論文の内容の要旨

論文題目 **Selberg Integral and Gauge/Toda Duality**

（ Selberg積分とゲージ／戸田双対性 ）

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String duality is crucial in the understanding of different types of string theories and their counterparts like quantum gravity and gauge theories, and serves as a powerful tool for studying strongly-coupled theories. For years most of this kind of research has been limited to AdS/CFT, but in 2002, Nekrsov performed a technique called Ω deformation in the reduction from 6D N = 1 gauge therory to 4D N = 2 gauge therory, and implied its connection with 2D conformal theory. He found exact formulae of the partition function (Nekrasov partition function) of the N = 2 gauge therory, and showed that it reproduces the prepotential as determined by the Seiberg-Witten curve. Later in 2009, some news attracts people 's attention. Alday, Gaiotto and Tachikawa presented an interesting observation that the Nekrasov partition functions of certain class of N=2 SU(2) gauge theories seem to coincide with the correlation function of 2D Liouville field theory(AGT conjecture). Soon later, Wyllard [5] and others have presented a generalization to SU(N) case.

In their observations, the correlation functions of Liouville (Toda) field theories are identified with the integral of the Nekrasov partition function Z_{Nek}, where the instanton part Z_{inst} in the gauge theory written in a form is identical to the conformal blocks, and the perturbative part Z_{1loop} corresponds to the (product of) three point functions. AGT conjecture is illuminating in showing a correspondence between 4D Yang-Mills and 2D integrable models and will be fundamental in the understanding of the duality of gauge theories. It will also be relevant to understanding strong coupling physics of multiple M5-branes. In this respect, it will be important to explore to what extent and how this conjecture holds. Especially, since the coincidence was found through the first few orders in the instanton expansion, the exact computation of conformal block is needed in the Liouville(Toda) side.

Since then there were many attempts on the interpretation of AGT conjecture, but no complete proof had been achieved. In 2011, A. Mironov et. al. had embarked on an interesting step toward this direction. They used the Dotsenko-Fateev method to calculate the conformal blocks. They analyzed the simplest example $SU(2)$, $N_f = 4$ and proved the AGT relation for a special choice of the Ω deformation parameter $\beta = -\epsilon_1/\epsilon_2 = 1$. The key step in their analysis is the reduction of the Dotsenko-Fateev (DF) formula to Selberg average with one or two Jack polynomial(s) which was computed explicitly by Kadell. This work is of great importance as the first direct proof, but still rigorous analysis is in demand. Recently, O. Schiffmann and E. Vasserot successfully introduced an algebra called $\mathbf{SH}_c$ to prove the AGT conjecture, yet their work is limited to pure super Yang-Mills theory.

We have been working on the proof of AGT conjecture through two different ways.

First we generalized A. Mironov et. al.'s idea to the much more complicated SU(N) case. We calculated the conformal block in the form of Dotsenko-Fateev integral and reduce it in the form of Selberg integral of N Jack polynomials. The old Dotsenko-Fateev integral and the choice of paths of the screening operators play key roles in this correspondence, and their significance to Matrix models and conformal blocks are pointed out. Itoyama and Oota calculated the SU(2) case of the reduction from DF integral to Selberg-Jack integral, and we performed the SU(N) version's proof.

Selberg integral is an n-dimensional generalization of the Euler beta integral, and Jack polynomial is a kind of symmetric polynomial labeled by Young diagram. Selberg integral showed its prominence, evidenced by its central role in random matrix theory, Calogero-Sutherland quantum many body systems, Knizhnik-Zamolodchikov equations, and multivariable orthogonal polynomial theory. By q-deformation, Jack polynomial will upgrade to MacDonald polynomial, whose application in five dimensions is in anticipation. These two subjects have long histories and are wide applied in both mathematical and physical fields. Yet the interaction of them has been a forbidding issue.

Surely, if we want to achieve a full direct proof for SU(N) case, the exact expression of SU(N) Selberg Jack integral will be required. No such formula is available in the mathematics literature, so we need to calculate this kind of integral by ourselves. Fortunately there are still some materials that for us to refer to. For SU(2) case, the relevant Selberg averages for one and two Jack polynomials were obtained by Kadell, and The one-Jack Selberg integral for SU(N) could be calculated by the formula offered by Warnaar. There works serve as a good hint for our calculation. Furthermore, another advantage we own is that, we already more or less know the deserved form of the Selberg Jack integral, from the expectation of AGT conjecture.

Though the actual process is much more complicated than expected, we manage to found a formula for such Selberg average which satisfies some nontrivial consistency conditions and showed that it reproduces the SU(N) version of AGT conjecture. Besides, we work out many technical details, including proofs of lemmas lacked in A. Mironov et. al.'s paper, which are essential to bring Selberg average into the form of Yang-Mills partition function. This work is the first direct approach of SU(N) AGT conjecture with $\beta = 1$.

Our recent method is based on recursion relations. We derive an infinite set of recursion formulae for Nekrasov instanton partition function for linear quiver U(N) supersymmetric gauge theories in 4D. They have a structure of a deformed version of $W_{1+\infty}$ algebra which is called $\mathbf{SH}_c$ algebra in the literature. The algebra contains W_N algebra with general central charge defined by a parameter β, which gives the Ω background in Nekrasov's analysis. Some parts of the formulae are identified with the conformal Ward identity for the conformal block function for Toda field theory. The SU(N) constraints give a direct support for AGT conjecture for general quiver gauge theories.

In detail, the instanton partition function for linear quiver gauge theories is decomposed into matrix like product with a factor Z, which depends on two sets of Young diagrams. Here the Young diagrams Y represents the fixed points of $U(N)$ instanton moduli space under localization. Z consists of contribution from one bifundamental hypermultiplet and vectormultiplets. We find that the building block Z satisfies an infinite series of recursion relations,

$$\delta_{\pm1,n} Z_{\vec{Y},\vec{W}} - U_{\pm1,n} Z_{\vec{Y},\vec{W}} = 0\,, \tag{1}$$

Then we give an interpretation of (1). We show that the variation in (1) can be understood as an action of an infinite-dimensional extended conformal algebra, the $\mathbf{SH}_c$ algebra. For this purpose, we construct an explicit representation where the basis of the Hilbert space is labeled by sets of N Young diagrams. Physically, it can be understood that these states correspond to instantons characterized by the same set of Young diagrams. In our previous paper, we showed a similar form of recursion formula under self-dual Ω-background ($\epsilon_1+\epsilon_2 = 0$) and discussed that it can be interpreted in terms of $\mathcal{W}_{1+\infty}$ algebra. The analysis here is a natural generalization to any Ω-deformation. SH^c algebra contains a parameter β, which is related to Ω-deformation parameters by $\beta = -\epsilon_1/\epsilon_2$. When we take $\beta = 1$, the action of SH^c algebra can be identified with the $\mathcal{W}_{1+\infty}$ algebra. We will also see SH^c algebra contains Heisenberg × Virasoro subalgebra and its central charge is the same as that of Heisenberg × $\mathcal{W}_N$ algebra with background charge $Q = \sqrt{\beta} - 1/\sqrt{\beta}$. The combination of Heisenberg algebra with $\mathcal{W}_N$ appears in some papers, where the authors formally construct a basis of Hilbert space of Heisenberg × $\mathcal{W}_N$ algebra which reproduces the factorized form of Nekrasov partition function. Such observation implies that one may regard the formula (1) as the conformal Ward identities which characterize the conformal block function.

We mention that there is another one parameter deformation of $\mathcal{W}_{1+\infty}$ algebra, $\mathcal{W}_\infty[\mu]$ in the context of higher spin supergravity. SH^c and $\mathcal{W}_\infty[\mu]$ share a property that they are generated by infinite higher spin generators and contains $\mathcal{W}_N$ algebra with general β as their reduction. Here we use SH^c since their action on a basis parametrized by sets of Young diagram is already known. It is natural to expect that these two algebras are identical although their appearances are very different. It should be also noted that the introduction of further deformation parameter is possible and was applied to a generalization of AGT conjecture.

As we will see later, it is tempting to speculate that identities from SH^c algebra fully reproduce the conformal block function. Because of a technical difficulty to characterize the vertex operator in SH^c, explicit demonstration of the relation is limited to the Heisenberg and Virasoro subalgebra. For these cases, the recursion for $n = 0, 1$ can be indeed interpreted as Ward identities. The algebra SH^c was introduced to prove the AGT conjecture for pure super Yang-Mills theory. Our analysis shows that it may be applied to linear quiver gauge theories as well.