Dissertation

The Spectrum of Classical String Theory and Integrability in the AdS/CFT Correspondence

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Abstract

Motivated by the open-closed duality in string theory, the AdS/CFT correspondence has been thoroughly investigated for more than a decade. A great advancement was made, among others, concerning the correspondence between $\mathcal{N} = 4$ super Yang-Mills theory and superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ with precision, after the discovery of integrability. The methods of integrability, like Bethe Ansatz and finite-gap solutions, allow us to find matching of the spectrum of both theories, by comparing their formulation as well as concrete examples. Remarkably, so-called asymptotic Bethe Ansatz equations have recently been proposed to all orders in the 't Hooft coupling, which reproduce a certain class of the spectrum of both gauge and string theories correctly.

In this thesis, we aim to comprehend this correspondence taking general examples of the spectrum, mainly focusing on its strong coupling region. To this aim, we construct a family of classical string solutions on $\mathbb{R}_{\mathrm{t}} \times \mathrm{S}^3$ subspace of $\mathrm{AdS}_5 \times \mathrm{S}^5$ background, which are related to Complex sine-Gordon solitons via the Pohlmeyer-Lund-Regge reduction. We obtain analytical expressions subject to periodic boundary conditions, which are shown to interpolate various classical spinning or oscillating string solutions known so far.

It is known that the asymptotic Bethe Ansatz equations have limited application for systems of finite size; they do not account for wrapping interactions in the weak coupling, nor they reproduce the exponential-type finite-size corrections in the strong coupling. To clarify the latter, we compute finite-size corrections to dyonic giant magnons, or magnon boundstates, in two ways. One is by examining the asymptotics of our general solutions in the limit where an angular momentum goes to infinity, and the other is by applying the generalized Lüscher formula to the situation in which incoming particles are boundstates. We find agreement of the two results, which makes possible to predict the (leading) finite-size correction for dyonic giant magnons to all orders in the 't Hooft coupling.

Contents

Introduction

Superstring theory is a consistent description of gravity and gauge theory at the Planck scale, free from ultraviolet divergence inherent in the quantum field theory of particles. This description necessitates extended objects called branes as well as fundamental strings. Interactions among them reduce to ordinary gravity and gauge interactions at the low energy scale. In particular, closed strings cause gravitational interaction, while open strings cause gauge interaction.

Huge symmetry resides in string theory, and part of which demonstrating surprising equivalence between apparently unrelated physical phenomena is referred to as duality. Open-closed duality, namely the one between open strings and closed strings, is one of the most prominent discoveries of dualities in string theory. The open-closed duality suggests a possibility that, in certain situations, gravity theory and gauge theory are dual descriptions of the same phenomenon.

AdS/CFT correspondence, first conjectured by Juan Maldacena in 1997 [1], is one realization of this open-closed duality. It dictates $\mathcal{N} = 4$ super Yang-Mills theory with $SU(N)$ gauge group is equivalent to superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ background with N units of RR flux, at least in large N limit. The Maldacena conjecture can be extended to the correspondence between a wide class of superconformal field theories and superstring on $\mathrm{AdS}_5 \times M$ background, with M a five-dimensional manifold. Succeedingly, Gubser, Klebanov, Polyakov and Witten proposed the correspondence between correlation functions of both gauge and gravity theories in more detail [2, 3].

We expect that study of the AdS/CFT correspondence will eventually elucidate the dual description of superstring theory with D-branes. However, soon it turned out that it is very hard to prove AdS/CFT correspondence. In large N limit, interactions in super Yang-Mills theory is governed by the 't Hooft coupling constant $\lambda \equiv N g_{\mathrm{YM}}^2$ while superstring theory contains the coupling $\lambda = R^4/\alpha'^2$, where R is the curvature scale of the spacetime and $(\alpha')^{1/2}$ is the length scale of strings. Under the AdS/CFT correspondence the perturbative region of super Yang-Mills, $\lambda \ll 1$, is mapped to the strong coupling region of superstring, $1/\lambda \gg 1$, and vice versa. In this way, one cannot predict the strong coupling behavior of either side, unless quantum corrections are tamed down by, for example, the use of supersymmetry. Fairly said, the AdS/CFT correspondence is still a well-tested conjecture.

One remarkable feature of $\mathcal{N} = 4$ super Yang-Mills is in that, besides the fact that this is maximally supersymmetric gauge field theory in four dimensions, it sits on the superconformal fixed point at tree level. It gives us a hope of uncovering its strong coupling dynamics and checking the validity of AdS/CFT correspondence in a qualitative manner.

Like many conformal field theories, conformal dimension of gauge-invariant local operators is an important physical quantity in $\mathcal{N} = 4$ super Yang-Mills. In general, the conformal dimension, or the eigenvalue of dilatation operator, receives quantum corrections to all orders in coupling constant. Furthermore, quantum effects can mix local operators of the same quantum number, giving different eigenvalues of dilatation operator to each eigenstate.

Solving the problem of operator mixing was considered as a formidable task without the aid of supersymmetry. Berenstein, Maldacena, and Nastase (BMN) considered dilatation eigenvalues for operators in near BPS sector, which are obtained by inserting a few elementary fields of $\mathcal{N} = 4$ theory to the half-BPS operator with sufficiently large length L [4]. The limit of large L reduces coupling constant λ to $\tilde{\lambda} \equiv \lambda/L^2$, and suppresses quantum corrections in a tractable manner. They found that these operators are dual to closed string modes on pp-wave background, which can be quantized to all orders of α'.

While trying to solve a general problem of operator mixing, Minahan and Zarembo found that the dilatation operator at one-loop in λ has the same form of an integrable spin chain [5]. It is known that for an integrable Hamiltonian, one can compute its general eigenstates by using methods of integrability like Bethe Ansatz equation. Through mapping from a super Yang-Mills operator to a spin chain, from the dilatation operator to the integrable Hamiltonian, one is able to study the spectrum even in non-BPS sectors of $\mathcal{N} = 4$ theory.

The Bethe Ansatz approach was generalized to higher orders of $su(2)$ sector (a set of gauge-invariant local operators made up of Z and W, two holomorphic scalars in $\mathcal{N} = 4$ theory) in [6]. However, in contrast to one-loop cases, this Bethe Ansatz is applicable only to long-range spin chains.

Long-range Bethe Ansatz equations are proposed to all orders of λ in $su(2)$ sector [7], in all three rank-one sectors [8], and in the full $psu(2,2|4)$ sector [9], by assuming all-order integrability as well as making use of some sophisticated guesses. Note that the all-loop Bethe Ansatz equations of [9] contained so-called dressing phase, which had been introduced in [10] to reconcile mismatch between the strong coupling limit of all-loop Bethe Ansatz equation [7] without dressing phase, and the integral equation derived from classical string theory [11]. Later it was also shown that the all-loop Bethe Ansatz equations of [9] are consistent with the $su(2|2)^2$-invariant S-matrix on which global symmetry imposes severe constraints [12].

There have been an increasing number of evidences and positive supports also for the dressing phase. At strong coupling, the dressing phase was generalized to incorporate one-loop results in $1/\sqrt{\lambda}$ [13]. On the analogy of S-matrix in relativistic quantum field theories, Janik argued the dressing phase should be crossing symmetric [14], which was later confirmed in [15].

By solving the requirement derived from the crossing symmetry, several all-order expressions for the dressing phase were proposed in [16]. In $sl(2)$ sector which is generated by a complex scalar Z and light-cone covariant derivative D_+, close relation between the dressing phase and the universal scaling function, also known as cusp anomalous dimension, is pointed out in [17]. Assuming further so-called transcendentality principle [18, 19, 20], the unique all-order expression of the dressing phase was presented in [21]. In summary, significant progress has been achieved in formulating the exact AdS/CFT Bethe Ansatz equation valid for all regions of λ. One should keep it in mind that the Bethe Ansatz description is believed to be exact only when the length of spin chain is infinite.

Perhaps for the moment we should explain our knowledge on the gravity side of AdS/CFT correspondence, which also exhibits integrability at least in the classical level.

Metsaev and Tseytlin constructed closed superstring action on $\mathrm{AdS}_5 \times \mathrm{S}^5$, in the Green-Schwarz formalism with coset target space $SU(2,2|4)/[SO(1,4) \times SO(5)]$ [22] (see also [23]). Classical integrability of Metsaev-Tseytlin action was found by Bena, Polchinski, and Roiban, where they explicitly constructed one-parameter family of flat conserved currents [24].

Classical integrability enables us to study classical string solutions from an algebro-geometric approach called finite-gap formulation. This line of study started from the work on $\mathbb{R}_\mathrm{t} \times \mathrm{S}^3$ subspace [11] (see also [25]), extended to other subspaces of $\mathrm{AdS}_5 \times \mathrm{S}^5$ [26, 27], and to the whole spacetime in [28, 29]. In this formulation, every string solution is characterized by a spectral curve endowed with an Abelian integral called quasimomentum. We only have to choose suitable algebro-geometric data (called a finite-gap solution) such that they reproduce the conserved charges and the mode numbers of classical string solutions of our concern; this is called Riemann-Hilbert problem. As discussed in [11], the finite-gap approach turned out quite useful for direct comparison of the spectrum at the level of algebraic curves.

In principle, one can reconstruct classical string solutions from given algebro-geometric data. The reconstruction of analytic profile of general finite-gap solutions on $\mathbb{R}_\mathrm{t} \times \mathrm{S}^3$ has been done in [30, 31].

One is also able to compute one-loop quantum correction to classical string theory using finite-gap formulation, as is thoroughly studied in [32, 33, 34].

We are now ready for introducing explicit examples of AdS/CFT correspondence. One of the prominent predictions of the AdS/CFT is the exact matching of the spectra on both sides, namely conformal dimension of individual super Yang-Mills operator and energy of the corresponding string state. Due to the strong/weak nature of this correspondence, one has to invent sophisticated ways of comparison, two of which we will briefly review on in what follows.

BMN scaling limit:

The essence of BMN scaling limit is to rescale the 't Hooft coupling constant by length of a spin chain L, which is total R-charge of the operator in $su(2)$ sector, or by total angular momenta of a classical string J. The effective coupling becomes $\tilde{\lambda} \equiv \lambda/L^2$ or $\tilde{\lambda} \equiv \lambda/J^2$, which can be taken arbitrarily small irrespective of the value of λ. We also scale momentum of an operator/worldsheet momentum as $p \sim 1/L$ or $1/J$ to keep anomalous dimension/classical energy finite.

We assume that string energy $E(J, \lambda)$ and conformal dimension $\Delta(J, \lambda)$ can be expanded in powers of $\tilde{\lambda}$ in both near-BPS (BMN) and far-from-BPS sectors, as

$$E = J + c_1(J)\,\tilde{\lambda} + c_2(J)\,\tilde{\lambda}^2 + \ldots \quad \text{and} \quad \Delta = L + a_1(L)\,\tilde{\lambda} + a_2(L)\,\tilde{\lambda}^2 + \ldots\,, \tag{0.0.1}$$

which is called BMN scaling hypothesis. Under this assumption we are able to test a proposal of the AdS/CFT quantitatively, that is, to check $a_k \overset{?}{=} c_k$ $(k = 1, 2, \ldots)$.

Concrete examples of such correspondence have been found. For instance, it was shown that certain long composite operators of $\mathcal{N} = 4$ theory, expressed as solutions to Bethe Ansatz equation in thermodynamic limit, are dual to semiclassical spinning/rotating string solutions [35, 36, 37, 38, 39, 40] or pulsating string solutions [41, 42, 43]. Much nontrivial examples of correspondence are found between (elliptic) folded strings and "double contour" configurations of Bethe roots; between (elliptic) circular string and "imaginary root" configurations of Bethe roots [44, 45].

By perturbatively expanding the energy of elliptic strings, they found remarkable agreement with the super Yang-Mills counterpart up to two-loop in $\tilde{\lambda}$. At the three-loop level, however, the coefficients start to disagree, *i.e.* $a_3 \neq c_3$, which is known as the "three-loop discrepancy" [46, 47]. The origin of this mismatch can be attributed to the breakdown of BMN scaling hypothesis at higher orders of $\tilde{\lambda}$ [17, 21] (see also [48]).

Hofman-Maldacena limit:

Beisert considered central extension of the $\mathcal{N} = 4$ superconformal symmetry algebra for spin chains of infinite length, also known as asymptotic spin chain [8], and derived a nontrivial dispersion relation valid to all orders in λ [12]. The corresponding limit on string theory side is invented by Hofman and Maldacena (HM), where J is again taken to infinity with λ and p kept fixed [49].

The ground state of asymptotic spin chain is 'ferromagnetic' vacuum of Z's, and excitations over it are called magnons. Magnons are classified according to representations of the $su(2|2)^2$ algebra. For example, the fundamental representation of $su(2|2)^2$ algebra is composed of sixteen 'impurity' fields of $\mathcal{N} = 4$ theory, and there are also BPS boundstates of elementary magnons [50].

Let $Q(\geq 1)$ be the number of constituent magnons for BPS boundstates, then they obey the dispersion relation

$$\Delta - J_1 = \sqrt{Q^2 + f(\lambda)\sin^2\left(\frac{p}{2}\right)} \qquad (\Delta\,,\; J_1 \to \infty)\,, \tag{0.0.2}$$

where p is the momentum of the magnon bound state along the spin-chain. The function $f(\lambda)$ is left undetermined from the BPS relation alone. In view of gauge theory, it should be $f(\lambda) = \lambda/\pi^2 + \mathcal{O}(\lambda^4)$. The dispersion relation (0.0.2) matches with the energy-spin relation of classical string solutions called (dyonic) giant magnons [49, 51, 52, 53, 54], if we set $f(\lambda) = \lambda/\pi^2$ and identify Q with the second angular momentum J_2.

There is close connection between (dyonic) giant magnons and (complex) sine-Gordon solitons. Under the reduction procedure found by Pohlmeyer, Lund, and Regge [55, 56, 57], giant magnon is mapped to the kink solution of sine-Gordon model, and dyonic giant magnon is to that of complex sine-Gordon model. The sine-Gordon point of view directs our attention to scattering of (dyonic) giant magnons taking place on worldsheet rather than in spacetime, and to compare S-matrix of worldsheet scattering with the S-matrix appearing in Bethe Ansatz equation discussed above [49, 58, 59].

It should be noted that one can compute S-matrix of worldsheet scattering from gauge-fixed sigma model on the whole $\mathrm{AdS}_5 \times \mathrm{S}^5$, and inspect symmetry governing the S-matrix of string theory such as factorization. In particular, Arutyunov, Frolov, and Zamaklar proposed string S-matrix which satisfies the standard Yang-Baxter equation [60], while gauge S-matrix of [12, 61] satisfies the twisted Yang-Baxter equation.

We have so far seen interesting examples in testing the correspondence between spin chains and classical strings, one is in BMN scaling limit and the other in Hofman-Maldacena limit. It would be then interesting to seek for more generic two-spin string solution interpolating both the BMN and the HM cases, which would give us further playground to test the AdS/CFT.

With this in mind, in Chapter 6 we construct a family of classical string solutions with large spins, by exploiting the relation between classical string action on $\mathbb{R}_t \times \mathrm{S}^3$ and complex sine-Gordon system. Starting from general elliptic solutions of complex sine-Gordon model, called helical-wave solutions, we construct analytical expression of the corresponding classical string solutions, which are shown to interpolate between two-spin folded/circular strings [38] and dyonic giant magnons [51].

Our solutions, which we will refer to as helical spinning strings, are written in terms of elliptic theta functions. From this fact one can foresee that they have clear interpretation from finite-gap point of view. Later helical strings are indeed reconstructed as finite-gap solutions, and it is shown that they are included in general two-cut finite-gap solutions in mathematical language [62]. In particular, it teaches us clearly how folded/circular strings and dyonic giant magnons are interpolated from the standpoint of algebraic curves.

Helical spinning strings are expected to cover a large part of strings dual to long composite operators in $su(2)$ sector, where the latter is characterized by large R-charges. Recall that in $\mathcal{N} = 4$ theory there are also operators of far smaller R-charges compared to its length. The string dual of such non-holomorphic operators are expected to have pulsating nature, as is understood from matching of the global charges or from an explicit example [42].

In Section 7, we investigate classical strings on $\mathbb{R}_t \times S^3$ with large winding numbers, rather than large spins. They are obtained by performing a transformation $\tau \leftrightarrow \sigma$, *i.e.* interchanging temporal and spatial coordinates of worldsheet, of helical spinning strings. We will refer to this transformation as the $\tau \leftrightarrow \sigma$ transformation, or just 2D transformation. As consequences of this $\tau \leftrightarrow \sigma$ map,

- Large spin states become large winding states.
- Rotating/spinning states become oscillating states.

Note that the first feature can be understood as analogue of T-duality, which exchanges (angular) momenta with winding numbers. We refer to the 2D-transformed helical spinning strings as helical oscillating strings, so as to remind us of the second feature. It turns out that helical oscillating strings also interpolate various classical string states of pulsating/oscillating nature known so far, such as pulsating strings [41, 42, 43] and single-spike solutions [63, 64].

Helical oscillating strings admit a finite-gap interpretation similar to helical spinning ones. The $\tau \leftrightarrow \sigma$ operation in conformal gauge corresponds to rearranging the configuration of cuts with respect to two singular points of the spectral parameter plane. An alternative description of $\tau \leftrightarrow \sigma$ operation is to swap the definition of quasi-momentum and so-called quasi-energy. Both helical spinning and oscillating strings thus exhaust all possible two-cut finite-gap solutions on $\mathbb{R}_t \times S^3$.

In Chapter 8, we also construct helical string solutions on $AdS_3 \times S^1$ by means of analytic continuation.

The current framework of all-loop Bethe Ansatz equations equipped with the dressing phase is not the full answer towards validation of AdS/CFT correspondence. A major limitation we face is that it correctly reproduce the super Yang-Mills result only when the length of spin chain L is large enough. For spin chains with finite size, the Bethe Ansatz equations do not account for wrapping interactions [7], which possibly arise from the order of λ^L as higher-genus diagrams [65]. In fact, the Bethe Ansatz prediction is found to disagree with the BFKL prediction [66, 67, 68] in [69]. Recently, it is found that the wrapping effects for the four-loop anomalous dimensions of certain short operators induce terms of higher degrees of transcendentality [70, 71]. At strong coupling, the Bethe Ansatz also fails to reproduce the exact expression for one-loop correction to energy-spin relation in string theory; when angular momenta are finite; there is deviation from the exact answer which is exponentially suppressed in angular momentum [72]. Note also that exponential correction to the energy-spin relation has already appeared at classical level, as finite-size correction to giant magnon solutions [52, 73].[1]

[1] In conformal gauge, the "size" can be interpreted also as the circumference of worldsheet.

It was argued in [74] that the exponential finite-size correction at strong coupling is related to the wrapping interaction at weak coupling, by using Thermodynamic Bethe Ansatz approach [75] or by the Lüscher formula [76, 77, 78]. Janik and Łukowski have elaborated this argument, assuming that Lüscher's argument can be applied to the non-relativistic dispersion relation

$$\varepsilon(p) = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)} \,. \tag{0.0.3}$$

They find that the generalized Lüscher formula reproduces the leading finite- J_1 correction to the dispersion relation of giant magnons correctly, after careful computation of the contribution from the dressing phase [79].

In Chapter 9, we extend their analysis and study the leading finite-size correction to magnon boundstates and dyonic giant magnons, using the dispersion relation (0.0.2) rather than (0.0.3). Firstly we analyze the asymptotic behavior of helical spinning strings in the limit when they approach an array of dyonic giant magnons, and determined the leading finite- J_1 correction to the energy-spin relation. Secondly, we apply the generalized Lüscher formula to the situation in which the incoming particle is magnon boundstate. Because the Lüscher formula applied to the all-loop S-matrix is valid at arbitrary value of λ, it will also reproduce quantum corrections to the 'finite-J dyonic giant magnons'.

The finite-size correction predicted by the Lüscher formula consists of what are called F-term and μ-term, and the latter is sensitive to the pole structure of the S-matrix in infinite-size theory. The study of pole structure of the BHL/BES dressing phase has started in [80, 81], but the analyticity of the dressing phase on the whole rapidity torus is not yet completely known. To determine the poles relevant to computation of μ-term, we use heuristic reasoning based on the argument similar to [81, 82]. Since our prescription reproduces those of classical strings, it provides rigid information on the location of poles of the conjectured S-matrix, albeit only around the nearest from the real axis.

Since there is a vast amount of literature around this subject, it would be helpful for readers to introduce several review articles. The AdS/CFT correspondence in general is reviewed, for example, in [83, 84, 85]. Application of integrability methods to AdS/CFT correspondence is summarized *e.g.* in [86, 87, 88, 89, 90, 91, 92, 93, 94, 95].

Organization of the thesis

This thesis is composed of three parts. The first part, Chapter 1 to 5, is a review on developments of AdS/CFT correspondence, discovery of integrability and application of it. The second part, Chapter 6 to 8, deals with construction of classical string solutions and its finite-gap interpretation. The third part, Chapter 9, discusses finite-size effects for dyonic giant magnons or magnon boundstates. Content of each Chapter is summarized as follows:

1. Notion of the AdS/CFT correspondence, also called Maldacena conjecture, is introduced. We explain how we arrive at such conjecture from a string theoretical point of view.

2. We discuss the integrability in $\mathcal{N} = 4$ super Yang-Mills theory, which arises when we diagonalize anomalous dimension matrix of gauge-invariant local operators.

3. We discuss the integrability in classical superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. Construction of finite-gap solutions is also reviewed.

4. Examples of finite-gap solutions are given. They can be regarded both as particular classical string solutions and as solutions of Bethe Ansatz equation in the thermodynamic limit.

5. Correspondence for the systems of infinite size is summarized. It is believed that S-matrix conjectured to all orders of the 't Hooft coupling can explain both sides exactly.

6. We study a family of classical string solutions (with large spins) on $\mathbb{R}_\mathrm{t} \times \mathrm{S}^3$ subspace of $\mathrm{AdS}_5 \times \mathrm{S}^5$ background, which we call helical strings, from perspective of Complex sine-Gordon model. We show they interpolate various known rigid configuration of strings with two spins.

7. We study a family of classical string solutions on $\mathbb{R}_\mathrm{t} \times \mathrm{S}^3$ subspace of $\mathrm{AdS}_5 \times \mathrm{S}^5$ background which have oscillating nature. They are obtained from helical (spinning) strings by interchanging worldsheet time and space coordinates.

8. We perform analytic continuation to make helical strings on $\mathrm{AdS}_3 \times \mathrm{S}^1$.

9. We compute finite-size corrections to dyonic giant magnons in two ways. One is to examine the asymptotic behavior of helical (spinning) strings and the other is to apply

generalized Lüscher formula of [79] to the case in which incoming particles are bound-states.

We then summarize our results, refer to some topics we do not incorporate in this thesis, and discuss open questions.

Appendix A is devoted to explanation of our notation for elliptic functions and elliptic integrals. Appendix B deals with the reduction between classical string on $\mathbb{R}_{\mathrm{t}} \times \mathrm{S}^3$ and complex sine-Gordon theory. In Appendix C, details for computation of finite-size correction are discussed.

Chapter 6 is based on the author's paper [96] done in collaboration with Keisuke Okamura. Chapter 7 is partially based on the paper [97], done in collaboration with Hirotaka Hayashi, Keisuke Okamura and Benoît Vicedo. Chapter 8 is based on appendix A of the paper [97]. Chapter 9 is based on the paper [98], done in collaboration with Yasuyuki Hatsuda. The review part is taken from various literature.

Chapter 1

The AdS/CFT correspondence

The idea of AdS/CFT correspondence was first proposed in the Maldacena's paper [1]. Among various works to check his proposal, one of the best studied version is about the one between $\mathcal{N} = 4$ super Yang-Mills theory and superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$ in the large N limit. In this chapter we briefly review these two theories in turn, and draw a rough sketch of Maldacena's proposal.

1.1 $\mathcal{N} = 4$ super Yang-Mills theory

The $\mathcal{N} = 4$ super Yang-Mills theory has the largest possible supersymmetry among supersymmetric gauge field theories in four dimensions. This theory is also an important example of superconformal field theory in four dimensions. We will summarize these features below.

1.1.1 Conformal field theory

As a preliminary, we review basic properties of conformal field theories [99, 100].

Let us start from Poincaré invariant field theories in d spacetime dimensions. Poincaré algebra contains two types of generators called momentum P_μ and angular momentum $M_{\mu\nu}$. They obey the following commutation relations of $so(1, d-1)$ algebra

$$[P_\mu\,, P_\nu] = 0\,, \tag{1.1.1}$$

$$[M_{\mu\nu}\,, P_\rho] = -i\,(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu)\,, \tag{1.1.2}$$

$$[M_{\mu\nu}\,, M_{\rho\sigma}] = -i\,\{\eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - (\mu \leftrightarrow \nu)\}\,. \tag{1.1.3}$$

When they act on a field $\phi(x)$, they can be realized as derivative operation

$$\hat{P}_\mu\, \phi(x) = i\partial_\mu \phi(x)\,, \qquad \hat{M}_{\mu\nu}\, \phi(x) = [i\,(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu}]\, \phi(x)\,. \tag{1.1.4}$$

Note that the first relation implies $\phi(x) = e^{-i\hat{P}x} \phi(0) e^{i\hat{P}x}$.

Conformal transformation is defined as the coordinate transformation that leaves the metric invariant up to overall scale,

$$x^\mu \ \mapsto\ y^\mu(x)\,, \qquad ds^2 \ \mapsto\ ds'^2 = \ \Omega(x)^2 ds^2\,. \tag{1.1.5}$$

Such transformation is generated by infinitesimal transformation $\delta x^\mu = \xi^\mu$ which satisfies conformal Killing equation

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{d}\,\eta_{\mu\nu}\,(\partial\cdot\xi)\,. \tag{1.1.6}$$

If $d \neq 2$ the solution of conformal Killing equation includes, besides (1.1.4),

$$D = ix\cdot\partial\,, \qquad K_\mu = i\left\{2x_\mu\,(x\cdot\partial) - x^2\partial_\mu\right\}, \tag{1.1.7}$$

where D is called dilatation and K_μ is special conformal transformation (or conformal boost). They act on a field $\phi(x)$ as

$$\hat{D}\,\phi(x) = \{ix\cdot\partial + \Delta\}\,\phi(x)\,, \tag{1.1.8}$$
$$\hat{K}_\mu\,\phi(x) = i\left\{2x_\mu\,(x\cdot\partial) - x^2\partial_\mu\right\}\phi(x) + e^{i\hat{P}x}\left(\hat{K}_\mu\,\phi(0)\right)e^{-i\hat{P}x}\,. \tag{1.1.9}$$

The commutation relations among $D, P_\mu\,, K_\mu\,, M_{\mu\nu}$ are computed from derivative representation (1.1.4) and (1.1.9). The d dimensional conformal symmetry is isomorphic to $so(2,d)$ algebra through identification

$$D = M_{d+1,d}\,, \qquad P_\mu = M_{d,\mu} + M_{d+1,\mu}\,, \qquad K_\mu = M_{d,\mu} - M_{d+1,\mu}\,. \tag{1.1.10}$$

Spectrum of conformal field theory is classified with regard to the representation of conformal algebra, and unitary representations are labeled by their spin and conformal dimension Δ of the highest weight state. In general, unitary representation of conformal algebra $so(d,2)$ is infinite dimensional. If we regard the superconformal generator K_μ as a raising operator and the momentum P_μ as a lowering operator, the highest state (or conformal primary) is defined by the condition

$$\hat{K}_\mu\,\mathcal{O}(0) = 0. \tag{1.1.11}$$

Descendants are obtained by acting P_μ on the primary state.

The invariance under the full conformal group severely restricts the form of correlation functions. Two point function of an operator of conformal dimension Δ is given by

$$\langle \mathcal{O}^\dagger(x)\mathcal{O}(y)\rangle = \frac{1}{|x-y|^{2\Delta}}\,, \tag{1.1.12}$$

and three point function is given by

$$\langle \mathcal{O}_1^\dagger(x)\mathcal{O}_2(y)\mathcal{O}_3(z)\rangle = \frac{C^1{}_{23}}{|y-z|^{\Delta_2+\Delta_3-\Delta_1}\,|z-x|^{-\Delta_2+\Delta_3+\Delta_1}\,|x-y|^{\Delta_2-\Delta_3+\Delta_1}}\,, \tag{1.1.13}$$

where $C^1{}_{23}$ is the leading OPE coefficient

$$\mathcal{O}_2(y)\mathcal{O}_3(z) = \sum_k \frac{C^k{}_{23}}{|y-z|^{\Delta_2+\Delta_3-\Delta_k}}\, \mathcal{O}_k(z) + \dots\,. \tag{1.1.14}$$

In conformal field theories, we also have the state-operator correspondence

$$|\mathcal{O}\rangle \equiv \lim_{x\to 0}\, \mathcal{O}(x)\, |0\rangle\,. \tag{1.1.15}$$

1.1.2 $\mathcal{N}=4$ Lagrangian

The $\mathcal{N}=4$ super Yang-Mills theory has the unique Lagrangian, including matter content and coupling. All elementary fields are in adjoint representation of the gauge group (which we take as $SU(N)$) and the coupling is proportional to the structure constant.[1]

The $\mathcal{N}=4$ Lagrangian reads [85]

$$\begin{aligned} S = -\frac{1}{2g_{\text{YM}}^2}\int d^4x\ \text{tr}\Big\{ &\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^i D^\mu\phi^i + \frac{1}{2}[\phi^i,\phi^j][\phi^i,\phi^j] \\ &+ i\left(\bar\lambda_A\,\bar\sigma^\mu D_\mu\lambda^A + \lambda^A\sigma^\mu D_\mu\bar\lambda_A\right) + [\bar\phi_{AB},\lambda^A]\lambda^B - [\phi^{AB},\bar\lambda_A]\bar\lambda_B\Big\}, \end{aligned} \tag{1.1.16}$$

where A_μ are gauge fields, ϕ^i are real scalars and λ^A are Weyl spinors. The fields $\bar\phi_{AB}$ and ϕ^{AB} are introduced via

$$\phi^i = \frac{1}{2}\,(\bar\tau^i)_{AB}\,\phi^{AB} = \frac{1}{2}\,(\tau^i)^{AB}\,\bar\phi_{AB} \qquad (i=1,\dots,6\,,\quad A,B=1,\dots,4) \tag{1.1.17}$$

where $(\bar\tau^i)_{AB}$ and $(\tau^i)^{AB}$ are the γ matrices of $so(6)$, antisymmetric with respect to A,B. The Lagrangian (1.1.16) also follows from trivial dimensional reduction of $d=10,\mathcal{N}=1$ free super Yang-Mills Lagrangian, through identification of

$$A_\mu^{(10)} = A_\mu^{(4)}\,, \qquad A_{3+i}^{(10)} = \phi_i^{(4)}\,. \tag{1.1.18}$$

The Lagrangian (1.1.16) can be rewritten using $\mathcal{N}=1$ superfield formalism as

$$\begin{aligned} S = \frac{1}{2g_{\text{YM}}^2}\int d^4x\ \text{tr}\Big\{ &\frac{1}{4}\int d^2\theta\, d^2\bar\theta\, \sum_{a=1}^3 \text{tr}\left(\bar\Phi^a e^{-2V}\Phi^a\right) + \frac{1}{8}\int d^2\theta\ \text{tr}\left(W_A^2\right) + \text{h.c.} \\ &+ \frac{i}{2}\int d^2\theta\ \text{tr}\left(\Phi^1\left[\Phi^2,\Phi^3\right]\right) + \text{h.c.}\Big\}, \end{aligned} \tag{1.1.19}$$

with Φ^a are chiral superfields, V is a gauge superfield, and $W_A \equiv \frac{i}{4}\bar D^2(e^{-V}D_A\, e^V)$ [102, 103].

The above Lagrangian is invariant under supersymmetry transformation

$$\delta\phi^i = (\bar\tau^i)_{AB}\,\lambda^{\alpha A}\eta_\alpha{}^\beta + (\tau^i)^{AB}\,\bar\eta_{\dot\alpha A}\bar\lambda^{\dot\alpha}{}_B \tag{1.1.20}$$

$$\delta\lambda_\alpha{}^A = -\frac{1}{2}\,F^-_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta\,\eta_\beta{}^B + i\,\not{D}_{\alpha\dot\alpha}\,\phi^{AB}\bar\eta^{\dot\alpha}{}_B + \frac{1}{2}\,[\phi^i,\phi^j]\,(\tau_{ij})^A{}_B\,\eta_\alpha{}^B \tag{1.1.21}$$

$$\delta A_\mu = -i\lambda^{\alpha A}(\sigma^\mu)_{\alpha\dot\alpha}\,\bar\eta^{\dot\alpha}{}_A - i\eta^{\alpha A}(\sigma^\mu)_{\alpha\dot\alpha}\,\bar\lambda^{\dot\alpha}{}_A\,. \tag{1.1.22}$$

[1]Some arguments on generalization of the $\mathcal{N}=4$ Lagrangian are found in [101].

Furthermore, this theory has vanishing one-loop β function, and thus lives exactly on the superconformal fixed point. It is believed that the theory is exactly conformal invariant as long as $\langle\phi^i\rangle = 0$. [2]

The superconformal symmetry of $\mathcal{N} = 4$ theory form $psu(2,2|4)$ Lie superalgebra, which is the global symmetry of this theory. The bosonic subgroup of $psu(2,2|4)$ is $su(2,2) \times su(4)_R$, and $su(4)_R \simeq so(6)_R$ is the R-symmetry. Under this R-symmetry, scalars transform in **6** representation, Weyl fermions in **4** representation, and gauge bosons in trivial representation.

The $\mathcal{N} = 4$ superconformal algebra is generated by supercharges Q_A, $\bar{Q}^A$, superconformal generators S^A, $\bar{S}_A$, and R-symmetry generators $T^A{}_B$, in addition to the generators of bosonic conformal algebra $\{D, P_\mu, K_\mu, M_{\mu\nu}\}$. Commutation relations are summarized as follows: The fermionic generators satisfy

$$\{Q_A, \bar{Q}^B\} = \delta^B_A\, \sigma^\mu P_\mu\,, \qquad \{S^A, \bar{S}_B\} = \delta^A_B\, \sigma^\mu K_\mu\,, \tag{1.1.23}$$

$$\{S^A, Q_B\} = \delta^A_B \left(\frac{1}{2}\,\sigma^{\mu\nu} M_{\mu\nu} + D\right) + T^A{}_B\,, \tag{1.1.24}$$

$$\{Q_A, Q_B\} = \{S^A, S^B\} = \{S^A, \bar{Q}^B\} = 0. \tag{1.1.25}$$

The $su(4)_R$ rotation $T^A{}_B$ commutes with all generators of bosonic conformal algebra, and the commutation between $T^A{}_B$ and fermionic generators are

$$[T^A{}_B, Q_C] = \delta^A_C\, Q_B - \frac{1}{4}\,\delta^A_B\, Q_C\,, \qquad [T^A{}_B, S^C] = \delta^C_B\, S^A - \frac{1}{4}\,\delta^A_B\, S^C\,. \tag{1.1.26}$$

Finally, commutation between bosonic and fermionic generators satisfy

$$[M_{\mu\nu}, Q_A] = \frac{1}{2}\,\sigma_{\mu\nu} Q_A\,, \qquad [K_\mu, Q_A] = \sigma_\mu \bar{S}_A\,, \qquad [D, Q_A] = \frac{1}{2}\, Q_A\,, \tag{1.1.27}$$

$$[M_{\mu\nu}, S^A] = \frac{1}{2}\,\sigma_{\mu\nu} S^A\,, \qquad [P_\mu, S^A] = \sigma_\mu \bar{Q}_A\,, \qquad [D, S^A] = -\frac{1}{2}\, S^A\,. \tag{1.1.28}$$

If one decomposes the $\mathcal{N} = 4$ multiplet in terms of $\mathcal{N} = 1$, it breaks up to three chiral multiplets and one gauge multiplets. Let us denote them by

$$(Z,\ \lambda_Z)\,, \quad (W,\ \lambda_W)\,, \quad (Y,\ \lambda_Y)\,, \quad (A_\mu,\ \lambda_A)\,,$$
$$\text{with} \qquad Z \equiv \phi^1 + i\phi^2\,, \qquad W \equiv \phi^3 + i\phi^4\,, \qquad Y \equiv \phi^5 + i\phi^6\,. \tag{1.1.29}$$

Let H_1, H_2, H_3 be Cartan generators of $so(6)_R$ and let J_1, J_2, J_3 be their eigenvalues. By looking upon the action of the Cartan generators onto $su(4)$ spinors, we can appropriately assign R-charge to the above fields in the following manner:

[2] See recent papers [104, 105], for vanishing of β functions at all orders of perturbation theory.

Fields	Z	W	Y	λ_Z	λ_W	λ_Y	A_μ	λ_A
J_1	1	0	0	1/2	-1/2	-1/2	0	-1/2
J_2	0	1	0	-1/2	1/2	-1/2	0	-1/2
J_3	0	0	1	-1/2	-1/2	1/2	0	-1/2

Table 1.1: R-charges of $\mathcal{N}=4$ elementary fields.

1.1.3 Large N limit

In [106], 't Hooft made an observation that $SU(N)$ Yang-Mills theory exhibits stringy behavior in the limit

$$N \to \infty\,, \qquad \text{with } \ \lambda \equiv N g_{\rm YM}^2 \ \text{ fixed.} \tag{1.1.30}$$

Planar diagrams give the dominant contribution in this limit, which is quite analogous to perturbation of string theory in terms of string coupling constant g_s . For this reason, the work of 't Hooft is considered as a remarkable precursor of the AdS/CFT correspondence.

In the $\mathcal{N}=4$ case, it is easy to compute the topology of Feynman diagrams. Recall that $\mathcal{N}=4$ theory has 3-point vertex of order $g_{\rm YM}$, and 4-point vertex of order ${g_{\rm YM}}^2$. Also, keep it in mind that all elementary fields of $\mathcal{N}=4$ are in adjoint representation, which is approximately regarded as product of fundamental and anti-fundamental representations at large N. If one draws a single line for propagation of fields with an (anti-)fundamental index, each propagator of adjoint field is drawn as a double line. Each loop of a single line indicates trace over fundamental representations, giving contribution of order N.

Suppose a diagram consists of V_3 3-point vertices, V_4 4-point vertices and L loops. From the above argument, this diagram is of order $N^L {g_{\rm YM}}^{V_3+2V_4}$. If we analyze this diagram from graphical point of view, we see that

$$V_3 + 2V_4 = \sum_{n=3}^{4} n V_n - 2\sum_{n=3}^{4} V_n = 2E - 2P\,, \qquad L = F \equiv \chi - P + E\,, \tag{1.1.31}$$

where P, E, F are the number of points, edges, faces, respectively; and χ is Euler number of the diagram. Now we can estimate the contribution of this diagram as

$$N^L {g_{\rm YM}}^{V_3+2V_4} = N^{\chi-P+E} {g_{\rm YM}}^{2E-2P} = N^\chi \lambda^{E-P}\,, \tag{1.1.32}$$

and therefore planar diagrams, $\chi = 2$, contributes the most in the 't Hooft limit (1.1.30).

1.2 Supergravity and AdS$_5 \times$ S^5 spacetime

We turn our attention to the other side of AdS/CFT correspondence. As we see in later sections, it is conjectured that $\mathcal{N}=4$ super Yang-Mills theory is a dual description of superstring on

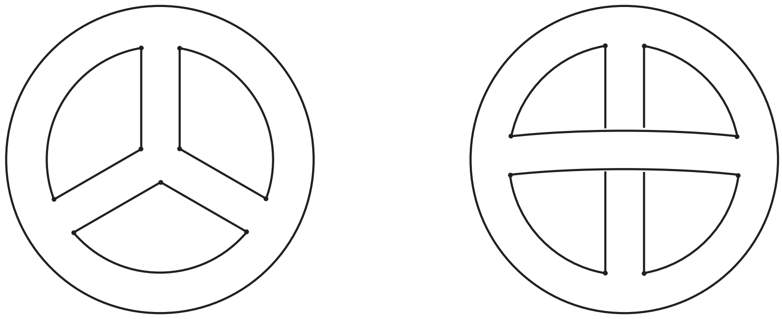


Figure 1.1: Left: An example of planar diagram of order $N^4 g_{\rm YM}^4 = N^2\lambda^2$. Right: An example of nonplanar diagram of order $N^2 g_{\rm YM}^4 = \lambda^2$.

$\mathrm{AdS}_5 \times \mathrm{S}^5$. Both descriptions can be understood as a particular limit of string theoretical description of D-branes. To support this way of understanding, in this section we look into how $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime arises as supergravity description of D3-branes.

1.2.1 Black 3-brane solution

Supergravity is an effective description of string theory when length scale of the system is much larger than string length $\ell_s \equiv \sqrt{\alpha'}$. D-branes are identified as extended black (having horizon) objects in supergravity description, and in many cases, BPS D-branes correspond to extremal black solutions.

We study so-called black 3-brane solution of type IIB supergravity in ten dimensions. Since we study the solution only classically, we can neglect fermions in the action. We also assume the solution does not have NS-NS 3-form flux, R-R 1-form and 3-form fluxes. Thus, the action we are going to extremize is, in string frame,

$$S = \frac{1}{\ell_s^8} \int d^{10}x \sqrt{-g}\, e^{-2\phi} \Big\{ R + 4\,(\nabla\phi)^2 \Big\} - \int F_{p+2} \wedge * F_{p+2}. \tag{1.2.1}$$

with $p = 3$. In addition, self-duality condition should be imposed on 5-form flux F_5. As discussed in [107], this system has the following extremal black 3-brane solutions:

$$ds^2 = f(\rho)^{1/2} \Big(- dt^2 + \sum_{i=1}^{3} dx_i^2 \Big) + \frac{d\rho^2}{f(\rho)^2} + \rho^2 d\Omega_5^2\,, \qquad f(\rho) = 1 - \left(\frac{r_H}{\rho}\right)^4 \tag{1.2.2}$$

$$F_5 = Q\,(\epsilon_5 + *\epsilon_5)\,, \qquad Q = \frac{2 r_H^4}{g_s \ell_s^4}\,, \tag{1.2.3}$$

$$\phi = \phi_0 \quad (\text{constant})\,, \qquad e^{\phi_0} \equiv g_s\,, \tag{1.2.4}$$

where ϵ_5 is the volume element on the unit 5-sphere, normalized as

$$\int_{S^5} \epsilon_5 = \frac{2\pi^3}{\Gamma(3)} = \big(\text{volume of } \mathrm{S}^5\big)\,. \tag{1.2.5}$$

Normalization of Q follows from the Einstein equation with constant dilaton

$$\frac{1}{{g_s}^2{\ell_s}^8}\, R_{MN} \sim F_{ML_1L_2L_3L_4}{F_N}^{L_1L_2L_3L_4}\,, \tag{1.2.6}$$

We have to impose the Dirac quantization condition on RR 5-form flux

$$\int_{S^5} *F_5 = \int_{S^5} F_5 = N\,,^3 \tag{1.2.7}$$

which determines the location of horizon as, up to some numerical constant,

$$r_H^4 \propto N g_s \ell_s^4 \qquad \Leftrightarrow \qquad N g_s \propto \frac{r_H^4}{\ell_s^4} = \frac{r_H^4}{\alpha'^2}\,. \tag{1.2.8}$$

By the change of coordinates $r^4 \equiv \rho^4 - r_H^4$, the metric (1.2.2) turns into

$$ds^2 = H(r)^{-1/2}\Big(-dt^2 + \sum_{i=1}^3 dx_i^2\Big) + H(r)^{1/2}\Big(dr^2 + r^2 d\Omega_5^2\Big), \qquad H(r) = 1 + \Big(\frac{r_H}{r}\Big)^4\,. \tag{1.2.9}$$

Let us consider near-horizon limit of this metric. Since the horizon is at $r = 0$, this limit is achieved by $r_H \gg r$, or by replacing $H(r)$ with $(r_H/r)^4$. The metric (1.2.9) then becomes

$$\begin{aligned} ds^2 &= \frac{r^2}{r_H^2}\Big(-dt^2 + \sum_{i=1}^3 dx_i^2\Big) + r_H^2\Big(\frac{dr^2}{r^2} + d\Omega_5^2\Big), \\ &= r_H^2\left[U^2\Big(-dt^2 + \sum_{i=1}^3 dx_i^2\Big) + \frac{dU^2}{U^2} + d\Omega_5^2\right], \qquad U \equiv \frac{r}{r_H^2}\,. \end{aligned} \tag{1.2.10}$$

As one finds below, this is the metric of $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime with the radius of AdS_5 and S^5 equal to r_H.

1.2.2 AdS spacetime

Here we summarize basic facts about AdS spacetime.

The simplest definition of AdS spacetime is by embedding $\mathrm{AdS}_d \subset \mathbb{R}^{2,d-1}$:

$$-(Y^0)^2 + \sum_{i=1}^{d-1}(Y^i)^2 - (Y^d)^2 = -R^2. \tag{1.2.11}$$

The parameter R is called radius of AdS_d, and this parametrization is called global coordinates. The metric has $so(2,d)$ isometry, and is given by

$$ds^2 = -(dY^0)^2 + \sum_{i=1}^{d-1}(dY^i)^2 - (dY^d)^2\,. \tag{1.2.12}$$

[3] The integral $\int_{\partial V} *F = \int_V d*F = \int_V j$ is often called electric charge by analogy with electrodynamics.

It is convenient to rewrite (1.2.12) in terms of polar coordinates which we define as

$$Y^0 + iY^d = R\cosh\rho\, e^{it}\,, \qquad Y^i = R\sinh\rho\,\Omega^i\,, \tag{1.2.13}$$

where Ω^i parametrizes S^{d-1} . The metric becomes

$$ds^2 = R^2\left(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega^2\right). \tag{1.2.14}$$

The boundary of AdS is located at $\rho \to \infty$.

The coordinate $t \in \mathbb{R}$ is called AdS time. If one regarded the parametrization (1.2.13) as imposing periodicity on t, one would encounter closed timelike curve. To maintain the causality of spacetime, we define the AdS time t by taking the universal covering in (1.2.13), so that two points t and $t + 2\pi$ refer to different points of spacetime.

Poincaré coordinates are useful for relating bulk theory with boundary theory. In Poincaré coordinates, the AdS metric is written as

$$ds^2 = R^2\left(\frac{dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu}{z^2}\right), \qquad \eta_{\mu\nu} = \mathrm{diag}\,(-1, +1, \ldots, +1)\,. \tag{1.2.15}$$

The region $z = 0$ is the boundary of AdS space and $z = \infty$ is called horizon. By change of coordinate $u = 1/z$, it becomes

$$ds^2 = R^2\left(\frac{du^2}{u^2} + u^2\,\eta_{\mu\nu}dx^\mu dx^\nu\right). \tag{1.2.16}$$

This is the metric we encountered in the last subsection (1.2.10).

The relation between global and Poincaré coordinates (1.2.16) is given as follows:

$$\begin{aligned}
Y^0 &= \frac{1}{2u}\Big\{1 + u^2\left(R^2 + \eta_{\mu\nu}x^\mu x^\nu\right)\Big\}\,,\\
Y^i &= Rux^i \qquad (i = 1, \ldots d-2)\,,\\
Y^{d-1} &= \frac{1}{2u}\Big\{1 - u^2\left(R^2 - \eta_{\mu\nu}x^\mu x^\nu\right)\Big\}\,,\\
Y^d &= Rut\,,
\end{aligned} \tag{1.2.17}$$

Note that the two points u and $-u$ are indistinguishable in Poincaré coordinates, while they are different in global coordinates as is clear from the relation $u = (Y^0 - Y^{d-1})/R^2$. Thus, Poincaré coordinates only covers one half of the hyperboloid (1.2.11).

1.3 Maldacena conjecture

In [108], Polchinski showed that D-branes, defined as extended objects at which open strings can end, carry RR charges. As we saw in Section 1.2.1, the black 3-brane solution has the background RR flux. It is natural to think of it as sourced by N-sheets of D3-branes.

This situation is quite interesting from purely closed string point of view. Because closed strings cannot distinguish D-branes with curved backgrounds (or gravitational potential) sourced by D-branes, they feel as if D-branes are dissolved into curved background with RR-flux. On the other hand, open strings are not so sensitive to the spacetime curvature, for no massless mode of open strings couple to gravity. Along this reasoning, one will be able to promote the above observation to the following (rather surprising) statement:

$$\begin{array}{ccc} \text{D-branes on flat space} & \multirow{2}{*}{=} & \text{Strings on curved spacetime} \\ \text{(open-like description)} & & \text{(closed-like description)} \end{array}$$

This relation can also be considered as realization of open/closed duality in string theory, or as two complementary points of view on physics of D-branes. In general, $U(N)$ gauge field theory is realized as massless open string excitations on N coincident D-branes. Hence, the above statement suggests the duality between gauge theory on flat spacetime and string theory on curved spacetime.

It is interesting to try taking $\alpha' \to 0$ limit in both ways of description. Pure gauge field theory can be realized on the D-brane side, because gravitational (or bulk-boundary) interaction decouples from 4-dimensional theory on D-branes in this limit. As a concrete example, in [1] Maldacena claimed that type IIB superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime is dual to $\mathcal{N} = 4$ super Yang-Mills theory in the large N limit, which is now referred to as the AdS/CFT correspondence.

Let us explain the decoupling limit of Maldacena in detail. On gauge theory side, we consider the limit

$$\alpha' \to 0 \qquad \langle \phi^i \rangle \ : \text{fixed}. \tag{1.3.1}$$

D-brane tension become infinite in this limit, and no massive closed string modes can be excited on the branes. In (1.3.1), we kept the vacuum expectation value of scalar fields finite. It corresponds to keeping finite the mass of open string modes stretched between branes, which is inversely proportional to separation between branes. Note that the $\mathcal{N} = 4$ theory is in superconformal phase when $\langle \phi^i \rangle = 0$.

On supergravity side, we take the corresponding limit

$$\alpha' \to 0 \qquad U \equiv \frac{r}{\alpha'} \ : \text{fixed}. \tag{1.3.2}$$

The parameter U has dimension of $(\text{mass})^1$, in agreement with the dimension of ϕ^i. Actually, this is the near-horizon limit of black 3-brane solution we took in (1.2.10). Note that there is a factor of α'^2 in the right hand side of (1.2.10), but this factor cancels out with the factor $1/\alpha'^4$ appearing in the action (1.2.1). Stringy excitations do not decouple in this limit; it just modifies the background spacetime to $\mathrm{AdS}_5 \times \mathrm{S}^5$.

Next let us compare the parameters of both theories. Recall that both string coupling and four dimensional Yang-Mills coupling are dimensionless constants. Since gauge bosons are equivalent to massless open string modes, we must have

$$g_s = g_{\rm YM}^2 \,. \tag{1.3.3}$$

If we introduce the 't Hooft coupling by (1.1.30), it follows that

$$\lambda \equiv N g_{\rm YM}^2 = N g_s = \frac{R^4}{\alpha'^2} \qquad R = (\text{radius of AdS}_5) = \big(\text{radius of S}^5\big) \ , \tag{1.3.4}$$

where we used the relation (1.2.8). Now it is clear that the AdS/CFT correspondence is strong/weak duality with respect to the 't Hooft coupling constant. Perturbative computation on gauge side is valid for $\lambda = N g_{\rm YM}^2 \ll 1$, whereas on gravity side classical approximation is valid for $\lambda = R^4/\alpha'^2 \gg 1$.

Since the above explanation of the AdS/CFT correspondence is quite intuitive, we can also give critical opinions. For instance, one cannot neglect backreaction of D-branes to the geometry when the number of D-branes N becomes large. Under such situation, it is not clear whether $U(N)$ gauge theory on flat space is realized on D-branes. It is also argued, taking into account the backreaction issue, that the dual gauge theory will live in the boundary of AdS_5, giving holographic description of bulk physics. From this point of view, relation to D-branes and open/closed duality cannot clearly be seen. Nonetheless, lots of evidences for Maldacena conjecture have been reported.

An important support for Maldacena conjecture is the correspondence of global symmetry. Both $\mathcal{N}=4$ super Yang-Mills and superstring on $\text{AdS}_5 \times \text{S}^5$ have $PSU(2,2|4)$ superconformal symmetry, whose bosonic subgroup is $SO(2,4) \times SO(6)$.

As we have already seen, Maldacena conjecture sounds very plausible. However, it is very difficulty to give a rigorous proof of Maldacena conjecture. One difficulty is that it is strong/weak duality, and other difficulties lie in:

- Quantizing superstring on $\text{AdS}_5 \times \text{S}^5$ exact in α', due to the background RR flux.
- Studying the property of $\mathcal{N}=4$ theory beyond a few orders of perturbation.
- Predicting how the two theories correspond with each other in a precise manner.

An answer to the last problem is proposed by Gubser, Klebanov, Polyakov and Witten [2, 3]. They interpreted AdS/CFT as the correspondence between bulk supergravity theory and CFT living on the boundary, and argued that correlation functions of both theories should obey certain relation.

To explain the GKP-Witten relation, let us consider a bulk supergravity field ϕ whose boundary value is fixed at $\phi = \phi_0$. We assume ϕ_0 couples to some operator $\mathcal{O}$ of boundary CFT, as $\int \phi_0 \, \mathcal{O}$. In the boundary theory, the quantity $\exp\big(\int \phi_0 \, \mathcal{O}\big)$ is regarded as generating functional for correlation functions of $\mathcal{O}$'s. In the bulk theory, such quantity can be regarded as a source term of ϕ in the effective action. Thus, we arrive at their proposal

$$\left\langle \exp\left(\int_{\mathrm{CFT}} d^4x\ \phi_0\,\mathcal{O}\right)\right\rangle = Z_{\mathrm{bulk}}\,(\phi)\,, \tag{1.3.5}$$

where the right hand side is the partition function of bulk supergravity. In the classical approximation, it becomes

$$Z_{\mathrm{bulk}}\,(\phi) = \exp\left(-I(\phi)\right) \qquad \phi\Big|_{\mathrm{boundary}} = \phi_0\,, \tag{1.3.6}$$

where $I(\phi)$ is classical supergravity action evaluated at its minimum. In superstring theory, the quantity Z_{bulk} should be considered as the partition function of target space (not of worldsheet).

The relation (1.3.5) sheds light on how to test Maldacena conjecture. However, there still remains a problem on how one can find the correspondence between ϕ and $\mathcal{O}$. It is true that one can compare the spectra based on representation theoretical arguments, but it is generally difficult to compare their physical quantity in both sides, unless they are BPS.

In later sections, we will try to give a partial answer to this problem. In particular, there has been a great progress on understanding the spectrum of both $\mathcal{N}=4$ and $\mathrm{AdS}_5\times\mathrm{S}^5$ theories based on integrability methods. Nontrivial examples of the correspondence have been found, which are now regarded as concrete and powerful evidences for the AdS/CFT correspondence.

Chapter 2

Integrability in $\mathcal{N} = 4$ theory

The AdS/CFT correspondence predicts individual string states are in one-to-one correspondence with gauge-invariant local operators of gauge theory. Comparison of global symmetry suggests that energy of a string state is equal to conformal dimension of the dual operator.

The $\mathcal{N} = 4$ super Yang-Mills theory is believed to sit on the superconformal fixed point to all orders of perturbation. Still, the conformal dimension (or anomalous dimension) of gauge-invariant local operators is not easy to compute. This is partly because operators with the same quantum numbers can mix through quantum effects. So only an appropriate linear combination of local operators becomes an eigenstate of the anomalous dimension matrix.

Studying diagonalization of anomalous dimension matrix has led to the discovery of integrability in $\mathcal{N} = 4$ theory by Minahan and Zarembo [5]. It enables us to compute anomalous dimension of local operators which are not necessarily BPS. Below we will review the discovery of integrability and succeeding development in $\mathcal{N} = 4$ super Yang-Mills theory.

2.1 Diagonalization of anomalous dimension matrix

Minahan and Zarembo considered action of dilatation operator of $\mathcal{N} = 4$ theory on general operators composed of scalar fields at one loop in λ, and discovered that the dilatation operator is of the same form as Hamiltonian of integrable spin chain [5]. In this section we follow their argument more in detail.

We are going to study renormalization of operators having the following form:

$$\mathcal{O}(x) = C_{i_1 i_2 \dots i_L} \,{}^{\circ}_{\circ}\, \mathrm{tr} \left[\phi^{i_1}(x) \phi^{i_2}(x) \cdots \phi^{i_L}(x) \right] {}^{\circ}_{\circ} \,, \tag{2.1.1}$$

where ${}^{\circ}_{\circ}\;{}^{\circ}_{\circ}$ denotes the normal ordering, and ϕ^i are scalars of the $\mathcal{N} = 4$ theory. If $\mathcal{O}$ is an eigenstate of dilatation operator, its two point function becomes

$$\langle \mathcal{O}^{\dagger}(y) \mathcal{O}(x) \rangle = \frac{\mathrm{const}}{|x-y|^{2\Delta_{\mathcal{O}}}} \equiv \frac{\mathrm{const}}{|x-y|^{2(L+\gamma_{\mathcal{O}})}} \,. \tag{2.1.2}$$

The eigenvalue of dilatation $\Delta_{\mathcal{O}}(\lambda)$ is a function of the 't Hooft coupling. The quantity $\Delta_{\mathcal{O}}(0) = L$ is called bare dimension and $\gamma_{\mathcal{O}}(\lambda)$ is called anomalous dimension.

Before we proceed, let us recall how we compute anomalous dimensions from wavefunction renormalization in (super) Yang-Mills [109]. We neglect operator mixing for the moment. Define an n point function of scalar fields ϕ evaluated at the scale μ, as

$$G^{(n)}\left(\{p_k\};\mu,\lambda\right) \equiv \langle \tilde{\phi}^{i_1}(p_1)\tilde{\phi}^{i_2}(p_2)\cdots\tilde{\phi}^{i_n}(p_n)\rangle\,, \tag{2.1.3}$$

where $\tilde{\phi}(p)$ is Fourier transform of $\phi(x)$. We introduce Z-factor for ϕ by

$$\phi\left(p,\mu\right) = Z_\phi\left(\mu\right)^{-1/2}\phi_{\text{bare}}\left(p,\mu_0\right). \tag{2.1.4}$$

The shift of the renormalization scale μ causes

$$\mu \mapsto \mu + \delta\mu, \qquad \tilde{\phi} \mapsto \tilde{\phi} + \frac{1}{2}\,\delta\left(\ln Z_\phi\right)\cdot\phi\,, \qquad G^{(n)} \mapsto G^{(n)} + \frac{n}{2}\,\delta\left(\ln Z_\phi\right)\cdot G^{(n)}\,. \tag{2.1.5}$$

The 't Hooft coupling λ is not renormalized in super Yang-Mills because it is protected by supersymmetry. The relation (2.1.5) implies that as a function of μ, $G^{(n)}$ obeys the equation

$$\frac{\partial G^{(n)}}{\partial \mu}\delta\mu = \frac{n}{2}\,\delta\left(\ln Z_\phi\right)\cdot G^{(n)}\,. \tag{2.1.6}$$

Suppose further that $G^{(2)}$ is of the form

$$G^{(2)}\left(p,\mu\right) = \frac{1}{p^2}\,f\left(\frac{\mu}{p}\right) \qquad \text{for} \quad p_1 = -p_2 = p. \tag{2.1.7}$$

It then follows

$$0 = \left[\frac{\partial}{\partial\ln\mu} - \frac{\partial\ln Z_\phi}{\partial\ln\mu}\right] f\left(\frac{\mu}{p}\right) \equiv \left[\frac{\partial}{\partial\ln\mu} - 2\gamma\right] f\left(\frac{\mu}{p}\right)\,. \tag{2.1.8}$$

When γ is constant, the last equation can be solved by

$$G^{(2)}\left(p,\mu\right) \sim \frac{\mu^{2\gamma}}{p^{2+2\gamma}}\,. \tag{2.1.9}$$

Thus we find that anomalous dimension γ is related to wavefunction renormalization Z via

$$\gamma = \frac{1}{2}\,\frac{\partial\ln Z}{\partial\ln\mu}\,, \qquad \text{for} \quad Z \equiv Z_\phi^2\,. \tag{2.1.10}$$

If we take into account the effect of operator mixing, this formula is slightly modified to

$$\Gamma^A{}_B = \frac{\partial Z^A{}_C}{\partial\ln\mu}\cdot\left(Z^{-1}\right)^C{}_B \qquad \text{for} \quad \mathcal{O}^A_{\text{ren}} = Z^A{}_B\,\mathcal{O}^B\,. \tag{2.1.11}$$

where we defined the renormalized operator $\mathcal{O}^A_{\text{ren}}$ so that the following quantity remains finite:

$$\left\langle \mathcal{O}^\dagger_{\text{ren},A}(x)\;\mathcal{O}^A_{\text{ren}}(y)\right\rangle\,. \tag{2.1.12}$$

In this way, the anomalous dimension $\gamma_{\mathcal{O}}$ in (2.1.2) is identified as an eigenvalue of anomalous dimension matrix Γ.

What kind of gauge-invariant local operators may mix with $\mathcal{O}$ of (2.1.1)? First of all, such operators should carry the same quantum numbers (including bare dimension) as $\mathcal{O}$. In addition, there should be nonzero amplitude of mixing when computed from the $\mathcal{N} = 4$ Lagrangian (1.1.16). It turns out that $\mathcal{O}$ can only mix with operators made up of scalars at one loop in λ. In other words, a set of gauge-invariant local operators made up of scalars form so-called $so(6)$ subsector of $\mathcal{N} = 4$ theory, which is closed at one loop in λ.

We can view this $so(6)$ sector from another angle. The operator $\mathcal{O}$ given in (2.1.1) can be thought of as the L-th order tensor product of $so(6)$ vectors. These states span a vector space of 6^L dimensions

$$\mathcal{H} = V_1 \otimes V_2 \otimes \cdots \otimes V_L \, . \tag{2.1.13}$$

It is convenient to think of this space as finite-dimensional Hilbert space for a spin chain of length L. Direction of the spin sitting at site k is interpreted as the $so(6)$ flavor ϕ^{i_k}, and the anomalous dimension matrix is identified as Hamiltonian of this spin chain. Note that one must impose a condition corresponding to the trace cyclicity. That is, the spin chain state should be invariant under the translation of index

$$|i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_L\rangle \quad \mapsto \quad |i_L\rangle \otimes |i_1\rangle \otimes \cdots \otimes |i_{L-1}\rangle \, . \tag{2.1.14}$$

Let us now compute the $so(6)$ anomalous dimension matrix at one loop. The calculation becomes simpler in the momentum space, so we consider the two point function of

$$\mathcal{O}(x) = C_{i_1 i_2 \ldots i_L} \,\mathop{\circ}_{\circ}\, \mathrm{tr} \left[\int \left(\prod_{k=1}^{L} \frac{d^4 p_k}{(2\pi)^4} \, e^{i p_k x} \, \tilde{\phi}^{i_k}(p_k) \right) \right] \mathop{\circ}_{\circ} \, , \tag{2.1.15}$$

instead of (2.1.1). The tree contribution is evaluated as

$$\left\langle \mathcal{O}^\dagger(y) \mathcal{O}(x) \right\rangle = \left(C_{i_1 i_2 \ldots i_L} \right)^2 \int \prod_{k=1}^{L} \frac{d^4 p_k}{(2\pi)^4} \, \frac{e^{i p_k (x-y)}}{p_k^2} \, . \tag{2.1.16}$$

Since we take large N limit, the interaction takes place in the nearest neighbor at one-loop. There are three kinds of Feynman diagrams which contribute to one-loop anomalous dimension matrix; gluon exchange, scalar 4 point interaction, and the self-energy of scalars.

The gluon exchange comes from the interaction

$$\mathcal{L} \;\ni\; \frac{1}{2} \, \mathrm{tr} \, D_\mu \phi^i D^\mu \phi^i \quad \longrightarrow \quad g_{\mathrm{YM}} \left(p^\mu - q^\mu \right) \, \mathrm{tr} \, \tilde{A}_\mu(p-q) \tilde{\phi}^i(p) \tilde{\phi}^i(q) \, , \tag{2.1.17}$$

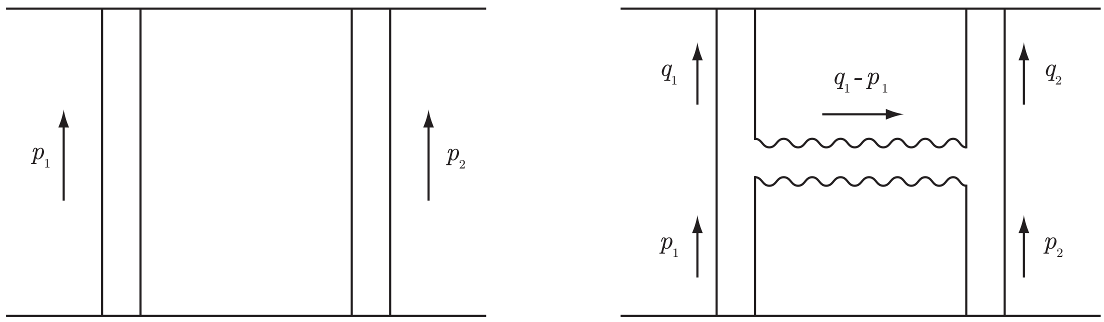


Figure 2.1: Left: The tree diagram. Right: Diagram of gluon exchange.

so this is flavor blind. The right figure of 2.1 yields

$$\sim -\frac{N g_{\mathrm{YM}}^2}{16\pi^2} \ln\Lambda \left(\int \prod_{k=1,2} \frac{d^4 p_k}{(2\pi)^4} \frac{e^{ip_k(x-y)}}{p_k^2} \right) . \tag{2.1.18}$$

Factor inside the parentheses is contribution of external propagators, and hence is neglected.

The scalar 4-point function comes from the interaction

$$\mathcal{L} \;\ni\; \frac{1}{4}\,\mathrm{tr}\,[\phi^i,\phi^j][\phi^i,\phi^j] \quad\longrightarrow\quad \frac{g_{\mathrm{YM}}^2}{4}\,\mathrm{tr}\left(2\tilde\phi^i\tilde\phi^j\tilde\phi^i\tilde\phi^j - \tilde\phi^i\tilde\phi^j\tilde\phi^j\tilde\phi^i - \tilde\phi^i\tilde\phi^i\tilde\phi^j\tilde\phi^j\right) , \tag{2.1.19}$$

and the diagram on the left of Figure 2.2 yields

$$\sim -\frac{N g_{\mathrm{YM}}^2}{16\pi^2} \ln\Lambda \left(2\delta_{i_1}^{j_2}\delta_{i_2}^{j_1} - \delta_{i_1}^{j_1}\delta_{i_2}^{j_2} - \delta_{i_1,i_2}\delta^{j_1,j_2} \right) , \tag{2.1.20}$$

neglecting contribution from external propagators.

Lastly, the scalar self-energy consists of the loops for gauge bosons and fermions. This diagram is computed in the paper [110], which reads

$$Z_\phi^{1/2} \sim 1 + \frac{N g_{\mathrm{YM}}^2}{8\pi^2} \ln\Lambda. \tag{2.1.21}$$

To sum up, the Z factor of wavefunction renormalization for the whole $\mathcal{O}$ is evaluated as

$$Z = \prod_{k=1}^{L} \left[I + \frac{\lambda}{16\pi^2} \ln\Lambda \left(2\delta_{i_k}^{j_k}\delta_{i_{k+1}}^{j_{k+1}} - 2\delta_{i_k}^{j_{k+1}}\delta_{i_{k+1}}^{j_k} + \delta_{i_k,i_{k+1}}\delta^{j_k,j_{k+1}} \right) \right] . \tag{2.1.22}$$

The Z factor is regularized by replacing UV cutoff Λ by the renormalization scale μ. Using (2.1.11), the anomalous dimension matrix is obtained as

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{k=1}^{L} \left(2I - 2P_{k,k+1} + K_{k,k+1} \right) , \tag{2.1.23}$$

where we defined permutation operator P and trace operator K by

$$P_{k,k+1} = \delta_{i_k}^{j_{k+1}}\delta_{i_{k+1}}^{j_k} , \qquad K_{k,k+1} = \delta_{i_k,i_{k+1}}\delta^{j_k,j_{k+1}} . \tag{2.1.24}$$

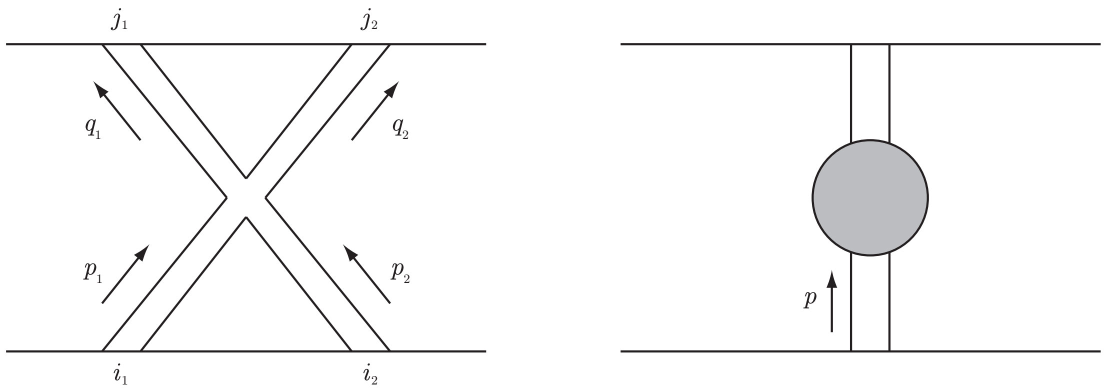


Figure 2.2: Left: Diagram of scalar 4 point interaction. Right: Self-energy of scalars.

As discussed in [5], this matrix Γ is equivalent to Hamiltonian of an integrable $so(6)$ spin chain.

We may also consider anomalous dimension matrix in the $su(2)$ sector. The $su(2)$ sector is composed of two holomorphic scalars $Z = \phi^1 + i\phi^2$ and $W = \phi^3 + i\phi^4$, and remains closed to all orders in λ. The $su(2)$ anomalous dimension matrix at one-loop is given by

$$\Gamma = \frac{\lambda}{8\pi^2} \sum_{k=1}^{L} \left(I_{k,k+1} - P_{k,k+1} \right), \tag{2.1.25}$$

because the trace operator $K_{k,k+1}$ vanishes on a holomorphic subsector of $SO(6)$. With the aid of the formula

$$P_{k,k+1} = \frac{1}{2} \left(I_k \otimes I_{k+1} + \vec{\sigma}_k \otimes \vec{\sigma}_{k+1} \right), \tag{2.1.26}$$

where Pauli matrices are defined by

$$\sigma_k^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_k, \quad \sigma_k^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_k, \quad \sigma_k^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_k, \quad \sigma_k^{\pm} = \frac{1}{2} \left(\sigma_k^x \pm i \sigma_k^y \right), \tag{2.1.27}$$

one finds that the $su(2)$ anomalous dimension matrix (2.1.25) is identical to the Hamiltonian of $\mathsf{XXX}_{1/2}$ spin chain (also known as XXX Heisenberg spin chain):

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{k=1}^{L} \left(I_k \otimes I_{k+1} - \vec{\sigma}_k \otimes \vec{\sigma}_{k+1} \right) \equiv \frac{\lambda}{8\pi^2} \mathcal{H}_{\mathsf{XXX}_{1/2}}. \tag{2.1.28}$$

With this interpretation, the operators Z and W are mapped to spin chain states $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively.

Higher-loop dilatation operator in $su(2)$ sector takes much more complicated form than (2.1.25), as given in [111, 112]. The one-loop dilatation operator in the full $psu(2,2|4)$ sector is obtained in [113], which is claimed to be the Hamiltonian of an integrable super spin chain at large N [114].

Since we are interested in comparison of the spectrum between gauge and string theory, it is often convenient to focus on closed subsectors of $\mathcal{N}=4$ theory. There are three rank-one closed subsectors in this theory. The first one is the $su(2)$ sector whose operators are of the form

$$\mathrm{tr}\, Z^{J_1} W^{J_2} + \cdots \, . \tag{2.1.29}$$

The second is the $sl(2)$ sector

$$\mathrm{tr}\, D_+^S Z^J + \cdots \, , \qquad (D_+ \text{ : covariant derivative in the lightcone direction})\, , \tag{2.1.30}$$

and the third is the $su(1|1)$ sector

$$\mathrm{tr}\, \psi^M Z^{J-M/2} + \cdots \, . \tag{2.1.31}$$

where $\psi = \lambda_A$ is an adjoint gaugino in $\mathcal{N}=1$ notation of Table 1.1. In particular, the fermionic $su(1|1)$ sector as well as the relationship among rank-one sectors are extensively studied in [8].

2.2 Diagonalization by Bethe Ansatz

In the last section, we identified the one-loop anomalous dimension matrix as Hamiltonian of integrable spin chains. This means that various mathematical techniques are applicable to the study of energy eigenstates of the system.

Among them, Bethe Ansatz is widely used to study the spectrum of exactly solvable models [115]. One famous example is Hubbard model in 1+1 dimensions, whose ground state energy and wavefunctions were determined from Bethe Ansatz approach [116, 117]. In this section we diagonalize the Hamiltonian of $\mathsf{XXX}_{1/2}$ spin chain using so-called coordinate Bethe Ansatz.

Let $\mathcal{H}_{\mathsf{XXX}_{1/2}}$ be the Hamiltonian of $\mathsf{XXX}_{1/2}$ spin chain given by

$$\mathcal{H}_{\mathsf{XXX}_{1/2}} \equiv \sum_{k=1}^{L} \left(I_{k,k+1} - P_{k,k+1} \right) = \frac{1}{2} \sum_{k=1}^{L} \left(I_k \otimes I_{k+1} - \vec{\sigma}_k \otimes \vec{\sigma}_{k+1} \right) . \tag{2.2.1}$$

We consider a periodic spin chain, so the positions $x = 1$ and $x = L+1$ are identical. The ground state of this Hamiltonian is ferromagnetic, and is given by

$$\left| 0 \right\rangle_L = \left| \uparrow \right\rangle_1 \otimes \left| \uparrow \right\rangle_2 \otimes \cdots \otimes \left| \uparrow \right\rangle_L \, . \tag{2.2.2}$$

In the $\mathcal{N}=4$ language, the ground state corresponds to half-BPS state

$$\left| 0 \right\rangle_L \equiv \mathrm{tr} \left[Z Z \ldots Z \right] \, , \tag{2.2.3}$$

which obeys the BPS relation $\Delta - L = 0$.[1]

[1] For the classification of 1/2, 1/4, and 1/8-BPS operators, see *e.g.* [84].

Impurities above the ferromagnetic vacuum are often called magnons. To describe spin chain states with magnons, we introduce the notation

$$|x_1\,,x_2\,,\cdots\,,x_M\rangle \equiv |{\uparrow}\rangle_1 \otimes \cdots |{\downarrow}\rangle_{x_1} \otimes \cdots |{\downarrow}\rangle_{x_M} \otimes \cdots |{\uparrow}\rangle_L \ , \tag{2.2.4}$$

where the spins turn downward at the positions $x = x_1\,, \ldots\,, x_M$, and upward otherwise. In the $\mathcal{N} = 4$ language, an up-spin corresponds to an operator Z and a down-spin to an operator W. One magnon state is constructed as superposition of plane waves:

$$|p\rangle \equiv \sum_{x=1}^{L} e^{ipx}\,|x\rangle\,. \tag{2.2.5}$$

The energy of one magnon is easily computed from

$$(I_{k,k+1} - P_{k,k+1})\,|p\rangle = \begin{cases} (1-e^{ip})\,|p\rangle & \text{for} \quad k = x \\ (1-e^{-ip})\,|p\rangle & \text{for} \quad k+1 = x \\ 0 & \text{otherwise} \end{cases} \ , \tag{2.2.6}$$

which gives

$$\mathcal{H}_{\mathsf{XXX}_{1/2}}\,|p\rangle = 4\sin^2\left(\frac{p}{2}\right)|p\rangle \equiv E(p)\,|p\rangle\,. \tag{2.2.7}$$

Recall that we have to impose cyclicity condition (2.1.14) in $\mathcal{N} = 4$ theory. This requires $p = 0$ for one magnon state. The one magnon state (2.2.5) then describes a half-BPS operator $\mathrm{tr}\,[WZZ\ldots Z]$.

Let us proceed to two magnon state. We make the following ansatz:

$$|p_1\,,p_2\rangle = \sum_{1\le x_1<x_2\le L} \psi(x_1\,,x_2)\,|x_1\,,x_2\rangle\,, \tag{2.2.8}$$

$$\psi(x_1\,,x_2) = e^{ip_1x_1+ip_2x_2} + S(p_2\,,p_1)\,e^{ip_2x_1+ip_1x_2}\,, \tag{2.2.9}$$

where $S(p_1\,,p_2)$ is called S-matrix which describes scattering of two particles of momentum (also called quasi-momentum) p_1 and p_2 , respectively. For $x_2 > x_1+1$, the Hamiltonian (2.2.1) returns the eigenvalue

$$E(p_1\,,p_2) = \sum_{k=1}^{2} E(p_k) = \sum_{k=1}^{2} 4\sin^2\left(\frac{p_k}{2}\right). \tag{2.2.10}$$

The S-matrix follows from the condition that the wavefunction (2.2.9) is an eigenstate with the eigenvalue (2.2.10) for $x_2 = x_1 + 1$. The result is

$$S(p_1\,,p_2) = -\frac{e^{-ip_2}\,(e^{ip_2}+e^{-ip_1}-2)}{e^{-ip_1}\,(e^{ip_1}+e^{-ip_2}-2)} = \frac{u_1-u_2+i}{u_1-u_2-i}\,, \tag{2.2.11}$$

where we introduced a rapidity variable

$$u_j \equiv \frac{1}{2}\cot\frac{p_j}{2} \qquad \text{or} \qquad e^{ip_j} = \frac{u_j+i/2}{u_j-i/2}\,. \tag{2.2.12}$$

We also require periodic boundary conditions on the wavefunction (2.2.9):

$$\psi(x_1\,,x_2) = \psi(x_2\,,x_1+L)\,, \tag{2.2.13}$$

which gives

$$e^{ip_2L} = S(p_2\,,p_1) \qquad \text{and} \qquad e^{ip_1L} = \frac{1}{S(p_2\,,p_1)} = S(p_1\,,p_2)\,. \tag{2.2.14}$$

These two relations together imply that the quasi-momenta are constrained as

$$\sum_{k=1}^{2} p_k L \equiv 0 \pmod{2\pi}\,. \tag{2.2.15}$$

Actually, from the cyclicity of trace (2.1.14) we must have

$$\sum_j p_j \equiv 0 \pmod{2\pi}\,. \tag{2.2.16}$$

An essential feature of integrable models is factorization of S-matrix. By factorization we mean that scattering of a particle a and particles $b_1\,,b_2\,,\ldots$ takes place elastically, so that the whole S-matrix is given by the product of two-body S-matrices

$$S_{\text{whole}}(a,\{b_k\}) = \prod_{k=1} S(a,b_k). \tag{2.2.17}$$

The $\mathsf{XXX}_{1/2}$ spin chain also has this property, from which one can generalize the above procedure to general M magnon states. Let $\{x_k\}$ be the lattice coordinates satisfying

$$1 \le x_1 < x_2 < \ldots < x_M \le L\,, \tag{2.2.18}$$

and suppose the wavefunction takes the form

$$|\{p_k\}\rangle = \sum_{\pi\in\mathcal{S}_M} a(\{p_{\pi(k)}\}) \exp\left(\sum_{k=1}^{M} ip_{\pi(k)}x_k\right), \tag{2.2.19}$$

where π is permutation and $\mathcal{S}_M$ is symmetric group of order M. The coefficients $a(\{p_{\pi(k)}\})$ are described by S-matrix up to normalization, as

$$\frac{a(\cdots,p_i\,,p_j\,,\cdots)}{a(\cdots,p_j\,,p_i\,,\cdots)} = S(p_i\,,p_j)\,, \tag{2.2.20}$$

where the two-body S-matrix is given by (2.2.11). When M $(\le L/2)$ magnons are all separated, the $\mathsf{XXX}_{1/2}$ Hamiltonian gives the eigenvalue

$$E_{\text{total}} = \sum_{k=1}^{M} E(p_k)\,, \qquad E(p_k) = 4\sin^2\left(\frac{p_k}{2}\right) = \frac{1}{\frac{1}{4}+u_k^2}\,. \tag{2.2.21}$$

The periodic boundary conditions impose the nonlinear constraint among quasi-momenta:

$$e^{ip_j L} = \left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k\neq j}^{M} S\left(p_j\,, p_k\right) = \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}\,. \tag{2.2.22}$$

This is the Bethe Ansatz equation. Note that the trace cyclicity requires an additional condition (2.2.16).

One can solve the Bethe Ansatz equation (2.2.22) for the simplest two magnon case. However, it is in general very difficult to find its solutions when the number of magnons becomes large. Even in such cases, the problem can be simplified if one considers thermodynamic limit. This is what we are going to discuss in Section 2.4.

2.3 Algebraic Bethe Ansatz for $\mathsf{XXX}_{1/2}$ spin chain

As though the coordinate Bethe Ansatz is intuitive and easy to understand, it is difficult to apply it to higher-loop anomalous dimension matrix, because non nearest-neighborhood interactions distort the wavefunction [8]. This difficulty can be overcome by using more abstract formulation called algebraic Bethe Ansatz. Below we explain main ideas of algebraic Bethe Ansatz applied to $\mathsf{XXX}_{1/2}$ spin chain. For rigorous argument, please consult various reviews or textbooks, for example [118, 119, 120, 121].

The starting point of algebraic Bethe Ansatz is R-matrix and L-operator satisfying Yang-Baxter relation. For $\mathsf{XXX}_{1/2}$ model, the R-matrix is given by

$$R(u) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad b(u) = \frac{\eta}{u+\eta}\,, \quad c(u) = \frac{u}{u+\eta}\,, \tag{2.3.1}$$

with $\eta \in \mathbb{C}$ an arbitrary constant, and the L-operator is given by

$$L_{0k}(u) \equiv (2u)\, I_0 \otimes I_k + \eta \sum_{a=1}^{3} \sigma_0^a \otimes \sigma_k^a\,, \qquad (k = 1, 2, \ldots, L)\,, \tag{2.3.2}$$

where '0' denotes a fictitious site for reference. By choosing appropriate basis of vector space at the site 0, we can express $L_{0k}(u) = [L_k(u)]^a_b$ as a matrix

$$L_{0k}(u) = \begin{pmatrix} [L_k(u)]^1_1 & [L_k(u)]^1_2 \\ [L_k(u)]^2_1 & [L_k(u)]^2_2 \end{pmatrix} = \begin{pmatrix} (2u) I_k + \eta\sigma_k^z & 2\eta\sigma_k^- \\ 2\eta\sigma_k^+ & (2u) I_k - \eta\sigma_k^z \end{pmatrix}. \tag{2.3.3}$$

These operators satisfy the Yang-Baxter relation:

$$R(u-v)\Big[L_{0k}(u) \underset{0}{\otimes} L_{0k}(v)\Big] = \Big[L_{0k}(v) \underset{0}{\otimes} L_{0k}(u)\Big] R(u-v)\,, \tag{2.3.4}$$

where $\otimes_0$ stands for tensor product over the referential vector space at 0. If we write indices of the vector space at 0 explicitly, (2.3.4) becomes

$$R(u-v)^{a_1,a_2}_{c_1,c_2} L_k(u)^{c_1}_{b_1} L_k(v)^{c_2}_{b_2} = L_k(v)^{a_1}_{c_1} L_k(u)^{a_2}_{c_2} R(u-v)^{c_1,c_2}_{b_1,b_2} \,. \tag{2.3.5}$$

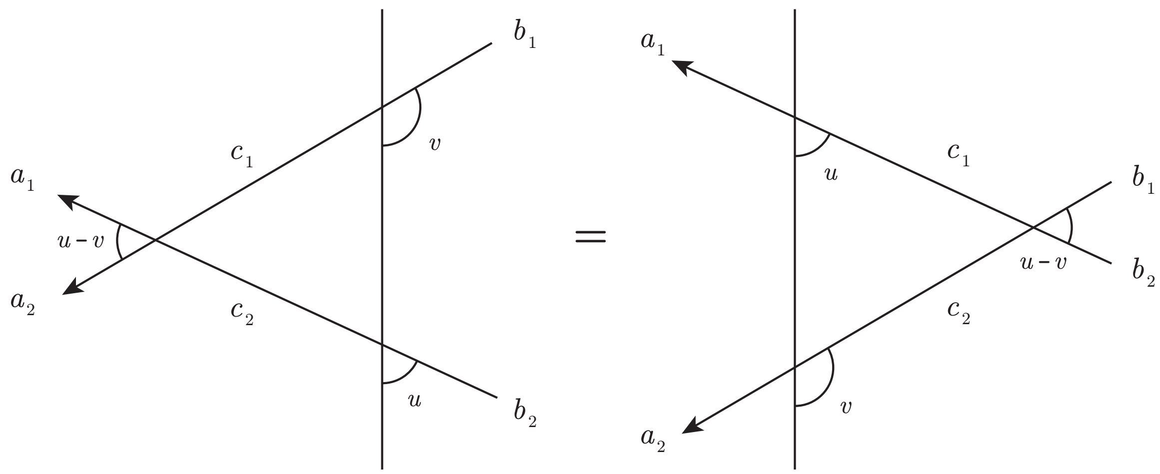


Figure 2.3: Graphical representation of Yang-Baxter relation for L-operator

From the Yang-Baxter relation we can deduce the existence of an infinite number of commuting charges, which is one of the important characterizations of integrable systems. To see it, let us define monodromy matrix by

$$\Omega(u) \equiv L_{0L}(u) \cdots L_{02}(u)\, L_{01}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} , \tag{2.3.6}$$

where we take the product of L-operators in the sense of (2.3.3). The trace of monodromy matrix is called transfer matrix:

$$T(u) = \mathrm{tr}\Omega(u) = A(u) + D(u) \,. \tag{2.3.7}$$

By using (2.3.4) repeatedly, one can show the Yang-Baxter relation for monodromy matrix:

$$R(u-v) \Big[\Omega(u) \underset{0}{\otimes} \Omega(v) \Big] = \Big[\Omega(v) \underset{0}{\otimes} \Omega(u) \Big] R(u-v) \,. \tag{2.3.8}$$

After multiplying $R(u-v)^{-1}$ from the left and taking the trace over the vector space at site 0, one finds

$$[T(u), T(v)] = 0 \,. \tag{2.3.9}$$

Thus, if we expand $T(u)$ in powers of u as $T(u) = \sum_{n=1} Q_n u^{n-1}$, this relation shows

$$[Q_m, Q_n] = 0 \,, \qquad (\forall\ m, n) \,, \tag{2.3.10}$$

as expected. The quantity Q_n is called the n-th conserved charge. From the Yang-Baxter relation (2.3.8), it also follows that

$$[B(u), B(v)] = 0\,, \tag{2.3.11}$$

where $B(u)$ is introduced in (2.3.6).

One advantage of algebraic Bethe Ansatz is that one can construct eigenvector of Bethe Ansatz equation (called Bethe vector) straightforwardly. In this formulation, the operator $B(u)$ plays the rôle of a creation operator, as is inferred from (2.3.3).

The M magnon state is given by

$$|M\rangle = B(u_1)B(u_2)\dots B(u_M)\,|0\rangle\;, \tag{2.3.12}$$

where $|0\rangle$ is the ground state appeared in (2.2.2). From the expression of L-operator (2.3.3), one easily finds that the M magnon state is the eigenstate of total spin operator

$$S^z\,|M\rangle \equiv \sum_{k=1}^{L} \frac{1}{2}\,\sigma_k^z\,|M\rangle = \left(\frac{L-2M}{2}\right)|M\rangle\,. \tag{2.3.13}$$

It can be shown that M magnon state is an eigenstate of transfer matrix with the eigenvalue

$$T(u;\{u_j\}) = (2u+\eta)^L \prod_{j=1}^{M} \frac{u-u_j-\eta}{u-u_j} + (2u-\eta)^L \prod_{j=1}^{M} \frac{u-u_j+\eta}{u-u_j}\,. \tag{2.3.14}$$

From the definition of transfer matrix (2.3.7), one sees that $T(u)$ is a polynomial of u of degree L, and therefore residues at the apparent poles at $u = u_j$ should vanish in (2.3.14). This consistency condition leads to the following equations

$$\left(\frac{u_j+\eta/2}{u_j-\eta/2}\right)^L = \prod_{k\neq j}^{M} \frac{u_j-u_k+\eta}{u_j-u_k-\eta}\,, \tag{2.3.15}$$

which is exactly same as the Bethe Ansatz equations (2.2.22) on setting $\eta = i$.

Before closing this section, let us make a few comments on the conserved charges. The transfer matrix $T(u)$ and the Hamiltonian of $\mathsf{XXX}_{1/2}$ spin chain (2.2.1) are related as

$$\frac{d}{du}\log T(u)\bigg|_{u=\eta/2} = \frac{1}{2\eta}\sum_{j=1}^{L}\left(I_j\otimes I_{j+1} + \sum_{a=1}^{3}\sigma_j^a\otimes\sigma_{j+1}^a\right) = \frac{1}{\eta}\Big(L\cdot\mathbf{1} - \mathcal{H}_{\mathsf{XXX}_{1/2}}\Big)\,. \tag{2.3.16}$$

To show the first equality, the following equality is useful:

$$L_{0k}\left(u=\frac{\eta}{2}\right) = \eta\Big(I_0\otimes I_k + \vec{\sigma}_0\otimes\vec{\sigma}_k\Big)\,. \tag{2.3.17}$$

Furthermore, it can also be shown that the transfer matrix is related to the total quasi-momentum as

$$\exp{(iP)} = \frac{1}{(2\eta)^L}\,T\left(u=\frac{\eta}{2}\right). \tag{2.3.18}$$

Looking carefully at the results (2.3.16) and (2.3.18), we notice that it is much convenient to redefine an infinite number of mutually commuting charges by

$$T(u) = (2u+\eta)^L \, e^{iP} \exp\left(\eta \sum_{n=1}^{\infty} \frac{(u-\eta/2)^n}{n} \, Q_{n+1}\right), \tag{2.3.19}$$

where $Q_2 = \mathcal{H}_{\mathsf{XXX}_{1/2}}$.

2.4 Thermodynamic limit of $\mathsf{XXX}_{1/2}$ spin chain

We look for solutions of the Bethe Ansatz equation (2.2.22) when there are a large number of magnons (or Bethe roots). To simplify the problem, we take thermodynamic limit of $\mathsf{XXX}_{1/2}$ spin chain, that is, to send $L \to \infty$. Specifically, we consider the situation in which the number of magnons M and the length of spin chain L become very large keeping the ratio M/L fixed, in order to compare with the spectrum of classical string theory in later chapters. Moreover, the rapidity u of individual magnons should run away to infinity in order to keep the energy (2.2.21) finite. To sum up, the limit we will take can be specified as

$$L \to \infty, \qquad \text{with} \quad \alpha \equiv \frac{M}{L}\,, \quad x \equiv \frac{u}{L} \quad \text{kept fixed.} \tag{2.4.1}$$

Let us apply the limit (2.4.1) to the Bethe Ansatz equation (2.2.22) following [11]. By taking the logarithm of both sides, we get

$$L \log\left(\frac{u_j + i/2}{u_j - i/2}\right) = \sum_{k \neq j}^{M} \log\left(\frac{u_j - u_k + i}{u_j - u_k - i}\right) - 2\pi i n_j\,, \qquad n_j \in \mathbb{Z}\,, \tag{2.4.2}$$

where mode number n_j specifies a branch of the logarithm. By taking the above limit (2.4.1), we find

$$\frac{1}{x_j} = \frac{2}{L} \sum_{k \neq j}^{M} \frac{1}{x_k - x_j} - 2\pi n_j\,. \tag{2.4.3}$$

The first term in the right hand side represents repulsive potential among Bethe roots.

For the moment, let us consider what happens if the first term is absent. One soon finds the solution in which Bethe roots are aligned along the real axis as $x_j = 1/(2\pi n_j)$. Now recall that the rapidity u is related to the quasi-momentum p by (2.2.12), then it follows

$$x_j L = u_j = \frac{1}{2} \cot \frac{p_j}{2} \quad \underset{\text{limit}}{\longrightarrow} \quad \frac{1}{p_j}\,, \qquad \therefore \;\; p_j = \frac{2\pi n_j}{L}\,, \tag{2.4.4}$$

which is a usual quantization condition of momentum. Here, any number of Bethe roots can occupy the same mode number.

Next, we turn on the first term with n_j fixed. The Bethe roots concentrated at $x_j = 1/(2\pi n_j)$ grow into the complex plane, symmetrically with respect to the real axis. Since we have taken the $L \to \infty$ limit, we can approximate a set of Bethe roots $\{x_j\}$ by a continuous segment $\xi \in \mathcal{C}$. In general, there can be several non-overlapping segments $\cup_k \mathcal{C}_k$.

These segments can be regarded as emergence of branch cuts on the complex rapidity plane. To describe them, we introduce the density of Bethe roots by

$$\rho(x) \equiv \frac{1}{L} \sum_j \delta(x - x_j) \,. \tag{2.4.5}$$

As the definition of δ-function is a bit ambiguous for $x \in \mathbb{C}$ (or $x \in \mathbb{CP}^1$), we may redefine it through the resolvent

$$G(x) \equiv \frac{1}{L} \sum_j \frac{1}{x - x_j} \equiv \oint_{\mathcal{C}} d\xi \, \frac{\rho(\xi)}{x - \xi} \,, \tag{2.4.6}$$

where $\mathcal{C}$ is a contour encircling all branch cuts $\cup_k \mathcal{C}_k$. By construction, the resolvent is an analytic function of x over the region $\mathbb{CP}^1 \setminus \{\cup_k \mathcal{C}_k\}$. Asymptotically, it behaves as

$$G(x) = \frac{\alpha}{x} + \mathcal{O}\Big(\frac{1}{x^2}\Big) \,, \qquad \alpha \equiv \sum_k \alpha_k \equiv \sum_k \oint_{\mathcal{C}_k} d\xi \; \rho(\xi) \,. \tag{2.4.7}$$

From the definition (2.4.5), one can identify α as the ratio M/L. The quantity α_k is called filling fractions.

The Bethe Ansatz equation (2.4.3) is rewritten using the resolvent as

$$\frac{1}{x} + 2\pi n_j = 2 \,⨍_{\mathcal{C}} d\xi \, \frac{\rho(\xi)}{x - \xi} = G\,(x + i\epsilon) + G\,(x - i\epsilon) \qquad \text{for} \quad x \in \mathcal{C}_j \,, \tag{2.4.8}$$

where we take the principal part to subtract the contribution from $k = j$. We can also rewrite other conditions in terms of resolvent. The trace cyclicity condition

$$P \equiv \sum_j p_j = \sum_j \frac{1}{x_j L} \equiv -2\pi m \,, \qquad m \in \mathbb{Z} \,, \tag{2.4.9}$$

is translated to

$$-\frac{1}{L} \sum_j \frac{1}{x_j} = G(0) = -\sum_k \oint_{\mathcal{C}_k} d\xi \, \frac{\rho(\xi)}{\xi} = \sum_j 2\pi n_j \oint_{\mathcal{C}_j} dx \, \rho(x) = 2\pi m \,, \tag{2.4.10}$$

where we used (2.4.3) and $\oint_{\mathcal{C}} ⨍_{\mathcal{C}} d\xi d\xi' \cdots = 0$. The last equality can be expressed in terms of filling fractions:

$$\sum_j n_j \alpha_j = m. \tag{2.4.11}$$

Similarly, the anomalous dimension is expressed by

$$\gamma = \frac{\lambda}{8\pi^2} \sum_{j=1}^{M} \frac{1}{\frac{1}{4} + u_j^2} \quad \xrightarrow[\text{limit}]{} \quad \gamma = \frac{\lambda}{8\pi^2 L} \sum_{j=1}^{M} \frac{1}{x_j^2} = \frac{\lambda}{8\pi^2 L} \oint_{\mathcal{C}} dx \, \frac{\rho(x)}{x^2} \,. \tag{2.4.12}$$

There is an additional comment on what is called condensate. If several Bethe roots are situated at the positions $u_k - u_{k+1} = i$, we have to take the thermodynamic limit of (2.4.2) carefully. Such configuration of Bethe roots is called condensate, and survives under the thermodynamic limit because

$$\log\left(u_j - u_{k+1} - i\right) = \log\left(u_j - u_k\right) = \log L + \log\left(x_j - x_k\right). \tag{2.4.13}$$

Thus, the condensate can be interpreted as an extra logarithmic cut with flat distribution $\rho(x) = 1$ (one root per the distance $1/L$). When the contour around a branch cut $\mathcal{C}_k$ passes condensate, there occurs a jump of mode number by 2π.

We want to rephrase the above formulation in the algebro-geometric language. We introduce two sheets of complex plane $\mathbb{CP}^1_\pm$ connected by a certain number of branch cuts $\cup_k \mathcal{C}_k$, and choose a- and b-periods as in Figure 2.4.

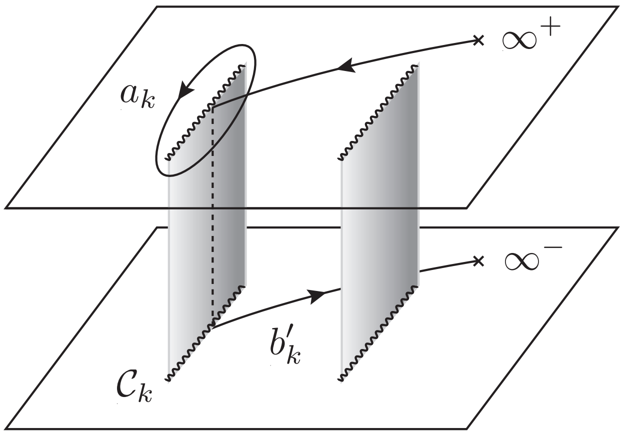


Figure 2.4: Choice of a- and b-cycles.

Let us define the function

$$\mathrm{p}(x) \equiv G(x) - \frac{1}{2x}, \tag{2.4.14}$$

which is called quasi-momentum in [11]. This should not be confused with the original momentum p_j appeared in (2.4.9). The Bethe Ansatz equation (2.4.8) is rewritten as

$$\mathrm{p}\left(x + i\epsilon\right) + \mathrm{p}\left(x - i\epsilon\right) = 2\pi n_j \qquad \text{for} \quad x \in \mathcal{C}_j. \tag{2.4.15}$$

The new variable $\mathrm{p}(x)$ shall define an Abelian integral ($\int^x d\mathrm{p}$) over the Riemann surface $\Sigma \simeq \mathbb{CP}^1_+ \cup \mathbb{CP}^1_-$ modulo $2\pi \times$ (integer). The differential $d\mathrm{p}(x)$ is nonsingular except for the location of double pole $x = 0$ (and for the location of condensate). Generally, Riemann surfaces of genus g have $K \equiv g + 1$ cuts and $2g$ independent cycles which we denote by $\{a_1, b_1, \ldots, a_g, b_g\}$. Let $b'_1, \ldots, b'_K$ be open paths with the endpoints at $\infty^\pm$, and define a number n'_∞ by

$$\mathrm{p}\left(\infty^+\right) - \mathrm{p}\left(\infty^-\right) = \int_{\infty^-}^{\infty^+} d\mathrm{p} \equiv 2\pi n'_\infty, \tag{2.4.16}$$

then the equation (2.4.15) can be interpreted as quantization of b-periods:

$$\oint_{b_j} d\mathrm{p} \equiv \oint_{b'_j} d\mathrm{p} - \oint_{b'_K} d\mathrm{p} = 2\pi\,(n_j - n_K)\,, \qquad (j = 1, 2, \ldots, g)\,. \tag{2.4.17}$$

Without condensate, a-periods of the Abelian integral ($\int^x d\mathrm{p}$) can be normalized to zero:

$$\oint_{A_j} d\mathrm{p} = 0, \tag{2.4.18}$$

and if condensate is present, we should modify this condition to

$$\oint_{A_j} d\mathrm{p} = 2\pi m_j\,. \tag{2.4.19}$$

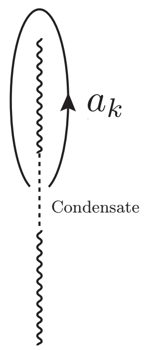


Figure 2.5: Modification of a-periods by condensate.

From the formula (2.4.7), we finds the function $\mathrm{p}(x)$ having asymptotics

$$\operatorname*{Res}_{x=\infty}(x\,d\mathrm{p}) = \alpha - \frac{1}{2}\,. \tag{2.4.20}$$

In this terminology, filling fractions are given by

$$\alpha_k = \oint_{\mathcal{C}_k} dx\ \rho(x) = \frac{1}{2\pi i}\oint_{\mathcal{C}_k} dx\,\mathrm{p}(x)\,, \qquad \sum_k \alpha_k = \alpha\,. \tag{2.4.21}$$

The trace cyclicity (2.4.10) is expressed as

$$G(0) = \operatorname*{Res}_{x=0}\left(\frac{\mathrm{p}(x)dx}{x}\right) = 2\pi m\,. \tag{2.4.22}$$

2.5 All-loop Bethe Ansatz conjecture

2.5.1 Towards the all-loop proposal

We would like to write now a short review on the study of spectrum in $su(2)$ sector at higher loops. We mainly focus our attention onto the work of Beisert, Dippel and Staudacher [7] and around.

Extension of one-loop anomalous dimension matrix (2.1.25) to two loops was first studied in [111]. They decided the form of dilatation operator without working on Feynman diagrams. Instead, they inspected all possible forms of Feynman diagrams which could contribute to two loop dilatation operator in $su(2)$ sector, and fixed coefficients by gauge invariance, non-renormalization theorem and consistency with the BMN limit [4].[2]

They also showed planar integrability of $su(2)$ sector at two loops. It was done by noting that the parity operation

$$P\,T^a P = (T^a)^{\mathsf{T}} = (T^a)^* \ , \qquad P\,\mathrm{tr}\,(\phi_1\phi_2\cdots\phi_L)\,P = \mathrm{tr}\,(\phi_L\cdots\phi_2\phi_1)\ , \tag{2.5.1}$$

is an exact symmetry of $SU(N)$ gauge theory, where T^a is the generator of $SU(N)$ gauge group. The parity-even sector is unrelated with parity-odd sector in general, but they observed that they have the same eigenvalue of dilatation operator if $N=\infty$, up to two loops. If the underlying theory is integrable, the second commuting conserved charge Q_2 (Q_1 being Hamiltonian as in (2.3.19)) is a parity-odd operator giving rise to degeneracy

$$Q_2\,\mathcal{O}_\pm \sim \mathcal{O}_\mp\,. \tag{2.5.2}$$

As it seems almost impossible to construct systems with $[Q_1\,,Q_2]=0$ which are not integrable, they concluded that the complete degeneracy signals the first sign of integrability beyond one loop.

They further conjectured three-loop planar dilatation generator in $su(2)$ sector assuming higher-loop integrability. Let us introduce the notation

$$\{n_1\,,n_2\,,\cdots\} = \sum_\ell P_{\ell+n_1-1\,,\ell+n_1} P_{\ell+n_2-1\,,\ell+n_2}\cdots\ , \tag{2.5.3}$$

where $P_{k,k+1}$ is the permutation operator introduced in (2.1.24), then their result reads

$$\mathcal{H} = \sum_{k=0}^\infty \left(\frac{\lambda}{16\pi^2}\right)^k \mathcal{H}_{2k}\,, \tag{2.5.4}$$

where

$$\begin{aligned}
\mathcal{H}_0 &= +\{\}\,,\\
\mathcal{H}_2 &= +2\{\} - 2\{1\}\,,\\
\mathcal{H}_4 &= -8\{\} + 12\{1\} - 2(\{1,2\}+\{2,1\})\,,\\
\mathcal{H}_6 &= +60\{\} - 104\{1\} + 4\{1,3\} + 24(\{1,2\}+\{2,1\})\\
&\quad - 4i\epsilon_2\{1,3,2\} + 4i\epsilon_2\{2,1,3\} - 4(\{1,2,3\}+\{3,2,1\})\,,
\end{aligned} \tag{2.5.5}$$

with ϵ_2, which does not alter the spectrum, set to zero.

[2]The BMN limit differs from the thermodynamic limit (2.4.1) in that M is kept finite as we take $L\to\infty$.

Applying this dilatation operator to length-four Konishi descendant [122]

$$\mathcal{O}_K = \mathrm{tr}(ZZWW) - \mathrm{tr}(ZWZW), \tag{2.5.6}$$

they found its anomalous dimension as

$$\Delta_K = 4 + 12\left(\frac{\lambda}{16\pi^2}\right) - 48\left(\frac{\lambda}{16\pi^2}\right)^2 + 336\left(\frac{\lambda}{16\pi^2}\right)^3 + \cdots . \tag{2.5.7}$$

In [123], Beisert confirmed the conjectured Hamiltonian (2.5.5) based on symmetry algebra and the BMN limit. He also proved the Hamiltonian and its integrability at three loops in $su(2|3)$ sector, which is the maximally compact closed subsector of $psu(2,2|4)$, along this line of study.

The anomalous dimension (2.5.7) agrees with field theoretical computation of [124]. Moreover, this result was shown to coincide with the anomalous dimension of twist-j operator ($j = 2$) up to three loops in [20] by applying BFKL method to $\mathcal{N} = 4$ theory.

Beyond one loop, Hamiltonian starts to acquire quite complicated structure. To obtain general spectrum of such Hamiltonian, we have to look for a systematic (or sophisticated) way of diagonalizing it. For this purpose, Serban and Staudacher studied a long range spin chain of Inozemtsev type, and found that it reproduces $su(2)$ dilatation operator up to three loops [6].

The Inozemtsev spin chain has the Hamiltonian

$$\mathcal{H}_{\text{Inozemtsev}} = \sum_{j=1}^{L}\sum_{n=1}^{L-1} \wp_{L,\pi/\kappa}(n)\left(1 - P_{j,j+n}\right), \tag{2.5.8}$$

where L is the length of spin chain, κ is coupling constant, and $P_{i,j}$ is permutation of site i and j. Recall that the Weierstrass $\wp$-function has the following series expansion:

$$\wp_{L,\pi/\kappa}(z) = \frac{1}{z^2} + \sum_{(m,n)\,\in\,\mathbb{Z}^2}{}' \left\{ \frac{1}{(z - mL - in\pi/\kappa)^2} - \frac{1}{(mL + in\pi/\kappa)^2} \right\}, \tag{2.5.9}$$

where the prime over the sum means that we omit $(m, n) = (0, 0)$. In [6], they considered long-range limit $L \to \infty$, in which elliptic functions reduce to hyperbolic ones:

$$\lim_{L\to\infty} \wp_{L,\pi/\kappa}(z) = \kappa^2\left(\frac{1}{\sinh^2 \kappa z} + \frac{1}{3}\right). \tag{2.5.10}$$

As the authors already noticed [6], Inozemtsev spin chain has a limited range of validity in the following sense:

1) The Inozemtsev Hamiltonian (2.5.8) contains two-spin interactions alone. However there can be more complicated interactions in the higher-loop dilatation operator.

2) They take the limit $L \to \infty$. When one wants to check if the agreement continues for finite L, one must solve the Bethe Ansatz equation expressed in terms of elliptic functions.

3) From four loops, thermodynamic limit of the Inozemtsev spin chain is not consistent with the perturbative BMN scaling [125]:

$$\Delta - J = \sum_{k=1}^{M} \sqrt{1 + \frac{\lambda}{J^2} n_k^2} + \mathcal{O}\left(\frac{1}{J}\right), \qquad \sum_{k=1}^{M} n_k = 0. \tag{2.5.11}$$

At this stage, there was still possibility that the Inozemtsev spin chain ceased to agree with $\mathcal{N} = 4$ theory, and the perturbative BMN scaling was valid beyond three loops. Beisert, Dippel and Staudacher pursued this matter, and proposed so-called BDS Bethe Ansatz which diagonalizes the Hamiltonian, and is compatible with the perturbative BMN scaling to all orders [7].

The precise form of the BDS Ansatz will be given in Section 2.5.2. Here we present several features of their proposal:

- The BDS Ansatz is a conjecture for all orders of λ.
- In contrast to the approach of [6], it is not clear how all-loop Hamiltonian operator looks like the BDS Ansatz is diagonalizing.
- Just like [6], the BDS Ansatz is supposed to be asymptotic. Namely, it is exact only when the length of spin chain L is infinite, and will breakdown at the loop order $\sim \lambda^L$ when L is finite [8].
- It diagonalizes five-loop Hamiltonian in $su(2)$ sector, that is, six-loop correction to conformal dimension.
- It also reproduces with the leading $1/J$ correction to BMN energy (2.5.11). However, it disagrees with the pp-wave limit of string theory.
- In the thermodynamic limit, it does not match again with classical string theory.

Concerning the last two problems, they speculated that these may be due to the 'order of limits' problem [7]. In gauge theory side, we first assume $L \gg 1,\ 1 \gg \lambda$ and then expand conserved charges in λ. In classical string side, we firstly take $\lambda \gg 1$ and then expand conserved charges in $1/J$. If there are terms like

$$f(\lambda, L) = \frac{\lambda^L}{(1+\lambda)^L}\,, \tag{2.5.12}$$

then these two limiting procedures return different values.

Although there is potentially such an order-of-limits problem, the 'discrepancy' between gauge and string theories is now understood in a different way. Much more convincing explanation is that neither the perturbative BMN scaling (2.5.11) nor the BMN scaling hypothesis

$$\Delta - L = L \left\{ \sum_{j \geq 1} \sum_{k \geq 1} a_{j,k} \left(\frac{\lambda}{L^2} \right)^j \left(\frac{1}{L} \right)^k \right\}, \tag{2.5.13}$$

remains valid from four loops in gauge theory. This is because the two-body S-matrix of gauge theory acquires a nontrivial phase factor starting at four loops. This factor, called dressing phase, induces terms with $k < 0$ in (2.5.13), and therefore completely destroys the BMN scaling.[3]

The existence of such phase at weak coupling was first found in the computation in the $sl(2)$ sector [127]. Later it has been confirmed by the field theory calculation of dilatation operator in $su(2)$ sector at four loops [112]. We postpone the discussion on the dressing phase until Section 5.3. For now, let us summarize the Bethe Ansatz approach in $su(2)$ sector more qualitatively.

2.5.2 All-loop Bethe Ansatz in $su(2)$ sector

We will explain the Bethe Ansatz equation which is believed to reproduce the spectrum of $su(2)$ sector up to the order $\sim \lambda^{L+1}$ where the wrapping interaction begins to take place.

It is convenient to introduce new rapidity parameters x and $x^{\pm}$ by [7, 128]

$$u = x + \frac{\lambda}{16\pi^2} \frac{1}{x}, \qquad u \pm \frac{i}{2} = x^{\pm} + \frac{\lambda}{16\pi^2} \frac{1}{x^{\pm}}. \tag{2.5.14}$$

The first equation can be easily inverted as

$$x(u) = \frac{1}{2} \left(u + u \sqrt{1 - \frac{\lambda}{4\pi^2} \frac{1}{u^2}} \right), \tag{2.5.15}$$

Note that x is an odd function of u. The second equation tells us that the variables $x^{\pm}$ are not independent, and constrained as

$$x^+ + \frac{\lambda}{16\pi^2} \frac{1}{x^+} - x^- - \frac{\lambda}{16\pi^2} \frac{1}{x^-} = i. \tag{2.5.16}$$

An alternative definition of $x^{\pm}$ is

$$u = \frac{1}{2} \left(x^+ + \frac{\lambda}{16\pi^2} \frac{1}{x^+} + x^- + \frac{\lambda}{16\pi^2} \frac{1}{x^-} \right), \tag{2.5.17}$$

with the constraint (2.5.16). We relate the $x^{\pm}$ variables to the magnon momentum p by

$$e^{ip} \equiv \frac{x^+}{x^-}. \tag{2.5.18}$$

[3]Breakdown of BMN scaling was also observed earlier in the plane-wave matrix theory, which is a truncation of Kaluza-Klein modes in $\mathcal{N} = 4$ super Yang-Mills on $\mathbb{R}_t \times S^3$ [126].

This allows us to express u and $x^{\pm}$ as functions of p:

$$u(p)=\frac{1}{2}\cot\left(\frac{p}{2}\right)\sqrt{1+\frac{\lambda}{\pi^2}\,\sin^2\left(\frac{p}{2}\right)}, \qquad x^{\pm}=e^{\pm ip/2}\left(\frac{1+\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)}}{4\sin\left(\frac{p}{2}\right)}\right). \tag{2.5.19}$$

The following identities are also useful in converting u to $x^{\pm}$:

$$u_j-u_k=(x_j-x_k)\left(1-\frac{g_B^2}{2x_jx_k}\right)=\left(x_j^{\pm}-x_k^{\pm}\right)\left(1-\frac{g_B^2}{2x_j^{\pm}x_k^{\pm}}\right), \tag{2.5.20}$$

$$u_j-u_k\pm\frac{i}{2}=\left(x_j^{\pm}-x_k\right)\left(1-\frac{g_B^2}{2x_j^{\pm}x_k}\right)=\left(x_j-x_k^{\mp}\right)\left(1-\frac{g_B^2}{2x_jx_k^{\mp}}\right), \tag{2.5.21}$$

$$u_j-u_k\pm i=\left(x_j^{\pm}-x_k^{\mp}\right)\left(1-\frac{g_B^2}{2x_j^{\pm}x_k^{\mp}}\right), \tag{2.5.22}$$

where we defined

$$g_B^2\equiv\frac{\lambda}{8\pi^2}\,. \tag{2.5.23}$$

The all-loop Bethe Ansatz in $su(2)$ sector is given by

$$e^{ip_jL}=\prod_{k\neq j}^{M}S\left(p_j\,,p_k\right)\,, \qquad S\left(p_j\,,p_k\right)\equiv\frac{u_k-u_j+i}{u_k-u_j-i}\,\sigma^2\left(p_j\,,p_k\,;\lambda\right). \tag{2.5.24}$$

The factor $\sigma^2\left(p_j\,,p_k\,;\lambda\right)$ is called dressing phase, which equals to the identity up to $\mathcal{O}(\lambda^3)$. Without this factor, the two-body S-matrix written in terms of u variable is same as the one-loop result (2.2.11). The $su(2)$ Bethe Ansatz without dressing phase was proposed in [7] and called BDS Ansatz.

The above equation can be reexpressed in terms of $x^{\pm}$ variables alone, as

$$\left(\frac{x_j^+}{x_j^-}\right)^L=\prod_{k\neq j}^{M}\frac{x_j^+-x_k^-}{x_j^--x_k^+}\,\frac{1-g_B^2/\left(2x_j^+x_k^-\right)}{1-g_B^2/\left(2x_j^-x_k^+\right)}\,\sigma^2\left(x_j^{\pm}\,,x_k^{\pm}\right). \tag{2.5.25}$$

An infinite number of commuting charges are proposed as follows [7]:

$$Q_r=\sum_{k=1}^{M}\frac{i}{r-1}\left\{\frac{1}{(x_k^+)^{r-1}}-\frac{1}{(x_k^-)^{r-1}}\right\}, \tag{2.5.26}$$

$$E\equiv g_B^2Q_2=\sum_{k=1}^{M}ig_B^2\left\{\frac{1}{x_k^+}-\frac{1}{x_k^-}\right\}=\sum_{k=1}^{M}\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p_k}{2}}-1\,, \tag{2.5.27}$$

with $E=\Delta-\Delta_0$. The motivation for defining charges as in (2.5.26), is to identify them with integral equation arising from finite-gap formulation of classical string theory. As discussed in the previous subsection, there are also various evidences for the expression for spin chain energy (2.5.27).

Let us make connection with the above formula with one-loop results. This can be done by the following reduction

$$u = x + \frac{\lambda}{16\pi^2}\frac{1}{x} = \frac{1}{2}\cot\left(\frac{p}{2}\right)\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p}{2}} \quad \underset{\text{one-loop}}{\longrightarrow} \quad u = x = \frac{1}{2}\cot\left(\frac{p}{2}\right), \qquad (2.5.28)$$

and

$$\Delta = \sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p}{2}} \quad \underset{\text{one-loop}}{\longrightarrow} \quad \Delta = 1 + \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2}\,, \qquad (2.5.29)$$

which agree with (2.2.12), (2.2.21) and (2.1.28). It is important to notice that the distinction between u and x starts from two loops in λ.

2.5.3 All-loop Bethe Ansatz in the full sector

The all-loop Bethe Ansatz in $su(2)$ sector (2.5.25) is part of all-loop Bethe Ansatz in the full $psu(2,2|4)$ sector [9]. Historically speaking, this is generalization of one-loop Bethe Ansatz in the full $psu(2,2|4)$ sector constructed in [114] based on earlier works on integrable aspects of QCD [129, 130, 131, 132, 133] and dilatation operator in the full sector [113]. In [9], Beisert and Staudacher proposed the expressions of Bethe Ansatz such that in thermodynamic limit they agree with the finite-gap formulation of superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$ [28, 29], which will be the issue of Chapter 3. We just cite their results in this section, so please consult the paper [9] for details and more justification.

We have several remarks on the all-loop Bethe Ansatz in the full sector:

- We need several species of Bethe roots when the rank of gauge group is greater than one. This means we have to use the nested Bethe Ansatz.

- It is convenient to work with $su(2,2|4)$ algebra with $u(1)$ constraint rather than $psu(2,2|4)$. Since the rank of $su(2,2|4)$ is 7, we need seven species of Bethe roots which are all independent.

- Cartan matrix of $su(2,2|4)$ superalgebra is not unique. We have to specify a particular expression of Cartan matrix.

- We denote the momentum-carrying roots by $x_{4,j}^{\pm}$. Other roots $(x_1\,, x_2\,, x_3\,, x_5\,, x_6\,, x_7)$ change the flavor of impurity. We also denote the number of the a-th roots by K_a. One can also construct Bethe vectors by analogy with (2.3.12). They are eigenstates of the Cartan subalgebra of $su(2,2|4)$, as they were in the $su(2)$ case (2.3.13).[4]

[4]Note that eigenstates of the Cartan subalgebra do not always belong to a single irreducible representation in general.

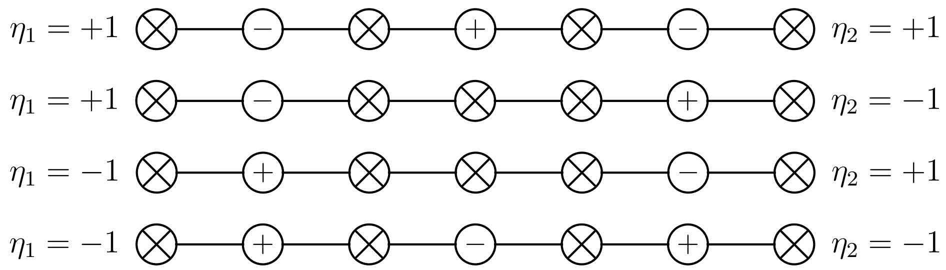


Figure 2.6: Dynkin diagrams of $su(2,2|4)$ for the gradings $\eta_1, \eta_2 = \pm 1$. Each node indicates that the corresponding diagonal element of the Cartan matrix is ± 2 or zero.

- If we set $\eta_1 = \eta_2$ and $K_1 = K_2 = K_3 = K_5 = K_6 = K_7 = 0$ in the following results, they reduce to the Bethe Ansatz equation in $su(2)$ sector.

We specify a Cartan matrix of $su(2,2|4)$ as

$$M = \begin{pmatrix} & +\eta_1 & & & & & \\ +\eta_1 & -2\eta_1 & +\eta_1 & & & & \\ & +\eta_1 & & -\eta_1 & & & \\ & & -\eta_1 & +\eta_1 + \eta_2 & -\eta_2 & & \\ & & & -\eta_2 & & +\eta_2 & \\ & & & & +\eta_2 & -2\eta_2 & +\eta_2 \\ & & & & & +\eta_2 & \end{pmatrix}, \tag{2.5.30}$$

and the corresponding Dynkin diagrams are shown in 2.6. The variables $\eta_1\,, \eta_2$ take values ± 1. Different choices of signs $\eta_{1,2} = \pm 1$ are related by duality transformations [114, 28, 9].

The all-loop Bethe Ansatz equations in the full $su(2,2|4)$ sector is given as follows:

$$1=\prod_{j=1}^{K_4}\frac{x^+_{4,j}}{x^-_{4,j}}\,, \tag{2.5.31}$$

$$1=\prod_{j=1}^{K_2}\frac{u_{1,k}-u_{2,j}+\frac{i}{2}\eta_1}{u_{1,k}-u_{2,j}-\frac{i}{2}\eta_1}\prod_{j=1}^{K_4}\frac{1-g_B^2/2x_{1,k}(x^+_{4,j})^{\eta_1}}{1-g_B^2/2x_{1,k}(x^-_{4,j})^{\eta_1}}\,, \tag{2.5.32}$$

$$1=\prod_{\substack{j=1\\ j\neq k}}^{K_2}\frac{u_{2,k}-u_{2,j}-i\eta_1}{u_{2,k}-u_{2,j}+i\eta_1}\prod_{j=1}^{K_3}\frac{u_{2,k}-u_{3,j}+\frac{i}{2}\eta_1}{u_{2,k}-u_{3,j}-\frac{i}{2}\eta_1}\prod_{j=1}^{K_1}\frac{u_{2,k}-u_{1,j}+\frac{i}{2}\eta_1}{u_{2,k}-u_{1,j}-\frac{i}{2}\eta_1}\,, \tag{2.5.33}$$

$$1=\prod_{j=1}^{K_2}\frac{u_{3,k}-u_{2,j}+\frac{i}{2}\eta_1}{u_{3,k}-u_{2,j}-\frac{i}{2}\eta_1}\prod_{j=1}^{K_4}\frac{x_{3,k}-(x^+_{4,j})^{\eta_1}}{x_{3,k}-(x^-_{4,j})^{\eta_1}}\,, \tag{2.5.34}$$

$$\begin{aligned}\left(\frac{x^+_{4,k}}{x^-_{4,k}}\right)^L=&\prod_{\substack{j=1\\ j\neq k}}^{K_4}\left(\frac{(x^+_{4,k})^{\eta_1}-(x^-_{4,j})^{\eta_1}}{(x^-_{4,k})^{\eta_2}-(x^+_{4,j})^{\eta_2}}\,\frac{1-g_B^2/2x^+_{4,k}x^-_{4,j}}{1-g_B^2/2x^-_{4,k}x^+_{4,j}}\,\sigma^2(x_{4,k},x_{4,j})\right)\\ &\times\prod_{j=1}^{K_1}\frac{1-g_B^2/2(x^-_{4,k})^{\eta_1}x_{1,j}}{1-g_B^2/2(x^+_{4,k})^{\eta_1}x_{1,j}}\prod_{j=1}^{K_3}\frac{(x^-_{4,k})^{\eta_1}-x_{3,j}}{(x^+_{4,k})^{\eta_1}-x_{3,j}}\prod_{j=1}^{K_5}\frac{(x^-_{4,k})^{\eta_2}-x_{5,j}}{(x^+_{4,k})^{\eta_2}-x_{5,j}}\prod_{j=1}^{K_7}\frac{1-g_B^2/2(x^-_{4,k})^{\eta_2}x_{7,j}}{1-g_B^2/2(x^+_{4,k})^{\eta_2}x_{7,j}}\,,\end{aligned} \tag{2.5.35}$$

$$1=\prod_{j=1}^{K_6}\frac{u_{5,k}-u_{6,j}+\frac{i}{2}\eta_2}{u_{5,k}-u_{6,j}-\frac{i}{2}\eta_2}\prod_{j=1}^{K_4}\frac{x_{5,k}-(x^+_{4,j})^{\eta_2}}{x_{5,k}-(x^-_{4,j})^{\eta_2}}\,, \tag{2.5.36}$$

$$1=\prod_{\substack{j=1\\ j\neq k}}^{K_6}\frac{u_{6,k}-u_{6,j}-i\eta_2}{u_{6,k}-u_{6,j}+i\eta_2}\prod_{j=1}^{K_5}\frac{u_{6,k}-u_{5,j}+\frac{i}{2}\eta_2}{u_{6,k}-u_{5,j}-\frac{i}{2}\eta_2}\prod_{j=1}^{K_7}\frac{u_{6,k}-u_{7,j}+\frac{i}{2}\eta_2}{u_{6,k}-u_{7,j}-\frac{i}{2}\eta_2}\,, \tag{2.5.37}$$

$$1=\prod_{j=1}^{K_6}\frac{u_{7,k}-u_{6,j}+\frac{i}{2}\eta_2}{u_{7,k}-u_{6,j}-\frac{i}{2}\eta_2}\prod_{j=1}^{K_4}\frac{1-g_B^2/2x_{7,k}(x^+_{4,j})^{\eta_2}}{1-g_B^2/2x_{7,k}(x^-_{4,j})^{\eta_2}}\,, \tag{2.5.38}$$

and higher charges are given by

$$Q_r=\frac{i}{r-1}\sum_{j=1}^{K_4}\left(\frac{1}{(x^+_{4,j})^{r-1}}-\frac{1}{(x^-_{4,j})^{r-1}}\right), \tag{2.5.39}$$

$$E=g_B^2Q_2=ig_B^2\sum_{j=1}^{K_4}\left(\frac{1}{x^+_{4,j}}-\frac{1}{x^-_{4,j}}\right), \tag{2.5.40}$$

with $E=\Delta-\Delta_0$.

As usual, Dynkin index is defined as the coefficient of weight vectors $\vec{\mu}$ expanded in terms of fundamental weights $\vec{\mu}^{(a)}$,

$$\vec{\mu} = \sum_{a=1}^{7} r_a \vec{\mu}^{(a)} \,, \qquad H_b \, |\vec{\mu}\rangle = \mu_b \, |\vec{\mu}\rangle \,, \tag{2.5.41}$$

where $\{H_b\}$ form Cartan subalgebra of $su(2,2|4)$. One can read off the Dynkin labels of a state by expanding the Bethe Ansatz equations around $u_{a,j} \sim x_{a,j} \sim \infty$, which reads:

$$\begin{aligned}
r_1 &= -\eta_1 K_2 - \frac{1}{2}\,\eta_1 E \,,\\
r_2 &= -\eta_1 K_3 + 2\eta_1 K_2 - \eta_1 K_1 \,,\\
r_3 &= +\eta_1 K_4 - \eta_1 K_2 + \frac{1}{2}\,\eta_1 E \,,\\
r_4 &= +L - (\eta_1 + \eta_2) K_4 + \eta_1 K_3 + \eta_2 K_5 + \frac{1}{4}\,(2 - \eta_1 - \eta_2) E \,,\\
r_5 &= +\eta_2 K_4 - \eta_2 K_6 + \frac{1}{2}\,\eta_2 E \,,\\
r_6 &= -\eta_2 K_5 + 2\eta_2 K_6 - \eta_2 K_7 \,,\\
r_7 &= -\eta_2 K_6 - \frac{1}{2}\,\eta_2 E \,.
\end{aligned} \tag{2.5.42}$$

To reduce $su(2,2|4)$ into $psu(2,2|4)$, we need to impose the following constraint among central elements:

$$\eta_1 r_1 - \eta_1 r_3 + \eta_2 r_5 - \eta_2 r_7 = 0. \tag{2.5.43}$$

Chapter 3

Classical string and integrability

We discuss superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ background under the classical approximation $\lambda \gg 1$. This theory is known to be integrable in the sense that their equations of motion can be rewritten in terms of a Lax pair [24]. This fact allows us to construct classical string solutions in an abstract manner known as finite-gap method. Remarkably, Kazakov, Marshakov, Minahan and Zarembo found that the finite-gap formulation is quite useful in comparing the spectrum of string and gauge theories [11], which is the main topic of this chapter.

3.1 Integrability of classical string on $\mathbf{AdS}_5 \times \mathbf{S}^5$

The $\mathrm{AdS}_5 \times \mathrm{S}^5$ space supported by RR flux is hard to quantize in the Neveu-Schwarz-Ramond formalism, due to the problem of defining RR vertex operator in curved backgrounds. Direct application of the Green-Schwarz formalism to this background is neither practical for the purpose of writing down the action in superspace coordinates (x, θ) and identifying supergravity fields.

To circumvent the problem, Metsaev and Tseytlin constructed the Green-Schwarz (κ-symmetric) superstring action on the coset superspace

$$\frac{SU(2,2|4)}{SO(1,4) \times SO(5)} \quad \sim \quad \left[\mathrm{AdS}_5 \times \mathrm{S}^5 \text{ background}\right] \times U(1)\,, \tag{3.1.1}$$

up to $\mathcal{O}\left(\theta^4\right)$ [22]. This was subsequently generalized to the full order of θ in [23]. The action of Metsaev and Tseytlin was further refined in [134] by performing Wick rotation

$$\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)} \quad \longrightarrow \quad \frac{PSL(4|4;\mathbb{R})}{Sp(4;\mathbb{R}) \times Sp(4;\mathbb{R})}\,, \tag{3.1.2}$$

such that the superconformal symmetry is manifest.

The classical integrability of the Metsaev-Tseytlin action on $\mathrm{AdS}_5 \times \mathrm{S}^5$ was discovered in [24]. Below we follow the discussion of [28] where they employed the strategy of [134]. The difference of signs between (3.1.1) and (3.1.2) is unimportant for showing the integrability.

We begin with a brief introduction of supermatrices. For details, see [28, 135] and references therein. Let η be the grading operator,

$$\eta \equiv \left(\begin{array}{c|c} +\mathbf{1}_d & \\ \hline & -\mathbf{1}_d \end{array}\right), \tag{3.1.3}$$

where $\mathbf{1}_d$ is the identity matrix in d dimensions. Define supertrace as

$$\mathrm{str}\, A = \mathrm{tr}\,(\eta A) = \mathrm{tr}\,(A\eta)\ , \tag{3.1.4}$$

where the right hand side is a regular trace. The superdeterminant is defined by

$$\mathrm{sdet}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \equiv \frac{\det(A - BD^{-1}C)}{\det D} = \frac{\det A}{\det(D - CA^{-1}B)}\,. \tag{3.1.5}$$

There are several identities on superdeterminant, like

$$\mathrm{sdet}\,(AB) = \mathrm{sdet}\,(A)\,\mathrm{sdet}\,(B)\,, \qquad \mathrm{sdet}\ \exp(A) = \exp\ \mathrm{str}\,(A)\,. \tag{3.1.6}$$

Note also

$$\mathrm{str}\,(\mathbf{1}_{2d}) = 0, \qquad \mathrm{sdet}\ (\xi\, \mathbf{1}_{2d}) = 1 \quad (\xi : \text{constant})\,. \tag{3.1.7}$$

The supertranspose is defined as

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)^{\mathsf{ST}} = \left(\begin{array}{c|c} A^{\mathsf{T}} & C^{\mathsf{T}} \\ \hline -B^{\mathsf{T}} & D^{\mathsf{T}} \end{array}\right). \tag{3.1.8}$$

The supertranspose is an operation of $\mathbb{Z}_4$ grading:

$$(A^{\mathsf{ST}})^{\mathsf{ST}} = \eta A \eta, \qquad (\eta A \eta)^2 = A. \tag{3.1.9}$$

The supergroup $SL(4|4;\mathbb{R})$ is parametrized by supermatrices of the form

$$g = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \qquad \mathrm{sdet}\, g = 1. \tag{3.1.10}$$

If we impose reality conditions, the matrices A and D are real while Grassmann matrices B and C are related by $B = C^\dagger$. The matrices in $PSL(4|4;\mathbb{R})$ is a quotient of (3.1.10) by matrices proportional to $\mathbf{1}_8$.

It is useful to introduce $\mathbb{Z}_4$ grading operator ω by

$$\omega \cdot g = \left(\begin{array}{c|c} EA^{\mathsf{T}}E & -EC^{\mathsf{T}}E \\ \hline EB^{\mathsf{T}}E & ED^{\mathsf{T}}E \end{array}\right), \qquad E = \begin{pmatrix} 0 & -1 & & \\ +1 & 0 & & \\ & & 0 & -1 \\ & & +1 & 0 \end{pmatrix}. \tag{3.1.11}$$

It then follows $E^2 = -1$, $\omega^4 = 1$, and $\omega \cdot [g_1 , g_2] = [\omega \cdot g_1 , \omega \cdot g_2]$. Since ω is an automorphism, elements of $PSL(4|4;\mathbb{R})$ can be classified with respect to its action, as

$$g = g^{(0)} + g^{(1)} + g^{(2)} + g^{(3)} , \qquad \omega \cdot g^{(k)} \equiv (-1)^{k/2} g^{(k)} . \tag{3.1.12}$$

By construction, E is invariant under the map $E \mapsto hEh^{\mathsf{ST}}$ for $h \in Sp(4;\mathbb{R}) \times Sp(4;\mathbb{R})$. It shows the denominator group of the coset is spanned by $g^{(0))}$.

To write down the action, let us introduce a supermatrix-valued function on the worldsheet $g(\tau, \sigma) \in PSL(4|4;\mathbb{R})$, with the periodicity

$$g(\tau, \sigma + 2\pi) = g(\tau, \sigma)h(\tau, \sigma), \qquad h(\tau, \sigma) \in Sp(4;\mathbb{R}) \times Sp(4;\mathbb{R}) . \tag{3.1.13}$$

We also introduce current of g by

$$J = -g^{-1}dg , \tag{3.1.14}$$

which is invariant under the left multiplication $g \mapsto Gg$ for $G \in PSL(4|4;\mathbb{R})$. One easily finds that this current obeys

$$dJ - J \wedge J = 0, \qquad \mathrm{str}\, J = 0. \tag{3.1.15}$$

The $\mathbb{Z}_4$ grading ω decomposes this current as

$$J = J^{(0)} + J^{(1)} + J^{(2)} + J^{(3)} \equiv H + Q_1 + P + Q_2 . \tag{3.1.16}$$

The flatness condition (or Bianchi identity) (3.1.15) can be decomposed similarly. This decomposition (3.1.16) as well as the condition $\mathrm{str}\, g = 1$ imply

$$\mathrm{str}\, H = \mathrm{str}\, Q_1 = \mathrm{str}\, P = \mathrm{str}\, Q_2 = 0. \tag{3.1.17}$$

The superstring action is given by [134, 28]

$$S = \frac{\sqrt{\lambda}}{4\pi} \int \mathrm{str}\,(P \wedge *P - Q_1 \wedge *Q_2) + \Lambda \wedge \mathrm{str}\, P , \tag{3.1.18}$$

where Λ is a Lagrange multiplier to guarantee the supertraceness of $g \in PSL(4|4;\mathbb{R})$. From infinitesimal variance $g \mapsto (1 + \delta G)g$, one can derive its equations of motion as

$$0 = P \wedge Q_2 - *P \wedge Q_2 + Q_2 \wedge P - Q_2 \wedge *P, \tag{3.1.19}$$

$$d*P = H \wedge *P + Q_1 \wedge Q_1 + *P \wedge H - Q_2 \wedge Q_2 + d\Lambda, \tag{3.1.20}$$

$$0 = P \wedge Q_1 + *P \wedge Q_1 + Q_1 \wedge P + Q_1 \wedge *P. \tag{3.1.21}$$

The above equations are concisely summarized as

$$d*K - J \wedge *K - *K \wedge J = 0, \qquad K \equiv P + \frac{1}{2} *Q_1 - \frac{1}{2} *Q_2 - *\Lambda. \tag{3.1.22}$$

If we introduce the left invariant current $k \equiv gKg^{-1}$,[1] the equation of motion (3.1.22) becomes

$$d * k = 0. \tag{3.1.23}$$

In search of classical integrability, Bena, Polchinski and Roiban tried an Ansatz for Lax connection [24]. In our language, it takes the form

$$a(z) \equiv \alpha(z)p + \beta(z)\left(*p - \Lambda\right) + \gamma(z)\left(q_1 + q_2\right) + \delta(z)\left(q_1 - q_2\right), \tag{3.1.24}$$

where we defined left invariant currents by $p \equiv gPg^{-1}$ and $q_{1,2} \equiv gQ_{1,2}g^{-1}$. The condition $da + a \wedge a = 0$ results in six equations for four functions with certain amount of redundancy. They found there exists one-parameter family of solutions given by

$$\alpha(z) = 1 - \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right), \qquad \beta(z) = \frac{1}{2}\left(z^2 - \frac{1}{z^2}\right), \tag{3.1.25}$$

$$\gamma(z) = 1 - \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad \delta(z) = \frac{1}{2}\left(z - \frac{1}{z}\right). \tag{3.1.26}$$

We can reexpress the connection (3.1.24) in terms of the right invariant currents, as

$$A(x) = H + \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)P - \frac{1}{2}\left(z^2 - \frac{1}{z^2}\right)\left(*P - \Lambda\right) + \frac{1}{z}\,Q_1 + z\,Q_2\,. \tag{3.1.27}$$

This connection is generalization of J in (3.1.15), and satisfy

$$dA(z) - A(z) \wedge A(z) = 0, \qquad \operatorname{str} A(z) = 0, \tag{3.1.28}$$

as well as $A(z = 1) = J$.

From Lax connection, one can construct an infinite number of conserved charges. To see it, let us define the Wilson loop around Lax connection

$$\Omega_0(z) \equiv \bar{P}\exp\left(\oint_0^{2\pi} d\sigma\, A_\sigma\left(\tau, \sigma\,; z\right)\right), \tag{3.1.29}$$

where $\bar{P}$ stands for the anti-path-ordering where larger values of σ stands to the left. Written explicitly, (3.1.29) is,

$$\bar{P}\exp\left(\oint_a^b d\sigma A_\sigma\left(\sigma\right)\right) \equiv 1 - \oint_a^b d\sigma_1 A_\sigma\left(\sigma_1\right) + \oint_a^b d\sigma_1 \oint_a^{\sigma_1} d\sigma_2\, A_\sigma\left(\sigma_1\right) A_\sigma\left(\sigma_2\right) + \cdots. \tag{3.1.30}$$

The monodromy matrix is then defined as

$$\Omega(z) \equiv \Omega_0(1)^{-1}\Omega_0(z). \tag{3.1.31}$$

[1] Note that the element K also transforms covariantly under $Sp(4;\mathbb{R}) \times Sp(4;\mathbb{R})$ [24].

Owing to the vanishing curvature condition (3.1.28), the monodromy matrix is independent of τ [136]. It tells us, in particular, that if we expand it around some point $z = z_0$

$$\Omega(z) \equiv \sum_n Q_n (z - z_0)^n \,, \tag{3.1.32}$$

all coefficients Q_n are independent of worldsheet time, and hence conserved.

For later use, we introduce x variable as

$$x \equiv \frac{1 + z^2}{1 - z^2} \qquad \text{or} \qquad z \equiv \sqrt{\frac{x - 1}{x + 1}} \,, \tag{3.1.33}$$

which satisfy

$$\frac{dx}{1 - 1/x^2} = \frac{dz}{z} \,. \tag{3.1.34}$$

This parameter x will be shown to correspond to x given in (2.5.14).

3.2 Polyakov action on $\mathrm{AdS}_5 \times \mathrm{S}^5$

Let us make a few remarks on the truncation of classical string action. In classical theory, we can freeze out the degrees of freedom in any particular directions and consider only string solutions which move in remaining directions. This amounts to truncation of classical string action on $\mathrm{AdS}_5 \times \mathrm{S}^5$ to its subspaces, like bosonic part of $\mathrm{AdS}_5 \times \mathrm{S}^5$, $\mathbb{R}_\mathrm{t} \times \mathrm{S}^3 \subset \mathrm{AdS}_5 \times \mathrm{S}^5$ or $\mathrm{AdS}_3 \times \mathrm{S}^1 \subset \mathrm{AdS}_5 \times \mathrm{S}^5$. By such truncation, the Lax pair formulation becomes much simplified. Note that it does not work at all in quantum theory, because loop integrals must involve all virtual particles of the theory.

The truncation is also useful in making comparison of the spectrum of gauge theory operators. In particular, from the correspondence of global charges, one may guess that some operators in $su(2)$ sector (2.1.29) would correspond to classical strings on $\mathbb{R}_\mathrm{t} \times \mathrm{S}^3$, and some operators in $sl(2)$ sector (2.1.30) to strings on $\mathrm{AdS}_3 \times \mathrm{S}^1$.

Now we fix our notation for classical string theory on the bosonic part of $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime. We define bosonic $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime by embedding into $\mathbb{C}^{1,2} \times \mathbb{C}^3$ whose coordinates are denoted by η_0, η_1, η_2 and ξ_1, ξ_2, ξ_3. We set the radius of AdS_5 and S^5 to unity, then

$$\vec{\eta}^{\,*} \cdot \vec{\eta} \equiv -\left|\eta_0\right|^2 + \left|\eta_1\right|^2 + \left|\eta_2\right|^2 = -1, \qquad \vec{\xi}^{\,*} \cdot \vec{\xi} \equiv \left|\xi_1\right|^2 + \left|\xi_2\right|^2 + \left|\xi_3\right|^2 = 1. \tag{3.2.1}$$

Expressed in polar coordinates,

$$\begin{aligned} &\eta_0 = \cosh\rho\, e^{it}, \quad \eta_1 = \sinh\rho\, \cos\theta\, e^{i\phi_1}, \quad \eta_2 = \sinh\rho\, \sin\theta\, e^{i\phi_2}, \\ &\xi_1 = \cos\gamma\, e^{i\varphi_1}, \quad \xi_2 = \sin\gamma\, \cos\psi\, e^{i\varphi_2}, \quad \xi_3 = \sin\gamma\, \sin\psi\, e^{i\varphi_3}. \end{aligned} \tag{3.2.2}$$

Polyakov action (bosonic classical string action) on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is[2]

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left[\gamma^{ab} \left(\partial_a \vec{\eta}^* \cdot \partial_b \vec{\eta} + \partial_a \vec{\xi}^* \cdot \partial_b \vec{\xi} \right) + \widetilde{\Lambda} \left(\vec{\eta}^* \cdot \vec{\eta} + 1 \right) + \Lambda \left(\vec{\xi}^* \cdot \vec{\xi} - 1 \right) \right]. \tag{3.2.3}$$

where $\widetilde{\Lambda}$ and Λ are Lagrange multipliers. The string equations of motion follow as

$$\partial_a \partial^a \vec{\eta} - \widetilde{\Lambda}\, \vec{\eta} = 0, \qquad \widetilde{\Lambda} = +\, \partial_a \vec{\eta}^* \cdot \partial^a \vec{\eta}\,, \tag{3.2.4}$$

$$\partial_a \partial^a \vec{\xi} - \Lambda\, \vec{\xi} = 0, \qquad \Lambda = -\, \partial_a \vec{\xi}^* \cdot \partial^a \vec{\xi}\,. \tag{3.2.5}$$

We take conformal gauge $\gamma^{\tau\tau} = -1, \gamma^{\sigma\sigma} = +1$ and $\gamma^{\sigma\tau} = \gamma^{\tau\sigma} = 0$. Then Virasoro constraints read

$$\begin{aligned} 0 \ &= \mathcal{T}_{\sigma\sigma} = \mathcal{T}_{\tau\tau} = \frac{\delta^{ab}}{2} \left(\partial_a \vec{\eta}^* \cdot \partial_b \vec{\eta} + \partial_a \vec{\xi}^* \cdot \partial_b \vec{\xi} \right), \\ 0 \ &= \mathcal{T}_{\tau\sigma} = \mathcal{T}_{\sigma\tau} = \mathrm{Re} \left(\partial_\tau \vec{\eta}^* \cdot \partial_\sigma \vec{\eta} + \partial_\tau \vec{\xi} \cdot \partial_\sigma \vec{\xi}^* \right). \end{aligned} \tag{3.2.6}$$

We define conserved charges by

$$E \equiv \frac{\sqrt{\lambda}}{\pi}\, \mathcal{E} \ = \frac{\sqrt{\lambda}}{2\pi} \oint_0^{2\pi} d\sigma\ \mathrm{Im} \left(\eta_0^*\, \partial_\tau \eta_0 \right), \tag{3.2.7}$$

$$S_j \equiv \frac{\sqrt{\lambda}}{\pi}\, \mathcal{S}_j \ = \frac{\sqrt{\lambda}}{2\pi} \oint_0^{2\pi} d\sigma\ \mathrm{Im} \left(\eta_j^*\, \partial_\tau \eta_j \right) \qquad (j = 1, 2)\,, \tag{3.2.8}$$

$$J_k \equiv \frac{\sqrt{\lambda}}{\pi}\, \mathcal{J}_k = \frac{\sqrt{\lambda}}{2\pi} \oint_0^{2\pi} d\sigma\ \mathrm{Im} \left(\xi_k^*\, \partial_\tau \xi_k \right) \qquad (k = 1, 2, 3)\,, \tag{3.2.9}$$

and winding numbers by

$$N_t \equiv \frac{1}{2\pi} \Big\{ t(\tau, \sigma + 2\pi) - t(\tau, \sigma) \Big\}\,, \tag{3.2.10}$$

$$N_{\phi_j} \equiv \frac{1}{2\pi} \Big\{ \phi_j(\tau, \sigma + 2\pi) - \phi_j(\tau, \sigma) \Big\}\,, \tag{3.2.11}$$

$$N_{\varphi_k} \equiv \frac{1}{2\pi} \Big\{ \varphi_k(\tau, \sigma + 2\pi) - \varphi_k(\tau, \sigma) \Big\}\,. \tag{3.2.12}$$

The angular momenta S_j and J_k are semiclassically quantized to integer values. For strings to be closed, the winding numbers N_{ϕ_j} and N_{φ_k} must be integers. The timelike winding N_t must vanish, namely $t\,(\tau, \sigma + 2\pi) = t\,(\tau, \sigma)$ (*not* mod 2π), because t and $t + 2\pi$ are not the same point of the AdS spacetime, as mentioned in Section 1.2.2.

In later sections, we will mainly discuss its subspaces $\mathbb{R}_{\mathrm{t}} \times \mathrm{S}^3$ and $\mathrm{AdS}_3 \times \mathrm{S}^1$. Their metrics are given by

$$ds^2_{\mathbb{R}_t \times \mathrm{S}^3} = -dt^2 + d\gamma^2 + \cos^2\!\gamma\, d\varphi_1^2 + \sin^2\!\gamma\, d\varphi_2^2\,, \tag{3.2.13}$$

$$d\widetilde{s}^2_{\mathrm{AdS}_3 \times \mathrm{S}^1} = -\cosh^2\!\tilde{\rho}\, d\tilde{t}^2 + d\tilde{\rho}^2 + \sinh^2\!\tilde{\rho}\, d\tilde{\phi}_1^2 + d\tilde{\varphi}_1^2\,. \tag{3.2.14}$$

The two metrics are related by an analytic continuation:

$$\tilde{\rho} = i\gamma, \quad \tilde{t} = \varphi_1, \quad \tilde{\phi}_1 = \varphi_2, \quad \tilde{\varphi}_1 = t \qquad \Longrightarrow \qquad d\widetilde{s}^2_{\mathrm{AdS}_3 \times \mathrm{S}^1} = -ds^2_{\mathbb{R}_t \times \mathrm{S}^3}\,. \tag{3.2.15}$$

[2]This action also has an infinite number of conserved charges [137].

3.3 Finite-gap formulation

We are going to review a method called finite-gap formulation, which makes full use of integrability of the theory. This helps to construct general classical string solutions in the language of algebraic geometry.

Originally, the term 'finite-gap' signifies the band structure in energy eigenvalues that often appears in the Schrödinger equation with periodic potential [138]. The (continuous part of) energy spectrum typically consists of a sequence of segments

$$\cdots\,,\ [E_{2k+1}\,, E_{2k}]\,,\ \cdots\,,\ [E_3\,, E_2]\,,\ [E_1\,, +\infty]\,. \tag{3.3.1}$$

The length of the k-th segment tends to shrink as k increases. When the number of segments with nonzero width is finite, the periodic potential is called finite-gap potential.

For a wide class of integrable models including classical string theory on $\mathbb{R}_{\rm t}\times\mathrm{S}^3$, the equation of motion is nonlinear. In such cases, one can separate the equation of motion into the kinetic term and the potential term, such that the potential itself depends on a particular choice of solution. For instance, the equation (3.2.5) can be regarded as

$$\partial_a\partial^a\vec{\xi}+\left(\partial_a\vec{\xi}^{\,*}\cdot\partial^a\vec{\xi}\right)\vec{\xi}=0 \quad\Leftrightarrow\quad \begin{cases}\partial_a\partial^a\vec{\psi}+V\vec{\psi}=0\\ \partial_a\vec{\xi}^{\,*}\cdot\partial^a\vec{\xi}\equiv V\end{cases}. \tag{3.3.2}$$

To solve these equations, one can firstly make an Ansatz for the potential V, and secondly solve the 'linear' Schrödinger equation. Of course, one must check if the solution is actually consistent with the defining equation of the potential.

It is known that integrable models usually have solutions which can be expressed by Riemann theta functions of genus $g\in\mathbb{Z}_{\geq 0}$. Roughly speaking the number of genus corresponds to the number of cuts (3.3.1), so one can classify these Ansätze with regard to the number of gaps the corresponding potential will produce. In this context, the term 'finite-gap solution' is used as algebro-geometric representation of 'Riemann theta' solutions for finite genus.

3.3.1 Lax pair and monodromy matrix

We review finite-gap formulation of classical string theory on $\mathbb{R}_{\rm t}\times\mathrm{S}^3$ [11, 31]. Polyakov action for a string staying at the center of AdS_5 reads, from (3.2.3),

$$S=\frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma\left[-\left(\partial_a t\right)^2+\partial_a\vec{\xi}^{*}\cdot\partial^a\vec{\xi}\right], \tag{3.3.3}$$

where we omit the term including the Lagrange multiplier. This action can be rewritten in an $SU(2)$-covariant manner. We introduce a group element $g\in SU(2)$ and Maurer-Cartan one-forms by

$$g=\begin{pmatrix}\xi_1 & -\xi_2^*\\ \xi_2 & \xi_1^*\end{pmatrix},\qquad j_a\equiv g^{-1}\partial_a g=\begin{pmatrix}A_a & -\bar{B}_a\\ B_a & \bar{A}_a\end{pmatrix},\qquad \ell_a\equiv\partial_a g\, g^{-1}=\begin{pmatrix}A_a & -\bar{C}_a\\ C_a & \bar{A}_a\end{pmatrix}. \tag{3.3.4}$$

where

$$A_a = \xi_1^* \partial_a \xi_1 + \xi_2^* \partial_a \xi_2 \,, \qquad B_a = \xi_1 \partial_a \xi_2 - \xi_2 \partial_a \xi_1 \,, \qquad C_a = \xi_1^* \partial_a \xi_2 - \xi_2 \partial_a \xi_1^* \,. \tag{3.3.5}$$

The action is rewritten as

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \left[(\partial_a t)^2 + \frac{1}{2} \operatorname{tr} \left(j_a^2 \right) \right] = -\frac{\sqrt{\lambda}}{4\pi} \int \left[dt \wedge *dt + \frac{1}{2} \operatorname{tr} \left(dj \wedge *dj \right) \right] . \tag{3.3.6}$$

In differential forms, the equation of motion and Bianchi identity are written as

$$d * t = 0 \,, \qquad d * j = 0 \,, \qquad dj + j \wedge j = 0 \,. \tag{3.3.7}$$

Written in components, they become

$$\partial_+ \partial_- t = 0, \qquad 0 = \partial_+ j_- + \partial_- j_+ = 0, \tag{3.3.8}$$

$$\partial_+ j_- - \partial_- j_+ + [j_+ \,, j_-] = 0 \,, \qquad \text{with} \quad j_\pm \equiv j_\tau \pm j_\sigma \,. \tag{3.3.9}$$

The equation of motion for t is solved as $t = \kappa\tau + \kappa'\sigma$. The condition of zero timelike winding (3.2.10) requires $\kappa' = 0$. Taking conformal gauge, Virasoro constraints read

$$\frac{1}{2} \operatorname{tr} \left(j_\pm^2 \right) = -\kappa^2 \,. \tag{3.3.10}$$

One can rewrite the action (3.3.6) in terms of ℓ_a , the right invariant current. It gives

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \left[(\partial_a t)^2 + \frac{1}{2} \operatorname{tr} \left(\ell_a^2 \right) \right] = -\frac{\sqrt{\lambda}}{4\pi} \int \left[dt \wedge *dt + \frac{1}{2} \operatorname{tr} \left(d\ell \wedge *d\ell \right) \right] . \tag{3.3.11}$$

The actions (3.3.6) or (3.3.11) are invariant under a global $SU(2)_L \times SU(2)_R$ symmetry,

$$g \quad \mapsto \quad U_L \, g \, U_R \,. \tag{3.3.12}$$

The corresponding Nöther charges are

$$\mathcal{Q}_j \equiv \frac{\sqrt{\lambda}}{4\pi} \oint d\sigma * j = -\frac{\sqrt{\lambda}}{4\pi} \oint d\sigma \, j_\tau \qquad \text{for} \quad SU(2)_R \,, \tag{3.3.13}$$

$$\mathcal{Q}_\ell \equiv \frac{\sqrt{\lambda}}{4\pi} \oint d\sigma * \ell = -\frac{\sqrt{\lambda}}{4\pi} \oint d\sigma \, \ell_\tau \qquad \text{for} \quad SU(2)_L \,, \tag{3.3.14}$$

which are indeed conserved because $d * j = 0$ and $d * \ell = 0$. From the parametrization (3.3.4), we see that the pair $(\xi_1 \,, -\xi_2^*)$ form a doublet under the right shift $g \mapsto g \, (1 + \epsilon\sigma^3)$, while the pair $(\xi_1 \,, \xi_2)$ form a doublet under the left shift $g \mapsto (1 + \epsilon\sigma^3) \, g$. Thus, if the solutions are the highest weight states of $SU(2)$, we have

$$\mathcal{Q}_j = -\frac{i\sigma^3}{2} \left(J_1 - J_2 \right) , \qquad \mathcal{Q}_\ell = -\frac{i\sigma^3}{2} \left(J_1 + J_2 \right) , \qquad \sigma^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.3.15}$$

It should be noted that we can also regard Polyakov action on $\mathbb{R}_t \times S^3$ as $O(4)$ sigma model [27]. We introduce a group element $h \in O(4)$ by

$$h_{ij} = \delta_{ij} - 2X_i X_j \,, \qquad \text{where} \;\; \sum_{i=1}^{4} X_i^2 = 1, \tag{3.3.16}$$

which obeys the relations $h = h^T = h^{-1}$ and the eigenvalues of h are $(-1, 1, 1, 1)$. One advantage of this parametrization is that two Maurer-Cartan one-forms defined by $j = h^{-1}dh$ and $\ell = dh\, h^{-1}$ coincide:

$$(j_a)_{ij} = 2\,(X_i\, \partial_a X_j - X_j\, \partial_a X_i) = -(\ell_a)_{ij} \,. \tag{3.3.17}$$

Following discussion similar to that of Section 3.1, the equation (3.3.7) can be extended to the flatness condition for one-parameter family of conserved currents

$$J(x) \equiv \frac{j - x * j}{1 - x^2} \qquad \text{or} \qquad J_\pm(x) \equiv \frac{j_\pm}{1 \mp x} \,,$$
$$dJ(x) - J(x) \wedge J(x) = 0. \tag{3.3.18}$$

Alternatively, we may introduce a pair of Lax connections (L, M) which satisfy the following auxiliary linear equations:

$$\partial_\sigma \psi - L\psi = \left[\partial_\sigma - \frac{1}{2}\left(\frac{j_-}{1+x} - \frac{j_+}{1-x} \right) \right] \psi = 0, \tag{3.3.19}$$

$$\partial_\tau \psi - M\psi = \left[\partial_\tau + \frac{1}{2}\left(\frac{j_-}{1+x} + \frac{j_+}{1-x} \right) \right] \psi = 0. \tag{3.3.20}$$

Then, one can show that the zero-curvature condition (3.3.18) is equivalent to the compatibility condition of these two equations

$$[\partial_\sigma - L \,,\; \partial_\tau - M] = 0. \tag{3.3.21}$$

As in (3.1.29), the Wilson loop operator (or monodromy matrix),

$$\Omega(x) \equiv \bar{P} \exp \left(\int_0^{2\pi} d\sigma\, L\, (\tau, \sigma\,; x) \right), \tag{3.3.22}$$

is independent of τ, because of the zero-curvature condition, or equivalently

$$[d - J(x), \Omega(x)] = 0. \tag{3.3.23}$$

In the right hand side of (3.3.22), $L\,(\tau, \sigma\,; x)$ is a traceless 2×2 matrix for the parametrization (3.3.4) and a traceless 4×4 matrix for the parametrization (3.3.16). By construction, $\Omega(x)$ is holomorphic in x except at $x = \pm 1$. From $\operatorname{tr} j_\pm = 0$, $\Omega(x)$ is unimodular, *i.e.* $\det \Omega = 1$. Suppose $L(\tau, \sigma)$ can be diagonalized at all values of σ by a gauge transformation

$$L(\tau, \sigma) \quad \mapsto \quad L_{\rm diag}(\tau, \sigma) \equiv U(\tau, \sigma) L(\tau, \sigma)\, U^{-1}(\tau, \sigma) + \partial_\sigma U(\tau, \sigma)\, U^{-1}(\tau, \sigma), \tag{3.3.24}$$

then $\Omega(x)$ is diagonalized as

$$\Omega(x) \sim \begin{cases} \text{diag}\,(e^{ip}, e^{-ip}) & \text{for } \ SU(2), \\ \text{diag}\,(e^{ip_L}, e^{-ip_L}, e^{ip_R}, e^{-ip_R}) & \text{for } \ O(4). \end{cases} \tag{3.3.25}$$

The eigenvalues $p(x)$ or $p_{L,R}(x)$ are called quasi-momentum.

As discussed in [27], the symmetry (3.3.17) between the left and the right current relates p_L and p_R in $O(4)$ case, as

$$p_L(x) + p_R\,(1/x) = 2\pi m\,, \qquad (m \in \mathbb{Z})\,. \tag{3.3.26}$$

Thus, any solution of classical strings on $\mathbb{R}_\mathrm{t}{\times}\mathrm{S}^3$ can be represented by the corresponding expression of quasi-momentum $p(x) = p_L(x)$. Hereafter we only consider the $SU(2)$ parametrization of the classical string action on $\mathbb{R}_\mathrm{t} \times \mathrm{S}^3$.

3.3.2 Asymptotic behaviors

From the definition of Lax connections (3.3.19), one can derive asymptotic behaviors of quasi-momentum $p(x)$. This allows us to compute the conserved charges of classical string solution solely from the behavior of $p(x)$.

Around $x = \infty$, the monodromy matrix behaves as

$$\Omega(x) = \bar{P}\exp\left[\oint d\sigma\,\frac{1}{x}\,*j + \mathcal{O}\left(\frac{1}{x^2}\right)\right], \tag{3.3.27}$$

$$= 1 - \frac{1}{x}\oint d\sigma\,*j + \mathcal{O}\left(\frac{1}{x^2}\right), \tag{3.3.28}$$

$$= 1 + \frac{1}{x}\frac{4\pi\mathcal{Q}_j}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{x^2}\right), \tag{3.3.29}$$

where we used (3.3.13). Assuming the solutions are the highest weight states of $SU(2)$ as in (3.3.15), we obtain

$$p(x) = -\frac{1}{x}\frac{2\pi}{\sqrt{\lambda}}\,(J_1 - J_2) + \mathcal{O}\left(\frac{1}{x^2}\right) \qquad \text{as } \ x \to \infty, \tag{3.3.30}$$

where we used the freedom to choose the branch of logarithm such that the term of $\mathcal{O}(x^0)$ vanishes. The states other than the highest weight, can be obtained by $SU(2)_L \times SU(2)_R$ transformation.

Around $x = 0$, the monodromy matrix behaves as

$$g(\tau,\sigma)\Omega(x)\,g^{-1}(\tau,\sigma) = \bar{P}\exp\left[\oint d\sigma\,\,-x*\ell + \mathcal{O}\left(x^2\right)\right], \tag{3.3.31}$$

$$= 1 + x\oint d\sigma\,*\ell + \mathcal{O}\left(x^2\right), \tag{3.3.32}$$

$$= 1 - x\,\frac{4\pi\mathcal{Q}_\ell}{\sqrt{\lambda}} + \mathcal{O}\left(x^2\right), \tag{3.3.33}$$

where we used (3.3.14). For the highest states, we obtain

$$p(x) = 2\pi m + x\,\frac{2\pi}{\sqrt{\lambda}}\,(J_1 + J_2) + \mathcal{O}\left(x^2\right) \qquad \text{as} \;\; x \to 0, \tag{3.3.34}$$

where $m \in \mathbb{Z}$ is same as the one appeared in (3.3.26).

Note that the asymptotic behaviors of $\Omega(x)$ given in (3.3.30), and $g\Omega(x)g^{-1}$ in (3.3.34) are already diagonal from the assumption of highest weight state.

We further investigate asymptotic behavior of $p(x)$ around $x = \pm 1$. For this purpose, we have to diagonalize $\Omega(x)$. This can be done by using the similarity transformation (3.3.24) at the leading order of $(x \mp 1)^{-1}$, as

$$\Omega(x) = \bar{P}\exp\left[\oint d\sigma\; \pm\frac{(j_{\mathrm{diag}})_\pm}{2\,(1 \mp x)} + \mathcal{O}\left((x \mp 1)^0\right)\right], \tag{3.3.35}$$

$$= \bar{P}\exp\left[\oint d\sigma\; -\frac{i\kappa\sigma^3}{1 \mp x} + \mathcal{O}\left((x \mp 1)^0\right)\right], \tag{3.3.36}$$

where the normalization of $(j_{\mathrm{diag}})_\pm$ is fixed by Virasoro constraints (3.3.10). At higher orders the monodromy matrix is diagonalized recursively [27].

Actually there exists sign ambiguity when we derive (3.3.36), which reflects the freedom to swap the first and the second eigenvalue of $\Omega(x) \sim \mathrm{diag}\,(e^{ip}\,, e^{-ip})$. We fix this ambiguity by demanding the quasi-momentum to behave as[3]

$$p(x) = -\frac{\pi\kappa}{x \mp 1} + \mathcal{O}\left((x \mp 1)^0\right) \qquad \text{as} \quad x \to \pm 1. \tag{3.3.37}$$

To account for the other possibility, we introduce another Abelian differential called quasi-energy, by

$$q(x) = \mp\frac{\pi\kappa}{x \mp 1} + \mathcal{O}\left((x \mp 1)^0\right) \qquad \text{as} \quad x \to \pm 1. \tag{3.3.38}$$

3.3.3 The spectral curve

The quasi-momentum defined by (3.3.25) need not be real, nor it must be an analytic function of $x \in \mathbb{C} \setminus \{x = \pm 1\}$. General solutions can have singularities such as marked points or branch cuts. To describe the singularity structure of $p(x)$, while avoiding the complexity of diagonalization of monodromy matrix, we are motivated to study the characteristic equation for the monodromy matrix,

$$\Gamma\;: \quad \det\left(y\mathbf{1}_2 - \Omega(x)\right) = 0. \tag{3.3.39}$$

The solution of this equation $y = y(x)$ defines what is known as the spectral curve. Since the characteristic equation (3.3.39) is quadratic in y, it defines the spectral curve as a 2-sheeted

[3]Strictly speaking, we have to specify a path on which $x = +1$ is connected with $x = -1$. We will return to issue in Section 7.3.

ramified cover of x-plane, $\Gamma \simeq \mathbb{CP}^1_+ \cup \mathbb{CP}^1_-$. There is a natural involution which swaps the two sheets, which acts on the quasi-momentum as

$$\hat{\sigma} : p(x) \ \mapsto \ -p(x), \qquad i.e. \quad (y, x) \ \mapsto \ (1/y, x)\,. \tag{3.3.40}$$

However, as discussed in [27, 31], the curve Γ has an infinitely many singular points at $e^{ip} = e^{-ip} = \pm 1$. Thus a better definition of the spectral curve is to take 'logarithm' of (3.3.39), as

$$\hat{\Sigma} : \quad \det\left(y \mathbf{1}_2 - L(x)\right) = 0,$$
$$U\, L(x)\, U^{-1} \equiv -i \frac{\partial}{\partial x} \log\left(U\, \Omega(x)\, U^{-1}\right). \tag{3.3.41}$$

One can further perform birational transformations and remove finitely many unphysical singularities from (3.3.41). Eventually the equation (3.3.41) is brought into the hyperelliptic form

$$\Sigma : \quad y^2 = \prod_{I=1}^{2K} (x - x_I)\,. \tag{3.3.42}$$

The number of cuts K is related to genus g of the hyperelliptic curve by $K = g + 1$.

Put it shortly, it is shown that the quasi-momentum $p(x)$ is a function over the hyperelliptic curve Σ expressed as 2-sheeted ramified cover of x-plane, $\Sigma \simeq \mathbb{CP}^1_+ \cup \mathbb{CP}^1_-$. When a solution can be described by an algebraic curve with finitely many branch cuts, it is called finite-gap solution. In what follows we will always assume K to be finite.

Alternatively, one can start from a hyperelliptic curve Σ and Abelian differential dp defined on it, together with the projection,

$$\pi : \quad \Sigma \quad \to \quad \mathbb{CP}^1\,, \tag{3.3.43}$$
$$\qquad \Uplus \qquad\quad \Uplus$$
$$x^{\pm} \quad \mapsto \quad x\,. \tag{3.3.44}$$

From this point of view, the spectral parameter $x \in \mathbb{CP}^1$ is identified as the one appeared in the flatness condition (3.3.18).[4]

Note that there is huge redundancy in expressing a classical string solution in terms of complex planes connected with branch cuts. In contrast to the thermodynamic limit of Bethe Ansatz equations, we have the freedom to connect branch points of (3.3.42) arbitrarily, because all of them define the same algebro-geometric data of a finite-gap solution.

Now we know that the quasi-momentum is a meromorphic function on Σ except at some singularities, we can speak of quantization of period integrals of an Abelian differential dp . Let $\{a_i\,, b_i\}$ $(i = 1, \ldots, g)$ be a basis of one-cycles with the canonical intersection

$$a_i \cap a_j = b_i \cap b_j = 0, \qquad a_i \cap b_j = \delta_{ij}\,. \tag{3.3.45}$$

[4]The superscript $\pm$ in $x^{\pm}$ is used to distinguish two sheets of $\mathbb{CP}^1_{\pm}$. This is unrelated with $x^{\pm}$ defined in (2.5.14).

Let ω_i be the holomorphic differentials on Σ, normalized such that

$$\int_{a_j} \omega_i = \delta_i{}^j \,, \qquad \int_{b_j} \omega_i = \Pi_i{}^j \,. \tag{3.3.46}$$

Since holomorphic differentials do not have any singularities, one may redefine $dp(x)$ by subtracting ω_i's to obtain normalized Abelian differential:

$$\int_{a_i} dp = 0. \tag{3.3.47}$$

As a consequence, the Abelian integral

$$p(P) \equiv \int_{\infty^+}^{P} dp, \tag{3.3.48}$$

is single-valued on the upper sheet $\mathbb{CP}^1_+$.

Next, consider discontinuity of $p(x)$ across one of the cuts $\mathcal{C}_k$. If $p(x+i\epsilon)$ $(\epsilon \ll 1)$ stands on the upper sheet, then $p(x-i\epsilon)$ stands on the lower sheet. When $\epsilon = 0$, we can regard $p(x \pm i\epsilon)$ as two independent solutions of the spectral curve equation (3.3.39), related by the involution $\hat{\sigma}$ of (3.3.40). Since $\Omega(x)$ is unimodular, we must have $e^{ip(x+i\epsilon)}\, e^{ip(x-i\epsilon)} = 1$. This condition demands the discontinuity across a cut to be multiple of 2π, as

$$p(x+i\epsilon) + p(x-i\epsilon) = 2\pi n_k \qquad \text{for} \quad x \in \mathcal{C}_k \,, \tag{3.3.49}$$

which is shown to be quantization of b-periods, following the same discussion in Section 2.4

Let us comment on the uniqueness of Abelian differentials. The normalized Abelian differentials (*i.e.* vanishing a-period (3.3.47)) are uniquely specified by their pole structure, as can be shown by using the Riemann bilinear identity [139, 138, 140]. In the case of quasi-momentum, its pole structure is described as

$$dp(x^\pm) \sim \mp d\left(\frac{\pi\kappa}{x-1}\right) \qquad \text{as} \quad x \to +1, \tag{3.3.50}$$

$$dp(x^\pm) \sim \mp d\left(\frac{\pi\kappa}{x+1}\right) \qquad \text{as} \quad x \to -1. \tag{3.3.51}$$

The same applies to the quasi-energy, which can be uniquely defined as a normalized Abelian differential (of the third kind) whose pole structure is given by (3.3.38).

Before closing, we explain a little on analytic profile (ξ_1, ξ_2) of general finite-gap solution on $\mathbb{R}_{\rm t} \times {\rm S}^3$. We have considered so far how to construct an algebraic curve with an Abelian differential (Σ, dp) when a consistent string solution is given. Conversely, one can consider the Riemann Hilbert problem, that is to determine the pair (Σ, dp) such that they reproduce mode numbers and conserved charges of a consistent classical string solution. Furthermore, one can also reconstruct analytic expression of classical string solution $(\xi_1\,,\xi_2)$ from the algebro-geometric data, which has been done by [30, 31].

The idea of the construction in [31] is to solve the auxiliary linear problem

$$(d - J(x))\,\boldsymbol{\psi}(x) = 0, \tag{3.3.52}$$

which is equivalent to (3.3.19) and (3.3.20). Remarkably, the solution $\boldsymbol{\psi}(P)$ $(P \in \Sigma)$ is uniquely specified by its analytic properties, and is expressed by so-called Baker-Akhiezer vector.

Given the solution of (3.3.52), one can immediately reconstruct the profile of classical string solution $(\xi_1\,,\xi_2)$. Note that the equation (3.3.52) is formally solved by

$$J(x) = \Psi^{-1}(x)\,d\Psi(x), \qquad \Psi(x) \equiv \left(\boldsymbol{\psi}(x^+), \boldsymbol{\psi}(x^-)\right). \tag{3.3.53}$$

Then, using $J(0) = j = -g\,dg^{-1}$, one can obtain g^{-1} from $\boldsymbol{\psi}$, as

$$g^{-1} = \begin{pmatrix} \xi_1^* & \xi_2^* \\ -\xi_2 & \xi_1 \end{pmatrix} = \frac{1}{\sqrt{\Psi(0)}}\,\Psi(0), \qquad \Psi(0) = \begin{pmatrix} \psi_1(0^+) & \psi_1(0^-) \\ \psi_2(0^+) & \psi_2(0^-) \end{pmatrix}. \tag{3.3.54}$$

Explicit expression of the Baker-Akhiezer vector in terms of Riemann θ functions is found, for example in [31], as

$$\psi_1(P,\sigma,\tau) = k_-(P)\,\frac{\theta\left(\boldsymbol{\mathcal{A}}(P) + \int_{\boldsymbol{b}} d\mathcal{Q} - \boldsymbol{\zeta}_{\gamma(0,0)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^+) - \boldsymbol{\zeta}_{\gamma(0,0)}\right)}{\theta\left(\boldsymbol{\mathcal{A}}(P) - \boldsymbol{\zeta}_{\gamma(0,0)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^+) + \int_{\boldsymbol{b}} d\mathcal{Q} - \boldsymbol{\zeta}_{\gamma(0,0)}\right)}\,\exp\left(i\int_{\infty^+}^{P} d\mathcal{Q}\right), \tag{3.3.55}$$

$$\psi_2(P,\sigma,\tau) = k_+(P)\,\frac{\theta\left(\boldsymbol{\mathcal{A}}(P) + \int_{\boldsymbol{b}} d\mathcal{Q} - \boldsymbol{\zeta}_{\gamma'(0,0)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^-) - \boldsymbol{\zeta}_{\gamma'(0,0)}\right)}{\theta\left(\boldsymbol{\mathcal{A}}(P) - \boldsymbol{\zeta}_{\gamma'(0,0)}\right)\theta\left(\boldsymbol{\mathcal{A}}(\infty^-) + \int_{\boldsymbol{b}} d\mathcal{Q} - \boldsymbol{\zeta}_{\gamma'(0,0)}\right)}\,\exp\left(i\int_{\infty^-}^{P} d\mathcal{Q}\right). \tag{3.3.56}$$

It is worth mentioning that the Baker-Akhiezer vector $\boldsymbol{\psi}\,(\tau,\sigma;P)$ depends on worldsheet coordinates solely through the differential form

$$d\mathcal{Q} := \frac{1}{2\pi}\,(\sigma dp + \tau dq)\,. \tag{3.3.57}$$

This expression infers that the quantities ‘quasi-momentum’ and ‘quasi-energy’ are nonlinear (and sophisticated) analogue of the Fourier transformation on worldsheet [28, 141].

3.3.4 Comparison with gauge theory

In this subsection, we will summarize the results derived so far using resolvent. Then we compare them with the thermodynamic limit of $\mathsf{XXX}_{1/2}$ spin chain discussed in Section 2.4 [11].

Just like gauge theory (2.4.14), we define resolvent $G(x)$ by subtracting pole singularities from the quasi-momentum, as

$$G(x) \equiv p(x) + \frac{\pi\kappa}{x-1} + \frac{\pi\kappa}{x+1}\,. \tag{3.3.58}$$

Similarly, the density is defined as

$$G(x) \equiv \oint_{\mathcal{C}} d\xi \, \frac{\rho(\xi)}{x-\xi} \,, \tag{3.3.59}$$

where $\mathcal{C} \equiv \cup_k \mathcal{C}_k$ surrounds all branch cuts.

From (3.3.30), the resolvent has an asymptotic behavior

$$G(x) \sim \frac{2\pi}{x} \left(\kappa - \frac{J_1 - J_2}{\sqrt{\lambda}} \right) \qquad \text{as} \;\; x \to \infty \,, \tag{3.3.60}$$

which translates into

$$\oint_{\mathcal{C}} dx \, \rho(x) = \frac{2\pi}{\sqrt{\lambda}} \left(E - J + 2J_2 \right) \,, \qquad J \equiv J_1 + J_2 \,, \tag{3.3.61}$$

where we used $E = \sqrt{\lambda}\kappa$, which follows from $t = \kappa\tau$ and (3.2.7). From (3.3.34) we obtain,

$$G(x) \sim 2\pi m - 2\pi x \left(\kappa - \frac{J_1 + J_2}{\sqrt{\lambda}} \right) \qquad \text{as} \;\; x \to 0 \,. \tag{3.3.62}$$

This gives

$$-\frac{1}{2\pi i} \oint_{c(0)} dx \, \frac{G(x)}{x} = \oint_{\mathcal{C}} dx \, \frac{\rho(x)}{x} = 2\pi m \,, \tag{3.3.63}$$

$$-\frac{1}{2\pi i} \oint_{c(0)} dx \, \frac{G(x)}{x^2} = \oint_{\mathcal{C}} dx \, \frac{\rho(x)}{x^2} = \frac{2\pi}{\sqrt{\lambda}} \left(E - J \right) , \tag{3.3.64}$$

where $c(z)$ is a small circle around $x = z$. The condition for discontinuity (3.3.49) is rewritten as

$$G(x + i\epsilon) + G(x - i\epsilon) = 2 \,⨍ d\xi \, \frac{\rho(\xi)}{x - \xi} = 2\pi n_k + \frac{2\pi\kappa}{x-1} + \frac{2\pi\kappa}{x+1} \qquad \text{for} \;\; x \in \mathcal{C}_k \,. \tag{3.3.65}$$

The main differences between the finite-gap formulation of classical string theory and the thermodynamic limit of Bethe Ansatz equation are:

- The quasi-momentum $p(x)$ in string theory (3.3.58) has single poles at $x = \pm 1$, while $\mathrm{p}(x)$ in the gauge theory (2.4.14) has a single pole at $x = 0$.
- The finite-gap formulation is valid under the classical approximation $\lambda \gg 1$, while in gauge theory side we have to take weak coupling limit $\lambda \ll 1$ together with the thermodynamic limit (2.4.1).

To compare both sides, we take a clever limit called BMN expansion [44, 45]. The idea is to take the limit $J \to \infty$ such that

$$J \to \infty, \qquad \alpha \equiv \frac{J_2}{J} \,, \quad \tilde{x} \equiv \frac{\sqrt{\lambda}\, x}{4\pi J} \quad \text{kept fixed,}^{5} \tag{3.3.66}$$

[5] An extra factor of $\sqrt{\lambda}/(4\pi)$ is needed to identify x_{gauge} with x_{string} .

together with the assumption of $\tilde{\lambda} \equiv \lambda/J^2$ being small. We further assume the energy is expanded as

$$E - J = J \left\{ \sum_{j\geq 1} \sum_{k\geq 1} c_{j,k} \left(\frac{\lambda}{J^2}\right)^j \left(\frac{1}{J}\right)^k \right\}, \tag{3.3.67}$$

by analogy with (2.5.13). This way of limiting procedure is, of course, motivated by thermodynamic limit in gauge theory (2.4.1). Furthermore, as we will see later, one can find several nontrivial examples of correspondence of the spectra between gauge and string theories in this limit, at least to a few orders in $\tilde{\lambda}$.

Let us apply the limit (3.3.66) to what we obtained in this section. In terms of $\tilde{x}$, the equations (3.3.61), (3.3.63), and (3.3.64) are rewritten as

$$\oint_{\mathcal{C}} d\tilde{x}\, \rho(\tilde{x}) = \frac{E - J + 2J_2}{2J} \tag{3.3.68}$$

$$\oint_{\mathcal{C}} d\tilde{x}\, \frac{\rho(\tilde{x})}{\tilde{x}} = 2\pi m\,, \tag{3.3.69}$$

$$\frac{\lambda}{8\pi^2 J} \oint_{\mathcal{C}} d\tilde{x}\, \frac{\rho(\tilde{x})}{\tilde{x}^2} = E - J\,, \tag{3.3.70}$$

These equations agree with the gauge theory results (2.4.7), (2.4.10), and (2.4.12), upon identification

$$(\Delta(\lambda), L, M) \qquad \leftrightarrow \qquad (E(\lambda), J, J_2)\,. \tag{3.3.71}$$

Moreover, the equation for discontinuity (3.3.65) becomes

$$2 \fint d\tilde{\xi}\, \frac{\rho(\tilde{\xi})}{\tilde{x} - \tilde{\xi}} = \frac{E}{J} \frac{x}{x^2 - \frac{\lambda}{16\pi^2 J^2}} + 2\pi n_k \qquad \text{for} \;\; x \in \mathcal{C}_k\,. \tag{3.3.72}$$

Since we compare our results with the one-loop results in gauge theory, the above equation can be evaluated at the zeroth order in λ. With the assumption (3.3.67) in mind, we can replace the first term in the right hand side with $1/x$. The resultant equation turns out exactly same as the Bethe Ansatz equation in gauge theory side (2.4.8).

We make a few remarks on further developments on the correspondence of algebraic curve.

One can further make such comparison at higher orders in $\tilde{\lambda}$, as already discussed in [11]. Later it turned out in [7, 10] that there is mismatch between integral equations of gauge and string theories, which can be reconciled partly by introducing the dressing phase.

Another remarkable progress is the correspondence of algebraic curve for sectors other than $su(2)$. Generalization to $sl(2)$ sector and $so(6)$ sector, and the full $psu(2,2|4)$ sector are discussed in [26], [27], and [28, 29], respectively.

Chapter 4

Solutions of the integral equations

We see several examples of solutions of the integral equations which arise as thermodynamic limit of $\mathsf{XXX}_{1/2}$ Bethe Ansatz equation, or as finite-gap formulation of classical string one loop in $\tilde{\lambda}$. As we saw in the previous chapter, the two formulation coincide after taking BMN scaling limit (2.5.13) or (3.3.67) at one loop.

We will confirm the above statement with close inspection on concrete examples. They also help us to understand how an algebraic curve with Abelian differential corresponds to a classical string solution.

4.1 Symmetric two-cut solutions

We consider the solutions of $\mathsf{XXX}_{1/2}$ Bethe Ansatz equation in thermodynamic limit with two cuts, each of which is located symmetrically with respect to imaginary axis [44, 45, 11]. We will then identify them as finite-gap interpretation of so-called Frolov-Tseytlin string solutions [38].

Let Σ be an elliptic curve defined by

$$\Sigma : \quad y^2 = (x^2 - x_1^2)(x^2 - x_2^2), \tag{4.1.1}$$

where we assumed branch points are located on the real axis. Later we will make analytic continuation of branch points to the complex plane. Recall that quasi-momentum $d\mathrm{p}$ given in (2.4.14) has double poles at $x = 0$, and has no other singularity elsewhere. Then quasi-momentum on the curve Σ will be given in general form

$$d\mathrm{p} = \frac{dx}{y} \left(\frac{a_{-2}}{x^2} + \frac{a_{-1}}{x} + a_0 \right). \tag{4.1.2}$$

Higher terms in x should vanish for it destroys asymptotics of $\mathrm{p}(x)$ as $x \to \infty$. The coefficients $a_{-2}\,, a_{-1}\,, a_0$ are determined from the conditions $d\mathrm{p} \sim dx/(2x^2) + \mathcal{O}(1)$ and (2.4.20), as

$$d\mathrm{p} = \frac{-dx}{\sqrt{(x_2^2 - x^2)(x^2 - x_1^2)}} \left(\frac{1 - 2\alpha}{2} - \frac{x_1 x_2}{2x^2} \right). \tag{4.1.3}$$

We define period integrals by

$$\oint_A d\mathrm{p} \;= -2i\int_{x_1}^{x_2}\frac{dx}{\sqrt{(x_2^2-x^2)(x^2-x_1^2)}}\left(\frac{1-2\alpha}{2}-\frac{x_1x_2}{2x^2}\right)\equiv 2\pi m, \tag{4.1.4}$$

$$\oint_B d\mathrm{p} \;= 4\int_{x_2}^{\infty}\frac{dx}{\sqrt{(x^2-x_2^2)(x^2-x_1^2)}}\left(\frac{1-2\alpha}{2}-\frac{x_1x_2}{2x^2}\right)\equiv 4\pi n. \tag{4.1.5}$$

These integrals can be expressed in terms of complete elliptic integrals (See Appendix A for definitions). After short calculation we obtain

$$1-2\alpha=\frac{1}{k}\,\frac{n\mathbf{E}'(k)+im\left(\mathbf{E}(k)-\mathbf{K}(k)\right)}{n\mathbf{K}'(k)-im\mathbf{K}(k)}\,, \tag{4.1.6}$$

and

$$x_1=\frac{1}{4}\,\frac{1}{n\mathbf{K}'(k)-im\mathbf{K}(k)}\,,\qquad x_2=\frac{1}{4\,k}\,\frac{1}{n\mathbf{K}'(k)-im\mathbf{K}(k)}\,, \tag{4.1.7}$$

where $\mathbf{E}'(k)\equiv\mathbf{E}(\sqrt{1-k^2})$ and $\mathbf{K}'(k)\equiv\mathbf{K}(\sqrt{1-k^2})$.

The anomalous dimension at one loop in λ is given by

$$\gamma=\frac{\lambda}{16\pi^2L}\left(\frac{1-2\alpha}{x_1x_2}-\frac{1}{2}\left(\frac{1}{x_1^2}+\frac{1}{x_2^2}\right)\right), \tag{4.1.8}$$

where $L\gg 1$ is the length of spin chain. The R-charges, or the angular momenta, are expressed as $J_1=(1-\alpha)L$ and $J_2=\alpha L$. Reality constraints restrict possible sets of (m,n). We are interested in the case either m or n is zero. It will turn out that the case $m=0$ corresponds to the folded string solution, while the case $n=0$ corresponds to the elliptic circular string solution.

Double Contour solution. We set $m=0$ and analytically continue x_1 and x_2 into complex values keeping $x_1=\bar{x}_2$. This is called double contour solution [44].

With $q\equiv\sqrt{1-k^2}$, the anomalous dimension is written as

$$\gamma=\frac{n^2\lambda}{\pi^2L}\,\mathbf{K}(q)\left\{\mathbf{E}(q)-\left(1-\frac{q^2}{2}\right)\mathbf{K}(q)\right\}. \tag{4.1.9}$$

The filling fraction and the R-charges are[1]

$$\alpha=\frac{1}{2}\left(1-\frac{\mathbf{E}(q)}{q'\,\mathbf{K}(q)}\right),\quad J_1=\frac{L}{2}\,\frac{q'\mathbf{K}(q)-\mathbf{E}(q)}{q'\,\mathbf{K}(q)},\quad J_2=\frac{L}{2}\,\frac{q'\mathbf{K}(q)+\mathbf{E}(q)}{q'\,\mathbf{K}(q)}. \tag{4.1.10}$$

The double contour solution is dual to folded string solutions found by Frolov and Tseytlin [38]. The conserved charges of folded string are given by

$$\widetilde{E}=\frac{n\sqrt{\lambda}}{\pi}\sqrt{x^2+u_2^2}\,\mathbf{K}(x)\,, \tag{4.1.11}$$

[1]Here we interchanged J_1 and J_2, so that $\alpha=J_1/L$.

and

$$\widetilde{J}_1 = \frac{n\sqrt{\lambda}}{\pi}\sqrt{1+u_2^2}\,(\mathbf{K}(x) - \mathbf{E}(x))\,, \quad \widetilde{J}_2 = \frac{n\sqrt{\lambda}}{\pi}\,u_2\,\mathbf{E}(x), \tag{4.1.12}$$

where $n \in \mathbb{Z}$ is the number of foldings, x and u_2 are parameters. If we consider the limit $u_2 \gg 1$, these conserved charges reduce to

$$\frac{\widetilde{J}_1}{\widetilde{J}} \approx 1 - \frac{\mathbf{E}(x)}{\mathbf{K}(x)}\,, \qquad \widetilde{E} = \frac{n\sqrt{\lambda}}{\pi}\,u_2\,\mathbf{K}(x) + \mathcal{O}\left(\frac{n\sqrt{\lambda}}{u_2}\right) = \widetilde{J} + \mathcal{O}\left(\frac{n\sqrt{\lambda}}{u_2}\right)\,, \tag{4.1.13}$$

with $\widetilde{J} \equiv \widetilde{J}_1 + \widetilde{J}_2$. Now we assume the conserved charges are expanded in powers of $\tilde{\lambda} \equiv \lambda/\widetilde{J}^2$ as (3.3.67), and try to compare the above results with (4.1.10). We may identify $\widetilde{J}_1/\widetilde{J}$ with the filling fraction of (4.1.10). The two elliptic moduli are then related as

$$x = \pm\frac{i(1-k)}{2\sqrt{k}}\,. \tag{4.1.14}$$

In fact, by performing modular transformation we find

$$\mathbf{K}(x) = \sqrt{k}\,\mathbf{K}'(k)\,, \quad \mathbf{E}(x) = \frac{1}{2}\left(\sqrt{k}\,\mathbf{K}'(k) + \frac{1}{\sqrt{k}}\,\mathbf{E}'(k)\right), \quad \frac{\widetilde{J}_1}{\widetilde{J}} \approx \frac{1}{2}\left(1 - \frac{\mathbf{E}(q)}{q'\mathbf{K}(q)}\right), \tag{4.1.15}$$

where we used $q = \sqrt{1-k^2}$ again. At the same time, we obtain relation

$$u_2 \approx \frac{1}{\sqrt{k}\,\mathbf{K}(k)}\,\frac{L\pi}{n\sqrt{\lambda}}\,, \tag{4.1.16}$$

which is indeed very large for $\tilde{\lambda} \ll 1$. Using (4.1.16) and (4.1.13), one finds that the correction term $\widetilde{E} - \widetilde{J}$ has the same order of magnitude as the anomalous dimension (4.1.9).

To compute subleading terms, it is useful to erase u_2 from (4.1.11) and (4.1.12) as

$$\left(\frac{\widetilde{E}}{\mathbf{K}(x)}\right)^2 - \left(\frac{\widetilde{J}_2}{\mathbf{E}(x)}\right)^2 = \frac{x^2 n^2 \lambda}{\pi^2}\,, \quad \left(\frac{\widetilde{J}_1}{\mathbf{K}(x) - \mathbf{E}(x)}\right)^2 - \left(\frac{\widetilde{J}_2}{\mathbf{E}(x)}\right)^2 = \frac{n^2\lambda}{\pi^2}\,. \tag{4.1.17}$$

By expanding x in series of $\tilde{\lambda}$ as $x = x_0 + x_1\,\tilde{\lambda} + \cdots$, one obtains the next correction term

$$\widetilde{E} - \widetilde{J} \approx \left(\frac{2\lambda}{\pi^2\widetilde{J}}\right)\mathbf{K}(x_0)\Big\{\mathbf{E}(x_0) - (1-x_0)\,\mathbf{K}(x_0)\Big\}, \qquad \frac{\widetilde{J}_1}{\widetilde{J}} \approx 1 - \frac{\mathbf{E}(x_0)}{\mathbf{K}(x_0)}\,, \tag{4.1.18}$$

By using the modular transformation (4.1.14), one can find that this expression equals to the one-loop anomalous dimension (4.1.9).

Imaginary root solution. We set $n = 0$, and bring the four branch points onto the imaginary axis keeping $x_{1,2} = -\bar{x}_{1,2}$.

The anomalous dimension is

$$\gamma = \frac{\lambda m^2}{\pi^2 L}\,\mathbf{K}(k)\left(\mathbf{E}(k) - \frac{1-k^2}{2}\mathbf{K}(k)\right). \tag{4.1.19}$$

The filling fraction and the R-charges are

$$\alpha = \frac{\mathbf{E}(k) - (1-k)\,\mathbf{K}(k)}{2k\,\mathbf{K}(k)}\,, \quad J_1 = \frac{L}{2}\,\frac{(1+k)\mathbf{K}(k) - \mathbf{E}(k)}{k\,\mathbf{K}(k)}\,, \quad J_2 = \frac{L}{2}\,\frac{-(1-k)\mathbf{K}(k) + \mathbf{E}(k)}{k\,\mathbf{K}(k)}\,. \tag{4.1.20}$$

The imaginary root solutions are dual to circular string solutions of Frolov and Tseytlin, whose conserved charges are given by

$$\widetilde{E} = \frac{m\sqrt{\lambda}}{\pi}\sqrt{1+u_2^2}\,\mathbf{K}(x)\,, \tag{4.1.21}$$

and

$$\widetilde{J}_1 = \frac{m\sqrt{\lambda}}{\pi}\,\frac{\sqrt{x^2+u_2^2}}{x^2}\left(\mathbf{K}(x) - \mathbf{E}(x)\right)\,, \quad \widetilde{J}_2 = \frac{m\sqrt{\lambda}}{\pi}\,\frac{u_2}{x^2}\Big(\mathbf{E}(x) - (1-x^2)\mathbf{K}\Big)\,, \tag{4.1.22}$$

where $n \in \mathbb{Z}$ is the number of winding. The comparison between γ and $\widetilde{E} - \widetilde{J}$ can be done in a similar manner [45].

Comparison of higher conserved charges has been done in [142, 42].

4.2 Pulsating and rotating strings

In general, it is not easy to compute the quasi-momentum of a given classical string solution explicitly in the manner described in Section 3.3. However, the pulsating string of [41], or the pulsating and rotating string of [42] are interesting examples whose quasi-momentum can be easily computed from the definition [11].

4.2.1 The profile

We follow [43] to obtain pulsating and rotating string solution.

We consider Polyakov action on $\mathbb{R}_t \times \mathrm{S}^3$ in conformal gauge,

$$S = \frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma\,\Big[-(\partial_a t)^2 + \partial_a \vec{\xi}^{\,*}\cdot\partial^a\vec{\xi}\,\Big]\,, \tag{4.2.1}$$

which is same as (3.3.3). Pulsating and rotating strings are obtained by an Ansatz

$$t = \kappa\tau, \quad \xi_1 = \cos\theta\, e^{i m_1 \sigma}\,, \quad \xi_2 = \sin\theta\, e^{i\phi_2}\,. \qquad \text{where} \quad \theta = \theta(\tau), \quad \phi_2 = \phi_2(\tau). \tag{4.2.2}$$

Equations of motion and Virasoro conditions are solved by

$$\mathcal{J}_2 = \sin^2\theta\,\dot{\phi}_2\,, \qquad \kappa^2 = \dot{\theta}^2 + m_1^2\cos\theta^2 + \left(\frac{\mathcal{J}_2}{\sin\theta}\right)^2. \tag{4.2.3}$$

where $\mathcal{J}_2 \equiv J_2/\sqrt{\lambda}$ is the angular momentum in ϕ_2 direction, and the dot ($\dot{}$) represents derivative with respect to τ. One can rewrite the second equation as

$$\tau = \int \frac{d\theta\,\sin\theta}{\sqrt{m_1^2\cos^4\theta - (m_1^2+\kappa^2)\cos^2\theta + \kappa^2 - \mathcal{J}_2^2}} \tag{4.2.4}$$

and integrate it out as

$$\cos\theta = a_- \,\mathrm{sn}\left(m_1 a_+ \tau,\ \frac{a_-}{a_+}\right)\,, \tag{4.2.5}$$

where

$$a_\pm^2 = \frac{m_1^2 + \kappa^2 \pm \sqrt{\left(m_1^2+\kappa^2\right)^2 - 4m_1^2\left(\kappa^2 - \mathcal{J}_2^2\right)}}{2m_1^2}\,. \tag{4.2.6}$$

There is another solution obtained by interchanging a_+ and a_-. As is clear from (4.2.5), interchange of $a_+ \leftrightarrow a_-$ induces modular transformation of Jacobi elliptic functions. Thus

$$\cos\theta = a_+ \,\mathrm{sn}\left(m_1 a_- \tau,\ \frac{a_+}{a_-}\right)\,, \tag{4.2.7}$$

is also a solution.

The first equation of (4.2.3) can be rewritten as

$$\frac{d\phi_2}{d\theta} = \frac{\mathcal{J}_2}{\sin\theta\sqrt{m_1^2\cos^4\theta - \left(m_1^2+\kappa^2\right)\cos^2\theta + \kappa^2 - \mathcal{J}_2^2}} \tag{4.2.8}$$

and integrated out by

$$\phi_2 = \frac{\mathcal{J}_2}{m_1 a_+}\,\mathbf{\Pi}\left(\mathrm{sn}\left(m_1 a_+\tau,\ \frac{a_-}{a_+}\right), a_-\,, \frac{a_-}{a_+}\right), \tag{4.2.9}$$

where

$$\mathbf{\Pi}\left(\varphi,\nu,k\right) \equiv \int_0^{\varphi} \frac{dt}{\left(1-\nu t^2\right)\sqrt{\left(1-t^2\right)\left(1-k^2t^2\right)}} \tag{4.2.10}$$

is the incomplete elliptic integral of the third kind. Notice again that

$$\phi_2 = \frac{\mathcal{J}_2}{m_1 a_-}\,\mathbf{\Pi}\left(\mathrm{sn}\left(m_1 a_-\tau,\ \frac{a_+}{a_-}\right), a_+\,, \frac{a_+}{a_-}\right), \tag{4.2.11}$$

is also a solution.

4.2.2 Finite-gap representation

Let us study the finite-gap representation of the pulsating and rotating string solution. Since $\Omega(x)$ is independent of τ, we may set $\tau = 0$ to evaluate the right hand side of

$$\Omega(x) \equiv \bar{P}\exp\left(\int_0^{2\pi} d\sigma\, L\left(\sigma,\tau\,;x\right)\right). \tag{4.2.12}$$

By using (4.2.2) and assuming $\theta(0) = \pi/2,\ \phi_2(0) = 0$, one obtains

$$L\left(\sigma,0\,;x\right) = \frac{x}{x^2-1}\begin{pmatrix} i\mathcal{J}_2 & \dot{\theta}\,e^{im_1\sigma} \\ -\dot{\theta}\,e^{-im_1\sigma} & -i\mathcal{J}_2 \end{pmatrix}. \tag{4.2.13}$$

It is easy to see that one needs σ-dependent special unitary transformation $U(\sigma)$ to diagonalize L.[2] Since the monodromy matrix (3.1.29) is gauge-invariant quantity, we perform an $SU(2)$ (or $O(4)$) gauge transformation on Lax connection L by

$$L \mapsto L' \equiv U L U^{-1} + \partial_\sigma U \, U^{-1}, \qquad U = \begin{pmatrix} e^{-im_1\sigma/2} & \\ & e^{im_1\sigma/2} \end{pmatrix}. \tag{4.2.14}$$

It turns out that L' is independent of σ, as

$$L' = \frac{x}{x^2-1} \begin{pmatrix} i\mathcal{J}_2 - i\nu & \dot{\theta} \\ -\dot{\theta} & -i\mathcal{J}_2 + i\nu \end{pmatrix}, \qquad \nu \equiv \frac{x^2-1}{2x}\, m_1 \, . \tag{4.2.15}$$

so that the quasi-momentum $p(x)$ equals to 2π times the eigenvalue of L'. Using $\dot{\theta}^2 = \kappa^2 - \mathcal{J}_2^2$ at $\tau = 0$, one finally gets

$$p(x) = -\frac{2\pi x}{x^2-1} \sqrt{\left(\frac{m_1}{2} \left(\frac{x^2-1}{x} \right) - \mathcal{J}_2 \right)^2 + \kappa^2 - \mathcal{J}_2^2} + \pi m_1 \, , \tag{4.2.16}$$

which is the result derived in [11]. We added an extra term $+\pi m_1$, which amounts to trivial redefinition of quasimomentum p. This result can be reexpressed in terms of $a_\pm$ variables defined in (4.2.6). By noting that

$$a_+^2 + a_-^2 = 1 + \frac{\kappa^2}{m_1^2} \, , \qquad a_+^2 a_-^2 = \frac{\kappa^2 - \mathcal{J}_2^2}{m_1^2} \, , \qquad \left(a_+^2 - 1\right)\left(1 - a_-^2\right) = \left(\frac{\mathcal{J}_2}{m_1} \right)^2 , \tag{4.2.17}$$

we obtain

$$p(x) = -\frac{2\pi m_1 x}{x^2-1} \sqrt{\left(\frac{x^2-1}{2x} - \sqrt{(a_+^2-1)(1-a_-^2)} \right)^2 + a_+^2 a_-^2} + \pi m_1 \, , \tag{4.2.18}$$

which is manifestly invariant under the interchange $a_+ \leftrightarrow a_-$.

The quasi-momentum (4.2.16) has the following asymptotic behaviors:

$$p(x) = -\frac{\pi\kappa}{x \mp 1} + O(1), \qquad \text{around } \; x = \pm 1, \tag{4.2.19}$$

$$p(x) = 2\pi m_1 + 2\pi m_1 \mathcal{J}_2 \, x + O(x^2), \qquad \text{as } \; x \to 0, \; x > 0, \tag{4.2.20}$$

$$p(x) = \frac{2\pi m_1 \mathcal{J}_2}{x} + O\left(\frac{1}{x^2}\right), \qquad \text{around } \; x = \infty. \tag{4.2.21}$$

Under the inversion $x \mapsto 1/x$, it transforms as

$$p(x) \mapsto p(1/x) = p(-x) = -p(x) + 2\pi m_1 \, . \tag{4.2.22}$$

When we set $\mathcal{J}_2 = 0$, the algebraic curve is represented as symmetric 2-cut solutions. In fact, the quasi-momentum (4.2.18) becomes

[2] "Special" is required to maintain the unimodularity of monodromy matrix.

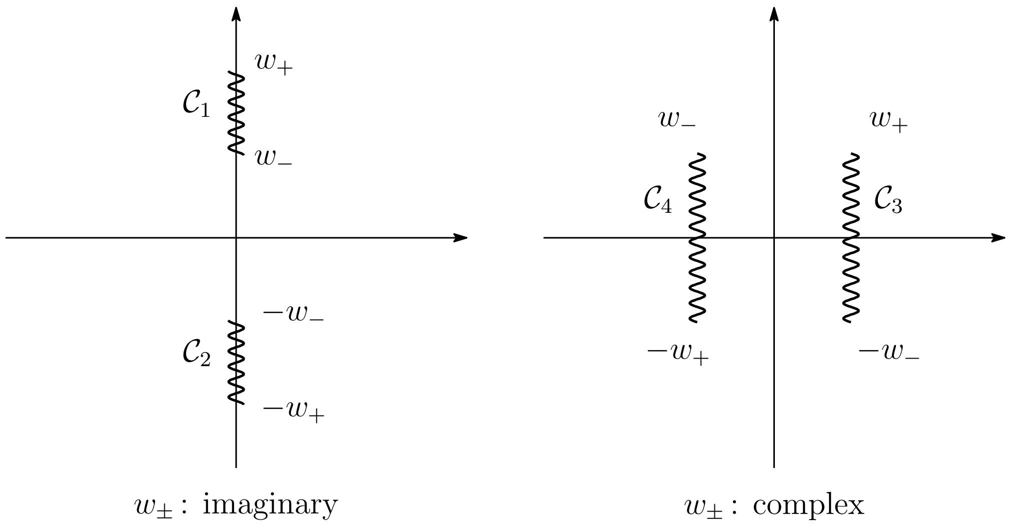


Figure 4.1: Choice of branch cuts. For $k > 1$, four branch points are connected as in Left Figure. For $0 < k < 1$, they are connected as in Right Figure.

$$
\begin{aligned}
p(x) &= -\frac{2\pi m_1 x}{x^2-1}\sqrt{\left(\frac{x^2-1}{2x}\right)^2 + k^2} + \pi m_1 \,, \\
&= -\frac{\pi m_1}{x^2-1}\sqrt{\left(x^2-1+2ikx\right)\left(x^2-1-2ikx\right)} + \pi m_1 \,, \\
&= -\frac{\pi m_1}{x^2-1}\sqrt{\left(x-w_+\right)\left(x-w_-\right)\left(x+w_+\right)\left(x+w_-\right)} + \pi m_1 \,, \qquad (4.2.23)
\end{aligned}
$$

where

$$
k = a_- = \frac{\kappa}{m_1} \,, \qquad w_\pm = ik \pm \sqrt{1-k^2}. \qquad (4.2.24)
$$

Thus, four branch points are located symmetrically with respect to the imaginary axis when $\mathcal{J}_2 = 0$.

Let us specify how to connect four branch points. The cases $k > 1$ and $0 < k < 1$ can be discussed separately. For $k > 1$ case, $w_\pm$ become purely imaginary. Thus we connect w_+ with w_-, and $-w_+$ with $-w_-$, and call it "imaginary cut" solution. The imaginary cut solution is included in type $(ii)'$ helical strings of Section 7.2.2. For $0 < k < 1$ case, we have $(w_+)^* = -w_-$. So we connect the complex conjugate pair of branch points, and call it "double contour" solution. The double contour solution is included in type $(i)'$ helical strings of Section 7.2.1.

4.2.3 On the gauge theory dual

Let us comment on the gauge theory dual. In [42], it was shown that the pulsating and rotating strings are dual to "singlet solutions" of $so(6)$ Bethe Ansatz equations. The solution presented in [42] was half-filled, *i.e.* $n_2 = n_3 = n_1/2$, where n_1 is the number of momentum-carrying

Bethe roots and $n_{2,3}$ are the number of auxiliary Bethe roots. Another expression of the $so(6)$ singlet solution is found in [27].

As explained in [11], we can recover the $so(6)$ singlet solution by taking weak coupling limit of the finite-gap solution. Firstly we rescale x by $x = 4\pi\kappa\tilde{x} = (4\pi E/\sqrt{\lambda})\,\tilde{x}$,[3] then the quasi-momentum (4.2.16) becomes

$$\begin{aligned} p(\tilde{x}) &= -\frac{1}{2\kappa\left(\tilde{x}^2 - \frac{1}{16\pi^2\kappa}\right)}\sqrt{\left(2\pi\kappa m_1\left(\tilde{x}^2 - \frac{1}{16\pi^2\kappa^2}\right) - \mathcal{J}_2\tilde{x}\right)^2 + (\kappa^2 - \mathcal{J}_2^2)\,\tilde{x}^2} + \pi m_1\,, \\ &= -\frac{1}{2\left(\tilde{x}^2 - \frac{1}{16\pi^2\kappa}\right)}\sqrt{\left(2\pi m_1\left(\tilde{x}^2 - \frac{1}{16\pi^2\kappa^2}\right) - \frac{\mathcal{J}_2}{\kappa}\,\tilde{x}\right)^2 + \left(1 - \frac{\mathcal{J}_2^2}{\kappa^2}\right)\tilde{x}^2} + \pi m_1 \end{aligned} \tag{4.2.25}$$

Secondly, recalling that $J_2 = \sqrt{\lambda}\,\mathcal{J}_2$ and $E = \sqrt{\lambda}\,\kappa$, we take the limit $\lambda \to 0$ and obtain

$$p_\pm(\tilde{x}) = \mp\frac{1}{2\tilde{x}}\sqrt{(2\pi m_1\tilde{x} - (1-\beta))^2 - \beta\,(\beta - 2)} + \pi m_1\,, \tag{4.2.26}$$

where $\beta = 1 - \mathcal{J}_2/\kappa = 1 - J_2/E$. Then, the resolvent

$$G_\pm(\tilde{x}) = \pm\frac{1}{2\tilde{x}} + p_\pm(\tilde{x}) \tag{4.2.27}$$

is exactly same as that of $so(6)$ singlet solution of [42]. Notice that by taking the weak coupling limit, the two-cut solution (4.2.25) shrinks to a pair of one-cut solutions (4.2.26). Hence, to reproduce the finite-gap representation of pulsating and rotating strings, one has to consider the sum of resolvents

$$G(\tilde{x}) = G_+(\tilde{x}) - G_+(-\tilde{x}) = -G_-(\tilde{x}) + G_-(-\tilde{x}) \tag{4.2.28}$$

in the gauge theory side.

The sum of all filling fractions in the region $x > 1$ is called "length" of the string L_{BKS} in [27]. This length agrees with the length of $so(6)$ spin chain at one-loop in $\tilde{\lambda}$, if there are no branch cuts passing across the unit circle.

Their argument is not applicable for general pulsating and rotating strings because branch cuts cross the unit circle. However, there is another quantity called "length" in strong coupling. In [43], they claim that $L_{\mathrm{KT}} = J + I_\theta$ should correspond to the length of spin chain, where I_θ is the action variable along theta direction

$$I_\theta = \frac{\sqrt{\lambda}}{2\pi}\oint d\tau\,\dot{\theta}^2 = \frac{2\kappa\sqrt{\lambda}}{\pi}\,\mathbf{E}\left(\frac{m_1}{\kappa}\right)\,. \tag{4.2.29}$$

They expanded the quantity $E - L_{\mathrm{KT}}$ in series of $\lambda/L_{\mathrm{KT}}^2$, and around $k = 0$ they found agreement with the gauge theory results [42, 143].

We do not understand well whether such nice agreement can be generalized to other examples in non-holomorphic sectors.

[3] This rescaling is almost same as what we did in (3.3.66).

Chapter 5

Correspondence for the systems of infinite size

It was Staudacher who pointed out that S-matrix is simpler object than Hamiltonian to study matching of the spectrum in AdS/CFT correspondence [8]. In fact, in ordinary quantum field theories in infinite volume the physical spectrum should appear as poles of the S-matrix. Thus, it is quite interesting to compare the S-matrix of a spin chain of infinite length, and that of string worldsheet whose spatial circumference is decompactified.

Surprisingly, it is shown that we can uniquely determine this S-matrix and the dispersion relation to all orders in λ, on the basis of existing results from perturbative computation, and the requirement that they agree with the results of string theory at strong coupling. They provide nontrivial examples of precise matching in AdS/CFT correspondence, which we are going to review below.

5.1 Asymptotic spin chain

In [12, 61], Beisert considered a spin chain of infinite length, called an asymptotic spin chain [8], and argued that if the length of a spin chain is strictly infinite, one can add extra central charges to its superconformal symmetry algebra. After the central extension, eigenvalue of the dilatation operator for BPS states becomes a nontrivial function of λ.

We start by defining the ground state of asymptotic spin chain as

$$|0\rangle_{L=\infty} \equiv [\dots ZZZ \dots] \,, \qquad \text{where} \qquad \Delta - L = 0, \quad (\Delta,\ L = \infty)\,. \tag{5.1.1}$$

We assume the ground state is invariant under the insertion or the removal of Z, because it has an infinite number of Z's. Accordingly we do not take trace in the right hand side, and neglect the trace cyclicity condition for the moment.

Excitations over the vacuum (5.1.1) can be classified according to representations of the superconformal symmetry algebra. Since the dilatation operator (or the Hamiltonian) is part of symmetry algebra and is not a central element in $\mathcal{N}=4$ theory, $psu(2,2|4)$ global symmetry is spontaneously broken down to $psu(2|2)^2 \ltimes \mathbb{R}$,[1]

$$\left(\begin{array}{c|c} \Delta & J_1 \\ S_1 & J_2 \\ S_2 & J_3 \end{array}\right) \xrightarrow[\text{broken}]{} \left(\begin{array}{c|c} & \\ S_1 & J_2 \\ S_2 & J_3 \end{array}\right). \tag{5.1.2}$$

The residual bosonic symmetry is $(SO(4)_{\text{AdS}} \times SO(4)_{\text{sphere}}) \ltimes \mathbb{R}$, where the central element corresponds to dilatation. We may discuss $psu(2|2)_L \ltimes \mathbb{R}$ and $psu(2|2)_R \ltimes \mathbb{R}$ separately, by identifying dilatation operator of the two algebra.

The algebra $su(2|2) \simeq psu(2|2) \ltimes \mathbb{R}$ is a part of the full $psu(2,2|4)$ algebra. It has bosonic subalgebra $su(2) \times su(2)$, whose generators are denoted by $\mathfrak{R}_a{}^b$ and $\mathfrak{L}^\alpha{}_\beta$ with $a, b, \alpha, \beta = 1, 2$. The supersymmetry and superconformal generators are denoted by $\mathfrak{Q}^\alpha{}_a$ and $\mathfrak{S}^b{}_\beta$, respectively. There is a central charge $\mathfrak{C}$ corresponding to the dilatation of $psu(2,2|4)$.

Being part of the $psu(2,2|4)$ symmetry, the commutation relations for $psu(2|2) \ltimes \mathbb{R}$ is given as follows:

$$[\mathfrak{R}_a{}^b, \mathfrak{J}^c] = \delta_a^c \, \mathfrak{J}^b - \frac{1}{2} \, \delta_a^b \, \mathfrak{J}^c \,, \tag{5.1.3}$$

$$[\mathfrak{L}^\alpha{}_\beta, \mathfrak{J}^\gamma] = \delta_\beta^\gamma \, \mathfrak{J}^\alpha - \frac{1}{2} \, \delta_\beta^\alpha \, \mathfrak{J}^\gamma \,, \tag{5.1.4}$$

$$\left\{\mathfrak{Q}^\alpha{}_a, \mathfrak{S}^b{}_\beta\right\} = \delta_a^b \, \mathfrak{L}^\alpha{}_\beta + \delta_\beta^\alpha \, \mathfrak{R}_a{}^b + \delta_a^b \delta_\beta^\alpha \, \mathfrak{C} \,, \tag{5.1.5}$$

which can be determined from the symmetry, and other commutation relations are trivial.

Now we extend the above algebra by adding two extra central elements as follows:

$$\left\{\mathfrak{Q}^\alpha{}_a, \mathfrak{Q}^\beta{}_b\right\} = \epsilon^{\alpha\beta} \epsilon_{ab} \mathfrak{P} \,, \tag{5.1.6}$$

$$\left\{\mathfrak{S}^a{}_\alpha, \mathfrak{S}^b{}_\beta\right\} = \epsilon_{\alpha\beta} \epsilon^{ab} \mathfrak{K} \,. \tag{5.1.7}$$

Since the generators $\mathfrak{Q}$ and $\mathfrak{S}$ have mass dimensions $+1/2$ and $-1/2$, the mass dimensions of $\mathfrak{P}$ and $\mathfrak{K}$ are $+1$ and -1, respectively. As we shall see below, they act as the insertion or the removal of Z.

5.1.1 The spectrum

We proceed to one magnon states. There are 16 magnons that have $\Delta_0 - J_1 = 1$ in $\mathcal{N}=4$ theory, where Δ_0 is conformal dimension at $\lambda = 0$. They constitute the fundamental representation of $su(2|2)^2$. Let us rewrite the indices of $\mathcal{N}=4$ bosonic fields as

$$\Phi^I = (\sigma^I)_{a\dot{a}} \Phi^{a\dot{a}} \,, \qquad D^\mu Z = (\sigma^\mu)_{a\dot{a}} D^{a\dot{a}} Z \,, \tag{5.1.8}$$

[1] The semidirect product means there are nonzero commutation relations between dilatation and the generators of $psu(2|2)^2$.

where the indices I or μ are raised or lowered by δ_{IJ} or by $\eta_{\mu\nu}$, respectively. Sixteen magnons are then decomposed as

$$\begin{array}{c|cccc} & \phi^1 & \phi^2 & \psi^1 & \psi^2 \\ \hline \bar{\phi}^{\,\dot{1}} & \Phi^{1\dot{1}} & \Phi^{2\dot{1}} & \Psi^{1\dot{1}} & \Psi^{2\dot{1}} \\ \bar{\phi}^{\,\dot{2}} & \Phi^{1\dot{2}} & \Phi^{2\dot{2}} & \Psi^{1\dot{2}} & \Psi^{2\dot{2}} \\ \bar{\psi}^{\,\dot{1}} & \overline{\Psi}^{\dot{1}1} & \overline{\Psi}^{\dot{1}2} & D^{1\dot{1}}Z & D^{2\dot{1}}Z \\ \bar{\psi}^{\,\dot{2}} & \overline{\Psi}^{\dot{2}1} & \overline{\Psi}^{\dot{2}2} & D^{1\dot{2}}Z & D^{2\dot{2}}Z \end{array} \tag{5.1.9}$$

Other fields such as $\overline{Z}$ and $F^{\mu\nu} = (\sigma^{\mu\nu})_{ab} F^{ab} + (\bar{\sigma}^{\mu\nu})_{\dot{a}\dot{b}} F^{\dot{a}\dot{b}}$ are realized as higher dimensional representations of $su(2|2)^2$.

From this decomposition table, one sees that the bosons ϕ^a have bare dimension 1/2 and the fermions ψ^α have bare dimension 1, which are equivalent to the mass dimensions of three-dimensional free field theories.

Let us focus again on one of the $su(2|2)^2$'s. If the cyclicity condition is relaxed, one magnon states can carry nonzero quasi-momentum:

$$|X(p)\rangle \equiv \sum_{n\in\mathbb{Z}} e^{ipn} \Big[\ldots ZZ \ldots \underset{\substack{\wedge \\ n}}{X} \ldots ZZ \ldots \Big] , \tag{5.1.10}$$

where $X \in \{\phi^1, \phi^2 \,|\, \psi^1, \psi^2\}$ is in the fundamental representation of $su(2|2)$ algebra, which we denote by $(\mathbf{2}|\mathbf{2})_p$ with p quasi-momentum of the magnon.

Let us see how the generators of $su(2|2)$ act on the fundamental representation $(\mathbf{2}|\mathbf{2})_p$. From (5.1.3) and (5.1.4). the rotation generators act as

$$\mathfrak{R}_a{}^b \,|\phi^c\rangle = \delta_a^c \,|\phi^b\rangle - \frac{1}{2}\,\delta_a^b \,|\phi^c\rangle , \tag{5.1.11}$$

$$\mathfrak{L}^\alpha{}_\beta \,|\psi^\gamma\rangle = \delta_\beta^\gamma \,|\psi^\alpha\rangle - \frac{1}{2}\,\delta_\beta^\alpha \,|\psi^\gamma\rangle , \tag{5.1.12}$$

and supersymmetry and superconformal generators act as

$$\mathfrak{Q}^\alpha{}_a \,|\phi^b\rangle = a\,\delta_a^b \,|\psi^\alpha\rangle , \tag{5.1.13}$$

$$\mathfrak{Q}^\alpha{}_a \,|\psi^\beta\rangle = b\,\epsilon^{\alpha\beta}\epsilon_{ab} \,|\phi^b\, Z^+\rangle , \tag{5.1.14}$$

$$\mathfrak{S}^a{}_\alpha \,|\phi^b\rangle = c\,\epsilon^{ab}\epsilon_{\alpha\beta} \,|\psi^\beta\, Z^-\rangle , \tag{5.1.15}$$

$$\mathfrak{S}^a{}_\alpha \,|\psi^\beta\rangle = d\,\delta_\alpha^\beta \,|\phi\rangle . \tag{5.1.16}$$

The symbols $Z^\pm$ signify insertion or removal of Z, which are needed to equate the mass dimension of both hand sides. Note that we do not distinguish the states with a different number of $Z^\pm$'s when we classify the excitations with respect to representations of $su(2|2)$.

Alternatively, these relations can be deduced from $psu(2,2|4)$ superconformal symmetry of the parent $\mathcal{N}=4$ theory using the decomposition of (5.1.9). One can find that insertion of Z is understood as $\delta_Q\psi \sim [\phi, Z]$ and removal of Z as taking OPE with $Z(0)$, like $\delta_S\phi(x)\cdot Z(0) \sim x\psi(x)\cdot Z(0) \sim \psi(0)$, at weak coupling [144].

Actions of $\mathfrak{P}$ and $\mathfrak{K}$ are determined from (5.1.6) and (5.1.7), as

$$\mathfrak{P}\,|X\rangle = ab\,\big|XZ^+\big\rangle \qquad \text{and} \qquad \mathfrak{K}\,|X\rangle = cd\,\big|XZ^-\big\rangle\,. \tag{5.1.17}$$

Using (5.1.5) to evaluate $\left\{\mathfrak{Q}^\alpha{}_a, \mathfrak{S}^b{}_\beta\right\}|X(p)\rangle$, one finds

$$ad - bc = 1 \qquad \text{and} \qquad \mathfrak{C}\,|X(p)\rangle = \frac{1}{2}\,(ad+bc)\,|X(p)\rangle\,. \tag{5.1.18}$$

If we impose the trace cyclicity condition on the $(\mathbf{2}|\mathbf{2})_p$ state, it must obey $\mathfrak{P} = \mathfrak{K} = 0$ as well as $p = 0$, giving $\mathfrak{C} = \pm 1/2$. Thus we can relate the central charge with the conformal dimension as $\mathfrak{C} = (\Delta - J_1)/2$ for the states with $\mathfrak{C} > 0$.

If the fundamental representation is unitary, we also have

$$a = \bar{d}\,, \quad b = \bar{c}\,, \quad p \in \mathbb{R}. \tag{5.1.19}$$

In other words, the generators $\mathfrak{Q}$ and $\mathfrak{S}$ are conjugate with each other.

We can obtain nontrivial results from centrally-extended supersymmetry algebra once we consider multi magnon states (or tensor products of $(\mathbf{2}|\mathbf{2})_{p}$,), because those generators act on the overall state.

Consider actions of $\mathfrak{P}$ and $\mathfrak{K}$ on the multi magnon state of the following form:

$$|X_1(p_1)\cdots X_M(p_M)\rangle \sim \sum_{n_1 \ll \cdots \ll n_M} e^{ip_1 n_1 + \cdots ip_M n_M}\Big[\ldots ZZ \ldots \underset{n_1}{\underset{\wedge}{X_1}} \ldots \underset{n_M}{\underset{\wedge}{X_M}} \ldots ZZ \ldots\Big]. \tag{5.1.20}$$

First, we have to notice difference between the states $|XZ^\pm\rangle$ and $|Z^\pm X\rangle$. If we insert Z^+ to the left of an impurity X, we get

$$\begin{aligned}\big|Z^+X(p)\big\rangle &= \sum_n e^{ipn}\Big[\ldots ZZ \ldots \underset{n+1}{\underset{\wedge}{X}} \ldots ZZ \ldots\Big] \\ &= \sum_n e^{ip(n-1)}\Big[\ldots ZZ \ldots \underset{n}{\underset{\wedge}{X}} \ldots ZZ \ldots\Big] = e^{-ip}\,\big|X(p)Z^+\big\rangle\,,\end{aligned} \tag{5.1.21}$$

and similarly we get $|Z^-X(p)\rangle = e^{+ip}\,|X(p)Z^-\rangle$. Next, by applying (5.1.17) successively to the state (5.1.20), we obtain

$$\mathfrak{P}\,|X_1(p_1)\cdots X_M(p_M)\rangle = \sum_{j=1}^{M} a_j b_j \,\exp\left(-i\sum_{k=j+1}^{M} p_k\right)|X_1(p_1)\cdots X_M(p_M)\rangle\,, \tag{5.1.22}$$

$$\mathfrak{K}\,|X_1(p_1)\cdots X_M(p_M)\rangle = \sum_{j=1}^{M} c_j d_j \,\exp\left(+i\sum_{k=j+1}^{M} p_k\right)|X_1(p_1)\cdots X_M(p_M)\rangle\,. \tag{5.1.23}$$

The factors $a_k b_k$ and $c_k d_k$ can be determined by the following argument. We require the conditions $\mathfrak{P} = \mathfrak{K} = 0$ must hold on the physical states that satisfy the trace cyclicity condition $\sum_{k=1}^M p_k = 0$ for any M. This requirement can be fulfilled if

$$a_k b_k = g_c \alpha \left(e^{-ip_k} - 1\right)\,, \qquad c_k d_k = g_c \beta \left(e^{ip_k} - 1\right)\,, \tag{5.1.24}$$

where constants α and β are independent of p_k, and $g_c \equiv \sqrt{\lambda}/(4\pi)$ are put for convenience.[2] Written in this way, it is clear that $\mathfrak{P}$ and $\mathfrak{K}$ generates gauge transformation corresponding to the insertion or the removal of Z,

$$\mathfrak{P} \;:\quad X \;\mapsto\; g_c\alpha\,[Z^+, X]\,, \qquad \mathfrak{K} \;:\quad X \;\mapsto\; g_c\beta\,[Z^-, X]\,. \tag{5.1.25}$$

Gathering the results (5.1.18) and (5.1.24), central charge for the multi magnon state is obtained as

$$\mathfrak{C}\,|X_1(p_1)\cdots X_M(p_M)\rangle = \sum_k \left(\pm\frac{1}{2}\sqrt{1+16\,g_c^2\alpha\beta\sin^2\left(\frac{p}{2}\right)}\right)|X_1(p_1)\cdots X_M(p_M)\rangle\,. \tag{5.1.26}$$

Since the factor $g_c^2\alpha\beta$ is a function of the 't Hooft coupling, we can rewrite this equation as

$$C = \frac{1}{2}\,(\Delta - J_1) = \frac{1}{2}\sqrt{1+f(\lambda)\sin^2\left(\frac{p}{2}\right)}. \tag{5.1.27}$$

Consistency with the BDS Ansatz [7] requires

$$f(\lambda) = \frac{\lambda}{\pi^2} + \mathcal{O}\left(\lambda^4\right)\,, \tag{5.1.28}$$

and comparison with the BMN/pp-wave limit [4], or the results of string theory (discussed in Section 5.2) dictates

$$f(\lambda) = \frac{\lambda}{16\pi^2} = g_c^2\,. \tag{5.1.29}$$

It is convenient to parametrize the four parameters a, b, c, d by another set of variables x^+, x^-, α, γ as follows:

$$a = \sqrt{g_c}\,\gamma\,, \quad b = \sqrt{g_c}\,\frac{\alpha}{\gamma}\left(1-\frac{x^+}{x^-}\right), \quad c = \sqrt{g_c}\,\frac{i\gamma}{\alpha x^+}\,, \quad d = \sqrt{g_c}\,\frac{x^+}{i\gamma}\left(1-\frac{x^-}{x^+}\right). \tag{5.1.30}$$

We have the consistency condition

$$ad - bc = 1 \quad\Longleftrightarrow\quad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g_c}\,. \tag{5.1.31}$$

The momentum p and the central charge C are expressed as

$$e^{ip} = \frac{x^+}{x^-}\,, \qquad C = \frac{g_c}{2i}\left(x^+ - \frac{1}{x^+} - x^- + \frac{1}{x^-}\right). \tag{5.1.32}$$

The new parameter γ controls normalization of the fermionic state $|\psi\rangle$ with respect to the bosonic state $|\phi\rangle$. We have $|x^+| = |x^-|$ for real values of p, and the unitarity (5.1.19) imposes the constraints

$$|\gamma|^2 = -i\left(x^+ - x^-\right), \qquad \left|\frac{\gamma}{\alpha}\right|^2 = -i\left(x^+ - x^-\right), \tag{5.1.33}$$

which can be solved by

$$|\gamma| = \left|-i\left(x^+ - x^-\right)\right|^{1/2}\,, \qquad |\alpha| = 1. \tag{5.1.34}$$

Their complex phases are left undetermined.

[2] Explicit forms of the coefficients, at one loop in λ, can also be obtained via the reduction of the superconformal transformation of the $psu(2,2|4)$ theory down to $psu(2|2)^2 \ltimes \mathbb{R}^3$. They are consistent with (5.1.24) [144].

5.1.2 The S-matrix

Next let us consider scattering of two magnons over asymptotic spin chain. The two magnon states transform as the representation $(\mathbf{2}|\mathbf{2})_{p_1} \times (\mathbf{2}|\mathbf{2})_{p_2}$, and appears like

$$
\begin{aligned}
&|X_1(p_1)X_2(p_2)\rangle \sim \Big[\ldots ZZ\ldots \underset{p_1}{\underset{\rightarrow}{X_1}} \ldots \underset{p_2}{\underset{\rightarrow}{X_2}} \ldots Z \ldots\Big] \\
&\quad + s_{\text{int}}\Big[\ldots ZZ\ldots \Big(\underset{p_1}{\underset{\rightarrow}{X_1}}\ \underset{p_2}{\underset{\rightarrow}{X_2}}\Big) \ldots Z\ldots\Big] + S(p_2,p_1)\Big[\ldots ZZ\ldots \underset{p_2}{\underset{\rightarrow}{X_1}} \ldots \underset{p_1}{\underset{\rightarrow}{X_2}} \ldots Z \ldots\Big]
\end{aligned}
\tag{5.1.35}
$$

The second term represents the state where two magnons get close to with each other. The coefficient s_{int} will be determined such that it is compatible with $su(2|2)$ symmetry. The last term contains $S(p_1, p_2)$, namely S-matrix of the asymptotic spin chain.

The S-matrix can be regarded as an operator interchanging two adjacent magnons,

$$
\mathcal{S}_{kl}\, |\cdots X_k X_l \cdots\rangle \quad \mapsto \quad (\text{coefficient})\, |\cdots X_l X_k \cdots\rangle\,. \tag{5.1.36}
$$

We require the S-matrix is compatible with the symmetry algebra, that is,

$$
[\mathfrak{J}_k + \mathfrak{J}_l, \mathcal{S}_{kl}] = 0, \tag{5.1.37}
$$

where $\mathfrak{J}$ is any generator of the $su(2|2)$ algebra. Noticeably, combining the last condition and the conjectured S-matrix in $su(2)$ subsector (with the dressing phase) together, one can uniquely determine the $su(2|2)$-invariant S-matrix. The results are listed below. We decompose $(\mathbf{2}|\mathbf{2})_{p_1} \times (\mathbf{2}|\mathbf{2})_{p_2}$ into irreducible representations of the $su(2|2)$ algebra, and regroup each element of the S-matrix with respect to these representations, as

$$
\mathcal{S}_{12}|\phi_1^a \phi_2^b\rangle = A_{12}|\phi_2^{\{a}\phi_1^{b\}}\rangle + B_{12}|\phi_2^{[a}\phi_1^{b]}\rangle + \frac{1}{2}\, C_{12}\epsilon^{ab}\epsilon_{\alpha\beta}|\psi_2^\alpha \psi_1^\beta Z^-\rangle, \tag{5.1.38}
$$

$$
\mathcal{S}_{12}|\psi_1^\alpha \psi_2^\beta\rangle = D_{12}|\psi_2^{\{\alpha}\psi_1^{\beta\}}\rangle + E_{12}|\psi_2^{[\alpha}\psi_1^{\beta]}\rangle + \frac{1}{2}\, F_{12}\epsilon^{\alpha\beta}\epsilon_{ab}|\phi_2^a \phi_1^b Z^+\rangle, \tag{5.1.39}
$$

$$
\mathcal{S}_{12}|\phi_1^a \psi_2^\beta\rangle = G_{12}|\psi_2^\beta \phi_1^a\rangle + H_{12}|\phi_2^a \psi_1^\beta\rangle, \tag{5.1.40}
$$

$$
\mathcal{S}_{12}|\psi_1^\alpha \phi_2^b\rangle = K_{12}|\psi_2^\alpha \phi_1^b\rangle + L_{12}|\phi_2^b \psi_1^\alpha\rangle. \tag{5.1.41}
$$

The coefficients from A_{12} to L_{12} are given by,

$$A_{12} = S^0_{12}\, \frac{x_2^+ - x_1^-}{x_2^- - x_1^+}\,, \tag{5.1.42}$$

$$B_{12} = S^0_{12}\, \frac{x_2^+ - x_1^-}{x_2^- - x_1^+} \left(1 - 2\, \frac{1 - 1/x_2^- x_1^+}{1 - 1/x_2^+ x_1^+}\, \frac{x_2^- - x_1^-}{x_2^+ - x_1^-} \right), \tag{5.1.43}$$

$$C_{12} = S^0_{12}\, \frac{\gamma_1 \gamma_2}{\alpha}\, \frac{2}{x_1^+ x_2^+}\, \frac{1}{1 - 1/x_1^+ x_2^+}\, \frac{x_2^- - x_1^-}{x_2^- - x_1^+}\,, \tag{5.1.44}$$

$$D_{12} = -S^0_{12}\,, \tag{5.1.45}$$

$$E_{12} = -S^0_{12} \left(1 - 2\, \frac{1 - 1/x_2^+ x_1^-}{1 - 1/x_2^- x_1^-}\, \frac{x_2^+ - x_1^+}{x_2^- - x_1^+} \right), \tag{5.1.46}$$

$$F_{12} = -S^0_{12}\, \frac{\alpha}{\gamma_1 \gamma_2}\, \frac{2(x_1^+ - x_1^-)(x_2^+ - x_2^-)}{x_1^- x_2^-}\, \frac{1}{1 - 1/x_1^- x_2^-}\, \frac{x_2^+ - x_1^+}{x_2^- - x_1^+}\,, \tag{5.1.47}$$

$$G_{12} = S^0_{12}\, \frac{x_2^+ - x_1^+}{x_2^- - x_1^+}\,, \tag{5.1.48}$$

$$H_{12} = S^0_{12}\, \frac{\gamma_1}{\gamma_2}\, \frac{x_2^+ - x_2^-}{x_2^- - x_1^+}\,, \tag{5.1.49}$$

$$K_{12} = S^0_{12}\, \frac{\gamma_2}{\gamma_1}\, \frac{x_1^+ - x_1^-}{x_2^- - x_1^+}\,, \tag{5.1.50}$$

$$L_{12} = S^0_{12}\, \frac{x_2^- - x_1^-}{x_2^- - x_1^+}\,. \tag{5.1.51}$$

Various formulae useful to derive the above result are listed in [12]. The scalar factor S^0_{12} are related to the dressing phase as [60]

$$\left(S^0_{12} \right)^2 = \frac{x_2^- - x_1^+}{x_1^- - x_2^+}\, \frac{1 - 1/x_1^- x_2^+}{1 - 1/x_1^+ x_2^-}\, \sigma^2(p_1\,, p_2)\,. \tag{5.1.52}$$

The representation of super-Lie algebra $su(2|2)^2$ has an unusual feature that the product of two irreducible representations can be irreducible. A good example is the tensor product of two short (4 dimensional) representations

$$(\mathbf{2}|\mathbf{2})_{p_1} \otimes (\mathbf{2}|\mathbf{2})_{p_2} \subset (\mathbf{8}|\mathbf{8})\,, \qquad \text{with} \quad C = \sum_{k=1}^{2} \sqrt{1 + 16\, g_c^2 \sin^2 \left(\frac{p_k}{2} \right)}\,, \tag{5.1.53}$$

which is in the long (16 dimensional) representation in general, depending on the value of central charge C. It can become short again if C can be rewritten in the single square-root form.

An important example of higher-dimensional short representation of $su(2|2)$ is supersymmetric extension of the totally symmetric representation of $su(2)$, which is called (BPS) magnon boundstates [50, 59, 58, 145]. For the two magnon case, the boundstate condition is given by

$$x_2^- = x_1^+\,. \tag{5.1.54}$$

Since the coefficient A_{12} diverges there, the S-matrix becomes the projector onto the symmetric product representation. The field content of two-magnon boundstate is explicitly obtained in [145]. In general, Q-magnon boundstate forms $16Q^2$-dimensional representation of $su(2|2)^2$.

Energy and charge of Q-magnon boundstate are given by

$$E_{\text{total}} = \sum_k E(x_k) , \qquad Q = \sum_k Q(x_k) . \tag{5.1.55}$$

By introducing the outermost rapidity variable by

$$X^+ \equiv x_Q^+ , \qquad X^- \equiv x_1^- , \tag{5.1.56}$$

and using the boundstate condition $x_j^+ = x_{j+1}^-$ for $j = 1, \ldots, Q-1$, the energy and the charge become

$$E = \frac{\sqrt{\lambda}}{4\pi i} \left\{ X^+ - \frac{1}{X^+} - X^- + \frac{1}{X^-} \right\} , \tag{5.1.57}$$

$$Q = \frac{\sqrt{\lambda}}{4\pi i} \left\{ X^+ + \frac{1}{X^+} - X^- - \frac{1}{X^-} \right\} . \tag{5.1.58}$$

We may further diagonalize the $su(2|2)$ spin chain using nested Bethe Ansatz, which was done in [12, 146]. The resultant Bethe Ansatz equations are a part of Beisert-Staudacher equations discussed in Section 2.5.3.

Let us comment on further development concerning the $su(2|2)$ invariant S-matrix. The above form of S-matrix is called spin chain frame. By a suitable definition of complex phase in (5.1.34) one can derive S-matrix in string frame, which naturally arises from worldsheet scattering in string theory [147, 60]. In addition, close connection between the $su(2|2)$ S-matrix and Shastry's R-matrix of one-dimensional Hubbard model, is pointed out in [61, 148].

5.2 Giant magnons and their scattering

Excitations over the asymptotic spin chain were characterized as the limit

$$\Delta,\ L \to \infty \qquad \text{while} \quad \Delta - L\,,\ p \ \text{ and } \ \lambda \ \text{ kept finite.} \tag{5.2.1}$$

In view of AdS/CFT correspondence, string states corresponding to these excitations should be found in the region

$$E,\ J \to \infty \qquad \text{while} \qquad E - J\,,\ p_{\text{str}} \ \text{ and } \ \lambda \ \text{ kept finite.} \tag{5.2.2}$$

Of course, classical string theory is valid only at strong coupling $\lambda \gg 1$.

Hofman and Maldacena considered classical string solutions with infinite angular momentum, and found a solution called giant magnon which is dual to one magnon state in the asymptotic spin chain [49]. By exploiting the relation between classical string on $\mathbb{R}_t \times S^2$ and sine-Gordon model, they computed scattering phase between two giant magnons, and found agreement with the conjectured all-loop S-matrix with the dressing phase in the limit $\lambda \to \infty$. We summarize their results in this section.

Giant magnon from Nambu-Goto approach

First, we follow construction of giant magnon solution in [49] where Nambu-Goto action is used. The metric on $\mathbb{R}_t \times S^2$ is written as

$$ds^2 = -dt^2 + d\theta^2 + \sin^2\theta d\varphi^2\,, \tag{5.2.3}$$

and Nambu-Goto action is given by

$$S = \frac{1}{2\pi\alpha'}\int d\tau d\sigma \sqrt{-\det G}\,, \tag{5.2.4}$$

with G_{ab} an induced metric. We fix the gauge by

$$t = a\tau, \quad \phi = b\,\sigma - a\,\tau, \tag{5.2.5}$$

with a and b constants. The induced metric is then written as

$$G_{ab} = \begin{pmatrix} -a^2 + a^2\sin^2\theta + \dot\theta^2 & \dot\theta\theta' - ab\sin^2\theta \\ \dot\theta\theta' - ab\sin^2\theta & \theta'^2 + b^2\sin^2\theta \end{pmatrix} \tag{5.2.6}$$

Assuming the Ansatz $\theta = \theta(\sigma)$, the Nambu-Goto action takes the form

$$S = \frac{\sqrt{\lambda}}{2\pi}\int d\tau' d\sigma' \sqrt{\cos^2\theta\theta'^2 + \sin^2\theta} \tag{5.2.7}$$

where $\tau' = a\tau,\ \sigma' = b\,\sigma$. The solutions to the equation of motion are

$$\sin\theta = 1, \qquad \text{or} \qquad \sin\theta = \frac{\Theta}{\cos\sigma'} \quad (\Theta \equiv \pm\sin\theta_c = \pm\cos\sigma'_c)\,, \tag{5.2.8}$$

where $-\sigma'_c \le \sigma' \le \sigma'_c$ and Θ is a constant. The latter solution is called giant magnon.

One advantage of the parametrization (5.2.5) is that it relates momentum on the worldsheet with the conserved charges in the spacetime, as

$$E - J = \frac{\sqrt{\lambda}}{2\pi}\int d\sigma' \left(\frac{\partial L}{\partial\left(\partial_{\tau'}t\right)} - \frac{\partial L}{\partial\left(\partial_{\tau'}\phi\right)}\right) = \frac{\sqrt{\lambda}}{2\pi}\int d\sigma'\ T_{\tau'}{}^{\tau'} \equiv P_{\tau'}\,, \tag{5.2.9}$$

$$J = \frac{\sqrt{\lambda}}{2\pi}\int d\sigma'\ \frac{\partial L}{\partial\left(\partial_{\tau'}\phi\right)} = \frac{\sqrt{\lambda}}{2\pi}\int d\sigma'\ T_{\sigma'}{}^{\tau'} \equiv P_{\sigma'}\,. \tag{5.2.10}$$

The canonical energy momentum tensor of giant magnon (5.2.8) is,[3]

$$\begin{aligned} T_{\tau'}{}^{\tau'} &= -\frac{\Theta}{\cos^2\sigma'}\,, & T_{\tau'}{}^{\sigma'} &= 0\,, \\ T_{\sigma'}{}^{\tau'} &= -\frac{\Theta^3\sin^2\sigma'}{\cos^2\sigma'\left(\cos^2\sigma' - \Theta^2\right)}\,, & T_{\sigma'}{}^{\sigma'} &= -\Theta. \end{aligned}$$

[3]Note that the ansatz $\theta = \theta(\sigma)$ should not be imposed before we obtain explicit expressions of $T_a{}^b$.

We immediately obtain

$$E - J = -\frac{\sqrt{\lambda}}{2\pi} \int_{-\sigma_c'}^{\sigma_c'} d\sigma' \; T_{\tau'}{}^{\tau'} \; = \; \frac{\sqrt{\lambda}}{\pi} \sin \frac{\Delta\varphi}{2} \,, \tag{5.2.11}$$

$$J = -\frac{\sqrt{\lambda}}{2\pi} \int_{-\sigma_c'}^{\sigma_c'} d\sigma' \; T_{\sigma'}{}^{\tau'} \approx \; \frac{\sqrt{\lambda}}{2\pi} \sin \frac{\Delta\varphi}{2} \left\{ 1 - \frac{1}{2} \ln \left(\frac{\sin \Delta\varphi}{\sigma_c' - \sigma'} \right) \right\} , \tag{5.2.12}$$

where we defined

$$\Delta\varphi = \Delta\sigma' \equiv 2\sigma_c' = \pi - 2\theta_c \,, \tag{5.2.13}$$

which is angular distance between two endpoints of an 'open' string. It is clear that $E - J$ remains finite while J diverges. With the identification of

$$\Delta\varphi = |p| \,, \tag{5.2.14}$$

the energy-spin relation (5.2.11) becomes

$$E - J = \frac{\sqrt{\lambda}}{\pi} \left| \sin \left(\frac{p}{2} \right) \right| , \tag{5.2.15}$$

which agrees with the strong coupling limit of one magnon state over the asymptotic spin chain,

$$\Delta - J_1 = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \left(\frac{p}{2} \right)}. \tag{5.2.16}$$

Giant magnon looks non-closed in the spacetime, in correspondence with the fact that one magnon state with $p \neq 0$ breaks the trace cyclicity condition. For the string to be closed, we have to add "the opposite piece" of a string with $\Delta\varphi = -p$.

Polyakov approach

We rewrite the giant magnon as a solution to classical string action on $\mathbb{R}_{\mathrm{t}} \times \mathrm{S}^2$ in conformal gauge, because it helps us to find connection with sine-Gordon solitons. In order to achieve the limit (5.2.2), we decompactify the string worldsheet as

$$(t, x) \equiv (\kappa\tau, \kappa\sigma) \,, \qquad \kappa \to \infty. \tag{5.2.17}$$

We identify the coordinate t with the AdS-time. The giant magnon solution is then rewritten as,

$$\xi_1 = \left\{ \cos \left(\frac{p}{2} \right) + i \tanh x_v \sin \left(\frac{p}{2} \right) \right\} e^{i\tau} , \quad \xi_2 = \frac{\sin \left(\frac{p}{2} \right)}{\cosh x_v} \,, \tag{5.2.18}$$

where we used the target space coordinates given in (3.2.2), and

$$x_v \equiv \frac{x - vt}{\sqrt{1 - v^2}} = \frac{x - \cos \left(\frac{p}{2} \right) t}{\sin \left(\frac{p}{2} \right)} \equiv x \cosh\theta - t \sinh\theta \,. \tag{5.2.19}$$

This solution obeys the boundary conditions

$$\xi_1 \to \exp \left(\pm \frac{ip}{2} + it \right) , \quad \xi_2 \to 0 \qquad \text{as} \;\; x \to \pm\infty. \tag{5.2.20}$$

Thus, the endpoints of a string move the equator of S^2 at the speed of light. The angular momentum density $\mathrm{Im}\,(\xi_1^* \partial_\tau \xi_1)$ is constant away from the origin $x_v = 0$. Thus the angular momentum around the equator diverges.

The giant magnon solution has natural interpretation from sine-Gordon point of view. Through the identification

$$\cos\phi = \sum_{j=1,2} \Big(- \partial_\tau \xi_j^* \partial_\tau \xi_j + \partial_\sigma \xi_j^* \partial_\sigma \xi_j \Big) \tag{5.2.21}$$

the field ϕ is same as the soliton solution of sine-Gordon equation

$$- \partial_\tau^2 \phi + \partial_\sigma^2 \phi - \sin\phi = 0, \qquad \phi = 4 \arctan\left(e^{-x_v}\right) . \tag{5.2.22}$$

Of course, this is not just coincidence. In Chapter 6, we will see that the any consistent solution of string theory on $\mathbb{R}_t \times S^3$ can be identified as the solution of Complex sine-Gordon system via the reduction procedure found by Pohlmeyer, Lund and Regge. Further, we will investigate the Pohlmeyer-Lund-Regge reduction thoroughly in Appendix B.

Soliton picture is quite useful to compute phase shift for the scattering of two giant magnons (GMs). This scattering takes place on the worldsheet rather than in spacetime, which corresponds to scattering of magnons on the asymptotic spin chain.

By the scattering of GMs we mean certain classical string solution which reduces to soliton scattering solution of sine-Gordon system via the map (5.2.21):

$$\tan\left(\frac{\phi}{4}\right) = \frac{1}{w} \frac{\sinh\left(\dfrac{w t_v}{\sqrt{1-w^2}}\right)}{\cosh\left(\dfrac{x_v}{\sqrt{1-w^2}}\right)} \qquad \text{for kink-kink scattering,} \tag{5.2.23}$$

$$\tan\left(\frac{\phi}{4}\right) = \frac{1}{w} \frac{\cosh\left(\dfrac{w t_v}{\sqrt{1-w^2}}\right)}{\sinh\left(\dfrac{x_v}{\sqrt{1-w^2}}\right)} , \qquad \text{for kink-antikink scattering.} \tag{5.2.24}$$

By comparing a kink solution (5.2.22) with scattering solutions (5.2.23) or (5.2.24), we find the spacetime profile is no longer rigid for scattering solutions. In fact, the GM scattering solution looks like two GMs placed next to each other at $t = -\infty$. As the time evolves, two GMs begin to collide while the center of mass moves along the equator at the speed of light. The "scattering" of GMs finishes at $t = \infty$, and the relative position of two GMs is interchanged.

To compute the phase shift of GM scattering, we do not need an explicit profile of the solution. We use the fact that under the map (5.2.21) both the scattering solutions of sine-Gordon and the GM scattering solution have the same dependence on the worldsheet coordinates, giving the same time delay.

For simplicity let us focus on kink-kink scattering solution (5.2.23). By taking the limit $t \to \pm\infty$ and comparing them with kink solution (5.2.22), one finds that for $v_1 > v_2$, the time delay that particle 1 experiences as it passes through particle 2 is given by

$$\Delta t_{12} = \frac{2\sqrt{1-v_1^2}}{v_1} \log w , \qquad v_j = \cos\left(\frac{p_j}{2}\right) = \tanh\theta_j \, , \tag{5.2.25}$$

where $w > 0$ is the relative velocity between particles 1 and 2 given by

$$w \equiv \tanh\theta_w = \tanh\left(\frac{\theta_1 - \theta_2}{2}\right) . \tag{5.2.26}$$

It can be expressed in terms of $p_{1,2}$ by using the definition of rapidity variable θ_j, as

$$w^2 = \frac{1-\cos\left(\frac{p_1-p_2}{2}\right)}{1-\cos\left(\frac{p_1+p_2}{2}\right)} , \qquad (p_1 \, , p_2 > 0) \, . \tag{5.2.27}$$

From (5.2.25) and (5.2.27) one can compute the phase shift by applying the following formula,[4]

$$\frac{\partial\delta_{12}\left(\epsilon_1 \, , \epsilon_2\right)}{\partial\epsilon_1} = \Delta t_{12} \, , \qquad \epsilon_j \equiv \frac{\sqrt{\lambda}}{\pi}\sin\left(\frac{p_j}{2}\right) \, . \tag{5.2.28}$$

By performing integration, one finds

$$\delta = \frac{\sqrt{\lambda}}{\pi}\left[-\cos\left(\frac{p_1}{2}\right) + \cos\left(\frac{p_2}{2}\right)\right] \log\left[\frac{1-\cos\left(\frac{p_1-p_2}{2}\right)}{1-\cos\left(\frac{p_1+p_2}{2}\right)}\right] - p_1\epsilon_2 \, . \tag{5.2.29}$$

The first term agrees with the strong coupling limit of AFS phase [10], which is classical part of the dressing phase. The second term comes from difference of gauge choice between gauge theory and string theory. Because GMs are excitations of nonzero size, the S-matrix depends on the gauge we choose. For classical string theory in conformal gauge, the unit length is chosen such that energy is constant, $\dot{t} = 1$. For spin chain theory, the unit length is chosen such that angular momentum J is constant. If the unit length differs by $E - J_1 = \epsilon$, then S-matrix $S = e^{i\delta}$ acquires an extra phase $e^{ip_1\epsilon_2}$.

We mention succeeding developments on generalization of giant magnon solutions. The giant magnon with the second spin J_2 is constructed in [51] and called dyonic giant magnon, which is dual to magnon boundstates found in [50]. The giant magnon with three spins $J_{1,2,3}$ is constructed in [150] via the generalized Neumann-Rosochatius Ansatz. The scattering solutions of giant magnons as well as dyonic ones are obtained explicitly by using the dressing method in [54, 151], and the phase shift for scattering of two dyonic giant magnons is studied in [58, 59]. Finite-J_1 extension of giant magnon is first discussed in [52] (see also [73]). The one-loop quantum correction to strings with an infinite spin is studied in [53], where they also studied finite-gap representation of (dyonic) giant magnon solutions.

[4] Recall there exists similar formula in quantum mechanics, which has been generalized to the case of solitons by Jackiw and Woo [149].

5.3 The dressing phase

5.3.1 Notation

There are two sorts of notation used in the literature. We will introduce them in turn.

Perturbative gauge theory (BDS) notation

The first one is particularly suited for perturbative gauge theory computation, and used, for example, in the paper of Beisert, Dippel, and Staudacher [7] as well as in Section 2.5.2.

As before, we introduced variables u and x through

$$u(p) = \frac{1}{2} \cot \frac{p}{2} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}\,, \tag{5.3.1}$$

$$u(x) = x + \frac{\lambda}{16\pi^2} \frac{1}{x}\,, \tag{5.3.2}$$

where p stands for momentum of a magnon. The relation (5.3.2) can be solved explicitly in x as

$$x(u) = \frac{1}{2} \left(u + \sqrt{u^2 - \frac{\lambda}{4\pi^2}} \right), \tag{5.3.3}$$

then it follows

$$\exp(ip) = \frac{x(u + i/2)}{x(u - i/2)} \equiv \frac{x^+}{x^-}\,. \tag{5.3.4}$$

We will also use the following coupling constant,

$$g_{\mathrm{B}}^2 \equiv \lambda / \left(8\pi^2\right). \tag{5.3.5}$$

Higher conserved charges are written as

$$q_r(x) = \frac{i}{r-1} \left\{ \frac{1}{(x^+)^{r-1}} - \frac{1}{(x^-)^{r-1}} \right\}, \tag{5.3.6}$$

and the dilatation operator Δ is written as

$$\Delta = \Delta_0 + \frac{\lambda}{8\pi^2} q_2\,. \tag{5.3.7}$$

By using (5.3.6), one can show the identity

$$\sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} - 1 = \frac{i\lambda}{8\pi^2} \left(\frac{1}{x^+} - \frac{1}{x^-} \right), \tag{5.3.8}$$

then (5.3.7) becomes

$$\Delta = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \qquad (\text{for} \;\; \Delta_0 = 1)\,. \tag{5.3.9}$$

String theory (crossing) notation

Next we introduce the notation which is suitable for expressing the functions invariant under Janik's crossing transformation [14].

We shall use

$$g_{\rm c}^2 \equiv \lambda / \left(16\pi^2\right), \tag{5.3.10}$$

instead of $g_{\rm B}$ defined in (5.3.5). We redefine variables u and $x^\pm$ by

$$u_{\rm old} \equiv g_{\rm c}\, u\,, \qquad x^\pm_{\rm old} \equiv g_{\rm c}\, x^\pm\,, \tag{5.3.11}$$

where $u_{\rm old}$, $x^\pm_{\rm old}$ are the variables used in the previous subsection. Note that we have encountered the same rescale of $x^\pm$ in (3.3.66). In terms of new variables, the relations (5.3.1) and (5.3.2) are rewritten as

$$u = \frac{1}{2g_{\rm c}} \cot\left(\frac{p}{2}\right) \sqrt{1 + 16 g_{\rm c}^2 \sin^2\left(\frac{p}{2}\right)} = x + \frac{1}{x}\,. \tag{5.3.12}$$

Equivalently, the functions $x^\pm = x^\pm(p)$ can be expressed as

$$x^\pm \equiv x\left(u \pm \frac{i}{2}\right) = e^{\pm ip/2}\, \frac{1 + \sqrt{1 + 16 g_{\rm c}^2 \sin^2\left(\frac{p}{2}\right)}}{4 g_{\rm c} \sin\left(\frac{p}{2}\right)}\,, \tag{5.3.13}$$

which are a solution of the constraint

$$x^+ - \frac{1}{x^+} - x^- + \frac{1}{x^-} = \frac{i}{g_{\rm c}}\,. \tag{5.3.14}$$

The last equation can be interpreted as the BPS condition for centrally-extended supersymmetry algebra $\mathrm{psu}(2|2)^2 \ltimes \mathbb{R}^3$. The parameter u can be reexpressed in terms of $x^\pm$, as

$$u = \frac{1}{2}\left(x^+ + \frac{1}{x^+} + x^- + \frac{1}{x^-}\right). \tag{5.3.15}$$

The constraint (5.3.14) defines a torus as the space of spectral parameters spanned by x^+ (or x^-), which is called rapidity torus. Uniformization of the rapidity torus is studied in [14, 152]. In the uniformized language, the crossing transformation $x^\pm \to 1/x^\pm$ can be mapped to the shift of half periods over the torus.

5.3.2 The dressing phase in gauge theory

All-loop asymptotic Bethe Ansatz equation in the rank-one subsectors of $\mathcal{N} = 4$ super Yang-Mills was proposed in [8].[5] With the notation introduced above, the Bethe Ansatz equations including the dressing phase are written as

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^K \left(\frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\right)^\eta \frac{1 - g_{\rm B}^2 / \left(2x_k^+ x_j^-\right)}{1 - g_{\rm B}^2 / \left(2x_k^- x_j^+\right)}\, \exp\Big(2i\theta(x_k, x_j)\Big)\,. \tag{5.3.16}$$

[5] The proposal of [8] was generalized to the full $psu(2,2|4)$ sector in [9].

where $\eta = 1$ for $su(2)$, $\eta = 0$ for $su(1|1)$, and $\eta = -1$ for $sl(2)$ subsector.

At weak coupling, four loop computation in $sl(2)$ subsector clarified the necessity of $\theta(x_k, x_j) \neq 0$ also in the gauge theory side [127]. The numerical result of [127] was made precise in [153].

To explain the four-loop results, we have to introduce the universal scaling function $f(g_{\mathrm{B}})$, also known as cusp anomalous dimension or soft anomalous dimension. The universal scaling function appears in several situations of AdS/CFT. For instance, they appear in the expression of anomalous dimension

$$\Delta_{\mathcal{O}} = S + f(g_{\mathrm{B}}) \log(S) + O(S^0), \qquad (S \gg 1)\,, \tag{5.3.17}$$

of the low-twist operators

$$\mathcal{O} = \mathrm{tr}\left(D_+^S Z^L\right) + (\text{permutations})\,, \qquad (S \gg L \sim O(1))\,, \tag{5.3.18}$$

where D_+ is covariant derivative in light-cone direction and Z is a complex scalar. In the large spin limit $S \gg L$, the universal scaling function satisfy the Eden-Staudacher equation [17]

$$f(g) = 4g^2 - 16g^4 \int_0^\infty dt\, \hat{\sigma}(t)\, \frac{J_1(\sqrt{2}gt)}{\sqrt{2}gt}\,, \tag{5.3.19}$$

$$\hat{\sigma}(t) = \frac{t}{e^t - 1}\left[\frac{J_1(\sqrt{2}gt)}{\sqrt{2}gt} - 2g^2 \int_0^\infty dt'\, \hat{K}(\sqrt{2}gt, \sqrt{2}gt')\, \hat{\sigma}(t')\right], \tag{5.3.20}$$

where $g = g_{\mathrm{B}}$, $J_{0,1}(t)$ are Bessel functions. As shown in [17], one can compute the integration kernel $\hat{K}(t, t')$ from the Bethe Ansatz equation in $sl(2)$ sector. If the dressing phase $\theta(x_k, x_j)$ is absent in (5.3.16), the kernel is given by

$$\hat{K}(t, t') = \frac{J_1(t)J_0(t') - J_0(t)J_1(t')}{t - t'}\,. \tag{5.3.21}$$

However, it turned out that the universal scaling function derived from the above kernel (5.3.21) disagrees with the results of four loop computation done by [127]. This suggests the dressing phase is nontrivial; $\theta(x_k, x_j) \neq 0$.

Let us consider the general form of dressing phase consistent with the integrability. It is argued that the dressing phase at weak coupling should take the form [10, 154]:

$$\begin{aligned}
\theta(x_k, x_j) &= \sum_{r=2}^\infty \sum_{s=r+1}^\infty \beta_{r,s}(g)\Big[q_r(x_k)q_s(x_j) - q_r(x_j)q_s(x_k)\Big], \\
\beta_{r,s}(g) &= \sum_{n=s-1}^\infty g^{2n}\, \beta_{r,s}^{(n)}, \qquad \beta_{r,r+2m}(g) = 0 \quad (m \in \mathbb{Z}),
\end{aligned} \tag{5.3.22}$$

where $q_r(x)$ are the higher conserved charges defined in (5.3.6).

The perturbative calculation at three and four loops revealed [7, 127, 153, 112]

$$\beta_{2,3}^{(2)} = 0, \qquad \beta_{2,3}^{(3)} = 4\zeta(3). \tag{5.3.23}$$

5.3.3 The BHL/BES proposal

Historically, the dressing phase $\theta(x_k, x_j)$ was first introduced on the classical string theory side [10]. It was then subsequently extended to incorporate the result one loop in $1/\sqrt{\lambda}$ [13]. The universality test of the dressing phase, namely to check independence from subsectors one chooses, was done up to one loop [155].

Janik argued in [14] that S-matrix equipped with the dressing phase should be crossing symmetric, by analogy with the S-matrix of relativistic quantum field theories. It was then shown that the dressing phase up to one loop in $1/\sqrt{\lambda}$ indeed satisfies Janik's crossing relation [15]. Beisert, Hernández, López constructed a general class of solutions to the crossing relation all order in $1/\sqrt{\lambda}$ [156, 16]. Beisert, Eden, and Staudacher picked up one of the BHL solutions, and proposed it as the exact form of the dressing phase. This is called BES (or BHL/BES) phase [21].

The BHL/BES phase at strong coupling takes the form

$$\theta(u_k, u_j) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{r,s}(g) \left(\tilde{q}_r(u_k)\, \tilde{q}_s(u_j) - \tilde{q}_s(u_k)\, \tilde{q}_r(u_j) \right), \tag{5.3.24}$$

where $\tilde{q}_r(u) = g_c^{r-1} q_r(u)$, and

$$c_{r,s}(g) = \sum_{n=0}^{\infty} c_{r,s}^{(n)} g^{1-n}\,, \tag{5.3.25}$$

$$c_{r,s}^{(n)} = \frac{(1-(-1)^{r+s})\,\zeta(n)}{2(-2\pi)^n \Gamma(n-1)} (r-1)(s-1) \frac{\Gamma[\frac{1}{2}(s+r+n-3)]\Gamma[\frac{1}{2}(s-r+n-1)]}{\Gamma[\frac{1}{2}(s+r-n+1)]\Gamma[\frac{1}{2}(s-r-n+3)]}\,. \tag{5.3.26}$$

Their proposal is based on the proposal of crossing symmetric phase [16] and on the transcendentality principle [18, 19, 20].

For $n = 0, 1$, they used the previously known results [10, 13] as an input:

$$c_{r,s}^{(0)} = \delta_{r+1,s}\,, \qquad c_{r,s}^{(1)} = -\frac{(1-(-1)^{r+s})}{\pi} \frac{(r-1)(s-1)}{(s+r-2)(s-r)}\,. \tag{5.3.27}$$

The term $n = 0$ is called AFS phase [10], and expected to capture classical string results. The term $n = 1$ is called HL phase [13], and corresponds to one-loop results in string theory.

Furthermore in [21], they found that 'analytic continuation of indices' gives weak coupling expansion of the dressing phase, consistent with the results (5.3.23). Suppose the function $c_{r,s}(g)$ in (5.3.25) can be analytically continued in the following manner:

$$c_{r,s}(g) = -\sum_{n=1}^{\infty} c_{r,s}^{(-n)} g^{1+n}\,, \tag{5.3.28}$$

then, after suitable regularization we find

$$c_{2,3}^{(-1)} = 0\,, \qquad c_{2,3}^{(-2)} = -4\,\zeta(3)\,. \tag{5.3.29}$$

The latter equation is indeed consistent with $\beta_{2,3}^{(3)} = 4\zeta(3)$ in (5.3.23). In general, one can relate $c_{r,s}^{(-n)}$ with $\beta_{r,s}^{(\ell)}$ as

$$\beta_{r,s}^{(\ell)} = -c_{r,s}^{(r+s-2\ell-1)}. \tag{5.3.30}$$

Putting this relation and (5.3.26) together, and using the identities

$$\zeta(1-z) = 2(2\pi)^{-z}\cos\left(\frac{\pi z}{2}\right)\Gamma(z)\,\zeta(z), \qquad \Gamma(1-z) = \frac{\pi}{\sin(\pi z)\Gamma(z)}\,, \tag{5.3.31}$$

we can deduce an all-order expression for $\beta_{r,s}^{(\ell)} \equiv \beta_{r,r+1+2\nu}^{(r+\mu+\nu)}$ as

$$\beta_{r,r+1+2\nu}^{(r+\mu+\nu)} \quad = \quad 2(-1)^{r+\mu+1}\,\frac{(r-1)(r+2\nu)}{2\mu+1}\left(\frac{2\mu+1}{\mu+1-r-\nu}\right)\left(\frac{2\mu+1}{\mu-\nu}\right)\zeta(2\mu\ +\ 1)\,. \tag{5.3.32}$$

The above result (5.3.32) can be rederived if we slightly modify the kernel of the Eden-Staudacher equation (5.3.20), which is called Beisert-Eden-Staudacher (BES) equation. To see this, we have to replace the undressed kernel (5.3.21) by

$$\hat{K}(t,t') \quad \longrightarrow \quad \hat{K}(t,t') + \hat{K}_d(t,t')\,, \tag{5.3.33}$$

where the dressing kernel $\hat{K}_d$ is given by

$$\hat{K}_d(t,t') \equiv 8\,g_c^2\int_0^\infty dt''\,\hat{K}_1(t,2\,g_c\,t'')\,\frac{t''}{e^{t''}-1}\,\hat{K}_0(2\,g_c\,t'',t')\,, \tag{5.3.34}$$

$$\hat{K}_0(t,t') \equiv \frac{tJ_1(t)J_0(t') - t'J_0(t)J_1(t')}{t^2-t'^2}\,, \tag{5.3.35}$$

$$\hat{K}_1(t,t') \equiv \frac{t'J_1(t)J_0(t') - tJ_0(t)J_1(t')}{t^2-t'^2}\,. \tag{5.3.36}$$

Note $\hat{K}_0$ is the even part of $\hat{K}$ under $(t,t') \to (-t,-t')$, while $\hat{K}_1$ is the odd part. After some calculation, the dressing kernel (5.3.34) can be rewritten as

$$\begin{aligned}\hat{K}_d(t,t') = -\frac{8}{t\,t'}\sum_{\mu=1}^{\infty} g^{2\mu+1}\sum_{k,l\geq 1}^{k+l\leq\mu+1} J_{2k}(t)J_{2l-1}(t')\,(-1)^{\mu+k+l}\,\times \\ \frac{(2k)\,(2l-1)}{2\mu+1}\left(\frac{2\mu+1}{\mu+1-k-l}\right)\left(\frac{2\mu+1}{\mu+1+k-l}\right)\zeta(2\mu+1)\,.\end{aligned} \tag{5.3.37}$$

As discussed in [17, 21], the dressing kernel and the dressing phase at weak coupling (5.3.22) are related as

$$\begin{aligned}\hat{K}_d(t,t') = \frac{4}{t\,t'}\sum_{\rho=1}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\mu=\nu}^{\infty} g^{2\mu+1}(-1)^{\nu}\Big(\beta_{2\rho,2\rho+1+2\nu}^{(2\rho+\nu+\mu)}J_{2\rho+2\nu}(t)J_{2\rho-1}(t') \\ + \beta_{2\rho+1,2\rho+2\nu+2}^{(2\rho+1+\nu+\mu)}J_{2\rho}(t)J_{2\rho+1+2\nu}(t')\Big)\,,\end{aligned} \tag{5.3.38}$$

One can easily check that the results (5.3.37) and (5.3.38) actually reproduce (5.3.32).

5.3.4 Breakdown of perturbative BMN scaling

It was shown in [17], the nontrivial dressing phase at four loop (5.3.23) violates BMN scaling hypothesis for anomalous dimensions in the gauge theory side. Let us observe that the term of $\beta_{2,3}^{(3)} = 4\zeta(3)$ diverges in the BMN limit.

The BMN limit is defined by

$$p = \frac{n}{L}\,, \qquad \tilde{\lambda} = \frac{\lambda}{L^2}\,, \qquad L \to \infty \qquad \text{with} \ \ n, \tilde{\lambda} \ \ \text{fixed.} \tag{5.3.39}$$

As can be seen from (5.3.13) and (5.3.11), the $x^{\pm}$ variables in the BDS notation scales as

$$x^{\pm} = L \left(1 \pm \frac{in}{2L} - \frac{1}{2} \left(\frac{n}{2L} \right)^2 + \cdots \right) \left(\frac{1 + \sqrt{1 + 4 \tilde{g}_{\mathrm{c}}^2 n^2}}{2n} \right) \qquad \text{for } L \gg 1. \tag{5.3.40}$$

Thus, for $L \gg 1$ we have

$$\frac{1}{(x^{\pm})^r} \sim \frac{1}{L^r \rho_n^r} \left(1 \mp \frac{inr}{2L} \right) \qquad \text{where} \quad \rho_n \equiv \frac{1 + \sqrt{1 + 4 \tilde{g}_{\mathrm{c}}^2 n^2}}{2n}\,, \tag{5.3.41}$$

and the higher conserved charges (5.3.6) behave as

$$q_r(x_n) = \frac{i}{r-1} \left\{ \frac{1}{\left(x_n^+\right)^{r-1}} - \frac{1}{\left(x_n^-\right)^{r-1}} \right\} \sim \frac{1}{L^r} \frac{n}{\rho_n^{r-1}}\,, \qquad \rho_n \equiv \frac{1 + \sqrt{1 + 4 \tilde{g}_{\mathrm{c}}^2 n^2}}{2n}\,. \tag{5.3.42}$$

Now it is easy to see that the first nontrivial term of BES phase behaves like

$$\begin{aligned} \theta(x_k, x_j) &= 4\zeta(3)\, g_{\mathrm{B}}^6 \Big[q_2(x_k) q_3(x_j) - q_2(x_j) q_3(x_k) \Big] \\ &\sim L \, \frac{4\zeta(3)\, \tilde{g}_{\mathrm{B}}^6 \, n_k n_j \left(\rho_k - \rho_j\right)}{\rho_k^2 \rho_j^2} \to \infty \qquad \text{as} \ \ L \to \infty, \end{aligned} \tag{5.3.43}$$

showing the breakdown of perturbative BMN scaling.

More generally, the BES phase behaves in the weak coupling region as

$$\begin{aligned} \theta(x_k, x_j) &= \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} \sum_{n=s-1}^{\infty} g_{\mathrm{B}}^{2n} \ \beta_{r,s}^{(n)} \Big[q_r(x_k) q_s(x_j) - q_r(x_j) q_s(x_k) \Big] \\ &= \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} \sum_{n=s-1}^{\infty} L^{2n-r-s} \, \frac{\beta_{r,s}^{(n)} \, \tilde{g}_{\mathrm{B}}^{2n} \, n_k n_j \left(\rho_k^{s-r} - \rho_j^{s-r} \right)}{\rho_k^{s-1} \rho_j^{s-1}} \qquad \text{for} \ \ L \gg 1. \end{aligned} \tag{5.3.44}$$

For fixed r, we relabel the index by $s = r + 1 + 2\nu$, $n = r + 2\nu + \mu$ where $\mu, \nu \geq 0$, then the sum becomes

$$\theta(x_k, x_j) = \sum_{r=2}^{\infty} \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} L^{2\mu+2\nu-1} \, \frac{\beta_{r,r+1+2\nu}^{(r+2\nu+\mu)} \, \tilde{g}_{\mathrm{B}}^{2n} \, n_k n_j \left(\rho_k^{1+2\nu} - \rho_k^{1+2\nu} \right)}{\rho_k^{r+2\nu} \rho_j^{r+2\nu}} \qquad \text{for} \ \ L \gg 1, \tag{5.3.45}$$

which implies the breakdown of perturbative BMN scaling at all order.[6]

Note also that one-loop quantum correction to energy-spin relation of classical strings neither obeys the BMN scaling hypothesis, as shown in [48].

[6] Interestingly, BES claims that the coefficient $\beta_{r,s}^{(n)}$ has transcendentality $2n - r - s + 2$ [21].

Chapter 6

Large spin strings

We study a family of classical string solutions with large spins on $\mathbb{R}_t \times S^3$ subspace of $AdS_5 \times S^5$ background, by pursuing connection with Complex sine-Gordon solitons. Via the reduction procedure of Pohlmeyer, Lund, and Regge, the equations of motion for the classical strings are cast into Lame equations and Complex sine-Gordon equations, which are solved under periodic boundary conditions. The general solution interpolates various kinds of known rigid configurations with two spins. The analytic profile of the solution is also reproduced as general 2-cut finite-gap solutions [62].

This chapter is mainly based on the author's paper with K. Okamura [96].

6.1 Classical strings as complex sine-Gordon solitons

In this section, we will briefly sketch how classical strings on $\mathbb{R}_t \times S^3$ are related to the solitons of Complex Sine-Gordon (CsG) equations.

We begin with the Polyakov action for a string which stays at the center of the AdS_5 and rotating on the three-sphere. From (3.2.3) or (3.3.3) it reads,

$$S_{\mathbb{R}_t \times S^3} = -\frac{\sqrt{\lambda}}{2} \int d\tau \int \frac{d\sigma}{2\pi} \left\{ \gamma^{ab} \left[-\partial_a \eta_0 \, \partial_b \eta_0 + \partial_a \vec{\xi} \cdot \partial_b \vec{\xi}^* \right] + \Lambda(|\vec{\xi}|^2 - 1) \right\} . \tag{6.1.1}$$

Taking the standard conformal gauge, Virasoro constraints read

$$\begin{aligned} 0 &= \mathcal{T}_{\sigma\sigma} = \mathcal{T}_{\tau\tau} = -\frac{1}{2} (\partial_\tau \eta_0)^2 - \frac{1}{2} (\partial_\sigma \eta_0)^2 + \frac{1}{2} |\partial_\tau \vec{\xi}|^2 + \frac{1}{2} |\partial_\sigma \vec{\xi}|^2 , \\ 0 &= \mathcal{T}_{\tau\sigma} = \mathcal{T}_{\sigma\tau} = \mathrm{Re} \left(\partial_\tau \vec{\xi} \cdot \partial_\sigma \vec{\xi}^* \right) . \end{aligned} \tag{6.1.2}$$

just as in (3.2.6). The equations of motion that follow from (6.1.1) are given by

$$\partial_a \partial^a \eta_0 = 0 \qquad \text{and} \qquad \partial_a \partial^a \vec{\xi} + (\partial_a \vec{\xi} \cdot \partial^a \vec{\xi}^*) \vec{\xi} = \vec{0} . \tag{6.1.3}$$

Now we are going to solve the equations (6.1.2) and (6.1.3) to find consistent string motions. Our strategy for that purpose is to make use of the trick invented by Pohlmeyer, Lund, and

Regge, that is, to relate $O(4)$ nonlinear sigma model with conformal gauge to CsG system [55, 56, 57]. With a solution of the CsG equations at hand, the problem of constructing corresponding string solutions will boil down to just solving a Schrödinger equation with a potential resulted from the CsG solution.

The recipe for the Pohlmeyer-Lund-Regge reduction for $O(4)$ sigma model is as follows. First, define worldsheet light-cone coordinates $\sigma^{\pm}$ by $\tau = \sigma^{+} + \sigma^{-}$, $\sigma = \sigma^{+} - \sigma^{-}$. Second, choose a basis of $O(4)$-covariant vectors as X_i, $\partial_+ X_i$, $\partial_- X_i$ and $\epsilon_{ijkl} X^j \partial_+ X^k \partial_- X^l \equiv K_i$ $(i, j, k, l = 1, \ldots, 4)$ so that any vectors can be written as a linear combination of them. We can then define two $O(4)$-invariants ϕ and χ through the relations

$$-\partial_+ \vec{X} \cdot \partial_- \vec{X} \equiv \cos\phi \,, \tag{6.1.4}$$

$$\partial_+^2 \vec{X} \cdot \vec{K} \equiv 2\,\partial_+ \chi \, \sin^2(\phi/2), \qquad \partial_-^2 \vec{X} \cdot \vec{K} \equiv -2\,\partial_- \chi \, \sin^2(\phi/2). \tag{6.1.5}$$

Third, by using the equations of motion, Virasoro constraints and the normalization condition $|\vec{\xi}|^2 = 1$, write the equations of motion for ϕ and χ as

$$\partial_a \partial^a \phi - \sin\phi - \frac{\sin(\phi/2)}{2\cos^3(\phi/2)} (\partial_a \chi)^2 = 0 \,, \quad \partial_a \partial^a \chi + \frac{2\,\partial_a \phi\, \partial^a \chi}{\sin\phi} = 0 \,. \tag{6.1.6}$$

They are nothing but the CsG equations. Finally, substitute (6.1.4) into (6.1.3) to get

$$\partial_a \partial^a \vec{\xi} + (\cos\phi)\, \vec{\xi} = \vec{0} \,. \tag{6.1.7}$$

This is the Schrödinger equation with a self-consistent potential mentioned above.

In [51], the authors utilized Pohlmeyer's reduction to obtain a family of classical string solutions called dyonic giant magnons, which were associated with *kink* solitons of CsG equations. In the same spirit, we are now going to exploit so-called *helical wave* solutions of CsG equations to find new, more general motions of strings on $\mathbb{R}_{\mathrm{t}} \times \mathrm{S}^3$.

Before doing so, let us end this section by making some additional notes on CsG system. The CsG equations (6.1.6) follow from the Lagrangian

$$\mathcal{L}_{\mathrm{CsG}} = \frac{1}{2} (\partial_a \phi)^2 + \frac{\tan^2(\phi/2)}{2} (\partial_a \chi)^2 - \cos\phi \,. \tag{6.1.8}$$

By introducing a complex field $\psi \equiv \sin(\phi/2) \exp(i\chi/2)$, we can rewrite it as

$$\mathcal{L}_{\mathrm{CsG}} = \frac{\tilde{\partial}_a \psi^* \, \tilde{\partial}^a \psi}{1 - |\psi|^2} + \mu^{-2} |\psi|^2 \,, \tag{6.1.9}$$

where we have also introduced a real parameter μ to rescale the worldsheet variables as $(\tilde{\tau}, \tilde{\sigma}) \equiv (\mu\tau, \mu\sigma)$. Then the equations of motion can be combined into

$$\tilde{\partial}_a \tilde{\partial}^a \psi + \psi^* \frac{(\tilde{\partial}_a \psi)^2}{1 - |\psi|^2} - \mu^{-2} \psi \left(1 - |\psi|^2\right) = 0 \,. \tag{6.1.10}$$

When $\chi =$ constant, this CsG system reduces to sG system with the sG field ϕ.

6.2 Helical string solutions with a single spin

To illustrate our strategy to find general classical string solutions, let us begin with a simple single-spin case. It should result from a so-called "helical wave" (or "kink train") of sG theory, which is a rigid array of kinks. An example of such helical solitons is given by

$$\phi_{\rm cn}(x,t) = 2\arcsin\left[\,\mathrm{cn}\left(\frac{(x-x_0)-v(t-t_0)}{k\sqrt{1-v^2}}, k\right)\right] , \qquad (6.2.1)$$

where v is the soliton velocity, (t_0, x_0) are initial values for (t, x) which will be set to zero in what follows, and cn is the Jacobian cn function.[1] The parameter k determines the spatial period (or "wavelength") of ϕ field with respect to $x - vt$ as $4k\,\mathbf{K}(k)\sqrt{1-v^2}$. Note that in the limit $k \to 1$, (6.2.1) reduces to an ordinary single-kink soliton with velocity v,

$$\phi(x,t) = 2\arcsin\left[1\Big/\cosh\left(\frac{x-vt}{\sqrt{1-v^2}}\right)\right] . \qquad (6.2.2)$$

As discussed before, our strategy to find periodic string solutions is to substitute (6.2.1) into (6.1.7) to obtain a Schrödinger equation. For a generic helical soliton, the string equation of motion (6.1.7) can be written in the form

$$\left\{-\partial_\tau^2 + \partial_\sigma^2 - \mu^2 k^2\left[2\,\mathrm{sn}^2\left(\frac{\mu(\sigma - v\tau)}{\sqrt{1-v^2}}, k\right) - 1\right]\right\}\vec{\xi} = \mu^2 U\,\vec{\xi}, \qquad (6.2.3)$$

with $(k\mu\tau, k\mu\sigma) \equiv (t,x)$. In particular, we have $U = 0$ for the cn-type helical soliton (6.2.1), but we will keep U general for the moment. Let us introduce boosted worldsheet coordinates,

$$T(\tau,\sigma) \equiv \frac{\tilde\tau - v\tilde\sigma}{\sqrt{1-v^2}}\,, \qquad X(\tau,\sigma) \equiv \frac{\tilde\sigma - v\tilde\tau}{\sqrt{1-v^2}}\,, \qquad (6.2.4)$$

with which we can rewrite the string equation of motion (6.2.3) as

$$\left[-\partial_T^2 + \partial_X^2 - k^2\left(2\,\mathrm{sn}^2(X,k) - 1\right)\right]\vec{\xi} = U\,\vec{\xi}. \qquad (6.2.5)$$

We can solve this equation under an Ansatz

$$\xi_j(T,X;w_j) = \mathcal{Y}_j(X;w_j)\,e^{iu_j(w_j)T} \qquad (j=1,2)\,. \qquad (6.2.6)$$

Here w_j are complex parameters and $\mathcal{Y}_j$ are independent of T. As for constraints on w, see Appendix A.3. The differential equation satisfied by $\mathcal{Y}_j$ then takes the form

$$\left[\frac{d^2}{dX^2} - k^2\Big(2\,\mathrm{sn}^2(X,k) - 1\Big) + u_j^2\right]\mathcal{Y}_j = U\,\mathcal{Y}_j\,, \qquad (6.2.7)$$

which is known as *Lamé equation.* General eigenfunctions of Lamé equations were found by Hermite and Halphen in the nineteenth century; see Chapter 23.7 of [157] for details. They are given by

$$\mathcal{Y}(X;w) \propto \frac{\Theta_1(X-w,k)}{\Theta_0(X,k)}\exp\left(Z_0(w,k)X\right) \quad \text{with} \quad u^2 = \mathrm{dn}^2(w,k) + U\,, \qquad (6.2.8)$$

[1] For our conventions of elliptic functions and elliptic integrals, see Appendix A.1.

where Θ_ν, Z_ν are the Jacobian theta and zeta functions defined in Appendix A.1, respectively.

The result (6.2.8) is a good starting point for us to construct string solutions that satisfy the string equation of motion (6.2.3), the consistency condition for Pohlmeyer's reduction (6.1.4) and the Virasoro conditions (6.1.2). Actually it turns out that, corresponding to several possibilities of choosing a helical soliton solution of (C)sG equation, there can be as many consistent string solutions. As it seems likely that all of them are related by appropriate reparametrization of the elliptic functions, in this thesis, we are only concerned with cn-type helical soliton of (6.2.1).

Recall that in Gubser-Klebanov-Polyakov (GKP) case [35], there were two possible configurations of closed strings moving on S^2: the folded and circular string. We will see, in our helical case also, there are two types of *rigid* string configurations possible. They will turn out to reduce, in certain limits, to each of two GKP configurations. The first type stays only one of the hemispheres about the equator, say the northern hemisphere (See Figure 6.1 below), while the second type sweeps in both hemispheres, crossing the equator several times (Figure 6.4). We will call the first type "type (i)" and the second "type (ii)" *helical* string solution, after the name "helical wave" in soliton theory.[2] Below we will demonstrate these two types in turn. We will only present the results, and the details will be presented in Section 6.3 and Appendix A.3.

6.2.1 Type (i) helical strings with a single spin

We begin with the type (i) case. The profile is given by[3]

$$t\,(T,X) = aT + bX \qquad \text{with} \quad a = k\,\mathrm{cn}(i\omega)\,, \quad b = -ik\,\mathrm{sn}(i\omega)\,, \tag{6.2.9}$$

$$\xi_1(T,X) = \frac{\sqrt{k}}{\mathrm{dn}(i\omega)}\frac{\Theta_0(0)}{\Theta_0(i\omega)}\frac{\Theta_1(X-i\omega)}{\Theta_0(X)}\,\exp\left[Z_0(i\omega)X + i\,\mathrm{dn}(i\omega)T\right]\,, \tag{6.2.10}$$

$$\xi_2(T,X) = \frac{\mathrm{dn}(X)}{\mathrm{dn}(i\omega)}\,, \tag{6.2.11}$$

with ω a real parameter. The soliton velocity v, which appeared in the definitions of T and X (6.2.4), is related to the parameters a and b in (6.2.9) as $v \equiv b/a$. Using various properties and identities listed in Appendices A.1 and A.3, one can check the proposed set of solutions (6.2.9)-(6.2.11) indeed satisfies the required physical constraints. Note that the AdS-time variable t can be rewritten as $t = k\tilde{\tau}$.

The spacetime profile of this kind of solutions is depicted in Figure 6.1. From its appearance, it looks quite similar to the one obtained in [158], which is known as a "spiky" string on S^2. Indeed, the single-spin limit of the type (i) helical spinning strings agrees with so-called "spiky strings" studied in [158, 52, 73]. Also, the authors of [150] argued both the "spiky" strings and

[2]Throughout this chapter, the term "helical strings" is used to mean helical spinning strings.

[3]We often omit the elliptic moduli k in the expressions of elliptic functions.

giant magnons can be obtained from a generalized Neumann-Rosochatius Ansatz on a string sigma model.

The type (i) single spin solution does not actually have singularities at the apparent spikes, as can be seen from $\partial_\sigma \vec{\xi}\Big|_{\sigma=\pm l} = \vec{0}$ with l defined in (6.2.12) below. Two-spin helical spinning strings are different from the spiky strings in that they have no singular points in spacetime. When embedded in $\mathbb{R}\times S^3$, the singular "cusps" of the spiky string that apparently existed on $\mathbb{R}\times S^2$ are all smoothed out to result in non-spiky profiles.

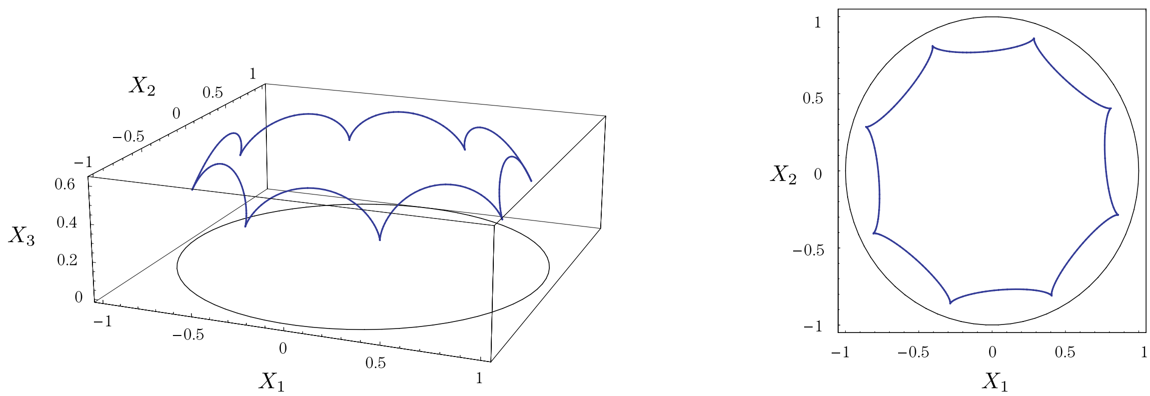


Figure 6.1: Type (i) helical solution with a single spin. The diagram shows $k = 0.68$ and $n = 8$ case. Each turning points are located away from the equator, and each segment curves *inwards*.

In order to make the string closed and rigid, we impose a periodic boundary condition. Since our solutions are quasi-periodic in X with the period $2\,\mathbf{K}$, we shall refer to the region

$$-\,l \le \sigma \le l\,, \qquad l \equiv \frac{\mathbf{K}\sqrt{1-v^2}}{\mu}\,, \tag{6.2.12}$$

at fixed τ as "one-hop". Just as (3.2.12), the periodicity of a closed string requires

$$\Delta\sigma\Big|_{\text{one-hop}} \equiv \frac{2\pi}{n} = \frac{2\mathbf{K}\sqrt{1-v^2}}{\mu}\,, \tag{6.2.13}$$

$$\Delta\varphi_1\Big|_{\text{one-hop}} \equiv \frac{2\pi N_{\varphi_1}}{n} = 2\mathbf{K}\left(-iZ_0(i\omega) + \frac{i\,\mathrm{sn}(i\omega)\,\mathrm{dn}(i\omega)}{\mathrm{cn}(i\omega)}\right) + (2n_1'+1)\,\pi, \tag{6.2.14}$$

with $n = 1, 2, \ldots$, and N_{φ_1}, n_1' being integers. When σ runs from 0 to 2π, an array of n hops winds N_{φ_1} times in φ_1-direction in the target space, thus making the string closed. The integer n_1' is related to periodicity with respect to ω. When we make a shift $\omega \mapsto \omega + 2\mathbf{K}'$, the integer n_1' increase by one while ξ_i and $\Delta\varphi_i$ are unchanged.

Let us compute the conserved charges for the type (i) strings. The energy E and the spin J_1 are defined as (3.2.7) and (3.2.9):

$$E \equiv \frac{\sqrt{\lambda}}{\pi}\,\mathcal{E} = \frac{n\sqrt{\lambda}}{2\pi}\int_{-l}^{l} d\sigma\,\partial_\tau t\,, \qquad J_1 \equiv \frac{\sqrt{\lambda}}{\pi}\,\mathcal{J}_1 = \frac{n\sqrt{\lambda}}{2\pi}\int_{-l}^{l} d\sigma\,\mathrm{Im}\left(\xi_1^*\partial_\tau\xi_1\right)\,. \tag{6.2.15}$$

Then the conserved charges for this type (i) solution are computed as

$$\mathcal{E} = \frac{nk\,\mathbf{K}}{\mathrm{cn}(i\omega)}\,, \qquad \mathcal{J}_1 = \frac{n(\mathbf{K}-\mathbf{E})}{\mathrm{dn}(i\omega)}\,. \tag{6.2.16}$$

In what follows, we will see two distinct limits that reduce the solution to two simple known examples; one is the folded string of GKP, and the other is the giant magnon of HM.

The GKP Case. In $\omega \to 0$ limit, a type (i) solution reduces to a folded string solution studied in [35]. See Figure 6.2 for the spacetime profile. In this limit, boosted worldsheet coordinates become $(T,X) \to (\tilde{\tau},\tilde{\sigma})$ defined in (6.2.4), and the fields (6.2.9)-(6.2.11) reduce to, respectively,

$$t \to k\tilde{\tau}\,, \qquad \xi_1 \to k\,\mathrm{sn}\,(\tilde{\sigma},k)\; e^{i\tilde{\tau}}\,, \qquad \xi_2 \to \mathrm{dn}\,(\tilde{\sigma},k)\;. \tag{6.2.17}$$

This solution corresponds to a kink-array of sG equation *at rest* $(v=0)$, and it spins around the northern pole of an S^2 with its center of mass fixed at the pole. The integer n counts the number of folding, which is related to μ via the boundary condition (6.2.13).

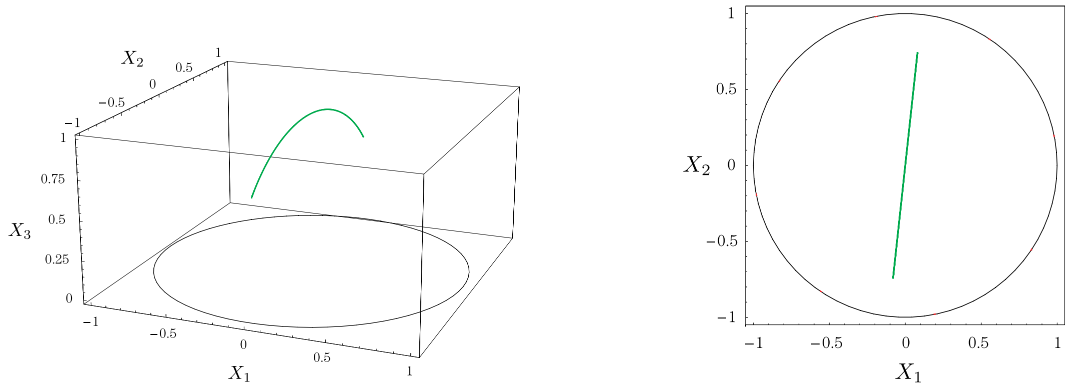


Figure 6.2: Type (i) helical solution with a single spin; $\omega = 0$ and $k = 0.75$. This can be regarded as a folded string of [38], in which case n represents the number of folds.

The HM Case. The limit $k \to 1$, $\mu \to \infty$ takes the type (i) solution to an array of giant magnons, each of which having the same soliton velocity of sG system [49]. The endpoints of the string move on the equator $\theta = \pi/2$ at the speed of light, see Figure 6.3. In this limit, boosted worldsheet coordinates become $T \to \tilde{\tau}/\cos\omega - (\tan\omega)\,\tilde{\sigma}$ and $X \to \tilde{\sigma}/\cos\omega - (\tan\omega)\,\tilde{\tau}$, and the fields (6.2.9)-(6.2.11) reduce to

$$t \to \tilde{\tau}\,, \quad \xi_1 \to \left[\tanh\left(\tfrac{\tilde{\sigma}-(\sin\omega)\tilde{\tau}}{\cos\omega}\right)\cos\omega - i\sin\omega\right] e^{i\tilde{\tau}}\,, \quad \xi_2 \to \frac{\cos\omega}{\cosh\left(\frac{\tilde{\sigma}-(\sin\omega)\tilde{\tau}}{\cos\omega}\right)}\,. \tag{6.2.18}$$

The following boundary conditions are imposed at each end of hops:

$$\xi_1 \to \exp\left(\pm i\Delta\varphi_1/2 + i\tilde{\tau}\right), \quad \xi_2 \to 0 \qquad \text{as} \quad \tilde{\sigma} \to \pm\infty\,, \tag{6.2.19}$$

in place of (6.2.13) and (6.2.14). One can see $\Delta\varphi_1$ is determined only by ω, which is further related to the magnon momentum p of the gauge theory as $\Delta\varphi_1 = p = \pi - 2\omega$ in view of the AdS/CFT [49].

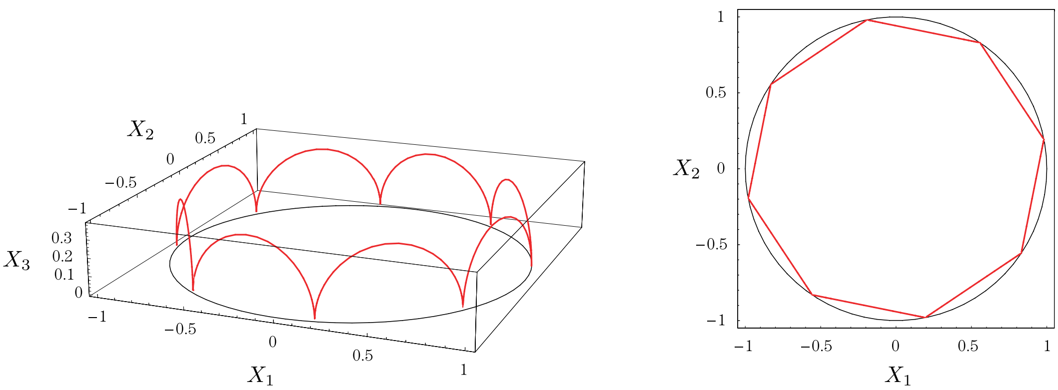


Figure 6.3: Type (i) helical solution with a single spin, in the limit $k \to 1$. The diagram shows $n = 8$ case, and it can be understood as an array of $n = 8$ giant magnons.

6.2.2 Type (ii) helical strings with a single spin

Let us turn to the type (ii) solution. In contrast to the type (i) case, it winds around the equator of S^2, waving up and down; see Figure 6.4. The profile is given by[4]

$$\hat{t}\,(T,X) = \hat{a}\,T + \hat{b}X\,, \qquad \text{with} \quad \hat{a} = \mathrm{dn}(i\omega)\,, \quad \hat{b} = -ik\,\mathrm{sn}(i\omega)\,, \tag{6.2.20}$$

$$\hat{\xi}_1(T,X) = \frac{1}{\sqrt{k}\;\mathrm{cn}(i\omega)}\frac{\Theta_0(0)}{\Theta_0(i\omega)}\frac{\Theta_1(X - i\omega)}{\Theta_0(X)}\exp\left[Z_0(i\omega)X + ik\,\mathrm{cn}(i\omega)T\right], \tag{6.2.21}$$

$$\hat{\xi}_2(T,X) = \frac{\mathrm{cn}(X)}{\mathrm{cn}(i\omega)}\,, \tag{6.2.22}$$

where ω is again a real parameter, and the soliton velocity is given by $\hat{v} \equiv \hat{b}/\hat{a}$. In this type ($ii$) case, the AdS-time can be written as $\hat{\eta}_0 = \tilde{\tau}$. Just as was the case with type (i) solutions, we need to impose the periodic boundary conditions for a type (ii) solution to be closed:

$$\Delta\sigma\Big|_{\text{one-hop}} \equiv \frac{2\pi}{m} = \frac{2\mathbf{K}\sqrt{1-v^2}}{\mu}\,, \tag{6.2.23}$$

$$\Delta\varphi_1\Big|_{\text{one-hop}} \equiv \frac{2\pi M_{\varphi_1}}{m} = 2\mathbf{K}\left(-iZ_0(i\omega) + \frac{ik^2\,\mathrm{sn}(i\omega)\,\mathrm{cn}(i\omega)}{\mathrm{dn}(i\omega)}\right) + (2m_1' + 1)\pi\,, \tag{6.2.24}$$

[4] We use a hat to indicate type (ii) quantities.

where $m = 1, 2, \ldots$ is the number of hops, M_{φ_1} is the winding number in φ_1-direction, and m'_1 is an integer.

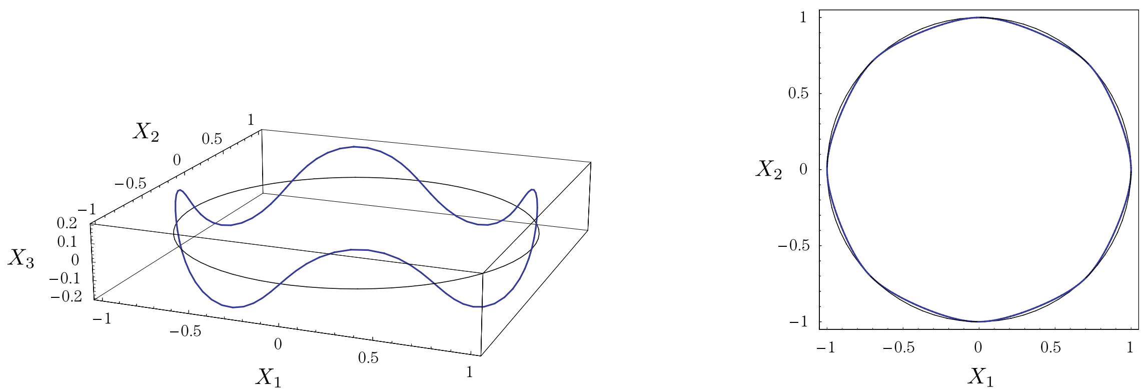


Figure 6.4: Type (ii) helical solution with a single spin. The diagram shows $k = 0.68$ and $m = 8$ case. As compared to the type (i) case, each segment curves *outwards* about the northern pole.

The conserved charges for the type (ii) solution are calculated in the same manner as in the type (i) case. They are given by

$$\hat{\mathcal{E}} = \frac{m\mathbf{K}}{\mathrm{dn}(i\omega)} \,, \qquad \hat{\mathcal{J}} = \frac{m(\mathbf{K} - \mathbf{E})}{k \, \mathrm{cn}(i\omega)} \,. \tag{6.2.25}$$

The GKP Case. In $\omega \to 0$ limit, a type (ii) solutions reduce to a circular string studied in [35]. See Figure 6.5 for a snapshot. Again, the boosted coordinates (6.2.4) become $(T, X) \to (\tilde{\tau}, \tilde{\sigma})$, and the profile reduces to

$$\hat{t} \to \tilde{\tau} \,, \qquad \hat{\xi}_1 \to \mathrm{sn}\,(\tilde{\sigma}, k) \; e^{i\tilde{\tau}} \,, \qquad \hat{\xi}_2 \to \mathrm{cn}\,(\tilde{\sigma}, k) \;. \tag{6.2.26}$$

The integer m counts the number of winding, which is related to μ via the boundary condition (6.2.23).

The HM Case. The limits $k \to 1$ and $\mu \to \infty$ reduce the type (ii) solution to an array of giant magnons and flipped giant magnons, one after the other. The shape of each giant magnon is same as (6.2.18), see Figure 6.6.

6.3 Helical string solutions with two spins

Let us now turn to the problem of finding generic helical string solutions with two spins. As discussed in Section 6.1, string solutions on $\mathbb{R}_t \times S^3$ of our concern are related to CsG solitons via Pohlmeyer's reduction. Therefore we begin with generalizing helical solitons of sG equation

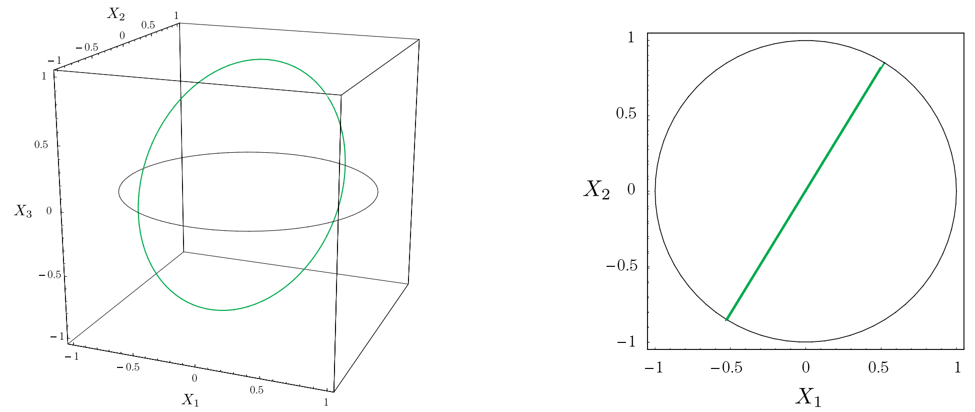


Figure 6.5: Type (*ii*) helical solution with a single spin, with $\omega = 0$. This can be regarded as a circular string of [38], in which case $m/2$ represents the winding number along a great circle.

(6.2.1) to those of CsG equations. One can easily confirm the following function is an example of such helical solutions of CsG equations:

$$\psi_{\rm cn} = ck\ \mathrm{cn}\,(cx_v, k)\exp\left(it_v\sqrt{(1-c^2k^2)(1+c^2(1-k^2))}\right), \tag{6.3.1}$$

where c takes the value in $-1/k < c < 1/k$ for $0 \le k \le 1$, and x_v, t_v are defined as

$$x_v \equiv \frac{x - vt}{\sqrt{1-v^2}}\,, \qquad t_v \equiv \frac{t - vx}{\sqrt{1-v^2}}\,. \tag{6.3.2}$$

Thus the periodic function (6.3.1) can be thought of a natural generalization of (6.2.1). We will use this solution to find the dyonic extended version of helical solutions.

The string equations of motion become the same as (6.2.5) under identifications $(\mu\tau, \mu\sigma) \equiv (ct, cx)$, and we can solve them with the same Ansatz (6.2.6). For the case of cn-type helical soliton (6.3.1), U is evaluated as $U_{\rm cn} = (1/c^2) - k^2 \ge 0$. If we started with other helical solitons such as of sn- or dn-type, they would give different ranges for U in general. Hence we will treat U as a controllable parameter.

We are interested in string configurations with two spins, which interpolate known string solutions in an obvious way.

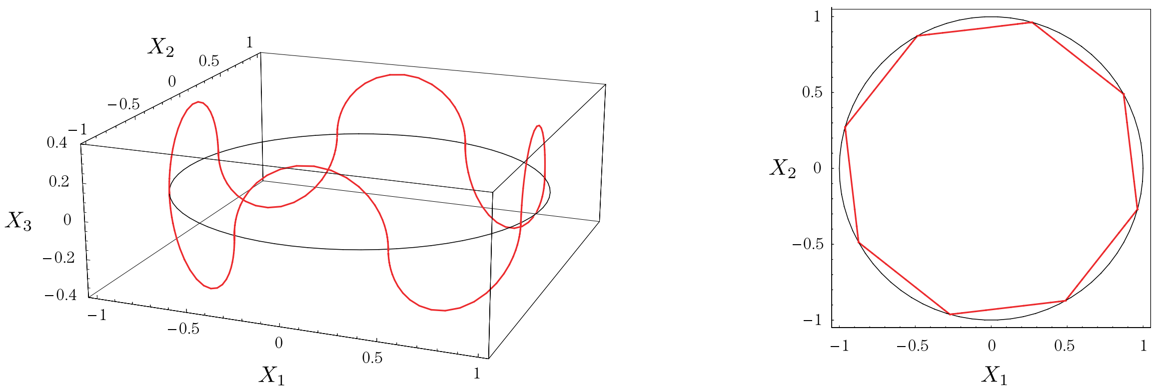


Figure 6.6: Type (ii) helical solution with single spin, in the limit $k \to 1$. The diagram shows $m = 8$ case, and it can be realized as an array of four giant magnons and four flipped giant magnons by turns. It can be regarded as the same configuration as that of Figure 6.3, which is made up of eight giant magnons; these two configurations can be switched to each other without energy costs.

6.3.1 Type (i) helical strings with two spins

First we will focus on the type (i) case. The solution can be written in the following form:

$$t = aT + bX \,, \tag{6.3.3}$$

$$\xi_1 = C \frac{\Theta_0(0)}{\sqrt{k}\,\Theta_0(i\omega_1)} \frac{\Theta_1(X - i\omega_1)}{\Theta_0(X)} \exp\Big(Z_0(i\omega_1)X + iu_1 T \Big) \,, \tag{6.3.4}$$

$$\xi_2 = C \frac{\Theta_0(0)}{\sqrt{k}\,\Theta_2(i\omega_2)} \frac{\Theta_3(X - i\omega_2)}{\Theta_0(X)} \exp\Big(Z_2(i\omega_2)X + iu_2 T \Big) \,. \tag{6.3.5}$$

Here ω_1 and ω_2 are real parameters. The normalization constant C is chosen as

$$C = \left(\frac{\mathrm{dn}^2(i\omega_2)}{k^2\,\mathrm{cn}^2(i\omega_2)} - \mathrm{sn}^2(i\omega_1) \right)^{-1/2} \,, \tag{6.3.6}$$

so that the sigma model condition $|\xi_1|^2 + |\xi_2|^2 = 1$ is satisfied. The parameters a and b in (6.3.3) are fixed by Virasoro conditions, which imply

$$a^2 + b^2 = k^2 - 2k^2\,\mathrm{sn}^2(i\omega_1) - U + 2u_2^2 \,, \tag{6.3.7}$$

$$ab = -i\,C^2 \left(u_1\,\mathrm{sn}(i\omega_1)\,\mathrm{cn}(i\omega_1)\,\mathrm{dn}(i\omega_1) - u_2\,\frac{1-k^2}{k^2}\,\frac{\mathrm{sn}(i\omega_2)\,\mathrm{dn}(i\omega_2)}{\mathrm{cn}^3(i\omega_2)} \right) . \tag{6.3.8}$$

Just as in the single spin cases, we can adjust the soliton velocity v so that the AdS-time is proportional to the worldsheet time variable. It then follows that $v \equiv b/a \le 1$ and $\eta_0 = \sqrt{a^2 - b^2}\,\tilde{\tau}$. Two angular velocities are constrained as

$$u_1^2 = U + \mathrm{dn}^2(i\omega_1) \,, \qquad u_2^2 = U - \frac{(1-k^2)\,\mathrm{sn}^2(i\omega_2)}{\mathrm{cn}^2(i\omega_2)} \,, \tag{6.3.9}$$

where the parameter U corresponds to the eigenvalue of the Lamé equation (6.2.5). From (6.3.9) we find the two angular velocities u_1 and u_2 satisfy

$$u_1^2 - u_2^2 = \operatorname{dn}^2(i\omega_1) + \frac{(1-k^2)\operatorname{sn}^2(i\omega_2)}{\operatorname{cn}^2(i\omega_2)} \,. \tag{6.3.10}$$

When $\omega_2 = u_2 = 0$, this reproduces the type (i) single spin solution of Section 6.2.1. The consistency condition (6.1.4) is indeed satisfied as

$$\frac{1}{\mu^2}\sum_{i=1}^{2}\left(|\partial_\sigma \xi_i|^2 - |\partial_\tau \xi_i|^2\right) = k^2 - 2k^2 \operatorname{sn}^2(X) - U \,, \tag{6.3.11}$$

from which we can deduce the equation of motion (6.2.5).

As in the single spin case, we can write down the conditions for a type (i) dyonic helical string to be closed. They read,

$$\Delta\sigma\Big|_{\text{one-hop}} \equiv \frac{2\pi}{n} = \frac{2\mathbf{K}\sqrt{1-v^2}}{\mu} \,, \tag{6.3.12}$$

$$\Delta\varphi_1\Big|_{\text{one-hop}} \equiv \frac{2\pi N_{\varphi_1}}{n} = 2\mathbf{K}\left(-iZ_0(i\omega_1) - v\,u_1\right) + (2n_1' + 1)\pi \,, \tag{6.3.13}$$

$$\Delta\varphi_2\Big|_{\text{one-hop}} \equiv \frac{2\pi N_{\varphi_2}}{n} = 2\mathbf{K}\left(-iZ_2(i\omega_2) - v\,u_2\right) + 2n_2'\pi \,. \tag{6.3.14}$$

As σ runs from 0 to 2π, the string hops n times in the target space, winding N_{φ_1} and N_{φ_2} times in φ_1- and φ_2-direction, respectively.

Global conserved charges can be computed just as was done in Section 6.2. The rescaled energy $\mathcal{E}$ and the spins $\mathcal{J}_j$ $(j = 1, 2)$ are evaluated after a little algebra to give

$$\mathcal{E} = n a \left(1 - v^2\right) \mathbf{K} \,, \tag{6.3.15}$$

$$\mathcal{J}_1 = \frac{nC^2 u_1}{k^2}\left[-\mathbf{E} + \left(\operatorname{dn}^2(i\omega_1) + \frac{v k^2}{u_1} i \operatorname{sn}(i\omega_1)\operatorname{cn}(i\omega_1)\operatorname{dn}(i\omega_1)\right)\mathbf{K}\right] \,, \tag{6.3.16}$$

$$\mathcal{J}_2 = \frac{nC^2 u_2}{k^2}\left[\mathbf{E} + (1-k^2)\left(\frac{\operatorname{sn}^2(i\omega_2)}{\operatorname{cn}^2(i\omega_2)} - \frac{v}{u_2}\frac{i\operatorname{sn}(i\omega_2)\operatorname{dn}(i\omega_2)}{\operatorname{cn}^3(i\omega_2)}\right)\mathbf{K}\right] . \tag{6.3.17}$$

6.3.2 Type (ii) helical strings with two spins

Next let us turn to the type (ii) solutions. We can reach them by shifting the parameter ω_2 of a type (i) solution by $\mathbf{K}'$.[5] The resulting expressions are

$$\hat{t} = \hat{a}T + \hat{b}X \,, \tag{6.3.18}$$

$$\hat{\xi}_1 = \hat{C}\frac{\Theta_0(0)}{\sqrt{k}\,\Theta_0(i\omega_1)}\frac{\Theta_1(X - i\omega_1)}{\Theta_0(X)} \exp\Big(Z_0(i\omega_1)X + iu_1T\Big)\,, \tag{6.3.19}$$

$$\hat{\xi}_2 = \hat{C}\frac{\Theta_0(0)}{\sqrt{k}\,\Theta_3(i\omega_2)}\frac{\Theta_2(X - i\omega_2)}{\Theta_0(X)} \exp\Big(Z_3(i\omega_2)X + iu_2T\Big)\,, \tag{6.3.20}$$

where $\hat{C}$ is the normalization constant given by

$$\hat{C} = \left(\frac{\mathrm{cn}^2(i\omega_2)}{\mathrm{dn}^2(i\omega_2)} - \mathrm{sn}^2(i\omega_1)\right)^{-1/2} . \tag{6.3.21}$$

The Virasoro conditions constrain the coefficients $\hat{a}$, $\hat{b}$ as

$$\begin{aligned} \hat{a}^2 + \hat{b}^2 \;&=\; k^2 - 2k^2\,\mathrm{sn}^2(i\omega_1) - U + 2u_2^2\,, & (6.3.22)\\ \hat{a}\,\hat{b} \;&=-i\,\hat{C}^2\left(u_1\,\mathrm{sn}(i\omega_1)\,\mathrm{cn}(i\omega_1)\,\mathrm{dn}(i\omega_1) + u_2\left(1-k^2\right)\frac{\mathrm{sn}(i\omega_2)\,\mathrm{cn}(i\omega_2)}{\mathrm{dn}^3(i\omega_2)}\right) . & (6.3.23) \end{aligned}$$

The soliton velocity is given by $\hat{v} \equiv \hat{b}/\hat{a} \le 1$ so that we have $\hat{\eta}_0 = \sqrt{\hat{a}^2 - \hat{b}^2}\,\tilde{\tau}$. The angular velocities u_1 and u_2 satisfy

$$u_1^2 = U + \mathrm{dn}^2(i\omega_1)\,, \qquad u_2^2 = U + \frac{1-k^2}{\mathrm{dn}^2(i\omega_2)}\,, \tag{6.3.24}$$

and are constrained as

$$u_1^2 - u_2^2 = \mathrm{dn}^2(i\omega_1) - \frac{1-k^2}{\mathrm{dn}^2(i\omega_2)}\,. \tag{6.3.25}$$

When $\omega_2 = u_2 = 0$, it reduces to the type (ii) single spin solution.

The closedness conditions for a type (ii) solution are given by

$$\Delta\sigma\Big|_{\text{one-hop}} \equiv \frac{2\pi}{m} = \frac{2\mathbf{K}\sqrt{1-\hat{v}^2}}{\mu}\,, \tag{6.3.26}$$

$$\Delta\varphi_1\Big|_{\text{one-hop}} \equiv \frac{2\pi M_{\varphi_1}}{m} = 2\mathbf{K}\left(-iZ_0(i\omega_1) - \hat{v}\,u_1\right) + (2m_1'+1)\pi\,, \tag{6.3.27}$$

$$\Delta\varphi_2\Big|_{\text{one-hop}} \equiv \frac{2\pi M_{\varphi_2}}{m} = 2\mathbf{K}\left(-iZ_3(i\omega_2) - \hat{v}\,u_2\right) + (2m_2'+1)\pi\,, \tag{6.3.28}$$

where $m = 1, 2, \ldots$ is again the number of hops for $0 \le \sigma \le 2\pi$, and M_{φ_1} and M_{φ_2} are winding numbers for φ_1- and φ_2-direction, respectively.

[5] The type (ii) solution can be also obtained by applying a transformation $k \to 1/k$ to the type (i) solution, just as for the cases with the Frolov-Tseytlin solutions. See, for example, [11].

The conserved charges of m hops can be evaluated as

$$\hat{\mathcal{E}} = m\hat{a}\left(1-\hat{v}^2\right)\,\mathbf{K}\,, \tag{6.3.29}$$

$$\hat{\mathcal{J}}_1 = \frac{m\hat{C}^2\,u_1}{k^2}\left[-\mathbf{E} + \left(\mathrm{dn}^2(i\omega_1) + \frac{\hat{v}\,k^2}{u_1}\,i\,\mathrm{sn}(i\omega_1)\,\mathrm{cn}(i\omega_1)\,\mathrm{dn}(i\omega_1)\right)\mathbf{K}\right]\,, \tag{6.3.30}$$

$$\hat{\mathcal{J}}_2 = \frac{m\hat{C}^2\,u_2}{k^2}\left[\mathbf{E} - (1-k^2)\left(\frac{1}{\mathrm{dn}^2(i\omega_2)} - \frac{\hat{v}\,k^2}{u_2}\,\frac{i\,\mathrm{sn}(i\omega_2)\,\mathrm{cn}(i\omega_2)}{\mathrm{dn}^3(i\omega_2)}\right)\mathbf{K}\right]\,. \tag{6.3.31}$$

6.4 Taking various limits

Now that we have obtained generic helical solutions with two spins, for both type (i) and (ii) dyonic solutions, we can reproduce known string configurations as their special limiting cases. Interesting limits are the "stationary" limit $\omega_i \to 0$, the "infinite spin" limit $k \to 1$ and the "uniform charge-density" limit $k \to 0$. We will see them in turn.

6.4.1 Stationary limit: Frolov-Tseytlin strings

In the stationary limit where both ω_i vanish, the soliton velocity tends to zero, thus reducing the solutions to the spinning strings of Frolov and Tseytlin [38].

As usual, let us begin with the type (i) case. In this limit, the boosted coordinates (6.2.4) become $(T, X) \to (\tilde{\tau}, \tilde{\sigma})$, and (6.3.3)-(6.3.5) reduce to

$$t = \sqrt{k^2+u_2^2}\;\tilde{\tau}\,, \qquad \xi_1 = k\,\mathrm{sn}(\tilde{\sigma},k)\,e^{iu_1\tilde{\tau}}\,, \qquad \xi_2 = \,\mathrm{dn}(\tilde{\sigma},k)\,e^{iu_2\tilde{\tau}}\,, \tag{6.4.1}$$

with a constraint $u_1^2 - u_2^2 = 1$. This is the folded spinning/rotating string of [38], which stretches over a great circle in the θ-direction and spinning around its center of mass with angular momentum J_2. The center of mass itself moves along another orthogonal great circle of S^5 with spin J_1. To compare our results with the one presented in [38], one should relate the parametrization as

$$\tilde{\tau} = \mu\tau_{\mathrm{FT}}\,, \quad \tilde{\sigma} = \mu\sigma_{\mathrm{FT}}\,, \quad \kappa_{\mathrm{FT}} = \mu\sqrt{k^2+u_2^2}\,, \quad w_i = \mu u_i \qquad \text{with} \quad \mu \equiv \sqrt{w_1^2-w_2^2}\,. \tag{6.4.2}$$

In this stationary limit, the conserved charges take the following simple form,

$$\mathcal{E} = n\sqrt{k^2+u_2^2}\,\mathbf{K}\,, \quad \mathcal{J}_1 = nu_1\left(\mathbf{K}-\mathbf{E}\right)\,, \quad \mathcal{J}_2 = nu_2\,\mathbf{E}\,, \tag{6.4.3}$$

with the hopping number n now represents the folding number.

As discussed in Section 4.1, by expanding the moduli k and the charges E and J_i in powers of λ/J^2 with $J = J_1 + J_2$, we can compare them with global charges of double-contour distribution of Bethe roots on the gauge side.

Circular strings of Frolov-Tseytlin [38] are also reproduced in much the same way, by taking the stationary limit for the type (*ii*) solutions. In this case (6.3.18)-(6.3.20) reduce to

$$\hat{t} = \sqrt{1+u_2^2}\,\tilde{\tau}\,, \qquad \hat{\xi}_1 = \mathrm{sn}(\tilde{\sigma},k)\,e^{iu_1\tilde{\tau}}\,, \qquad \hat{\xi}_2 = \mathrm{cn}(\tilde{\sigma},k)\,e^{iu_2\tilde{\tau}}\,, \tag{6.4.4}$$

with a constraint $u_1^2 - u_2^2 = k^2$. This string wraps around a great circle of S^5 and rotates both in X^1-X^2 and X^3-X^4 planes. The conserved charges are given by

$$\hat{\mathcal{E}} = m\sqrt{1+u_2^2}\,\mathbf{K}\,, \quad \hat{\mathcal{J}}_1 = \frac{mu_1}{k^2}\left(\mathbf{K}-\mathbf{E}\right)\,, \quad \hat{\mathcal{J}}_2 = \frac{mu_2}{k^2}\left(\mathbf{E}-(1-k^2)\mathbf{K}\right)\,, \tag{6.4.5}$$

with m now represents the winding number for θ-angle.

Again, the moduli k and the charges can be expanded in powers of λ/J^2 to obtain c_k of (0.0.1). This time, they can be compared to the a_k for a imaginary root distribution of Bethe roots on the gauge side.

6.4.2 Infinite spin limit : dyonic giant magnons

When the moduli parameter k goes to unity, both type (*i*) and (*ii*) solutions become an array of dyonic giant magnons. The relation (6.3.10) (or (6.3.25)) implies that the ω_2-dependence of the solutions disappears in this limit. We will therefore write ω in place of ω_1. The relation $u_1^2 - u_2^2 = 1 + \tan^2\omega$ implies $a = u_1$ and $b = \tan\omega$ in view of (6.3.7) and (6.3.8) (or (6.3.22) and (6.3.23)), and the profiles of both types of strings become

$$t = \sqrt{1+u_2^2}\,\tilde{\tau}\,, \qquad \xi_1 = \frac{\sinh(X-i\omega)}{\cosh(X)}\,e^{i\tan(\omega)X+iu_1T}\,, \qquad \xi_2 = \frac{\cos(\omega)}{\cosh(X)}\,e^{iu_2T}\,. \tag{6.4.6}$$

Let us impose the same boundary conditions as in the single spin case (6.2.19), then it requires $\mu\to\infty$ as well as the relation $\Delta\varphi_1 = \pi - 2\omega$.

The conserved charges for one-hop (*i.e.*, single giant magnon) are given by

$$\mathcal{E} = u_1\left(1-\tfrac{\tan^2\omega}{u_1^2}\right)\mathbf{K}(1)\,, \quad \mathcal{J}_1 = u_1\left[\left(1-\tfrac{\tan^2\omega}{u_1^2}\right)\mathbf{K}(1)-\cos^2\omega\right], \quad \mathcal{J}_2 = u_2\cos^2\omega\,, \tag{6.4.7}$$

where $\mathbf{K}(1)$ is divergent, *i.e.*, $\mathcal{E}$, $\mathcal{J}_1\to\infty$. Energy-spin relation then becomes

$$\mathcal{E}-\mathcal{J}_1 = \sqrt{\mathcal{J}_2^2+\cos^2\omega}\,. \tag{6.4.8}$$

By comparing (6.4.8) with (0.0.2) with an identification $Q \equiv J_2 = (\sqrt{\lambda}/\pi)\mathcal{J}_2$, we find $p = \pi-2\omega$ as we mentioned earlier. It would be useful to note that, one can match the expressions above with the ones presented in [51], by redefining the parameters as

$$T = |\cos\alpha|\,\widetilde{T}\,, \quad X = |\cos\alpha|\,\widetilde{X} \qquad \text{and} \qquad u_2 \equiv \tan\alpha\,, \tag{6.4.9}$$

where $\widetilde{T}$ and $\widetilde{X}$ are the boosted worldsheet variables used in [51].

6.4.3 Uniform charge-density limit

Another interesting limit is $k \to 0$, where the densities of J_i tend to distribute uniformly along the worldsheet space variable σ in our gauge choice.

As for the type (i) case, the parameters a and b go to $a \to u_2 = \pm\sqrt{U + \tanh^2 \omega_2}$ and $b \to -\tanh \omega_2$, and the fields become

$$t = \tilde{\tau}\,, \qquad \xi_1 = 0\,, \qquad \xi_2 = e^{i\sqrt{U}\tilde{\tau}}\,, \tag{6.4.10}$$

and the conserved charges for one-hop are $E = \sqrt{\lambda}/2$, $J_1 = 0$ and $J_2 = \sqrt{\lambda}/2$. This is a point-like, BPS ($E - J_2 = 0$) string, rotating along the great circle in the X^3-X^4 plain.

For the type (ii) case, the profile becomes

$$\hat{t} = \sqrt{\hat{a}^2 - \hat{b}^2}\,\tilde{\tau}\,, \quad \hat{\xi}_1 = \hat{C}\,\sin(X - i\omega_1)\,e^{iu_1 T}\,, \quad \hat{\xi}_2 = \hat{C}\,\cos(X - i\omega_2)\,e^{iu_2 T}\,, \tag{6.4.11}$$

where $\hat{C} = \left(\cosh^2 \omega_2 + \sinh^2 \omega_1\right)^{-1/2}$. The angular velocities satisfy $u_1^2 = u_2^2 = U + 1$. The parameters $\hat{a}$ and $\hat{b}$ (with $\hat{a} \ge \hat{b}$) are determined by

$$\hat{a}^2 + \hat{b}^2 = -U + 2u_2^2\,, \tag{6.4.12}$$

$$\hat{a}\,\hat{b} = \hat{C}^2\sqrt{U+1}\,(\sinh\omega_1 \cosh\omega_1 \mp \sinh\omega_2 \cosh\omega_2)\,, \tag{6.4.13}$$

where $\mp$ reflects the sign ambiguity of angular momenta. The conserved charges for one-hop are evaluated as

$$\hat{\mathcal{E}} = \frac{\pi\hat{a}\,(1 - \hat{v}^2)}{2}\,, \tag{6.4.14}$$

$$\hat{\mathcal{J}}_1 = -\frac{\pi\hat{C}^2\hat{v}}{2}\,\sinh\omega_1 \cosh\omega_1\,, \tag{6.4.15}$$

$$\hat{\mathcal{J}}_2 = \frac{\pi\hat{C}^2\hat{v}}{2}\,\sinh\omega_2 \cosh\omega_2\,. \tag{6.4.16}$$

As we are assuming $\hat{a} \ge \hat{b} \ge 0$, the situation $\hat{b} = 0$ can be realized when $\omega_1 = \omega_2$ with "$-$" sign of (6.4.13), or when $\omega_1 = -\omega_2$ with "+" sign. In both cases, the soliton velocity $\hat{v} \equiv \hat{b}/\hat{a}$ vanishes, which then implies the equal spin relation $J_1 = J_2$ in view of (6.4.15) and (6.4.16). This equal two-spin (or "rational") solution can also be realized as $J_1 = J_2$ case of a so-called constant-radii string solution, which follows from an Ansatz $\xi_j = a_j\,e^{i(w_j\tau + n_j\sigma)}$ $(j = 1, 2)$ with a_j constants [38]. From the viewpoint of a finite-gap problem, an equal two-spin case mentioned above corresponds to a single-cut limit of the symmetric two-cut imaginary root solution, that is, the limit when the outer two branch points of the cuts go to $\pm i\infty$, thus making it a single-cut. This situation can also be realized as a certain limiting configuration of a single cut distribution of Bethe roots, that is, when the filling fraction of the spin-chain (the ratio of the number of impurities to the number of sites) goes to 1/2.

6.5 On the moduli space of helical solutions

The profile of helical string solutions contains many parameters under several constraints. For the sake of completeness, we count the number of independent parameters and show several numerical examples which solve all constraints explicitly.

6.5.1 Number of independent parameters

Computing the moduli space of solutions, namely the number of independent parameters, is easy. There are four parameters corresponding to four conserved charges, say

$$(k\,, U\,, \omega_1\,, \omega_2) \quad \leftrightarrow \quad (N_{\varphi_1}\,, N_{\varphi_2}\,, J_1\,, J_2)\,. \tag{6.5.1}$$

In addition, the parameter μ, that is spatial scale of worldsheet, controls the number of hops n.

Any classical solutions are characterized by two real parameters and three integers $(n, N_1\,, N_2)$, so the moduli space of solutions is real two-dimensional. If one imposes the semiclassical quantization conditions on J_1 and J_2 . then the moduli space becomes zero-dimensional, specified by five integers $(n, N_1\,, N_2\,, J_1\,, J_2)$.

From Complex sine-Gordon point of view, there are three parameters $(k\,, U\,, v)$ which characterize helical-wave solutions (6.3.1). Helical-wave solutions of the real sine-Gordon model do not depend on U.

There are other constraints which should be kept in mind when we look for consistent solutions.

- The reality of ω_j is required for the normalizability of ξ_j as well as the equation of motion.
- The condition $v = b/a$ is required for the timelike winding number to vanish. The parameters a and b, which are chosen as a solution of Virasoro constraints, must of course be real.
- The parameters u_1 and u_2 must be real, which imposes the lower limit on U.

6.5.2 Numerical results

We tried to find a pair of real parameters $(\omega_1\,, \omega_2)$ which can solve the closedness conditions for given (k, U) and $(n, N_{\varphi_1}\,, N_{\varphi_2})$, by computer-aided search.

Because periodic boundary conditions like (6.3.13), (6.3.14) contain the ambiguity of $n'_{1,2}$, we had to look for solutions up to

$$N_{\varphi_1} \equiv N_{\varphi_1} + n\,, \qquad \text{and} \qquad N_{\varphi_2} \equiv N_{\varphi_2} + n\,. \tag{6.5.2}$$

k	U	n	$N^{\circ}_{\varphi_1}$	$N^{\circ}_{\varphi_2}$	ω_1	ω_2	v	μ	$\mathcal{E}$	$\mathcal{J}_1$	$\mathcal{J}_2$
0.7	3	6	1	-1	0.8953	2.914	0.4460	3.155	18.52	11.56	6.284
		6	1	-2	1.118	3.195	0.5317	2.986	17.76	12.17	4.852
		6	1	-3	1.267	3.436	0.6073	2.801	17.16	12.43	3.784
		6	1	-4	1.376	3.676	0.6767	2.595	16.47	12.35	2.930
		6	1	-5	1.443	0.2194	0.7241	2.431	15.92	12.14	2.357
		6	1	-6	1.465	0.5698	0.7404	2.369	15.80	12.20	1.998
		6	-1	5	2.282	3.506	-0.7245	2.431	15.91	12.13	2.355
		6	2	-1	0.6285	3.540	0.2411	3.421	20.28	8.394	11.10
		6	3	-1	0.3723	0.2639	0.05333	3.520	21.22	7.162	13.08
		6	-3	1	3.353	3.459	-0.05283	3.520	21.23	7.167	13.07
0.7	50	6	1	-5	1.344	3.338	0.2392	3.423	76.73	64.01	12.38
0.7	0.1386	6	1	-1	1.057	0	0.7626	2.280	6.146	3.042	1.458
0.7	1.127	6	1	-6	1.143	1.846	0.6686	2.621	12.00	8.511	1.648

Table 6.1: List of numerical values of parameters $(k,\, U, \omega_1\,, \omega_2)$ that make strings to be closed.

Once we find a consistent pair of parameters $(\omega_1\,, \omega_2)$, we can compute the physical winding number $(N^{\circ}_{\varphi_1}\,, N^{\circ}_{\varphi_2})$.

The results are listed as follows:

There are several interesting features of this result:

- As far as we studied, there is only one solution $(N^{\circ}_{\varphi_1}\,, N^{\circ}_{\varphi_2})$ for any winding number $(N_{\varphi_1}\,, N_{\varphi_2})$ defined modulo n. In other words, the winding numbers $|N_{\varphi_1}|$ and $|N_{\varphi_2}|$ are bounded from above.
- It seems that the bound on $|N_{\varphi_1}|$ is stronger. In fact, the above table suggests $|N_1| \leq n/2$. Existence of such bound was also consistent with the argument in the Nambu-Goto approach of [158].
- The inequality $\mathcal{E} > \mathcal{J}_1 + \mathcal{J}_2$ is satisfied for all solutions. This is interpreted as the counterpart of unitarity bound imposed on gauge theory side.

Obviously, the numerical results listed above are neither comprehensive nor satisfactory. One should not draw any conclusion from it, except that there indeed exist lots of solutions to both Virasoro constraints and periodicity conditions.

6.6 General 2-cut finite-gap solutions

Helical (spinning) string solutions are identified as general 2-cut finite-gap solutions in [62], which we will summarize below.

In Section 3.3.3, we see that general finite-gap solutions on $\mathbb{R}_t \times S^3$ constructed in [31] are expressed in terms of the Riemann θ functions. General 2-cut solutions can be obtained when the Riemann θ functions reduce to the Jacobi θ functions, that is, when the genus is one.

A genus-one algebraic curve is called elliptic. Let us define an elliptic curve by

$$y^2 := (x - x_1)\,(x - \bar{x}_1)\,(x - x_2)\,(x - \bar{x}_2)\,. \tag{6.6.1}$$

The hermiticity of flat currents requires that the branch points should be located symmetrically with respect to the real axis. We introduce the normalized holomorphic differential on this elliptic curve by

$$\omega := \nu \Big/ \int_a \nu\,, \qquad \nu := \frac{dx}{y}\,, \tag{6.6.2}$$

where the integral over a stands for the a-period chosen as in Figure 6.7. Then, the parameters $\tilde{\rho}_\pm$ are given by

$$i\omega_1 = i\tilde{\rho}_- \equiv 2\mathbf{K}(k) \left(\int_{\infty^-}^{0^+} \omega - \frac{i\mathbf{K}'(k)}{2\mathbf{K}(k)} \right), \qquad i\omega_2 = i\tilde{\rho}_+ \equiv 2\mathbf{K}(k) \left(\int_{\infty^+}^{0^+} \omega - \frac{1}{2} \right). \tag{6.6.3}$$

By using Riemann's bilinear identity, one can express the integral $\int_{\infty^\mp}^{0^+} \omega$ in terms of the location of the branch points. The results are

$$\int_{\infty^\mp}^{0^+} \omega = \frac{iF\,(\varphi_\pm, k')}{2\mathbf{K}(k)}\,, \qquad \text{with} \quad \tan\left(\frac{\varphi_\pm}{2}\right) = \frac{\left(\sqrt{\bar{x}_2} \pm \sqrt{x_1}\right)\left(\sqrt{\bar{x}_1} + \sqrt{x_2}\right)}{|x_1 - \bar{x}_2|}\,, \tag{6.6.4}$$

where $F\,(\varphi, k)$ is the normal (or incomplete) elliptic integral of the first kind given in Appendix A.1. From (6.6.3) and (6.6.4), we obtain the relation between the parameters $\omega_{1,2}$ of helical strings and the location of the branch points:

$$\omega_1 = F\,(\varphi_+, k') - \mathbf{K}'(k)\,, \qquad \omega_2 = F\,(\varphi_-, k') + i\mathbf{K}(k). \tag{6.6.5}$$

It can be shown that the right hand side of the second equation is always real. So we may redefine ω_2 as

$$\omega_2 = \begin{cases} \mathrm{Re}\,[F\,(\varphi_-, k')] & (\text{for} \;\; k < 1,\;\; k \to 1)\,, \\ \mathrm{Re}\,[F\,(\varphi_-, k')] - \dfrac{\pi}{2} & (\text{for} \;\; k > 1,\;\; k \to 1)\,. \end{cases} \tag{6.6.6}$$

This expression is more useful than (6.6.5) for studying the behavior of ω_2 near $k = 1$.

The profile of 2-cut finite-gap solutions obtained in [62] reads

$$\begin{aligned} Z_1 &= C\,\frac{\Theta_3\big(\tilde{X} - i\tilde{\rho}_+\big)}{\Theta_2\big(i\tilde{\rho}_+\big)\Theta_0\big(\tilde{X}\big)}\, \exp\Big(Z_2(i\tilde{\rho}_+, k)\tilde{X} + iv_+\tilde{T} + i\varphi_1^0 \Big)\,, \\ Z_2 &= C\,\frac{\Theta_1\big(\tilde{X} - i\tilde{\rho}_-\big)}{\Theta_0\big(i\tilde{\rho}_-\big)\Theta_0\big(\tilde{X}\big)}\, \exp\Big(Z_0(i\tilde{\rho}_-, k)\tilde{X} + iv_-\tilde{T} + i\varphi_2^0 \Big)\,, \end{aligned} \tag{6.6.7}$$

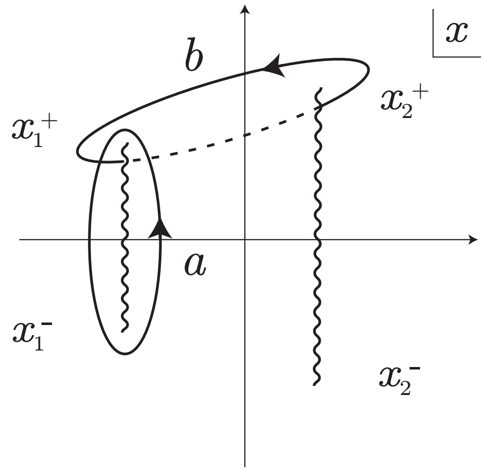


Figure 6.7: Choice of a- and b- cycles for an elliptic curve.

where C and $\varphi^0_{1,2}$ are real constants. It has exactly the same form as (6.3.4) and (6.3.5) after trivial interchange of $Z_1 \leftrightarrow Z_2$. Let us explain a little more about parameters used here in terms of algebro-geometric data.

The elliptic modulus k is determined from the location of branch points, as

$$k' = \sqrt{1-k^2} = \left| \frac{x_1 - x_2}{x_1 - \bar{x}_2} \right| , \tag{6.6.8}$$

The boosted space coordinate X is defined as

$$X \equiv X_0 + \frac{1}{2\pi} \int_b d\mathcal{Q} = X_0 + \frac{1}{2\pi} \int_b (\sigma dp + \tau dq) \, . \tag{6.6.9}$$

The integral over b-cycle gives

$$\tilde{X} \equiv 2\mathbf{K}(k) X = \tilde{X}_0 + \frac{x - vt}{\sqrt{1-v^2}} \, , \qquad v \equiv \frac{y_+ - y_-}{y_+ + y_-} \, , \tag{6.6.10}$$

where (x, t) is given by

$$(x, t) \equiv (\mu\sigma \, , \mu\tau) \, , \qquad \mu \equiv \kappa \frac{|x_1 - \bar{x}_2|}{\sqrt{y_+ y_-}} , \tag{6.6.11}$$

and $y_\pm = y(x)|_{x=\pm 1}$. The tilde stands for rescaling by $2\mathbf{K}(k)$, like $\tilde{\rho}_\pm = 2\mathbf{K}(k)\rho_\pm$, and so on.

The differential $d\mathcal{Q}$ appeared also in the exponential term of the general formula (3.3.55) and (3.3.56). A part of exponential term gives the boosted time coordinate T multiplied by angular velocities $v_\pm$, which are given by

$$\tilde{T} = \frac{t - vx}{\sqrt{1-v^2}} \qquad \text{and} \qquad v_\pm = \frac{y(0) \pm 1}{|x_1 - \bar{x}_2|} \, . \tag{6.6.12}$$

In finite-gap solutions, the b-period of quasi-momentum is quantized. In the present situation, the mode number

$$n \equiv \frac{1}{2\pi} \int_b dp \in \mathbb{Z} \, , \tag{6.6.13}$$

is identified as the number of hops. The periodic boundary conditions for a closed string are expressed as

$$\frac{1}{2\pi}\int_{\infty^{\pm}}^{0^{+}} dp \equiv -N_{\pm} \in \mathbb{Z}\,. \tag{6.6.14}$$

Global conserved charges are computed as

$$E - J_1 = 2\,\mathrm{Re}\left[\frac{1}{2\pi i}\oint_b \tilde{\alpha}\right], \qquad J_2 = 2\,\mathrm{Re}\left[\frac{1}{2\pi i}\oint_b \alpha\right], \tag{6.6.15}$$

where the differentials α and $\tilde{\alpha}$ are defined as

$$\alpha \equiv \frac{\sqrt{\lambda}}{4\pi}\left(x + \frac{1}{x}\right) dp\,, \qquad \tilde{\alpha} \equiv \frac{\sqrt{\lambda}}{4\pi}\left(x - \frac{1}{x}\right) dp\,. \tag{6.6.16}$$

Recall that an array of dyonic giant magnon solution is obtained by taking $k \to 1$ limit of helical spinning string. In the finite-gap formulation, from the relation (6.6.8) one finds the limit $k \to 1$ is equivalent to $x_1 \to x_2$.

The quasi-momentum $dp(x)$ on the upper sheet $\mathbb{CP}^1_+$ in this singular curve limit is given by

$$dp(x) = \frac{\pi\kappa\, dx}{(x - x_1)(x - \bar{x}_1)}\left(\frac{|1 - x_1|^2}{(x-1)^2} + \frac{|1 + x_1|^2}{(x+1)^2}\right). \tag{6.6.17}$$

We substitute this expression into (6.6.15). Because the integral over b-cycle picks up a pole at $x = x_1$, global conserved charges are expressed as functions of x_1, as

$$E - J_1 = \frac{n\sqrt{\lambda}}{4\pi}\left|\left(x_1 - \frac{1}{x_1}\right) - \left(\bar{x}_1 - \frac{1}{\bar{x}_1}\right)\right|, \tag{6.6.18}$$

$$J_2 = \frac{n\sqrt{\lambda}}{4\pi}\left|\left(x_1 + \frac{1}{x_1}\right) - \left(\bar{x}_1 + \frac{1}{\bar{x}_1}\right)\right|. \tag{6.6.19}$$

For the case of a single dyonic giant magnon, *i.e.* $n = 1$, we obtain the famous square-root formula

$$E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)}\,. \tag{6.6.20}$$

The results (6.6.18) and (6.6.19) exactly match with the conserved charges of magnon bound-states (5.1.57) and (5.1.58) upon identification of the parameters $x_1^{\pm} = X^{\pm}$.

Chapter 7

Large winding strings

We study a family of classical strings on $\mathbb{R}_{\rm t}\times {\rm S}^3$ background which has large winding numbers and oscillates in spacetime. They are obtained from helical spinning strings discussed in the previous chapter by interchanging worldsheet time and space coordinates. They interpolate various classical strings whose energy is roughly given by the length times the tension of a string, like pulsating strings and single-spike strings. From a finite-gap perspective, this transformation is realized as an interchange of quasi-momentum and quasi-energy defined for the algebraic curve.

This chapter is mainly based on the author's paper with H. Hayashi, K. Okamura, and B. Vicedo [97].

7.1 2D-transforming classical strings on $\mathbb{R}_{\rm t}\times {\rm S}^3$

We consider classical string theory on $\mathbb{R}_{\rm t}\times {\rm S}^3$ and relate it to CsG system via the Pohlmeyer-Lund-Regge reduction procedure, just as in Section 6.1.

We are interested in how the 2D transformation acts on classical strings and solutions of Complex sine-Gordon equations, respectively. Let us first look at the string equations of motion (6.1.3) and the Virasoro constraints (6.1.2). In view that they are invariant under the $\tau \leftrightarrow \sigma$ flip, any string solution is mapped to another solution under this map. On closer inspection of the Virasoro constraints (6.1.2), one actually finds that the $\tau \leftrightarrow \sigma$ operation can be applied independently to the $\mathbb{R} \subset AdS_5$ and $S^3 \subset S^5$ parts. We will use this observation to generate new string solutions from known solutions on $\mathbb{R} \times S^3$, by transforming only the S^3 part while retaining the gauge $t \propto \tau$. In order to satisfy other consistency conditions such as closedness of the string, one needs to care about the periodicity in the new σ direction (that used to be the τ direction before the flip).

Before discussing the CsG counterparts of such $\tau \leftrightarrow \sigma$ transformed string solutions, it would be useful to review some relevant aspects of the (C)sG $\leftrightarrow$ string correspondence before the transformation. A good starting point is a single-spin helical spinning string. From the standpoint of sG theory, the helical string corresponds to the following helical wave ("kink-train") solution of sG equation,

$$\phi(t,x) = 2\arcsin\left[\mathrm{cn}\left(\frac{(x-x_0)-v(t-t_0)}{k\sqrt{1-v^2}}, k\right)\right]. \tag{7.1.1}$$

via the PLR procedure. The single-spin helical string thus has two controllable parameters derived from the sG soliton (7.1.1); one is the soliton velocity v and the other is the elliptic moduli parameter k that controls the period of the kink-array. In the $k \to 1$ limit, it reduces to an array of giant magnons, while as $v \to 0$, it reduces to a folded/circular string of [35].

Actually there is another periodic solution of sG equation, namely a periodic instanton. Generally, one can interpret a static, finite energy classical solution of sG theory in $(1+1)$-dimensions as a finite action Euclidean solution in $(1+0)$-dimension that interpolates between different vacua of the theory. Such a sG instanton solution is known in the literature (see, *e.g.*, [162]) and is given by

$$\phi(t') = 2\arcsin\left[\mathrm{cn}\left(\frac{t'-t'_0}{k}, k\right)\right]. \tag{7.1.2}$$

Here $t' = it$ is the Euclidean time. One can see that a static kink soliton of sG equation $-\partial_x^2\phi = \sin\phi$ (set $v = 0$ in (7.1.1)) is related to the instanton (7.1.2) of the Euclidean sG equation $\partial_{it}^2\phi = -\partial_{t'}^2\phi = \sin\phi$ by a formal translation $x \leftrightarrow t'$ (*i.e.*, space-like motion turns into "time-like" motion), which amounts to swapping worldsheet variables $\tau \leftrightarrow \sigma$. Starting from the instanton solution (7.1.2), and boosting it by a parameter v, we obtain a one parameter family of sG solutions of the form

$$\phi(t',x') = 2\arcsin\left[\mathrm{cn}\left(\frac{(t'-t'_0)-v(x'-x'_0)}{k\sqrt{1-v^2}}, k\right)\right] \tag{7.1.3}$$

with $(t',x') = (it, ix)$, which is related to the sG helical wave (7.1.1) by $\tau \leftrightarrow \sigma$.

Via the PLR map, each periodic instanton corresponds to a point-like segment, or "string-bit", and an infinite series of such periodic sG instantons (7.1.2) arrayed in the σ-direction make up the corresponding classical string. Note that for the boosted instanton (7.1.3), v no longer represents a velocity, rather it should be viewed as a parameter that controls the difference between time-origins t'_0 for each bits. A pulsating string corresponds to the $v = 0$ case, when the timing of the pulsation of each string-bits is perfectly right. When the pulsation timing of the bits is off in a coherent manner, a symmetric "spike" comes into being, reflecting the staggered motions of bits.[1] In the limit $k \to 1$, the oscillation period of each bit becomes infinite, and the bits stay in the vicinity of the equator for an infinite amount of time, except

[1] The situation is much the same as the case of familiar transverse waves, where oscillation in the medium takes place in a perpendicular direction to its own motion. This direction of motion corresponds to, in our case, the circumferential direction along the equator of the sphere.

during a short sudden jump away from the equator — this is one way to interpret the single-spin single-spike string of [63] from the sG point of view.[2]

We have just discussed the way to realise the oscillating solutions resulting from a $\tau \leftrightarrow \sigma$ transformation in terms of a collection of sG instantons. We gave this interpretation because it is very intuitive. Actually one cannot generalize this argument to the CsG case directly, since in this case the argument requires χ to be imaginary. So for the CsG case, it would be convenient instead to interpret the effect of the $\tau \leftrightarrow \sigma$ operation as flipping the sign of the "mass" term in the Lagrangian as

$$\mathcal{L}_{\rm CsG} = \frac{\partial_a \psi^* \, \partial^a \psi}{1 - \psi^* \psi} + \psi^* \psi \quad \mapsto \quad \frac{\partial_a \psi^* \, \partial^a \psi}{1 - \psi^* \psi} - \psi^* \psi \, .$$

In this way one can easily understand how one solution of CsG is related to another via the $\tau \leftrightarrow \sigma$ transformation, keeping ϕ and χ real.

Notice also, as in the soliton cases, that there are two classes of "boosted" instantons possible; the first is an instanton that oscillates about one of the barriers of the periodic potential with fixed finite oscillation range, while the other no longer oscillates back and forth but goes on from one barrier to the neighboring one. A similar kind of distinction exists for what we call type $(i)'$ and type $(ii)'$ strings.

7.2 Helical oscillating strings

We are now in a position to discuss the 2D transformed helical strings. We first study the type $(i)'$ case in the following section 7.2.1. The results on the type $(ii)'$ solutions will be collected in section 7.2.2.

7.2.1 Type $(i)'$ helical strings

We start from the profile of helical spinning strings (6.3.3)-(6.3.5), and swap τ and σ of $\xi_i(\tau, \sigma)$ $(i = 1, 2)$ while keeping the relation $t\,(\tau, \sigma) = aT + bX$ as it is. One then obtains the 2D-transformed version of the type (i) two-spin helical (spinning) strings, which we call type $(i)'$

[2] As is noticed in [63], for sG case, it is also possible to argue that the $\tau \leftrightarrow \sigma$ transformation results in the change of sG kink soliton from $\phi = 2 \arcsin(1/\cosh x_v)$ to $\phi = 2 \arcsin(\tanh x_v)$. However, it seems this interpretation cannot be directly applied to CsG case.

helical (oscillating) strings,

$$t = aT + bX\,, \tag{7.2.1}$$

$$\xi_1 = C\frac{\Theta_0(0)}{\sqrt{k}\,\Theta_0(i\omega_1)}\frac{\Theta_1(T-i\omega_1)}{\Theta_0(T)}\,\exp\Big(Z_0(i\omega_1)T + iu_1X\Big)\,, \tag{7.2.2}$$

$$\xi_2 = C\frac{\Theta_0(0)}{\sqrt{k}\,\Theta_2(i\omega_2)}\frac{\Theta_3(T-i\omega_2)}{\Theta_0(T)}\,\exp\Big(Z_2(i\omega_2)T + iu_2X\Big)\,. \tag{7.2.3}$$

The coordinates (T, X) and the normalization constant C are same as before. Virasoro constraints fix the parameters a and b as in (6.3.7) and (6.3.8), and the PLR reduction relation (6.1.7) relates u_1 and u_2 as (6.3.9). We adjust the parameter v such that the AdS time is proportional to the worldsheet time variable, namely $\eta_0 = \sqrt{a^2-b^2}\,\tilde{\tau}$ with $v \equiv b/a \le 1$.

We are interested in closed string solutions, which means we need to consider the periodicity conditions. The period in σ-direction is defined such that it leaves the theta functions in (6.3.4) and (6.3.5) invariant, namely it is given by

$$-\ell \le \sigma \le \ell, \qquad \ell = \frac{\mathbf{K}\sqrt{1-v^2}}{v\mu}\,, \qquad (v>0)\;. \tag{7.2.4}$$

Then, closedness of the string requires

$$\Delta\sigma \equiv \frac{2\pi}{n} \qquad = \frac{2\mathbf{K}\sqrt{1-v^2}}{v\mu}\,, \tag{7.2.5}$$

$$\Delta\varphi_1 \equiv \frac{2\pi N_{\varphi_1}}{n} = 2\mathbf{K}\left(\frac{u_1}{v} + iZ_0(i\omega_1)\right) + (2n_1'+1)\pi\,, \tag{7.2.6}$$

$$\Delta\varphi_2 \equiv \frac{2\pi N_{\varphi_2}}{n} = 2\mathbf{K}\left(\frac{u_2}{v} + iZ_2(i\omega_2)\right) + 2n_2'\pi\,, \tag{7.2.7}$$

where $n = 1, 2, \ldots$ counts the number of periods in $0 \le \sigma \le 2\pi$, and N_{φ_1,φ_2} are the winding numbers in $\varphi_{1,2}$-directions respectively. The integers $n_{1,2}'$ specify the ranges of $\omega_{1,2}$ respectively.

The energy and angular momenta of a string, defined in (6.2.15), yields

$$\mathcal{E} = \frac{na(1-v^2)}{v}\,\mathbf{K} = \frac{n(a^2-b^2)}{b}\,\mathbf{K}\,, \tag{7.2.8}$$

$$\mathcal{J}_1 = \frac{nC^2\,u_1}{k^2}\left[\mathbf{E} - \left(\mathrm{dn}^2(i\omega_1) + \frac{ik^2}{vu_1}\,\mathrm{sn}(i\omega_1)\,\mathrm{cn}(i\omega_1)\,\mathrm{dn}(i\omega_1)\right)\mathbf{K}\right]\,, \tag{7.2.9}$$

$$\mathcal{J}_2 = \frac{nC^2\,u_2}{k^2}\left[-\mathbf{E} - (1-k^2)\left(\frac{\mathrm{sn}^2(i\omega_2)}{\mathrm{cn}^2(i\omega_2)} - \frac{i}{vu_2}\frac{\mathrm{sn}(i\omega_2)\,\mathrm{dn}(i\omega_2)}{\mathrm{cn}^3(i\omega_2)}\right)\mathbf{K}\right]\,. \tag{7.2.10}$$

It is meaningful to compare the above expressions with those of type (i) helical spinning strings, (6.3.15)-(6.3.17) :

$$\mathcal{E}^{(i)} = na\left(1-v^2\right)\mathbf{K} = \frac{n(a^2-b^2)}{a}\,\mathbf{K}\,, \tag{7.2.11}$$

$$\mathcal{J}_1^{(i)} = \frac{nC^2\,u_1}{k^2}\left[-\mathbf{E} + \left(\mathrm{dn}^2(i\omega_1) + \frac{ivk^2}{u_1}\,\mathrm{sn}(i\omega_1)\,\mathrm{cn}(i\omega_1)\,\mathrm{dn}(i\omega_1)\right)\mathbf{K}\right]\,, \tag{7.2.12}$$

$$\mathcal{J}_2^{(i)} = \frac{nC^2\,u_2}{k^2}\left[\mathbf{E} + (1-k^2)\left(\frac{\mathrm{sn}^2(i\omega_2)}{\mathrm{cn}^2(i\omega_2)} - \frac{iv}{u_2}\frac{\mathrm{sn}(i\omega_2)\,\mathrm{dn}(i\omega_2)}{\mathrm{cn}^3(i\omega_2)}\right)\mathbf{K}\right]\,. \tag{7.2.13}$$

If we regard $\mathcal{E}$ and $\mathcal{J}_i$ as functions of $v = b/a$, the global charges of the transformed solutions are related to the original ones by $\mathcal{E}(a,b) = -\mathcal{E}^{(i)}(b,a)$ and $\mathcal{J}_i(v) = -\mathcal{J}_i^{(i)}(-1/v)$. Similar relations are also true for the winding numbers given in (7.2.6) and (7.2.7), $N_{\varphi_i}(v) = -N_{\varphi_i}^{(i)}(-1/v)$ $(i=1,2)$. They are just a consequence of the symmetry $a \leftrightarrow b$ the Virasoro constraints possess. For example, if $(a,b) = (a_0,b_0)$ solves (6.3.7) and (6.3.8), then $(a,b) = (b_0,a_0)$ gives another solution.

Notice that in the limit $v \to 0$ $(\omega_{1,2} \to 0)$, all the winding numbers in (7.2.5)-(7.2.7) become divergent (and so ill-defined), due to the fact that the θ defined in (3.2.2) becomes independent of σ . Therefore, in this limiting case, we may choose μ arbitrarily without the need of solving (7.2.5), provided that N_{φ_1} and N_{φ_2} are both integers.

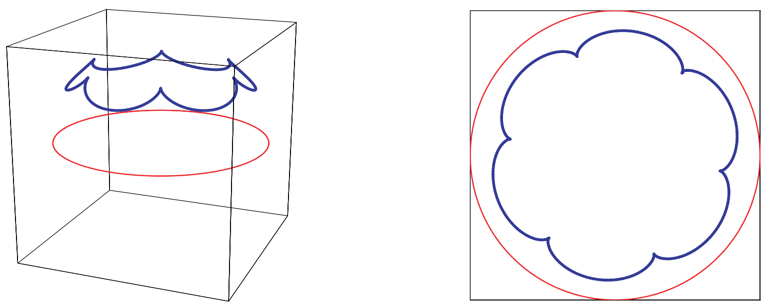


Figure 7.1: Type $(i)'$ helical string $(k=0.68\,, n=6)$, projected onto S^2 . The figure shows a single-spin case $(u_2=\omega_2=0)$. The (red) circle indicates the $\theta=0$ line (referred to as the "equator" in the main text).

The type $(i)'$ helical strings contains both pulsating strings and single-spike strings in particular limits. Below we will consider various limits including them.

- $\omega_{1,2} \to 0$ **limit : Pulsating strings**

Let us first consider the $\omega_{1,2} \to 0$ limit. In this limit, the boosted coordinates (6.2.4) reduce to $(T,X) \to (\tilde\tau,\tilde\sigma)$, and (7.2.1)-(7.2.3) become

$$t = \sqrt{k^2+u_2^2}\;\tilde\tau\,, \qquad \xi_1 = k\,\mathrm{sn}(\tilde\tau,k)\,e^{iu_1\tilde\sigma}\,, \qquad \xi_2 = \mathrm{dn}(\tilde\tau,k)\,e^{iu_2\tilde\sigma}\,, \tag{7.2.14}$$

with the constraint $u_1^2 - u_2^2 = 1$. Since the radial direction is independent of σ , we may treat μ as a free parameter satisfying $N_{\varphi_1} = \mu u_1$ and $N_{\varphi_2} = \mu u_2$. Then the conserved charges for a period become

$$\mathcal{E} = \pi k \sqrt{N_{\varphi_1}^2 + \left(\frac{1}{k^2} - 1\right) N_{\varphi_2}^2} \,, \qquad \mathcal{J}_1 = \mathcal{J}_2 = 0 \,. \tag{7.2.15}$$

Left of Figure 7.2 shows the time evolution of the type $(i)'$ pulsating string. It stays above the equator, and sweeps back and forth between the pole ($\theta = \frac{\pi}{2}$) and the turning latitude determined by k .

When we set $u_2 = 0$, this string becomes identical to the simplest pulsating solution studied in [41] (the zero-rotation limit of rotating and pulsating strings studied in [42, 43]).[3]

• $k \to 1$ limit : Single-spike strings

When the moduli parameter k goes to unity, type $(i)'$ helical string becomes an array of single-spike strings studied in [63, 64]. Dependence on ω_2 drops out in this limit, so we write ω instead ω_1 . The Virasoro constraints can be explicitly solved by setting $a = u_1$ and $b = \tan\omega$. The profile of the string then becomes

$$t = \sqrt{1 + u_2^2}\, \tilde{\tau} \,, \qquad \xi_1 = \frac{\sinh(T - i\omega)}{\cosh(T)}\; e^{i\tan(\omega)T + iu_1 X} \,, \qquad \xi_2 = \frac{\cos(\omega)}{\cosh(T)}\; e^{iu_2 X} \,. \tag{7.2.16}$$

with the constraint $u_1^2 - u_2^2 = 1 + \tan^2\omega$.[4] The conserved charges are computed as

$$\mathcal{E} = \left(\frac{u_1^2 - \tan^2\omega}{\tan\omega}\right) \mathbf{K}(1) \,, \qquad \mathcal{J}_1 = u_1 \cos^2\omega \,, \qquad \mathcal{J}_2 = u_2 \cos^2\omega \,, \tag{7.2.17}$$

where $\mathbf{K}(1)$ is a divergent constant. For $n = 1$ case (single spike), the expressions (7.2.17) result in

$$\mathcal{J}_1 = \sqrt{\mathcal{J}_2^2 + \cos^2\omega} \,, \qquad i.e., \quad J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\cos^2\omega} \,. \tag{7.2.18}$$

Since the winding number $\Delta\varphi_1$ also diverges as $k \to 1$, this limit can be referred to as the "infinite winding" limit,[5] which can be viewed as the 2D-transformed version of the infinite spin limit of [49]. By examining the periodicity condition carefully, one finds that both of the divergences come from the same factor $\mathbf{K}(k)|_{k\to 1}$. Using the formula (A.4.24), one can deduce that

$$\mathcal{E} - \frac{\Delta\varphi_1}{2}\bigg|_{k\to 1} = -\left(\omega - \frac{(2n_1' + 1)\,\pi}{2}\right) \equiv \bar{\theta} \,. \tag{7.2.19}$$

Using the $\bar{\theta}$ variable introduced above, which is the same definition as used in [63], one can see (7.2.18) precisely reproduces the relation between spins obtained in [63].

[3] The type $(i)'$ pulsating solution studied here and also the type $(ii)'$ pulsating string discussed later are qualitatively different solutions from the so called "rotating pulsating string" [42], so that the finite-gap interpretation and the gauge theory interpretation of type $(i)'$ and $(ii)'$ are also different from those of [42].

[4] Here $u_{1,2}$ and ω are related to γ used in [63] (see their Eq. (6.23)) by $u_1 = \frac{1}{\cos\gamma\cos\omega}$ and $u_2 = \frac{\tan\gamma}{\cos\omega}$.

[5] Notice, however, that the string wraps very close to the equator but touches it only once every period (every "cusp").

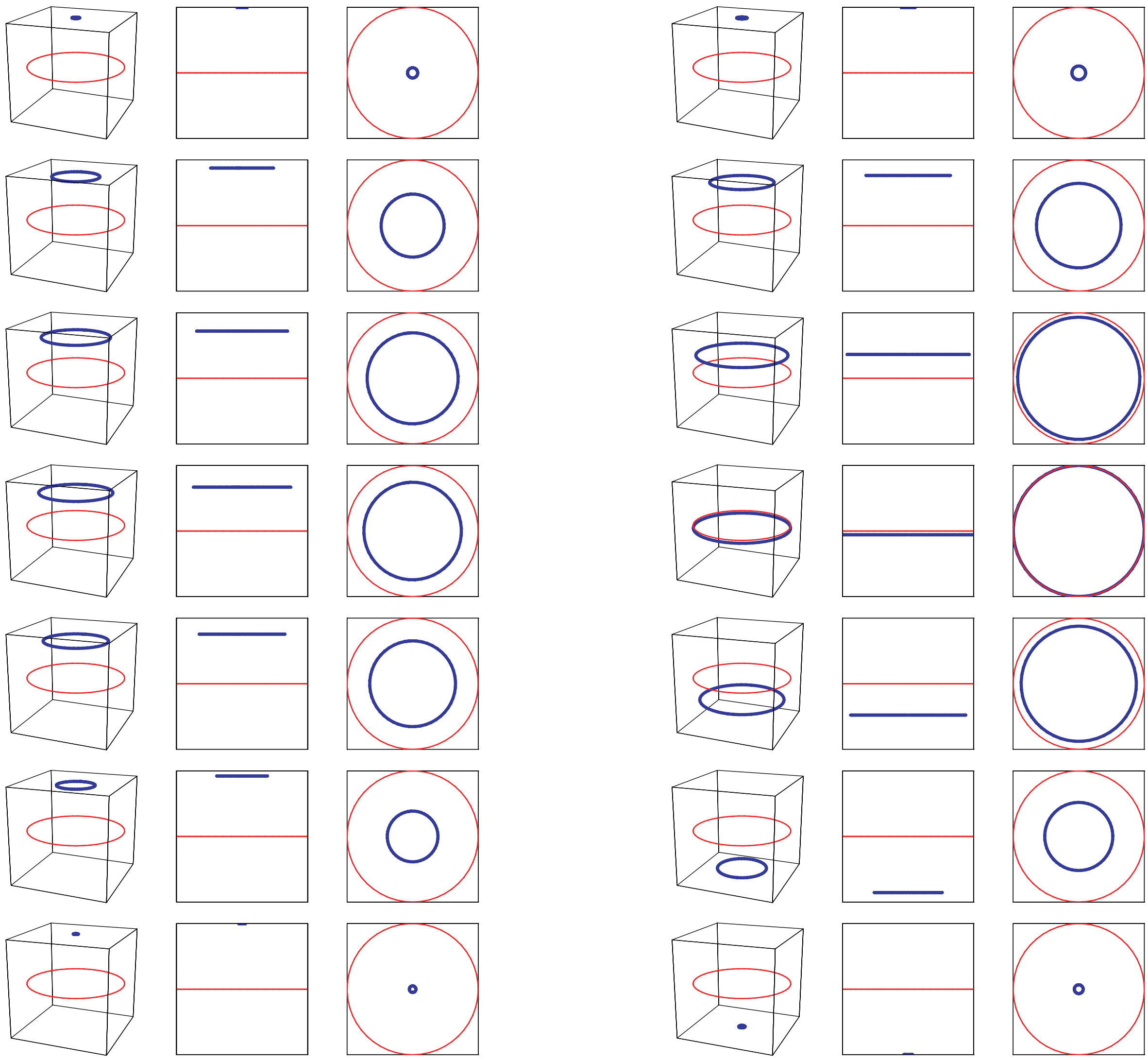


Figure 7.2: In the $\omega_{1,2} \to 0$ limit, type $(i)'$ (Left figure) and type $(ii)'$ (Right figure) helical strings reduce to different types of pulsating strings. Their behaviors are different in that the type $(i)'$ sweeps back and forth only in the top hemisphere with turning latitude controlled by the elliptic modulus, while the type $(ii)'$ pulsates on the entire sphere, see Section 7.2.2. For the type $(ii)'$ case, we only showed half of the oscillation period (for the other half, it sweeps back from the south pole to the north pole).

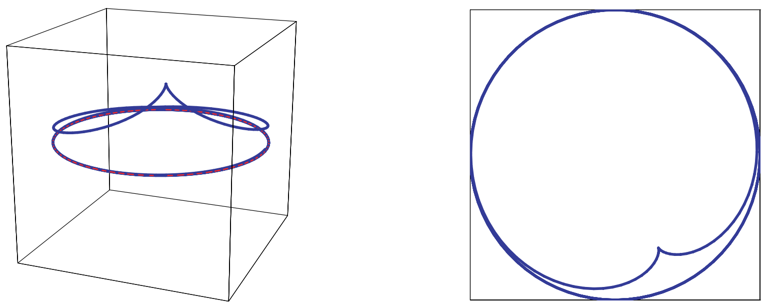


Figure 7.3: The $k \to 1$ limit of type $(i)'$ helical string: single-spike string ($\omega = 0.78$). The figure shows the single-spin case ($u_2 = \omega_2 = 0$).

Let us comment on a subtly about $v \to 0$ (or equivalently $\omega \to 0$) limit of a single spike string. It is easy to see the profile of single-spike solution (7.2.16) with $\omega = 0$ agrees with that of pulsating string solution (7.2.14) with $k = 1$, however, due to a singular nature of the $v \to 0$ limit, the angular momenta of both solutions (7.2.18) and (7.2.15) do not agree if we just naively take the limits on both sides.

- $k \to 0$ **limit: Rational circular (static) strings**

Another interesting limit is to send k to zero, where elliptic functions reduce to rational functions. The Virasoro conditions become

$$a^2 + b^2 = u_2^2 + \tanh^2 \omega_2 \quad \text{and} \quad ab = \pm u_2 \tanh \omega \,, \tag{7.2.20}$$

where $u_2 = \sqrt{U + \tanh^2 \omega}$. This can be solved by $a = u_2$ and $b = \tanh \omega$ (assuming $U > 0$). The profile is given by

$$t = \sqrt{U} \tilde{\tau} \,, \qquad \xi_1 = 0 \,, \qquad \xi_2 = e^{i\sqrt{U}\tilde{\sigma}} \,. \tag{7.2.21}$$

This is an unstable string that has no spins and just wraps around one of the great circles, and can be viewed as the $\tau \leftrightarrow \sigma$ transformed version of a point-like, BPS string with $E - (J_1 + J_2) = 0$. The conserved charges for one period reduce to

$$\mathcal{E} = \pi \mu \sqrt{U} \,, \qquad \mathcal{J}_1 = \mathcal{J}_2 = 0 \,. \tag{7.2.22}$$

The winding number for the φ_2-direction becomes $N_{\varphi_2} = \mu \sqrt{U}$, so the energy can also be written as

$$E = N_{\varphi_2} \sqrt{\lambda} = (2\pi N_{\varphi_2}) \times \left(\frac{\sqrt{\lambda}}{2\pi} \right) . \tag{7.2.23}$$

Thus the energy of rational circular strings is given by (length) $\times$ (tension).

• $u_2,\, \omega_2 \to 0$ **: Single-spin limit**

A single-spin type $(i)'$ helical string is obtained by setting $u_2 = \omega_2 = 0$, which results in $J_2 = N_{\varphi_2} = 0$.[6] In view of (6.3.9), the condition $u_2 = \omega_2 = 0$ requires $U = 0$, $u_1 = \mathrm{dn}(i\omega)$ and $C = \sqrt{k}/\,\mathrm{dn}(i\omega)$, and the Virasoro constraints (6.3.7) and (6.3.8) are solved by setting $a = k\,\mathrm{cn}(i\omega)$, $b = -ik\,\mathrm{sn}(i\omega)$ and $v = -i\,\mathrm{sn}(i\omega)/\,\mathrm{cn}(i\omega)$. Periodicity conditions then become

$$\Delta\sigma = \frac{2\pi}{n} = \frac{2i\mathbf{K}}{\mu\,\mathrm{sn}(i\omega)}\,, \qquad \frac{2\pi N_{\varphi_2}}{n} = 0\,, \tag{7.2.24}$$

$$\Delta\varphi_1 = \frac{2\pi N_{\varphi_1}}{n} = 2i\mathbf{K}\left(\frac{\mathrm{cn}(i\omega)\,\mathrm{dn}(i\omega)}{\mathrm{sn}(i\omega)} + Z_0(i\omega)\right) + (2n_1' + 1)\,\pi\,, \tag{7.2.25}$$

and the conserved charges for one period are

$$\mathcal{E} = \frac{ik}{\mathrm{sn}(i\omega)}\mathbf{K}\,, \qquad \mathcal{J}_1 = \frac{1}{k\,\mathrm{dn}(i\omega)}\Big[\mathbf{E} - \big(1-k^2\big)\,\mathbf{K}\Big]\,, \qquad \mathcal{J}_2 = 0\,. \tag{7.2.26}$$

7.2.2 Type $(ii)'$ helical strings

The type $(ii)'$ solution can be obtained from the type $(i)'$ solutions, either by shifting $\omega_2 \mapsto \omega_2 + \mathbf{K}'$ or by transforming k to $1/k$. The profile is given by[7]

$$\hat{t} = \hat{a}T + \hat{b}X\,, \tag{7.2.27}$$

$$\hat{\xi}_1 = \hat{C}\frac{\Theta_0(0)}{\sqrt{k}\,\Theta_0(i\omega_1)}\frac{\Theta_1(T - i\omega_1)}{\Theta_0(T)}\,\exp\Big(Z_0(i\omega_1)T + iu_1X\Big)\,, \tag{7.2.28}$$

$$\hat{\xi}_2 = \hat{C}\frac{\Theta_0(0)}{\sqrt{k}\,\Theta_3(i\omega_2)}\frac{\Theta_2(T - i\omega_2)}{\Theta_0(T)}\,\exp\Big(Z_3(i\omega_2)T + iu_2X\Big)\,, \tag{7.2.29}$$

The normalization constant C is same as that of Section 6.3.2. The equations of motion imposes the relation (6.3.24) on u_1 and u_2. Virasoro conditions are equivalent to (6.3.22) and (6.3.23). As in the type $(i)'$ case, we can set $\hat{t} = \sqrt{\hat{a}^2 - \hat{b}^2}\,\tilde{\tau}$ with $\hat{v} \equiv \hat{b}/\hat{a} \le 1$.

The periodicity conditions for the type $(ii)'$ solutions become

$$\Delta\sigma \equiv \frac{2\pi}{m} = \frac{2\mathbf{K}\sqrt{1-\hat{v}^2}}{\hat{v}\,\mu}\,, \tag{7.2.30}$$

$$\Delta\varphi_1 \equiv \frac{2\pi M_{\varphi_1}}{m} = 2\mathbf{K}\left(\frac{u_1}{\hat{v}} + iZ_0(i\omega_1)\right) + (2m_1' + 1)\,\pi\,, \tag{7.2.31}$$

$$\Delta\varphi_2 \equiv \frac{2\pi M_{\varphi_2}}{m} = 2\mathbf{K}\left(\frac{u_2}{\hat{v}} + iZ_3(i\omega_2)\right) + (2m_2' + 1)\,\pi\,, \tag{7.2.32}$$

where $m = 1, 2, \ldots$ counts the number of periods in $0 \le \sigma \le 2\pi$, and M_{φ_1,φ_2} are the winding numbers in the $\varphi_{1,2}$-directions respectively, and $m'_{1,2}$ are integers. The conserved charges are

[6] It turns out the other single-spin limit $u_1\,, \omega_1 \to 0$, which gives $J_1 = 0$, does not result in real solutions for this type $(i)'$ case.

[7] We use a hat to indicate type $(ii)'$ variables.

given by

$$\hat{\mathcal{E}} = \frac{ma(1-v^2)}{v}\,\mathbf{K} = \frac{n(a^2-b^2)}{b}\,\mathbf{K}\,, \tag{7.2.33}$$

$$\hat{\mathcal{J}}_1 = \frac{m\hat{C}^2\,u_1}{k^2}\left[\mathbf{E} - \left(\mathrm{dn}^2(i\omega_1) + \frac{i\,k^2}{\hat{v}\,u_1}\,\mathrm{sn}(i\omega_1)\,\mathrm{cn}(i\omega_1)\,\mathrm{dn}(i\omega_1)\right)\mathbf{K}\right]\,, \tag{7.2.34}$$

$$\hat{\mathcal{J}}_2 = \frac{m\hat{C}^2\,u_2}{k^2}\left[-\mathbf{E} + (1-k^2)\left(\frac{1}{\mathrm{dn}^2(i\omega_2)} - \frac{i\,k^2}{\hat{v}\,u_2}\,\frac{\mathrm{sn}(i\omega_2)\,\mathrm{cn}(i\omega_2)}{\mathrm{dn}^3(i\omega_2)}\right)\mathbf{K}\right]\,. \tag{7.2.35}$$

Just as in the type $(i) \leftrightarrow (i)'$ case, the winding numbers and the conserved charges of the original type (ii) and $(ii)'$ are related by $\hat{\mathcal{E}}(\hat{a},\hat{b}) = -\mathcal{E}^{(ii)}(\hat{b},\hat{a})$, $\hat{\mathcal{J}}_i(\hat{v}) = -\hat{\mathcal{J}}_i^{(ii)}(-1/\hat{v})$ and $M_{\varphi_i}(\hat{v}) = -M_{\varphi_i}^{(ii)}(-1/\hat{v})$.

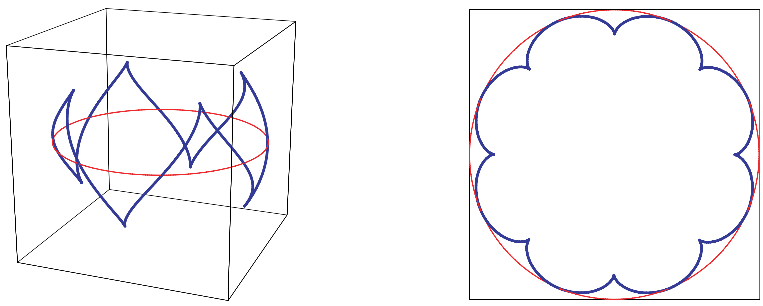


Figure 7.4: Type $(ii)'$ helical string ($k = 0.40\,, m = 8$). The figure shows a single-spin case ($u_2 = \omega_2 = 0$).

As in the type $(i)'$ case, we can take various limits.

• $\omega_{1,2} \to 0$ **limit : Pulsating strings**

The profiles (6.3.18)-(6.3.20) reduce to

$$\hat{t} = \sqrt{1+u_2^2}\,\tilde{\tau}\,, \qquad \hat{\xi}_1 = \mathrm{sn}(\tilde{\tau},k)\,e^{iu_1\tilde{\sigma}}\,, \qquad \hat{\xi}_2 = \mathrm{cn}(\tilde{\tau},k)\,e^{iu_2\tilde{\sigma}}\,, \tag{7.2.36}$$

with constraint $u_1^2 - u_2^2 = k^2$. The conserved charges for a period become

$$\mathcal{E} = \frac{\pi}{k}\sqrt{M_{\varphi_1}^2 + (k^2-1)\,M_{\varphi_2}^2}\,, \qquad \mathcal{J}_1 = \mathcal{J}_2 = 0\,. \tag{7.2.37}$$

Right of Figure 7.2 shows the time evolution of the type $(ii)'$ pulsating string. Again, when we set $u_2 = 0$, this string reduces to the simplest pulsating solution studied in [41].

• $k \to 1$ **limit : Single-spike strings**

This limit results in essentially the same solution as the type $(i)'$ case, that is an array of single-spike strings. The only difference is that while in the type $(i)'$ case every cusp appears in the same side about the equator, say the northern hemisphere, in the type $(ii)'$ case cusps appear in both the northern and southern hemispheres in turn, each after an infinite winding.

• $k \to 0$ **limit : Rational circular strings**

In the $k \to 0$ limit, the profile becomes

$$\hat{t} = \sqrt{\hat{a}^2 - \hat{b}^2}\,\tilde{\tau}\,, \quad \hat{\xi}_1 = \hat{C}\ \sin(T - i\omega_1)\, e^{iu_1 X}\,, \quad \hat{\xi}_2 = \hat{C}\ \cos(T - i\omega_2)\, e^{iu_2 X}\,, \tag{7.2.38}$$

with $\hat{C} = \left(\cosh^2\omega_2 + \sinh^2\omega_1\right)^{-1/2}$ and $u_1^2 = u_2^2 = U + 1$. Virasoro constraints imply the following set of relations between the parameters $\hat{a}$ and $\hat{b}$ (with $\hat{a} \ge \hat{b}$):

$$\hat{a}^2 + \hat{b}^2 = -U + 2u_2^2\,, \tag{7.2.39}$$

$$\hat{a}\,\hat{b} = \hat{C}^2\sqrt{U+1}\,(\sinh\omega_1\cosh\omega_1 \mp \sinh\omega_2\cosh\omega_2)\ . \tag{7.2.40}$$

Here $\mp$ reflects the sign ambiguity in the angular momenta. The periodicity conditions become

$$\Delta\sigma \equiv \frac{2\pi}{m} = \frac{\pi\sqrt{1-\hat{v}^2}}{\hat{v}\,\mu}\,, \tag{7.2.41}$$

$$\Delta\varphi_1 \equiv \frac{2\pi M_{\varphi_1}}{m} = \frac{\pi\, u_1}{\hat{v}} + (2m_1' + 1)\,\pi\,, \tag{7.2.42}$$

$$\Delta\varphi_2 \equiv \frac{2\pi M_{\varphi_2}}{m} = \frac{\pi\, u_2}{\hat{v}} + (2m_2' + 1)\,\pi\,. \tag{7.2.43}$$

The conserved charges for a single period are evaluated as

$$\hat{\mathcal{E}} = \frac{\pi\hat{a}\,(1-\hat{v}^2)}{2\hat{v}}\,, \qquad \hat{\mathcal{J}}_1 = \frac{\pi\hat{C}^2}{2\hat{v}}\,\sinh\omega_1\cosh\omega_1\,, \qquad \hat{\mathcal{J}}_2 = -\frac{\pi\hat{C}^2}{2\hat{v}}\,\sinh\omega_2\cosh\omega_2\,. \tag{7.2.44}$$

• $u_2\,, \omega_2 \to 0$ **: Single-spin limit**

As in the type $(i)'$ case, we obtain the type $(ii)'$ helical strings with $J_2 = M_{\varphi_2} = 0$ by setting $u_2 = \omega_2 = 0$.[8] Then we find $U = -1 + k^2$, $u_1 = k\,\mathrm{cn}(i\omega)$ and $\hat{C} = 1/\,\mathrm{cn}(i\omega)$. The Virasoro conditions require $\hat{a} = \mathrm{dn}(i\omega)$, $\hat{b} = -ik\,\mathrm{sn}(i\omega)$ and $\hat{v} = -ik\,\mathrm{sn}(i\omega)/\,\mathrm{dn}(i\omega)$. The periodicity conditions become

$$\Delta\sigma = \frac{2\pi}{m} = \frac{2i\mathbf{K}}{\mu k\,\mathrm{sn}(i\omega)}\,, \qquad \frac{2\pi M_{\varphi_2}}{m} = 0\,, \tag{7.2.45}$$

$$\Delta\varphi_1 = \frac{2\pi M_{\varphi_1}}{m} = 2i\mathbf{K}\left(\frac{\mathrm{cn}(i\omega)\,\mathrm{dn}(i\omega)}{\mathrm{sn}(i\omega)} + Z_0(i\omega)\right) + (2m_1' + 1)\,\pi\,, \tag{7.2.46}$$

[8] For the type $(ii)'$ case, the other single-spin limit $u_1 = \omega_1 = 0$ results in $U = -1$, $u_2^2 = -1 + (1 - k^2)/\,\mathrm{dn}^2(i\omega_2)$ and $\hat{C} = \mathrm{dn}(i\omega_2)/\,\mathrm{cn}(i\omega_2)$. It turns out equivalent to the $\omega_{1,2} \to 0$ limit, because u_2 must be real, and thus the second condition implies $\omega_2 = 0$.

and the conserved charges for a single period are given by

$$\hat{\mathcal{E}} = \frac{i}{k\,\mathrm{sn}(i\omega)}\,\mathbf{K}\,, \qquad \hat{\mathcal{J}}_1 = \frac{1}{k\,\mathrm{cn}(i\omega)}\,\mathbf{E}\,, \qquad \hat{\mathcal{J}}_2 = 0\,. \tag{7.2.47}$$

7.3 Finite-gap interpretation

The helical strings (6.3.4), (6.3.5) of [96] were shown in [62] to be equivalent to the most general elliptic ("two-cut") finite-gap solution on $\mathbb{R} \times S^3 \subset AdS_5 \times S^5$, with both cuts intersecting the real axis within the interval $(-1, 1)$ (see Figure 7.5 (a)). The aim of this section is to present the corresponding finite-gap description of the $\tau \leftrightarrow \sigma$ transformed helical string (7.2.2), (7.2.3) obtained in the previous section.

Recall first from [62] that the (σ, τ)-dependence of the general finite-gap solution enters solely through the differential form

$$d\mathcal{Q}(\sigma, \tau) = \frac{1}{2\pi}\left(\sigma dp + \tau dq\right)\,, \tag{7.3.1}$$

where dp and dq are the differentials of the quasi-momentum and quasi-energy defined below by their respective asymptotics near the points $x = \pm 1$. The differential multiplying σ in $d\mathcal{Q}(\sigma, \tau)$ (namely dp) is related to the eigenvalues of the monodromy matrix, which by definition is the parallel transporter along a closed loop $\sigma \in [0, 2\pi]$ on the worldsheet. This is because the Baker-Akhiezer vector $\boldsymbol{\psi}(P, \sigma, \tau)$, whose (σ, τ)-dependence also enters solely through the differential form $d\mathcal{Q}(\sigma, \tau)$ in (7.3.1), satisfies [31]

$$\boldsymbol{\psi}(P, \sigma + 2\pi, \tau) = \exp\left\{ i \int_{\infty^+}^{P} dp \right\} \boldsymbol{\psi}(P, \sigma, \tau)\,.$$

Now it is clear from (7.3.1) that the $\sigma \leftrightarrow \tau$ operation can be realised on the general finite-gap solution by simply interchanging the quasi-momentum with the quasi-energy,

$$dp \;\leftrightarrow\; dq\,. \tag{7.3.2}$$

However, since we wish dp to always denote the differential related to the eigenvalues of the monodromy matrix, by the above argument it must always appear as the coefficient of σ in $d\mathcal{Q}(\sigma, \tau)$. Therefore equation (7.3.2) should be interpreted as saying that the respective definitions of the differentials dp and dq are interchanged, but $d\mathcal{Q}(\sigma, \tau)$ always takes the same form as in (7.3.1).

Before proceeding let us recall the precise definitions of these differentials dp and dq . Consider an algebraic curve Σ , which admits a hyperelliptic representation with cuts. For what follows it will be important to specify the position of the different cuts relative to the points $x = \pm 1$, *i.e.*, Figures 7.5 (a) and 7.5 (b) are to be distinguished for the purpose of defining dp and dq . We could make this distinction by specifying an equivalence relation on representations of Σ in terms of cuts, where two representations are equivalent if the cuts of one can be deformed into the cuts of the other within $\mathbb{CP}^1 \setminus \{\pm 1\}$. It is straightforward to see that there are only two such equivalence classes for a general algebraic curve Σ. For example, in the case of an elliptic curve Σ the representatives of these two equivalence classes are given in Figures 7.5 (a) and 7.5 (b). Now with respect to a given equivalence class of cuts, the differentials dp and dq can be uniquely defined on Σ as in [31] by the following conditions:

(1) their $\mathcal{A}$-period vanishes.

(2) their respective poles at $x = \pm 1$ are of the following form, up to a trivial overall change of sign (see [62]),

$$dp(x^{\pm}) \underset{x\to+1}{\sim} \mp \frac{\pi\kappa dx}{(x-1)^2}, \qquad dp(x^{\pm}) \underset{x\to-1}{\sim} \mp \frac{\pi\kappa dx}{(x+1)^2}, \tag{7.3.3}$$

$$dq(x^{\pm}) \underset{x\to+1}{\sim} \mp \frac{\pi\kappa dx}{(x-1)^2}, \qquad dq(x^{\pm}) \underset{x\to-1}{\sim} \pm \frac{\pi\kappa dx}{(x+1)^2}, \tag{7.3.4}$$

where $x^{\pm} \in \Sigma$ denotes the pair of points above x, with x^+ being on the physical sheet, and x^- on the other sheet.

Once the differentials dp and dq have been defined by (7.3.3) and (7.3.4) with respect to a given equivalence class of cuts, one can move the cuts around into the other equivalence class (by crossing say $x = -1$ with a single cut) to obtain a representation of dp and dq with respect to the other equivalence class of cuts. So for instance, if we define dp and dq by (7.3.3) and (7.3.4) with respect to the equivalence class of cuts in Figure 7.5 (a), then with respect to the equivalence class of cuts in Figure 7.5 (b) the definition of dp will now be (7.3.4) and that of dq will now be (7.3.3).

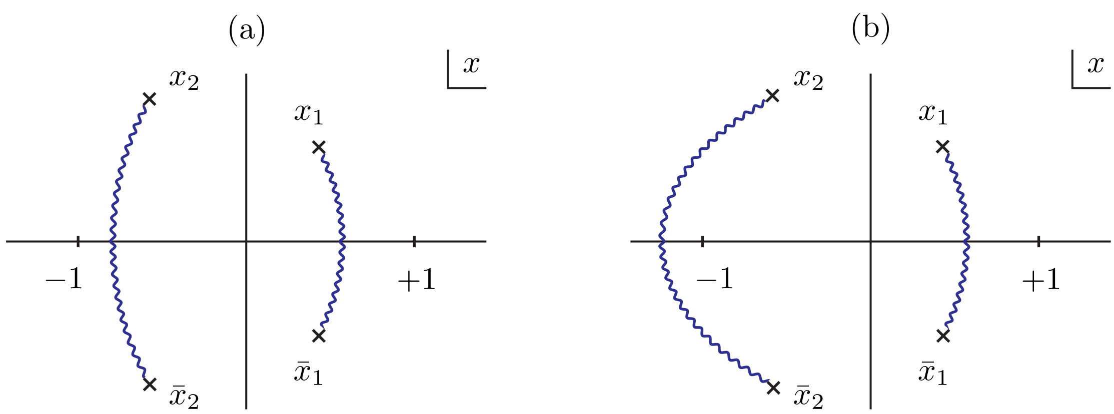


Figure 7.5: Different possible arrangements of cuts relative to $x = \pm 1$: (a) corresponds to the helical string, (b) corresponds to the $\tau \leftrightarrow \sigma$ transformed helical string.

In summary, both equivalence classes of cuts represents the very same algebraic curve Σ, but each equivalence class gives rise to a different definition of dp and dq. So the two equivalence classes of cuts give rise to two separate finite-gap solutions but which can be related by a $\tau \leftrightarrow \sigma$ transformation (7.3.2). Indeed, if in the construction of [62], it is assumed the generic configuration of cuts given in Figure 7.5 (b), instead of Figure 7.5 (a), then the resulting solution is the generic helical string but with

$$X \leftrightarrow T$$

namely the 2D transformed helical string (7.2.2), (7.2.3). Therefore, with dp and dq defined as above by their respective asymptotics (7.3.3) and (7.3.4) at $x = \pm 1$, the helical string of [96, 62] is the general finite-gap solution corresponding to the class represented by Figure 7.5 (a), whereas the 2D transformed helical string corresponds to the most general elliptic finite-gap solution on $\mathbb{R} \times S^3$ with cuts in the other class represented in Figure 7.5 (b).

As is clear from the above, a given finite-gap solution is not associated with a particular equivalence class of cuts; since dp and dq are defined relative to an equivalence class of cuts, one can freely change equivalence class provided one also changes the definitions of dp and dq with respect to this new equivalence class according to (7.3.2), so that in the end dp and dq define the same differentials on Σ in either representation. For example, we can describe the 2D transformed helical string in two different ways: either we take the configuration of cuts in Figure 7.5 (b) with dp and dq defined as usual by their asymptotics (7.3.3) and (7.3.4) at $x = \pm 1$, or we take the configuration of cuts in Figure 7.5 (a) but need to swap the definitions of dp and dq in (7.3.3) and (7.3.4). In the following we will use the latter description of Figure 7.5 (a) in order to take the singular limit $k \to 1$ where the cuts merge into a pair of singular points.

We can obtain expressions for the global charges $J_1 = (J_L + J_R)/2$, $J_2 = (J_L - J_R)/2$ along the same lines as in [62] for the helical string. In terms of the differential form

$$\alpha \equiv \frac{\sqrt{\lambda}}{4\pi}\left(x + \frac{1}{x}\right) dp\,, \qquad \tilde{\alpha} \equiv \frac{\sqrt{\lambda}}{4\pi}\left(x - \frac{1}{x}\right) dp\,, \tag{7.3.5}$$

we can write

$$J_1 = -\mathrm{Res}_{0^+}\alpha + \mathrm{Res}_{\infty^+}\alpha = \mathrm{Res}_{0^+}\tilde{\alpha} + \mathrm{Res}_{\infty^+}\tilde{\alpha}\,, \tag{7.3.6}$$

$$J_2 = -\mathrm{Res}_{0^+}\alpha - \mathrm{Res}_{\infty^+}\alpha\,. \tag{7.3.7}$$

Note that α and $\tilde{\alpha}$ both have simple poles at $x = 0\,, \infty$ but $\tilde{\alpha}$ also has simple poles at $x = \pm 1$ coming from the double poles in dp at $x = \pm 1$. It follows that we can rewrite (7.3.6), (7.3.7) as

$$J_1 = -\sum_{I=1}^{2} \frac{1}{2\pi i}\int_{\mathcal{A}_I} \tilde{\alpha} - \mathrm{Res}_{(+1)^+}\tilde{\alpha} - \mathrm{Res}_{(-1)^+}\tilde{\alpha}\,, \tag{7.3.8}$$

$$J_2 = \sum_{I=1}^{2} \frac{1}{2\pi i}\int_{\mathcal{A}_I} \alpha\,, \tag{7.3.9}$$

where $\mathcal{A}_I$ is the $\mathcal{A}$-cycle around the I-th cut. Whereas in [62] the residues of $\tilde{\alpha}$ at $x = \pm 1$ were of the same sign (as a consequence of $p(x)$ having equal residues at $x = \pm 1$) so that their sum gave the energy E of the string, in the present 2D-transformed helical case the residues of $\tilde{\alpha}$ at $x = \pm 1$ are now opposite (since $p(x)$ now has opposite residues at $x = \pm 1$) and therefore cancel in the above expression for J_1, resulting in the following expressions

$$- J_1 = \sum_{I=1}^{2} \frac{1}{2\pi i} \int_{\mathcal{A}_I} \tilde{\alpha} \,, \qquad J_2 = \sum_{I=1}^{2} \frac{1}{2\pi i} \int_{\mathcal{A}_I} \alpha \,. \tag{7.3.10}$$

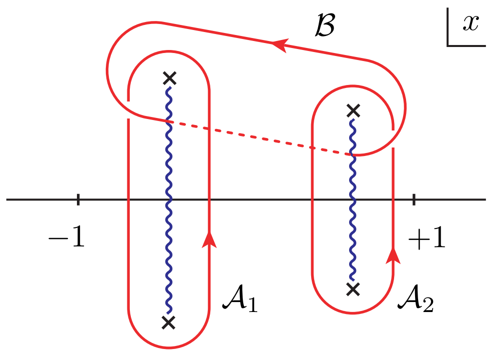


Figure 7.6: Definitions of cycles.

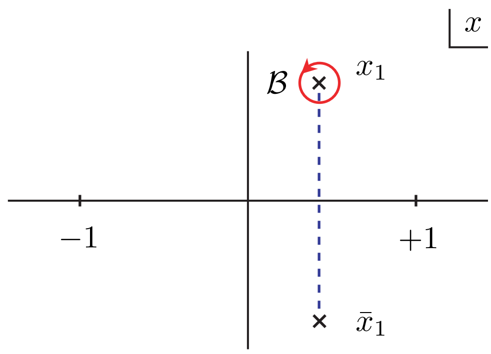


Figure 7.7: $k \to 1$ limit of cuts.

In parallel to the discussion of the helical string case in [62], there are two types of limits one can consider: the symmetric cut limit (where the curve acquires the extra symmetry $x \leftrightarrow -x$) which corresponds to taking $\omega_{1,2} \to 0$ in the finite-gap solution, or the singular curve limit which corresponds to taking the moduli of the curve to one, $k \to 1$. In the symmetric cut limit the discussion is identical to that in [62] (when working with the configuration of cuts in Figure 7.5 (a)), in particular there are two possibilities corresponding to the type $(i)'$ and type $(ii)'$ cases, for which the cuts are symmetric with $x_1 = -\bar{x}_2$ and imaginary with $x_1 = -\bar{x}_1$, $x_2 = -\bar{x}_2$ respectively (see Figure 2 of [62]).

In the singular limit $k \to 1$ where both cuts merge into a pair of singular points at $x = x_1$, $\bar{x}_1$ [62], the sum of $\mathcal{A}$-cycles turns into a sum of cycles around the points $x_1, \bar{x}_1$, so that (7.3.10) yields in this limit

$$- J_1 = \mathrm{Res}_{x_1} \tilde{\alpha} + \overline{\mathrm{Res}_{x_1} \tilde{\alpha}} \,, \qquad J_2 = \mathrm{Res}_{x_1} \alpha + \overline{\mathrm{Res}_{x_1} \alpha} \,. \tag{7.3.11}$$

Moreover, in the singular limit dp acquires simple poles at $x = x_1$, $\bar{x}_1$ so that the periodicity condition about the $\mathcal{B}$-cycle, $\int_{\mathcal{B}} dp = 2\pi n$, implies

$$\mathrm{Res}_{x_1} dp = \frac{n}{i} \,.$$

Let us set $n=1$ (n can be easily recovered at any moment). Then (7.3.11) simplifies to

$$-J_1 = \frac{\sqrt{\lambda}}{4\pi}\left|\left(x_1 - \frac{1}{x_1}\right) - \left(\bar{x}_1 - \frac{1}{\bar{x}_1}\right)\right| \,, \tag{7.3.12}$$

$$J_2 = \frac{\sqrt{\lambda}}{4\pi}\left|\left(x_1 + \frac{1}{x_1}\right) - \left(\bar{x}_1 + \frac{1}{\bar{x}_1}\right)\right| \,. \tag{7.3.13}$$

The energy $E = \sqrt{\lambda}\,\kappa = (n\sqrt{\lambda}/\pi)\,\mathcal{E}$ diverges in the singular limit $k \to 1$, but this divergence can be related to the one in $\Delta\varphi_1$. In the present case the σ-periodicity condition $\int_{\mathcal{B}} dp \in 2\pi\mathbb{Z}$ can be written as

$$-\frac{2\mathbf{K}\sqrt{1-v^2}}{v} = \frac{2\pi}{n}\,\mu \equiv \frac{2\pi\kappa|x_1 - \bar{x}_2|}{n\sqrt{y_+ y_-}} \,,$$

where we used (6.6.11). Using this σ-periodicity condition the energy can be expressed in the $k \to 1$ limit as

$$\mathcal{E} = \frac{u_1}{v}\left(1 - v^2\right)\mathbf{K}(1) \,.$$

We can relate this divergent expression with the expression (7.2.6) for $\Delta\varphi_1$ which also diverge in the limit $k \to 1$, making use of the relation $u_1 v = \tan\omega_1$,[9] and find

$$\mathcal{E} - \frac{\Delta\varphi_1}{2} = -\left(\omega_1 - \frac{(2n_1' + 1)\pi}{2}\right) \equiv \bar{\theta} \,. \tag{7.3.14}$$

Comparing this scenario with the one for helical strings in [62] we can write an expression for $\bar{\theta}$ in terms of the spectral data x_1 of the singular curve. Identifying

$$\bar{\theta} = -\frac{i}{2}\ln\left(\frac{x_1}{\bar{x}_1}\right) \,, \tag{7.3.15}$$

the expressions (7.3.12), (7.3.13) and (7.3.15) together imply the relation[10]

$$-\,J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\sin^2\bar{\theta}} \,. \tag{7.3.16}$$

[9] The notation in Section 6.6 is $u_1 = v_-$ and $\omega_1 = \tilde{\rho}_-$.

[10] The sign difference between (7.2.18) and here is not essential.

Chapter 8

AdS helical strings

This chapter is devoted to construction of helical string solutions on $\mathrm{AdS}_3 \times \mathrm{S}^1 \subset \mathrm{AdS}_5 \times \mathrm{S}^1$. The construction almost parallels that in previous two chapters. However, non-compactness of the AdS space leads to new non-trivial features compared to the $\mathbb{R}_\mathrm{t} \times \mathrm{S}^3$ case.

8.1 Classical strings on $\mathrm{AdS}_3 \times \mathrm{S}^1$ and Complex sinh-Gordon model

A string theory on $AdS_3 \times S^1 \subset AdS_5 \times S^5$ spacetime is described by an $O(2,2) \times O(2)$ sigma model. Let us denote the coordinates of the embedding space as η_0, η_1 (for AdS_3) and ξ_1 (for S^1) and set the radii of AdS_3 and S^1 both to unity,

$$\vec{\eta}^* \cdot \vec{\eta} \equiv - |\eta_0|^2 + |\eta_1|^2 = -1 \,, \qquad |\xi_1|^2 = 1 \,. \tag{8.1.1}$$

In the standard polar coordinates, the embedding coordinates are expressed as

$$\eta_0 = \cosh\rho \, e^{it} \,, \quad \eta_1 = \sinh\rho \, e^{i\phi_1} \,, \quad \xi_1 = e^{i\varphi_1} \,, \tag{8.1.2}$$

and all the charges of the string states are defined as Nöther charges associated with shifts of the angular variables. The bosonic Polyakov action for the string on $AdS_3 \times S^1$ is given by

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \left[\gamma^{ab} \left(\partial_a \vec{\eta}^* \cdot \partial_b \vec{\eta} + \partial_a \xi^* \cdot \partial_b \xi \right) + \widetilde{\Lambda} \Big(\vec{\eta}^* \cdot \vec{\eta} + 1 \Big) + \Lambda \Big(\xi_1^* \cdot \xi_1 - 1 \Big) \right] \,, \tag{8.1.3}$$

and we take the same conformal gauge as in the $\mathbb{R}_\mathrm{t} \times \mathrm{S}^3$ case. From the action (8.1.3) we get the equations of motion

$$\partial_a \partial^a \vec{\eta} - \left(\partial_a \vec{\eta}^* \cdot \partial^a \vec{\eta} \right) \vec{\eta} = 0 \,, \qquad \partial_a \partial^a \xi_1 + \left(\partial_a \xi_1^* \cdot \partial^a \xi_1 \right) \xi_1 = 0 \,, \tag{8.1.4}$$

and Virasoro constraints

$$0 = \mathcal{T}_{\sigma\sigma} = \mathcal{T}_{\tau\tau} = \frac{\delta^{ab}}{2} \left(\partial_a \vec{\eta}^* \cdot \partial_b \vec{\eta} + \partial_a \xi_1^* \cdot \partial_b \xi_1 \right) \,, \tag{8.1.5}$$

$$0 = \mathcal{T}_{\tau\sigma} = \mathcal{T}_{\sigma\tau} = \mathrm{Re} \left(\partial_\tau \vec{\eta}^* \cdot \partial_\sigma \vec{\eta} + \partial_\tau \xi_1 \cdot \partial_\sigma \xi_1^* \right) \,. \tag{8.1.6}$$

We assume $\delta^{ab}\partial_a\xi_1^* \cdot \partial_b\xi_1 = 1$ and $\partial_\tau\xi_1 \cdot \partial_\sigma\xi_1^*$ throughout this section. In contrast to the $\mathbb{R}_t \times S^3$ case, however, it is possible to construct string solutions with $\delta^{ab}\partial_a\xi_1^* \cdot \partial_b\xi_1 = 0$ as was done in [163].

The PLR reduction procedure, which we made use of in obtaining the $O(4)$ sigma model solutions from Complex sine-Gordon solution, also works for the current case in much the same way. The $O(2,2)$ sigma model in conformal gauge is now related to what we call Complex sinh-Gordon (CshG) model, which is defined by the Lagrangian

$$\mathcal{L}_{\text{CshG}} = \frac{\partial^a\psi^*\partial_a\psi}{1+\psi^*\psi} + \psi^*\psi\,, \tag{8.1.7}$$

with $\psi = \psi(\tau,\sigma)$ being a complex field. It can be viewed as a natural generalization of the well-known sinh-Gordon model in the sense we describe below. By defining two real fields α and β of the CshG model through $\psi \equiv \sinh(\alpha/2)\exp(i\beta/2)$, the Lagrangian (8.1.7) is rewritten as

$$\mathcal{L}_{\text{CshG}} = \frac{1}{4}\,(\partial_a\alpha)^2 + \frac{\tanh^2(\alpha/2)}{4}\,(\partial_a\beta)^2 + \sinh^2(\alpha/2)\,. \tag{8.1.8}$$

The equations of motion that follow from the Lagrangian are

$$\partial^a\partial_a\psi - \psi^*\frac{\partial^a\psi\,\partial_a\psi}{1+\psi^*\psi} - \psi\,(1+\psi^*\psi) = 0\,, \tag{8.1.9}$$

$$i.e., \quad \left\{ \begin{aligned} &\partial^a\partial_a\alpha - \frac{\sinh(\alpha/2)}{2\cosh^3(\alpha/2)}\,(\partial_a\beta)^2 - \sinh\alpha = 0\,, \\ &\partial^a\partial_a\beta + \frac{2\,\partial_a\alpha\,\partial^a\beta}{\sinh\alpha} = 0\,. \end{aligned} \right. \tag{8.1.10}$$

We refer to the coupled equations (8.1.10) as Complex sinh-Gordon (CshG) equations. If β is a constant field, the first equation in (8.1.10) reduces to

$$\partial_a\partial^a\alpha - \sinh\alpha = 0\,. \tag{8.1.11}$$

which is the ordinary sinh-Gordon equation. As readers familiar with the PLR reduction can easily imagine, it is this field α that gets into a self-consistent potential in the Schrödinger equation this time. Namely, we can write the string equations of motion given in (8.1.4) as

$$\partial_a\partial^a\vec{\eta} - (\cosh\alpha)\,\vec{\eta} = 0\,, \qquad \cosh\alpha \equiv \partial_a\vec{\eta}^* \cdot \partial^a\vec{\eta}\,, \tag{8.1.12}$$

with the same field α we introduced as the real part of the CshG field ψ. What this means is that if $\{\vec{\eta},\xi\}$ is a consistent string solution which satisfies Virasoro conditions (8.1.5) and (8.1.6), then $\psi = \sinh(\alpha/2)\exp(i\beta/2)$ defined via (8.1.12) and (8.1.16) solves the CshG equations.

The derivation of this fact parallels the usual PLR reduction procedure. Let us define worldsheet light-cone coordinates as $\sigma^\pm = \tau\pm\sigma$, and the embedding coordinates as $\eta_0 = Y_0 + iY_5$ and $\eta_1 = Y_1 + iY_2$. Then consider the equations of motion of the $O(2,2)$ nonlinear sigma model through the constraints

$$\vec{Y}\cdot\vec{Y} = -1\,, \quad (\partial_+\vec{Y})^2 = -1\,, \quad (\partial_-\vec{Y})^2 = -1\,, \quad \partial_+\vec{Y}\cdot\partial_-\vec{Y} \equiv -\cosh\alpha\,, \tag{8.1.13}$$

where $\vec{Y}\cdot\vec{Y} \equiv (\vec{Y})^2 \equiv -(Y_0)^2 + (Y_1)^2 + (Y_2)^2 - (Y_5)^2$. A basis of $O(2,2)$-covariant vectors can be given by Y_i, $\partial_+ Y_i$, $\partial_- Y_i$ and $K_i \equiv \epsilon_{ijkl} Y^j \partial_+ Y^k \partial_- Y^l$. By defining a pair of scalar functions u and v as

$$u \equiv \frac{\vec{K}\cdot\partial_+^2\vec{Y}}{\sinh\alpha}\,, \qquad v \equiv \frac{\vec{K}\cdot\partial_-^2\vec{Y}}{\sinh\alpha}\,, \tag{8.1.14}$$

the equations of motion of the $O(2,2)$ sigma model are recast in the form

$$\partial_-\partial_+\alpha + \sinh\alpha + \frac{uv}{\sinh\alpha} = 0\,, \qquad \partial_- u = \frac{v\,\partial_+\alpha}{\sinh\alpha}\,, \qquad \partial_+ v = \frac{u\,\partial_-\alpha}{\sinh\alpha}\,. \tag{8.1.15}$$

One can easily confirm that this set of equations is equivalent to the pair of equations (8.1.10) of CshG theory, under the identifications

$$u = (\partial_+\beta)\,\tanh\frac{\alpha}{2}\,, \qquad v = -(\partial_-\beta)\,\tanh\frac{\alpha}{2}\,. \tag{8.1.16}$$

Thus there is a (classical) equivalence between the $O(2,2)$ sigma model $\leftrightarrow$ CshG as in the $O(4) \leftrightarrow$ CsG case. Making use of the equivalence, one can construct classical string solutions on $AdS_3 \times S^1$ by the following recipe:

1. Find a solution ψ of CshG equation (8.1.9).
2. Identify $\cosh\alpha \equiv \partial_a\vec{\eta}^* \cdot \partial^a\vec{\eta}$, where α appears in the real part of the solution ψ, and η are the embedding coordinates of the corresponding string solution in AdS_3.
3. Solve the "Schrödinger equation" (8.1.12) together with the Virasoro constraints (8.1.5) and (8.1.6), under appropriate boundary conditions.
4. Resulting set of $\vec{\eta}$ ("wavefunction") and ξ_1 gives the corresponding string profile in $AdS_3 \times S^1$.

Let us start with step 1. From the similarities between the CshG equation and the CsG equation, it is easy to find helical-wave solutions of the CshG equation. Here we give two such solutions that will be important later. The first one is given by

$$\psi_{\rm cd} = kc\frac{\mathrm{cn}(cx_v)}{\mathrm{dn}(cx_v)} \exp\left(i\sqrt{(1+c^2)(1+k^2c^2)}\;t_v\right), \tag{8.1.17}$$

and the second one is

$$\psi_{\rm ds} = c\frac{\mathrm{dn}(cx_v)}{\mathrm{sn}(cx_v)} \exp\left(i\sqrt{(1-k^2c^2)(1+c^2-k^2c^2)}\;t_v\right). \tag{8.1.18}$$

By substituting the solution (8.1.18) into the string equations of motion (8.1.12), we obtain

$$\left[-\partial_T^2 + \partial_X^2 - k^2\left(\frac{2}{k^2\,\mathrm{sn}^2(X,k)} - 1\right)\right]\vec{\eta} = U\vec{\eta}\,, \tag{8.1.19}$$

under the identification of $(\mu\tau, \mu\sigma) \equiv (ct, cx)$. The "eigenenergy" U can be treated as a free parameter as was the case in [96]. Different choices of helical-waves of CshG equation simply correspond to taking different ranges of U.

We are now at the stage of constructing the corresponding string solution by following the steps 2 - 4 listed before. However, we do not need to do this literally. Since the metrics of $\mathrm{AdS}_3 \times \mathrm{S}^1$

string solutions on both manifolds are related by a sort of analytic continuation of global coordinates. Therefore, the simplest way to obtain helical string solutions on $\mathrm{AdS}_3 \times \mathrm{S}^1$ is to perform analytic continuation of helical string solutions on $\mathbb{R}_{\mathrm{t}} \times \mathrm{S}^3$, as will be done in the following sections. Large parts of the calculation parallel the $\mathbb{R} \times S^3$ case. The most significant difference lies in the constraints imposed on the solution of the equations of motion, such as the periodicity conditions.

8.2 Helical strings on $\mathrm{AdS}_3 \times \mathrm{S}^1$ with two spins

In this section, we consider the analytic continuation of helical strings on $\mathbb{R}_{\mathrm{t}} \times \mathrm{S}^3$ to those on $\mathrm{AdS}_3 \times \mathrm{S}^1$. Among various possible solutions, we will concentrate on two particular examples that have clear connections with known string solutions of interest to us. The first example, called type (iii) helical string, is a helical generalization of the folded string solution on $\mathrm{AdS}_3 \times \mathrm{S}^1$ [164]. The second one, called type (iv), reproduces the $SL(2)$ "giant magnon" solution [53, 165] in the infinite-spin limit.

8.2.1 Type (iii) helical strings

In [45], it was pointed out that (S, J) folded strings can be obtained from (J_1, J_2) folded strings by analytic continuation of the elliptic modulus squared, from $k^2 \geq 0$ to $k^2 \leq 0$. Here we apply the same analytic continuation to type (i) helical strings to obtain solutions on $\mathrm{AdS}_3 \times \mathrm{S}^1$, which we call type (iii) strings. For notational simplicity, it is useful to introduce a new moduli parameter q through the relation

$$k \equiv \frac{iq}{q'} \equiv \frac{iq}{\sqrt{1-q^2}}\,. \tag{8.2.1}$$

If k is located on the upper half of the imaginary axis, *i.e.*, $k = i\kappa$ with $0 \leq \kappa$, then q is a real parameter in the interval $[0, 1]$.

As shown in Appendix A.2.2, the transformation (8.2.1) can be regarded as a T-transformation of the modulus τ. Hence, by performing a T-transformation on the profile of type (i) helical strings (6.3.3)-(6.3.5), we obtain type (iii) string solutions:

$$\eta_0 = \frac{C}{\sqrt{qq'}} \frac{\Theta_3(0)\,\Theta_0(\tilde{X} - i\tilde{\omega}_0)}{\Theta_2(i\tilde{\omega}_0)\,\Theta_3(\tilde{X})} \exp\Big(Z_2(i\tilde{\omega}_0)\tilde{X} + i\tilde{u}_0\tilde{T}\Big)\,, \tag{8.2.2}$$

$$\eta_1 = \frac{C}{\sqrt{qq'}} \frac{\Theta_3(0)\,\Theta_1(\tilde{X} - i\tilde{\omega}_1)}{\Theta_3(i\tilde{\omega}_1)\,\Theta_3(\tilde{X})} \exp\Big(Z_3(i\tilde{\omega}_1)\tilde{X} + i\tilde{u}_1\tilde{T}\Big)\,, \tag{8.2.3}$$

$$\xi_1 = \exp\Big(i\tilde{a}\tilde{T} + i\tilde{b}\tilde{X}\Big)\,, \tag{8.2.4}$$

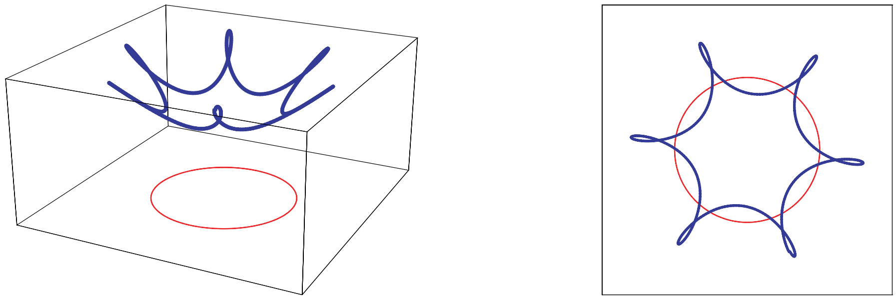


Figure 8.1: Type (iii) helical string ($q = 0.700$, $U = 12.0$, $\tilde{\omega}_0 = -0.505$, $\tilde{\omega}_1 = 0.776$, $n = 6$), projected onto AdS_2 spanned by $(\mathrm{Re}\,\eta_1, \mathrm{Im}\,\eta_1, |\eta_0|)$. The circle represents a unit circle $|\eta_1| = 1$ at $\eta_0 = 0$.

where we rescaled various parameters as

$$\tilde{X} = X/q', \quad \tilde{T} = T/q', \quad \tilde{\omega}_j = \omega_j/q', \quad \tilde{a} = aq', \quad \tilde{b} = bq', \quad \tilde{u}_j = u_j q'. \tag{8.2.5}$$

We choose the constant C so that they satisfy $|\eta_0|^2 - |\eta_1|^2 = 1$. One such possibility is to choose[1]

$$C = \left(\frac{1}{q^2 \,\mathrm{cn}^2(i\tilde{\omega}_0)} + \frac{\mathrm{sn}^2(i\tilde{\omega}_1)}{\mathrm{dn}^2(i\tilde{\omega}_1)} \right)^{-1/2}. \tag{8.2.6}$$

With the help of various formulae on elliptic functions, one can check that $\vec{\eta}$ in (8.2.2), (8.2.3) certainly solves the string equations of motion as

$$\left[-\partial_{\tilde{T}}^2 + \partial_{\tilde{X}}^2 + q^2 \left(2(1-q^2)\, \frac{\mathrm{sn}^2}{\mathrm{dn}^2}(\tilde{X}, q) - 1 \right) \right] \vec{\eta} = \tilde{U} \vec{\eta}, \tag{8.2.7}$$

if the parameters are related as

$$\tilde{u}_0^2 = \tilde{U} - (1-q^2) \frac{\mathrm{sn}^2(i\tilde{\omega}_0)}{\mathrm{cn}^2(i\tilde{\omega}_0)}, \qquad \tilde{u}_1^2 = \tilde{U} + \frac{1-q^2}{\mathrm{dn}^2(i\tilde{\omega}_1)}. \tag{8.2.8}$$

As is clear from (8.2.7), the type (iii) solution is related to the helical-wave solution of the CshG equation given in (8.1.17). The Virasoro constraints (8.1.5) and (8.1.6) impose constraints on $\tilde{a}$ and $\tilde{b}$ in (8.2.4):[2]

$$\tilde{a}^2 + \tilde{b}^2 = -q^2 - \tilde{U} - \frac{2(1-q^2)}{\mathrm{cn}^2(i\omega_0)} + 2\tilde{u}_1^2, \tag{8.2.9}$$

$$\tilde{a}\tilde{b} \;= i\, C^2 \left(\frac{\tilde{u}_0}{q^2} \frac{\mathrm{sn}(i\omega_0)\,\mathrm{dn}(i\omega_0)}{\mathrm{cn}^3(i\omega_0)} + \tilde{u}_1 \frac{\mathrm{sn}(i\omega_1)\,\mathrm{cn}(i\omega_1)}{\mathrm{dn}^3(i\omega_1)} \right). \tag{8.2.10}$$

[1] In contrast to the $\mathbb{R}_t \times S^3$ case, the RHS of (8.2.6) is not always real for arbitrary real values of $\tilde{\omega}_0$ and $\tilde{\omega}_1$. If $C^2 < 0$, we have to interchange η_0 and η_1 to obtain a solution properly normalized on AdS_3.

[2] Note that the Virasoro constraints require neither $a \geq b$ nor $a \leq b$. This means that both $\xi_1 = \exp\big(i\tilde{a}_0\tilde{T} + i\tilde{b}_0\tilde{X}\big)$ and $\exp\big(i\tilde{b}_0\tilde{T} + i\tilde{a}_0\tilde{X}\big)$ are consistent string solutions. It can be viewed as the $\tau \leftrightarrow \sigma$ transformation applied only to the $S^1 \subset S^5$ part while leaving the AdS_3 part intact.

The reality of $\tilde{a}$ and $\tilde{b}$ must also hold.

Since we are interested in closed string solutions, we should impose periodic boundary conditions. Let us define the period in the σ direction by

$$\Delta\sigma = \frac{2\mathbf{K}(k)\sqrt{1-v^2}}{\mu} = \frac{2q'\,\mathbf{K}(q)\sqrt{1-v^2}}{\mu} \equiv 2l \equiv \frac{2\pi}{n}\,, \tag{8.2.11}$$

which is equivalent to $\Delta\tilde{X} = 2\mathbf{K}(q)$ and $\Delta\tilde{T} = -2v\mathbf{K}(q)$. The closedness conditions for the AdS variables are written as

$$\Delta t = 2\mathbf{K}(q)\left\{-iZ_2(i\tilde{\omega}_0) - v\tilde{u}_0\right\} + 2n'_{\rm time}\pi \equiv \frac{2\pi N_t}{n}\,, \tag{8.2.12}$$

$$\Delta\phi_1 = 2\mathbf{K}(q)\left\{-iZ_3(i\tilde{\omega}_1) - v\tilde{u}_1\right\} + (2n'_1+1)\,\pi \equiv \frac{2\pi N_{\phi_1}}{n}\,. \tag{8.2.13}$$

And from the periodicity in φ_1 direction, we have

$$N_{\varphi_1} = \mu\,\frac{\tilde{b}-v\tilde{a}}{\sqrt{1-v^2}} \in \mathbb{Z}\,. \tag{8.2.14}$$

We must further require the timelike winding N_t to be zero. Just as in the $\mathbb{R}_{\rm t}\times {\rm S}^3$ case, one can adjust the value of v to fulfill this requirement.[3] The integer $n'_{\rm time}$ is evaluated as

$$2n'_{\rm time}\pi = \frac{1}{2i}\int_{-\mathbf{K}}^{\mathbf{K}} d\tilde{X}\,\frac{\partial}{\partial\tilde{X}}\left[\log\left(\frac{\Theta_0(\tilde{X}-i\tilde{\omega}_0)}{\Theta_0(\tilde{X}+i\tilde{\omega}_0)}\right)\right]. \tag{8.2.15}$$

Then, by solving the equation $N_t = 0$, one finds an appropriate value of $v = v_t$. The absolute value of the worldsheet boost parameter v_t may possibly exceed one (the speed of light). In such cases, we have to perform the 2D transformation $\tau \leftrightarrow \sigma$ on the AdS space to get $v_t \mapsto -1/v_t$.

As usual, conserved charges are defined by

$$E \equiv \frac{\sqrt{\lambda}}{\pi}\,\mathcal{E} = \frac{n\sqrt{\lambda}}{2\pi}\int_{-l}^{l} d\sigma\;{\rm Im}\left(\eta_0^*\,\partial_\tau\eta_0\right)\,, \tag{8.2.16}$$

$$S \equiv \frac{\sqrt{\lambda}}{\pi}\,\mathcal{S} = \frac{n\sqrt{\lambda}}{2\pi}\int_{-l}^{l} d\sigma\,{\rm Im}\left(\eta_1^*\,\partial_\tau\eta_1\right)\,, \tag{8.2.17}$$

$$J \equiv \frac{\sqrt{\lambda}}{\pi}\,\mathcal{J} = \frac{n\sqrt{\lambda}}{2\pi}\int_{-l}^{l} d\sigma\,{\rm Im}\left(\xi_1^*\,\partial_\tau\xi_1\right)\,. \tag{8.2.18}$$

which are evaluated as, for the current type (iii) case,

$$\mathcal{E} = \frac{nC^2\,\tilde{u}_0}{q^2(1-q^2)}\left[\mathbf{E} + (1-q^2)\left\{\frac{{\rm sn}^2(i\tilde{\omega}_0)}{{\rm cn}^2(i\tilde{\omega}_0)} - \frac{iv}{\tilde{u}_0}\,\frac{{\rm sn}(i\tilde{\omega}_0)\,{\rm dn}(i\tilde{\omega}_0)}{{\rm cn}^3(i\tilde{\omega}_0)}\right\}\mathbf{K}\right], \tag{8.2.19}$$

$$\mathcal{S} = \frac{nC^2\,\tilde{u}_1}{q^2(1-q^2)}\left[\mathbf{E} - (1-q^2)\left\{\frac{1}{{\rm dn}^2(i\tilde{\omega}_1)} - \frac{ivq^2}{\tilde{u}_1}\,\frac{{\rm sn}(i\tilde{\omega}_1)\,{\rm cn}(i\tilde{\omega}_1)}{{\rm dn}^3(i\tilde{\omega}_1)}\right\}\mathbf{K}\right], \tag{8.2.20}$$

$$\mathcal{J} = n\left(\tilde{a} - v\tilde{b}\right)\mathbf{K}\,. \tag{8.2.21}$$

[3] Note in $\mathbb{R}\times S^3$ case, the vanishing-N_t condition was trivially solved by $v = b/a$.

It is interesting to see some of the limiting behaviors of this type (*iii*) helical string in detail.[4]

• $\tilde{\omega}_{1,2} \to 0$ limit : folded strings on $\mathrm{AdS}_3 \times \mathrm{S}^1$

In the $\tilde{\omega}_{1,2} \to 0$ the timelike winding condition (8.2.12) requires $v = 0$, so the boosted worldsheet coordinates $(\tilde{T}, \tilde{X})$ become

$$(\tilde{T}, \tilde{X}) \to \left(\frac{\mu\tau}{q'} , \frac{\mu\sigma}{q'} \right) \equiv (\tilde{\mu}\tau, \tilde{\mu}\sigma) \equiv (\tilde{\tau}, \tilde{\sigma}) \, . \tag{8.2.22}$$

The periodicity condition (8.2.11) allows $\tilde{\mu}$ to take only a discrete set of values.

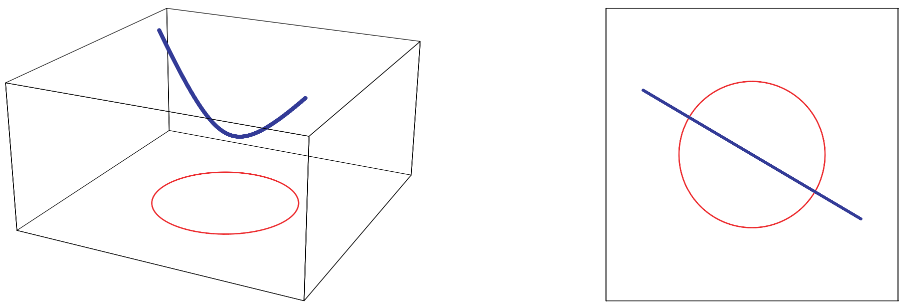


Figure 8.2: $\tilde{\omega}_{1,2} \to 0$ limit of type (*iii*) helical string becomes a folded string studied in [164].

The profile of type (*iii*) strings now reduces to

$$\eta_0 = \frac{1}{\mathrm{dn}(\tilde{\sigma}, q)} \, e^{i\tilde{u}_0 \tilde{\tau}} \, , \qquad \eta_1 = \frac{q \, \mathrm{sn}(\tilde{\sigma}, q)}{\mathrm{dn}(\tilde{\sigma}, q)} \, e^{i\tilde{u}_1 \tilde{\tau}} \, , \qquad \xi_1 = \exp\left(i\sqrt{\tilde{U} - q^2} \, \tilde{\tau} \right) \, , \tag{8.2.23}$$

where $\tilde{u}_0^2 = \tilde{U}$ and $\tilde{u}_1^2 = \tilde{U} + 1 - q^2$. This solution is equivalent to T-transformation of (J_1, J_2) folded strings of [38], namely, (S, J) folded strings.[5] The conserved charges of (8.2.23) are computed as

$$\mathcal{E} = \frac{n\tilde{u}_0}{1 - q^2} \, \mathbf{E}(q) \, , \quad \mathcal{S} = \frac{n\tilde{u}_1}{1 - q^2} \Big(\mathbf{E}(q) - (1 - q^2)\mathbf{K}(q) \Big) \, , \quad \mathcal{J} = n\sqrt{\tilde{U} - q^2} \, \mathbf{K}(q) \, . \tag{8.2.24}$$

Rewriting these expressions in terms of the original imaginary modulus k, we find the following relations among conserved charges :

$$\left(\frac{\mathcal{J}}{\mathbf{K}(k)} \right)^2 - \left(\frac{\mathcal{E}}{\mathbf{E}(k)} \right)^2 = n^2 k^2 \, , \qquad \left(\frac{\mathcal{S}}{\mathbf{K}(k) - \mathbf{E}(k)} \right)^2 - \left(\frac{\mathcal{J}}{\mathbf{K}(k)} \right)^2 = n^2 (1 - k^2) \, , \tag{8.2.25}$$

as obtained in [45].

[4] It seems the original "spiky string" solution of [166] is also contained in the type (*iii*) class, although we have not been able to reproduce it analytically.

[5] Note the set, $\eta_{0,1} =$ the same as (8.2.23) and $\xi_1 = \exp[i\sqrt{\tilde{U} - q^2} \, \tilde{\sigma}]$, also gives a solution.

• $q \to 1$ limit : logarithmic behavior

Another interesting limit is to send the elliptic modulus q to unity. In this limit, the spikes of the type (iii) string attach to the AdS boundary, and the energy E and AdS spin S become divergent. Again, the condition of vanishing timelike winding is fulfilled by $v = 0$, and the periodicity condition (8.2.11) implies that $\tilde{\mu}$ given in (8.2.22) goes to infinity. The profile becomes

$$\eta_0 = C \cosh(\tilde{\sigma} - i\tilde{\omega}_0)\, e^{i\tilde{u}_0\tilde{\tau}} \,, \quad \eta_1 = C \sinh(\tilde{\sigma} - i\tilde{\omega}_1)\, e^{i\tilde{u}_1\tilde{\tau}} \,, \quad \xi_1 = \exp\Big(i\tilde{a}\tilde{\tau} + i\tilde{b}\tilde{\sigma} \Big) \,, \quad (8.2.26)$$

where

$$C = \left(\cos^2\tilde{\omega}_1 - \sin^2\tilde{\omega}_0\right)^{-1/2} \,, \quad \tilde{u}_0^2 = \tilde{u}_1^2 = \tilde{U} \,. \quad (8.2.27)$$

The constants $\tilde{a}$ and $\tilde{b}$ satisfy the constraints

$$\tilde{a}^2 + \tilde{b}^2 = -1 + \tilde{U} \,, \qquad \tilde{a}\tilde{b} = C^2 \left(\tilde{u}_0 \sin\tilde{\omega}_0 \cos\tilde{\omega}_0 + \tilde{u}_1 \sin\tilde{\omega}_1 \cos\tilde{\omega}_1 \right) \,. \quad (8.2.28)$$

The conserved charges are computed as

$$\mathcal{E} = nC^2\tilde{u}_0 \Big(\Lambda - \sin^2\tilde{\omega}_0\, \mathbf{K}(1) \Big) \,, \quad \mathcal{S} = nC^2\tilde{u}_1 \Big(\Lambda - \cos^2\tilde{\omega}_1\, \mathbf{K}(1) \Big) \,, \quad \mathcal{J} = n\tilde{a}\, \mathbf{K}(1) \,, \quad (8.2.29)$$

where we defined a cut-off $\Lambda \equiv 1/(1-q^2)$.

Let us pay special attention to the $\tilde{u}_0 = \tilde{u}_1 = \sqrt{\tilde{U}}$ case. For this case the energy-spin relation reads

$$\mathcal{E} - \mathcal{S} = n\sqrt{\tilde{U}}\, \mathbf{K}(1) \,. \quad (8.2.30)$$

Obviously the RHS is divergent, and careful examination reveals it is logarithmic in $\mathcal{S}$. This can be seen by first noticing, on one hand, that the complete elliptic integral of the first kind $\mathbf{K}(q) \equiv \mathbf{K}(e^{-r})$ has asymptotic behavior

$$\mathbf{K}(e^{-r}) = -\frac{1}{2} \ln\left(\frac{r}{8}\right) + \mathcal{O}(r\ln r) \,, \quad (8.2.31)$$

while on the other, the degree of divergence for Λ is

$$\Lambda = \frac{1}{1-q^2} = \frac{1}{1-e^{-2r}} \sim \frac{1}{2r} \,, \qquad (\text{as } r \to 0) \,. \quad (8.2.32)$$

Since the most divergent part of $\mathcal{S}$ is governed by Λ rather than $\mathbf{K}(1)$, it follows that

$$\mathbf{K}(e^{-r}) \sim \mathbf{K}(1-r) \sim -\frac{1}{2} \ln\left(\frac{nC^2\tilde{u}_1}{16\mathcal{S}}\right) \,, \qquad (\text{as } r \to 0) \,, \quad (8.2.33)$$

at the leading order. Then it follows that

$$\mathcal{E} - \mathcal{S} \sim -\frac{n\sqrt{\tilde{U}}}{2} \ln\left(\frac{16\mathcal{S}}{nC^2\tilde{u}_1}\right) \,, \qquad (\text{as } r \to 0) \,, \quad (8.2.34)$$

as promised.

Let us consider the particular case $\tilde{U}=1$, which is equivalent to $\tilde{a}=\tilde{b}=0$ and $\tilde{\omega}_0=-\tilde{\omega}_1$. The above dispersion relation (8.2.34) now reduces to

$$E-S\sim\frac{n\sqrt{\lambda}}{2\pi}\,\ln S\,, \tag{8.2.35}$$

omitting the finite part. This result was first obtained in [35] for the $n=2$ case, and generalised to generic n case in [166].

One can also reproduce the double logarithm behavior of [164] (see also [45, 167, 168, 169]). To see this, let us set $\tilde{b}=0$ and $\tilde{a}=\sqrt{\tilde{U}-1}$, and rewrite the relation (8.2.30) as

$$\mathcal{E}-\mathcal{S}=\sqrt{\mathcal{J}^2+n^2\,\mathbf{K}(1)^2}\sim\left[\mathcal{J}^2+\frac{n^2}{4}\ln^2\left(\frac{2\mathcal{S}}{nC^2\sqrt{\tilde{U}}}\right)\right]^{1/2}. \tag{8.2.36}$$

There are two limits of special interest. The "slow long string" limit of [168], is reached by $\sqrt{U}\ll\lambda$, so that in the strong coupling regime $\lambda\gg1$ the RHS of (8.2.36) becomes

$$\mathcal{E}-\mathcal{S}\sim\sqrt{\mathcal{J}^2+\frac{n^2}{4}\ln^2\mathcal{S}}\,. \tag{8.2.37}$$

Similarly, the "fast long string" of [168] is obtained by taking $\sqrt{U}\sim\lambda\gg1$, resulting in

$$\mathcal{E}-\mathcal{S}\sim\left[\mathcal{J}^2+\frac{n^2}{4}\left(\ln\left(\frac{\mathcal{S}}{\mathcal{J}}\right)+\ln\left(\ln r\right)\right)^2\right]^{1/2}\sim\sqrt{\mathcal{J}^2+\frac{n^2}{4}\ln^2\left(\frac{\mathcal{S}}{\mathcal{J}}\right)}\,, \tag{8.2.38}$$

where we neglected a term $\ln(\ln r)$ which is relatively less divergent in the limit $r\to0$.

8.2.2 Type (iv) helical strings

Let us finally present another AdS helical solution which incorporates the $sl(2)$ "(dyonic) giant magnon" of [53, 165]. This solution, which we call the type (iv) string, is obtained by applying a shift $X\to X+i\mathbf{K}'(k)$ to the type (i) helical string. Its profile is given by

$$\eta_0=\frac{C}{\sqrt{k}}\,\frac{\Theta_0(0)\,\Theta_0(X-i\omega_0)}{\Theta_0(i\omega_0)\,\Theta_1(X)}\,\exp\Big(Z_0(i\omega_0)X+iu_0T\Big)\,, \tag{8.2.39}$$

$$\eta_1=\frac{C}{\sqrt{k}}\,\frac{\Theta_0(0)\,\Theta_3(X-i\omega_1)}{\Theta_2(i\omega_1)\,\Theta_1(X)}\,\exp\Big(Z_3(i\omega_1)X+iu_1T\Big)\,, \tag{8.2.40}$$

$$\xi_1=\exp\left(iaT+ibX\right)\,. \tag{8.2.41}$$

We omit displaying all the constraints among the parameters (they can be obtained in a similar manner as in the type (i) case). The type (iv) solution corresponds to the helical-wave solution given in (8.1.18), and satisfy the string equations of motion of the form (8.1.19). [6]

[6] This can be easily checked by using a relation $1/k^2\,\mathrm{sn}^2(x,k)=\,\mathrm{sn}^2\left(x+i\mathbf{K}'(k),k\right)$.

• $k \to 1$ limit : $sl(2)$ "dyonic giant magnon"

The $sl(2)$ "dyonic giant magnon" is reproduced in the limit $k \to 1$, as

$$\eta_0 = \frac{\cosh(X - i\omega_0)}{\sinh X}\, e^{i(\tan\omega_0)X + iu_0 T}\,, \quad \eta_1 = \frac{\cos\omega_0}{\sinh X}\, e^{iu_1 T}\,, \quad \xi_1 = e^{i\hat{a}T + i\hat{b}X}\,, \tag{8.2.42}$$

where

$$u_0^2 = u_1^2 + \frac{1}{\cos^2\omega_0}\,, \qquad (\hat{a}, \hat{b}) = (u_1, \tan\omega_0) \ \text{ or } \ (\tan\omega_0, u_1)\,. \tag{8.2.43}$$

Due to the non-compactness of AdS space, the conserved charges are divergent. This is a UV divergence, and we regularize it by the following prescription. First change the integration range for the charges (see (8.2.16) - (8.2.18)) from $\int_0^{2l} d\sigma$ to $\int_\epsilon^{2l-\epsilon} d\sigma$, with $\epsilon > 0$, to obtain

$$\mathcal{E} = u_0 \cos^2\omega_0 \left(\epsilon^{-1} - 1\right) + \mathbf{K}(1)(u_0 - v\tan\omega_0)\,, \tag{8.2.44}$$

$$\mathcal{S} = u_1 \cos^2\omega_0 \left(\epsilon^{-1} - 1\right)\,, \tag{8.2.45}$$

$$\mathcal{J} = \mathbf{K}(1)(u_0 - v\tan\omega_0)\,, \tag{8.2.46}$$

then drop the terms proportional to ϵ^{-1} by hand. This prescription yields a regularized energy and an S^5 spin which are still IR divergent due to the non-compactness of the worldsheet. However, their difference becomes finite, leading to the energy-spin relation

$$(\mathcal{E} - \mathcal{J})_{\rm reg} = -\sqrt{(\mathcal{S})^2_{\rm reg} + \cos^2\omega_0}\,. \tag{8.2.47}$$

Note that in view of the AdS/CFT correspondence, $\mathcal{E} - \mathcal{J}$ must be positive, which in turn implies $(\mathcal{E} - \mathcal{J})_{\rm reg}$ is negative.

Let us take $v = \tan\omega_0/u_0$ in (8.2.42), and consider a rotating frame $\eta_0^{\rm new} = e^{-i\tilde{\tau}}\eta_0 \equiv \tilde{Y}_0 + i\tilde{Y}_5$. We then find $\tilde{Y}_5 = -i\sin\omega_0$ is independent of $\tilde{\tau}$ and $\tilde{\sigma}$, showing that the "shadow" of the $sl(2)$ "dyonic giant magnon" projected onto the $\tilde{Y}_0$-$\tilde{Y}_5$ plane is just given by two semi-infinite straight lines on the same line. Namely, the shadow is obtained by removing a finite segment from an infinitely long line, where the two endpoints of the segment are on the unit circle $|\eta_0| = 1$ with angular difference $\Delta t = \pi - 2\omega_0$. Figure 8.3 shows the snapshot of the $sl(2)$ "dyonic giant magnon", projected onto the plane spanned by $(\operatorname{Re}\eta_0, \operatorname{Im}\eta_0, |\eta_1|)$.

It is interesting to compare this situation with the usual giant magnon on $\mathbb{R}\times S^3$. In the sphere case, the "shadow" of the giant magnon is just a straight line segment connecting two endpoints on the equatorial circle $|\xi_1| = 1$. So the "shadows" of $su(2)$ and $sl(2)$ giant magnons are just complementary. Using this picture of "shadows on the LLM plane", one can further discuss the "scattering" of two $sl(2)$ "(dyonic) giant magnons" in the similar manner as in the $su(2)$ case.[7]

These "shadow" pictures remind us of the corresponding finite-gap representations of both solutions, resulting from the $su(2)$ and $sl(2)$ spin-chain analyses. While in the $su(2)$ case, a

[7] Scattering $sl(2)$ (dyonic) giant magnon solutions can be constructed from the scattering $su(2)$ (dyonic) giant magnon solutions $\xi_i(u_1, u_2; v_1, v_2)$ [54] by performing $(u_1, u_2) \mapsto (u_1 + i\pi/2, u_2 + i\pi/2)$.

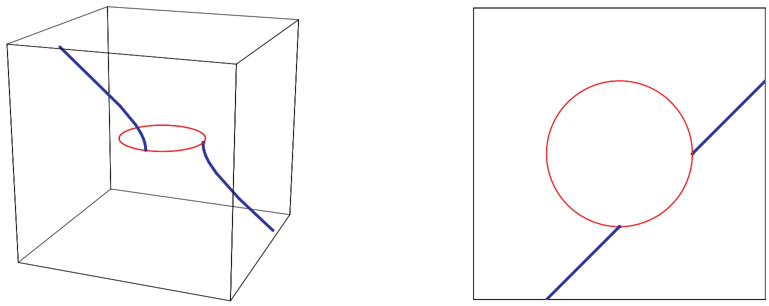


Figure 8.3: $k \to 1$ limit of type (iv) helical string ($\omega_0 = 0.785\,,\ u_0 = 1.41\,,\ u_1 = 0$) : "giant magnon" solution in AdS space.

condensate cut, or a Bethe string, has finite length in the imaginary direction of the complex spectral parameter plane, for the $sl(2)$ case, they are given by two semi-infinite lines in the same imaginary direction [53]. This complementary feature reflects the structural symmetry between the BDS parts of S-matrices, $S_{su(2)} = S_{sl(2)}^{-1}$.

These "shadow" pictures also show up in matrix model context [170, 171, 172, 173]. In a reduced matrix quantum mechanics setup obtained from $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$, a "string-bit" connecting eigenvalues of background matrices forming $\frac{1}{2}$-BPS circular droplet can be viewed as the shadow of the corresponding string. For the $su(2)$ sector, it is true even for the boundstate (bound "string-bits") case [172]. It would be interesting to investigate the $sl(2)$ case along similar lines of thoughts.

Chapter 9

Finite-size effects for dyonic giant magnons

We compute finite-size corrections to dyonic giant magnons in two ways. One is to examine the asymptotic behavior of helical spinning strings as elliptic modulus k goes to unity, and the other is to apply generalized Lüscher formula of [79] to the case in which incoming particles are boundstates. We find agreement of the two results in special cases, confirming the validity of generalized Lüscher formula, which captures the leading finite-size correction to the energy solely from the infinite-size information for general dispersion relation.

9.1 Overview

There are two types of finite-size corrections which are well studied in the context of AdS/CFT correspondence: one is $1/J$-type and the other is e^{-J}-type.

Recall in the BMN scaling limit we keep $\tilde{\lambda} = \lambda/J^2$ fixed and small, so $1/J$ correction can be regarded, via $1/J \sim 1/\sqrt{\lambda}$, as one-loop quantum corrections to classical strings. In the dual Bethe Ansatz framework, we take thermodynamic limit where the number of Bethe roots are of order L. Thus the finite-size correction is regarded as fluctuation of a few number of Bethe roots. For literature on an interplay between finite size effects in Bethe Ansatz equations and one-loop corrections, see [169, 174].

The exponential-type correction appears when we consider finite-J_1 extension of (dyonic) giant magnons, where J_1 is the angular momentum along a great circle of S^5. The finite-J_1 extension of giant magnons is constructed in [52, 73], where they find the energy-spin relation receives correction of the form e^{-cJ_1}, with c a constant. The exponential correction e^{-cJ} also shows up in the one-loop computation of string theory, for the case of $su(2)$ sector [175] as well as of $sl(2)$ sector [72]. In [72], they further discovered that quantum string Bethe Ansatz cannot reproduce such terms.

It is argued in [74] that the exponential finite-size correction at strong coupling is related to the wrapping interaction at weak coupling, based on Thermodynamic Bethe Ansatz approach [75, 176, 177] and the Lüscher formula [76, 77, 78]. Recently, Janik and Łukowski have elaborated this argument [79], assuming that Lüscher's argument can be applied to the non-relativistic dispersion relation

$$\varepsilon(p) = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)} \,. \tag{9.1.1}$$

Their "generalized Lüscher formula" computes finite-J_1 correction to the energy-spin relation of giant magnons from the S-matrix and the dispersion relation (9.1.1) of infinite-J_1 system. Since we know the conjectured S-matrix and dispersion relation of infinite-size system, the generalized Lüscher formula will in principle give the finite-size correction valid at arbitrary values of λ. However, just like the original Lüscher formula, it is only sensitive to the leading part of corrections exponentially suppressed in L (or J_1), that is the first term in the following expansion:

$$\delta\varepsilon(p) = \alpha(p, \lambda, L)\, e^{-c(p,\lambda)L} + \mathcal{O}\left(e^{-c'(p,\lambda)L}\right) \qquad \text{with} \quad c'(p,\lambda) > c(p,\lambda), \tag{9.1.2}$$

where $\alpha(p, \lambda, L)$ contains no factor exponentially dependent on L. According to the (generalized) Lüscher formula, the leading finite-size correction arises from exchanging virtual particles going around the worldsheet cylinder once, and is written as

$$\delta\varepsilon(p) = \delta\varepsilon^{\mu}(p) + \delta\varepsilon^{F}(p) \,. \tag{9.1.3}$$

The first term is called μ-term and the second one is called F-term, which have different diagrammatic interpretation as shown in Figure 9.1.

Janik and Łukowski computed the μ-term of their generalized formula and found, after taking contributions from the BHL/BES dressing phase [16, 21] into account, that

$$\alpha(p, \lambda, L)\, e^{-cL}\Big|_{\mu-\text{term}} \approx -\frac{4\sqrt{\lambda}}{\pi} \sin^3\left(\frac{p}{2}\right) \exp\left[-\frac{2\pi L}{\sqrt{\lambda}\sin\left(\frac{p}{2}\right)} - 2\right] \quad (\text{as } \lambda, L \to \infty)\,, \tag{9.1.4}$$

which correctly reproduces the leading finite-J_1 correction to the dispersion relation of giant magnons in conformal gauge, with $L = J_1$ [52, 73].[1]

In this chapter, we extend their analysis and study the leading finite-size correction to magnon boundstates and dyonic giant magnons. Firstly, we analyze the asymptotic behavior of helical strings of [96] in the limit when they nearly reduce to an array of dyonic giant magnons, and determined the leading finite-J_1 correction to the energy-spin relation. Secondly, we apply the generalized Lüscher formula for μ-term to the situation in which the incoming particle is magnon boundstate.

[1] What corresponds to the F-term in string theory, is not discussed in [79]. Indeed, the exponential part of F-term seems to be different from that of μ-term, so we do not discuss F-term in the main text.

Figure 9.1: Diagrams for the leading finite-size corrections. The left is called μ-term, and the right F-term. a is an incoming physical particle, and b, c are virtual (but on-shell) particles.

Since the generalized Lüscher formula of Janik and Łukowski is applicable only to incoming elementary magnons, we slightly generalize their argument, assuming there exists an effective field theory such that it reproduces the non-relativistic dispersion

$$\varepsilon_Q(p) = \sqrt{Q^2 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)}\,, \tag{9.1.5}$$

and the S-matrix which is given by the product of the conjectured two-body S-matrices. Our results serve as a consistency check between the generalized Lüscher formula and the results from string theory. It is desirable if one can give further justification of these formulae from other methods of computing the finite-size corrections.

Also we would like to stress that evaluation of the formula is not straightforward. Evaluation of the μ-term requires information of residue at the poles that are located at the nearest from the real axis. Thus, to compute the μ-term correctly, we have to determine which poles of the $su(2|2)^2$ S-matrix are relevant.

Singularity structure of the $su(2|2)^2$ S-matrix with the BHL/BES dressing phase has been studied in [80, 81]. Particularly in [81], they discussed where in the spectral parameter torus one can find the singularity of magnon S-matrix corresponding to exchanges of physical particle. In [81] they determined the location of simple and double poles when incoming particles are elementary magnons. This result is extended in [82] to the case where incoming particles are magnon boundstates.

To pick up the relevant poles for μ-term, we use heuristic reasoning based on the arguments similar to [81, 82]. It should be noticed that the generalized Lüscher formula is sensitive to residue at the (simple) poles, while kinematical (or diagrammatic) arguments of [81, 82] probed only the location of poles.

9.2 Finite-J correction to dyonic giant magnons

Remember that dyonic giant magnon is a classical string solution on $\mathbb{R}_t \times S^3$ obeying the square-root type energy-spin relation:

$$E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \sin^2 \left(\frac{p_1}{2} \right)} \,, \qquad E, \ J_1 \to \infty \,. \tag{9.2.1}$$

Two-spin helical spinning strings can be regarded as finite-J_1 generalization of this solution. Thus, the finite-J_1 correction to the above energy-spin relation can be computed quite straightforwardly.

9.2.1 Dyonic giant magnons

We begin with the review on $J_1 = \infty$ case: the dyonic giant magnons. Dyonic giant magnons can be obtained by taking k, the elliptic modulus of helical string, to unity.

As shown in (6.4.7), the conserved charges for one-hop (a single dyonic giant magnon) are given by

$$\mathcal{E} = u_1 \left(1 - \frac{\tan^2 \omega_1}{u_1^2} \right) \mathbf{K}(1) \,, \quad \mathcal{J}_1 = u_1 \left[\left(1 - \frac{\tan^2 \omega_1}{u_1^2} \right) \mathbf{K}(1) - \cos^2 \omega_1 \right] , \quad \mathcal{J}_2 = u_2 \cos^2 \omega_1 \,, \tag{9.2.2}$$

where $\mathbf{K}(1)$ is a divergent constant. Then, the relation (9.2.1) follows by setting $\Delta\varphi_1 \equiv p_1$.

One can estimate exponential part of the finite-J_1 corrections to the leading order, only from the above information. This is because the correction term is of order $(k')^2$, while k' can also be expressed by the angular momenta.

Let us first relate k' with the complete elliptic integral of the first kind $\mathbf{K}(k)$. As shown in Appendix A.4.2, $\mathbf{K}(k)$ has the asymptotic form

$$\mathbf{K}(k) = \ln \left(\frac{4}{k'} \right) + \mathcal{O}\left(k'^2 \ln k' \right) , \qquad (\text{as } k \to 1) \,. \tag{9.2.3}$$

Inverting this relation, we obtain $k' = 4 \exp\left[-\mathbf{K}(1) \right]$. We express a divergent constant $\mathbf{K}(1)$ by angular momenta $\mathcal{J}_1$ and $\mathcal{J}_2$. The expressions (9.2.2) tell us

$$\mathbf{K}(1) = \frac{1}{1 - \frac{\tan^2 \omega_1}{u_1^2}} \left(\frac{\mathcal{J}_1}{u_1} + \cos^2 \omega_1 \right) , \qquad \text{where} \quad u_1 = \frac{\sqrt{\mathcal{J}_2^2 + \cos^2 \omega_1}}{\cos^2 \omega_1} \,. \tag{9.2.4}$$

Eliminating u_1 from the first equation, we get

$$\begin{aligned} \mathbf{K}(1) &= \frac{\mathcal{J}_2^2 + \cos^2 \omega_1}{\mathcal{J}_2^2 + \cos^4 \omega_1} \left(\frac{\mathcal{J}_1 \cos^2 \omega_1}{\sqrt{\mathcal{J}_2^2 + \cos^2 \omega_1}} + \cos^2 \omega_1 \right) , \\ &\approx \frac{\mathcal{J}_2^2 + \sin^2 \frac{p_1}{2}}{\mathcal{J}_2^2 + \sin^4 \frac{p_1}{2}} \left(\frac{\mathcal{J}_1 \sin^2 \frac{p_1}{2}}{\sqrt{\mathcal{J}_2^2 + \sin^2 \frac{p_1}{2}}} + \sin^2 \frac{p_1}{2} \right) , \end{aligned} \tag{9.2.5}$$

where we neglected higher-order corrections to the relation $2\omega_1 = \pi - p_1 + \mathcal{O}\left(k'^2\right)$ in the second line.

If we take the limit $\mathcal{J}_2 \to 0$ within this expression, we get

$$\mathbf{K}(1) \to \frac{\mathcal{J}_1}{\cos\omega_1} + 1 \approx \frac{\mathcal{J}_1}{\sin\frac{p_1}{2}} + 1\,, \tag{9.2.6}$$

which is the single-spin result.

9.2.2 Helical strings with two spins near $k = 1$

For general value of k, helical strings have two finite angular momenta $J_1\,, J_2$ and two finite winding numbers $N_1\,, N_2$. Correspondingly, there are four controllable parameters $(k, U, \omega_1\,, \omega_2)$. Other parameters which appear in the profile of helical strings can be expressed as functions of those four parameters. Below, we are going to investigate the precise form of these functions when k is near 1, and determine finite-J_1 correction to the energy-spin relation of dyonic giant magnons.

We collect the results of Section 6.3 again for convenience. The profile of type (i) helical string is shown in Figure 9.2, and takes the form:

$$\eta_0 = aT + bX\,, \tag{9.2.7}$$

$$\xi_1 = C\frac{\Theta_0(0)}{\sqrt{k}\,\Theta_0(i\omega_1)}\frac{\Theta_1(X - i\omega_1)}{\Theta_0(X)}\,\exp\Big(Z_0(i\omega_1)X + iu_1 T\Big)\,, \tag{9.2.8}$$

$$\xi_2 = C\frac{\Theta_0(0)}{\sqrt{k}\,\Theta_2(i\omega_2)}\frac{\Theta_3(X - i\omega_2)}{\Theta_0(X)}\,\exp\Big(Z_2(i\omega_2)X + iu_2 T\Big)\,. \tag{9.2.9}$$

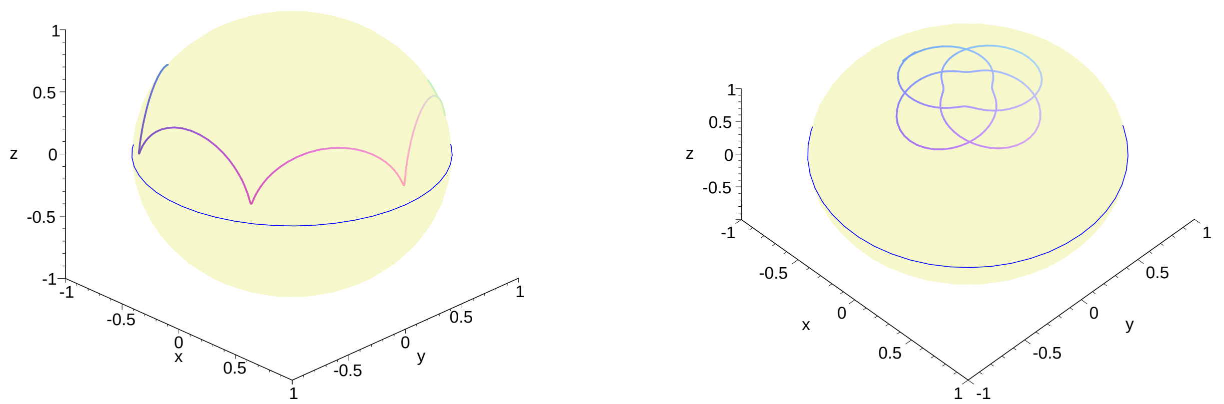


Figure 9.2: Left: Type (i) helical spinning string solution with two spins, where the (x, y, z) axes show $(\mathrm{Re}\,\xi_1\,, \mathrm{Im}\,\xi_1\,, |\xi_2|)$. Right: The same string solution with $(x, y, z) = (\mathrm{Re}\,\xi_2\,, \mathrm{Im}\,\xi_2\,, |\xi_1|)$.

The normalization constant C is given by

$$C = \left(\frac{\mathrm{dn}^2(i\omega_2)}{k^2\,\mathrm{cn}^2(i\omega_2)} - \mathrm{sn}^2(i\omega_1)\right)^{-1/2}. \tag{9.2.10}$$

Two angular velocities are constrained as

$$u_1^2 = U + \mathrm{dn}^2(i\omega_1)\,, \qquad u_2^2 = U - \frac{(1-k^2)\,\mathrm{sn}^2(i\omega_2)}{\mathrm{cn}^2(i\omega_2)}\,. \tag{9.2.11}$$

Then, the parameters a and b are fixed by Virasoro conditions, and given by

$$a^2 + b^2 = k^2 - 2k^2\,\mathrm{sn}^2(i\omega_1) - U + 2u_2^2\,, \tag{9.2.12}$$

$$ab = -i\,C^2 \left(u_1\,\mathrm{sn}(i\omega_1)\,\mathrm{cn}(i\omega_1)\,\mathrm{dn}(i\omega_1) - u_2\,\frac{1-k^2}{k^2}\,\frac{\mathrm{sn}(i\omega_2)\,\mathrm{dn}(i\omega_2)}{\mathrm{cn}^3(i\omega_2)} \right)\,. \tag{9.2.13}$$

The velocity v is chosen so that $v \equiv b/a \le 1$.

All quantities given above can be expanded in powers of $k' \equiv \sqrt{1-k^2}\,$. Let us see the leading k' corrections by turns. The angular velocities become

$$u_1 = \frac{\sqrt{U\cos^2\omega_1 + 1}}{\cos\omega_1} - \frac{k'^2}{4}\,\frac{\sin\omega_1\,(\omega_1 + \sin\omega_1\cos\omega_1)}{\cos^2\omega_1\sqrt{U\cos^2\omega_1+1}} + \mathcal{O}(k'^4), \tag{9.2.14}$$

$$u_2 = \sqrt{U} + \frac{k'^2}{2}\,\frac{\sin^2\omega_2}{\sqrt{U}} + \mathcal{O}(k'^4). \tag{9.2.15}$$

The normalization constant is

$$C = \cos(\omega_1) + \frac{k'^2}{4}\Big\{ \big(1 - 2\,\cos^2\omega_2\big)\cos^3\omega_1 - \cos\omega_1 + \omega_1\sin\omega_1 \Big\} + \mathcal{O}(k'^4). \tag{9.2.16}$$

The parameters a, b and $v = b/a$ become, at the next-to-leading order,

$$a \approx \frac{\sqrt{U + \cos^2\omega_1}}{\cos\omega} + k'^2\,a^{(2)}\,, \quad b \approx \tan\omega_1 + k'^2\,b^{(2)}\,, \quad v \approx \frac{\sin\omega_1}{\sqrt{U+\cos^2\omega_1}} + k'^2\,v^{(2)}\,, \tag{9.2.17}$$

where the exact expressions of $a^{(2)}\,, b^{(2)}$ and $v^{(2)}$ are shown in Appendix A.5.

From (6.3.12)-(6.3.14), the conditions for the type (i) helical string to be closed read,

$$\Delta\sigma\Big|_{\text{one-hop}} \equiv \frac{2\pi}{n} = \frac{2\mathbf{K}(k)\sqrt{1-v^2}}{\mu}\,, \tag{9.2.18}$$

$$\Delta\varphi_1\Big|_{\text{one-hop}} \equiv \frac{2\pi N_1}{n} = 2\mathbf{K}(k)\,(-iZ_0(i\omega_1) - v\,u_1) + (2n'_1 + 1)\pi\,, \tag{9.2.19}$$

$$\Delta\varphi_2\Big|_{\text{one-hop}} \equiv \frac{2\pi N_2}{n} = 2\mathbf{K}(k)\,(-iZ_2(i\omega_2) - v\,u_2) + 2n'_2\pi\,. \tag{9.2.20}$$

The finite J_1 effects on the periodicity conditions can be evaluated in a similar manner. Let $p_{1,2} \equiv \Delta\varphi_{1,2}$, then the equations (9.2.19) and (9.2.20) are rewritten as, at the next-to-leading order,

$$p_1 \equiv \pi - 2\,\omega_1 + \frac{k'^2}{2}\,p_1^{(2)} + \mathcal{O}\left(k'^4\right)\,, \tag{9.2.21}$$

$$p_2 \equiv -\frac{2\,\ell_k\sin\omega_1\sqrt{U}}{\sqrt{U\cos^2\omega_1+1}} - 2\,\omega_2 + \frac{k'^2}{2}\,p_2^{(2)} + \mathcal{O}\left(k'^4\right)\,. \tag{9.2.22}$$

where $\ell_k \equiv \ln(4/k')$ and $p_{1,2}^{(2)}$ are given in Appendix A.5. By inverting the relation (9.2.21), one can express ω_1 in terms of p_1. For instance, we obtain

$$\cos^2\omega_1 \equiv \sin^2\left(\frac{p_1}{2}\right) - \frac{k'^2}{2}\,\sin\left(\frac{p_1}{2}\right)\cos\left(\frac{p_1}{2}\right)\,W_1^{(2)} + \mathcal{O}\left(k'^4\right), \tag{9.2.23}$$

where $W_1^{(2)}$ can be obtained by formally substituting $\omega_1 = (\pi - p_1)/2$ into the expression of $p_1^{(2)}$, at this order of validity. However, since p_2 is generally divergent as $k \to 1$, we cannot invert the relation (9.2.22). We will return to this issue in Section 9.2.3.

The rescaled energy $\mathcal{E}$ and the spins $\mathcal{J}_j$ $(j = 1, 2)$ were evaluated in (6.3.15)-(6.3.17). There we can find

$$\mathcal{E} = n a\left(1 - v^2\right)\mathbf{K}(k), \tag{9.2.24}$$

$$\mathcal{J}_1 = \frac{nC^2 u_1}{k^2}\left[-\mathbf{E}(k) + \left(\mathrm{dn}^2(i\omega_1) + \frac{v k^2}{u_1} i\,\mathrm{sn}(i\omega_1)\,\mathrm{cn}(i\omega_1)\,\mathrm{dn}(i\omega_1)\right)\mathbf{K}(k)\right], \tag{9.2.25}$$

$$\mathcal{J}_2 = \frac{nC^2 u_2}{k^2}\left[\mathbf{E}(k) + (1-k^2)\left(\frac{\mathrm{sn}^2(i\omega_2)}{\mathrm{cn}^2(i\omega_2)} - \frac{v}{u_2}\frac{i\,\mathrm{sn}(i\omega_2)\,\mathrm{dn}(i\omega_2)}{\mathrm{cn}^3(i\omega_2)}\right)\mathbf{K}(k)\right]. \tag{9.2.26}$$

We may set $n = 1$, since a single dyonic giant magnon corresponds to this case. By expanding the conserved charges in $\ell_k = \ln(4/k')$ and k', we obtain

$$\mathcal{E} = \frac{\ell_k\,(U+1)\cos\omega_1}{\sqrt{U\cos^2\omega_1 + 1}} + \frac{k'^2}{4}\,\mathcal{E}^{(2)} + \mathcal{O}\left(k'^4\right), \tag{9.2.27}$$

$$\mathcal{J}_1 = \frac{\ell_k\,(U+1)\cos\omega_1}{\sqrt{U\cos^2\omega_1 + 1}} - \sqrt{U\cos^2\omega_1 + 1}\,\cos\omega_1 + \frac{k'^2}{4}\,\mathcal{J}_1^{(2)} + \mathcal{O}\left(k'^4\right), \tag{9.2.28}$$

$$\mathcal{J}_2 = \sqrt{U}\cos^2\omega_1 + \frac{k'^2}{4}\,\mathcal{J}_2^{(2)} + \mathcal{O}\left(k'^4\right), \tag{9.2.29}$$

where $\mathcal{E}^{(2)}, \mathcal{J}_1^{(2)}$ and $\mathcal{J}_2^{(2)}$ are functions of ω_1, ω_2, U and ℓ_k. We want to rewrite (9.2.27)-(9.2.29) in terms of $p_1 = \Delta\varphi_1$, because this parameter has a clearer physical meaning than ω_1. By using (9.2.23), we obtain

$$\mathcal{E} = \frac{\ell_k\,(U+1)\sin\left(\frac{p_1}{2}\right)}{\sqrt{U\sin^2\left(\frac{p_1}{2}\right) + 1}} + \frac{k'^2}{4}\,\mathcal{E}^{(2')} + \mathcal{O}\left(k'^4\right), \tag{9.2.30}$$

$$\mathcal{J}_1 = \frac{\ell_k\,(U+1)\sin\left(\frac{p_1}{2}\right)}{\sqrt{U\sin^2\left(\frac{p_1}{2}\right) + 1}} - \sqrt{U\sin^2\left(\frac{p_1}{2}\right) + 1}\,\sin\left(\frac{p_1}{2}\right) + \frac{k'^2}{4}\,\mathcal{J}_1^{(2')} + \mathcal{O}\left(k'^4\right), \tag{9.2.31}$$

$$\mathcal{J}_2 = \sqrt{U}\,\sin^2\left(\frac{p_1}{2}\right) + \frac{k'^2}{4}\,\mathcal{J}_2^{(2')} + \mathcal{O}\left(k'^4\right). \tag{9.2.32}$$

It follows that

$$\mathcal{E} - \mathcal{J}_1 \approx \sqrt{\mathcal{J}_2^2 + \sin^2\left(\frac{p_1}{2}\right)} + \frac{k'^2}{4}\left(\mathcal{E}^{(2')} - \mathcal{J}_1^{(2')} - \frac{\sqrt{U}\sin\left(\frac{p_1}{2}\right)}{\sqrt{U\sin^2\left(\frac{p_1}{2}\right) + 1}}\,\mathcal{J}_2^{(2')}\right). \tag{9.2.33}$$

where we assumed $\sin(p_1/2) > 0$.

The precise form of the next-to-leading terms appearing in (9.2.33) is computed in Appendix A.5. With those expressions, we finally obtain a quite simple result

$$\mathcal{E}^{(2')} - \mathcal{J}_1^{(2')} - \frac{\sqrt{U}\sin\left(\frac{p_1}{2}\right)}{\sqrt{U\sin^2\left(\frac{p_1}{2}\right)+1}}\,\mathcal{J}_2^{(2')} \;\approx\; \sin^3\left(\frac{p_1}{2}\right)\frac{\left(1-2\cos^2\omega_2\right)}{\sqrt{U\sin^2\left(\frac{p_1}{2}\right)+1}}\,, \tag{9.2.34}$$

At this order of validity, it can also be reexpressed as

$$\mathcal{E}^{(2')} - \mathcal{J}_1^{(2')} - \frac{\mathcal{J}_2}{\sqrt{\mathcal{J}_2^2+\sin^2\left(\frac{p_1}{2}\right)}}\,\mathcal{J}_2^{(2')} \;\approx\; \sin^4\left(\frac{p_1}{2}\right)\frac{\left(1-2\cos^2\omega_2\right)}{\sqrt{\mathcal{J}_2^2+\sin^2\left(\frac{p_1}{2}\right)}}\,. \tag{9.2.35}$$

For later purpose, let us introduce a new 'rapidity' variable θ by

$$\tanh\left(\frac{\theta}{2}\right) = \frac{\mathcal{J}_2}{\sqrt{\mathcal{J}_2^2+\sin^2\left(\frac{p_1}{2}\right)}} = \frac{\sqrt{U}\sin\left(\frac{p_1}{2}\right)}{\sqrt{U\sin^2\left(\frac{p_1}{2}\right)+1}} + \mathcal{O}\left(k'^2\right)\,, \tag{9.2.36}$$

then it follows

$$\cosh\left(\frac{\theta}{2}\right) = \frac{\sqrt{\mathcal{J}_2^2+\sin^2\left(\frac{p_1}{2}\right)}}{\sin\left(\frac{p_1}{2}\right)} \approx \sqrt{U\sin^2\left(\frac{p_1}{2}\right)+1}\,. \tag{9.2.37}$$

Using this rapidity variable, (9.2.35) is rewritten as

$$\mathcal{E}^{(2')} - \mathcal{J}_1^{(2')} - \tanh\left(\frac{\theta}{2}\right)\,\mathcal{J}_2^{(2')} = \sin^3\left(\frac{p_1}{2}\right)\frac{\left(1-2\cos^2\omega_2\right)}{\cosh\left(\frac{\theta}{2}\right)}\,, \tag{9.2.38}$$

which is the prefactor of the leading finite-J_1 correction.

For the exponential part, recall that k' is related to J_1 as in (9.2.5):

$$\begin{aligned} k' &\approx 4\,\exp\left[-\frac{\sin^2\left(\frac{p_1}{2}\right)}{\mathcal{J}_2^2+\sin^4\left(\frac{p_1}{2}\right)}\sqrt{\mathcal{J}_2^2+\sin^2\left(\frac{p_1}{2}\right)}\left(\mathcal{J}_1+\sqrt{\mathcal{J}_2^2+\sin^2\left(\frac{p_1}{2}\right)}\right)\right], \\ &= 4\,\exp\left[-\frac{\sin^2\left(\frac{p_1}{2}\right)\cosh^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{p_1}{2}\right)+\sinh^2\left(\frac{\theta}{2}\right)}\left(\frac{\mathcal{J}_1}{\sin\left(\frac{p_1}{2}\right)\cosh\left(\frac{\theta}{2}\right)}+1\right)\right], \end{aligned} \tag{9.2.39}$$

Collecting the results (9.2.38) and (9.2.39), the energy-spin relation (9.2.33) becomes

$$\begin{aligned} \mathcal{E} - \mathcal{J}_1 &\approx \sqrt{\mathcal{J}_2^2+\sin^2\left(\frac{p_1}{2}\right)} \\ &\quad - 4\cos\left(2\omega_2\right)\frac{\sin^3\left(\frac{p_1}{2}\right)}{\cosh\left(\frac{\theta}{2}\right)}\,\exp\left[-\frac{2\sin^2\left(\frac{p_1}{2}\right)\cosh^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{p_1}{2}\right)+\sinh^2\left(\frac{\theta}{2}\right)}\left(\frac{\mathcal{J}_1}{\sin\left(\frac{p_1}{2}\right)\cosh\left(\frac{\theta}{2}\right)}+1\right)\right]. \end{aligned} \tag{9.2.40}$$

This is consistent with the finite-J_1 correction to giant magnons in the literature [52, 73] if we set $\theta = 0$ and $\cos(2\omega_2) = 1$. In other words, their results are equivalent to the asymptotic behavior of single-spin type (i) helical strings near $k = 1$.

Single-spin type (*ii*) helical strings corresponds to the case $\cos(2\omega_2) = -1$. For two-spin case, the finite-J_1 correction is essentially same as (9.2.40), because type (*ii*) solution can be obtained via the operation

$$\omega_2 \;\mapsto\; \omega_2 + \mathbf{K}'(1) = \omega_2 + \frac{\pi}{2}\,. \tag{9.2.41}$$

9.2.3 Finite-gap interpretation

Results in the last subsection revealed that the finite-J_1 correction to the energy-spin relation of dyonic giant magnons depends on the parameter ω_2 that has not appeared in the $J_1 = \infty$ case.[2] Unfortunately we are unable to fix ω_2 from the periodicity conditions of closed strings, because the winding number N_2 becomes ill-defined as $k \to 1$ as we saw in (9.2.22). To clarify the situation, we reconsider the rôle of the parameter ω_2 from a finite-gap point of view.

As discussed in Section 6.6, two-spin helical strings are equivalent to general elliptic finite-gap solutions of classical string action on $\mathbb{R}_t \times S^3$, and the limit $k \to 1$ corresponds to the situation in which the algebraic curve becomes singular. Written explicitly, the functions $Z_1\,, Z_2$ of [62] correspond to $\xi_2\,, \xi_1$ given in (9.2.9), (9.2.8), and the parameters $\tilde{\rho}_+\,, \tilde{\rho}_-$ of [62] correspond to $\omega_2\,, \omega_1$, respectively. The parameters $\omega_{1,2}$ and the location of branch points are related as

$$\omega_1 = F(\varphi_+, k') - \mathbf{K}'(k)\,, \qquad \omega_2 = \begin{cases} \mathrm{Re}\left[F(\varphi_-, k')\right] & (\text{for } \; k<1,\; k \to 1)\,, \\ \mathrm{Re}\left[F(\varphi_-, k')\right] - \dfrac{\pi}{2} & (\text{for } \; k>1,\; k \to 1)\,, \end{cases} \tag{9.2.42}$$

where $F(\varphi, k)$ is the normal elliptic integral of the first kind, and the angles $\varphi_\pm$ are given by

$$\tan\left(\frac{\varphi_\pm}{2}\right) = \frac{\left(\sqrt{\bar{x}_2} \pm \sqrt{x_1}\right)\left(\sqrt{\bar{x}_1} + \sqrt{x_2}\right)}{|x_1 - \bar{x}_2|}\,. \tag{9.2.43}$$

Let us take the $k \to 1$ limit of the relation (9.2.42), which is equivalent to $x_2 \to x_1$. From the definition of $\varphi_\pm$ in (9.2.43), one finds

$$\tan\left(\frac{\varphi_+}{2}\right) \to \pm\cot\left(\frac{p}{4}\right)\,, \quad \tan\left(\frac{\varphi_-}{2}\right) \to \mp i\,, \qquad \text{with} \quad x_1 \equiv \exp\left(\frac{ip - \theta}{2}\right)\,. \tag{9.2.44}$$

If we choose the upper sign in each equation, we find

$$\varphi_+ = -\frac{p}{2} + n_+\pi\,, \qquad \varphi_- = -i\infty + r\,. \tag{9.2.45}$$

with n_+ being an integer and r a real number. Applying the formula (A.1.14) to (6.6.5) and setting $n_+ = 1$, we can reproduce the results in the previous subsection $\omega_1 = (\pi - p)/2$. Similarly we have $\omega_2 = r$ or $\omega_2 = r - \pi/2$; in the latter case we may redefine r to have $\omega_2 = r$.

[2] When a two-spin helical string reduces to an array of dyonic giant magnons in $k \to 1$ limit, the dependence of ω_2 naturally disappears whatever value it has.

To study the case k is close but not equal to unity, one has to pull x_2 off from x_1. What matters here is that the direction in which x_2 is to be pulled off. If we write

$$x_2 = e^{i\alpha}\, x_1\,, \qquad \alpha \equiv a + ib, \qquad \text{with} \quad |\alpha| \ll 1, \tag{9.2.46}$$

then the former expressions (9.2.44) are modified into

$$\tan\left(\frac{\varphi_+}{2}\right) = \pm\left\{\cot\left(\frac{p}{4}\right) - \frac{a}{4\sin^2\left(\frac{p}{4}\right)}\right\} + \mathcal{O}\left(|\alpha|^2\right), \tag{9.2.47}$$

$$\tan\left(\frac{\varphi_-}{2}\right) = \mp\left\{i + \frac{b}{2\sin\left(\frac{p}{2}\right)}\right\} + \mathcal{O}\left(|\alpha|^2\right). \tag{9.2.48}$$

Note that the parameters a and b should be of order k', as follows from the expression of elliptic modulus in terms of the location of branch points:

$$k' = \left|\frac{x_1 - x_2}{x_1 - \bar{x}_2}\right| \approx \left|\frac{\alpha}{2\sin\left(\frac{p}{2}\right)}\right| \geq \left|\frac{\alpha}{2}\right|. \tag{9.2.49}$$

Substituting these results into (6.6.5) and (6.6.6), one finds

$$\omega_1 = \left(n_+ + \frac{1}{2}\right)\pi - \frac{p}{2} + \mathcal{O}\left(|\alpha|\right), \qquad \omega_2 = r + \mathcal{O}\left(|\alpha|\right). \tag{9.2.50}$$

This result suggests that ω_2 is left undetermined again in this finite-gap method.

9.3 Review of the generalized Lüscher formula

In this section, we give a brief review on the generalized Lüscher formula proposed by Janik and Łukowski [79]. The original Lüscher formula is a method to compute finite-size mass corrections from infinite-volume information of relativistic field theories [76, 77]. In [79], this formula was generalized to the non-relativistic theory, in which an elementary particle has the dispersion relation

$$\varepsilon_1(p) = \sqrt{1 + 16g^2\sin^2\left(\frac{p}{2}\right)}\,, \tag{9.3.1}$$

with $g \equiv \sqrt{\lambda}/(4\pi)$ and they reproduced the correct finite-size corrections to giant magnons. Here we consider a little more general situation where a particle satisfies the dispersion relation of a magnon boundstate

$$\varepsilon_Q(p) = \sqrt{Q^2 + 16g^2\sin^2\left(\frac{p}{2}\right)}\,, \tag{9.3.2}$$

with Q an arbitrary integer. In other words, we draw a *single* propagator for a set of particles among whose spectral parameters satisfy the boundstate conditions $x_j^- = x_{j-1}^+$.

Before deriving the generalized Lüscher formula, let us make our position clearer. We start from a two-dimensional effective Lagrangian describing the worldsheet theory in the decompactified limit. To fix the 2-point function, we use the dispersion relations (9.3.1) and (9.3.2) that are conjectured to all-loop orders in the 't Hooft coupling. We also assume the existence of 3- and higher point vertices, chosen so that they reproduce the conjectured two-body S-matrices. Our treatment grounds on the following Lüscher's argument [77]. The non-perturbative nature of his formula suggests that the leading finite-size correction can be captured only by kinematics rather than dynamics, once the exact dispersion relation and S-matrix are known. Therefore, if we regard the magnon boundstates as a composite particle obeying the dispersion relation (9.3.2), we can expect generalization of Lüscher formula to the dispersion relation (9.3.2) should reproduce the correct finite-size corrections to dyonic giant magnons.[3]

Now let us see derivation of the Lüscher formula. We begin with the two-point function for bosonic excitations in the 2d *infinite* volume theory:

$$\langle \phi_a(x)\phi_b(0)\rangle = \delta_{ab}\int \frac{d^2p}{(2\pi)^2} e^{ipx} G_{a,Q}(p), \tag{9.3.3}$$

$$G_{a,Q}(p) = \frac{1}{\varepsilon_E^2 + \varepsilon_Q^2(p^1) - \Sigma(p)}\,, \tag{9.3.4}$$

where $\varepsilon_E = ip^0$ is the Euclidean energy and $\Sigma(p)$ is the self-energy of ϕ_a. The self-energy $\Sigma(p)$ and its derivative with respective to p^μ vanish on the mass shell:

$$\Sigma(p)\big|_{\text{on-shell}} = \frac{\partial\Sigma(p)}{\partial p^\mu}\bigg|_{\text{on-shell}} = 0. \tag{9.3.5}$$

The latter condition fixes the normalization of ϕ_a, and the former condition fixes the residue of the Green function as

$$\operatorname*{Res}_{\varepsilon_E^2} G_{a,Q}(p) = 1\,. \tag{9.3.6}$$

Regarded as a function of p^1, (9.3.6) is equivalent to

$$\operatorname*{Res}_{p^1=p_*} G_{a,Q}(p) = \frac{1}{\varepsilon_Q^2(p_*)'}\,. \tag{9.3.7}$$

Next let us proceed to the theory on a cylinder of *finite*-circumference L. We impose the periodic boundary condition on ϕ_a,

$$\phi_a^{(L)}(x^0,x^1) = \phi_a^{(L)}(x^0,x^1+mL), \qquad \text{for } \forall m\in\mathbb{Z}\,, \tag{9.3.8}$$

so the Green function is given by

$$\langle \phi_a^{(L)}(x)\phi_b^{(L)}(0)\rangle = \delta_{ab}\frac{1}{L}\sum_{p^1}\int \frac{dp^0}{2\pi} e^{ipx} G_{a,Q}^{(L)}(p), \tag{9.3.9}$$

$$G_{a,Q}^{(L)}(p) = \frac{1}{\varepsilon_E^2 + \varepsilon_Q^2(p^1) - \Sigma_L(p)}\,. \tag{9.3.10}$$

[3]More generally, such method will be applicable to string states corresponding to asymptotic spin chains [8, 12], but generic states which are not dual to asymptotic spin chains, may not be described in a simple way using particle-like picture. We thank the reviewer of *Nuclear Physics B* for a valuable comment.

Note that the integral over p^1 in (9.3.3) is replaced with summation over discrete values. Combining the on-shell conditions for both infinite and finite volume cases (9.3.4), (9.3.10), and assuming

$$\Big[\Sigma(p) - \Sigma_L(p)\Big]_{\text{on-shell}} \sim \mathcal{O}\left(e^{-cL}\right), \qquad \left[\frac{\partial\Sigma(p)}{\partial p^\mu} - \frac{\partial\Sigma_L(p)}{\partial p^\mu}\right]_{\text{on-shell}} \sim \mathcal{O}\left(e^{-cL}\right), \tag{9.3.11}$$

for some constant c, we obtain the equation

$$\delta\varepsilon_L(p) \approx -\frac{\Sigma_L(p)}{2\,\varepsilon_Q(p^1) + i\left(\dfrac{\partial\Sigma_L(p)}{\partial\,\varepsilon_E}\right)} \approx -\frac{1}{2\,\varepsilon_Q(p^1)}\,\Sigma_L(p), \tag{9.3.12}$$

where $\delta\varepsilon_L(p)$ is the finite-size energy correction defined by $\varepsilon_E = i(\varepsilon_Q(p) + \delta\varepsilon_L(p))$. Thus we get the finite-size energy correction if we can calculate the finite-size self-energy.

An important fact is that the finite-size two-point function can be related to the infinite one as follows:

$$\langle\phi_a^{(L)}(x)\phi_b^{(L)}(0)\rangle = \sum_{m\in\mathbb{Z}}\langle\phi_a(x^0, x^1 + mL)\phi_b(0)\rangle\,. \tag{9.3.13}$$

In the momentum space language, the Green function is given by

$$G_{a,Q}^{(L)}(p) = \sum_{m\in\mathbb{Z}} e^{ip^1 mL} G_{a,Q}(p)\,. \tag{9.3.14}$$

Following Lüscher [76, 77], we consider only the case that $|m| = 1$ below because the leading finite-size correction arises from $|m| = 1$ as in the relativistic case.

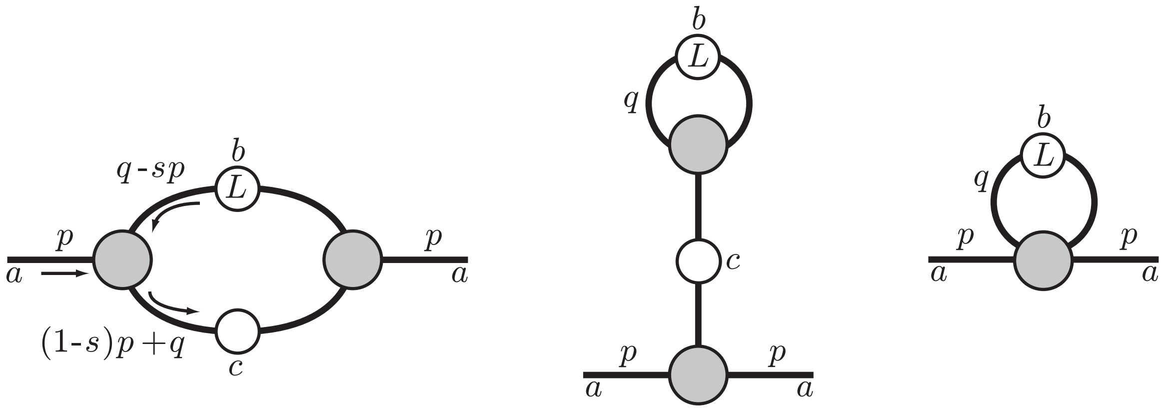
There are three types of diagrams shown in Figure 9.3 contributing to the self-energy of particle a whose charge is Q:

$$(\Sigma_L)_a = \frac{1}{2}\left(\sum_{b,c} I_{abc} + \sum_{b,c} J_{abc} + \sum_{b} K_{ab}\right). \tag{9.3.15}$$

The term I_{abc} consists of odd-point vertices, K_{ab} consists of even-point vertices, and J_{abc} consists of tadpole diagrams. They are given by

$$\begin{aligned} I_{abc} &= \sum_{Q_b\neq 0}\sum_{Q_c\neq 0}\int\frac{d^2q}{(2\pi)^2}\,2e^{-iq^1L} G_{b,Q_b}(q - sp) G_{c,Q_c}(q + (1-s)p)\times \\ &\qquad \Gamma_{abc}(-p, -q + sp, (1-s)p + q)\Gamma_{acb}(p, -(1-s)p - q, q - sp)\,, &(9.3.16)\\ J_{abc} &= \sum_{Q_b\neq 0}\sideset{}{'}\sum_{Q_c\neq 0}\int\frac{d^2q}{(2\pi)^2}\,2e^{-iq^1L} G_{b,Q_b}(q)\Gamma_{bbc}(q, -q, 0) G_{c,Q_c}(0)\Gamma_{aac}(-p, p, 0)\,, &(9.3.17)\\ K_{ab} &= \sum_{Q_b\neq 0}\int\frac{d^2q}{(2\pi)^2}\,2e^{-iq^1L} G_{b,Q_b}(q)\Gamma_{aabb}(p, -p, q, -q)\,, &(9.3.18) \end{aligned}$$

where G is the (infinite-size) Green function, *e.g.* given by $G_{b,Q_b}(q) = ((q_E^0)^2 + \varepsilon_{Q_b}^2(q^1) - \Sigma(q))^{-1}$, and the Γ's are effective 3- and 4-point vertices. We replaced $e^{iq^1L} + e^{-iq^1L}$ with $2e^{-iq^1L}$ by an



(a) (b) (c)

Figure 9.3: Diagrams which contribute to the finite-size self-energy Σ_L . The propagator carrying the exponential correction is marked with L. The diagram (a), (b), and (c) represents the term I_{abc} , J_{abc} , and K_{ab} given in (9.3.16), (9.3.17) , and (9.3.18), respectively.

appropriate change of the loop momentum q, and assigned the multiplet number Q_b, Q_c to the particle b, c respectively, which travel around the world (see Figure 9.1). The prime over $\sum$ in (9.3.17) means we sum over particles having no global $psu(2|2)^2$ charges (if such particles exist).

Assuming the analyticity of propagators and vertices, now we shift the contour of integration over q^1 to imaginary values, $\kappa \equiv \text{Im } q^1 < 0$.[4] The integral over κ is suppressed by $e^{-\kappa L}$, so we are able to neglect it in the limit $L \to \infty$. We cannot however neglect the contribution from poles of the Green function. The momentum vector $(q_E^0, q^1) = (\tilde{q}, \tilde{q}^1)$ at the pole of $G_{b,Q_b}(q)$ satisfy the condition

$$\tilde{q}^2 + \varepsilon_{Q_b}^2(\tilde{q}^1) = 0\,, \tag{9.3.19}$$

and using the dispersion relation $\varepsilon_Q(p) = \sqrt{Q^2 + 16g^2 \sin^2(\frac{p}{2})}$, we obtain

$$\tilde{q}^1 = -2i \,\text{arcsinh}\left(\frac{\sqrt{Q_b^2 + \tilde{q}^2}}{4g}\right). \tag{9.3.20}$$

The integrand of I_{abc} has two poles coming from $G_{b,Q_b}(q - sp)$ and $G_{c,Q_c}(q + (1 - s)p)$. We denote the contribution from $G_{b,Q_b}(q - sp)$ by I^+_{abc} and from $G_{c,Q_c}(q + (1 - s)p)$ by I^-_{abc} following [79]. As for I^+_{abc}, we shift the integration variable as

$$q \;\mapsto\; q + sp, \qquad G_{b,Q_b}(q - sp)G_{c,Q_c}(q + (1 - s)p) \;\mapsto\; G_{b,Q_b}(q)G_{c,Q_c}(q + p), \tag{9.3.21}$$

[4] Alternatively, one may deform the contour of integration to Im $q^1 > 0$. But the final results are independent of this choice.

and obtain the momentum-vector (9.3.20). Similarly for I^-_{abc}, we perform

$$q \mapsto q-(1-s)p, \qquad G_{b,Q_b}(q-sp)G_{c,Q_c}(q+(1-s)p) \mapsto G_{b,Q_b}(q-p)G_{c,Q_c}(q). \quad (9.3.22)$$

Since the term I_{abc} (9.3.16) is symmetric under the interchange of b and c, we obtain the same momentum-vector as in (9.3.20) for both (9.3.21) and (9.3.22).

Now using Eq. (9.3.7), we can perform integration over q^1 and get the expression

$$(\Sigma_L)_a = i\sum_n \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{-iq_*L}}{\varepsilon_n^2(q_*)'} \mathcal{I}_a(p,q) \quad (9.3.23)$$

where $\mathcal{I}_a$ is the integrand coming from the sum $I^+_{abc}+I^-_{abc}+J_{abc}+K_{ab}$ and explicitly given by

$$\begin{aligned}\mathcal{I}_a(p,q) =& \sum_b\sum_c \Big\{\Gamma_{abc}(-p,-q,p+q)G_{c,Q_c}(p+q)\Gamma_{acb}(p,-p-q,q) \\ &+\Gamma_{acb}(-p,p-q,q)G_{c,Q_c}(q-p)\Gamma_{abc}(p,-q,q-p)+\Gamma_{aabb}(p,-p,q,-q)\Big\} \\ &+\sum_b\sum_c{}' \Gamma_{aac}(p,-p,0)G_{c,Q_c}(0)\Gamma_{bbc}(q,-q,0), \end{aligned} \quad (9.3.24)$$

where the momentum vectors p and q are both on-shell. Lüscher's remarkable observation is that the integrand $\mathcal{I}_a$ is just the connected 4-point forward Green function $G_{abab}(-p,-q,p,q)$ between on-shell particles [76, 77, 78]. Furthermore, this 4-point Green function is related to the S-matrix element as follows:

$$G_{abab}(-p,-q,p,q) = -4i\varepsilon_Q(p)\varepsilon_n(q_*)(\varepsilon'_n(q_*)-\varepsilon'_Q(p))(S^{ba}_{ba}(q,p)-1) \quad (9.3.25)$$

We finally obtain the finite-size energy correction called F-term

$$\delta\varepsilon^F_a(p) = -\sum_{Q_b}\int_{-\infty}^{\infty}\frac{d\tilde{q}}{2\pi}\left(1-\frac{\varepsilon'_Q(p)}{\varepsilon'_{Q_b}(\tilde{q}^1)}\right)e^{-i\tilde{q}^1L}\sum_b(S^{ba}_{ba}(\tilde{q},p)-1)\,, \quad (9.3.26)$$

where $\tilde{q}^1$ is given by Eq. (9.3.20).

There is another type of the finite-size correction called μ-term, which comes from the integral in $I^\pm_{abc}$. The shifts of the integration variable made in (9.3.21), (9.3.22) push the contour of integration over q into the complex plane, because q is Euclidean while p is Minkowskian. When we deform the contour back again onto the real axis, one may encounter new poles from the S-matrix. If we denote the location of pole by $\tilde{q}^1 = q^1_*$, we obtain the generalized μ-term formula

$$\delta\varepsilon^\mu_a(p) = -i\sum_{Q_b}\left(1-\frac{\varepsilon'_Q(p)}{\varepsilon'_{Q_b}(q^1_*)}\right)e^{-iq^1_*L}\operatorname*{Res}_{\tilde{q}=\tilde{q}_*}\sum_b S^{ba}_{ba}(\tilde{q},p)\,. \quad (9.3.27)$$

The expression (9.3.27) is not real-valued in general. This problem can be attributed to the replacement $\cos(iq^1L)$ by $2e^{-iq^1L}$ to obtain the formula (9.3.16)-(9.3.18). If we analytically continue q^1 to the upper half plane, we obtain the result that is complex conjugate to (9.3.27). By undoing such replacement and adding the two contributions, we obtain the real part of the above result. Consequently, the generalized μ-term formula becomes

$$\delta\varepsilon_a^\mu = \mathrm{Re}\left\{-i\sum_{Q_b>0}\left(1-\frac{\varepsilon_Q'(p)}{\varepsilon_{Q_b}'(q_*^1)}\right)e^{-iq_*^1 L}\ \underset{\tilde{q}=\tilde{q}_*}{\mathrm{Res}}\sum_b S_{ba}^{ba}(\tilde{q},p)\right\}, \tag{9.3.28}$$

in place of (9.3.27).

9.4 Finite-size corrections to magnon boundstates

In this section, we calculate finite-size corrections to magnon boundstates by using the Lüscher formula known in quantum field theory, relating finite-size correction to the single-particle energy with the S-matrix of infinite-size system. In the infinite-size limit, (dyonic) giant magnons correspond to solitons of (complex) sine-Gordon system, which are localized excitations of a two-dimensional theory. Thus we can think of a (dyonic) giant magnon as the particle of an effective field theory, and use the Lüscher formula to compute the finite-size effects of it. More generally, such method will be applicable to string states corresponding to asymptotic spin chains [8, 12], but generic states which are not dual to asymptotic spin chains, may not be described in a simple way using particle-like picture.

Here we focus ourselves on considering the μ-term correction, which is given by[5]

$$\delta\varepsilon_a^\mu = \mathrm{Re}\left\{-i\sum_{Q_b>0}\left(1-\frac{\varepsilon_Q'(p)}{\varepsilon_{Q_b}'(q_*^1)}\right)e^{-iq_*^1 L}\ \underset{\tilde{q}=\tilde{q}_*}{\mathrm{Res}}\sum_b S_{ba}^{ba}(\tilde{q},p)\right\}, \tag{9.4.1}$$

where p, q_*^1 are the momenta of particles a, b respectively and Q_b is multiplet number of b.

There is possible contribution from the F-term. We expect that they do not contribute to the leading finite-size correction because the exponential part of the F-term seems different from that of the μ-term, or negligibly small if S-matrix behaves regularly over the path of integration. We will discuss this point in Appendix C.2.

9.4.1 The $su(2|2)^2$ S-matrix and its singularity

Before applying the generalized Lüscher formula to our case, let us briefly summarize some facts about the $su(2|2)^2$ S-matrix. Recall that elementary magnons appearing here are in the fundamental BPS representation of the $su(2|2)^2$ superconformal symmetry.

There are 16 kinds of such elementary magnons, among which scalar fields can form a part of boundstate multiplet. The Q-magnon boundstate also belongs to a $16\,Q^2$-dimensional BPS

[5] At the time of writing the version 5 of this paper, it is known that the correct formula is given by $\sum_b (-1)^{F_b} S_{ba}^{ba}$ rather than $\sum_b S_{ba}^{ba}$ [179, 180]. Here we neglect this sign because fermionic terms are subleading in our computation.

representation of $su(2|2)^2$ [145, 61]. We refer to the number of magnons Q as the multiplet number.

Let us first consider the scattering of two elementary magnons. The two-body S-matrix has the following form:

$$S(y,x) = S_0(y,x)[S_{su(2|2)}(y,x) \otimes S_{su(2|2)}(y,x)]\,, \tag{9.4.2}$$

where S_0 is the scalar factor expressed as

$$S_0(y,x) = \frac{y^- - x^+}{y^+ - x^-} \cdot \frac{1 - \frac{1}{x^- y^+}}{1 - \frac{1}{x^+ y^-}} \cdot \sigma^2(y,x)\,, \tag{9.4.3}$$

and $S_{su(2|2)}$ is the $su(2|2)$ invariant S-matrix and determined only by the symmetry algebra [12]. The dressing phase $\sigma^2(y,x)$ takes the following form,

$$\sigma^2(y,x) = \exp\Big[2i\left(\chi(y^-,x^-) - \chi(y^+,x^-) + \chi(y^+,x^+) - \chi(y^-,x^+)\right)\Big]\,, \tag{9.4.4}$$

where $\chi(x,y) = \tilde{\chi}(x,y) - \tilde{\chi}(y,x)$, and

$$\tilde{\chi}(x,y) = \sum_{n=0}^{\infty} \frac{\tilde{\chi}^{(n)}(x,y)}{g^{n-1}}\,, \qquad \tilde{\chi}^{(n)}(x,y) = \sum_{r=2}^{\infty}\sum_{s=r+1}^{\infty} \frac{-c_{r,s}^{(n)}}{(r-1)(s-1)x^{r-1}y^{s-1}}\,, \tag{9.4.5}$$

with the coefficients $c_{r,s}^{(n)}$ are given in [21].

When considering one of the two scattering bodies belongs to the $su(2)$ subsector, we just have to extract matrix elements of the form $E_i^j \otimes E_1^1$ from the S-matrix of [60]. Written explicitly, they are given by

$$S(y,x) = S_0(y,x)\left[a_1 E_1^1 \otimes E_1^1 + (a_1 + a_2) E_2^2 \otimes E_1^1 + a_6\left(E_3^3 \otimes E_1^1 + E_4^4 \otimes E_1^1\right)\right]^2\,, \tag{9.4.6}$$

where

$$a_1(y,x) \equiv \frac{y^+ - x^-}{y^- - x^+}\,\frac{\eta(x)\eta(y)}{\tilde{\eta}(x)\tilde{\eta}(y)}\,, \tag{9.4.7}$$

$$a_2(y,x) \equiv \frac{(y^- - y^+)(x^- - x^+)(y^+ + x^-)}{(y^- - x^+)(y^- x^- - y^+ x^+)}\,\frac{\eta(x)\eta(y)}{\tilde{\eta}(x)\tilde{\eta}(y)}\,, \tag{9.4.8}$$

$$a_6(y,x) \equiv \frac{y^+ - x^+}{y^- - x^+}\,\frac{\eta(x)}{\tilde{\eta}(x)}\,, \tag{9.4.9}$$

The $su(2|2)$ invariant S-matrix does depend on the choice of frame η. For instance, if we take the *string* frame of [60], we will obtain

$$\frac{\eta(x)}{\tilde{\eta}(x)} = \sqrt{\frac{x^+}{x^-}}\,, \qquad \frac{\eta(y)}{\tilde{\eta}(y)} = \sqrt{\frac{y^-}{y^+}}\,. \tag{9.4.10}$$

As for the *spin chain* frame, we obtain $\eta(x)/\tilde{\eta}(x) = \eta(y)/\tilde{\eta}(y) = 1$.

If two magnons are in the same $su(2)$ sector, the corresponding S-matrix without the dressing phase in the spin chain frame is called BDS S-matrix and given by

$$S_{\mathrm{BDS}}(y,x) = \frac{y^- - x^+}{y^+ - x^-} \cdot \frac{1 - \frac{1}{x^- y^+}}{1 - \frac{1}{x^+ y^-}} \, a_1(y,x)^2 = \frac{(y^+ - x^-)(1 - \frac{1}{y^+ x^-})}{(y^- - x^+)(1 - \frac{1}{y^- x^+})} \,. \tag{9.4.11}$$

It is important to notice that the S-matrix of two *boundstates* factorizes into the product of the two-body S-matrix between elementary magnons, as the consequence of integrability. Q-magnon boundstate has spectral parameters $x_k^\pm$ $(k = 1, \dots, Q)$, which satisfy the boundstate conditions

$$x_k^- = x_{k-1}^+ \quad (k = 2, \dots, Q). \tag{9.4.12}$$

The magnon boundstate is thus characterized by the outermost variables

$$X^- \equiv x_1^- \qquad \text{and} \qquad X^+ \equiv x_Q^+ \,. \tag{9.4.13}$$

The BDS S-matrix between boundstate $\{x_j^\pm\}$ and elementary magnon $y^\pm$ is given by

$$\begin{aligned} \prod_{k=1}^{Q} S_{\mathrm{BDS}}(y, x_k) &= \prod_{j=1}^{Q} \frac{(y^+ - x_k^-)(1 - \frac{1}{y^+ x_k^-})}{(y^- - x_k^+)(1 - \frac{1}{y^- x_k^+})} \\ &= \frac{(y^+ - X^-)(1 - \frac{1}{y^+ X^-})}{(y^- - X^+)(1 - \frac{1}{y^- X^+})} \, \frac{(y^- - X^-)(1 - \frac{1}{y^- X^-})}{(y^+ - X^+)(1 - \frac{1}{y^+ X^+})} \equiv S_{\mathrm{BDS}}(y, X) \,, \end{aligned} \tag{9.4.14}$$

where we used (9.4.12) and (9.4.13) [58, 59].

Recall that the $su(2|2)$ invariant S-matrix given in (9.4.6) is also written as

$$S(y, x_k) = S_{\mathrm{BDS}}(y, x_k) \left(\frac{\eta(x)\eta(y)}{\tilde\eta(x)\tilde\eta(y)} \right)^2 \sum_{i,j=1}^{4} \frac{a_i(y,x_k) a_j(y,x_k)}{a_1(y,x_k)^2} \, (E_i^i \otimes E_1^1) \otimes (E_j^j \otimes E_1^1). \tag{9.4.15}$$

Since the flavors i or j remain unchanged during each of the two-body scatterings, one can easily execute the product over k in this expression. Thus we obtain the elementary-boundstate S-matrix as

$$S(y, X) = S_{\mathrm{BDS}}(y, X) \, \Sigma^2(y, X) \left[\sum_{b=1}^{4} s_b(y, X) \, {E_b}^b \otimes {E_{(1\dots1)}}^{(1\dots1)} \right]^2 , \tag{9.4.16}$$

where $\Sigma(y, X)$ and $s_b(y, X)$ are given by

$$\Sigma(y, X) \equiv \prod_{k=1}^{Q} \sigma(y, x_k) \, \frac{\eta(x_k)\eta(y)}{\tilde\eta(x_k)\tilde\eta(y)} = \sigma(y, X) \, \frac{\eta(X)}{\tilde\eta(X)} \left(\frac{\eta(y)}{\tilde\eta(y)} \right)^Q , \tag{9.4.17}$$

$$s_1(y, X) = 1 \,, \quad s_2(y, X) = \prod_{k=1}^{Q} \left(1 + \frac{a_2(y, x_k)}{a_1(y, x_k)} \right) , \quad s_3(y, X) = s_4(y, X) = \prod_{k=1}^{Q} \frac{a_6(y, x_k)}{a_1(y, x_k)} \,. \tag{9.4.18}$$

Interestingly, the following formula holds[6]

$$s_2(y,X) = \frac{y^+ - X^+}{y^+ - X^-}\,\frac{1 - \frac{1}{y^- X^+}}{1 - \frac{1}{y^- X^-}}\,, \qquad s_3(y,X) = \frac{y^+ - X^+}{y^+ - X^-}\,\frac{\tilde{\eta}(X)}{\eta(X)}\,, \tag{9.4.20}$$

which agree with the recent results of [181, 182].

In order to compute the μ-term (9.4.1), we have to evaluate the residue at poles of the S-matrix. Then which poles should we pick up? If one follows derivation of the μ-term formula discussed in Section 9.3, one finds that the following criteria need to be satisfied for a given pole to contribute to the μ-term:

1. The L-dependent exponential factor of (9.4.1) damps.

2. Gives the leading (or the largest) contribution.

3. Comes from the I_{abc}-type diagram.[7]

The first two criteria will be used to derive the leading exponential term (9.4.29), where we will consider splitting of an on-shell particle with charge Q into two on-shell particles with ± 1 and $Q \mp 1$.

The third criterion is related to the fact that, in quantum field theories, poles of S-matrix correspond to the scattering processes where intermediate particles become on-shell. For a given pole, one must be able to find a scattering process such that the on-shell condition for its intermediate states is equivalent to the pole condition of the S-matrix. The relation between poles of the $su(2|2)^2$ S-matrix and scattering processes are investigated in detail in [81, 82].

The third criterion states that we should pick up only the poles related to the scattering process of I_{abc}-type. This is so severe that various complicated processes of splitting drop out from the μ-term formula. For instance, from analysis of the S-matrix singularity alone, the splitting process depicted in Figure 9.4 seems possible. However, this process should be classified as a K_{ab}-type diagram, and hence does not contribute to the μ-term.

9.4.2 Locating relevant poles

In this section, we investigate the third criterion in detail, in order to select the poles that contribute to the μ-term. As will be discussed in Section 9.3, during the I_{abc}-type process an incoming particle a splits into two particles b, c and these two recombine into the original one

[6] There is an identity for the spectral parameters of elementary magnons:

$$\frac{y^+ - x^-}{y^- - x^+}\left(1 - \frac{y^+ - x^+}{y^+ - x^-}\,\frac{1 - \frac{1}{y^- x^+}}{1 - \frac{1}{y^- x^-}}\right) = \frac{(y^- - y^+)(x^- - x^+)(y^+ + x^-)}{(y^- - x^+)(y^+ x^+ - y^- x^-)}\,. \tag{9.4.19}$$

[7] For classification of the Feynman diagrams, see Section 9.3.

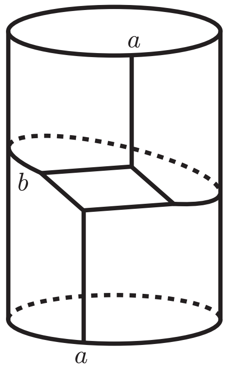


Figure 9.4: The splitting process of box type.

after going around the worldsheet cylinder as shown in Figure 9.1 (Left). Importantly, the three particles a, b and c are all on-shell, and consequently for such processes to happen they must satisfy the conditions:

3-1. Energy and momentum are conserved.

3-2. There is a Landau-Cutkosky diagram corresponding to the process $a \to b + c$.

Let us first consider the conservation of energy and momentum for an on-shell splitting process $a \to b + c$. By on-shell we mean that the energy, the multiplet number, and the momentum of a (boundstate) particle are given by functions of spectral parameters $X^{\pm} \equiv e^{(\pm ip+\theta)/2}$ as

$$E(X^{\pm}) = \frac{g}{i}\left(X^{+} - \frac{1}{X^{+}} - X^{-} + \frac{1}{X^{-}}\right) = 4g\cosh\left(\frac{\theta}{2}\right)\sin\left(\frac{p}{2}\right), \tag{9.4.21}$$

$$Q(X^{\pm}) = \frac{g}{i}\left(X^{+} + \frac{1}{X^{+}} - X^{-} - \frac{1}{X^{-}}\right) = 4g\sinh\left(\frac{\theta}{2}\right)\sin\left(\frac{p}{2}\right), \tag{9.4.22}$$

$$p(X^{\pm}) = \log\left(\frac{X^{+}}{X^{-}}\right), \tag{9.4.23}$$

where $g = \sqrt{\lambda}/(4\pi)$. The last two equations are solved as

$$X^{\pm} \equiv e^{(\pm ip+\theta)/2} = e^{\pm ip/2}\,\frac{Q + \sqrt{Q^2 + 16g^2\sin^2(\frac{p}{2})}}{4g\sin(\frac{p}{2})} = e^{\pm ip/2}\,\frac{\mathcal{Q} + \sqrt{\mathcal{Q}^2 + \sin^2(\frac{p}{2})}}{\sin(\frac{p}{2})}\,, \tag{9.4.24}$$

where $\mathcal{Q} \equiv Q/(4g)$, and the parameter θ introduced above is identical to (9.2.36) with $\mathcal{J}_2 \leftrightarrow \mathcal{Q}$.

Suppose the incoming particle a has the multiplet number $Q = Q(X^{\pm})$, the R-charge $r_a = Q$, and the momentum $p = p(X^{\pm})$. We denote the multiplet number, and the momentum of the split particle b by Q_b, p_b, respectively; and similarly for the other split particle c. Then, the conservation of energy and momentum imposes the relation:

$$\sqrt{Q^2 + 16g^2\sin^2\left(\frac{p}{2}\right)} = \sqrt{Q_b^2 + 16g^2\sin^2\left(\frac{p_b}{2}\right)} + \sqrt{Q_c^2 + 16g^2\sin^2\left(\frac{p-p_b}{2}\right)}\,. \tag{9.4.25}$$

We are interested in its solution that gives the smallest value of $|\mathrm{Im}\, p_b|$, with $\mathrm{Im}\, p_b < 0$. Such situation occurs when $Q_b = 1$ or $Q_c = 1$, and we may choose $Q_b = 1$ without loss of generality. Further, we can constrain the multiplet number Q_c by the following argument. In order that the splitting process takes place invariantly under the $su(2|2)^2$ symmetry, one should be able to contract the product of the representation of particle b and that of particle c with the representation of particle a, leaving us the singlet. In particular, if we define $\gamma \equiv Q - Q_c$, we should have $|\gamma| \leq 1$.[8]

Let us now solve (9.4.25) in the region $|p_b| \ll 1$ and $Q \gg 1$. The right hand side of (9.4.25) can be evaluated as

$$\text{R.H.S.} \approx \sqrt{1 + 4g^2 p_b^2} + \sqrt{Q^2 + 16g^2 \sin^2\left(\frac{p}{2}\right)} - \frac{8g^2 p_b \sin(\frac{p}{2})\cos(\frac{p}{2}) + \gamma Q}{\sqrt{Q^2 + 16g^2 \sin^2\left(\frac{p}{2}\right)}}\,, \tag{9.4.26}$$

where we used $Q \gg 1$. Inserting Eq. (9.4.26) into Eq. (9.4.25), we obtain

$$p_b \approx \frac{2\gamma Q \cos(\frac{p}{2})\sin(\frac{p}{2}) - \frac{i}{2g}\sqrt{(1-\gamma^2)Q^2 + 16g^2 \sin^4(\frac{p}{2})}\sqrt{Q^2 + 16g^2\sin^2(\frac{p}{2})}}{Q^2 + 16g^2 \sin^4(\frac{p}{2})} \equiv q_{\text{split},\gamma}\,, \tag{9.4.27}$$

where we choose the branch $\mathrm{Im}\, p_b < 0$. It is easy to see that $\mathrm{Im}\, q_{\text{split},\gamma}$ reaches its minimum when $\gamma = \pm 1$,

$$p_b = q_{\text{split},\pm} = \frac{\pm 2Q\cos(\frac{p}{2})\sin(\frac{p}{2}) - 2i\sin^2(\frac{p}{2})\sqrt{Q^2 + 16g^2\sin^2(\frac{p}{2})}}{Q^2 + 16g^2\sin^4(\frac{p}{2})}. \tag{9.4.28}$$

From Eq. (9.4.1), we obtain the exponential factor

$$|e^{-iq_{\text{split},\pm}L}| = e^{(\mathrm{Im}\, q_{\text{split},\pm})L} \approx \exp\left[-\frac{2\sin^2(\frac{p}{2})\sqrt{Q^2 + 16g^2\sin^2(\frac{p}{2})}}{Q^2 + 16g^2\sin^4(\frac{p}{2})}\, L\right]. \tag{9.4.29}$$

One can easily see that the coefficient of L is same as that of J_1 given in (9.2.40) or (9.2.39).

Next, we turn our attention to the condition 3-2. Firstly, we regard the self-energy diagrams of I_{abc}-type as the Landau-Cutkosky diagram of s- or t-type using the following argument (See Figure 9.5). If we set the particle travelling around the world, namely b particle, on-shell, then self-energy diagrams of I_{abc}-type become equivalent to $2 \to 2$ scattering processes between particles a and b exchanging particle c, where the momenta of a and b remain the same after scattering. If we further put particle c on-shell, this process can be expressed in terms of the Landau-Cutkosky diagram of s-type or t-type.

Secondly, for any scattering processes $a(p_a) + b(p_b) \to c(p_c) \to a(p_a) + b(p_b)$ to be kinematically allowed, it must satisfy the conservation of energy, momentum, and R-charge at each

[8]This argument is essentially same as in [178].

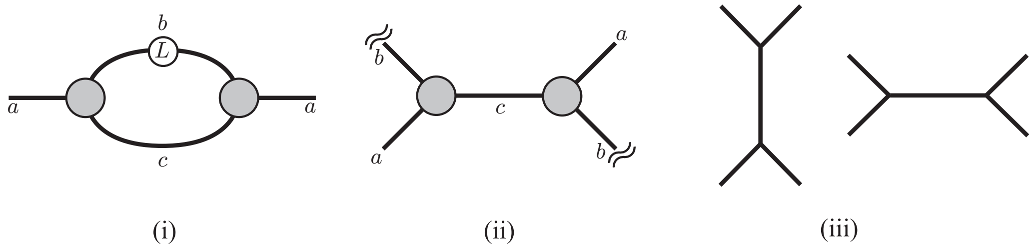


Figure 9.5: (i): Self-energy diagram of I_{abc}-type. (ii): Diagram of $ab \to ab$ scattering made from the diagram (i). (iii): The diagram (ii) can be viewed in two ways: s-type diagram as shown in the left, and t-type diagram as shown in the right.

Table 9.1: All possible combinations of scattering processes coming from I_{abc}-type diagrams which gives a damping exponential factor, namely $\operatorname{Im} p_b < 0$. Note that the crossing transformation $X^\pm \mapsto 1/X^\pm$ within this table maps the momentum with $\operatorname{Im} p_b < 0$ to the one with $\operatorname{Im} p_b > 0$. The combinations $y^- = 1/X^+$ and $y^- = X^-$ are realized as t-type diagram, while the ones $y^+ = 1/X^+$ and $y^+ = X^-$ are as s-type.

	s-type		t-type	
Pole Condition	$y^- = X^+$	$y^- = 1/X^-$	$y^+ = X^+$	$y^+ = 1/X^-$
In S_{BDS}	pole	zero	pole	zero
$E(Z^\pm)$	$E(X^\pm) + E(y^\pm)$	$E(X^\pm) + E(y^\pm)$	$E(X^\pm) - E(y^\pm)$	$E(X^\pm) - E(y^\pm)$
$Q(Z^\pm)$	$Q(X^\pm) + Q(y^\pm)$	$Q(X^\pm) - Q(y^\pm)$	$Q(X^\pm) - Q(y^\pm)$	$Q(X^\pm) + Q(y^\pm)$
p_b	$\dfrac{-i}{2g\sin\left(\frac{p-i\theta}{2}\right)}$	$\dfrac{-i}{2g\sin\left(\frac{p+i\theta}{2}\right)}$	$\dfrac{-i}{2g\sin\left(\frac{p-i\theta}{2}\right)}$	$\dfrac{-i}{2g\sin\left(\frac{p+i\theta}{2}\right)}$

point of interaction. Classification of the consistent Landau-Cutkosky diagrams of s- or t-type has essentially been done in [81, 82]. By following similar arguments, one can easily exhaust all consistent Landau-Cutkosky diagrams of s- or t-type. Let $X^\pm$ be the spectral parameters of the particle a, and $y^\pm$ be those of b with $Q_b = 1$, which satisfy the equation

$$y^+ + \frac{1}{y^+} - y^- - \frac{1}{y^-} = \frac{i}{g}\,. \tag{9.4.30}$$

Then we find four possible combinations of $\{X^\pm, y^\pm\}$ which reproduce $p_b = q_{\mathrm{split},\pm}$, as listed in Table 9.1. The corresponding Landau-Cutkosky diagrams of s- or t-type are shown in Figure 9.6.

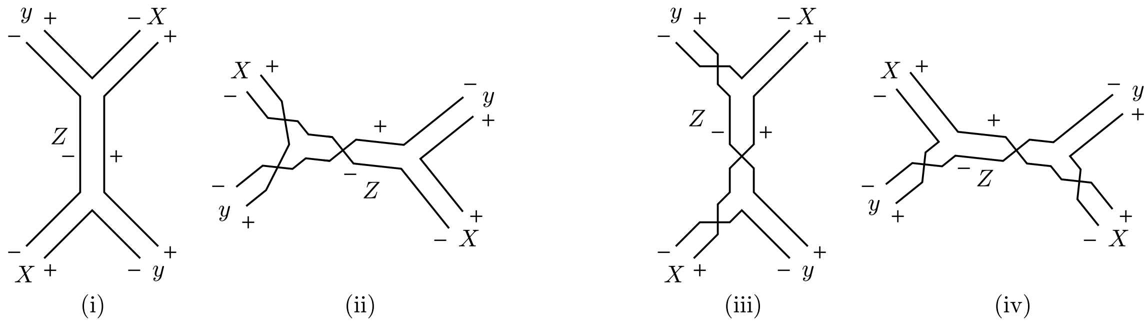


Figure 9.6: The scattering processes which correspond to (i) $y^- = X^+$, (ii) $y^+ = X^+$, (iii) $y^- = 1/X^-$, (iv) $y^+ = 1/X^-$. We follow the convention of the diagrams in [81].

The processes corresponding to $y^\mp = X^+$ satisfy $p_b \approx q_{\text{split},+}$ and solve the condition (9.4.25) with $Q_c = Q - 1$ at strong coupling. The ones corresponding to $y^\mp = 1/X^-$ have $p_b \approx q_{\text{split},-}$ and solve (9.4.25) with $Q_c = Q + 1$.[9] Note that this result disagrees with the classification of Table 9.1. This is not contradictory, because the analyses of [81, 82] are valid for arbitrary values of g while ours are restricted to the case $g \to \infty$ where the solutions to the splitting condition (9.4.25) are degenerate.

Out of the four conditions, only the ones $y^\mp = X^+$ appear as poles of the BDS S-matrix (9.4.14), and the conditions $y^\mp = 1/X^-$ appear as the zeroes. The latter two actually become the poles of the full S-matrix because the AFS phase bring double poles at these locations. In this case, however, the spectral parameters $y^\pm$ do not lie inside the physical region $|y^\pm| > 1$, so we should not pick up the residues at $y^\mp = 1/X^-$.[10]

In summary, we conclude that solutions to all criteria are exhausted by the two poles at $y^\mp = X^+$.

9.4.3 Evaluation of residues

We are going to evaluate the residue of each pole for the two cases $Q \sim \mathcal{O}(g) \gg 1$ and $Q \sim \mathcal{O}(1) \ll g$. Note that the orientation of the contour needs to be specified to fix the sign of the residue. It will turn out that the sum of two residues with the same orientation does not reproduce the results of classical string, so we will argue how the contour should be shifted to obtain the desired results.

The case $Q \sim \mathcal{O}(g) \gg 1$

Let us first consider the condition $y^- = X^+$. Because $1/(y^+ - X^+) \sim \mathcal{O}(g)$ around this pole, the term proportional to s_2 and s_3 in (9.4.16) are negligible at strong coupling. The residue of

[9] There is no clear interpretation as such when $Q \sim \mathcal{O}(1) \ll g$.

[10] We thank S. Frolov for a comment on physicality issue.

S_{BDS} is given by

$$\operatorname*{Res}_{\tilde{q}=\tilde{q}_*} S_{\text{BDS}}(y,X) \approx \frac{(X^+ - X^-)}{(y^-)'} \frac{\left(1-\frac{1}{X^+X^-}\right)}{\left(1-\frac{1}{(X^+)^2}\right)} \frac{(X^+ - X^-)}{iq_*^1 X^+} \frac{\left(1-\frac{1}{X^+X^-}\right)}{\left(1-\frac{1}{(X^+)^2}\right)}, \tag{9.4.31}$$

where $(y^-)'$ is the Jacobian given by (C.1.5), and we used

$$y^+ = \left(1+iq_*^1\right) y^- + \mathcal{O}\left((q_*^1)^2\right). \tag{9.4.32}$$

Next we evaluate the dressing phase. By using

$$\chi(y^+, X^\pm) \;\approx\; \chi(y^-, X^\pm) + iq_*^1 y^- \chi_{1,0}(y^-, X^\pm), \tag{9.4.33}$$

we find

$$\sigma^2(y,X) \;\approx\; \exp\Big[2q_*^1 y^- \left(\chi_{1,0}(y^-, X^-) - \chi_{1,0}(y^-, X^+)\right)\Big], \tag{9.4.34}$$

where $\chi_{1,0}(y,x) \equiv \partial_y \chi(y,x) = \partial_x \tilde{\chi}(y,x) - \partial_y \tilde{\chi}(x,y)$. A crucial fact is that $\chi_{1,0}^{(n)}(X^+, X^+)$ and $\chi_{1,0}^{(n)}(X^+, X^-)$ are the order $1/g^{n-1}$ quantities if $Q \sim \mathcal{O}(\lambda^{1/2}) \gg 1$. The dressing phase with $n \geq 1$ does not contribute at strong coupling, which is remarkable distinction from the elementary magnon case [79]. Thus, it suffices to consider the contribution of $\chi^{(0)}$, namely the AFS phase [10]. The series (9.4.5) with $c_{r,s}^{(0)} = \delta_{r+1,s}$ sums up to give

$$\chi^{(0)}(y,x) = -g\left(\frac{1}{x} - \frac{1}{y}\right)\left(1 - (1-xy)\log\left(1 - \frac{1}{xy}\right)\right). \tag{9.4.35}$$

It follows that

$$\chi_{1,0}^{(0)}(y,x) = -\frac{g}{y}\left(\frac{1}{x} + \left(y - \frac{1}{y}\right)\log\left(1 - \frac{1}{xy}\right)\right). \tag{9.4.36}$$

Using this equation, the contribution of the AFS phase becomes

$$\sigma_{\text{AFS}}^2(y,X) \approx \exp\left[-\frac{2}{\left(X^+ - \frac{1}{X^+}\right)}\left(\frac{1}{X^-} - \frac{1}{X^+}\right) - 2\ln\left(\frac{1-\frac{1}{y^-X^-}}{1-\frac{1}{y^-X^+}}\right)\right]. \tag{9.4.37}$$

By combining (9.4.31) and (9.4.37), we find

$$\operatorname*{Res}_{\tilde{q}=\tilde{q}_*} S_{\text{BDS}}(y,X)\,\sigma_{\text{AFS}}^2(y,X) \approx -8ig\frac{\sin^2\left(\frac{p}{2}\right)}{\sin\left(\frac{p-i\theta}{2}\right)} \exp\left[-ip - \frac{\epsilon_Q(p) - Q}{2g\sin\left(\frac{p-i\theta}{2}\right)}\right]. \tag{9.4.38}$$

To compute the μ-term, one just has to multiply the prefactor

$$-i\left(1 - \frac{\epsilon_Q'(p^1)}{\epsilon_1'(q_*^1)}\right)e^{-iq_*^1 L} = -i\,\frac{\sin\left(\frac{p}{2}\right)\sin\left(\frac{p-i\theta}{2}\right)}{\cosh\left(\frac{\theta}{2}\right)} \exp\left[-\frac{L}{2g\sin\left(\frac{p-i\theta}{2}\right)}\right], \tag{9.4.39}$$

as well as the factor from the string frame

$$\frac{X^+}{X^-}\left(\frac{y^-}{y^+}\right)^Q \approx \exp\left[ip - \frac{Q}{2g\sin\left(\frac{p-i\theta}{2}\right)}\right]. \tag{9.4.40}$$

In total, the μ-term from the pole $y^- = X^+$ is evaluated as

$$\delta E^\mu \Big|_{y^-=X^+} = -8g \, \frac{\sin^3\left(\frac{p}{2}\right)}{\cosh\left(\frac{\theta}{2}\right)} \, \exp\left[-\frac{L+\epsilon_Q(p)}{2g\sin\left(\frac{p-i\theta}{2}\right)}\right]. \tag{9.4.41}$$

Next, we study the pole $y^+ = X^+$. Now the coefficients $s_2(y,X)$ and $s_3(y,X)$ vanish due to (9.4.20), and only the term $s_1(y,X)$ can contribute to the μ-term. The residue of S_{BDS} is

$$\operatorname*{Res}_{\tilde{q}=\tilde{q}_*} S_{\text{BDS}}(y,X) \approx \frac{(X^+ - X^-)}{-iq^1_* X^+} \frac{\left(1-\frac{1}{X^+X^-}\right)}{\left(1-\frac{1}{(X^+)^2}\right)} \frac{(X^+ - X^-)}{(y^+)'} \frac{\left(1-\frac{1}{X^+X^-}\right)}{\left(1-\frac{1}{(X^+)^2}\right)} . \tag{9.4.42}$$

Since $(y^+)' \approx (y^-)'$ as shown in (C.1.5), this result is just the minus of (9.4.31). The AFS phase at $y^+ = X^+$ becomes

$$\sigma^2_{\text{AFS}}(y,X) \approx \exp\left[-\frac{2}{\left(X^+ - \frac{1}{X^+}\right)}\left(\frac{1}{X^-} - \frac{1}{X^+}\right) - 2\ln\left(\frac{1-\frac{1}{y^+X^-}}{1-\frac{1}{y^+X^+}}\right)\right], \tag{9.4.43}$$

which is equal to (9.4.37). Hence we conclude

$$\delta E^\mu \Big|_{y^+=X^+} = 8g \, \frac{\sin^3\left(\frac{p}{2}\right)}{\cosh\left(\frac{\theta}{2}\right)} \, \exp\left[-\frac{L+\epsilon_Q(p)}{2g\sin\left(\frac{p-i\theta}{2}\right)}\right]. \tag{9.4.44}$$

Here we neglected the orientation of contour when deriving the above results. We will discuss this issue in Section 9.4.3.

The case $Q \sim \mathcal{O}(1) \ll g$

Let us now study the case $Q > 1$ with $Q \ll g$, and compute the residues of (9.4.16) at $y^\pm = X^+$. We have to evaluate the dressing phase carefully, because the terms higher order in $1/g$ contribute to the μ-term, as discussed in [79].

Computation of the residue of the BDS S-matrix is straightforward, so let us focus on the dressing phase. It is useful to introduce new variables α^{ab} by

$$\frac{\alpha^{ab}}{2g\sin\left(\frac{p}{2}\right)} = 1 - \frac{1}{y^a X^b} \qquad \text{if} \quad y^a X^b \to 1 \quad \text{as} \quad g \to \infty. \tag{9.4.45}$$

We can neglect the higher-order terms in the dressing phase when $y^a X^b$ is not close to unity. The values of α^{ab} around the pole conditions are listed in Table 9.2.

The AFS phase [10] can be easily computed from the following expressions:

$$\sigma^2_{\text{AFS}}(y,X) = \left(\frac{1-\frac{1}{y^-X^-}}{1-\frac{1}{y^+X^-}}\right)^{2Q} \left(\frac{1-\frac{1}{y^-X^+}}{1-\frac{1}{y^-X^-}}\right)^{2}, \tag{9.4.46}$$

Table 9.2: List of α^{ab} at strong coupling, corresponding to the pole with $\operatorname{Im} q^1 < 0$.

Pole Condition	α^{++}	α^{--}	α^{+-}	α^{-+}
$y^- = X^+$		Q	$Q+1$	
$y^+ = X^+$		$Q-1$	Q	

which are derived in Appendix C.1. The Hernández-López phase [13] can be computed by employing the results of [79],

$$\chi^{(1)}(y^a, X^b) \approx \mp \frac{i}{2} \log \left(\frac{\alpha^{ab}}{2g \sin\left(\frac{p}{2}\right)} \right), \tag{9.4.47}$$

where the sign ambiguity comes from the choice of a logarithmic branch. As shown in Appendix C.1, the rest of the BES phase [21] is summarized as

$$\sigma^2_{n\geq 2}(y, X) \approx \exp\left[2\left(\alpha^{--} - \alpha^{+-}\right)\right] \left(\frac{\alpha^{+-}}{\alpha^{--}}\right)^{\alpha^{--}+\alpha^{+-}}, \quad \left(\text{for} \;\; y \sim e^{ip/2}\right), \tag{9.4.48}$$

Note that $\chi^{(2m+1)}(y^a, X^b) \approx 0$. By combining the results (9.4.47) and (9.4.48), the higher-order dressing phase is evaluated as

$$\sigma^2(y, X) \approx -\frac{16g^2 \sin^2\left(\frac{p}{2}\right)}{Q(Q+1)} \, e^{-ip-2} \left(\frac{Q+1}{Q}\right)^{\pm 1} \qquad \text{for} \;\; y^- = X^+, \tag{9.4.49}$$

$$\sigma^2(y, X) \approx -\frac{16g^2 \sin^2\left(\frac{p}{2}\right)}{Q(Q-1)} \, e^{-ip-2} \left(\frac{Q}{Q-1}\right)^{\pm 1} \qquad \text{for} \;\; y^+ = X^+. \tag{9.4.50}$$

We will choose the $+$ sign for (9.4.49) and the $-$ sign for (9.4.50) for consistency with the $Q = 1$ case.[11]

One can calculate the remaining part of the S-matrix in the same manner as before. One should take care that the coefficient $s_2(y, X)$ is non-zero for $y^- = X^+$. The final results in string frame are summarized as

$$\delta E^\mu \Big|_{y^-=X^+} = -8g \left(1 + \frac{1}{Q}\right) \sin^3\left(\frac{p}{2}\right) \exp\left[-\frac{L + \epsilon_Q(p)}{2g \sin\left(\frac{p}{2}\right)}\right], \tag{9.4.51}$$

$$\delta E^\mu \Big|_{y^+=X^+} = +8g \left(1 - \frac{1}{Q}\right) \sin^3\left(\frac{p}{2}\right) \exp\left[-\frac{L + \epsilon_Q(p)}{2g \sin\left(\frac{p}{2}\right)}\right], \tag{9.4.52}$$

where $\epsilon_Q(p) \approx 4g \sin(p/2)$.

[11] Consistency for the latter is only formal, for there is no pole at $y^+ = X^+$ when $Q = 1$.

Comparison with classical string

Now we check if the Lüscher μ-term can reproduce the results of classical string theory, which was given in (9.2.40) as

$$\delta(E-J_1)=-16g\,\cos(2\omega_2)\,\frac{\sin^3\left(\frac{p_1}{2}\right)}{\cosh\left(\frac{\theta}{2}\right)}\,\exp\left[-\frac{\sin\left(\frac{p_1}{2}\right)\cosh\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{p_1}{2}\right)+\sinh^2\left(\frac{\theta}{2}\right)}\,\frac{J_1+\epsilon_Q(p_1)}{2g}\right]. \tag{9.4.53}$$

We begin with the case $Q\sim\mathcal{O}(g)\gg 1$. Here, the poles $y^\pm=X^+$ are located around $\tilde{q}=\cot\left(\frac{p-i\theta}{2}\right)$, and the residues obey the relation

$$\delta E^\mu\Big|_{y^-=X^+}=-\delta E^\mu\Big|_{y^+=X^+}. \tag{9.4.54}$$

It suggests that the sum of μ-term will vanish if we simply sum up the residues of all poles on the upper half plane. In order to obtain a nonvanishing result, for instance, we should take the difference of two residues.

We can flip the relative sign of them if we modify the contour of $\tilde{q}$ integration in the F-term formula (9.3.26) as shown in Figure 9.7, where $\tilde{q}$ is the Euclidean energy of the particle travelling around the cylinder. As discussed in Section 9.3, we obtain the μ-term from the shifts of the contour. When we set $s=1/2$ in (9.3.21) and (9.3.22), we find a clockwise contour shifted upward and a counterclockwise contour shifted downward. Note that it is possible to have a clockwise contour shifted downward and a counterclockwise upward, if we choose the other branch of square root in (C.1.2), which flips the overall sign. Thus, the modified and shifted contours provide us with an additional minus sign in front of the residue at $y^+=X^+$, giving us

$$\delta E^\mu\Big|_{y^-=X^+}-\delta E^\mu\Big|_{y^+=X^+}=-16g\,\cos(\alpha)\,\frac{\sin^3\left(\frac{p}{2}\right)}{\cosh\left(\frac{\theta}{2}\right)}\,\exp\left[-\frac{L+\epsilon_Q(p)}{2g\sin\left(\frac{p-i\theta}{2}\right)}\right]. \tag{9.4.55}$$

Since the μ-term (9.4.1) is given by the real part of the last expression, we obtain

$$\delta E^\mu=\mp16g\,\cos(\alpha)\,\frac{\sin^3\left(\frac{p}{2}\right)}{\cosh\left(\frac{\theta}{2}\right)}\,\exp\left[-\frac{\sin\left(\frac{p}{2}\right)\cosh\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{p}{2}\right)+\sinh^2\left(\frac{\theta}{2}\right)}\,\frac{L+\epsilon_Q(p)}{2g}\right], \tag{9.4.56}$$

where

$$\alpha=\frac{\cos\left(\frac{p}{2}\right)\sinh\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{p}{2}\right)+\sinh^2\left(\frac{\theta}{2}\right)}\,\frac{L+\epsilon_Q(p)}{2g}, \tag{9.4.57}$$

for $Q\sim\mathcal{O}(g)\gg 1$. This agrees with (9.4.53) upon identifying $J_1\leftrightarrow L$, $p_1\leftrightarrow p$ and $2\omega_2\leftrightarrow\alpha$.

Next, let us consider the case $Q\sim\mathcal{O}(1)\ll g$. As shown in (C.1.4), both poles are located on the upper half plane of the $\tilde{q}$ plane, namely

$$\tilde{q}=\cot\left(\frac{p}{2}\right)+\frac{i(Q\pm1)}{2g\sin^3\left(\frac{p}{2}\right)}\qquad\text{for}\quad y^\mp=X^+. \tag{9.4.58}$$

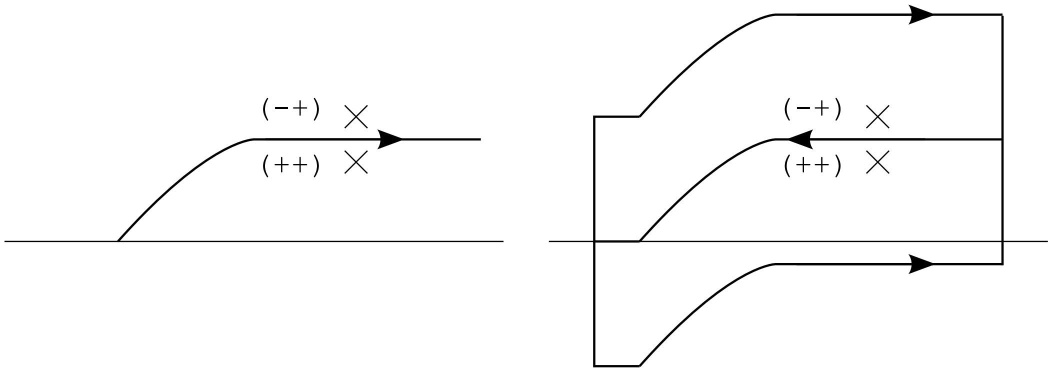


Figure 9.7: The contour of integration in $\tilde{q}$ is deformed as in the left figure. By considering the difference between the F-term contour and the shifted contours (9.3.21), (9.3.22), we can pick up the μ-term as depicted in the right figure. By $(-+), (++)$ we denote the location of poles $y^- = X^+, y^+ = X^+$, respectively.

By making the same deformation of the contour as in Figure 9.7, we find

$$\pm\left(\delta E^{\mu}\Big|_{y^-=X^+} - \delta E^{\mu}\Big|_{y^+=X^+}\right) = \mp 16 g \sin^3\left(\frac{p}{2}\right) \exp\left[-\frac{L+\epsilon_Q(p)}{2g\sin\left(\frac{p}{2}\right)}\right], \tag{9.4.59}$$

for $Q \sim \mathcal{O}(1) \ll g$. This result is already real, and agrees with (9.4.53) if we set $\theta = \omega_2 = 0$ and identify J_1 with L, p_1 with p.

Finally let us comment on computation in the spin chain frame. The result of the spin chain frame differ from that of the string frame by the factor (9.4.40). As a consequence, the μ-term for $Q \sim \mathcal{O}(g)$ in (9.4.56) turns into

$$\delta E^{\mu} = \mp 16 g \cos(\alpha_p) \frac{\sin^3\left(\frac{p}{2}\right)}{\cosh\left(\frac{\theta}{2}\right)} \exp\left[-\frac{\sin\left(\frac{p}{2}\right)\cosh\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{p}{2}\right)+\sinh^2\left(\frac{\theta}{2}\right)} \frac{L-Q+\epsilon_Q(p)}{2g}\right], \tag{9.4.60}$$

where

$$\alpha_p = p + \frac{\cos\left(\frac{p}{2}\right)\sinh\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{p}{2}\right)+\sinh^2\left(\frac{\theta}{2}\right)} \frac{L-Q+\epsilon_Q(p)}{2g}\,. \tag{9.4.61}$$

This expression also agrees with the result of classical string (9.4.53) if we identify $L - Q \leftrightarrow J_1$, $p_1 \leftrightarrow p$ and $2\omega_2 \leftrightarrow \alpha_p$.[12] Also, the expression (9.4.61) is the same as the one found in [178].

Thus, the μ-term of the generalized Lüscher formula can capture the leading finite-size (or finite angular momentum) correction to dyonic giant magnons.

[12]It appears that what we call length depends on the choice of frame.

Conclusions and Outlook

This dissertation can be divided into three parts.

In the first part, we reviewed recent developments of AdS/CFT correspondence between $\mathcal{N}=4$ super Yang-Mills theory and superstring on $\mathrm{AdS}_5\times\mathrm{S}^5$ after the discovery of integrability.

On gauge theory side, the central idea is to diagonalize anomalous dimension matrix by using integrability method called Bethe Ansatz. On string theory side, we can construct algebro-geometric solutions based on Lax-pair formulation of the equations of motion. Singular integral equations appeared in both sides. They enabled us to compare the spectrum in terms of algebraic curves and Abelian differentials.

In the second part, we investigated families of classical string solutions on $\mathbb{R}_\mathrm{t}\times\mathrm{S}^3$ and on $\mathrm{AdS}_3\times\mathrm{S}^1$ from sine-Gordon perspective. We show they interpolate various rigid and spinning/oscillating and winding string solutions known so far. Put it schematically, for helical spinning strings on $\mathbb{R}_\mathrm{t}\times\mathrm{S}^3$, we obtain

$$\mathrm{I}\ :\quad \begin{array}{l}\text{Type } (i) \text{ helical string}\\ \text{with generic } (k,U,\omega_{1,2})\end{array} \to \begin{cases} \text{- Point-like (BPS), rotating string} & (k\to 0)\\ \text{- Array of dyonic giant magnons} & (k\to 1)\\ \text{- Elliptic, spinning folded string} & (\omega_{1,2}\to 0)\end{cases},$$

$$\mathrm{II}\ :\quad \begin{array}{l}\text{Type } (ii) \text{ helical string}\\ \text{with generic } (k,U,\omega_{1,2})\end{array} \to \begin{cases} \text{- Rational, spinning circular string} & (k\to 0)\\ \text{- Array of dyonic giant magnons} & (k\to 1)\\ \text{- Elliptic, spinning circular string} & (\omega_{1,2}\to 0)\end{cases}.$$

and for helical oscillating strings on $\mathbb{R}_\mathrm{t}\times\mathrm{S}^3$, we obtain

$$\mathrm{I}'\ :\quad \begin{array}{l}\text{Type } (i)' \text{ helical string}\\ \text{with generic } (k,U,\omega_{1,2})\end{array} \to \begin{cases} \text{- Rational, static circular string} & (k\to 0)\\ \text{- Array of single-spike strings} & (k\to 1)\\ \text{- Elliptic, type } (i)' \text{ pulsating string} & (\omega_{1,2}\to 0)\end{cases},$$

$$\mathrm{II}'\ :\quad \begin{array}{l}\text{Type } (ii)' \text{ helical string}\\ \text{with generic } (k,U,\omega_{1,2})\end{array} \to \begin{cases} \text{- Rational circular string} & (k\to 0)\\ \text{- Array of single-spike strings} & (k\to 1)\\ \text{- Elliptic, type } (ii)' \text{ pulsating string} & (\omega_{1,2}\to 0)\end{cases}.$$

It is also discussed that these two families of classical string solutions are characterized general two-cut finite-gap solutions. In algebro-geometric language, the $\tau \leftrightarrow \sigma$ operation is translated into either the interchange of quasi-momentum with quasi-energy, or the reconnection of branch cuts passing through the singularities at $x = \pm 1$.

For Cases I and II, the gauge theory duals are well-known. All of them are of the form

$$\mathcal{O} \sim \mathrm{Tr}\left(Z^{L-M} W^{M}\right) + \dots\,, \tag{9.4.62}$$

with L very large. Though operators look simple as such, their configuration on rapidity plane will be in general quite complicated.

In contrast, gauge theory duals for Cases I′ and II′ are not yet clearly known. From comparison of global charges, they should be dual to certain non-holomorphic operators with little R-charges. In the paper [97] we conjectured excitations over the singlet state in $so(6)$ sector will be a good candidate. The correspondence involving such operators has not been much studied so far, due to large quantum corrections on super Yang-Mills side. Assuming all-order integrability and using all-loop Bethe Ansatz, one may be able to deduce strong coupling prediction of non-holomorphic sector as in [183]. More works are certainly needed to establish correspondence for the case of large winding strings.

For further check of AdS/CFT correspondence we have to find sophisticated way of comparison, or to compare both sides from general perspective. In either way, we have to know the spectrum of both sides in great detail. We expect our solutions serve as a catalyst for opening new region of comparison.

In the third part, we considered application of our solutions to study the finite-size effects in AdS/CFT correspondence. In particular, we have computed finite-size corrections to giant magnons with two angular momenta from two points of view:

(i) Studying the asymptotic behavior of helical strings as $k \to 1$

(ii) Applying the generalized Lüscher formula to the case in which incoming particles are boundstates

We found that two results exactly match taking into consideration the finite-gap interpretation of [62]. This result supports the validity of generalized Lüscher formula for the case of boundstates.

In contrast to the work of [79], it turned out that the leading term is only sensitive to the AFS phase in the strong coupling limit. Nevertheless, our results coincide with those in [79] in the limit $\mathcal{Q} \to 0$.

Towards computation of the finite-size corrections exact in L, several approaches have been known in the theory of integrable systems, such as Thermodynamic Bethe Ansatz (TBA) [75, 176, 177], nonlinear integral equations (NLIE) [184, 185, 186, 187], and functional relations among commuting transfer-matrices [188]. Recently, Arutyunov and Frolov have studied TBA formulation of the finite-size system by double Wick rotation on the worksheet, and determined S-matrix of the "mirror" model [189]. Moreover, they obtained the finite-size exponential factor which is identical to the giant magnon's, by considering (two-magnon) boundstates of the mirror model. It will be very interesting to reanalyze our results from the TBA approach for multi-magnon boundstates.

Unmentioned topics and Open questions

There are several important topics we have not discussed in this dissertation, some of which are listed as follows.

Quantum correction to classical strings. Contrary to classical computation in string theory where we are able to truncate action to its arbitrary subspaces, the one-loop correction requires the full information of superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$. In this respect, it is quite important to extend classical analysis into the quantum level. Note that in general there is non-zero correction to the energy-spin relation, because worldsheet supersymmetry is spontaneously broken when we fix certain classical background.

One basic approach to compute such effects is to sum up fluctuations over the given classical background. For instance, one-loop correction to Frolov-Tseytlin strings is computed in [164, 36, 37, 38], and one-loop correction to (dyonic) giant magnons is discussed in [53, 190, 191, 34].

Another direction is to make use of Bethe Ansatz. Although we do not know whether quantum superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is integrable or not, a brave proposal was first made in [10] that Bethe Ansatz with the dressing phase will capture quantum string spectrum. At one-loop level, the string S-matrix acquires so-called Hernandez-Lopez phase [13].

Since the dressing phase appears as a scalar factor, it must be independent of subsectors one chooses. The Hernandez-Lopez phase passes this test, and its universality is confirmed in [155].

Worldsheet computation at two loops in $1/\sqrt{\lambda}$ has recently been done in [192, 193, 194].

Landau-Lifshitz effective action. It is well known that $\mathsf{XXX}_{1/2}$ spin chain model has an effective description as σ-model called Landau-Lifshitz effective action, which can be obtained from a coherent-state path-integral of $\mathsf{XXX}_{1/2}$ Hamiltonian. Kruczenski pointed out that by taking an appropriate "large J" limit at the level of classical string action, it agrees with the Landau-Lifshitz effective action derived above from $\mathsf{XXX}_{1/2}$ spin chain[195].

One advantage of considering effective action is in that it provides an intuitive map between $\mathcal{N} = 4$ operators and classical strings; for instance, the position of impurity is identified as the spatial coordinate of worldsheet. From the standpoint of effective action, folded and circular strings are interpreted as periodic solitons of Landau-Lifshitz equation, and dyonic giant magnons can be compared to pulse-like solitons in an infinite spin-chain [196, 197]. It will be interesting to study how the helical strings are interpreted in a coherent-state picture.

Interestingly, continuum limit of the half-filled one-dimensional Hubbard model was compared with string effective action in "small J" (or slow-string) limit in [198]. Although there was mismatch of a numerical factor [199], such approach would pave the way to correct understanding of $\mathcal{N} = 4$ operator dual to large winding strings.

Worldsheet scattering on $\mathbf{AdS}_5 \times \mathbf{S}^5$. In quantum field theory, the spectrum of physical particles can be read off from some poles of S-matrix of the theory. If we are interested in excitations of a string which look like particles, then studying the S-matrix gives more detailed information on that theory than the spectrum itself.

By taking so-called uniform light-cone gauge, we can spontaneously break conformal symmetry on worldsheet, and obtain a theory with massive excitations [52, 60]. Worldsheet scattering on $\mathrm{AdS}_5 \times \mathrm{S}^5$ has been studied in such a way to probe its integrable structure [200, 147]. The worldsheet scattering is also studied in the near-flat-space limit [80], in which the theory becomes facilitated while the BHL/BES dressing phase kept nontrivial [201, 202, 203].[13]

We would like to conclude this dissertation by posing some open questions.

- Integrability of the $\mathcal{N} = 4$ theory is the cornerstone of recent developments in AdS/CFT correspondence. We must ask ourselves if the $\mathcal{N} = 4$ theory is integrable to all-orders in λ for general L, or to what extent the integrability remains valid. On string theory side, the proof of (even one-loop) quantum integrability still remains an open problem [204, 205, 206].

- Even if we assume all-order integrability, we still do not understand what exactly is the Hamiltonian operator we are diagonalizing when we use the conjectured all-loop Bethe Ansatz. Furthermore, it is not clear how the dynamics of gauge or string theories favors or disfavors BES choice of the dressing phase from BHL solutions. Works such as [183, 207, 208] can be thought of as a trial for answering this question. In addition, intricate relation between Hubbard model and the $su(2|2)^2$ asymptotic spin chain has been found in [209, 61, 148], which may give a clue to the above question.

- It is known that there is close relation between the derivative sector of $\mathcal{N} = 4$ theory and that of large N QCD, such as the conjecture of transcendentality principle. There is possibility of applying methods of integrability to other conformal or superconformal theories.

[13]Strictly speaking, however, the near-flat-space limit is not the limit of infinite-J, which may possibly invalidate the BHL/BES phase.

Hopefully answers to these problems will lead to new surprising ideas, interesting observations, or useful techniques concerning strong coupling dynamics of gauge, gravity and string theories.

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Appendix A

Elliptic functions

A.1 Definitions of elliptic functions

Our conventions for the elliptic functions, elliptic integrals are presented below.

Elliptic theta functions

Let $Q = \prod_{n=1}^{\infty}(1 - e^{2\pi i n\tau})$. We define elliptic theta functions by

$$\vartheta_0(z,\tau) := Q\prod_{n=1}^{\infty}\left(1 - 2\,e^{\pi i(2n-1)\tau}\cos(2\pi nz) + e^{2\pi i(2n-1)\tau}\right), \tag{A.1.1}$$

$$\vartheta_1(z,\tau) := 2\,Q\,e^{i\pi\tau/4}\sin(2\pi z)\prod_{n=1}^{\infty}\left(1 - 2\,e^{2\pi in\tau}\cos(2\pi nz) + e^{4\pi in\tau}\right), \tag{A.1.2}$$

$$\vartheta_2(z,\tau) := 2\,Q\,e^{i\pi\tau/4}\cos(2\pi z)\prod_{n=1}^{\infty}\left(1 + 2\,e^{2\pi in\tau}\cos(2\pi nz) + e^{4\pi in\tau}\right), \tag{A.1.3}$$

$$\vartheta_3(z,\tau) := Q\prod_{n=1}^{\infty}\left(1 + 2\,e^{\pi i(2n-1)\tau}\cos(2\pi nz) + e^{2\pi i(2n-1)\tau}\right). \tag{A.1.4}$$

We also use an abbreviation $\vartheta_\nu^0 \equiv \vartheta_\nu(0,k)$. The following functions are known as Jacobi theta and zeta functions, respectively:

$$\Theta_\nu(z,k) \equiv \vartheta_\nu\left(\frac{z}{2\mathbf{K}}, \tau = \frac{i\mathbf{K}'}{\mathbf{K}}\right), \qquad Z_\nu(z,k) \equiv \frac{\partial_z\Theta_\nu(z,k)}{\Theta_\nu(z,k)}. \tag{A.1.5}$$

Complete elliptic integrals

Complete elliptic integral of the first kind and its complement are defined as, respectively,

$$\mathbf{K}(k) := \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}, \qquad \mathbf{K}'(k) := \mathbf{K}(\sqrt{1-k^2}). \tag{A.1.6}$$

We often write $\mathbf{K}(k)$ as $\mathbf{K}$. Likewise, we omit the moduli parameter k of other elliptic functions or elliptic integrals as well. There are alternative expressions for $\mathbf{K}$ and $\mathbf{K}'$ in terms of elliptic theta functions:

$$\mathbf{K}(k) = \frac{\pi(\vartheta_3^0)^2}{2}\,, \qquad \mathbf{K}'(k) = -i\mathbf{K}\tau = \frac{\pi i\tau(\vartheta_3^0)^2}{2}\,. \tag{A.1.7}$$

Complete elliptic integral of the second kind is defined as

$$\mathbf{E}(k) := \int_0^1 \sqrt{\frac{1-k^2z^2}{1-z^2}}dz = \int_0^{\mathbf{K}} \mathrm{dn}^2(u)du\,, \qquad \mathbf{E}'(k) := \mathbf{E}(\sqrt{1-k^2})\,. \tag{A.1.8}$$

Jacobi elliptic functions

Jacobi sn, dn and cn functions are defined as

$$\mathrm{sn}(z) := \frac{\vartheta_3^0}{\vartheta_2^0}\frac{\vartheta_1(w)}{\vartheta_0(w)}\,, \quad \mathrm{dn}(z) := \frac{\vartheta_0^0}{\vartheta_3^0}\frac{\vartheta_3(w)}{\vartheta_0(w)}\,, \quad \mathrm{cn}(z) := \frac{\vartheta_0^0}{\vartheta_2^0}\frac{\vartheta_2(w)}{\vartheta_0(w)}\,, \tag{A.1.9}$$

where $z = \pi\left(\vartheta_3^0\right)^2 w = 2\mathbf{K}w$. In terms of Jacobi theta functions, they can be written as

$$\mathrm{sn}(z) = \frac{\Theta_3(0)}{\Theta_2(0)}\frac{\Theta_1(z)}{\Theta_0(z)}\,, \quad \mathrm{dn}(z) = \frac{\Theta_0(0)}{\Theta_3(0)}\frac{\Theta_3(z)}{\Theta_0(z)}\,, \quad \mathrm{cn}(z) = \frac{\Theta_0(0)}{\Theta_2(0)}\frac{\Theta_2(z)}{\Theta_0(z)}\,. \tag{A.1.10}$$

The moduli k and $k' \equiv \sqrt{1-k^2}$ are related to the elliptic theta functions by

$$k \equiv \left(\frac{\vartheta_2^0}{\vartheta_3^0}\right)^2, \qquad k' \equiv \left(\frac{\vartheta_0^0}{\vartheta_3^0}\right)^2. \tag{A.1.11}$$

The Jacobi elliptic functions satisfy the following relations:

$$\begin{gathered}\mathrm{sn}^2(z,k) + \mathrm{cn}^2(z,k) = 1, \qquad k^2\,\mathrm{sn}^2(z,k) + \mathrm{dn}^2(z,k) = 1,\\ \mathrm{dn}^2(z,k) - k^2\,\mathrm{cn}^2(z,k) = 1-k^2.\end{gathered} \tag{A.1.12}$$

Normal (or incomplete) elliptic integrals

$$F(\phi,k) = \int_0^{\phi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin\phi}\frac{dt}{\sqrt{(1-t^2)\,(1-k^2t^2)}} \tag{A.1.13}$$

is called the normal elliptic integral of the first kind. At special values, it reduces to

$$F(0,k) = 0, \qquad F\left(\frac{\pi}{2},k\right) = \mathbf{K}(k), \qquad F(\phi,0) = \phi, \qquad F(\phi,1) = \mathrm{arctanh}\,\phi. \tag{A.1.14}$$

The normal elliptic integral of the first kind is related to the inverse of an elliptic function. If one regards $F(\phi,k)$ as a function of $y = \sin\phi$, then $f(y,k) \equiv F(\sin^{-1}y,k)$ obeys the differential equation

$$\left(\frac{\partial f}{\partial y}\right)^2 = \frac{1}{(1-t^2)\,(1-k^2t^2)}\,. \tag{A.1.15}$$

showing that

$$F(\phi,k) = f(y,k) = \mathrm{sn}^{-1}(y,k). \tag{A.1.16}$$

And if one regards $F(\phi, k)$ as a function of ϕ, its inverse defines Jacobi amplitude function by

$$F(\phi, k) = u \qquad \Longleftrightarrow \qquad \phi = \mathrm{am}(u, k). \tag{A.1.17}$$

From (A.1.16) and (A.1.17), it follows

$$\mathrm{sn}(u, k) = y = \sin\phi = \sin\left(\mathrm{am}(u, k)\right). \tag{A.1.18}$$

As corollaries,

$$\mathrm{cn}(u, k) = \cos\phi, \qquad \mathrm{dn}(u, k) = \sqrt{1 - k^2 \sin^2\phi} \qquad \text{for} \quad \phi = \mathrm{am}(u, k). \tag{A.1.19}$$

We also use the notation

$$\mathbf{F}(z, k) \equiv F(\phi, k), \qquad \text{for} \quad \phi = \mathrm{am}(z, k). \tag{A.1.20}$$

The normal (or incomplete) elliptic integral of the second kind is defined by

$$E(\phi, k) = \int_0^{\phi} d\theta \, \sqrt{1 - k^2 \sin^2\theta} = \int_0^{\sin\phi} dt \, \sqrt{\frac{1 - k^2 t^2}{1 - t^2}}\,. \tag{A.1.21}$$

We also use the notation

$$\mathbf{E}(z, k) \equiv E(\phi, k), \qquad \text{for} \quad \phi = \mathrm{am}(z, k). \tag{A.1.22}$$

At special values, it reduces to

$$E(0, k) = 0, \qquad E\left(\frac{\pi}{2}, k\right) = \mathbf{E}(k), \qquad E(\phi, 0) = \phi, \qquad E(\phi, 1) = \sin\phi. \tag{A.1.23}$$

The normal elliptic integral of the second kind is related to the integral of an elliptic function, as

$$E(\phi, k) = \mathbf{E}(z, k) = \int_0^{z} dw \; \mathrm{dn}^2(w, k) \qquad \text{for} \quad \phi = \mathrm{am}(z, k). \tag{A.1.24}$$

Using (A.1.17), and (A.1.24), one can rewrite Jacobi Zeta function as

$$Z_0(z, k) = E(\phi, k) - F(\phi, k)\,\frac{\mathbf{E}(k)}{\mathbf{K}(k)} \qquad \left(\phi = \mathrm{am}(z, k)\right), \tag{A.1.25}$$

or equivalently,

$$Z_0(z, k) = \mathbf{E}(z, k) - z\,\frac{\mathbf{E}(k)}{\mathbf{K}(k)}\,. \tag{A.1.26}$$

Other functions

Below we describe the definitions of other functions and integrals which will be used in Appendix A.4.

The digamma function is defined by

$$\psi(x) := \frac{d \ln \Gamma(x)}{dx}, \qquad \Gamma(x) := \int_0^\infty dt\, t^{x-1} e^{-t}. \tag{A.1.27}$$

At special values, it behaves as

$$\psi(1) = -\gamma, \qquad \psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2, \tag{A.1.28}$$

with γ the Euler-Mascheroni constant. The digamma function obeys the recurrence relation

$$\psi(x+1) = \psi(x) + \frac{1}{x}. \tag{A.1.29}$$

As corollaries,

$$\psi(n) = -\gamma + H_{n-1}, \qquad \psi\left(n + \frac{1}{2}\right) = -\gamma - 2\ln 2 + \sum_{k=1}^{n} \frac{2}{2k-1}, \qquad \text{for } \; n \in \mathbb{Z}_{\geq 1}, \tag{A.1.30}$$

where H_n is the harmonic number defined by The harmonic number:

$$H_n \equiv \sum_{k=1}^{n} \frac{1}{k}. \tag{A.1.31}$$

We also use the standard definitions of Pochhammer's symbol :

$$(a)_0 \equiv 1, \qquad (a)_n \equiv a(a+1)\cdots(a+n-1) \qquad \text{for } \; n \in \mathbb{Z}_{\geq 1}. \tag{A.1.32}$$

and binomial coefficients:

$$\binom{a}{0} = 1, \qquad \binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{1\cdot 2\cdot 3\cdots n} = (-1)^n \frac{(-a)_n}{n!}. \tag{A.1.33}$$

A.2 Mathematical facts on elliptic functions

This appendix provides some properties and formulae useful for computation involving Jacobi elliptic functions and elliptic integrals.

A.2.1 Useful properties

Parity

$$\text{Odd functions:} \qquad \mathrm{sn}(z), \qquad \Theta_1(z), \qquad Z_\nu(z) \quad (\nu = 0, \cdots, 3), \tag{A.2.1}$$

$$\text{Even functions:} \qquad \mathrm{dn}(z),\; \mathrm{cn}(z), \qquad \Theta_0(z),\; \Theta_2(z),\; \Theta_3(z). \tag{A.2.2}$$

All functions listed above are meromorphic, so we have

$$f(\bar{z}) = \overline{f(z)} \qquad \text{and} \qquad f_{\rm odd}(ix) \in \sqrt{-1}\,\mathbb{R}\,, \qquad f_{\rm even}(ix) \in \mathbb{R}\,, \qquad \text{for} \;\; x \in \mathbb{R}\,. \tag{A.2.3}$$

Special values

$\Theta_\nu(z)$ have no poles at finite values of z, but have zeros at

$$\Theta_0(z): \quad z = 2m\mathbf{K} + (2n+1)\,i\mathbf{K}'\,, \tag{A.2.4}$$

$$\Theta_1(z): \quad z = 2m\mathbf{K} + 2ni\mathbf{K}'\,, \tag{A.2.5}$$

$$\Theta_2(z): \quad z = (2m+1)\mathbf{K} + 2ni\mathbf{K}'\,, \tag{A.2.6}$$

$$\Theta_3(z): \quad z = (2m+1)\mathbf{K} + (2n+1)\,i\mathbf{K}'\,, \tag{A.2.7}$$

where $m,\ n$ are integers.

For Jacobi sn, cn and dn functions, we have

$$\mathrm{sn}(0,k) = 0, \quad \mathrm{sn}(\mathbf{K},k) = 1, \quad \mathrm{sn}(i\mathbf{K}') = \infty, \tag{A.2.8}$$

$$\mathrm{dn}(0,k) = 1, \quad \mathrm{dn}(\mathbf{K},k) = k', \quad \mathrm{dn}(i\mathbf{K}') = \infty, \tag{A.2.9}$$

$$\mathrm{cn}(0,k) = 1, \quad \mathrm{cn}(\mathbf{K},k) = 0, \quad \mathrm{cn}(i\mathbf{K}') = \infty. \tag{A.2.10}$$

We also have

$$\mathrm{sn}(z,k) = z - \left(\frac{1+k^2}{6}\right) z^3 + \left(\frac{1+14k^2+k^4}{120}\right) z^5 + \mathcal{O}\left(z^7\right), \qquad \text{as} \;\; z \to 0. \tag{A.2.11}$$

The complete elliptic integrals $\mathbf{K}(k),\ 1/\mathbf{K}'(k),\ 1/\mathbf{E}(k)$ are monotonically increasing function of k. They take the values

$$\mathbf{K}(0) = \frac{\pi}{2}, \quad \mathbf{K}(1) = \infty, \quad \mathbf{K}'(0) = \infty, \quad \mathbf{K}'(1) = \frac{\pi}{2}, \quad \mathbf{E}(0) = \frac{\pi}{2}, \quad \mathbf{E}(1) = 1. \tag{A.2.12}$$

We also have

$$\mathbf{K}(k) - \mathbf{E}(k) = \frac{\pi}{4}\,k^2 + \frac{3\pi}{32}\,k^4 + \mathcal{O}(k^6)\,. \tag{A.2.13}$$

The zeros and the poles of $Z_0(z)$ are located at

$$Z_0(z) = 0 \quad \text{at} \;\; z = m\mathbf{K} + 2ni\mathbf{K}', \qquad Z_0(z) = \infty \quad \text{at} \;\; z = 2m\mathbf{K} + (2n+1)i\mathbf{K}'\,. \tag{A.2.14}$$

where $m,\ n$ are integers. Asymptotically, $Z_\nu(z)$ behave as

$$Z_0(z) \sim \frac{1}{z - i\mathbf{K}'} + \frac{i\pi}{2\mathbf{K}} + \mathcal{O}(z - i\mathbf{K}'), \qquad Z_1(z) \sim \frac{1}{z} + \mathcal{O}(z). \tag{A.2.15}$$

To derive them, the following identity is useful:

$$\frac{\partial^2 \vartheta_j\,(z,\tau)}{\partial z^2} = 4\pi i\,\frac{\partial \vartheta_j\,(z,\tau)}{\partial \tau}\,, \tag{A.2.16}$$

We also have

$$Z_0(\mathbf{K}+i\mathbf{K}') = -\frac{\pi i}{2\mathbf{K}}\,, \quad Z_2(0) = 0. \tag{A.2.17}$$

Periodicity

Jacobi sn, cn, dn functions have the following periodicity:

$$\begin{aligned}
&\mathrm{sn}(z+\mathbf{K}) = \frac{\mathrm{cn}z}{\mathrm{dn}z}\,, && \mathrm{sn}(z+i\mathbf{K}') = \frac{1}{k\,\mathrm{sn}z}\,, && \mathrm{sn}(z+2\mathbf{K}) = -\,\mathrm{sn}(z)\,, && \mathrm{sn}(z+2i\mathbf{K}') = \,\mathrm{sn}z,\\
&\mathrm{dn}(z+\mathbf{K}) = \frac{k'}{\mathrm{dn}z}\,, && \mathrm{dn}(z+i\mathbf{K}') = \frac{\mathrm{cn}z}{i\,\mathrm{sn}z}\,, && \mathrm{dn}(z+2\mathbf{K}) = \,\mathrm{dn}z\,, && \mathrm{dn}(z+2i\mathbf{K}') = -\,\mathrm{dn}z,\\
&\mathrm{cn}(z+\mathbf{K}) = -\frac{k'\,\mathrm{sn}z}{\mathrm{dn}z}\,, && \mathrm{cn}(z+i\mathbf{K}') = \frac{\mathrm{dn}z}{ik\,\mathrm{sn}z}\,, && \mathrm{cn}(z+2\mathbf{K}) = -\,\mathrm{cn}z\,, && \mathrm{cn}(z+2i\mathbf{K}') = -\,\mathrm{cn}z.
\end{aligned}$$

As corollaries, we have

$$\mathrm{sn}(z-\mathbf{K}) = -\frac{\mathrm{cn}z}{\mathrm{dn}z}\,, \quad \mathrm{sn}(z-i\mathbf{K}') = \frac{1}{k\,\mathrm{sn}z}\,, \quad \mathrm{sn}(z-\mathbf{K}-i\mathbf{K}') = -\frac{\mathrm{dn}z}{k\,\mathrm{cn}z}\,, \tag{A.2.18}$$

$$\mathrm{dn}(z-\mathbf{K}) = \frac{k'}{\mathrm{dn}z}\,, \quad \mathrm{dn}(z-i\mathbf{K}') = i\,\frac{\mathrm{cn}z}{\mathrm{sn}z}\,, \quad \mathrm{dn}(z-\mathbf{K}-i\mathbf{K}') = \frac{-ik'\,\mathrm{sn}z}{\mathrm{cn}z}\,, \tag{A.2.19}$$

$$\mathrm{cn}(z-\mathbf{K}) = \frac{k'\,\mathrm{sn}z}{\mathrm{dn}z}\,, \quad \mathrm{cn}(z-i\mathbf{K}') = i\,\frac{\mathrm{dn}z}{k\,\mathrm{sn}z}\,, \quad \mathrm{cn}(z-\mathbf{K}-i\mathbf{K}') = \frac{-ik'}{k\,\mathrm{cn}z}\,. \tag{A.2.20}$$

For Jacobi theta functions, we have

$$\begin{aligned}
&\Theta_0(z+\mathbf{K}) = \Theta_3(z)\,, && \Theta_0(z+i\mathbf{K}') = iN\Theta_1(z)\,, && \Theta_0(z+\mathbf{K}+i\mathbf{K}') = N\Theta_2(z)\,,\\
&\Theta_1(z+\mathbf{K}) = \Theta_2(z)\,, && \Theta_1(z+i\mathbf{K}') = iN\Theta_0(z)\,, && \Theta_1(z+\mathbf{K}+i\mathbf{K}') = N\Theta_3(z)\,,\\
&\Theta_2(z+\mathbf{K}) = -\Theta_1(z)\,, && \Theta_2(z+i\mathbf{K}') = N\Theta_3(z)\,, && \Theta_2(z+\mathbf{K}+i\mathbf{K}') = -iN\Theta_0(z)\,,\\
&\Theta_3(z+\mathbf{K}) = \Theta_0(z)\,, && \Theta_3(z+i\mathbf{K}') = N\Theta_2(z)\,, && \Theta_3(z+\mathbf{K}+i\mathbf{K}') = iN\Theta_1(z)\,,
\end{aligned}$$

where

$$N = N(z) \equiv \exp\left(-\frac{i\pi}{2\mathbf{K}}\,(z+i\mathbf{K}'/2)\right). \tag{A.2.21}$$

As corollaries,

$$\begin{aligned}
&\Theta_0(z+2\mathbf{K}) = \Theta_0(z)\,, && \Theta_0(z+2i\mathbf{K}') = -M\,\Theta_0(z)\,,\\
&\Theta_1(z+2\mathbf{K}) = -\Theta_1(z)\,, && \Theta_1(z+2i\mathbf{K}') = -M\,\Theta_1(z)\,,\\
&\Theta_2(z+2\mathbf{K}) = -\Theta_2(z)\,, && \Theta_2(z+2i\mathbf{K}') = M\,\Theta_2(z)\,,\\
&\Theta_3(z+2\mathbf{K}) = \Theta_3(z)\,, && \Theta_3(z+2i\mathbf{K}') = M\,\Theta_3(z)\,,
\end{aligned}$$

where

$$M = M(z) \equiv \exp\left(-\frac{i\pi}{K}\,(z+i\mathbf{K}')\right). \tag{A.2.22}$$

Periodicity for $Z_\nu(z)$ can be derived from the one for $\Theta_\nu(z)$:

$$Z_0(z+2\mathbf{K}) = Z_0(z), \quad Z_0(z+2i\mathbf{K}') = Z_0(z) - \frac{i\pi}{\mathbf{K}}\,. \tag{A.2.23}$$

We also have

$$Z_1(z) = Z_0(z+i\mathbf{K}') + \frac{i\pi}{2\mathbf{K}}\,, \qquad Z_1(z) = Z_0(z-i\mathbf{K}') - \frac{i\pi}{2\mathbf{K}}\,, \tag{A.2.24}$$

$$Z_2(z) = Z_0(z+\mathbf{K}+i\mathbf{K}') + \frac{i\pi}{2\mathbf{K}}\,, \qquad Z_2(z) = Z_0(z-\mathbf{K}-i\mathbf{K}') - \frac{i\pi}{2\mathbf{K}}\,, \tag{A.2.25}$$

$$Z_3(z) = Z_0(z+\mathbf{K})\,, \qquad Z_3(z) = Z_0(z-\mathbf{K})\,. \tag{A.2.26}$$

See also (A.2.35), (A.2.36), (A.2.37) for the relation among Jacobin Zeta functions,

Derivative

Derivative of elliptic functions with respect to z is summarized as follows.

$$\frac{\partial}{\partial z}\,\mathrm{sn}z = \mathrm{cn}z\,\mathrm{dn}z, \quad \frac{\partial}{\partial z}\,\mathrm{dn}z = -k^2\,\mathrm{sn}z\,\mathrm{cn}z, \quad \frac{\partial}{\partial z}\,\mathrm{cn}z = -\,\mathrm{sn}z\,\mathrm{dn}z, \tag{A.2.27}$$

$$\frac{\partial}{\partial z} Z_0(z) = \mathrm{dn}^2 z - \frac{\mathbf{E}(k)}{\mathbf{K}(k)}\,, \qquad \frac{\partial}{\partial z} Z_1(z) = -\frac{\mathrm{cn}^2 z}{\mathrm{sn}^2 z} - \frac{\mathbf{E}(k)}{\mathbf{K}(k)}\,. \tag{A.2.28}$$

A.2.2 Useful formulae

We collect useful formulae to perform calculation in Section A.3. For details, please consult textbooks such as [157, 210].

Addition and multiplication formulae

For Jacobi sn function, we have

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn}u\ \mathrm{cn}v\,\mathrm{dn}v + \mathrm{sn}v\ \mathrm{cn}u\,\mathrm{dn}u}{1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v}, \quad \mathrm{sn}(u+v)\,\mathrm{sn}(u-v) = \frac{\mathrm{sn}^2 u - \mathrm{sn}^2 v}{1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v}\,, \tag{A.2.29}$$

and in particular,

$$\mathrm{sn}(2u) = \frac{2\,\mathrm{sn}u\,\mathrm{cn}u\,\mathrm{dn}u}{1-k^2\,\mathrm{sn}^4 u}\,. \tag{A.2.30}$$

For Jacobi Zeta function,

$$Z_0(u+v) = Z_0(u) + Z_0(v) - k^2\,\mathrm{sn}(u)\,\mathrm{sn}(v)\,\mathrm{sn}(u+v)\,. \tag{A.2.31}$$

As corollaries, by putting $u = x + i\mathbf{K}',\ v = y + i\mathbf{K}'$ we get

$$\frac{1}{2}\Big(Z_1(x+y) + Z_1(x-y)\Big) = Z_0(x) + \frac{\mathrm{sn}x\,\mathrm{cn}x\,\mathrm{dn}x}{\mathrm{sn}^2 x - \mathrm{sn}^2 y}\,. \tag{A.2.32}$$

And by putting $u = x + i\mathbf{K}'$, $v = y - x$ we get

$$\frac{1}{2}\Big(Z_1(x+y) - Z_1(x-y)\Big) = Z_0(y) - \frac{\mathrm{sn}y\,\mathrm{cn}y\,\mathrm{dn}y}{\mathrm{sn}^2 x - \mathrm{sn}^2 y}\,. \tag{A.2.33}$$

Other formulae

Below are the results listed in [210]. Another expression of Jacobi Zeta is,

$$Z_0(z,k) = \frac{k^2\,\mathrm{sn}(z,k)\,\mathrm{cn}(z,k)\,\mathrm{dn}(z,k)}{\mathbf{K}(k)} \int_0^{\mathbf{K}(k)} \frac{du\ \mathrm{sn}^2(u,k)}{1 - k^2\,\mathrm{sn}^2(z,k)\,\mathrm{sn}^2(u,k)}\,. \tag{A.2.34}$$

Using the addition formula (A.2.31), one can express Jacobi Zeta's solely by Z_0, as

$$Z_1(z,k) = Z_0(z,k) + \frac{\mathrm{cn}(z,k)\,\mathrm{dn}(z,k)}{\mathrm{sn}(z,k)}\,, \tag{A.2.35}$$

$$Z_2(z,k) = Z_0(z,k) - \frac{\mathrm{sn}(z,k)\,\mathrm{dn}(z,k)}{\mathrm{cn}(z,k)}\,, \tag{A.2.36}$$

$$Z_3(z,k) = Z_0(z,k) - \frac{k^2\,\mathrm{sn}(z,k)\,\mathrm{cn}(z,k)}{\mathrm{dn}(z,k)}\,. \tag{A.2.37}$$

Trigonometric limits

By taking $k \to 0$ or $k \to 1$ in (A.1.11), elliptic functions reduce to trigonometric functions.

$$\mathrm{sn}(z,0) = \sin(z), \qquad \mathrm{dn}(z,0) = 1, \qquad \mathrm{cn}(z,0) = \cos(z), \tag{A.2.38}$$

$$\mathrm{sn}(z,1) = \tanh(z), \qquad \mathrm{dn}(z,1) = \frac{1}{\cosh(z)}\,, \qquad \mathrm{cn}(z,1) = \frac{1}{\cosh(z)}\,. \tag{A.2.39}$$

Jacobi zeta and theta functions become

$$Z_0(z,0) = 0, \qquad \Longrightarrow \qquad \Theta_0(z,0) = A'\,, \tag{A.2.40}$$

$$Z_0(z,1) = \tanh(z), \qquad \Longrightarrow \qquad \Theta_0(z,1) = A\cosh(z)\,, \tag{A.2.41}$$

where A and A' are possibly divergent constants. We can find the trigonometric limit of other Jacobi zeta and theta functions from the definitions of Jacobi sn, cn and dn functions (A.1.9):

$$\Theta_1(z,0) = \sqrt{k}\,\Theta_0(z,k)\,\mathrm{sn}(z,k)\,, \tag{A.2.42}$$

$$\Theta_2(z,0) = \sqrt{\frac{k}{k'}}\,\Theta_0(z,k)\,\mathrm{cn}(z,k)\,, \tag{A.2.43}$$

$$\Theta_3(z,0) = \frac{1}{\sqrt{k'}}\,\Theta_0(z,k)\,\mathrm{dn}(z,k)\,. \tag{A.2.44}$$

In the $k \to 0$ limit, they become

$$\Theta_1(z,0) = \sqrt{k}\,A'\sin(z)\,, \qquad \Theta_2(z,0) = \sqrt{k}\,A'\cos(z)\,, \qquad \Theta_3(z,0) = A'\,. \tag{A.2.45}$$

$$Z_1(z,0) = \cot(z)\,, \qquad Z_2(z,0) = -\tan(z)\,, \qquad Z_3(z,0) = 0\,. \tag{A.2.46}$$

In the $k \to 1$ limit, we have

$$\Theta_1(z,1) = A\sinh(z)\,, \qquad \Theta_2(z,1) = \frac{A}{\sqrt{k'}}\,, \qquad \Theta_3(z,1) = \frac{A}{\sqrt{k'}}\,. \tag{A.2.47}$$

$$Z_1(z,1) = \coth(z)\,, \qquad Z_2(z,1) = Z_3(z,1) = 0\,. \tag{A.2.48}$$

A.2.3 Moduli transformations

We collect some formulae for $SL(2,\mathbb{Z})$ transformations acting on elliptic functions.

Elliptic theta functions transform under the T-transformation as

$$\vartheta_0(z|\tau+1) = \vartheta_3(z|\tau)\,, \qquad \vartheta_1(z|\tau+1) = e^{\pi i/4}\,\vartheta_1(z|\tau)\,, \tag{A.2.49}$$

$$\vartheta_2(z|\tau+1) = e^{\pi i/4}\,\vartheta_2(z|\tau)\,, \qquad \vartheta_3(z|\tau+1) = \vartheta_0(z|\tau)\,, \tag{A.2.50}$$

and complete elliptic integrals with $q \geq 0$ transform as

$$\mathbf{K}(q) = k'\mathbf{K}(k)\,, \qquad \mathbf{K}'(q) = k'\left(\mathbf{K}'(k) - i\mathbf{K}(k)\right)\,, \qquad \mathbf{E}(q) = \mathbf{E}(k)/k'\,. \tag{A.2.51}$$

Jacobi theta functions, defined by

$$\Theta_\nu(z,k) \equiv \vartheta_\nu\left(\frac{z}{2\mathbf{K}(k)},\, \tau = \frac{i\mathbf{K}'(k)}{\mathbf{K}(k)}\right)\,, \quad (\nu = 0,1,2,3) \tag{A.2.52}$$

transform as

$$\Theta_0(z|\tau+1) = \Theta_3(z/k'|\tau)\,, \qquad \Theta_1(z|\tau+1) = e^{\pi i/4}\,\Theta_1(z/k'|\tau)\,, \tag{A.2.53}$$

$$\Theta_2(z|\tau+1) = e^{\pi i/4}\,\Theta_2(z/k'|\tau)\,, \qquad \Theta_3(z|\tau+1) = \Theta_0(z/k'|\tau)\,, \tag{A.2.54}$$

and Jacobi zeta functions defined by $Z_\nu(z,k) \equiv \partial_z \ln \Theta_\nu(z,k)$ transform as

$$Z_0(z|\tau+1) = Z_3(z/k'|\tau)/k'\,, \qquad Z_1(z|\tau+1) = Z_1(z/k'|\tau)/k'\,, \tag{A.2.55}$$

$$Z_2(z|\tau+1) = Z_2(z/k'|\tau)/k'\,, \qquad Z_3(z|\tau+1) = Z_0(z/k'|\tau)/k'\,. \tag{A.2.56}$$

Therefore, the T-transformation acts on the elliptic modulus k as

$$q \equiv \left(\frac{\Theta_2(0|\tau+1)}{\Theta_3(0|\tau+1)}\right)^2 = i\left(\frac{\Theta_2(0|\tau)}{\Theta_0(0|\tau)}\right)^2 = \frac{ik}{k'}\,, \tag{A.2.57}$$

$$q' \equiv \left(\frac{\Theta_0(0|\tau+1)}{\Theta_3(0|\tau+1)}\right)^2 = \left(\frac{\Theta_3(0|\tau)}{\Theta_0(0|\tau)}\right)^2 = \frac{1}{k'}\,. \tag{A.2.58}$$

In terms of the modulus q defined in (8.2.1), the Jacobi sn, cn and dn functions are written as

$$\mathrm{sn}(z,q) = k'\frac{\mathrm{sn}(z/k',k)}{\mathrm{dn}(z/k',k)}\,, \quad \mathrm{cn}(z,q) = \frac{\mathrm{cn}(z/k',k)}{\mathrm{dn}(z/k',k)}\,, \quad \mathrm{dn}(z,q) = \frac{1}{\mathrm{dn}(z/k',k)}\,. \tag{A.2.59}$$

Normal elliptic integrals behaves under reciprocal modular transformation, as

$$\mathbf{F}\left(z,\frac{1}{k}\right) = k\mathbf{F}\left(\frac{z}{k},k\right)\,, \qquad \mathbf{E}\left(z,\frac{1}{k}\right) = \frac{1}{k}\left\{\mathbf{E}\left(\frac{z}{k},k\right) - \left(1-k^2\right)\mathbf{F}\left(\frac{z}{k},k\right)\right\}\,. \tag{A.2.60}$$

A.3 Some details of calculations

In this appendix we will collect some key formulae that would be useful in checking the calculation involving the function

$$\begin{aligned}\Xi(X,T,w) &= \frac{\Theta_1(X-X_0-w+w_0)}{\Theta_0(X-X_0)\,\Theta_0(w-w_0)}\exp\Big(Z_0(w-w_0)(X-X_0)+iu(T-T_0)\Big),\\ u^2 &= U+\,\mathrm{dn}^2(w-w_0),\end{aligned} \tag{A.3.1}$$

where X, X_0, T and T_0 are all real. For the moment we assume w and w_0 to be purely imaginary. The degrees of freedom of (T_0, X_0) correspond to the initial values for the phases of ξ_j, and in what follows, we will set them as zero. We will also set $w_0 = 0$.

As a preliminary, we shall collect several useful formulae from the ones presented in the last section.

- One can express $Z_0(z,k)$ in terms of Jacobi dn function and complete elliptic integrals as

$$Z_0(z,k) = \int_0^z \mathrm{dn}^2(u,k)du - z\,\frac{\mathbf{E}}{\mathbf{K}}\,. \tag{A.3.2}$$

- By using an addition theorem

$$Z_0(u+v) = Z_0(u) + Z_0(v) - k^2\,\mathrm{sn}(u)\,\mathrm{sn}(v)\,\mathrm{sn}(u+v)\,, \tag{A.3.3}$$

one can verify the following identities:

$$\frac{1}{2}\Big(Z_1(x+y)+Z_1(x-y)\Big) = Z_0(x) + \frac{\mathrm{sn}x\,\mathrm{cn}x\,\mathrm{dn}x}{\mathrm{sn}^2x-\,\mathrm{sn}^2y}\,, \tag{A.3.4}$$

$$\frac{1}{2}\Big(Z_1(x+y)-Z_1(x-y)\Big) = Z_0(y) - \frac{\mathrm{sn}y\,\mathrm{cn}y\,\mathrm{dn}y}{\mathrm{sn}^2x-\,\mathrm{sn}^2y}\,. \tag{A.3.5}$$

- Concerning the absolute value of $\Xi(X,T,w)$, one can show that

$$\frac{\Theta_1(z-w)\,\Theta_1(z+w)}{\Theta_0^2(z)\,\Theta_0^2(w)} = \frac{k}{\Theta_0^2(0)}\Big(\,\mathrm{sn}^2z-\,\mathrm{sn}^2w\Big)\,. \tag{A.3.6}$$

With the help of those formulae, we can easily arrived at the following relations:

$$\left|\frac{\partial_X\Xi}{\Xi}\right|^2 = \frac{\mathrm{sn}^2(X)\,\mathrm{cn}^2(X)\,\mathrm{dn}^2(X)-\,\mathrm{sn}^2(w)\,\mathrm{cn}^2(w)\,\mathrm{dn}^2(w)}{(\,\mathrm{sn}^2(X)-\,\mathrm{sn}^2(w))^2}\,, \tag{A.3.7}$$

$$\mathrm{Re}\left(\frac{\partial_T\Xi^*}{\Xi}\,\frac{\partial_X\Xi}{\Xi}\right) = -iu\,\frac{\mathrm{sn}(w)\,\mathrm{cn}(w)\,\mathrm{dn}(w)}{\mathrm{sn}^2(X)-\,\mathrm{sn}^2(w)}\,, \tag{A.3.8}$$

$$\mathrm{Im}\left(\frac{\partial_X\Xi}{\Xi}\right) = \frac{1}{i}\,\frac{\mathrm{sn}(w)\,\mathrm{cn}(w)\,\mathrm{dn}(w)}{\mathrm{sn}^2(X)-\,\mathrm{sn}^2(w)}\,, \tag{A.3.9}$$

which should be useful in evaluating the consistency condition, Virasoro conditions and conserved charges in the main text.

We can now discuss a generalization of the Ansatz (6.2.6). In order for $\Xi(X, T, w)$ to be normalizable for all range of X, $Z_0(w,k)$ must be purely imaginary. When k is real, this can be achieved if and only if $w = m\mathbf{K} + i\omega$ with $m \in \mathbb{Z}$ and $\omega \in \mathbb{R}$. Therefore, with the Ansatz (6.2.6), general solutions of (8.1.12) are given by

$$\Xi^0 = \frac{\Theta_1(X - i\omega)}{\Theta_0(X)\Theta_0(i\omega)} \exp\Big(Z_0(i\omega)X + iuT\Big), \quad u^2 = U + \mathrm{dn}^2(i\omega)\,, \tag{A.3.10}$$

$$\Xi^1 = \frac{\Theta_0(X - i\omega)}{\Theta_0(X)\Theta_1(i\omega)} \exp\Big(Z_1(i\omega)X + iuT\Big), \quad u^2 = U - \frac{\mathrm{cn}^2(i\omega)}{\mathrm{sn}^2(i\omega)}\,, \tag{A.3.11}$$

$$\Xi^2 = \frac{\Theta_3(X - i\omega)}{\Theta_0(X)\Theta_2(i\omega)} \exp\Big(Z_2(i\omega)X + iuT\Big), \quad u^2 = U - \frac{(1-k^2)\,\mathrm{sn}^2(i\omega)}{\mathrm{cn}^2(i\omega)}\,, \tag{A.3.12}$$

$$\Xi^3 = \frac{\Theta_2(X - i\omega)}{\Theta_0(X)\Theta_3(i\omega)} \exp\Big(Z_3(i\omega)X + iuT\Big), \quad u^2 = U + \frac{1-k^2}{\mathrm{dn}^2(i\omega)}\,. \tag{A.3.13}$$

These four functions are mutually related by a shift of w as

$$\begin{aligned} &\Xi^0(X,T;w) = \Xi(X,T;w = i\omega)\,, && \Xi^1(X,T;w) = -\Xi(X,T;w = i\omega - i\mathbf{K}')\,, \\ &\Xi^2(X,T;w) = \Xi(X,T;w = i\omega - \mathbf{K} - i\mathbf{K}')\,, && \Xi^3(X,T;w) = \Xi(X,T;w = i\omega - \mathbf{K})\,. \end{aligned} \tag{A.3.14}$$

Note that in $\omega \to 0$ limit, the functions Ξ^0, Ξ^2 and Ξ^3 reduce to $\mathrm{sn}(X)$, $\mathrm{dn}(X)$ and $\mathrm{cn}(X)$ with the angular velocity satisfying $u^2 = U + 1$, U and $U + 1 - k^2$, respectively.

It would also be useful to note the properties of Ξ^i given in (A.3.14). They are doubly periodic with respect to w:

$$\Xi^i \to -\Xi^i \quad (w \to w + 2\mathbf{K})\,, \qquad \Xi^i \to \Xi^i \quad (w \to w + 2i\mathbf{K}')\,, \tag{A.3.15}$$

and quasi-periodic with respect to X:

$$\begin{aligned} &\Xi^0(X + 2\mathbf{K}) = -e^{2Z_0(w)\mathbf{K}}\,\Xi^0(X)\,, && \Xi^1(X + 2\mathbf{K}) = e^{2Z_1(w)\mathbf{K}}\,\Xi^1(X)\,, \\ &\Xi^2(X + 2\mathbf{K}) = e^{2Z_2(w)\mathbf{K}}\,\Xi^2(X)\,, && \Xi^3(X + 2\mathbf{K}) = -e^{2Z_3(w)\mathbf{K}}\,\Xi^3(X)\,. \end{aligned} \tag{A.3.16}$$

A.4 Expansions around $k = 1$

Behavior of Jacobi elliptic functions around $k = 1$ is discussed below.

A.4.1 Jacobi sn, cn and dn functions

Jacobi sn, cn, and dn functions can be expanded in power series of $k'^2 \equiv 1 - k^2$ around $k = 1$. We want to know the expansion up to the order of k'^4 for later use. We follow the method of [211], where they computed asymptotics around $k = 0$.

The Jacobi sn function obeys an equation

$$u = \int_0^{\mathrm{sn}(u,k)} \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\,. \tag{A.4.1}$$

Differentiating both sides with respect to k, one finds

$$\frac{\partial\,\mathrm{sn}(u,k)}{\partial k} = -\,\mathrm{cn}(u,k)\,\mathrm{dn}(u,k) \int_0^{\mathrm{sn}(u,k)} \frac{k t^2\,dt}{\sqrt{1-t^2}\,(1-k^2t^2)^{3/2}}\,. \tag{A.4.2}$$

Taking the limit $k \to 1$ and substituting $u = i\omega$, we obtain

$$\left.\frac{\partial\,\mathrm{sn}(u,k)}{\partial k}\right|_{k\to 1} = \frac{i\,(\omega - \sin\omega\cos\omega)}{2\cos^2\omega}\,, \tag{A.4.3}$$

which is the first term in the expansion of the Jacobi sn function around $k=1$. The asymptotics of the Jacobi cn and dn functions can be determined by the relations

$$\mathrm{sn}^2(u,k) + \mathrm{cn}^2(u,k) = 1, \qquad \mathrm{dn}^2(u,k) + k^2\,\mathrm{sn}^2(u,k) = 1\,. \tag{A.4.4}$$

We write down the results at higher orders in k':[1]

$$\begin{aligned}
\mathrm{sn}(i\omega,k) &\approx i\tan(\omega) + \frac{i(1-k^2)}{4\cos^2(\omega)}\,(\sin\omega\cos\omega - \omega) \\
&\quad + \frac{i(1-k^2)^2}{64\cos^3(\omega)}\Big(-9\omega\cos\omega + \sin\omega\left(4\omega^2 + 9 - 7\sin^2\omega - 2\sin^4\omega\right)\Big),
\end{aligned} \tag{A.4.5}$$

$$\begin{aligned}
\mathrm{cn}(i\omega,k) &\approx \frac{1}{\cos\omega} + \frac{1-k^2}{4\cos^2(\omega)}\left(\cos\omega\sin^2\omega - \omega\sin\omega\right) \\
&\quad + \frac{(1-k^2)^2}{64\cos^3(\omega)}\Big(2\omega^2\left(1+\sin^2\omega\right) - \omega\sin\omega\cos\omega\left(13 - 4\sin^2\omega\right) + 11\sin^2\omega\cos^2\omega\Big),
\end{aligned} \tag{A.4.6}$$

$$\begin{aligned}
\mathrm{dn}(i\omega,k) &\approx \frac{1}{\cos\omega} - \frac{1-k^2}{4\cos^2(\omega)}\left(\cos\omega\sin^2\omega + \omega\sin\omega\right) \\
&\quad + \frac{(1-k^2)^2}{64\cos^3(\omega)}\Big(2\omega^2\left(1+\sin^2\omega\right) + \omega\sin\omega\cos\omega\left(3 - 4\sin^2\omega\right) - 5\sin^2\omega\cos^2\omega\Big).
\end{aligned} \tag{A.4.7}$$

A.4.2 Elliptic Integrals and Jacobi Zeta function

The expansion of elliptic integrals and Jacobi Zeta functions around $k=1$ is not polynomial in k', because it involves $\ln k'$. Here we borrow the general results from the textbook [210],

Normal elliptic integrals

Normal elliptic integral of the first kind behaves as

$$F(\phi,k) = \sum_{m=0}^{\infty} \binom{-1/2}{m} k'^{2m} \varrho_{2m}(\phi) \qquad \text{for} \;\; (0 < k'^2\tan^2\phi < 1,\;\; k<1), \tag{A.4.8}$$

[1] These results can be checked also by `Mathematica 6`.

where

$$\varrho_0(\phi) = \ln\left(\frac{1+\sin\phi}{\cos\phi}\right), \qquad \varrho_2(\phi) = \frac{1}{2}\left[\frac{\sin\phi}{\cos^2\phi} - \ln\left(\frac{1+\sin\phi}{\cos\phi}\right)\right], \tag{A.4.9}$$

$$\varrho_4(\phi) = \frac{1}{8}\left[\frac{2\sin^3\phi}{\cos^4\phi} - \frac{3\sin\phi}{\cos^2\phi} + 3\ln\left(\frac{1+\sin\phi}{\cos\phi}\right)\right]. \tag{A.4.10}$$

And in general

$$\varrho_{2m}(\phi) = \frac{1}{2m}\left\{-(2m-1)\varrho_{2m-2}(\phi) + \frac{\sin^{2m-1}\phi}{\cos^{2m}\phi}\right\}, \qquad (m \geq 1). \tag{A.4.11}$$

First few terms are written as

$$F(\phi,k) = \ln\left(\frac{1+\sin\phi}{\cos\phi}\right) - \frac{k'^2}{4}\left[\frac{\sin\phi}{\cos^2\phi} - \ln\left(\frac{1+\sin\phi}{\cos\phi}\right)\right] + \frac{3k'^4}{64}\left[\frac{2\sin^3\phi}{\cos^4\phi} - \frac{3\sin\phi}{\cos^2\phi} + 3\ln\left(\frac{1+\sin\phi}{\cos\phi}\right)\right] + \cdots. \tag{A.4.12}$$

Normal elliptic integral of the second kind behaves as

$$E(\phi,k) = \sum_{m=0}^{\infty}\binom{1/2}{m}k'^{2m}d_{2m}(\phi) \qquad \text{for} \ \ (0 < k'^2\tan^2\phi < 1,\ k < 1), \tag{A.4.13}$$

where

$$d_0(\phi) = \sin\phi, \qquad d_2(\phi) = -\sin\phi + \ln\left(\frac{1+\sin\phi}{\cos\phi}\right), \tag{A.4.14}$$

$$d_4(\phi) = \frac{1}{2}\left[\frac{\sin^3\phi}{\cos^2\phi} + 3\sin\phi - 3\ln\left(\frac{1+\sin\phi}{\cos\phi}\right)\right]. \tag{A.4.15}$$

And in general

$$d_{2m}(\phi) = \frac{1}{2(m-1)}\left\{-(2m-1)d_{2m-2}(\phi) + \frac{\sin^{2m-1}\phi}{\cos^{2m-2}\phi}\right\}, \qquad (m \geq 2). \tag{A.4.16}$$

First few terms are

$$E(\phi,k) = \sin\phi + \frac{k'^2}{2}\left[-\sin\phi + \ln\left(\frac{1+\sin\phi}{\cos\phi}\right)\right] - \frac{k'^4}{16}\left[\frac{\sin^3\phi}{\cos^2\phi} + 3\sin\phi - 3\ln\left(\frac{1+\sin\phi}{\cos\phi}\right)\right] + \cdots. \tag{A.4.17}$$

Complete elliptic integrals

Complete elliptic integral of the first kind behaves as

$$\mathbf{K}(k) = \sum_{m=0}^{\infty}\binom{-1/2}{m}^2 k'^{2m}\left[\ln\left(\frac{4}{k'}\right) - b_m\right], \qquad (k < 1), \tag{A.4.18}$$

where

$$b_0 = 0, \quad b_m = 2\sum_{j=1}^{2m} \frac{(-1)^j}{j} = b_{m-1} + \frac{2}{2m\,(2m-1)}\,. \tag{A.4.19}$$

First few terms are written as

$$\mathbf{K}(k) = \ln\left(\frac{4}{k'}\right) + \frac{k'^2}{4}\left[\ln\left(\frac{4}{k'}\right) - 1\right] + \frac{9k'^4}{64}\left[\ln\left(\frac{4}{k'}\right) - \frac{7}{6}\right] + \cdots\,. \tag{A.4.20}$$

Another but equivalent expression of this expansion is

$$\mathbf{K}(k) = \sum_{m=0}^{\infty}\left(\frac{\left(\frac{1}{2}\right)_m}{m!}\right)^2 k'^{2m}\left[-\ln k' + \psi\,(m+1) - \psi\left(m+\frac{1}{2}\right)\right], \qquad (k<1), \tag{A.4.21}$$

where ψ is the digamma function defined in (A.1.27).

Complete elliptic integral of the second kind behaves as:

$$\begin{aligned}\mathbf{E}(k) = 1 + \frac{1}{4}\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}\right)_m\left(\frac{3}{2}\right)_m}{m!(m+1)!}\ k'^{2m}\times \\ \left[-2\ln k' + \psi\,(m+2) + \psi\,(m+1) - \psi\left(m+\frac{3}{2}\right) - \psi\left(m+\frac{1}{2}\right)\right], \qquad (k<1).\end{aligned} \tag{A.4.22}$$

First few terms become

$$\mathbf{E}(k) = 1 + \frac{k'^2}{2}\left[\ln\left(\frac{4}{k'}\right) - \frac{1}{2}\right] + \frac{3k'^4}{16}\left[\ln\left(\frac{4}{k'}\right) - \frac{13}{12}\right] + \cdots\,. \tag{A.4.23}$$

Substituting the expansion of elliptic integrals (A.4.17), (A.4.20), (A.4.23) and Jacobi sn and cn functions (A.4.5), (A.4.6), into the expression of Jacobi Zeta (A.1.26), one obtains its asymptotic behavior near $k=1$:

$$\begin{aligned}Z_0\,(i\omega,k) = i\tan\omega - \frac{i\omega}{\ell_k} - \frac{i\,k'^2}{4}\left[\frac{\omega+\sin\omega\cos\omega}{\cos^2\omega} - \omega\left(\frac{2}{\ell_k} - \frac{1}{\ell_k^2}\right)\right] \\ + \frac{i\,k'^4}{128}\left[\frac{-2\omega\cos\omega + 2\sin\omega\,(4\omega^2 - 5\cos^2\omega + 2\cos^4\omega)}{\cos^3\omega} + 3\omega\left(\frac{4}{\ell_k} + \frac{1}{\ell_k^2}\right)\right] \\ + O\left(k'^6\right) + O\left(\frac{1}{\ell_k^3}\right),\end{aligned} \tag{A.4.24}$$

where $\ell_k \equiv \ln\,(4/k')$.

A.5 The asymptotic behavior of helical strings near $k=1$

Below we show the asymptotic behavior of helical strings in detail by using the formula derived in Appendix A.4.

The parameters a, b and $v = b/a$, which appeared in (9.2.12) and (9.2.13), behave as

$$\begin{aligned} a &= \frac{\sqrt{U+\cos^2\omega_1}}{\cos\omega} && + k'^2\, a_{(2)} + \mathcal{O}(k'^4)\,, \\ b &= \tan\omega_1 && + k'^2\, b_{(2)} + \mathcal{O}(k'^4)\,, \\ v &= \frac{\sin\omega_1}{\sqrt{U+\cos^2\omega_1}} && + k'^2\, v_{(2)} + \mathcal{O}(k'^4)\,, \end{aligned}$$

where

$$\begin{aligned} a_{(2)} = \Big(&-(U+1)\,\omega_1 \sin\omega_1 - 4\left(U\cos^2\omega_1 + 1\right)\cos^3\omega_1\cos^2\omega_2 \\ &+ 4\sqrt{U}\sqrt{U\cos^2\omega_1+1}\,\sin\omega_1\cos^3\omega_1\sin\omega_2\cos\omega_2 \\ &\quad + 2\,U\cos^5\omega_1 + (3+U)\cos^3\omega_1 - (U+1)\cos\omega_1\Big) \Big/ \left[4\sqrt{U\cos^2\omega_1+1}\,(U+1)\cos^2\omega_1\right], \\ b_{(2)} = \Big(&-(U+1)\,\omega_1\sin\omega_1 - 4\,U\sin^2\omega_1\cos^3\omega_1\cos^2\omega_2 \\ &- 4\,\sqrt{U}\sqrt{U\cos^2\omega_1+1}\,\sin\omega_1\cos^3\omega_1\sin\omega_2\cos\omega_2 \\ &\quad - 2\,U\cos^5\omega_1 + (3\,U+1)\cos^3\omega_1 - (U+1)\cos\omega_1\Big) \Big/ \left[4\,(U+1)\cos^2\omega_1\sin\omega_1\right], \\ v_{(2)} = \Big(&-(U+1)^2\,\omega_1\sin\omega_1\cos\omega_1 - 4\,(U-1)\left(U\cos^2\omega_1+1\right)\sin^2\omega_1\cos^2\omega_1\cos^2\omega_2 \\ &- 4\,\sqrt{U}\sqrt{U\cos^2\omega_1+1}\left(2-\cos^2\omega_1+U\cos^2\omega_1\right)\sin\omega_1\cos^2\omega_1\sin\omega_2\cos\omega_2 \\ &+ \sin^2\omega_1\cos^2\omega_1\left\{\left(2\,\cos^2\omega_1-1\right)U^2 - 2\,U\cos^2\omega_1 - 3\right\} \Big/ \left[4\left(U\cos^2\omega_1+1\right)^{3/2}(U+1)\sin\omega_1\right]. \end{aligned}$$

The conserved charges, which appeared in (9.2.24), (9.2.25) and (9.2.26), become

$$\mathcal{E} = \frac{\ell_k\,(U+1)\sin\left(\frac{p_1}{2}\right)}{\sqrt{U\sin^2\left(\frac{p_1}{2}\right)+1}} + \frac{k'^2}{4}\,\mathcal{E}^{(2')} + \mathcal{O}\left(k'^4\right), \tag{A.5.1}$$

$$\mathcal{J}_1 = \frac{\ell_k\,(U+1)\sin\left(\frac{p_1}{2}\right)}{\sqrt{U\sin^2\left(\frac{p_1}{2}\right)+1}} - \sqrt{U\sin^2\left(\frac{p_1}{2}\right)+1}\,\sin\left(\frac{p_1}{2}\right) + \frac{k'^2}{4}\,\mathcal{J}_1^{(2')} + \mathcal{O}\left(k'^4\right), \tag{A.5.2}$$

$$\mathcal{J}_2 = \sqrt{U}\,\sin^2\left(\frac{p_1}{2}\right) + \frac{k'^2}{4}\,\mathcal{J}_2^{(2')} + \mathcal{O}\left(k'^4\right), \tag{A.5.3}$$

where the next-to-leading terms in (9.2.33) are given by

$$\begin{aligned}
\mathcal{E}^{(2')} - \mathcal{J}_1^{(2')} = \sin^3\left(\frac{p_1}{2}\right) \Bigg\{ & 2\,\ell_k \left[U\left(U\sin^2\left(\frac{p_1}{2}\right)+1\right)\left((2-2\,U)\cos^2\left(\frac{p_1}{2}\right)+1+U\right)\right. \\
& \times\left(-1+2\,\cos^2\omega_2\right) + 2\sqrt{U\left(U\sin^2\left(\frac{p_1}{2}\right)+1\right)}\left(\left(2\,U^2-2\,U\right)\cos^2\left(\frac{p_1}{2}\right)-2\,U^2-U+1\right) \\
& \left.\times\cos\left(\frac{p_1}{2}\right)\sin\omega_2\cos\omega_2\right] + (U+1)\left(U\sin^2\left(\frac{p_1}{2}\right)+1\right) \\
& \times\left[-4\left(U\sin^2\left(\frac{p_1}{2}\right)+1\right)\cos^2\omega_2 + 3 + \left(-2\,\cos^2\left(\frac{p_1}{2}\right)+3\right)U\right]\Bigg\} \\
& \Big/ \left[\left(U\sin^2\left(\frac{p_1}{2}\right)+1\right)^{3/2}(U+1)\right],
\end{aligned}$$

$$\begin{aligned}
\mathcal{J}_2^{(2')} = \sin^2\left(\frac{p_1}{2}\right) \Bigg\{ & 2\,\ell_k \left[U\sqrt{U\sin^2\left(\frac{p_1}{2}\right)+1}\left((2-2\,U)\cos^2\left(\frac{p_1}{2}\right)+1+U\right)\right. \\
& \left.\times\left(-1+2\,\cos^2\omega_2\right) + 2\sqrt{U}\left(\left(2\,U^2-2\,U\right)\cos^2\left(\frac{p_1}{2}\right)-2\,U^2-U+1\right)\cos\left(\frac{p_1}{2}\right)\sin\omega_2\cos\omega_2\right] \\
& + \sqrt{U\sin^2\left(\frac{p_1}{2}\right)+1}\,(U+1)\left[-4\left(U\sin^2\left(\frac{p_1}{2}\right)+\frac{1}{2}\right)\cos^2\omega_2 + 2 + \left(-2\,\cos^2\left(\frac{p_1}{2}\right)+3\right)U\right]\Bigg\} \\
& \Big/ \left[\sqrt{U}\,(U+1)\sqrt{U\sin^2\left(\frac{p_1}{2}\right)+1}\right].
\end{aligned}$$

Thus, the next-to-leading term in (9.2.33) is,

$$\mathcal{E}^{(2')} - \mathcal{J}_1^{(2')} - \frac{\sqrt{U}\sin\left(\frac{p_1}{2}\right)}{\sqrt{U\sin^2\left(\frac{p_1}{2}\right)+1}}\,\mathcal{J}_2^{(2')} = \sin^3\left(\frac{p_1}{2}\right)\frac{\left(1-2\cos^2\omega_2\right)}{\sqrt{U\sin^2\left(\frac{p_1}{2}\right)+1}}\,. \tag{A.5.4}$$

Appendix B

The Pohlmeyer-Lund-Regge reduction

The spectrum of Complex sine-Gordon system is investigated through the Pohlmeyer-Lund-Regge reduction of classical string theory on $\mathbb{R}_t \times S^3$. We study one- and two-soliton solutions and the mapping of spectral parameters of two theories.

We further show that so-called 'Complex sine-Gordon breather' is exactly identical to the kink-antikink scattering solution, by making use of invariance under the exchange of spectral parameters as first suggested in [81]. This result is consistent with the semiclassical quantization of Complex sine-Gordon theory, where no boundstates of kink and antikink are found.

B.1 Brief introduction

Complex sine-Gordon is an integrable 1+1 dimensional field theory, and can be regarded as generalization of sine-Gordon system with additional $U(1)$ charge [212]. Pohlmeyer, Lund, and Regge found that classical string theory on $\mathbb{R}_t \times S^3$ in conformal gauge admits reduction to another integrable model known as Complex sine-Gordon (CsG) system [55, 56, 57]. Moreover, one can reconstruct classical string solutions from those of CsG system at least locally.

An example of such correspondence is giant magnon solution of classical strings on $\mathbb{R}_t \times S^2$ and kink solution of sine-Gordon system [49]. The two-spin generalization of giant magnon solution, called dyonic giant magnon is related to kink solution of CsG system [51]. We will see this correspondence more in detail below.

sine-Gordon kink	$\leftrightarrow$	Giant Magnon	CsG kink	$\leftrightarrow$	Dyonic Giant Magnon
sine-Gordon scattering	$\leftrightarrow$	GM scattering	CsG scattering	$\leftrightarrow$	DGM scattering

Table B.1: Examples of the Pohlmeyer-Lund-Regge reduction.

Soliton solutions of CsG system can be constructed by various techniques such as inverse scattering method. One can also study its spectrum from poles of S-matrix.

Semiclassical analysis of CsG S-matrix was initiated by [213, 214]. About a decade later, Bakas reformulated CsG system as $SU(2)/U(1)$ gauged WZW model perturbed by its first thermal operator [215]. Based on the result, an exact quantum S-matrix was conjectured [216]. They found that the spectrum of Complex sine-Gordon consists of boundstate of same charges alone. The solitons of opposite charges do not form boundstate.

The situation is completely different from sine-Gordon system, where analytic continuation of 2-kink solution gives the boundstate solution known as "breather". In CsG system, breather solution becomes exactly identical to the kink-antikink scattering solution. In fact, Dorey, Hofman, and Maldacena showed that analytic continuation of dyonic giant magnon scattering solution does not give another new solution of classical string on $\mathbb{R}_t \times S^3$ [81].

In this appendix, we also establish the equivalence of CsG breather and CsG kink-antikink, by means of the Pohlmeyer-Lund-Regge reduction of dyonic giant magnon scattering. To show this equivalence, we make use of the exchange symmetry of the classical string solution found in [81].

B.2 Multisoliton solutions of sine-Gordon system

We begin with collecting facts on the soliton solutions of sine-Gordon system before looking into the connection with classical string theory.

Multi-soliton solutions of sine-Gordon system can be obtained by using inverse scattering method [217]. The following expression is known for K-soliton solutions of sine-Gordon system:

$$-\partial_t^2\phi + \partial_x^2\phi - \sin\phi = 0, \qquad \phi = -2i\log\left(\frac{\det\left(I_K + V\right)}{\det\left(I_K - V\right)}\right) \tag{B.2.1}$$

where I_K is $K \times K$ identity matrix, and the components of the $K \times K$ matrix V are given by

$$V_{jk} = \frac{im_j}{\lambda_j + \lambda_k}\exp\Big(i\left(\mu_j + \mu_k\right)x + 2i\nu_j t\Big), \qquad \mu_j \equiv \lambda_j - \frac{1}{16\lambda_j}\,, \quad \nu_j \equiv \lambda_j + \frac{1}{16\lambda_j}\,. \tag{B.2.2}$$

For $\phi = \phi(x,t)$ to be real, V must satisfy

$$\det\left(I + V\right) = \det\left(I - V^*\right). \tag{B.2.3}$$

One-soliton case. By substituting $K = 1$, we have

$$V = \frac{im}{2\lambda}\,e^{2i(\mu x + \nu t)}, \qquad \tan\left(\frac{u}{4}\right) = \frac{m}{2\lambda}\,e^{2i(\mu x + \nu t)}\,. \tag{B.2.4}$$

Plugging them back to (B.2.1), we can identify the result as the famous expression of 1-soliton solution of sine-Gordon system

$$\tan\left(\frac{\phi}{4}\right) = \exp\left(-\frac{x - vt}{\sqrt{1 - v^2}} + x_0\right), \tag{B.2.5}$$

by setting

$$m = 2\lambda\, e^{x_0}, \quad v = -\frac{\nu}{\mu} = \frac{1+16\lambda^2}{1-16\lambda^2}\,. \tag{B.2.6}$$

For this solution to be real, we must have

$$v < 1 \quad \Longleftrightarrow \quad \lambda = i\,a, \quad a \in \mathbb{R}. \tag{B.2.7}$$

Thus, in the one-soliton case, the imaginary part of λ controls the Lorentz boost v.

Two-soliton case. Consider the reality constraint (B.2.3) imposed on V. By writing

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \qquad \tilde{V} = \begin{pmatrix} V_{21} & V_{21} \\ V_{12} & V_{11} \end{pmatrix}, \tag{B.2.8}$$

we can see that $V^* = -\tilde{V}$ solves the constraint. Thus, after the redefinition of the parameters as

$$\lambda \equiv \lambda_1 = -\lambda_2^*\,, \quad m \equiv m_1 = m_2^*\,, \quad \mu \equiv \mu_1 = -\mu_2^*\,, \quad \nu \equiv \nu_1 = -\nu_2^*\,, \tag{B.2.9}$$

the matrix V given in (B.2.2) becomes

$$V = \begin{pmatrix} \dfrac{im}{2\lambda}\, e^{2i\mu x + 2i\nu t} & \dfrac{im}{\lambda - \bar{\lambda}}\, e^{i(\mu-\bar{\mu})x + 2i\nu t} \\ \dfrac{i\bar{m}}{\lambda - \bar{\lambda}}\, e^{i(\mu-\bar{\mu})x - 2i\nu t} & \dfrac{-i\bar{m}}{2\bar{\lambda}}\, e^{-2i\bar{\mu}x - 2i\bar{\nu}t} \end{pmatrix}. \tag{B.2.10}$$

Then, we substitute V into (B.2.1) to obtain the expression of $\tan(\phi/4)$. By setting

$$m = \rho\lambda e^{i\theta}, \quad \lambda = \xi + i\eta, \tag{B.2.11}$$

we have

$$\tan\left(\frac{\phi}{4}\right) = \frac{-i\rho \sin(T+\theta)}{e^X \left[1 - \left(\dfrac{\xi\rho}{2\eta}\right)^2 e^{-2X}\right]}\,, \tag{B.2.12}$$

where T and X are defined by

$$T = \frac{\xi}{8\,(\xi^2+\eta^2)}\Big\{\big(1+16\,\xi^2+16\,\eta^2\big)\,t - \big(1-16\,\xi^2-16\,\eta^2\big)\,x\Big\}, \tag{B.2.13}$$

$$X = \frac{\eta}{8\,(\xi^2+\eta^2)}\Big\{\big(1+16\,\xi^2+16\,\eta^2\big)\,x - \big(1-16\,\xi^2-16\,\eta^2\big)\,t\Big\}. \tag{B.2.14}$$

We further set

$$\rho = \frac{2\eta}{\xi}\, e^{-\chi}, \quad v = \frac{1-16\,(\xi^2+\eta^2)}{1+16\,(\xi^2+\eta^2)}\,, \tag{B.2.15}$$

and introduce $(t_v\,, x_v)$ as the Lorentz boost of (t,x) by velocity v, then (B.2.12) becomes

$$\tan\left(\frac{\phi}{4}\right) = \frac{-i\eta}{\xi}\, \frac{\sin\left(\dfrac{\xi\, t_v}{\sqrt{\xi^2+\eta^2}} + \theta\right)}{\sinh\left(\dfrac{\eta\, x_v}{\sqrt{\xi^2+\eta^2}} + \chi\right)}\,. \tag{B.2.16}$$

We can identify (B.2.16) as the famous expression of breather solution of sine-Gordon system

$$\tan\left(\frac{\phi}{4}\right) = \frac{1}{w}\,\frac{\sin\left(\dfrac{w t_v}{\sqrt{1+w^2}} + t_0\right)}{\cosh\left(\dfrac{x_v}{\sqrt{1+w^2}} + x_0\right)}\,, \tag{B.2.17}$$

by choosing

$$w = \frac{\xi}{\eta}\,, \quad t_0 = \theta, \quad x_0 = \chi + \frac{i\pi}{2} \tag{B.2.18}$$

From (B.2.11), (B.2.15), and (B.2.18), one finds that the absolute value of λ controls the period of breathing, the phase of λ controls the Lorentz boost, and m controls the initial values of t and x.

There are other two scattering solutions known in sine-Gordon theory. One is the kink-kink solution

$$\tan\left(\frac{\phi}{4}\right) = \frac{1}{w}\,\frac{\sinh\left(\dfrac{w t_v}{\sqrt{1-w^2}} + t_0\right)}{\cosh\left(\dfrac{x_v}{\sqrt{1-w^2}} + x_0\right)}\,, \tag{B.2.19}$$

which can be obtained by setting $\lambda_1 = ia_1\,, \lambda_2 = ia_2$ with $a_1\,, a_2 \in \mathbb{R}$, or by setting w to be purely imaginary. The other is the kink-antikink solution

$$\tan\left(\frac{\phi}{4}\right) = \frac{1}{w}\,\frac{\cosh\left(\dfrac{w t_v}{\sqrt{1-w^2}} + t_0\right)}{\sinh\left(\dfrac{x_v}{\sqrt{1-w^2}} + x_0\right)}\,. \tag{B.2.20}$$

which can be obtained by the shift of t_0 and x_0 in the kink-kink solution. The kink-kink and the kink-antikink solutions have different topological charge,

$$Q_{\rm top} \equiv \int_{-\infty}^{\infty} dx\,\frac{\partial\phi}{\partial x}\,. \tag{B.2.21}$$

That is, $Q_{\rm top}$ is nonzero for the kink-kink solution, while it is zero for the kink-antikink solution.

B.3 Review of the Pohlmeyer-Lund-Regge reduction

We review the reduction procedure invented by Pohlmeyer, Lund, and Regge [55, 56, 57], by partly repeating argument in Section 6.1. We then discuss how dyonic giant magnon solution is reduced to kink solution of Complex sine-Gordon (CsG) system.

We follow the discussion of Section 6.1. There we find that the equations of motion of classical strings on $\mathbb{R}_{\rm t} \times {\rm S}^3$ take the form

$$\partial_a\partial^a\vec{\xi} + (\partial_a\vec{\xi}\cdot\partial^a\vec{\xi}^*)\vec{\xi} = \vec{0}\,, \tag{B.3.1}$$

and Virasoro constraints read

$$|\partial_t \vec{\xi}|^2 + |\partial_x \vec{\xi}|^2 = 1, \qquad \mathrm{Re}\left(\partial_t \vec{\xi} \cdot \partial_x \vec{\xi}^*\right) = 0. \tag{B.3.2}$$

We can construct a solution of Complex sine-Gordon out of any solution of classical string on $\mathbb{R}_t \times S^3$ in conformal gauge using the Pohlmeyer-Lund-Regge reduction. The reduction procedure goes as follows. First, let us define $O(4)$-invariant variables ϕ and χ by

$$-\partial_+ \vec{X} \cdot \partial_- \vec{X} \equiv \cos\phi\,, \tag{B.3.3}$$

$$\partial_+^2 \vec{X} \cdot \vec{K} \equiv 2\,\partial_+ \chi\, \sin^2(\phi/2), \qquad \partial_-^2 \vec{X} \cdot \vec{K} \equiv -2\,\partial_- \chi\, \sin^2(\phi/2), \tag{B.3.4}$$

where $x^\pm$ are the light-cone coordinates defined by $t = x^+ + x^-$, $x = x^+ - x^-$, and $K_i \equiv \epsilon_{ijkl} X^j \partial_+ X^k \partial_- X^l$ $(i,j,k,l = 1,\ldots,4)$. Then, we can derive differential equations for ϕ and χ

$$\partial_a \partial^a \phi - \sin\phi - \frac{\sin(\phi/2)}{2\cos^3(\phi/2)}\,(\partial_a \chi)^2 = 0\,, \quad \partial_a \partial^a \chi + \frac{2\,\partial_a \phi\, \partial^a \chi}{\sin\phi} = 0\,. \tag{B.3.5}$$

by using the equations of motion (B.3.1), Virasoro constraints (B.3.2), and the normalization condition $|\vec{\xi}|^2 = 1$. The equations (B.3.5) are called Complex sine-Gordon equations. If we introduce a complex variable $\psi \equiv \sin(\phi/2)\exp(i\chi/2)$, the equations (B.3.5) are rewritten as

$$\partial_a \partial^a \psi + \psi^* \frac{(\partial_a \psi)^2}{1 - |\psi|^2} - \psi\left(1 - |\psi|^2\right) = 0\,. \tag{B.3.6}$$

An interesting problem is whether we can reconstruct the classical string solution on $\mathbb{R}_t \times S^3$ from the solution of Complex sine-Gordon system. The problem is more precisely formulated as follows. Take any solution of CsG system, call it $\psi_0 = \psi_0(t,x)$, and substitute it into the string equation of motion (B.3.1) with the identification (B.3.3). Then we have to solve the 'reduced' equation of motion

$$\partial_a \partial^a \vec{\xi} + \left(1 - 2\,|\psi_0|^2\right) \vec{\xi} = \vec{0}\,. \tag{B.3.7}$$

under suitable boundary conditions. Throughout this section, we impose the following Dirichlet boundary conditions:

$$\xi_1 \to \exp(it \pm ip/2)\,, \qquad \xi_2 \to 0, \qquad (\text{as}\ \ x \to \pm\infty)\,. \tag{B.3.8}$$

After having found a solution to (B.3.7), we have to check its consistency; that is, the solution must satisfy the relation (B.3.3) and (B.3.4) with the right hand sides determined by $\psi_0 = \sin(\phi_0/2)\exp(i\chi_0/2)$.

Let us illustrate how to deal with this problem with a simple example. The CsG kink solution is given by

$$\psi_{\text{kink}} = \frac{\cos A}{\cosh U}\, e^{iV}\,, \tag{B.3.9}$$

where coordinates U, V are defined by

$$U = \cos A \left(x \cosh\theta - t \sinh\theta \right), \quad V = \sin A \left(t \cosh\theta - x \sinh\theta \right). \tag{B.3.10}$$

We further introduce a pair of auxiliary variables $\lambda^{\pm}$ used in [54, 151, 81], as[1]

$$U = \frac{-i\left(\lambda^+ - \lambda^-\right)}{\left(1-(\lambda^+)^2\right)\left(1-(\lambda^-)^2\right)} \Big[x \left(1 + \lambda^+\lambda^-\right) - t\left(\lambda^+ + \lambda^-\right) \Big], \tag{B.3.11}$$

$$V = \frac{1 - \lambda^+\lambda^-}{\left(1-(\lambda^+)^2\right)\left(1-(\lambda^-)^2\right)} \Big[t \left(1 + \lambda^+\lambda^-\right) - x\left(\lambda^+ + \lambda^-\right) \Big], \tag{B.3.12}$$

where A and θ are reparametrized as

$$\cos A = \frac{-i\left(\lambda^+ - \lambda^-\right)}{\sqrt{\left(1-(\lambda^+)^2\right)\left(1-(\lambda^-)^2\right)}}, \qquad \sin A = \frac{1 - \lambda^+\lambda^-}{\sqrt{\left(1-(\lambda^+)^2\right)\left(1-(\lambda^-)^2\right)}}, \tag{B.3.13}$$

$$\cosh\theta = \frac{1 + \lambda^+\lambda^-}{\sqrt{\left(1-(\lambda^+)^2\right)\left(1-(\lambda^-)^2\right)}}, \qquad \sinh\theta = \frac{\lambda^+ + \lambda^-}{\sqrt{\left(1-(\lambda^+)^2\right)\left(1-(\lambda^-)^2\right)}}. \tag{B.3.14}$$

Now we want to solve the reduced equation of motion (B.3.7) with $\psi_0 = \psi_{\rm kink}$, under the boundary conditions (B.3.8). The solution is nothing but the dyonic giant magnon solution obtained in [51],

$$\begin{aligned} \xi_1 &= \frac{e^{it}}{2\sqrt{\lambda^+\lambda^-}} \frac{\lambda^+ e^{U} + \lambda^- e^{-U}}{\cosh U} &&= e^{it}\left[\cos\left(\frac{p}{2}\right) + i \sin\left(\frac{p}{2}\right)\tanh U\right] \\ \xi_2 &= \frac{-i}{2\sqrt{\lambda^+\lambda^-}} \frac{\left(\lambda^+ - \lambda^-\right) e^{iV}}{\cosh U} &&= \frac{\sin(p/2)}{\cosh U}\, e^{iV}, \end{aligned} \tag{B.3.15}$$

where we used

$$\lambda^+ \equiv \left(\lambda^-\right)^* \equiv e^{(ip+q)/2}. \tag{B.3.16}$$

We can easily check that the Pohlmeyer-Lund-Regge reduction of the dyonic giant magnon (B.3.15) is indeed consistent with the CsG kink (B.3.9).

The conserved charges of Complex sine-Gordon are defined by

$$E_{\rm CsG} \equiv \int_{-\infty}^{\infty} dx \left(\frac{\left|\partial_t\psi\right|^2 + \left|\partial_x\psi\right|^2}{1 - \left|\psi\right|^2} + \left|\psi\right|^2 \right), \quad Q_{\rm u(1)} \equiv \int_{-\infty}^{\infty} dx \left(\frac{{\rm Im}\, \left(\psi^* \partial_t \psi\right)}{1 - \left|\psi\right|^2} \right). \tag{B.3.17}$$

For the kink solution, they become

$$E_{\rm CsG} = 4\cosh\theta \left|\cos A\right| = \frac{4\left(\lambda^+ - \lambda^-\right)\left(1 + \lambda^+\lambda^-\right)}{\left(1-(\lambda^+)^2\right)\left(1-(\lambda^-)^2\right)}, \tag{B.3.18}$$

$$\begin{aligned} Q_{\rm u(1)} &= \frac{\sin 2A}{\left|\sin 2A\right|}\Big(\pi - 2\left|A\right|\Big) \\ &= {\rm sign}\left(\frac{\left(\lambda^+ - \lambda^-\right)\left(1 - \lambda^+\lambda^-\right)}{\left(1-(\lambda^+)^2\right)\left(1-(\lambda^-)^2\right)} \right) \left\{ \pi - 2 \left| {\rm arccot}\left(\frac{-i\left(\lambda^+ - \lambda^-\right)}{1 - \lambda^+\lambda^-} \right) \right| \right\}. \end{aligned} \tag{B.3.19}$$

[1] The variables $\lambda^{\pm}$ are aliases of $x^{\pm}$ variables used so far.

Note that the CsG kink has no topological charge unless $\cos A = 1$ which is sine-Gordon limit (see Section B.2). In this connection, the $U(1)$ charge $Q_{\mathrm{u}(1)}$ has discontinuity around $A = 0$.

The conserved charges of classical strings on $\mathbb{R}_{\mathrm{t}} \times \mathrm{S}^3$ are defined by

$$E - J_1 \equiv \frac{\sqrt{\lambda}}{2\pi} (\mathcal{E} - \mathcal{J}_1) = \frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^{\infty} dx \left\{ 1 - \mathrm{Im}\,(\xi_1^* \partial_t \xi_1) \right\}, \tag{B.3.20}$$

$$J_2 \equiv \frac{\sqrt{\lambda}}{2\pi} \mathcal{J}_2 = \frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^{\infty} dx \; \mathrm{Im}\,(\xi_2^* \partial_t \xi_2)\,. \tag{B.3.21}$$

In the case of dyonic giant magnon, they become

$$\begin{aligned} \mathcal{E} - \mathcal{J}_1 &= \frac{s(\lambda^\pm)}{2i} \left(\lambda^+ - \frac{1}{\lambda^+} - \lambda^- + \frac{1}{\lambda^-} \right) = \cosh \frac{q}{2} \left| \sin \frac{p}{2} \right|, \\ \mathcal{J}_2 &= \frac{s(\lambda^\pm)}{2i} \left(\lambda^+ + \frac{1}{\lambda^+} - \lambda^- - \frac{1}{\lambda^-} \right) = \sinh \frac{q}{2} \left| \sin \frac{p}{2} \right|, \end{aligned} \tag{B.3.22}$$

where $s(\lambda^\pm)$ is the sign function of the form

$$s(\lambda^\pm) \equiv \frac{-i\,(\lambda^+ - \lambda^-)\,(1 + \lambda^+ \lambda^-)}{(1 - (\lambda^+)^2)\,(1 - (\lambda^-)^2)} \left| \frac{(1 - (\lambda^+)^2)\,(1 - (\lambda^-)^2)}{(\lambda^+ - \lambda^-)\,(1 + \lambda^+ \lambda^-)} \right|. \tag{B.3.23}$$

The crossing transformation $\lambda^\pm \to 1/\lambda^\pm$ of Janik [14] induces the transformation

$$\psi_{\mathrm{kink}} \to (\psi_{\mathrm{kink}})^* \,, \quad (\xi_1\,,\ \xi_2) \;\to\; (\xi_1\,,\ -(\xi_2)^*)\,, \quad (\mathcal{E} - \mathcal{J}_1\,,\ \mathcal{J}_2) \;\to\; (\mathcal{E} - \mathcal{J}_1\,,\ -\mathcal{J}_2)\,, \tag{B.3.24}$$

which can be regarded as transformation from kink to *antikink* in Complex sine-Gordon system.

Let us make a few comments on mapping the spectral parameters of CsG model and classical string theory. The relation between $\lambda^\pm$ variables of classical strings and the parameters of CsG solutions, namely $\tan A$, $\tanh \theta$ given in (B.3.13) and (B.3.14), is given as follows. The velocity of a soliton is given by

$$\tanh \theta = \frac{\lambda^+ + \lambda^-}{1 + \lambda^+ \lambda^-} = \frac{4g \sin \frac{p}{2} \cos \frac{p}{2}}{\sqrt{n^2 + 16 g^2 \sin^2 \frac{p}{2}}} = \frac{1}{2g} \frac{dE}{dp}\,, \tag{B.3.25}$$

where the right hand side is normalized so that the speed of light is 1. The $U(1)$ parameter of a soliton is written as

$$\tan A = \frac{i\,(1 - \lambda^+ \lambda^-)}{\lambda^+ - \lambda^-} = \frac{-n}{4g \sin^2 \frac{p}{2}} = \frac{1}{2g} \frac{dJ_2}{dp}\,. \tag{B.3.26}$$

We can also relate the spectral parameter of sine-Gordon model λ and $\lambda^\pm$ variables of classical strings.[2] To this end, we identify the velocity of sG kink (B.2.6) as $\tanh \theta$, then it follows

$$v = \tanh \theta \quad \Longleftrightarrow \quad \frac{1 + 16\lambda^2}{1 - 16\lambda^2} = \frac{\lambda^+ + \lambda^-}{1 + \lambda^+ \lambda^-}\,, \qquad 4\lambda = \pm i \sqrt{\frac{(1 - \lambda^+)\,(1 - \lambda^-)}{(1 + \lambda^+)\,(1 + \lambda^-)}}\,. \tag{B.3.27}$$

[2]Note that the spectral parameter of Complex sine-Gordon is same as that of sine-Gordon.

B.4 Analytic continuation of 2-soliton solutions

First, we summarize the construction of 2-soliton solutions in CsG system by Bäcklund transformation method done in [218, 219], Let us redefine the (U, V) coordinates (B.3.10) as

$$\begin{aligned} U_k &= \cos A_k \left[\frac{x_v}{2} \left(\delta_k + \frac{1}{\delta_k} \right) + \frac{t_v}{2} \left(\delta_k - \frac{1}{\delta_k} \right) \right] , \\ V_k &= \sin A_k \left[\frac{t_v}{2} \left(\delta_k + \frac{1}{\delta_k} \right) + \frac{x_v}{2} \left(\delta_k - \frac{1}{\delta_k} \right) \right] , \end{aligned} \tag{B.4.1}$$

where t_v , x_v are Lorentz-boosted coordinates

$$t_v = \frac{t - vx}{\sqrt{1-v^2}} \,, \qquad x_v = \frac{x - vt}{\sqrt{1-v^2}} \,. \tag{B.4.2}$$

Define two auxiliary functions by

$$\mathbf{u}_k = \frac{\cos A_k}{\cosh U_k} \, e^{iV_k} \,, \quad \mathbf{v}_k = -e^{i\Omega} \Big(\cos(A_k) \tanh(U_k) + i \sin(A_k) \Big) \,. \tag{B.4.3}$$

The general form of 2-soliton solution of CsG system is given by

$$\psi_{2-\text{soliton}} = \frac{(-\delta_1 \mathbf{v}_2^* + \delta_2 \mathbf{v}_1^*) \, e^{i\Omega} \, (\delta_1 \mathbf{u}_1 - \delta_2 \mathbf{u}_2) + (-\delta_1 \mathbf{u}_2 + \delta_2 \mathbf{u}_1) \, e^{-i\Omega} \, (-\delta_1 \mathbf{v}_1 + \delta_2 \mathbf{v}_2)}{(\delta_1)^2 + (-\mathbf{u}_1^* \mathbf{u}_2 - \mathbf{u}_2^* \mathbf{u}_1 - \mathbf{v}_1^* \mathbf{v}_2 - \mathbf{v}_2^* \mathbf{v}_1) \, \delta_2 \delta_1 + (\delta_2)^2} \,, \tag{B.4.4}$$

which satisfies the CsG equation (B.3.6). Note that the solution (B.4.4) is independent of Ω.

The two-soliton solutions are completely specified by the choice of $\delta_{1,2}$ and $A_{1,2}$. The parameters $\delta_{1,2}$ determines the relative velocities of each soliton and the parameters $A_{1,2}$ determines their amplitude. In general, there will be $2K$ independent parameters for K-soliton solutions.

Using terminology of the paper [218, 219], the 2-kink scattering solution is given by

$$\delta_1 = -\frac{1}{\delta_2} \equiv \sqrt{\frac{1-W}{1+W}} \,, \qquad \text{and} \qquad A_1 \,, A_2 \in \mathbb{R}, \tag{B.4.5}$$

the kink-antikink scattering solution is

$$\delta_1 = \frac{1}{\delta_2} \equiv \sqrt{\frac{1-W}{1+W}} \,, \qquad \text{and} \qquad A_1 \,, A_2 \in \mathbb{R}, \tag{B.4.6}$$

and the breather solution is

$$\delta_1 = \frac{1}{\delta_2} \equiv \sqrt{\frac{1-iW}{1+iW}} \,, \qquad \text{and} \qquad A_1 = (A_2)^* \equiv A \in \mathbb{C}, \tag{B.4.7}$$

where $W \in \mathbb{R}$ for each of them. Note that the choice

$$\delta_1 = -\frac{1}{\delta_2} \equiv \sqrt{\frac{1-iW}{1+iW}} \,, \qquad \text{and} \qquad A_1 = (A_2)^* \equiv A \in \mathbb{C}, \tag{B.4.8}$$

violates the reality condition for ϕ; that is, $|\psi| = \sin(\phi/2) > 1$.

It turns out quite useful to introduce auxiliary variables $\lambda^{\pm}_{1,2}$ as in (B.3.13), (B.3.14). So we rewrite the four parameters v, W, and $A_{1,2}$ as

$$\cot A_k \equiv \frac{-i\left(\lambda_k^+ - \lambda_k^-\right)}{1-\lambda_k^+\lambda_k^-}\,, \qquad v \equiv \tanh\nu, \qquad W \equiv \tanh\omega, \tag{B.4.9}$$

where

$$\tanh(\nu+\omega) \equiv \frac{\lambda_1^+ + \lambda_1^-}{1+\lambda_1^+\lambda_1^-}\,, \quad \tanh(\nu-\omega) \equiv \frac{\lambda_2^+ + \lambda_2^-}{1+\lambda_2^+\lambda_2^-}\,. \tag{B.4.10}$$

Care should be taken in solving $(\tanh\nu,\ \tanh\omega)$ in terms of $\Theta_\pm \equiv \tanh\left(\nu\pm\omega\right)$ due to sign ambiguity. Written explicitly, the solutions are

$$\begin{aligned} \tanh\nu_\pm &= \frac{1+\Theta_-\Theta_+ \pm \sqrt{(1-\Theta_-^2)\,(1-\Theta_+^2)}}{\Theta_+ + \Theta_-}\,,\\ \tanh\omega_\pm &= \frac{1-\Theta_-\Theta_+ \pm \sqrt{(1-\Theta_-^2)\,(1-\Theta_+^2)}}{\Theta_+ - \Theta_-}\,. \end{aligned} \tag{B.4.11}$$

Only the combinations $(\tanh\nu_+\,,\ \tanh\omega_+)$ or $(\tanh\nu_-\,,\ \tanh\omega_-)$ can recover the original relation (B.4.9). We will concentrate on the combination $(\nu_-,\ \omega_-)$, because $\tanh\nu_+ \geq 1$, thus making ν_+ complex. Below we will denote the parameters $(\nu_-\,,\ \omega_-)$ by $(\nu,\ \omega)$.[3]

In terms of the variables $\lambda_k^\pm$, the 2-kink solution is redefined as

$$\psi_{\rm kk}\left(\lambda_1^+,\lambda_2^+,\lambda_1^-,\lambda_2^-\right) \quad\Longleftrightarrow\quad \delta_1 = -\frac{1}{\delta_2} = e^{-\omega}, \quad A_1\,,A_2\,,v \ \text{ are given in (B.4.9)}, \tag{B.4.12}$$

the kink-antikink solution is

$$\psi_{\rm ka}\left(\lambda_1^+,\lambda_2^+,\lambda_1^-,\lambda_2^-\right) \quad\Longleftrightarrow\quad \delta_1 = \frac{1}{\delta_2} = e^{-\omega}, \quad A_1\,,A_2\,,v \ \text{ are given in (B.4.9)}, \tag{B.4.13}$$

and the breather solution is redefined by the analytic continuation of the kink-antikink solution[4]

$$\psi_{\rm br}\left(\lambda_1^+,\lambda_2^+,\lambda_1^-,\lambda_2^-\right) \equiv \psi_{\rm ka}\left(\lambda_1^+,\lambda_2^+,\lambda_2^-,\lambda_1^-\right). \tag{B.4.14}$$

Let us discuss the properties of the functions $\psi_{\rm kk}\,,\psi_{\rm ak}$ and $\psi_{\rm br}$. They change their signs under the simultaneous complex conjugation

$$\begin{aligned} \psi_{\rm kk}\left(\lambda_1^+,\lambda_2^+,\lambda_1^-,\lambda_2^-\right) &= -\psi_{\rm kk}\left(\lambda_1^-,\lambda_2^-,\lambda_1^+,\lambda_2^+\right),\\ \psi_{\rm ka}\left(\lambda_1^+,\lambda_2^+,\lambda_1^-,\lambda_2^-\right) &= -\psi_{\rm ka}\left(\lambda_1^-,\lambda_2^-,\lambda_1^+,\lambda_2^+\right),\\ \psi_{\rm br}\left(\lambda_1^+,\lambda_2^+,\lambda_1^-,\lambda_2^-\right) &= -\psi_{\rm br}\left(\lambda_1^-,\lambda_2^-,\lambda_1^+,\lambda_2^+\right). \end{aligned} \tag{B.4.15}$$

Also, $\psi_{\rm kk}$ and $\psi_{\rm ak}$ are invariant under crossing

$$\lambda_1^\pm \to 1/\lambda_1^\pm\,, \quad \lambda_2^\pm : \text{fixed}, \qquad \text{or} \qquad \lambda_2^\pm \to 1/\lambda_2^\pm\,, \quad \lambda_1^\pm : \text{fixed}, \tag{B.4.16}$$

[3] We can nevertheless obtain the 2-soliton solution satisfying the reality constraint $|\psi| \leq 1$ for the combination (ν_+, ω_+). Such consideration complicates the classification of the 2-soliton solutions, but the statement (B.4.18) remains unchanged.

[4] As explained in (B.4.8), the analytic continuation of 2-kink solution does not work at all.

and $\psi_{\rm br}$ is invariant under the simultaneous crossing

$$\lambda_1^\pm \to 1/\lambda_1^\pm \,, \quad \lambda_2^\pm \to 1/\lambda_2^\pm \,. \tag{B.4.17}$$

A surprising is fact that the kink-antikink (B.4.13) and the breather solution (B.4.14) written in terms of $\lambda_k^\pm$ variables are *identical*

$$\psi_{\rm br}\left(\lambda_1^+, \lambda_2^+, \lambda_1^-, \lambda_2^-\right) \equiv \psi_{\rm ka}\left(\lambda_1^+, \lambda_2^+, \lambda_2^-, \lambda_1^-\right) = \psi_{\rm ka}\left(\lambda_1^+, \lambda_2^+, \lambda_1^-, \lambda_2^-\right), \tag{B.4.18}$$

which clearly shows that there are no 'boundstate' solution in the Complex sine-Gordon system. This identity can be proven either by direct computation, or by the exchange symmetry in the scattering solution of dyonic giant magnons, which will be discussed in the next section.

B.5 Relation to dyonic giant magnon scattering

The scattering solution of dyonic giant magnons (DGMs) were explicitly constructed in [54, 151] using dressing method. Here we consider the Pohlmeyer-Lund-Regge reduction of these solutions.

The profile of the DGM scattering solution is

$$\begin{aligned} \xi_1 &= \frac{e^{it}}{2D\sqrt{\lambda_1^+\lambda_1^-\lambda_2^+\lambda_2^-}} \Big\{ R + \lambda_1^+\lambda_1^-\lambda_{11}^{+-}\lambda_{22}^{+-} e^{i(v_1-v_2)} + \lambda_2^+\lambda_2^-\lambda_{11}^{+-}\lambda_{22}^{+-} e^{-i(v_1-v_2)} \Big\}, \\ \xi_2 &= \frac{-i}{2D\sqrt{\lambda_1^+\lambda_1^-\lambda_2^+\lambda_2^-}} \Big\{ \lambda_{11}^{+-} e^{iv_1} \left[\lambda_{12}^{++}\lambda_{12}^{-+}\lambda_2^- e^{u_2} + \lambda_{12}^{--}\lambda_{12}^{+-}\lambda_2^+ e^{-u_2} \right] + (1 \leftrightarrow 2) \Big\}, \end{aligned} \tag{B.5.1}$$

where

$$R = \lambda_{12}^{++}\lambda_{12}^{--}\left[\lambda_1^+\lambda_2^+ e^{u_1+u_2} + \lambda_1^-\lambda_2^- e^{-u_1-u_2}\right] + \lambda_{12}^{-+}\lambda_{12}^{+-}\left[\lambda_1^+\lambda_2^- e^{u_1-u_2} + \lambda_1^-\lambda_2^+ e^{-u_1+u_2}\right], \tag{B.5.2}$$

$$D = \lambda_{12}^{++}\lambda_{12}^{--}\cosh(u_1+u_2) + \lambda_{12}^{+-}\lambda_{12}^{-+}\cosh(u_1-u_2) + \lambda_{11}^{+-}\lambda_{22}^{+-}\cos(v_1-v_2), \tag{B.5.3}$$

and $\lambda_{jk}^{\pm\pm}$ are defined by

$$\lambda_{jk}^{++} = \lambda_j^+ - \lambda_k^+ \,, \quad \lambda_{jk}^{+-} = \lambda_j^+ - \lambda_k^- \,, \quad \lambda_{jk}^{-+} = \lambda_j^- - \lambda_k^+ \,, \quad \lambda_{jk}^{--} = \lambda_j^- - \lambda_k^- \,. \tag{B.5.4}$$

As pointed out in [54, 81], this solution is invariant under the 'exchange' transformation

$$\lambda_1^- \leftrightarrow \lambda_2^- \,, \quad \lambda_k^+ : \text{fixed}, \qquad \text{or} \quad \lambda_1^+ \leftrightarrow \lambda_2^+ \,, \quad \lambda_k^- : \text{fixed}.$$

Consider the Pohlmeyer-Lund-Regge reduction (B.3.3), (B.3.4) of the above solution, and call the corresponding CsG solution as $\psi_{\rm string}$. The function $|\psi_{\rm string}|$ is invariant under

simultaneous complex conjugation	$\lambda_1^+ \leftrightarrow \lambda_1^-\,, \lambda_2^+ \leftrightarrow \lambda_2^-$	(B.5.5)
simultaneous crossing	$\lambda_1^\pm \to 1/\lambda_1^\pm\,, \lambda_2^\pm \to 1/\lambda_2^\pm$	(B.5.6)
exchange	$\lambda_1^- \leftrightarrow \lambda_2^-\,, \quad \lambda_k^+ : \text{fixed},$	
	or $\lambda_1^+ \leftrightarrow \lambda_2^+\,, \quad \lambda_k^- : \text{fixed}.$	(B.5.7)

Let $x_k \equiv e^{(iP_k+Q_k)/2}$ $(k=1,2)$ be two complex numbers with $Q_1 > 0$ and $Q_2 < 0$. By suitably identifying $\lambda_k^{\pm}$ in the scattering solution of dyonic giant magnons with the ones in CsG 2-soliton solutions (B.4.9), we get identities

$$\begin{aligned} &\left|\psi_{\rm string}\left(\lambda_1^+ = x_1,\ \lambda_2^+ = x_2,\ \lambda_1^- = (x_1)^*,\ \lambda_2^- = (x_2)^*\right)\right| \\ &\qquad = \quad \left|\psi_{\rm kk}\left(\lambda_1^+ = x_1,\ \lambda_2^+ = x_2,\ \lambda_1^- = (x_1)^*,\ \lambda_2^- = (x_2)^*\right)\right|, \end{aligned} \tag{B.5.8}$$

$$\begin{aligned} &\left|\psi_{\rm string}\left(\lambda_1^+ = x_1,\ \lambda_2^+ = 1/x_2,\ \lambda_1^- = (x_1)^*,\ \lambda_2^- = 1/(x_2)^*\right)\right| \\ &\qquad = \quad \left|\psi_{\rm ka}\left(\lambda_1^+ = x_1,\ \lambda_2^+ = x_2,\ \lambda_1^- = (x_1)^*,\ \lambda_2^- = (x_2)^*\right)\right|. \end{aligned} \tag{B.5.9}$$

We have checked the identities (B.5.8) and (B.5.9) by numerically evaluating both sides. The phase of ψ must also agree up to overall signs, from the uniqueness of the solution to the CsG equations (B.3.6).

If we recall the discussion around (B.3.24), the transformation $(x_1\,,\ x_2) \to (x_1\,,\ 1/x_2)$ should turn the CsG 2-kink solution into the CsG kink-antikink solution, which explains the difference between (B.5.8) and (B.5.9). Also, with the identification (B.5.9), the exchange symmetry (B.5.7) of DGM scattering solution provides a proof of the equivalence between the CsG kink-antikink and the CsG breather posed at (B.4.18).

Appendix C

Details of calculation for finite-size effects

C.1 S-matrix contribution

C.1.1 The spectral parameters and Jacobian

The Lüscher F-term formula (9.3.26) contains an integration over $\tilde{q}$, while the S-matrix is written in terms of the spectral parameters $y^\pm$. Thus in order to compute the Jacobian, we need to rewrite $y^\pm$ as functions of $\tilde{q}$.

The spectral parameters $y^\pm$ as functions of q^1 is defined by

$$y^\pm(q^1) = e^{\pm i q^1/2} \, \frac{Q_b + \sqrt{Q_b^2 + 16 g^2 \sin^2\left(\frac{q^1}{2}\right)}}{4 g \sin\left(\frac{q^1}{2}\right)} \,, \tag{C.1.1}$$

and the momentum q^1 is related to $\tilde{q}$ via (9.3.20). There are two branches of the square root, corresponding to $E(y^\pm) = \pm i\tilde{q}$. If we choose $E(y^\pm) = -i\tilde{q}$, we obtain

$$y^\pm(\tilde{q}) = \frac{\sqrt{16 g^2 + Q_b^2 + \tilde{q}^2} \pm \sqrt{Q_b^2 + \tilde{q}^2}}{4g} \, \frac{i Q_b + \tilde{q}}{\sqrt{Q_b^2 + \tilde{q}^2}} \,. \tag{C.1.2}$$

If we introduce another parameter by $\tilde{q} \equiv Q_b \cot(r/2)$, they translate into

$$y^\pm(\tilde{q}) = \frac{\sqrt{Q_b^2 + 16 g^2 \sin^2 \frac{r}{2}} \pm Q_b}{4 g \sin \frac{r}{2}} \, e^{ir/2} \,. \tag{C.1.3}$$

Roughly speaking, the Wick rotation (9.3.20) with $\tilde{q} = i q^0$ is equivalent to the transformation $(y^+, y^-) \mapsto (y^+, 1/y^-)$. When we set $Q_b = 1$ and use (C.1.3), we can solve the condition $y^\pm = X^+$ to the next order of $1/g$ as

$$y^\pm = X^+ \equiv e^{(ip+\theta)/2} \qquad \Longleftrightarrow \qquad r_* \approx p - i\theta \pm \frac{i}{2g \sin\left(\frac{p-i\theta}{2}\right)} + \mathcal{O}\left(\frac{1}{g^2}\right) . \tag{C.1.4}$$

Note that $\theta \approx Q/[2g\sin(p/2)]$ if $Q \ll g$.

It is easy to compute the Jacobian between $\tilde{q}$ and $y^{\pm}$ from (C.1.2). They read

$$\frac{dy^{\pm}(\tilde{q})}{d\tilde{q}} \approx \frac{i}{(i-\tilde{q})\sqrt{1+\tilde{q}^2}} = -i\sin^2\left(\frac{r}{2}\right)\, e^{ir/2}\,, \tag{C.1.5}$$

for $g \gg 1$. Note in particular that both $(y^+)'$ and $(y^-)'$ are equal for this case.

C.1.2 Dressing phase

We will evaluate the dressing phase (9.4.4) for the case $Q \sim \mathcal{O}(1) \ll 1$.

AFS phase. The AFS phase is given in (9.4.35). Since the first term sums up to zero, the following expression is more useful:

$$\chi^{(0)}(y,x) = -g(y-x)\left(1-\frac{1}{yx}\right)\log\left(1-\frac{1}{yx}\right). \tag{C.1.6}$$

By using the relations

$$(y^+ - X^{\pm})\left(1-\frac{1}{y^+X^{\pm}}\right) = (y^- - X^{\pm})\left(1-\frac{1}{y^-X^{\pm}}\right) + \frac{i}{g}\,, \tag{C.1.7}$$

we find

$$\begin{aligned}&\chi^{(0)}(y^-,X^{\pm}) - \chi^{(0)}(y^+,X^{\pm})\\ &\qquad = -g(y^- - X^{\pm})\left(1-\frac{1}{y^-X^{\pm}}\right)\log\left(\frac{1-\frac{1}{y^-X^{\pm}}}{1-\frac{1}{y^+X^{\pm}}}\right) + i\log\left(1-\frac{1}{y^+X^{\pm}}\right).\end{aligned} \tag{C.1.8}$$

We can relate the terms with X^+ to those with X^- via

$$(y^- - X^+)\left(1-\frac{1}{y^-X^+}\right) = (y^- - X^-)\left(1-\frac{1}{y^-X^-}\right) - \frac{iQ}{g}\,. \tag{C.1.9}$$

Thus we obtain

$$\sigma^2_{\rm AFS}(y,X) = \left(\frac{1-\frac{1}{y^-X^-}}{1-\frac{1}{y^+X^-}}\right)^{2Q}\left(\frac{1-\frac{1}{y^-X^+}}{1-\frac{1}{y^-X^-}}\right)^2, \tag{C.1.10}$$

which is equal to (9.4.46).

Higher dressing phase. We reconsider the sum of even part of the dressing phase higher order in $1/g$. As shown in [79], there are contributions to the μ-term from the terms $\chi^{(2m)}(y^a,X^b)$ with $y^aX^b \sim 1$ at strong coupling. If we use the variable α^{ab} defined by (9.4.45), the higher dressing phase can be written as

$$\chi^{(2m)}(\alpha^{ab}) = \pm 2i\alpha^{ab}(2m-2)!\,\frac{\zeta(2m)}{(2\pi i\alpha^{ab})^{2m}}\,, \tag{C.1.11}$$

where we take the upper sign for $y^a \sim e^{ip/2}$ and the lower sign for $y^a \sim e^{-ip/2}$. By means of Borel resummation, we can compute the summation of $\chi^{(2m)}$ over m as

$$\begin{aligned}\sum_{m=1}^{\infty} \chi^{(2m)}(\alpha^{ab}) &= \pm 2i\alpha^{ab} \sum_{m=1}^{\infty} \int_0^{\infty} dt\, e^{-t}\, \frac{t^{2m-2}\zeta(2m)}{(2\pi i\alpha^{ab})^{2m}} \\ &= \pm i \int_0^{\infty} dt\, e^{-t} \left[\frac{\alpha^{ab}}{t^2} - \frac{\coth\left(\frac{t}{2\alpha^{ab}}\right)}{2t} \right] . \end{aligned} \tag{C.1.12}$$

The last expression can be simplified further with the help of the following formula:[1]

$$\begin{aligned} 2\int_0^{\infty} dt\, e^{-t} &\left[\frac{(\alpha^{ab}-\alpha^{cd})}{t^2} - \frac{1}{2t}\coth\left(\frac{t}{2\alpha^{ab}}\right) + \frac{1}{2t}\coth\left(\frac{t}{2\alpha^{cd}}\right) \right] \\ &= (\alpha^{ab}+\alpha^{cd}) \log\left(\frac{\alpha^{ab}}{\alpha^{cd}}\right) - 2(\alpha^{ab}-\alpha^{cd}) \qquad \left(\text{if} \quad \alpha^{ab}-\alpha^{cd} = \pm 1\right) . \end{aligned} \tag{C.1.13}$$

The dressing phase can be computed by collecting terms with nonvanishing α^{ab}. According to Table 9.2, we find

$$\sigma^2_{n\geq 2}(y, X) \approx \exp\left[2\left(\alpha^{--} - \alpha^{+-}\right) \right] \left(\frac{\alpha^{+-}}{\alpha^{--}}\right)^{\alpha^{--}+\alpha^{+-}} , \quad \left(\text{for} \;\; y \sim e^{ip/2}\right), \tag{C.1.14}$$

which is (9.4.48).

C.2 Discussion on F-term

We show that F-term becomes negligibly small when we can avoid singularities of the S-matrix.

Let us first rewrite the expression for F-term (9.3.26) by changing integration variable. We introduce another variable κ by

$$q^2 = 16g^2 \sinh^2\left(\frac{\kappa}{2}\right) - Q_b^2 , \qquad \left(q^1 = q_* \equiv -i\kappa\right), \tag{C.2.1}$$

where Q_b is the multiplet number of particle b. The F-term can be rewritten as

$$\delta\varepsilon_a^F(p) = -\sum_{Q_b \geq 1} \int_{C_{Q_b}} \frac{d\kappa}{2\pi}\, \frac{4g^2 \sinh\kappa}{\sqrt{16g^2\sinh^2\left(\frac{\kappa}{2}\right) - Q_b^2}} \left(1 - \frac{\varepsilon'_Q(p)}{\varepsilon'_{Q_b}(q_*)}\right) e^{-\kappa L} \sum_b (S_{ba}^{ba}(q,p) - 1) , \tag{C.2.2}$$

where the contour C_Q is defined as

$$C_Q = \left\{ \kappa \in \mathbb{R} \;\middle|\; \kappa \geq \kappa_{\rm cr}^{(Q)} \right\} , \qquad \kappa_{\rm cr}^{(Q)} = 2\,\mathrm{arcsinh}\left(\frac{Q}{4g}\right) . \tag{C.2.3}$$

Because each term within the sum at most gives the contribution $\sim e^{-\kappa_{\rm cr}^{(Q_b)} L}$, we may focus on the leading term $Q_b = 1$ and rewrite it as

$$\delta\varepsilon_a^F(p)\Big|_{Q_b=1} \equiv -\int_{\kappa_{\rm cr}^{(1)}}^{\infty} d\kappa\, \frac{e^{-\kappa L}}{\sqrt{\sinh\left(\frac{\kappa}{2}\right) - \sinh\left(\frac{\kappa_{\rm cr}^{(1)}}{2}\right)}}\, f\,(q,p) . \tag{C.2.4}$$

[1]We checked this equality numerically.

At large L the dominant contribution comes from $\kappa = \kappa_{\rm cr}^{(1)}$. If one finds singularity of S-matrix along the integration path, one can slightly deform the contour assuming the analyticity of integrand. Thus, if S-matrix behaves regularly around $\kappa = \kappa_{\rm cr}^{(1)}$, we can approximate the integral (C.2.4) as

$$\delta\varepsilon_a^F(p)\Big|_{Q_b=1} \approx -\int_0^\infty dk\, \frac{e^{-(k+\kappa_{\rm cr}^{(1)})L}}{\sqrt{k}} \cdot \frac{f(-i\kappa_{\rm cr}^{(1)}, p)}{\cosh^{1/2}\left(\frac{\kappa_{\rm cr}^{(1)}}{2}\right)} = \frac{e^{-\kappa_{\rm cr}^{(1)}L}}{\sqrt{L}} \cdot \frac{f(-i\kappa_{\rm cr}^{(1)}, p)}{\cosh^{1/2}\left(\frac{\kappa_{\rm cr}^{(1)}}{2}\right)}\,, \tag{C.2.5}$$

which is subleading in the limit $L \to \infty$, because of the factor $L^{-1/2}$.

Singularities of the S-matrix appear at the position depending on the value of $X^\pm$ and g. And if there is a singularity at $q_* = -i\kappa_{\rm cr}^{(1)}$ which is different from single poles of the BDS S-matrix, the above argument will break down. We will consider a few particular cases in which the $su(2|2)^2$ S-matrix may possibly have singularity at $q^1 = -i\kappa_{\rm cr}^{(1)}$ in what follows.[2]

Using the expression of $y^\pm$ given in Appendix C.1.1, one can find that the zeroes or the poles of the BDS S-matrix are found at

$$q^1 = \frac{-i}{2g\sin\left(\frac{p\pm i\theta}{2}\right)} \quad \text{for } \ \mathrm{Im}\, q^1 < 0, \qquad q^1 = \frac{+i}{2g\sin\left(\frac{p\pm i\theta}{2}\right)} \quad \text{for } \ \mathrm{Im}\, q^1 > 0, \tag{C.2.6}$$

and they do not hit the path (C.2.3) unless $p = \pi, \theta = 0$. Also, by looking at (9.4.16), one sees that the coefficients $s_2(y, X)$ and $s_3(y, X)$ do not bring new poles.

As discussed in [81, 82], the BHL/BES dressing phase contains an infinite number of double poles located at

$$X^+ + \frac{1}{X^+} - Y^- - \frac{1}{Y^-} = -\frac{im}{g} \qquad (m = 1, 2, \ldots)\,, \tag{C.2.7}$$

where either one of X^+ or Y^- must be inside the unit circle, while the other be outside. These double poles are interpreted as the kinematical constraint for the Landau-Cutkosky diagram of box type (Figure 9.4). Below we will analytically continue $Y^\pm$ keeping particle a real, $X^+ = (X^-)^*$, and study if both (C.2.7) and $q_* = -i\kappa_{\rm cr}^{(1)}$ can be solved at a particular value of $X^\pm$.

First of all, with $q_* = -i\kappa_{\rm cr}^{(1)}$ and $Q(Y^\pm) = 1$, we evaluate $Y^\pm$ as,

$$Y^\pm = e^{\pm\frac{iq^1}{2}} \left(\frac{1 + \sqrt{1 + 16g^2\sin^2(\frac{q^1}{2})}}{4g\sin(\frac{q^1}{2})} \right) \Bigg|_{q^1 = -i\kappa_{\rm cr}^{(1)}} = i\, e^{\pm\frac{\kappa_{\rm cr}^{(1)}}{2}} = i\left(\frac{1 \pm \sqrt{1+16g^2}}{4g} \right), \tag{C.2.8}$$

showing $|Y^+| > 1$ and $|Y^-| < 1$. Plugging (C.2.8) into (C.2.7), we find

$$X^+ + \frac{1}{X^+} = -\frac{i}{2g}\,(2m+1)\,, \tag{C.2.9}$$

[2]Note that the condition $p(Y^\pm) \equiv q^1 = -i\kappa_{\rm cr}^{(1)}$ implies $E(Y^\pm) = 0$.

which has the solutions

$$X^{+} = i\left(\frac{-\left(2m+1\right) \pm \sqrt{\left(2m+1\right)^{2} + 16g^{2}}}{4g}\right). \tag{C.2.10}$$

Note that we must choose the lower sign so that X^{+} stays outside the unit circle. By using the definition $X^{\pm} \equiv e^{(\pm ip+\theta)/2}$ as in (9.4.24), we can identify this solution as $p = -\pi$ and $\sinh(\theta/2) = (2m+1)/4g$, which implies

$$X^{-} = -i\left(\frac{-\left(2m+1\right) - \sqrt{\left(2m+1\right)^{2} + 16g^{2}}}{4g}\right). \tag{C.2.11}$$

However, it turns out that the spectral parameters given by (C.2.10) and (C.2.11) give rise to $Q(X^{\pm}) = -\left(2m+1\right) < 0$, which is impossible. Therefore, we conclude that there are no real values of p and θ which are consistent with the double pole condition (C.2.7), $q_{*} = -i\kappa_{\mathrm{cr}}^{(1)}$, $Q(Y^{\pm}) = 1$, and $Q(X^{\pm}) > 0$.

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