapsto 0.$$

Now we similarly define the quotient supermodules $\underline{M}^c_{\lambda;\mu} \twoheadrightarrow \underline{M}^{c(\nu)}_{\lambda;\mu}$ and $\underline{S}^c_{\lambda;\mu} \coloneqq \underline{M}^{c(\lambda)}_{\lambda;\mu}$. For these supermodule we obtain the following theorems.

THEOREM 10.7.

(1) $\underline{S}^c_{\lambda;\lambda}$ *is spanned by* $\underline{m}_\lambda$ *over* $\underline{\Gamma}_\lambda$.
(2) $\underline{S}^c_{\lambda;\mu} = 0$ *unless* $\lambda \geq \mu$. □

We say that a fakepartition $\lambda$ is *strict* if $\lambda'$ is strict, and the sets $\underline{\mathrm{STab}}^c_{\lambda;\mu}$ and $\underline{\mathrm{STab}}^{c'}_{\lambda;\mu}$ are defined by direct limits similarly as before. Then the standardly based structure of the category of parabolic fakemodules is obtained by the same proofs as for the ordinal case.

THEOREM 10.8. *Assume* $\Bbbk = \mathbb{Q}(a,q)$. *Then*

(1) $\underline{M}^c_{\lambda;\mu}$ *has a basis*
$$\bigsqcup_{\nu\colon \text{strict fakepartition}} \{\underline{m}_{\mathsf{S}} \circ_\nu \underline{m}_{\mathsf{T}^*} \mid \mathsf{S} \in \underline{\mathrm{STab}}^c_{\nu;\mu}, \mathsf{T} \in \underline{\mathrm{STab}}^{c'}_{\nu;\lambda}\}.$$
(2) $\underline{S}^c_{\lambda;\mu}$ *has a basis* $\{\underline{m}_{\mathsf{T}} \mid \mathsf{T} \in \underline{\mathrm{STab}}^c_{\lambda;\mu}\}$ *so*
$$\underline{S}^c_{\lambda;\lambda} = \begin{cases} \underline{\Gamma}_\lambda & \textit{if } \lambda \textit{ is a strict fakepartition,} \\ 0 & \textit{otherwise.} \end{cases}$$
(3) *The product*
$$\circ_\nu \colon \underline{S}^c_{\nu;\mu} \otimes_{\underline{\Gamma}_\lambda} \underline{S}^{c*}_{\nu;\lambda} \to \underline{M}^{c(\nu)}_{\lambda;\mu}$$
*is injective.* □

CHAPTER 11

# Cellular structure on the Hecke–Clifford superalgebra, II

We return to our subject, the classification of simple modules of $H_n^c$. Throughout in this chapter, we assume $q \in \Bbbk^\times$.

## 1. Identification of the quotient superalgebras

In order to classify the simple modules, we first determine the quotient superalgebra $M^c_{\lambda;\lambda} \twoheadrightarrow S^c_{\lambda;\lambda}$. In this computation the superalgebras $\underline{M}^c_{\lambda;\lambda} \twoheadrightarrow \underline{S}^c_{\lambda;\lambda}$ of fakemorphisms are used.

LEMMA 11.1. *Let $\lambda = (m,k)$ with $m > k$. Then in $S^c_{\lambda;\lambda}$ we have*

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&①&2&\cdots&2\\ \hline 1&\cdots&1 \\ \cline{1-3}\end{array} \equiv (-1)^k q^{\binom{k}{2}}(\gamma_{\lambda;1} - q^{m-k}\gamma_{\lambda;2})m_\lambda.$$

PROOF. By 5.14, we have

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&2&\cdots&2\\ \hline 1&\cdots&1 \\ \cline{1-3}\end{array} \equiv (-1)^k q^{\binom{k}{2}} m_\lambda.$$

Hence the equation is implied by

$$\gamma_{\lambda;2}\cdot \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&2&\cdots&2\\ \hline 1&\cdots&1 \\ \cline{1-3}\end{array} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&2&\cdots&②\\ \hline 1&\cdots&1 \\ \cline{1-3}\end{array}$$

and

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&2&\cdots&2\\ \hline 1&\cdots&1 \\ \cline{1-3}\end{array} \cdot \gamma_{\lambda;1} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&①&2&\cdots&2\\ \hline 1&\cdots&1 \\ \cline{1-3}\end{array} + q^{m-k} \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&1&2&\cdots&②\\ \hline 1&\cdots&1 \\ \cline{1-3}\end{array}.$$

□

COROLLARY 11.2. *For a fakepartition $\lambda = (t-k,k)$, in $\underline{S}^c_{\lambda;\lambda}$ we have*

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&\cdots&①&2&\cdots&2\\ \hline 1&\cdots&1 \\ \cline{1-3}\end{array} \equiv (-1)^k q^{\binom{k}{2}}(\underline{\gamma}_{\lambda;1} - q^{t-2k}\gamma_{\lambda;2})\underline{m}_\lambda.$$

LEMMA 11.3. *Let $\lambda$ be a partition. Then*

(1) *if $\lambda_1 > \lambda_2$, we have $\underline{S}^c_{\lambda;\lambda} \simeq S^c_{\lambda;\lambda}$,*
(2) *if $\lambda_1 = \lambda_2$, $\operatorname{Ker}(\underline{S}^c_{\lambda;\lambda} \twoheadrightarrow S^c_{\lambda;\lambda})$ is generated by $(\underline{\gamma}_{\lambda;1} - \gamma_{\lambda;2})m_\lambda$ as a 2-sided ideal.*

PROOF. For the case (2), by the computation in Lemma 9.18 we have $(\gamma_{\lambda;1} - \gamma_{\lambda;2})m_\lambda \equiv 0$ in $S^c_{\lambda;\lambda}$ so the kernel contains $(\underline{\gamma}_{\lambda;1} - \gamma_{\lambda;2})\underline{m}_\lambda$. We prove the converse inclusions.

$\operatorname{Ker}(\underline{M}^c_{\lambda;\lambda} \twoheadrightarrow M^c_{\lambda;\lambda})$ is spanned by $\underline{m}_{\mathsf{T}}$ for $\mathsf{T} \in \underline{\mathrm{Tab}}^c_{\lambda;\lambda}$ which satisfies either of the condition that $\#_{11}(\mathsf{T}^\times) < 0$ or that $\#_{11}(\mathsf{T}^\times) = 0$ with ① in its first row. If $\lambda_1 > \lambda_2$, we have $\lambda_1 - \#_{11}(\mathsf{T}^\times) > \lambda_2$ for such $\mathsf{T}$ so that $\underline{m}_{\mathsf{T}} \equiv 0$ in $\underline{S}^c_{\lambda;\lambda}$ as we did in the proof of Lemma 5.20, which implies that $\operatorname{Ker}(\underline{M}^c_{\lambda;\lambda} \twoheadrightarrow M^c_{\lambda;\lambda})$ is already zero in $\underline{S}^c_{\lambda;\lambda}$; in other words, $\underline{S}^c_{\lambda;\lambda} \simeq S^c_{\lambda;\lambda}$. In the other case $\lambda_1 = \lambda_2$, we also have $\underline{m}_{\mathsf{T}} \equiv 0$ if $\#_{11}(\mathsf{T}^\times) < 0$. Otherwise by applying local transformation on the second row or below $\underline{m}_{\mathsf{T}}$ can be transformed into a linear combination of tableaux $\mathsf{S}$ such that $\mathsf{T}(1,j) = \mathsf{S}(1,j)$ for all $j$ and $\#_{21}(\mathsf{S}^{\times}) = \lambda_2$, that is, which is in the form

$$\begin{ytableau} 1 & 1 & 1 & 1 & \cdots & \textcircled{1} & \; & \; & \; & \; \\ 1 & 1 & 1 & 1 \\ \; & \; & \; & \; \\ \; & \; \end{ytableau} \qquad \text{or} \qquad \begin{ytableau} 1 & 1 & 1 & 1 & \cdots & \textcircled{1} & \; & \; & \; & \; \\ 1 & 1 & 1 & \textcircled{1} \\ \; & \; & \; & \; \\ \; & \; \end{ytableau}\,.$$

By the corollary above, for the special case we have

$$\begin{ytableau} 1 & 1 & 1 & 1 & \cdots & \textcircled{1} & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 \\ 4 & 4 \end{ytableau} \equiv (-1)^{\lambda_2} q^{\binom{\lambda_2}{2}} (\underline{\gamma}_{\lambda;1} - \gamma_{\lambda;2}) \underline{m}_{\lambda}$$

in $\underline{S}^{c}_{\lambda;\lambda}$, so that

$$\begin{ytableau} 1 & 1 & 1 & 1 & \cdots & \textcircled{1} & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & \textcircled{1} \\ 3 & 3 & 3 & 3 \\ 4 & 4 \end{ytableau} \equiv (-1)^{\lambda_2} q^{\binom{\lambda_2}{2}} (\underline{\gamma}_{\lambda;1} - \gamma_{\lambda;2}) \gamma_{\lambda;2} \underline{m}_{\lambda}.$$

Every such $\mathsf{S}$ above can be made by multiplying elements to these tableaux from left. Hence the image of $\operatorname{Ker}(\underline{M}^{c}_{\lambda;\lambda} \twoheadrightarrow M^{c}_{\lambda;\lambda})$ in $\underline{S}^{c}_{\lambda;\lambda}$ is generated by $(\underline{\gamma}_{\lambda;1} - \gamma_{\lambda;2})\underline{m}_{\lambda}$ as a 2-sided ideal. $\square$

LEMMA 11.4. *For a fakepartition $\lambda = (\lambda_1, \lambda')$, we have $\underline{S}^{c}_{\lambda;\lambda} \simeq C_1(a[\![\lambda_1]\!]) \otimes S^{c}_{\lambda';\lambda'}$.*

PROOF. Since these modules are preserved by extension of scalars, it suffices to prove for the universal ring $\Bbbk = \mathbb{Z}[a, q^{\pm}]$. Let

$$V \coloneqq \sum_{\nu > \lambda,\, \nu_1 > \lambda_1} \underline{M}^{c\,\nu}_{\lambda;\lambda} \qquad \text{and} \qquad W \coloneqq \sum_{\nu > \lambda,\, \nu_1 = \lambda_1} \underline{M}^{c\,\nu}_{\lambda;\lambda}$$

so that $\underline{M}^{c\,>\nu}_{\lambda;\lambda} = V + W$. On the other hand, let

$$T \coloneqq \{\mathsf{T} \in \mathrm{Tab}^{c}_{\lambda;\lambda} \mid \mathsf{T}^{\times}(1,j) = 1 \text{ for all } j\}$$

and

$$X \coloneqq \Bbbk\{\underline{m}_{\mathsf{T}} \mid \mathsf{T} \in \mathrm{Tab}^{c}_{\lambda;\lambda} \setminus T\}, \qquad Y \coloneqq \Bbbk\{\underline{m}_{\mathsf{T}} \mid \mathsf{T} \in T\}$$

so that $\underline{M}^{c}_{\lambda;\lambda} = X \oplus Y$. Since $\underline{\Gamma}_{\lambda}\underline{m}_{\lambda}, W \subset Y$ we have $\underline{M}^{c}_{\lambda;\lambda} = V + Y$. Hence

$$\underline{S}^{c}_{\lambda;\lambda} = \underline{M}^{c}_{\lambda;\lambda} / \underline{M}^{c\,>\nu}_{\lambda;\lambda} = (V+Y)/(V+W) \simeq Y/((V \cap Y) + W).$$

For a $\mathbb{Z}[a, q^{\pm}]$-module $M$, let $\tilde{M} \coloneqq M \otimes_{\mathbb{Z}[a,q^{\pm}]} \mathbb{Q}(a,q)$ be its localization. By the cellular basis theorem, we have $\dim \tilde{V} + \dim \tilde{W} = \dim \underline{\tilde{M}}^{c}_{\lambda;\lambda}$ and

$$\dim \tilde{W} = \sum_{\nu > \lambda,\, \nu_1 = \lambda_1} \#\underline{\mathrm{STab}}^{c}_{\nu;\lambda} \cdot \#\underline{\mathrm{STab}}^{c\prime}_{\nu;\lambda}.$$

On the other hand, since we can view $\lambda_1$ as a sufficiently large number, we have a natural one-to-one correspondence

$$\{\text{strict fakepartition } \nu \mid \nu > \lambda,\, \nu_1 = \lambda_1\} \xleftrightarrow{1:1} \{\text{strict partition } \nu' \mid \nu' > \lambda'\}$$

and for such $\nu$,

$$\underline{\mathrm{STab}}^{c}_{\nu;\lambda} \xleftrightarrow{1:1} \{1, \textcircled{1}\} \times \mathrm{STab}^{c}_{\nu';\lambda'} \qquad \text{and} \qquad \underline{\mathrm{STab}}^{c\prime}_{\nu;\lambda} \xleftrightarrow{1:1} \mathrm{STab}^{c\prime}_{\nu';\lambda'}.$$

Then by using the shifted Knuth correspondence for $\lambda'$, we obtain $\dim \tilde{Y} = \dim \tilde{W}$. Since localization of modules is exact, from the exact sequence

$$0 \to V \cap Y \to V \oplus Y \to \underline{M}^{c}_{\lambda;\lambda} \to 0$$

we obtain

$$0 \to \widetilde{V \cap Y} \to \tilde{V} \oplus \tilde{Y} \to \underline{\tilde{M}}^{c}_{\lambda;\lambda} \to 0.$$

By comparison of dimensions we have $\widetilde{V \cap Y} = 0$. Since $V \cap Y \subset Y$ is a torsion-free module over the integral domain $\mathbb{Z}[a, q^{\pm}]$, it implies $V \cap Y = 0$. Hence we have

$$\underline{S}^{c}_{\lambda;\lambda} \simeq Y/W \simeq C_1(a[\![\lambda_1]\!]) \otimes S^{c}_{\lambda';\lambda'}.$$

$\square$

Using these two lemmas we obtain the following identification of the quotient superalgebras $S^{c}_{\lambda;\lambda}$ and $\underline{S}^{c}_{\lambda;\lambda}$.

THEOREM 11.5. *Recall the assumption $q \in \Bbbk^{\times}$.*

(1) *For a partition $\lambda$, the 2-sided ideal $\mathrm{Ker}(\Gamma_\lambda \twoheadrightarrow S^{c}_{\lambda;\lambda})$ is generated by $\gamma_i - \gamma_j$ for all $i, j$ such that $\lambda_i = \lambda_j$.*
(2) *For a fakepartition $\lambda$, the 2-sided ideal $\mathrm{Ker}(\underline{\Gamma}_\lambda \twoheadrightarrow \underline{S}^{c}_{\lambda;\lambda})$ is generated by $\gamma_i - \gamma_j$ for all $i, j$ such that $\lambda_i = \lambda_j$ and $i, j \geq 2$.* $\square$

PROOF. We use a mutual induction for (1) and (2) on the number of components of $\lambda$. First let $\lambda$ be a fakepartition and suppose that (1) holds for $\lambda'$. Then

$$\mathrm{Ker}(\underline{\Gamma}_\lambda \twoheadrightarrow \underline{S}^{c}_{\lambda;\lambda}) \simeq C_1(a[\![\lambda_1]\!]) \otimes \mathrm{Ker}(\Gamma_{\lambda'} \twoheadrightarrow S^{c}_{\lambda';\lambda'})$$

has a generating set above. Next let $\lambda$ be a partition of $n > 0$ and suppose (2) holds for $\lambda$. We have a commutative square

$$\begin{array}{ccc} \underline{\Gamma}_\lambda & \twoheadrightarrow & \underline{S}^{c}_{\lambda;\lambda} \\ \big\downarrow & & \big\downarrow \\ \Gamma_\lambda & \twoheadrightarrow & S^{c}_{\lambda;\lambda} \end{array}$$

where $\underline{\Gamma}_\lambda \simeq \Gamma_\lambda$ since $\lambda_1 > 0$. Hence as a generating set of the kernel of $\Gamma_\lambda \twoheadrightarrow S^{c}_{\lambda;\lambda}$ we can take the union of that of $\Gamma_\lambda \simeq \underline{\Gamma}_\lambda \twoheadrightarrow \underline{S}^{c}_{\lambda;\lambda}$ and that of $\underline{S}^{c}_{\lambda;\lambda} \twoheadrightarrow S^{c}_{\lambda;\lambda}$. $\square$

Consequently we obtain the following classification of simple modules of $\mathscr{S}^{c}_{r,n}$. We remark that $S^{c}_{\lambda;\mu}$ is not free over $\Bbbk$ in general even if in this case $q \in \Bbbk^{\times}$.

THEOREM 11.6. *Suppose $q \in \Bbbk^{\times}$. For a partition $\lambda$, let $\Theta_\lambda$ be the 2-sided ideal generated by $\gamma_{\lambda;i} - \gamma_{\lambda;j}$ above. Then there is a one-to-one correspondence*

$$\mathrm{Irr}(\mathscr{S}^{c}_{r,n}) \overset{1:1}{\longleftrightarrow} \bigsqcup_{\nu=(\nu_1,\ldots,\nu_r);\text{partition}} \mathrm{Irr}(\Gamma_\lambda/\Theta_\lambda).$$

Note that for a partition $\lambda = (k, k)$, we have

$$\Gamma_\lambda/\Theta_\lambda \simeq \Gamma_k \otimes (\Bbbk/2a[\![k]\!]\Bbbk)$$

since $a[\![k]\!] = \gamma^2_{\lambda;1} \equiv \gamma_{\lambda;1}\gamma_{\lambda;2} = -\gamma_{\lambda;2}\gamma_{\lambda;1} \equiv -\gamma^2_{\lambda;1} = -a[\![k]\!]$. In addition, it is clear that $2a[\![k]\!]\Bbbk + 2a[\![l]\!]\Bbbk = 2a[\![\gcd\{k,l\}]\!]\Bbbk$. Thus for a general partition $\lambda$, let $\mu$ be the strict partition obtained by removing duplicate components of $\lambda$ and let $k_1, \ldots, k_r$ be such components, then

$$S^{c}_{\lambda;\lambda} \simeq \Gamma_\lambda/\Theta_\lambda \simeq \Gamma_\mu \otimes (\Bbbk/2a[\![\gcd\{k_1, \ldots, k_n\}]\!]\Bbbk).$$

In particular, $S^{c}_{\lambda;\lambda} = 0$ if and only if $2a[\![\gcd\{k_1, \ldots, k_n\}]\!] \in \Bbbk^{\times}$.

Remember that when $\Bbbk$ is a field the Clifford superalgebra $\Gamma_\lambda$ has a unique simple module up to parity change. For $e \geq 2$, we say that a partition $\lambda$ is *e-strict* if $\lambda_i = \lambda_j$, $i \neq j$ implies $e|\lambda_i$. For convention the word $\infty$-strict stands for strict. For a superalgebra $A$, let $\mathrm{Irr}(A)/\Pi$ be a quotient set of $\mathrm{Irr}(A)$ on which $V \in \mathrm{Irr}(A)$ is identified with its parity change $\Pi V$.

Corollary 11.7. *Suppose that $\Bbbk$ is a field and $2aq \in \Bbbk^\times$. Let $e_2$ be the $q^2$-characteristic of $\Bbbk$. Then there is a one-to-one correspondence*

$$\mathrm{Irr}(\mathscr{S}^c_{r,n})/\Pi \xleftrightarrow{1:1} \{\nu = (\nu_1, \dots, \nu_r); e_2\text{-strict partition}\}.$$

The case $2a = 0$ is easier.

Corollary 11.8. *Suppose that $\Bbbk$ is a field and $q \in \Bbbk^\times$, $2a = 0$. Then there is a one-to-one correspondence*

$$\mathrm{Irr}(\mathscr{S}^c_{r,n})/\Pi \xleftrightarrow{1:1} \{\nu = (\nu_1, \dots, \nu_r); \mathrm{partition}\}.$$

□

## 2. Identification of the ideals

We keep assuming that $q \in \Bbbk^\times$. Finally we reach to the classification of simple modules of the Hecke–Clifford superalgebra $H^c_n$. Now let $J^c_\lambda \subset \Gamma_\lambda$ be the pullback of the 2-sided ideal $m_\lambda \cdot S^c_\lambda \subset S^c_{\lambda;\lambda}$ via the surjective map $\Gamma_\lambda \twoheadrightarrow \Gamma_\lambda/\Theta_\lambda \simeq S^c_{\lambda;\lambda}$. We determine this ideal as follows.

For $n \in \mathbb{N}$, let $K_n \subset \Bbbk$ be the ideal generated by the elements

$$\left\{ \left(\tfrac{a(q-1)}{[2]}\right)^s [n]! \,\middle|\, 0 \le s \le n/2 \right\}.$$

Then by Lemma 9.7, we have $m_n C_n m_n = K_n m_n \oplus K_{n-1}\gamma^L_n m_n$. Since it is a 2-sided ideal of $\Gamma_n m_n$, the following statement holds.

Lemma 11.9. *There are inclusions $a[\![n]\!]K_{n-1} \subset K_n \subset K_{n-1}$.* □

Lemma 11.10. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a partition. For each $i$, let $\Delta_{\lambda;i} \subset \Gamma_\lambda$ be the supermodule*

$$\begin{aligned}\Delta_{\lambda;i} \coloneqq K_{\lambda_i-\lambda_{i+1}} \oplus K_{\lambda_i-\lambda_{i+1}}\gamma_{\lambda;i+1} \oplus K_{\lambda_i-\lambda_{i+1}-1}(\gamma_{\lambda;i} - q^{\lambda_i-\lambda_{i+1}}\gamma_{\lambda;i+1})\\ \oplus K_{\lambda_i-\lambda_{i+1}-1}(\gamma_{\lambda;i} - q^{\lambda_i-\lambda_{i+1}}\gamma_{\lambda;i+1})\gamma_{\lambda;i+1}\end{aligned}$$

*and let $\Delta_\lambda \coloneqq \Delta_{\lambda;r}\cdots\Delta_{\lambda;2}\Delta_{\lambda;1}$. Then $\Delta_\lambda \subset \Gamma_\lambda$ is a 2-sided ideal.*

Proof. For simplicity we write $\Delta = \Delta_\lambda$, $\Delta_i = \Delta_{\lambda;i}$ and $\gamma_i = \gamma_{\lambda;i}$. First we prove

$$\Delta_i\gamma_i \subset \Delta_i + \gamma_{i+1}\Delta_i, \qquad \Delta_i\gamma_{i+1} \subset \Delta_i, \qquad \Delta_i\gamma_j = \gamma_j\Delta_i \quad \text{for } j \ne i, i+1.$$

The second and the third inclusions are clear so we prove the first one. Since $K_n \subset K_{n-1}$, the inclusion $K_{\lambda_i-\lambda_{i+1}}\gamma_i \subset \Delta_i$ is also obvious. We also have

$$\begin{aligned}(\gamma_i - q^{\lambda_i-\lambda_{i+1}}\gamma_{i+1})\gamma_i - q^{\lambda_i-\lambda_{i+1}}\gamma_{i+1}(\gamma_i - q^{\lambda_i-\lambda_{i+1}}\gamma_{i+1}) &= \gamma_i^2 - q^{2(\lambda_i-\lambda_{i+1})}\gamma_{i+1}^2\\ &= a[\![\lambda_i - \lambda_{i+1}]\!]\end{aligned}$$

so that $K_{\lambda_i-\lambda_{i+1}-1}(\gamma_i - q^{\lambda_i-\lambda_{i+1}}\gamma_{i+1})\gamma_i \subset \Delta_i + \gamma_{i+1}\Delta_i$ by $a[\![n]\!]K_{n-1} \subset K_n$. Putting them together we obtain $\Delta_i\gamma_i \subset \Delta_i + \gamma_{i+1}\Delta_i$ as desired. Then

$$\Delta\gamma_i = \Delta_r\cdots\Delta_{i-1}\gamma_i\cdots\Delta_1 \subset \Delta_r\cdots\Delta_{i-1}\cdots\Delta_1 = \Delta$$

for $i \ge 2$, and

$$\Delta\gamma_1 \subset \Delta + \Delta_r\cdots\Delta_2\gamma_2\Delta_1 \subset \Delta + \Delta_r\cdots\gamma_3\Delta_2\Delta_1 \subset \cdots \subset \Delta$$

so $\Delta$ is a right ideal. By the equation above we also have inclusions

$$\gamma_i\Delta_i \subset \Delta_i, \qquad \gamma_{i+1}\Delta_i \subset \Delta_i + \Delta_i\gamma_i$$

which imply that $\Delta$ is also a left ideal in a similar manner. □

Lemma 11.11. *For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ above, we have $\Delta^r_\lambda + \Theta_\lambda \subset J^c_\lambda \subset \Delta_\lambda + \Theta_\lambda$.*

PROOF. Parallel to the proof of Lemma 5.20. So first we prove $J^c_\lambda \subset \Delta_\lambda + \Theta_\lambda$. Take an arbitrary $\mathsf{T} \in \mathrm{Tab}^c_\lambda$. Let $\mu := (\lambda_1, 1^{n-\lambda_1})$ and define $\mathsf{S} \in \mathrm{Tab}^c_{\lambda;\mu}$ which has underlying tableau $\mathsf{S}^\times = \mathsf{T}^\times|_\mu$ and for each its bar $\begin{array}{|c|c|c|c|}\hline 1 & 1 & \cdots & 1 \\ \hline\end{array}$ it has a circle if and only if there are odd number of circles in the corresponding boxes in $\mathsf{T}$. Let $k := \#_{11}(\mathsf{S}^\times)$ and let $p := 0$ if $\mathsf{S}$ does not have $\textcircled{1}$ in its first row, and otherwise $p := 1$. Then by Lemma 9.7 we have

$$m_\mu \cdot m_{\mathsf{T}} \in K_{k-p} m_{\mathsf{S}}.$$

If $k < \lambda_1 - \lambda_2$, we have $m_{\mathsf{S}} \equiv 0$. If $k = \lambda_1 - \lambda_2$ and $p = 1$, $m_{\mathsf{S}}$ can be transformed into a linear combination of tableaux generated by

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & \cdots & \textcircled{1} & 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 \\ \cline{1-4} 3 & 3 & 3 & 3 \\ \cline{1-4} 4 & 4 \\ \cline{1-2}\end{array} \equiv (-1)^{\lambda_2} q^{-\binom{\lambda_2}{2}} m_\lambda (\gamma_{\lambda;1} - q^{\lambda_1-\lambda_2}\gamma_{\lambda;2})$$

or

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & \cdots & \textcircled{1} & 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & \textcircled{1} \\ \cline{1-4} 3 & 3 & 3 & 3 \\ \cline{1-4} 4 & 4 \\ \cline{1-2}\end{array} \equiv (-1)^{\lambda_2} q^{-\binom{\lambda_2}{2}} m_\lambda (\gamma_{\lambda;1} - q^{\lambda_1-\lambda_2}\gamma_{\lambda;2})\gamma_{\lambda;2}$$

as we did before. Hence

$$K_{k-p} m_{\mathsf{S}} \subset (\mathbb{1}^c_{\lambda_1} * S^c_{\lambda'}) \cdot \Delta_{\lambda;1}.$$

In the other cases we have $K_{k-p} \subset K_{\lambda_1-\lambda_2}$ so the inclusion above also holds. By induction we may assume that $m_{\lambda'} \cdot S^c_{\lambda'} \subset m_{\lambda'}\Delta_{\lambda'}$. This implies $m_\lambda \cdot m_{\mathsf{T}} \in m_\lambda \Delta_\lambda$ in $S^c_\lambda$.

We can prove the other inclusion by using circled tableaux whose underlying tableau is $\mathsf{R}$ in the proof of Lemma 5.20. By putting circles on suitable boxes of $\mathsf{R}_\downarrow$ we can make arbitrary elements of

$$(\Delta_{\lambda;r} \cdots \Delta_{\lambda;2}\Delta_{\lambda;1})(\Delta_{\lambda;r} \cdots \Delta_{\lambda;2}) \cdots (\Delta_{\lambda;r}\Delta_{\lambda;r-1})\Delta_{\lambda;r} \supset \Delta^r_\lambda.$$

For example, for $\lambda = (6,4,1)$

$$m_\lambda \begin{array}{|c|c|c|c|c|c|}\hline 1 & \textcircled{2} & 7 & 8 & 9 & \textcircled{10} \\ \hline 3 & 4 & \textcircled{5} & \textcircled{9} \\ \cline{1-4} 6 \\ \cline{1-1}\end{array} = [2][3]! \begin{array}{|c|c|c|c|c|c|}\hline 1 & \textcircled{1} & 2 & 2 & 2 & \textcircled{3} \\ \hline 1 & 1 & \textcircled{1} & \textcircled{2} \\ \cline{1-4} 1 \\ \cline{1-1}\end{array} = -[2][3]! \begin{array}{|c|c|c|c|c|c|}\hline 1 & \textcircled{1} & 2 & 2 & \textcircled{2} & \textcircled{3} \\ \hline 1 & 1 & 1 & 1 \\ \cline{1-4} 2 \\ \cline{1-1}\end{array} \gamma_3(\gamma_2 - q^3\gamma_3)$$
$$= \cdots = q^6 m_\lambda \cdot \gamma_3 \cdot [3]! \cdot (\gamma_1 - q^2\gamma_2) \cdot \gamma_3 \cdot [2](\gamma_2 - q^3\gamma_3) \cdot 1$$

where $\gamma_1 - q^2\gamma_2 \in \Delta_{\lambda;1}$, $[3]!, [2](\gamma_2 - q^3\gamma_3) \in \Delta_{\lambda;2}$ and $\gamma_3, \gamma_3, 1 \in \Delta_{\lambda;3}$. Hence we conclude that $m_\lambda \cdot S^c_{\lambda;\lambda} \supset m_\lambda \Delta^r_\lambda$. $\square$

We state again the main theorem of this paper.

THEOREM 11.12. *When $q \in \Bbbk^\times$, there is a one-to-one correspondence*

$$\mathrm{Irr}(H^c_n) \overset{1:1}{\longleftrightarrow} \bigsqcup_{\lambda:\ \text{partition}} \mathrm{Irr}^{\Delta_\lambda+\Theta_\lambda}_{\Theta_\lambda}(\Gamma_\lambda).$$

$\square$

Now assume that $\Bbbk$ is a field. By specializing this theorem we obtain several classifications. First consider the case $q \neq -1$. In this case simply $K_n = [n]!\,\Bbbk$. Let $e$ (resp. $e_2$) be a $q$-characteristic (resp. $q^2$-) of $\Bbbk$. Then we have

$$e_2 = \begin{cases} e & \text{if } e \text{ is odd}, \\ e/2 & \text{if } e \text{ is even}. \end{cases}$$

Let $\lambda$ be a partition. If $\lambda_i > \lambda_{i+1} + e$, we have $\Delta_\lambda = 0$ as before. On the other hand if $\lambda_i < \lambda_{i+1} + e$ we have $\Delta_\lambda = \Gamma_\lambda$. So suppose $\lambda_i = \lambda_{i+1} + e$ so that $K_{\lambda_i - \lambda_{i+1}} = 0$

but $K_{\lambda_i-\lambda_{i+1}-1} = \Bbbk$. If $e_2|\lambda_i$ then $\gamma_{\lambda_i}$ and $\gamma_{\lambda;i+1}$ are central nilpotent so that they are contained in the Jacobson radical of $\Gamma_\lambda$. Otherwise

$$(\gamma_{\lambda;1} - q^{\lambda_1-\lambda_2}\gamma_{\lambda;2})^2 = a[\![\lambda_1]\!] + aq^{2(\lambda_1-\lambda_2)}[\![\lambda_2]\!] = 2a[\![\lambda_1]\!]$$

is invertible if and only if $2a \neq 0$. When $2a = 0$, $K_{\lambda_i-\lambda_{i+1}-1}(\gamma_{\lambda;1} - q^{\lambda_1-\lambda_2}\gamma_{\lambda;2})$ generates a nilpotent ideal so is in the Jacobson radical also in this case.

Summarizing the above, we obtain the following results. We say that an $e_2$-strict partition $\lambda$ is *e-restricted* if

$$\begin{cases} \lambda_i - \lambda_{i+1} < e & \text{if } e_2|\lambda_i, \\ \lambda_i - \lambda_{i+1} \leq e & \text{otherwise.} \end{cases}$$

Corollary 11.13. *Suppose $\Bbbk$ is a field and $2aq[2] \neq 0$. Then there is a one-to-one correspondence*

$$\mathrm{Irr}(H_n^c)/\Pi \xleftrightarrow{1:1} \{e\text{-restricted } e_2\text{-strict partition}\}.$$

The result is now coincides with the crystal $B(\Lambda_0)$ of type $\mathsf{A}^{(2)}_{e-1}$ for odd $e$ or of type $\mathsf{D}^{(2)}_{e/2}$ for even $e$ whose descriptions are obtained by Kang [**Kan03**] and Hu [**Hu06**] respectively.

Corollary 11.14. *Suppose $\Bbbk$ is a field and $q[2] \neq 0$, $2a = 0$. Then there is a one-to-one correspondence*

$$\mathrm{Irr}(H_n^c)/\Pi \xleftrightarrow{1:1} \{e\text{-restricted partition}\}.$$

Next consider the case $q = -1$. First assume that $2a \neq 0$. Let $p$ be the (ordinary) characteristic of $\Bbbk$. Then we have $K_n = \Bbbk$ if $n < p$ and otherwise $K_n = 0$. Hence by a similar arguments as above we obtain the following.

Corollary 11.15. *Suppose $\Bbbk$ is a field of characteristic $p \neq 2$ and $q = -1$, $a \neq 0$. Then there is a one-to-one correspondence*

$$\mathrm{Irr}(H_n^c)/\Pi \xleftrightarrow{1:1} \{p\text{-restricted } p\text{-strict partition}\}.$$

Now finally let $q = -1$ and $2a = 0$, so that $K_0 = K_1 = \Bbbk$ and $K_n = 0$ for $n \geq 2$. Similar to the case above for $2a = 0$, we obtain the following.

Corollary 11.16. *Suppose $\Bbbk$ is a field and $q = -1$, $2a = 0$. Then there is a one-to-one correspondence*

$$\mathrm{Irr}(H_n^c)/\Pi \xleftrightarrow{1:1} \{2\text{-restricted partition}\}.$$

In fact, the two results for $2a = 0$ are already obtained in Remark 9.6.

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