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Pressure-driven and ionosphere-driven modes of magnetospheric interchange instability

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[1] A general stability criterion for magnetospheric interchange instability, which includes an ionospheric destabilizing contribution, is derived for an arbitrary finite-β magnetospheric model satisfying the magnetohydrostatic force balance. The derivation is based on the magnetospheric energy principle. Unperturbed field-aligned currents in finite-β nonaxisymmetric magnetospheric models are assumed to close via diamagnetic currents in the magnetosphere or in the ionosphere. By exploiting the limit of a very large perpendicular wave number and the eikonal representation for the perpendicular plasma displacement, the magnetospheric interchange mode is shown to be compressible. In this limit the kink mode makes no contribution to the change in the magnetospheric potential energy. By using magnetospheric flux coordinates, the explicit form of the magnetospheric potential energy change is calculated for interchange perturbations, which do not bend magnetospheric magnetic fields. For a nonaxisymmetric finite-β magnetospheric model, a combined effect of the pressure gradient and field line curvature, not only in the meridional plane but also in the plane parallel to the longitudinal direction, is responsible for pressure-driven interchange instability. For an axisymmetric, north-south symmetric and low-β magnetospheric model, in which the magnetic field is approximated by a dipole field, the $m = 1$ or $m = 2$ ionosphere-driven mode, where m is the azimuthal mode number, has an upper critical equatorial β value for instability in the order of 1. Thus a substantial region of the inner magnetosphere or the near-Earth magnetosphere may be unstable against ionosphere-driven interchange instability caused by a horizontal plasma displacement on the spherical ionospheric surface.

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1. Introduction

[2] The possibility of spontaneous large-scale interchange motion of magnetic flux tubes and the plasma contained in them by pressure-driven interchange instability in the magnetosphere has been discussed by *Gold* [1959]. Such an interchange perturbation does not bend magnetic field lines in the magnetosphere and allows a class of motions to occur without the need of overcoming magnetic tension force. Since then, many studies have been devoted to clarify characteristics of magnetospheric interchange instability [e.g., *Sonnerup and Laird*, 1963; *Cheng*, 1985; *Rogers and Sonnerup*, 1986; *Southwood and Kivelson*, 1987; *Ferrière et al.*, 2001]. In particular, stability criteria for magnetospheric interchange instability have been discussed widely for several limited magnetospheric models. *Gold* [1959] discussed the stability of low-β magnetospheric configuration assuming a dipole magnetic field. He showed that for pressure gradients steeper than $R^{-20/3}$, which corresponds to the adiabatic gradient for the magnetohydrostatics of a plasma in a dipole field with R being the distance from the earth's center in the equatorial plane, the magnetospheric plasma is unstable against fast adiabatic convection. *Gold* [1959] suggested that a better understanding of the processes of magnetic storms and auroras and of the Van Allen radiation belts would all require better estimates of the interchange motions. The interchange instability has also been widely discussed as a potentially important mechanism for the redistribution of mass in planetary magnetospheres. Therefore, to know accurately the conditions necessary for interchange instability is very important in clarifying the dynamics of the magnetosphere.

[3] The primary objective of the present study is to derive a general stability criterion for interchange instability, which includes an ionospheric destabilizing contribution, for an arbitrary finite-β and nonaxisymmetric three-dimensional magnetospheric plasma, on the basis of a magnetospheric energy principle [*Miura*, 2007]. The unperturbed magnetospheric plasma is assumed to satisfy the magnetohydrostatic force balance $\mathbf{J} \times \mathbf{B} = \nabla p$ and is bounded by ideal ionospheric boundaries. The magnetospheric energy principle is an extension of the energy principle [*Bernstein et al.*, 1958] valid for any finite-β magnetohydrostatic equilibrium

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configuration used in fusion plasmas to a magnetospheric system with ideal ionospheric boundary conditions, which satisfy the self-adjointness of the force operator.

[4] A stability criterion for the interchange instability has been discussed by *Bernstein et al.* [1958] for a finite-β axisymmetric system with periodicity in the direction of the axis of symmetry. *Hameiri et al.* [1991] derived a stability criterion for finite-β axisymmetric magnetosphere with closed field lines, which was originally derived by *Spies* [1971]. They assumed that field lines are closed loops without any boundaries such as ionospheres and thus avoided complicated problems arising from taking into account ionospheric boundary conditions in the real magnetospheric plasma. In these criteria the plasma pressure and the specific volume of a magnetic flux tube U are considered to be a function of only one variable representing the radial coordinate. Here,

$$U = \oint \frac{d\ell}{B}$$

and the integration is taken for one period or for a closed field line loop.

[5] These studies assume an axisymmetric three-dimensional system. Therefore, there is no unperturbed field-aligned current. When the plasma pressure p or the specific flux tube volume U is nonaxisymmetric, however, an unperturbed field-aligned current apppears and stability is different from that without a field-aligned current. When there is a field-aligned current in the magnetosphere, the current must be closed three-dimensionally either in the ionosphere or in the magnetosphere to satisfiy current continuity. When there is an unperturbed field-aligned current and when this is closed by Pedersen current in the ionosphere, *Volkov and Mal'tsev* [1986] calculated the growth rate of the interchange instability for a finite-β plasma by using a perturbation stability analysis in a local Cartesian coordinate system.

[6] However, when the field-aligned current is closed by ionospheric Pedersen current, there arises Joule dissipation in the ionosphere and the zeroth-order state assumed for instability is no longer a steady state but decays with a time constant larger than the Alfvén transit time [*Miura*, 1996]. Hence, the sum of the magnetic energy, kinetic energy and the internal energy in ideal magnetohydrodynamics (MHD) is not conserved. Therefore, any stability analysis assuming a current closure by Pedersen current in the ionosphere cannot be a stability analysis of the steady state in the strictest sense, because the zeroth-order state is not a steady state. In other words, when the field-aligned current is closed by ionospheric conduction currents, the electric field is set up for current closure and this electric field sets the magnetospheric plasma in motion by $\mathbf{E} \times \mathbf{B}$ drift. Therefore, the magnetospheric plasma cannot be a static equilibrium satisfying $\mathbf{J} \times \mathbf{B} = \nabla p$. In order to avoid this decaying nature of the zeroth-order state, which is incompatible with the magnetohydrostatic force balance, it is assumed in the present study that the unperturbed field-aligned current is closed by a diamagnetic current perpendicular to the unperturbed magnetic field. Therefore, the framework of ideal MHD is retained strictly and the unperturbed state remains a steady state. Since in ideal MHD the sum of the kinetic energy and potential energy consisting of magnetic and internal energy is conserved, a magnetospheric energy principle is applicable. The present approach based on the variational magnetospheric energy principle is applicable for any magnetospheric equilibium model satisfiying magnetohydrostatic force balance with and without axisymmetry.

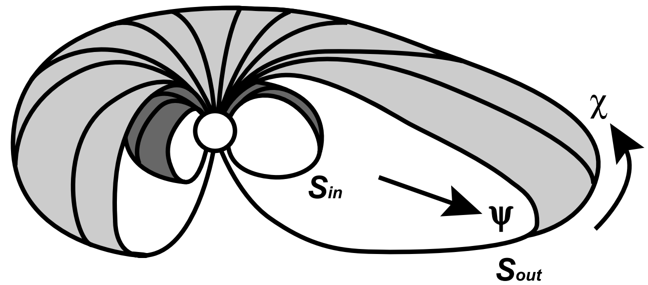

Figure 1. A three-dimensional view of the plasma volume P surrounded by an ideal ionospheric boundary and by the outer flux surface S_{out} and the inner flux surface S_{in} in the magnetohydrostatic magnetospheric equilibrium, which are shown by the light grey and the dark grey surfaces, respectively.

[7] Since magnetospheric interchange instability is not driven by the solar wind-magnetosphere interaction, one is interested in the stability of a plasma with a finite volume P in the magnetosphere. The plasma volume P is surrounded by an ideal ionospheric boundary and two flux surfaces in a magnetohydrostatic equilibrium. Shown in Figure 1 are two flux surfaces S_{out} and S_{in} in the magnetohydrostatic magnetospheric equilibrium, which are the light grey surface and the dark grey surface, respectively. Those two flux surfaces S_{out} and S_{in} are virtual boundaries located far from disturbed field lines in the plasma volume P. The outer flux surface S_{out} is taken inside the magnetopause. Therefore, energy conservation holds in the plasma volume P and one can thus apply the magnetospheric energy principle to study the magnetospheric interchange instability in the plasma volume P.

[8] The energy conservation necessary for the magnetospheric energy principle requires that the ideal MHD force operator must be self-adjoint. *Miura* [2007] obtained four ideal ionospheric boundary conditions, which are compatible with the self-adjoint property of the force operator and satisfying $\mathbf{J} \times \mathbf{B} = \nabla p$ in the magnetosphere. Since the interchange of flux tubes involves the motion of a whole flux tube, the interchange of magnetospheric flux tubes requires horizontal displacement of magnetic field lines in the ionosphere. According to the magnetospheric energy principle there are two ideal ionospheric boundary conditions allowing the horizontal displacement of magnetic field lines in the ionosphere. One is the horizontally free boundary condition for compressible perturbation and the other is the free boundary condition, which requires that the perturbation is incompressible. Between these two ionospheric boundary conditions, the horizontally free boundary condition with a compressible perturbation is shown to be necessary for magnetospheric interchange instability.

[9] When the horizontal displacement $\boldsymbol{\xi}_{\perp}$ is nonzero at the ionosphere and the ionosphere is a spherical surface, there arises a change in the potential energy in the

ionopshere δW_I according to the magnetospheric energy principle [*Miura*, 2007]. This term is negative. Thus the plasma displacement on the spherical ionospheric surface gives a negative contribution to the change of the potential energy for interchange perturbation. The detailed calculation in the present study shows a possibility of ionosphere-driven interchange instability due to a nonzero ionospheric horizontal plasma displacement on the spherical ionospheric surface. Such a possibility has never been pointed out previously and the instability criterion for an ionosphere-driven interchange mode is discussed in detail in the present study on the basis of the magnetospheric energy principle.

[10] In representing the magnetospheric equilibrium model, choice of the coordinate system is very important. By choosing a proper coordinate system, the stability analysis based on the energy principle becomes particularly simple and tractable as has been demonstrated in fusion plasmas [see, e.g., *Freidberg*, 1987]. Therefore, a flux coordinate is introduced to simplify the stability analysis in the present study of magnetospheric interchange instability.

[11] The organization of the present paper is as follows. The definition of an unperturbed state and a flux coordinate is given in section 2. Magnetospheric energy principle and ionospheric boundary conditions used for the analysis of magnetospheric interchange instability are reviewed in section 3. Eikonal ansatz, compressibility of the magnetospheric interchange mode and an expression for the variational change in the potential energy in the magnetosphere δW_F are explained in section 4. The change in the potential energy δW_F is further reduced for interchange perturbation in section 5. The total change of the potential energy $\delta W = \delta W_F + \delta W_I$ is calculated in section 6. Stability criteria for different magnetospheric models are presented in section 7. Discussion is presented and a realistic evaluation of the criterion for ionosphere-driven interchange instability is given in section 8. Summary and conclusion are presented in section 9. The four ideal ionospheric boundary conditions, which are compatible with the magnetospheric energy principle, are derived physically from the requirement of energy conservation in Appendix A. Field line bending in perturbations at the ionosphere is clarified in Appendix B.

2. Unperturbed State, Current Closure, and Flux Coordinates

[12] In the present study, the interchange instability of a magnetohydrostatic magnetospheric configuration satisfying

$$\mathbf{J} \times \mathbf{B} = \nabla p \tag{1}$$

is investigated, where $\mathbf{J}$, $\mathbf{B}$, and p are an unperturbed current, the magnetic field, and the pressure, respectively. As long as a magnetospheric configuration is in a magnetohydrostatic equilibrium, any magnetospheric configuration, whether it is dipole-like or tail-like, can be used in the following analysis.

[13] Although steady flows are often present in the magnetospheric regions of interest, a magnetohydrostatic equilibrium must be assumed in the present analysis on the basis of the energy principle. This is because the energy principle is based on the self-adjoint property of the force operator, but the existence of steady flows causes the appearance of a nonself-adjoint operator. Therefore, the stability of a system with steady flows cannot be studied by a powerful minimizing principle, i.e., the energy principle.

[14] In order to represent a magnetospheric magnetohydrostatic equilibrium satisfying equation (1), some coordinate system must be specified. The unperturbed magnetic field satisfying equation (1) can generally be written as

$$\mathbf{B} = \nabla\psi \times \nabla\chi, \tag{2}$$

where $\psi(\mathbf{r})$ and $\chi(\mathbf{r})$ are scalar functions of position. They are Clebsch potentials, which are also called Euler potentials [*Stern*, 1970]. In general, $\nabla\psi$ and $\nabla\chi$ are not orthogonal. In the magnetospheric plasma ψ is chosen as the magnetic flux and represents a "radial-like" variable. The other, defined as χ, describes a toroidal "angle-like" variable. The third coordinate that needs to be defined is a "length-like" variable measuring distance along the magnetic line. This coordinate is denoted by ζ. It is convenient to treat the general perturbation problems in these coordinates and the use of ψ, χ, and ζ makes the variational stability analysis particularly simple. Figure 1 shows that ψ increases outwardly and χ increases counterclockwise.

[15] For closed magnetic field lines (i.e., closed loops of lines of force), which are used in the axisymmetric magnetospheric model of *Hameiri et al.* [1991], the specific flux tube volume is defined by

$$U(\psi) = \oint \frac{d\ell}{B}, \tag{3}$$

where the integration is taken for one period along the field line loop and ℓ is the distance along the field line. This is a function of the only ψ and $U(\psi)$ is a flux label. However, in nonaxisymmetric magnetospheric model, the specific volume of a magnetic tube must be defined by

$$U(\psi, \chi) = \int_S^N \frac{d\ell}{B}, \tag{4}$$

where S is the footpoint of a field line at the southern ionosphere and N is the foot point at the northern hemisphere. For the nonaxisymmetric magnetospheric model, U is generally a function of ψ and χ, i.e., $U = U(\psi, \chi)$. The unperturbed plasma pressure p is also generally a function of ψ and χ, i.e., $p = p(\psi, \chi)$. In such a nonaxisymmetric magnetospheric model, there is an unperturbed field-aligned current $J_\parallel$. This can be seen from the following equation, which is derived from $\nabla \cdot \mathbf{J} = 0$ [*Schindler*, 2007]:

$$\left[\frac{J_\parallel}{B}\right]_N - \left[\frac{J_\parallel}{B}\right]_S = -\left[\frac{\mathbf{J}_\perp \cdot \nabla s}{B}\right]_N + \left[\frac{\mathbf{J}_\perp \cdot \nabla s}{B}\right]_S + \frac{\partial p}{\partial \chi}\frac{\partial U}{\partial \psi} - \frac{\partial p}{\partial \psi}\frac{\partial U}{\partial \chi}, \tag{5}$$

where ∇s is the vector along the unperturbed magnetic field and $\mathbf{J}_\perp$ is the current perpendicular to the unperturbed magnetic field. Since $\mathbf{J}_\perp$ is perpendicular to ∇s, the first two terms on the right hand side of equation (5) vanish.

Therefore, only the third and fourth terms are left on the right hand side. Thus, it is obvious that any nonaxisymmetry of $p = p(\psi, \chi)$ or $U = U(\psi, \chi)$ gives rise to a net unperturbed field-aligned current at the ionospheric height. For an axisymmetric magnetospheric model characterized by $p = p(\psi)$ or $U = U(\psi)$, $[J_{\|}/B]_N - [J_{\|}/B]_S$ vanishes. The unperturbed field-aligned current affects the stability property of the magnetospheric interchange instability. Notice that when the unperturbed magnetic field is incident obliquely on the ionospheric surface, the plane perpendicular to the unperturbed magnetic field at N or S intersects the neutral atmosphere. Therefore, $\mathbf{J}_{\perp}$ at N or S in equation (5) cannot flow in an arbitrary direction. This may restrict current closure in the system of the magnetosphere and the ionosphere and thus an unperturbed state in that system may be somewhat restricted. However, if one assumes that the unperturbed magnetic field is incident vertically on the ionospheric surface, $\mathbf{J}_{\perp}$ at N or S in equation (5) can flow in any direction in the plane perpendicular to the unperturbed magnetic field and thus the above restriction caused by the oblique incidence of the unperturbed magnetic field on the ionospheric surface is removed.

[16] When there is a field-aligned current in the magnetosphere, it must close in the magnetosphere or in the ionosphere to satisfy $\nabla \cdot \mathbf{J} = 0$. Since no unperturbed electric field is allowed in the magnetospheric energy principle, this field-aligned current cannot be closed in the ionosphere via conduction currents in the present study. Therefore, one assumes in the present study that the field-aligned current is closed by diamagnetic currents

$$\mathbf{J}_{\perp} = \frac{\mathbf{B} \times \nabla p}{B^2} \tag{6}$$

in the magnetosphere or in the ionosphere.

[17] From Ampère's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, one obtains

$$\mathbf{J}_{\perp} = \mu_0^{-1} \mathbf{B} \times \boldsymbol{\kappa} - \mu_0^{-1} \mathbf{b} \times \nabla B \tag{7}$$

and

$$J_{\|} = \mu_0^{-1} B \mathbf{b} \cdot (\nabla \times \mathbf{b}), \tag{8}$$

where $\mathbf{b} = \mathbf{B}/|\mathbf{B}| = \mathbf{B}/B$ is the unit vector parallel to the unperturbed magnetic field and $\boldsymbol{\kappa} = (\mathbf{b} \cdot \nabla)\mathbf{b}$. Since ∇B, $\boldsymbol{\kappa}$ and $\nabla \times \mathbf{b}$ are all finite quantities at the ionosphere, both $\mathbf{J}_{\perp}$ and $J_{\|}$ remain finite at the ionosphere. The perpendicular current (7) in the ionosphere, where there are no conduction currents, is provided by diamagnetic current (6), which is obviously a finite quantity.

[18] Since p is constant along the field line, the diamagnetic current given by equation (6) does not change direction along the field line. However, from equation (6) one obtains

$$\nabla \cdot \mathbf{J}_{\perp} = \frac{2}{B} \mathbf{B} \cdot \left[\nabla p \times \nabla \left(\frac{1}{B} \right) \right]. \tag{9}$$

Therefore, $\nabla \cdot \mathbf{J}_{\perp}$ can change sign at the ionosphere according to the change of $\nabla \cdot B^{-1}$ and therefore $\nabla (J_{\|} \mathbf{b})$ can also change sign at different places in the ionosphere. Thus, in the present ionosphere without conduction currents the field-aligned current at the ionosphere can close via diamagnetic currents. In other words, the distribution of B in the magnetohydrostatic model of the magnetosphere and the ionosphere is determined, so that the current closure or $\nabla \cdot \mathbf{J} = 0$ is satisfied in the unperturbed configuration of the magnetosphere and the ionosphere.

[19] The following variational stability analysis based on the magnetospheric energy principle is valid for any finite-β magnetohydrostatic equilibrium. Although no specific finite-β magnetohydrostatic magnetospheric equilibrium model is used in the present study, there are several numerical nonaxisymmetric magnetospheric models satisfying equation (1). For example, there are nonaxisymmetric, north-south symmetric magnetospheric models characterized by $p = p(\psi)$ and $U = U(\psi, \chi)$ [*Cheng*, 1995], and by $p = p(\psi, \chi)$ and $U = U(\psi, \chi)$ [*Zaharia et al.*, 2004].

3. Magnetospheric Energy Principle and Ideal Ionospheric Boundary Conditions

[20] The magnetospheric energy principle states that a plasma equilibrium is stable if and only if $\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = \delta W_F + \delta W_I \geq 0$ for all allowable displacements $\boldsymbol{\xi}$. Here, δW_F is the variational change of the potential energy for the magnetospheric plasma, which is calculated for the unperturbed plasma volume P. The volume P is surrounded by two lateral boundaries S_{out} and S_{in}, which are taken to be flux surfaces in a magnetohydrostatic magnetospheric equilibrium (see Figure 1), and is also bounded by an unperturbed ionospheric surface. The boundaries S_{out} and S_{in} are located far enough from disturbed field lines in the three-dimensional magnetospheric model. Here, δW_I is the ionospheric surface contribution to the variational change of the potential energy and is calculated for the unperturbed spherical ionospheric surface. The single assumption of this magnetospheric energy principle is that the unperturbed magnetic field at the ionosphere is perpendicular to the spherical ionospheric surface [*Miura*, 2007].

[21] The specific form of δW_F for the magnetospheric energy principle is the same as δW_F as given by *Freidberg* [1987] and can be written as

$$\begin{aligned} \delta W_F = \frac{1}{2} \int_P d\mathbf{r} \Big[& \mu_0^{-1} |\mathbf{Q}_{\perp}|^2 + \mu_0^{-1} B^2 |\nabla \cdot \boldsymbol{\xi}_{\perp} + 2\boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\kappa}|^2 \\ & + \gamma p |\nabla \cdot \boldsymbol{\xi}|^2 - 2(\boldsymbol{\xi}_{\perp} \cdot \nabla p)(\boldsymbol{\kappa} \cdot \boldsymbol{\xi}_{\perp}^*) \\ & - J_{\|} (\boldsymbol{\xi}_{\perp}^* \times \mathbf{b}) \cdot \mathbf{Q}_{\perp} \Big], \end{aligned} \tag{10}$$

where γ is the ratio of specific heats. This is the intuitive form of δW_F originally suggested by *Furth et al.* [1965, p. 103] and *Greene and Johnson* [1968]. Here,

$$\mathbf{Q} \equiv \mathbf{B}_1 = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}), \tag{11}$$

where the subscript "1" represents the perturbed quantity.

[22] The last two terms in the integrand of equation (10) can be positive or negative and thus can drive instabilities. The first of these is proportional to $\nabla p \sim \mathbf{J}_{\perp} \times \mathbf{B}$ while the

second is proportional to $J_\parallel$. Thus, either perpendicular or parallel currents represent potential sources of instability. The former type are sometimes referred to as pressure-driven modes, which are subdivided into interchange and ballooning modes, and the latter as current-driven modes or kink modes.

[23] The plasma displacement vector $\boldsymbol{\xi}$ at the ionospheric boundary must satisfy one of the following boundary conditions, i.e.,

$$\xi_\parallel = 0 \text{ and } (\mathbf{b}\cdot\nabla)\boldsymbol{\xi}_\perp = 0, \tag{12}$$

$$\nabla\cdot\boldsymbol{\xi} = 0 \text{ and } (\mathbf{b}\cdot\nabla)\boldsymbol{\xi}_\perp = 0, \tag{13}$$

$$\nabla\cdot\boldsymbol{\xi} = 0 \text{ and } \boldsymbol{\xi}_\perp = 0, \tag{14}$$

$$\xi_\parallel = 0 \text{ and } \boldsymbol{\xi}_\perp = 0, \tag{15}$$

where $\boldsymbol{\xi} = \boldsymbol{\xi}_\perp + \xi_\parallel\mathbf{b}$. For these combinations of the boundary conditions at the ideal ionosphere, the force operator becomes self-adjoint. Since the self-adjointness of the force operator is equivalent to energy conservation of the system under consideration [*Bernstein et al.*, 1958; *Freidberg*, 1987; *Miura*, 2007], these four ideal ionospheric boundary conditions can also be derived from the requirement of energy conservation in the system of the magnetosphere and the ideal ionosphere (see Appendix A for details).

[24] It is obvious that among the above four boundary conditions, the interchange mode requires nonzero $\boldsymbol{\xi}_\perp$ at the ideal ionosphere. Therefore, only boundary conditions (12) and (13) satisfy the requirement for the interchange mode. The boundary conditions (12) and (13) are called the horizontally free boundary condition and the free boundary condition, respectively. It is important to point out here that the horizontally free boundary condition (12) becomes an insulating boundary condition for a flat ionosphere, since there is no finite $\mathbf{B}_{1\perp}$ at the ionosphere and hence no ionospheric surface current for a flat ionospheric surface [*Miura*, 2007].

[25] The specific form of δW_I for the three-dimensional configuration assuming a spherical ionospheric surface is

$$\delta W_I = -\frac{1}{2\mu_0}\left(\int_{\text{North}}\frac{B^2|\boldsymbol{\xi}_\perp|^2}{R_I}dS + \int_{\text{South}}\frac{B^2|\boldsymbol{\xi}_\perp|^2}{R_I}dS\right), \tag{16}$$

where R_I is the sum of the Earth's radius R_E and the ionospheric height h ($R_I = R_E + h \sim R_E$) and "North" and "South" denote unperturbed ionospheric surfaces in the Northern Hemisphere and the Southern Hemisphere, respectively.

[26] The surface contribution δW_I is negative for both horizontally free and free boundary conditions. Therefore, this term is destabilizing for these boundary conditions. Since R_I appears in the denominator of equation (16), this destabilizing effect by δW_I occurs for a spherical ionospheric surface for the three-dimensional configuration. The existence of negative δW_I suggests that a magnetospheric plasma can be MHD unstable under proper conditions even without potential sources of pressure-driven modes or current-driven modes.

4. Eikonal Ansatz, Compressibility of the Magnetospheric Interchange Mode, and Expression for δW_F

[27] In this section an explicit form of δW_F is derived for an arbitrary magnetospheric equilibrium by assuming an eikonal ansatz for $\boldsymbol{\xi}_\perp$. Retaining compressibility is shown to be essential for magnetospheric interchange instability.

[28] Significant simplifications occur in the stability analysis of an arbitrary magnetospheric equilibrium by using magnetospheric energy principle if one focuses attention on interchange modes. The most unstable modes in the interchange instability are usually characterized by a highly localized $\mathbf{k}_\perp \to \infty$ perturbation, where $\mathbf{k}_\perp$ is the perpendicular wave number. In the following, δW is reduced from its original three-dimensional form involving the three components of $\boldsymbol{\xi}$ into a more tractable one-dimensional form involving only the normal component of $\boldsymbol{\xi}$. In order to exploit the $\mathbf{k}_\perp \to \infty$ limit, an eikonal representation is used for $\boldsymbol{\xi}_\perp$ [*Freidberg*, 1987]

$$\boldsymbol{\xi}_\perp = \boldsymbol{\eta}_\perp e^{iS}, \tag{17}$$

where $\mathbf{k}_\perp$ is defined using the eikonal S as

$$\mathbf{k}_\perp = \nabla S \tag{18}$$

$$\mathbf{B}\cdot\nabla S = 0. \tag{19}$$

[29] The quantity $\boldsymbol{\eta}_\perp$ is assumed to vary "slowly" on the equilibrium length scale a: $|a\nabla\boldsymbol{\eta}_\perp|/|\boldsymbol{\eta}_\perp| \sim 1$. In contrast, the assumption $\mathbf{k}_\perp \to \infty$ implies that the variation of S is rapid: $|a\nabla S| \gg 1$. Notice that there is no assumption about the parallel component $\xi_\parallel$.

[30] Substitution of (17) into (11) yields

$$\mathbf{Q}_\perp = e^{iS}[\nabla\times(\boldsymbol{\eta}_\perp\times\mathbf{B})]_\perp. \tag{20}$$

By substituting equations (17) and (20) into equation (10) the exact form of δW_F can be expressed as

$$\delta W_F = \frac{1}{2\mu_0}\int d\mathbf{r}\Big[|(\nabla\times(\boldsymbol{\eta}_\perp\times\mathbf{B}))_\perp|^2 + B^2|i\mathbf{k}_\perp\times\boldsymbol{\eta}_\perp + \nabla\times\boldsymbol{\eta}_\perp + 2\boldsymbol{\kappa}\times\boldsymbol{\eta}_\perp|^2 + \mu_0\gamma p|\nabla\cdot\boldsymbol{\xi}|^2 - 2\mu_0(\boldsymbol{\eta}_\perp\cdot\nabla p)(\boldsymbol{\eta}_\perp^*\cdot\boldsymbol{\kappa}) - \mu_0 J_\parallel(\boldsymbol{\eta}_\perp^*\times\mathbf{b})\cdot(\nabla\times(\boldsymbol{\eta}_\perp\times\mathbf{B}))_\perp\Big]. \tag{21}$$

[31] In order to reduce δW_F further, precise knowledge of $\nabla\cdot\boldsymbol{\xi}$ is necessary. The minimization condition of δW_F with respect to $\xi_\parallel$ is obtained from equation (21) and can be written as [*Miura*, 2007]

$$(\mathbf{B}\cdot\nabla)\nabla\cdot\boldsymbol{\xi} = 0. \tag{22}$$

This means that $\nabla \cdot \boldsymbol{\xi}$ is constant along the unperturbed magnetic field line. Therefore, in order to calculate $\nabla \cdot \boldsymbol{\xi}$ in the magnetosphere, one needs to specify the ionospheric boundary condition on $\nabla \cdot \boldsymbol{\xi}$. As was shown in the previous section, there are two ideal ionospheric boundary conditions allowing the horizontal displacement of the field line at the ionosphere. One is the horizontally free boundary condition (12), which requires $\xi_\| = 0$ at $\ell = \ell_S$ and $\ell = \ell_N$, where ℓ_N and ℓ_S are end points of a field line in the Northern Hemisphere and Southern Hemisphere, respectively. The other is the free boundary condition (13), which requires $\nabla \cdot \boldsymbol{\xi} = 0$ at $\ell = \ell_S$ and $\ell = \ell_N$.

[32] For the free boundary condition (13), it is obvious that $\nabla \cdot \boldsymbol{\xi}$ is zero everywhere in the magnetosphere owing to the minimization condition (22). So, there is no need to take into account the $\mu_0 \gamma p |\nabla \boldsymbol{\xi}|^2$ term in equation (21). For the horizontally free boundary condition (12), the integration of equation (22) along the field line from $\ell = \ell_S$ to $\ell = \ell_N$ yields

$$\nabla \cdot \boldsymbol{\xi} = \frac{\int_S^N B^{-1} \nabla \cdot \boldsymbol{\xi}_\perp d\ell}{\int_S^N B^{-1} d\ell}. \tag{23}$$

It follows that for both ionospheric boundary conditions (12) and (13), δW_F can be written by using only the normal component of $\boldsymbol{\eta}$.

[33] Notice that in many fusion plasma applications, the plasma compressibility term $\nabla \cdot \boldsymbol{\xi}$ can be neglected [e.g., *Freidberg*, 1987]. Therefore, the general reduction of δW_F proceeds without taking into account the $\mu_0 \gamma p |\nabla \cdot \boldsymbol{\xi}|^2$ term in equation (21). The calculation of δW_F for the incompressible case is described in detail by *Freidberg* [1987]. However, in the magnetospheric case, *Gold* [1959] showed intuitively that, for a dipole field, the specific flux tube volume is proportional to R^4, where R is the distance from the earth's center to the flux tube in the equatorial plane. Therefore, for an interchange perturbation in the dipole field, the compressibility must be retained, since the specific flux tube volume changes with the change in R caused by the interchange motion. Whether or not the compressibility is essential is not clear for the interchange perturbation for an arbitrary finite-β magnetospheric model, since the specific flux tube volume cannot be calculated explicitly as a function of ψ and χ for an arbitrary finite-β magnetospheric model. Therefore, in the following, one performs calculations for both the ideal ionospheric boundary conditions (12) and (13), and then one checks aposteriori whether $\nabla \cdot \boldsymbol{\xi}$ is really important for magnetospheric interchange mode.

[34] An examination of equation (21) indicates that the only explicit appearance of S (i.e., $\mathbf{k}_\perp$) occurs in the magnetic compression term, i.e., $B^2|i\mathbf{k}_\perp \cdot \boldsymbol{\eta}_\perp + \nabla \cdot \boldsymbol{\eta}_\perp + 2\boldsymbol{\kappa} \cdot \boldsymbol{\eta}_\perp|^2$. Following *Freidberg* [1987], one is now motivated to consider the limit $\mathbf{k}_\perp \to \infty$ (geometrical optics limit) since δW_F can be systematically minimized by expanding

$$\boldsymbol{\eta}_\perp = \boldsymbol{\eta}_{\perp 0} + \boldsymbol{\eta}_{\perp 1} + \cdots, \tag{24}$$

with $|\boldsymbol{\eta}_{\perp 1}|/|\boldsymbol{\eta}_{\perp 0}| \sim 1/k_\perp a$.

[35] Let us first consider the case of the free boundary condition (13). In this case there is no $\mu_0 \gamma p |\nabla \cdot \boldsymbol{\xi}|^2$ term in equation (21). Therefore, in the $\mathbf{k}_\perp \to \infty$ limit, the zeroth-order contribution to δW_F, which is written as δW_0, reduces to

$$\delta W_0 = \frac{1}{2\mu_0} \int d\mathbf{r} B^2 |\mathbf{k}_\perp \cdot \boldsymbol{\eta}_{\perp 0}|^2. \tag{25}$$

Clearly, the perturbation which minimizes δW_0 satisfies $\mathbf{k}_\perp \cdot \boldsymbol{\eta}_{\perp 0} = 0$ and therefore $\boldsymbol{\eta}_{\perp 0}$ can be written as

$$\boldsymbol{\eta}_{\perp 0} = Y \mathbf{b} \times \mathbf{k}_\perp. \tag{26}$$

Here, Y is a scalar quantity, varying on the "slow" equilibrium-scale length.

[36] The first nonvanishing contribution to δW_F occurs in second-order proportional to $(\mathbf{k}_\perp \cdot \boldsymbol{\eta}_{\perp 1})^2$. That is, $\delta W_F = \delta W_0 + \delta W_2 + \cdots = \delta W_2 + \cdots$. In this expression, the only appearance of the quantity $\boldsymbol{\eta}_{\perp 1}$ is in the magnetic compression term, which is written as

$$\delta W_2(\mathrm{comp}) = \frac{1}{2\mu_0} \int \mathrm{d}\mathbf{r} \mathrm{B}^2 |\mathrm{i}\mathbf{k}_\perp \cdot \boldsymbol{\eta}_{\perp 1} + \nabla \cdot \boldsymbol{\eta}_{\perp 0} + 2\boldsymbol{\kappa} \cdot \boldsymbol{\eta}_{\perp 0}|^2. \tag{27}$$

Obviously, the minimum of $\delta W_2(\mathrm{comp})$ occurs when

$$Z + 2\boldsymbol{\eta}_{\perp 0} \cdot \boldsymbol{\kappa} = 0, \tag{28}$$

where

$$Z = i\mathbf{k}_\perp \cdot \boldsymbol{\eta}_{\perp 1} + \nabla \cdot \boldsymbol{\eta}_{\perp 0}. \tag{29}$$

[37] Now one can check aposteriori whether $\boldsymbol{\nabla} \cdot \xi$ term can really be neglected for the free boundary condition (13) in the magnetospheric interchange mode. In the lowest order the $\nabla \cdot \boldsymbol{\xi}$ term in equation (21) can be written as

$$\nabla \cdot \boldsymbol{\xi} = -\frac{\xi_\|}{B} \mathbf{b} \cdot \nabla B + \mathbf{b} \cdot \nabla \xi_\| + \nabla \cdot \boldsymbol{\xi}_\perp, \tag{30}$$

where

$$\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp = e^{iS} Z = e^{iS}(\nabla \cdot \boldsymbol{\eta}_{\perp 0} + i\mathbf{k}_\perp \cdot \boldsymbol{\eta}_{\perp 1}) = e^{iS}(-2\boldsymbol{\eta}_{\perp 0} \cdot \boldsymbol{\kappa}). \tag{31}$$

Since $\boldsymbol{\eta}_{\perp 0} \to \infty$ as $\mathbf{k}_\perp \to \infty$ from equation (26), $\nabla \cdot \boldsymbol{\xi}_\perp \to \infty$ as $\mathbf{k}_\perp \to \infty$ from equation (31). Since $\xi_\|$ is finite (bounded by the field line length from the southern ionosphere to the northern ionosphere) and $\mathbf{b} \cdot \nabla \xi_\|$ is considered to be small for the magnetospheric interchange mode, the first two terms on the right hand side of equation (30) cannot cancel out the $\nabla \cdot \boldsymbol{\xi}_\perp$ term in the $\mathbf{k}_\perp \to \infty$ limit. Therefore, in the $\mathbf{k}_\perp \to \infty$ limit, $\nabla \cdot \boldsymbol{\xi}$ cannot be zero. This contradicts the original assumption of the free boundary condition (13) that means $\nabla \cdot \boldsymbol{\xi} = 0$ everywhere in the magnetosphere. Therefore, the boundary condition (13) is discarded for the magnetospheric interchange mode.

[38] One needs, therefore, to use the horizontally free boundary condition (12) for the magnetospheric interchange mode and to retain $\nabla \cdot \boldsymbol{\xi}$ in equation (21). Since

$$\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp = e^{iS}(\nabla \cdot \boldsymbol{\eta}_\perp + i\mathbf{k}_\perp \cdot \boldsymbol{\eta}_\perp), \tag{32}$$

one obtains from equation (23)

$$|\nabla\cdot\boldsymbol{\xi}|^2 = \frac{|\int_S^N B^{-1}(\nabla\cdot\boldsymbol{\eta}_\perp + i\mathbf{k}_\perp\cdot\boldsymbol{\eta}_\perp)d\ell|^2}{\left(\int_S^N B^{-1}d\ell\right)^2}. \quad (33)$$

Therefore, in the $\mathbf{k}_\perp \to \infty$ limit, the zeroth-order contribution to δW_F reduces to

$$\delta W_0 = \frac{1}{2\mu_0}\int d\mathbf{r}\left[B^2|\mathbf{k}_\perp\cdot\boldsymbol{\eta}_{\perp 0}|^2 + \gamma p\mu_0\frac{|\int_S^N B^{-1}i\mathbf{k}_\perp\cdot\boldsymbol{\eta}_{\perp 0}d\ell|^2}{\left(\int_S^N B^{-1}d\ell\right)^2}\right]. \quad (34)$$

Again, the perturbation which minimizes δW_0 satisfies $\mathbf{k}_\perp\cdot\boldsymbol{\eta}_{\perp 0} = 0$ and therefore $\boldsymbol{\eta}_{\perp 0}$ can be written as

$$\boldsymbol{\eta}_{\perp 0} = Y\mathbf{b}\times\mathbf{k}_\perp. \quad (35)$$

[39] The first nonvanishing contribution to δW_F occurs in second-order proportional to $(\mathbf{k}_\perp\cdot\boldsymbol{\eta}_{\perp 1})^2$. This second-order contribution to δW_F is denoted by δW_2. In this expression the only appearance of the quantity $\boldsymbol{\eta}_{\perp 1}$ is in the $\delta W_2{}'(\boldsymbol{\eta}_{\perp 1})$ term, which is the integral of the sum of the magnetic compression term, $B^2|i\mathbf{k}_\perp\cdot\boldsymbol{\eta}_{\perp 1} + \nabla\cdot\boldsymbol{\eta}_{\perp 0} + 2\boldsymbol{\kappa}\cdot\boldsymbol{\eta}_{\perp 0}|^2$, and the plasma compression term $\mu_0\gamma p|\nabla\cdot\boldsymbol{\xi}|^2$. That is,

$$\delta W_2'(\boldsymbol{\eta}_{\perp 1}) = \frac{1}{2\mu_0}\int d\psi d\chi W_2, \quad (36)$$

where

$$W_2 = \int_S^N \left[B^2|Z + 2\boldsymbol{\eta}_{\perp 0}\cdot\boldsymbol{\kappa}|^2 + \mu_0\gamma p|\langle Z\rangle|^2\right]\frac{d\ell}{B} \quad (37)$$

and

$$\boldsymbol{\eta}_{\perp 0} = (X/B)(\mathbf{b}\times\mathbf{k}_\perp), \quad (38)$$

$$\langle Z\rangle = \frac{1}{U}\int_S^N \frac{Z}{B}d\ell, \quad (39)$$

where $X \equiv YB$. Here, by setting $\zeta = \ell$, where ℓ is the length along the field line, one finds that $d\mathbf{r} = Jd\psi d\chi d\zeta = d\psi\, d\chi d\ell/B$, since J is the Jacobian of the transformation

$$J = \frac{1}{(\nabla\psi\times\nabla\chi)\cdot\nabla\zeta} = \frac{1}{\mathbf{B}\cdot\nabla\zeta} = \frac{1}{\mathbf{B}\cdot\nabla\ell} = \frac{1}{B}. \quad (40)$$

[40] Therefore, in order to minimize W_2 with respect to Z, one has to obtain Z, which satisfies $\delta(W_2)_Z = W_2(Z + \delta Z) - W_2(Z) = 0$. The solution for this equation is shown to be

$$B^2(Z + 2\boldsymbol{\eta}_{\perp 0}\cdot\boldsymbol{\kappa}) = -\mu_0\gamma p\langle Z\rangle. \quad (41)$$

Therefore, one obtains

$$\langle Z\rangle = -\frac{2\langle\boldsymbol{\eta}_{\perp 0}\cdot\boldsymbol{\kappa}\rangle}{1 + \mu_0\gamma p\langle\frac{1}{B^2}\rangle}. \quad (42)$$

[41] From equations (31) and (41), one obtains

$$\nabla\cdot\boldsymbol{\xi}_\perp = e^{iS}Z = e^{iS}\left(-2\boldsymbol{\eta}_{\perp 0}\cdot\boldsymbol{\kappa} + \frac{\mu_0\gamma p}{B^2}\frac{2\langle\boldsymbol{\eta}_{\perp 0}\cdot\boldsymbol{\kappa}\rangle}{1 + \gamma\mu_0 p\langle\frac{1}{B^2}\rangle}\right). \quad (43)$$

In the $\mathbf{k}_\perp \to \infty$ limit, $\nabla\cdot\boldsymbol{\xi}_\perp \to \infty$. Equation (30) also holds for the horizontally free ionospheric boundary condition (12). Since $\xi_\parallel$ is finite and $\mathbf{b}\cdot\nabla\xi_\parallel$ is considered to be small for the magnetospheric interchange mode, $\nabla\cdot\boldsymbol{\xi}$ cannot be zero in the $\mathbf{k}_\perp \to \infty$ limit. Therefore, indeed, the $\nabla\cdot\boldsymbol{\xi}$ term cannot be neglected for the horizontally free boundary condition (12). This is consistent with the original assumption of the ionospheric boundary condition (12).

[42] In the evaluation of other terms in δW_2 it is useful to simplify the quantity $[\nabla\times(\boldsymbol{\eta}_\perp\times\mathbf{B})]_\perp$ as follows [*Freidberg*, 1987]:

$$[\nabla\times(\boldsymbol{\eta}_\perp\times\mathbf{B})]_\perp = (\nabla X\times\mathbf{k}_\perp)_\perp. \quad (44)$$

If one now writes $\nabla X = \nabla_\perp X + \mathbf{b}(\mathbf{b}\cdot\nabla X)$, then the perpendicular component of $[\nabla\times(\boldsymbol{\eta}_\perp\times\mathbf{B})]_\perp$ can be expressed as

$$[\nabla\times(\boldsymbol{\eta}_\perp\times\mathbf{B})]_\perp = (\mathbf{b}\cdot\nabla X)\mathbf{b}\times\mathbf{k}_\perp. \quad (45)$$

[43] Using this relation one can show that the kink contribution to δW_2, which is written as δW_2(kink) is given by [*Freidberg*, 1987]

$$\begin{aligned}\delta W_2(\text{kink}) &= -\frac{1}{2}\int d\mathbf{r}J_\parallel(\boldsymbol{\eta}_\perp^*\times\mathbf{b})\cdot[\nabla\times(\boldsymbol{\eta}_\perp\times\mathbf{B})]_\perp \\ &= -\frac{1}{2}\int d\mathbf{r}\left[(J_\parallel/B)X^*(\mathbf{b}\cdot\nabla X)\right]\cdot[\mathbf{k}_\perp\cdot(\mathbf{b}\times\mathbf{k}_\perp)] = 0.\end{aligned} \quad (46)$$

Thus, in the $\mathbf{k}_\perp \to \infty$ limit, the kink term makes no contribution to stability. Therefore, by adding all the contributions to δW_2 one obtains from equation (21) the full expression of the minimized form of δW_2

$$\delta W_2 = \frac{1}{2\mu_0}\int d\psi d\chi W, \quad (47)$$

where

$$W = \int_S^N\left(\mathbf{k}_\perp^2|\mathbf{b}\cdot\nabla X|^2 - \frac{2\mu_0}{B^2}[(\mathbf{b}\times\mathbf{k}_\perp)\cdot\nabla p][(\mathbf{b}\times\mathbf{k}_\perp)\cdot\boldsymbol{\kappa}]|X|^2\right)\cdot\frac{d\ell}{B} + \gamma p\mu_0\frac{4|\int_S^N\frac{X}{B^2}\boldsymbol{\kappa}\cdot(\mathbf{b}\times\mathbf{k}_\perp)d\ell|^2}{\int_S^N\frac{d\ell}{B} + \gamma p\mu_0\int_S^N\frac{d\ell}{B^3}}. \quad (48)$$

[44] *Miura* [2007] has shown in the magnetospheric energy principle that for the rigid ionospheric boundary condition (15), compressible ballooning modes occur in the magnetosphere. The potential energy change δW_2 for the compressible ballooning mode is given by the same δW_2 as equation (47) for the horizontally free boundary condition, since both boundary conditions satisfy $\xi_\parallel = 0$ at the ionosphere and $\nabla\cdot\boldsymbol{\xi}$ is given by equation (23). For the two-dimensional cylindrical magnetospheric configuration, in which the dawn-dusk direction is parallel to the cylindrical axis, a similar form of δW_2 has been derived for compressible

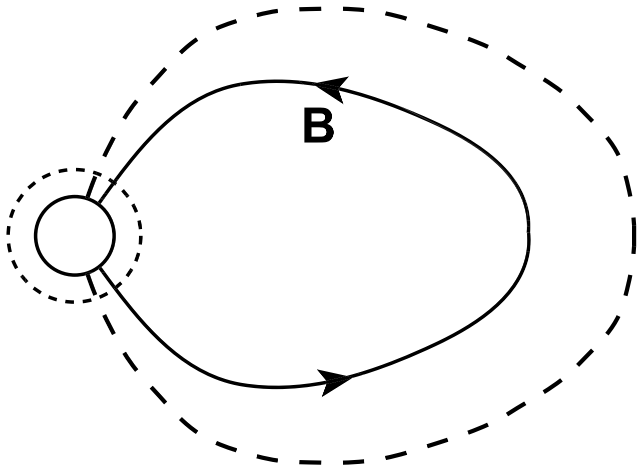

Figure 2. The solid line shows an unperturbed field line in the magnetosphere. The short-dashed circle is the ionospheric boundary. Shown schematically by a long-dashed line is a field line without line bending, which is perturbed by an interchange mode. The amplitude of perturbation is exaggerated. Adapted from *Miura* [2007].

ballooning modes using different methods [e.g., *Lee and Wolf*, 1992; *Bhattacharjee et al.*, 1998; *Miura*, 2000; *Schindler and Birn*, 2004].

[45] One observes in equation (48) that the derivative appearing on X involves only ℓ. Hence, ψ and χ enter W solely as parameters. Stability can thus be tested one magnetic line at a time (i.e., for fixed ψ and χ). The three-dimensional stability problem has thus been reduced to the solution of a sequence of one-dimensional problems, representing an enormous reduction in effort.

5. Reduction of δW_2 for Interchange Perturbations

[46] In this section the variational magnetospheric potential energy change δW_2 obtained in the previous section is further reduced to a simple form for interchange perturbations.

[47] Attention is now focused on the stability of interchange perturbations. This type of disturbance corresponds to a trial function of the form $X(\psi, \chi, \zeta) = \hat{X}(\psi - \psi_0, \chi - \chi_0)$ where $\hat{X}$ is a localized function about $\psi = \psi_0$, $\chi = \chi_0$, the magnetic line under consideration. Since X is assumed to be independent of ζ, the first term in the integrand of W in equation (48) vanishes. That is,

$$\mathbf{Q}_\perp = \mathbf{B}_{1\perp} = e^{iS}(\mathbf{b}\cdot\nabla X)\mathbf{b}\times\mathbf{k}_\perp = 0 \tag{49}$$

in the magnetosphere. Thus, the interchange perturbation sets $|\mathbf{Q}_\perp|^2 = 0$, thereby eliminating the stabilizing effects of the line bending. Although there is no line bending in the magnetosphere as equation (49) shows, there is line bending in perturbations at the ionosphere (see Appendix B for details). Figure 2 is a schematic of the midnight meridian plane and shows an unperturbed field line (solid line) and a field line, which shows no line bending (dashed line), perturbed by an interchange mode. The amplitude of perturbation is exaggerated in Figure 2.

[48] For the interchange trial function $\hat{X}$ without any ζ dependence, W in equation (48) reduces to

$$W(\psi,\chi) = 2\mu_0|\hat{X}|^2\cdot\left[-\int_S^N \frac{1}{B^2}[(\mathbf{b}\times\mathbf{k}_\perp)\cdot\nabla p][(\mathbf{b}\times\mathbf{k}_\perp)\cdot\boldsymbol{\kappa}]\frac{d\ell}{B} + 2\gamma p\frac{\left(\int_S^N \frac{1}{B^2}\boldsymbol{\kappa}\cdot(\mathbf{b}\times\mathbf{k}_\perp)d\ell\right)^2}{U\left(1+\gamma p\mu_0\langle\frac{1}{B^2}\rangle\right)}\right]. \tag{50}$$

The requirement $\mathbf{B}\cdot\nabla S = 0$ implies that

$$S(\psi,\chi,\zeta) = S(\psi,\chi). \tag{51}$$

Therefore, the wave vector $\mathbf{k}_\perp$ immediately follows:

$$\mathbf{k}_\perp = \nabla S = \frac{\partial S}{\partial\psi}\nabla\psi + \frac{\partial S}{\partial\chi}\nabla\chi. \tag{52}$$

Since $\mathbf{B}\cdot\nabla p = 0$, ∇p can be expressed as

$$\nabla p = \frac{\partial p}{\partial\psi}\nabla\psi + \frac{\partial p}{\partial\chi}\nabla\chi. \tag{53}$$

Similarly, since the curvature vector, $\boldsymbol{\kappa} = (\mathbf{b}\cdot\nabla)\mathbf{b}$, satisfies $\mathbf{b}\cdot\boldsymbol{\kappa} = 0$, it is convinient to expand the curvature vector $\boldsymbol{\kappa}$ as

$$\boldsymbol{\kappa} = \kappa_\psi\nabla\psi + \kappa_\chi\nabla\chi, \tag{54}$$

where

$$\kappa_\psi = -\frac{(\mathbf{b}\times\nabla\chi)\cdot\boldsymbol{\kappa}}{B} \tag{55}$$

$$\kappa_\chi = \frac{(\mathbf{b}\times\nabla\psi)\cdot\boldsymbol{\kappa}}{B}. \tag{56}$$

From equations (52) and (53), one obtains

$$(\mathbf{b}\times\mathbf{k}_\perp)\cdot\nabla p = B\left(\frac{\partial S}{\partial\psi}\frac{\partial p}{\partial\chi} - \frac{\partial S}{\partial\chi}\frac{\partial p}{\partial\psi}\right). \tag{57}$$

Substitution of equation (57) into equation (50) yields

$$W(\psi,\chi) = 2\mu_0|\hat{X}|^2\int_S^N \frac{1}{B^2}(\mathbf{b}\times\mathbf{k}_\perp)\cdot\boldsymbol{\kappa}d\ell \cdot\left[-\left(\frac{\partial S}{\partial\psi}\frac{\partial p}{\partial\chi} - \frac{\partial S}{\partial\chi}\frac{\partial p}{\partial\psi}\right) + 2\gamma p\frac{\int_S^N\frac{1}{B^2}\boldsymbol{\kappa}\cdot(\mathbf{b}\times\mathbf{k}_\perp)d\ell}{U\left(1+\gamma p\mu_0\langle\frac{1}{B^2}\rangle\right)}\right]. \tag{58}$$

From equations (52) and (54) one obtains

$$\mathbf{b}\times\mathbf{k}_\perp\cdot\boldsymbol{\kappa} = B\left(\frac{\partial S}{\partial\psi}\kappa_\chi - \frac{\partial S}{\partial\chi}\kappa_\psi\right). \tag{59}$$

Therefore, one obtains

$$\int_S^N \frac{1}{B^2}(\mathbf{b}\times\mathbf{k}_\perp)\cdot\boldsymbol{\kappa}d\ell = \frac{\partial S}{\partial\psi}\int_S^N\frac{\kappa_\chi}{B}d\ell - \frac{\partial S}{\partial\chi}\int_S^N\frac{\kappa_\psi}{B}d\ell. \tag{60}$$

Therefore, in order to calculate W one is now left to calculate $\int_S^N \frac{\kappa_\chi}{B}\, d\ell$ and $\int_S^N \frac{\kappa_\psi}{B}\, d\ell$.

[49] One obtains from the pressure balance equation

$$\boldsymbol{\kappa} = \frac{1}{B^2}\left[\mu_0 \nabla p + \nabla\left(\frac{B^2}{2}\right)\right] - \frac{\mathbf{b}}{B^2}(\mathbf{B}\cdot\nabla B). \tag{61}$$

Therefore, one obtains

$$\kappa_\chi = -\boldsymbol{\kappa}\cdot\left(\frac{\nabla\psi\times\mathbf{B}}{B^2}\right) = -\frac{1}{B^2}\left[\mu_0\nabla p + \nabla\left(\frac{B^2}{2}\right)\right]\cdot\left(\frac{\nabla\psi\times\mathbf{B}}{B^2}\right). \tag{62}$$

Using vector formulae, this is further reduced to

$$\kappa_\chi = -\frac{\mu_0}{2B^2}\nabla\psi\cdot\mathbf{J}_\perp + \frac{1}{2}\nabla\cdot\left(\frac{\nabla\psi\times\mathbf{B}}{B^2}\right). \tag{63}$$

Using equation (53) one obtains

$$\mathbf{J}_\perp = \mathbf{J} - J_\parallel \mathbf{b} = \frac{1}{B^2}(\mathbf{B}\times\nabla\psi)\frac{\partial p}{\partial\psi} + \frac{1}{B^2}(\mathbf{B}\times\nabla\chi)\frac{\partial p}{\partial\chi}. \tag{64}$$

Taking the divergence of this equation yields

$$\nabla\cdot\left(J_\parallel\mathbf{b}\right) = \left[\nabla\cdot\left(\frac{\nabla\psi\times\mathbf{B}}{B^2}\right)\right]\frac{\partial p}{\partial\psi} + \left[\nabla\cdot\left(\frac{\nabla\chi\times\mathbf{B}}{B^2}\right)\right]\frac{\partial p}{\partial\chi}. \tag{65}$$

[50] From the static pressure balance equation one has

$$\nabla p = (\mathbf{J}\cdot\nabla\chi)\nabla\psi - (\mathbf{J}\cdot\nabla\psi)\nabla\chi. \tag{66}$$

The comparison of this with equation (53) yields

$$\frac{\partial p}{\partial\chi} = -\nabla\psi\cdot\mathbf{J}_\perp,\quad \frac{\partial p}{\partial\psi} = \nabla\chi\cdot\mathbf{J}_\perp. \tag{67}$$

[51] Therefore, the use of (67) and substitution of equation (65) into equation (63) yield

$$\kappa_\chi = \frac{1}{2\frac{\partial p}{\partial\psi}}\nabla\cdot\left(J_\parallel\mathbf{b}\right) - \left[-\frac{\mu_0}{2B^2} + \frac{1}{2\frac{\partial p}{\partial\psi}}\nabla\cdot\left(\frac{\nabla\chi\times\mathbf{B}}{B^2}\right)\right]\frac{\partial p}{\partial\chi}. \tag{68}$$

By dividing this equation by B and then integrating along the field line one obtains

$$\int_S^N \frac{\kappa_\chi}{B}d\ell = \frac{1}{2\frac{\partial p}{\partial\psi}}\left(\frac{J_{\parallel N}}{B_N} - \frac{J_{\parallel S}}{B_S}\right) + \frac{\partial p}{\partial\chi}\left[\frac{\mu_0}{2}U\left\langle\frac{1}{B^2}\right\rangle - \frac{1}{2\frac{\partial p}{\partial\psi}}\int_S^N \frac{1}{B}\nabla\cdot\left(\frac{\nabla\chi\times\mathbf{B}}{B^2}\right)d\ell\right]. \tag{69}$$

[52] Using equations (61) and (67), one obtains

$$\kappa_\psi = \frac{\mu_0}{2B^2}\frac{\partial p}{\partial\psi} - \frac{1}{2}\nabla\cdot\left(\frac{\nabla\chi\times\mathbf{B}}{B^2}\right). \tag{70}$$

By dividing this equation by B and then integrating along the field line, one obtains

$$\int_S^N \frac{1}{B}\nabla\cdot\left(\frac{\nabla\chi\times\mathbf{B}}{B^2}\right)d\ell = \mu_0\frac{\partial p}{\partial\psi}U\left\langle\frac{1}{B^2}\right\rangle - 2\int_S^N \frac{\kappa_\psi}{B}d\ell. \tag{71}$$

Substitution of equation (71) into equation (69) yields

$$\int_S^N \frac{\kappa_\chi}{B}d\ell = \frac{1}{2\frac{\partial p}{\partial\psi}}\left(\frac{J_{\parallel N}}{B_N} - \frac{J_{\parallel S}}{B_S}\right) + \frac{\frac{\partial p}{\partial\chi}}{\frac{\partial p}{\partial\psi}}\int_S^N \frac{\kappa_\psi}{B}d\ell. \tag{72}$$

Substitution of equation (72) into equation (60) yields

$$\int_S^N \frac{1}{B^2}(\mathbf{b}\times\mathbf{k}_\perp)\cdot\boldsymbol{\kappa}d\ell = \frac{\frac{\partial S}{\partial\psi}}{2\frac{\partial p}{\partial\psi}}\left(\frac{J_{\parallel N}}{B_N} - \frac{J_{\parallel S}}{B_S}\right) + \left(\frac{\partial S}{\partial\psi}\frac{\frac{\partial p}{\partial\chi}}{\frac{\partial p}{\partial\psi}} - \frac{\partial S}{\partial\chi}\right)\int_S^N \frac{\kappa_\psi}{B}d\ell. \tag{73}$$

[53] On the other hand, Appendix B and equation (49) of *Hameiri et al.* [1991] give

$$\oint \frac{\kappa_\psi}{B}d\ell = \frac{1}{2}\left[\mu_0\frac{dp}{d\psi}U\left\langle\frac{1}{B^2}\right\rangle - \frac{dU}{d\psi}\right] \tag{74}$$

for a closed field line loop used in their calculation. This equation is obtained by taking a volume integral of $\nabla\cdot(\nabla\chi\times\mathbf{B}/B^2)$ in a volume surrounded by closed field lines (see Appendix B of *Hameiri et al.* [1991] for details). If one calculates a volume integral of $\nabla\cdot(\nabla\chi\times\mathbf{B}/B^2)$ in a volume surrounded by ionospheric surfaces and field lines from S to N, one obtains a similar equation

$$\int_S^N \frac{\kappa_\psi}{B}d\ell = \frac{1}{2}\left[\mu_0\frac{\partial p}{\partial\psi}U\left\langle\frac{1}{B^2}\right\rangle - \frac{\partial U}{\partial\psi}\right] \tag{75}$$

by using a similar calculation as in Appendix B of *Hameiri et al.* [1991]. The only difference of such a calculation from that in Appendix B of *Hameiri et al.* [1991] is that a surface integral of $\nabla\chi\times\mathbf{B}/B^2$ over the ionospheric surface arises. However, this ionospheric contribution vanishes owing to the assumption that unperturbed field lines are incident vertically on the ionospheric surface. Therefore, a complete similarity between equations (74) and (75) arises. Substitution of equation (73) and (75) into equation (58) yields

$$W(\psi,\chi) = \frac{\mu_0|\hat{X}|^2}{U\left(1+\gamma p\mu_0\langle\frac{1}{B^2}\rangle\right)\left(\frac{\partial p}{\partial\psi}\right)^2}\left[\frac{\partial S}{\partial\psi}\left(\frac{J_{\parallel N}}{B_N} - \frac{J_{\parallel S}}{B_S}\right) + \left(\frac{\partial S}{\partial\psi}\frac{\partial p}{\partial\chi} - \frac{\partial S}{\partial\chi}\frac{\partial p}{\partial\psi}\right)\times\left(\mu_0 U\frac{\partial p}{\partial\psi}\left\langle\frac{1}{B^2}\right\rangle - \frac{\partial U}{\partial\psi}\right)\right] \cdot\left[\gamma p\frac{\partial S}{\partial\psi}\left(\frac{J_{\parallel N}}{B_N} - \frac{J_{\parallel S}}{B_S}\right) - \left(\frac{\partial S}{\partial\psi}\frac{\partial p}{\partial\chi} - \frac{\partial S}{\partial\chi}\frac{\partial p}{\partial\psi}\right)\times\left(U\frac{\partial p}{\partial\psi} + \gamma p\frac{\partial U}{\partial\psi}\right)\right]. \tag{76}$$

6. Calculation of $\delta W = \delta W_F + \delta W_I$

[54] The ionospheric contribution to the change of the potential energy δW_I is given by equation (16). Here,

$$|\boldsymbol{\xi}_\perp|^2 = |\boldsymbol{\eta}_{\perp 0}|^2 = \frac{1}{B^2}|\hat{X}|^2\mathbf{k}_\perp^2, \tag{77}$$

where

$$\mathbf{k}_\perp^2 = \left(\frac{\partial S}{\partial \psi}\right)^2 (\nabla\psi)^2 + \left(\frac{\partial S}{\partial \chi}\right)^2 (\nabla\chi)^2 + 2\frac{\partial S}{\partial \psi}\frac{\partial S}{\partial \chi}\nabla\psi\cdot\nabla\chi. \quad (78)$$

Notice that $\nabla\psi$ and $\nabla\chi$ are not orthogonal in general. Since the unperturbed field lines are assumed to be incident vertically on the ionospheric surface, $d\mathbf{r} = dSd\ell$ at the ionospheric surface. Therefore, one obtains

$$dS = \frac{d\psi d\chi}{B}. \quad (79)$$

Therefore, equation (16) can be written as

$$\delta W_I = -\frac{1}{2\mu_0}\left(\int_{\text{North}} \frac{\mathbf{k}_\perp^2 |\hat{X}|^2}{R_I B} d\psi d\chi + \int_{\text{South}} \frac{\mathbf{k}_\perp^2 |\hat{X}|^2}{R_I B} d\psi d\chi\right). \quad (80)$$

From equations (47) and (80) one obtains

$$\delta W = \delta W_2 + \delta W_I = \frac{1}{2\mu_0}\int d\psi d\chi W(\psi,\chi) - \frac{1}{2\mu_0} \times \left(\int_{\text{North}} \frac{\mathbf{k}_\perp^2 |\hat{X}|^2}{R_I B} d\psi d\chi + \int_{\text{South}} \frac{\mathbf{k}_\perp^2 |\hat{X}|^2}{R_I B} d\psi d\chi\right). \quad (81)$$

[55] When there is north-south symmetry, one obtains from equation (81)

$$\delta W = \frac{1}{2\mu_0}\int d\psi d\chi W'(\psi,\chi) \quad (82)$$

$$W' = W(\psi,\chi) - 2\frac{\mathbf{k}_\perp^2 |\hat{X}|^2}{R_I B_I}, \quad (83)$$

where $B_N = B_S = B_I$. Since equation (5) can be written as

$$\frac{J_{\parallel N}}{B_N} - \frac{J_{\parallel S}}{B_S} = \frac{\partial p}{\partial \chi}\frac{\partial U}{\partial \psi} - \frac{\partial p}{\partial \psi}\frac{\partial U}{\partial \chi}, \quad (84)$$

substitution of equation (84) into equation (83) yields

$$W'(\psi,\chi) = |\hat{X}|^2\Bigg[\frac{\mu_0}{U\left(1+\gamma p\mu_0\langle\frac{1}{B^2}\rangle\right)}\Bigg(\frac{\partial S}{\partial \psi}\left(\mu_0 U\left\langle\frac{1}{B^2}\right\rangle\frac{\partial p}{\partial \chi} - \frac{\partial U}{\partial \chi}\right) + \frac{\partial S}{\partial \chi}\left(\frac{\partial U}{\partial \psi} - \mu_0 U\left\langle\frac{1}{B^2}\right\rangle\frac{\partial p}{\partial \psi}\right)\Bigg)\Bigg(\frac{\partial S}{\partial \chi}\left(\gamma p\frac{\partial U}{\partial \psi} + U\frac{\partial p}{\partial \psi}\right) - \frac{\partial S}{\partial \psi}\left(\gamma p\frac{\partial U}{\partial \chi} + U\frac{\partial p}{\partial \chi}\right)\Bigg) - \frac{2\mathbf{k}_\perp^2}{R_I B_I}\Bigg]. \quad (85)$$

7. Stability Criteria for Different Magnetospheric Models

7.1. Axisymmetric and North-South Symmetric Magnetospheric Models

7.1.1. High-β Plasma

[56] When the unperturbed magnetospheric states are axisymmetric and north-south symmetric, $p(\psi, \chi)$ and $U(\psi, \chi)$ become $p(\psi)$ and $U(\psi)$, respectively. Under this condition, W' in equation (85) can be written as

$$W'(\psi,\chi) = |\hat{X}|^2\left[\frac{\mu_0}{U\left(1+\gamma p\mu_0\langle\frac{1}{B^2}\rangle\right)}\times\left(\frac{\partial S}{\partial \chi}\right)^2 \times\left(\frac{dU}{d\psi} - \mu_0 U\left\langle\frac{1}{B^2}\right\rangle\frac{dp}{d\psi}\right)\left(\gamma p\frac{dU}{d\psi} + U\frac{dp}{d\psi}\right) - \frac{2\mathbf{k}_\perp^2}{R_I B_I}\right]. \quad (86)$$

It is obvious that $W' \geq 0$ is the stability condition for interchange perturbations. This stability criterion for interchange perturbations is

$$\frac{\mu_0}{U\left(1+\gamma p\mu_0\langle\frac{1}{B^2}\rangle\right)}\left(\frac{\partial S}{\partial \chi}\right)^2\left[\frac{dU}{d\psi} - \mu_0 U\left\langle\frac{1}{B^2}\right\rangle\frac{dp}{d\psi}\right] \times\left(\gamma p\frac{dU}{d\psi} + U\frac{dp}{d\psi}\right) - \frac{2\mathbf{k}_\perp^2}{R_I B_I} \geq 0. \quad (87)$$

[57] In the limit of planar ionospheric surface ($R_I \to \infty$) or in the absence of ionospheric destabilizing contribution, equation (87) can be written as

$$\left[\frac{dU}{d\psi} - \mu_0 U\left\langle\frac{1}{B^2}\right\rangle\frac{dp}{d\psi}\right]\left(\gamma p\frac{dU}{d\psi} + U\frac{dp}{d\psi}\right) \geq 0. \quad (88)$$

This stability criterion for interchange instability is the same as that derived by *Spies* [1971] and *Hameiri et al.* [1991] for closed field line configurations, although their $\langle 1/B^2\rangle$ is defined by the integral along the closed field line instead of the integral from S to N. Although they assume that there are no boundaries along the field line and the field line is a closed loop without any boundary, the present stability criterion (88) was derived for the realistic magnetospheric configuration with horizontally free ionospheric boundary condition. Since this criterion includes high-β terms expressed by $\langle 1/B^2\rangle dp/d\psi$, this criterion is a fully general criterion for interchange stability in the finite-β axisymmetric and north-south symmetric magnetospheric model.

7.1.2. Low-β Plasma

[58] For low-β plasma, the $\langle 1/B^2\rangle dp/d\psi$ term in equation (86) can be neglected and equation (86) becomes particularly simple as

$$W'(\psi,\chi) = |\hat{X}|^2\left[\frac{\mu_0}{U}\left(\frac{\partial S}{\partial \chi}\right)^2\frac{dU}{d\psi}\left(\gamma p\frac{dU}{d\psi} + U\frac{dp}{d\psi}\right) - \frac{2\mathbf{k}_\perp^2}{R_I B_I}\right]. \quad (89)$$

In the low-β limit, the unperturbed magnetic field may be represented by a dipole field for interchange perturbations. Therefore,

$$B(R,\Phi) = \frac{\mu_0 M}{4\pi R^3}\frac{\sqrt{1+3\sin^2\Phi}}{\cos^6\Phi}, \quad (90)$$

where

$$R = R_E\cos^{-2}\Phi_0 = LR_E. \quad (91)$$

Here, M is the magnetic moment of the earth, Φ is the latitude, and Φ_0 is the latitude at which the field line intersects the earth's surface. Since $d\ell = R\cos\Phi\sqrt{1+3\sin^2\Phi}d\Phi$, the specific flux tube volume $U(\psi)$ is written as

$$U = \int_S^N \frac{d\ell}{B} = \frac{8\pi}{\mu_0 M} R^4 f(\Phi_0), \tag{92}$$

where

$$\begin{aligned} f(\Phi_0) = f(L) &= \int_0^{\Phi_0} \cos^7 \Phi d\Phi \\ &= \frac{16}{35}\sin\Phi_0\left(1+\frac{1}{2L}+\frac{3}{8L^2}+\frac{5}{16L^3}\right) \\ &= \frac{16}{35}\left(1-\frac{35}{128L^4}-\frac{7}{64L^5}-\cdots\right). \end{aligned} \tag{93}$$

Therefore, neglecting terms of $O(\frac{1}{L^4})$ and smaller, the specific flux tube volume U is proportional to R^4. Notice that in an intuitive argument of *Gold* [1959], he derived $U \propto R^4$, since in the dipole field the magnetic field strength is proportional to R^{-3} and the cross section of the flux tube is proportional to R^3, and he assumed that the flux tube length is proportional to R. The extra R dependence contained in $f(L)$ in the exact relation (93) is due to the fact that the field line end points are accurately taken into account in equation (92).

[59] In the dipole field, it is more convenient to use the dipole coordinate (R, ϕ, Φ), where ϕ is the longitude, instead of the flux coordinate (ψ, χ, ζ). Therefore, one needs to transform (ψ, χ) to (R, ϕ). Here, $\psi = \psi(R)$ and $\chi = \chi(\phi)$ in the dipole field. The flux function ψ should be proportional to the poloidal flux ψ_p, which is defined by

$$\psi_p = \int \mathbf{B}_p \cdot d\mathbf{A}. \tag{94}$$

The stream function ψ' can be defined in the equatorial plane as

$$B_z = \frac{1}{R}\frac{\partial\psi'}{\partial R}. \tag{95}$$

[60] Since one has

$$B_z(R) = B(R) = \frac{\mu_0 M}{4\pi R^3}, \tag{96}$$

one obtains from equations (95) and (96)

$$\psi'(R) = \frac{\mu_0 M}{4\pi}\left(\frac{1}{R_E}-\frac{1}{R}\right), \tag{97}$$

where the integration constant was chosen so that $\psi'(R = R_E) = 0$. It is then straightforward to show that

$$\psi_p = \int_{R_E}^R B_z 2\pi R dR = 2\pi\psi'. \tag{98}$$

Since the stream function ψ' is proportional to the poloidal magnetic flux ψ_p, one can choose ψ' as the flux function ψ. That is,

$$\psi(R) = \frac{\mu_0 M}{4\pi}\left(\frac{1}{R_E}-\frac{1}{R}\right). \tag{99}$$

[61] In the dipole coordinate, the magnetic field in the equatorial plane is expressed as

$$\mathbf{B}(R) = \frac{\mu_0 M}{4\pi R^3}\hat{R}\times\mathbf{e}_\phi, \tag{100}$$

where $\hat{R}$ is the unit vector in the radial direction in the equatorial plane and $\mathbf{e}_\phi$ is the unit vector in the azimuthal direction. Since

$$\nabla\psi = \frac{\mu_0 M}{4\pi R^2}\hat{R}, \tag{101}$$

the comparison of equations (2) and (100) shows that

$$\nabla\chi = \frac{1}{R}\mathbf{e}_\phi. \tag{102}$$

Since $\chi = \chi(\phi)$, one obtains

$$\nabla\chi = \nabla\chi(\phi) = \frac{1}{R}\frac{d\chi}{d\phi}\mathbf{e}_\phi. \tag{103}$$

Therefore, $d\chi/d\phi = 1$ and one can take $\chi = \phi$.

[62] For the axisymmetric equilibrium configuration, one can Fourier analyze with respect to ϕ

$$\boldsymbol{\xi}(\psi,\chi,\zeta) = \boldsymbol{\xi}(\psi,\zeta)\exp(-im\phi), \tag{104}$$

with m being the azimuthal mode number. The ϕ dependence is explicitly accounted for in the interchange mode analysis by writing

$$S(\psi,\chi) = S(R,\phi) = -m\phi + \tilde{S}(\psi). \tag{105}$$

No other ϕ dependence appears in the analysis. Therefore, $X = X(\psi)$. In the dipole coordinate, $\nabla\psi$ and $\nabla\chi$ are orthogonal. Therefore, from equation (78) one obtains

$$\mathbf{k}_\perp^2 = \left(\frac{\mu_0 M}{4\pi R^2}\right)^2\left(\frac{d\tilde{S}}{d\psi}\right)^2 + \frac{m^2}{R^2}. \tag{106}$$

Substitution of equation (106) into equation (89) yields

$$\begin{aligned} W'(\psi,\chi) = |\hat{X}|^2\Bigg[&m^2\frac{\mu_0}{U}\frac{dU}{d\psi}\left(\gamma p\frac{dU}{d\psi}+U\frac{dp}{d\psi}\right) \\ &-\frac{2}{R_I B_I}\left(\left(\frac{\mu_0 M}{4\pi R^2}\right)^2\left(\frac{d\tilde{S}}{d\psi}\right)^2+\frac{m^2}{R^2}\right)\Bigg]. \end{aligned} \tag{107}$$

Therefore, for $m = 0$ mode, one obtains

$$W'(\psi,\chi) = |\hat{X}|^2\left[-\frac{2}{R_I B_I}\left(\frac{\mu_0 M}{4\pi R^2}\right)^2\left(\frac{d\tilde{S}}{d\psi}\right)^2\right] < 0. \tag{108}$$

Since $W' < 0$ is the condition for interchange instability, this inequality means that the $m = 0$ mode is unconditionally unstable.

[63] When m is nonzero, the stability criterion $W' \geq 0$ becomes

$$\frac{\mu_0}{U}\frac{dU}{d\psi}\left(\gamma p\frac{dU}{d\psi}+U\frac{dp}{d\psi}\right)-\frac{2}{R_I B_I R^2}\left[1+\frac{R^2}{m^2}\left(\frac{d\tilde{S}}{dR}\right)^2\right]\geq 0. \tag{109}$$

After some calculation, this inequality is reduced to

$$\gamma\frac{d\ell nU}{dR}+\frac{d\ell np}{dR}-\left(\frac{\mu_0 M}{4\pi R^3}\right)^2\frac{1}{\mu_0 p}\frac{2}{R_I B_I}\frac{dR}{dU}\left[1+\frac{R^2}{m^2}\left(\frac{d\tilde{S}}{dR}\right)^2\right]\geq 0. \tag{110}$$

From equations (92) and (93) one obtains by neglecting terms of $O(\frac{1}{L^4})$ and smaller

$$\frac{d\ell nU}{dR}\simeq\frac{4}{R}. \tag{111}$$

Substituting equation (111) and B_I from equation (90) into equation (110) yields

$$-\frac{d\ell np}{d\ell nR}+\frac{R}{2R_E\beta_{eq}}\frac{\cos^6\Phi_0}{\sqrt{1+3\sin^2\Phi_0}f_0}\left[1+\frac{R^2}{m^2}\left(\frac{d\tilde{S}}{dR}\right)^2\right]\leq 4\gamma, \tag{112}$$

where β_{eq} is the plasma β in the equatorial plane at $R = LR_E$ and $f_0 = 16/35$, for stability.

[64] The first term on the left hand side of equation (112) comes from the pressure-driven destabilizing term and the second term represents the ionosphere-driven destabilizing term. The right hand side of equation (112) represents the stabilizing contribution by compressibility.

[65] When

$$\frac{d\ell np}{d\ell nR}+4\gamma<0, \tag{113}$$

the stability criterion (112) is not satisfied for any β_{eq}. Therefore, the system is unstable. When the plasma pressure distribution has an inverse power law distribution such as $p \propto R^{-q}$, $d\ell np/d\ell nR = -q$. Therefore, this instability criterion means that the $q > 4\gamma$ case is unstable. This is consistent with the intuitive argument of *Gold* [1959]. However, unlike *Gold* [1959], the $q = 4\gamma$ case could also be unstable, since the ionosphere gives additional destabilizing contribution.

[66] When

$$\frac{d\ell np}{d\ell nR}+4\gamma=-q+4\gamma>0, \tag{114}$$

the magnetosphere is not interchange unstable by pressure-driven mechanism. For this case, the stability criterion (112) becomes

$$\beta_{eq}\geq\beta_{cr}=\frac{1}{4\gamma+\frac{d\ell np}{d\ell nR}}\frac{L}{2}\frac{\cos^6\Phi_0}{\sqrt{1+3\sin^2\Phi_0}f_0}\left[1+\frac{R^2}{m^2}\left(\frac{d\tilde{S}}{dR}\right)^2\right]. \tag{115}$$

By neglecting terms of $O(\frac{1}{L^4})$ and smaller, β_{cr} is further reduced to

$$\beta_{cr}=\frac{1}{16\gamma f_0(1-q/(4\gamma))L^2}\left(1-\frac{3}{8L}-\frac{9}{128L^2}-\frac{27}{1029L^3}\right)^{-1}\cdot\left[1+\frac{R^2}{m^2}\left(\frac{d\tilde{S}}{dR}\right)^2\right]. \tag{116}$$

Therefore, when the plasma β at the equator is larger than or equal to some critical β (β_{cr}), which is given by the right hand side of equation (115), the magnetospheric plasma is stable.

[67] However, when $\beta_{eq} < \beta_{cr}$, the magnetosphere is unstable for interchange perturbations because of the destabilizing ionospheric contribution. Therefore, when $4\gamma - q > 0$, $\beta_{eq} < \beta_{cr}$ is the interchange instability criterion. When $\frac{d\ell np}{d\ell nR} + 4\gamma > 0$ and $\beta_{eq} < \beta_{cr}$, $\delta W_I < 0$, $\delta W_F > 0$ and $\delta W_F + \delta W_I < 0$. Therefore, all the energy to destabilize the magnetospheric interchange perturbation satisfying $X(\psi, \chi, \ell) = \hat{X}(\psi - \psi_0, \chi - \chi_0)$ comes from the ionosphere. Therefore, the unstable interchange perturbation for $\frac{d\ell np}{d\ell nR} + 4\gamma > 0$ and $\beta_{eq} < \beta_{cr}$ is different from the normal pressure-driven interchange perturbation, in which $\delta W_F < 0$. Therefore, to differentiate this mode, which is driven by ionospheric potential energy $\delta W_I < 0$, from the normal pressure-driven interchange mode, this mode is appropriately called ionosphere-driven interchange mode. Therefore, ionosphere-driven interchange instability occurs for low-β plasma. Notice that unlike pressure-driven interchange instability for $q > 0$, the ionosphere-driven interchange instability occurs even for $q < 0$ or $dp/dR > 0$ when $\beta_{eq} < \beta_{cr}$ is satisfied.

7.2. Nonaxisymmetric and North-South Symmetric Magnetospheric Model

[68] The magnetospheric energy principle gives the general expression of $W'(\psi, \chi)$ for arbitrary finite-β and nonaxisymmetric magnetospheric models represented by $p(\psi, \chi)$ and $U(\psi, \chi)$. Therefore, the stability criterion for this case becomes

$$\frac{\mu_0}{U(1+\gamma p\mu_0\langle 1/B^2\rangle)}\left[\frac{\partial S}{\partial\psi}\left(\mu_0 U\left\langle\frac{1}{B^2}\right\rangle\frac{\partial p}{\partial\chi}-\frac{\partial U}{\partial\chi}\right)+\frac{\partial S}{\partial\chi}\left(\frac{\partial U}{\partial\psi}-\mu_0 U\left\langle\frac{1}{B^2}\right\rangle\frac{\partial p}{\partial\psi}\right)\right]\left[\frac{\partial S}{\partial\chi}\left(\gamma p\frac{\partial U}{\partial\psi}+U\frac{\partial p}{\partial\psi}\right)-\frac{\partial S}{\partial\psi}\left(\gamma p\frac{\partial U}{\partial\chi}+U\frac{\partial p}{\partial\chi}\right)\right]-\frac{2\mathbf{k}_\perp^2}{R_I B_I}\geq 0. \tag{117}$$

8. Discussion

8.1. Realistic Evaluation of the Criterion for Ionosphere-Driven Interchange Instability in an Axisymmetric, North-South Symmetric and Low-β Magnetospheric Model

[69] In order to evaluate the ionosphere-driven interchange instability criterion $\beta_{eq} < \beta_{cr}$ for $m \neq 0$ and $4\gamma - q > 0$, where β_{cr} is given by equation (116), one needs to calculate $(d\tilde{S}/dR)^2$. Since the variation of $\tilde{S}$ with R is considered to be very rapid in the present $\mathbf{k}_\perp \to \infty$ ap-

proximation and for a realistic evaluation of the criterion one needs to calculate the order of magnitude of $d\tilde{S}/dR$, let us simply assume

$$\tilde{S} \simeq k_R R = \frac{2\pi}{\lambda_R} R \tag{118}$$

instead of trying to obtain $\tilde{S}(R)$ accurately.

[70] From the original assumption of $|a\nabla S| \gg 1$, λ_R must be smaller than the perpendicular inhomogeneity scale length a in the R direction. If one assumes $\lambda_R \sim 1\ R_E$ on the basis of the assumption of a being a few R_E, substitution of equation (118) into $\beta_{eq} < \beta_{cr}$ yields

$$\beta_{eq} < \frac{1}{16\gamma f_0}\left(\frac{2\pi}{m}\right)^2 \frac{1}{1-q/(4\gamma)}\left(1-\frac{3}{8L}-\frac{9}{128L^2}-\frac{27}{1029L^3}\right)^{-1} \cdot \left[1+\left(\frac{m}{2\pi L}\right)^2\right]. \tag{119}$$

Notice that for $L \gg 1$ and $O(m) \sim 1$, the right hand side of equation (119) has only a small dependence on L. For the adiabatic case (γ = 5/3) and Φ_0 = 60° (L = 4), this ionosphere-driven interchange instability criterion for low β magnetospheric plasma becomes

$$\beta_{eq} < \frac{3.64}{m^2}\frac{1}{1-3q/20}\left[1+\left(\frac{m}{8\pi}\right)^2\right]. \tag{120}$$

[71] For $|q| \ll 4\gamma = 20/3$ and $m = 1$, $\beta_{eq} < 3.65$. For $|q| \ll 4\gamma = 20/3$ and $m = 2$, $\beta_{eq} < 0.916$. For $|q| \ll 4\gamma = 20/3$ and $m = 3$, $\beta_{eq} < 0.410$. Since the dipole field is assumed, the present ionosphere-driven interchange instability criterion is considered to be applicable to the inner magnetosphere covering the plasmasphere, radiation belts and the ring current region. Notice that for Φ_0 = 45°, 60°, 70°, L = 2, 4, 8.5, respectively, and the angles Θ between the unperturbed magnetic field vector and the horizontal ionospheric plane, which are given by $\tan(\pi/2 - \Theta) = (2\tan\Phi_0)^{-1}$, are 63°, 74°, 80°, respectively. Therefore, for $L > 2$ the assumption of the normal incidence of the unperturbed magnetic field on the ionospheric surface in the magnetospheric energy principle may well be justified.

[72] It is known that the quiet time radial profile of the pressure p is peaked around $L \sim 3$. However, it is very improbable that with increasing radius measured in the equatorial plane, the pressure diminishes more rapidly than $R^{-20/3}$ in the $q > 0$ or $dp/dR < 0$ region in the inner magnetosphere, which lies typically in $L > 3$. Therefore, by assuming $|q| \ll 4\gamma = 20/3$ in this region, one can apply the above criterion (120). In the inner magnetosphere in $L > 3$, $\beta_{eq} < 3.65$ is easily satisfied and $\beta_{eq} < 0.916$ may at times be satisfied, but $\beta_{eq} < 0.41$ may not be satisfied, One conjectures, therefore, that the ionosphere-driven interchange mode with m = 1 or 2 would be destabilized in the inner magnetosphere even if the inverse power law index q is much smaller than the critical value of 20/3 for pressure-driven interchange instability. Notice that even for $q < 0$ or $dp/dR > 0$, ionosphere-driven interchange instability occurs when $\beta_{eq} < \beta_{cr}$, although β_{cr} is reduced from that for $0 < q \ll 4\gamma$. This may suggest that the $q < 0$ or $dp/dR > 0$ region in $L < 3$ is also unstable against ionosphere-driven interchange instability. Therefore, while pressure-driven interchange instability [*Gold*, 1959] requires $q > 4\gamma$, which is not easily satisfied in the inner magnetosphere, the instability condition for ionosphere-driven interchange instability is easily satisfied in the inner magnetosphere. Thus, ionosphere-driven interchange instability may cause magnetohydrodynamic disturbances in the inner magnetosphere.

[73] One notes that the m = 0 ionosphere-driven mode is unconditionally unstable. Note that the m = 0 mode has $\mathbf{k}_\perp$ parallel to $\nabla\psi$, and hence $\boldsymbol{\eta}_{\perp 0}$ is parallel to the direction of $\nabla\phi$. Since one has not calculated growth rates of different m modes in the present magnetospheric energy principle, it is difficult to determine which mode is dominant. However, if one considers that the real magnetosphere is nonaxisymmetric, the m = 0 mode would be strongly influenced by such a nonaxisymmetry. Therefore, one conjectures that m = 1 or m = 2 mode would survive in the presence of nonaxisymmetry and m = 1 or m = 2 mode would be a dominant mode.

[74] The ionosphere-driven interchange instability may be viable even in a high-β region such as the near-Earth tail at quiet times. However, a realistic evaluation of the upper critical β value for such a high-β region is difficult, because an accurate high-β magnetospheric model must be used to obtain the criterion. Therefore, it is only suggested in this study that the ionosphere-driven interchange instability in such a region would be important when the equatorial plasma β remains small and the magnetic field remains dipole-like such as before the growth phase or in the recovery phase of a substorm. In the near-Earth high-β region, the ionosphere-driven interchange instability would be quenched as the plasma β increases in the equatorial plane with progress of the growth phase.

8.2. Relevance to Previous Stability Analyses Without Ionospheric Destabilizing Contribution

[75] Using a local Cartesian coordinate system, *Volkov and Mal'tsev* [1986] performed a stability analysis of pressure-driven interchange instability when there is an unperturbed field-aligned current in the z direction in a finite-β plasma. They assumed a perturbation with the form $\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$, where $\mathbf{k} = k_x\hat{x} + k_y\hat{y}$. They assumed an unperturbed field-aligned current J_z

$$J_z = 2\mathbf{e}_z \cdot (\nabla V \nabla p), \tag{121}$$

where $\mathbf{e}_z$ is the unit vector along the magnetic field and

$$V = \int_0^{\ell_N} \frac{dz}{B}, \tag{122}$$

where the integration is from the equator (ℓ = 0) to the ionosphere ($\ell = \ell_N$) in the Northern Hemisphere. Since north-south symmetry is assumed in their model, $V = U/2$. Equation (121) becomes

$$J_z = 2\left(\frac{\partial p}{\partial y}\frac{\partial V}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial V}{\partial y}\right). \tag{123}$$

They assumed that the field-aligned current is closed by Pedersen current in the ionosphere.

[76] Their growth rate ω_i (imaginary part of ω) of the instability is given by

$$\omega_i = -\frac{2Vp}{\Sigma_p B_i k^2(1+\beta\gamma/2)}\left[\mathbf{k}\times\nabla\ell n\left(p^{-\frac{\beta}{2}}V\right)\right]\cdot\left[\mathbf{k}\times\nabla\ell n(pV^{\gamma})\right], \tag{124}$$

where B_i is the magnetic field strength at the ionosphere and

$$\beta = \frac{1}{V}\int_0^{\ell_N}\frac{2\mu_0 p}{B^2}\frac{dz}{B} = \left\langle\frac{2\mu_0 p}{B^2}\right\rangle = 2\mu_0 p\left\langle\frac{1}{B^2}\right\rangle \tag{125}$$

is the average plasma β along the flux tube. It is straightforward from equation (124) to show that

$$\begin{aligned}\omega_i = & -\frac{2}{\Sigma_p B_i k^2(1+\mu_0\gamma p\langle 1/B^2\rangle)}\left[k_x\left(\frac{\partial V}{\partial y}-\mu_0 V\left\langle\frac{1}{B^2}\right\rangle\frac{\partial p}{\partial y}\right)\right.\\ & \left.-k_y\left(\frac{\partial V}{\partial x}-\mu_0 V\left\langle\frac{1}{B^2}\right\rangle\frac{\partial p}{\partial x}\right)\right]\left[k_x\left(\frac{\partial p}{\partial y}+\gamma p V^{-1}\frac{\partial V}{\partial y}\right)\right.\\ & \left.-k_y\left(\frac{\partial p}{\partial x}+\gamma p V^{-1}\frac{\partial V}{\partial x}\right)\right].\end{aligned} \tag{126}$$

[77] On the other hand, when the ionospheric destabilizing contribution to $W'(\psi, \chi)$ is neglected, $W'(\psi, \chi) = W(\psi, \chi)$ in equation (85) can be written as

$$\begin{aligned}W(\psi,\chi) = |\hat{X}|^2 & \frac{\mu_0}{1+\mu_0\gamma p\langle 1/B^2\rangle}\left[\frac{\partial S}{\partial\psi}\left(\frac{\partial U}{\partial\chi}-\mu_0 U\left\langle\frac{1}{B^2}\right\rangle\frac{\partial p}{\partial\chi}\right)\right.\\ & \left.-\frac{\partial S}{\partial\chi}\left(\frac{\partial U}{\partial\psi}-\mu_0 U\left\langle\frac{1}{B^2}\right\rangle\frac{\partial p}{\partial\psi}\right)\right]\left[\frac{\partial S}{\partial\psi}\left(\frac{\partial p}{\partial\chi}+\gamma p U^{-1}\frac{\partial U}{\partial\chi}\right)\right.\\ & \left.-\frac{\partial S}{\partial\chi}\left(\frac{\partial p}{\partial\psi}+\gamma p U^{-1}\frac{\partial U}{\partial\psi}\right)\right].\end{aligned} \tag{127}$$

By changing $\psi \to x$ and $\chi \to y$ and assuming the lowest-order approximation $\partial S/\partial\psi \to k_x$ and $\partial S/\partial\chi \to k_y$ in equation (127), one sees that $W(\psi, \chi)$ is proportional to $-\omega_i$. The pressure-driven interchange instability criterion is $W < 0$. This corresponds to $\omega_i > 0$ and hence to an unstable perturbation in the stability analysis. Therefore, although the magnitude of the growth rate is inversely proportional to Σ_p in equation (126), the instability criterion is the same in both equations (126) and (127). That is, there is agreement concerning the instability condition between the perturbation stability analysis with the closure of unperturbed currents via Pedersen current and the criterion derived from the magnetospheric energy principle.

[78] For a standard pressure-driven interchange instability envisaged by *Gold* [1959], $p = p(\psi)$ and $U = U(\psi)$. Therefore, there is no unperturbed field-aligned current. From equation (127) it is obvious that when there is an unperturbed field-aligned current and hence when p or U is also a function of χ, $\partial p/\partial\chi$ and $\partial U/\partial\chi$ play the same roles in equation (127) as $\partial p/\partial\psi$ and $\partial U/\partial\psi$. Therefore, just as a combined effect of $\partial p/\partial\psi$ and κ_ψ causes a pressure-driven interchange instability, a combined effect of $\partial p/\partial\chi$ and κ_χ also causes a pressure-driven interchange instability. Thus, a general pressure-driven interchange instability involves field line curvature both in the meridional plane and in the plane parallel to the longitudinal direction. The pressure-driven interchange instability considered by *Gold* [1959], which is caused by the combined effect of $dp/d\psi$ and κ_ψ, is a special case of such a general pressure-driven interchange instability.

8.3. Relevance to Magnetospheric Interchange Instability When the Unperturbed Field-Aligned Current is Closed Via Pedersen Current

[79] In the real magnetosphere an unperturbed field-aligned current generated in the magnetosphere is more likely to be closed via conduction currents such as Pedersen currents in the ionosphere. Equation (84), which is derived from equation (5), gives $J_\|$ at the ionosphere. This relation has also been derived previously [*Vasyliunas*, 1970]. When $J_\|$ at the ionosphere is assumed to close via Pedersen current in the ionosphere, it is driven by an unperturbed electric field. In such a case the static pressure balance equation $\mathbf{J}\times\mathbf{B} = \nabla p$ is no longer valid in the magnetosphere, since the electric field sets the magnetospheric plasma in motion. Therefore, the actual magnetospheric equilibrium state in such a case must be determined by solving

$$\rho(\mathbf{V}\cdot\nabla)\mathbf{V} = \mathbf{J}\times\mathbf{B} - \nabla\mathbf{p}, \tag{128}$$

where $\mathbf{V}$ is the unperturbed flow velocity equal to the $\mathbf{E}\times\mathbf{B}$ drift velocity. This equilibrium state is not a steady state, because it decays with a decay time constant τ_d, which is given by *Miura* [1996] by

$$\left(\frac{\tau_d}{\tau_A}\right)^{-1} = \frac{2\mu_0\Sigma_p V_A}{1+\left(\mu_0\Sigma_p V_A\right)^2}, \tag{129}$$

where $\tau_A = 2\ell/V_A$ is the Alfvén transit time with 2ℓ being the field line length from the ionosphere in one hemisphere to the ionosphere in the opposite hemisphere and V_A being the average Alfvén speed in the magnetosphere.

[80] The horizontally free ionospheric boundary condition necessary for magnetospheric interchange instability means that the field line has a finite horizontal displacement on the spherical ionospheric surface. This means that Σ_p is very small, since otherwise the field line is more or less tied to the ionosphere and cannot move freely in the horizontal direction. In the small Σ_p limit, one obtains from equation (129)

$$\frac{\tau_d}{\tau_A} \simeq \frac{1}{2\mu_0\Sigma_p V_A}. \tag{130}$$

[81] In order for the unperturbed state to remain a steady state, $\tau_d \gg \tau_A$ is necessary. Therefore, $\mu_0 V_A\Sigma_p \ll 1$ is necessary. Thus, the present analysis of the magnetospheric interchange instability based on the magnetospheric energy principle would be applicable to the real magnetosphere, when Σ_p is small or $\mu_0 V_A\Sigma_p \ll 1$. This means that the present analysis would be more applicable to the nightside magnetosphere, since Σ_p in the nightside is smaller than the dayside.

[82] For a typical nightside quiet ionosphere in the high latitude one may have $\Sigma_p \simeq 0.5$ mho and an average Alfvén speed $V_A \simeq 500$ km/s. Therefore, one has $\mu_0 V_A\Sigma_p \simeq 0.31$. This may validate $\mu_0 V_A\Sigma_p \ll 1$, which is necessary for the application of the present analysis. However, the ring current and the near-Earth plasma sheet in the nightside are more likely to be mapped onto the auroral oval, where

conductivity is greatly enhanced by energetic particle precipitation. The present analysis is based on the magnetospheric energy principle, which assumes no unperturbed flow and hence no unperturbed electric field. Therefore, in applying it to vaious regions of the real magnetosphere, the realistic evaluation of $\mu_0 V_A \Sigma_P$ seems to be important.

8.4. Closure of Current Perturbations

[83] Taking the perturbation of Ampère's law $\nabla \times \mathbf{B}_1 = \mu_0 \mathbf{J}_1$, one obtains

$$\mathbf{J}_{1\perp} = \mu_0^{-1}(\nabla \times \mathbf{B}_{1\perp}) + \mu_0^{-1} B_{1\|}\mathbf{b} \times \boldsymbol{\kappa} - \mu_0^{-1}\mathbf{b} \times \left(\nabla B_{1\|}\right) \quad (131)$$

$$J_{1\|} = \mathbf{b} \cdot \mathbf{J}_1 = \mu_0^{-1}\mathbf{b} \cdot \nabla \times \mathbf{B}_{1\perp} + \mu_0^{-1} B_{1\|}\mathbf{b} \cdot (\nabla \times \mathbf{b}). \quad (132)$$

From $\nabla \cdot \mathbf{B}_1 = 0$, one also obtains

$$\nabla \cdot \mathbf{B}_{1\perp} - B_{1\|}(\mathbf{b} \cdot \nabla \ell n B) + \mathbf{b} \cdot \nabla B_{1\|} = 0. \quad (133)$$

In the magnetosphere, $\mathbf{B}_{1\perp}$ is zero, but $\mathbf{B}_{1\perp}$ is finite at the ionosphere (see Appendix B). Therefore, from equation (131) there is a nonzero $\mathbf{J}_{1\perp}$ at the ionosphere. Since there is a jump of $\mathbf{B}_{1\perp}$ at the ionosphere, the first term of equation (131) and hence $\mathbf{J}_{1\perp}$ shows a δ function-like behavior at the ionosphere. This peculiar distribution of $\mathbf{J}_{1\perp}$ seems to be unavoidable because of the assumption of an infinitely thin ionosphere and the normal incidence of the unperturbed magnetic field on the ionospheric surface in the magnetospheric energy principle [*Miura*, 2007].

[84] From the perturbed equation of motion (equation (12) of *Miura* [2007]), one obtains in the ideal ionosphere

$$\mathbf{J}_{1\perp} = -B^{-1}\omega^2 \rho \mathbf{b} \times \boldsymbol{\xi}_\perp - B^{-2}\left(B_{1\|}\mathbf{J}_\perp - J_\|\mathbf{B}_{1\perp}\right) - B^{-1}\mathbf{b} \times \nabla(\boldsymbol{\xi} \cdot \nabla p + \gamma p \nabla \cdot P\boldsymbol{\xi}), \quad (134)$$

where the first term represents the inertia current and the last term represents the perturbed diamagnetic current. Thus, the ionospheric surface current $\mathbf{J}_{1\perp}$ given by equation (131) can be provided by equation (134) in the ideal ionosphere.

[85] In the actual ionosphere, where there are also neutral components, the perpendicular current perturbation for constant plasma density and stationary neutral components is usually expressed by

$$\mathbf{J}_{1\perp} = \sigma_P \mathbf{E}_1 - \sigma_H \mathbf{E}_1 \times \mathbf{b}, \quad (135)$$

where σ_P and σ_H are Pedersen and Hall conductivities, respectively. For the typical E-layer ionosphere in the high latitude, one has $\sigma_P \simeq n_0 e^2/(m_i \nu_{\text{in}})$ and $\sigma_H \simeq n_0 e/B$, where ν_{in} is the ion neutral collision frequency and m_i is the ion mass. When σ_P and σ_H are height integrated, they yield Σ_P and Σ_H. It is obvious that even in the ideal MHD ionosphere, where there are no conduction currents given by equation (135), there are perpendicular currents given by equation (134), which help close the current required by Ampère's law.

9. Summary and Conclusion

[86] A general criterion for magnetospheric interchange instability, which does not bend magnetic fields in the magnetosphere, is derived for an arbitrary finite-β magnetospheric model satisfying the magnetohydrostatic force balance. The derivation is based on the magnetospheric energy principle [*Miura*, 2007], the only assumption of which is that the unperturbed magnetic field is incident vertically on the spherical ionospheric surface. The criterion includes the effect of an unperturbed field-aligned current, which exists in finite-β nonaxisymmetric magnetospheric models, and the ionospheric destabilizing contribution caused by a finite horizontal plasma displacement on the spherical ionospheric surface. The unperturbed field-aligned current is assumed to close via diamagnetic currents in the magnetosphere or in the ionosphere, so that the ideal MHD and the magnetospheric energy principle are applicable.

[87] By exploiting the $\mathbf{k}_\perp \to \infty$ limit and thus using the eikonal representation for $\boldsymbol{\xi}_\perp$, it is shown that the magnetospheric interchange mode is compressible. Using the horizontally free ionospheric boundary condition for compressible perturbations, the explicit form of δW_F is calculated by using magnetospheric flux coordinates. By choosing $X(\psi, \chi, \zeta) = \hat{X}(\psi - \psi_0, \chi - \chi_0)$ and thus assuming no line bending in the magnetosphere ($\mathbf{Q}_\perp = \mathbf{B}_{1\perp} = 0$), δW_F is further reduced for interchange perturbations. In the $\mathbf{k}_\perp \to \infty$ limit the kink mode makes no contribution to δW_F.

[88] For arbitrary magnetospheric models, the general stability condition for interchange instability becomes $\delta W = \delta W_F + \delta W_I \geq 0$, where δW is given by equation (81) and δW_I is the ionospheric contribution. Here, δW_I is negative when $\boldsymbol{\xi}_\perp \neq 0$ at the ionosphere and thus the ionosphere gives a destabilizing contribution. Notice that when there is north-south symmetry, this stability can be tested one magnetic line at a time and the stability condition can be written as $W'(\psi, \chi) \geq 0$, where $W'(\psi, \chi)$ is given by equation (85). The general stability criterion is valid for arbitrary finite-β and nonaxisymmetric magnetospheric models and is not restricted to any particular magnetospheric models. This general stability criterion shows that in the general pressure-driven interchange instability a combined effect of $\partial p/\partial\psi$ and κ_ψ or a combined effect of $\partial p/\partial\chi$ and κ_χ destabilizes pressure-driven interchange instability.

[89] If one specialises to an axisymmetric finite-β magnetospheric model in the absence of ionospheric contribution, the stability criterion tested for one magnetic field line becomes similar to the stability criterion derived by *Spies* [1971] and *Hameiri et al.* [1991].

[90] If one further specialises to an axisymmetric, north-south symmetric and low-β magnetospheric model, in which the magnetic field is approximated by a dipole field, a stability criterion by *Gold* [1959], i.e., $q < 4\gamma$ is recovered by neglecting terms of $O(\frac{1}{L^4})$ and smaller and the ionospheric destabilizing contribution. Furthermore, for this axisymmetric, north-south symmetric and low-β magnetospheric model, the existence of ionosphere-driven interchange mode is shown, when the ionospheric destabilizing contribution is included in the criterion for instability. The $m = 0$ ionosphere-driven interchange mode is unconditionally unstable. For $|q| \ll 4\gamma$, thus for a stable case in the pressure-driven mechanism, the $m = 1$ or $m = 2$ ionosphere-driven mode has an upper ciritical equatorial β value for the instability in the order of 1 for a reasonble parameter set. Thus, such a mode with $m = 1$ or $m = 2$ would be a viable instability in the inner magnetosphere, where the magnetic

field remains dipole-like. Unlike the pressure-driven interchange mode for $q > 0$, the ionosphere-driven interchange mode becomes unstable even for $q < 0$ or $dp/dR > 0$, which may occur in the quiet time in $L < 3$, when $\beta_{eq} < \beta_{cr}$ is satisfied.

[91] When one specializes to a local Cartesian coordinate system and when the ionospheric destabilizing contribution is neglected and there is no translational symmetry in a finite-β magnetospheric plasma, the expression for the lowest-order approximation of W given by equation (127) becomes proportional to $-\omega_i$. Here, ω_i is derived from a perturbation analysis in a local Cartesian coordinate system [*Volkov and Mal'tsev*, 1986], in which the unperturbed field-aligned current is assumed to close via Pedersen current. Therefore, for such a case the stability criterion derived from the magnetospheric energy principle becomes the same as the stability criterion derived from a perturbation analysis in a local Cartesian coordinate system with the closure of unperturbed field-aligned currents via Pedersen current.

[92] The general stability criterion for magnetospheric interchange instability derived in the present study provides a framework for the stability analysis of interchange modes in realistic arbitrary finite-β magnetospheric equilibria. It is very improbable that in the $q > 0$ or $dp/dR < 0$ region in the inner magnetosphere the plasma pressure diminishes more rapidly than $R^{-20/3}$ and thus pressure-driven interchange instability may not be easily destabilized. However, a substantial region of the inner magnetosphere or the near-Earth magnetosphere may be unstable against the ionosphere-driven interchange instability caused by a horizontal plasma displacement on the spherical ionospheric surface.

Appendix A: Physical Derivation of Ideal Ionospheric Boundary Conditions

[93] The physical results in the present study are obtained for a specific choice of boundary conditions given in equations (12) to (15). Therefore, the understanding of how those ionospheric boundary conditions are derived is essential for understanding the physical results in the present study. Although those ideal ionospheric boundary conditions are derived from the requirement of the self-adjointness of the force operator by *Miura* [2007], the self-adjointness of the force operator is equivalent to the energy conservation of the system under consideration. Therefore, in this appendix, those ideal ionospheric boundary conditions are derived directly from the requirement of energy conservation, that is, the conservation of $H(t) = K(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_1) + \delta W(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}) = K + \delta W_F + \delta W_I$ in the system of the magnetosphere and the ideal ionosphere, where $K(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_1) = 1/2\int_P \rho\tilde{\mathbf{v}}_1^2 d\mathbf{r}$ and $\delta W(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}) = -1/2\int_P \tilde{\boldsymbol{\xi}} \cdot \mathbf{F}(\tilde{\boldsymbol{\xi}}) d\mathbf{r}$. In the following, K, δW, δW_F, and δW_I denote $K(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_1)$, $\delta W(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}})$, $\delta W_F(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}})$, and $\delta W_I(\tilde{\boldsymbol{\xi}}_\perp, \tilde{\boldsymbol{\xi}}_\perp)$, respectively. The mathematical details of the calculation are described in Appendix B of *Miura* [2007].

[94] Notice that, contrary to the notation in the main text, in this appendix the subscript 0 is added explicitly to the unperturbed quantity in order to avoid confusion. Subscripts 1 and 2 denote the first-order linear perturbation and the second-order linear perturbation, respectively, and the tilde on the perturbation means that the calculation is done in a real time domain and the perturbation is a function of position $\mathbf{r}$ and time t.

[95] One starts from a rigorous local energy conservation equation of ideal MHD describing the time evolution of total energy at any point inside the magnetosphere and the ideal ionosphere

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{v}^2 + \frac{p}{\gamma-1} + \frac{B^2}{2\mu_0}\right) = -\nabla\cdot\left[\left(\frac{1}{2}\rho\mathbf{v}^2 + p + \frac{p}{\gamma-1}\right)\mathbf{v} + \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right], \tag{A1}$$

where, contrary to the notation used in the main text, ρ, p, B, $\mathbf{v}$, $\mathbf{E}$, and $\mathbf{B}$ are all total quantities, which are functions of the position $\mathbf{r}$ and time t and not unperturbed quantities.

[96] Taking the second-order perturbation of (A1) and then integrating the resultant equation over the unperturbed plasma volume P, one obtains

$$\frac{\partial}{\partial t}\int_P\left(\frac{1}{2}\rho_0\tilde{\mathbf{v}}_1^2 + \tilde{w}_2\right)d\mathbf{r} = -\int_S \tilde{\mathbf{u}}_2\cdot\mathbf{n}dS, \tag{A2}$$

where S is the unperturbed surface surrounding the unperturbed plasma volume P, $\tilde{w}_2$ is the sum of the second-order perturbations of internal energy and magnetic energy, i.e.,

$$\tilde{w}_2 = \frac{\tilde{p}_2}{\gamma-1} + \frac{1}{2\mu_0}\left(2\mathbf{B}_0\cdot\tilde{\mathbf{B}}_2 + \tilde{\mathbf{B}}_1^2\right) \tag{A3}$$

and $\tilde{\mathbf{u}}_2$ is the second-order perturbation of the energy flux density, i.e.,

$$\tilde{\mathbf{u}}_2 = \frac{\gamma}{\gamma-1}(\tilde{p}_1\tilde{\mathbf{v}}_1) - \frac{1}{\mu_0}\left[\left(\tilde{\mathbf{v}}_1\times\tilde{\mathbf{B}}_1\right)\mathbf{B}_0 + (\tilde{\mathbf{v}}_1\times\mathbf{B}_0)\times\tilde{\mathbf{B}}_1\right]. \tag{A4}$$

Since $\tilde{\boldsymbol{\xi}}_\perp = 0$ on $S_{\rm out}$ and $S_{\rm in}$, $\tilde{\mathbf{u}}_2\cdot\mathbf{n} = 0$ on $S_{\rm out}$ and $S_{\rm in}$, where $\mathbf{n}$ is the outward normal vector on $S_{\rm out}$ and $S_{\rm in}$. Therefore, only the integral over the ionospheric surface contributes to the right hand side of equation (A2). Owing to the assumption of normal incidence of the unperturbed magnetic field on the ionospheric surface, i.e., $\mathbf{n} = \mathbf{b}$ on the ionosphere of the Northern Hemisphere, one obtains from equation (A4)

$$\begin{aligned}\tilde{\mathbf{u}}_2\cdot\mathbf{n} &= -\frac{\gamma}{\gamma-1}\left(\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla p_0 + \gamma p_0\nabla\cdot\tilde{\boldsymbol{\xi}}\right)\tilde{v}_{1\parallel} + \tilde{\mathbf{s}}_2\cdot\mathbf{n}\\ &= -\frac{\gamma}{\gamma-1}\left(\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla p_0 + \gamma p_0\nabla\cdot\tilde{\boldsymbol{\xi}}\right)\tilde{v}_{1\parallel} - \frac{B_0^2}{\mu_0}\Big[\tilde{\mathbf{v}}_{1\perp}\cdot\left((\mathbf{b}\cdot\nabla)\tilde{\boldsymbol{\xi}}_\perp\right)\\ &\quad - \tilde{\mathbf{v}}_{1\perp}\cdot\left(\left(\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla\right)\mathbf{b}\right)\Big],\end{aligned} \tag{A5}$$

where $\tilde{\mathbf{s}}_2$ is the second-order Poynting vector (see Appendix B of *Miura* [2007] for details).

[97] Since $(\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla)\mathbf{b}$ is equal to $-\tilde{\boldsymbol{\xi}}_\perp/R_I$ on the ionospheres of the Northern Hemisphere owing to the assumption of $\mathbf{n} = \mathbf{b}$ (see Appendix A of *Miura* [2007])

$$\tilde{\mathbf{s}}_2\cdot\mathbf{n} = -\frac{B_0^2}{\mu_0}\tilde{\mathbf{v}}_{1\perp}\cdot\left((\mathbf{b}\cdot\nabla)\tilde{\boldsymbol{\xi}}_\perp\right) - \frac{1}{2\mu_0}\frac{B_0^2}{R_I}\frac{\partial}{\partial t}\tilde{\boldsymbol{\xi}}_\perp^2. \tag{A6}$$

Therefore, one obtains from equation (A2)

$$\frac{\partial}{\partial t}(K + \delta W_F + \delta W_I) = -\int_S \tilde{\mathbf{u}}_2' \cdot \mathbf{n} dS, \quad \text{(A7)}$$

where $\delta W_F \equiv \int_P \tilde{w}_2 d\mathbf{r}$ and

$$\begin{aligned}\tilde{\mathbf{u}}_2' \cdot \mathbf{n} &= -\frac{\gamma}{\gamma - 1}\left(\tilde{\boldsymbol{\xi}}_\perp \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \tilde{\boldsymbol{\xi}}\right)\tilde{v}_{1\parallel} - \frac{B_0^2}{\mu_0}\left[\tilde{\mathbf{v}}_{1\perp} \cdot \left((\mathbf{b} \cdot \nabla)\tilde{\boldsymbol{\xi}}_\perp\right)\right] \\ &= \tilde{\mathbf{u}}_2 \cdot \mathbf{n} + \frac{1}{2\mu_0}\frac{B_0^2}{R_I}\frac{\partial}{\partial t}\tilde{\boldsymbol{\xi}}_\perp^2. \end{aligned} \quad \text{(A8)}$$

In order for energy conservation in the present system of the magnetosphere and the ideal ionosphere to hold, $H = K + \delta W_F + \delta W_I$ must be conserved. For boundary conditions (12), (14) and (15), $\tilde{\mathbf{u}}_2' \cdot \mathbf{n}$ vanishes on the ionosphere and therefore H is conserved. For the boundary condition (13), one obtains

$$\tilde{\mathbf{u}}_2' \cdot \mathbf{n} = -\frac{\gamma}{\gamma - 1}\tilde{v}_{1\parallel}\tilde{\boldsymbol{\xi}}_\perp \cdot \nabla p_0. \quad \text{(A9)}$$

Therefore, a finite term is left on the right hand side of equation (A7) after integration of equation (A9). However, the right hand side of equation (A7) in this case is shown to be much smaller than $\partial \delta W_I / \partial t$ on the left hand side for low-β ionospheric plasma (see Appendix B of *Miura* [2007] for details). Therefore, energy conservation is also valid for the boundary condition (13), when the ionosphere is a low-β plasma. Thus, for all the boundary conditions (12) to (15), H is conserved. There are no other combinations of boundary conditions, which validate the constancy of H. Therefore, one finds that the four ideal ionospheric boundary conditions (12) to (15) are derived from the condition of $\tilde{\mathbf{u}}_2' \cdot \mathbf{n} = 0$ at the ionosphere or the requirement of the conservation of H.

[98] Notice that for a flat ionosphere ($R_I \to \infty$) $\tilde{\mathbf{u}}_2' \cdot \mathbf{n} = \tilde{\mathbf{u}}_2 \cdot \mathbf{n}$. Since $\tilde{\mathbf{u}}_2 \cdot \mathbf{n}$ at the ionosphere is the normal component of the total second-order energy flux density on the ionospheric surface, $\tilde{\mathbf{u}}_2' \cdot \mathbf{n} = \tilde{\mathbf{u}}_2 \cdot \mathbf{n} = 0$ at the ionosphere means that there is no second-order energy exchange between the magnetosphere and the neutral atmosphere across a flat ionospheric surface. One also notes that for all ionospheric boundary conditions (12)–(15), first-order energy conservation is well satisfied for low-β ionospheric plasma in the present system of the magnetosphere and the ideal ionosphere (see Appendix B of *Miura* [2007] for details). Thus, for the four ideal ionospheric boundary conditions (12) to (15), the constancy of the total energy H in the system of the magnetosphere and the ideal ionosphere is guaranteed without the need to take into account the energy in the neutral atomosphere under the ionosphere.

Appendix B: Field Line Bending at the Ionosphere

[99] In this appendix, it is shown that although there is no field line bending in the magnetosphere in interchange perturbations, at the ionosphere, however, there is indeed a line bending in interchange perturbations. From equation (84) of *Miura* [2007] one has

$$\mathbf{B}_1 = -B\mathbf{b}(\nabla \cdot \boldsymbol{\xi}_\perp) + B(\mathbf{b} \cdot \nabla)\boldsymbol{\xi}_\perp - B(\boldsymbol{\xi}_\perp \cdot \nabla)\mathbf{b} - \mathbf{b}(\boldsymbol{\xi}_\perp \cdot \nabla)B. \quad \text{(B1)}$$

Let us define

$$\mathbf{B}_1' = B(\mathbf{b} \cdot \nabla)\boldsymbol{\xi}_\perp - B(\boldsymbol{\xi}_\perp \cdot \nabla)\mathbf{b} \quad \text{(B2)}$$

then by using vector formulae one obtains

$$\mathbf{b} \cdot \mathbf{B}_1' = -B\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}. \quad \text{(B3)}$$

Therefore, $\mathbf{B}_1'$ contains a component parallel to $\mathbf{b}$. One obtains, therefore, from equations (B2) and (B3)

$$\mathbf{B}_{1\perp} = \mathbf{B}_1' - \left(\mathbf{b} \cdot \mathbf{B}_1'\right)\mathbf{b} = B(\mathbf{b} \cdot \nabla)\boldsymbol{\xi}_\perp - B(\boldsymbol{\xi}_\perp \cdot \nabla)\mathbf{b} + B(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})\mathbf{b}. \quad \text{(B4)}$$

Substitution of equation (35) into equation (20) yields

$$\mathbf{Q}_\perp = \mathbf{B}_{1\perp} = e^{iS}(\mathbf{b} \cdot \nabla X)\mathbf{b} \times \mathbf{k}_\perp. \quad \text{(B5)}$$

Since the interchange trial function $X = \hat{X}(\psi - \psi_0, \chi - \chi_0)$ does not have ζ dependence,

$$\mathbf{B}_{1\perp} = 0 \quad \text{(B6)}$$

in the magnetosphere. Therefore, from equation (B4)

$$(\mathbf{b} \cdot \nabla)\boldsymbol{\xi}_\perp - (\boldsymbol{\xi}_\perp \cdot \nabla)\mathbf{b} + (\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})\mathbf{b} = 0 \quad \text{(B7)}$$

must be satisfied in the magnetosphere. This gives a constraint on eikonal S.

[100] Since the horizontally free boundary condition is adopted for interchange mode, $(\mathbf{b} \cdot \nabla)\boldsymbol{\xi}_\perp = 0$ at the ionosphere. Therefore, at the ionosphere one obtains from equation (B4)

$$\mathbf{B}_{1\perp} = -B(\boldsymbol{\xi}_\perp \cdot \nabla)\mathbf{b} + B(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})\mathbf{b}. \quad \text{(B8)}$$

At the ionosphere in the Northern Hemisphere, Appendix A of *Miura* [2007] gives

$$(\boldsymbol{\xi}_\perp \cdot \nabla)\mathbf{b} = -\frac{\boldsymbol{\xi}_\perp}{R_I}. \quad \text{(B9)}$$

Therefore, $\mathbf{B}_{1\perp} \neq 0$ at the ionosphere. This means that there is a field line bending in perturbations at the ionosphere.

[101] Let us assume that $\ell = 0$ at the ionosphere in the Northern Hemisphere and $\ell > 0$ in the magnetosphere above the ionosphere, and $\ell < 0$ below the ionosphere. From equations (B8) and (B9) one obtains

$$\mathbf{B}_{1\perp}(\ell = 0) = B\frac{\boldsymbol{\xi}_\perp}{R_I}. \quad \text{(B10)}$$

Thus, from equations (B5) and (B10) one obtains at $\ell = 0$

$$\frac{B}{R_I}\boldsymbol{\xi}_\perp = e^{iS}\left(\mathbf{b} \cdot \nabla \hat{X}\right)\mathbf{b} \times \mathbf{k}_\perp. \quad \text{(B11)}$$

Since

$$\boldsymbol{\xi}_\perp = \boldsymbol{\eta}_{\perp 0} e^{iS}, \quad \text{(B12)}$$

one obtains from equation (B11)

$$\frac{B}{R_I}\boldsymbol{\eta}_{\perp 0} = \left(\mathbf{b}\cdot\nabla\hat{X}\right)\mathbf{b}\times\mathbf{k}_\perp. \quad \text{(B13)}$$

Substituting

$$\boldsymbol{\eta}_{\perp 0} = Y\mathbf{b}\times\mathbf{k}_\perp = \frac{\hat{X}}{B}\mathbf{b}\times\mathbf{k}_\perp \quad \text{(B14)}$$

into equation (B13) yields

$$(\mathbf{b}\cdot\nabla)\ell n\hat{X} = \frac{1}{R_I}. \quad \text{(B15)}$$

Since $\hat{X}$ is not a function of ℓ in the magnetosphere ($\ell > 0$), equation (B15) is considered to define $(\mathbf{b}\cdot\nabla)\ell n\hat{X}$ at $\ell = 0$. If one assumes ficticiously $\hat{X}(\ell < 0)$, which satisfies

$$\frac{\partial}{\partial\ell}\ell n\hat{X} = \frac{1}{R_I} \quad \text{(B16)}$$

at $\ell = 0_-$, which is just below the ionosphere, this equation is formally integrated to give

$$\hat{X}(\psi-\psi_0,\chi-\chi_0,\ell) = \hat{X}(\psi-\psi_0,\chi-\chi_0,0)\exp\left(\frac{\ell}{R_I}\right) \quad \text{(B17)}$$

at $\ell = 0_-$. Therefore, although $\hat{X}$ is continuous at $\ell = 0$ at the ionosphere, $[(\mathbf{b}\cdot\nabla)\hat{X}]_{\ell=0_-} \neq [(\mathbf{b}\cdot\nabla)\hat{X}]_{\ell=0_+} = 0$. This means that although $\boldsymbol{\xi}_\perp$ is continuous at the ionosphere, $\mathbf{B}_{1\perp}$ is discontinuous at the ionosphere, if one defines $[(\mathbf{b}\cdot\nabla)\hat{X}]_{\ell=0} \equiv [(\mathbf{b}\cdot\nabla)\hat{X}]_{\ell=0_-}$. Therefore, there is a field line bending at the ionosphere.

[102] Since $(\mathbf{b}\cdot\nabla)\hat{X}$ is nonzero only at $\ell = \ell_S$ and $\ell = \ell_N$ in equation (48) and it is not infinite, the finite $(\mathbf{b}\cdot\nabla)\hat{X}$ term at $\ell = \ell_S$ and $\ell = \ell_N$ in the integrand of W in equation (48) does not give any finite contribution to W. In other words, the field line bending in perturbations at the ionosphere does not affect the value of the variational magnetospheric potential energy change δW_F.

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