学位論文

Mean-Lagrangian renormalization theory of inhomogeneous turbulent flow

（非一様乱流に対する平均ラグランジュ的繰り込み理論）

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Thesis

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Abstract

An analytical method for inhomogeneous turbulent flow has been proposed on the basis of the *mean-Lagrangian-coordinate system* with an emphasis on the general covariance under the coordinate transformation. The mean-Lagrangian-coordinate system, which is defined to be a coordinate system convected by the mean flow, is excel in describing the turbulence-statistical quantities in generally covariant manner. By rewriting all the physical quantities and laws from the Eulerian frame to the mean-Lagrangian coordinate system, the *renormalized perturbation theory* has been developed to be applicable to generally-covariant turbulence correlations.

As an example, the present theory has been applied to the investigation of the Reynolds stress. As a result, a temporally non-local representation has been obtained in generally covariant form, which is a generalized result of multiple-time analysis in homogeneous-turbulence methodology. Applying the temporal-locality approximation, an algebraic representation of the Reynolds stress has been obtained with some new effects such as the *Oldroyd derivative* of the strain rate, spatial derivatives of the strain rate and the absolute vorticity.

Contents

Part I
Introduction

Turbulent flow

Fluid is a physical model applied to deformable material such as gasses or liquids; atmosphere or water are typical examples which always appear in any circumstances in our daily lives. These typical fluids have two oppositional properties; one is the nonlinearity which promotes complex and fine motion in space and time, and the other is the viscosity acting as resistivity against the complex motion in space. The ratio of these two effects can be most simply characterized by what is called the *Reynolds number*; as the Reynolds number increases, the nonlinearity exceeds the viscosity and makes flow more complex, disordered and unstable, and finally flow reaches what we call *turbulent flow* which is the exact target of our interest (Reynolds 1883). Under the low-Reynolds-number condition, the turbulent motion is suppressed by the viscosity and flow tends to show ordered and stable behaviors. In contrast to the terminology "turbulent flow", these ordered flows are called the *laminar flow*.

One of the most typical features of turbulence is the strong mixing which causes huge effective diffusion of momentum, temperature or chemical substances, while the smaller mixing is driven by molecular diffusion in non-turbulent flow. These turbulence diffusion rates are often Reynolds-number times as large as their molecular-diffusion counterparts. Thus, in most of real-world turbulent flow such that the Reynolds number exceeds several thousands or millions, the turbulence effect governs the transportation phenomena; it effectively promotes the mixing of the fuel and air in engines, it dramatically enhances the drag force on the surfaces of cars, ships or airplanes, it helps the oxygen to dissolve into ocean, it equalizes the temperature of the atmosphere, it diffuses the gigantic magnetic flux in the sun, interstellar gasses or galaxies. Namely, almost everywhere in our universe, there are lots of phenomena which can never be explained without appropriate knowledges about turbulence. Because of its universality and wide applicability, fluid turbulence has been the targets of various scientific fields, and the disclosure of its essence should have a huge impact on wide variety of fields of both pure and applied sciences.

Statistical approach to turbulent flow

Because turbulent flow is stochastic, some kinds of statistical averaging are often applied to it. The spatial and time averaging are available for the spatially homogeneous and stationary systems respectively. When the system is homogeneous in certain directions, we may be able to average quantities in those directions; channel turbulence is a typical example whose statistical averaging is often taken in the plane parallel to the wall. In more general case where flow has neither homogeneity nor stationarity, one may take average about the ensemble of realizations, which is called the "*ensemble average*".

The most fundamental problem of the statistical analysis of turbulence is the *closure problem*; because of the nonlinearity, statistically-averaged quantities cannot form a closed set of finite-number equations despite its original is closed. So far, various types of closure models have been proposed by researchers according to their purposes. We can roughly divide these modelings into two strategies; homogeneous-turbulence and inhomogeneous-turbulence closures. For industrial applications where physical properties vary in space, the mean flow is the primary interest so that the closure model is desired to be easily handled with spatially-varying mean quantities. Thus these models are usually closed only in terms of one-point one-time quantities. On the other hand, researchers, who are interested in the fundamental mechanism in the smaller-scale flow rather than the mean flow, may choose the homogeneous turbulence since they do not have to consider about the mean flow nor boundaries, which makes mathematical analysis remarkably simple. In the homogeneous-turbulence methodologies we often use multiple-point multiple-time quantities which are connected to detailed structures such as eddies, vortex tubes, vortex sheets, etc. Both homogeneous-turbulence and inhomogeneous-turbulence closures have their merits and problems respectively.

Homogeneous and inhomogeneous turbulence methodologies

Homogeneous-turbulence closures enable us to analyze the small-scale structures in details, and there are successful models enough able to reproduce some of fundamental real features with few empirical constants.

However, the homogeneity is no more than an ideal situation and lack of reality. On the other hand, inhomogeneous-turbulence closure is directly connected to the practical applications. However, inhomogeneous-turbulence models often include lots of empirical functions or constants in order to make the model applicable to as wide range of situations as possible. Namely, theoretical supports and wide applicability are often not compatible. Thus *theoretically well-supported methodology for inhomogeneous turbulent flow* may be desired for the further development of this field.

For several decades, theoretical researches for turbulent flow had been conducted mainly on homogeneous turbulence with taking advantages of its statistical homogeneity, and remarkable successes have been made in extracting some of primitive and fundamental properties such as Kolmogorov's scale-similarity laws in time and space (Kolmogorov 1941). Direct-interaction approximation (DIA), renormalized perturbation theory (RPT) and renormalization group (RNG) theory are the major groups of these works and all enable us to derive approximated laws based on mathematical structures of the Navier-Stokes equation (accompanied by the incompressibility condition) which is the exact law of nature (Kraichnan 1959, Wyld 1961, Foster et al. 1976, 1977). Although homogeneous turbulence is rather ideal situation considerably simplified in contrast to the real turbulence, these studies tell us detailed information in fine structures via multiple-point multiple-time quantities such as the second-order velocity correlation; Lagrangian-history direct-interaction approximation (LHDIA), abridged LHDIA (ALHDIA), Lagrangian renormalization approximation (LRA) or Lagrangian direct-interaction approximation (LDIA) are good examples which have successfully derived the Kolmogorov's scale similarity and other qualitative and quantitative agreements with experiments at least in terms of the lower order moments without using any empirical parameters (Kraichnan 1965, Kaneda 1981, Kida & Goto 1997).

On the contrary, for inhomogeneous turbulent flow which is observed in the real world, self-consistent methodologies have not been well-established. This originates from immutable differences between the ideal and real turbulence; in the real world turbulence is inevitably associated with anisotropy and inhomogeneity which drive turbulent motion via energy-cascading process. For anisotropic case, we cannot characterize the dynamics of turbulence only by scalar variables, unlike homogeneous isotropic case, and we cannot avoid to deal with complex equations for multiple-component variables. Besides, due to inhomogeneity, it is difficult to discriminate clearly the scale of fluctuation and that of the global structure of the mean field so that we have to understand the dynamics of those two simultaneously. Although inhomogeneous turbulence studies are necessary in explaining the real phenomena, these mathematical complexities prevent us from approaching to it in analytical manners.

Thus, in the inhomogeneous-turbulence studies, phenomenological understanding is very important. The closure model in industrial fields is referred to as the "*turbulence model*", which is often based on phenomenology and some mathematical constraints such as tensor analysis or dimensional analysis. Turbulence models are roughly classified into first-order (algebraic) and second-order models. The first-order model is an attempt to represent turbulence fluxes in algebraic ways; linear-eddy-viscosity model of the Reynolds stress (momentum flux due to turbulence) is representative of this which is originally based on the analogy of molecular viscosity of the Newtonian fluid. Second-order model is more elaborate and challenging attempt to construct the transport equation of the turbulence fluxes themselves, which seems to be more precise than the former since it incorporates into the model the detailed physical processes of them such as convection, production, redistribution, diffusion and dissipation due to turbulence motion. Strategies in turbulence modeling have been gradually improved by enduring trial and error, and now these models are applied to many practical cases such as homogeneous anisotropic flows, plane mixing layer, plane jet, symmetric and asymmetric channel flows (Hanjalić & Launder 2012).

In spite of these remarkable successes, turbulence models are still under the development and researchers have to continue the endless effort still now. The main cause of the situation may be attributed to its empirical and intuitive processes. Since these models inevitably include some uncertainties in functional forms of the modeled terms and constants attached to them which are usually optimized in order that the model is consistent with experiments or simulations of some canonical flows. For each target, one may construct elaborate models which can reproduce the real features as accurately as possible by tuning them. However, as long as one pursues the universal model applicable to as various cases as possible, these conventional techniques may be indirect approaches to the goal.

Another negative aspect of turbulence modeling is that it lacks direct connection with homogeneous-turbulence methodology. Turbulence models are usually based on one-point one-time quantities while homogeneous-turbulence closures on multiple-point multiple-time quantities. However, it is needed to bridge the gap between the turbulence modeling and homogeneous-turbulence methodology since the successes of ideal-turbulence theories clearly indicate the fundamental importance of the multiple-point multiple-time variables in the real turbulence. Although this bridging has been partially achieved by Leslie (1973) in channel-flow by comparing the Reynolds-stress transport equation and DIA energy equation, there is still enough room to search for more general correspondence between them.

Theoretical approach to inhomogeneous turbulent flow

Some of pioneering attempts for shear turbulence had been done from theoretical viewpoints. Kraichnan (1964) applied his DIA methodology to shear turbulence and obtained analytical expressions for the Reynolds stress and the scalar flux. Leslie (1973) also applied DIA to channel flow with a coordinate-decomposition technique and theoretically derived the transport term of the turbulence-energy and Reynolds-stress equations which correspond to the model by Hanjalić and Launder (1972). Unfortunately, these remarkable works are accompanied by unavoidable complexities in mathematical manipulations and it seems to be far difficult to extend these strategies to more general flow with complex boundaries. More manageable strategies have evolved from RNG; Yakhot and Orszag (1986) extended the RNG procedures to inhomogeneous turbulence and derived the K-ϵ model including numerical coefficients. Rubinstein and Barton (1990) utilized RNG to obtain so called the *nonlinear K-ϵ model* where the Reynolds stress is expressed as the quadratic function of the velocity gradient.

Another type of inhomogeneous-turbulence theories has evolved from RPT strategy; *two-scale direct-interaction approximation* (TSDIA) which is a combination technique of multiple-sale expansion in the singular perturbation method and DIA (Yoshizawa 1984). TSDIA is mainly based on the following physical insight. The mean fields often have larger-scale variations than those of fluctuations in both space and time so that the spatial and time derivatives of the mean fields, which are causes of turbulence motion, may be evaluated to be small relative to those of the instantaneous fields highly fluctuating in space and time. Thus TSDIA incorporates these mean-field effects as perturbations, and homogeneous-turbulence field, to which we can apply the homogeneous-turbulence theories, is taken as the non-perturbative field. Unlike the traditional inhomogeneous-turbulence modelings, TSDIA enables us to investigate various statistical quantities on the basis of the mathematical structures of the governing equations.

This strategy has been applied to various turbulence phenomena so far; non-linear turbulence viscosity effect (Yoshizawa 1984, 1985a, Nishizima & Yoshizawa 1987, Okamoto 1994), turbulence diffusion of scalars passively convected by fluids (Yoshizawa 1985b), chemical reaction in turbulent flow (Hamba 1987), turbulence of compressible fluid (Yoshizawa 1990a, 1991, 1992, 2003), frame-rotation effect (Yokoi & Yoshizawa 1993, Okamoto 1995) or turbulence dynamo and magnetic reconnection effects in magneto-hydrodynamic turbulent flow (Yoshizawa 1985c, 1990b, Yokoi & Yoshizawa 1993, Guo et al. 2012). Especially for the charge-neutral incompressible fluid, TSDIA theoretically derived some of *two-equation algebraic models*, which are practically-used strategies in mechanical engineering society very often, without any empirical parameter. Furthermore, TSDIA is able to derive even new effects that have never been expected from the dimensional nor tensor analyses. Although TSDIA still contains some unclear assumptions, these successes should be emphasized as great advantages over the traditional modeling.

Despite TSDIA has shown prominent successes as theoretical methodology in the above senses, it has not achieved the entire goal of the inhomogeneous-turbulence closure. One of the most obvious shortfalls is incapability of quantitative (or sometimes even qualitative) agreement with some real cases; models directly derived from TSDIA are sometime inconsistent with the real features, and thus it is needed in its practical use to modify some of constants and model forms derived as purely theoretical results. For example, the second-order non-linear algebraic model derived from TSDIA cannot reproduce the proper inequality of the turbulence intensities of simple shear flow. Thus the theoretical results from TSDIA are not always validated even in some canonical flows.

Another problematic fact, which has not been well-recognized so far, is that TSDIA contains inevitable inconsistencies in the rule of coordinate transformation. This may be rephrased that TSDIA breaks the covariance under several types of coordinate transformations. As will be shown later, the general covariance under the coordinate transformation is one of the most fundamental properties of turbulence, and gives a strict mathematical constraints to its dynamics. Thus we should take the general covariance as the fundamental property for the turbulence theory, which will provide us a strong guideline for the construction of the physical models. However, the models of such generally covariant quantities derived by TSDIA cannot satisfy the general covariance, mainly because it is formulated in Eulerian framework. As a consequence, the results of the TSDIA cannot reproduce the proper rule of the coordinate transformation. Therefore we have to reconstruct the theory in alternative ways, since the rule of coordinate transformation is one of the most fundamental properties in any theory of physics and the inclusion of such fundamental features often brings us some remarkable improvements in the physical modeling.

Composition of thesis

It is the objective of this thesis to show the importance of the general covariance and to formulate the new theoretical approach to the inhomogeneous-turbulence closure consistent with the general covariance. For this purpose, we first review some of fundamentals in mathematical treatments, phenomenologies and the conventional strategies of turbulence physics in Part II. Part III is the backbone of the thesis. In §5 the importance of the covariance under the general coordinate transformations is explained. In §6, the importance of the mean-Lagrangian formalism is emphasized from the view point of both multiple-time closure and the general covariance. In §7 the details of the formulation of the present theory are given. In §8, the present theory is applied to the Reynolds stress as an example. In particular the temporally non-local representation of the Reynolds stress clearly shows how the mean-Lagrangian representation is favorable in generally covariant formulation. As the result of the temporal-locality approximation, an algebraic representation of the Reynolds stress is derived in covariant form.

Part II
Conventional methodologies

In this part, we will review fundamentals of the statistical-turbulence methodologies. First the concept of ensemble average and the Reynolds decomposition, which are very standard descriptions of turbulence, will be introduced. Next we will review the phenomenologies, mathematical treatments and conventional modelings of homogeneous and inhomogeneous turbulence. Finally we will see a brief summary of TSDIA which is a theoretical approach to inhomogeneous turbulence.

We mainly use the orthonormal coordinate system as a frame of reference so that we do not have to distinguish the covariant and contravariant components of tensors. Thus we attach indices on the bottom-right side of each symbol to describe tensor components. Spatial derivative operation $\partial/\partial x_i$ on an arbitrary tensor $C_{abd\dots}$ is abbreviated as $C_{abd\dots,i}$ if necessary. Laplacian $\partial^2/\partial x_i \partial x_i$ is written as $\triangle$.

1 Governing law of fluid

In the inertial frame of reference, continuum, including fluid, is governed by the conservation laws of momentum, energy and mass, which are given respectively by

$$\rho \frac{\mathsf{d}}{\mathsf{d}t} v_i = \sigma_{ij,j}, \tag{1.1}$$

$$\rho \frac{\mathsf{d}}{\mathsf{d}t} u = \frac{1}{2}\sigma_{ij} s_{ij} - q_{j,j}, \tag{1.2}$$

$$\frac{\mathsf{d}}{\mathsf{d}t}\rho + \rho v_{j,j} = 0, \tag{1.3}$$

where $\mathbf{v}$, ρ, $\boldsymbol{\sigma}$, u and $\mathbf{q}$ are velocity, density, stress, internal energy per unit mass and heat-flux density respectively. s is the strain rate given by $\mathsf{s}_{ij} = v_{i,j} + v_{j,i}$. The operator $\mathsf{d}/\mathsf{d}t = \partial/\partial t + v_j \partial/\partial x_j$ is the Lagrangian derivative. The above equation holds not only for fluid but also for general continuum so that we should introduce some restrictions to u, $\boldsymbol{\sigma}$ and $\mathbf{q}$ to obtain the fluid equations. Newtonian fluid is defined by the following constitutive equation where stress is represented as an isotropic-linear function of strain rate.

$$\sigma_{ij} = -p\delta_{ij} + \frac{1}{2}\lambda s_{kk}\delta_{ij} + \mu s_{ij}, \tag{1.4}$$

where p is the pressure, λ and μ are the first and second molecular-viscosity coefficients. We apply the Fourier's heat-conduction law to the heat-flux density $\mathbf{q}$ and obtain

$$q_i = -\kappa T_{,i}\ , \tag{1.5}$$

where T is the temperature, κ is heat-conduction coefficient. Here we introduced new scalar variables p, T, λ, μ and κ which are all thermodynamical variables. Generally speaking, thermodynamical state of the single-chemical-species fluid is determined only by two thermodynamical variables so that the above equations (1.1)-(1.5) form the closed system for $\mathbf{v}$ and two of thermodynamical variables. If we choose the mass density and the temperature as the representative thermodynamical variable, the rests are represented as $p = p(\rho, T)$, $u = u(\rho, T)$, $\mu = \mu(\rho, T)$ and $\kappa = \kappa(\rho, T)$ (In case of the ideal gasses, we have $u = C_v \rho T$, $p = R\rho T$ where T is the temperature, C_v and R are constants). By assuming the incompressibility, the system of equations is simplified into the following;

$$\rho \frac{\mathsf{d}}{\mathsf{d}t} v_i = -p_{,i} + \left(\mu_{ij}\right)_{,j}\ , \tag{1.6}$$

$$\rho \frac{\mathsf{d}}{\mathsf{d}t} u = \frac{1}{2}\mu s_{ij} s_{ij} + \left(\kappa T_{,j}\right)_{,j}\ , \tag{1.7}$$

$$v_{j,j} = 0. \tag{1.8}$$

If we neglect the temperature dependence of μ and κ, (1.6) and (1.7) are reduced to

$$\rho \frac{\mathsf{d}}{\mathsf{d}t} v_i = -p_{,i} + \mu \triangle v_i, \tag{1.9}$$

$$\rho \frac{\mathsf{d}}{\mathsf{d}t} u = \frac{1}{2} \mu s_{ij} s_{ij} + \kappa \triangle T. \tag{1.10}$$

Here we should recognize that (1.9) and (1.8) form the closed system for $\mathbf{v}$ and p without T. Namely the temperature gives no contribution to fluid dynamics. (1.9) is called Navier-Stokes equation. In this thesis we discard the energy equation (1.10) and deal with the following two equations as the fundamental laws;

$$\frac{\mathsf{d}}{\mathsf{d}t} v_i = -p_{,i} + \nu \triangle v_i, \tag{1.11}$$

$$v_{j,j} = 0. \tag{1.12}$$

Here we divided the pressure by the constant density and put $p/\rho \to p$ for simplicity. Although $\nu = \mu/\rho$ is often called the kinematic-viscosity coefficient, we call ν as the molecular viscosity to emphasis the contrast between the molecular-viscous and turbulence-viscous effects.

2 Statistical methodology

The motion of fluid is governed by a deterministic system of equations (1.11) and (1.12) which had already been established in 19th century (Stokes 1845). Thus one might think that there was no need to pay special attentions to turbulent flow since it is totally determined by (1.11) and (1.12) with proper boundary and initial conditions. However, this is not true. Because of the strong nonlinearity, even a slight disturbance given at certain time can cause the huge departure from its original solution within finite time, which is often referred to as the "*sensitive dependence on the initial condition*" of nonlinear systems. If we had a proper way to deal with differential equations without any error or approximation, we could predict turbulent flow in details enough. However, in reality, we do not have general analytical solutions to these highly complex phenomena, and thus we have to rely on the numerical analysis which always contains finite errors of discretization. Thus, so far, we have no way to predict nor control the detail motion of turbulent flow at every location and time. Besides, it is also impossible to give the detailed predictions to the real phenomena since our observations always contain finite errors. Thus we should start everything with recognizing that turbulent flow is *actually stochastic* in spite of its fundamental determinism.

Fortunately researchers had found another possibility of turbulence research; the statistical regularities. Although instantaneous properties of turbulence look totally at random, there are lots of universal regularities in the mean properties so that researchers hope there will be some simple and beautiful mathematical laws behind the complexities and chaos, which has been the pure scientific motivation of turbulence researches. Besides, in contrast to each realization of turbulent flow, the averaged properties can be reproduced and observed experimentally so that they can be the target of controls in industrial applications. For example, although we cannot predict nor control the instantaneous drag force at every point on a wall, we can do it for drag force averaged on the wall. Kolmogorov (1941) had established statistical phenomenologies of small-scale turbulence and found some beautiful scale similarities, which had been confirmed in various experiments and computations in later works. These discoveries of universal regularities motivate researchers to establish the statistical methodology for turbulence and this thesis is also about a branch of these attempts.

2.1 Ensemble average and the Reynolds decomposition

In statistical approach, ensemble averaging is the fundamental operation to establish its formalism. First we denote the ensemble-averaging operation by $\langle\ \rangle$; the average value F of an arbitrary dynamical variable f in time and space is supposed to be given by $F = \langle f \rangle$. Obviously averaging is linear operation so that for arbitrary dynamical variable f and g we have

$$\langle C_f f + C_g g \rangle = C_f \langle f \rangle + C_g \langle g \rangle, \tag{2.1}$$

where C_f and C_g are constants. Here we should add an important insight of this operation; averaging of average is identical to averaging once. Namely we have

$$\langle \langle f \rangle \rangle = \langle f \rangle = F. \tag{2.2}$$

The fluctuation f' of f is given as the deviation from its average; namely $f' = f - \langle f \rangle$. Because of (2.2) we obtain

$$\langle f' \rangle = \langle f - \langle f \rangle \rangle = \langle f \rangle - \langle \langle f \rangle \rangle = F - F = 0, \tag{2.3}$$

i.e. any fluctuation is zero-mean. This decomposition operation between the mean and fluctuation based on the ensemble averaging is called the "*Reynolds decomposition*" (Tennekes & Lumley 1972).

2.2 Statistical unclosedness and need for turbulence closure

By applying the above Reynolds decomposition to (1.11) and (1.12), we obtain the governing equation for the mean fields and fluctuations which are given by

$$\frac{\mathrm{D}}{\mathrm{D}t} V_i = -P_{,i} + \nu \triangle V_i - R_{ij,j}, \tag{2.4}$$

$$V_{j,j} = 0, \tag{2.5}$$

$$\left(\frac{\mathrm{D}}{\mathrm{D}t} - \nu \triangle \right) v'_i + \left(v'_i v'_j \right)_{,j} + p'_{,i} = -V_{i,j} v'_j + R_{ij,j}, \tag{2.6}$$

$$v'_{j,j} = 0, \tag{2.7}$$

where $\mathrm{D}/\mathrm{D}t = \partial/\partial t + V_j \partial/\partial x_j$ is the Lagrangian derivative based on the mean velocity. $R_{ij} = \langle v'_i v'_j \rangle$ is called the *Reynolds stress* which represents the effective momentum flux due to turbulence motion and this gives the direct contribution of the turbulence to the mean flow. For four unknowns $\mathbf{V}$, P, $\mathbf{v}'$ and p' the above set of equations are necessary and sufficient so that it is exactly the closed system. In constructing the statistically closed system of equations, however, we are supposed to close the system only in terms of statistically-averaged values so that we have to calculate $\mathbf{R}$ without using fluctuation $\mathbf{v}'$ explicitly. The most straightforward way may be to construct the equation of $\mathbf{R}$, which is written as

$$\left(\frac{\mathrm{D}}{\mathrm{D}t} - \nu \triangle \right) R_{ij} = P^{R}_{ij} - \epsilon_{ij} + \Pi_{ij} - T_{ijk,k} \ , \tag{2.8}$$

where

$$P^{R}_{ij} = -V_{i,k} R_{kj} - V_{j,k} R_{ik} \ , \tag{2.9}$$

$$\epsilon_{ij} = 2\nu \langle v'_{i,k} v'_{j,k} \rangle \ , \tag{2.10}$$

$$\Pi_{ij} = \langle p' \left(v'_{i,j} + v'_{j,i} \right) \rangle \ , \tag{2.11}$$

$$T_{ijk} = \langle p' (v'_j \, \delta_{ik} + v'_i \, \delta_{jk}) \rangle + \langle v'_i v'_j v'_k \rangle \ . \tag{2.12}$$

$\mathbf{P}^{R}$, $\boldsymbol{\epsilon}$, $\boldsymbol{\Pi}$ and $\mathbf{T}$ are called production, dissipation, redistribution and transportation flux respectively. Our equation contains new unknowns, especially third-order moments in redistribution and transportation term. In exact treatment of this equation we need to solve these third-order correlations, whose equations include fourth-order properties as well. This is the immortal problem of nonlinear system including fluid turbulence; namely equation for a moment at certain order inevitably includes higher-order moments and thus we always strike into infinite hierarchy which prevents us from investigating the system in exact procedures. In practical analysis we need to truncate this hierarchy by representing the higher-order correlations in terms of the lower quantities. This is referred to as the "*closure problem*" and it has been the most fundamental target of the researches of this field until now.

3 Homogeneous turbulence

In statistical treatments of turbulence, we have to introduce statistical dynamical variables which characterize the nature of turbulence. In primitive stage, one may introduce the correlation length or time in the physical space, with which we can easily imagine the picture of turbulence eddies traveling in the physical space. However they lack the information about the smaller-scale eddies; since the correlation length and time scales given in physical space are of the largest-eddy motion and the smaller-scale structures are hidden behind these properties. In order to investigate the fine-scale physics of turbulence, let us introduce a concept of *homogeneous turbulence* where all the turbulence quantities except for the mean velocity are spatially uniform. Homogeneous turbulence was first introduced by Taylor (1935) as the idealized situation of the real turbulence; in the region enough apart from the wall, turbulence may be taken as locally homogeneous so that homogeneous turbulence may be able to illustrate some of real features. Although this statement is not completely validated in fluid turbulence, homogeneous turbulence gives us canonical model of small-scale turbulence in the real nature which can be utilized in the later studies of inhomogeneous turbulence.

Homogeneous turbulence is an ideal turbulence. The whole statistical quantities are identical under spatial translation and hence the degree of freedom is greatly reduced. As a result of this simplifications, we can analyze the fine structures in detail without interference of non-uniform structures in larger scale. In order to reveal the smaller-scale structures, the Fourier transformation is available. Although the Fourier analysis is often incapable of illustrating the spatial structures in physical space, it enable us to investigate each size of motion separately.

3.1 Representative dynamical variables

First of all we set up an orthonormal coordinate system $\{x_1, x_2, x_3\}$ as an inertial frame of reference. In general, the correlation of dynamical variables $f(\mathbf{x}, t)$, $g(\mathbf{x}', t')$, $h(\mathbf{x}'', t'')$ are the functions of $\mathbf{x}$, $\mathbf{x}'$, $\mathbf{x}''$, t, t' and t'' all of which are independent variables. In our situation, however, the system is now statistically homogeneous so that any correlations are identical under arbitrary translations. For example the correlation of dynamical variables $f(\mathbf{x}, t)$, $g(\mathbf{x}', t')$, $h(\mathbf{x}'', t'')$, should depends on their relative positions; namely

$$\langle f(\mathbf{x}, t) g(\mathbf{x}', t') h(\mathbf{x}'', t'') \rangle = C(\mathbf{x}' - \mathbf{x}, \mathbf{x}'' - \mathbf{x}, t, t', t''), \tag{3.1}$$

which depends on only two spatial variables. Thus the functional form is simplified very much which makes later analysis very much simpler than the general inhomogeneous cases. In particular, the one-point correlations such as the turbulence energy $K(\equiv \langle \mathbf{v}' \cdot \mathbf{v}' \rangle / 2)$ or its dissipation rate $\epsilon(\equiv \nu \langle \nabla \mathbf{v}' \cdot \nabla \mathbf{v}' \rangle)$ are all uniform in the space.

Next, let us see the Fourier spectra of statistical quantities. Providing there is no boundary and Fourier integrals are convergent, we obtain the spectrum of arbitrary dynamical variable $f(\mathbf{x}, t)$ as

$$f(\mathbf{k}, t) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3 x \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}, t) = \mathcal{F}|_x^k f(\mathbf{x}, t), \tag{3.2}$$

where we define the Fourier-transformation operator $\mathcal{F}$ as follows in this thesis;

$$\mathcal{F}|_x^k \times \equiv \frac{1}{(2\pi)^3} \int \mathrm{d}^3 x \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} \times . \tag{3.3}$$

Here we introduce the second-order correlation of velocity spectra which is given by

$$U_{ij}(\mathbf{k}, t; \mathbf{k}', t') = \langle v_i'(\mathbf{k}, t) v_j'(\mathbf{k}', t') \rangle. \tag{3.4}$$

This is the general representation which holds for inhomogeneous cases. Here we add the statistical homogeneity as a constraint and obtain

$$\begin{aligned} U_{ij}(\mathbf{k}, t; \mathbf{k}', t') &= \frac{1}{(2\pi)^3} \int \mathrm{d}^3 x \mathrm{e}^{-\mathbf{k}\cdot\mathbf{x}} \frac{1}{(2\pi)^3} \int \mathrm{d}^3 x' \mathrm{e}^{-\mathbf{k}'\cdot\mathbf{x}'} \langle v_i'(\mathbf{x}, t) v_j'(\mathbf{x}', t') \rangle \\ &= \frac{1}{(2\pi)^3} \int \mathrm{d}^3 x' \mathrm{e}^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}'} \frac{1}{(2\pi)^3} \int \mathrm{d}^3 x \, \mathrm{e}^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \, U_{ij}(\mathbf{x} - \mathbf{x}'; t, t'), \end{aligned} \tag{3.5}$$

where $\mathbf{U}(\mathbf{x} - \mathbf{x}'; t, t')(\equiv \langle \mathbf{v}'(\mathbf{x}, t) \otimes \mathbf{v}'(\mathbf{x}', t') \rangle)$ is the second-order velocity correlation in physical space. For statistical homogeneity, the integral $\int \mathrm{d}^3 x \, \exp[-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')] \, U_{ij}(\mathbf{x} - \mathbf{x}'; t, t')/(2\pi)^3$ does not depend on $\mathbf{x}'$ so that we can conduct the two volume integrations independently. Thus we obtain

$$U_{ij}(\mathbf{k}, t; \mathbf{k}', t') = \delta^3(\mathbf{k} + \mathbf{k}') U_{ij}(\mathbf{k}; t, t'), \tag{3.6}$$

where $U_{ij}(\mathbf{k}; t, t') = \mathcal{F}|_{x-x'}^k U_{ij}(\mathbf{x} - \mathbf{x}'; t, t')$. Let us think the simple physical meaning of $U_{ij}(\mathbf{k}; t, t')$. Using $U_{ij}(\mathbf{k}; t, t')$, the turbulence energy K is given by the following;

$$K(t) = \frac{1}{2} \int \mathrm{d}^3 k \, U_{ii}(\mathbf{k}; t, t). \tag{3.7}$$

Thus $U_{ii}(\mathbf{k};t,t)/2$ represents the contribution of each mode $\mathbf{k}$ to the total energy. Likewise we can analyze various properties of turbulence motion of each size separately. For homogeneous and isotropic case, $U_{ij}(\mathbf{k};t,t')$ is simply represented only by three real scalar functions $Q(k;t,t')$, $R(k;t,t')$ and $S(k;t,t')$ as

$$U_{ij}(\mathbf{k};t,t') = \delta_{ij}Q(k;t,t') + \frac{k_i k_j}{k^2}R(k;t,t') + \epsilon_{ijl}k_l S(k;t,t'). \tag{3.8}$$

In addition, for incompressible case we have $k_j\, v_j(\mathbf{k},t) = 0$ and thus we have $k_j\, U_{ij}(\mathbf{k};t,t') = 0$ as well, which yields

$$\begin{aligned} k_j\, U_{ij}(\mathbf{k};t,t') &= \delta_{ij}k_j Q(k;t,t') + \frac{k_i k_j}{k^2}k_j R(k;t,t') + i\epsilon_{ijl}k_l k_j S(k;t,t') \\ &= k_i Q(k;t,t') + k_i R(k;t,t') \\ &= 0. \end{aligned} \tag{3.9}$$

Thus we have $R(k;t,t') = -Q(k;t,t')$. Finally we obtain

$$U_{ij}(\mathbf{k};t,t') = \frac{1}{4\pi k^2}\left\{P_{ij}(\mathbf{k})E(k;t,t') + i\epsilon_{ijl}\frac{k_l}{2k^2}H(k;t,t')\right\}\ , \tag{3.10}$$

which is represented by only two real scalar functions $E(k;t,t') = 4\pi k^2 Q(k;t,t')$ and $H(k;t,t') = 8\pi k^4 S(k;t,t')$ each of which is related to turbulence energy and helicity. Indeed wavenumber integral of each function gives the total turbulence energy $K(t)$ and helicity $H(t) \equiv \langle \mathbf{v}' \cdot \mathrm{rot}\mathbf{v}' \rangle$ respectively as $K(t) = \int_0^\infty E(k;t,t)\,\mathrm{d}k$ and $H(t) = \int_0^\infty H(k;t,t)\,\mathrm{d}k$. $P_{ij} = \delta_{ij} - k_i k_j/k^2$ is a projector to solenoidal component[1].

3.2 Dynamical equation

In homogeneous isotropic turbulence, the two-point velocity correlation is represented only by two scalar functions $E(k;t,t')$ or $H(k;t,t')$. And we can extract the length, velocity and time scales of turbulence only by $E(k,t)$. In this sense $E(k,t)$ could be a representative dynamical variable which describes simple and basic characters of turbulence. Thus let us construct its dynamical equation. The exact analysis of homogeneous isotropic turbulence requires the absence of inhomogeneity and anisotropy from the system, i.e. the non-uniform distribution of the Reynolds stress or the mean-flow gradient should be removed from (2.6). However, as has been stated above, these inhomogeneity and anisotropy are the exact energy sources in the real phenomena. The equation for the turbulence energy is given by taking the contraction of indices i and j of (2.8) and dividing the both sides by 2, that is

$$\left(\frac{\mathrm{D}}{\mathrm{D}t} - \nu\triangle\right)K = -\frac{1}{2}R_{ij}S_{ij} - \epsilon - \left\langle \left(p' + \frac{1}{2}v'_j v'_j\right)v'_i \right\rangle_{,i}\ , \tag{3.11}$$

which means that energy is produced by anisotropic stress and transported by the gradients of pressure-velocity and triple correlations. For purely homogeneous isotropic case, the above is reduced to $\partial K/\partial t = -\epsilon$, where turbulence is monotonically decaying. Thus we cannot sustain turbulence without inhomogeneity and anisotropy. We can avoid this trade-off problem between homogeneous-isotropy and sustainability by adding an external random force whose spectrum is distributed only in lower-wavenumber region. Under this assumption the governing laws (2.6) and (2.7) are reduced to

$$\left(\frac{\partial}{\partial t} - \nu\triangle\right)v'_i(\mathbf{x},t) + \left(v'_i v'_j\right)_{,j}(\mathbf{x},t) + p'_{,i}(\mathbf{x},t) = f_i(\mathbf{x},t), \tag{3.12}$$

$$v'_{j,j}(\mathbf{x},t) = 0. \tag{3.13}$$

The following equation is derived from the Fourier transformation of the above two equations.

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right)Q(k,t) = \frac{1}{i}M_{j.ab}(\mathbf{k})[\mathbf{k};\mathbf{p},\mathbf{q}]T_{abj}(\mathbf{p},\mathbf{q};t) + \langle f_l v'_l\rangle(\mathbf{k},t), \tag{3.14}$$

[1] Arbitrary vector $\mathbf{w}$ in wave-number space can be represented as a linear combination of $\mathbf{k}$ and another vector, say $\check{\mathbf{k}}$, orthogonal to $\mathbf{k}$. Let A and B be arbitrary complex-number constants and put $\mathbf{w} = A\mathbf{k} + B\check{\mathbf{k}}$. By multiplying $\mathbf{w}$ by P, we obtain

$$P_{ij}w_j = A\,P_{ij}k_j + B\,P_{ij}\check{k}_j = A\,(k_i - k_i) + B\,(\check{k}_i - 0) = B\check{k}_i\ ,$$

where we used the orthogonality $\mathbf{k}\cdot\check{\mathbf{k}} = 0$. Thus P projects wave-number-space tensors into solenoidal components.

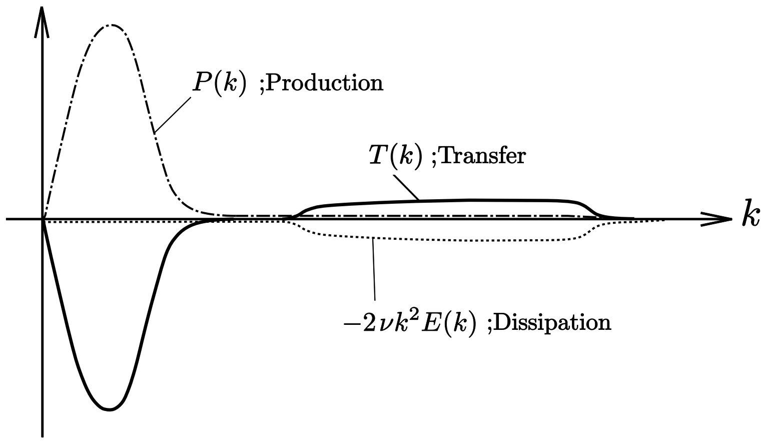

Figure 1: Budget of (3.15); The outlines of production spectrum $P(k)$ (chain line), dissipation spectrum $2\nu k^2 E(k)$ (dotted line) and energy-transfer function $T(k)$ (thick solid line) are illustrated. In the fully-developed turbulence, the dissipation spectrum $2\nu k^2 E(k)$ locates in higher-wavenumber range (finer scale in physical space) than the production-rate spectrum $P(k)$, since the dissipation rate originates from the higher derivative term. Thus there is a certain scale gap between these two spectra. The energy transfer function $T(k)$ is supposed to take negative value in the production range while it is to be positive in the dissipation range so as to transfer energy from the production to dissipation ranges through the scale gap.

where $T_{abi}(\mathbf{p},\mathbf{q};t) = \int d^3k' \, \langle v'_a(\mathbf{p},t)v'_b(\mathbf{q},t)v'_i(\mathbf{k}',t)\rangle$ and $\langle f_l v'_i\rangle(\mathbf{k},t) = \mathcal{F}|^k_{x-x'}\langle f_l v'_i\rangle(\mathbf{x}-\mathbf{x}',t)$. By multiplying the both sides by $4\pi k^2$ we obtain

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) E(k,t) = T(k,t) + P(k,t). \tag{3.15}$$

Here,

$$T(k,t) = \frac{1}{i} M_{j.ab}(\mathbf{k})[\mathbf{k};\mathbf{p},\mathbf{q}]T_{abj}(\mathbf{p},\mathbf{q};t)$$

is called the *energy-transfer function* which represents the energy-injection rate from other modes via non-linear interaction.

$$P(k,t) = 4\pi k^2 \langle f^l v'_l\rangle(\mathbf{k},t)$$

is the energy-production-rate spectrum which represents the production rate of the energy due to the external force at each mode k. Due to statistical isotropy of the external force, $P(k,t)$ is supposed to be an isotropic function.

Note that by integrating the both sides of (3.15) by k from 0 to ∞, we obtain the K equation in the physical space as follows;

$$\frac{\partial K}{\partial t} = \langle f_j v'_j\rangle - \epsilon, \tag{3.16}$$

where the first term of the right-hand side is the energy-production rate, ϵ is the energy-dissipation rate.

3.3 Phenomenology

In the previous discussion we have found that the scalar-binary-correlation function $Q(k,t)$ or the energy spectrum $E(k,t)$ tells us some important features of homogeneous isotropic turbulence, and for further deductive argument we have to deal with the dynamical equations of $Q(k,t)$ or $E(k,t)$. Unfortunately, as has been stated previously, there is always the infinite-hierarchy problem in our moment equations so that it is still difficult to go to the next stage without some closure methodologies. In spite of this unpleasant situation, we still have an alternative way to obtain some universal laws of this canonical turbulence from a combination of simple phenomenological assumptions, which had been first discussed by Kolmogorov (1941).

In physical space, according to (3.16), homogeneous isotropic turbulence can be most simply explained by the two processes; the energy-production and dissipation processes of energy, each of which has a characteristic length scale. The scale where the energy production occurs often contains the most of energy and thus it is referred to as the *energy-containing range* (or simply *containing range*). In a fully-developed turbulence, the dissipation process occurs in finer scale than that of the energy production since the Navier-Stokes equation has a second-order-derivative term as the dissipative term. The energy budgets in the wavenumber space is depicted in the figure 1. Because of this scale gap between these two processes, the nature requires another process bridging the gap; namely there is a process in the intermediate scale which transfers energy from the energy-containing scale to the dissipative scale. In the physical space, this process may be expressed by turbulence eddies which are colliding and destructed into small pieces, which may be interpreted as the energy-transfer from the larger-scale motion to the smaller. Here let us see a physical insight called "*locality*" of the nonlinear interaction, which is essential statement in the later discussions. Imagine what happens when eddies of totally different length scales meet together. From the view point of the finer-scale eddies, the large eddy's motion may look like a uniform flow so that the smaller eddy would be simply convected without destructed. Thus the nonlinear interaction between wavenumbers of huge scale gap may be relatively small to that between closer wavenumbers; the nonlinear interaction is expected to be locally effective in wavenumber space. Because of this locality, small-scale motion should be the result of multiple eddy-destruction processes, not by the direct energy injection from the larger scale. This is referred to as the *energy-cascading process*, which cuts off the direct connection between the injection-scale structure and small-scale turbulence and suggests the existence of some universal laws in small scale. On the basis of this idea, Kolmogorov (1941) had proposed two hypothesis which may be rephrased by the followings.

1. Arbitrary statistical properties of small-scale motion include only ν and ϵ as the scale-determining parameters.

2. In far larger scale than $\eta = \nu^{3/4}\epsilon^{-1/4}$ arbitrary statistical properties of small-scale motion include only ϵ as the scale-determining parameter.

η is called Kolmogorov length which characterizes the length scale of dissipation process. We can construct other characteristic quantities such as velocity $\mathcal{V}_\eta = \nu^{1/4}\epsilon^{1/4}$ or time $\mathcal{T}_\eta = \nu^{1/2}\epsilon^{-1/2}$. The Reynolds number of the dissipation scale is calculated as $\mathrm{Re}_\eta = \mathcal{V}_\eta \eta/\nu = 1$, which indicates the balance between inertia and viscosity. In smaller scale than this scale, the viscosity excess the inertia and turbulence motion may be suppressed. Thus the Kolmogorov length represents the smallest length scale of turbulence. According to the first hypothesis, the energy spectrum of high-wavenumber range is supposed to be given by $E(k) = \nu^{5/4}\epsilon^{1/4}\hat{E}(\hat{k})$ where $\hat{E}(\hat{k})$ is the universal dimensionless function of dimensionless wavenumber $\hat{k} = \eta k$. In fully-developed turbulence, we observe a huge scale gap between the energetic and dissipative scales and there exists a wavenumber band called the *inertial range* where the inertia (nonlinear interaction) plays the dominant role in the energy transfer between while the dissipation effect is negligibly small. According to the second hypothesis, the only possible choice for $\hat{E}$ in this range is a power function $\hat{E}(\hat{k}) = K_o \hat{k}^{-5/3}$ and we obtain $E(k) = K_o \epsilon^{2/3} k^{-5/3}$, where K_o is called the Kolmogorov constant which is one of the most fundamental universal constant of fluid turbulence. Experiments and computations suggest $K_o = 1.62 \pm 0.17$ (Sreenivasan 1995)

Let us investigate further insight with (3.15) in a semi-phenomenological manner. First we divide the Fourier spectra into two parts; the wavenumber range higher than k and the lower range. In addition, we assume that only the range lower than k_p has non-zero energy injection $P(k,t)$, namely $P(k,t) = 0$ for $k > k_p$. Thus the total-energy-injection rate $\epsilon_I(t) = \int_0^\infty \mathrm{d}k' P(k',t)$ now coincides with $\int_0^k \mathrm{d}k' P(k',t)$. By integrating (3.15) from 0 to $k (> k_p)$, we obtain the equation of lower-range energy $K_{<k}(t) = \int_0^k E(k',t)\mathrm{d}k'$ which is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} K_{<k}(t) = -2\nu \int_0^k E(k',t) k'^2 \, \mathrm{d}k' - \Pi(k,t) + \epsilon_I(t), \tag{3.17}$$

where $\Pi(k,t) = -\int_0^k T(k',t)\mathrm{d}k'$. On the other hand, by integrating from k to ∞ we obtain the equation of the higher-range energy $K_{>k}(t) = \int_k^\infty E(k',t)\mathrm{d}k'$ as

$$\frac{\mathrm{d}}{\mathrm{d}t} K_{>k}(t) = -2\nu \int_k^\infty E(k',t) k'^2 \, \mathrm{d}k' + \Pi(k,t). \tag{3.18}$$

In the derivation of $\Pi(k,t)$ we used the relation $\int_0^\infty \mathrm{d}k' T(k') = 0$ which guarantees the total-energy conservation for the inviscid limit $\nu \to 0$. On the right-hand side of (3.18), the only positive term which can provide energy to the higher range is $\Pi(k,t)$. Thus $\Pi(k,t)$ represents the energy-transfer rate from the lower range through the nonlinear interaction. Let us see next the equilibrium state where (3.17) is reduced to

$$-2\nu \int_0^k k'^2 E(k')\, \mathrm{d}k' - \Pi(k) + \epsilon_{I} = 0. \tag{3.19}$$

In the limit case $k \to \infty$ we obtain $-2\nu \int_0^\infty k'^2 E(k')\, \mathrm{d}k' + \epsilon_{I} = -\epsilon + \epsilon_{I} = 0$ and thus $\epsilon_{I} = \epsilon$. Although (3.19) includes only ν and ϵ in explicit form, we should notice that there is still possibility that the second term includes the information of the energy-injection scale in general. Assuming the locality of the nonlinear interaction, $\Pi(k)$ at $k > k_p$ is supposed to be free from the large-scale structures. Then the above equation have only two independent parameters ν and ϵ. Providing we obtain a closure approximation truncated at second order, $\Pi(k)$ would be represented as a functional of $E(k)$; namely we have

$$-2\nu \int_0^k k'^2 E(k')\, \mathrm{d}k' - \Pi[E](k) + \epsilon = 0. \tag{3.20}$$

This is a consistent functional equation for $E(k)$ so that this exactly determines the energy spectrum of the small scale which has only ν and ϵ as scale-determining factors. Thus $E(k)$ is expected to include only ν and ϵ as scale-determining factors, which corresponds to Kolmogorov's first hypothesis, and hence the solution should be again in the form of $E(k) = \nu^{5/4}\epsilon^{1/4}\hat{E}(\hat{k})$. Likewise the functional $\Pi[E](k)$ is also characterized only by ν and ϵ so that $\Pi[E](k)$ should be represented as $\epsilon\hat{\Pi}[\hat{E}](\hat{k})$ where $\hat{\Pi}$ is a non-dimensional functional of $\hat{E}$. Thus (3.20) reduces to

$$-2 \int_0^{\hat{k}} \hat{k}'^2\, \hat{E}(\hat{k}')\, \mathrm{d}\hat{k}' + \hat{\Pi}[\hat{E}](\hat{k}) + 1 = 0. \tag{3.21}$$

Thus $\hat{E}(\hat{k})$ is firmly a universal function. In the band $k_p < k \ll 1/\eta$ we shall neglect the first term and obtain

$$\hat{\Pi}[\hat{E}](\hat{k}) + 1 = 0 \tag{3.22}$$

Due to the locality of Π, ϵ is the only scale-determining factor, which corresponds to the Kolmogorov's second hypothesis. Hence we obtain again the Kolmogorov's power law $E(k) = K_o\epsilon^{2/3}k^{-5/3}$. Second-moment-closure theories such as LHDIA, LRA or LDIA provide us the functional forms for $\Pi[E](k)$ which enable us to analytically calculate K_o from (3.22); LHDIA concludes $K_o = 1.77$ while LRA and LDIA yield $K_o = 1.722$. Eulerian DIA had failed to reproduce the $-5/3$-power law since it could not eliminate the energy-containing velocity from Π; namely DIA provides us

$$\Pi[E;U](k) + \epsilon = 0, \tag{3.23}$$

where U is the characteristic velocity of the energy-containing range, which results in $E(k) \propto U^{1/2}\epsilon^{1/2}k^{-3/2}$. The actual failure of this result comes from its Eulerian formulation. The locality of the nonlinear interaction originally arises from the convection of smaller eddies by the larger which can be properly described only by Lagrangian view. In the Eulerian view, however, the small-scale properties are always swept by the larger eddies and this sweeping effect is erroneously accounted into the energy-transfer process. Namely larger-scale structure directly interferes the small-scale eddies and this is the reason why DIA spectrum includes U.

Let us see more details by applying the simplification to the energy-containing and dissipative ranges. We simply replace the exact spectral forms of the both ranges by simple cut-off wavenumbers k_c and k_d (see figure 2). Under this simplification we can calculate the bulk quantities K and ϵ as

$$K = \int_0^\infty E(k)\mathrm{d}k \approx K_o\epsilon^{2/3}\int_{k_c}^{k_d} k^{-5/3}\mathrm{d}k = \frac{3}{2}K_o\epsilon^{2/3}\left(k_c^{-2/3} - k_d^{-2/3}\right) \approx \frac{3}{2}K_o\epsilon^{2/3}k_c^{-2/3}, \tag{3.24}$$

$$\epsilon = 2\nu\int_0^\infty k^2E(k)\mathrm{d}k \approx 2K_o\nu\epsilon^{2/3}\int_{k_c}^{k_d} k^{1/3}\mathrm{d}k = \frac{3}{2}K_o\nu\epsilon^{2/3}\left(k_d^{4/3} - k_c^{4/3}\right) \approx \frac{3}{2}K_o\nu\epsilon^{2/3}k_d^{4/3}, \tag{3.25}$$

where we assumed $k_c \ll k_d$ which holds in the fully-developed turbulence. We can solve k_c and k_d, the spectral properties, in terms of K and ϵ, the properties in the physical space; namely we have

$$k_c = \left(\frac{3}{2}K_o\right)^{3/2} K^{-3/2}\epsilon, \tag{3.26}$$

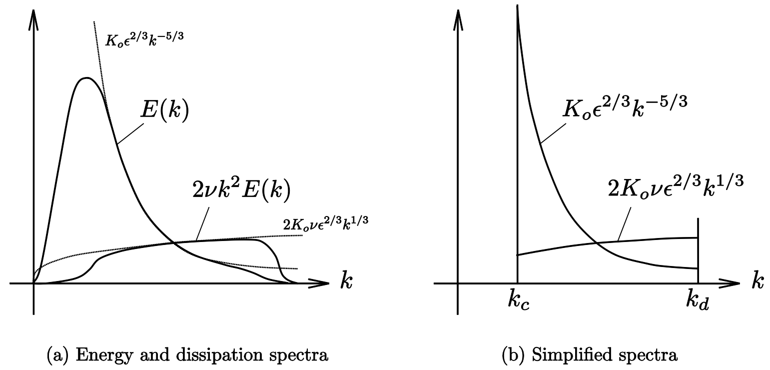

Figure 2: Schematic views of the energy and dissipation spectra; (a) The outlines of the energy and dissipation spectra are illustrated. Energy is most contained in the lower-wavenumber range (energy-containing range) while it rapidly decreases in the dissipation range. Between these two, there is the inertial range where the energy spectrum is given in $-5/3$-power form. (b) We avoid to deal with the detailed structures of the energy-containing and dissipation ranges, but instead, we simply replace these two ranges with a simple cutoff wavenumbers k_c and k_d.

$$k_d = \left(\frac{3}{2}K_o\right)^{3/4} \nu^{-3/4}\epsilon^{1/4} = \left(\frac{3}{2}K_o\right)^{3/4} \eta^{-1}. \tag{3.27}$$

Thus the energy-containing and dissipative scales are given by three factors K, ϵ and ν. It is fundamentally important fact for the turbulence modeling that the basic structures of the turbulence motion are roughly described by two bulk quantities K and ϵ, which partially validates the two-equation modeling such as K-ϵ, K-l, or K-ω models. The ratio of these scales k_d/k_c is given by

$$\begin{aligned}\frac{k_d}{k_c} &= \left(\frac{3}{2}K_o\right)^{-3/2} \times \left(\frac{K^2}{\nu\epsilon}\right)^{3/4} \\ &= 2^{-3/2}\,3^{3/8}\,\pi^{-3/4}\,K_o^{3/8} \times \left(\frac{UL}{\nu}\right)^{3/4} \\ &= 2^{-7/4}\,3^{3/4}\,\pi^{-1} \times \frac{L}{\eta}\ ,\end{aligned} \tag{3.28}$$

where $U(=\sqrt{2K})$ and $L(=2\pi/k_c)$ are the characteristic scales of velocity and length of the energy-containing eddies. Namely we have

$$\frac{UL}{\nu} \sim \left(\frac{L}{\eta}\right)^{4/3} \sim \left(\frac{k_d}{k_c}\right)^{4/3} \sim \frac{K^2}{\nu\epsilon}. \tag{3.29}$$

Thus the Reynolds number of the energy-containing scale is directly related to the gap between the energy-containing and dissipative scales, which is also prescribed by K, ϵ and ν. From a different viewpoint, we can extract another physical meaning of the Reynolds number. The mixing of scalars and momentum are basically due to the energy-containing eddies whose characteristic scales of velocity and length are given by $U(\sim K^{1/2})$ and $L(\sim K^{3/2}/\epsilon)$. Thus, according to an analogy of molecular viscosity, the effective viscosity ν_T (or diffusivity κ_T) is supposed to be given by $U \times L \sim K^2/\epsilon$. Thus the turbulence Reynolds number UL/ν also means the ratio of turbulence and molecular viscosities ν_T/ν.

3.4 Renormalized-perturbation theory of turbulence

Although we have obtained several basic and universal features of homogeneous turbulence from the above discussion in a remarkably simple manner, it basically depends on physical hypothesis and dimensional analysis. In order to derive them consistently, we should directly deal with the exact governing equation (3.15) which is not closed yet. Among lots of closure theories, we pick up here the *renormalized-perturbation theory* (RPT) which is now a day one of the most frequently-used theories of homogeneous isotropic turbulence. The theory is based on two dynamical variables. One is the spectrum of two-point two-time correlation $Q(k;t,t')$, the other is obtained from the *response against infinitesimal disturbance* introduced as follows.

Let us think the forcing term $\mathbf{f}$ in (3.12) as a small disturbance. The solution $\mathbf{v}'$ totally depends on this disturbance so that $\mathbf{v}'$ may be represented as a functional of $\mathbf{f}$. Thus we rewrite this as $\mathbf{v}'(\mathbf{k},t\|\mathbf{f})$. By assuming $\mathbf{f}$ be a small factor, we can expand $\mathbf{v}'(\mathbf{k},t\|\mathbf{f})$ in terms of $\mathbf{f}$ around the non-disturbed field $\mathbf{v}'(\mathbf{k},t\|\mathbf{0})$ as

$$v_i'(\mathbf{k},t\|\mathbf{f}) = v_i'(\mathbf{k},t\|\mathbf{0}) + \int_{-\infty}^{t} \mathrm{d}t' \int \mathrm{d}^3k'\, G_{ij}'(\mathbf{k},t;\mathbf{k}',t')\, f_j(\mathbf{k}',t') + O(\mathbf{f}^2). \tag{3.30}$$

G' is what is called the response against infinitesimal disturbance. By taking the ensemble average of this, we obtain

$$\left\langle G_{ij}'(\mathbf{k},t;\mathbf{k}',t')\right\rangle = \delta^3(\mathbf{k}-\mathbf{k}')\, P_{ij}(\mathbf{k})\, G(k;t,t'), \tag{3.31}$$

where we used homogeneity, incompressibility and isotropy of turbulence. The isotropic scalar function $G(k;t,t')$ is chosen as the other dynamical variable of RPT. One of the prominent superiorities of RPT to the previous works is that the strong nonlinearity of turbulence is effectively included through G.

LRA, which is one of the most successful closure theories, is a branch of RPT methodology. Incorporating the Lagrangian view into the formulation, LRA shows good agreements with experiments and simulations at least with respect to the second-order moments (Kaneda 1981, Kaneda 1986, Kraichnan 1965, Kraichnan 1977).

4 Inhomogeneous turbulence

In the real turbulence, the statistical properties are not uniform in space; some area may be more fluctuating than other areas, or may have different anisotropic aspect from others. Thus turbulent flow in the real world is accompanied by non-uniform mean fields, unlike the homogeneous turbulence. The more fluctuating area has larger K, the more dissipative area has larger ϵ. Thus the non-uniform variations of K or ϵ become the topic of debates.

The closure model is expected to be the system of field equations for these statistical quantities regarded as the field variables. One of the most important field variables is of course the mean velocity field V, whose governing equations are given by (2.4) and (2.5). In order to solve these equation, the Reynolds stress R should be solved. Thus the modeling of the Reynolds stress is a major topic of the inhomogeneous-turbulence closure. Note that the author will have the discussions about the inhomogeneous turbulence with emphasis on the modeling of the Reynolds stress.

In this section, some attempts of inhomogeneous-turbulence closure are reviewed in two categories; one is based on the phenomenological understandings, the other is on the theoretical understandings.

4.1 Turbulence modeling

As has been stated previously, we should truncate the moment hierarchy at finite order for inhomogeneous turbulence. The most primitive but common strategy is to represent the Reynolds stress in the simple viscosity form as

$$R_{ij} = \frac{2}{3}K\delta_{ij} - \nu_T S_{ij}, \tag{4.1}$$

where ν_T is effective viscosity due to the turbulence motion, $S_{ij}(= V_{i,j} + V_{j,i})$ is the mean strain rate. ν_T should reflect the contribution from the turbulence motion. The oldest type of this model is the *mixing-length model* where the distribution of ν_T is supposed to be given before calculating the mean velocity field. One of more practical strategies is to employ K and ϵ which can describe the small-scale structures in simple way (see §3.3). As has been pointed out in the §3.3, the turbulence viscosity is estimated as $U \times L \sim K^2/\epsilon$. By attaching a constant C_ν we obtain

$$\nu_T = C_\nu \frac{K^2}{\epsilon}. \tag{4.2}$$

In order to close the whole system, we have to solve K and ϵ both of whose equations are the targets of the closure. The equation for K is given by

$$\left(\frac{\mathrm{D}}{\mathrm{D}t} - \nu\triangle\right) K = -\frac{1}{2}R_{ij}S_{ij} - \epsilon - \left\langle \left(p' + \frac{1}{2}v_j'v_j'\right) v_i' \right\rangle_{,i}, \tag{4.3}$$

which includes the triple moment that represents the transportation of K due to the turbulence motion. Usually these turbulence fluxes are modelled as gradient-diffusion effect; namely turbulence intensity is transported from the highly-fluctuating region to the lower. Thus the turbulence-energy flux $\langle (p' + v'_j v'_j/2) v'_i \rangle$ is modelled as $-\nu_T/\sigma_K K_{,i}$ where σ_K is a positive constant. Thus we obtain the following model equation for K.

$$\frac{\mathrm{D}K}{\mathrm{D}t} = -\frac{1}{2} R_{ij} S_{ij} - \epsilon + \left\{ \left(\frac{\nu_T}{\sigma_K} + \nu \right) K_{,i} \right\}_{,i}, \tag{4.4}$$

Comparing to K equation, on the contrary, it is generally difficult to give physical interpretation to each unknown term of the exact ϵ equation. Instead it is often modelled as similar equation to K equation; namely, highly turbulent region is expected to be highly dissipative. This linkage between K and ϵ is very important since it prevents them from taking negative value. The standard model equation of ϵ is written as

$$\frac{\mathrm{D}\epsilon}{\mathrm{D}t} = -\frac{1}{2} C_{\epsilon 1} \frac{\epsilon}{K} R_{ij} S_{ij} - C_{\epsilon 2} \frac{\epsilon^2}{K} + \left\{ \left(\frac{\nu_T}{\sigma_\epsilon} + \nu \right) \epsilon_{,i} \right\}_{,i}, \tag{4.5}$$

where $C_{\epsilon 1}$, $C_{\epsilon 2}$ and σ_ϵ are positive constants (more detailed derivation can be seen in Hanjalić & Launder 1972). Equations (2.4), (2.5), (4.1), (4.2), (4.4) and (4.5) form a closed system and it is called the standard K-ϵ model. These two-equation linear-eddy-viscosity models have been the most popular and standard strategy to practical cases. K-ϵ model, K-l model or K-ω model are representative examples which are now a day widely used in industrial, meteorological or medical science, where $l(\propto K^{3/2}/\epsilon)$ and $\omega^{-1}(\propto \epsilon/K)$ are the characteristic length and time respectively (Launder & Spalding 1974, Wilcox 1988). These two-equation linear-eddy-viscosity models have enough power to mimic the simple flows such as wall-bounded or plane-symmetric-channel flows where only the shear-Reynolds stress plays a principal role in the mean-flow dynamics, and thus they are now incorporated into some commercial software packages for computations.

Another strategy is to mimic the exact generation process of the Reynolds stress itself (2.8), which is called the *Reynolds-stress modeling* (RSM). In this strategy the Reynolds stress itself is taken as the dynamical variable. For example, we model the unknown terms of (2.8) as

$$\begin{aligned} \frac{\mathrm{D}R_{ij}}{\mathrm{D}t} =& P^{R}_{ij} - \frac{2}{3}\epsilon\delta_{ij} - C_R \frac{\epsilon}{K}\left(R_{ij} - \frac{2}{3} K \delta_{ij} \right) - C_{IP}\left(P^{R}_{ij} - \frac{1}{3} P^{R}_{kk} \delta_{ij} \right) \\ &+ C'_{TR} \left\{ \frac{K}{\epsilon} \left(R_{ia} R_{jk,a} + R_{ja} R_{ki,a} + R_{ka} R_{ij,a} \right) \right\}_{,k} + \nu \triangle R_{ij}. \end{aligned} \tag{4.6}$$

(2.4), (2.5), (4.5) and (4.6) form a closed set of equations for $\mathbf{V}$, P, $\mathbf{R}$ and ϵ. This set of equations, first proposed by Launder, Reece and Rodi (1975), shows preferable performances in relatively wide varieties of flow to the simple two-equation first-order models. The second-order modelings begin to show remarkable power in their prediction, not only for elementary flows with simple boundaries, but also for more complex flows in industrial cases; homogeneous anisotropic flows, plane mixing layer, plane jet, symmetric and asymmetric channel flows. However, they could not reach the perfect agreements with all the experiments; they recognized that its main cause comes from the shortfall in the modeling of the redistribution process. Before and after this standard model, a number of works have been done in terms of the redistribution term. Two-component-limit (TCL) modelings for the rapid term is now a day one of the most successful strategies in this context which have noticeable power in the description of strong anisotropic case such as near-wall turbulence (Launder & Li 1994, Craft & Lien 1995, Iacovides et al. 1996, Kidger 1999, Craft et al. 2000, Craft & Launder 2001, Suga 2003).

Although RSM have achieved many successful results in various practical flows, there is still a huge problem in the detached flow behind bluff bodies where the velocity-pressure correlation becomes dominant. Besides, the model contains multiple components of the Reynolds stress so that it sometime suffers from the numerical instability. Considering these fact, RSM strategy is still under the development now a days.

4.2 Two-scale direct-interaction approximation

So far we have seen the modelings based on relatively weak constraints such as dimensional analysis or physical intuitions. Here we will see an example of theoretical approach, two-scale direct-interaction approximation (TSDIA), which provides us theoretical tools to analyze various turbulence quantities based on the Navier-Stokes equation. The original work has been done by Yoshizawa (1984) for the theoretical analysis of the

Reynolds stress, which was modified later by Hamba (1987) in the treatment of incompressibility.

TSDIA is based mainly on the scale-separation concept; in the region apart from boundaries, the mean quantities usually have much smoother variations in both space and time than the fluctuations, and thus the contribution from the mean quantities to the fluctuation dynamics may be taken as small perturbations. Let us see again (2.6) which governs the velocity fluctuation. If the mean velocity and the Reynolds stress have only small variations in space, their gradients may be evaluated as small. Furthermore, as long as we consider the small-scale dynamics of the fluctuation, the variations of mean fields may be negligible. Under these assumptions, (2.6) reduces to

$$\left(\frac{\partial}{\partial t}+V_j\frac{\partial}{\partial x_j}-\nu\triangle\right)v'_i(\mathbf{x},t)+\left(v'_iv'_j\right)_{,j}(\mathbf{x},t)+p'_{,i}(\mathbf{x},t)=-\delta V_{i,j}v'_j(\mathbf{x},t)+\delta R_{ij,j}\ , \tag{4.7}$$

where δ is an expediential parameter indicating that $V_{i,j}$ and $R_{ij,j}$ are small. Here we neglect spatial variations of V_j, $V_{i,j}$ and $R_{ij,j}$. This equation has two prominent advantages over the original; one is that this can be taken as statistically homogeneous equation, and the other is that we can apply perturbation methodology to this equation by regarding the δ-related terms as small perturbation. Thus one may apply the homogeneous-turbulence methodology to this equation with incorporating the anisotropic and inhomogeneous contributions from $V_{i,j}$ and $R_{ij,j}$ via perturbation technique. However, the scale separation is not successful in reality; the scales of mean fields and fluctuations sometime overlap each other and they are not clearly separated.

In order to describe both fine and smoother variations, we introduce two types of coordinates and time parameters; $\boldsymbol{\xi}(=\mathbf{x})$, $\mathbf{X}(=\delta\mathbf{x})$, $\tau(=t)$ and $T(=\delta t)$. Using these new parameters, we rewrite arbitrary physical quantity $f(\mathbf{x},t)$ as $f(\boldsymbol{\xi},\tau;\mathbf{X},T)$. Besides, we assume that all the mean fields are described by global coordinates $\mathbf{X}$ and time T. Thus the Reynolds decomposition may be rewritten as $f(\boldsymbol{\xi},\tau;\mathbf{X},T)=f'(\boldsymbol{\xi},\tau;\mathbf{X},T)+F(\mathbf{X},T)$. In addition, we assume the statistical homogeneity in terms of $\boldsymbol{\xi}$, which corresponds to the local homogeneity. Under these assumptions the original equations (2.6) and (2.7) are rewritten as

$$\begin{aligned}&\left(\frac{\partial}{\partial\tau}+V_j(\mathbf{X},T)\frac{\partial}{\partial\xi_j}-\nu\frac{\partial^2}{\partial\xi_j\,\partial\xi_j}\right)v'_i(\boldsymbol{\xi},\tau;\mathbf{X},T)+\frac{\partial}{\partial\xi_j}\left(v'_iv'_j\right)(\boldsymbol{\xi},\tau;\mathbf{X},T)+\frac{\partial}{\partial\xi_i}p'(\boldsymbol{\xi},\tau;\mathbf{X},T)\\&=-\delta\frac{\mathrm{D}}{\mathrm{D}T}v'_i(\boldsymbol{\xi},\tau;\mathbf{X},T)-\delta\frac{\partial V_i}{\partial X_j}(\mathbf{X},T)v'_j(\boldsymbol{\xi},\tau;\mathbf{X},T)-\delta\frac{\partial}{\partial X_i}p'(\boldsymbol{\xi},\tau;\mathbf{X},T)-\delta\frac{\partial}{\partial X_j}\left(v'_iv'_j\right)(\boldsymbol{\xi},\tau;\mathbf{X},T)\\&\quad+\delta\frac{\partial R_{ij}}{\partial X_j}(\mathbf{X},T)+\delta^2\nu\frac{\partial^2}{\partial X_j\,\partial X_j}v'_i(\boldsymbol{\xi},\tau;\mathbf{X},T),\end{aligned} \tag{4.8}$$

$$\left(\frac{\partial}{\partial\xi_j}+\delta\frac{\partial}{\partial X_j}\right)v'_j(\boldsymbol{\xi},\tau;\mathbf{X},T)=0. \tag{4.9}$$

These form a closed set of equations of $v'_i(\boldsymbol{\xi},\tau;\mathbf{X},T)$ and $p'(\boldsymbol{\xi},\tau;\mathbf{X},T)$. In the later analysis we apply the perturbation techniques; we expand v'_i and p' as

$$v'_i(\boldsymbol{\xi},\tau;\mathbf{X},T)=v'^{0}_i(\boldsymbol{\xi},\tau;\mathbf{X},T)+\delta v'^{1}_i(\boldsymbol{\xi},\tau;\mathbf{X},T)+O(\delta^2)\ , \tag{4.10}$$

$$p'(\boldsymbol{\xi},\tau;\mathbf{X},T)=p'^{0}(\boldsymbol{\xi},\tau;\mathbf{X},T)+\delta p'^{1}(\boldsymbol{\xi},\tau;\mathbf{X},T)+O(\delta^2)\ . \tag{4.11}$$

By substituting these two into the dynamical equations (4.8) and (4.9), and collecting the terms of each order, we obtain the equations for the quantities of each order. What is essential for the later formulation is that the zeroth fields v'^{0}_i and p'^{0} are governed by totally the same equations as the homogeneous turbulence. Thus we assume that the zeroth fields behave as homogeneous turbulence. The total turbulence field can be expanded in terms of zeroth field so that all the statistical properties are supposed to be represented by those of zeroth field.

Applying the TSDIA to the Reynolds stress yields

$$R_{ij}=\frac{2}{3}\delta_{ij}\int Q^{0}(k;\tau,\tau)\mathrm{d}^3k-\delta\frac{7}{15}\left(\frac{\partial V_i}{\partial X_j}+\frac{\partial V_j}{\partial X_i}\right)\int\mathrm{d}^3\,k\int_{-\infty}^{\tau}\mathrm{d}\,\tau' Q^{0}(k;\tau,\tau')G^{0}(k;\tau,\tau')+O(\delta^2)\ , \tag{4.12}$$

where Q^{0} and G^{0} are the binary correlation and the infinitesimal response function of the zeroth field. By the replacement $X_i\to\delta x_i$, we obtain the eddy-viscosity representation as the lowest-order analysis, and we may put

$$K=\int\mathrm{d}^3k\,Q^{0}(k;\tau,\tau), \tag{4.13}$$

$$\nu_T = \frac{7}{15}\int \mathrm{d}^3 k \int_{-\infty}^{\tau} \mathrm{d}\,\tau' Q^{\mathrm{o}}(k;\tau,\tau')G^{\mathrm{o}}(k;\tau,\tau'). \tag{4.14}$$

This explains the eddy viscosity as the accumulation of contributions from various scale motions each of which is given by the energy multiplied by the time-scale. Thus, we have obtained here an organic connection between the eddy viscosity of turbulence modeling and small-scale turbulence. In theoretical sense, this is quite natural consequence since the class of inhomogeneous turbulence includes that of homogeneous one as special case and inhomogeneous-turbulence theory could be an extended form of homogeneous version; namely TSDIA had succeeded in giving us a bridge between homogeneous and inhomogeneous turbulence methods.

By substituting the exact spectral forms of Q^{o} and G^{o} into (4.14), we can investigate ν_T in more detail. Especially, when turbulence is developed enough to have clear inertial subrange in small scale, we learned that the small-scale structures are roughly characterized by only three factors; K, ϵ and ν (§3.3), and hence we can expect to obtain simple explanation of ν_T. Indeed, by assuming the exact spectral form as

$$Q^{\mathrm{o}}(k;\tau,\tau') = C_\sigma\,\epsilon^{2/3}k^{-11/3}G^{\mathrm{o}}(k;\tau,\tau'), \quad G^{\mathrm{o}}(k;\tau,\tau') = \exp[-C_\omega \epsilon^{1/3}k^{2/3}(\tau-\tau')],$$

and applying the cut-off wavenumber k_c of the energy-injection range we utilized in §3.3 to Q^{o}, we reach

$$\nu_T = C_\nu \frac{K^2}{\epsilon}, \quad C_\nu = \frac{7}{360\pi C_\sigma C_\omega}. \tag{4.15}$$

Now the coefficient C_ν of K-ϵ eddy viscosity is explicitly given in terms of C_σ and C_ω which are the universal constants of homogeneous isotropic turbulence. Using a homogeneous-turbulence closure, Yoshizawa (1978) estimated $C_\sigma = 0.118$ and $C_\omega = 0.419$ which conclude $C_\nu = 0.123$, while $C_\nu = 0.09$ is often chosen in practical applications (Okamoto 1994). The original TSDIA (Yoshizawa 1984) concludes $C_\nu = 0.0785$ while its treatment of incompressibility was modified by Hamba (1987) yielding the present result. Since the whole procedure is totally based on the Navier-Stokes equation (some unclear assumptions are included though), TSDIA needs no empirical constants in principle[2]. This point should be emphasized as a great advantage over the traditional methodology.

Furthermore, not only the acknowledged types but also entirely-new types of models that have never been expected in the conventional frameworks, TSDIA can derive in general. In higher-order analysis of the Reynolds stress, TSDIA yields

$$\begin{aligned} R_{ij} = \frac{2}{3}K\delta_{ij} &- \left(0.123\frac{K^2}{\epsilon} - 0.147\frac{K^2}{\epsilon^2}\frac{\mathrm{D}K}{\mathrm{D}t} + 0.0933\frac{K^3}{\epsilon^3}\frac{\mathrm{D}\epsilon}{\mathrm{D}t}\right)\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) \\ &+ 0.0427\frac{K^3}{\epsilon^2}\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) \\ &+ 0.0542\frac{K^3}{\epsilon^2}\left(\frac{\partial V_i}{\partial x_a}\frac{\partial V_j}{\partial x_a} - \frac{1}{3}\delta_{ij}\frac{\partial V_a}{\partial x_b}\frac{\partial V_a}{\partial x_b}\right) \\ &+ 0.0297\frac{K^3}{\epsilon^2}\left(\frac{\partial V_i}{\partial x_a}\frac{\partial V_a}{\partial x_j} + \frac{\partial V_j}{\partial x_a}\frac{\partial V_a}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial V_a}{\partial x_b}\frac{\partial V_b}{\partial x_a}\right) \\ &+ 0.00525\frac{K^3}{\epsilon^2}\left(\frac{\partial V_a}{\partial x_i}\frac{\partial V_a}{\partial x_j} - \frac{1}{3}\delta_{ij}\frac{\partial V_a}{\partial x_b}\frac{\partial V_a}{\partial x_b}\right) \\ &+ \dots\,, \end{aligned} \tag{4.16}$$

which includes not only non-linear strain-rate terms but also $\mathrm{D}/\mathrm{D}t$-related terms that represent the non-equilibrium effects (Yoshizawa 1984, Okamoto 1994). Yokoi and Yoshizawa (1993) developed TSDIA analysis and obtained a prediction that the gradient of turbulence helicity H causes the vortical motion, which leads to K-ϵ-H model. Yoshizawa (1985c) applied TSDIA to the turbulence electromotive force of the magnetohydrodynamic (MHD) turbulence and discovered the cross-helicity dynamo stating that the alignment of the velocity and magnetic-field fluctuations causes the effective electromotive force parallel to the vorticity. Now we can summarize the basic features of TSDIA as follows.

[2]In practical computations, however, these theoretical constants have often been tuned so as to reproduce experimental results.

1. It is a self-consistent approach to inhomogeneous turbulence along with organic connections to the Navier-Stokes equation. (It does not require any empirical parameter in principle.)

2. It bridges a gap between the inhomogeneous-turbulence models and the homogeneous turbulence theories.

3. It provides theoretical supports for conventional models and suggestions of new types of models which is hardly estimated only from intuitions and experiences.

Although these successes sound very attractive in theoretical study of general turbulent flows, TSDIA is inconsistent with several classes of coordinate transformation; it causes some contradictions, at least, when we consider the time-dependent rotation of the reference frame, which may be rephrased as follows; TSDIA *breaks the covariance* under several types of coordinate transformations. It has been wellknown and discussed for some decades that turbulence has covariance under the translation and time-dependent rotation and this has been one of important guidelines of the turbulence modeling (Speziale 1979). In the next part, the author will present further generalized statement i.e. turbulence has the covariance under far wider group of transformation.

The main reason of the covariance breakdown of TSDIA is that the bases of perturbative expansions in TSDIA are Eulerian which are not covariant even under translation and time-dependent rotation. Although it may provide covariant results at the infinite order, it always breaks the covariance as long as we truncate the perturbation expansion at a finite order. Thus the most natural solution for this problem may be to find another base for the perturbative expansion with which we may always obtain covariant results at arbitrary orders of truncation. In the next part, we will see the importance of the general covariance and our new strategy in more details.

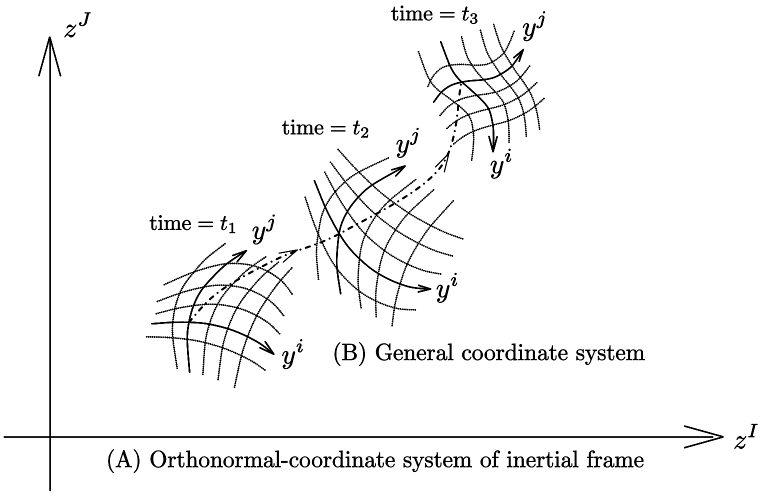

Figure 3: General coordinate system; The general coordinate system $\{\mathbf{y}\}$ (B) is illustrated in contrast to an inertial reference frame (A) (In general, $\{z\}$ does not have to be orthonormal.). In the thesis, we define the class of general coordinate system as those freely moving as time passes.

Part III

The mean-Lagrangian formalism

In this part the author will present a new formalism of inhomogeneous turbulent flow. First, in §5, we shall recognize the importance of the covariance under the general coordinate transformation in inhomogeneous turbulent flows[3]. In §6, according to the analogy of the continuum physics, we will introduce the mean-Lagrangian coordinate system. In §7, we will extend the RPT to the inhomogeneous turbulence on the basis of the mean-Lagrangian representation. In §8, the theory will be applied to the Reynolds stress in order to see both relation and difference between the present theory and the conventional modeling.

In the formulation, we inevitably employ curvilinear coordinate systems and thus we have to discriminate covariant and contravariant components of tensors. In this part, we denote the indices of covariant components on lower-right side of the tensor symbol while the contravariant counterparts on the upper-right side. We will see physics in curved coordinate system so that we shall treat two types of spatial derivatives. One is the ordinary partial derivative; for an arbitrary multi-component quantity ${C^{ab\cdots}}_{cd\cdots}$, j-component partial derivative is supposed to be represented as ${C^{ab\cdots}}_{cd\cdots,j}$. The other is the covariant derivative whose application to ${C^{ij\cdots}}_{kl\ldots}$ is given by

$$\nabla_a {C^{ij\ldots}}_{kl\ldots} = {C^{ij\ldots}}_{kl\ldots,a} + \Gamma^i_{ma} {C^{mj\cdots}}_{kl\ldots} + \Gamma^j_{ma} {C^{im\cdots}}_{kl\ldots} + \cdots - \Gamma^n_{ka} {C^{ij\ldots}}_{nl\ldots} - \Gamma^n_{la} {C^{ij\cdots}}_{kn\ldots} - \ldots.$$

Γ^i_{ab} is the Christoffel symbol of the second kind which is given by

$$\Gamma^a_{ij} = \frac{1}{2} g^{ab} \left(g_{bj,i} + g_{bi,j} - g_{ij,b} \right),$$

where $\mathbf{g}$ is the metric tensor. If necessary, we use the abbreviated form of the covariant derivative given by $\nabla_a {C^{ij\cdots}}_{kl\ldots} = {C^{ij\cdots}}_{kl\ldots;a}$.

5 Need for the general covariance

5.1 General covariance in continuum physics

Let us begin with the discussion about the coordinate systems for reviewing what kind of covariance we treat. The physical space is assumed to be a three-dimensional space flat in the sense of the metric geometry.

[3]Note that we only treat the general transformation in the 3-dimensional space, since we do not consider the general-relativistic effects. In the non-relativistic limit, our transformation group is larger than one has ever seen in classical mechanics. We should remark that what we call the general transformation group is asymptotically extended to the subgroup of that in general relativity.

First we discriminate the inertial frame from the other frames of reference. We introduce an orthonormal-coordinate system $\{z^1, z^2, z^3\} = \{\mathbf{z}\}$ as an inertial frame of reference as (A) in figure 3 [4]. Note that we employ the capital roman letters for the indices of Eulerian-coordinate representation. For example, we can represents the coordinate variables as $z^I(I = 1, 2, 3)$.

For the next step we introduce the curvilinear-coordinate system $\{\mathbf{y}\}$ which is moving relatively to the inertial frame $\{\mathbf{z}\}$ by providing the condition such as

$$y^i = y^i\,(z^1, z^2, z^3, t) = y^i\,(\mathbf{z}, t)\,, \tag{5.1}$$

where $i = 1, 2, 3$. We employ small roman letters for the indices of the general-coordinate representation. The simplest example of the above relation is the linear relation between $\mathbf{y}$ and $\mathbf{z}$, which may be written as

$$y^i = T^i_J(t)\, z^J + Y^i(t). \tag{5.2}$$

If we put T as the unit matrix and $\mathbf{Y}$ as linear function of t, (5.2) represents the Galilean transformation. If $\mathbf{Y}$ is nonlinear in t, (5.2) is transformation to an accelerated frame. If T is a time-dependent orthogonal matrix, (5.2) represents the transformation to a rotating frame.

Generally speaking, coordinate system $\{\mathbf{y}\}$ may have rotation and distortion non-uniform in time and space which is depicted by (B) in figure 3. In the non-relativistic sense, the coordinate system $\{\mathbf{y}\}$ defined by (5.1) forms the largest class of the coordinate systems. Thus, in this thesis, we call coordinate systems such as $\{\mathbf{y}\}$ given by (5.1) the *general coordinate system*, and we call the transformation between general coordinate systems the *general coordinate transformation*. For example, transformation from $\{\mathbf{y}\}$ to another general-coordinate system $\{\tilde{\mathbf{y}}\}$ is given as follows;

$$\tilde{y}^{\tilde{a}} = \tilde{y}^{\tilde{a}}(y^1, y^2, y^3, t) = \tilde{y}^{\tilde{a}}(\mathbf{y}, t), \tag{5.3}$$

where $\tilde{a} = 1, 2, 3$; the coordinate representation in $\{\tilde{\mathbf{y}}\}$ is discriminated from that of $\{\mathbf{y}\}$ by indices. Under the general-coordinate transformation group, we consider the covariance. We require the tensor fields to satisfy the transformation rule given by

$$C^{ij\cdots}{}_{kl\cdots}(\mathbf{y}, t) \to C^{\tilde{a}\tilde{b}\cdots}{}_{\tilde{c}\tilde{d}\cdots}(\tilde{\mathbf{y}}, t) = \frac{\partial \tilde{y}^{\tilde{a}}}{\partial y^i}(\mathbf{y}, t)\, \frac{\partial \tilde{y}^{\tilde{b}}}{\partial y^j}(\mathbf{y}, t) \cdots \frac{\partial y^k}{\partial \tilde{y}^{\tilde{c}}}(\tilde{\mathbf{y}}, t)\, \frac{\partial y^l}{\partial \tilde{y}^{\tilde{d}}}(\tilde{\mathbf{y}}, t)\, \cdots C^{ij\cdots}{}_{kl\cdots}(\mathbf{y}, t)\,,$$

where $(\mathbf{y}, t)$ and $(\tilde{\mathbf{y}}, t)$ show the same space-time point (If we consider the transformation between different times, we cannot do these abbreviations). In the later analysis, we sometime employ abbreviations in notations; we employ the same main symbols for physical quantity in any coordinate system, since we can recognize its coordinate representation by tilde on the indices. For example $\tilde{C}^{\tilde{a}\tilde{b}\cdots}{}_{\tilde{c}\tilde{d}\cdots}$ is supposed to be rewritten as $C^{\tilde{a}\tilde{b}\cdots}{}_{\tilde{c}\tilde{d}\cdots}$. In the same manner, we rewrite the coordinate variable $\tilde{y}^{\tilde{a}}$ as $y^{\tilde{a}}$ since we can discriminate it from y^i by tilde on the index [5]. Following the abbreviation, the above transformation rule is rewritten as follows [6];

$$C^{ij\cdots}{}_{kl\cdots} \to C^{\tilde{a}\tilde{b}\cdots}{}_{\tilde{c}\tilde{d}\cdots} = y^{\tilde{a}}{}_{,i}\, y^{\tilde{b}}{}_{,j} \cdots y^k{}_{,\tilde{c}}\, y^l{}_{,\tilde{d}} \cdots C^{ij\cdots}{}_{kl\cdots}. \tag{5.4}$$

In the continuum physics, every physically objective quantity should be represented as a tensor field at certain rank. One of the typical examples of non-covariant quantities is the velocity field of continuum. The transformation rule of the velocity field is given by

$$v^{\tilde{a}} = y^{\tilde{a}}{}_{,i}\, v^i + y^{\tilde{a}}{}_{,t}, \tag{5.5}$$

[4] In the figure, an orthonormal-coordinate system is used for simplicity. However, $\{\mathbf{z}\}$ does not have to be orthonormal in general.

[5] This strategy is extended to the inertial-frame and the mean-Lagrangian-frame variables (see §6.2); the inertial-coordinate and the mean-Lagrangian-coordinate representations of C are written respectively as $C^{IJ\cdots}{}_{KL\cdots}$ and $C^{\mu\nu\cdots}{}_{\rho\sigma\cdots}$, since we can recognize the coordinate representations by indices of the capital roman and the Greek alphabet.

[6] The covariant derivatives of covariant quantities transform as tensor; for example we have

$$C^{ij\cdots}{}_{kl;m\cdots} \to C^{\tilde{a}\tilde{b}\cdots}{}_{\tilde{c}\tilde{d}\cdots;\tilde{e}} = y^{\tilde{a}}{}_{,i}\, y^{\tilde{b}}{}_{,j} \cdots y^k{}_{,\tilde{c}}\, y^l{}_{,\tilde{d}} \cdots y^m{}_{,\tilde{e}} C^{ij\cdots}{}_{kl\cdots;m}.$$

which is accompanied by an extra term (see appendix A). Thus we must be careful enough in extracting the covariant properties of material motion. For example, in the Eulerian-coordinate system, we often use a quantity given by

$$s_{ij} = v_{i;j} + v_{j;i}, \tag{5.6}$$

which is called the strain rate in classical hydrodynamics. However this quantity is not generally covariant since the velocity field transforms as (5.5) [7]. In the modern theory of continuum physics, the strain rate is given by

$$s_{ij} = \frac{\mathrm{o}}{\mathrm{o}t} g_{ij} = \frac{\partial}{\partial t} g_{ij} + v^k{}_{;i}\, g_{kj} + v^k{}_{;j}\, g_{ik}, \tag{5.7}$$

which is consistent with the general covariance (Eringen 1975, also see appendixB)[8]. The operator $\mathrm{o}/\mathrm{o}t$ is called the Oldroyd derivative, which is introduced in order to measure the pure time derivative in the Lagrangian frame (see appendix B). In the Eulerian frame, $\partial g_{IJ}/\partial t = \mathbf{0}$ so that (5.7) coincides with (5.6) [9]. The operation of $\mathrm{o}/\mathrm{o}t$ for the general tensor C is given by

$$\begin{aligned} \frac{\mathrm{o}}{\mathrm{o}t} C^{ij\ldots}{}_{kl\ldots} = \frac{\mathsf{d}}{\mathsf{d}t} C^{ij\ldots}{}_{kl\ldots} - v^i{}_{;a} C^{aj\ldots}{}_{kl\ldots} - v^j{}_{;a} C^{ia\ldots}{}_{kl\ldots} - \ldots \\ + v^b{}_{;k} C^{ij\ldots}{}_{bl\ldots} + v^b{}_{;l} C^{ij\ldots}{}_{kb\ldots} + \ldots, \end{aligned} \tag{5.8}$$

where the operator $\mathsf{d}/\mathsf{d}t$ is called the Lagrangian derivative, which has been often used in conventional fluid dynamics.

$$\frac{\mathsf{d}}{\mathsf{d}t} = \frac{\partial}{\partial t} + v^j \nabla_j\ . \tag{5.9}$$

It is noticeable that the Lagrangian derivative is non-covariant operation in general, excepting for when it operates on scalar field, since the Lagrangian derivative does not contain the effect of the distortion or rotation of the coordinate frame while they are taken into account in the Oldroyd derivative.

5.2 General covariance of turbulence quantities

So far we have discussed about the exact velocity field. Next let us turn to the mean flow and fluctuation, which are of our interest. By taking the fluctuating part of (5.5), we obtain

$$v'^{\tilde{a}} \equiv v^{\tilde{a}} - \langle v^{\tilde{a}} \rangle = y^{\tilde{a}}{}_{,i} \left\{ v^i - \langle v^i \rangle \right\} = y^{\tilde{a}}{}_{,i}\, v'^i\,. \tag{5.10}$$

In this thesis we use the angular bracket $\langle \cdots \rangle$ for the ensemble averaging. (5.10) is the same as the rule of generally covariant vectors. Thus the velocity fluctuation $\mathbf{v}'$, which is the most fundamental quantity of turbulence, is a covariant quantity and therefore we should take the general covariance as one of the most fundamental properties of turbulence. As a natural consequence, we notice that many quantities constructed from the velocity fluctuation are covariant. For example, by taking the moments of (5.10) we obtain

$$\langle v'^{\tilde{a}}\, v'^{\tilde{b}} \rangle = y^{\tilde{a}}{}_{,i}\, y^{\tilde{b}}{}_{,j} \langle v'^i\, v'^j \rangle, \tag{5.11}$$

$$\langle v'^{\tilde{a}}\, v'^{\tilde{b}}\, v'^{\tilde{c}} \rangle = y^{\tilde{a}}{}_{,i}\, y^{\tilde{b}}{}_{,j}\, y^{\tilde{c}}{}_{,k} \langle v'^i\, v'^j\, v'^k \rangle, \tag{5.12}$$

[7] Note that the covariant derivative of non-covariant quantities such as $v_i \equiv g_{ik}\, v^k$ does not yield covariant result; namely,

$$v_{\tilde{a};\tilde{b}} \neq y^{\tilde{a}}{}_{,i}\, y^{\tilde{b}}{}_{,j}\ v_{i;j}.$$

[8] Following the exact definition of the Oldroyd derivative, s_{ij} is written as

$$s_{ij} = \frac{\mathrm{o}}{\mathrm{o}t} g_{ij} = \frac{\partial}{\partial t} g_{ij} + v^k\, g_{ij;k} + v^k{}_{;i}\, g_{kj} + v^k{}_{;j}\, g_{ik}.$$

However, the covariant derivative of the metric is identically zero; namely

$$g_{ij;k} = g_{ij,k} - \Gamma^a_{ik}\, g_{aj} - \Gamma^a_{jk}\, g_{ia} = \left(\Gamma_{i.jk} + \Gamma_{j.ik}\right) - \Gamma_{j.ik} - \Gamma_{i.jk} = 0,$$

which yields (5.7).

[9] Note that the strain rate is often defined as the half of the above; namely $\frac{1}{2}\mathrm{o}g_{ij}/\mathrm{o}t$, so that it coincides with that of *infinitesimal deformation theory of continuum.*

and so on. (5.11) proves that the Reynolds stress $\mathsf{R} \equiv \langle \mathbf{v}' \otimes \mathbf{v}' \rangle$, which plays principle roles in the mean-flow dynamics, transforms as a covariant tensor.

In addition to the fluctuation, we find another remarkable character in the mean flow. By taking the ensemble average of (5.5), we obtain

$$V^{\tilde{a}} = y^{\tilde{a}}{}_{,i}\, v'^{i} + y^{\tilde{a}}{}_{,t}\,, \tag{5.13}$$

where $\mathbf{V} = \langle \mathbf{v} \rangle$ is the mean velocity. This indicates that the mean velocity field $\mathbf{V}$ transforms in the same manner as the original velocity field $\mathbf{v}$. Thus the derivative operation given by

$$\begin{aligned} \frac{\mathrm{O}}{\mathrm{O}t} C^{ij\cdots}{}_{kl\ldots} = \frac{\mathrm{D}}{\mathrm{D}t} C^{ij\cdots}{}_{kl\ldots} &- C^{aj\cdots}{}_{kl\ldots} V^{i}{}_{;a} - C^{ia\cdots}{}_{kl\ldots} V^{j}{}_{;a} - \cdots \\ &+ C^{ij\cdots}{}_{bl\ldots} V^{b}{}_{;k} + C^{ij\cdots}{}_{kb\ldots} V^{b}{}_{;l} + \cdots, \end{aligned} \tag{5.14}$$

is also a covariant operation as well as (5.8) since $\mathbf{V}$ transforms in the same manner as $\mathbf{v}$ (see appendix B). Here, the operator $\mathrm{D}/\mathrm{D}t = \partial/\partial t + V^{j}\nabla_{j}$ is the Lagrangian derivative based on the mean flow, which is a non-covariant operation as well as $\mathsf{d}/\mathsf{d}t$ given by (5.9).

So far we have obtained two insights; one is the general covariance of the fluctuation $\mathbf{v}'$ and the other is the equivalence between the original velocity field $\mathbf{v}$ and the mean velocity field $\mathbf{V}$ in terms of the transformation rule. Here we should recognize the point in common between the continuum physics and the turbulence closure. In continuum physics, covariant quantities such as stress or heat flux have certain relations with the motion of the continuum and to clarify these relations as the constitutive equations is required for the closure of the continuum theory. The covariance imposes on those relations a mathematical constraint; *When we constitute the physical model for the constitutive equation, we must employ only the generally covariant quantities.* This has been used as a strong guideline in determining the constitutive equations.

On the other hand, in turbulence closure, we have to clarify the relations between the statistical quantities and the motion of the mean flow. The statistical quantities which give major contributions to the mean-flow dynamics are often covariant (Reynolds stress is a typical example). Thus, following the continuum physics, we reach a strict mathematical constraint for the inhomogeneous turbulence as follows; *When we constitute the physical model of the generally-covariant correlations, we have to employ only the generally-covariant quantities of turbulence.*

This constraint can be applied as guideline to the actual turbulence modeling in what follows. When we constitute the physical model of the Reynolds stress, for a concrete example, the model should be represented as a function or functional of covariant quantities such as

$$\mathsf{R} = \mathsf{F}\left\{K, \epsilon, \ldots, \mathsf{S}, \frac{\mathrm{O}\mathsf{S}}{\mathrm{O}t}, \frac{\mathrm{O}^2\mathsf{S}}{\mathrm{O}t^2}, \ldots, \boldsymbol{\Theta}, \frac{\mathrm{O}\boldsymbol{\Theta}}{\mathrm{O}t}, \frac{\mathrm{O}^2\boldsymbol{\Theta}}{\mathrm{O}t^2}, \ldots\right\}, \tag{5.15}$$

where $K \equiv \frac{1}{2}\langle \mathbf{v}' \cdot \mathbf{v}' \rangle$ and $\epsilon \equiv \nu \langle \nabla\mathbf{v}' \cdot \nabla\mathbf{v}' \rangle$ are turbulence energy and its dissipation rate, S is the strain rate of the mean flow given by

$$S_{ij} = \frac{\mathrm{O}}{\mathrm{O}t} g_{ij}, \tag{5.16}$$

$\boldsymbol{\Theta}$ is the absolute vorticity of the mean flow whose definition will be given by (7.11)[10]. In other words we cannot accept the model such as

$$\mathsf{R} = \mathsf{F}\left\{K, \epsilon, \ldots, \nabla\mathbf{V}, \frac{\mathrm{D}\nabla\mathbf{V}}{\mathrm{D}t}, \frac{\mathrm{D}^2\nabla\mathbf{V}}{\mathrm{D}t^2}, \ldots\right\}, \tag{5.17}$$

which contains non-covariant factors. The Reynolds stress (4.16) derived by TSDIA is an example of (5.17), which contradict the general covariance of the real nature.

Generally speaking, the results derived by TSDIA are not consistent even with the rule under transformations between Euclidean coordinate systems rotating relatively to each other. Yokoi and Yoshizawa (1993) investigated the effect of the frame rotation by applying perturbative expansion about the frame rotation

[10]Instead of the Oldroyd derivative, it is also possible to employ other objective derivatives such as the *Jauman* derivative (Oldroyd 1958).

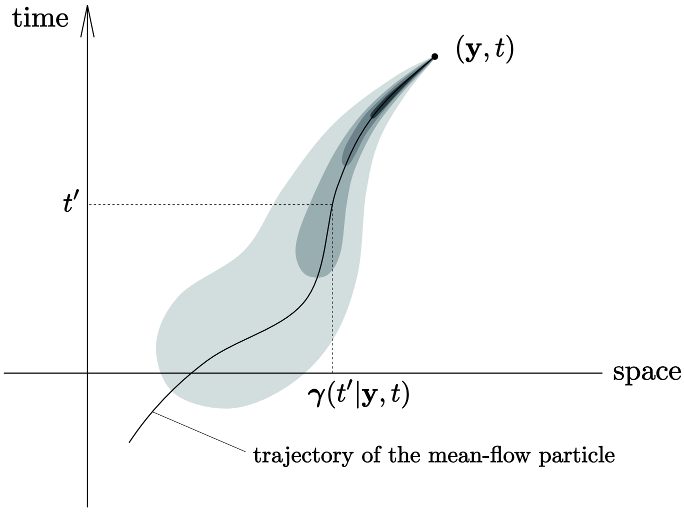

Figure 4: The space-time area contributing to the evolution of fluctuation at a space-time point $(\mathbf{y}, t)$ is illustrated by contours. In this figure, three-dimensional space is represented in one dimension for simplicity. The darker area is supposed to give more contribution. The area around the trajectory of the mean-flow particle often gives important contribution.

and obtained a preferable representation of the Reynolds stress in which the mean vorticity is accompanied by the frame rotation (namely vorticity is retained as absolute vorticity), these results are valid only when the frame rotation is enough small. In other words, these models lose the theoretical validity as the reference frame differs from the inertial frame, which is unsatisfactory in terms of physical objectivity. Strictly speaking, perturbative expansion method based on the rotation of the reference frame is not preferable in the first place since the perturbative interactions depend on the observer so that the perterbative solution at each order cannot be objective and, accordingly, the results contradict the general covariance at any order.

Not only the Reynolds stress, as is mentioned above, but also other dynamical quantities such as (5.12) and the relations between them are supposed to have covariant structures. Thus we have to take the general covariance into account in the formulation of the whole theory.

Needs for the physical objectivity have already been pointed out by some pioneering works in the context of covariance under some limited transformation groups. Weis & Hutter (2002) claimed the importance of the covariance under the time-dependent rotational transformation and represented it as “Euclidean invariance”. Hamba (2006) also took it as an indispensable factor and showed its availability in practical modelings by using so called “co-rotational derivative” which is consistent with the Euclidean invariance. Hamba and Sato (2008) employed the co-rotational derivative and reformulated TSDIA to be consistent with the Euclidean invariance.

The general covariance, on the contrary, have not been recognized so far, despite it is a fundamental property of turbulence. Hence it is enough expected that the inclusion of the general covariance will offer more accurate guideline which lead us to more accurate physical models. Thus it would be very important phase in the development of turbulence research to incorporate the general covariance into the base of the formulation.

6 Need for the mean-Lagrangian formalism

6.1 Multiple-time analysis and covariance

Generally speaking, the physical state of non-local area in space and time gives the contribution to the evolution of fluctuation. However, because of the mean-flow convection, some localized area in the space-time is especially important. In order to give more clear explanation, we introduce a concept of the *mean-flow particle* as follows.

Providing the mean velocity $\mathbf{V}(\mathbf{y},t) \equiv \langle \mathbf{v}(\mathbf{y},t)\rangle$ is known in every point and time in a general coordinate system $\{\mathbf{y}\}$, we can construct a differential equation given by

$$\frac{\mathrm{d}}{\mathrm{d}t'}\gamma^i(t'|\mathbf{y},t) = V^i\left(\boldsymbol{\gamma}(t'|\mathbf{y},t),t'\right), \tag{6.1}$$

where $\boldsymbol{\gamma}(t|\mathbf{y},t)=\mathbf{y}$. (6.1) describes the trajectory of a point convected by the mean flow which locates at $\mathbf{y}$ at time t. The solution is uniquely identified so that we can regard $\boldsymbol{\gamma}(t'|\mathbf{y},t)$ as a virtual particle which never split into more than two pieces nor unite with others. Let us call this the *mean-flow particle.*

Because of the convection by the mean flow, the area around the trajectory of the mean-flow particle often gives the main contribution to the evolution of turbulence at point $\mathbf{y}$ at time t (see figure 4). Thus it is often meaningful to discuss about the turbulence quantities on the trajectory of the mean-flow particle. Hence, when we consider a scalar field (say $f(\mathbf{y},t)$) relating to the actual turbulent flow, the following quantity is supposed to be important to the dynamics on the space-time point $(\mathbf{y},t)$;

$$^{\gamma}f(\mathbf{y},t;t') = f\left(\boldsymbol{\gamma}(t'|\mathbf{y},t),t'\right). \tag{6.2}$$

The quantity $^{\gamma}f(\mathbf{y},t;t')$ is also important from the viewpoint of the general covariance; $^{\gamma}f(\mathbf{y},t;t')$ behaves as scalar field at point $\mathbf{y}$ at time t. Using these quantities, we can take algebraic operations between quantities of different times [11]. This is very important fact in terms of the requirement for the physical modeling of turbulence quantities discussed in §5. According to the requirement, we can incorporate the turbulence energy in the past times $t',t'',t''',\ldots(<t)$ into the model of the Reynolds stress such as

$$\mathbf{R}(\mathbf{y},t) = \mathbf{F}\left\{\ldots,{}^{\gamma}K(\mathbf{y},t;t'),{}^{\gamma}K(\mathbf{y},t;t''),{}^{\gamma}K(\mathbf{y},t;t'''),\ldots\right\}, \tag{6.3}$$

where $\mathbf{F}$ is a function including sum or product of arguments. This is exactly an example of multiple-time closure consistent with the general covariance.

In case of general tensor, we need more complex operations. We cannot simply extend (6.2) to general tensor, namely $C^{ij\cdots}{}_{kl\cdots}(\boldsymbol{\gamma}(t'|\mathbf{y},t),t')$, since the frame of reference has deformation and rotation which cannot be removed by a simple translation. Thus we should cancel out the deformation and rotation by applying linear transformations. The counterpart of (6.2) in case of general tensor is given by

$$^{\gamma}C^{ij\cdots}{}_{kl\cdots}(\mathbf{y},t;t') = Z^i_a(t'|\mathbf{y},t)Z^j_b(t'|\mathbf{y},t)\cdots\bar{Z}^c_k(t'|\mathbf{y},t)\bar{Z}^d_l(t'|\mathbf{y},t)\cdots C^{ab\cdots}{}_{cd\cdots}(\boldsymbol{\gamma}(t'|\mathbf{y},t),t'), \tag{6.4}$$

where the transformation coefficient $Z^i_a(t'|\mathbf{y},t)$ satisfies

$$\frac{\partial}{\partial t'}Z^i_a(t'|\mathbf{y},t) = -Z^i_k(t'|\mathbf{y},t)V^k_{,a}(\boldsymbol{\gamma}(t'|\mathbf{y},t),t'), \tag{6.5}$$

with $Z^i_a(t|\mathbf{y},t)=\delta^i_a$, $\bar{\mathbf{Z}}(t'|\mathbf{y},t)$ is the inverse of $\mathbf{Z}(t'|\mathbf{y},t)$. On this stage, we can take both addition and multiplication [12] [13]. In the same context of (6.3), we can construct the physical model of the Reynolds stress

[11] For example, for scalar fields $f(\mathbf{y},t)$, $g(\mathbf{y},t)$, $h(\mathbf{y},t)$, $\ldots$, we can take sum and product such as

$$^{\gamma}f(\mathbf{y},t;t') + {}^{\gamma}g(\mathbf{y},t;t'') + {}^{\gamma}h(\mathbf{y},t;t''') + \ldots$$

or

$$^{\gamma}f(\mathbf{y},t;t')\,{}^{\gamma}g(\mathbf{y},t;t'')\,{}^{\gamma}h(\mathbf{y},t;t''') + \ldots$$

both of which behave as scalar fields at point $\mathbf{y}$ at time t.

[12] For example, we can take addition and multiplication of tensor fields $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\ldots$ at different times t, t', t'' $\ldots$ such as

$$^{\gamma}A^{ij\cdots}{}_{kl\cdots}(\mathbf{y},t;t') + {}^{\gamma}B^{ij\cdots}{}_{kl\cdots}(\mathbf{y},t;t'') + {}^{\gamma}C^{ij\cdots}{}_{kl\cdots}(\mathbf{y},t;t''') + \cdots,$$

or

$$^{\gamma}A^{ab\cdots}{}_{cd\cdots}(\mathbf{y},t;t')\,{}^{\gamma}B^{ef\cdots}{}_{gh\cdots}(\mathbf{y},t;t'')\,{}^{\gamma}C^{ij\cdots}{}_{kl\cdots}(\mathbf{y},t;t''')\cdots,$$

both of which transforms as tensors. In continuum physics, the tensor given by

$$E_{ij} = g_{ij}(\mathbf{y},t) - {}^{\gamma}g_{ij}(\mathbf{y},t;t') \quad (t'<t)$$

sometime plays an important role in investigating the geometrical change of material, which is called the strain tensor (Sometime it is defined as the half of the above).

[13] The following limit also behaves as a tensor field.

$$\lim_{t'\to t}\frac{{}^{\gamma}C^{ij\cdots}{}_{kl\cdots}(\mathbf{y},t;t') - C^{ij\cdots}{}_{kl\cdots}(\mathbf{y},t)}{t'-t} = \frac{\mathrm{D}C^{ij\cdots}{}_{kl\cdots}}{\mathrm{D}t}(\mathbf{y},t) - V^i{}_{;a}(\mathbf{y},t)C^{aj\cdots}{}_{kl\cdots}(\mathbf{y},t) - V^j{}_{;a}(\mathbf{y},t)C^{ia\cdots}{}_{kl\cdots}(\mathbf{y},t) - \cdots$$
$$+ V^b{}_{;k}(\mathbf{y},t)C^{ij\cdots}{}_{bl\cdots}(\mathbf{y},t) + V^b{}_{;l}(\mathbf{y},t)C^{ij\cdots}{}_{kb\cdots}(\mathbf{y},t) + \cdots,$$

such as

$$\mathbf{R}(\mathbf{y},t) = \mathbf{F}\left\{\dots, {}^{\gamma}\mathbf{S}(\mathbf{y},t;t'), {}^{\gamma}\mathbf{S}(\mathbf{y},t;t''), {}^{\gamma}\mathbf{S}(\mathbf{y},t;t'''), \dots\right\}, \tag{6.6}$$

which incorporates the strain rate at the past times $t', t'', t''', \dots (< t)$.

(6.3) and (6.6) may look discrete about the past times. We can extend these functions to continuous ones by using a special integration in time. Since ${}^{\gamma}\mathbf{C}(\mathbf{y},t;t')$ behaves as tensor, the following integration does also behave as tensor;

$$\begin{aligned}
&\int^{t} \mathrm{d}t' \; {}^{\gamma}C^{ij\cdots}{}_{jk\dots}(\mathbf{y},t;t') \\
&= \int^{t} \mathrm{d}t' \; Z^{i}_{a}(t'|\mathbf{y},t) Z^{j}_{b}(t'|\mathbf{y},t) \cdots \bar{Z}^{c}_{k}(t'|\mathbf{y},t) \bar{Z}^{d}_{l}(t'|\mathbf{y},t) \cdots C^{ab\cdots}{}_{cd\dots}(\boldsymbol{\gamma}(t'|\mathbf{y},t),t').
\end{aligned} \tag{6.7}$$

In the continuum physics, this is called the *convected integration* which is often used to represent the memory effect observed in viscoelastic materials (Oldroyd 1950). In turbulence modeling, the Reynolds stress is sometime explained by the relaxation effect of strain rate where the relaxation-time scale is given by K/ϵ. By using the convected integration, we may put the Reynolds-stress model as

$$\mathbf{R}(\mathbf{y},t) = \mathbf{F}\left\{\dots, \int_{-\infty}^{t} \mathrm{d}t' \exp\left[-\int_{t'}^{t} \mathrm{d}\tau \frac{{}^{\gamma}\epsilon(\mathbf{y},t;\tau)}{{}^{\gamma}K(\mathbf{y},t;\tau)}\right] {}^{\gamma}S^{ij}(\mathbf{y},t;t'), \dots\right\}, \tag{6.8}$$

which is permitted as a generally covariant model[14]. Here we should notice that the integration is taken over the trajectory of the mean flow shown in figure 4. The value of exponential part tends to zero as $t - t'$ increases so that (6.8) indicates the present strain rate is more relevant in explaining the Reynolds stress than that of the past.

In (6.3), (6.6) and (6.8), only the quantities on the trajectory $\boldsymbol{\gamma}$ are considered. In more accurate treatment, however, we have to incorporate the area around the trajectory (see figure 4). In the present theory, we will represent the spatially-nonlocal effect by the derivative expansion in space; namely we should incorporate the spatial derivatives such as ∇K, $\nabla\epsilon$, $\nabla\mathbf{S}$, $\nabla\boldsymbol{\Theta}$ or higher-order derivatives of them into our model.

6.2 Mean-Lagrangian representation

In the present theory, we should pay special attention to the treatment of multiple-time quantities in order to keep the whole theory consistent with the general covariance; namely we have to calculate both the mean-flow trajectory $\boldsymbol{\gamma}$ and matrix $\mathbf{Z}$ from the mean velocity field. However, these procedures are so complex that their direct use would make the later work too much cumbersome.

In order to avoid these complexities, we introduce coordinate system of a special kind where the mean velocity field is globally cancelled. We denote this new coordinate variables as x^{μ} ($\mu = 1,2,3$). We use x for the main symbol and the Greek alphabet for indices in the coordinate representation. According to (5.13),

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which gives another definition of the Oldroyd derivative. By applying this to the metric tensor, we obtain

$$\lim_{t'\to t} \frac{g_{ij}(\mathbf{y},t) - {}^{\gamma}g_{ij}(\mathbf{y},t;t')}{t-t'} = \lim_{t'\to t} \frac{E(\mathbf{y},t;t')}{t-t'},$$

which represents the change rate of strain. This is the exact reason why the strain rate is defined by (5.16).

[14]By transforming the model equation (4.6) into covariant form and integrating it, we obtain

$$\begin{aligned}
R^{ij}(\mathbf{y},t) = &\frac{2}{3}K(\mathbf{y},t)g^{ij}(\mathbf{y},t) \\
&- \frac{2}{3}(1-C_{IP})\left(\delta^{i}_{a}\delta^{j}_{b} - \frac{1}{3}g^{ij}g_{ab}\right)(\mathbf{y},t)\int_{-\infty}^{t}\mathrm{d}t'\exp\left[-C_R\int_{t'}^{t}\mathrm{d}\tau\frac{{}^{\gamma}\epsilon(\mathbf{y},t;\tau)}{{}^{\gamma}K(\mathbf{y},t;\tau)}\right]{}^{\gamma}K(\mathbf{y},t;t')\,{}^{\gamma}S^{ab}(\mathbf{y},t;t') + \cdots.
\end{aligned}$$

Thus RSM may be categorized in the models such as (6.8).

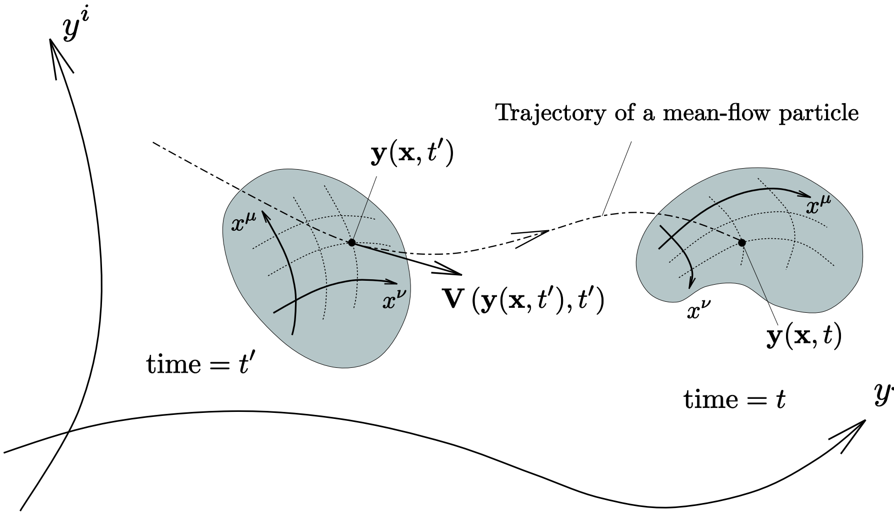


Figure 5: Configuration of the mean-Lagrangian coordinate system; The coordinate system $\{\mathbf{x}\}$ convected by the mean-flow is the mean-Lagrangian coordinate system. The set of coordinate variables is interpreted as the label for the mean-flow particle. We should notice that $\mathbf{y}(\mathbf{x}, t)$ alone represents the trajectory of mean-flow particle labelled as $\mathbf{x}$.

the transformation of the mean velocity field from $\{\mathbf{y}\}$ to $\{\mathbf{x}\}$ is written as follows [15] [16];

$$\begin{aligned} V^\mu(\mathbf{x}, t) = 0 &= \frac{\partial x^\mu}{\partial y^i}(\mathbf{y}(\mathbf{x}, t), t)\ V^i(\mathbf{y}(\mathbf{x}, t), t) + \frac{\partial x^\mu}{\partial t}(\mathbf{y}(\mathbf{x}, t), t) \\ &= \frac{\partial x^\mu}{\partial y^i}(\mathbf{y}(\mathbf{x}, t), t) \left\{ V^i(\mathbf{y}(\mathbf{x}, t), t) - \frac{\partial}{\partial t} y^i(\mathbf{x}, t) \right\}. \end{aligned}$$

Thus we have

$$\frac{\partial}{\partial t} y^i(\mathbf{x}, t) = V^i(\mathbf{y}(\mathbf{x}, t), t). \tag{6.9}$$

This equation has the same form as (6.1). Thus $y^i(\mathbf{x}, t)$ represents the trajectory of a mean-flow particle for constant $\mathbf{x}$; $\mathbf{x}$ is the label for the mean flow particle which is the exact counterpart of the Lagrangian coordinate system in the continuum physics. In this thesis we call the coordinate system convected by the mean flow as the *mean-Lagrangian coordinate system* (see figure 5).

Since the mean velocity is totally cancelled out in the mean-Lagrangian coordinate system, $\boldsymbol{\gamma}(t'|\mathbf{x}, t) = \mathbf{x}$ and $\mathbf{Z}(t'|\mathbf{x}, t) = \mathbf{1}$ so that

$$Z^\mu_\alpha(t'|\mathbf{x}, t) Z^\nu_\beta(t'|\mathbf{x}, t) \cdots \bar{Z}^\gamma_\rho(t'|\mathbf{x}, t) \bar{Z}^\delta_\sigma(t'|\mathbf{x}, t) \cdots C^{\alpha\beta\cdots}{}_{\gamma\delta\cdots}(\gamma(t'|\mathbf{x}, t), t') = C^{\mu\nu\cdots}{}_{\rho\sigma\cdots}(\mathbf{x}, t'). \tag{6.10}$$

[15] When we discuss about the coordinate transformation in the continuum physics, we often deal with the composed function such as

$$f(\mathbf{y}(\mathbf{x}, t), t).$$

In this representation, we should notice that the derivative function such as $\partial f/\partial t\ (\mathbf{y}(\mathbf{x}, t), t)$ is discriminated from the simple time-derivative operation. The former is obtained by substituting $\mathbf{y} = \mathbf{y}(\mathbf{x}, t)$ into the derivative function $\partial f/\partial t\ (\mathbf{y}, t)$, while the latter is given by

$$\frac{\partial}{\partial t} f(\mathbf{y}(\mathbf{x}, t), t) = \frac{\partial f}{\partial y^i}(\mathbf{y}(\mathbf{x}, t), t)\ \frac{\partial y^i}{\partial t}(\mathbf{x}, t) + \frac{\partial f}{\partial t}(\mathbf{y}(\mathbf{x}, t), t) \neq \frac{\partial f}{\partial t}(\mathbf{y}(\mathbf{x}, t), t),$$

[16] By differentiating $x^\mu(\mathbf{y}(\mathbf{x}, t), t)$ by t with fixing $\mathbf{x}$, we obtain

$$\frac{\partial}{\partial t} x^\mu(\mathbf{y}(\mathbf{x}, t), t) = 0 = \frac{\partial x^\mu}{\partial y^i}(\mathbf{y}(\mathbf{x}, t), t)\ \frac{\partial y^i}{\partial t}(\mathbf{x}, t) + \frac{\partial x^\mu}{\partial t}(\mathbf{y}(\mathbf{x}, t), t).$$

Thus we have

$$\frac{\partial x^\mu}{\partial t}(\mathbf{y}(\mathbf{x}, t), t) = -\frac{\partial x^\mu}{\partial y^i}(\mathbf{y}(\mathbf{x}, t), t)\ \frac{\partial y^i}{\partial t}(\mathbf{x}, t).$$

Thus in the mean-Lagrangian frame we can take both sum and product of quantities at arbitrary times [17] [18]. Due to this feature, we can construct the covariant model in remarkably simpler manners. For example, the model (6.8) is supposed to be rewritten as follows [19];

$$\mathbf{R}(\mathbf{x}, t) = \mathbf{F}\left\{\ldots, \int_{-\infty}^{t} \mathrm{d}t' \exp\left[-\int_{t'}^{t} \mathrm{d}\tau \frac{\epsilon(\mathbf{x}, \tau)}{K(\mathbf{x}, \tau)}\right] S^{\mu\nu}(\mathbf{x}, t'), \ldots\right\}. \tag{6.12}$$

This is much simpler form than (6.8) since we do not have to calculate $\boldsymbol{\gamma}$ nor $\mathbf{Z}$; we only need the simple time integration. In the mean-Lagrangian coordinate system, we have to consider neither the convection of the mean-flow particle nor the frame deformation, and thus multiple-time quantities are represented in the simplest forms. In the following formulation, we shall employ lots of multiple-time quantities in a manner similar to the homogeneous-turbulence theory so that the mean-Lagrangian representation is very effective to avoid cumbersome calculations of $\boldsymbol{\gamma}$ and $\mathbf{Z}$. Note that the strain rate $\mathbf{S}$ is given by

$$S_{\mu\nu}(\mathbf{x}, t) = \frac{\partial}{\partial t} g_{\mu\nu}(\mathbf{x}, t), \tag{6.13}$$

instead of (5.16). This is also due to the absence of the mean velocity field in the mean-Lagrangian representation.

There is another feature of the mean-Lagrangian representation. Comparing to the general coordinate representation, the coordinate-space area contributing to the fluctuation dynamics is effectively localized in the mean-Lagrangian coordinate representation. In the general coordinate system, we have to consider wide range of spatial-coordinate area since the mean-flow particle travels in the coordinate space, as is shown in figure 6 (a). In the mean-Lagrangian coordinate system, on the contrary, the area we have to consider is localized around the actual coordinate point $\mathbf{x}$, as shown in figure 6 (b). In the later discussion, we will treat spatially nonlocal effect by spatial derivative expansions. As long as we truncate the derivative expansion at lower order, the mean-Lagrangian representation is suitable choice since the spatial-coordinate area we have to consider is expected to be minimized.

7 Details of the formulation

In this section we will see the detailed formulation of the present theory. In the whole procedure, we will investigate the turbulence dynamics with tracing the mean flow. By extending the renormalized perturbation theory and applying it to our system, we will reach a series of method to represent varieties of general correlations in terms of homogeneous isotropic properties in a manner consistent with the general covariance.

7.1 Covariant form of the dynamical equation

First we start from the inertial coordinate system $\{\mathbf{z}\}$ for simplicity. In the inertial frame, the motion of the incompressible Newtonian fluid is governed by the Navier-Stokes equation and the incompressibility condition

[17] For example, sum of tensors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, ... at different times t, t', t'', ..., namely

$$A^{\mu\nu\cdots}{}_{\rho\sigma\cdots}(\mathbf{x}, t) + B^{\mu\nu\cdots}{}_{\rho\sigma\cdots}(\mathbf{x}, t') + C^{\mu\nu\cdots}{}_{\rho\sigma\cdots}(\mathbf{x}, t'') + \cdots ,$$

is a tensor. Product of them, namely

$$A^{\mu\nu\cdots}{}_{\rho\sigma\cdots}(\mathbf{x}, t) B^{\alpha\beta\cdots}{}_{\gamma\delta\cdots}(\mathbf{x}, t') C^{\omega\zeta\cdots}{}_{\xi\eta\cdots}(\mathbf{x}, t'') \cdots ,$$

is also a tensor. These operations are clearly simpler than those in the general coordinate representation.

[18] The Oldroyd derivative is reduced to

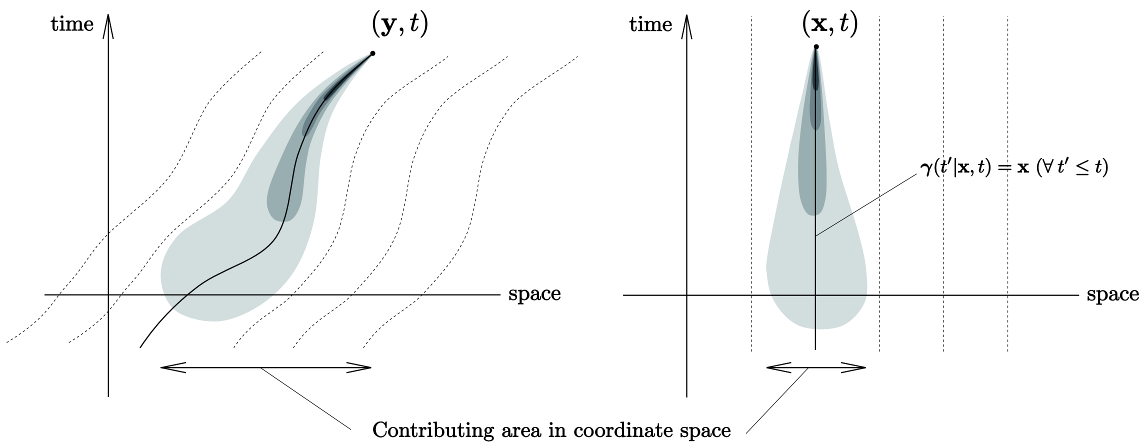
$$\frac{\mathrm{O}C^{\mu\nu\cdots}{}_{\rho\sigma\cdots}}{\mathrm{O}t}(\mathbf{x}, t) = \lim_{t' \to t} \frac{C^{\mu\nu\cdots}{}_{\rho\sigma\cdots}(\mathbf{x}, t') - C^{\mu\nu\cdots}{}_{\rho\sigma\cdots}(\mathbf{x}, t)}{t' - t} = \frac{\partial C^{\mu\nu\cdots}{}_{\rho\sigma\cdots}}{\partial t}(\mathbf{x}, t).$$

Thus the Oldroyd derivative is represented by simple time derivative, which can also be understood from (5.14) since the mean velocity field is cancelled in the mean Lagrangian coordinate system.

[19] Because of (6.10), the convected integration of $\mathbf{C}$ becomes

$$\int^{t} \mathrm{d}t' \; C^{\mu\nu\cdots}{}_{\rho\sigma\cdots}(\mathbf{x}, t'), \tag{6.11}$$

which is just a simple integration.



(a) General coordinate representation (b) Mean-Lagrangian representation

Figure 6: The space-time area contributing to the fluctuation is depicted in two coordinate representations; the general coordinate representation and the mean-Lagrangian representation. Dotted and solid lines are the trajectories of the mean flow. (a) In the general coordinate system, we have to consider wide range of spatial-coordinate variable since the mean-flow particle travels in the coordinate space. (b) In the mean-Lagrangian coordinate system, on the contrary, the area we have to consider is localized in the neighborhood of the coordinate point $\mathbf{x}$.

written as

$$\frac{\mathsf{d}}{\mathsf{d}t} v^I(\mathbf{z}, t) = -\mathrm{p}^{;I}(\mathbf{z}, t) + \nu \triangle v^I(\mathbf{z}, t), \tag{7.1}$$

$$v^J{}_{;J}(\mathbf{z}, t) = 0, \tag{7.2}$$

where p is the pressure normalized by constant density, ν is the kinematic viscosity. Note that the sets of arguments $(\mathbf{z}, t)$ and $(\mathbf{y}, t)$ are abbreviated in the following transformations. The ensemble averages of the above equations yield the equations of the mean flow; Reynolds-Averaged Navier-Stokes equation (RANS eq.) and its incompressibility condition given by

$$\frac{\mathrm{D}}{\mathrm{D}t} V^I = -P^{;I} + \nu \triangle V^I - R^{IJ}{}_{;J}, \tag{7.3}$$

$$V^J{}_{;J} = 0, \tag{7.4}$$

where $P \equiv \langle \mathrm{p} \rangle$ is the mean pressure. By subtracting the mean-equations (7.3) and (7.4) from the originals (7.1) and (7.2) respectively, we obtain the set of equations for the velocity fluctuation $\mathbf{v}'$ and the pressure fluctuation $\mathrm{p}' \equiv \mathrm{p} - P$;

$$\left(\frac{\mathrm{D}}{\mathrm{D}t} - \nu \triangle \right) v'^I + \left(v'^I v'^J \right)_{;J} + \mathrm{p}'^{;I} = -V^I{}_{;J} v'^J + R^{IJ}{}_{;J}, \tag{7.5}$$

$$v'^J{}_{;J} = 0. \tag{7.6}$$

(7.5) is obviously non-covariant form, containing Lagrangian derivative and mean velocity gradient both of which are non-covariant. As was shown in the previous section, velocity fluctuation is covariant vector and its dynamical equation should be represented in a covariant form. By adding $-V^I{}_{;J} v'^J$ to the both sides of (7.5), we transform the time derivative term into a covariant form;

$$\left(\frac{\mathrm{O}}{\mathrm{O}t} - \nu \triangle \right) v'^I + \left(v'^I v'^J \right)_{;J} + \mathrm{p}'^{;I} = -2 V^I{}_{;J}\, v'^J + R^{IJ}{}_{;J}\, . \tag{7.7}$$

Except for the first term of the right-hand side, all terms are written in covariant form. Multiplying the above by transformation coefficients $y^i{}_{,I}$ and transforming it yields

$$-2 y^i{}_{,I}\ z^J{}_{,j}\ V^I{}_{;J}\ v'^j = \left(\frac{\mathrm{O}}{\mathrm{O}t} - \nu \triangle \right) v'^i + \left(v'^i v'^j \right)_{;j} + \mathrm{p}'^{;i} - R^{ij}{}_{;j}. \tag{7.8}$$

All quantities on the right-hand side are obviously covariant so that the left-hand side should also be covariant. Let us define a 2-rank tensor given by

$$\Sigma^i{}_j \equiv y^i{}_{,I}\ z^J{}_{,j}\ V^I{}_{;J}. \tag{7.9}$$

$\boldsymbol{\Sigma}$ is an objective measure of departure of the mean-flow motion from the inertial motion, and behaves as a generally covariant tensor (see appendix C). The symmetric part of $\Sigma_{ij} = g_{ik}\,\Sigma^k{}_j$ is rewritten as

$$\begin{aligned}\Sigma_{ij} + \Sigma_{ji} &= z^I{}_{,i}\, z^I{}_{,j}\,(V_{I;J} + V_{J;I}) = z^I{}_{,i}\, z^I{}_{,j}\left(g_{IJ,t} + V^K\, g_{IJ;K} + V^K{}_{;I}\, g_{KJ} + V^K{}_{;J}\, g_{IK}\right)\\ &= z^I{}_{,i}\, z^I{}_{,j}\frac{\mathrm{O}}{\mathrm{O}t} g_{IJ} = \frac{\mathrm{O}}{\mathrm{O}t} g_{ij} = S_{ij},\end{aligned} \tag{7.10}$$

where we used $g_{IJ,t} = 0$ and $g_{IJ;K} = 0$. Anti-symmetric part is supposed to be called the absolute vorticity of the mean flow, which is written in this thesis as

$$\Theta_{ij} = \Sigma_{ij} - \Sigma_{ji}. \tag{7.11}$$

As a consequence, we obtain generally covariant form of the dynamical equation for the velocity fluctuation in the general coordinate system $\{\mathbf{y}\}$ as follows;

$$\left(\frac{\mathrm{O}}{\mathrm{O}t} - \nu\triangle\right) v'^i + \left(v'^i v'^j\right)_{;j} + \mathrm{p}'^{;i} = -\left(S^i_j + \Theta^i{}_j\right)\ v'^j + R^{ij}{}_{;j}. \tag{7.12}$$

Finally, in the mean-Lagrangian coordinate system $\{\mathbf{x}\}$, we have the following set of governing equations;

$$\left(\frac{\partial}{\partial t} - \nu\triangle\right) v'^\mu(\mathbf{x},t) + \left(v'^\mu v'^\nu\right)_{;\nu}(\mathbf{x},t) + \mathrm{p}'^{;\mu}(\mathbf{x},t) = -\left(S^\mu_\rho + \Theta^\mu{}_\rho\right)(\mathbf{x},t)\ v'^\rho(\mathbf{x},t) + R^{\mu\rho}{}_{;\rho}(\mathbf{x},t), \tag{7.13}$$

$$v'^\rho{}_{;\rho}(\mathbf{x},t) = 0. \tag{7.14}$$

7.2 Homogeneity-inhomogeneity decomposition

In the region apart from boundaries, the variations in time and space of the mean fields are usually smoother than those of fluctuations. Based on this intuition, an assumption had been proposed; inhomogeneous turbulence may be treated as homogeneous one in local area (Local homogeneity: Taylor 1935). Following this idea, Yoshizawa (1984) introduced the two-scale representation for Eulerian coordinate variables and time respectively, which is the synonym of the multiple-scale variables used in a singular perturbation method; Yoshizawa employed his two-scale representation to discriminate the characteristic scales of the fluctuations and those of the mean fields. In the present work, on the other hand, we will introduce two coordinate systems in order to separate the dependence on the coordinate variables into *homogeneous* and *inhomogeneous* parts.

First, for every dynamical variable, we introduce a corresponding function of two sets of coordinate variables each of which forms a three dimensional space, namely $\boldsymbol{\xi}$ and $\mathbf{X}$, and we assume they are independent of each other. For example we introduce $f(\boldsymbol{\xi},t|\mathbf{X})$ corresponding to an arbitrary dynamical variable $f(\mathbf{x},t)$. Next, we have to set the relation between the original quantity and its dual-coordinate counterpart. We assume that the real dynamical variable is given by substituting $\mathbf{x}$ and $\delta\mathbf{x}$ into $\boldsymbol{\xi}$ and $\mathbf{X}$ of the corresponding function, where δ is an expediential parameter whose meaning will be mentioned later. Thus the real dynamical variable $f(\mathbf{x},t)$ is reproduced from its dual-coordinate counterpart as

$$f(\mathbf{x},t|\delta\mathbf{x}) = f(\mathbf{x},t). \tag{7.15}$$

Next we set up a non-trivial hypothesis stating that any fluctuating variables are statistically homogeneous with respect to $\boldsymbol{\xi}$. Note that the inhomogeneity of the original quantity is denoted by variable $\mathbf{X}$. In this sense let us call this operation the *homogeneity-inhomogeneity decomposition* (HID).

HID should form a mapping from the class of functions in physical space such as $f(\mathbf{x},t)$ to the other class of the dual-coordinate functions such as $f(\boldsymbol{\xi},t|\mathbf{X})$. In order to apply HID to all the dynamical variables without contradiction, it should conserve the structures of both addition and multiplication. Thus HID of the sum and product of arbitrary dynamical variables f and g should be given by

$$(f+g)(\mathbf{x},t) = f(\mathbf{x},t) + g(\mathbf{x},t) \overset{\text{HID}}{\longmapsto} (f+g)(\boldsymbol{\xi},t|\mathbf{X}) = f(\boldsymbol{\xi},t|\mathbf{X}) + g(\boldsymbol{\xi},t|\mathbf{X}), \tag{7.16}$$

$$(fg)(\mathbf{x},t) = f(\mathbf{x},t)g(\mathbf{x},t) \overset{\text{HID}}{\longmapsto} (fg)(\boldsymbol{\xi},t|\mathbf{X}) = f(\boldsymbol{\xi},t|\mathbf{X})g(\boldsymbol{\xi},t|\mathbf{X}). \tag{7.17}$$

Under these assumptions, the following relation should hold for arbitrary fluctuating variables f and g;

$$\langle f(\boldsymbol{\xi},t|\mathbf{X})g(\boldsymbol{\xi}',t'|\mathbf{X})\rangle = C(\boldsymbol{\xi}-\boldsymbol{\xi}',t,t'|\mathbf{X}). \tag{7.18}$$

As a special case, we have

$$\langle f(\boldsymbol{\xi},t|\mathbf{X})g(\boldsymbol{\xi},t|\mathbf{X})\rangle = C(\mathbf{0},t,t|\mathbf{X}) = C(t|\mathbf{X}), \tag{7.19}$$

which is independent of $\boldsymbol{\xi}$. Thus, HID counterparts of any one-point correlations such as turbulence energy, its dissipation rate and the Reynolds stress are independent of $\boldsymbol{\xi}$ as follows;

$$K(\mathbf{x},t),\ \epsilon(\mathbf{x},t),\ R^{\mu\nu}(\mathbf{x},t) \overset{\text{HID}}{\longmapsto}\ K(t|\mathbf{X}),\ \epsilon(t|\mathbf{X}),\ R^{\mu\nu}(t|\mathbf{X}), \tag{7.20}$$

where the non-uniform variations of them are denoted by $\mathbf{X}$. Generally speaking, the ensemble average of the fluctuating quantity on the mean-Lagrangian coordinate system should be independent from $\boldsymbol{\xi}$ for the homogeneity in $\boldsymbol{\xi}$ space. Thus we have the following relation for averaged fluctuating quantities;

$$F(\mathbf{x},t) = \langle f(\mathbf{x},t)\rangle \overset{\text{HID}}{\longmapsto} \langle f(\boldsymbol{\xi},t|\mathbf{X})\rangle = F(t|\mathbf{X}), \tag{7.21}$$

where F is the averaged value of f. Note that we cannot apply the relation (7.21) to some of non-fluctuating quantities such as the metric $\mathbf{g}$, the strain rate $\mathbf{S}$ and the absolute vorticity $\boldsymbol{\Theta}$ of the mean flow, since they are not the average of the fluctuating quantities on the mean-Lagrangian-coordinate system. However, it will be proved in the following discussion that they are also independent of $\boldsymbol{\xi}$ in different contexts.

In our mean-Lagrangian view, the coordinate system may be curvilinear since it is moving with the mean flow. Thus it is needed to recognize the duality between covariant and contravariant tensors. For example, for arbitrary fluctuating contravariant vector h^μ there exists the conjugate quantity given by

$$h_\mu(\mathbf{x},t) = g_{\mu\nu}(\mathbf{x},t)h^\nu(\mathbf{x},t). \tag{7.22}$$

By applying HID to this relation, we obtain

$$h_\mu(\boldsymbol{\xi},t|\mathbf{X}) = g_{\mu\nu}(\boldsymbol{\xi},t|\mathbf{X})h^\nu(\boldsymbol{\xi},t|\mathbf{X}). \tag{7.23}$$

By taking the ensemble average of the both sides, we obtain

$$\begin{aligned} H_\mu(t|\mathbf{X}) &= \langle h_\mu(\boldsymbol{\xi},t|\mathbf{X})\rangle \\ &= \langle g_{\mu\nu}(\boldsymbol{\xi},t|\mathbf{X})h^\nu(\boldsymbol{\xi},t|\mathbf{X})\rangle \\ &= g_{\mu\nu}(\boldsymbol{\xi},t|\mathbf{X})\langle h^\nu(\boldsymbol{\xi},t|\mathbf{X})\rangle \\ &= g_{\mu\nu}(\boldsymbol{\xi},t|\mathbf{X})H^\nu(t|\mathbf{X}), \end{aligned} \tag{7.24}$$

where $\mathbf{H}$ is the averaged value of $\mathbf{h}$. Note that the metric tensor $\mathbf{g}$ is not fluctuating. Now it is obvious that the metric is independent of $\boldsymbol{\xi}$ since the left-hand side is independent of $\boldsymbol{\xi}$. The same situation holds also for the contravariant metric. Thus we have

$$g_{\mu\nu}(\mathbf{x},t) \overset{\text{HID}}{\longmapsto} g_{\mu\nu}(t|\mathbf{X}), \tag{7.25}$$

$$g^{\mu\nu}(\mathbf{x},t) \overset{\text{HID}}{\longmapsto} g^{\mu\nu}(t|\mathbf{X}). \tag{7.26}$$

So far we have required the statistical homogeneity in $\boldsymbol{\xi}$ for both covariant and contravariant quantities equivalently. This sounds quite natural since covariant and contravariant quantities are conjugate and they should be treated in an equivalent way. This is the reason why (7.25) and (7.26) hold. In other words, the relations (7.25) and (7.26) ensure the compatibility between the homogeneity and the duality. As a consequence, the HID representation of (6.13) is given by

$$S_{\mu\nu}(\boldsymbol{\xi},t|\mathbf{X}) = \frac{\partial}{\partial t}g_{\mu\nu}(t|\mathbf{X}). \tag{7.27}$$

Thus the HID representation of the strain rate is supposed to be independent of $\boldsymbol{\xi}$ in order that the metric tensor is consistently independent of $\boldsymbol{\xi}$. Thus we have

$$S_{\mu\nu}(\mathbf{x},t) \overset{\text{HID}}{\longmapsto} S_{\mu\nu}(t|\mathbf{X}). \tag{7.28}$$

Accordingly we have

$$\begin{aligned} &S^{\mu\nu}(\mathbf{x},t) = g^{\mu\rho}(\mathbf{x},t)g^{\nu\sigma}(\mathbf{x},t)S_{\rho\sigma}(\mathbf{x},t) \\ \overset{\text{HID}}{\longmapsto} &S^{\mu\nu}(t|\mathbf{X}) = g^{\mu\rho}(t|\mathbf{X})g^{\nu\sigma}(t|\mathbf{X})S_{\rho\sigma}(t|\mathbf{X}). \end{aligned} \tag{7.29}$$

Next let us think about the spatial derivative. Taking into account that the condition (7.15) must hold, we have the following rule for the spatial-partial derivative [20];

$$\frac{\partial}{\partial x^\mu} f(\mathbf{x},t) \overset{\text{HID}}{\longmapsto} \left(\frac{\partial}{\partial \xi^\mu} + \delta \frac{\partial}{\partial X^\mu}\right) f(\boldsymbol{\xi},t|\mathbf{X}). \tag{7.30}$$

In our formalism, we use the covariant derivative which is accompanied by the Christoffel symbols. Taking into account that the metric tensor becomes independent of $\boldsymbol{\xi}$, we notice that the Christoffel symbols of the first kind should be evaluated as δ-related quantities as

$$\begin{aligned} \Gamma_{\rho.\mu\nu}(\mathbf{x},t) \overset{\text{HID}}{\longmapsto} &\frac{1}{2}\delta\left\{\frac{\partial}{\partial X^\nu} g_{\rho\mu}(t|\mathbf{X}) + \frac{\partial}{\partial X^\mu} g_{\rho\nu}(t|\mathbf{X}) - \frac{\partial}{\partial X^\rho} g_{\mu\nu}(t|\mathbf{X})\right\} \\ = &\delta\Gamma_{\rho.\mu\nu}(t|\mathbf{X}). \end{aligned} \tag{7.31}$$

Thus, for that of the second kind, we have

$$\Gamma^\rho_{\mu\nu}(\mathbf{x},t) = g^{\rho\sigma}(\mathbf{x},t)\Gamma_{\rho.\mu\nu}(\mathbf{x},t) \overset{\text{HID}}{\longmapsto} g^{\rho\sigma}(t|\mathbf{X})\delta\Gamma_{\rho.\mu\nu}(t|\mathbf{X}) = \delta\Gamma^\rho_{\mu\nu}(t|\mathbf{X}). \tag{7.32}$$

Thus, in the HID representation, the covariant derivative is supposed to be represented as

$$\nabla_\kappa C^{\mu\nu\cdots}{}_{\rho\sigma\ldots}(\mathbf{x},t) \overset{\text{HID}}{\longmapsto} \left(\frac{\partial}{\partial \xi^\kappa} + \delta\, {}^{X}\nabla_\kappa\right) C^{\mu\nu\cdots}{}_{\rho\sigma\ldots}(\boldsymbol{\xi},t|\mathbf{X}), \tag{7.33}$$

where the operator ${}^{X}\boldsymbol{\nabla}$, which may be called the covariant derivative of $\mathbf{X}$ or simply the covariant derivative as far as it does not cause any confusion with the original, is given as follows;

$$\begin{aligned} &{}^{X}\nabla_\kappa C^{\mu\nu\cdots}{}_{\rho\sigma\ldots}(\boldsymbol{\xi},t|\mathbf{X}) \\ &\quad= \frac{\partial}{\partial X^\kappa} C^{\mu\nu\cdots}{}_{\rho\sigma\ldots}(\boldsymbol{\xi},t|\mathbf{X}) \\ &\quad+ \Gamma^\mu_{\alpha\kappa}(t|\mathbf{X})C^{\alpha\nu\cdots}{}_{\rho\sigma\ldots}(\boldsymbol{\xi},t|\mathbf{X}) + \Gamma^\nu_{\alpha\kappa}(t|\mathbf{X})C^{\mu\alpha\cdots}{}_{\rho\sigma\ldots}(\boldsymbol{\xi},t|\mathbf{X}) + \cdots \\ &\quad- \Gamma^\beta_{\rho\kappa}(t|\mathbf{X})C^{\mu\nu\cdots}{}_{\beta\sigma\ldots}(\boldsymbol{\xi},t|\mathbf{X}) - \Gamma^\beta_{\sigma\kappa}(t|\mathbf{X})C^{\mu\nu\cdots}{}_{\rho\beta\ldots}(\boldsymbol{\xi},t|\mathbf{X}) - \cdots . \end{aligned} \tag{7.34}$$

Let us see the dynamical equation for the strain rate in order to confirm the consistency of (7.28) and (7.29). The HID representation of the dynamical equation of $S_{\mu\nu}$ is

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \nu\, {}^{X}\nabla^2\right) S_{\mu\nu}(t|\mathbf{X}) = &-\frac{2}{3}S^\rho_\mu(t|\mathbf{X})S_{\rho\nu}(t|\mathbf{X}) + \frac{1}{2}\Theta^\rho{}_\mu(\boldsymbol{\xi},t|\mathbf{X})\Theta_{\rho\nu}(\boldsymbol{\xi},t|\mathbf{X}) \\ &+ \frac{1}{2}\{S^\rho_\mu(t|\mathbf{X})\Theta_{\nu\rho}(\boldsymbol{\xi},t|\mathbf{X}) + S^\rho_\nu(t|\mathbf{X})\Theta_{\mu\rho}(\boldsymbol{\xi},t|\mathbf{X})\} \\ &- 2P_{;\mu\nu}(t|\mathbf{X}) - \delta(R^\rho_{\mu;\rho\nu} + R^\rho_{\nu;\rho\mu})(t|\mathbf{X}). \end{aligned} \tag{7.35}$$

Thus we should put $\boldsymbol{\Theta}$ as follows in order to guarantee the independence of the mean-strain rate from $\boldsymbol{\xi}$;

$$\Theta_{\mu\nu}(\mathbf{x},t) \overset{\text{HID}}{\longmapsto} \Theta_{\mu\nu}(\boldsymbol{\xi},t|\mathbf{X}) = \Theta_{\mu\nu}(t|\mathbf{X}). \tag{7.36}$$

This relation is consistent with the dynamical equation of $\boldsymbol{\Theta}$ shown below, which does not contain the dependence of $\boldsymbol{\xi}$ at all and allows the existence of the solution for $\boldsymbol{\Theta}$ free from $\boldsymbol{\xi}$;

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \nu\, {}^{X}\nabla^2\right) \Theta_{\mu\nu}(t|\mathbf{X}) = &S^\rho_\mu(t|\mathbf{X})\Theta_{\rho\nu}(t|\mathbf{X}) - S^\rho_\nu(t|\mathbf{X})\Theta_{\rho\mu}(t|\mathbf{X}) \\ &+ \delta(R^\rho_{\nu;\rho\mu} - R^\rho_{\mu;\rho\nu})(t|\mathbf{X}). \end{aligned} \tag{7.37}$$

[20] In reproducing the spatial derivative $\partial f/\partial x^\mu\,(\mathbf{x},t)$, we replace $\boldsymbol{\xi}$ and $\mathbf{X}$ with $\mathbf{x}$ and $\delta\mathbf{x}$ of the derivative function $(\partial f/\partial \xi^\mu + \delta\partial f/\partial X^\mu)\,(\boldsymbol{\xi},t|\mathbf{X})$, namely

$$\frac{\partial f}{\partial x^\mu}(\mathbf{x},t) = \left(\frac{\partial}{\partial \xi^\mu} + \delta\frac{\partial}{\partial X^\mu}\right) f(\boldsymbol{\xi},t|\mathbf{X})\bigg|_{\boldsymbol{\xi}=\mathbf{x},\,\mathbf{X}=\delta\mathbf{x}} = \frac{\partial f}{\partial \xi^\mu}(\mathbf{x},t|\delta\mathbf{x}) + \delta\frac{\partial f}{\partial X^\mu}(\mathbf{x},t|\delta\mathbf{x}).$$

Finally, by applying HID to (7.13) and (7.14), we reach a set of equations for the velocity and pressure fluctuations give by

$$\begin{aligned}&\left\{\frac{\partial}{\partial t} - \nu g^{\rho\sigma}(t|\mathbf{X})\frac{\partial}{\partial \xi^\rho}\frac{\partial}{\partial \xi^\sigma} - 2\delta\nu\frac{\partial}{\partial \xi^\rho}{}^X\nabla^\rho - \delta^2\nu{}^X\nabla^2\right\} v'^\mu(\boldsymbol{\xi}, t|\mathbf{X}) \\ &+ \left(\frac{\partial}{\partial \xi^\rho} + \delta{}^X\nabla_\rho\right)(v'^\mu v'^\rho)(\boldsymbol{\xi}, t|\mathbf{X}) + g^{\mu\nu}(t|\mathbf{X})\left(\frac{\partial}{\partial \xi^\nu} + \delta{}^X\nabla_\nu\right)\mathrm{p}'(\boldsymbol{\xi}, t|\mathbf{X}) \\ &= -\left(S^\mu_\rho + \Theta^\mu{}_\rho\right)(t|\mathbf{X})v'^\rho(\boldsymbol{\xi}, t|\mathbf{X}) + \delta R^{\mu\nu}{}_{;\nu}(t|\mathbf{X}),\end{aligned} \tag{7.38}$$

$$\left(\frac{\partial}{\partial \xi^\nu} + \delta{}^X\nabla_\nu\right) v'^\nu(\boldsymbol{\xi}, t|\mathbf{X}) = 0. \tag{7.39}$$

We obtain the solution for the original equations (7.13) and (7.14) by replacing $\boldsymbol{\xi}$ and $\mathbf{X}$ of the solutions $\mathbf{v}'(\boldsymbol{\xi}, t|\mathbf{X})$ and $p'(\boldsymbol{\xi}, t|\mathbf{X})$ of (7.38) and (7.39) with $\mathbf{x}$ and $\delta\mathbf{x}$ respectively.

Here we should notice that our equations (7.38) and (7.39) permit the solution homogeneous with respect to $\boldsymbol{\xi}$, since their forms are identical under the translation;

$$\boldsymbol{\xi} \mapsto \boldsymbol{\xi}' = \boldsymbol{\xi} + \mathbf{a}, \tag{7.40}$$

where $\mathbf{a}$ is a constant vector independent of $\boldsymbol{\xi}$ and t. In our formulation, all the fluctuating quantities are constructed from $\mathbf{v}'$ and p' both of which are permitted to be homogeneous about $\boldsymbol{\xi}$. Therefore all of the fluctuating quantities are permitted to be homogeneous about $\boldsymbol{\xi}$.

It should be noted here that HID performs as the synonym of two-scale decomposition employed in TSDIA in the present analysis(Yoshizawa 1984, also see §4.2). As is mentioned at the beginning of section, the variations of the statistical quantities are often smoother than those of fluctuations. In such cases δ is supposed to be evaluated as small, which makes the dependence on $\mathbf{X}$ much smaller than that of $\boldsymbol{\xi}$. In this sense, we can call $\boldsymbol{\xi}$ and $\mathbf{X}$ local coordinates and global coordinates respectively, since $\boldsymbol{\xi}$ plays a principal roles in describing the local structure while $\mathbf{X}$ denotes the global variation of the statistical quantities.

In the case that the variations of the statistical quantities are so steep as to be comparable to that of the fluctuation such as near-wall region, we cannot take δ as small parameter. In such cases, we need some non-perturbative treatment of δ-effect.

7.3 Fourier transformation

As is often used in the homogeneous turbulence theories, we apply the Fourier analysis to $\boldsymbol{\xi}$ in order to discuss in the spectrum space. First we define the Fourier transformation as

$$\mathcal{F}\times \equiv (2\pi)^{-3}\int \mathrm{d}vol_\xi \exp(-ik_\rho\xi^\rho)\times, \tag{7.41}$$

where $\mathrm{d}vol_\xi = \mathrm{d}^3\xi\sqrt{\mathcal{G}}$, $\mathcal{G} \equiv \det[g_{\rho\sigma}]$. Under this definition, the delta function is given as follows;

$$\delta^3_c(\mathbf{k}, t|\mathbf{X}) = (2\pi)^{-3}\int \mathrm{d}vol_\xi \exp(-ik_\rho\xi^\rho)\,. \tag{7.42}$$

Nonlinear term is transformed in our representation as follows;

$$\begin{aligned}&f(\boldsymbol{\xi}, t|\mathbf{X})g(\boldsymbol{\xi}, t|\mathbf{X}) \\ &\quad\overset{\mathcal{F}}{\longmapsto} (2\pi)^{-3}\int \mathrm{d}vol_\xi \exp(-ik_\alpha\xi^\alpha) \\ &\qquad\qquad \times \int \mathrm{d}vol_p \int \mathrm{d}vol_q \exp(ip_\beta\xi^\beta)\exp(iq_\gamma\xi^\gamma) f(\mathbf{p}, t|\mathbf{X})g(\mathbf{q}, t|\mathbf{X}) \\ &\quad= \int \mathrm{d}vol_p \int \mathrm{d}vol_q \\ &\qquad\qquad \times (2\pi)^{-3}\int \mathrm{d}vol_\xi \exp\left\{i(p_\alpha + q_\alpha - k_\alpha)\xi^\alpha\right\} f(\mathbf{p}, t|\mathbf{X})g(\mathbf{q}, t|\mathbf{X}) \\ &\quad= \int \mathrm{d}vol_p \int \mathrm{d}vol_q \delta^3_c\left(\mathbf{k} - \mathbf{p} - \mathbf{q}, t|\mathbf{X}\right) f(\mathbf{p}, t|\mathbf{X})g(\mathbf{q}, t|\mathbf{X}),\end{aligned} \tag{7.43}$$

where $\mathrm{d}vol_k = \mathrm{d}^3k\sqrt{\mathcal{G}^{-1}}$. For simplicity, we use the following symbol for the convolution;

$$[\mathbf{k};\mathbf{p},\mathbf{q}]\, f(\mathbf{p},t|\mathbf{X})g(\mathbf{q},t|\mathbf{X}) \equiv \int \mathrm{d}vol_p \int \mathrm{d}vol_q \delta_c^3(\mathbf{k}-\mathbf{p}-\mathbf{q},t|\mathbf{X}) f(\mathbf{p},t|\mathbf{X})g(\mathbf{q},t|\mathbf{X}).$$

It is important to see the commutation relations between the modified Fourier transformation and the spatial covariant derivative and time derivative. For this purpose, we investigate $\mathcal{G}$ which is the only factor dependent on the global coordinates and time in the modified-Fourier-transformation operator. Taking the covariant derivative of $\mathcal{G}$, we obtain

$$^X\nabla_\rho \mathcal{G}(t|\mathbf{X}) = \mathcal{G}(t|\mathbf{X}) g^{\alpha\beta}(t|\mathbf{X})\, ^X\nabla_\rho g_{\alpha\beta}(t|\mathbf{X}). \tag{7.44}$$

Here we used a formula $\partial \det[M_{\mu\nu}]/\partial M_{\alpha\beta} = \det[M_{\mu\nu}]\,(M^{-1})^{\alpha\beta}$, where $\mathbf{M}$ is an arbitrary two-rank matrix. Since the covariant derivative of the metric is identically zero, we obtain

$$^X\nabla_\rho \mathcal{G}(t|\mathbf{X}) = 0. \tag{7.45}$$

In the same manner, taking the time derivative of $\mathcal{G}$ yields

$$\frac{\partial}{\partial t}\mathcal{G}(t|\mathbf{X}) = \mathcal{G}(t|\mathbf{X}) g^{\alpha\beta}(t|\mathbf{X}) \frac{\partial}{\partial t} g_{\alpha\beta}(t|\mathbf{X}) = \mathcal{G}(t|\mathbf{X}) g^{\alpha\beta}(t|\mathbf{X}) S_{\alpha\beta}(t|\mathbf{X}). \tag{7.46}$$

Since we treat only incompressible fluid in this thesis, the strain rate is traceless. Therefore we obtain

$$\frac{\partial}{\partial t}\mathcal{G}(t|\mathbf{X}) = 0. \tag{7.47}$$

Thus the modified Fourier transformation $\mathcal{F}$ commutes with both spatial and time derivatives as follows[21];

$$\left(^X\nabla, \frac{\partial}{\partial t}\right)\mathcal{F} = \mathcal{F}\left(^X\nabla, \frac{\partial}{\partial t}\right). \tag{7.48}$$

Finally, by applying the modified Fourier transformation to both (7.38) and (7.39), we obtain the following set of equations;

$$\begin{aligned} &\left\{\frac{\partial}{\partial t} + \nu g^{\rho\sigma}(t|\mathbf{X}) k_\rho k_\sigma - 2i\delta\nu k_\rho\, ^X\nabla^\rho - \delta^2\, ^X\nabla^2\right\} v'^\mu(\mathbf{k},t|\mathbf{X}) \\ &\quad + \left(ik_\nu + \delta\, ^X\nabla_\nu\right)[\mathbf{k};\mathbf{p},\mathbf{q}]\, v'^\mu(\mathbf{p},t|\mathbf{X}) v'^\nu(\mathbf{q},t|\mathbf{X}) \\ &\quad + \left(ik^\mu + \delta\, ^X\nabla^\mu\right) \mathrm{p}'(\mathbf{k},t|\mathbf{X}) \\ &= -\left(S^\mu_\nu + \Theta^\mu{}_\nu\right)(t|\mathbf{X}) v'^\nu(\mathbf{k},t|\mathbf{X}) + \delta R^{\mu\nu}{}_{;\nu}(t|\mathbf{X})\delta_c^3(\mathbf{k},t|\mathbf{X}), \end{aligned} \tag{7.49}$$

$$\left(ik_\rho + \delta\, ^X\nabla_\rho\right) v'^\rho(\mathbf{k},t|\mathbf{X}) = 0. \tag{7.50}$$

7.4 Elimination of the pressure fluctuation

In the master equations (7.49) and (7.50) we have two dynamical variables; $\mathbf{v}'$ and p'. Here we are aiming to eliminate the pressure from our equations. Let us separate the pressure related part and the others in (7.49) as

$$\left(ik^\mu + \delta\, ^X\nabla^\mu\right)\mathrm{p}'(\mathbf{k},t|\mathbf{X}) = {}^p f^\mu(\mathbf{k},t|\mathbf{X}), \tag{7.51}$$

where

$$\begin{aligned} {}^p f^\mu(\mathbf{k},t|\mathbf{X}) = &-\left\{\frac{\partial}{\partial t} + \nu g^{\rho\sigma}(t|\mathbf{X}) k_\rho k_\sigma - 2i\delta\nu k_\rho\, ^X\nabla^\rho - \delta^2\, ^X\nabla^2\right\} v'^\mu(\mathbf{k},t|\mathbf{X}) \\ &- \left(ik_\nu + \delta\, ^X\nabla_\nu\right)[\mathbf{k};\mathbf{p},\mathbf{q}]\, v'^\mu(\mathbf{p},t|\mathbf{X}) v'^\nu(\mathbf{q},t|\mathbf{X}) \\ &- \left(S^\mu_\nu + \Theta^\mu{}_\nu\right)(t|\mathbf{X}) v'^\nu(\mathbf{k},t|\mathbf{X}) + \delta R^{\mu\nu}{}_{;\nu}(t|\mathbf{X})\delta_c^3(\mathbf{k},t|\mathbf{X}). \end{aligned} \tag{7.52}$$

[21] In TSDIA, on the contrary, the spectra depend on the mean velocity and, consequently, the spectra of spatial/time derivative terms based on TSDIA are accompanied by spatial/time derivatives of the mean velocity field which are apparently non-covariant quantities. To make matters worse, these terms cause divergence which cannot be removed from appropriate reason (Okamoto 1994). In the mean-Lagrangian formulation, on the contrary, the dynamical equations do not contain the mean velocity from the very beginning and thus it is totally free from the above difficulties.

By multiplying the both sides of (7.51) by k_μ/ik^2 we obtain

$$ {}^{p}\hat{L}\mathrm{p}'(\mathbf{k},t|\mathbf{X}) = \frac{k_\nu}{ik^2}\,{}^{p}f^\nu(\mathbf{k},t|\mathbf{X}), \tag{7.53}$$

where ${}^{p}\hat{L} = 1 - \delta i k_\rho\,{}^{X}\nabla^\rho/k^2$, $k = \sqrt{g^{\alpha\beta}k_\alpha k_\beta}$. Using the inverse operator of ${}^{p}\hat{L}$, we obtain

$$ \mathrm{p}'(\mathbf{k},t|\mathbf{X}) = {}^{p}\hat{L}^{-1}\frac{k_\nu}{ik^2}\,{}^{p}f^\nu(\mathbf{k},t|\mathbf{X}), \tag{7.54}$$

where the operator ${}^{p}\hat{L}^{-1}$ is given by a series expansion as

$$ {}^{p}\hat{L}^{-1} = 1 + \delta\frac{ik_\rho}{k^2}\,{}^{X}\nabla^\rho + \delta^2\left(\frac{ik_\rho}{k^2}\,{}^{X}\nabla^\rho\right)^2 + \delta^3\left(\frac{ik_\rho}{k^2}\,{}^{X}\nabla^\rho\right)^3 + \cdots\,. \tag{7.55}$$

By substituting (7.54) into (7.51), we obtain

$$ \tilde{P}^\mu_\nu(\mathbf{k},t,{}^{X}\nabla|\mathbf{X})\,{}^{p}f^\nu(\mathbf{k},t|\mathbf{X}) = 0, \tag{7.56}$$

where the operator $\tilde{\mathbf{P}}$ is defined by

$$ \tilde{P}^\mu_\nu(\mathbf{k},t,{}^{X}\nabla|\mathbf{X}) = \delta^\mu_\nu - \left(ik^\mu + \delta\,{}^{X}\nabla^\mu\right){}^{p}\hat{L}^{-1}\frac{k_\nu}{ik^2}. \tag{7.57}$$

(7.56) does not contain the pressure, which is the objective of this subsection.

In the later analysis, we will take the similar procedure to that of the incompressible-turbulence theory. Thus it is useful to introduce the solenoidal (incompressible) part of the velocity fluctuation. Therefore we introduce

$$ {}^{s}v^\mu(\mathbf{k},t|\mathbf{X}) = P^\mu_\nu(\mathbf{k},t|\mathbf{X})v'^\nu(\mathbf{k},t|\mathbf{X}), \tag{7.58}$$

where $P^\mu_\nu(\mathbf{k},t|\mathbf{X}) = \delta^\mu_\nu - k^\mu k_\nu/k^2$ is a projection operator for extracting the solenoidal part. By making use of (7.50), we obtain

$$ \begin{aligned} {}^{s}v^\mu(\mathbf{k},t|\mathbf{X}) &= \left(\delta^\mu_\nu - \frac{k^\mu k_\nu}{k^2}\right)v'^\nu(\mathbf{k},t|\mathbf{X}) \\ &= v'^\mu(\mathbf{k},t|\mathbf{X}) - \delta\frac{ik^\mu\,{}^{X}\nabla_\nu}{k^2}v'^\nu(\mathbf{k},t|\mathbf{X}) \\ &= {}^{s}\hat{L}^\mu_\nu v'^\nu(\mathbf{k},t|\mathbf{X}), \end{aligned} \tag{7.59}$$

where ${}^{s}\hat{L}^\mu_\nu = \delta^\mu_\nu - \delta i k^\mu\,{}^{X}\nabla_\nu/k^2$. In the same manner as we have done for the pressure, we can also express the above result as

$$ v'^\mu(\mathbf{k},t|\mathbf{X}) = ({}^{s}\hat{L}^{-1})^\mu_\nu\,{}^{s}v^\nu(\mathbf{k},t|\mathbf{X}), \tag{7.60}$$

$$ ({}^{s}\hat{L}^{-1})^\mu_\nu = \delta^\mu_\nu + \delta\frac{ik^\mu\,{}^{X}\nabla_\nu}{k^2} + \delta^2\frac{ik^\mu\,{}^{X}\nabla_\rho}{k^2}\frac{ik^\rho\,{}^{X}\nabla_\nu}{k^2} + \cdots\,. \tag{7.61}$$

Thus we have the equation for the solenoidal velocity by substituting (7.60) into $\mathbf{v}'$ in (7.56). It is also convenient for the later discussion to multiply the modified projector (7.57) by the solenoidal projector to investigate only the solenoidal component. Then we have the following equation for the solenoidal velocity;

$$ \hat{P}^\mu_\nu(\mathbf{k},t,{}^{X}\nabla|\mathbf{X})\,{}^{p}f^\nu(\mathbf{k},t|\mathbf{X}) = 0, \tag{7.62}$$

$$ \begin{aligned} {}^{p}f^\mu(\mathbf{k},t|\mathbf{X}) = &-\left\{\frac{\partial}{\partial t} + \nu g^{\rho\sigma}(t|\mathbf{X})k_\rho k_\sigma - 2i\delta\nu k_\rho\,{}^{X}\nabla^\rho - \delta^2\,{}^{X}\nabla^2\right\}({}^{s}\hat{L}^{-1})^\mu_\nu\,{}^{s}v^\nu(\mathbf{k},t|\mathbf{X}) \\ &- \left(ik_\rho + \delta\,{}^{X}\nabla_\rho\right)[\mathbf{k};\mathbf{p},\mathbf{q}]\,({}^{s}\hat{L}^{-1})^\mu_\nu\,{}^{s}v^\nu(\mathbf{p},t|\mathbf{X})({}^{s}\hat{L}^{-1})^\rho_\sigma\,{}^{s}v^\sigma(\mathbf{q},t|\mathbf{X}) \\ &- \left(S^\mu_\nu + \Theta^\mu{}_\nu\right)(t|\mathbf{X})({}^{s}\hat{L}^{-1})^\nu_\rho\,{}^{s}v^\rho(\mathbf{k},t|\mathbf{X}) + \delta R^{\mu\nu}{}_{;\nu}(t|\mathbf{X})\delta^3_c(\mathbf{k},t|\mathbf{X}), \end{aligned} \tag{7.63}$$

where $\hat{\mathbf{P}}$ is a modified projection operator given by

$$ \hat{P}^\mu_\nu(\mathbf{k},t,{}^{X}\nabla|\mathbf{X}) = P^\mu_\rho\tilde{P}^\rho_\nu(\mathbf{k},t,{}^{X}\nabla|\mathbf{X}) = P^\mu_\nu - \delta P^\mu_\rho\,{}^{X}\nabla^\rho\,{}^{p}\hat{L}^{-1}\frac{k_\nu}{ik^2}. \tag{7.64}$$

By expanding the operators ${}^p\hat{L}$ and ${}^s\hat{\mathbf{L}}$ in (7.62), we obtain the perturbative representation of the dynamical equation as

$$\begin{aligned}\left\{\frac{\partial}{\partial t}+\nu g^{\rho\sigma}(t|\mathbf{X})k_\rho k_\sigma\right\}{}^s v^\mu(\mathbf{k},t|\mathbf{X}) =&\frac{1}{i}M^\mu_{\rho\sigma}(\mathbf{k},t|\mathbf{X})[\mathbf{k};\mathbf{p},\mathbf{q}]\,{}^s v^\mu(\mathbf{p},t|\mathbf{X})\,{}^s v^\mu(\mathbf{q},t|\mathbf{X})\\ &-P^\mu_\nu(\mathbf{k},t|\mathbf{X})(S^\nu_\rho+\Theta^\nu{}_\rho)(t|\mathbf{X})\,{}^s v^\rho(\mathbf{q},t|\mathbf{X})\\ &+O(\delta),\end{aligned} \tag{7.65}$$

where $M^\mu_{\rho\sigma}=\frac{1}{2}(P^\mu_\rho k_\sigma+P^\mu_\sigma k_\rho)$ is another solenoidal projector. In principle we can obtain the exact $\mathbf{v}'$ and p' by solving (7.65) and using (7.60) and (7.54) in a perturbative manner.

7.5 Static-metric representation

In the following process we will solve the velocity-fluctuation field in perturbative manner on the basis of the homogeneous isotropic turbulence. At first glance, from (7.65), one may consider the homogeneous-isotropic-turbulence equation is given by

$$\left\{\frac{\partial}{\partial t}+\nu g^{\rho\sigma}(t|\mathbf{X})k_\rho k_\sigma\right\}v^\mu(\mathbf{k},t|\mathbf{X})=\frac{1}{i}M^\mu_{\rho\sigma}(\mathbf{k},t|\mathbf{X})[\mathbf{k};\mathbf{p},\mathbf{q}]v^\mu(\mathbf{p},t|\mathbf{X})v^\mu(\mathbf{q},t|\mathbf{X}).$$

If this were true, one could regard the second term on the right-hand side of (7.65) as perturbative interaction and would straightly apply the perturbation analysis. However this is not true since the metric in this coordinate representation clearly depends on time[22]. Thus, in this subsection, we are aiming to transform (7.65) into a proper form where the perturbative and non-perturbative parts are separated according to our purpose.

First we introduce a new representation in which the metric is static. In the following discussion, we attach "ˇ" to the static-metric-represented wavenumber. In the static-metric frame, indices are denoted by capital roman letters. The transformation from the mean-Lagrangian to static-metric representation is given by

$$\check{k}_I={}^x a^\mu_I(t)k_\mu, \tag{7.66}$$

$$\begin{aligned}&C^{IJ\cdots}{}_{KL\ldots}(\check{\mathbf{k}},t|\mathbf{X})\\ &\quad={}^x a^I_\mu(t)\,{}^x a^J_\nu(t)\ldots{}^x a^\rho_K(t)\,{}^x a^\sigma_L(t)\ldots C^{\mu\nu\cdots}{}_{\rho\sigma\ldots}({}^x a^A_\alpha(t)\check{k}_A,t|\mathbf{X}),\end{aligned} \tag{7.67}$$

where ${}^x\mathsf{a}$'s are the transformation coefficients which obey the following relations;

$${}^x a^I_\mu(t)\,{}^x a^\mu_J(t)=\delta^I_J\,, \tag{7.68}$$

$${}^x a^I_\mu(t)\,{}^x a^\nu_I(t)=\delta^\nu_\mu\,. \tag{7.69}$$

The metric tensors of the static-metric frame are given by

$${}^x a^\mu_I(t){}^x a^\nu_J(t)g_{\mu\nu}(t|\mathbf{X})=g_{IJ}, \tag{7.70}$$

$${}^x a^I_\mu(t)\,{}^x a^J_\nu(t)g^{\mu\nu}(t|\mathbf{X})=g^{IJ}, \tag{7.71}$$

where g_{IJ} and g^{IJ} are both constant in time. By transforming (7.70), we obtain

$${}^x a^I_\mu(t)\,{}^x a^J_\nu(t)\,g_{IJ}=g_{\mu\nu}(t|\mathbf{X}). \tag{7.72}$$

[22]This can be understood by recognizing the operator

$$\frac{\partial}{\partial t}+\nu g^{\rho\sigma}(t|\mathbf{X})k_\rho k_\sigma$$

is different from what we see in the homogeneous-turbulence equation in Eulerian frame;

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)v_i(\mathbf{k},t)=\frac{1}{i}M_{i.ab}(\mathbf{k},t)[\mathbf{k};\mathbf{p},\mathbf{q}]v_a(\mathbf{p},t)v_b(\mathbf{q},t).$$

In our coordinate representation, the time-derivative is the Oldroyd derivative where the deformation of the coordinate frame is considered.

Taking the time derivative of both sides of the above leads to

$$\frac{\mathrm{d}\,{}^{X}\!a^I_\mu(t)}{\mathrm{d}t}\;{}^{X}\!a^J_\nu(t)\,g_{IJ} + \frac{\mathrm{d}\,{}^{X}\!a^J_\nu(t)}{\mathrm{d}t}\;{}^{X}\!a^I_\mu(t)\,g_{IJ} = S_{\mu\nu}(t|\mathbf{X}). \tag{7.73}$$

This indicates that the symmetric part of $(\mathrm{d}\,{}^{X}\!a^I_\mu(t)/\mathrm{d}t)\,{}^{X}\!a^J_\nu(t)\,g_{IJ}$ coincides with $\frac{1}{2}\mathbf{S}$. Thus the above relation yields the following relation;

$$\frac{\mathrm{d}\,{}^{X}\!a^I_\mu(t)}{\mathrm{d}t}\;{}^{X}\!a^J_\nu(t)\,g_{IJ} = \frac{1}{2}(S_{\mu\nu} + A_{\mu\nu})(t|\mathbf{X}). \tag{7.74}$$

Here, $\mathbf{A}$ is an arbitrary 2-rank anti-symmetric tensor and this is an extra dynamical variable corresponding to the angular velocity of the static-metric frame relative to the mean-Lagrangian frame (see appendix D). The determination of $\mathbf{A}$ will be discussed later with its role in the perturbation analysis (see §7.7).

By transforming (7.74), we obtain the differential equations for the transformation coefficients as

$$\frac{\mathrm{d}\,{}^{X}\!a^I_\mu(t)}{\mathrm{d}t} = \frac{1}{2}(S^\nu_\mu + A_\mu{}^\nu)(t|\mathbf{X})\,{}^{X}\!a^I_\nu(t), \tag{7.75}$$

$$\frac{\mathrm{d}\,{}^{X}\!a^\nu_J(t)}{\mathrm{d}t} = -\frac{1}{2}(S^\nu_\mu + A_\mu{}^\nu)(t|\mathbf{X})\,{}^{X}\!a^\mu_J(t). \tag{7.76}$$

By using them, we obtain the relation between the time derivative in the static-metric representation and the counterpart in the mean-Lagrangian representation. Especially for the solenoidal vector $\mathbf{C}$, we obtain the following relation (see appendix E);

$$\begin{aligned}\frac{\partial C^I}{\partial t}(\check{\mathbf{k}},t|\mathbf{X}) &= P^I_J(\check{\mathbf{k}})\frac{\partial C^J}{\partial t}(\check{\mathbf{k}},t|\mathbf{X})\\ &= {}^{X}\!a^I_\mu(t)\left\{P^\mu_\nu\frac{\partial}{\partial t} + \frac{1}{2}P^\mu_\rho(S^\rho_\nu + A_\nu{}^\rho) + \frac{1}{2}P^\mu_\nu(S^\rho_\sigma + A_\sigma{}^\rho)k_\rho\frac{\partial}{\partial k_\sigma}\right\}{}^{X}\!a^\nu_J(t)\;C^J(\check{\mathbf{k}},t|\mathbf{X}).\end{aligned} \tag{7.77}$$

The viscous-decaying operator in the static-metric frame is given by

$$\begin{aligned}\hat{L}^I_J(\check{\mathbf{k}},t|\mathbf{X}) \equiv& P^I_J(\check{\mathbf{k}})\left(\frac{\partial}{\partial t} + \nu\;g^{AB}\check{k}_A\check{k}_B\right)\\ =& {}^{X}\!a^I_\mu(t)\left\{P^\mu_\nu\frac{\partial}{\partial t} + \frac{1}{2}P^\mu_\rho(S^\rho_\nu + A_\nu{}^\rho) + \frac{1}{2}P^\mu_\nu(S^\rho_\sigma + A_\sigma{}^\rho)k_\rho\frac{\partial}{\partial k_\sigma} + P^\mu_\nu \nu g^{\rho\sigma}k_\rho k_\sigma\right\}\,{}^{X}\!a^\nu_J(t),\end{aligned} \tag{7.78}$$

whose mean-Lagrangian version is given by

$$\begin{aligned}\hat{L}^\mu_\nu(\mathbf{k},t|\mathbf{X}) \equiv& {}^{X}\!a^\mu_I(t)\hat{L}^I_J(\check{\mathbf{k}},t|\mathbf{X})\,{}^{X}\!a^J_\nu(t)\\ =& P^\mu_\nu\frac{\partial}{\partial t} + \frac{1}{2}P^\mu_\rho(S^\rho_\nu + A_\nu{}^\rho) + \frac{1}{2}P^\mu_\nu(S^\rho_\sigma + A_\sigma{}^\rho)k_\rho\frac{\partial}{\partial k_\sigma} + P^\mu_\nu \nu g^{\rho\sigma}k_\rho k_\sigma.\end{aligned} \tag{7.79}$$

Thus the homogeneous-isotropic-field equation is written in the mean-Lagrangian representation as

$$\hat{L}^\mu_\nu\,v^\nu(\mathbf{k},t|\mathbf{X}) = \frac{1}{i}M^\mu_{\rho\sigma}(\mathbf{k},t|\mathbf{X})[\mathbf{k};\mathbf{p},\mathbf{q}]v^\mu(\mathbf{p},t|\mathbf{X})v^\nu(\mathbf{q},t|\mathbf{X}).$$

Therefore we transform the dynamical equation (7.65) into the following form.

$$\begin{aligned}\hat{L}^\mu_\nu\,{}^{s}v^\nu(\mathbf{k},t|\mathbf{X}) =& \frac{1}{i}M^\mu_{\rho\sigma}[\mathbf{k};\mathbf{p},\mathbf{q}]\,{}^{s}v^\rho(\mathbf{p},t|\mathbf{X})\,{}^{s}v^\sigma(\mathbf{q},t|\mathbf{X})\\ &+ \hat{F}^\mu_\nu(\mathbf{k},t|\mathbf{X})\,{}^{s}v^\nu(\mathbf{k},t|\mathbf{X})\\ &+ O(\delta),\end{aligned} \tag{7.80}$$

where $\hat{\mathbf{F}}$ is a linear operator defined by

$$\hat{F}^\mu_\nu = P^\mu_\rho\left(-\frac{1}{2}S^\rho_\nu - \Theta^\rho_\nu + \frac{1}{2}A_\nu{}^\rho\right) + \frac{1}{2}P^\mu_\nu(S^\rho_\sigma + A_\sigma{}^\rho)k_\rho\frac{\partial}{\partial k_\sigma}. \tag{7.81}$$

The objective of this subsection has been achieved by (7.80).

So far we have used ${}^{x}\mathsf{a}$ as the main tool in this subsection. This convenient representation method is, however, apparently non-objective since it totally depends on how we choose the static-metric frame. Thus we introduce here time-evolution coefficients which express the time evolution of ${}^{x}\mathsf{a}$ in a covariant manner. First we define the time-evolution coefficient of the first kind as follows;

$$\Lambda^{\mu}_{\nu}(t;t'|\mathbf{X}) = {}^{x}a^{\mu}_{J}(t)\, {}^{x}a^{J}_{\nu}(t'). \tag{7.82}$$

Multiplying this by ${}^{x}a^{\nu}_{I}(t')$ and using (7.68), we obtain

$${}^{x}a^{\mu}_{I}(t) = \Lambda^{\mu}_{\nu}(t;t'|\mathbf{X})\, {}^{x}a^{\nu}_{I}(t'). \tag{7.83}$$

Substituting this into (7.76) yields the equation for the time-evolution coefficient as

$$\frac{\partial}{\partial t}\Lambda^{\mu}_{\nu}(t;t'|\mathbf{X}) = -\frac{1}{2}(S^{\mu}_{\rho} + A_{\rho}{}^{\mu})(t|\mathbf{X})\Lambda^{\rho}_{\nu}(t;t'|\mathbf{X}), \tag{7.84}$$

with

$$\Lambda^{\mu}_{\nu}(t';t'|\mathbf{X}) = \delta^{\mu}_{\nu}. \tag{7.85}$$

Note that the time-evolution coefficient is determined independently of the choice of the static metric representation, as is clear from (7.84) and (7.85). Using the time-evolution coefficient, we can reach the covariant result without using the static-metric representation explicitly as will be discussed later (see (7.107) and (7.108) in §7.7). In the same manner, we introduce the time-evolution coefficient of the second kind as

$$\bar{\Lambda}^{\mu}_{\nu}(t;t'|\mathbf{X}) = {}^{x}a^{J}_{\mu}(t)\, {}^{x}a^{\nu}_{J}(t') \tag{7.86}$$

which satisfies the followings;

$${}^{x}a^{I}_{\mu}(t) = \bar{\Lambda}^{\nu}_{\mu}(t;t'|\mathbf{X})\, {}^{x}a^{I}_{\nu}(t'), \tag{7.87}$$

$$\frac{\partial}{\partial t}\bar{\Lambda}^{\nu}_{\mu}(t;t'|\mathbf{X}) = \frac{1}{2}(S^{\rho}_{\mu} + A_{\mu}{}^{\rho})(t|\mathbf{X})\bar{\Lambda}^{\nu}_{\rho}(t;t'|\mathbf{X}), \tag{7.88}$$

$$\bar{\Lambda}^{\nu}_{\mu}(t';t'|\mathbf{X}) = \delta^{\nu}_{\mu}. \tag{7.89}$$

(7.82) and (7.86) yields $\Lambda^{\mu}_{\nu}(t;t'|\mathbf{X}) = \bar{\Lambda}^{\mu}_{\nu}(t';t|\mathbf{X})$.

7.6 Partial renormalization

So far we have transformed the set of dynamical equations into (7.80) which has a manageable form for our purpose, and now is the time to apply the statistical analysis to the master equation (7.80). Our basic strategy is based on RPT. In this context we have to specify the perturbative terms, so that we introduce perturbative parameters λ and μ for the nonlinear self-interaction and S, $\mathbf{\Theta}$-related terms respectively. Thus we rewrite (7.80) as

$$\begin{aligned}\hat{L}^{\mu}_{\nu}\, {}^{s}v^{\nu}(\mathbf{k},t|\mathbf{X}) =& \lambda\frac{1}{i}M^{\mu}_{\rho\sigma}[\mathbf{k};\mathbf{p},\mathbf{q}]\, {}^{s}v^{\rho}(\mathbf{p},t|\mathbf{X})\, {}^{s}v^{\sigma}(\mathbf{q},t|\mathbf{X}) \\ &+ \mu\hat{F}^{\mu}_{\nu}(\mathbf{k},t|\mathbf{X})\, {}^{s}v^{\nu}(\mathbf{k},t|\mathbf{X}) \\ &+ O(\delta).\end{aligned} \tag{7.90}$$

We assume these three parameters λ, μ and δ as perturbative parameters representing the magnitude of nonlinearity, anisotropy and inhomogeneity respectively. Thus we regard the following solenoidal field $\tilde{\mathbf{v}}$ as the non-perturbative field;

$$\hat{L}^{\mu}_{\nu}\tilde{v}^{\nu}(\mathbf{k},t|\mathbf{X}) = 0. \tag{7.91}$$

In addition, we introduce the propagator of $\tilde{\mathbf{v}}$ as

$$\hat{L}^{\mu}_{\rho}\tilde{G}^{\rho}_{\nu}(\mathbf{k};t,t'|\mathbf{X}) = P^{\mu}_{\nu}(\mathbf{k},t|\mathbf{X})\delta(t-t'). \tag{7.92}$$

$\tilde{\mathbf{v}}$ and $\tilde{\mathsf{G}}$ are to be referred to as the bare field and bare propagator in the context of RPT. By using $\tilde{\mathsf{G}}$, we can integrate (7.90) as

$$\begin{aligned}{}^{s}v^{\mu}(\mathbf{k},t|\mathbf{X}) = &\tilde{v}^{\mu}(\mathbf{k},t|\mathbf{X}) \\ &+ \lambda\int_{-\infty}^{t}\mathrm{d}t'\ \tilde{G}^{\mu}_{\nu}(\mathbf{k};t,t'|\mathbf{X})\frac{1}{i}M^{\nu}_{\rho\sigma}[\mathbf{k};\mathbf{p},\mathbf{q}]\, {}^{s}v^{\rho}(\mathbf{p},t|\mathbf{X})\, {}^{s}v^{\sigma}(\mathbf{q},t|\mathbf{X}) \\ &+ \mu\int_{-\infty}^{t}\mathrm{d}t'\ \tilde{G}^{\mu}_{\nu}(\mathbf{k};t,t'|\mathbf{X})\hat{F}^{\nu}_{\rho}(\mathbf{k},t'|\mathbf{X})\, {}^{s}v^{\rho}(\mathbf{k},t'|\mathbf{X}) \\ &+ O(\delta).\end{aligned} \tag{7.93}$$

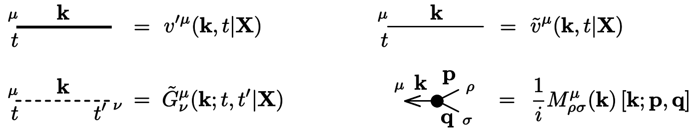


Figure 7: Diagrammatic representations: The total fluctuation $\mathbf{v}'$, the non-perturbative field $\tilde{\mathbf{v}}$, its propagator $\tilde{\mathbf{G}}$ and the convolution are represented by a thick line, a thin line, a dotted line and a vertex respectively.

(a) Convolution (b) Convolution with propagator

Figure 8: Examples of vertex representation; (a) the convolution of two velocity fluctuations is represented by vertex with two thick lines attached to the two tail terminals, and (b) its integration with the bare propagator is represented by attaching a dotted line to the arrow terminal.

By substituting the above expansion into ${}^{s}\mathbf{v}$ on the right-hand side iteratively, we obtain the solution for the equation (7.90) as a series expansion. Using (7.60), we obtain the perturbative expansion of the total velocity fluctuation;

$$\begin{aligned}
v'^{\mu}(\mathbf{k},t|\mathbf{X}) = \tilde{v}^{\mu}(\mathbf{k},t|\mathbf{X})& \\
&+ \lambda \int_{-\infty}^{t} \mathrm{d}t'\ \tilde{G}^{\mu}_{\nu}(\mathbf{k};t,t'|\mathbf{X})\frac{1}{i}M^{\nu}_{\rho\sigma}[\mathbf{k};\mathbf{p},\mathbf{q}]\tilde{v}^{\rho}(\mathbf{p},t|\mathbf{X})\tilde{v}^{\sigma}(\mathbf{q},t|\mathbf{X}) \\
&+ \mu \int_{-\infty}^{t} \mathrm{d}t'\ \tilde{G}^{\mu}_{\nu}(\mathbf{k};t,t'|\mathbf{X})\hat{F}^{\nu}_{\rho}(\mathbf{k},t'|\mathbf{X})\tilde{v}^{\rho}(\mathbf{k},t'|\mathbf{X}) \\
&+ \delta \frac{ik^{\mu}\,{}^{X}\nabla_{\nu}}{k^2}\,\tilde{v}^{\nu}(\mathbf{k},t|\mathbf{X}) \\
&+ \cdots .
\end{aligned} \tag{7.94}$$

In order to make the discussion simple and clear, we employ the symbolic method first introduced by Wyld (1961). The total fluctuation $\mathbf{v}'$, the non-perturbative field $\tilde{\mathbf{v}}$, its propagator $\tilde{\mathbf{G}}$ and the convolution are represented respectively by a thick line, a thin line, a dotted line and a vertex shown in figure 7 [23].

By using them, we express the perturbative expansion of $\mathbf{v}'$ with a series of diagrams shown in figure 9, where $\hat{\mathbf{F}}$ operates on the side of "$*$". Here we have a particular group of diagrams free from μ and δ as (A) in figure 9. This group is obviously the solution of the following equation for ${}^{B}\mathbf{v}$;

$$\hat{L}^{\mu}_{\nu}\,{}^{B}v^{\nu}(\mathbf{k},t|\mathbf{X}) = \lambda\frac{1}{i}M^{\mu}_{\rho\sigma}[\mathbf{k};\mathbf{p},\mathbf{q}]\,{}^{B}v^{\rho}(\mathbf{p},t|\mathbf{X})\,{}^{B}v^{\sigma}(\mathbf{q},t|\mathbf{X}). \tag{7.95}$$

Let us call ${}^{B}\mathbf{v}$ the basic field which will play the fundamental role in our closure method.

Then we introduce the other key factor: the response of the basic field for the infinitesimal disturbance. Let us consider the above equation with infinitesimal external force $\boldsymbol{\Upsilon}$;

$$\begin{aligned}
\hat{L}^{\mu}_{\nu}\,{}^{B}v^{\nu}(\mathbf{k},t|\mathbf{X}\|\boldsymbol{\Upsilon}) = \lambda\frac{1}{i}M^{\mu}_{\rho\sigma}[\mathbf{k};\mathbf{p},\mathbf{q}]\,{}^{B}v^{\rho}(\mathbf{p},t|\mathbf{X}\|\boldsymbol{\Upsilon})\,{}^{B}v^{\sigma}(\mathbf{q},t|\mathbf{X}\|\boldsymbol{\Upsilon})& \\
+ P^{\mu}_{\nu}(\mathbf{k},t|\mathbf{X})\Upsilon^{\nu}(\mathbf{k},t|\mathbf{X}).&
\end{aligned} \tag{7.96}$$

[23] The vertex has three terminals; arrow terminal is for propagator integration, the other two of them are for convolution. For example, convolution of two velocity fluctuations, namely

$$\frac{1}{i}M^{\mu}_{\rho\sigma}[\mathbf{k};\mathbf{p},\mathbf{q}]v'^{\rho}(\mathbf{p},t|\mathbf{X})v'^{\sigma}(\mathbf{q},t|\mathbf{X}),$$

is represented as figure 8(a), while its integration with the propagator, namely

$$\int_{-\infty}^{t}\mathrm{d}t'\,\tilde{G}^{\mu}_{\nu}(\mathbf{k};t,t'|\mathbf{X})\frac{1}{i}M^{\mu}_{\rho\sigma}[\mathbf{k};\mathbf{p},\mathbf{q}]v'^{\rho}(\mathbf{p},t'|\mathbf{X})v'^{\sigma}(\mathbf{q},t'|\mathbf{X}),$$

is represented by figure 8(b).

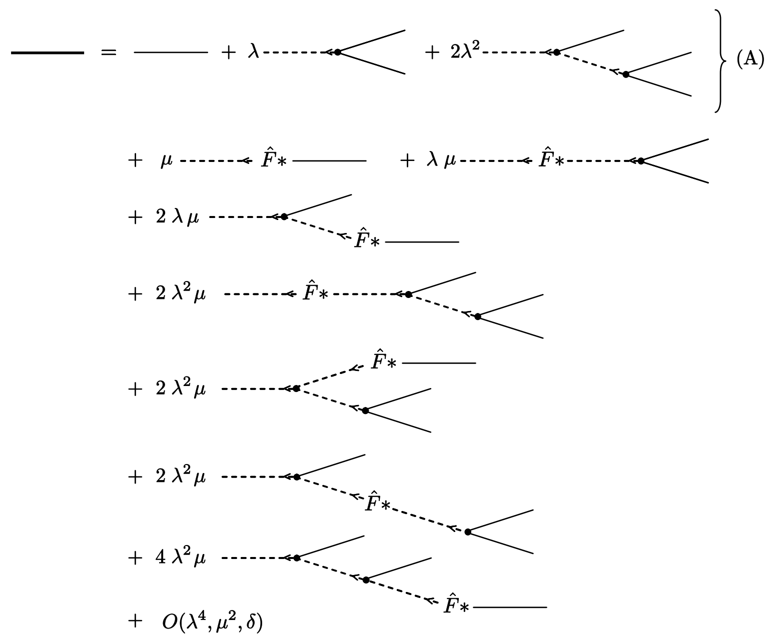


Figure 9: Perturbative expansion of the total velocity fluctuation; We have shown the diagrammatic expansion up to the order of $\lambda^2\,\mu^1$ for simplicity. (A) is a group of diagrams free from both μ and δ, which is defined as the basic field ${}^{B}\mathbf{v}$.

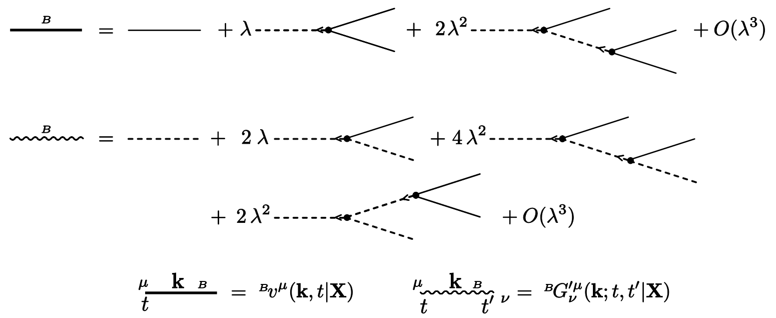


Figure 10: Perturbative expansion of the basic field and its one-wave-number response.

The external force causes the infinitesimal variation of ${}^{B}\mathbf{v}$ from its original value. This variation may be expanded in terms of $\boldsymbol{\Upsilon}$ as

$$ {}^{B}v^{\mu}(\mathbf{k},t|\mathbf{X}\|\boldsymbol{\Upsilon}) = {}^{B}v^{\mu}(\mathbf{k},t|\mathbf{X}) + \int \mathrm{d}vol_{k'} \int_{-\infty}^{t} \mathrm{d}t'\ {}^{B}G'^{\mu}_{\nu}(\mathbf{k},t;\mathbf{k}',t'|\mathbf{X})\Upsilon^{\nu}(\mathbf{k}',t'|\mathbf{X}) + O(\boldsymbol{\Upsilon}^2). \tag{7.97} $$

${}^{B}\mathbf{G}'$ is the response function against infinitesimal disturbance $\boldsymbol{\Upsilon}$, which has two wave numbers $\mathbf{k}$ and $\mathbf{k}'$; the former for response and the latter for disturbance. Here we define the one-wave-number response by

$$ {}^{B}G'^{\mu}_{\nu}(\mathbf{k};t,t'|\mathbf{X}) \equiv \int \mathrm{d}vol_{k'}\ {}^{B}G'^{\mu}_{\nu}(\mathbf{k},t;\mathbf{k}',t'|\mathbf{X}), \tag{7.98} $$

which is governed by

$$ \begin{aligned} \hat{L}^{\mu}_{\rho}\,{}^{B}G'^{\rho}_{\nu}(\mathbf{k};t,t'|\mathbf{X}) =& P^{\mu}_{\nu}(\mathbf{k},t|\mathbf{X})\delta(t-t') \\ &+ 2\lambda\frac{1}{i}M^{\mu}_{\rho\sigma}[\mathbf{k};\mathbf{p},\mathbf{q}]\,{}^{B}v^{\rho}(\mathbf{p},t|\mathbf{X})\,{}^{B}G'^{\sigma}_{\nu}(\mathbf{q};t,t'|\mathbf{X}). \end{aligned} \tag{7.99} $$

The diagrammatic representations for the basic field ${}^{B}\mathbf{v}$ and its one-wave-number response ${}^{B}\mathbf{G}'$ are given in figure 10. Now we can expand the velocity fluctuation on the basis of $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{G}}$. The pressure fluctuation is expressed by (7.54) so that it is also expanded with $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{G}}$, and thus all of the fluctuating quantities are

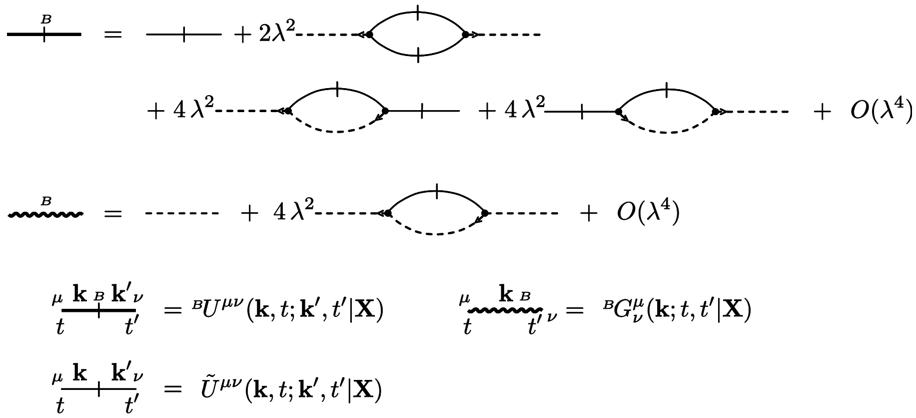


Figure 11: Perturbative expansion of the binary correlation and the averaged response of the basic field

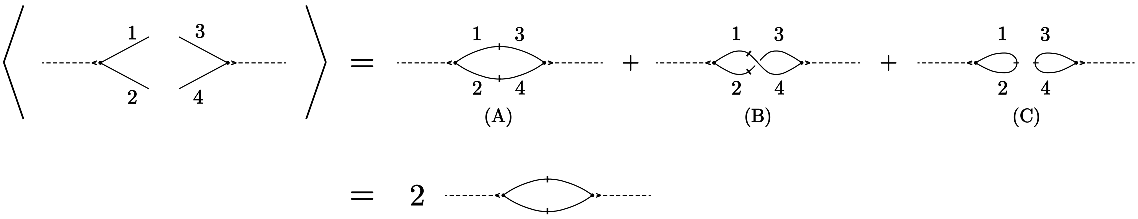


Figure 12: An example of self correlation of a diagram in figure 10 is shown. According to the Normality assumption, we have three choices of pairs; $(1,3)(2,4)$, $(1,4)(2,3)$, and $(1,2)(3,4)$, each of which yields the diagrams (A), (B) and (C). Reminding the vertex is symmetric under exchange of its two tails, we notice that (A) and (B) are identical to each other. The rest (C) is a self-connected-loop diagram which automatically vanishes under the given assumptions.

expanded in terms of $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{G}}$.

Here we assume that the velocity fluctuation is Gaussian-random factor in the far past, which indicates that the bare field itself is also Gaussian-random equivalently at arbitrary time. Under this assumption, arbitrary correlation of $\tilde{\mathbf{v}}$ is to be represented in terms of its binary correlations. For example, the fourth-order correlation of $\tilde{\mathbf{v}}$ is calculated as

$$\begin{aligned}
&\langle \tilde{v}^\alpha(\mathbf{k}^{(1)}, t^{(1)})\tilde{v}^\beta(\mathbf{k}^{(2)}, t^{(2)})\tilde{v}^\gamma(\mathbf{k}^{(3)}, t^{(3)})\tilde{v}^\delta(\mathbf{k}^{(4)}, t^{(4)})\rangle \\
&\quad=\langle \tilde{v}^\alpha(\mathbf{k}^{(1)}, t^{(1)})\tilde{v}^\beta(\mathbf{k}^{(2)}, t^{(2)})\rangle\ \langle \tilde{v}^\gamma(\mathbf{k}^{(3)}, t^{(3)})\tilde{v}^\delta(\mathbf{k}^{(4)}, t^{(4)})\rangle \\
&\qquad+\langle \tilde{v}^\alpha(\mathbf{k}^{(1)}, t^{(1)})\tilde{v}^\gamma(\mathbf{k}^{(3)}, t^{(3)})\rangle\ \langle \tilde{v}^\beta(\mathbf{k}^{(2)}, t^{(2)})\tilde{v}^\delta(\mathbf{k}^{(4)}, t^{(4)})\rangle \\
&\qquad+\langle \tilde{v}^\alpha(\mathbf{k}^{(1)}, t^{(1)})\tilde{v}^\delta(\mathbf{k}^{(4)}, t^{(4)})\rangle\ \langle \tilde{v}^\beta(\mathbf{k}^{(2)}, t^{(2)})\tilde{v}^\gamma(\mathbf{k}^{(3)}, t^{(3)})\rangle.
\end{aligned} \tag{7.100}$$

In general, for even positive integer n, the correlation of the nth order is reduced to

$$\begin{aligned}
&\langle \tilde{v}^\alpha(\mathbf{k}^{(1)}, t^{(1)})\tilde{v}^\beta(\mathbf{k}^{(2)}, t^{(2)})\cdots\tilde{v}^\eta(\mathbf{k}^{(n-1)}, t^{(n-1)})\tilde{v}^\zeta(\mathbf{k}^{(n)}, t^{(n)})\rangle \\
&\quad=\langle \tilde{v}^\alpha(\mathbf{k}^{(1)}, t^{(1)})\tilde{v}^\beta(\mathbf{k}^{(2)}, t^{(2)})\rangle\cdots\langle \tilde{v}^\eta(\mathbf{k}^{(n-1)}, t^{(n-1)})\tilde{v}^\zeta(\mathbf{k}^{(n)}, t^{(n)})\rangle \\
&\qquad+\langle \tilde{v}^\alpha(\mathbf{k}^{(1)}, t^{(1)})\tilde{v}^\zeta(\mathbf{k}^{(n)}, t^{(n)})\rangle\cdots\langle \tilde{v}^\eta(\mathbf{k}^{(n-1)}, t^{(n-1)})\tilde{v}^\beta(\mathbf{k}^{(2)}, t^{(2)})\rangle \\
&\qquad+\cdots\ \text{all the rest of combinations}\quad(\text{for even}\, n),
\end{aligned} \tag{7.101}$$

while it vanishes for odd n. We write this bare binary correlation $\langle \tilde{v}^\mu(\mathbf{k}, t|\mathbf{X})\tilde{v}^\nu(\mathbf{k}', t'|\mathbf{X})\rangle$ as $\tilde{U}^{\mu\nu}(\mathbf{k}, t; \mathbf{k}', t'|\mathbf{X})$. Thus we can represent any correlations in terms of the bare correlation $\tilde{\mathbf{U}}$ and the bare propagator $\tilde{\mathbf{G}}$.

Following the Normality assumption of the bare field, we obtain the binary correlation and the averaged one-wave-number response of the basic field as the series expansions shown in figure 11 [24], each of which are

[24] As an example, let us calculate the self correlation of λ^1-order term in figure 10 (the second term of the right-hand side of ${}^B\mathbf{v}$'s expansion). In the diagrammatic representation, each binary correlation of the bare field is obtained by connecting a

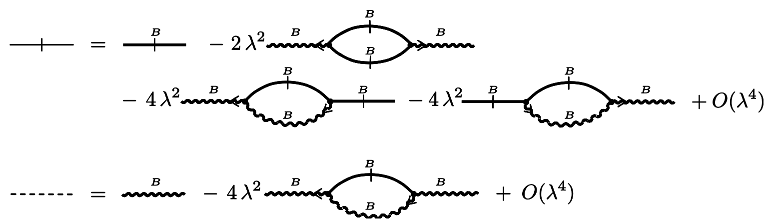


Figure 13: Reverse expansions of the basic quantities

defined by

$$^{B}U^{\mu\nu}(\mathbf{k},t;\mathbf{k}',t'|\mathbf{X}) = \langle {}^{B}v^{\mu}(\mathbf{k},t|\mathbf{X})\, {}^{B}v^{\nu}(\mathbf{k}',t'|\mathbf{X})\rangle \tag{7.102}$$

and

$$^{B}G^{\mu}_{\nu}(\mathbf{k};t,t'|\mathbf{X}) = \langle\, {}^{B}G'^{\mu}_{\nu}(\mathbf{k};t,t'|\mathbf{X})\rangle. \tag{7.103}$$

Unlike the bare counterparts, these basic-field quantities include infinite order of λ terms and exactly incorporate the strong nonlinear effect of the Navier-Stokes turbulence. What follows is to utilize these ${}^{B}\mathbf{U}$ and ${}^{B}\mathbf{G}$ to incorporate the strong nonlinearity into the simple perturbation analysis. We will conduct this by replacing the bases of the expansion $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{G}}$ by ${}^{B}\mathbf{U}$ and ${}^{B}\mathbf{G}$. The later procedure may be summarized as the following steps.

1. Due to the Gaussian nature of the bare field, arbitrary correlation F of our concern can be expanded in terms of λ, μ and δ and is represented as a series expansion $F[\tilde{\mathbf{U}},\tilde{\mathbf{G}};\lambda,\mu,\delta]$.

2. On the other hand, the basic quantities ${}^{B}\mathbf{U}$ and ${}^{B}\mathbf{G}$ are to be expanded only by λ and are represented as series expansions ${}^{B}\mathbf{U}[\tilde{\mathbf{U}},\tilde{\mathbf{G}};\lambda]$ and ${}^{B}\mathbf{G}[\tilde{\mathbf{U}},\tilde{\mathbf{G}};\lambda]$ (see figure 11).

3. Then we *invert* these expansion and obtain the series expansions of bare quantities $\tilde{\mathbf{U}}[{}^{B}\mathbf{U},{}^{B}\mathbf{G};\lambda]$ and $\tilde{\mathbf{G}}[{}^{B}\mathbf{U},{}^{B}\mathbf{G};\lambda]$ (see figure 13).

4. By substituting these inverted expansions into $F[\tilde{\mathbf{U}},\tilde{\mathbf{G}};\lambda,\mu,\delta]$, we obtain the series expansion: $F[{}^{B}\mathbf{U},{}^{B}\mathbf{G};\lambda,\mu,\delta] = F[\tilde{\mathbf{U}}\{{}^{B}\mathbf{U},{}^{B}\mathbf{G};\lambda\},\tilde{\mathbf{G}}\{{}^{B}\mathbf{U},{}^{B}\mathbf{G};\lambda\};\lambda,\mu,\delta]$.

5. By taking the proper truncation with respect to λ, we obtain the demanded approximation of F.

Under this procedure the nonlinear interaction is firmly incorporated in the series expansion even at the lowest order of λ. We may call this procedure "*renormalization*" following some pioneering works (Wyld 1961, Kraichnan 1977, Kaneda 1981). In this context the key factors ${}^{B}\mathbf{U}$ and ${}^{B}\mathbf{G}$ are referred to as the renormalized correlation and propagator respectively. We should recognize an important fact of the renormalization we have introduced here; we employed ${}^{B}\mathbf{U}$ and ${}^{B}\mathbf{G}$ which include only the contributions from λ as the renormalized quantities. Thus the *infinite partial summation* has been done only for the nonlinearity and we merely performed simple perturbative expansions for μ and δ, which may lead us to undesirable result as anisotropy or inhomogeneity become prominent. In this sense we call our procedure *partial renormalization.*

In the diagrammatic expansions in figures 11 and 13, readers may realize that they lack of some diagrams which have loops each of which is closed with only one vertex. These diagrams, however, automatically vanish due to the assumptions we have set up (see appendix F).

pair of thin lines. Here we label the four thin lines as 1-4. According to the Normality assumption, we have three choices of pairs; $(1,3)(2,4)$, $(1,4)(2,3)$, and $(1,2)(3,4)$ each of which yields the diagrams (A), (B) and (C) in figure 12. Reminding the vertex is symmetric under exchange of its two tails, we notice that (A) and (B) are identical to each other. The rest (C) is a self-connected-loop diagram which automatically vanish under our assumptions (see appendix F). Finally we obtain a λ^2-order diagram (the second term of ${}^{B}\mathbf{U}$'s expansion in figure 11).

7.7 Statistical properties of the basic field

So far we have developed a series of methods to express the statistical quantities in terms of ${}^{B}\mathbf{U}$ and ${}^{B}\mathbf{G}$. In order to obtain further insights, we need more specific information on the basic field. For the later convenience, we assume the basic field as isotropic and parity-symmetric in the static-metric representation, which may be partly justified by the fact that the basic field is governed by the same equation as that of the isotropic field as is indicated by (7.95) (we will see another justification in §7.8). Under this assumption, the basic-field quantities ${}^{B}\mathbf{U}$ and ${}^{B}\mathbf{G}$ reduce to

$$ {}^{B}U^{IJ}(\check{\mathbf{k}};t,t'|\mathbf{X}) \equiv \int \mathrm{d}vol_{\check{k}'}\,{}^{B}U^{IJ}(\check{\mathbf{k}},t;\check{\mathbf{k}}',t'|\mathbf{X}) = P^{IJ}(\check{\mathbf{k}})\,{}^{B}Q(k;t,t'|\mathbf{X}), \tag{7.104} $$

$$ {}^{B}G^{I}_{J}(\check{\mathbf{k}};t,t'|\mathbf{X}) = P^{I}_{J}(\check{\mathbf{k}})\,{}^{B}G(k;t,t'|\mathbf{X}), \tag{7.105} $$

where ${}^{B}Q$ and ${}^{B}G$ are isotropic scalar functions. k is the norm of the wave-number vector $\mathbf{k}$ and is represented in the same form in both of mean-Lagrangian and static-metric frames as

$$ k = \sqrt{g^{\mu\nu}k_{\mu}k_{\nu}} = \sqrt{g^{IJ}\check{k}_{I}\check{k}_{J}}. \tag{7.106} $$

The relations (7.83) and (7.87) yield the correlation and propagator in the mean-Lagrangian representation as follows;

$$ \begin{aligned} {}^{B}U^{\mu\nu}(\mathbf{k};t,t'|\mathbf{X}) &= {}^{X}a^{\mu}_{I}(t)\,{}^{X}a^{\nu}_{J}(t')\,{}^{B}U^{IJ}(\check{\mathbf{k}};t,t'|\mathbf{X}) \\ &= {}^{X}a^{\mu}_{I}(t)\,{}^{X}a^{\nu}_{J}(t')P^{IJ}(\check{\mathbf{k}}){}^{B}Q(k;t,t'|\mathbf{X}) \\ &= {}^{X}a^{\mu}_{I}(t)\Lambda^{\nu}_{\rho}(t';t|\mathbf{X}){}^{X}a^{\rho}_{J}(t)P^{IJ}(\check{\mathbf{k}}){}^{B}Q(k;t,t'|\mathbf{X}) \\ &= \Lambda^{\nu}_{\rho}(t';t|\mathbf{X})P^{\mu\rho}(\mathbf{k},t|\mathbf{X}){}^{B}Q(k;t,t'|\mathbf{X}), \end{aligned} \tag{7.107} $$

$$ \begin{aligned} {}^{B}G^{\mu}_{\nu}(\mathbf{k};t,t'|\mathbf{X}) &= {}^{X}a^{\mu}_{I}(t){}^{X}a^{J}_{\nu}(t')\,{}^{B}G^{I}_{J}(\check{\mathbf{k}};t,t'|\mathbf{X}) \\ &= {}^{X}a^{\mu}_{I}(t){}^{X}a^{J}_{\nu}(t')P^{I}_{J}(\check{\mathbf{k}})\,{}^{B}G(k;t,t'|\mathbf{X}) \\ &= {}^{X}a^{\mu}_{I}(t)\bar{\Lambda}^{\rho}_{\nu}(t';t|\mathbf{X}){}^{X}a^{J}_{\rho}(t)P^{I}_{J}(\check{\mathbf{k}})\,{}^{B}G(k;t,t'|\mathbf{X}) \\ &= \bar{\Lambda}^{\rho}_{\nu}(t';t|\mathbf{X})P^{\mu}_{\rho}(\mathbf{k},t|\mathbf{X})\,{}^{B}G(k;t,t'|\mathbf{X}), \end{aligned} \tag{7.108} $$

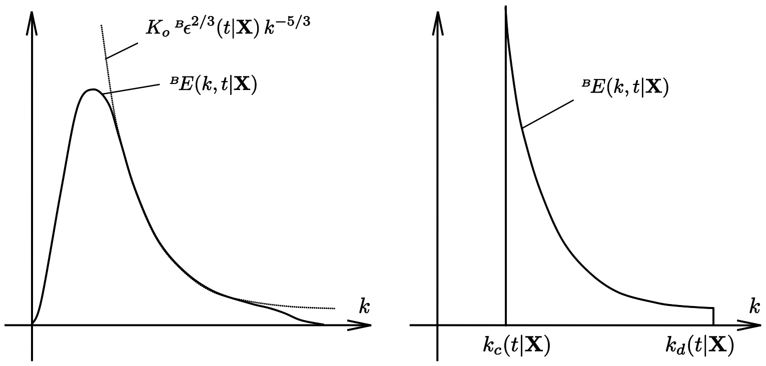
where the relations (G·4) and (G·7) in appendix G are used. We note that ${}^{X}\mathbf{a}$'s do not appear in both (7.107) and (7.108). Now any types of correlations are expressed in terms of a set of scalar functions ${}^{B}Q$ and ${}^{B}G$, and we are about to analyze these two dynamical variables.

By applying the renormalized-perturbation method explained in §7.6 to equations for ${}^{B}Q$ and ${}^{B}G$, one can obtain a closed system of equations for them and, consequently, one may close the whole system involving ${}^{B}Q$ and ${}^{B}G$ as dynamical variables. However, these equations are the same as what had already been obtained for the homogeneous isotropic case using DIA (Kraichnan 1959), and it is wellknown that the DIA equations cannot reproduce the proper features of developed turbulence at high Reynolds number such as the $-5/3$-power law of the energy spectrum in the inertial range. Thus we take a tentative method which has been proposed by Yoshizawa (1978, 1984). He had introduced an artificial damping around the infra-red region to remove the infra-red divergence in the equation of the propagator and reconstructed the DIA equations to be consistent with Kolmogorov's power law. By using the modified DIA equations, spectral forms were successfully obtained for the inertial range with numerical coefficients. In this thesis, we make full use of his result.

We expect the basic field to behave as the homogeneous isotropic field at high Reynolds number. In such a case there exists a characteristic $-5/3$-power law in the energy spectrum ${}^{B}E(k,t|\mathbf{X}) \equiv 4\pi k^{2}\sigma(k,t|\mathbf{X})$ as is shown in the figure 14(a). Thus we simplify the energy spectrum as figure 14(b); namely we replace the detailed structure of the energy-containing and dissipative ranges with simple cut-off wave numbers $k_c(t|\mathbf{X})$ and $k_d(t|\mathbf{X})$. Besides we assume the exponential decay for the time evolution of both ${}^{B}Q$ and ${}^{B}G$. Thus the inertial-range forms of ${}^{B}Q$ and ${}^{B}G$ are given by

$$ {}^{B}Q(k;t,t'|\mathbf{X}) = \sigma(k,t|\mathbf{X})\exp\left[-\omega(k,t|\mathbf{X})|t-t'|\right], \tag{7.109} $$

$$ {}^{B}G(k;t,t'|\mathbf{X}) = \exp\left[-\omega(k,t|\mathbf{X})(t-t')\right], \tag{7.110} $$



(a) Energy spectrum of the basic field (b) Simplified spectrum

Figure 14: Simplification of the energy spectrum; (a) the outlines of the energy and dissipation spectra are illustrated. (b) We avoid to deal with the detailed structures of the energy-containing and dissipation ranges, but instead, we simply replace these two ranges with a simple cutoff wavenumbers k_c and k_d. Due to this simplification, the basic field is characterized by only three factors; k_c, k_d and ${}^B\epsilon$.

$$\sigma(k,t|\mathbf{X}) = C_\sigma k^{-11/3}\,{}^B\epsilon^{2/3}(t|\mathbf{X}), \tag{7.111}$$

$$\omega(k,t|\mathbf{X}) = C_\omega k^{2/3}\,{}^B\epsilon^{1/3}(t|\mathbf{X}), \tag{7.112}$$

where C_σ is proportional to the Kolmogorov constant, C_ω characterizes the decaying time due to turbulence. ${}^B\epsilon$ is the dissipation rate of the basic-field energy. In the high-Reynolds-number limit ($k_c \to 0$ and $k_d \to \infty$), the modified DIA with the above simplification gives the following set of constants (Yoshizawa 1978);

$$C_\sigma = 0.118\ , \quad C_\omega = 0.419. \tag{7.113}$$

This yields the Kolmogorov's constant $K_o = 4\pi C_\sigma = 1.48$ which is consistent with experimental results $K_o = 1.62 \pm 0.17$ (Sreenivasan 1995). In this approximation all the spectral information of the basic field is characterized only by three factors, k_c, k_d and ${}^B\epsilon$, and this fact enables us to investigate the basic-field quantities in a remarkably simple way. In reality, the actual number of degrees of freedom is only two instead of three, since ${}^B\epsilon$ is theoretically given by ${}^B\epsilon = 2\nu \int \mathrm{d}vol_k k^2\,{}^BQ(k;t,t|\mathbf{X})$ for arbitrary spectral form of ${}^BQ(k;t,t|\mathbf{X})$. Under the approximation, it is reduced to

$$\begin{aligned}
{}^B\epsilon &= 2\nu \int \mathrm{d}vol_k k^2\,{}^BQ(k;t,t|\mathbf{X}) \\
&= 2\nu \int_{k_c}^{k_d} \mathrm{d}k\, 4\pi k^2\, C_\sigma\,{}^B\epsilon^{2/3} k^{-11/3} \\
&= 8\pi C_\sigma \nu\,{}^B\epsilon^{2/3} \int_{k_c}^{k_d} \mathrm{d}k\, k^{1/3} \\
&= 6\pi C_\sigma \nu\,{}^B\epsilon^{2/3} \left(k_d^{4/3} - k_c^{4/3}\right)\ ,
\end{aligned} \tag{7.114}$$

and thus we obtain

$${}^B\epsilon = (6\pi C_\sigma)^3\ \nu^3 \left(k_d^{4/3} - k_c^{4/3}\right)^3 . \tag{7.115}$$

Thus all the properties of the basic field can be described by any two of the three factors k_c, k_d and ${}^B\epsilon$.

7.8 Fixing of **A**

In (7.107) and (7.108), there is still uncertainty of **A** introduced in §7.5 as an extra dynamical variable through $\boldsymbol{\Lambda}$ and $\bar{\boldsymbol{\Lambda}}$ (see (7.84) and (7.88)). Since the whole procedure of the renormalization discussed above should presuppose determined **A**, we have to prescribe **A** in an independent manner of §7.6.

Here we pay attention to the global motion. There exists vortical motion prescribed by $\boldsymbol{\Theta}$ in the large scale from which the energy cascades into the smaller scale. The motion in the small scale is expected to be affected by the global rotation through the energy cascading process. If the global motion has rotation in the static-metric frame, the isotropic assumption represented as (7.104) and (7.105) cannot be accepted since both ${}^{B}\mathbf{U}$ and ${}^{B}\mathbf{G}$ would include some anisotropy owing to the global rotation. On the other hand, if we take the static metric frame to be rotating together with the mean flow, the mean flow has no rotation relative to the static metric frame and the basic field is expected to be independent from the global rotation in the static metric representation. Therefore we choose the static-metric frame rotating with the mean flow, which is equivalent to put $\mathbf{A}$ as zero (see appendix D);

$$A_{\mu\nu}(t|\mathbf{X}) = 0. \tag{7.116}$$

One may think the static-metric frame should not be rotating against the inertial frame. In this choice the small-scale turbulence is supposed to be more strongly affected by the inertia than the rotation of the large eddies which drives turbulence motion. However, in reality, the inertial effect such as Coriolis force decreases as the length scale gets small. Thus the author thinks that the mean-flow determines the rotational state of the small-scale motion at a primary stage; the basic field should be rotating with the mean flow.

8 Application to the Reynolds stress

Here we apply the above method to the Reynolds stress as a specific example. Using the HID representation, the Reynolds stress is denoted as

$$R^{\mu\nu}(t|\mathbf{X}) = \langle v'^{\mu}(\boldsymbol{\xi}, t|\mathbf{X}) v'^{\nu}(\boldsymbol{\xi}, t|\mathbf{X})\rangle = \int \mathrm{d}vol_k \int \mathrm{d}vol_{k'} \langle v'^{\mu}(\mathbf{k}, t|\mathbf{X}) v'^{\nu}(\mathbf{k}', t|\mathbf{X})\rangle. \tag{8.1}$$

By substituting $\delta\mathbf{x} = \mathbf{X}$ and replacing all the perturbative parameters as unity (δ, μ, λ=1), we obtain the Reynolds stress in the real space; namely $R^{\mu\nu}(\mathbf{x}, t) = R^{\mu\nu}(t|\mathbf{x})$. Thus we have to investigate the total binary correlation $\langle v'^{\mu}(\mathbf{k}, t|\mathbf{X}) v'^{\nu}(\mathbf{k}', t|\mathbf{X})\rangle$. Following §7.6, we expand the above correlation by $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{G}}$ as illustrated in figure 15 where each of M.R. denotes the mirror-reversed counterpart of the left diagram. By substituting the reverse expansions shown in figure 13 into the expansion in figure 15, we obtain the renormalized expansion shown in figure 16 where some loops arising from λ terms have been effectively absorbed into the thick lines of the no-loop diagrams.

Unfortunately, we cannot determine clearly the proper order of λ for the truncation of the expansion, which is expected to be a proper approximation, since the nonlinearity cannot be assumed as small in turbulence phenomena. However, it is wellknown empirically that the truncation at even lower order still retains important features of real phenomena. Thus it is still valuable to truncate at the lowest order of λ to extract the major contributions from μ- and δ- related terms. Following this idea, we eliminate one-loop corrections below. Thus the renormalized expansion in figure 16 results in figure 17.

The present analysis includes μ^1-, μ^2- and $\mu^1\delta^2$-order terms. Since $\mu^2\delta^2$-order analysis produces an enormous number of terms, these higher-order analyses are left for the future works. Note that δ^1-order terms automatically vanish since these terms include odd numbers of wavenumbers which have no contribution to the wavenumber integrations.

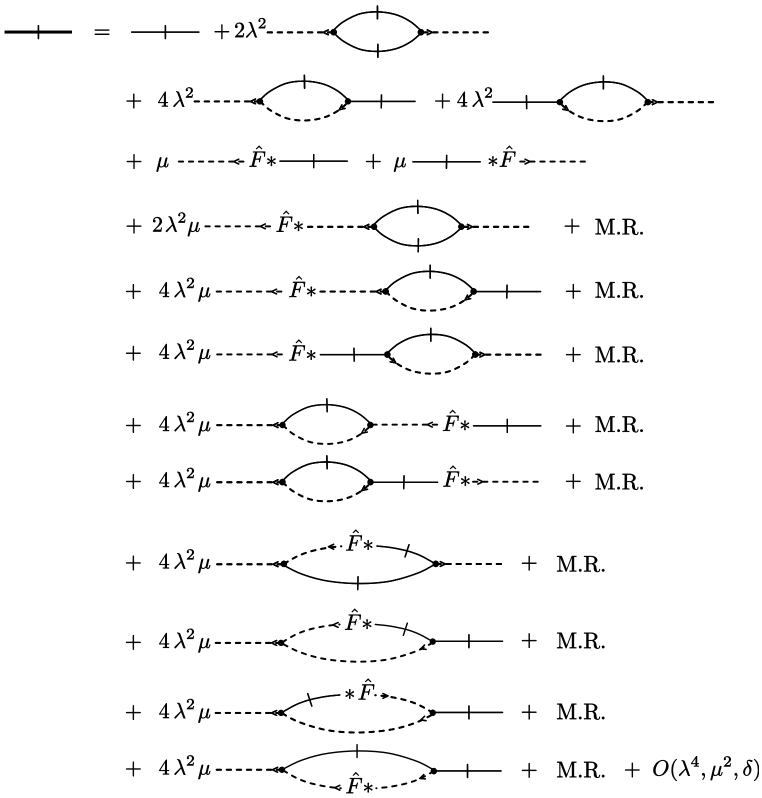


Figure 15: Perturbative expansion for the binary correlation. Each M.R. denotes the mirror-reversed counterpart of the left diagram.

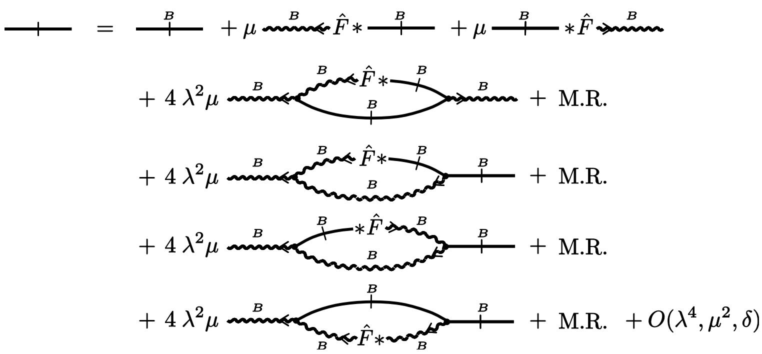


Figure 16: Renormalized expansion of the binary correlation up to first order of μ.

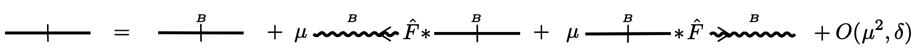


Figure 17: Truncated renormalized expansion for the binary correlation.

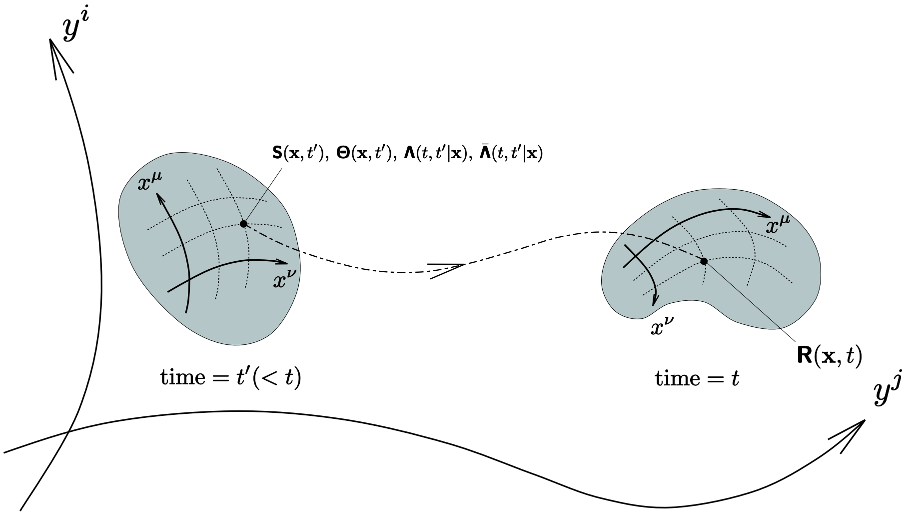


Figure 18: Configuration of the point of integrand in (8.2) is depicted with a mean-flow element and the mean-Lagrangian frame on it; In explaining $\mathbf{R}$, history of $\mathbf{S}$, $\mathbf{\Theta}$, $\mathbf{\Lambda}$ and $\bar{\mathbf{\Lambda}}$ on the mean-flow particle is considered over its trajectory. Note that the trajectory is not a point in general coordinate system $\{\mathbf{y}\}$.

8.1 Temporal nonlocality of the Reynolds stress

By substituting into (8.1) the renormalized truncated expansion obtained in the previous section, we can calculate the Reynolds stress as follows;

$$\begin{aligned}
R^{\mu\nu}(\mathbf{x},t) =& \frac{2}{3} g^{\mu\nu}(\mathbf{x},t) \int \mathrm{d}vol_k\, {}^{B}Q(k;t,t|\mathbf{x}) \\
&- \frac{7}{15}\int_{-\infty}^{t} \mathrm{d}t' \left\{ \Lambda^{\mu}_{\rho}(t;t'|\mathbf{x}) \bar{\Lambda}^{\sigma}_{\kappa}(t;t'|\mathbf{x}) g^{\kappa\nu}(\mathbf{x},t) + \Lambda^{\nu}_{\rho}(t;t'|\mathbf{x}) \bar{\Lambda}^{\sigma}_{\kappa}(t;t'|\mathbf{x}) g^{\kappa\mu}(\mathbf{x},t) \right\} \\
&\qquad\qquad \times \left(\frac{1}{2} S^{\rho}_{\sigma} + \Theta^{\rho}{}_{\sigma} \right)(\mathbf{x},t') \int \mathrm{d}vol_k\, {}^{B}G(k;t,t'|\mathbf{x})\, {}^{B}Q(k;t,t'|\mathbf{x}) \\
&- \frac{1}{10}\int_{-\infty}^{t} \mathrm{d}t' \left\{ g^{\mu\rho}(\mathbf{x},t) S^{\nu}_{\rho}(\mathbf{x},t') + g^{\nu\rho}(\mathbf{x},t) S^{\mu}_{\rho}(\mathbf{x},t') \right\} \\
&\qquad\qquad \times \int \mathrm{d}vol_k\, {}^{B}G(k;t,t'|\mathbf{x})\, {}^{B}Q(k;t,t'|\mathbf{x}) \\
&- \frac{1}{15}\int_{-\infty}^{t} \mathrm{d}t' \left\{ g^{\mu\rho}(\mathbf{x},t) S^{\nu}_{\rho}(\mathbf{x},t') + g^{\nu\rho}(\mathbf{x},t) S^{\mu}_{\rho}(\mathbf{x},t') \right\} \\
&\qquad\qquad \times \int \mathrm{d}vol_k\, {}^{B}G(k;t,t'|\mathbf{x}) k \frac{\partial}{\partial k}\, {}^{B}Q(k;t,t'|\mathbf{x}) \\
&+ O(\mu^2, \mu\delta^2),
\end{aligned} \tag{8.2}$$

where up to μ^1-order diagrams are explicitly shown for simplicity. Note that the present analysis includes even higher-order effect of δ. This result is very important in the following two senses.

First, the time evolutions of the mean-flow properties ($\mathbf{S}$, $\mathbf{\Theta}$, and $\mathbf{\Lambda}$) and the fluctuation properties (${}^{B}Q$ and ${}^{B}G$) coexist in the time integration. Thus (8.2) can illustrate the system where the time scales of the mean flow and the fluctuation are not separated. In this sense (8.2) is more generalized form of the conventional algebraic Reynolds stress models where the Reynolds stress is determined only by the present information [25]. This is clearly the generalization of the RPT of homogeneous turbulence which explains phenomena in

[25] The counterpart of (8.2) in TSDIA is (4.12), where the history effect of the velocity gradient is not considered. To be precise, Lagrangian derivatives of the velocity gradient appears in higher order analysis of TSDIA. Thus, TSDIA may include the history effects as time-derivative expansions. However, because of the time-scale separation between mean flow and fluctuation, TSDIA cannot perform the time integration of mean-flow quantities and fluctuation simultaneously.

terms of multiple-time quantities of fluctuation [26].

Second, all the integrals in the (8.2) are the convected integrations on the mean-flow particle labelled by $\mathbf{x}$. Thus, in general coordinate representation, (8.2) integrates the history of $\mathbf{S}$, $\mathbf{\Theta}$, $\mathbf{\Lambda}$ and $\bar{\mathbf{\Lambda}}$ on the nonlocal trajectory in the coordinate space in generally covariant manner (see figure 18). Thus (8.2) successfully describe the non-local contribution of the mean-flow trajectory to the Reynolds stress in covariant manner due to the mean-Lagrangian representation, whose importance has been stated in §6.

8.2 Temporal-locality approximation

We expand $\mathbf{S}(\mathbf{x},t')$, $\mathbf{\Theta}(\mathbf{x},t')$, $\mathbf{\Lambda}(t';t|\mathbf{x})$ and $\bar{\mathbf{\Lambda}}(t';t|\mathbf{x})$ around t as follows [27];

$$\begin{aligned}
S^\rho_\sigma(\mathbf{x},t') &= S^\rho_\sigma(\mathbf{x},t) + (t'-t)\frac{\partial S^\rho_\sigma}{\partial t}(\mathbf{x},t) + \frac{1}{2}(t'-t)^2\frac{\partial^2 S^\rho_\sigma}{\partial t^2}(\mathbf{x},t) + \cdots\ , \\
\Theta^\rho{}_\sigma(\mathbf{x},t') &= \Theta^\rho{}_\sigma(\mathbf{x},t) + (t'-t)\frac{\partial \Theta^\rho{}_\sigma}{\partial t}(\mathbf{x},t) + \frac{1}{2}(t'-t)^2\frac{\partial^2 \Theta^\rho{}_\sigma}{\partial t^2}(\mathbf{x},t) + \cdots\ , \\
\Lambda^\rho_\sigma(t';t|\mathbf{x}) &= \delta^\rho_\sigma - \frac{1}{2}(t'-t)S^\rho_\sigma(\mathbf{x},t) - \frac{1}{4}(t'-t)^2\left(S^\rho_\alpha S^\alpha_\sigma + \frac{\partial S^\rho_\sigma}{\partial t}\right)(\mathbf{x},t) + \cdots\ , \\
\bar{\Lambda}^\rho_\sigma(t';t|\mathbf{x}) &= \delta^\rho_\sigma + \frac{1}{2}(t'-t)S^\rho_\sigma(\mathbf{x},t) - \frac{1}{4}(t'-t)^2\left(S^\rho_\alpha S^\alpha_\sigma + \frac{\partial S^\rho_\sigma}{\partial t}\right)(\mathbf{x},t) + \cdots\ .
\end{aligned} \tag{8.3}$$

By substituting (8.3) into (8.2), we obtain

$$\begin{aligned}
R^{\mu\nu} = \frac{2}{3}\,{}^B\!K g^{\mu\nu} - \nu_T S^{\mu\nu} &+ \gamma_t\left(\frac{\partial S^{\mu\nu}}{\partial t} + S^\mu_\rho S^{\nu\rho}\right) \\
&+ N_I\ \mathbf{S}\cdot\mathbf{S}g^{\mu\nu} \\
&+ N_{II}\ \mathbf{\Theta}\cdot\mathbf{\Theta}g^{\mu\nu} \\
&+ N_{III}\ S^\mu_\rho S^{\nu\rho} \\
&+ N_{IV}\ \Theta^\mu{}_\rho\Theta^{\nu\rho} \\
&+ N_V\left(S^\mu_\rho\Theta^{\nu\rho} + S^\nu_\rho\Theta^{\mu\rho}\right) \\
&+ D_I\left(S^{\mu;\nu\rho}_\rho + S^{\nu;\mu\rho}_\rho\right) \\
&+ D_{II}\ S^{\alpha\beta}{}_{;\alpha\beta}g^{\mu\nu} \\
&+ D_{III}\ (\Theta^\mu{}_\rho{}^{;\nu\rho} + \Theta^\mu{}_\rho{}^{;\nu\rho}) \\
&+ D_{IV}\ \triangle S^{\mu\nu},
\end{aligned} \tag{8.4}$$

where up to second-order terms of $\mathbf{S}$ and $\mathbf{\Theta}$ are retained[28]. ${}^B\!K$, ν_T, γ_t, N_I-N_V and D_I-D_{IV} are all scalars explained by the fluctuation properties as follows [29];

[26] Another profound connection between the present and homogeneous-turbulence theories is discussed in §9.2, which indicates that the present theory may illustrate the interaction between fluctuation and the mean flow in almost the same manner as one of the most reliable theory of homogeneous turbulence.

[27] Higher-order terms in (8.3) produce cubic terms and even higher-order terms. For example, the first-order derivative of $\mathbf{S}$ can be evaluated as $\mathbf{S}^2$ since

$$g_{\mu\nu}\frac{\partial S^{\mu\nu}}{\partial t} = -\mathbf{S}\cdot\mathbf{S},$$

while the second-order derivative can be evaluated as third since

$$g_{\mu\nu}\frac{\partial^2 S^{\mu\nu}}{\partial t^2} = -2S^\alpha_\beta S^\beta_\gamma S^\gamma_\alpha + S_{\mu\nu}\frac{\partial S^{\mu\nu}}{\partial t}.$$

[28] Retaining the higher order terms of (8.3) produces cubic terms such as $\mathbf{SSS}$, $\mathbf{SS\Theta}$, $\mathbf{S\Theta\Theta}$ and $\mathbf{\Theta\Theta\Theta}$, and even higher-order terms. However, the present work does not include cubic and even higher-order analysis of μ which include higher-order effects of $\mathbf{S}$ and $\mathbf{\Theta}$, so that we retained up to second-order terms of $\mathbf{S}$ and $\mathbf{\Theta}$.

[29] The wave-number integrations were conducted along with the following formulae.

$$\int \mathrm{d}vol_k\,\frac{k_\mu k_\nu}{k^2}\times = \frac{1}{3}g_{\mu\nu}\int \mathrm{d}vol_k\,\times$$

$$ {}^{B}\!K = \int \mathrm{d}vol_k \, {}^{B}\!Q(k;t,t), \tag{8.5} $$

$$ \nu_T = \frac{1}{15}\int_{-\infty}^{t} \mathrm{d}t' \int \mathrm{d}vol_k \left\{ 10\, {}^{B}\!G(k;t,t')\, {}^{B}\!Q(k;t,t') + 2\, {}^{B}\!G(k;t,t') k \frac{\partial}{\partial k} {}^{B}\!Q(k;t,t') \right\}, \tag{8.6} $$

$$ \gamma_t = \frac{1}{15}\int_{-\infty}^{t} \mathrm{d}t' (t-t') \int \mathrm{d}vol_k \left\{ 10\, {}^{B}\!G(k;t,t')\, {}^{B}\!Q(k;t,t') + 2\, {}^{B}\!G(k;t,t') k \frac{\partial}{\partial k} {}^{B}\!Q(k;t,t') \right\}. \tag{8.7} $$

$$ \begin{aligned} N_I = & \frac{2}{35}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t'} \mathrm{d}t''\, {}^{B}\!G(k;t,t')\, {}^{B}\!G(k;t',t'')\, {}^{B}\!Q(k;t,t'') \\ & + \frac{26}{105}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t'} \mathrm{d}t''\, k\frac{\partial {}^{B}\!G}{\partial k}(k;t,t')\, {}^{B}\!G(k;t',t''){}^{B}\!Q(k;t,t'') \\ & + \frac{2}{105}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t'} \mathrm{d}t''\, {}^{B}\!G(k;t,t')\, k\frac{\partial {}^{B}\!G}{\partial k}(k;t',t'')\, {}^{B}\!Q(k;t,t'') \\ & + \frac{2}{35}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t'} \mathrm{d}t''\, k^2\frac{\partial}{\partial k}\left\{\frac{\partial {}^{B}\!G}{\partial k}(k;t,t')\, {}^{B}\!G(k;t',t'')\right\} {}^{B}\!Q(k;t,t'') \\ & + \frac{1}{21}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t} \mathrm{d}t''\, {}^{B}\!G(k;t,t')\, {}^{B}\!G(k;t,t'')\, {}^{B}\!Q(k;t',t'') \\ & + \frac{2}{105}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t} \mathrm{d}t''\, k\frac{\partial {}^{B}\!G}{\partial k}(k;t,t')\, {}^{B}\!G(k;t,t'')\, {}^{B}\!Q(k;t',t'') \\ & + \frac{1}{35}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t} \mathrm{d}t''\, k^2\frac{\partial {}^{B}\!G}{\partial k}(k;t,t')\, \frac{\partial {}^{B}\!G}{\partial k}(k;t,t'')\, {}^{B}\!Q(k;t',t''), \end{aligned} \tag{8.8} $$

$$ \begin{aligned} N_{I\!I} = & -\frac{2}{15}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t'} \mathrm{d}t''\, {}^{B}\!G(k;t,t')\, {}^{B}\!G(k;t',t'')\, {}^{B}\!Q(k;t,t'') \\ & + \frac{1}{15}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t} \mathrm{d}t''\, {}^{B}\!G(k;t,t')\, {}^{B}\!G(k;t,t'')\, {}^{B}\!Q(k;t',t''), \end{aligned} \tag{8.9} $$

$$ \begin{aligned} N_{I\!I\!I} = & -\frac{5}{21}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t'} \mathrm{d}t''\, {}^{B}\!G(k;t,t')\, {}^{B}\!G(k;t',t'')\, {}^{B}\!Q(k;t,t'') \\ & - \frac{11}{35}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t'} \mathrm{d}t''\, k\frac{\partial {}^{B}\!G}{\partial k}(k;t,t')\, {}^{B}\!G(k;t',t''){}^{B}\!Q(k;t,t'') \\ & - \frac{13}{105}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t'} \mathrm{d}t''\, {}^{B}\!G(k;t,t')\, k\frac{\partial {}^{B}\!G}{\partial k}(k;t',t'')\, {}^{B}\!Q(k;t,t'') \\ & - \frac{1}{105}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t'} \mathrm{d}t''\, k^2\frac{\partial}{\partial k}\left\{\frac{\partial {}^{B}\!G}{\partial k}(k;t,t')\, {}^{B}\!G(k;t',t'')\right\} {}^{B}\!Q(k;t,t'') \\ & + \frac{1}{42}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t} \mathrm{d}t''\, {}^{B}\!G(k;t,t')\, {}^{B}\!G(k;t,t'')\, {}^{B}\!Q(k;t',t'') \\ & - \frac{13}{105}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t} \mathrm{d}t''\, k\frac{\partial {}^{B}\!G}{\partial k}(k;t,t')\, {}^{B}\!G(k;t,t'')\, {}^{B}\!Q(k;t',t'') \\ & + \frac{2}{105}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t} \mathrm{d}t''\, k^2\frac{\partial {}^{B}\!G}{\partial k}(k;t,t')\, \frac{\partial {}^{B}\!G}{\partial k}(k;t,t'')\, {}^{B}\!Q(k;t',t''), \end{aligned} \tag{8.10} $$

$$ \int \mathrm{d}vol_k \, \frac{k_\alpha k_\beta k_\gamma k_\delta}{k^4} \times = \frac{1}{15}\left(g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma}\right) \int \mathrm{d}vol_k \times $$

$$\begin{aligned} N_{IV} = &-\frac{4}{15}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t'} \mathrm{d}t'' \; {}^{B}G(k;t,t')\, {}^{B}G(k;t',t'')\, {}^{B}Q(k;t,t'') \\ &+ \frac{2}{15}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t} \mathrm{d}t'' \; {}^{B}G(k;t,t')\, {}^{B}G(k;t,t'')\, {}^{B}Q(k;t',t''), \end{aligned} \tag{8.11}$$

$$\begin{aligned} N_{V} = &\frac{1}{15}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t'} \mathrm{d}t'' \; {}^{B}G(k;t,t')\, {}^{B}G(k;t',t'')\, {}^{B}Q(k;t,t'') \\ &+ \frac{1}{10}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t} \mathrm{d}t'' \; {}^{B}G(k;t,t')\, {}^{B}G(k;t,t'')\, {}^{B}Q(k;t',t''), \end{aligned} \tag{8.12}$$

$$D_{I} = \frac{1}{7}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \; \frac{1}{k^2} {}^{B}G(k;t,t')\, {}^{B}Q(k;t,t'), \tag{8.13}$$

$$D_{II} = \frac{4}{105}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \; \frac{1}{k^2} {}^{B}G(k;t,t')\, {}^{B}Q(k;t,t'), \tag{8.14}$$

$$D_{III} = -\frac{1}{15}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \; \frac{1}{k^2} {}^{B}G(k;t,t')\, {}^{B}Q(k;t,t'), \tag{8.15}$$

$$D_{IV} = -\frac{2}{21}\int \mathrm{d}vol_k \int_{-\infty}^{t} \mathrm{d}t' \; \frac{1}{k^2} {}^{B}G(k;t,t')\, {}^{B}Q(k;t,t'). \tag{8.16}$$

We can easily transform the above result into the general coordinate representation as follows[30];

$$\begin{aligned} R^{ij} = \frac{\partial y^i}{\partial x^\mu}\frac{\partial y^j}{\partial x^\nu} R^{\mu\nu} = \frac{2}{3}\, {}^{B}K g^{ij} - \nu_T S^{ij} + \gamma_t \left(\frac{\mathrm{O}S^{ij}}{\mathrm{O}t} + S^i_a S^{ja} \right) & \\ + N_I \; \mathbf{S}\cdot\mathbf{S} g^{ij} & \\ + N_{II} \; \boldsymbol{\Theta}\cdot\boldsymbol{\Theta} g^{ij} & \\ + N_{III} \; S^i_a S^{ja} & \\ + N_{IV} \; \Theta^i{}_a \Theta^{ia} & \\ + N_{V} \left(S^i_a \Theta^{ja} + S^j_a \Theta^{ja} \right) & \\ + D_{I} \left(S^{i;ja}_a + S^{j;ia}_a \right) & \\ + D_{II} \; S^{ab}{}_{;ab} g^{ij} & \\ + D_{III} \left(\Theta^i{}_a{}^{;ja} + \Theta^i{}_a{}^{;ja} \right) & \\ + D_{IV} \; \triangle S^{ij}. & \end{aligned} \tag{8.17}$$

All the terms on the right-hand side of (8.17) are generally covariant so that we have successfully derived the covariant representation for the Reynolds stress in a temporally-localized form. ${}^{B}K$-, ν_T-, and N-related terms of (8.17) give us a theoretical support for the conventional quadratic-algebraic models. The other

[30] Here the Oldroyd derivative of the strain rate is given by

$$\frac{\mathrm{O}S^{ij}}{\mathrm{O}t} = \frac{\mathrm{D}S^{ij}}{\mathrm{D}t} - V^i{}_{;k} S^{kj} - V^j{}_{;k} S^{ki}.$$

The mean velocity field in the above equation is the one observed in the general coordinate system $\{\mathbf{y}\}$.

terms are unusual in previous works; γ_t-related term and D-related terms represent nonlocal effect in space and time.

Note that (8.17) does not contain any adjustable parameter, unlike the conventional algebraic models. By prescribing the statistical information of fluctuation through ${}^{B}Q$ and ${}^{B}G$, (8.17) can determine the Reynolds stress.

Somebody may say the present result (8.17) could be obtained only by replacing non-covariant terms of (4.16) derived from TSDIA with covariant counterparts. However, we cannot simply replace the $\mathrm{D}/\mathrm{D}t$ term of (4.16) with the Oldroyd derivative, since there exist several covariant time derivatives other than the Oldroyd derivative. Thus we cannot obtain (9.31) from TSDIA with the simple replacement. On the other hand, in the present formulation, the Oldroyd derivative in the delay-response term (the C_T-related term) is obtained as the natural consequence since it is the simple time derivative in the mean-Lagrangian representation. Moreover, we should notice that even if we choose the Oldroyd derivative, the present form, namely

$$\frac{\mathrm{O}S^{ij}}{\mathrm{O}t} + S^{i}_{a}S^{ja},$$

is not trivial. Instead of the present form, one may well employ the deviatoric part of $\mathrm{O}S^{ij}/\mathrm{O}t$, namely

$$\frac{\mathrm{O}S^{ij}}{\mathrm{O}t} + \frac{1}{3}\mathbf{S}\cdot\mathbf{S}\,g^{ij}.$$

9 Discussions

9.1 General covariance and temporal nonlocality

At a glance of (8.17), one may think that the present theory could be utilized only for the algebraic models, which are described only by one-point one-time quantities. If this were true, one may think that the mean-Lagrangian representation were unnecessarily complex and cumbersome to obtain (8.17), since some similar forms to (8.17) have already been obtained by some pioneering methodologies such as TSDIA in simpler formalism (Note that they are not formulated in as a wide range of covariance as (8.17)). However, we should notice that the present theory is not only for deriving algebraic models of the Reynolds stress, but it is capable of deriving temporally-nonlocal effect.

By looking at (8.2) carefully, we notice that the multiple-time dynamics of both mean flow and fluctuation are firmly considered by the present theory. Indeed, in all the RPT procedures (see §7.6), we have integrated the quantities of both mean flow and fluctuation in time. Thus we can surely say that the present theory is a multiple-time closure theory of both mean flow and fluctuation, which is the generalization of the homogeneous-turbulence RPT where the multiple-time quantities of fluctuation are essential in the formulation. We should note that (8.17) is a result of temporal-locality approximation, and that multiple-time analysis is essential for the long derivation of (8.17) [31].

The present theory has two prominent advantages over TSDIA; one is the consistency with the general covariance and the other is the illustration of multiple-time dynamics of both mean field and fluctuation. In order to combine these two features, the mean-Lagrangian representation is the most natural and suitable choice, as has been stated in §6. This combination may be depicted most simply by (8.2). Indeed, all the time integrations in (8.2) are the convected integrations due to the mean-Lagrangian representation, which successfully incorporate the history of $\mathbf{S}$, $\boldsymbol{\Theta}$ and spatial derivatives of them in generally covariant manner [32] [33].

[31] The counterpart of (8.2) in TSDIA is (4.12). In this representation, the history of the velocity gradient is not considered since the time scales of the mean flow and the fluctuation are separated from the begining.

[32] Note that all the time integrations in RPT procedures of §7.6 are also convected integration due to the mean-Lagrangian representation. Thus, not only for the theoretical results (8.2) and (8.17), but also for every step of mathematical manipulation, the mean-Lagrangian formalism plays indispensable roles for covariant multiple-time analysis.

[33] Temporally nonlocal representations such as (6.8) also incorporate the historical effect of the mean flow. However, the

9.2 Analogy with LRA theory

By assuming so called the "*fluctuation-dissipation relation*" for the basic properties; namely

$$ {}^{B}Q(k;t,t') = {}^{B}Q(k;t',t'){}^{B}G(k;t,t'), \tag{9.1}$$

we obtain

$$\nu_T = \frac{1}{15}\int_{-\infty}^{t} \mathrm{d}t' \int \mathrm{d}vol_k \left\{ 7\, {}^{B}G(k;t,t')\, {}^{B}Q(k;t,t') + k\, {}^{B}G(k;t,t')\, {}^{B}G(k;t,t') \frac{\partial}{\partial k} {}^{B}Q(k;t',t') \right\}. \tag{9.2}$$

In the framework of LRA of homogeneous isotropic turbulence, it has been shown that far-smaller-scale motion acts on the larger as turbulence viscosity which is given by a quite similar expression to (9.2) (Kaneda 1986). In the wavenumber space, the turbulence viscosity on the mode much lower than k_0 may be represented as

$$\nu_T(k_0) = \frac{1}{15}\int_{-\infty}^{t} \mathrm{d}t' \int_{k_0}^{\infty} 4\pi\, k^2 \mathrm{d}k \left\{ 7\, {}^{L}G(k;t,t')\, {}^{L}Q(k;t,t') + k\, {}^{L}G(k;t,t')\, {}^{L}G(k;t,t') \frac{\partial}{\partial k} {}^{L}Q(k;t',t') \right\}, \tag{9.3}$$

where ${}^{L}Q(k;t,t')$ and ${}^{L}G(k;t,t')$ are "*Lagrangian velocity correlation*" and "*Lagrangian response function*" respectively [34]. The present result (9.2) corresponds to the case $k_0 \to 0$ of (9.3), which may be explained that all the modes of fluctuation are considered in deriving (9.2). Although ${}^{B}Q$ and ${}^{B}G$ are not based on the Lagrangian view in $\boldsymbol{\xi}$ space, this analogy indicates that the present theory can describe the interactions between the mean flow and the fluctuation at close level of the LRA theory. Of course, the eddy-viscosity stress of our result is just an approximation and it represents only a limited part of the total energy dissipation, which has been also pointed out in homogeneous turbulence theories (Kaneda 1986, Kraichnan 1976). In the present work, deviation from linear-eddy-viscosity stress may be explained by the nonlinear-eddy-viscosity stress and inhomogeneity terms.

On the other hand, the counter part of TSDIA given by (4.14), namely

$$\nu_T = \frac{7}{15}\int_{-\infty}^{\tau} \mathrm{d}\tau' \int \mathrm{d}^3\, k G^0(k;\tau,\tau')\, Q^0(k;\tau,\tau'),$$

lacks k-derivative term appears in (9.2) and (9.3). The exact cause of this is from two-scale-represented convection term; $V_j(\mathbf{X},T)\ \partial/\partial\xi_j$ in the first bracket of the left-hand side of (4.8). As a result of two-scale decomposition of the term, the non-uniform variation of the mean velocity is neglected in $\boldsymbol{\xi}$-scale dynamics, while the velocity gradient is not neglected in the right-hand side. As a consequence, in the homogeneous turbulence subjected to the uniform shear, (4.8) of TSDIA yields different type of fluctuation equation from the exact Navier-Stokes equation. In the present work, on the contrary, the master equation (7.38) in case of homogeneous shear flow is identical to the exact Navier-Stokes equation.

9.3 Relation with the conventional K-ϵ model

By applying the simplification discussed in §7.7, the relation between (8.17) and K-ϵ type of models is suggested. Under this simplification, (8.5)-(8.16) reduce to

$${}^{B}K = 6\pi C_\sigma k_c^{-2/3} \left(1 - {}^{B}\mathrm{Re}^{-1/2}\right)\, {}^{B}\epsilon^{2/3}, \tag{9.4}$$

$$\nu_T = \frac{7}{15}\pi \frac{C_\sigma}{C_\omega} k_c^{-4/3} \left(1 - {}^{B}\mathrm{Re}^{-1}\right)\, {}^{B}\epsilon^{1/3}, \tag{9.5}$$

memory function (the exponential function) is based on K and ϵ both of which are one-time correlations while (8.2) is based on the two-time correlations. Thus, (8.2) given by the present result is clearly distinct from the temporally nonlocal models such as (6.8); RSM can be categorized in these models.

[34] In the original work of Kaneda (1986), (9.3) is given by

$$\nu_T(k_0) = \frac{1}{15}\int_{-\infty}^{t} \mathrm{d}t' \int_{k_0}^{\infty} 2\pi\, k^2 \mathrm{d}k \left\{ 7\, {}^{L}G(k;t,t')\, {}^{L}Q(k;t,t') + k\, {}^{L}G(k;t,t')\, {}^{L}G(k;t,t') \frac{\partial}{\partial k} {}^{L}Q(k;t',t') \right\};$$

namely, it looks like the half of (9.3). This is simply because ${}^{L}Q$ in the original work is defined as the double of the above (Kaneda 1981, 1986).

$$\gamma_t = \frac{8}{45}\pi\frac{C_\sigma}{C_\omega^2}k_c^{-2}\left(1 - {}^B\mathrm{Re}^{-3/2}\right), \tag{9.6}$$

$$N_I = \frac{2}{45}\pi\frac{C_\sigma}{C_\omega^2}k_c^{-2}\left(1 - {}^B\mathrm{Re}^{-3/2}\right), \tag{9.7}$$

$$N_{II} = 0, \tag{9.8}$$

$$N_{III} = \frac{4}{135}\pi\frac{C_\sigma}{C_\omega^2}k_c^{-2}\left(1 - {}^B\mathrm{Re}^{-3/2}\right), \tag{9.9}$$

$$N_{IV} = 2N_{II} = 0, \tag{9.10}$$

$$N_V = \frac{2}{15}\pi\frac{C_\sigma}{C_\omega^2}k_c^{-2}\left(1 - {}^B\mathrm{Re}^{-3/2}\right), \tag{9.11}$$

$$D_I = \frac{3}{35}\pi\frac{C_\sigma}{C_\omega}k_c^{-10/3}\left(1 - {}^B\mathrm{Re}^{-5/2}\right)\ {}^B\epsilon^{1/3}, \tag{9.12}$$

$$D_{II} = \frac{4}{175}\pi\frac{C_\sigma}{C_\omega}k_c^{-10/3}\left(1 - {}^B\mathrm{Re}^{-5/2}\right)\ {}^B\epsilon^{1/3}, \tag{9.13}$$

$$D_{III} = -\frac{1}{25}\pi\frac{C_\sigma}{C_\omega}k_c^{-10/3}\left(1 - {}^B\mathrm{Re}^{-5/2}\right)\ {}^B\epsilon^{1/3}, \tag{9.14}$$

$$D_{IV} = -\frac{2}{35}\pi\frac{C_\sigma}{C_\omega}k_c^{-10/3}\left(1 - {}^B\mathrm{Re}^{-5/2}\right)\ {}^B\epsilon^{1/3}. \tag{9.15}$$

Here ${}^B\mathrm{Re} = (k_d/k_c)^{4/3}$ corresponds to the Reynolds number of the basic field. In the high-Reynolds-number flow, ${}^B\mathrm{Re}$ is expected to be much larger than unity. In the present analysis, therefore, we omit all the negative power of ${}^B\mathrm{Re}$.

Here we introduce the basic field energy given by

$${}^BK = 6\pi C_\sigma\ {}^B\epsilon^{2/3}k_c^{-2/3}. \tag{9.16}$$

Using (9.16) we can represent (9.5)-(9.15) by ${}^BK$ and ${}^B\epsilon$. Therefore (8.17) is reduced to

$$\begin{aligned}
R^{ij} = \frac{2}{3}\,{}^BKg^{ij} &- C_\nu\frac{{}^BK^2}{{}^B\epsilon}S^{ij} \\
&+ C_T\frac{{}^BK^3}{{}^B\epsilon^2}\left(\frac{\mathrm{O}S^{ij}}{\mathrm{O}t} + S^i_aS^{ja}\right) \\
&+ C'_S\frac{{}^BK^3}{{}^B\epsilon^2}\mathbf{S}\cdot\mathbf{S}g^{ij} \\
&+ C_S\frac{{}^BK^3}{{}^B\epsilon^2}S^i_aS^{ja} \\
&+ C_C\frac{{}^BK^3}{{}^B\epsilon^2}\left(S^i_a\Theta^{ja} + S^j_a\Theta^{ia}\right) \\
&+ C_{\mathrm{MD1}}\frac{{}^BK^5}{{}^B\epsilon^3}\left(S^{i;ja}_a + S^{j;ia}_a\right) \\
&+ C'_{\mathrm{MD1}}\frac{{}^BK^5}{{}^B\epsilon^3}S^{ab}{}_{;ab}\,g^{ij} \\
&- C_{\mathrm{MD2}}\frac{{}^BK^5}{{}^B\epsilon^3}\left(\Theta^i{}_a{}^{;ja} + \Theta^j{}_a{}^{;ia}\right) \\
&- C_{\mathrm{MD3}}\frac{{}^BK^5}{{}^B\epsilon^3}\triangle S^{ij},
\end{aligned} \tag{9.17}$$

where the constants are given as follows;

$$C_\nu = \frac{7}{540\pi C_\sigma C_\omega} = 0.0835, \tag{9.18}$$

$$C_T = \frac{1}{1215\pi^2 C_\sigma^2 C_\omega^2} = 0.0341, \tag{9.19}$$

$$C'_S = \frac{1}{4860\pi^2 C_\sigma^2 C_\omega^2} = 0.00853, \tag{9.20}$$

$$C_S = \frac{1}{7290\pi^2 C_\sigma^2 C_\omega^2} = 0.00569, \tag{9.21}$$

$$C_C = \frac{1}{1620\pi^2 C_\sigma^2 C_\omega^2} = 0.0256, \tag{9.22}$$

$$C_{\text{MD1}} = \frac{1}{90720\pi^4 C_\sigma^4 C_\omega} = 0.00139, \tag{9.23}$$

$$C'_{\text{MD1}} = \frac{1}{340200\pi^4 C_\sigma^4 C_\omega} = 0.000371, \tag{9.24}$$

$$C_{\text{MD2}} = \frac{1}{194400\pi^4 C_\sigma^4 C_\omega} = 0.000650, \tag{9.25}$$

$$C_{\text{MD3}} = \frac{1}{136080\pi^4 C_\sigma^4 C_\omega} = 0.000929. \tag{9.26}$$

All the transport coefficients in (9.17) are expressed in terms of the basic-field quantities ${}^{B}K$ and ${}^{B}\epsilon$, which are not the total quantities observed in the real flow. Thus let us employ observable quantities; total turbulence energy K and its dissipation rate ϵ. First the turbulence energy is obtained only by taking the trace of the Reynolds stress (9.17) as follows;

$$K = \frac{1}{2} g_{ab} R^{ab} = {}^{B}K + \frac{1}{2}(3C'_S + C_S)\frac{{}^{B}K^3}{{}^{B}\epsilon^2}\mathbf{S}\cdot\mathbf{S} + \frac{1}{2}\left(3C'_{\text{MD1}} + 2C_{\text{MD1}}\right)\frac{{}^{B}K^5}{{}^{B}\epsilon^3} S^{ab}{}_{;ab}. \tag{9.27}$$

Through the same procedure we applied to the Reynolds stress, we can calculate the full dissipation rate expanded in terms of the basic-field quantities as follows;

$$\epsilon = {}^{B}\epsilon + C_\epsilon \frac{\log {}^{B}\mathrm{Re}}{{}^{B}\mathrm{Re}} \frac{{}^{B}K^2}{{}^{B}\epsilon}\mathbf{S}\cdot\mathbf{S} + O({}^{B}\mathrm{Re}^{-1}), \tag{9.28}$$

where C_ϵ is a positive constant. In high-Reynolds-number limit ${}^{B}\mathrm{Re} \to \infty$, it reduces to

$$\epsilon = {}^{B}\epsilon, \tag{9.29}$$

where we used a wellknown relation; $\log r/r \to 0$ as $r \to \infty$ [35]. By inverting the expansion (9.27) we obtain ${}^{B}K$ expanded by the real turbulence energy K and its dissipation rate ϵ as

$${}^{B}K = K - \frac{1}{2}(3C'_S + C_S)\frac{K^3}{\epsilon^2}\mathbf{S}\cdot\mathbf{S} - \frac{1}{2}\left(3C'_{\text{MD1}} + 2C_{\text{MD1}}\right)\frac{K^5}{\epsilon^3} S^{ab}{}_{;ab} + O(\mathbf{S}^4). \tag{9.30}$$

[35] Namely inhomogeneity does not affect the energy-dissipation rate in high-Reynolds-number case, that is consistent with Kolmogorov's theory (Kolmogorov 1941)

By substituting (9.29) and (9.30) into (9.17) and collecting terms up to second-order of $\mathbf{S}$, we obtain

$$\begin{aligned} R^{ij} = \frac{2}{3}Kg^{ij} - C_\nu \frac{K^2}{\epsilon} S^{ij} \\ + C_T \frac{K^3}{\epsilon^2} \left(\frac{\mathrm{O}S^{ij}}{\mathrm{O}t} + S^i_a S^{ja} \right) \\ + C_S \frac{K^3}{\epsilon^2} \left(S^i_a S^{ja} - \frac{1}{3} \mathbf{S} \cdot \mathbf{S} g^{ij} \right) \\ + C_C \frac{K^3}{\epsilon^2} \left(S^i_a \Theta^{ja} + S^j_a \Theta^{ia} \right) \\ + C_{\mathrm{MD1}} \frac{K^5}{\epsilon^3} \left(S^{i;ja}_a + S^{j;ia}_a - \frac{2}{3} S^{ab}{}_{;ab}\, g^{ij} \right) \\ - C_{\mathrm{MD2}} \frac{K^5}{\epsilon^3} \left(\Theta^i{}_a{}^{;ja} + \Theta^j{}_a{}^{;ia} \right) \\ - C_{\mathrm{MD3}} \frac{K^5}{\epsilon^3} \triangle S^{ij}. \end{aligned} \tag{9.31}$$

Note that C'_S- and C'_{MD1} related terms in (9.17) are completely absolved into the isotropic term $\frac{2}{3}Kg^{\mu\nu}$ through the second and third terms of (9.30). (9.31) should be compared with the explicit algebraic models of high-Reynolds-number K-ϵ type where all the transport coefficients are represented in terms of K and ϵ. What is derived here is discriminated from the conventional models since (9.31) contains nonlocal effect in space and time (C_T-, C_{MD1}-, C_{MD2}- and C_{MD3}-related terms).

C_ν-, C_T-, C_S- and C_C-related terms of (9.31) are derived from the μ^1- and μ^2-order diagrams. These terms take non-zero value when the strain rate or the absolute vorticity are non-zero in the actual point. The basic functions of these terms are briefly explained in appendix H. Note that the constant $C_\nu = 0.0835$ attached to the linear-viscosity term is close to the counterpart of the standard K-ϵ models; usually it is optimized as $C_\nu = 0.09$ from observations of the near-wall flows [36].

C_{MD1}-, C_{MD2}- and C_{MD3}-related terms are from $\mu^1\delta^2$-order diagrams which may be non-zero even when neither the strain rate nor the absolute vorticity exist. Simple applications of (9.31) are given in appendix I and J, where some basic features of these new effects are briefly given.

Note that the K-ϵ type of model (9.31) is not the only result of the present theory. In the derivation of (9.31), we utilize the simplified spectrum discussed in §7.7. In the simplification procedure, we simply replace the detailed structure of both the energy-containing and dissipation ranges with cutoff wavenumbers k_c and k_d. In general, however, these two ranges can take various forms in different cases. Especially, the energy containing range depends on information of global structures. The result (8.2) and (8.17) may be capable of reflecting such a detailed correction from both global and local structures. Starting from more general form (8.17), we may be able to extend the simple K-ϵ form (9.31) to more generalized version. These analysis can hardly be conducted by conventional empirical modeling.

9.4 Approach to the total closure model

In this thesis we applied the present theory only to the Reynolds stress which is merely one of all the targets. However, it should be emphasized that the present theory can be applied to arbitrary correlation in hydrodynamic turbulence. This wide applicability come from the original RPT of homogeneous turbulence. Since the present theory is a natural generalization of RPT, we can investigate not only the Reynolds stress but also various correlations. Furthermore, for any covariant correlations, the present theory always derives the results consistent with the general covariance, since all of the quantities in the formulation are generally covariant. By applying the present theory to necessary number of correlations, we can obtain the closed system of equations.

[36] Constants (9.18)-(9.26) totally depends on the choice of values C_σ and C_ω. Note that the choice in the present work can be modified by means of more accurate treatment of the basic field, which changes the values given by (9.18)-(9.26) (see §9.5).

For example, for the closure based on turbulence energy K and its dissipation rate ϵ, we need to close equations for them besides (9.31), which are written in general-coordinate representation as follows;

$$\frac{\mathrm{D}K}{\mathrm{D}t} = -\frac{1}{2}R^{ij}S_{ij} - \epsilon + \left\langle \left(\frac{1}{2} g_{ab} v'^a v'^b + p' \right) v'^j \right\rangle_{;j}, \tag{9.32}$$

$$\begin{aligned} \frac{\mathrm{D}\epsilon}{\mathrm{D}t} = & -2\nu \left\langle v'^i{}_{;j} v'_{i;k} v'^{j;k} \right\rangle - 2\nu^2 \left\langle v'^{i;jk} v'_{i;jk} \right\rangle - \nu \left(S^i_j + \Theta^i{}_j \right) \left\langle v'_{i;k} v'^{j;k} + v'_{k;i} v'^{k;j} \right\rangle \\ & - \nu \left(S_{ij;k} + \Theta_{ij;k} \right) \left\langle v'^j v'^{i;k} \right\rangle - \left(\nu \left\langle v'^{a;b} v'_{a;b} v'^j \right\rangle + 2\nu \left\langle p'_{,a} v'^{j;a} \right\rangle \right)_{;j} + \nu \triangle \epsilon, \end{aligned} \tag{9.33}$$

where all the correlations in (9.32) and (9.33) are generally covariant. Note that the present theory can be applied to all the correlations given here and we can obtain the closed set of equations which are consistent with the general covariance [37]. Thus the ability of the present theory to construct the covariant model is surely important.

9.5 Need for fine-Lagrangian view

Although we employed the mean-Lagrangian view which cancels out the sweeping effect caused by the mean flow, it is still unable to remove the sweeping effect caused by fine structure since our formulation is fundamentally Eulerian framework for smaller scale, so that we have employed the tentative method in §7.7 where we avoided to deal with the exact dynamics of the basic field and put the spectral form consistent with the Kolmogorov's theory, the time scale employed in this method completely contradicts what it should be in Eulerian framework though; namely Eulerian propagator we employed in this formulation should depend on the properties in larger scale while (7.112) does not. In this sense the present work is still apart from self-consistent method on the stage of the simplification. Besides, the constants $C_\sigma = 0.118$ and $C_\omega = 0.419$, which all the constants C_ν, C_T, C_S and C_C of our algebraic result of the Reynolds stress based on, are derived by the modified Eulerian DIA so that we cannot fully rely on these values. Indeed the Kolmogolov's constant K_o predicted by fine-Lagrangian frameworks is often larger than $K_o = 4\pi C_\sigma = 1.48$ obtained by the present methodology; LHDIA predicts $K_o = 1.77$ (Kraichnan 1966), LRA and LDIA predict $K_o = 1.722$ (Kaneda 1986, Kida & Goto 1997).

Nevertheless, the author thinks it is still valuable to make this simplification as far as we pick up the principal corrections due to anisotropy and inhomogeneity effects, since the differences of diagrams between Eulerian and Lagrangian framework appear in the one-loop correction for the first time and there is no fatal difference in the no-loop diagrams. Thus, even if we involve the fine-Lagrangian view, we will obtain the same result as the present work as long as we discard the contributions from the loop diagrams. It is also noticeable that the Eulerian formalism is considerably simpler than the Lagrangian formalism that the former has remarkable power in analyzing more complex system than isotropic Navier-Stokes turbulence such as the MHD system which have more perturbative interactions other than simple nonlinear interaction. Thus it is effective to some degree to employ the mean-Lagrangian framework with constants C_σ and C_ω re-optimized by fine-Lagrangian theories such as LHDIA, LRA or LDIA.

On the contrary, the fine-Lagrangian formulation will be crucial on the stage where we deal with the dynamics of the binary correlation directly with its dynamical equation, since we will see the essential contribution from the loop diagrams even at the lowest order of λ, which will not be so much complex and will make whole procedure self-consistent within feasible effort.

9.6 A priori test in a channel flow

In order to see the validity of the present theory, we shall investigate our theoretical result by using a DNS data of a channel flow (Moser et al. 1999). Here, we compare the Reynolds stress calculated by DNS and algebraic representation $\mathbf{R}\{K, \epsilon, \boldsymbol{S}, \boldsymbol{\Theta}\}$ into which the computed values of K, ϵ, $\mathbf{S}$ and $\boldsymbol{\Theta}$ are substituted. It is also worthwhile to compare the present result with the previous works. Here we employ the second-order algebraic model derived by TSDIA given by (4.16) as a target for the comparison.

[37] We should remark that the higher-order-derivative terms of (9.33) may not be precisely treated by the simplification explained in §7.7. Because of higher-order differentiation, the higher-wavenumber properties of the basic field gives important contributions in the analysis. In this treatment, we need more precise information than the simple cutoff wavenumber k_d.

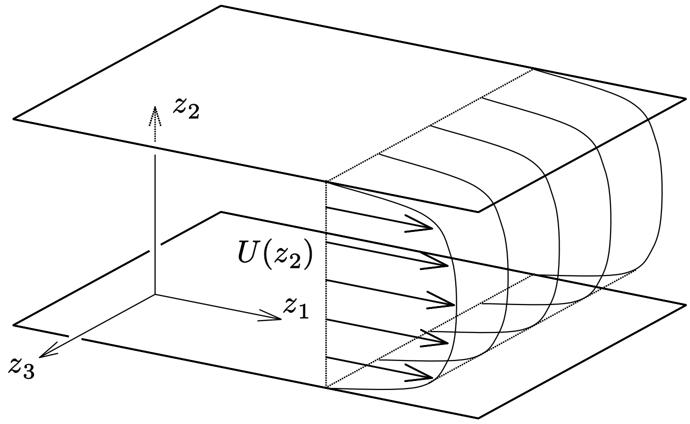


Figure 19: Channel flow with coordinate system

Due to the symmetry of the system, every statistical quantity is uniform in the direction parallel to the wall. Thus we have only to investigate the distribution of the each quantity in wall-normal direction.

We set up the coordinate system shown in the figure 19; z_1 for the stream-wise direction, z_2 for the wall-normal direction and z_3 for span-wise direction respectively [38]. Flow is characterized by the following Reynolds number;

$$\mathrm{Re}_\tau = \frac{U_\tau H}{\nu} = 587.19\,, \tag{9.34}$$

where H is the half width of the channel and U_τ is the friction velocity defined by

$$U_\tau = \sqrt{-\nu \left.\frac{dU}{dz_2}\right|_{wall}}\,. \tag{9.35}$$

In the following discussion, every quantity is normalized by H and U_τ.

9.6.1 Anisotropic distribution of turbulence intensity

The turbulence intensity of the DNS is shown in figures 20. Over the whole region of the channel, the anisotropic distribution $R^{11} > R^{33} > R^{22}$ is obtained. The expansion up to the second term of (9.31), namely

$$R^{ij} = \frac{2}{3}Kg^{ij} - C_\nu \frac{K^2}{\epsilon}S^{ij}, \tag{9.36}$$

cannot reproduce the anisotropic distribution as is shown in figure 21. Higher-order representation coupled by C_T-, C_S- and C_C-term corrections, namely

$$\begin{aligned} R^{ij} = \frac{2}{3}Kg^{ij} - C_\nu \frac{K^2}{\epsilon}S^{ij} &+ C_T\frac{K^3}{\epsilon^2}\left(\frac{\mathrm{O}S^{ij}}{\mathrm{O}t} + S^i_a S^{ja}\right) \\ &+ C_S\frac{K^3}{\epsilon^2}\left(S^i_a S^{ja} - \frac{1}{3}\mathbf{S}\cdot\mathbf{S}g^{ij}\right) \\ &+ C_C\frac{K^3}{\epsilon^2}\left(S^i_a\Theta^{ja} + S^j_a\Theta^{ia}\right), \end{aligned} \tag{9.37}$$

can reproduce the anisotropic distribution $R^{11} > R^{33} > R^{22}$ as is shown in figure 22. (4.16) derived by TSDIA, on the contrary, erroneously yields $R^{11} > R^{22} > R^{33}$ which contradicts the reality (see figure 23). Although the anisotropic distribution gives no contribution to the dynamics of channel flow, it is critically important for inducing the secondary flows observed in flows with more complex geometry such as flows in

[38] Although we have employed the contra-variant coordinate system in all the procedures we have done so far, we employ here the covariant coordinate variables z_I $(= g_{IJ}\, z^J)$ since contravariant coordinates z^1, z^2 and z^3 are likely to be confounded with powers of z.

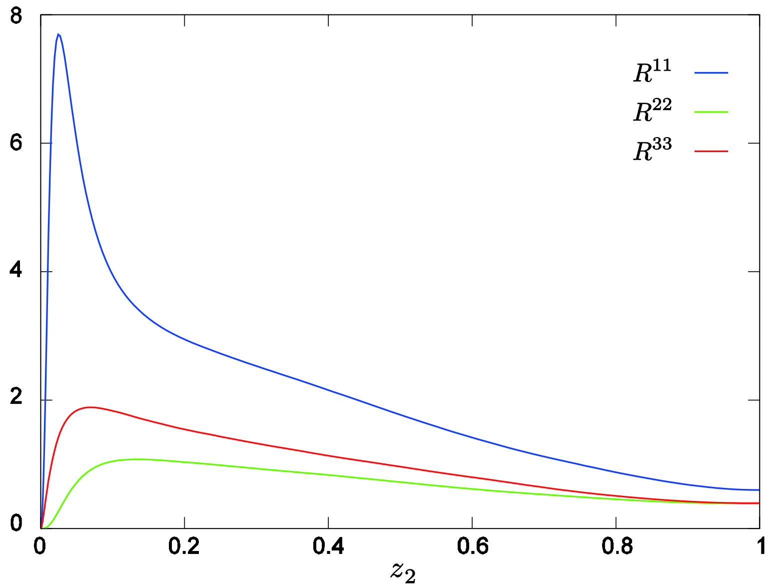


Figure 20: Turbulence intensity of the DNS: A wellknown distribution $R^{11} > R^{33} > R^{22}$ is obtained.

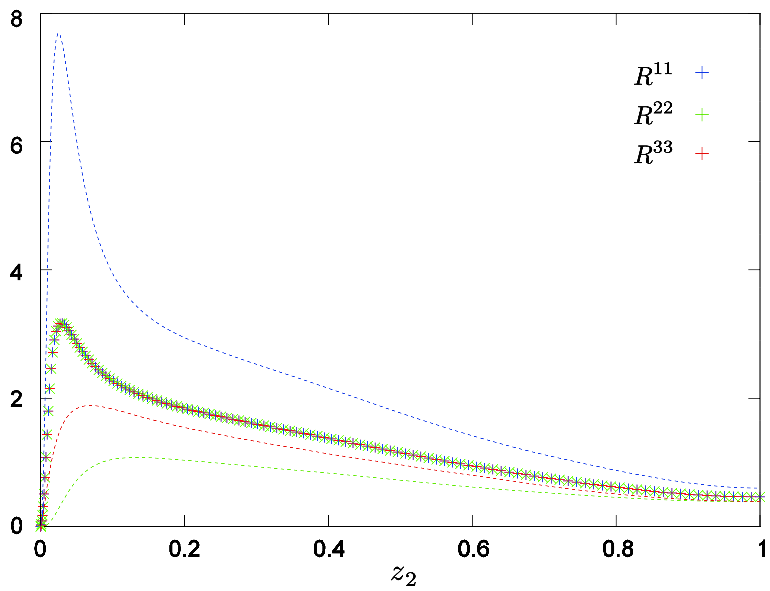


Figure 21: Turbulence intensity of linear-eddy-viscosity representation (9.36): All the components R^{11}, R^{22} and R^{33} take the same value and the anisotropic distribution $R^{11} > R^{33} > R^{22}$ cannot be reproduced only by the C_ν-term correction in (9.36). Here DNS values are plotted by dotted lines.

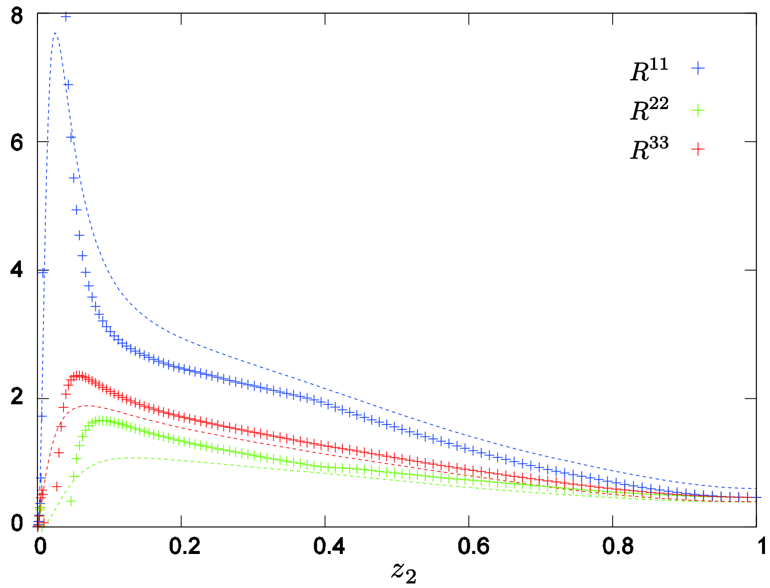


Figure 22: Turbulence intensity with nonlinear-viscosity correction (9.37): Due to the correction, the correct distribution $R^{11} > R^{33} > R^{22}$ of DNS is reproduced.

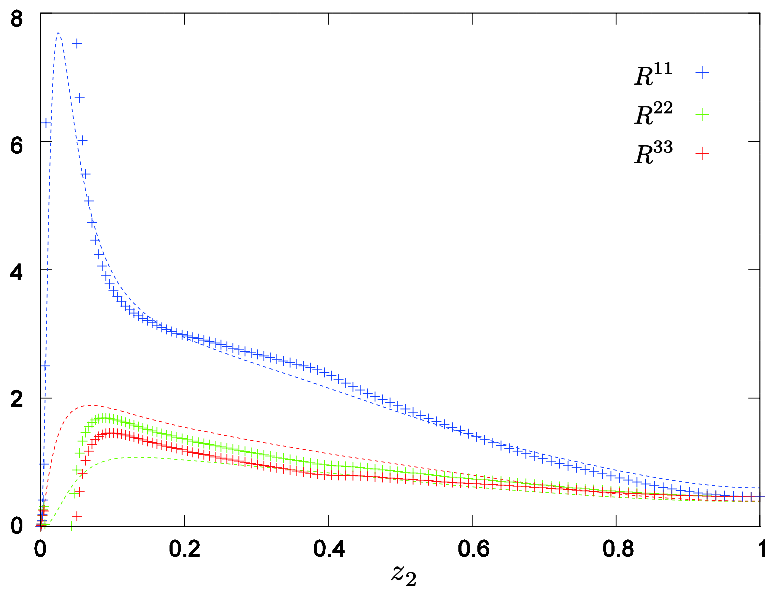


Figure 23: Turbulence intensity given by TSDIA: A wrong distribution $R^{11} > R^{22} > R^{33}$ is obtained.

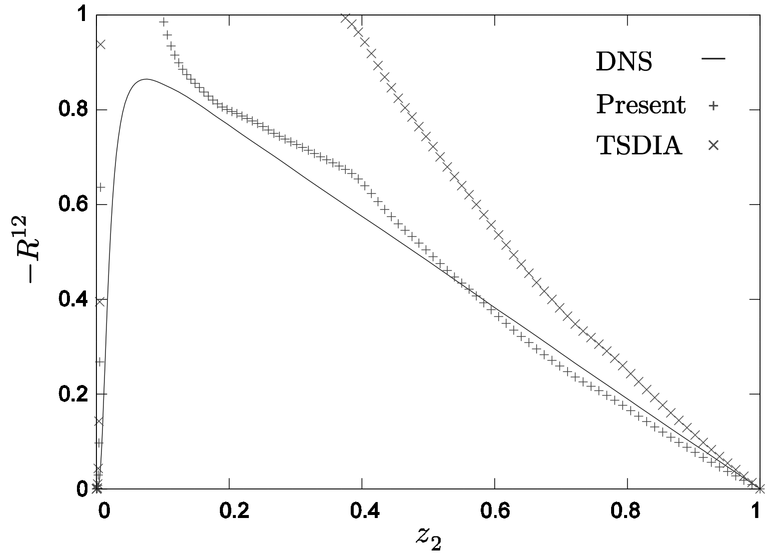


Figure 24: Comparison of shear stresses: Reynolds shear stress obtained by the DNS, the present work and TSDIA are shown.

pipes of non-circular cross-section (Speziale 1982).

Note that we cannot discuss about the validity from how much the models coincide with DNS values, since the values of the present model totally depend on the constants C_σ and C_ω given by (7.113). As has been discussed in §9.5, we cannot fully rely on these constants, and these may be re-optimized by fine-Lagrangian theories in future works. However, the relation $R^{11} > R^{33} > R^{22}$ always consists for any positive C_σ and C_ω.

We should remark that the higher-order-derivative terms ($C_{\rm MD1}$-, $C_{\rm MD2}$- and $C_{\rm MD3}$- related terms) do not give any correction to the turbulence intensity, which is analytically shown by (I·1) in appendix I.

9.6.2 Shear stress

The comparison of shear stresses obtained by (9.37) and (4.16) is shown in figure 24. Comparing to TSDIA, the present work shows better agreement with the DNS data. Besides, the comparison of turbulence-viscosity coefficients $\nu_T \equiv -R^{12}/(dU/dz_2)$ are shown in figure 25. As well as the shear stress, the present work shows better agreement with DNS data. We should note, however, that the results shown here also totally depend on the constants C_σ and C_ω, and thus these comparison are not the exact evidence for validity of the present work.

Note that only C_ν-related term gives measurable contribution to the shear stress. The nonlinear terms (C_T-, C_S- and C_C-terms) analytically vanishes and the higher-order-derivative terms ($C_{\rm MD1}$-, $C_{\rm MD2}$- and $C_{\rm MD3}$-related terms) does not contribute so much in simple shear flows, which is indicated by (I·9) in appendix I.

9.6.3 Near-wall region

Although we have made certain success in the area enough apart from the wall in both qualitative and quantitative senses, we have to refer to the defects in the near-wall region; theoretical results for both normal and shear components anomalously exceed the real values. Adding to this, the normal components R^{22} and R^{33} violate the realizability $R^{22}, R^{33} > 0$. We may be able to overcome these difficulties by incorporating higher order terms of perturbations since the contributions from inhomogeneity and anisotropy become rapidly prominent as one gets close to the wall as is shown by figure 20. However, the simple expansion will bring us more complexity in terms of tensorial form and the algebraic representation will have more than ten of independent terms, as is seen in some pioneering works for higher-order nonlinear models based on TSDIA (Yoshizawa 1984, Okamoto 1994).

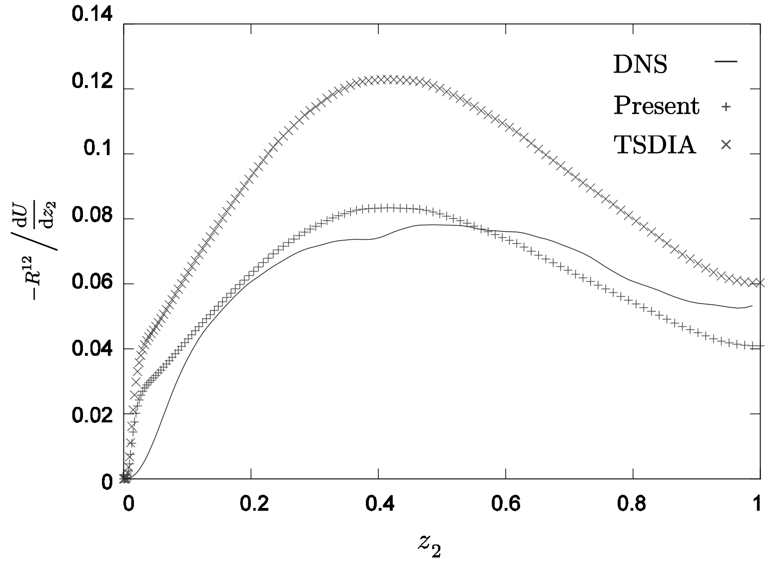


Figure 25: Comparison of eddy viscosities: The comparison of turbulence-viscosity coefficient $\nu_T \equiv -R^{12}/(dU/dz_2)$ is shown. Comparing to the TSDIA result, the present work shows better agreement with DNS.

Here let us see another possibility that would be a hint to solve this problem. The transport coefficients in (9.17) are originally represented by properties of the basic field which are only isotropic part of the whole fluctuation field and has limited energy and dissipation; ${}^{B}K$ and ${}^{B}\epsilon$. In other words, not whole energy and its dissipation but only energy and dissipation from the isotropic part give the actual contributions to the transport coefficients. In our algebraic representation, however, we truncated the renormalized expansion at the order of μ^2 so that ${}^{B}K$ and ${}^{B}\epsilon$ in the transport coefficients are simply replaced by K and ϵ, both of which seem to differ from ${}^{B}K$ and ${}^{B}\epsilon$, especially K would be considerably higher than ${}^{B}K$ when turbulence has strong anisotropy, since the large scale motion which is easily affected by anisotropy gives the major contribution to the turbulence energy. This may be one of the reasons why our result (9.31) has shown erroneous behaviors in the near-wall region.

10 Conclusions

The author has reached the following as the conclusion of the present study.

1. It was proved that the fluctuation field is generally-covariant quantity, and that the various turbulence quantities including the Reynolds stress are generally covariant.

2. The mean-Lagrangian-coordinate representation is newly introduced, and its advantage for the combination of the multiple-time analysis and the general covariance was shown.

3. By taking the advantage of the mean-Lagrangian formalism, a theory of inhomogeneous turbulence on the basis of multiple-point multiple-time quantities were developed in agreement with the general covariance.

4. A temporally non-local representation of the Reynolds stress was derived in the form of the convected integration, which clearly includes the history of both fluctuation and mean field along the mean-flow trajectory in a generally covariant manner.

5. An algebraic representation of the Reynolds stress was derived which contains new effects such as the Oldroyd derivative of the strain rate, spatial derivative of the strain rate and absolute vorticity. These represent the non-local effect in both space and time in a generally covariant manner.

Acknowledgement

The author should emphasize here that this study could not have been done if the author had been working in other place than Hamba group in the Institute of Industrial Science, the University of Tokyo, where TSDIA, the unique methodology has been developed for a long period with continuous and conscientious efforts of the great pioneers since its birth by Prof. Akira Yoshizawa. The author is very grateful to Dr. Nobumitsu Yokoi and Prof. Akira Yoshizawa for giving the author a great deal of meaningful advice. The author is especially grateful to Prof. Fujihiro Hamba for sincerely taking care of the author since the very beginning of the author's master program.

It is needless to say that the author can never stand on the present place alone. Family, friends, and a bit of plants have been essential for all the way since his birth. Family tell the author how much important the loneliness is. Friends tell him how much fun unreasonableness is. And even plants on the desk tell him how much peaceful his noisy life is.

Appendix

A Transformation rule of the velocity field

Let an element of continuum be P whose position at time t is given by ${}^{P}y^{i}(t)$ in a coordinate system $\{\mathbf{y}\}$. The velocity of P in this coordinate representation is given by

$$ {}^{P}v^{i}(t) = \frac{\mathrm{d}}{\mathrm{d}t}{}^{P}y^{i}(t). \tag{A·1}$$

We introduce another coordinate system $\{\tilde{\mathbf{y}}\}$ whose relation with $\{\mathbf{y}\}$ is given by $\tilde{y}^{\tilde{a}} = \tilde{y}^{\tilde{a}}(\mathbf{y},t)$ ($y^{\tilde{a}} = y^{\tilde{a}}(\mathbf{y},t)$ in the abbreviation form). In the new coordinate system, the position of P is given by

$$ {}^{P}y^{\tilde{a}}(t) = y^{\tilde{a}}\left({}^{P}\mathbf{y}(t),t\right). \tag{A·2}$$

Thus the velocity is given by

$$ {}^{P}v^{\tilde{a}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}{}^{P}y^{\tilde{a}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}y^{\tilde{a}}\left({}^{P}\mathbf{y}(t),t\right) = \frac{\partial y^{\tilde{a}}}{\partial y^{i}}\left({}^{P}\mathbf{y}(t),t\right){}^{P}v^{i}(t) + \frac{\partial y^{\tilde{a}}}{\partial t}\left({}^{P}\mathbf{y}(t),t\right) \tag{A·3}$$

Here we should notice that the last term on the right-hand side is obtained by substituting $\mathbf{y} = {}^{P}\mathbf{y}(t)$ into the derivative function $\partial y^{\tilde{a}}/\partial t\,(\mathbf{y},t)$. The element velocity ${}^{P}\mathbf{v}(t)$ is given by $\mathbf{v}\left({}^{P}\mathbf{y}(t),t\right)$, so that the above is reduced to

$$ v^{\tilde{a}}\left({}^{P}\tilde{\mathbf{y}}(t),t\right) = \frac{\partial y^{\tilde{a}}}{\partial y^{i}}\left({}^{P}\mathbf{y}(t),t\right)v^{i}\left({}^{P}\mathbf{y}(t),t\right) + \frac{\partial y^{\tilde{a}}}{\partial t}\left({}^{P}\mathbf{y}(t),t\right). \tag{A·4}$$

This relation holds for arbitrary element of continuum. Thus, by replacing $({}^{P}\mathbf{y}(t),t)$ and $({}^{P}\tilde{\mathbf{y}}(t),t)$ with $(\mathbf{y},t)$ and $(\tilde{\mathbf{y}},t)$, we obtain

$$ v^{\tilde{a}}(\tilde{\mathbf{y}},t) = \frac{\partial y^{\tilde{a}}}{\partial y^{i}}(\mathbf{y},t)\; v^{i}(\mathbf{y},t) + \frac{\partial y^{\tilde{a}}}{\partial t}(\mathbf{y},t). \tag{A·5}$$

B Oldroyd derivative

Oldroyd derivative is originally derived from the pure time derivative in the Lagrangian frame. Let us derive the Oldroyd derivative of a two-rank tensor $C^{i}{}_{j}(\mathbf{y},t)$ from its Lagrangian representation $C^{\mu}{}_{\nu}(\mathbf{x},t)$. For simplicity, we denote the time differentiation for fixed Lagrangian variable $\mathbf{x}$ by a dot; namely,

$$ \dot{f} \equiv \left(\frac{\partial f}{\partial t}\right)_{\mathbf{x}} $$

for arbitrary quantity.

By taking the time derivative of $C^{\mu}{}_{\nu}(\mathbf{x},t)$, we obtain

$$ \begin{aligned} C^{\mu}{}_{\nu,t} = \dot{C}^{\mu}{}_{\nu} &= \left(x^{\mu}{}_{,i}\; y^{j}{}_{,\nu}\; C^{i}{}_{j}\right)^{\cdot} \\ &= \dot{x}^{\mu}{}_{,i}\; y^{j}{}_{,\nu}\; C^{i}{}_{j} + x^{\mu}{}_{,i}\; \dot{y}^{j}{}_{,\nu}\; C^{i}{}_{j} + x^{\mu}{}_{,i}\; y^{j}{}_{,\nu}\; \dot{C}^{i}{}_{j} \end{aligned} \tag{B·1}$$

The third term of the right-hand side of (B·1) reduces to [39]

$$ x^{\mu}{}_{,i}\; y^{j}{}_{,\nu}\; \dot{C}^{i}{}_{j} = x^{\mu}{}_{,i}\; y^{j}{}_{,\nu}\; \left(C^{i}{}_{j,t} + C^{i}{}_{j,k}y^{k}{}_{,t}\right) = x^{\mu}{}_{,i}\; y^{j}{}_{,\nu}\; \left(C^{i}{}_{j,t} + v^{k}\, C^{i}{}_{j,k}\right), $$

where we used $y^{k}{}_{,t} = y^{k}{}_{,t}(\mathbf{x},t) = v^{k}$. The second term of the right-hand side of (B·1) is rewritten as

$$ \begin{aligned} x^{\mu}{}_{,i}\; \dot{y}^{j}{}_{,\nu}\; C^{i}{}_{j} &= x^{\mu}{}_{,i}\; y^{j}{}_{,\nu t}\; C^{i}{}_{j} = x^{\mu}{}_{,i}\; y^{j}{}_{,t\nu}\; C^{i}{}_{j} \\ &= x^{\mu}{}_{,i}\; v^{j}{}_{,k}\; y^{k}{}_{,\nu}\; C^{i}{}_{j} = x^{\mu}{}_{,i}\; y^{j}{}_{,\nu}\; v^{k}{}_{,j}\; C^{i}{}_{k}. \end{aligned} \tag{B·2}$$

[39] In a full notation $\dot{C}^{i}{}_{j}$ can be calculated as

$$ \left(\frac{\partial}{\partial t}C^{i}{}_{j}(\mathbf{y}(\mathbf{x},t),t)\right)_{\mathbf{x}} = C^{i}{}_{j,k}(\mathbf{y}(\mathbf{x},t),t)\; y^{k}{}_{,t}(\mathbf{x},t) + C^{i}_{j,t}(\mathbf{y}(\mathbf{x},t),t). $$

The first term of the right-hand side of (B·1) is transformed by using $x^\mu{}_{,i}\ y^i{}_{,\nu} = \delta^\mu_\nu$. First, by taking the dot derivative of the both sides, we obtain

$$\dot{\delta^\mu_\nu} = 0 = \dot{x}^\mu{}_{,i}\ y^i{}_{,\nu} + x^\mu{}_{,i}\ \dot{y}^i{}_{,\nu}. \tag{B·3}$$

Thus we have

$$\begin{aligned} \dot{x}^\mu{}_{,i} &= -x^\mu{}_{,k}\ \dot{y}^k{}_{,\rho}\ x^\rho{}_{,i} \\ &= -x^\mu{}_{,k}\ y^k{}_{,\rho t}\ x^\rho{}_{,i} \\ &= -x^\mu{}_{,k}\ y^k{}_{,t\rho}\ x^\rho{}_{,i} \\ &= -x^\mu{}_{,k}\ v^k{}_{,l}\ y^l{}_{,\rho}\ x^\rho{}_{,i}, \end{aligned} \tag{B·4}$$

from which the first term is transformed as follows;

$$\dot{x}^\mu{}_{,i}\ y^j{}_{,\nu}\ C^i{}_j = -x^\mu{}_{,i}\ y^j{}_{,\nu}\ v^i{}_{,k}\, C^k{}_j. \tag{B·5}$$

Thus (B·1) becomes

$$\begin{aligned} \dot{C}^\mu{}_\nu &= x^\mu{}_{,i}\ y^j{}_{,\nu} \left(C^i{}_{j,t} + v^k\, C^i{}_{j,k} - v^i{}_{,k}\, C^k{}_j + v^k{}_{,j}\, C^i{}_k\right) \\ \Leftrightarrow & C^i{}_{j,t} + v^k\, C^i{}_{j,k} - v^i{}_{,k}\, C^k{}_j + v^k{}_{,j}\, C^i{}_k = y^i{}_{,\mu}\ x^\nu{}_{,j} \dot{C}^\mu{}_\nu. \end{aligned} \tag{B·6}$$

Using an identity;

$$v^k \left(\Gamma^i_{lk} C^l{}_j - \Gamma^l_{jk} C^i{}_l\right) - \Gamma^i_{lk} v^l C^k{}_j + \Gamma^k_{lj} v^l C^i{}_k = 0,$$

(B·6) results in

$$C^i{}_{j,t} + v^k \left(C^i{}_{j,k} + \Gamma^i_{lk} C^l{}_j - \Gamma^l_{jk} C^i{}_l\right) - \left(v^i{}_{,k} + \Gamma^i_{lk} v^l\right) C^k{}_j + \left(v^k{}_{,j} + \Gamma^k_{lj} v^l\right) C^i{}_k = y^i{}_{,\mu}\ x^\nu{}_{,j} \dot{C}^\mu{}_\nu. \tag{B·7}$$

Here we should note that $\dot{C}^\mu{}_\nu = \dot{C}^\mu{}_\nu(\mathbf{x}, t) = C^\mu{}_{\nu,t}$. The left-hand side is rewritten as

$$\frac{\mathrm{o}}{\mathrm{o}t} C^i{}_j = C^i{}_{j,t} + v^k\, C^i{}_{j;k} - v^i{}_{;k}\, C^k{}_j + v^k{}_{;j}\, C^i{}_k \tag{B·8}$$

which is what we call the Oldroyd derivative of $C^i{}_j$. The Oldroyd derivative of a general tensor $C^{ij\cdots}{}_{kl\cdots}$ is obtained in totally the same manner as we discussed above;

$$\begin{aligned} \frac{\mathrm{o}}{\mathrm{o}t} C^{ij\cdots}{}_{kl\cdots} = C^{ij\cdots}{}_{kl\cdots,t} + v^m\, C^{ij\cdots}{}_{kl\cdots;m} - v^i{}_{;m}\, C^{mj\cdots}{}_{kl\cdots} - v^j{}_{;m}\, C^{im\cdots}{}_{kl\cdots} - \cdots \\ + v^m{}_{;k}\, C^{ij\cdots}{}_{ml\cdots} + v^m{}_{;l}\, C^{ij\cdots}{}_{km\cdots} + \cdots \end{aligned} \tag{B·9}$$

The counterpart of (B·7) is given by

$$\frac{\mathrm{o}}{\mathrm{o}t} C^{ij\cdots}{}_{kl\cdots} = y^i{}_{,\mu}\ y^j{}_{,\nu} \cdots\ x^\rho{}_{,k}\ x^\sigma{}_{,l} \cdots\ C^{\mu\nu\cdots}{}_{\rho\sigma\cdots,t} \tag{B·10}$$

Using the Lagrangian coordinate system, the general covariance of the Oldroyd derivative is shown by following steps. First we transform (B·10) as follows;

$$C^{\mu\nu\cdots}{}_{\rho\sigma\cdots,t} = x^\mu{}_{,i}\ x^\nu{}_{,j} \cdots\ y^k{}_{,\rho}\ y^l{}_{,\sigma} \cdots\ \frac{\mathrm{o}}{\mathrm{o}t} C^{ij\cdots}{}_{kl\cdots}. \tag{B·11}$$

Following (B·10), we obtain the Oldroyd derivative in another coordinate system $\{\tilde{\mathbf{y}}\}$ as

$$\frac{\mathrm{o}}{\mathrm{o}t} C^{\tilde{a}\tilde{b}\cdots}{}_{\tilde{c}\tilde{d}\cdots} = y^{\tilde{a}}{}_{,\mu}\ y^{\tilde{b}}{}_{,\nu} \cdots\ x^\rho{}_{,\tilde{c}}\ x^\sigma{}_{,\tilde{d}} \cdots\ C^{\mu\nu\cdots}{}_{\rho\sigma\cdots,t}. \tag{B·12}$$

By substituting (B·11) into (B·12) yields

$$\begin{aligned} \frac{\mathrm{o}}{\mathrm{o}t} C^{\tilde{a}\tilde{b}\cdots}{}_{\tilde{c}\tilde{d}\cdots} &= y^{\tilde{a}}{}_{,\mu}\ y^{\tilde{b}}{}_{,\nu} \cdots\ x^\rho{}_{,\tilde{c}}\ x^\sigma{}_{,\tilde{d}} \cdots \left(x^\mu{}_{,i}\ x^\nu{}_{,j} \cdots\ y^k{}_{,\rho}\ y^l{}_{,\sigma} \cdots\ \frac{\mathrm{o}}{\mathrm{o}t} C^{ij\cdots}{}_{kl\cdots}\right) \\ &= y^{\tilde{a}}{}_{,i}\ y^{\tilde{b}}{}_{,j} \cdots\ y^k{}_{,\tilde{c}}\ y^l{}_{,\tilde{d}} \cdots\ \frac{\mathrm{o}}{\mathrm{o}t} C^{ij\cdots}{}_{kl\cdots}, \end{aligned} \tag{B·13}$$

which proves that $\mathrm{o}C^{\tilde{a}\tilde{b}\cdots}{}_{\tilde{c}\tilde{d}\cdots}/\mathrm{o}t$ obeys the general tensor rule.

Since $\mathbf{V}$ transforms in totally the same manner as $\mathbf{v}$ (see (5.13) and (5.5)), all the above discussions hold for the mean-flow version; namely,

$$\frac{\mathrm{O}}{\mathrm{O}t}C^{ij\cdots}{}_{kl\cdots} = C^{ij\cdots}{}_{kl\cdots,t} + V^m C^{ij\cdots}{}_{kl\cdots;m} - V^i{}_{;m} C^{mj\cdots}{}_{kl\cdots} - V^j{}_{;m} C^{im\cdots}{}_{kl\cdots} - \cdots + V^m{}_{;k} C^{ij\cdots}{}_{ml\cdots} + V^m{}_{;l} C^{ij\cdots}{}_{km\cdots} + \cdots \tag{B$\cdot$14}$$

satisfies

$$\frac{\mathrm{O}}{\mathrm{O}t}C^{\tilde{a}\tilde{b}\cdots}{}_{\tilde{c}\tilde{d}\cdots} = y^{\tilde{a}}{}_{,i}\, y^{\tilde{b}}{}_{,j} \cdots\; y^k{}_{,\tilde{c}}\, y^l{}_{,\tilde{d}} \cdots\; \frac{\mathrm{O}}{\mathrm{O}t}C^{ij\cdots}{}_{kl\cdots}. \tag{B$\cdot$15}$$

C General covariance of the tensor $\boldsymbol{\Sigma}$

Covariance of $\boldsymbol{\Sigma}$ introduced by (7.9)may be confirmed by the following steps. First by transforming (7.9) we obtain

$$V^I{}_{;J} = z^I{}_{,i}\, y^j{}_{,J} \Sigma^i{}_j. \tag{C$\cdot$1}$$

In arbitrary coordinate system $\{\mathbf{y}\}$, its definition is given by (7.9). Thus, in another coordinate system $\{\tilde{\mathbf{y}}\}$, $\boldsymbol{\Sigma}$ is given by

$$\Sigma^{\tilde{a}}{}_{\tilde{b}} = y^{\tilde{a}}{}_{,I}\, z^J{}_{,\tilde{b}}\, V^I{}_{;J}. \tag{C$\cdot$2}$$

By substituting (C·1) into (C·2), we obtain

$$\Sigma^{\tilde{a}}{}_{\tilde{b}} = y^{\tilde{a}}{}_{,I}\, z^J{}_{,\tilde{b}} \left(z^I{}_{,i}\, y^j{}_{,J} \Sigma^i{}_j\right) = y^{\tilde{a}}{}_{,i}\, y^j{}_{,\tilde{b}}\, \Sigma^i{}_j, \tag{C$\cdot$3}$$

which is consistent with the general tensor rule (5.4). Covariant and contravariant representations of $\boldsymbol{\Sigma}$ are obtained respectively as follows;

$$\Sigma_{ij} \equiv g_{ik}\Sigma^k{}_j = g_{ik}\, y^k{}_{,I}\, z^J{}_{,j} V^I{}_{;J} = g_{IK}\, z^K{}_{,i}\, z^J{}_{,j}\, V^I{}_{;J} = z^K{}_{,i}\, z^J{}_{,j}\, V_{K;J}, \tag{C$\cdot$4}$$

$$\Sigma^{ij} \equiv g^{jk}\, \Sigma^i{}_k = g^{jk}\, y^i{}_{,I}\, z^J{}_{,k}\, V^I{}_{;J} = y^i{}_{,I}\, y^j{}_{,A}\, g^{JA}\, V^I{}_{;A} = y^i{}_{,I}\, y^j{}_{,A}\, V^{I;A}, \tag{C$\cdot$5}$$

where we used identities; $g_{ik}\, y^k{}_{,I} = g_{IK}\, z^K{}_{,i}$ and $g^{jk}\, z^J{}_{,k} = y^j{}_{,A}\, g^{JA}$. Thus $\boldsymbol{\Sigma}$ is completely determined by the mean-velocity gradient in the inertial frame.

It is noticeable that we have to identify the inertial frame of reference in the first place. Since the law of fluid treated in the present work is based on non-relativistic framework, inertial frame has a special meaning comparing to any other frame of references. In this context, $\boldsymbol{\Sigma}$ is not only a velocity gradient in the inertial frame, but an objective measure of how much the mean flow deviates from the inertial motion.

D Physical meaning of the antisymmetric field $\mathbf{A}$

To clarify the physical meaning of $\mathbf{A}$, let us think about a vector ${}^{\mathbf{x}}\mathbf{h}$ fixed to the static-metric frame on a mean-flow particle labeled by $\mathbf{X}$. By using (7.75), we obtain the equation for ${}^{\mathbf{x}}\mathbf{h}$ in the mean-Lagrangian representation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\,{}^{x}h_\mu(t) = \frac{\mathrm{d}\,{}^{x}a^J_\mu(t)}{\mathrm{d}t}\,{}^{x}\check{h}_J = \frac{1}{2}\left(S^\nu_\mu + A_\mu{}^\nu\right)(t|\mathbf{X})\,{}^{x}a^J_\nu(t)\,{}^{x}\check{h}_J = \frac{1}{2}\left(S^\nu_\mu + A_\mu{}^\nu\right)(t|\mathbf{X})\,{}^{x}h_\nu \tag{D$\cdot$1}$$

In an Eulerian coordinate system (it may be rotating relative to the inertial frame), this is rewritten as

$$\frac{\mathrm{O}}{\mathrm{O}t}\,{}^{x}h_i(t) = \frac{\mathrm{D}}{\mathrm{D}t}\,{}^{x}h_i(t) + {}^{x}h_j(t)V^j{}_{;i} = \frac{1}{2}\left(S^j_i + A_i{}^j\right)\,{}^{x}h_j(t). \tag{D$\cdot$2}$$

By dividing the mean-velocity gradient into strain and vorticity, we obtain

$$\frac{\mathrm{D}}{\mathrm{D}t}\,{}^{x}h_i(t) = \frac{1}{2}\left(A_i{}^j - \Omega_i{}^j\right)\,{}^{x}h_j(t), \tag{D$\cdot$3}$$

while a constant-length vector ${}^{x}\mathbf{J}$ rotating with the mean flow satisfies the following equation;

$$\frac{\mathrm{D}}{\mathrm{D}t}\,{}^{x}J_i(t) = -\frac{1}{2}\Omega_i{}^j(t|\mathbf{X})\,{}^{x}J_j(t). \tag{D$\cdot$4}$$

Thus $\mathbf{A}$ represents the angular velocity of the static-metric frame relative to the mean flow and, accordingly, the condition (7.116) makes (D·3) coincide with (D·4), which means that the static-metric frame is to be rotating with the mean flow without distortion.

E Time derivatives in the static metric representation and the mean-Lagrangian representation

By taking the time derivative of (7.67), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t} C^{IJ\dots}{}_{KL\dots}(\check{\mathbf{k}}, t|\mathbf{X}) \\
=&\frac{\mathrm{d}}{\mathrm{d}t}\, {}^{x}a^{I}_{\mu}(t)\, {}^{x}a^{J}_{\nu}(t) \dots {}^{x}a^{\rho}_{K}(t)\, {}^{x}a^{\sigma}_{L}(t) \dots C^{\mu\nu\dots}{}_{\rho\sigma\dots}\left({}^{x}a^{A}_{\alpha}(t)\check{k}_A, t|\mathbf{X}\right) \\
=&\frac{\mathrm{d}\, {}^{x}a^{I}_{\mu}}{\mathrm{d}t}(t)\, {}^{x}a^{J}_{\nu}(t) \dots {}^{x}a^{\rho}_{K}(t)\, {}^{x}a^{\sigma}_{L}(t) \dots C^{\mu\nu\dots}{}_{\rho\sigma\dots}\left({}^{x}a^{A}_{\alpha}(t)\check{k}_A, t|\mathbf{X}\right) \\
&+ {}^{x}a^{I}_{\mu}(t)\frac{\mathrm{d}\, {}^{x}a^{J}_{\nu}}{\mathrm{d}t}(t) \dots {}^{x}a^{\rho}_{K}(t)\, {}^{x}a^{\sigma}_{L}(t) \dots C^{\mu\nu\dots}{}_{\rho\sigma\dots}\left({}^{x}a^{A}_{\alpha}(t)\check{k}_A, t|\mathbf{X}\right) \\
&+ \dots \\
&+ {}^{x}a^{I}_{\mu}(t)\, {}^{x}a^{J}_{\nu}(t) \dots \frac{\mathrm{d}\, {}^{x}a^{\rho}_{K}}{\mathrm{d}t}(t)\, {}^{x}a^{\sigma}_{L}(t) \dots C^{\mu\nu\dots}{}_{\rho\sigma\dots}\left({}^{x}a^{A}_{\alpha}(t)\check{k}_A, t|\mathbf{X}\right) \\
&+ {}^{x}a^{I}_{\mu}(t)\, {}^{x}a^{J}_{\nu}(t) \dots {}^{x}a^{\rho}_{K}(t)\frac{\mathrm{d}\, {}^{x}a^{\sigma}_{L}}{\mathrm{d}t}(t) \dots C^{\mu\nu\dots}{}_{\rho\sigma\dots}\left({}^{x}a^{A}_{\alpha}(t)\check{k}_A, t|\mathbf{X}\right) \\
&+ \dots \\
&+ {}^{x}a^{I}_{\mu}(t)\, {}^{x}a^{J}_{\nu}(t) \dots {}^{x}a^{\rho}_{K}(t)\, {}^{x}a^{\sigma}_{L}(t) \dots \frac{\mathrm{d}\, {}^{x}a^{A}_{\alpha}}{\mathrm{d}t}(t)\check{k}_A \frac{\partial C^{\mu\nu\dots}{}_{\rho\sigma\dots}}{\partial k_\alpha}\left({}^{x}a^{A}_{\alpha}(t)\check{k}_A, t|\mathbf{X}\right) \\
&+ {}^{x}a^{I}_{\mu}(t)\, {}^{x}a^{J}_{\nu}(t) \dots {}^{x}a^{\rho}_{K}(t)\, {}^{x}a^{\sigma}_{L}(t) \dots \frac{\partial C^{\mu\nu\dots}{}_{\rho\sigma\dots}}{\partial t}\left({}^{x}a^{A}_{\alpha}(t)\check{k}_A, t|\mathbf{X}\right) \\
=& {}^{x}a^{I}_{\mu}(t)\, {}^{x}a^{J}_{\nu}(t) \dots {}^{x}a^{\rho}_{K}(t)\, {}^{x}a^{\sigma}_{L}(t) \dots \\
&\times \left\{ \frac{1}{2}(S^{\mu}_{\beta} + A_{\beta}{}^{\mu})C^{\beta\nu\dots}{}_{\rho\sigma\dots} + \frac{1}{2}(S^{\nu}_{\beta} + A_{\beta}{}^{\nu})C^{\mu\beta\dots}{}_{\rho\sigma\dots} + \dots \right. \\
&- \frac{1}{2}(S^{\gamma}_{\rho} + A_{\rho}{}^{\gamma})C^{\mu\nu\dots}{}_{\gamma\sigma\dots} - \frac{1}{2}(S^{\gamma}_{\sigma} + A_{\sigma}{}^{\gamma})C^{\mu\nu\dots}{}_{\rho\gamma\dots} + \dots \\
&\left. + \frac{1}{2}(S^{\beta}_{\gamma} + A_{\gamma}{}^{\beta})k_\beta \frac{\partial C^{\mu\nu\dots}{}_{\rho\sigma\dots}}{\partial k_\gamma} + \frac{\partial C^{\mu\nu\dots}{}_{\rho\sigma\dots}}{\partial t} \right\} \left({}^{x}a^{A}_{\alpha}(t)\check{k}_A, t|\mathbf{X}\right).
\end{aligned}$$

Thus the time derivative in static-metric representation transforms into the following form in the mean-Lagrangian representation.

$$\begin{aligned}
&\frac{\partial}{\partial t} C^{IJ\dots}{}_{KL\dots}(\check{\mathbf{k}}, t|\mathbf{X}) \\
&\quad\mapsto \left\{ \frac{\partial C^{\mu\nu\dots}{}_{\rho\sigma\dots}}{\partial t} \right. \\
&\qquad + \frac{1}{2}(S^{\mu}_{\beta} + A_{\beta}{}^{\mu})C^{\beta\nu\dots}{}_{\rho\sigma\dots} + \frac{1}{2}(S^{\nu}_{\beta} + A_{\beta}{}^{\nu})C^{\mu\beta\dots}{}_{\rho\sigma\dots} + \dots \\
&\qquad - \frac{1}{2}(S^{\gamma}_{\rho} + A_{\rho}{}^{\gamma})C^{\mu\nu\dots}{}_{\gamma\sigma\dots} - \frac{1}{2}(S^{\gamma}_{\sigma} + A_{\sigma}{}^{\gamma})C^{\mu\nu\dots}{}_{\rho\gamma\dots} + \dots \\
&\qquad \left. + \frac{1}{2}(S^{\beta}_{\gamma} + A_{\gamma}{}^{\beta})k_\beta \frac{\partial C^{\mu\nu\dots}{}_{\rho\sigma\dots}}{\partial k_\gamma} \right\} (\mathbf{k}, t|\mathbf{X})
\end{aligned} \tag{E·1}$$

Especially for a vector $\mathbf{C}$, we have

$$\frac{\partial C^I}{\partial t}(\check{\mathbf{k}}, t|\mathbf{X}) \mapsto \left\{ \frac{\partial C^\mu}{\partial t} + \frac{1}{2}(S^{\mu}_{\rho} + A_{\rho}{}^{\mu})C^\rho + \frac{1}{2}(S^{\rho}_{\sigma} + A_{\sigma}{}^{\rho})k_\rho \frac{\partial C^\mu}{\partial k_\sigma} \right\} (\mathbf{k}, t|\mathbf{X}). \tag{E·2}$$

F Self-connected-loop diagram

In constructing the λ^2 diagrams in §7.6, we have omitted another type of diagrams which have loops each of which is closed with only one vertex as is shown in figure 26. These *self-connected loops*, however, do not contribute to the final result as long as we perform the lowest-order-λ truncation. The self-connected loop

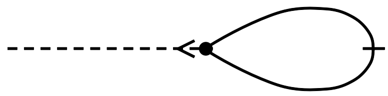


Figure 26: Self-connected loop appearing in simple perturbative expansions

shown in figure 26 is calculated as

$$\begin{aligned}
&\int_{-\infty}^{t} \mathrm{d}t'\ \tilde{G}^{\mu}_{\nu}(\mathbf{k};t,t'|\mathbf{X})\,\frac{1}{i}M^{\nu}_{\rho\sigma}(\mathbf{k},t'|\mathbf{X})\iint \mathrm{d}vol_p\,\mathrm{d}vol_q\,\delta^3_c(\mathbf{k}-\mathbf{p}-\mathbf{q},t'|\mathbf{X})\,\tilde{U}^{\rho\sigma}(\mathbf{p},t';\mathbf{q},t')\\
&\xrightarrow{\text{Renormalization}} \int_{-\infty}^{t} \mathrm{d}t'\ {}^{B}G^{\mu}_{\nu}(\mathbf{k};t,t'|\mathbf{X})\\
&\qquad\times\frac{1}{i}M^{\nu}_{\rho\sigma}(\mathbf{k},t'|\mathbf{X})\iint \mathrm{d}vol_p\,\mathrm{d}vol_q\,\delta^3_c(\mathbf{k}-\mathbf{p}-\mathbf{q},t'|\mathbf{X})\,{}^{B}U^{\rho\sigma}(\mathbf{p},t';\mathbf{q},t')\\
&\qquad+O(\lambda^3)
\end{aligned} \tag{F$\cdot$1}$$

Now the basic field is zero-mean random factor. Thus, by taking the average of (7.95) we obtain

$$\begin{aligned}
&\frac{1}{i}M^{\nu}_{\rho\sigma}(\mathbf{k},t|\mathbf{X})\iint \mathrm{d}vol_p\,\mathrm{d}vol_q\,\delta^3_c(\mathbf{k}-\mathbf{p}-\mathbf{q},t|\mathbf{X})\,{}^{B}U^{\rho\sigma}(\mathbf{p},t;\mathbf{q},t)\\
&\qquad=\hat{L}^{\mu}_{\nu}\,\langle\,{}^{B}v^{\mu}(\mathbf{k},t)\rangle\\
&\qquad=0
\end{aligned} \tag{F$\cdot$2}$$

and hence (F$\cdot$1) $=0+O(\lambda^3)$. Thus the renormalization of any diagrams which include self-connected loops give no contribution at the lowest order. By assuming the isotropy of the bare field, the self-connected loop takes zero value; namely we obtain

$$\begin{aligned}
&\frac{1}{i}M^{\mu}_{\rho\sigma}(\mathbf{k},t|\mathbf{X})\iint \mathrm{d}vol_p\,\mathrm{d}vol_q\,\delta^3_c(\mathbf{k}-\mathbf{p}-\mathbf{q},t|\mathbf{X})\,\langle\tilde{v}^{\rho}(\mathbf{p},t)\tilde{v}^{\sigma}(\mathbf{q},t|\mathbf{X})\rangle\\
&\quad=\frac{1}{i}M^{\mu}_{\rho\sigma}(\mathbf{k},t|\mathbf{X})\iint \mathrm{d}vol_p\,\mathrm{d}vol_q\,\delta^3_c(\mathbf{k}-\mathbf{p}-\mathbf{q},t|\mathbf{X})\tilde{U}^{\rho\sigma}(\mathbf{p},t|\mathbf{X})\delta^3_c(\mathbf{p}+\mathbf{q},t|\mathbf{X})\\
&\quad=\frac{1}{i}M^{\mu}_{\rho\sigma}(\mathbf{k},t|\mathbf{X})\delta^3_c(\mathbf{k},t|\mathbf{X})\int \mathrm{d}vol_p\,\tilde{U}^{\rho\sigma}(\mathbf{p},t|\mathbf{X})\\
&\quad=\frac{1}{i}M^{\mu}_{\rho\sigma}(\mathbf{k},t|\mathbf{X})\delta^3_c(\mathbf{k},t|\mathbf{X})\,\frac{2}{3}g^{\rho\sigma}(t|\mathbf{X})\int \mathrm{d}vol_p\,\tilde{Q}(q,t|\mathbf{X})\\
&\quad=0,
\end{aligned} \tag{F$\cdot$3}$$

where the relation $M^{\mu}_{\rho\sigma}(\mathbf{k},t|\mathbf{X})\,k_{\rho}g^{\rho\sigma}(t|\mathbf{X})=P^{\mu\rho}(\mathbf{k},t|\mathbf{X})\,k_{\rho}=0$ is used.

G Proof of (7.107) and (7.107)

One-wavenumber correlation in the mean-Lagrangian frame is given by

$$\begin{aligned}
{}^{B}U^{\mu\nu}(\mathbf{k};t,t'|\mathbf{X}) &= \int \mathrm{d}vol_{k'}\,{}^{B}U^{\mu\nu}(\mathbf{k},t;\mathbf{k}',t'|\mathbf{X})\\
&= \int \mathrm{d}vol_{k'}\,\langle{}^{B}v^{\mu}(\mathbf{k},t|\mathbf{X}){}^{B}v^{\nu}(\mathbf{k}',t'|\mathbf{X})\rangle\,.
\end{aligned} \tag{G$\cdot$1}$$

Thus, using (7.66) and (7.67), we obtain

$${}^{B}U^{\mu\nu}(\mathbf{k};t,t'|\mathbf{X}) = {}^{X}a^{\mu}_{I}(t){}^{X}a^{\nu}_{J}(t')\int \mathrm{d}vol_{k'}\,\left\langle{}^{B}\check{v}^{I}({}^{X}a^{\alpha}_{A}(t)k_{\alpha},t)\,{}^{B}\check{v}^{J}({}^{X}a^{\beta}_{B}(t')k'_{\beta},t')\right\rangle. \tag{G$\cdot$2}$$

Under the coordinate transformation (7.66), the volume element is transformed as

$$
\begin{aligned}
\mathrm{d}vol_{k'} &= \sqrt{\det\{g^{\alpha\beta}(t'|\mathbf{X})\}}\ \mathrm{d}^3k' \\
&= \sqrt{\det\{g^{\alpha\beta}(t'|\mathbf{X})\}}\ \det\left(\frac{k'_\beta}{\check{k}'_B}\right)\mathrm{d}^3\check{k}' \\
&= \sqrt{\det\{g^{\alpha\beta}(t'|\mathbf{X})\}}\ \det\left\{{}^{X}a^B_\beta(t')\right\}\mathrm{d}^3\check{k}' \\
&= \sqrt{\det\{g^{\alpha\beta}(t'|\mathbf{X})\}\ \det^2\left\{{}^{X}a^B_\beta(t')\right\}}\ \mathrm{d}^3\check{k}' \\
&= \sqrt{\det\left\{g^{\alpha\beta}(t'|\mathbf{X})\,{}^{X}a^A_\alpha(t')\,{}^{X}a^B_\beta(t')\right\}}\ \mathrm{d}^3\check{k}' \\
&= \sqrt{\det\{\check{g}^{AB}\}}\ \mathrm{d}^3\check{k}' = \mathrm{d}vol_{\check{k}'}.
\end{aligned}
\tag{G·3}
$$

Thus we obtain

$$
\begin{aligned}
{}^{B}U^{\mu\nu}(\mathbf{k};t,t'|\mathbf{X}) &= {}^{X}a^\mu_I(t)\,{}^{X}a^\nu_J(t')\int \mathrm{d}vol_{\check{k}'}\langle {}^{B}\check{v}^I(\check{\mathbf{k}},t)\,{}^{B}\check{v}^J(\hat{\mathbf{k}}',t')\rangle \\
&= {}^{X}a^\mu_I(t)\,{}^{X}a^\nu_J(t')\,{}^{B}\check{U}^{IJ}(\check{\mathbf{k}};t,t'|\mathbf{X}).
\end{aligned}
\tag{G·4}
$$

Next, let us turn to the infinitesimal response of the basic field in order to investigate the renormalized propagator. The infinitesimal response in the static-metric field is given by

$$
\begin{aligned}
{}^{B}v^I(\check{\mathbf{k}},t|\mathbf{X}\|\mathbf{\Upsilon}) =& {}^{B}v^I(\check{\mathbf{k}},t|\mathbf{X}) + \int \mathrm{d}vol_{\check{k}'}\int_{-\infty}^{t}\mathrm{d}t'\ {}^{B}G'^I_J(\check{\mathbf{k}},t;\check{\mathbf{k}}',t'|\mathbf{X})\Upsilon^J(\check{\mathbf{k}}',t'|\mathbf{X}) + O(\mathbf{\Upsilon}^2) \\
=& {}^{X}a^I_\mu(t)\,{}^{B}v^\mu(\mathbf{k},t|\mathbf{X}) + {}^{X}a^I_\mu(t)\int \mathrm{d}vol_{k'}\int_{-\infty}^{t}\mathrm{d}t'\ {}^{X}a^\mu_I(t)\,{}^{B}G'^I_J(\check{\mathbf{k}},t;\check{\mathbf{k}}',t'|\mathbf{X})\,{}^{X}a^J_\nu(t')\Upsilon^\nu(\mathbf{k}',t'|\mathbf{X}) \\
&+ O(\mathbf{\Upsilon}^2).
\end{aligned}
\tag{G·5}
$$

Comparing this with (7.97) we have

$$
{}^{B}G'^\mu_\nu(\mathbf{k},t;\mathbf{k}',t'|\mathbf{X}) = {}^{X}a^\mu_I(t)\,{}^{X}a^J_\nu(t')\,{}^{B}G'^I_J(\check{\mathbf{k}},t;\check{\mathbf{k}}',t'|\mathbf{X}). \tag{G·6}
$$

Thus we have

$$
\begin{aligned}
{}^{B}G^\mu_\nu(\mathbf{k};t,t'|\mathbf{X}) &= \int \mathrm{d}vol_{k'}\ \langle {}^{B}G'^\mu_\nu(\mathbf{k},t;\mathbf{k}',t'|\mathbf{X})\rangle \\
&= {}^{X}a^\mu_I(t)\,{}^{X}a^J_\nu(t')\int \mathrm{d}vol_{\check{k}'}\ \left\langle {}^{B}G'^I_J(\check{\mathbf{k}},t;\check{\mathbf{k}}',t'|\mathbf{X})\right\rangle \\
&= {}^{X}a^\mu_I(t)\,{}^{X}a^J_\nu(t')\,{}^{B}G^I_J(\check{\mathbf{k}};t,t'|\mathbf{X}).
\end{aligned}
\tag{G·7}
$$

H Physical meaning of μ^1- and μ^2-order terms

Let us see the physical meaning of C_ν-, C_T-, C_S- and C_C-related terms of our result (9.31), which may be effectively classified in terms of the turbulence-energy production $P_K = -\frac{1}{2}R^{\mu\nu}S_{\mu\nu}$. First, turbulence-energy production rate caused by the eddy-viscosity stress is given by

$$
\frac{1}{2}C_\nu\frac{K^2}{\epsilon}S^a_bS^b_a\ (\geq 0)\,,
$$

which never takes negative value, and thus the eddy-viscosity stress only reduces the mean-flow energy, never enhances. Thus the eddy-viscosity stress is *dissipative*. Next, in order to see the C_T related term, we calculate the time-derivative of $\mathbf{S}\cdot\mathbf{S}$, that is

$$
\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t}S^{ij}S_{ij} &= \frac{\mathrm{O}}{\mathrm{O}t}\left(S^{ij}g_{ia}g_{jb}S^{ab}\right) = \frac{\mathrm{O}S^{ij}}{\mathrm{O}t}g_{ia}g_{jb}S^{ab} + S^{ij}\frac{\mathrm{O}g_{ia}}{\mathrm{O}t}g_{jb}S^{ab} + S^{ij}g_{ia}\frac{\mathrm{O}g_{jb}}{\mathrm{O}t}S^{ab} + S^{ij}g_{ia}g_{jb}\frac{\mathrm{O}S^{ab}}{\mathrm{O}t} \\
&= 2S_{ab}\frac{\mathrm{O}S^{ab}}{\mathrm{O}t} + 2S^a_bS^b_cS^c_a,
\end{aligned}
$$

Thus we have

$$S_{ij}\left(\frac{\mathrm{O}S^{ij}}{\mathrm{O}t}+S^i_aS^{ja}\right)=\frac{1}{2}\frac{\mathrm{D}}{\mathrm{D}t}S^{ij}S_{ij}, \tag{H·1}$$

where we should notice that the Lagrangian derivative on *scalar* is covariant. The C_T-related stress yields the following production rate;

$$\frac{1}{2}C_T\frac{K^3}{\epsilon^2}\frac{\mathrm{D}}{\mathrm{D}t}S^{ij}S_{ij}\,. \tag{H·2}$$

By coupling this with that of the eddy-viscosity stress, we obtain

$$\frac{1}{2}C_\nu\frac{K^2}{\epsilon}\left(1-\frac{C_T}{2C_\nu}\frac{K}{\epsilon}\frac{\mathrm{D}}{\mathrm{D}t}\right)\mathbf{S}\cdot\mathbf{S}\,. \tag{H·3}$$

Thus C_T-related stress can be interpreted as *the delay-response of the eddy-viscosity dissipation against the acceleration of the straining motion.* The C_S-related stress yields the following production rate;

$$-\frac{1}{2}C_S\frac{K^3}{\epsilon^2}S^a_bS^b_cS^c_a. \tag{H·4}$$

We employ the proper-orthogonal representation to investigate the triple product in a simple manner. We put the principal values of $\mathbf{S}$ as $\alpha,\ \beta$, and γ so that the triple product is to be given by $\alpha^3+\beta^3+\gamma^3$. In general we have

$$(\alpha+\beta+\gamma)^3=\alpha^3+\beta^3+\gamma^3+3\alpha^2(\beta+\gamma)+3\beta^2(\gamma+\alpha)+3\gamma^2(\alpha+\beta)+6\alpha\beta\gamma.$$

Using the incompressibility condition $\alpha+\beta+\gamma=0$, we obtain

$$0^3=\alpha^3+\beta^3+\gamma^3+3\alpha^2(-\alpha)+3\beta^2(-\beta)+3\gamma^2(-\gamma)+6\alpha\beta\gamma, \tag{H·5}$$

which results in $\alpha^3+\beta^3+\gamma^3=3\alpha\beta\gamma=3I\!I\!I_S$, where $I\!I\!I_S$ is the third invariant of $\mathbf{S}$. Thus turbulence-energy production rate caused by C_S-related stress is

$$-\frac{3}{2}C_S\frac{K^3}{\epsilon^2}I\!I\!I_S\,. \tag{H·6}$$

$I\!I\!I_S$ can be both positive and negative; it is positive when one of three principal values ($\alpha,\ \beta$, and γ) is positive and the others negative, it is negative when one of them is negative and the others positive, it is zero when one of them is zero. Thus we can summarize the role of C_S-related stress as follows.

1. It enhances the mean-flow energy when the mean-flow is 1-elongation 2-contraction straining (figure 27(a)).
2. It reduces the mean-flow energy when the mean-flow is 2-elongation 1-contraction straining (figure 27(b)).
3. It neither reduces nor enhances the mean-flow energy when the mean-flow is 2-dimensionally straining.

Thus we may refer to C_S-related stress as the *reversible stress.*

The rest is the $\mathbf{S}$-$\boldsymbol{\Theta}$-cross term which may be referred to as *orthogonal stress* since its scalar product with strain rate is zero as follows;

$$\left(S^i_l\Theta^{jl}+S^j_l\Theta^{il}\right)S_{ij}=0. \tag{H·7}$$

Thus this term does not give any contribution to the mean-flow-energy cascade. Because of its *orthogonality* to the strain rate, this term produces totally different tensorial form from the others which are related to the strain rate. In order to obtained a simple representation of $\mathbf{S}$-$\boldsymbol{\Theta}$-cross term, we take the proper-orthogonal representation of the strain rate. We write the matrix representations of $\mathbf{S}$, $\boldsymbol{\Theta}$ and $\mathbf{g}$ as

$$\left[S^{ij}\right]=\begin{bmatrix}\alpha&&\\&\beta&\\&&\gamma\end{bmatrix},\quad\left[\Theta^{ij}\right]=\begin{bmatrix}&-\zeta&\eta\\\zeta&&-\xi\\-\eta&\xi&\end{bmatrix},\quad\left[g_{ij}\right]=\left[g^{ij}\right]=\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix}, \tag{H·8}$$

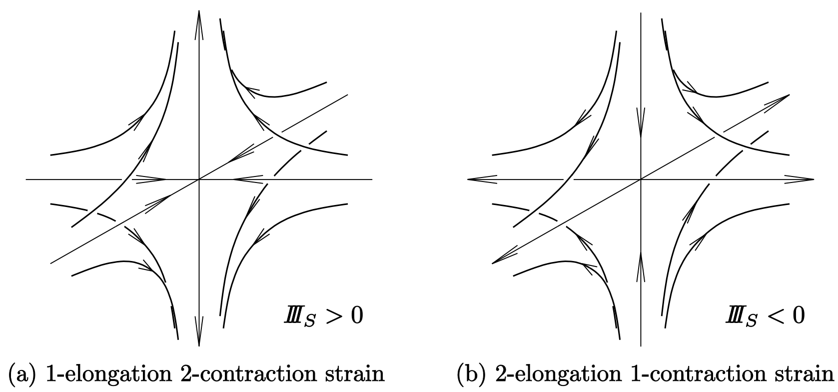


Figure 27: Strain types and III_S; (a) If the mean-flow has a straining motion of 1-elongation 2-contraction, one of three eigen value is positive and the other two negative so that III_S is to be positive. (b) On the other hand if the mean-flow has a straining motion of 2-elongation 1-contraction, one of three eigen values is negative and the others positive so that III_S is to be negative.

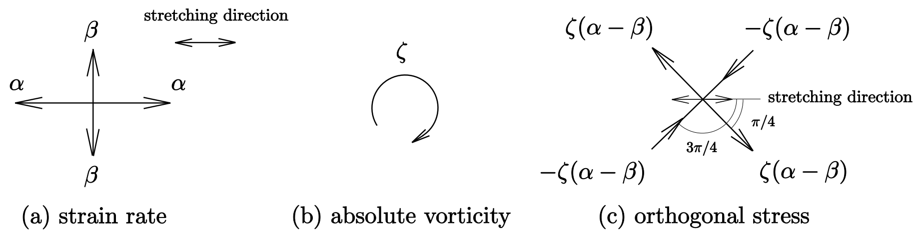


Figure 28: Orthogonal stress; the strain rate and the orthogonal stress are represented in terms of the principal axes and the principal values. Providing the mean flow is more stretching in one direction than the other (a) and the absolute vorticity takes non-zero value (b), the orthogonal stress produces the positive (negative) normal stress in the direction turned $\pi/4$ ($3\pi/4$) along the absolute vorticity from the stretching direction (c).

where α, β and γ are again the eigenvalues of $\mathbf{S}$. $\xi(=-\Theta^{23})$, $\eta(=-\Theta^{31})$ and $\zeta(=-\Theta^{12})$ are the positive-oriented absolute vorticities of 1, 2 and 3 axis. In this representation, we obtain

$$\left[S^i_a\Theta^{ja}\right]=\left[S^{ia}\right]\left[g_{ab}\right]\left(-\left[\Theta^{jb}\right]^T\right)=\begin{bmatrix} & \alpha\zeta & -\alpha\eta\\ -\beta\zeta & & \beta\xi\\ \gamma\eta & -\gamma\xi & \end{bmatrix}, \tag{H·9}$$

and finally we reach

$$\left[S^i_a\Theta^{ja}+S^j_a\Theta^{ia}\right]=\begin{bmatrix} & (\alpha-\beta)\zeta & (\gamma-\alpha)\eta\\ (\alpha-\beta)\zeta & & (\beta-\gamma)\xi\\ (\gamma-\alpha)\eta & (\beta-\gamma)\xi & \end{bmatrix}. \tag{H·10}$$

Let us investigate the above result in 1-2 plane. If we assume $\alpha>\beta$ and $\xi>0$, the 1-2 component $(\alpha-\beta)\zeta$ is positive. By applying rotational transformation at the angle of $\pi/4$ in 1-2 plane along the absolute vorticity, we obtain

$$\begin{bmatrix}\cos\pi/4 & \sin\pi/4\\ -\sin\pi/4 & \cos\pi/4\end{bmatrix}\begin{bmatrix} & (\alpha-\beta)\zeta\\ (\alpha-\beta)\zeta & \end{bmatrix}\begin{bmatrix}\cos\pi/4 & -\sin\pi/4\\ \sin\pi/4 & \cos\pi/4\end{bmatrix}=\begin{bmatrix}(\alpha-\beta)\zeta & \\ & -(\alpha-\beta)\zeta\end{bmatrix}, \tag{H·11}$$

which may be depicted in the figure 28. In general the orthogonal stress causes the positive/negative normal stress in the direction turned $\pi/4$/$3\pi/4$ along the absolute vorticity from the stretching direction.

I Application to the simple shear flow

Let us see an application of (9.31) to the simple shear flow. We employ a Cartesian inertial coordinate system $\{z_1, z_2, z_3\}$. Let z^1, z^2 and z^3 be of the streamwise, normalwise and spanwise directions respectively, whose

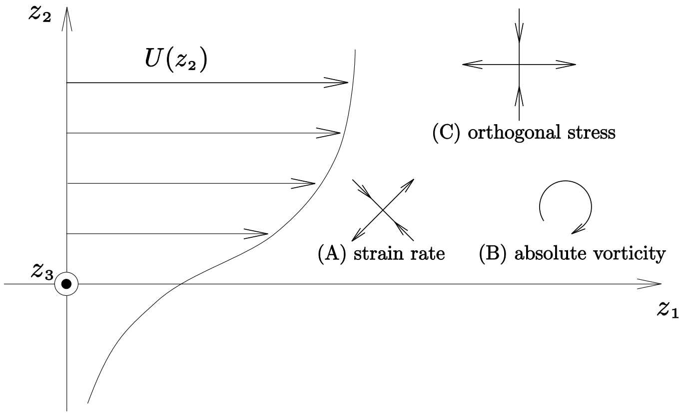


Figure 29: Configuration of simple shear flow; Mean velocity has only the horizontal component. The strain rate and orthogonal stress are represented in terms of the principal axes and the signs of principal values (the outward arrows show positive principal value while the inward negative). The mean flow is stretched in $\pi/4$-rotated direction from the streamwise direction as (A), and has the right-handed absolute vorticity as (B). The orthogonal stress produces positive/negative normal stress in streamwise/normalwise direction as (C).

configuration is depicted in figure 29. In this situation, the mean flow is stretched in $\pi/4$-rotated direction from the stream-wise direction and has the right-handed absolute vorticity. Thus the orthogonal stress may produce positive/negative stress in the stream-wise/perpendicular-wise direction. Indeed the simple application of our result (9.31) to this system yields

$$
\begin{aligned}
R^{11} &= \frac{2}{3}K + C_{R1}\frac{K^3}{\epsilon^2}D^2, \\
R^{22} &= \frac{2}{3}K + C_{R2}\frac{K^3}{\epsilon^2}D^2, \\
R^{33} &= \frac{2}{3}K + C_{R3}\frac{K^3}{\epsilon^2}D^2,
\end{aligned} \tag{I$\cdot$1}
$$

where $D = \mathrm{d}U/\mathrm{d}z_2$ is the velocity gradient,

$$
\begin{aligned}
C_{R1} &= -C_T + \frac{1}{3}C_S + 2C_C = 4.57 \times 10^{-4} \times (\pi C_\sigma C_\omega)^{-2}, \\
C_{R2} &= C_T + \frac{1}{3}C_S - 2C_C = -3.66 \times 10^{-4} \times (\pi C_\sigma C_\omega)^{-2}, \\
C_{R3} &= -\frac{2}{3}C_S = -9.14 \times 10^{-5} \times (\pi C_\sigma C_\omega)^{-2},
\end{aligned} \tag{I$\cdot$2}
$$

are all dimensionless constants. According to our result, $C_{R1} > C_{R3} > C_{R2}$ so that we obtain

$$
R^{11} > R^{33} > R^{22} \tag{I$\cdot$3}
$$

which is consistent result with experiments and simulations [40]. Let us see more detailed cause of this consistency. (I·3) can be rewritten as $R^{11} > R^{33}$ and $R^{33} > R^{22}$, which can be rephrased in our representation as $C_{R1} - C_{R3} > 0$ and $C_{R3} - C_{R2} > 0$. Here we should notice that $C_{R1} - C_{R3} = C_{R3} - C_{R2} + C_S > C_{R3} - C_{R2}$ which indicates only $C_{R3} - C_{R2} > 0$ is needed. Thus the following inequality is necessary and sufficient for the proper relation (I·3);

$$
C_{R3} - C_{R2} = 2C_C - C_T - C_S > 0 \tag{I$\cdot$4}
$$

Since both C_T and C_S are positive, this indicates that the C_C related term, namely orthogonal stress, plays the indispensable role in the anisotropic intensity distribution (I·3). On the contrary, the result from TSDIA (4.16) yields

$$
R^{11} > R^{22} > R^{33}, \tag{I$\cdot$5}
$$

[40] Note that this consistent condition (I·3) always holds irrespective of the values of C_σ and C_ω.

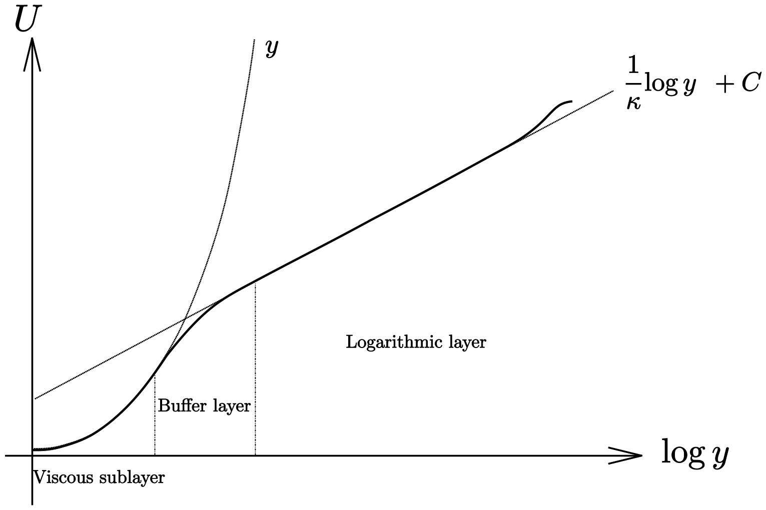


Figure 30: Out line of the mean-velocity profile of the wall-bounded turbulent flow: The universal structure of high-Reynolds-number flow in the near-wall region is depicted. In the closest area from the wall, there exist the *viscous sublayer* where the molecular viscosity is dominant in the momentum diffusion. In the outer region, the *logarithmic layer* exists where the molecular viscosity is negligible. The logarithmic velocity profile is universally observed in both experiments and simulations.

which contradicts the real feature. The cause of this failure can be explained as follows; (4.16) has smaller orthogonal stress and larger reversible stress than those of the present result. Although (4.16) is not generally covariant form, we can rewrite this as follows as long as we remain in the orthonormal Eulerian coordinate system;

$$\begin{aligned} R_{IJ} = \frac{2}{3}K\delta_{IJ} - \tilde{C}_\nu \frac{K^2}{\epsilon} S_{IJ} + \tilde{C}_T \frac{K^3}{\epsilon^2}\frac{\mathrm{D}S_{IJ}}{\mathrm{D}t} \\ + \tilde{C}_S \frac{K^3}{\epsilon^2}\left(S_{ik}S_{jk} - \frac{1}{3}\mathbf{S}\cdot\mathbf{S}\delta_{IJ}\right) \\ + \tilde{C}_C \frac{K^3}{\epsilon^2}\left(S_{ik}\Omega_{jk} + S_{jk}\Omega_{ik}\right), \end{aligned} \tag{I·6}$$

where the constants $\tilde{C}_\nu$-$\tilde{C}_C$ given by

$$\tilde{C}_\nu = 0.123,\ \tilde{C}_T = 0.0427,\ \tilde{C}_S = 0.0297,\ \tilde{C}_C = 0.0122,$$

do not take the same values as C_ν-C_C of the present result (Okamoto 1994). The necessary and sufficient condition for (I·3) is $2\tilde{C}_C - \tilde{C}_S > 0$. However, it does not hold since $\tilde{C}_C(= 0.0122)$ is less than half of $C_C(= 0.0256)$ and $\tilde{C}_S(= 0.0297)$ is almost five times larger than $C_S(= 0.00569)$. Thus, the weaker orthogonal stress and the stronger reversible stress cause the unrealistic result (I·5).

It is remarkable that the inhomogeneity terms in (9.31) give no contribution to the normal components. The inhomogeneity effect can be seen in the shear component which is given by

$$R^{12} = -C_\nu \frac{K^2}{\epsilon}\frac{\mathrm{d}U}{\mathrm{d}z_2} - C_\alpha \frac{K^5}{\epsilon^3}\frac{\mathrm{d}^3U}{\mathrm{d}z_2^3}, \tag{I·7}$$

where C_α is a positive constant given by

$$C_\alpha = -C_{\mathrm{MD1}} + C_{\mathrm{MD2}} + C_{\mathrm{MD3}} = 1.89 \times 10^{-4}. \tag{I·8}$$

In order to avoid confusion in spatial differentiation, we utilize here the covariant component of the coordinate variable z_2.

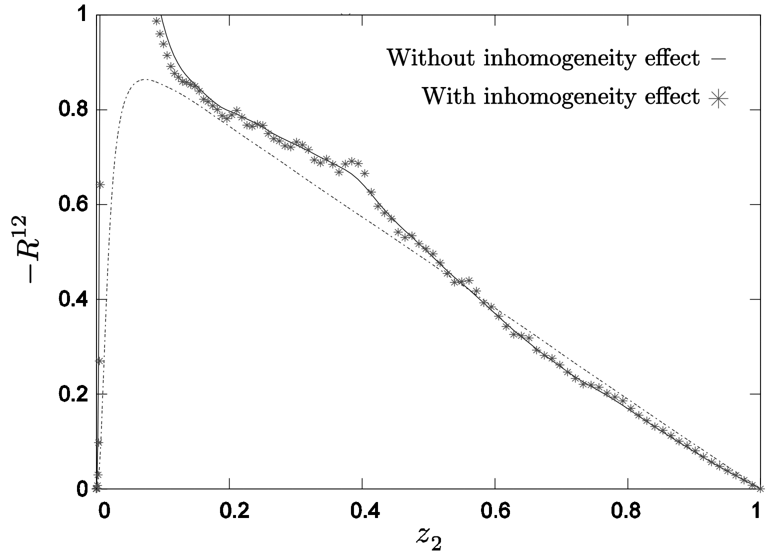


Figure 31: Comparison of the shear stresses with and without inhomogeneity effect. The maximum value of $|-C_\alpha(K^5/\epsilon^3)(\mathrm{d}^3U/\mathrm{d}^3z_2)|$ (the magnitude of the inhomogeneity effect on shear stress) reaches 0.028 at $z_2 = 0.39$ in the region $0.2 \leq z_2 \leq 1$. Although the roughness of the third-order derivative of the mean velocity is reflected in the profile, the inhomogeneity term does not contribute to the total shear stress so much as the linear-viscosity term.

It is wellknown that the standard K-ϵ model is consistent with the logarithmic velocity profile typically observed in wall-bounded turbulent flows (figure 30) and that the standard K-ϵ form $R^{12} = -C_\nu K^2/\epsilon$ has the asymptotic solution of the logarithmic velocity profile. In the present work, however, we derived (I·7) which contains an extra term. We should remark that our result (I·7) does not contradict this velocity profile. In the logarithmic layer physical quantities are approximately given by

$$U \approx \frac{1}{\kappa}\log z_2 + const.\ ,\ \epsilon \approx \frac{1}{\kappa\, z_2},\ K \approx 3,\ R^{12} \approx -1,$$

where all the quantities are normalized by a velocity scale $U_\tau \equiv (\nu \mathrm{d}U/dz_2)^{1/2}$ and a length scale ν/U_τ. κ is a non-dimensional universal constant called the *Karman constant* which is estimated as 0.41 from observations. Under the above approximation, C_α term is reduced to

$$-C_\alpha \frac{K^5}{\epsilon^3}\frac{\mathrm{d}^3U}{\mathrm{d}z_2^3} \approx -C_\alpha 3^5 (\kappa\, z_2)^3\, \frac{1}{\kappa\, z_2^3} = -C_\alpha 3^5 \kappa^2 \approx -0.016, \tag{I·9}$$

which gives only little effect on the total shear stress $R^{12} \approx -1$. Thus, in the wall-bounded turbulence, C_α-related term may not contribute so much comparing to the linear-eddy-viscosity term and, consequently, it may not affect the nature of logarithmic layer.

The inclusion of inhomogeneity term is examined in terms of the shear stress in a priori test (see §9.6.2), whose result is shown in figure 31. Although the roughness caused by the third-order derivative of the mean velocity appears, the inhomogeneity term does not contributes to the total shear stress so much as the linear-viscosity term. The roughness may be removed by taking the average over larger ensemble.

J Application to the axisymmetric flow

Axisymmetric flows are typical flows appearing in various industrial situations, and these are still important target of turbulence-modeling studies. These flows have curvatures in their streamlines or isosurfaces of scalars so that we can expect new physical effect distinguished from those in the simple shear flows. Let us see here an axially-uniform axisymmetric flows for simplicity.

Here we employ the cylindrical coordinate system $\{r, \theta, z\}$ whose configuration is given by figure 32. For simplicity we assume the system is uniform in axial direction. We put the velocity components as

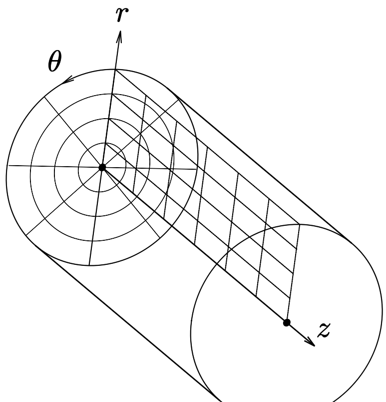


Figure 32: Configuration of the cylindrical coordinate system

$\big(0, V^\theta(r), V^z(r)\big)$; the radial component should vanish because of the incompressibility. In this coordinate representation, r-θ and r-z components of (9.31) are given by

$$R^{r\theta} = -C_\nu \frac{K^2}{\epsilon}\frac{\mathrm{d}V^\theta}{\mathrm{d}r} - C_\alpha \frac{K^5}{\epsilon^3}\frac{\mathrm{d}^3V^\theta}{\mathrm{d}r^3} + C_\beta \frac{K^5}{\epsilon^3}\frac{1}{r}\frac{\mathrm{d}^2V^\theta}{\mathrm{d}r^2} + C_\gamma \frac{K^5}{\epsilon^3}\frac{1}{r^2}\frac{\mathrm{d}V^\theta}{\mathrm{d}r}, \tag{J·1}$$

$$R^{rz} = -C_\nu \frac{K^2}{\epsilon}\frac{\mathrm{d}V^z}{\mathrm{d}r} - C_\alpha \frac{K^5}{\epsilon^3}\frac{\mathrm{d}^3V^z}{\mathrm{d}r^3} + C'_\beta \frac{K^5}{\epsilon^3}\frac{1}{r}\frac{\mathrm{d}^2V^\theta}{\mathrm{d}r^2} + C'_\gamma \frac{K^5}{\epsilon^3}\frac{1}{r^2}\frac{\mathrm{d}V^\theta}{\mathrm{d}r}, \tag{J·2}$$

where C_α-C'_γ are all positive constants given by

$$\begin{aligned}
C_\alpha &= -C_{\mathrm{MD1}} + C_{\mathrm{MD2}} + C_{\mathrm{MD3}}\,,\\
C_\beta &= 3C_{\mathrm{MD1}} - 3C_{\mathrm{MD2}} - 2C_{\mathrm{MD3}}\,,\\
C_\gamma &= C_{\mathrm{MD1}} - C_{\mathrm{MD2}}\,,\\
C'_\beta &= 3C_{\mathrm{MD1}} - 3C_{\mathrm{MD2}} - 2C_{\mathrm{MD3}}\,,\\
C'_\gamma &= -C_{\mathrm{MD1}} + C_{\mathrm{MD2}} + 2C_{\mathrm{MD3}}\,.
\end{aligned} \tag{J·3}$$

First of all, we should remark that nonlinear-eddy-viscosity effects do not contribute to r-θ and r-z components, which are the shear stresses on the cylindrical surface. Up to second-order nonlinear-eddy viscosity, the first departure from the linear-eddy viscosity is achieved by the inhomogeneity effect. Next, the third and fourth terms on the right-hand sides of (J·1) and (J·2) originate from the curvature of the stream line and these terms do not appear in the simple shear flow in (I·7). Especially the fourth term can be combined with the first term as

$$R^{r\theta} = -C_\nu \frac{K^2}{\epsilon}\left(1 - \frac{C_\gamma}{C_\nu}\frac{K^3}{\epsilon^2}r^{-2}\right)\frac{\mathrm{d}V^\theta}{\mathrm{d}r} + C_\beta \frac{K^5}{\epsilon^3}\frac{1}{r}\frac{\mathrm{d}^2V^\theta}{\mathrm{d}r^2} - C_\alpha \frac{K^5}{\epsilon^3}\frac{\mathrm{d}^3V^\theta}{\mathrm{d}r^3}, \tag{J·4}$$

$$R^{rz} = -C_\nu \frac{K^2}{\epsilon}\left(1 - \frac{C'_\gamma}{C_\nu}\frac{K^3}{\epsilon^2}r^{-2}\right)\frac{\mathrm{d}V^z}{\mathrm{d}r} + C'_\beta \frac{K^5}{\epsilon^3}\frac{1}{r}\frac{\mathrm{d}^2V^z}{\mathrm{d}r^2} - C_\alpha \frac{K^5}{\epsilon^3}\frac{\mathrm{d}^3V^z}{\mathrm{d}r^3}. \tag{J·5}$$

Thus, in the core region, the eddy viscosity may be effectively reduced by the inhomogeneity effect. This feature is actually needed in the swirling flow in a circular pipe. Let us suppose that strongly swirling flow is imposed from the inlet and there exist the pressure gap between inlet and outlet. It is wellknown from experiments that the streamwise velocity is effectively reduced in the core region, and the velocity reduction continues for long along the pipe. In the simple eddy-viscosity or nonlinear eddy-viscosity models, however, this velocity reduction breaks down soon after the inlet. This is often attributed to the overestimated shear stresses $R^{r\theta}$ and R^{rz} which diffuse both swirling motion and velocity reduction of the core. In the present results (J·4) and (J·5), however, $R^{r\theta}$ and R^{rz} are effectively reduced in core region, which may contribute to the better prediction of velocity reduction.

References

[1] Craft, T. J., Kidger, J. W. & Launder, B. E. 2000 Second-moment modelling of developing and self-similar three-dimensional turbulent free-surface jets. *Int. J. Heat Fluid Flow* **21**, 338-344.

[2] Craft, T. J. & Launder, B. E. 2001 Principles and performance of TCL-based second-moment closures. *Flow, Turbulence Combustion* **66**, 355-372.

[3] Craft, T. J. & Lien, F. S. 1995 Computation of flow through a circular to rectangular transition duct using advanced turbulence models. *Proceedings 4th ERCOFTAC/IAHR Refined Flow Modelling*, Karlsruhe, Germany.

[4] Eringen, C. 1975 *Continuum Physics vol. II.* ACADEMIC PRESS, New york.

[5] Foster, D. Nelson, D. R. & Stephan, M. J. 1976 Long-time tails and the large-eddy behavior of a randomly stirred fluid. *Phys. Rev. Lett.* **36**, 867-870.

[6] Foster, D. Nelson, D. R. & Stephan, M. J. 1977 Large-distance and long-time properties of a randomly stirred fluid. *Phys. Rev. A* **16**, 732-749.

[7] Guo, Z. B., Diamond, P. H. & Wang, X. G. 2012 Magnetic reconnection, helicity dynamics, and hyper diffusion. *APJ* **757**, 173-187.

[8] Hanjalić, K. & Launder, B. E. 1972 A Reynolds stress model of turbulence and its application to thin shear flows. *J. Fluid Mech.* **52**, 609-638.

[9] Hanjalić, K. & Launder, B. E. 2011 *Modelling Turbulence in Engineering and the Environment.* 60-142, Cambridge University press.

[10] Hamba, F. 1987 Statistical analysis of chemically reacting passive scalars in turbulent shear flows. *J. Phys. Soc. Jpn.* **56**, 79-96.

[11] Hamba, F. 2006 Euclidean invariance and weak-equilibrium condition for the algebraic Reynolds stress model. *J. Fluid Mech.* **569**, 399-408.

[12] Hamba, F. & Sato, H. 2008 Turbulent transport coefficients and residual energy in mean-field dynamo theory. *Phys. Plasmas* **15**, 022302.

[13] Kaneda, Y. 1981 Renormalized expansion in the theory of turbulence with the use of the Lagrangian position function. *J. Fluid Mech.* **107**, 131-145.

[14] Kaneda, Y. 1986 Inertial range structure of turbulent velocity and scalar fields in a Lagrangian renormalized approximation. *Phys. Fluids* **29**, 701-708.

[15] Kida, S. & Goto, S. 1997 A Lagrangian direct-interaction approximation for homogeneous isotropic turbulence. *J. Fluid Mech.* **345** 307-345.

[16] Kidger, J. W. 1999 Turbulence modelling for stably stratified flows and free surface jets. Ph.D. Thesis, University of Manchester Institute of Science and Technology.

[17] Kolmogorov, A. N. 1941 The local structure of turbulence in incompressive viscous fluid for very large Reynolds numbers. *C. R. Acad. Sci. USSR* **30** 301-305.

[18] Kraichnan, R. H. 1959 The structure of isotropic turbulence at very high Reynolds numbers. *J. Fluid Mech.* **5** 497-543.

[19] Kraichnan, R. H. 1964 Direct-interaction approximation for shear and thermally driven turbulence. *Phys. Fluids* **7** 1048-1062.

[20] Kraichnan, R. H. 1965 Lagrangian-history closure approximation for turbulence. *Phys. Fluids* **8** 575-598.

[21] Kraichnan, R. H. 1966 Isotropic turbulence and inertial-range structure. *Phys. Fluids* **9** 1728-1725.

[22] Kraichnan, R. H. 1977 Eulerian and Lagrangian renormalization in turbulence theory. *J. Fluid Mech.* **83** 349-374.

[23] Launder, B. E., Reece, G. J. & Rodi, W. 1975 Progress in the development of a Reynolds-stress turbulence closure. *J. Fluid Mech.* **68** 537-566.

[24] Launder, B. E. & Spalding, D. B. 1974 The numerical computation of turbulent flows. *Comput. Methods Appl. Mech. Engng* **3** 269-289.

[25] Launder, B. E. & Li, S. P. 1994 On the elimination of wall-topography parameters from second-moment closure. *Phys. Fluids* **6** 999-1006.

[26] Lesieur, M. 1987 *Turbulence in Fluids, second edition.* 213-283, Kluwer Academic Publishers.

[27] Leslie, D. C. 1973 *Developments in the theory of turbulence.* 312-355, Oxford University press.

[28] Moser, R. Kim, J. & Mansour, N. N. 1999 Direct numerical simulation of turbulent channel flow up to $Re_\tau = 590$. *Phys. Fluid* **11**, 943-945.

[29] Naot, D., Shavit, A. & Wolfshtein, M. 1970 Interactions between components of the turbulent correlation tensor. *I*sr. J. Technol. **8**, 259-269.

[30] Nisizima, S. & Yoshizawa, A. 1987 Turbulent channel and Couette flows using an anisotropic K-ϵ model. *AIAA Journal* **25**, 414-420.

[31] Okamoto, M. 1994 Theoretical investigation of an eddy-viscosity-type representation of the Reynolds stress. *J. Phys. Soc. Jpn* **63**, 2102-2122.

[32] Okamoto, M. 1995 Theoretical turbulence modelling of homogeneous decaying flow in a rotating frame. *J. Phys. Soc. Jpn* **64**, 2854-2867.

[33] Oldroyd, J. G. 1950 On the formulation of rheological equations of state. *Proc. Roy. Soc. A* **200**, 523-541.

[34] Oldroyd, J. G. 1959 Non-Newtonian effects in stready motion of some idealized elastic-viscous liquids. *Proc. Roy. A* **295**, 278.

[35] Orszag, S. A. 1977 Statistical theory of turbulence. In *Fluid Dynamics 1973*, Les Houches Summer School of Theoretical Physics, R. Balian and J. L. Peube eds, Gordon and Breach, 237-374.

[36] Pouquet, A., Lesieur, M. & André 1975 Evolution of high Reynolds number two-dimensional turbulence. *J. Fluid Mech.* **72**, 305-319.

[37] Reynolds, O. 1883 An experimental investigation of the circumstances which determine whether the motion of water shall be direct and sinuous, and the law of resistance in parallel channels. *Phil. Trans. Roy. Soc.*, 51-105.

[38] Rubinstein, R. & Barton, J. M. 1990 Nonlinear Reynolds stress models and the Renormalization group. *Phys. Fluid* A **2**, 1472-1476.

[39] Speziale, C. G. 1979 Invariance of turbulent closure models. *Phys. Fluids* **22**, 1033-1037.

[40] Speziale, C. G. 1982 On turbulent secondary flow in pipes of noncircular cross-section. *Int. J. Engng Sci.* **20**, 863-872.

[41] Speziale, C. G. 1993 A Consistency condition for non-linear algebraic Reynolds stress models in turbulence. *Int. J. Non-linear Mechanics* **33**, 579-584.

[42] Sreenivasan, K. R. 1995 On the universality of the Kolmogorov constant. *Phys. Fluids* **7**, 2778-2784.

[43] Stokes, G. G. 1845 On the theories of the internal friction of fluids in motion, and of equilibrium and motion of elastic solids. *T*rans. Camb. Phil. Soc. **8**, 287-319.

[44] Suga, K. 2003 Predicting turbulence and heat transfer in 3-D curved ducts by near-wall second moment closures. *Int. J. Heat Mass Transfer* **46**, 161-173.

[45] Taylor, G. I. 1935 Statistical theory of turbulence. Part1-4 *Proc. Roy. Soc.* **A 151**, 421-478.

[46] Tennekes, H. & Lumley, J. L. 1972 *A first course in turbulence.* pp300, MIT press

[47] Weis, J. & Hutter, K. 2003 On Euclidean invariance of algebraic Reynolds stress models in turbulence. *J. Fluid. Mech.* **476**, 63-68.

[48] Wilcox, D. C. 1988 Reassessment of the scale-determining equation for advanced turbulence models *AIAA J.* **26**, 1299-1310.

[49] Wyld, H. W. 1961 Formulation of the theory of turbulence in an incompressible fluid. *Ann. phys.* **14**, 143-165.

[50] Yakhot, V. M. & Orszag, S. A. 1986 Renormalization group analysis of turbulence. I. Basic theory *J. Sci. Comput.* **1**, 3-52.

[51] Yokoi, N. & Yoshizawa, A. 1993 Statistical analysis of the effects of helicity in inhomogeneous turbulence. *Phys. Fluids* **A5**, 464-477.

[52] Yoshizawa, A. 1978 A governing equation for the small-scale turbulence. II. Modified DIA approach and Kolmogorov's -5/3 power law. *J. Phys. Soc. Jpn* **45**, 1734-1740.

[53] Yoshizawa, A. 1984 Statistical analysis of the deviation of the Reynolds stress from its eddy-viscosity representation. *Phys. Fluids* **27**, 1377-1387.

[54] Yoshizawa, A. 1985a A statistically derived system of equations for turbulent shear flows. *Phys. Fluids* **28**, 59-63.

[55] Yoshizawa, A. 1985 Statistical theory of anisotropy of scalar diffusion in turbulent shear flows. *Phys. Fluids* **28**, 3226-3231.

[56] Yoshizawa, A. 1985c Statistical theory for magnetohydrodynamic turbulent shear flows. *Phys. Fluids* **28**, 3313-3320.

[57] Yoshizawa, A. 1990a Three equation modeling of inhomogeneous compressible turbulence based on a two-scale direct-interaction approximation. *Phys. Fluids* A**2**, 838-850.

[58] Yoshizawa, A. 1990b Self-consistent turbulent dynamo modeling of reversed field pinches and planetary magnetic field. *Phys. Fluids* B**2**, 1589-1600.

[59] Yoshizawa, A. 1991 Statistical modeling of compressible turbulence: Shock-wave/turbulence interactions and Buoyancy effects. *J. Phys. Soc. Jpn.* **60**, 2500-2504.

[60] Yoshizawa, A. 1992 Statistical analysis of compressible turbulent flows, with special emphasis on turbulence modeling. *Phys. Rev. A* **46**, 3292-3306.

[61] Yoshizawa, A. 1998 *Hydrodynamic and Magnetohydrodynamic Turbulent Flow: modelling and Statistical Theory.* 173-253, Kluwer Academic Publishers.

[62] Yoshizawa, A. 2003 Statistical theory of compressible turbulence based on mass-weighted averaging, with special emphasis on a cause of counter-gradient diffusion. *Phys. Fluids* **20**, 585-596.