

論文の内容の要旨

Mean-Lagrangian renormalization theory of inhomogeneous turbulent flow

(非一様乱流に対する平均ラグランジュ的繰り込み理論)

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1 Background

In nature, fluids often show stochastic and disordered behaviors both in time and space. These phenomena are inclusively called the “*turbulent flows*”. In various circumstances in the real world such that the Reynolds number exceeds some thousands or millions, fluid turbulence plays important roles; it promotes the mixing of the fuel and air in engines, it enhances the drag force on the surfaces of cars, ships or airplanes, it helps the oxygen to dissolve into ocean, it equalizes the temperature of the atmosphere, it diffuses the gigantic magnetic flux in the sun, interstellar gasses or galaxies. Namely, almost everywhere in our universe, there are lots of phenomena which can never be explained without appropriate knowledges about turbulence. Because of its universality and wide applicability, fluid turbulence has been the targets of various scientific fields and the disclosure of its essence should give huge impacts to wide variety of field of both pure and applied science.

The incompressible Newtonian fluids, which we simply call “fluids” in this thesis, are governed by the Navier-Stokes equation and incompressibility condition. Because of its chaotic and unpredictable behavior, we apply *ensemble averaging* (let us denote averaging by $\langle \dots \rangle$) to the Navier-Stokes equation, which yield

$$\frac{D}{Dt} V_i = -P_{,i} + \nu \Delta V_i - \langle v'_i v'_j \rangle_{,j}, \quad (1)$$

where \mathbf{V} and P are the mean velocity and pressure, $D/Dt = \partial/\partial t + V_j \partial_j$ is the Lagrangian derivative based on the mean velocity, ν is the molecular viscosity coefficient. The two-rank tensor $\langle v'_i v'_j \rangle (= R_{ij})$, represented as the second-order moment of the velocity fluctuation \mathbf{v}' , is called the *Reynolds stress*, which acts as an effective momentum flux due to turbulence motion. In exact treatment of the Reynolds stress, we shall deal with its transport equation. However, because of the nonlinearity of the Navier-Stokes equation, the Reynolds-stress equation contains the third-order moments, and we meet the fourth-order moments in the third-order-moment equation. Namely, we inevitably strike into the infinite hierarchy of moment equations. In practical analysis we need to truncate this hierarchy by representing the higher-order correlations in terms of the lower quantities. This is referred to as the “*closure problem*” and it has been the most fundamental target of the researches of this field until now. Especially in the inhomogeneous-turbulence studies, this is termed as the “*turbulence modeling*”. In turbulence modelings, the phenomenological modelings have been the main approaches to the goal, and these strategies have developed to some commercial soft-ware packages in computations and are applied to the various practical cases. On the contrary, there are some of theoretical attempts based on the exact governing laws. One of these branches is *two-sale direct-interaction approximation* (TSDIA: Yoshizawa 1984); a combination technique of a singular perturbation method and *direct-interaction approximation* (DIA: Kraichnan 1959). Unlike the traditional inhomogeneous-turbulence modellings, TSDIA enables us to investigate various statistical quantities based on the mathematical structures of the governing equations and, besides, it is applicable not only to the charge-neutral fluid but also to more complex situation such as magnetohydrodynamic turbulence (Yoshizawa 1998).

2 General covariance of turbulence

In spite of its theoretical consistency and wide applicability, it is known that TSDIA is inconsistent with some classes of coordinate transformations, at least it has been confirmed so far that it is inconsistent with time-dependent rotation (Hamba & Sato 2008). This may be rephrased as follows; TSDIA includes the *covariance breakages* under some classes of coordinate transformations. The covariances under the Galilean

transformation, time-independent and time-dependent rotation have already been discussed often in turbulence community (Speziale 1979). In the thesis, the author has extended the covariance to more general statement; *turbulence is strictly covariant under the general coordinate transformations*. This is clearly shown by considering the transformation rule of the velocity field. The transformation of the exact velocity \mathbf{v} from a coordinate system $\{\mathbf{y}\}$ to another system $\{\tilde{\mathbf{y}}\}$ is given by

$$v^{\tilde{a}} = y^{\tilde{a}}_{,i} v^i + y^{\tilde{a}}_{,t}. \quad (2)$$

By taking the average and fluctuation we obtain

$$V^{\tilde{a}} = y^{\tilde{a}}_{,i} V^i + y^{\tilde{a}}_{,t}, \quad (3)$$

$$v'^{\tilde{a}} = v^{\tilde{a}} - V^{\tilde{a}} = y^{\tilde{a}}_{,i} (v^i - V^i) = y^{\tilde{a}}_{,i} v'^i. \quad (4)$$

(4) is clearly meaning that turbulence is generally covariant. From (4), we obtain $R^{\tilde{a}\tilde{b}} = y^{\tilde{a}}_{,i} y^{\tilde{b}}_{,j} R^{ij}$ which means that the Reynolds stress is also generally covariant. Thus the turbulence modeling should be conducted in consistent manner with the general covariance.

3 Mean-Lagrangian formalism

From (3), the author has found an important suggestion; the mean velocity transforms in totally the same manner as the exact velocity field. This indicates that the mathematical equivalence between the exact and mean flow. In the continuum physics, the coordinate system convected by the exact flow is utilize in the description of long-time historical effect in covariant manner. From its analogy, we can introduce the coordinate system convected by the mean flow in covariant description of the turbulence quantities, which we call the *mean-Lagrangian coordinate system*. The later procedures may be summarized as follows; (I) Rewrite the fluctuation equation into generally covariant form. (II) Rewrite it in the mean-Lagrangian coordinate system. (III) Separate the coordinate dependence into homogeneous and inhomogeneous parts. (IV) Apply the *renormalized perturbation theory* (RPT; Wyld 1961, Kraichnan 1977, Kaneda 1981) to the calculations of statistical quantities. These procedures has been applied to the Reynolds stress and the author obtained the following result;

$$\begin{aligned} R^{\mu\nu}(\mathbf{x}, t) &= \frac{2}{3} g^{\mu\nu}(\mathbf{x}, t) \int d\text{vol}_k {}^B Q(k; t, t|\mathbf{x}) \\ &\quad - \frac{7}{15} \int_{-\infty}^t dt' \{ \Lambda_{\rho}^{\mu}(t; t'|\mathbf{x}) \bar{\Lambda}_{\kappa}^{\sigma}(t; t'|\mathbf{x}) g^{\kappa\nu}(\mathbf{x}, t) + \Lambda_{\rho}^{\nu}(t; t'|\mathbf{x}) \bar{\Lambda}_{\kappa}^{\sigma}(t; t'|\mathbf{x}) g^{\kappa\mu}(\mathbf{x}, t) \} \\ &\quad \quad \quad \times \left(\frac{1}{2} S_{\sigma}^{\rho} + \Theta^{\rho}_{\sigma} \right) (\mathbf{x}, t') \int d\text{vol}_k {}^B G(k; t, t'|\mathbf{x}) {}^B Q(k; t, t'|\mathbf{x}) \\ &\quad - \frac{1}{10} \int_{-\infty}^t dt' \{ g^{\mu\rho}(\mathbf{x}, t) S_{\rho}^{\nu}(\mathbf{x}, t') + g^{\nu\rho}(\mathbf{x}, t) S_{\rho}^{\mu}(\mathbf{x}, t') \} \\ &\quad \quad \quad \times \int d\text{vol}_k {}^B G(k; t, t'|\mathbf{x}) {}^B Q(k; t, t'|\mathbf{x}) \\ &\quad - \frac{1}{15} \int_{-\infty}^t dt' \{ g^{\mu\rho}(\mathbf{x}, t) S_{\rho}^{\nu}(\mathbf{x}, t') + g^{\nu\rho}(\mathbf{x}, t) S_{\rho}^{\mu}(\mathbf{x}, t') \} \\ &\quad \quad \quad \times \int d\text{vol}_k {}^B G(k; t, t'|\mathbf{x}) k \frac{\partial}{\partial k} {}^B Q(k; t, t'|\mathbf{x}) \\ &\quad + O(\mu^2, \mu\delta^2), \end{aligned} \quad (5)$$

where up to lowest-order diagrams are explicitly calculated for simplicity. This result is very distinct from the conventional modelings in the following senses; the time evolutions of the mean-flow properties (\mathbf{S} , Θ , and $\mathbf{\Lambda}$) and the fluctuation properties (${}^B Q$ and ${}^B G$) coexist in the time integration. Thus (5) can illustrate the system where the time scales of the mean flow and the fluctuation are not clearly separated. In this sense (5) is more generalized form of the conventional algebraic Reynolds stress models where the Reynolds stress is determined only by the present information. This is clearly the generalization of the RPT of homogeneous turbulence which explains phenomena in terms of multiple-time quantities of fluctuation. On the other hand, the temporally-localized approximation can be obtained as follows;

$$\begin{aligned} R^{ij} &= \frac{2}{3} {}^B K g^{ij} - \nu_T S^{ij} + \gamma_t \left(\frac{O S^{ij}}{O t} + S_a^i S^{ja} \right) \\ &\quad + N_I \mathbf{S} \cdot \mathbf{S} g^{ij} + N_{II} \Theta \cdot \Theta g^{ij} + N_{III} S_a^i S^{ja} + N_{IV} \Theta_a^i \Theta^{ia} + N_V (S_a^i \Theta^{ja} + S_a^j \Theta^{ia}) \\ &\quad + D_I (S_a^{i;ja} + S_a^{j;ia}) + D_{II} S^{ab}{}_{;ab} g^{ij} + D_{III} (\Theta_a^i{}^{;ja} + \Theta_a^i{}^{;ja}) + D_{IV} \Delta S^{ij}, \end{aligned} \quad (6)$$

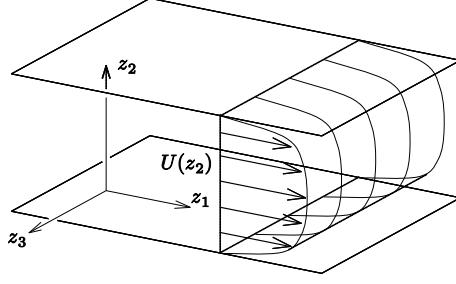


Figure 1: Channel flow with coordinate system

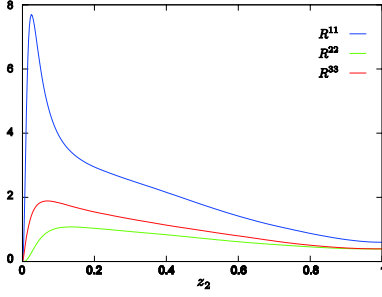


Figure 2: Intensity of DNS

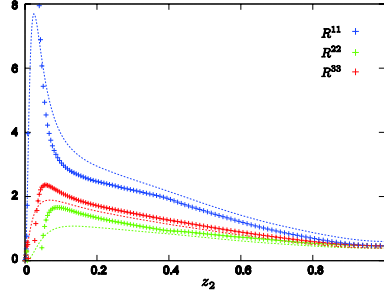


Figure 3: Intensity of the present

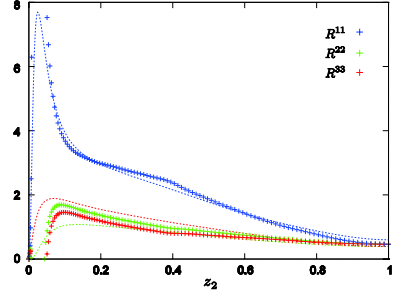


Figure 4: Intensity of TSDIA

where ${}^B K$, ν_T , γ_t , N_s and D_s are scalar coefficients expressed by the statistical properties of fluctuation. O/O_t is the *Oldroyd's derivative* which is generally covariant operation. It is noticeable that D/Dt is not generally covariant (except when it is applied to scalar fields). Note that γ_t - and D -related terms originate from the nonlocal effect in time and space respectively, which suggests a generalization of conventional algebraic Reynolds stress models (ARSM). Besides, (6) is covariant under the general coordinate transformation.

4 Discussions

4.1 Relation with the conventional K - ϵ model

By assuming the renormalized field as fully-developed isotropic turbulence, (6) reduces to a generalized algebraic representation of K - ϵ type as follows;

$$\begin{aligned}
 R^{ij} = & \frac{2}{3} K g^{ij} - C_\nu \frac{K^2}{\epsilon} S^{ij} \\
 & + C_T \frac{K^3}{\epsilon^2} \left(\frac{O S^{ij}}{O t} + S_a^i S^{ja} \right) + C_S \frac{K^3}{\epsilon^2} \left(S_a^i S^{ja} - \frac{1}{3} \mathbf{S} \cdot \mathbf{S} g^{ij} \right) + C_C \frac{K^3}{\epsilon^2} (S_a^i \Theta^{ja} + S_a^j \Theta^{ia}) \quad (7) \\
 & + C_{MD1} \frac{K^5}{\epsilon^3} \left(S_a^{i;ja} + S_a^{j;ia} - \frac{2}{3} S^{ab}{}_{;ab} g^{ij} \right) - C_{MD2} \frac{K^5}{\epsilon^3} (\Theta^i{}_a{}^{;ja} + \Theta^j{}_a{}^{;ia}) - C_{MD3} \frac{K^5}{\epsilon^3} \Delta S^{ij}.
 \end{aligned}$$

Here K , ϵ , \mathbf{g} , \mathbf{S} and Θ are turbulence energy, its dissipation rate, metric, strain rate and absolute vorticity. The standard K - ϵ model, which has been widely employed in many engineering fields, is obtained as the lowest-order truncation of the present analysis. Again, C_T - and C_{MD} - related terms are from the nonlocal effect in time and space, which are newly introduced by the present analysis.

4.2 Application to a plane channel flow

In order to see the validity of the present theory, the author has tested (7) in comparison with a direct-numerical simulation of the plane-channel turbulence (Moser et al. 1999). The author substituted the simulation data of K , ϵ , \mathbf{S} and Θ into (7) and compared this with the simulated \mathbf{R} in each component. Generally speaking, the present theory has shown good agreement in every component in the region apart from the wall. Especially the present theory reproduced the proper distribution of the turbulence intensity i.e. $R^{22} < R^{33} < R^{11}$ (see figures 2 and 3), which is well-known phenomena in simple shear flows such as wall-bounded flows, while TSDIA cannot (see figure 4).

5 Conclusion

The author has proposed a new analytical approach to inhomogeneous turbulent flow based on the mean-Lagrangian formulation with an emphasis on the general covariance under the coordinate transformation. As a consequence, the author has obtained the followings as the conclusion of the present study.

1. It was proved that the fluctuation field is generally-covariant quantity, and that the various turbulence quantities including the Reynolds stress are generally covariant.
2. The mean-Lagrangian-coordinate representation is newly introduced, and its advantage for the combination of the multiple-time analysis and the general covariance was shown.
3. By taking the advantage of the mean-Lagrangian formalism, a theory of inhomogeneous turbulence on the basis of multiple-point multiple-time quantities were developed in agreement with the general covariance.
4. A temporally non-local representation of the Reynolds stress was derived in the form of the convected integration, which clearly includes the history of both fluctuation and mean field along the mean-flow trajectory in a generally covariant manner.
5. An algebraic representation of the Reynolds stress was derived which contains new effects such as the Oldroyd derivative of the strain rate, spatial derivative of the strain rate and absolute vorticity. These represent the non-local effect in both space and time in a generally covariant manner.

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