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A magnetospheric energy principle for hydromagnetic stability problems

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[1] A magnetospheric energy principle is formulated to study hydromagnetic stability of a magnetospheric plasma. The magnetospheric plasma is either in a two-dimensional or three-dimensional static equilibrium. It is surrounded by lateral perfectly conducting walls and ideal ionospheres in both cases and also by dawn-dusk periodic boundaries in the two-dimensional case. The two-dimensional case has a translational symmetry and has no unperturbed magnetic field component in the dawn-dusk direction. Unlike the conventional energy principle for a plasma surrounded by a perfectly conducting wall, field lines are assumed to vertically thread the ionospheric boundary, which is not a perfectly conducting rigid wall. Ideal ionospheric boundary conditions are obtained, so that the force operator becomes self-adjoint and the magnetospheric energy principle is valid. There are four ideal ionospheric boundary conditions to satisfy these requirements: horizontally free, conducting, free, and rigid. A change in the potential energy becomes equal to the sum of the change in the fluid energy and an ionospheric surface contribution, which is negative and thus destabilizing for horizontally free and free ionospheric boundary conditions. A minimization condition for the change in the potential energy is obtained. When an unperturbed field-aligned current vanishes, the horizontally free, conducting, free, and rigid boundary conditions allow interchange, incompressible ballooning, incompressible ballooning, and compressible ballooning modes, respectively. Different characteristics of those three pressure-driven modes are clarified. Existing interchange stability criteria are compared and results of several different numerical stability analyses of ballooning instabilities for different magnetospheric equilibria are discussed systematically in light of the present magnetospheric energy principle.

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1. Introduction

[2] Magnetospheric regions of interest are characterized by finite-β plasmas. In such regions, field line-curvature influences the dynamics of magnetospheric plasmas by its potentially unstable interaction with plasma pressure gradients. Indeed, the most dangerous electromagnetic instabilities usually involve curvature of the field. Such ideal magnetohydrodynamic (MHD) instabilities driven by pressure gradients or currents perpendicular to the magnetic field are called pressure-driven instabilities, which are subdivided into interchange and ballooning instabilities. There are subtle kinetic effects due to the field line curvature in collisionless magnetospheric plasmas [*Hurricane et al.*, 1994; *Horton et al.*, 2001], which invalidate a fluid description. However, the possibility of ballooning growth rate being larger than the bounce frequency of the bulk of ions in the near-Earth magnetosphere within $15R_E$ from the Earth has recently been shown for a specific magnetospheric model [*Miura*, 2004]. This validates a fluid description of ballooning instability in the near-Earth magnetosphere, where a substorm onset occurs, and indicates that under such circumstances, the magnetohydrodynamics (MHD) are not only references but also real possibilities.

[3] However, even within ideal MHD, the study of pressure-driven instabilities in the magnetosphere has been difficult because of their awkward dependence on the explicit form of the equilibrium. In order to overcome this difficulty, an energy principle [*Bernstein et al.*, 1958] has been applied to pressure-driven modes in magnetospheric plasmas. However, from the point of validity of the application of the energy principle to the magnetosphere, a fundamental point has never been addressed previously, in particular, concerning proper specification of ionospheric boundary conditions compatible with the energy principle.

[4] The energy principle [*Bernstein et al.*, 1958] was originally used in fusion plasmas to study plasma stability and is a rather general approach based on the variational principle. A brief review of the energy principle can be

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found in the work of *Kadomtsev* [1965] and a very comprehensive review can be found in the work of *Freidberg* [1987]. When the plasma is surrounded by a perfectly conducting wall, magnetic field lines do not thread the wall and the energy principle becomes particularly simple. This type of energy principle with a perfectly conducting wall boundary has been widely used in the context of fusion plasma applications. However, in the magnetosphere the plasma is bounded at the earthward ends by ionospheres. Field lines can thread the ionospheric boundaries. This is an important difference from a case of perfectly conducting walls, where the normal component of the magnetic field at the plasma-wall interface must vanish. Therefore as noted by *Hameiri et al.* [1991], boundary conditions are very different in the magnetosphere from those used in fusion applications, for example, torus-like tokamaks, although previous applications of the energy principle to the magnetosphere have not seriously taken into account this important difference. Furthermore, the ionospheric boundary is not a fixed boundary but a free boundary. Therefore the Lagrangian displacement $\boldsymbol{\xi} = 0$ (rigid boundary condition) is not the only allowed ionospheric boundary condition. The purpose of this paper is to formulate a magnetospheric energy principle by taking into account these fundamental differences peculiar to the magnetospheric plasma. Although both interchange and ballooning instabilities are described by three-dimensional ideal MHD equations, they are quite different and have their own distinct characteristics. This study clarifies that both instabilities are characterized by different ionospheric boundary conditions.

[5] When magnetospheric field lines are connected to real ionospheres with finite conductivities, pressure-driven modes in the magnetosphere are no longer ideal MHD modes because of finite ionospheric energy dissipation, and the energy principle does not hold. Therefore instead of realistically taking into account ionospheric boundary conditions due to finite ionospheric conductivities [*Southwood and Kivelson*, 1989; *Hameiri et al.*, 1991], this study pursues to formulate an energy principle for the magnetospheric stability by obtaining ideal ionospheric boundary conditions that are necessary for the force operator to be self-adjoint. In this sense the present ionosphere is ideal and not realistic. Nevertheless, a consideration of such ionospheric boundary conditions is helpful for numerical models of ideal MHD instabilities in the magnetosphere, in which idealized boundary conditions are imposed at ionospheric boundaries. Although previous studies [e.g., *Hameiri et al.*, 1991] have not sought ideal ionospheric boundary conditions to satisfy the self-adjointness of the ideal MHD force operator, this study pursues ideal ionospheric boundary conditions so that the force operator becomes self-adjoint.

[6] The present study does not depend on any specific model of the magnetospheric equilibrium as long as the magnetospheric plasma satisfies a static balance. The present magnetospheric energy principle is also applicable to kink modes driven by field-aligned currents, if an equilibrium magnetospheric model with currents parallel to the magnetic field is properly specified. Since the energy principle obtains a condition by making the variation of the change in the potential energy zero, the obtained condition is valid for the most unstable modes.

[7] In his seminal paper, *Gold* [1959] pointed out the possibility of spontaneous large-scale interchange motions of magnetic flux tubes and the plasma contained in them occuring in the Earth's magnetosphere. This possibility has been widely discussed as a potentially important mechanism for the redistribution of mass in planetary magnetospheres. The further discussion of interchange instability and its extension to magnetospheres of rapidly rotating planets such as Jupiter can be found in literature [e.g., *Chandrasekhar*, 1960; *Sonnerup and Laird*, 1963; *Melrose*, 1967; *Hill*, 1976; *Cheng*, 1985; *Rogers and Sonnerup*, 1986; *Southwood and Kivelson*, 1987; *Ferrière et al.*, 2001].

[8] *Miura et al.* [1989] did a numerical eigenmode analysis of ballooning instability in the geomagnetic tail. They have shown that the geomagnetic tail is ballooning unstable where the plasma β at the equator exceeds a critical β value, which is calculated approximately. Their numerical analysis has also shown that the ballooning instability in the tail grows very rapidly (or exponentiate) in a timescale of the field line curvature radius divided by the Alfvén speed at the equator [*Miura*, 2004], which becomes several to tens of seconds in the geomagnetic tail, compatible with the rapid substorm onset timescale. Their numerical stability analysis and other numerical stability analyses of ballooning instability solved one-dimensional eigenmode equations numerically [e.g., *Miura et al.*, 1989; *Lee*, 1998; *Cheng and Zaharia*, 2004] or used an initial value approach [*Wu et al.*, 1998; *Zhu et al.*, 2004] for different boundary conditions, which are imposed at the ionosphere, and for different tail or magnetospheric models. Other numerical models [*Lee and Wolf*, 1992; *Schindler and Birn*, 2004] calculated a potential energy functional for a rigid ionospheric boundary condition and for different tail models to study stability of the geomagnetic tail. Although *Miura et al.* [1989] studied incompressible ballooning modes, most other studies considered compressible ballooning modes. Therefore whether a ballooning instability related to a substorm onset occurs in an incompressible manner or in a compressible manner is controversial.

[9] It is obvious from these observations of previous studies that complete understanding of pressure-driven instabilities in the magnetosphere, such as interchange and ballooning instabilities, is crucial for understanding dynamics of magnetospheric plasmas.

[10] As a somewhat related subject, the existence of centrifugally driven instabilities in the Jovian outer magnetosphere has been discussed by *Melrose* [1967] and *Hill* [1976] and these studies have shown that when the density of the outer magnetosphere falls off rapidly enough with distance from the planet, centrifugally driven instabilities occur. The existence of such centrifugally driven modes in magnetospheres of rapidly rotating planets means rich nature in parameter space. However, even in such a rapidly rotating magnetosphere, purely pressure-driven modes can be postulated. *McNutt et al.* [1987] point out that in Jupiter's magnetosphere a pressure-driven incompressible ballooning mode occurs and is responsible for plasma depletion. Therefore the present study is focused on formulation of a magnetospheric energy principle for ideal MHD instabilities in a static plasma.

[11] In the following, a model of the magnetospheric physical region of interest, for which an energy principle

is formulated, is constructed in section 2. Linear stability equations are presented and the normal-mode formulation of the linearized MHD stability problem is given in section 3. Boundary conditions on lateral boundaries of the physical region of interest in the magnetosphere are given in section 4. Ideal ionospheric boundary conditions, that are necessary for the ideal MHD force operator to be self-adjoint, are obtained in section 5. A magnetospheric energy principle is formulated in section 6. The condition for minimization of the change in the potential energy with respect to $\xi_{\|}$ is obtained in section 7. Discussion is presented in section 8. Summary and conclusion are presented in section 9. The calculation of two terms appearing in the proof of the self-adjointness of the force operator is given in Appendix A. The condition for the validity of the magnetospheric energy principle and physical meaning of the change in the potential energy are clarified in Appendix B.

2. Model of the Physical Region of Interest

[12] In fusion plasmas, the region of interest can often be considered surrounded by a rigid, perfectly conducting wall. However, in the magnetospheric case, the region of interest is bounded by ionospheres at the earthward ends. Therefore in this study the region of interest is assumed to be surrounded by boundaries in the magnetosphere and at the ionospheres. In the present magnetospheric region of interest the plasma must be under a static force balance. As long as a magnetospheric configuration is in a static equilibrium, any magnetospheric configuration, whether it is dipole-like or tail-like, can be used in the following analysis. Although steady flows are often present in the magnetospheric regions of interest, a static equilibrium is assumed in the present analysis, because the energy principle is based on the self-adjoint property of the force operator, which leads to the result that ω^2 (the square of the frequency of the characteristic oscillation) is real. This is in contrast to instabilities driven by an equilibrium flow, such as Kelvin-Helmholtz instability, which has a complex frequency.

[13] To make a magnetospheric energy principle general, two types of magnetospheric configurations, which have often been used for stability analyses of pressure-driven modes, are used. One is a two-dimensional magnetospheric model, which is two-dimensional in the midnight meridian plane and has translational symmetry in the cross-tail or dawn-dusk direction. There is no magnetic field component in the dawn-dusk direction. This two-dimensional model includes, for example, an analytic tail-like equilibrium model of *Kan* [1973] and a magnetostatic MHD model of *Voigt* [1986]. Since a ballooning mode wavelength in the dawn-dusk direction is much shorter than the magnetospheric size in the dawn-dusk direction, the two-dimensional model with a periodic perturbation in the dawn-dusk direction is valid for magnetospheric ballooning instability. The other configuration is a full three-dimensional magnetospheric configuration. The three-dimensional model includes, for example, a low-β limit of the magnetospheric configuration, that is, the dipole field configuration. The dipole model has been used to study the low-β stability of interchange modes in the magnetosphere [*Gold*, 1959]. Another example of the three-dimensional model is a numerical three-dimensional force-balanced magnetospheric model of *Zaharia and Cheng* [2003], which is not axisymmetric.

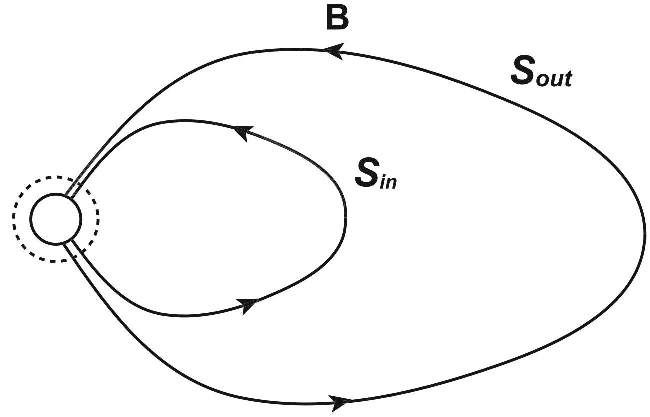

Figure 1. A cross section of the magnetospheric region of interest in the midnight meridian plane. The solid circle is the Earth. The dotted circle is the ionospheric boundary. Two lateral boundaries, S_{out} and S_{in}, which are shown by solid lines, are considered to be thin perfectly conducting walls.

[14] Figure 1 shows a cross section of the magnetospheric region of interest in the midnight meridian plane. The solid circle is the Earth. The dotted circle is the ionospheric boundary. For the three-dimensional configuration, the region of interest is surrounded by two lateral boundaries, S_{out} and S_{in}, which are shown by solid lines. Both boundary surfaces surround a part of the magnetospheric plasma, which is in a static equilibrium. These virtual boundaries are taken to be flux surfaces of the static magnetospheric equilibrium and are located far enough from disturbed field lines. Hence these boundaries are not perturbed and there is no magnetic field component threading these boundaries. Therefore these virtual boundaries, S_{out} and S_{in}, are considered to be thin perfectly conducting walls. For the two-dimensional configuration with translational symmetry in the cross-tail direction, there are two other lateral boundaries S_{dawn} and S_{dusk} at the dawn and dusk ends. At these boundaries, periodic boundary conditions are imposed on perturbations. For both two-dimensional and three-dimensional configurations, earthward ends of the region of interest are bounded by ideal ionospheres, which are shown by a dotted line in Figure 1. Unlike S_{out} and S_{in}, magnetic field lines thread the ionospheric boundaries.

3. Linearized Stability Equations

[15] In this section, following *Freidberg* [1987], the stability problem is cast into the form of a normal-mode eigenvalue problem.

[16] To begin, assume that a static ideal MHD equilibrium, satisfying

$$\mathbf{J}_0 \times \mathbf{B}_0 = \nabla p_0, \tag{1}$$

$$\mu_0 \mathbf{J}_0 = \nabla \times \mathbf{B}_0, \tag{2}$$

$$\nabla \cdot \mathbf{B}_0 = 0, \tag{3}$$

$$\mathbf{v}_0 = 0, \tag{4}$$

is given. All quantities are linearized about this background state: $\mathbf{Q}(\mathbf{r}, t) = \mathbf{Q}_0(\mathbf{r}) + \tilde{\mathbf{Q}}_1(\mathbf{r}, t)$ with $\tilde{\mathbf{Q}}_1/\mathbf{Q}_0 << 1$. All perturbed quantities denoted by subscript 1 are expressed in terms of a displacement vector $\tilde{\boldsymbol{\xi}}$ defined by

$$\tilde{\mathbf{v}}_1 = \frac{\partial \tilde{\boldsymbol{\xi}}}{\partial t}. \tag{5}$$

Using $\tilde{\boldsymbol{\xi}}$ the general linearized equations of motion are cast into an initial value problem. In this formulation one needs to specify appropriate initial data. A very convenient choice of initial data for stability problems is as follows:

$$\tilde{\boldsymbol{\xi}}(\mathbf{r}, 0) = \tilde{\mathbf{B}}_1(\mathbf{r}, 0) = \tilde{\rho}_1(\mathbf{r}, 0) = \tilde{p}_1(\mathbf{r}, 0) = 0, \tag{6}$$

$$\frac{\partial \tilde{\boldsymbol{\xi}}(\mathbf{r}, 0)}{\partial t} \equiv \tilde{\mathbf{v}}_1(\mathbf{r}, 0) \neq 0. \tag{7}$$

This corresponds to the situation where at $t = 0$, the plasma is in its exact equilibrium position but is moving away with a small velocity $\tilde{\mathbf{v}}_1(\mathbf{r}, 0)$. The linearized momentum equation, subject to $\tilde{\boldsymbol{\xi}}(\mathbf{r}, 0) = 0$, $\partial\tilde{\boldsymbol{\xi}}(\mathbf{r}, 0)/\partial t = \tilde{\mathbf{v}}_1(\mathbf{r}, 0)$, plus appropriate boundary conditions (as discussed later) constitute the formulation of the general linearized stability equation as an initial value problem.

[17] A more efficient way to investigate linear stability is to reformulate the initial value problem as a normal mode problem. To do this, all perturbed quantities are assumed to vary as follows:

$$\tilde{\mathbf{Q}}_1(\mathbf{r}, t) = \mathbf{Q}_1(\mathbf{r}) \exp(-i\omega t), \tag{8}$$

where $\mathbf{Q}_1(\mathbf{r})$ is a complex quantity in general, whereas $\tilde{\mathbf{Q}}_1(\mathbf{r}, t)$ is defined in the real time domain. Then, the linearized form of the mass conservation equation, energy relation, and Faraday's law becomes

$$\rho_1 = -\nabla \cdot (\rho \boldsymbol{\xi}), \tag{9}$$

$$p_1 = -\boldsymbol{\xi} \cdot \nabla p - \gamma p \nabla \cdot \boldsymbol{\xi}, \tag{10}$$

$$\mathbf{Q} \equiv \mathbf{B}_1 = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}). \tag{11}$$

In equations (9), (10), and (11) and hereafter, the zero subscript has been dropped from all equilibrium quantities. Upon substituting these relations into the momentum equation, one finds

$$-\omega^2 \rho \boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi}), \tag{12}$$

where the force operator $\mathbf{F}(\boldsymbol{\xi})$ is given by

$$\mathbf{F}(\boldsymbol{\xi}) = \mu_0^{-1}(\nabla \times \mathbf{Q}) \times \mathbf{B} + \mu_0^{-1}(\nabla \times \mathbf{B}) \times \mathbf{Q} + \nabla(\boldsymbol{\xi} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{\xi}). \tag{13}$$

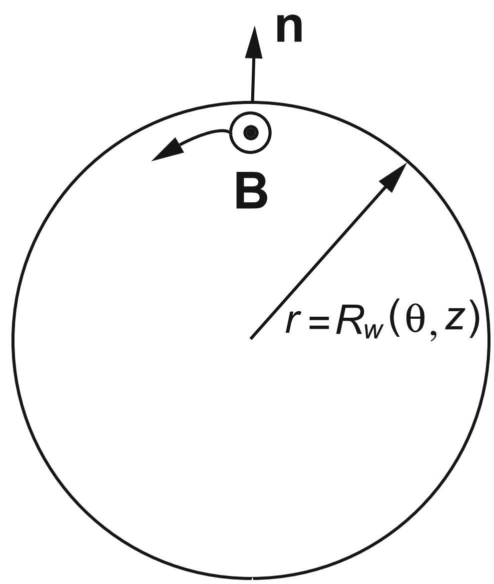

Figure 2. A cross section of plasma surrounded by a perfectly conducting wall, which is shown by a large solid circle. **n** is the outward-pointing normal vector on the wall boundary. The magnetic field **B** does not thread the wall.

Equations (12) and (13) represent the normal-mode formulation of the linearized MHD stability problem. In this approach only appropriate boundary conditions on $\boldsymbol{\xi}$ are required.

4. Boundary Conditions on Lateral Boundaries

[18] The energy principle [*Bernstein et al.*, 1958] has been typically applied to fusion plasmas surrounded by a perfectly conducting wall. Figure 2 shows a cross section of plasma surrounded by a perfectly conducting wall. As shown in this figure, the plasma extends out to a stationary, perfectly conducting wall located at $r = R_w(\theta, z)$. The unperturbed magnetic field **B** does not thread the conducting wall. The electromagnetic boundary conditions require that the tangential electric field and normal magnetic field vanish at the conducting wall

$$\mathbf{n} \times \mathbf{E}|_{R_w} = 0, \tag{14}$$

$$\mathbf{n} \cdot \mathbf{B}|_{R_w} = 0. \tag{15}$$

Here **n** is the outward-pointing normal vector on the boundary. It then follows from the ideal Ohm's law that $\mathbf{n} \times \mathbf{E} + (\mathbf{n} \cdot \mathbf{B})\mathbf{v} - (\mathbf{n} \cdot \mathbf{v})\mathbf{B} = 0$; that is, the normal component of velocity also automatically vanishes at the wall:

$$\mathbf{n} \cdot \mathbf{v}|_{R_w} = 0. \tag{16}$$

[19] In both two-dimensional and three-dimensional magnetospheric configurations, the physical region of interest is assumed to be surrounded in the meridian plane by the perfectly conducting walls S_{out} and S_{in} as shown in Figure 1.

Therefore on these lateral boundaries $S_{\rm out}$ and $S_{\rm in}$ at $\mathbf{r} = \mathbf{R}_{\rm w}$, the following boundary conditions are valid.

$$\mathbf{n} \times \mathbf{E}|_{\mathbf{R}_{\rm w}} = 0, \tag{17}$$

$$\mathbf{n} \cdot \mathbf{B}|_{\mathbf{R}_{\rm w}} = 0, \tag{18}$$

$$\mathbf{n} \cdot \mathbf{v}|_{\mathbf{R}_{\rm w}} = 0. \tag{19}$$

In the two-dimensional magnetospheric configuration defined in section 2, the physical region of interest is also surrounded by $S_{\rm dawn}$ and $S_{\rm dusk}$ at the dawn and dusk ends. In the two-dimensional configuration the periodic boundary condition is imposed on perturbations at $S_{\rm dawn}$ and at $S_{\rm dusk}$. Normal vectors $\mathbf{n}$ on the boundary surfaces at the dawn and dusk ends are opposite each other.

5. Ionospheric Boundary Conditions Necessary for Self-Adjointness of the Force Operator F

[20] Before deriving ideal ionospheric boundary conditions, it is important to note that a neutral atmosphere exists below the ionosphere and is bounded by the solid Earth at the lowest end. This magnetospheric situation is somewhat similar to a more general configuration in magnetic confinement devices, in which the plasma is separated from a perfectly conducting wall by a vacuum region [*Bernstein et al.*, 1958; *Kadomtsev*, 1965; *Freidberg*, 1987]. For such a general configuration in magnetic confinement devices, the validity of the energy principle has been shown by proving the self-adjointness of the force operator $\mathbf{F}$. That is, the vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ must satisfy

$$\int_P \boldsymbol{\eta} \cdot \mathbf{F}(\boldsymbol{\xi}) d\mathbf{r} = \int_P \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\eta}) d\mathbf{r}, \tag{20}$$

where P is the unperturbed plasma volume, which is surrounded by a plasma-vacuum boundary. In showing equation (20) three boundary conditions are used [*Bernstein et al.*, 1958; *Kadomtsev*, 1965; *Freidberg*, 1987]. One boundary condition is obtained by expanding the pressure balance condition at the perturbed plasma-vacuum boundary, $\mathbf{r}_{\rm s} = \mathbf{r}_{\rm p} + \boldsymbol{\xi}$, where $\mathbf{r}_{\rm p}$ is the unperturbed plasma-vacuum boundary and $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{r}_{\rm p})$, in powers of the small quantity and retaining linear terms based on the assumption that the acceleration of the boundary remains finite. The other two boundary conditions are boundary conditions on vector potential at the outer perfectly conducting wall and at the unperturbed plasma-vacuum boundary.

[21] In spite of the apparent similarity, the magnetospheric case is more difficult than the above general magnetic confinement configuration, because the solid Earth is not a perfect conductor and the neutral atmosphere is not a vacuum region. In order to avoid this difficulty, the existence of a neutral atmosphere below the ionosphere has been neglected and some boundary conditions have been imposed at fixed ionospheric boundaries [*Miura et al.*, 1989; *Lee and Wolf*, 1992; *Bhattacharjee et al.*, 1998; *Lee*, 1998; *Wu et al.*, 1998; *Cheng and Zaharia*, 2004; *Schindler and Birn*, 2004; *Zhu et al.*, 2004]. However, even with such a simplification, ideal ionospheric boundary conditions compatible with the energy principle have so far been unknown. The purpose of this and following sections is to obtain ideal ionospheric boundary conditions on two arbitrary vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ at the unperturbed ionospheric boundaries, for which the force operator $\mathbf{F}$ becomes self-adjoint. That is, the vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ must satisfy equation (20), where the integral is calculated for the unperturbed plasma volume P surrounded by $S_{\rm out}$ and $S_{\rm in}$ in the magnetosphere and the unperturbed ionospheres in the three-dimensional case and by $S_{\rm out}$ and $S_{\rm in}$, the unperturbed ionospheres, and unperturbed $S_{\rm dawn}$ and $S_{\rm dusk}$ in the two-dimensional case. For the sake of simplicity, the unperturbed magnetic field $\mathbf{B}$ at the ionospheric surface is assumed to be everywhere perpendicular to the local unperturbed ionospheric surface. Therefore $\mathbf{n} = \mathbf{b}$, where $\mathbf{b} = \mathbf{B}/|\mathbf{B}|$, is assumed on the ionospheric surface in the Northern Hemisphere and $\mathbf{n} = -\mathbf{b}$ is assumed on the ionospheric surface in the Southern Hemisphere. This normal incidence of the unperturbed magnetic field on the ionosphere is the only assumption in the present magnetospheric energy principle and the validity of this assumption is verified in section 8.6. On the lateral boundaries $S_{\rm out}$ and $S_{\rm in}$, which are thin perfectly conducting walls, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, satisfy $\mathbf{n} \cdot \boldsymbol{\xi} = \mathbf{n} \cdot \boldsymbol{\eta} = 0$ from equation (19). In the two-dimensional configuration $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are periodic at $S_{\rm dawn}$ and $S_{\rm dusk}$.

[22] When the plasma is assumed to be surrounded by a perfectly conducting wall, all the boundary terms (surface integrals) arising from the volume integral of equation (20) vanish and the force operator $\mathbf{F}$ becomes self-adjoint. This is shown in detail in Appendix A of *Freidberg* [1987]. For the above more general magnetic confinement configuration, the self-adjointness of the force operator $\mathbf{F}$ is proven by using the known boundary conditions. The situation is opposite in the present magnetospheric case, however, and one is interested in obtaining unknown boundary conditions at the ideal ionospheric boundary necessary for the force operator $\mathbf{F}$ to become self-adjoint. These additional boundary conditions at the ideal ionospheric boundary are obtained below. Some parts of the derivation below are parallel with the derivation in Appendix A of *Freidberg* [1987].

[23] The integrand of equation (20) can be written as

$$\begin{aligned}\boldsymbol{\eta} \cdot \mathbf{F}(\boldsymbol{\xi}) = \boldsymbol{\eta} \cdot [\mu_0^{-1}(\nabla \times \mathbf{Q}) \times \mathbf{B} + \mu_0^{-1}(\nabla \times \mathbf{B}) \\ \times \mathbf{Q} + \nabla(\boldsymbol{\xi} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{\xi})].\end{aligned} \tag{21}$$

[24] One now writes $\boldsymbol{\xi} = \boldsymbol{\xi}_\perp + \xi_\| \mathbf{b}$, $\boldsymbol{\eta} = \boldsymbol{\eta}_\perp + \eta_\| \mathbf{b}$. Following Appendix A of *Freidberg* [1987] and using vector identity one can write

$$\boldsymbol{\eta} \cdot \mathbf{F}(\boldsymbol{\xi}) = \nabla \cdot [\gamma p \boldsymbol{\eta}(\nabla \cdot \boldsymbol{\xi})] - \gamma p (\nabla \cdot \boldsymbol{\xi})(\nabla \cdot \boldsymbol{\eta}) + I, \tag{22}$$

where I is a function only of the perpendicular components of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$:

$$\begin{aligned}I(\boldsymbol{\xi}_\perp, \boldsymbol{\eta}_\perp) = \boldsymbol{\eta}_\perp \cdot [\mu_0^{-1}(\nabla \times \mathbf{Q}) \times \mathbf{B} + \mu_0^{-1}(\nabla \times \mathbf{B}) \\ \times \mathbf{Q} + \nabla(\boldsymbol{\xi}_\perp \cdot \nabla p)].\end{aligned} \tag{23}$$

By rewriting the last term of equation (23) using vector identity and by using standard vector identity, equation (23) can be rewritten as

$$I = \mu_0^{-1}\boldsymbol{\eta}_\perp \cdot [(\mathbf{Q}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{Q} - \nabla(\mathbf{Q}\cdot\mathbf{B})] + \nabla\cdot[(\boldsymbol{\xi}_\perp\cdot\nabla p)\boldsymbol{\eta}_\perp] - (\boldsymbol{\xi}_\perp\cdot\nabla p)\nabla\cdot\boldsymbol{\eta}_\perp. \quad (24)$$

If one uses vector identity,

$$(\mathbf{c}\cdot\nabla)(d\mathbf{e}) = [(\mathbf{c}\cdot\nabla)d]\mathbf{e} + d(\mathbf{c}\cdot\nabla)\mathbf{e}, \quad (25)$$

one obtains, after a lengthy but straightforward calculation,

$$\boldsymbol{\eta}_\perp\cdot[(\mathbf{Q}\cdot\nabla)\mathbf{B}] = \boldsymbol{\eta}_\perp\cdot[(((\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp)\cdot\nabla)\mathbf{B} - (((\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{B})\cdot\nabla)\mathbf{B}] - B^2(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa})\nabla\cdot\boldsymbol{\xi}_\perp, \quad (26)$$

where $\boldsymbol{\kappa} = (\mathbf{b}\cdot\nabla)\mathbf{b}$.

[25] Next, one rewrites the second term of equation (24). Using vector identity

$$(\mathbf{c}\cdot\nabla)(\mathbf{d}\cdot\mathbf{e}) = \mathbf{d}\cdot[(\mathbf{c}\cdot\nabla)\mathbf{e}] + \mathbf{e}\cdot[(\mathbf{c}\cdot\nabla)\mathbf{d}], \quad (27)$$

one obtains, after a straightforward calculation,

$$\boldsymbol{\eta}_\perp\cdot[(\mathbf{B}\cdot\nabla)\mathbf{Q}] = \nabla\cdot[(\boldsymbol{\eta}_\perp\cdot\mathbf{Q})\mathbf{B}] - [(\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp]\cdot[(\mathbf{B}\cdot\nabla)\boldsymbol{\eta}_\perp] + [(\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{B}]\cdot[(\mathbf{B}\cdot\nabla)\boldsymbol{\eta}_\perp] - B^2(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa})\nabla\cdot\boldsymbol{\xi}_\perp. \quad (28)$$

Since

$$\mathbf{B}\cdot\mathbf{Q} = -B^2(\nabla\cdot\boldsymbol{\xi}_\perp) - B^2(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}) - \boldsymbol{\xi}_\perp\cdot\nabla\left(B^2/2\right) \quad (29)$$

is valid, the third term of equation (24) can be rewritten as

$$-\boldsymbol{\eta}_\perp\cdot[\nabla(\mathbf{B}\cdot\mathbf{Q})] = -\nabla\cdot[(\mathbf{B}\cdot\mathbf{Q})\boldsymbol{\eta}_\perp] - B^2(\nabla\cdot\boldsymbol{\xi}_\perp)(\nabla\cdot\boldsymbol{\eta}_\perp) - [\boldsymbol{\xi}_\perp\cdot\nabla(B^2/2) + B^2(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})]\nabla\cdot\boldsymbol{\eta}_\perp. \quad (30)$$

[26] Therefore I can be written by using equations (26), (28), and (30) as

$$\begin{aligned} I = {} & \mu_0^{-1}\nabla\cdot[(\boldsymbol{\eta}_\perp\cdot\mathbf{Q})\mathbf{B}] - \mu_0^{-1}\nabla\cdot[(\mathbf{B}\cdot\mathbf{Q})\boldsymbol{\eta}_\perp] \\ & + \nabla\cdot[(\boldsymbol{\xi}_\perp\cdot\nabla p)\boldsymbol{\eta}_\perp] - \mu_0^{-1}B^2(\nabla\cdot\boldsymbol{\xi}_\perp)(\nabla\cdot\boldsymbol{\eta}_\perp) \\ & - \mu_0^{-1}[(\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp]\cdot[(\mathbf{B}\cdot\nabla)\boldsymbol{\eta}_\perp] \\ & - 2\mu_0^{-1}B^2(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa})\nabla\cdot\boldsymbol{\xi}_\perp \\ & - \left[\boldsymbol{\xi}_\perp\cdot\nabla\left(p+\mu_0^{-1}B^2/2\right) + \mu_0^{-1}B^2(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})\right]\nabla\cdot\boldsymbol{\eta}_\perp + R, \end{aligned} \quad (31)$$

where

$$\mu_0 R = \boldsymbol{\eta}_\perp\cdot[(((\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp)\cdot\nabla)\mathbf{B} - (((\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{B})\cdot\nabla)\mathbf{B}] + ((\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{B})\cdot((\mathbf{B}\cdot\nabla)\boldsymbol{\eta}_\perp). \quad (32)$$

[27] In rewriting the first term of equation (32) one notes that

$$\nabla\cdot[(\boldsymbol{\eta}_\perp\cdot((\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{B}))\mathbf{B}] = [(\mathbf{B}\cdot\nabla)\boldsymbol{\eta}_\perp]\cdot[(\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{B}] + \boldsymbol{\eta}_\perp\cdot[(((\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp)\cdot\nabla)\mathbf{B}] + \boldsymbol{\eta}_\perp\cdot(\mathbf{B}\boldsymbol{\xi}_\perp:\nabla\nabla)\mathbf{B}, \quad (33)$$

where,

$$\boldsymbol{\eta}_\perp\cdot(\mathbf{B}\boldsymbol{\xi}_\perp:\nabla\nabla)\mathbf{B} \equiv \sum_i \eta_{\perp i}\left(\sum_j\sum_k B_j\xi_{\perp k}\partial^2/\partial x_j\partial x_k B_i\right). \quad (34)$$

In rewriting the second term of equation (32) one also notes that

$$\boldsymbol{\eta}_\perp\cdot[(\boldsymbol{\xi}_\perp\cdot\nabla)((\mathbf{B}\cdot\nabla)\mathbf{B})] = \boldsymbol{\eta}_\perp\cdot[(((\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{B})\cdot\nabla)\mathbf{B}] + \boldsymbol{\eta}_\perp\cdot(\mathbf{B}\boldsymbol{\xi}_\perp:\nabla\nabla)\mathbf{B}. \quad (35)$$

Substitution of equations (33) and (35) into equation (32) yields

$$\mu_0 R = \nabla\cdot[(\boldsymbol{\eta}_\perp\cdot((\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{B}))\mathbf{B}] - \boldsymbol{\eta}_\perp\cdot[(\boldsymbol{\xi}_\perp\cdot\nabla)((\mathbf{B}\cdot\nabla)\mathbf{B})]. \quad (36)$$

[28] From equation (1) and

$$(\mathbf{B}\cdot\nabla)\mathbf{B} = (\mathbf{B}\cdot\nabla B)\mathbf{b} + B^2\boldsymbol{\kappa} \quad (37)$$

one obtains

$$(\mathbf{B}\cdot\nabla B)\mathbf{b} + B^2\boldsymbol{\kappa} - B\nabla B = \mu_0\nabla p. \quad (38)$$

Therefore equation (36) can be written as

$$R = \mu_0^{-1}\nabla\cdot[(\boldsymbol{\eta}_\perp\cdot((\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{B}))\mathbf{B}] - (\boldsymbol{\eta}_\perp\boldsymbol{\xi}_\perp:\nabla\nabla)\left(p+\mu_0^{-1}B^2/2\right), \quad (39)$$

where

$$(\boldsymbol{\eta}_\perp\boldsymbol{\xi}_\perp:\nabla\nabla)\left(p+\mu_0^{-1}B^2/2\right) \equiv \sum_i\sum_k \eta_{\perp i}\xi_{\perp k}\partial^2/\partial x_i\partial x_k\left(p+\mu_0^{-1}B^2/2\right). \quad (40)$$

Substituting equations (31) and (39) into equation (22) yields

$$\begin{aligned} \boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi}) = T & - \mu_0^{-1}\nabla\cdot[(\mathbf{B}\cdot\mathbf{Q})\boldsymbol{\eta}_\perp] \\ & + \nabla\cdot[\gamma p\boldsymbol{\eta}(\nabla\cdot\boldsymbol{\xi})] + \nabla\cdot[(\boldsymbol{\xi}_\perp\cdot\nabla p)\boldsymbol{\eta}_\perp] \\ & - \mu_0^{-1}B^2(\nabla\cdot\boldsymbol{\xi}_\perp)(\nabla\cdot\boldsymbol{\eta}_\perp) \\ & - \mu_0^{-1}[(\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp]\cdot[(\mathbf{B}\cdot\nabla)\boldsymbol{\eta}_\perp] \\ & - 2\mu_0^{-1}B^2(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa})\nabla\cdot\boldsymbol{\xi}_\perp \\ & - \left[\boldsymbol{\xi}_\perp\cdot\nabla\left(p+\mu_0^{-1}B^2/2\right) + \mu_0^{-1}B^2(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})\right]\nabla\cdot\boldsymbol{\eta}_\perp \\ & - (\boldsymbol{\eta}_\perp\boldsymbol{\xi}_\perp:\nabla\nabla)\left(p+\mu_0^{-1}B^2/2\right) \\ & - \gamma p(\nabla\cdot\boldsymbol{\xi})(\nabla\cdot\boldsymbol{\eta}), \end{aligned} \quad (41)$$

where

$$T = \mu_0^{-1}\nabla\cdot[(\boldsymbol{\eta}_\perp\cdot((\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{B}))\mathbf{B}] + \mu_0^{-1}\nabla\cdot[(\boldsymbol{\eta}_\perp\cdot\mathbf{Q})\mathbf{B}]. \quad (42)$$

[29] Since

$$\mathbf{Q} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}) = -\mathbf{B}(\nabla\cdot\boldsymbol{\xi}_\perp) + [(\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp] - (\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{B} \quad (43)$$

is valid, one obtains

$$\boldsymbol{\eta}_\perp \cdot \mathbf{Q} = \boldsymbol{\eta}_\perp \cdot [(\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp] - B\boldsymbol{\eta}_\perp \cdot [(\boldsymbol{\xi}_\perp \cdot \nabla)\mathbf{b}]. \quad (44)$$

[30] Substitution of equation (44) into equation (42) yields

$$T = \mu_0^{-1}\nabla\cdot[(\boldsymbol{\eta}_\perp\cdot((\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{B}))\mathbf{B}] + \mu_0^{-1}\nabla\cdot[(\boldsymbol{\eta}_\perp\cdot((\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp)\mathbf{B}) - B(\boldsymbol{\eta}_\perp\cdot((\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{b}))\mathbf{B}]. \quad (45)$$

Since $\boldsymbol{\eta}_\perp\cdot((\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{B}) = B\boldsymbol{\eta}_\perp\cdot((\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{b})$ is valid, the first and third terms in T cancel each other.

[31] In calculating the integral of equation (41) over the unperturbed plasma volume P, the contribution from the second and fourth terms in equation (41) gives

$$\int_S \left[\boldsymbol{\xi}_\perp\cdot\nabla p - \mu_0^{-1}(\mathbf{B}\cdot\mathbf{Q})\right]\boldsymbol{\eta}_\perp\cdot\mathbf{n}dS, \quad (46)$$

where $d\mathbf{S} = dS\mathbf{n}$ and dS is the surface area element, and the integral is calculated for the unperturbed surface S surrounding the unperturbed plasma volume P. The vector $\mathbf{n}$ is the outward-pointing normal vector on the unperturbed surface S. The contributions to the integral (46) from $S_{\rm out}$ and $S_{\rm in}$ vanish because of $\boldsymbol{\eta}\cdot\mathbf{n} = \boldsymbol{\eta}_\perp\cdot\mathbf{n} = 0$ on $S_{\rm out}$ and $S_{\rm in}$. The ionospheric contributions to integral (46) vanish because of $\boldsymbol{\eta}_\perp\cdot\mathbf{n} = 0$ on the ionospheric boundary. For the two-dimensional configuration, $\boldsymbol{\eta}\cdot\mathbf{n}$ changes sign at $S_{\rm dawn}$ and $S_{\rm dusk}$ because of the periodic condition and, therefore, the contributions to the integral from $S_{\rm dawn}$ and $S_{\rm dusk}$ cancel out. Therefore equation (46) vanishes.

[32] The integration of the first and third terms in equation (41) over the unperturbed plasma volume P gives

$$\int_S \left[\gamma p(\nabla\cdot\boldsymbol{\xi})(\boldsymbol{\eta}\cdot\mathbf{n}) + \mu_0^{-1}(\boldsymbol{\eta}_\perp\cdot((\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp))(\mathbf{B}\cdot\mathbf{n})\right]dS. \quad (47)$$

For both the three-dimensional and two-dimensional configurations, the contributions from $S_{\rm out}$ and $S_{\rm in}$ in equation (47) vanish, because $\boldsymbol{\eta}\cdot\mathbf{n} = \mathbf{B}\cdot\mathbf{n} = 0$ on $S_{\rm out}$ and $S_{\rm in}$. For the two-dimensional configuration, the contributions from $S_{\rm dawn}$ and $S_{\rm dusk}$ cancel out in equation (47). Therefore only the integral over the ionospheric surface contributes to equation (47). Therefore equation (47) can be written as

$$\int_{\rm North}\left[\gamma p\eta_\parallel(\nabla\cdot\boldsymbol{\xi}) + \mu_0^{-1}B\boldsymbol{\eta}_\perp\cdot((\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp)\right]dS - \int_{\rm South}\left[\gamma p\eta_\parallel(\nabla\cdot\boldsymbol{\xi}) + \mu_0^{-1}B\boldsymbol{\eta}_\perp\cdot((\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp)\right]dS, \quad (48)$$

where "North" and "South" denote unperturbed ionospheric surfaces in the Northern Hemisphere and the Southern Hemisphere, respectively.

[33] It is shown here that equation (48) does not vanish for the special case when there is a symmetry property of displacement vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ with respect to the equatorial plane. Such a symmetric case arises when the distribution of the magnetic field strength B and the pressure p is symmetric with respect to the equatorial plane. Since the force operator $\mathbf{F}$ in equation (13) does not change by replacing $\mathbf{B}$ with $-\mathbf{B}$, the force operator $\mathbf{F}$ is a function of B and not $\mathbf{B}$ and is symmetric with respect to the equatorial plane in such a case. Then, by using a well-known mathematical theorem for a case when there is only one eigenvector $\boldsymbol{\xi}$ for each eigenvalue of a symmetric operator or the eigenvalue is not degenerate, each component of the displacement eigenvector of $\mathbf{F}$ is shown to be either symmetric or antisymmetric with respect to the equatorial plane. Therefore when B and p are symmetric with respect to the equatorial plane, there is also a restriction that each component of the displacement vector $\boldsymbol{\xi}$ or $\boldsymbol{\eta}$ be either symmetric or antisymmetric with respect to the equatorial plane through the symmetry of the force operator $\mathbf{F}$ in addition to the restriction on $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ by boundary conditions.

[34] For the three-dimensional configuration, a field-aligned coordinate system (s, θ, ϕ), where s increases along the field line with the distance from the ionosphere in the Southern Hemisphere, is defined (see also Appendix A). Unit vectors of the coordinate system are $\mathbf{b}$, $\hat{\theta}$, and $\hat{\phi}$, where $\hat{\phi} = \mathbf{b}\times\hat{\theta}$ directs eastward. In this coordinate system

$$\boldsymbol{\eta} = \eta_\parallel\mathbf{b} + \eta_\theta\hat{\theta} + \eta_\phi\hat{\phi}. \quad (49)$$

[35] Then a symmetric mode, with physically symmetric displacements in both hemispheres, has an antisymmetric $\eta_\parallel$ and a symmetric $\nabla\cdot\boldsymbol{\eta}$ and an antisymmetric mode, with physically antisymmetric displacements in both hemispheres, has a symmetric $\eta_\parallel$ and an antisymmetric $\nabla\cdot\boldsymbol{\eta}$. Therefore for both symmetric and antisymmetric modes, the contribution to equation (48) from $\gamma p\eta_\parallel(\nabla\cdot\boldsymbol{\xi})$ does not vanish for either symmetric or antisymmetric modes.

[36] Appendix A shows that

$$\boldsymbol{\eta}_\perp\cdot[(\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp] = B\left(\eta_\theta\frac{\partial\xi_\theta}{\partial s} + \eta_\phi\frac{\partial\xi_\phi}{\partial s}\right) \quad (50)$$

in both hemispheres. A symmetric mode, with symmetric displacements in both hemispheres, has symmetric η_θ and η_ϕ, and antisymmetric $\partial\xi_\theta/\partial s$ and $\partial\xi_\phi/\partial s$. An antisymmetric mode, with antisymmetric displacements in both hemispheres, has antisymmetric η_θ and η_ϕ, and symmetric $\partial\xi_\theta/\partial s$ and $\partial\xi_\phi/\partial s$. Therefore from equation (50), the contribution to equation (48) from $\mu_0^{-1}B\boldsymbol{\eta}_\perp\cdot((\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp)$ does not vanish for either symmetric or antisymmetric modes.

[37] For the two-dimensional configuration, a field-aligned coordinate system (s, θ, z), where z is the dawn-dusk direction, is defined (see also Appendix A). Unit vectors of the coordinate system are $\mathbf{b}$, $\hat{\theta}$, and $\hat{\mathbf{z}}$, where $\hat{\mathbf{z}} = \mathbf{b}\times\hat{\theta}$ directs eastward. In this coordinate system

$$\boldsymbol{\eta} = \eta_\parallel\mathbf{b} + \eta_\theta\hat{\theta} + \eta_z\hat{\mathbf{z}}. \quad (51)$$

Then, using the same argument as used for the three-dimensional configuration above and a result in Appendix A that

$$\boldsymbol{\eta}_\perp\cdot[(\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp] = B\left(\eta_\theta\frac{\partial\xi_\theta}{\partial s} + \eta_z\frac{\partial\xi_z}{\partial s}\right) \quad (52)$$

for the two-dimensional configuration, it is shown that equation (48) does not vanish for either symmetric or antisymmetric modes for the two-dimensional configuration.

[38] Therefore for the special case when there is a symmetric property in both the three-dimensional and the two-dimensional configurations, the contributions to equation (48) from ionospheres in both the Southern Hemisphere and the Northern Hemisphere do not cancel out.

[39] In general, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are arbitrary except for boundary conditions and they are independent. Therefore equation (48) vanishes only when

$$\eta_{\|} = 0 \text{ or } \nabla \cdot \boldsymbol{\xi} = 0 \tag{53}$$

and

$$\boldsymbol{\eta}_{\perp} = 0 \text{ or } (\mathbf{b} \cdot \nabla)\boldsymbol{\xi}_{\perp} = 0 \tag{54}$$

are satisfied on the unperturbed ionospheric surfaces in both the hemispheres.

[40] When equations (53) and (54) are satisfied, one obtains by integrating equation (41)

$$\begin{aligned}\int_P \boldsymbol{\eta} \cdot \mathbf{F}(\boldsymbol{\xi}) d\mathbf{r} = -\int_P d\mathbf{r}\big[\mu_0^{-1}((\mathbf{B} \cdot \nabla)\boldsymbol{\xi}_{\perp}) \cdot ((\mathbf{B} \cdot \nabla)\boldsymbol{\eta}_{\perp}) \\ + \gamma p(\nabla \cdot \boldsymbol{\xi})(\nabla \cdot \boldsymbol{\eta}) + \mu_0^{-1} B^2 (\nabla \cdot \boldsymbol{\xi}_{\perp} \\ + 2\boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\kappa})(\nabla \cdot \boldsymbol{\eta}_{\perp} + 2\boldsymbol{\eta}_{\perp} \cdot \boldsymbol{\kappa}) \\ - 4\mu_0^{-1} B^2 (\boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\kappa})(\boldsymbol{\eta}_{\perp} \cdot \boldsymbol{\kappa}) \\ + (\boldsymbol{\eta}_{\perp}\boldsymbol{\xi}_{\perp} : \nabla\nabla)\big(p + \mu_0^{-1} B^2/2\big)\big],\end{aligned} \tag{55}$$

which is clearly a self-adjoint form by inspection.

[41] In summary, when the vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfy the boundary conditions (53) and (54) at the unperturbed ionospheric boundaries, the force operator **F** becomes self-adjoint for both the two-dimensional and the three-dimensional configurations. This conclusion is valid even for the special case when B and p are symmetric with respect to the equatorial plane and the eigenvalue of the symmetric force operator **F** is not degenerate and thus the displacement eigenvector is either symmetric or antisymmetric with respect to the equatorial plane.

6. A Magnetospheric Energy Principle

6.1. Expression for the Change in the Potential Energy δW

[42] The physical basis for the energy principle is the fact that energy is exactly conserved in the ideal MHD model. The conservation of energy is proven by the following calculation [*Bernstein et al.*, 1958; *Freidberg*, 1987], which is carried out in the real time domain.

[43] Consider the energy $H(t)$, which is the sum of the kinetic energy K and the change in the potential energy δW, given by

$$\begin{aligned} H &= K\left(\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t}, \frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t}\right) + \delta W\left(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}\right) \\ &= \frac{1}{2}\int_P d\mathbf{r}\left[\rho\left(\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t}\right)^2 - \tilde{\boldsymbol{\xi}} \cdot \mathbf{F}\left(\tilde{\boldsymbol{\xi}}\right)\right],\end{aligned} \tag{56}$$

where $\tilde{\boldsymbol{\xi}} = \tilde{\boldsymbol{\xi}}(\mathbf{r}, t)$ and P represents the unperturbed plasma volume. A simple calculation that makes use of the self-adjoint property of **F** yields

$$\frac{dH}{dt} = \int_P d\mathbf{r}\frac{\partial\tilde{\boldsymbol{\xi}}}{\partial t} \cdot \left[\rho\frac{\partial^2\tilde{\boldsymbol{\xi}}}{\partial t^2} - \mathbf{F}\left(\tilde{\boldsymbol{\xi}}\right)\right] = 0, \tag{57}$$

where it is used that $\mathbf{F}(\tilde{\boldsymbol{\xi}})$ is linear with respect to $\tilde{\boldsymbol{\xi}}$. This equation corresponds to energy conservation. The validity of this energy conservation is further discussed on the basis of a rigorous local energy conservation equation in Appendix B.

[44] On physical grounds one expects that if $\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi})$ can be made negative, then the system is unstable. The energy principle for plasma surrounded by a perfectly conducting wall is the following: A plasma equilibrium is stable if and only if

$$\delta W\left(\boldsymbol{\xi}^*, \boldsymbol{\xi}\right) \geq 0 \tag{58}$$

for all allowable displacements (i.e., $\boldsymbol{\xi}$ bounded in energy and satisfying appropriate boundary conditions); that is, if the minimum value of the change in the potential energy is positive for all displacements, the system is stable. If it is negative for any of the displacements, the system is unstable. The proof of this energy principle is given by *Bernstein et al.* [1958]. A proof of the energy principle based on the conservation of energy is given by *Laval et al.* [1965].

[45] Since the change in the potential energy δW can be calculated by replacing $\boldsymbol{\eta}$ with $\boldsymbol{\xi}$ in equation (20), $\boldsymbol{\eta}$ in the boundary conditions (53) and (54) must be replaced with $\boldsymbol{\xi}$. In section 5 the self-adjointness of the force operator **F** has been shown to be valid for boundary conditions (53) and (54). Therefore the energy is conserved in the magnetosphere and a magnetospheric energy principle is also valid if the displacement $\boldsymbol{\xi}$ satisfies the ionospheric boundary conditions (53) and (54).

[46] Because of the self-adjointness of **F**, the normal mode problem, equations (12) and (13), can be easily cast into the form of a variational principle [*Bernstein et al.*, 1958; *Freidberg*, 1987]. That is, the dot product of equation (13) with $\boldsymbol{\xi}^*$ is formed and then integrated over the plasma volume, yielding

$$\omega^2 = \frac{\delta W\left(\boldsymbol{\xi}^*, \boldsymbol{\xi}\right)}{K\left(\boldsymbol{\xi}^*, \boldsymbol{\xi}\right)}, \tag{59}$$

where the change in the potential energy $\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi})$ can be written as

$$\begin{aligned}\delta W\left(\boldsymbol{\xi}^*, \boldsymbol{\xi}\right) &= -\frac{1}{2}\int_P \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi}) d\mathbf{r} \\ &= -\frac{1}{2}\int_P \boldsymbol{\xi}^* \cdot \left[\mu_0^{-1}(\nabla \times \mathbf{Q}) \times \mathbf{B} + \mu_0^{-1}(\nabla \times \mathbf{B}) \times \mathbf{Q}\right. \\ &\quad \left. + \nabla(\boldsymbol{\xi} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{\xi})\right] d\mathbf{r},\end{aligned} \tag{60}$$

and $K(\boldsymbol{\xi}^*, \boldsymbol{\xi})$, which is proportional to the kinetic energy, is written as

$$K\left(\boldsymbol{\xi}^*, \boldsymbol{\xi}\right) = \frac{1}{2}\int_P \rho|\boldsymbol{\xi}|^2 d\mathbf{r}. \tag{61}$$

It should be noted that the volume integral appearing in calculation of δW is calculated for the unperturbed plasma volume P. Notice in equation (60) that $\boldsymbol{\xi}(\mathbf{r})$ is treated as complex in anticipation of cases where, because of symmetry, several spatial coordinates can be Fourier analyzed, for example, in the azimuthal direction for the three-dimensional configuration and in the dawn-dusk direction for the two-dimensional configuration. From the self-adjointness of $\mathbf{F}$, ω^2 is shown to be real. Therefore from equation (59), it is obvious that negative δW gives an unstable solution.

[47] The change in the potential energy δW given by equation (60) depends on ionospheric boundary values. In order to see this, δW is reduced by taking account of the boundary conditions. Since

$$\nabla\cdot\left[\left(\boldsymbol{\xi}^*\times\mathbf{B}\right)\times\mathbf{Q}\right]=\mathbf{Q}\cdot\left[\nabla\times\left(\boldsymbol{\xi}^*\times\mathbf{B}\right)\right]-\left(\boldsymbol{\xi}^*\times\mathbf{B}\right)\cdot(\nabla\times\mathbf{Q}) \tag{62}$$

is valid, one obtains

$$\boldsymbol{\xi}^*\cdot[(\nabla\times\mathbf{Q})\times\mathbf{B}]=\nabla\cdot\left[\left(\boldsymbol{\xi}^*\times\mathbf{B}\right)\times\mathbf{Q}\right]-|\mathbf{Q}|^2. \tag{63}$$

One also notes

$$\boldsymbol{\xi}^*\cdot[\nabla(\gamma p\nabla\cdot\boldsymbol{\xi})]=\nabla\cdot\left[\gamma p\boldsymbol{\xi}^*\nabla\cdot\boldsymbol{\xi}\right]-\gamma p|\nabla\cdot\boldsymbol{\xi}|^2. \tag{64}$$

Substituting equations (62) and (63) into equation (60), one obtains

$$\begin{aligned}\delta W\left(\boldsymbol{\xi}^*,\boldsymbol{\xi}\right)=&\frac{1}{2}\int_P d\mathbf{r}\left[\mu_0^{-1}|\mathbf{Q}|^2+\gamma p|\nabla\cdot\boldsymbol{\xi}|^2\right.\\&\left.-\boldsymbol{\xi}^*\cdot(\mathbf{J}\times\mathbf{Q}+\nabla(\boldsymbol{\xi}\cdot\nabla p))\right]\\&-\frac{1}{2}\int_S\gamma p(\nabla\cdot\boldsymbol{\xi})\boldsymbol{\xi}^*\cdot d\mathbf{S}\\&-\frac{1}{2\mu_0}\int_S\left[\left(\boldsymbol{\xi}^*\times\mathbf{B}\right)\times\mathbf{Q}\right]\cdot d\mathbf{S}.\end{aligned} \tag{65}$$

In equation (65) one writes

$$\left(\boldsymbol{\xi}^*\times\mathbf{B}\right)\times\mathbf{Q}=-\left[(\mathbf{Q}\cdot\mathbf{B})\boldsymbol{\xi}_\perp^*-\left(\mathbf{Q}\cdot\boldsymbol{\xi}_\perp^*\right)\mathbf{B}\right]. \tag{66}$$

Substitution of equation (66) into equation (65) yields

$$\begin{aligned}\delta W\left(\boldsymbol{\xi}^*,\boldsymbol{\xi}\right)=&\frac{1}{2}\int_P d\mathbf{r}\left[\mu_0^{-1}|\mathbf{Q}|^2+\gamma p|\nabla\cdot\boldsymbol{\xi}|^2\right.\\&\left.-\boldsymbol{\xi}^*\cdot(\mathbf{J}\times\mathbf{Q}+\nabla(\boldsymbol{\xi}\cdot\nabla p))\right]\\&-\frac{1}{2}\int_S\left(\boldsymbol{\xi}^*\cdot\mathbf{n}\right)\gamma p(\nabla\cdot\boldsymbol{\xi})dS\\&+\frac{1}{2\mu_0}\int_S(\mathbf{B}\cdot\mathbf{Q})\boldsymbol{\xi}_\perp^*\cdot\mathbf{n}dS\\&-\frac{1}{2\mu_0}\int_S\left(\mathbf{Q}\cdot\boldsymbol{\xi}_\perp^*\right)\mathbf{n}\cdot\mathbf{B}dS,\end{aligned} \tag{67}$$

where the second, third, and fourth integrals are surface integrals depending on the ionospheric boundary values.

[48] Since $\boldsymbol{\xi}=\boldsymbol{\xi}_\perp+\xi_\parallel\mathbf{b}$, one obtains

$$\begin{aligned}\boldsymbol{\xi}^*\cdot[\mathbf{J}\times\mathbf{Q}+\nabla(\boldsymbol{\xi}\cdot\nabla p)]=&\boldsymbol{\xi}_\perp^*\cdot(\mathbf{J}\times\mathbf{Q})+\boldsymbol{\xi}_\perp^*\cdot\nabla(\boldsymbol{\xi}\cdot\nabla p)\\&+\xi_\parallel^*\mathbf{b}\cdot[\mathbf{J}\times\mathbf{Q}+\nabla(\boldsymbol{\xi}\cdot\nabla p)].\end{aligned} \tag{68}$$

From Appendix A of *Freidberg* [1987] one obtains

$$\mathbf{b}\cdot[\mathbf{J}\times\mathbf{Q}+\nabla(\boldsymbol{\xi}\cdot\nabla p)]=0. \tag{69}$$

From equations (68) and (69), equation (67) can be rewritten as

$$\delta W\left(\boldsymbol{\xi}^*,\boldsymbol{\xi}\right)=\delta W_F+B.T., \tag{70}$$

where

$$\begin{aligned}\delta W_F=&\frac{1}{2}\int_P d\mathbf{r}\left[\mu_0^{-1}|\mathbf{Q}|^2+\gamma p|\nabla\cdot\boldsymbol{\xi}|^2\right.\\&\left.-\boldsymbol{\xi}_\perp^*\cdot(\mathbf{J}\times\mathbf{Q})+(\boldsymbol{\xi}_\perp\cdot\nabla p)\nabla\cdot\boldsymbol{\xi}_\perp^*\right],\end{aligned} \tag{71}$$

$$\begin{aligned}\mathrm{B.T.}=&-\frac{1}{2}\int_S\left(\boldsymbol{\xi}^*\cdot\mathbf{n}\right)\gamma p(\nabla\cdot\boldsymbol{\xi})dS\\&+\frac{1}{2}\int_S\left[\mu_0^{-1}\mathbf{B}\cdot\mathbf{B}_1-\boldsymbol{\xi}_\perp\cdot\nabla p\right]\boldsymbol{\xi}_\perp^*\cdot\mathbf{n}dS\\&-\frac{1}{2\mu_0}\int_S\left[\left(\mathbf{B}_1\cdot\boldsymbol{\xi}_\perp^*\right)\right]\mathbf{n}\cdot\mathbf{B}dS.\end{aligned} \tag{72}$$

The B.T. in equation (72) denotes the boundary term. Here the second integral vanishes because $\boldsymbol{\xi}\cdot\mathbf{n}=\boldsymbol{\xi}_\perp\cdot\mathbf{n}=0$ on S_{out} and S_{in} and $\boldsymbol{\xi}_\perp\cdot\mathbf{n}=0$ at the ionosphere, and for the two-dimensional configuration the normal vectors $\mathbf{n}$ are opposite each other at S_{dawn} and S_{dusk}.

[49] Substitution of

$$\mathbf{B}_1\cdot\boldsymbol{\xi}_\perp^*=\boldsymbol{\xi}_\perp^*\cdot[(\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp]-B\boldsymbol{\xi}_\perp^*\cdot[(\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{b}]. \tag{73}$$

into equation (72) yields

$$\begin{aligned}\mathrm{B.T.}=&-\frac{1}{2}\int_S\left(\boldsymbol{\xi}^*\cdot\mathbf{n}\right)\gamma p(\nabla\cdot\boldsymbol{\xi})dS-\frac{1}{2}\mu_0^{-1}\int_S(\mathbf{B}\cdot\mathbf{n})\\&\cdot\left[\boldsymbol{\xi}_\perp^*\cdot((\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp)-B\boldsymbol{\xi}_\perp^*\cdot((\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{b})\right]dS.\end{aligned} \tag{74}$$

When the plasma volume is surrounded by a perfectly conducting wall such as fusion plasmas, B.T. is obviously zero because of $\mathbf{n}\cdot\boldsymbol{\xi}=\mathbf{B}\cdot\mathbf{n}=0$ at the boundary. However, for the magnetospheric configuration, B.T. is not necessarily zero, because $\mathbf{n}\cdot\boldsymbol{\xi}=\mathbf{B}\cdot\mathbf{n}=0$ is not generally satisfied on the ionospheric surface.

[50] In equation (74), let us denote the contribution from the last term by δW_{I}. Then

$$\delta W_{\mathrm{I}}=\frac{1}{2}\mu_0^{-1}\int_S B\left[\boldsymbol{\xi}_\perp^*\cdot((\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{b})\right](\mathbf{B}\cdot\mathbf{n})dS \tag{75}$$

[51] Appendix A shows that for the three-dimensional configuration $(\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{b}=\boldsymbol{\xi}_\perp/R_{\mathrm{I}}$ in the ionosphere of the Southern Hemisphere and $(\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{b}=-\boldsymbol{\xi}_\perp/R_{\mathrm{I}}$ in the

ionosphere of the Northern Hemisphere, where R_I is the sum of the Earth's radius R_E and the ionospheric height h ($R_I = R_E + h \sim R_E$). Therefore equation (75) can be written as

$$\delta W_I = -\frac{1}{2\mu_0}\left(\int_{North}\frac{B^2|\boldsymbol{\xi}_\perp|^2}{R_I}dS + \int_{South}\frac{B^2|\boldsymbol{\xi}_\perp|^2}{R_I}dS\right). \quad (76)$$

For the two-dimensional configuration, using the results of Appendix A shows that

$$\delta W_I = -\frac{1}{2\mu_0}\left(\int_{North}\frac{B^2|\boldsymbol{\xi}_{\perp m}|^2}{R_I}dS + \int_{South}\frac{B^2|\boldsymbol{\xi}_{\perp m}|^2}{R_I}dS\right), \quad (77)$$

where $\boldsymbol{\xi}_{\perp m}$ is a component of $\boldsymbol{\xi}_\perp$ in the meridian plane.

6.2. Possible Ideal Ionospheric Boundary Conditions Compatible With the Magnetospheric Energy Principle

[52] There are four possible combinations of the boundary conditions, i.e., combinations of either one of the equations (53) and (54), for which the force operator **F** becomes self-adjoint. In equations (53) and (54), $\boldsymbol{\eta}$ must be replaced with $\boldsymbol{\xi}$. In the following, it is clarified that B.T. becomes equal to δW_I for all combinations of the boundary conditions.

[53] Both integrands of equation (74) vanish at the perfectly conducting walls S_{out} and S_{in}. The contribution to B.T. in equation (74) from S_{dawn} and S_{dusk} also vanishes for the two-dimensional configuration. Therefore only the ionospheric boundary contributes to B.T. in equation (74). Equation (74) can be written as

$$\text{B.T.} = -\frac{1}{2}\int_S \left[\gamma p(\nabla\cdot\boldsymbol{\xi})\xi_\parallel^* + \mu_0^{-1}B\boldsymbol{\xi}_\perp^* \cdot((\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp)\right](\mathbf{b}\cdot\mathbf{n})dS + \delta W_I. \quad (78)$$

[54] From equation (78) it is obvious that B.T. becomes equal to δW_I for all combinations of either one of the equations (53) and (54), i.e.,

$$\xi_\parallel = 0 \text{ and } (\mathbf{b}\cdot\nabla)\boldsymbol{\xi}_\perp = 0, \quad (79)$$

$$\nabla\cdot\boldsymbol{\xi} = 0 \text{ and } (\mathbf{b}\cdot\nabla)\boldsymbol{\xi}_\perp = 0, \quad (80)$$

$$\nabla\cdot\boldsymbol{\xi} = 0 \text{ and } \boldsymbol{\xi}_\perp = 0, \quad (81)$$

$$\xi_\parallel = 0 \text{ and } \boldsymbol{\xi}_\perp = 0 \quad (82)$$

on the unperturbed ionospheric surface in both hemispheres.

[55] For the above four ideal ionospheric boundary conditions, the self-adjointness of the force operator defined for the magnetospheric plasma volume P is satisfied and the energy is conserved within the magnetospheric system. Therefore the problem is closed without considering the existence of a neutral atmosphere below the ionosphere.

[56] Let us consider the physical meaning of each of the boundary conditions (79), (80), (81), and (82). From the linearization of the ideal Ohm's law one obtains

$$\mathbf{E}_{1\perp} + \mathbf{v}_{1\perp}\times\mathbf{B} = 0, \quad (83)$$

where the subscript 1 denotes a perturbed part. Therefore the boundary condition (81) means that $\mathbf{E}_{1\perp} = 0$. For this reason the boundary condition (81) is called a conducting ionospheric boundary condition. A similar usage of the conducting boundary condition was made in the study of the stability of a pressure-driven mode in tandem mirrors [*Nevins and Pearlstein*, 1988]. Note that the conducting boundary condition (81) is different from the conducting boundary condition used in *Hameiri et al.* [1991], which is defined by using ionospheric conductivity.

[57] On the other hand, from equation (43) one obtains

$$\mathbf{B}_1 = -B\mathbf{b}(\nabla\cdot\boldsymbol{\xi}_\perp) + B(\mathbf{b}\cdot\nabla)\boldsymbol{\xi}_\perp - B(\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{b} - \mathbf{b}(\boldsymbol{\xi}_\perp\cdot\nabla)B. \quad (84)$$

For the boundary condition (79), equation (84) becomes

$$\mathbf{B}_1 = -B\mathbf{b}(\nabla\cdot\boldsymbol{\xi}_\perp) - B(\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{b} - \mathbf{b}(\boldsymbol{\xi}_\perp\cdot\nabla)B. \quad (85)$$

Here the second term, which is equal to $B(\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{b}$, is of a much smaller order than $B\mathbf{b}(\nabla\cdot\boldsymbol{\xi}_\perp)$, because $(\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{b}$ is $O(|\boldsymbol{\xi}_\perp|/R_I)$ for the three-dimensional configuration and $O(|\boldsymbol{\xi}_{\perp m}|/R_I)$ for the two-dimensional configuration and $\mathbf{b}(\nabla\cdot\xi_\perp)$ is $O(|\boldsymbol{\xi}_\perp|/\triangle\ell)$ with $\triangle\ell$ being a typical width of the auroral zone and $\triangle\ell$ (~500 km) $\ll R_I$ (~6500 km). Therefore the $\mathbf{B}_{1\perp}$ component arising from $B(\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{b}$ is small. If $\mathbf{B}_{1\perp}$ is exactly zero, there is no surface current in the ionosphere and this boundary condition should be called an insulating boundary condition as used by *Nevins and Pearlstein* [1988]. However, since $\mathbf{B}_{1\perp}$ is nonzero for the present spherical or cylindrical ionospheric boundary surface and is essential for nonzero δW_I as shown by equation (75), when the ionospheric $\boldsymbol{\xi}_\perp$ is nonzero, the boundary condition (79) is called a horizontally free boundary condition.

[58] The boundary condition (80) is called a free boundary condition, because $\xi_\parallel \neq 0$ and $\boldsymbol{\xi}_\perp \neq 0$. The boundary condition (82) also means $\mathbf{E}_{1\perp} = 0$. However, this condition is called a rigid or fixed boundary condition, because $\boldsymbol{\xi} = 0$ means mechanically that the ionosphere is a rigid surface. The rigid boundary condition has been used to model a photospheric boundary by *Schindler et al.* [1983]. The rigid boundary condition and an approximate boundary condition of $\boldsymbol{\xi}_\perp = 0$ and $\xi_\parallel \neq 0$, which is somewhat similar to the conducting boundary condition, have also been used by *Hood* [1986] to model a photospheric boundary for ballooning instability. Since the timescale of a change in the equilibrium state is much longer than the timescale of perturbations, an equilibrium state with a normal component of the equilibrium field $\mathbf{B}_0$ is considered to be given a priori at the ionospheric boundary regardless of the boundary condition for the perturbation as has been assumed for both magnetospheric and solar plasmas. For both conducting, horizontally free, free, and rigid ionospheric boundary conditions, the boundary conditions (53) and (54), for which the force operator **F** becomes self-adjoint, are satisfied. Therefore for both conducting, horizontally free, free, and rigid boundary conditions, the magnetospheric energy principle is valid without the need to consider the existence of the neutral atmosphere below the ionosphere and the change in the potential energy is simply given by $\delta W_F + \delta W_I$.

6.3. Magnetospheric Energy Principle Expressed by an Intuitive Form of δW_F

[59] According to *Freidberg* [1987], one obtains

$$\delta W_F = \frac{1}{2}\int_P d\mathbf{r}\left[\mu_0^{-1}|\mathbf{Q}_\perp|^2 + \mu_0^{-1}B^2|\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}|^2 + \gamma p|\nabla\cdot\boldsymbol{\xi}|^2 - 2(\boldsymbol{\xi}_\perp\cdot\nabla p)(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*) - J_\parallel(\boldsymbol{\xi}_\perp^*\times\mathbf{b})\cdot\mathbf{Q}_\perp\right]. \quad (86)$$

This is the intuitive form of δW_F originally suggested by *Furth et al.* [1965] and *Greene and Johnson* [1968].

[60] The last two terms in the integrand of equation (86) can be positive or negative and thus can drive instabilities. The first of these is proportional to $\nabla p \sim \mathbf{J}_\perp \times \mathbf{B}$ while the second is proportional to $J_\parallel$. Thus either perpendicular or parallel currents represent potential sources of instability. The former type are sometimes referred to as pressure-driven modes and the latter as current-driven modes or kink modes.

[61] The surface contribution δW_I is negative for both horizontally free and free boundary conditions when the curved finite ionospheric surface area is taken into account. Therefore this term is destabilizing for these boundary conditions. Since R_I appears in equations (76) and (77), this destabilizing effect by δW_I occurs for a spherical ionospheric surface for the three-dimensional configuration or for a cylindrical ionospheric surface for the two-dimensional configuration. The surface contribution δW_I obviously vanishes for a fictitious flat ionospheric surface without any curvature (or in the limit of $R_I \to \infty$).

[62] In summary, the magnetospheric energy principle states that a plasma equilibrium is stable if and only if $\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = \delta W_F + \delta W_I \geq 0$ for all allowable displacements (i.e., $\boldsymbol{\xi}$ bounded in energy and satisfying one of the appropriate boundary conditions, i.e., (79), (80), (81), or (82)), where δW_F is calculated for the unperturbed plasma volume P for both the two-dimensional and three-dimensional configurations and δW_I is calculated for the unperturbed ionospheric surfaces. The single assumption of this magnetospheric energy principle is that the unperturbed magnetic field is perpendicular to the ionospheric surface over the whole ionosphere, the validity of which is verified in section 8.6.

7. Minimization of the Change in the Potential Energy δW with Respect to $\xi_\parallel$

[63] In order to obtain a condition, which is satisfied by the most unstable mode, the change in the potential energy $\delta W = \delta W_F + \delta W_I$ is minimized with respect to $\xi_\parallel$ for the magnetospheric geometry shown in Figure 1. The minimizing condition follows from letting $\xi_\parallel \to \xi_\parallel + \delta\xi_\parallel$ in $\delta W = \delta W_F + \delta W_I$ and then setting the corresponding variation $\delta(\delta W)_{\xi_\parallel}$ to zero. Since there are four possible boundary conditions at the ionosphere, i.e., horizontally free, conducting, free, and rigid boundary conditions, careful attention must be paid to the definition of the variation of $\xi_\parallel$, i.e., $\delta\xi_\parallel$, at the ionospheric boundary for each case. For horizontally free and rigid boundary conditions $\xi_\parallel = 0$ at the ionospheric boundary. Therefore it is necessary to choose $\delta\xi_\parallel$ so as to satisfy $\delta\xi_\parallel = 0$ at the ionospheric boundary for these two ionospheric boundary conditions. Except for that constraint on $\delta\xi_\parallel$ at the ionospheric boundary, $\delta\xi_\parallel$ is arbitrary elsewhere. For the conducting and free ionospheric boundary conditions, $\xi_\parallel$ is not fixed at the ionospheric boundary. Therefore $\delta\xi_\parallel$ is arbitrary everywhere including the ionospheric boundary.

[64] Keeping these constraints on $\delta\xi_\parallel$ in mind, let us calculate the variation of δW with respect to $\xi_\parallel$. Following *Freidberg* [1987] and noting that $\xi_\parallel$ appears only in the $\gamma p|\nabla\cdot\boldsymbol{\xi}|^2$ term, one finds

$$\begin{aligned}\delta(\delta W)_{\xi_\parallel} &= \delta W_F\left(\xi_\parallel + \delta\xi_\parallel\right) - \delta W_F\left(\xi_\parallel\right)\\ &= \frac{1}{2}\int_P d\mathbf{r}\gamma p\left[|\nabla\cdot\boldsymbol{\xi}_\perp + \nabla\cdot\left(\left(\xi_\parallel + \delta\xi_\parallel\right)\mathbf{b}\right)|^2 - |\nabla\cdot\boldsymbol{\xi}_\perp + \nabla\cdot\left(\xi_\parallel\mathbf{b}\right)|^2\right]\\ &= \frac{1}{2}\int_P d\mathbf{r}\gamma p\left(|\nabla\cdot\boldsymbol{\xi} + \nabla\cdot\left(\delta\xi_\parallel\mathbf{b}\right)|^2 - |\nabla\cdot\boldsymbol{\xi}|^2\right)\\ &\cong \frac{1}{2}\int_P d\mathbf{r}\gamma p\left[(\nabla\cdot\boldsymbol{\xi})\nabla\cdot\left(\delta\xi_\parallel^*\mathbf{b}\right) + \left(\nabla\cdot\boldsymbol{\xi}^*\right)\nabla\cdot\left(\delta\xi_\parallel\mathbf{b}\right)\right]. \quad (87)\end{aligned}$$

In equation (87) one obtains

$$\left(\nabla\cdot\boldsymbol{\xi}^*\right)\left[\nabla\cdot\left(\frac{\delta\xi_\parallel}{B}\mathbf{B}\right)\right] = \nabla\cdot\left[\left(\nabla\cdot\boldsymbol{\xi}^*\right)\frac{\delta\xi_\parallel}{B}\mathbf{B}\right] - \left[\nabla\left(\nabla\cdot\boldsymbol{\xi}^*\right)\right]\cdot\frac{\delta\xi_\parallel}{B}\mathbf{B}. \quad (88)$$

It follows from equations (87) and (88) that

$$\begin{aligned}\delta(\delta W)_{\xi_\parallel} = &-\frac{1}{2}\int_P d\mathbf{r}\gamma p\left[\frac{\delta\xi_\parallel}{B}(\mathbf{B}\cdot\nabla)\left(\nabla\cdot\boldsymbol{\xi}^*\right) + \frac{\delta\xi_\parallel^*}{B}(\mathbf{B}\cdot\nabla)(\nabla\cdot\boldsymbol{\xi})\right]\\ &+\frac{1}{2}\int_S \gamma p\left[\frac{\delta\xi_\parallel}{B}\left(\nabla\cdot\boldsymbol{\xi}^*\right) + \frac{\delta\xi_\parallel^*}{B}(\nabla\cdot\boldsymbol{\xi})\right]\mathbf{B}\cdot\mathbf{n}dS. \quad (89)\end{aligned}$$

In equation (89), the second integral represents the boundary term. The contribution to this boundary term from perfectly conducting walls vanishes because of $\mathbf{B}\cdot\mathbf{n} = 0$ at that boundary for the two-dimensional and three-dimensional configurations. For the two-dimensional configuration, contributions to the integrals from the dawn and dusk boundaries cancel out because of periodic conditions. For the horizontally free and rigid ionospheric boundary conditions this boundary term becomes zero because $\delta\xi_\parallel = 0$ at the ionospheric boundary. For the conducting and free ionospheric boundary conditions this boundary term also becomes zero because $\nabla\cdot\boldsymbol{\xi} = 0$ at the ionospheric boundary. Therefore for all the possible ionospheric boundary conditions

$$\delta(\delta W)_{\xi_\parallel} = -\frac{1}{2}\int_P d\mathbf{r}\gamma p\left[\frac{\delta\xi_\parallel}{B}(\mathbf{B}\cdot\nabla)\left(\nabla\cdot\boldsymbol{\xi}^*\right) + \frac{\delta\xi_\parallel^*}{B}(\mathbf{B}\cdot\nabla)(\nabla\cdot\boldsymbol{\xi})\right]. \quad (90)$$

Since $\delta\xi_{\|}$ and $\delta\xi_{\|}^{*}$ are arbitrary in the integrand of equation (90), equation (90) becomes zero only when

$$(\mathbf{B}\cdot\nabla)\nabla\cdot\boldsymbol{\xi}=0 \tag{91}$$

within the unperturbed plasma volume *P*.

[65] The condition (91) represents the minimizing condition of the change in the potential energy δW with respect to $\xi_{\|}$. Equation (91) means that for the most unstable mode $\nabla\cdot\boldsymbol{\xi}$ = const. along the field line. For the conducting and free ionospheric boundary conditions, $\nabla\cdot\boldsymbol{\xi}=0$ on the ionospheric boundary. Therefore $\nabla\cdot\boldsymbol{\xi}=0$ holds along the entire field line for the conducting and free ionospheric boundary conditions. For the horizontally free and rigid ionospheric boundary conditions, $\nabla\cdot\boldsymbol{\xi}$ = const. $\neq 0$ holds along the entire field line.

8. Discussion

8.1. Energy Principle and Pressure-Driven Modes

[66] Although the present energy principle has sought ionospheric boundary conditions to satisfy the self-adjointness of the force operator, it might be possible that there are other physically realizable boundary conditions for which the operator $\mathbf{F}(\boldsymbol{\xi})$ is not self-adjoint, so that no energy criterion exists and eigenvalues become complex. Such a case might nevertheless be relevant for magnetospheric stability or instability. For example, when there is an unperturbed flow in the magnetosphere and there is a shear in the flow velocity, Kelvin-Helmholtz instability occurs and the instability has a complex frequency. Such a case cannot have a self-adjoint force operator but the magnetospheric plasma may still be subject to such an ideal MHD instability. As an other example, let us consider a more realistic ionosphere with a finite Pedersen conductivity Σ_P. For such a case, an energy principle does not exist obviously, because there is a finite energy dissipation in the ionosphere, and the force operator cannot be self-adjoint. However, the magnetosphere might still be subject to MHD instability. Although there may be other examples, the mentioning of these two examples would be sufficient to show that there are ionospheric boundary conditions, which do not satisfy the self-adjointness of the force operator, but nevertheless are relevant for magnetospheric stability or instability. Since these cases cannot be studied by an energy principle, the following discussions focus on ideal MHD instabilities, which are able to be studied by an energy principle.

[67] Let us consider the stability of magnetospheric plasma in the physical region of interest shown in Figure 1. For the sake of simplicity, magnetospheric models with $J_{\|}$ = 0 are considered, although such magnetospheric models may be atypical [*Birn*, 1991]. If one assumes that the static equilibrium has no $J_{\|}$ in equation (86), then the kink mode is excluded. Therefore apart from the destabilizing influence of δW_{I} for horizontally free and free ionospheric boundary conditions, the only possible instability modes for such an equilibrium are pressure-driven modes such as interchange and ballooning modes. It is shown in the following three subsections that the four different ionospheric boundary conditions, i.e., horizontally free, conducting, free, and rigid boundary conditions, correspond to interchange, incompressible ballooning, incompressible ballooning, and compressible ballooning modes, respectively. Since equation (91) is used, the discussion in the following subsections is focused on the most unstable modes.

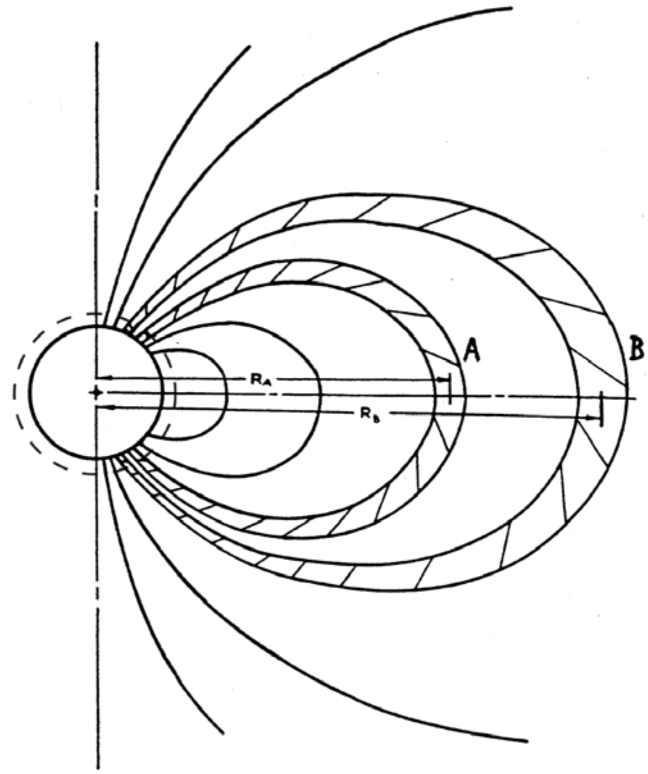

Figure 3. A slight modification of Figure 1 by *Gold* [1959]. The ionosphere is shown by a dashed circle. Two flux tubes A and B change their positions by interchange instability.

[68] Except for the study of a low-β limit of the magnetospheric stability, which can be studied by using the dipole field, it is necessary to properly specify a finite β magnetospheric equilibrium in order to study the stability of pressure-driven modes in the finite-β magnetosphere. There have been several different stability analyses for different magnetospheric equilibria. Since the present magnetospheric energy principle is valid for any magnetospheric equilibrium, these models are discussed in light of the present magnetospheric energy principle. Some numerical stability analyses using ballooning eigenmode equations have studied the stability of a single field line and thus are different from the present analysis studying the stability of a finite volume plasma in the magnetosphere. However, it is expected that knowledge obtained about boundary conditions and the minimizing condition by the present study is applicable to those stability analyses.

[69] In the following two subsections, the destabilizing influence by δW_{I} is neglected for horizontally free and free ionospheric boundary conditions on the assumption that a finite $\boldsymbol{\xi}_{\perp}$ at the ionosphere is confined in a narrow latitudinal region, so that the surface integral (76) or (77) is negligible. Such an assumption is strictly valid for stability discussion of a single field line. The destabilizing influence by δW_{I} is discussed in section 8.7.

8.2. Interchange Mode

[70] *Gold* [1959] first studied interchange instability in magnetospheric plasmas. Figure 3 is a slight modification of

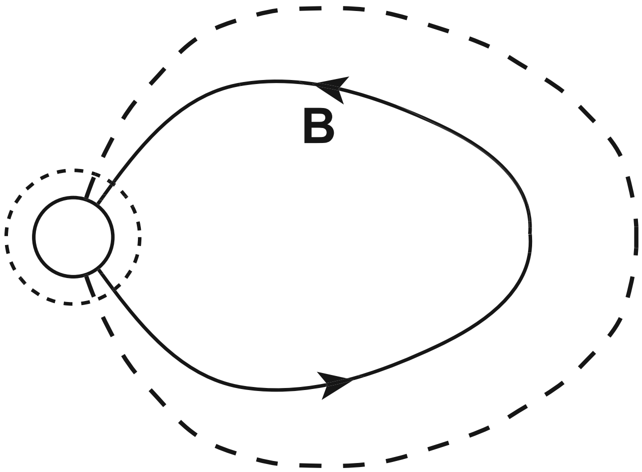

Figure 4. The solid line shows an unperturbed field line in the magnetosphere. The dotted circle is the ionospheric boundary. Shown schematically by a dashed line is a field line perturbed by an interchange mode. The amplitude of perturbation is exaggerated.

an original figure in the work of *Gold* [1959]. The dashed line is the ionospheric boundary. In this figure, it is shown that two flux tubes A and B, which are assumed to contain the same magnetic flux, are changing their positions by interchange instability. Since the interchange mode involves the motion of a flux tube as a whole, as shown in this figure, this instability requires $\boldsymbol{\xi}_\perp \neq 0$ at the ionospheric boundaries. Otherwise, the field lines are fixed at the ionospheric boundaries and the interchange of flux tubes is not possible. Among the four possible ionospheric boundary conditions, only the horizontally free and free boundary conditions [equations (79) and (80)] satisfy this requirement. Notice, in particular, that the rigid ionospheric boundary condition, i.e., $\boldsymbol{\xi} = 0$ at the ionospheric boundary, is not valid for interchange mode.

[71] However, in the magnetosphere, there is no line bending or $\mathbf{B}_{1\perp} \approx 0$ for interchange instability [*Freidberg*, 1987] and the perturbed magnetic field of the interchange mode virtually has only a component parallel to the equilibrium magnetic field. This kind of perturbation corresponds to the definition of an interchange motion in the sense that the geometrical shape of the field line is unchanged [*Hameiri et al.*, 1991]. From equation (84) one has

$$B_{1\|} = \mathbf{b} \cdot \mathbf{B}_1 = -B\nabla \cdot \boldsymbol{\xi}_\perp + \frac{\mu_0}{B}\boldsymbol{\xi}_\perp \cdot \nabla p - 2B\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}, \tag{92}$$

where the first term originates from the $\nabla \cdot \boldsymbol{\xi}$ term. Therefore in order to have $B_{1\|}$ of the interchange mode in the low-altitude region, where the plasma β is much smaller than unity and the curvature of the field line is weak, the nonzero $\nabla \cdot \boldsymbol{\xi}$ is necessary. Since the most unstable mode satisfies equation (91), this means that the interchange mode, which is the most unstable, is compressible in the magnetospheric plasma. Therefore between the horizontally free and free boundary conditions only the horizontally free ionospheric boundary condition allows the existence of interchange modes. Figure 4 is a schematic of the midnight meridian plane and shows an unperturbed field line (solid line) and a field line (dashed line) perturbed by an interchange mode. The amplitude of perturbation is exaggerated in this figure.

[72] Since $\nabla \cdot \xi$ is nonzero, this mode suffers a stabilizing influence of compressibility due to the $\gamma p|\nabla \cdot \boldsymbol{\xi}|^2$ term in equation (86). When the destabilizing term $2(\boldsymbol{\xi}_\perp \cdot \nabla p)(\boldsymbol{\kappa} \cdot \boldsymbol{\xi}^*_\perp)$ in equation (86) overcomes this stabilizing term, interchange instability occurs. For an axisymmetric system with periodicity in the direction of the axis of symmetry, *Bernstein et al.* [1958] showed that the system is stable if and only if $\Lambda > 0$ for all values of the flux function ψ, where

$$\Lambda = \frac{\gamma p(V'' - p'L')(V''/V' + p'/(\gamma p))}{V' + \gamma pL'}. \tag{93}$$

Here L' and V' are both positive and the prime denotes $d/d\psi$. The flux function ψ represents the flux out to a given flux surface, so ψ is a flux label, and $V(\psi)$ is the volume enclosed by the surface in one period along the symmetry axis. The specific volume of a magnetic tube or the volume of a flux tube per flux $U(\psi)$ is defined by

$$U(\psi) = \oint \frac{d\ell}{B}, \tag{94}$$

where the integration is taken for one period along the symmetry axis and ℓ is the distance along the field line. It is shown that $V' = U(\psi)$ [e.g., *Kulsrud*, 2005].

[73] For a general system with closed field lines, *Hameiri et al.* [1991] obtained that a sufficient condition for stability is

$$\left(U' - Up'\left\langle\frac{1}{B^2}\right\rangle\right)(\gamma pU' + p'U) > 0. \tag{95}$$

Here $U(\psi)$ is given by equation (94), where now the integration is taken for a closed field line and $\langle f\rangle \equiv \oint fB^{-1}d\ell/U$. The full expression (95) was originally derived by *Spies* [1971]. It is easily shown that $\Lambda > 0$ is equivalent to equation (95) in the axisymmetric system considering that $V' = U$.

[74] In the limit of small plasma pressure, the first parenthesis in equation (95) may be replaced by U' and the expression becomes the familiar low-β interchange stability criterion [*Rosenbluth and Longmire*, 1957]

$$\delta U\delta(pU^\gamma) > 0. \tag{96}$$

Here the integration in the calculation of $U(\psi)$ is taken for one period because an axisymmetric system case, which is periodic along the axis of symmetry, is considered.

[75] In the above three calculations there is no boundary where the field line is anchored. This assumption was necessary in the above three calculations to allow the existence of interchange modes. Therefore the integration in equation (94) must strictly be taken for either one period or a closed field line. However, the present magnetospheric energy principle shows that the horizontally free ionospheric boundary condition gives an interchange mode. Therefore if the horizontally free boundary condition is

imposed at the ionosphere, the specific volume $U(\psi)$ can be defined as

$$U(\psi) = \int \frac{d\ell}{B}, \tag{97}$$

where now the integration is taken from the ionosphere in the Southern Hemisphere to the ionosphere in the Northern Hemisphere. By this reasoning, it is possible to apply the above stability criteria to a magnetosphere with ionospheric boundaries.

[76] From the equivalence of $\Lambda > 0$ and equation (95), it is obvious that the stability criterion $\Lambda > 0$ is also valid for a two-dimensional system. *Hurricane et al.* [1996] have shown that an approximate two-dimensional magnetostatic equilibrium tail model of *Lembège and Pellat* [1982] gives $\Lambda < 0$ and thus is unstable. More specifically, they have shown that $p' + \gamma p V''/V' < 0$ in their tail model, noting that $V'' > p'L$. Although they do not discuss how $U(\psi)$ should be calculated to avoid a boundary problem inhibiting interchange modes, the integration in equation (97) can be taken between the two ionospheres, if the horizontally free boundary condition is imposed at the ionospheric boundaries.

[77] For a point dipole, the magnetic field strength $B_{\rm eq}$ at the equator is proportional to R^{-3} and thus the specific flux tube volume $U \propto B_{\rm eq}^{-1} R$ is proportional to R^4 [*Gold*, 1959], where R is the distance from the Earth's center in the equatorial plane. In the low-β limit, the stability criterion can be written as equation (96), which is also valid for closed field lines like equation (95). Since $dU/dR > 0$, equation (96) becomes $d(pU^\gamma)/dR > 0$ for the instability. Substitution of $U \propto R^4$ into $d(pU^\gamma)/dR > 0$ yields

$$\frac{d\ell np(R)}{d\ell nR} > -4\gamma \tag{98}$$

as an interchange stability criterion in the low-β limit of the magnetosphere, i.e., in the dipole field configuration [*Kadomtsev*, 1965]. Although *Kadomtsev* [1965] calculated $U(\psi)$ for a closed field line by using equation (94), $U(\psi)$ can be calculated by using equation (97), if a horizontally free boundary condition is imposed at the ionosphere. Therefore the three-dimensional system will be stable when the energy density or the pressure falls off with increasing radius more slowly than $R^{-20/3}$ for the adiabatic case ($\gamma = 5/3$), or R^{-4} for the isothermal case ($\gamma = 1$) as has been shown by *Gold* [1959].

[78] By defining entropy S_e by

$$S_e = \ell n(pU^\gamma), \tag{99}$$

Schindler and Birn [2004] showed that the sufficient condition for interchange stability in the geomagnetic tail can be written as

$$\frac{dS_e}{dp} < 0. \tag{100}$$

Since ψ increases as R increases in the tail, this is equivalent to $dS_e/d\psi > 0$ because of $dp/d\psi < 0$. By dividing both sides of equation (95) by pU^γ, this stability criterion is valid when

$$U' > Up'\left\langle \frac{1}{B^2} \right\rangle. \tag{101}$$

For a typical tail configuration $B_{\rm eq}$ is decreasing with increasing R and the specific flux tube volume $U(\psi)$, which is nearly proportional to $B_{\rm eq}^{-1}$, increases with the distance from the Earth or $U' = V'' > 0$. Also, $p' < 0$ holds for a typical tail configuration. Therefore for a typical geomagnetic tail configuration the interchange stability criterion expressed by the flux tube entropy in equation (100) is a sufficient stability criterion for interchange instability as *Schindler and Birn* [2004] argue. However, for an atypical tail configuration as used by *Wu et al.* [1998], in which $V'' = U' < 0$, equation (100) is a sufficient stability criterion only when equation (101) is valid.

[79] The high-β stabilization of interchange mode by compressibility in the tail flux tube has been discussed by using the MHD potential energy functional δW by *Horton et al.* [1999] and *Miura* [2000, 2001] for specific tail field models. They assumed no line bending or $\mathbf{B}_{1\perp} \approx 0$ for interchange instability [*Freidberg*, 1987]. Both studies have shown that the tail flux tube is stable when β is larger than a critical β value. A slight difference in their critical β values is due to the difference of the specific field models.

8.3. Incompressible Ballooning Mode

[80] When either the conducting or free ionospheric boundary condition is used, the most unstable mode becomes incompressible along the entire field line because of equation (91), which is equivalent to a condition given in the Appendix of *Schindler and Birn* [2004]. Since the interchange mode is compressible, this mode is not an interchange mode but an incompressible ballooning mode.

[81] The incompressible ballooning mode does not need to satisfy the interchange instability criterion and it occurs even when the interchange instability criterion is not satisfied. As *Kulsrud* [2005] argues, if the interchange instability criterion is not satisfied, then altering the displacement from the pure interchange to emphasize the curvature dominated regions of the magnetic field can lead to a negative change in the potential energy. If this is large enough to overcome the change in the magnetic energy associated with this altered magnetic field then the plasma becomes unstable.

[82] *Miura et al.* [1989] solved an incompressible ballooning mode equation by imposing $\boldsymbol{\xi}_\perp = 0$ at the ionospheric boundary. This is equivalent to solving ideal MHD equations for the conducting ionospheric boundary condition $\boldsymbol{\xi}_\perp = \nabla \cdot \boldsymbol{\xi} = 0$, since the magnetospheric energy principle requires $\nabla \cdot \boldsymbol{\xi} = 0$ along the entire field line. For this mode, the $\mu_0^{-1}|\mathbf{Q}_\perp|^2$ term in equation (86) gives a stabilizing influence due to the field line tension. They used a tail-like equilibrium model of *Kan* [1973] and found that all tail field lines are unstable and the growth rate decreases with increasing distance from the Earth. For any tail-like equilibrium, in which the pressure gradient force is balanced with the field line curvature force, the field line becomes unstable when the plasma β at the equator satisfies approximately

$$\beta > \beta_{\rm cr} = L_{\rm p}/R_{\rm c}, \tag{102}$$

where $L_{\rm p}$ is the pressure gradient scale length at the equator and $R_{\rm c}$ is the field line curvature radius at the equator [*Miura*, 2004].

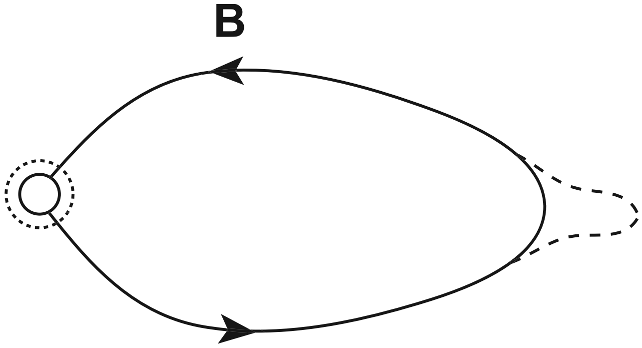

Figure 5. The solid line shows an unperturbed field line in the magnetosphere. The dotted circle is the ionospheric boundary. Shown schematically by a dashed line is a stretched field line perturbed by an incompressible ballooning mode. The amplitude of the perturbation is exaggerated. Only the perturbed field line projected on the midnight meridian plane is shown in this figure.

[83] According to *Miura et al.* [1989], the unstable mode is bound near the equator (see Figure 7 of *Miura et al.* [1989]) and is exponentially decreasing toward the ionospheric boundary. Therefore this mode strucure is compatible with $\boldsymbol{\xi}_\perp = 0$ at the ionosphere. Figure 5 is a schematic of the midnight meridian plane and shows a stretched field line (dashed line) perturbed by an incompressible ballooning mode. The solid circle is the Earth and the dotted circle is the ionospheric boundary. The highly localized nature of this incompressible ballooning perturbation near the equatorial plane is inferred from the results of Figure 7 in *Miura et al.* [1989]. The solid line in this figure shows an unperturbed field line. In Figure 5 only the perturbed field line projected on the midnight meridian plane is shown. The amplitude of perturbation is exaggerated in this figure. Since the perturbed region near the equator is so remote from the ionosphere, the incompressible ballooning mode is expected to be rather insensitive to the ionospheric boundary condition on $\boldsymbol{\xi}_\perp$, although it is sensitive to the ionospheric boundary condition on $\nabla \cdot \boldsymbol{\xi}$ or $\xi_\parallel$ through equation (91). Therefore it is expected that both the conducting and free boundary conditions allow the existence of an incompressible ballooning mode. This insensitivity of the incompressible ballooning mode to the ionospheric boundary condition on $\boldsymbol{\xi}_\perp$ means that this mode is robust.

[84] *Lee and Wolf* [1992] argue, based on the consideration of perturbations in the magnetosphere and the neglect of a magnetic field deformation near the equator, that the $\xi_\parallel \neq 0$ condition of conducting and free ionospheric boundary conditions is inapplicable. However, this specification of the most realistic ionospheric boundary condition cannot be made by considering the behavior of the magnetospheric plasma. That is, a boundary condition on partial differential equations cannot be determined by the consideration of the equations themselves.

[85] Employing a growth phase magnetospheric equilibrium including $J_\parallel \neq 0$ [*Zaharia and Cheng*, 2003], *Cheng and Zaharia* [2004] solved ballooning eigenmode equations numerically. Their ballooning mode equations were obtained by the WKB-ballooning formalism [*Dewar and Glasser*, 1983], which does not contain the kink term in the lowest order. They found that all field lines beyond $\simeq 6R_\mathrm{E}$ radius down the tail in the nightside, where the equatorial $\beta_\mathrm{eq} \geq 1$, are unstable. The most unstable region is located in the strong cross-tail current sheet region, which maps into the ionosphere in the transition area between Region-1 and Region-2 currents. They used the conducting ionospheric boundary condition, i.e., $\boldsymbol{\xi}_\perp = \nabla \cdot \boldsymbol{\xi} = 0$. However, whether or not the most unstable mode between Region-1 and Region-2 currents is a ballooning mode is not clear, because the effect of kink term $J_\parallel\,(\boldsymbol{\xi}_\perp^* \times \mathbf{b}) \cdot \mathbf{Q}_\perp$ in equation (86) cannot be neglected when $J_\parallel$ is nonzero. Their ballooning eigenmode equations do not contain the kink term due to $J_\parallel$. Their results show that for the same boundary conditions the inner region is stable. This is simply explained by the fact that ballooning modes are stabilized by field line tension represented by the term $\mu_0^{-1}|\mathbf{Q}_\perp|^2$ in equation (86). Since *Miura et al.* [1989] treated only tail regions with distance from the Earth greater than a certain distance, where the field line tension is not strong enough to stabilize the mode, they could not find any stable modes.

[86] It is important to note here that in discussing ballooning modes in a magnetic confinement device with an axisymmetric toroidal configuration, *Coppi* [1977] takes into account only the convective or incompressible contribution $\boldsymbol{\xi} \cdot \nabla p$ in equation (10) by neglecting the compressible term $\gamma p \nabla \cdot \boldsymbol{\xi}$ in equation (10). The same assumption has also been made by *Miura et al.* [1989] in deriving the incompressible ballooning eigenmode equation for a two-dimensional configuration.

8.4. Compressible Ballooning Mode

[87] When the rigid ionospheric boundary condition is imposed, $\nabla \cdot \boldsymbol{\xi} \neq 0$ at the ionosphere and the most unstable mode becomes compressible along the entire field line because of equation (91). Since $\boldsymbol{\xi}_\perp = 0$ is used at the ionosphere, this mode is not an interchange mode but a compressible ballooning mode.

[88] Assuming that $\nabla \cdot \boldsymbol{\xi}$ is constant along the field line and $\xi_\parallel = 0$ at the ionosphere, the compressibility factor $\nabla \cdot \boldsymbol{\xi}$ can be calculated with integration along the field line [see equation (104)]. The compressibility factor so obtained enables one to arrive at an eigenmode equation for compressible ballooning modes and a corresponding ideal MHD potential energy functional [e.g., *Schindler et al.*, 1983; *Lee and Wolf*, 1992; *Bhattacharjee et al.*, 1998; *Miura*, 2000]. The obtained compressible ballooning eigenmode equation has been used in some stability analyses of compressible ballooning modes. For compressible ballooning modes, stability analyses have been made for two basic configurations as shown below. One is a tail configuration with stretched field lines and the other is a near-Earth tail configuration including the dipole field.

[89] *Lee and Wolf* [1992] could not find an unstable solution for Kan's two-dimensional tail model. *Wu et al.* [1998] showed, by a linearized ideal MHD simulation (initial value approach), that a pressure-driven mode grows in the region where an interchange instability criterion $\Lambda < 0$ is satisfied. They employed, as an equilibrium state, a current sheet structure obtained by particle simulations, which is an approximate magnetostatic equilibrium. In their case B_eq is increasing with increasing distance and $V'' = U'$

< 0 unlike a typical tail. However, since their tail is high-β, $-p'L'$ factor dominates in the first parenthesis in the numerator of equation (93) and the system becomes unstable. Although their numerical outer boundaries are not flux surfaces such as S_{out} and S_{in}, their boundaries are located far enough from the disturbed region, so that the present magnetospheric energy principle can reasonably be applied. Since the rigid boundary condition is used at the earthward boundary in their calculation, their unstable mode is a compressible ballooning mode. They found that the growth rate becomes smaller as the wavelength in the dawn-dusk direction becomes larger. They obtained a rapid growth time of about 29 s for compressible ballooning modes in a highly stretched tail field configuration, which is comparable to a rapid ballooning mode growth time obtained for the tail by *Miura et al.* [1989]. Therefore it needs to be clarified whether such a rapid growth rate of the compressible ballooning mode is due to their atypical tail configuration, where B_{eq} is increasing with distance, or due to the fact that an incompressible limit is attained in their configuration.

[90] *Schindler and Birn* [2004] studied the stability of a two-dimensional equilibrium tail configuration with stretched field lines. By numerically minimizing the corresponding MHD potential energy functional for three different equilibrium models, they found that the stability of the symmetric mode was governed by the interchange criterion based on entropy. They showed for specific numerical examples that the interchange criterion based on entropy [equation (100)] is not only a sufficient but also a necessary condition for stability of compressible ballooning modes, at least within an excellent approximation. They showed by numerical minimization that for specific highly stretched tail models, in which the flux tube entropy decreases with distance from the Earth, the tail is ballooning unstable.

[91] *Lee* [1998] obtained an unstable compressible ballooning mode in the near-earth region by using an equilibrium model of *Voigt* [1986] including the two-dimensional (line) dipole field. *Zhu et al.* [2004] studied linear characteristics of compressible ballooning modes by employing the same initial value approach and the same rigid boundary conditions as used by *Wu et al.* [1998]. However, they used an exact two-dimensional magnetostatic equilibrium of *Voigt* [1986] including the two-dimensional dipole field in the near-Earth region. They found unstable compressible ballooning modes where the condition, $\Lambda < 0$, is satisfied. However, unlike the atypical tail treated by *Wu et al.* [1998], where B_{eq} is increasing with distance, their case has B_{eq} decreasing with distance. They found the same wavelength dependence of the growth rate as that of *Wu et al.* [1998]. They also found a dependence of the instability growth rate on plasma β and the ratio of specific heats γ.

[92] Figure 6 is a schematic of the midnight meridian plane and shows a field line (dashed line) perturbed by a compressible ballooning mode. The solid circle is the Earth and the dotted circle is the ionospheric boundary. The solid line is an unperturbed field line. Unlike the case of an incompressible ballooning mode shown in Figure 5, the compressible ballooning mode is not strongly localized near the equator and has a broad distribution along the field line with fixed ends at the ionospheres (see, for example, Figure 4 of *Schindler and Birn* [2004]). The amplitude of perturbation is exaggerated in this figure.

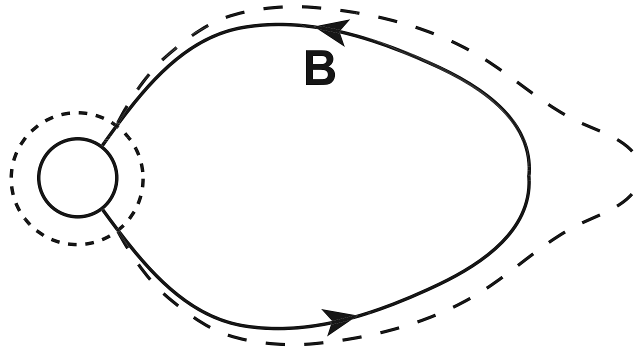

Figure 6. The solid line shows an unperturbed field line in the magnetosphere. The dotted circle is the ionospheric boundary. Shown schematically by a dashed line is a field line perturbed by a compressible ballooning mode. The amplitude of perturbation is exaggerated.

8.5. Symmetric Versus Antisymmetric Mode

[93] A special case, when the distribution of B and p is symmetric with respect to the equatorial plane, is considered here. When an eigenvalue ω^2 of the symmetric force operator $\mathbf{F}$ in such a case is not degenerate, the displacement eigenvector $\boldsymbol{\xi}$ of the eigenmode equation (12) is either symmetric or antisymmetric with respect to the equatorial plane.

[94] Let us consider the conducting and free ionospheric boundary conditions, in which $\nabla \cdot \boldsymbol{\xi} = 0$ is satisfied at the unperturbed ionospheric surface. The most unstable mode, which satisfies equation (91), is incompressible along the entire field line. Therefore only the field line tension is a stabilizing force for this mode. Since the field line tension is the weakest at the equator and the equator is the curvature dominated region, the field line displacement is likely to peak at the equator and therefore, the most unstable mode is likely to be a symmetric mode. For the two-dimensional configuration in the limit of a large wave number in the dawn-dusk direction, this property of the most unstable incompressible mode being a symmetric mode is rigorously proven by *Schindler and Birn* [2004].

[95] Next, let us consider the horizontally free and rigid boundary conditions, in which $\xi_\parallel = 0$ is satisfied at the unperturbed ionospheric surface. By dividing by B and then integrating both sides of an identity

$$\nabla \cdot \boldsymbol{\xi} = \nabla \cdot \boldsymbol{\xi}_\perp + (\mathbf{B} \cdot \nabla)\left(\xi_\parallel / B\right) \tag{103}$$

from $s = -s_b$ to $s = s_b$, where s is the distance along the field line and $s = -s_b$ is the ionosphere in the Southern Hemisphere and $s = s_b$ is the ionosphere in the Northern Hemisphere, one obtains for the most unstable mode

$$\nabla \cdot \boldsymbol{\xi} = \frac{\int_{-s_b}^{s_b} B^{-1} \nabla \cdot \boldsymbol{\xi}_\perp ds}{\int_{-s_b}^{s_b} B^{-1} ds}. \tag{104}$$

If the most unstable mode is an antisymmetric mode, this mode is shown to be incompressible from equation (104) by using $\boldsymbol{\xi}_\perp(-s) = -\boldsymbol{\xi}_\perp(s)$. However, this is unlikely, because for $\nabla \cdot \boldsymbol{\xi} = 0$ at the ionosphere, the most unstable mode is likely to be the symmetric mode as has been discussed above for conducting and free ionospheric boundary conditions. Therefore for the horizontally free and rigid boundary conditions, this possibility of the most unstable mode being an incompressible antisymmetric mode is excluded. Hence for the horizontally free and rigid boundary conditions the most unstable mode is a symmetric mode. This mode is an interchange mode for horizontally free boundary conditions and a compressible ballooning mode for rigid boundary conditions. This is compatible with an intuition that the interchange mode is a symmetric mode by definition (see Figures 3 and 4).

[96] In short, when the distribution of B and p is symmetric with respect to the equatoral plane, the most unstable mode is likely to be a symmetric mode for all the possible ideal ionospheric boundary conditions.

8.6. Condition for Validity of the Assumption of the Normal Incidence of the Unperturbed Magnetic Field on the Ionospheric Surface

[97] The present energy principle is applicable to any static magnetospheric equilibrium. The only assumption about the equilibrium is that the unperturbed magnetic field is incident vertically on the unperturbed ionospheric surface. This assumption imposes some restrictions on a magnetospheric equilibrium model near the ionospheric surface.

[98] Let us assume, for the sake of simplicity, that an unperturbed magnetic field is axisymmetric near the ionospheric surface for the three-dimensional configuration. Then, one can assume that the unperturbed magnetic field at the ionosphere of the Northern Hemisphere is given by $\mathbf{B}_0 = B_0(r, \lambda)\mathbf{b} = -B_0(r, \lambda)\mathbf{e}_r$ at $r = R_\mathrm{I}$ and at $\lambda_\mathrm{in} \leq \lambda \leq \lambda_\mathrm{out}$, where r is the geocentric distance, $\mathbf{e}_r$ is the unit vector in the radial direction, λ is the latitude, and λ_in and λ_out are the latitudes, where S_in and S_out intersect the spherical ionospheric surface. From the continuity of the normal component of the magnetic field, one can also assume that the unperturbed magnetic field near the ionosphere of the Northern Hemisphere is given by $\mathbf{B}_0 = B_0(r, \lambda)\mathbf{b} = -B_0(r, \lambda)\mathbf{e}_r$ in the neighborhood of $r \sim R_\mathrm{I}$ and at $\lambda_\mathrm{in} \leq \lambda \leq \lambda_\mathrm{out}$. Substitution of this magnetic field into $\nabla \cdot \mathbf{B}_0 = 0$ yields $B_0(r) = a(\lambda)r^{-2}$, where $a(\lambda)$ is a function of λ. Therefore $\mathbf{B}_0 = -a(\lambda)r^{-2}\mathbf{e}_r$ at $r \sim R_\mathrm{I}$ and $\lambda_\mathrm{in} \leq \lambda \leq \lambda_\mathrm{out}$. Since the region between lateral boundaries at $\lambda = \lambda_\mathrm{in}$ and $\lambda = \lambda_\mathrm{out}$ is a narrow latitutinal zone, one can assume further that $a(\lambda)$ is a constant. This means that $\mathbf{B}_0$ is a monopolar field at $r \sim R_\mathrm{I}$ and $\lambda_\mathrm{in} \leq \lambda \leq \lambda_\mathrm{out}$. Since this is a potential field, $\mathbf{J}_0 = \mu_0^{-1}\nabla \times \mathbf{B}_0 = 0$ near the ionospheric surface.

[99] For the two-dimensional configuration with translational symmetry in the dawn-dusk direction, a similar argument leads to $B_0(r) = a(\lambda)r^{-1}$ at $r \sim R_\mathrm{I}$ and $\lambda_\mathrm{in} \leq \lambda \leq \lambda_\mathrm{out}$. Therefore $\mathbf{B}_0 = -a(\lambda)r^{-1}\mathbf{e}_r$. This is also a potential field and $\mathbf{J}_0 = 0$, when $a(\lambda)$ is a constant.

[100] Therefore for both the two-dimensional and three-dimensional configurations, the assumption of the normal incidence of the unperturbed magnetic field on the ionosphere with a narrow latitudinal width requires $\mathbf{J}_0 \times \mathbf{B}_0 = 0$ at $r = R_\mathrm{I}$ and $\lambda_\mathrm{in} \leq |\lambda| \leq \lambda_\mathrm{out}$. Since p_0 is nonzero and nonuniform in the direction perpendicular to the unperturbed magnetic field for pressure-driven instabilities, this is compatible with the magnetostatic equilibrium satisfying $\mathbf{J}_0 \times \mathbf{B}_0 = \nabla p_0$ only when the ionospheric plasma beta, β_I, is much smaller than one and the ∇p_0 term is neglected in the force balance equation.

[101] By assuming $\beta \sim 1$ at $r \sim 10R_\mathrm{E}$ in the equatorial plane and also a dipole field with $B_0 \propto r^{-3}$ in the magnetosphere, the ionospheric β becomes $\sim 10^{-6}$ on the same field line. Therefore the present assumption of the normal incidence of the unperturbed magnetic field on the ionospheric surface is well justified owing to this low-β nature of the ionosphere in the present magnetospheric energy principle.

8.7. Energy Balance

[102] There are three possible combinations of $\delta W_\mathrm{F}(\boldsymbol{\xi}^*, \boldsymbol{\xi})$, $\delta W_\mathrm{I}(\boldsymbol{\xi}_\perp^*, \boldsymbol{\xi}_\perp)$, and $\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi})$: (1) $\delta W_\mathrm{F}(\boldsymbol{\xi}^*, \boldsymbol{\xi}) < 0$, $\delta W_\mathrm{I}(\boldsymbol{\xi}_\perp^*, \boldsymbol{\xi}_\perp) \leq 0$, $\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = \delta W_\mathrm{F} + \delta W_\mathrm{I} < 0$, (2) $\delta W_\mathrm{F}(\boldsymbol{\xi}^*, \boldsymbol{\xi}) > 0$, $\delta W_\mathrm{I}(\boldsymbol{\xi}_\perp^*, \boldsymbol{\xi}_\perp) \leq 0$, $\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = \delta W_\mathrm{F} + \delta W_\mathrm{I} > 0$, (3) $\delta W_\mathrm{F}(\boldsymbol{\xi}^*, \boldsymbol{\xi}) > 0$, $\delta W_\mathrm{I}(\boldsymbol{\xi}_\perp^*, \boldsymbol{\xi}_\perp) < 0$, $\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = \delta W_\mathrm{F} + \delta W_\mathrm{I} < 0$. Energy balance of each of these cases is briefly discussed below.

[103] Let us first consider case 1. This case includes an unstable mode, which grows exponentially, as is obvious from equation (59). For a single field line with $\delta W_\mathrm{I} = 0$, the existence of such pressure-driven unstable modes have been shown either analytically or numerically as has been discussed in detail in sections 8.2 through 8.4.

[104] When the curved finite ionospheric surface area is taken into account in case 1 and $\tilde{\boldsymbol{\xi}}_\perp^2$ at the ionospheric surface increases with time for nonzero $\tilde{\boldsymbol{\xi}}_\perp$ ionospheric boundary conditions, $\partial \delta W_\mathrm{I}(\tilde{\boldsymbol{\xi}}_\perp, \tilde{\boldsymbol{\xi}}_\perp)/\partial t$ is negative. Therefore equation (B30) shows that there is a net upward parallel Poynting flux into the magnetosphere across the ionospheric surface (notice from equation (B38) that the first order Poynting vector across the ionospheric surface vanishes). In this sense, the ionospheric contribution is destabilizing, although the instability itself may be pressure-driven. In other words, the kinetic energy of the magnetospheric plasma increases with a decrease in the potential energy of the magnetospheric system (see Appendix B for details).

[105] Since the relationship between Poynting vector across the ionospheric surface and the temporal change of $\tilde{\boldsymbol{\xi}}_\perp$ is important to understand the energy balance in the magnetospheric system, let us consider a physical picture of the relationship. One first notes that the second order linear perturbation of Poynting vector $\tilde{\mathbf{s}}_2$ is expressed by

$$\tilde{\mathbf{s}}_2 = \mu_0^{-1}\left(\tilde{\mathbf{E}}_1 \times \tilde{\mathbf{B}}_1 + \tilde{\mathbf{E}}_2 \times \mathbf{B}_0\right), \quad (105)$$

where the subscripts 1 and 2 denote the first order linear perturbation and the second order linear perturbation, respectively, and the subscript 0 denotes an unperturbed quantity. Since one is interested in the parallel component of Poynting vector, only the first term in equation (105) is relevant in the following discussion.

[106] Let us consider an interchange mode, which occurs for the horizontally free ionospheric boundary condition. Figure 7 shows a physical picture to explain the direction of Poynting vector across the ionospheric surface when the amplitude of $\tilde{\boldsymbol{\xi}}_\perp$ is increasing. The solid arc in this figure represents an ionospheric surface in the meridian plane of the Northern Hemisphere. Field lines and field vectors in the neighborhood of point A on the ionospheric surface are shown in this figure. An enlarged plot is also shown in the left-bottom corner of this figure. The straight line AB

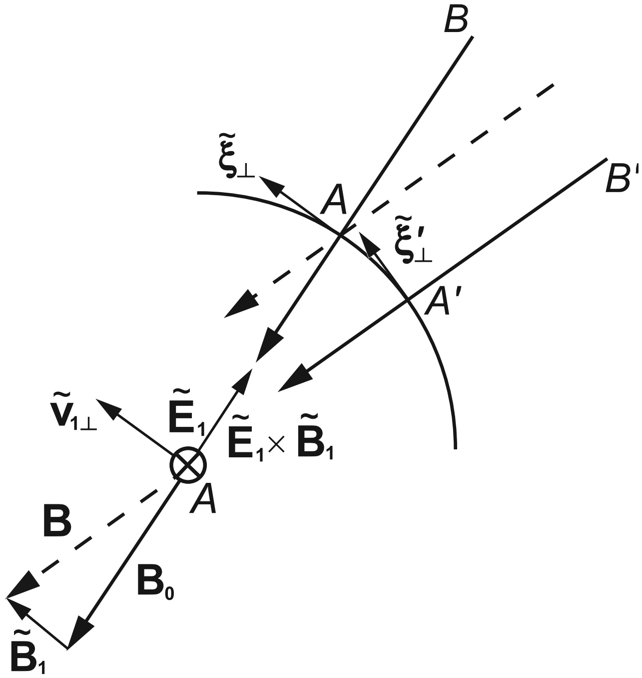

Figure 7. A physical picture showing that Poynting vector across the ionospheric surface is upward into the magnetosphere when the amplitude of the displacement vector $\tilde{\boldsymbol{\xi}}_\perp$ is increasing with time in the interchange instability and the curved finite ionospheric surface area is taken into account.

represents an unperturbed field line incident vertically on the ionospheric surface at point A. Assume that a straight line $A'B'$ represents an unperturbed field line in the neighborhood of field line AB. Since $\mathbf{n} = \mathbf{b}$ is assumed on the entire ionospheric surface in the Northern Hemisphere, line $A'B'$ is also incident vertically on the ionospheric surface at point A'. Let us assume that $\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_A, t)$ is directed toward a higher latitude as shown in Figure 7 and its amplitude is increasing. Then, $\tilde{\boldsymbol{\xi}}'_\perp = \tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}'_A, t)$ is also directed toward a higher latitude and its amplitude is increasing, since the plasma moves as a whole in interchange instability. Point A' is taken, so that the distance between A and A' along the ionospheric surface is nearly equal to $|\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_{A'}, t)|$. Since the amplitude of $\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_A, t)$ is increasing, $\tilde{\mathbf{v}}_{1\perp}(\mathbf{r}_A, t)$ is parallel to $\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_A, t)$ as shown in Figure 7. Therefore $\tilde{\mathbf{E}}_1(\mathbf{r}_A, t) = -\tilde{\mathbf{v}}_{1\perp} \times \mathbf{B}_0$ is directed into the page as shown in Figure 7, where $\mathbf{B}_0$ is the unperturbed magnetic field at point A. If time t is not very different from the time of the start of the instability, the unperturbed field line $A'B'$ has been moving toward line AB by time t in the interchange instability. Therefore at time t, line $A'B'$ takes a position represented by the dashed line passing through point A. Since the boundary condition $(\mathbf{b} \cdot \nabla)\tilde{\boldsymbol{\xi}}_\perp = 0$ is also valid at point A', the dashed line passing through point A is parallel with line $A'B'$. Therefore a perturbed magnetic field at point A at time t is shown by the dashed vector denoted by $\mathbf{B} = \mathbf{B}(\mathbf{r}_A, t)$ in Figure 7. Since

$$\tilde{\mathbf{B}}_1(\mathbf{r}_A, t) = \mathbf{B}(\mathbf{r}_A, t) - \mathbf{B}_0(\mathbf{r}_A), \tag{106}$$

$\tilde{\mathbf{B}}_1(\mathbf{r}_A, t)$ is shown by $\tilde{\mathbf{B}}_1$ in Figure 7. Thus, it is obvious that $\tilde{\mathbf{E}}_1 \times \tilde{\mathbf{B}}_1$, which has a component parallel to the unperturbed magnetic field, is directed upward into the magnetosphere at point A as shown in Figure 7. Therefore one notes physically that the second order linear perturbation of Poynting vector at point A has an upward component when the amplitude of $\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_A, t)$ is increasing. Thus when the curved finite ionospheric surface area is taken into account in the interchange instability, the magnetospheric and ionospheric plasmas participate cooperatively in an increase in the kinetic energy of the magnetospheric plasma in the present magnetospheric system.

[107] For case 2, ω is pure real as is obvious from equation (59). Therefore $\omega = \omega_r$, where the subscript r denotes the real part, and the magnetosphere is stable. In this case, $\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}, t)$ at the ionosphere can be written as

$$\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}, t) = \mathbf{c}(\mathbf{r}) \cos(\omega_r t). \tag{107}$$

Therefore,

$$\tilde{\xi}^2_\perp(\mathbf{r}, t) = \mathbf{c}^2(\mathbf{r}) \cos^2(\omega_r t) = \frac{1}{2}\mathbf{c}^2(\mathbf{r})[1 + \cos(2\omega_r t)]. \tag{108}$$

By substituting this into equation (B30) one notes that the total parallel Poynting flux across the ionosphere surface changes sign periodically. When $\tilde{\xi}^2_\perp$ at the ionosphere is decreasing, the parallel Poynting flux across the ionospheric surface, which is integrated over the whole ionospheric surface, is downward from the magnetosphere. The time average of the total Poynting flux across the ionospheric surface vanishes in this case.

[108] Figure 8 shows a physical picture to explain the direction of Poynting vector across the ionospheric surface when the amplitude of $\tilde{\boldsymbol{\xi}}_\perp$ is decreasing in the same format

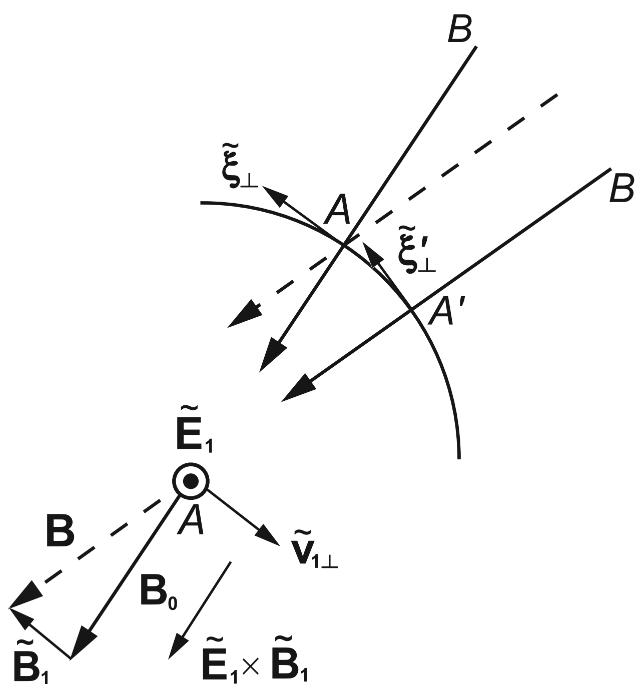

Figure 8. A physical picture showing that Poynting vector across the ionospheric surface is downward from the magnetosphere when the amplitude of the displacement vector $\tilde{\boldsymbol{\xi}}_\perp$ is decreasing with time and the curved finite ionospheric surface area is taken into account.

as Figure 7. Since the amplitude of $\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_A, t)$ is decreasing, $\tilde{\mathbf{v}}_{1\perp}(\mathbf{r}_A, t)$ is directed opposite of $\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_A, t)$ as shown in Figure 8. Therefore $\tilde{\mathbf{E}}_1(\mathbf{r}_A, t)$ is directed out of page as shown in this figure. Hence the direction of $\tilde{\mathbf{E}}_1(\mathbf{r}_A, t)$ is opposite to that shown in Figure 7. Since $\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_A, t)$ is still directed toward a higher latitude as shown in Figure 8, $\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_{A'}, t)$ is also directed toward a higher latitude. Therefore at time t, the dashed line, which is parallel to field line $A'B'$, represents a perturbed field line passing through point A. Hence the direction of $\tilde{\mathbf{B}}_1(\mathbf{r}_A, t)$ is the same as that shown in Figure 7. Thus the Poyning vector $\mu_0^{-1}\tilde{\mathbf{E}}_1 \times \tilde{\mathbf{B}}_1$ is directed downward at point A. Thus one notes physically that Poynting vector $\mu_0^{-1}\tilde{\mathbf{E}}_1 \times \tilde{\mathbf{B}}_1$ is directed downward from the magnetosphere when the amplitude of $\tilde{\boldsymbol{\xi}}_\perp$ is decreasing at point A.

[109] In case 3, the energy balance equation shows that the entire energy that goes into the increase of kinetic energy comes from the ionospheric term. However, unlike case 1 as discussed above, it is not certain in the present study whether there is indeed such a perturbation $\boldsymbol{\eta}(\mathbf{r})$ that makes $\delta W_F(\boldsymbol{\eta}^*, \boldsymbol{\eta}) > 0$, $\delta W_I(\boldsymbol{\eta}_\perp^*, \boldsymbol{\eta}_\perp) < 0$, $\delta W(\boldsymbol{\eta}^*, \boldsymbol{\eta}) = \delta W_F + \delta W_I < 0$ possible. Since the displacement $\boldsymbol{\eta}(\mathbf{r})$ must be nonzero at the ionosphere, magnetic field lines just above the ionosphere must be distorted. Since B is very large in the low-altitude region, a stabilizing term in δW_F, i.e., $\mu_0^{-1}|\mathbf{Q}_\perp|^2$ term in equation (86), becomes very large. Therefore it would be very difficult to make $\delta W_I + \delta W_F < 0$. Thus it is not certain whether there is indeed such a perturbation $\boldsymbol{\eta}(\mathbf{r})$, which makes $\delta W_I + \delta W_F < 0$ possible. Therefore the feasibility of case 3 is not clear in the present study, which has mainly focused on pressure-driven instabilities in the magnetosphere. Thus case 3 is not discussed further in the present study and the main focus in the present study in previous subsections has been case 1 as discussed above, i.e., pressure-driven modes.

9. Summary and Conclusion

[110] A magnetospheric energy principle has been formulated to study hydromagnetic stability problems in a general magnetospheric equilibrium by taking into account the fact that field lines thread the ionospheric boundary, which is not a perfectly conducting rigid wall. This is peculiar to the magnetospheric system. The formulation is based on the assumption that the unperturbed magnetic field is perpendicular to the ionospheric surface over the whole ionosphere. This is justified by the low-β nature of the ionospheric plasma. Ideal ionospheric boundary conditions at the unperturbed ionospheric boundary are obtained, so that the force operator $\mathbf{F}$ becomes self-adjoint. There are four possible ionospheric boundary conditions which satisfy this requirement: horizontally free, conducting, free, and rigid boundary conditions. For these ionospheric boundary conditions the energy, $H(t) = K(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_1) + \delta W(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}})$, is conserved within the unperturbed magnetospheric plasma volume P and the stability problem is closed without the need to consider the existence of a neutral atmosphere below the ionosphere. The magnetospheric energy principle states that for all allowable plasma displacements $\boldsymbol{\xi}$ satisfying either a horizontally free, conducting, free, or rigid ionospheric boundary condition, an equilibrium is stable if and only if $\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = \delta W_F(\boldsymbol{\xi}^*, \boldsymbol{\xi}) + \delta W_I(\boldsymbol{\xi}_\perp^*, \boldsymbol{\xi}_\perp) \geq 0$. That is, if the minimum value of the change in the potential energy is positive for all displacements, the system is stable. If it is negative for any of the displacements, the system is unstable. The ionospheric surface contribution δW_I to the change in the potential energy is negative and thus destabilizing for horizontally free and free ionospheric boundary conditions. The Poynting flux across the ionospheric surface is expressed by the temporal change of $\delta W_I(\tilde{\boldsymbol{\xi}}_\perp, \tilde{\boldsymbol{\xi}}_\perp)$ after integration over the ionospheric surface under the assumption that the unperturbed magnetic field is incident vertically on the ionospheric surface. In particular, when the curved finite ionospheric surface area is taken into account, the Poynting flux across the ionospheric surface directs upward into the magnetosphere in interchange instability. Therefore the magnetospheric and ionospheric plasmas move as a whole and participate cooperatively in the instability. Thus the kinetic energy of the magnetospheric plasma increases with a decrease in the potential energy of the magnetospheric system. By minimizing δW with respect to $\xi_\parallel$, it is found that the most unstable mode satisfies $\nabla \cdot \boldsymbol{\xi} =$ const. along the field line and $\nabla \cdot \boldsymbol{\xi}$ is equal to its ionospheric value for all ionospheric boundary conditions.

[111] Next, by setting $J_\parallel = 0$ in the magnetosphere, and thus excluding kink modes, pressure-driven modes in the magnetosphere such as interchange and ballooning modes are studied. The δW_I contribution to δW is neglected by assuming that a nonzero $\boldsymbol{\xi}_\perp$ is limited to a very narrow latitudinal region in the ionosphere for both horizontally free and free boundary conditions.

[112] Interchange modes satisfy the horizontally free ionospheric boundary condition and $\nabla \cdot \boldsymbol{\xi} = \text{const.} \neq 0$ along the field line. Incompressible ballooning modes satisfy the conducting or free ionospheric boundary condition and $\nabla \cdot \boldsymbol{\xi} = 0$ along the field line. Compressible ballooning modes satisfy the rigid ionospheric boundary condition. Existing interchange stability criteria have been compared and it is shown that the integral appearing in the calculation of a specific flux tube volume can be calculated by integrating between the two ionospheric ends. The results of several different numerical stability analyses of ballooning instabilities for different magnetospheric equilibria have been discussed systematically in light of the present magnetospheric energy principle. In particular, the incompressible ballooning mode does not need to satisfy the interchange instability criterion.

[113] The present magnetospheric energy principle is an extension of the ideal MHD energy principle to more general boundary conditions, which may apply to the ionospheric boundary of magnetospheric configurations. Whether or not a magnetospheric plasma is stable is determined by magnetospheric conditions, for example, by interchange stability criteria for interchange modes or by an approximate critical β criterion for incompressible ballooning modes in a tail-like configuration, when the ionospheric surface contribution δW_I is negligible. However, the magnetospheric energy principle leads to the result that the ionospheric boundary condition plays a very important role in limiting the possible modes of pressure-driven ideal MHD instabilities in the magnetosphere.

[114] Since the ionospheric boundary conditions obtained in this study are ideal ionospheric boundary conditions, the value of the present energy priniciple extended to a mag-

netospheric plasma depends on the physical feasibility of those different ionospheric boundary conditions, which satisfy the self-adjointness of the force operator.

Appendix A: Calculation of $\boldsymbol{\eta}_\perp \cdot [(\mathbf{B} \cdot \nabla)\boldsymbol{\xi}_\perp]$ and $(\boldsymbol{\xi}_\perp \cdot \nabla)\mathbf{b}$ on the Ionospheric Surface

[115] Let us assume that the ionospheric unperturbed magnetic field is vertical to the unperturbed ionospheric plane and the ionospheric surface is a spherical surface of radius $\boldsymbol{R}_\mathrm{I}$ for the three-dimensional configuration, and a cylindrical surface of radius R_I for the two-dimensional configuration. In order to calculate $\boldsymbol{\eta}_\perp \cdot [(\mathbf{B} \cdot \nabla)\boldsymbol{\xi}_\perp]$ and $(\boldsymbol{\xi}_\perp \cdot \nabla)\mathbf{b}$ on the ionospheric surface of both hemispheres, a common coordinate system must be used in both hemispheres.

[116] For the three-dimensional configuration, let us define a field-aligned coordinate system (s, θ, ϕ), where s increases along the field line with the distance from the ionosphere in the Southern Hemisphere. Unit vectors of the coordinate system are $\mathbf{b}$, $\hat{\theta}$, and $\hat{\phi}$, where $\hat{\phi} = \mathbf{b} \times \hat{\theta}$ directs eastward. In this coordinate system

$$\boldsymbol{\xi} = \xi_\parallel \mathbf{b} + \xi_\theta \hat{\theta} + \xi_\phi \hat{\phi}. \tag{A1}$$

[117] In the ionosphere of the Southern Hemisphere, this coordinate system agrees with the spherical coordinate system represented by (r, θ, ϕ). Therefore,

$$\boldsymbol{\xi} = \xi_\parallel \mathbf{e}_r + \xi_\theta \mathbf{e}_\theta + \xi_\phi \mathbf{e}_\phi \tag{A2}$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$ are unit vectors in the spherical coordinate system. Thus, $\boldsymbol{\xi}_\perp = \xi_\theta \mathbf{e}_\theta + \xi_\phi \mathbf{e}_\phi$ and $\boldsymbol{\eta}_\perp = \eta_\theta \mathbf{e}_\theta + \eta_\phi \mathbf{e}_\phi$. Therefore,

$$\begin{aligned}(\boldsymbol{\xi}_\perp \cdot \nabla)\mathbf{b} &= (\boldsymbol{\xi}_\perp \cdot \nabla)\mathbf{e}_r \\ &= \frac{\boldsymbol{\xi}_\perp}{R_\mathrm{I}}\end{aligned} \tag{A3}$$

and

$$\begin{aligned}\boldsymbol{\eta}_\perp \cdot [(\mathbf{B} \cdot \nabla)\boldsymbol{\xi}_\perp] &= B\left(\eta_\theta \frac{\partial \xi_\theta}{\partial r} + \eta_\phi \frac{\partial \xi_\phi}{\partial r}\right) \\ &= B\left(\eta_\theta \frac{\partial \xi_\theta}{\partial s} + \eta_\phi \frac{\partial \xi_\phi}{\partial s}\right)\end{aligned} \tag{A4}$$

in the ionosphere in the Southern Hemisphere, since $\mathbf{b} \cdot \nabla = \mathbf{e}_r \cdot \nabla = \partial/\partial r = \partial/\partial s$.

[118] In the Northern Hemisphere, $\mathbf{b} = -\mathbf{e}_r$, $\hat{\theta} = -\mathbf{e}_\theta$, and $\hat{\phi} = \mathbf{e}_\phi$. Therefore $\boldsymbol{\xi}_\perp = -\xi_\theta \mathbf{e}_\theta + \xi_\phi \mathbf{e}_\phi$ and $\boldsymbol{\eta}_\perp = -\eta_\theta \mathbf{e}_\theta + \eta_\phi \mathbf{e}_\phi$. Therefore,

$$\begin{aligned}(\boldsymbol{\xi}_\perp \cdot \nabla)\mathbf{b} &= -(\boldsymbol{\xi}_\perp \cdot \nabla)\mathbf{e}_r \\ &= -\frac{\boldsymbol{\xi}_\perp}{R_\mathrm{I}}\end{aligned} \tag{A5}$$

and

$$\begin{aligned}\boldsymbol{\eta}_\perp \cdot [(\mathbf{B} \cdot \nabla)\boldsymbol{\xi}_\perp] &= -B\left(\eta_\theta \frac{\partial \xi_\theta}{\partial r} + \eta_\phi \frac{\partial \xi_\phi}{\partial r}\right) \\ &= B\left(\eta_\theta \frac{\partial \xi_\theta}{\partial s} + \eta_\phi \frac{\partial \xi_\phi}{\partial s}\right)\end{aligned} \tag{A6}$$

in the ionosphere in the Northern Hemisphere, since $\mathbf{b} \cdot \nabla = -\mathbf{e}_r \cdot \nabla = -\partial/\partial r = \partial/\partial s$.

[119] For the two-dimensional configuration, let us define a field-aligned coordinate system (s, θ, z), where z is the dawn-dusk direction. Unit vectors of the coordinate system are $\mathbf{b}$, $\hat{\theta}$, and $\hat{\mathbf{z}}$, where $\hat{\mathbf{z}} = \mathbf{b} \times \hat{\theta}$ directs eastward. In this coordinate system

$$\boldsymbol{\xi} = \xi_\parallel \mathbf{b} + \xi_\theta \hat{\theta} + \xi_z \hat{\mathbf{z}}. \tag{A7}$$

[120] In the ionosphere of the Southern Hemisphere, this coordinate system agrees with the cylindrical coordinate system represented by (r, θ, z), where z is the dawn-dusk direction. Since there is no variation of $\mathbf{b}$ in the z direction,

$$\begin{aligned}(\boldsymbol{\xi}_\perp \cdot \nabla)\mathbf{b} &= (\boldsymbol{\xi}_\perp \cdot \nabla)\mathbf{e}_r \\ &= \frac{\boldsymbol{\xi}_{\perp m}}{R_\mathrm{I}},\end{aligned} \tag{A8}$$

where $\boldsymbol{\xi}_{\perp m}$ is the component of $\boldsymbol{\xi}_\perp$ in the meridian plane. Also,

$$\begin{aligned}\boldsymbol{\eta}_\perp \cdot [(\mathbf{B} \cdot \nabla)\boldsymbol{\xi}_\perp] &= B\left(\eta_\theta \frac{\partial \xi_\theta}{\partial r} + \eta_z \frac{\partial \xi_z}{\partial r}\right) \\ &= B\left(\eta_\theta \frac{\partial \xi_\theta}{\partial s} + \eta_z \frac{\partial \xi_z}{\partial s}\right).\end{aligned} \tag{A9}$$

[121] In the Northern Hemisphere, $\mathbf{b} = -\mathbf{e}_r$ and $\hat{\theta} = -\mathbf{e}_\theta$. Therefore

$$\begin{aligned}(\boldsymbol{\xi}_\perp \cdot \nabla)\mathbf{b} &= -(\boldsymbol{\xi}_\perp \cdot \nabla)\mathbf{e}_r \\ &= -\frac{\boldsymbol{\xi}_{\perp m}}{R_\mathrm{I}}\end{aligned} \tag{A10}$$

and

$$\begin{aligned}\boldsymbol{\eta}_\perp \cdot [(\mathbf{B} \cdot \nabla)\boldsymbol{\xi}_\perp] &= -B\left(\eta_\theta \frac{\partial \xi_\theta}{\partial r} + \eta_z \frac{\partial \xi_z}{\partial r}\right) \\ &= B\left(\eta_\theta \frac{\partial \xi_\theta}{\partial s} + \eta_z \frac{\partial \xi_z}{\partial s}\right)\end{aligned} \tag{A11}$$

in the ionosphere in the Northern Hemisphere, since $\mathbf{b} \cdot \nabla = -\mathbf{e}_r \cdot \nabla = -\partial/\partial r = \partial/\partial s$.

Appendix B: Condition for Validity of the Magnetospheric Energy Principle and Physical Meaning of the Change in the Potential Energy $\delta W(\tilde{\xi}, \tilde{\xi})$

[122] The energy conservation equation (57) is derived from the self-adjointness of the force operator and it means that $H(t) = K(\tilde{\mathbf{v}}_\mathrm{I}, \tilde{\mathbf{v}}_\mathrm{J}) + \delta W(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}) = K + \delta W_\mathrm{F} + \delta W_\mathrm{I}$, where $\delta W(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}) = -1/2 \int_P \tilde{\boldsymbol{\xi}} \cdot \mathbf{F}(\tilde{\boldsymbol{\xi}}) d\mathbf{r}$, is conserved. This is the basis of the present magnetospheric energy principle. In this Appendix, the validity of the sufficiency of the self-adjointness of the force operator for energy conservation is discussed on the basis of a rigorous nonlinear MHD equation describing the time evolution of total energy. Thus the relevance of the change in the potential energy $\delta W(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}) = \delta W_\mathrm{F} + \delta W_\mathrm{I}$ to the sum of the second order perturbations of internal energy and magnetic energy is clarified (see, also, question 5.3.1 of *Bateman* [1978]). In the following, K, δW, δW_F, and δW_I

denote $K(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_1)$, $\delta W(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}})$, $\delta W_{\rm F}(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}})$, and $\delta W_{\rm I}(\tilde{\boldsymbol{\xi}}_\perp, \tilde{\boldsymbol{\xi}}_\perp)$, respectively.

[123] Since a perturbation of kinetic energy is a quadratic of $\tilde{\mathbf{v}}_1$ in a static equilibrium, let us first calculate the temporal change of a second order linear perturbation $\tilde{w}_2$, which is the sum of the second order perturbations of internal energy and magnetic energy, i.e.,

$$\tilde{w}_2 = \frac{\tilde{p}_2}{\gamma - 1} + \frac{1}{2\mu_0}\left(2\mathbf{B}_0 \cdot \tilde{\mathbf{B}}_2 + \tilde{\mathbf{B}}_1^2\right), \tag{B1}$$

where

$$\frac{\partial \tilde{p}_2}{\partial t} = -\tilde{\mathbf{v}}_1 \cdot \nabla \tilde{p}_1 - \gamma \tilde{p}_1 \nabla \cdot \tilde{\mathbf{v}}_1, \tag{B2}$$

$$\frac{\partial \tilde{\mathbf{B}}_1}{\partial t} = \nabla \times (\tilde{\mathbf{v}}_1 \times \mathbf{B}_0), \tag{B3}$$

$$\frac{\partial \tilde{\mathbf{B}}_2}{\partial t} = \nabla \times \left(\tilde{\mathbf{v}}_1 \times \tilde{\mathbf{B}}_1\right). \tag{B4}$$

Notice that, contrary to the notation in the main text, the subscript 0 is added explicitly to the unperturbed quantity in this Appendix in order to avoid confusion. Subscripts 1 and 2 denote the first order linear perturbation and the second order linear perturbation, respectively, and the tilde on the perturbation means that the calculation is done in the real time domain and the perturbation is a function of position **r** and time t.

[124] Using equations (10) and (11) and vector identities, it is straightforward to show after some algebra that

$$\frac{\partial \tilde{w}_2}{\partial t} = -\tilde{\mathbf{v}}_1 \cdot \mathbf{F}(\tilde{\boldsymbol{\xi}}) - \frac{\gamma}{\gamma - 1}\nabla \cdot (\tilde{p}_1 \tilde{\mathbf{v}}_1) + \frac{1}{\mu_0}\nabla \cdot \left[\left(\tilde{\mathbf{v}}_1 \times \tilde{\mathbf{B}}_1\right) \times \mathbf{B}_0 + (\tilde{\mathbf{v}}_1 \times \mathbf{B}_0) \times \tilde{\mathbf{B}}_1\right], \tag{B5}$$

where $\mathbf{F}(\tilde{\boldsymbol{\xi}})$ is given by replacing $\boldsymbol{\xi}$ with $\tilde{\boldsymbol{\xi}}$ in equation (13). Here,

$$\tilde{\mathbf{v}}_1 \cdot \mathbf{F}(\tilde{\boldsymbol{\xi}}) = \rho_0 \frac{\partial \tilde{\boldsymbol{\xi}}}{\partial t} \cdot \frac{\partial^2 \tilde{\boldsymbol{\xi}}}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 \tilde{\mathbf{v}}_1^2\right). \tag{B6}$$

Therefore equation (B5) can be rewritten as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 \tilde{\mathbf{v}}_1^2 + \tilde{w}_2\right) = -\frac{\gamma}{\gamma - 1}\nabla \cdot (\tilde{p}_1 \tilde{\mathbf{v}}_1) + \frac{1}{\mu_0}\nabla \cdot \left[\left(\tilde{\mathbf{v}}_1 \times \tilde{\mathbf{B}}_1\right) \times \mathbf{B}_0 + (\tilde{\mathbf{v}}_1 \times \mathbf{B}_0) \times \tilde{\mathbf{B}}_1\right]. \tag{B7}$$

This equation is also obtained by taking the second order linear perturbation of a nonlinear MHD equation describing the time evolution of total energy at any point in space

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho \mathbf{v}^2 + \frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0}\right) = -\nabla \cdot \left[\left(\frac{1}{2}\rho \mathbf{v}^2 + p + \frac{p}{\gamma - 1}\right)\mathbf{v} + \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}\right], \tag{B8}$$

where, contrary to the notation used in the main text, ρ, p, B, **v**, **E**, and **B** are all total quantities, which are functions of position **r** and time t and not unperturbed quantities. Therefore equation (B7) represents the second order linear perturbation of a rigorous local energy conservation equation (B8).

[125] Let us define the second order perturbation $\tilde{\mathbf{u}}_2$ of the energy flux density by

$$\tilde{\mathbf{u}}_2 = \frac{\gamma}{\gamma - 1}(\tilde{p}_1 \tilde{\mathbf{v}}_1) - \frac{1}{\mu_0}\left[\left(\tilde{\mathbf{v}}_1 \times \tilde{\mathbf{B}}_1\right) \times \mathbf{B}_0 + (\tilde{\mathbf{v}}_1 \times \mathbf{B}_0) \times \tilde{\mathbf{B}}_1\right], \tag{B9}$$

then, equation (B7) can be written as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 \tilde{\mathbf{v}}_1^2 + \tilde{w}_2\right) = -\nabla \cdot \tilde{\mathbf{u}}_2. \tag{B10}$$

By integrating equation (B10) over the unperturbed plasma volume P, one obtains

$$\frac{\partial}{\partial t}\int_P \left(\frac{1}{2}\rho_0 \tilde{\mathbf{v}}_1^2 + \tilde{w}_2\right) d\mathbf{r} = -\int_S \tilde{\mathbf{u}}_2 \cdot \mathbf{n} dS. \tag{B11}$$

[126] Equation (B9) can be rewritten as

$$\tilde{\mathbf{u}}_2 = \frac{\gamma}{\gamma - 1}(\tilde{p}_1 \tilde{\mathbf{v}}_1) + \frac{1}{\mu_0}\left[2\tilde{\mathbf{v}}_1\left(\mathbf{B}_0 \cdot \tilde{\mathbf{B}}_1\right) - \mathbf{B}_0\left(\tilde{\mathbf{v}}_1 \cdot \tilde{\mathbf{B}}_1\right) - \tilde{\mathbf{B}}_1(\tilde{\mathbf{v}}_1 \cdot \mathbf{B}_0)\right]. \tag{B12}$$

Using vector identities, it is straightforward to show that

$$\begin{aligned} &2\tilde{\mathbf{v}}_1\left(\mathbf{B}_0 \cdot \tilde{\mathbf{B}}_1\right) - \mathbf{B}_0\left(\tilde{\mathbf{v}}_1 \cdot \tilde{\mathbf{B}}_1\right) - \tilde{\mathbf{B}}_1(\tilde{\mathbf{v}}_1 \cdot \mathbf{B}_0) \\ &= 2\tilde{\mathbf{v}}_{1\perp}\left[-B_0^2\left(\nabla \cdot \tilde{\boldsymbol{\xi}}_\perp\right) - B_0^2 \tilde{\boldsymbol{\xi}}_\perp \cdot \boldsymbol{\kappa} - B_0\left(\left(\tilde{\boldsymbol{\xi}}_\perp \cdot \nabla\right)B_0\right)\right] \\ &\quad - \tilde{v}_{1\|}B_0^2 \mathbf{b}\left(\tilde{\boldsymbol{\xi}}_\perp \cdot \boldsymbol{\kappa}\right) - B_0 \mathbf{b}\left[\tilde{\mathbf{v}}_{1\perp} \cdot \left((\mathbf{B}_0 \cdot \nabla)\tilde{\boldsymbol{\xi}}_\perp\right)\right. \\ &\quad \left. - B_0 \tilde{\mathbf{v}}_{1\perp} \cdot \left(\left(\tilde{\boldsymbol{\xi}}_\perp \cdot \nabla\right)\mathbf{b}\right)\right] - B_0^2 \tilde{v}_{1\|}\left[(\mathbf{b} \cdot \nabla)\tilde{\boldsymbol{\xi}}_\perp - \left(\tilde{\boldsymbol{\xi}}_\perp \cdot \nabla\right)\mathbf{b}\right]. \end{aligned} \tag{B13}$$

Therefore on $S_{\rm out}$ and $S_{\rm in}$, one obtains

$$\tilde{\mathbf{u}}_2 \cdot \mathbf{n} = -\mu_0^{-1} B_0^2 \tilde{v}_{1\|}\left[-\tilde{\boldsymbol{\xi}}_\perp \cdot ((\mathbf{b} \cdot \nabla)\mathbf{n}) - \mathbf{n} \cdot \left(\left(\tilde{\boldsymbol{\xi}}_\perp \cdot \nabla\right)\mathbf{b}\right)\right], \tag{B14}$$

where $\tilde{\mathbf{v}}_1 \cdot \mathbf{n} = \tilde{\mathbf{v}}_{1\perp} \cdot \mathbf{n} = 0$ on $S_{\rm out}$ and $S_{\rm in}$ was used. Since $\tilde{\boldsymbol{\xi}}_\perp = 0$ on $S_{\rm out}$ and $S_{\rm in}$, $\tilde{\mathbf{u}}_2 \cdot \mathbf{n} = 0$ on $S_{\rm out}$ and $S_{\rm in}$. For the two-dimensional configuration, contributions to the right hand side of equation (B11) from $S_{\rm dawn}$ and $S_{\rm dusk}$ cancel each other owing to the periodic condition in the dawn-dusk direction. Therefore only the integral over the ionospheric surface contributes to the right hand side of equation (B11).

[127] For the rigid ionospheric boundary condition satisfying $\tilde{\mathbf{v}}_1 = 0$ at the unperturbed ionosphere, equation (B12) shows that $\tilde{\mathbf{u}}_2 \cdot \mathbf{n} = 0$ at the ionospheric boundary. Therefore from equation (B11), $K + W$, where $W \equiv \int_P \tilde{w}_2 d\mathbf{r}$, is conserved. Comparing this with equation (57), one obtains

$$\delta W\left(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}\right) = \delta W_F\left(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}\right) = W + const. \tag{B15}$$

If one takes the const., so that $\delta W = W$ at $t = 0$, one obtains const. = 0. Therefore one has $\delta W(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}) = \delta W_{\mathrm{F}}(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}) = W$ for the rigid boundary condition.

[128] However, the assumption of normal incidence of the unperturbed magnetic field on the ionospheric surface also validates the energy conservation for other ionospheric boundary conditions as shown below. Since $\mathbf{n} = \mathbf{b}$ is assumed on the ionosphere of the Northern Hemisphere in the present magnetospheric energy principle, one obtains from equation (B13) that

$$\begin{aligned}\left[2\tilde{\mathbf{v}}_1\left(\mathbf{B}_0\cdot\tilde{\mathbf{B}}_1\right) - \mathbf{B}_0\left(\tilde{\mathbf{v}}_1\cdot\tilde{\mathbf{B}}_1\right) - \tilde{\mathbf{B}}_1\left(\tilde{\mathbf{v}}_1\cdot\mathbf{B}_0\right)\right]\cdot\mathbf{n} \\ = -B_0^2\left[\tilde{\mathbf{v}}_{1\perp}\cdot\left((\mathbf{b}\cdot\nabla)\tilde{\boldsymbol{\xi}}_\perp\right) - \tilde{\mathbf{v}}_{1\perp}\cdot\left(\left(\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla\right)\mathbf{b}\right)\right]\end{aligned} \quad \text{(B16)}$$

by using vector identities. Therefore on the ionosphere of the Northern Hemisphere one obtains

$$\begin{aligned}\tilde{\mathbf{u}}_2\cdot\mathbf{n} = -\frac{\gamma}{\gamma-1}\left(\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla p_0 + \gamma p_0\nabla\cdot\tilde{\boldsymbol{\xi}}\right)\tilde{v}_{1\parallel} \\ -\frac{B_0^2}{\mu_0}\left[\tilde{\mathbf{v}}_{1\perp}\cdot\left((\mathbf{b}\cdot\nabla)\tilde{\boldsymbol{\xi}}_\perp\right) - \tilde{\mathbf{v}}_{1\perp}\cdot\left(\left(\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla\right)\mathbf{b}\right)\right].\end{aligned} \quad \text{(B17)}$$

On the ionosphere of the Southern Hemisphere, the sign of the right hand side of equation (B17) reverses.

[129] For the conducting ionospheric boundary condition, equation (B17) shows that $\tilde{\mathbf{u}}_2\cdot\mathbf{n}$ vanishes on the ionospheric surface. Therefore as in the case of a rigid boundary condition, $K + W$ is conserved and one has $\delta W(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}) = \delta W_{\mathrm{F}}(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}) = W$ for the conducting ionospheric boundary condition.

[130] For the horizontally free ionospheric boundary condition

$$\tilde{\mathbf{u}}_2\cdot\mathbf{n} = \mu_0^{-1}B_0^2\tilde{\mathbf{v}}_{1\perp}\cdot\left[\left(\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla\right)\mathbf{b}\right] \quad \text{(B18)}$$

is valid on the ionosphere of the Northern Hemisphere. On the ionosphere of the Southern Hemisphere, the sign of the right hand side reverses. Since $(\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla)\mathbf{b}$ is equal to $-\tilde{\boldsymbol{\xi}}_\perp/R_{\mathrm{I}}$ and $\tilde{\boldsymbol{\xi}}_\perp/R_{\mathrm{I}}$ on the ionospheres of the Northern Hemisphere and Southern Hemisphere, respectively, for the three-dimensional configuration, substitution of equation (B18) into equation (B11) yields

$$\frac{\partial}{\partial t}\int_P\left(\frac{1}{2}\rho_0\tilde{\mathbf{v}}_1^2 + \tilde{w}_2\right)d\mathbf{r} = -\frac{\partial}{\partial t}\delta W_{\mathrm{I}}\left(\tilde{\boldsymbol{\xi}}_\perp, \tilde{\boldsymbol{\xi}}_\perp\right), \quad \text{(B19)}$$

where

$$\delta W_{\mathrm{I}}\left(\tilde{\boldsymbol{\xi}}_\perp, \tilde{\boldsymbol{\xi}}_\perp\right) = -\frac{1}{2\mu_0}\left(\int_{\mathrm{North}}\frac{B_0^2\tilde{\boldsymbol{\xi}}_\perp^2}{R_{\mathrm{I}}}dS + \int_{\mathrm{South}}\frac{B_0^2\tilde{\boldsymbol{\xi}}_\perp^2}{R_{\mathrm{I}}}dS\right). \quad \text{(B20)}$$

For the two-dimensional configuration $\tilde{\boldsymbol{\xi}}_\perp$ must be replaced with $\tilde{\boldsymbol{\xi}}_{\perp m}$ in equation (B20). This means that for the horizontally free boundary condition, a second order contribution from Poynting vector given by equation (B18) is expressed by the temporal change of δW_{I} after integration over the ionospheric surface. Therefore in this case, equation (B19) shows that $K + W + \delta W_{\mathrm{I}}$ is conserved. The comparison of equation (B19) with the energy conservation equation (57) resulting from the self-adjointness of the force operator yields $\delta W_{\mathrm{F}}(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}_\perp) \equiv \delta W - \delta W_{\mathrm{I}} = W$ for the horizontally free ionospheric boundary condition.

[131] For the free ionospheric boundary condition

$$\tilde{\mathbf{u}}_2\cdot\mathbf{n} = -\frac{\gamma}{\gamma-1}\tilde{v}_{1\parallel}\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla p_0 + \mu_0^{-1}B_0^2\tilde{\mathbf{v}}_{1\perp}\cdot\left[\left(\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla\right)\mathbf{b}\right] \quad \text{(B21)}$$

is valid on the ionosphere of the Northern Hemisphere. On the ionosphere of the Southern Hemisphere, the sign of the right hand side reverses. Substitution of equation (B21) into equation (B11) yields

$$\begin{aligned}&\frac{\partial}{\partial t}\int_P\left(\frac{1}{2}\rho_0\tilde{\mathbf{v}}_1^2 + \tilde{w}_2\right)d\mathbf{r} \\ &= \frac{\partial}{\partial t}(K + W) \\ &= \frac{\gamma}{\gamma-1}\left(\int_{\mathrm{North}}\frac{\partial\tilde{\xi}_\parallel}{\partial t}\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla p_0 dS - \int_{\mathrm{South}}\frac{\partial\tilde{\xi}_\parallel}{\partial t}\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla p_0 dS\right) \\ &\quad -\frac{\partial}{\partial t}\delta W_{\mathrm{I}}\left(\tilde{\boldsymbol{\xi}}_\perp, \tilde{\boldsymbol{\xi}}_\perp\right).\end{aligned} \quad \text{(B22)}$$

Therefore in this case $K + W + \delta W_{\mathrm{I}}$ is not conserved. But as is obvious from equation (57), $H = K + \delta W = K + \delta W_{\mathrm{F}} + \delta W_{\mathrm{I}}$ is conserved. Comparing equation (B22) with the energy conservation equation (57), one obtains

$$\begin{aligned}\frac{\partial}{\partial t}\delta W_{\mathrm{F}} = \frac{\partial W}{\partial t} - \frac{\gamma}{\gamma-1}\left(\int_{\mathrm{North}}\frac{\partial\tilde{\xi}_\parallel}{\partial t}\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla p_0 dS\right. \\ \left. - \int_{\mathrm{South}}\frac{\partial\tilde{\xi}_\parallel}{\partial t}\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla p_0 dS\right)\end{aligned} \quad \text{(B23)}$$

for the free boundary condition. Therefore for the free boundary condition, δW_{F} is not simply equal to W as is the case for the other three ionospheric boundary conditions. This means that for the free boundary condition, no form of energy conservation is obtained from the local energy conservation in equation (B8). This seems to contradict the conservation of $H = K + \delta W = K + \delta W_{\mathrm{F}} + \delta W_{\mathrm{I}}$ derived from the self-adjointness of the force operator.

[132] This contradiction in the case of the free ionospheric boundary condition is due to the fact that, as has been discussed in section 8.6, the normal incidence of the unperturbed magnetic field on the entire ionospheric surface is valid only when $\mathbf{J}_0 \times \mathbf{B}_0 \sim 0$ at the ionosphere. Since the force operator is defined by assuming $\mathbf{J}_0 \times \mathbf{B}_0 = \nabla p_0$ everywhere including the ionospheric surface, the sufficiency of the self-adjointness of the force-operator for the energy conservation for the free ionospheric boundary condition is valid only when the ionospheric plasma beta is much smaller than one and $\mathbf{J}_0 \times \mathbf{B}_0 \sim 0$ is satisfied at the ionosphere. For such a low-β ionosphere, equation (B23) shows that the ionospheric surface contribution by nonzero ∇p_0 on the right hand side can be neglected and $\delta W_{\mathrm{F}} \equiv \delta W - \delta W_{\mathrm{I}} \simeq W$ holds for the free boundary condition.

[133] This condition for the validity of $\delta W_{\mathrm{F}} \simeq W$ for the free boundary condition can be obtained more rigorously as follows. When the first term can be neglected in comparison with the second term on the right hand side of equation (B21), the last two terms on the right hand side of equation (B23)

can be neglected. The condition, for which the first term can be neglected in comparison with the second term in equation (B21), can be written as

$$\beta_{\rm I} \ll 2\frac{\ell_{0\perp}}{R_{\rm I}}\frac{\ell_\perp}{\ell_\parallel}, \tag{B24}$$

where $\beta_{\rm I}$ is the plasma beta at the ionosphere, $\ell_{0\perp}$ is the perpendicular scale length of p_0 in the ionosphere, $\ell_\parallel$ and $\ell_\perp$ are a parallel scale length of $\tilde{v}_{1\parallel}$ and a perpendicular scale length of $\tilde{\mathbf{v}}_{1\perp}$ at the ionosphere, respectively, and $\nabla\cdot\tilde{\mathbf{v}}_1 = 0$ at the ionosphere is used. For reasonable parameters, the ionospheric beta, $\beta_{\rm I} \sim 10^{-6}$, obtained in section 8.6 is small enough to satisfy equation (B24). Therefore for such a low-β ionosphere, $\delta W_{\rm F} \simeq W$ holds even for the free ionospheric boundary condition and energy conservation is also obtained from the rigorous equation (B8) describing the time evolution of total energy and thus the above contradiction in the case of the free ionospheric boundary condition is resolved.

[134] Since a second order perturbation of Poynting flux is related to the temporal change of $\delta W_{\rm I}(\tilde{\boldsymbol{\xi}}_\perp, \tilde{\boldsymbol{\xi}}_\perp)$, an explicit form of Poynting vector perturbation is given here. A second order perturbation of Poynting vector $\tilde{\mathbf{s}}_2$ is expressed from equation (B12) as

$$\tilde{\mathbf{s}}_2 = \mu_0^{-1}\left[2\tilde{\mathbf{v}}_1\left(\mathbf{B}_0\cdot\tilde{\mathbf{B}}_1\right) - \mathbf{B}_0\left(\tilde{\mathbf{v}}_1\cdot\tilde{\mathbf{B}}_1\right) - \tilde{\mathbf{B}}_1(\tilde{\mathbf{v}}_1\cdot\mathbf{B}_0)\right]. \tag{B25}$$

Therefore one has

$$\begin{aligned}\tilde{\mathbf{s}}_2\cdot\mathbf{n} = \mu_0^{-1}&\left[2(\tilde{\mathbf{v}}_1\cdot\mathbf{n})\left(\mathbf{B}_0\cdot\tilde{\mathbf{B}}_1\right) - (\mathbf{B}_0\cdot\mathbf{n})\right.\\ &\left.\cdot\left(\tilde{\mathbf{v}}_1\cdot\tilde{\mathbf{B}}_1\right) - \left(\tilde{\mathbf{B}}_1\cdot\mathbf{n}\right)(\tilde{\mathbf{v}}_1\cdot\mathbf{B}_0)\right].\end{aligned} \tag{B26}$$

By using conditions (18) and (19), one finds that $\tilde{\mathbf{s}}_2\cdot\mathbf{n} = 0$ on $S_{\rm out}$ and $S_{\rm in}$. Also, for the two-dimensional configuration, contributions to $\int_S \tilde{\mathbf{s}}_2\cdot\mathbf{n}dS$ from $S_{\rm dawn}$ and $S_{\rm dusk}$ cancel each other owing to the periodic condition in the dawn-dusk direction.

[135] From equation (B26) one finds that $\tilde{\mathbf{s}}_2\cdot\mathbf{n}$ vanishes exactly on the unperturbed ionospheric surface for the rigid ionospheric boundary condition. Since $\mathbf{n} = \mathbf{b}$ is assumed on the ionospheric surface of the Northern Hemisphere in the present energy principle, one obtains from equation (B17)

$$\tilde{\mathbf{s}}_2\cdot\mathbf{n} = \tilde{\mathbf{s}}_2\cdot\mathbf{b} = -\mu_0^{-1}B_0^2\left[\tilde{\mathbf{v}}_{1\perp}\cdot\left((\mathbf{b}\cdot\nabla)\tilde{\boldsymbol{\xi}}_\perp\right) - \tilde{\mathbf{v}}_{1\perp}\cdot\left(\left(\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla\right)\mathbf{b}\right)\right]. \tag{B27}$$

Therefore one also obtains $\tilde{\mathbf{s}}_2\cdot\mathbf{n} = 0$ on the ionospheric surface for the conducting ionospheric boundary condition. For the horizontally free and free boundary conditions, the first term on the right hand side of equation (B27) vanishes. Thus for all ionospheric boundary conditions, one obtains

$$\tilde{\mathbf{s}}_2\cdot\mathbf{n} = \mu_0^{-1}B_0^2\left[\tilde{\mathbf{v}}_{1\perp}\cdot\left(\left(\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla\right)\mathbf{b}\right)\right] \tag{B28}$$

on the ionospheric surface of the Northern Hemisphere. On the ionospheric surface of the Southern Hemisphere, the sign of the right hand side of equation (B28) reverses. Therefore one obtains

$$\tilde{\mathbf{s}}_2\cdot\mathbf{n} = -\frac{1}{2\mu_0}\frac{B_0^2}{R_{\rm I}}\frac{\partial}{\partial t}\tilde{\boldsymbol{\xi}}_\perp^2 \tag{B29}$$

on the ionospheric surfaces of both hemispheres. By integrating over the ionospheric surface and adding contributions from both hemispheres, one obtains

$$\int_{\rm North}\tilde{\mathbf{s}}_2\cdot\mathbf{n}dS + \int_{\rm South}\tilde{\mathbf{s}}_2\cdot\mathbf{n}dS = \frac{\partial}{\partial t}\delta W_{\rm I}\left(\tilde{\boldsymbol{\xi}}_\perp, \tilde{\boldsymbol{\xi}}_\perp\right). \tag{B30}$$

[136] From equation (B29) it is obvious that when $\tilde{\boldsymbol{\xi}}_\perp^2$ is increasing at the ionosphere, the parallel component of Poynting vector across the ionospheric surface is upward 2220 into the magnetosphere. Equation (B30) means that when $\delta W_{\rm I}(\tilde{\boldsymbol{\xi}}_\perp, \tilde{\boldsymbol{\xi}}_\perp)$ is decreasing, the net parallel Poynting flux integrated over the ionospheric surface is upward into the magnetosphere.

[137] While first order energy conservation has no direct effect on the second order terms and the discussion of stability based on these, first order terms are lower order than second order terms. Therefore it is necessary to check whether or not the first order energy conservation is valid. If one takes a first order perturbation of equation (B8), one obtains

$$\frac{\partial\tilde{w}_1}{\partial t} = -\nabla\cdot\tilde{\mathbf{u}}_1, \tag{B31}$$

where $\tilde{\mathbf{u}}_1$ is the first order perturbation of the energy flux density expressed by

$$\tilde{\mathbf{u}}_1 = \frac{\gamma}{\gamma-1}p_0\tilde{\mathbf{v}}_1 - \frac{1}{\mu_0}(\tilde{\mathbf{v}}_1\times\mathbf{B}_0)\times\mathbf{B}_0, \tag{B32}$$

and $\tilde{w}_1$ is the first order perturbation of the sum of the internal energy and magnetic energy expressed by

$$\tilde{w}_1 = \frac{\tilde{p}_1}{\gamma-1} + \frac{1}{\mu_0}\mathbf{B}_0\cdot\tilde{\mathbf{B}}_1. \tag{B33}$$

[138] It is straightforward to show that

$$\tilde{\mathbf{u}}_1 = \frac{\gamma}{\gamma-1}p_0\tilde{\mathbf{v}}_1 + \frac{1}{\mu_0}B_0^2\tilde{\mathbf{v}}_{1\perp}. \tag{B34}$$

Therefore on the ionospheric surface, one has

$$\tilde{\mathbf{u}}_1\cdot\mathbf{n} = \frac{\gamma}{\gamma-1}p_0\tilde{\mathbf{v}}_1\cdot\mathbf{n} + \frac{1}{\mu_0}B_0^2\tilde{\mathbf{v}}_{1\perp}\cdot\mathbf{n}. \tag{B35}$$

Since

$$\frac{\partial}{\partial t}\int_P\tilde{w}_1 d\mathbf{r} = -\int_S\tilde{\mathbf{u}}_1\cdot\mathbf{n}dS \tag{B36}$$

holds, $\int_P \tilde{w}_1 d\mathbf{r}$ is conserved for the rigid boundary condition. Since $\mathbf{n} = \mathbf{b}$ is assumed on the ionospheric surface of the Northern Hemisphere, one obtains

$$\tilde{\mathbf{u}}_1\cdot\mathbf{n} = \tilde{\mathbf{u}}_1\cdot\mathbf{b} = \frac{\gamma}{\gamma-1}p_0\tilde{v}_{1\parallel}. \tag{B37}$$

Therefore for the horizontally free ionospheric boundary condition $\tilde{\mathbf{u}}_1\cdot\mathbf{n}$ also vanishes on the ionospheric surface and $\int_P \tilde{w}_1 d\mathbf{r}$ is conserved. For the conducting and free ionospheric boundary conditions $\tilde{\mathbf{u}}_1 \cdot \mathbf{n}$ does not vanish and $\int_P \tilde{w}_1 d\mathbf{r}$ is not precisely conserved. However, if one assumes $O(\tilde{\mathbf{v}}_1) \sim O(\tilde{\mathbf{v}}_{1\perp})$ in equation (B34), the first term is $O(\gamma\beta_I/2(\gamma - 1))$ smaller than the second term of the right hand side of equation (B34). Since $\beta_I \sim 10^{-6}$, one obtains

$$\tilde{\mathbf{u}}_1 \simeq \mu_0^{-1} B_0^2 \tilde{\mathbf{v}}_{1\perp} = \tilde{\mathbf{s}}_1 \qquad \text{(B38)}$$

on the ionospheric surface. This means that at the ionospheric surface the first order energy flux density is nearly equal to the first order perturbation of Poynting vector $\tilde{\mathbf{s}}_1$.

[139] Since $\mathbf{n} = \mathbf{b}$ is assumed on the ionospheric surface in the Northern Hemisphere, one obtains $\tilde{\mathbf{u}}_1 \cdot \mathbf{n} \simeq \tilde{\mathbf{s}}_1 \cdot \mathbf{n} = 0$ from equation (B38). This means that the normal component of the first order Poynting vector on the ionospheric surface vanishes. Therefore the low-β nature of the ionosphere leads to the result that $\int_P \tilde{w}_1 d\mathbf{r}$ is also nearly conserved for the conducting and free ionospheric boundary conditions. Although in this approximation procedure, one neglects the first term on the right hand side of equation (B34) compared with the second term, which eventually vanishes after taking the dot product with $\mathbf{n} = \mathbf{b}$, the very small value of β_I compared with unity would validate the above approximation procedure. Therefore one notes that for all ionospheric boundary conditions, the first order energy conservation (conservation of $\int_P \tilde{w}_1 d\mathbf{r}$) is well satisfied.

[140] In summary, the first order energy conservation is nearly satisfied for a low-β ionospheric plasma. In the second order, the energy conservation derived from the self-adjointness of the force operator is consistent with the rigorous MHD nonlinear equation (B8) describing the time evolution of total energy, when the ionospheric plasma beta is much smaller than one. For such a low-β ionospheric plasma, $H = K + \delta W$ is conserved for all ideal ionospheric boundary conditions. Here $\delta W = \delta W_F = W$ for the rigid and conducting ionospheric boundary conditions, but $\delta W = \delta W_F + \delta W_I = W + \delta W_I$ for the horizontally free and free ionospheric boundary conditions.

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