Minimum energy state and minimum angle rotation of the magnetic field in a current sheet with sheared magnetic field

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[1] In order to answer why and how "the minimum angle rotation of the magnetic field" is realized in a current sheet with a sheared magnetic field, Taylor's helicity constraint, which is valid for low-β plasmas, is applied to a one-dimensional planar current sheet with a sheared magnetic field. A single constant helicity is defined for the total rectangular volume surrounding the current sheet and is shown to be gauge-invariant. The minimization of the magnetic energy with the constraint of the constant total helicity shows that the field is described by the constant α force-free equation and that the current sheet is a special class of tangential discontinuities with a constant field strength, or a "perpendicular rotational discontinuity." The total rotational angle of the magnetic field across the current sheet is proportional to the ratio of the total magnetic energy/helicity in the force-free state. It is proposed that among an infinite number of force-free states the current sheet relaxes into a unique force-free state with the absolute minimum ratio of energy/helicity and thus into the absolute minimum energy state for a given constant helicity. Therefore, in the relaxed state the total rotational angle of the magnetic field across the current sheet is minimum and less than 180°. Although the present study of the relaxed state is applicable only to a tangential discontinuity, a qualitative resemblance of the model prediction with observations in situ and simulations of quasi-perpendicular rotational discontinuities suggests that the observed minimum angle rotation of the magnetic field in a current sheet with a sheared magnetic field is an emergence of plasma relaxation or self-organization in space plasmas. *INDEX TERMS:* 7811 Space Plasma Physics: Discontinuities; 2109 Interplanetary Physics: Discontinuities; 2724 Magnetospheric Physics: Magnetopause, cusp, and boundary layers; 7524 Solar Physics, Astrophysics, and Astronomy: Magnetic fields; 2708 Magnetospheric Physics: Current systems (2409); *KEYWORDS:* current sheet, discontinuities, self-organization, helicity, force-free magnetic field, minimum energy state

1. Introduction

[2] The current sheet in magnetized plasmas is a typical example of a nonequilibrium system in the thermodynamic sense and it is observed in space, astrophysical, and laboratory plasmas. Current sheets occurring in nature have a finite thickness and internal field structure, because the zero thickness creates an infinite current density, which is unphysical, or because intrinsic kinetic scale lengths such as the ion Larmor radius are finite. The most well studied current sheet is the Harris sheet [*Harris*, 1962], which is the one-dimensional current sheet with a one-directional field and a hyperbolic-tangent field profile. Concerning the field structure in current sheets, a remarkable finding of *Sonnerup and Ledley* [1979] is the continuous rotation of the magnetic field across the current sheet at the magnetopause. They found in outbound magnetopause crossings, in which the magnetosheath side of the magnetopause is approached, that when the sheath field is almost antiparallel to that in the magnetosphere (northward field), the field reverses direction across the magnetopause by a rotation through the magnetopause rather than by a decrease of the northward field to zero followed by an increasing southward field as expected for the Harris sheet. *Berchem and Russell* [1982a] further showed that the sense of the magnetic field rotation through the current layer in the magnetopause is simply controlled by the relative orientation of the magnetosheath and magnetospheric fields and that the rotation angle is minimized when the magnetic field changes from one orientation to the other, i.e., that the magnetic field rotates so that the total rotational angle becomes minimum or less than 180°. These findings of "the minimum angle rotation of the magnetic field" have been further confirmed by other magnetopause observations and the field rotations with rotation angles larger than 180° have seldom been observed. These findings have also been confirmed by field observations of rotating field structures in the solar wind [e.g., *Neugebauer and Buti*, 1990] and in the heliospheric current sheet [e.g., *Smith*, 2001]. In spite of these extensive observations of the minimum angle rotation of the magnetic field, the physics of the field rotation with the minimum field rotation or shear has not been clear. The primary objective of this paper is to present a new approach based on Taylor's helicity constraint or self-organization principle to answer

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why and how the minimum angle rotation must be realized among an infinite number of possible rotations in a current sheet with a sheared magnetic field. More specifically, we show that the field rotation with the minimum rotation angle in the current sheet is explained simply by requiring that the magnetic field there is force-free and the absolute value of the ratio of the magnetic energy/helicity is absolutely minimum or that the magnetic energy is absolutely minimum for a given constant helicity. Therefore, the present study enables us to answer the fundamental question of how the external magnetic field outside the magnetopause adjusts itself to the magnetic field inside the magnetopause in the transition region from the magnetosheath to the magnetosphere. Although it has been suggested that the heliospheric current sheet is force-free [*Smith*, 2001] and, as a related problem, there are models of interplanetary magnetic clouds solved by force-free fields [*Burlaga*, 1988], this paper specifically applies Taylor's helicity constraint and force-free fields to explanation of the minimum angle rotation of the magnetic field.

[3] Self-organization processes always occur in systems far from thermodynamic equilibrium, and by these processes a plasma relaxes into a self-organized state. Since the self-organization needs dissipation, the self-organization and dissipation are interconnected. Therefore, the complete understanding of the transport processes at the current sheet requires the understanding of self-organization processes. The study of the plasma relaxation into a force-free state [*Taylor*, 1974], based on an earlier study of *Woltjer* [1958], is a remarkable example of the self-organization of a plasma [*Hasegawa*, 1985]. According to the plasma relaxation the turbulence, allied with finite but small enough resistivity, allows the plasma rapid access to a particular minimum-energy state, where the plasma is stable. In that relaxation process the magnetic energy becomes minimum with a constraint that the global magnetic helicity is invariant. The concept of the magnetic helicity was first discussed for astrophysical plasmas by *Woltjer* [1958]. Since the helicity in general depends on the choice of the gauge function [*Berger and Field*, 1984; *Heyvaerts and Priest*, 1984; *Finn and Antonsen*, 1985], the helicity has a physical significance only when it is gauge-invariant. The use of the magnetic helicity in the general theory of magnetic reconnection in space plasmas has been discussed by *Schindler et al.* [1988]. *Taylor* [1974, 1986] showed that the spontaneous generation of reversed fields in toroidal plasmas is a consequence of such a relaxation process under helicity constraints. Such a theory of the plasma relaxation or self-organization into a force-free state has been successfully applied to plasmas in many different laboratory systems and to astrophysical plasmas, in particular, to the solar corona [e.g., *Heyvaerts and Priest*, 1984].

[4] Although a "discontinuity" observed in space plasmas has a finite thickness, the MHD jump conditions across the ideal "discontinuity" are also valid for a finite thick "discontinuity" as long as the "discontinuity" is one-dimensional, since the contributions to the surface or line integrals of quantities from the portion within the discontinuity, from which the jump conditions of the quantities are obtained, cancel out owing to the one-dimensional assumption. Therefore, an important magnetic field characteristic of the tangential discontinuity that the tangential field can change its magnitude and direction across the discontinuity is also applicable to the finite thick tangential "discontinuity". Also, the important field characteristics of the rotational discontinuity that the normal field component B_n and the magnitude of the tangential field are equal across the discontinuity [*Landau and Lifshitz*, 1984] are applicable to the finite thick rotational "discontinuity". Following the conventional definition, we call the rotational discontinuity having a normal vector nearly perpendicular to the total magnetic field a "quasi-perpendicular rotational discontinuity" [*Neugebauer and Buti*, 1990]. Furthermore, it may be appropriate to call a special rotational discontinuity having $B_n = 0$ and a constant tangential field strength (with the normal vector perpendicular to the total magnetic field) a "perpendicular rotational discontinuity", which is also a special class of tangential discontinuities with a constant field strength.

[5] Since the magnetic field structure in discontinuities at the magnetopause has traditionally been studied using a kinetic approach and those results are very relevant to the present study, it is necessary here to describe a brief history of the kinetic approach. *Su and Sonnerup* [1968] showed that in the case of a rotational discontinuity, which is a large amplitude standing Alfvén wave, the tangential magnetic field has a polarization, which is the same as that of the electron whistler mode. *Swift and Lee* [1983] investigated the magnetic field rotation by a one-dimensional particle simulation for the rotational discontinuities whose normal vectors are nearly perpendicular to the total field direction and found that the initial rotational discontinuity with the rotational angle larger than 180° becomes unstable, while the rotational discontinuity with the rotational angle smaller than 180° is stable. *Richter and Scholer* [1989] investigated the stability of symmetric rotational discontinuities in which the magnetic field rotates by 180° via hybrid simulations. They found that those rotational discontinuities with large angle Θ_{bn} between the discontinuity normal vector and the total magnetic field (quasi-perpendicular rotational discontinuity) are stable, while those with small Θ_{bn} are not stable. *Goodrich and Cargill* [1991] investigated the structure of rotational discontinuities by using hybrid simulations for a range of Θ_{bn} and plasma β. On the basis of simulations of rotational discontinuities with small ion β, *Vasquez and Cargill* [1993] examined the evolution of rotational discontinuities with various Θ_{bn}, plasma β, and the ion to electron temperature ratio. *Krauss-Varban et al.* [1995] studied the properties of rotational discontinuities with Θ_{bn} = 60° and 80° by using hybrid simulations, which adopt a dynamic formation mechanism with a piston method, and found that the rotational discontinuities choose the sense of rotation that corresponds to the minimum angle between the upstream and downstream field vector. *Lin and Lee* [2000] studied, by using hybrid simulations, the evolution of magnetic field rotations in rotational discontinuities whose initial field rotation angles are larger than 180° for different Θ_{bn} and found that the rotational discontinuities evolve until smaller rotation angles less than 180° are reached. *Omidi* [1992] extended hybrid simulations of quasi-perpendicular rotational discontinuities to include a temperature anisotropy in the magnetosheath without a priori assumptions regarding the initial structure of the rotational discontinuity. The results are in agreement with both the observations [e.g., *Berchem and Russell*, 1982a]

and previous simulation studies that show the tendency of the minimum angle rotation of the magnetic field. Therefore, for quasi-perpendicular rotational discontinuities many hybrid simulations have confirmed the minimum angle rotation of the magnetic field.

[6] The field rotation in the tangential discontinuity has also been investigated using a kinetic approach. *Roth* [1978] generalized a Vlasov-Maxwell approach introduced by *Sestero* [1964, 1966] and showed that the arbitrary angle of the magnetic field rotation is created by suitably choosing distribution functions. On the basis of Vlasov-Maxwell equations, *Lee and Kan* [1979] showed for the tangential discontinuity that the presence of trapped particle populations inside the magnetopause is required in order to allow the magnetic field to rotate more than a certain critical angle ($\sim 90°$). *Kuznetsova et al.* [1994] considered the effect of the velocity shear at the magnetopause on the structure of the tangential discontinuity based on Vlasov-Maxwell equations and such a kinetic analysis has been further extended to more realistic configurations by *De Keyser and Roth* [1997]. *De Keyser and Roth* [1998] further give precise predictions concerning the sense and the magnitude of the magnetic field rotation based on assumptions concerning the velocity distributions. Although these kinetic analyses based on Vlasov-Maxwell equations are self-consistent, they remain complicated and not completely conclusive as they depend on the features of the particle distributions, boundary conditions, and many unavoidable simplifying assumptions. Therefore, in spite of these detailed kinetic studies of the field rotation, the physics of the field rotation has not been clarified, specifically concerning why the minimum angle rotation must occur.

[7] When the internal energy of plasma and hence ∇p are neglected in the current sheet, an assumption valid for low-β plasmas, the requirement of a minimum magnetic energy with the constraint of the constant magnetic helicity, yields that the magnetic field in the steady state is force-free and is characterized by a single constant α [*Taylor*, 1974]. Then, only the magnetic field rotation is allowed inside the one-dimensional current sheet, because the magnetic field strength must be uniform. Therefore, the solution of the internal structure and rotation of the magnetic field appears to be uniquely determined and trivial. However, this is not the case, because the rotation angles of any multiples of 360° can be added to the field rotation angle without violating the field directions imposed at both ends of the current sheet. Furthermore, the clockwise and counter clockwise rotations are allowed as different solutions. Therefore, there are an infinite number of possible force-free solutions with different field rotations within the current sheet. Among such infinite solutions, nature must realize a unique solution and a current sheet with the minimum rotation angle has been predominantly observed. Since the force-free equation itself, which allows an infinite number of solutions, cannot determine a unique rotation among the infinite number of solutions, which unique solution of the field rotation is realized in the current sheet must be determined by an independent physical discussion. In this study we show that there is a clear relationship between the magnetic energy, the helicity, and the total rotational angle of the magnetic field in a current sheet with a sheared magnetic field in the constant α force-free states and that a physical consideration about the relationship between the helicity and the magnetic energy leads to the unique determination of the minimum energy state for the given helicity. Although the present study is a basic study clarifying the field structure inside the ideally simplified current sheet with spatially constant field strength, the present results are applicable to magnetopause current layers, current layers in the solar wind, and current layers in space and astrophysical plasmas at least when the magnetic field strength in those current layers is constant and $B_n = 0$. Such a magnetopause crossing with a nearly constant field strength and $B_n \sim 0$ has often been observed at the magnetopause [e.g., *Sonnerup and Ledley*, 1979].

[8] The structure of the present paper is as follows. In section 2 we summarize briefly Taylor's helicity constraint. In section 3 a model of the one-dimensional current sheet with sheared magnetic field is described and a constant α force-free state is defined. In section 4 the physical meaning of the constant α is clarified and multiple solutions of the constant α force-free equations are obtained. In section 5 the magnetic energy and the helicity in a specific volume in the current sheet, when it is a force-free state, are defined and the relationship between the total magnetic energy and the gauge-invariant helicity is clarified. In section 6 hodograms of the force-free magnetic field in the current sheet including the minimum energy state are shown for a representative case. In section 7 a discussion is given and the possible relevance of the present results to observations and simulations is discussed. In section 8 a summary is given. In Appendix A the constant α force-free field is given by using the present notation of variables.

2. Taylor's Helicity Constraint

[9] *Taylor* [1974] conjectured that when the plasma is not in stable equilibrium and when it is released, the plasma will move and dissipate energy before coming to rest. Only when its energy is a minimum is it incapable of further rapid movement. Hence the final state must be one which makes the energy a minimum subject to any constraints which are imposed on the allowed motion. The major problem, of course, lies in determining and applying these constraints. *Taylor* [1974] has shown that the ideal MHD frozen-in constraint can be replaced by an infinite set of integral constraints involving the field line helicity defined for arbitrary individual flux tube volume. Therefore, when the plasma is perfectly conducting and the internal energy is neglected, the plasma state, in which the total magnetic energy is a minimum with a constraint of the invariance of the field line helicity defined for arbitrary individual flux tube volume, is given by [*Taylor*, 1974; *Freidberg*, 1987]

$$\nabla \times \mathbf{B} = \lambda \mathbf{B} \tag{1}$$

where λ is a function of each field line, i.e., a function of position, and has a different value on different field lines ($\mathbf{B} \cdot \nabla\lambda = 0$). Hence when all the constraints appropriate to a perfectly conducting fluid are observed, the state of the minimum magnetic energy is some force-free configuration. In other words, if the magnetic energy is a minimum, the field can produce no motions, and therefore the Lorentz force must vanish, or the field must be force-free.

[10] Now let us consider how the situation is modified by small departures from the perfect-conductivity approximation. The main consequence of any small departure from perfect conductivity is that topological properties of the magnetic field are no longer preserved and lines of force may break and coalesce. The field line helicity defined for each flux tube is no longer an invariant for each line of force. In such systems field line topology need no longer be preserved as the plasma moves so that a much wider class of lower energy states is now accessible. *Taylor* [1974] assumed that an infinite number of individual field line helicities, conserved in the case of infinite conductivity, are reduced to the single global helicity integral defined for the total volume V of the system due to finite but small enough resistivity in the system. Then minimizing the energy under constant global helicity defined for the total volume V of the system using the appropriate Lagrange multiplier function [*Woltjer*, 1958] yields

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} \tag{2}$$

where α is now a single constant having the same value on all field lines. Thus, when topological constraints are relaxed, the final state is no longer any force-free configuration but a specific one completely determined once the constant α and the boundary conditions are known.

3. Model of the One-Dimensional Current Sheet With Force-Free Sheared Magnetic Field

[11] Let us assume that the current sheet is planar and has a thickness of 2b. Figure 1 shows a model of the one-dimensional current sheet used in the present calculation. The current sheet exists between x = $-$b and x = b. Two planes shown in Figure 1 are parallel to the y-z plane. Magnetic field vectors at x = $-$b and x = b, which lie in the y-z plane, are shown by **B** in Figure 1. We use the one-dimensional assumption, i.e., that the magnetic field **B**(x) is the function of only x. The total volume V used in the present problem is a region with $|x| \leq b$, $0 \leq y \leq 1$, and $0 \leq z \leq 1$. By minimizing the magnetic energy under constraint of constant global helicity defined for V using the appropriate Lagrange multiplier function, we obtain (2). Substitution of $\mathbf{B} = (B_x, B_y, B_z)$ into (2) and the one-dimensional assumption, i.e., that the quantities are functions of only x, yield

$$B_x = 0 \tag{3}$$

Next, by taking the cross product of **B** with (2) and using the one-dimensional assumption we obtain $\nabla B^2(x) = 0$, which yields

$$B_y^2(x) + B_z^2(x) = B_0^2 = \text{const.}(B_0 > 0) \tag{4}$$

where B_0 is the constant magnetic field strength, which is independent of x, but is in general a function of α. Therefore, the present one-dimensional model of the current layer with a force-free magnetic field is the tangential discontinuity with a constant field strength or the rotational discontinuity with zero normal magnetic field component ("perpendicular rotational discontinuity").

[12] Let us define the phase angle θ of the magnetic field, which is the angle between the magnetic field vector and the *z* axis. We assume that $\theta = \theta_1$ at x = $-$b and $\theta = \theta_2$ at x = b. Then the magnetic fields at x = b and $-$b are

$$\mathbf{B}(-b) = B_0(0, \sin\theta_1, \cos\theta_1) \tag{5}$$

$$\mathbf{B}(b) = B_0(0, \sin\theta_2, \cos\theta_2) \tag{6}$$

We obtain by taking curl of (2)

$$\nabla^2\mathbf{B} + \alpha^2\mathbf{B} = 0 \tag{7}$$

We obtain d/dx (cos θ) = $-\alpha$ sin θ from the y-component of (2). Therefore, we obtain

$$\frac{d\theta}{dx} = \alpha \tag{8}$$

which means that the field rotation occurs with a constant rotation rate. Therefore, when θ_1 and θ_2 are different, the possible solution of the field configuration is the rotating magnetic field, which has a uniform strength and rotates continuously from θ_1 to θ_2 with the constant rotation rate α.

4. Physical Meaning of Constant α and Multiple Solutions of Constant α Force-Free Equation

[13] We obtain from (2) that

$$\alpha = \frac{\mathbf{B} \cdot (\nabla \times \mathbf{B})}{B^2} = \frac{\mu_0 j_\parallel}{B} \tag{9}$$

where $j_\parallel$ is the field-aligned current density. Therefore, the constant α, which is the constant rotation rate of the magnetic field, is also proportional to the field-aligned current density. The quantity αb gives the eigenvalue α^2 in the present eigenvalue problem (7). Since any multiples of 2π can be added to θ_2 at x = b, we obtain in general

$$\theta = \theta_2 + 2n\pi(n = 0, \pm1, \pm2, \ldots) \text{ at } x = b \tag{10}$$

By integrating (8) from x = $-$b to x = b we obtain

$$\theta_2 - \theta_1 + 2n\pi = 2\alpha b \; (n = 0, \pm1, \pm2, \ldots) \tag{11}$$

This means that for the same θ_1 and θ_2 there are an infinite number of constant α force-free fields characterized by different n values. This gives the general solution of α. The equation (11) means that αb is equal to half of the total rotation angle of the magnetic field as x changes from $-$b to b. Notice that the negative αb indicates that the rotation of the magnetic field through the current sheet as x increases from x = $-$b to x = b is counter clockwise in the B_y-B_z plane. The positive αb indicates that the rotation of the magnetic field is clockwise in the B_y-B_z plane. We obtain from (8) by using $\theta = \theta_1$ at x = $-$b

$$\theta = \alpha(x + b) + \theta_1 \; (n = 0, \pm1, \pm2, \ldots) \tag{12}$$

The solutions of the force-free equation (2) for constant α, which is reduced to (7), are given in Appendix A by using the present notation of variables. The force-free solution ((A1) and (A2)) appears to be unique. However, this is not the case, because there are an infinite number of possible

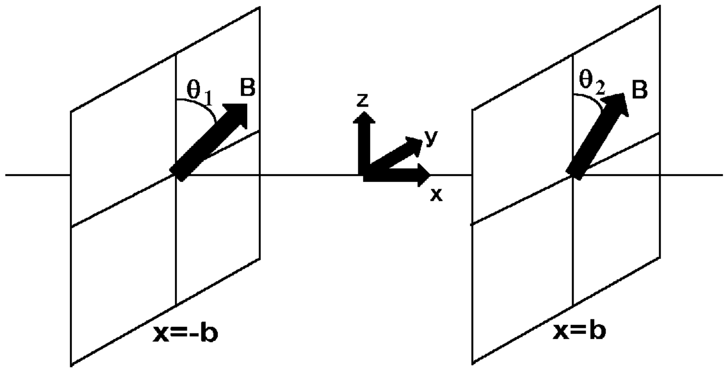

Figure 1. One-dimensional model of a current sheet with a sheared magnetic field. The magnetic field **B** is a function of x.

eigenvalues α characterized by different n values, which characterize the clockwise and counterclockwise rotations as different solutions. Therefore, there are an infinite number of possible force-free solutions with local minimum energies. The force-free equation (2) itself cannot determine which particular n is realized in nature and an independent physical consideration is necessary to determine a unique solution, which is realized in nature.

5. Relationship Between the Magnetic Energy, Helicity, and Constant α in the Force-Free State

[14] In the calculus of variations of *Woltjer* [1958], the total magnetic energy and the helicity are calculated in a specific volume V. Let us use the vector potential **A**, which satisfies

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{13}$$

We define the total magnetic energy W_M in the volume V by

$$W_M = \iiint_v \frac{B^2}{2\mu_0} dV = \int_{-b}^{b} \frac{B^2}{2\mu_0} dx \tag{14}$$

We also define the helicity K_M in this volume by

$$K_M = \frac{1}{2} \iiint_v \mathbf{A} \cdot \mathbf{B} dV = \frac{1}{2} \int_{-b}^{b} \mathbf{A} \cdot \mathbf{B} dx \tag{15}$$

Since the definition of the vector potential **A** allows the choice of an arbitrary gauge function, the helicity K_M is in general gauge-dependent, when $\mathbf{B} \cdot \hat{\mathbf{n}} \neq 0$ at the surface of the volume V, where $\hat{\mathbf{n}}$ is the surface normal vector.

[15] From (2) and (13) we obtain

$$\nabla \times \mathbf{A} = \frac{1}{\alpha} \nabla \times \mathbf{B} \tag{16}$$

at $-b \leq x \leq b$, when $\alpha \neq 0$. Since α is constant, this yields

$$\mathbf{A} - \frac{1}{\alpha} \mathbf{B} = \nabla \varphi(x) \tag{17}$$

Here, $\varphi(x)$ is an arbitrary function of x and the appearance of $\nabla\varphi(x)$ in the right hand side reflects the fact that **A** is gauge-dependent. By substituting (17) into (15) we obtain

$$K_M = \frac{1}{2} \int_{-b}^{b} \left[\frac{1}{\alpha} \mathbf{B} + \nabla \varphi(x) \right] \cdot \mathbf{B} \, dx = \frac{1}{2\alpha} \int_{-b}^{b} B^2 dx \tag{18}$$

where the gauge-dependent term vanishes and the helicity K_M is gauge-invariant in the present one-dimensional planar model of the force-free current sheet.

[16] Using (14), (18) becomes simply

$$\frac{W_M}{K_M} = \frac{\alpha}{\mu_0} \tag{19}$$

This important relationship means that the ratio of the total magnetic energy W_M and the helicity K_M of each force-free state is gauge-invariant and is proportional to the constant α. Since the total magnetic energy W_M is positive, α and the helicity K_M in the same force-free state must have the same signs. This relationship can determine a unique solution of α in the force-free equation in a physically meaningful way as follows. *Taylor* [1974] conjectures that the plasma will evolve to that particular state which has the lowest absolute value of the magnetic energy. Therefore, by assuming that different rotation states have the same K_M, we find from (19) that the state with the absolute minimum $|\alpha|$ has the lowest W_M. On the basis of this conjecture we propose in the present problem of the field rotation in a force-free current sheet with a sheared magnetic field that the state with the absolute minimum energy or the minimum rotation angle is the final unique force-free rotation state, which is realized in nature among an infinite number of force-free states.

6. Example of Hodograms for the Force-Free Field

[17] We assume that θ_1 is an arbitrary angle and that θ_2 is equal to 0. This corresponds to the ideal magnetopause situation, in which θ_1 is an arbitrary polar angle of the magnetosheath magnetic field and θ_2 is the polar angle of

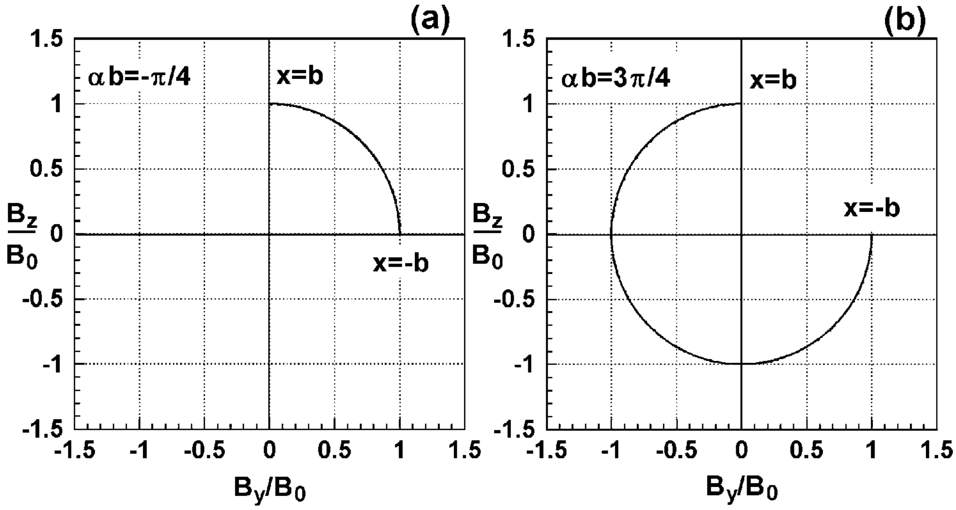

Figure 2. (a) Hodogram of the magnetic field for $\alpha b = -\pi/4$ when $\theta_1 = \pi/2$ and $\theta_2 = 0$. (b) Hodogram of the magnetic field for $\alpha b = 3\pi/4$ when $\theta_1 = \pi/2$ and $\theta_2 = 0$.

the magnetospheric field, which is due north. Without loss of generality we can assume that $0 \leq \theta_1 - \theta_2 < 2\pi$. In the following we obtain force-free fields for a particular case with $\theta_1 = \pi/2$. In this case we obtain from (11) that

$$\alpha b = -\frac{\pi}{4} + n\pi \ (n = 0, \pm 1, \pm 2, \ldots) \tag{20}$$

6.1. Negative Helicity States With n = 0, −1, −2, ...

[18] The state with $n = 0$ gives $\alpha b = -\pi/4$ and has the minimum absolute value of the ratio of energy/helicity. This state has the absolute minimum rotation angle of the magnetic field within the current sheet and is therefore the rotation state, which is realized according to the discussion in section 5. The eigenmode solutions of B_y (x) and B_z (x) in this case is obtained by substituting $\alpha b = -\pi/4$ into (A1) and (A2). The hodogram of the magnetic field at $-b \leq x \leq b$ is shown in Figure 2a. The tip of the normalized magnetic field at $x = -b$ is located at (1, 0) in the normalized B_y–B_z plane. The tip of the magnetic field vector at $x = b$ is located at (0, 1) in the same plane. The trace of the tip of the magnetic field in the B_y–B_z plane is a quarter circle, which has the minimum rotation angle for transition from $x = -b$ to $x = b$.

6.2. Positive Helicity States With n = 1, 2, ...

[19] The smallest positive value of the ratio of energy/helicity in this case occurs for $n = 1$ in (20), i.e., $\alpha b = 3\pi/4$. Figure 2b shows the hodogram of the magnetic field in this state. The total rotation angle for this state is $3\pi/2$ and the rotation occurs clockwise from (1, 0) to (0, 1) in the normalized B_y–B_z plane. As is obvious, this is not the state with the minimum absolute ratio of the magnetic energy/helicity. The total rotation angle of the magnetic field is larger than π in this case and therefore is not the rotation state, which is realized in nature, according to the discussion in section 5.

7. Discussion

[20] The present analysis is an eigenmode analysis obtaining the lowest energy state among an infinite number of force-free eigenstates for given boundary conditions. For a given constant helicity the lowest energy state is the force-free state with the minimum rotation angle of the sheared magnetic field. We obtained the minimum rotation angle $|\alpha|$ and the corresponding minimum magnetic energy according to equation (19) for a given constant helicity. It is obvious that we give only θ_1 and θ_2 at the boundaries and the field strength B_0 is not given at the boundaries and hence the magnetic energy (14), which is proportional to B_0^2, is a quantity to be obtained. This is more evidently seen by writing the present eigenmode equation (2) with the constant field strength explicitly in the form not including B_0 as follows;

$$\nabla \times \mathbf{b} = \alpha \mathbf{b} \tag{21}$$

where $\mathbf{b}$ is the magnetic field normalized by the constant B_0, i.e., $\mathbf{b} = \mathbf{B}/B_0$.

7.1. Stability of Force-Free States

[21] The stability argument of force-free states shows that for higher energy states there does exist an infinitesimal helicity-preserving perturbation that decreases the magnetic energy [e.g., *Krüger*, 1976; *Bondeson et al.*, 1981; *Berger*, 1985]. Therefore, the system relaxes eventually to the absolute minimum energy state. The small changes in the helicity or boundary conditions cannot trigger an instability from the minimum energy state and, therefore, the minimum energy state is stable [*Berger*, 1985]. Although the present analysis is not a time dependent analysis, the above stability argument supports the present conjecture in section 5 that the current sheet with a sheared field relaxes into a unique force-free state with the absolute minimum magnetic energy for a given constant helicity and hence with the minimum rotation angle of the sheared field. In other words, the magnetic field rotation in a current sheet with a sheared field occurs so that the total rotation angle becomes minimum or less than 180°. Since the total rotational angle is proportional to the field-aligned current density, this also means that the magnetic field rotation occurs to realize the

minimum absolute value of the field-aligned current density in the current sheet with a sheared magnetic field. However, this conjecture would not be contradictory to a field rotation of more than 180°, because such a state can be a higher energy state, which may eventually relax into the lowest energy state, although such a higher energy state has seldom been observed in situ.

7.2. Relevance to Observations and Simulations

[22] In considering the relevance of the present self-organization or plasma relaxation model to field rotations observed in situ and in simulations, we should note that the present model current sheet with a constant field strength B_0 and $B_n = 0$ is a special class of tangential discontinuities with a constant field strength or a perpendicular rotational discontinuity. Therefore, the present results are strictly applicable only to those rotational field structures with $B_n = 0$. The present results are supported by preliminary observations that the principle of the minimum angle rotation is valid for several magnetopause crossings when the magnetopause is clearly a tangential discontinuity without normal magnetic field components (H. Kawano, personal communication, 1999). There is also a suggestion based on analyses of solar wind discontinuities that tangential discontinuities with constant field strength or perpendicular rotational discontinuities resemble rotational discontinuities with small B_n in some of their characteristics [*Neugebauer et al.*, 1984; *De Keyser et al.*, 1998]. If this suggestion is the case and there is indeed a similarity between the tangential discontinuities with constant field strength (or perpendicular rotational discontinuities) and rotational discontinuities with small B_n, the present analysis for the self-organization in the one-dimensional force-free current sheet with $B_n = 0$ might be relevant to observational results of rotating field structures at the magnetopause and in the solar wind when B_n is small.

[23] Concerning the simulation results described in section 1, special attention must be paid to the boundary condition used in those simulations. In most simulations, except the simulations of *Omidi* [1992] and *Krauss-Varban et al.* [1995] using a piston method, the magnetic fields at the boundaries ($x = \pm b$ in the present notation) are fixed. In those one-dimensional simulations, there is a freedom in which the magnetic field strength $|\mathbf{B}(x)|$ is not constant in $-b < x < b$ and is allowed to change as a function of x. This is indeed observed in hodograms in those simulations. For example, hodograms in Figure 7 of *Swift and Lee* [1983] and Figures 1 and 3 of *Lin and Lee* [2000] show that although $|\mathbf{B}|$ is fixed at the two boundaries throughout their simulations, the field strength $|\mathbf{B}|$ is reduced substantially in $-b < x < b$ from its initial strength and finally the minimum angle rotation is realized in those regions of reduced $|\mathbf{B}|$ at the last stage of their simulations. Therefore, although the magnetic field strength is not allowed to relax near the boundaries in these simulations, the total magnetic energy can relax to lower energy so as to become consistent with the plasma relaxation theory of *Taylor* [1974].

7.3. Self-Organization and Dissipation

[24] According to the self-organization theory [*Taylor*, 1974; *Hasegawa*, 1985] the plasma relaxation into the force-free state is realized by the selective dissipation process [*Riyopoulos et al.*, 1982]. *Taylor* [1974, 1986] conjectured that the turbulence, allied with finite but small enough resistivity, allows the plasma rapid access to a particular minimum energy state in the toroidal fusion plasma. In space plasmas such as the magnetopause and the solar wind, the plasma is collisionless and a required resistivity may be due to wave-particle interactions, which involve momentum transfer between ions and electrons and can be expressed as an effective resistivity, which is a fluid term in the generalized Ohm's law or the equation of motion of the electron fluid. Another possibility may be that the required dissipation is due to electron inertia. While *Taylor* [1986] suggested that reconnection is involved in such a decay (dissipation) process for the toroidal plasma, the observations of the current layer at the magnetopause suggest that the magnetopause is not always a rotational discontinuity as expected from the reconnection, but often a tangential discontinuity [*Sonnerup and Cahill*, 1968]. Therefore, it is not certain whether such a selective decay process in the current layer at the magnetopause may always involve reconnection or may simply be a resistive dissipation process.

[25] The present study based on the one-dimensional assumption shows that the relaxed state is only possible for $B_n = 0$ (or $B_x = 0$ in the present notation) and therefore the observed rotating structures with non-zero B_n cannot be the relaxed states. It may be conjectured that the observed rotating field structures tending to have minimum rotation angle and a small B_n are still in the process of relaxation into the minimum energy state with $B_n = 0$, if the field line breaking and coalescence during the relaxation process require non-zero B_n.

7.4. Limitation of the Model

[26] Taylor's helicity constraint is derived by assuming that the plasma β is small and the internal energy can be neglected. Concerning the role of plasma pressure and ∇p in Taylor's theory we should mention the following reasoning by *Ortolani and Schnack* [1993]. According to the general force balance equation $\nabla p = \mathbf{j} \times \mathbf{B}$, the pressure becomes constant along flux tubes, but can vary from tube to tube in a way determined by the initial conditions. There are again many possible equilibrium states associated with a given set of externally applied parameters. However, when finite resistivity is introduced, the breaking and connection of field lines associated with the relaxation process allow the pressure to equilibrate over the entire plasma volume. Thus relaxed states with uniform pressure ($\nabla p = 0$) are obtained even when the plasma β is not small. Although this reasoning by *Ortolani and Schnack* [1993] seems persuasive, direct proof of *Taylor*'s [1974] conjecture for finite pressure plasmas is still lacking and therefore the effect of the nonzero pressure in the present problem should be investigated in future. In this regard the applicability of the present simple model to the subsolar magnetopause is limited to a particular case when the plasma pressure outside the subsolar magnetopause is balanced with the plasma pressure inside the magnetopause, because the $\mathbf{j} \times \mathbf{B}$ force is nearly zero in such a case. Although such a case is not predominant, a magnetic field rotation with a systematic variation of both B_y and B_z with $B_y^2 + B_z^2$ approximately constant even for total rotation angles close to 180° as the

satellite passes through the magnetopause, is often observed at the magnetopause (see, for example, Figure 5 of *Sonnerup and Ledley* [1979]).

[27] Since the finite pressure or high-β effect also means the importance of resonant particles and kinetic effects, the self-organization for high-β finite pressure system may be more complicated partly owing to the kinetic effects.

7.5. Further Extension of the Model

[28] The present approach may be extended in several ways to include other essential features at the current sheet in space plasmas. In the real magnetopause or in discontinuities in the solar wind, the bulk velocity is nonzero and it varies. In such a case the present self-organization model neglecting flow seems not applicable. However, it is possible to generalize the present variational approach to include the variations of both magnetic field and velocity. In such a case we need two Lagrange multipliers, corresponding to the fact that there are now two global invariants, i.e., the magnetic helicity and the cross helicity [e.g., *Biskamp*, 1993]. Since the magnetic helicity is still an invariant in such a general configuration, we expect that the magnetic field in such a case is still described by the force-free equation. Therefore, the minimum angle rotation of the magnetic field is still valid even in the presence of a sheared flow velocity. Furthermore, the constraint on the current sheet thickness in the present model may be unnecessary. In the present problem, the thickness of the current sheet 2b is given arbitrarily. According to observations at the magnetopause, this thickness of the current sheet at the magnetopause is several times larger than the ion Larmor radius [e.g., *Russell and Elphic*, 1978; *Paschmann et al.*, 1978; *Berchem and Russell*, 1982b]. Whereas the present analysis has clarified the self-organizing internal structure of the magnetic field when the magnetic field directions at two boundaries of the current sheet with given thickness are specified, it might be possible to construct a variational approach to determine also the current sheet thickness at the magnetopause by including the spatial magnetic field variation in the magnetosphere and minimizing a quantity related to magnetic energy.

8. Summary

[29] The field rotation across the one-dimensional current sheet with a sheared magnetic field was investigated by using Taylor's helicity constraint (self-organization principle). When the topological properties of the magnetic field are no longer preserved owing to any small departures from the perfect-conductivity approximation and when the total helicity defined for the total volume surrounding the current sheet is conserved, the plasma relaxes into a constant-α force-free state, which is a tangential discontinuity with a constant field strength (perpendicular rotational discontinuity) for the present one-dimensional planar configuration. For the absolute minimum energy force-free state and force-free states with higher energies, the magnetic energy is proportional to the total rotation angle of the magnetic field and thus only discrete values of the magnetic energy are allowed in the force-free states, because only discrete values of the total rotation angle are possible (see equation (11)). It is proposed that among an infinite number of possible force-free states, a one-dimensional current sheet with a sheared magnetic field relaxes into a unique force-free state with $B_n = 0$ and having the absolute minimum magnetic energy, in which the minimum angle rotation of the magnetic field occurs and the absolute value of the field-aligned current density becomes minimum. It is conjectured that the observed rotating field structures tending to have minimum rotation angle and a small B_n may still be in the process of relaxation into the minimum energy state with $B_n = 0$. In order to understand completely the relevance of plasma relaxation or self-organization to the observed minimum angle rotation of the magnetic field in the current sheet with a sheared magnetic field, further study of the self-organization process for high-β (finite pressure) space plasmas is necessary.

Appendix A

[30] Since the phase angle θ of the magnetic field, which is measured from the *z* axis, is given by (12), B_y and B_z of the constant α solution of (7) can be written as

$$B_y(x) = B_0 \sin\theta = B_0 \sin(\alpha x + \alpha b + \theta_1) \tag{A1}$$

$$B_z(x) = B_0 \cos\theta = B_0 \cos(\alpha x + \alpha b + \theta_1) \tag{A2}$$

Since every solution of the second order differential equation (7) is not necessarily a solution of the force-free equation (2) [*Chandrasekhar and Kendall*, 1957], we inserted the solutions (A1), (A2) into (2) and found that the solutions (A1), (A2) are indeed solutions of the force-free equation (2).

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