博 士 論 文

論文題目 :

On the C^1 stabilization of homoclinic tangencies for diffeomorphisms in dimension three

（3次元の微分同相写像に対する
ホモクリニック接触の C^1 安定化について）

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On the C^1 stabilization of homoclinic tangencies for diffeomorphisms in dimension three

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Abstract

For C^1 diffeomorphisms of three dimensional closed manifolds and their periodic saddles with non-real eigenvalues, we provide sufficient conditions for stabilizing a homoclinic tangency within a given perturbation range. In the main step, suppose p has stable index two, if the sum of the largest two Lyapunov exponents is more than $\log(1-\delta)$, then δ-weak contracting eigenvalues are obtained by an arbitrarily small perturbation. As an application, a consequence on the existence of dominated splittings for R-robustly entropy-expansive homoclinic classes is also shown.

Key words homoclinic classes, weak eigenvalues, C^1 generic properties, Franks Lemma, homoclinic tangencies, dominated splittings, heterodimensional cycles.

AMS Mathematics Subject Classification 37C20, 37C29, 37D25, 37D30.

1 Introduction

For diffeomorphisms of closed smooth manifolds, homoclinic tangencies and heterodimensional cycles are realized as two basic sorts of bifurcations beyond uniform hyperbolic systems. They are defined as follows: (Let Λ and Γ be transitive hyperbolic sets of a diffeomorphism f and throughout the paper, the *index* of a transitive hyperbolic set Γ, denoted by $\mathrm{ind}(\Gamma)$, is defined as the dimension of its stable subspace.)

- f has a *cycle* associated to Λ and Γ if the stable manifold $W^s(\Lambda)$ of Λ intersects the unstable manifold $W^u(\Gamma)$ of Γ and the same holds for $W^u(\Lambda)$ and $W^s(\Gamma)$. The cycle is called *heterodimensional* if the indices of Λ and Γ are different. In particular, the cycle is said to be *co-index one* if $\mathrm{ind}(\Lambda) = \mathrm{ind}(\Gamma) \pm 1$.
- f has a *homoclinic tangency* associated to Γ if there exist $x, y \in \Gamma$ such that $W^s(x)$ intersects $W^u(y)$ non-transversally.

(Obviously, by definition, heterodimensional cycles only exist on manifolds of dimension at least three.) Lots of interesting phenomena, for instance, super exponential growth of the number of periodic points [BDF], existence of infinitely many sinks or sources [N1], non-hyperbolic robust transitivity [BDPR] and entropy-expansiveness [LVY], are closely related to them. It is conjectured by Palis that these two are typical mechanisms beyond uniform hyperbolicity, especially in the C^1 topology (See [B] for a brief introduction on this topic).

C^1 Palis Conjecture Every non-hyperbolic C^1 diffeomorphism can be approximated, in the C^1 topology, by diffeomorphisms exhibiting heterodimensional cycles or homoclinic tangencies.

In particular, we can consider heterodimensional cycles and homoclinic tangencies associated to hyperbolic periodic saddles. Note that both of heterodimensional cycles and homoclinic tangencies associated to periodic points contain non-transversal intersections, which will be easily destroyed by small perturbations.

Towards the study of Palis Conjecture, if one wants to develop perturbations while keeping these bifurcations, he needs to consider the robust version of them. More precisely, if there is a neighborhood $\mathcal{U}$ of f such that for all $g \in \mathcal{U}$, the hyperbolic continuation Γ_g of Γ for g exhibits homoclinic tangencies, then, we say that f has a *robust homoclinic tangency* associated to Γ. *Robust heterodimensional cycles* are defined in a similar way. Obviously, robust homoclinic tangencies and robust heterodimensional cycles must be associated to non-trivial hyperbolic sets. Concrete examples of them can be found in [A] and [AS], for instance. A natural question arises immediately: Starting from a homoclinic tangency (resp. heterodimensional cycle) associated to a hyperbolic periodic saddle p (resp. hyperbolic periodic saddles p and q) of f, is there an arbitrarily small perturbation g of f, admitting robust homoclinic tangencies (resp. robust heterodimensional cycles)? This problem is called the *stabilization* of homoclinic tangencies (resp. heterodimensional cycles).

In the C^2 topology, Newhouse gave a positive answer to the stabilization of homoclinic tangencies [N2]. In the C^1 topology, for heterodimensional cycles, the first result was obtained by Bonatti and Díaz by introducing a model of blender horseshoe, a kind of thick hyperbolic set. They proved that every co-index one heterodimensional cycle can be stabilized [BD1]. Later, this result was improved by Bonatti, Díaz and Kiriki in [BDK] (see Lemma 2.7). Comparing to [BD1], the stabilization in [BDK] is stronger in the following sense: the hyperbolic sets Γ and Λ (to which the robust heterodimensional cycle of g is associated) contain the continuation p_g and q_g respectively. Moreover, examples (called *fragile cycles*) were constructed which cannot be stabilized in this sharp sense [BD2].

Through the above observations, we propose the following question: In the C^1 topology, is it possible to stabilize a homoclinic tangency? In fact, this question only make sense when the dimension of M (denoted by $\dim M$) is larger or equal to three. Since according to [Mo], for surface diffeomorphisms, C^1 robust homoclinic tangency does not exist. In higher dimensional case, thanks to Bonatti and Díaz who built the so-called *folding manifolds* in a blender horseshoe which exhibit robust tangent intersections in a natural setting [BD3]. Based on their result, the main theorem of this paper is as follows: (Let M be a compact smooth Riemannian manifold without boundary. In particular, write M^d if it is necessary to emphasize the dimension d of M. Denote by $\mathrm{Diff}^1(M)$ the space of C^1 diffeomorphisms of M endowed with the C^1 topology. Let $\chi_1(p) \leq \chi_2(p) \leq \chi_3(p)$ be the Lyapunov exponents of p, counting with multiplicities and write $\|Df^{\pm}(p)\| = \max\{\|Df^{\beta}(x)\| : \beta = \pm 1,\ x \in \mathrm{orb}(p)\}$, where $\|A\|$ denote the operator norm of a linear map A.)

Theorem A. For any $a > 1$, there exists $\delta_0(a) > 0$ with $\delta_0(a) \to 0$ as $a \to 1$, such that if $0 < \delta < \delta_0(a)$ and $f \in \mathrm{Diff}^1(M^3)$ exhibits a homoclinic tangency associated to a hyperbolic periodic saddle p having non-real contracting eigenvalues satisfying $\chi_2(p) + \chi_3(p) > \log(1 - \delta)$, then, there exists g with $\mathrm{dist}_{C^1}(f, g) < a\delta\|Df^{\pm}(p)\|$, exhibiting a robust heterodimensional cycle and a robust homoclinic tangency.

Remark 1.1.

- When $\mathrm{ind}(p) = 1$, replacing f by its inverse, the symmetric version of this theorem is also valid.
- If $\chi_2(p) + \chi_3(p) > 0$, then $\mathrm{dist}_{C^1}(f, g)$ can be required arbitrarily small, which also follows from [BCDG, Theorem 1].

It is worth mentioning another related result by Bonatti, Crovisier, Díaz and Gourmelon, which dealt with the stabilization of homoclinic tangencies in case of $\dim M \geq 3$ (An earlier version is due to Shinohara in [S]). Within a fixed perturbation range, Theorem A says that stabilization of homoclinic tangencies, at least in the weak sense, can be realized if $\chi_2(p) + \chi_3(p) > \log(1-\delta)$ comparing to [BCDG] which requires $\chi_2(p) + \chi_3(p) > -\delta$.

As a corollary, the following result is available if one wants to stabilize a homoclinic tangency in the strong sense (see Definition 2.4 for the definition of dominated splittings and their dimensions).

Corollary B. For any $a > 1$, suppose $f \in \mathrm{Diff}^1(M^3)$ exhibits a homoclinic tangency associated to a hyperbolic periodic point p such that

- $H(p_g)$ does not admit dominated splittings of dimension $\mathrm{ind}(p_g)$ for all g in a neighborhood $\mathcal{U}_f$ of f; and
- p has non-real contracting eigenvalues satisfying $\chi_2(p) + \chi_3(p) > \log(1-\delta)$,

where $0 < \delta < \delta_0(a)$ is sufficiently small, depending on $\mathcal{U}_f$. Then, there exists g with $\mathrm{dist}_{C^1}(f,g) < a\delta\|Df^{\pm}(p)\|$, exhibiting a robust heterodimensional cycle and a robust homoclinic tangency associated to a hyperbolic set Γ containing p_g.

Actually, in the proof of Theorem A, the main part is devoted to the creation of weak contracting eigenvalues. Let us be more precise. Suppose $H(p)$ is a homoclinic class of some hyperbolic periodic saddle p of f, then the set of hyperbolic periodic saddles of f which are homoclinically related to p is a dense subset of $H(p)$, which is denoted by $\pitchfork(p)$. We say that $H(p)$ has *weak eigenvalues* associated to periodic points homoclinically related to p if for any $\epsilon > 0$, there exists $q \in \pitchfork(p)$ such that q has some contracting eigenvalue $\lambda^s(q)$ satisfying $|\lambda^s(q)| > (1-\epsilon)^{\pi(q)}$ or q has some expanding eigenvalue $\lambda^u(q)$ satisfying $|\lambda^u(q)| < (1+\epsilon)^{\pi(q)}$, where $\pi(q)$ is the period of q. Such an eigenvalue is called *ϵ-weak*. It is not hard to show that if $H(p)$ does not admit dominated splittings of dimension $\mathrm{ind}(p)$, then we can obtain arbitrarily weak eigenvalues associated to p_g by an arbitrarily small perturbation g of f. However, unless additional assumptions are given, in general, we cannot designate such a weak eigenvalue to be contracting or expanding in advance. For example, when $\dim M = 3$ and $\mathrm{ind}(p) = 2$, if we want to use folding manifolds and blender horseshoe to construct robust homoclinic tangencies by small perturbations, as a preliminary step, we should find a periodic point $q \in \pitchfork(p)$ with sufficiently weak *contracting* eigenvalues and then decrease $\mathrm{ind}(q)$ by stretching Df over $TM|\mathrm{orb}(q)$. Otherwise, if the weak eigenvalue of $q \in \pitchfork(p)$ we obtain is always expanding, we might thus get nothing but a sink after C^1small perturbations, which of course escape the continuation of the original homoclinic class.

By this observation, we see that designating the type of a weak eigenvalue is very important in some situation. Along this direction, Bochi and Bonatti developed a method which said, in rough terms, that one can mix two consecutive Lyapunov exponents of some periodic point such that both of them move continuously towards their midpoint [BB, Theorem 4.1 and Proposition 3.1]. As a result, under the same setting as above (i.e. $\dim M = 3$ and $\mathrm{ind}(p) = 2$), if we want to get δ-weak contracting eigenvalues by using [BB], the assumption of $\chi_2 + \chi_3 > -\delta$ is necessary. For otherwise, along the parameter curve, χ_3 decreases to zero before χ_2 increases to $-\delta$. But according to the so-called isotopic Franks Lemma (Lemma 2.8), in order to guarantee the above perturbation does not make the periodic point outside the continuation of the original homoclinic class, none of the Lyapunov exponents is permitted to pass through zero. [BCDG] directly borrowed [BB] to obtain δ-weak contracting eigenvalues. In this paper, by using a new approach, we also give a sufficient condition for getting δ-weak contracting eigenvalues, which is better than [BB] when p has non-real eigenvalues.

Theorem C. Given $\delta > 0$. Suppose p is a hyperbolic periodic saddle of $f \in \mathrm{Diff}^1(M^3)$ satisfying:

- p has non-real contracting eigenvalues satisfying $\chi_2(p) + \chi_3(p) > \log(1-\delta)$; and
- f exhibits a homoclinic tangency associated to p.

Then, there exists g arbitrarily close to f and a hyperbolic periodic saddle q of g, homoclinically related to p_g, having δ-weak contracting eigenvalues.

Remark 1.2.

- When $\mathrm{ind}(p) = 1$, replacing f by its inverse, we can give the symmetric version of this theorem.
- According to its proof, this theorem is still valid when $\delta = 0$ (which also follows from [BCDG]). In this case, *δ-weak* should be read as *arbitrarily weak*. That is, for any $\epsilon > 0$, there exists g arbitrarily close to f and $q \in \pitchfork(p_g)$ admitting ϵ-weak contracting eigenvalues.

When δ is positive, to get δ-weak contracting eigenvalues, our assumption on Lyapunov exponents is weaker than that of [BCDG] which comes from [BB]. Indeed, the mixing process of Lyapunov exponents in [BB] is obtained by induction on dimensions, thus can be reduced to planar dynamics. In a periodic orbit $\mathrm{orb}(q)$, once there exists some $r \in \mathrm{orb}(q)$ with small angle θ between its two eigendirections, a rotation in the tangent space at r with size less than θ is enough to mix the Lyapunov exponents of $\mathrm{orb}(q)$ (see [BDP, Lemma 3.2] for instance). But in our perturbations, under the weaker assumption, only rotating at a single point is not sufficient, we also need additional perturbations on tangent spaces over many points in $\mathrm{orb}(q)$ with relatively large angles. These points are so many that the number of them take a positive proportion in $\mathrm{orb}(q)$ especially when q has a large period. The additional perturbations at the many points should make some effect on the exponential growth of tangent vectors which assists the eigenvalues condition of p, causing the weaker assumption of inequality than [BCDG]'s. The selection of such periodic orbit heavily relies on the delicate constructions of a horseshoe model in Section 4.

As mentioned before, the proof of Theorem C occupied the central position of this paper. Independent of [BB], we adopt a different way which is somewhat geometric. Let E^u (resp. E^s) denote the unstable (resp. stable) subspace of a periodic point. Our proof involves looking at the interplay between $\angle(E^s, E^u)$ and contracting rate of vectors in E^s (shortly, E^s-rate). Roughly speaking, for a sequence of periodic saddles $q_n \in \pitchfork(p)$, if $\angle(E^s, E^u)$ decrease to zero more rapidly than E^s-rate, then weak contracting eigenvalues can be created by C^1 small perturbations inside the homoclinic class. In fact, in order to apply the isotopic Franks Lemma, we need to find a continuous path C_t ($t \in [0, 1]$) of matrices which connects the derivative of the original first return map and a matrix with weak contracting eigenvalues. In general, finding such a path is not so difficult, while ensuring its hyperbolicity is much harder and more important. Our strategy is the following: Choose a path $Dg^n(q) \circ C_t$ without paying attention to its hyperbolicity for a while. Then, modify this path by adding another matrix, say D_t, which aims to recover the expanding eigenvector. As a consequence, the expanding eigenvalue survives all the time along the modified path, which indicates that the weak eigenvalue we obtain must be contracting. Let us remark that although it is necessary only in theoretic sense, the introduction of D_t is the main reason for assuming $\chi_2(p) + \chi_3(p) > \log(1-\delta)$. In the foreseeable future, this assumption is difficult to be removed, see [B, Conjecture 8].

In the last part of this introduction, we present an application of Theorem A to R-robustly entropy-expansive diffeomorphisms. Let us recall some relative definitions.

Definition 1.3. ([Bo]) Let Λ be a compact f-invariant subset of M. For any $x \in \Lambda$, denote

$$\Gamma_\varepsilon(x, f) = \{y \in M : \mathrm{dist}(f^n(x), f^n(y)) \le \varepsilon \quad (\forall n \in \mathbb{Z})\},$$

which will be written simply as $\Gamma_\varepsilon(x)$ when no confusion arises. We say that $f|\Lambda$ is *entropy-expansive* if there exists $\varepsilon > 0$ such that

$$\sup_{x\in\Lambda} h(f|\Gamma_\varepsilon(x)) = 0,$$

where h denotes the topological entropy.

Definition 1.4. For $f \in \mathrm{Diff}^1(M)$ and a homoclinic class $H(p)$ of f, if there exists a neighborhood $\mathcal{U}_f$ of f in $\mathrm{Diff}^1(M)$ and a residual subset $\mathcal{R}_f$ of $\mathcal{U}_f$ such that $g|H(p_g)$ is entropy-expansive for all $g \in \mathcal{R}_f$, then we say that $f|H(p)$ is *R-robustly entropy-expansive* in $\mathcal{U}_f$.

Here, the letter R stands for the word *residual*. As an extension of robustly entropy-expansiveness [PV], this notion was introduced in [L] where it was shown that in an open and dense subset of $\mathcal{U}_f$, the continuation of $H(p)$ admits a dominated splitting of dimension $\mathrm{ind}(p)$ (see [L, Theorem A]). Thus, it is interesting to know whether $H(p)$ itself also admits such a splitting.

Theorem D. Suppose $f|H(p)$ is R-robustly entropy-expansive in an open ball centered at $f \in \mathrm{Diff}^1(M^3)$ with radius ρ. Let $\sigma = \dfrac{\rho}{\rho + \|Df^{\pm}(p)\|}$. If p has non-real contracting eigenvalues and $\chi_2(p) + \chi_3(p) > \log(1-\sigma)$, then, $H(p)$ admits a dominated splitting of dimension $\mathrm{ind}(p)$.

This paper is organized as follows. In Section 2, through a brief review of some basic facts and background of this topic, we summarize without proofs some basic properties as the set-up of notation and terminology. In Section 3, we provide a sufficient condition for getting weak contracting eigenvalues inside homoclinic classes by arbitrarily small perturbations. Theorem C will be proved in Section 4 by building a horseshoe model near a homoclinic tangency. Theorem A and Corollary B are proved in Section 5 where index change is shown through weak eigenvalues. Finally, we prove Theorem D in Section 6.

2 Preliminaries

For $f \in \mathrm{Diff}^1(M)$ and a hyperbolic periodic point p of f, let $\pi(p)$ denote the period of p and $\mathrm{orb}(p)$ be the orbit of p. Suppose the eigenvalues of $Df^{\pi(p)}(p)$, counting with multiplicities, satisfy $|\lambda^1(p)| \le \cdots \le |\lambda^s(p)| < 1 < |\lambda^{s+1}(p)| \le \cdots \le |\lambda^d(p)|$ where $d = \dim M$ and $s = \mathrm{ind}(p)$, then, $\lambda^s(p)$ is called the *central contracting eigenvalue*. In particular, if $|\lambda^{s-1}(p)| < |\lambda^s(p)|$, we say that $\lambda^s(p)$ has multiplicity one. For any $1 \le i \le s$, we say that $Df^{\pi(p)}(p)$ has i-strong stable direction if $|\lambda^i(p)| < |\lambda^{i+1}(p)|$. In this case, one can define the i-strong stable manifold of p, denoted by $W_i^{ss}(p)$, as the unique submanifold in the stable manifold $W^s(p)$ of p, which is tangent to the i-strong stable direction of $Df^{\pi(p)}(p)$. Symmetric definitions can also be given for unstable eigenvalues.

A set is *residual* in $\mathrm{Diff}^1(M)$ if it can be written as a countable intersection of open and dense subsets of $\mathrm{Diff}^1(M)$. In particular, residual sets of $\mathrm{Diff}^1(M)$ are dense. One can easily verify the following fact: if $\mathcal{A}$ is residual in $\mathcal{B}$ and $\mathcal{B}$ is residual in $\mathrm{Diff}^1(M)$, then $\mathcal{A}$ is residual in $\mathrm{Diff}^1(M)$. Throughout the paper, we say that a property holds *generically* in $\mathrm{Diff}^1(M)$ if it is satisfied by diffeomorphisms contained in a residual subset of $\mathrm{Diff}^1(M)$.

Generically in $\mathrm{Diff}^1(M)$, homoclinic classes exhibit many good properties which are similar to the basic sets in the Spectral Decomposition Theorem of Axiom A diffeomorphisms. For this reason, we will mainly focus on the dynamics of C^1 diffeomorphisms restricted to homoclinic classes. Recall that the *homoclinic class* of a hyperbolic periodic saddle p of f, denoted by $H(p)$, is defined as the closure of transversal intersections of the stable and unstable manifolds of p. We can equivalently define $H(p)$

as the closure of all hyperbolic periodic saddles q homoclinically related to p (i.e. the stable manifold manifold of p transversally meets the unstable manifold of q and vice versa). When talking about the homoclinic class of a generic C^1 diffeomorphism, the following useful results of [CMP] and [ABCDW] assert that, two homoclinic classes are either coincide or disjoint, and the collection of indices of all periodic points in $H(p)$, denoted by $\text{ind}(H(p))$, form an interval in $\mathbb{N}$. For this reason, we will use the terminology of *index-interval* of $H(p)$.

Lemma 2.1. ([ABCDW, Lemma 2.1]) There exists a residual subset $\mathcal{G}_1$ of $\text{Diff}^1(M)$, such that for every f in $\mathcal{G}_1$ and every pair of saddles p, q of f, there is a neighborhood $\mathcal{U}_f$ of f in $\text{Diff}^1(M)$, such that *either*

- $H(p_g) = H(q_g)$ for all $g \in \mathcal{U}_f \cap \mathcal{G}_1$; *or*
- $H(p_g) \cap H(q_g) = \emptyset$ for all $g \in \mathcal{U}_f \cap \mathcal{G}_1$.

Lemma 2.2. ([ABCDW, Theorem 1.1]) There is a residual subset $\mathcal{G}_2$ of $\text{Diff}^1(M)$, such that for every $f \in \mathcal{G}_2$, every homoclinic class $H(p)$ of f containing hyperbolic saddles of indices a and b also contains a dense subset of saddles of index i for all $i \in [a, b] \cap \mathbb{N}$.

Another sort of elementary dynamical pieces which are closely related to homoclinic classes are chain recurrent classes. In general, a homoclinic class is a proper subset of a chain recurrent class [BCGP]. However, It was shown by Bonatti and Crovisier that as long as periodic points are involved, C^1 generically, these two notions coincide.

Lemma 2.3. ([BC]) Generically in $\text{Diff}^1(M)$, every homoclinic class is a chain recurrent class; Equivalently, every chain recurrent class containing a periodic point p coincide with the homoclinic class of p.

Recall that an *ε-pseudo-orbit* of f is a sequence $x_i \in M$ such that all the *jumps* $\text{dist}(f(x_i), x_{i+1})$ are less than ε. A point $x \in M$ is called *chain recurrent* if for every $\epsilon > 0$, there exists ϵ-pseudo orbit starting and ending at x. The *chain recurrent class* of x, denoted by $C(x)$, is the collection of all points $y \in M$ such that there are pseudo orbits of arbitrarily small jumps from x to y and from y to x. The following fact is straightforward: Suppose f has a heterodimensional cycle associated to transitive hyperbolic sets Λ and Γ, then Λ and Γ are contained in the same chain recurrent class of f.

Definition 2.4. Let $f \in \text{Diff}^1(M)$ and let $\Lambda \subset M$ be a compact f-invariant subset. A continuous splitting $T_\Lambda M = E \oplus F$ of the tangent bundle over Λ is called *dominated* if it is Df-invariant and there exists $N \in \mathbb{N}$ such that for all $x \in \Lambda$, one has

$$\frac{\|Df^N(x)u\|}{\|Df^N(x)v\|} < \frac{1}{2},$$

where u and v are any unit vectors in E and F respectively. The *dimension* of this dominated splitting is defined as $\dim E$. More generally, a Df-invariant splitting $T_\Lambda M = E_1 \oplus \cdots \oplus E_k$ ($k \geq 2$) is dominated if for each $l = 1, \cdots, k-1$, the splitting $T_\Lambda M = (E_1 \oplus \cdots \oplus E_l) \oplus (E_{l+1} \oplus \cdots \oplus E_k)$ is dominated.

Dominated splittings persist in the following sense:

Lemma 2.5. ([BDV, Appendix B.1.2]) Let Λ be a compact f-invariant set with a dominated splitting, then there is a neighborhood $U \subset M$ of Λ such that for any g sufficiently C^1 close to f, the maximal g-invariant set contained in the closure of U admits a dominated splitting having the same dimensions of subbundles as the initial dominated splitting of f over Λ.

Nowadays, homoclinic tangencies are known to be closely related to the absence of some particular type of dominated splitting (see Lemma 4.2). For the existence of robust tangencies, the following criterion is quite useful.

Lemma 2.6. ([BD3, Theorem 1.2]) Let M be a compact manifold with $\dim M \geq 3$. There is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that, for every $f \in \mathcal{R}$ and every periodic saddle p of f such that

- $H(p)$ has a periodic saddle q with $\mathrm{ind}(p) \neq \mathrm{ind}(q)$; and
- $H(p)$ does not admit dominated splittings of dimension $\mathrm{ind}(p)$.

The saddle p belongs to a transitive hyperbolic set having a C^1 robust homoclinic tangency.

As another kind of homoclinic bifurcation, heterodimensional cycles can be stabilized in most cases: (See also [BD1] for an earlier result on that.)

Lemma 2.7. ([BDK, Theorem 1]) Let f be a C^1 diffeomorphism with a co-index one heterodimensional cycle associated to periodic saddles p and q. Suppose that at least one of the homoclinic classes of these saddles is non-trivial. Then there exist an arbitrarily small perturbation g of f and hyperbolic sets $\Lambda \ni p_g$, $\Gamma \ni q_g$ such that g admits a robust heterodimensional cycle associated to Λ and Γ.

Now, let us introduce the basic tool which will be used in our perturbation. Usually, Franks Lemma ([F, Lemma 1.1]) is well known as a simple but helpful result which allow us to realize a linear perturbation of Df along a finite set of M by perturbing f itself in an arbitrarily small neighborhood of that finite set. However, this result has an inherent disadvantage, especially when someone wants to perturb Df along some periodic orbit while keeping its homoclinic (resp. heteroclinic) relation with another periodic point. In other words, unless additional assumptions, for instance isolation, are given, a periodic point might escape the continuation of the original homoclinic class (resp. chain recurrent classes). However, Gourmelon's result in [G1] gave a sufficient condition for controlling the behavior of stable/unstable manifolds. By applying this isotopic version of Franks Lemma, we are allowed to give perturbations inside a homoclinic class.

Lemma 2.8. (Isotopic Franks Lemma [G1, G2]) Given $f \in \mathrm{Diff}^1(M)$, let Q be a periodic point of f with period n. Consider $\epsilon > 0$ and $i, j \in \mathbb{N}$. Suppose $(A_{l,t})_{\substack{l=0,\dots,n-1\\ t\in[0,1]}}$ is a parameter linear cocycle in $\mathrm{GL}(\mathbb{R}, d)$ satisfying

- $A_{l,0} = Df(f^l(Q))$ for $l = 0, ..., n-1$;
- The radius of the curve, defined by
$$\max_{\substack{l=0,\dots,n-1\\ t\in[0,1]}} \left\{ \|A_{l,t} - A_{l,0}\|, \|A_{l,t}^{-1} - A_{l,0}^{-1}\| \right\},$$
is less than ϵ;
- For any $t \in [0,1]$, the product $\prod_{l=0}^{n-1} A_{l,t} = A_{n-1,t} \circ \cdots \circ A_{0,t}$ admits i-strong stable direction and j-strong unstable direction.

Then, for any neighborhood V of $\mathrm{orb}_f(Q)$ in M, there exists $g \in \mathrm{Diff}^1(M)$ such that

- $\mathrm{dist}_{C^1}(g, f) < \epsilon$;
- $g = f$ on $\mathrm{orb}_f(Q)$ and on $M \setminus V$, in particular, $Q_g = Q$;
- $Dg(g^l(Q)) = A_{l,1}$ for $l = 0, \dots, n-1$;

- g preserves the local i-strong stable manifold of Q outside V and the local j-strong unstable manifold outside V.

where the local i-strong stable manifold of Q outside V is the set of points contained in $W_i^{ss}(f^l(Q)) \cap (M \setminus V)$ whose positive iteration enter V without leaving it.

Since we will make a systematic application of this result which, especially preserving some particular homoclinic or heteroclinic relations, the following version of Lemma 2.8 is convenient.

Lemma 2.9. Under the hypothesis of Lemma 2.8, if we assume further that the i-strong stable manifold $W_i^{ss}(Q)$ of Q intersect the unstable manifold of another periodic point R of f, then the perturbed diffeomorphism g also satisfies $W_i^{ss}(Q_g) \cap W^u(R_g) \neq \emptyset$.

Note that the existence of R_g is guaranteed since $\mathrm{orb}_f(R)$ is outside the support of the perturbation. The proof of this lemma is similar in spirit to [S, Lemma 4.6], just noting that the statement there includes the transversality of the intersection, but in the proof, transversality is not used at all.

In the application of Lemma 2.9, we often give the perturbation of Df separately on invariant subspaces, say, E and F with $E \oplus F = TM$. At this moment, we should be very careful because the angle between E and F might cause some trouble. When this angle is small, even if perturbations of $Df|_E$ and $Df|_F$ are both small, the total size of the perturbation probably becomes pretty large. For subspaces E and F of $\mathbb{R}^d$ with $E \cap F = \{0\}$, let $\mathrm{Angle}(E, F) \in [0, \pi/2]$ denote the Euclidean angle between E and F, and define $\angle(E, F) \in [0, +\infty]$ as $\tan \mathrm{Angle}(E, F)$. Obviously, when $\mathrm{Angle}(E, F)$ goes to zero, these two quantities become almost the same since $\lim_{\theta \to 0} \dfrac{\theta}{\tan \theta} = 1$. The following lemma will be frequently used when estimating the sizes of perturbations.

Lemma 2.10. ([M, lemma II.10]) Let $\mathbb{R}^d = E \oplus F$, and $T : \mathbb{R}^d \to \mathbb{R}^d$ is a linear map having E and F as its invariant subspaces, then, the operator norm of T has an upper bound:

$$\|T\| \leq \frac{1 + \angle(E, F)}{\angle(E, F)} \Big(\|T|_E\| + \|T|_F\| \Big).$$

3 Weak contracting eigenvalues

In this section, we give a sufficient condition for getting weak contracting eigenvalues in a homoclinic class, which will be used in the proof of Theorem C.

Lemma 3.1. Let $f \in \mathrm{Diff}^1(M^3)$ and a hyperbolic periodic point p of f with index 2 be given. Suppose there exist sequences λ_k, $g_k \to f$ and $q_k \in \pitchfork(p_{g_k})$ of period n_k with $\lambda_k^{n_k} \to 0$ $(k \to \infty)$ such that, letting ξ_k be the unit vector in the image of orthogonal projection of $E^u(q_k)$ into $E^s(q_k)$, the following properties hold:

(i) $\limsup_{k \to \infty} \angle(Dg_k^{n_k}(q_k)\xi_k, \xi_k) > 0$;

(ii) $\lim_{k \to \infty} \dfrac{\|Dg_k^{n_k}|_{E^s}(q_k)\|}{\lambda_k^{n_k}} = 0$;

(iii) $\lim_{k \to \infty} \dfrac{\lambda_k^{n_k} \angle(E^s(q_k), E^u(q_k))}{\|Dg_k^{n_k}(q_k)\xi_k\|} = 0$.

Then, there exists an arbitrarily small perturbation h of f, admitting $(1 - \lambda_k)$-weak contracting eigenvalues associated to some $q_k \in \pitchfork(p_h)$ for arbitrarily large k.

Remark 3.2.

(1) Conditions (ii) and (iii) imply $\angle(E^s(q_k), E^u(q_k)) \to 0$ as $k \to \infty$.

(2) Replace f by f^{-1}, we obtain the symmetric version of Lemma 3.1 which provides arbitrarily weak expanding eigenvalues.

(3) Moreover, we can require this weak eigenvalue is real, central, and has multiplicity one. This is because we have the following:

Lemma 3.3. ([YG, Lemma 2.3]) For generic f in $\mathrm{Diff}^1(M)$ and any hyperbolic periodic point p of f, if f has a periodic point q homoclinically related to p, having an ϵ-weak eigenvalue, then f has a periodic point p_1 homoclinically related to p with an ϵ-weak eigenvalue, whose eigenvalues are all real.

Although it is stated as a property for C^1 generic diffeomorphisms, this lemma is actually a perturbation result. Moreover, by checking its proof, we see that if the weak eigenvalue in the hypothesis is contracting (resp. expanding), then after the perturbation, one gets also contracting (resp. expanding) weak eigenvalues. Once a real weak eigenvalue is obtained by Lemma 3.3, which is associated to some $q \in \pitchfork(p_g)$, then an additional arbitrarily small perturbation using the isotopic Franks Lemma will help us to split each eigenvalue such that all of them have multiplicity one. This last perturbation still preserves the homoclinic relation because $Dg^{\pi(q)}(q)$ keeps its hyperbolicity in the process.

Now, let us turn to the proof of Lemma 3.1. The main idea is as follows: Firstly, find a perturbation which induces weak contracting eigenvalues by modifying [M, Lemma II.9] to solve a linear equation. Secondly, create a parameter curve $C_t \in \mathrm{GL}(\mathbb{R}, 3)$ which connects the identity and the perturbation obtained in the previous step. But this isotopic perturbation cannot be used directly, since hyperbolicity might be destroyed until it arrives its endpoint. To avoid this happens, thirdly, we recover the unstable eigenvector by adding another isotopic perturbation D_t before C_t. In this step, we will make use of the own dynamics in the 2-dimensional subspace $E^s(q)$ to guarantee, that the additional perturbation D_t can be given separately in two invariant subspaces which have a relatively large angle. This will help us to control the size of the perturbation. Finally, we apply the isotopic Franks Lemma to the new perturbation $Dg^n(q) \circ C_t \circ D_t$ of $Dg^n(q)$, obtaining weak contracting eigenvalues.

Proof of Lemma 3.1. For any $\epsilon > 0$ fixed, we are going to construct an ϵ-perturbation h of f, having a periodic point homoclinically related to p_h with $(1-\lambda_k)$-weak contracting eigenvalues for an arbitrarily large $k \in \mathbb{N}$. First, by assumption, we are allowed to select $k \in \mathbb{N}$ large, satisfying the following conditions:

(K1)
$$\mathrm{dist}_{C^1}(g_k, f) < \frac{\epsilon}{2};$$

(K2)
$$\mathrm{Angle}(Dg_k^{n_k}(q_k)\xi_k, \xi_k) \geq \sigma,$$

where $0 < \tan\sigma < \limsup_{k\to\infty} \angle(Dg_k^{n_k}(q_k)\xi_k, \xi_k)$, independent of k.

(K3)
$$\max\left\{\|Dg_k^{n_k}|_{E^s}(q_k)\|, \frac{\|Dg_k^{n_k}|_{E^s}(q_k)\|}{\lambda_k^{n_k}}\right\} = \frac{\|Dg_k^{n_k}|_{E^s}(q_k)\|}{\lambda_k^{n_k}} < \frac{1}{2};$$

(K4)
$$\max\left\{6\theta_{q_k}, \frac{600\lambda_k^{n_k}\theta_{q_k}}{\|Dg_k^{n_k}(q_k)\xi_k\|}\right\} < \frac{\epsilon}{8D},$$

where $D = \sup\left\{\|Dg\| + \|Dg^{-1}\| : \mathrm{dist}_{C^1}(f, g) \leq 1\right\}$ and $\theta_{q_k} = \angle(E^s(q_k), E^u(q_k))$;

(K5) For any $A:\mathbb{R}\to\mathbb{R}$ with $\|A^{-1}\|<1$, one has

$$\frac{1}{2}\|v\|\leq\left\|\left(I-\lambda_k^{n_k}A^{-1}\right)v\right\|\leq 2\|v\|$$

for any $v\in\mathbb{R}$;

(K6) For $B=Dg_k^{n_k}|_{E^s}(q_k)$, one has

$$\frac{1}{2}\|u\|\leq\left\|\left(I-\lambda_k^{-n_k}B\right)^{-1}u\right\|\leq 2\|u\| \text{ and } \mathrm{Angle}\left(\left(I-\lambda_k^{-n_k}B\right)^{-1}u,u\right)<\frac{\sigma}{2}$$

for any $u\in\mathbb{R}^2$.

From now on, fix the integer k satisfying (K1)-(K6) above, for notation simplicity, let us denote λ_k, g_k, q_k and n_k by λ, g, q and n respectively. Take orthogonal coordinate chart $\{(E^s(q))^\perp,E^s(q)\}$ of T_qM. Since $E^s(q)$ is $Dg^n(q)$-invariant, we write $Dg^n(q)=\begin{pmatrix}A & 0\\ P & B\end{pmatrix}$ in this coordinate, where

$$\begin{aligned}A&=Dg^n|_{(E^s)^\perp}(q)\in\mathbb{R},\\ B&=Dg^n|_{E^s}(q)\in\mathrm{GL}(\mathbb{R},2).\end{aligned}$$

Clearly, $\|A^{-1}\|=|A|^{-1}<1$ since $\dim(E^s(q))^\perp=1$. Define a linear map $L:(E^s(q))^\perp\to E^s(q)$ such that

$$E^u(q)=\mathrm{graph}(L)=\{v+Lv:v\in(E^s(q))^\perp\}.$$

Thus $\theta_q=\angle(E^s(q),E^u(q))=\|L\|^{-1}$. Since $E^u(q)$ is $Dg^n(q)$-invariant, we obtain $LA=P+BL$. According to (K3),

$$\|L\|\leq\|PA^{-1}\|+\|BLA^{-1}\|\leq\|PA^{-1}\|+\|B\|\cdot\|L\|\leq\|PA^{-1}\|+\frac{1}{2}\|L\|,$$

which implies $\|PA^{-1}\|^{-1}\leq 2\|L\|^{-1}=2\theta_q$. Now, consider the following linear equation:

$$\begin{pmatrix}A & 0\\ P & B\end{pmatrix}\begin{pmatrix}I & C\\ 0 & I\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\lambda^n\begin{pmatrix}x\\ y\end{pmatrix}\tag{1}$$

which is equivalent to

$$\begin{cases}x=-(I-\lambda^nA^{-1})^{-1}Cy & (2)\\ y=(\lambda^n-B)^{-1}P(I-(I-\lambda^nA^{-1})^{-1})Cy. & (3)\end{cases}$$

Notice that

$$I=(I-\lambda^nA^{-1})(I-\lambda^nA^{-1})^{-1}=(I-\lambda^nA^{-1})^{-1}-\lambda^nA^{-1}(I-\lambda^nA^{-1})^{-1},$$

then (3) can be rewritten as

$$y=-\lambda^n(\lambda^n-B)^{-1}PA^{-1}(I-\lambda^nA^{-1})^{-1}Cy.$$

To solve (1), take $v\in(E^s(q))^\perp$, such that $\|v\|=\|PA^{-1}\|^{-1}$ and $\|PA^{-1}v\|=1$. Let

$$y=\lambda^n(\lambda^n-B)^{-1}PA^{-1}v\in E^s(q).$$

This definition does make sense, because $\lambda^n-B=\lambda^n(I-\lambda^{-n}B)$ and by (K3), $\|\lambda^{-n}B\|\leq\lambda^{-n}\|B\|<\frac{1}{2}$, which implies that (λ^n-B) is invertible ([PdM, Lemma 2.4.2]). Take norms of the equality $\lambda^nPA^{-1}v=(\lambda^n-B)y$, we have

$$\lambda^n=\|\lambda^nPA^{-1}v\|=\|(\lambda^n-B)y\|\leq\frac{3}{2}\lambda^n\|y\|,$$

which gives $\|y\| \geq \frac{2}{3}$. Let $w = -(I - \lambda^n A^{-1})v$, we conclude from (K5) that $\|w\| \leq 2\|v\|$. Define C as a linear map from $E^s(q)$ to $(E^s(q))^{\perp}$ satisfying $Cy = w$ and $\|C\| = \frac{\|w\|}{\|y\|}$. By the previous estimations,

$$\|C\| \leq \frac{2\|v\|}{2/3} = 3\|v\| = 3\|PA^{-1}\|^{-1} \leq \frac{6}{\|L\|} = 6\theta_q \leq \frac{\epsilon}{8D},$$

where in the last inequality we used (K4). It is easy to verify that for C defined above,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v \\ \lambda^n(\lambda^n - B)^{-1}PA^{-1}v \end{pmatrix}$$

is exactly the solution of (1). Now, consider isotopic perturbation $(A_{l,t})_{\substack{l=0,\dots,n-1 \\ t\in[0,1]}}$ of Dg on $\mathrm{orb}_g(q)$ as follows:

- $A_{0,t} = Dg(q) \circ C_t$, where $C_t = \begin{pmatrix} I & tC \\ 0 & I \end{pmatrix}$;
- $A_{l,t} = Dg(g^l(q))$ for $l = 1, \dots, n-1$.

By the previous construction, $\prod_{l=0}^{n-1} A_{l,1}$ admits an eigenvector with eigenvalue λ^n as we desired, hence we are intend to apply Lemma 2.9 to this parameter curve. However, in general, when t moves from 0 to 1, $\prod_{l=0}^{n-1} A_{l,t}$, although begins as a hyperbolic matrix $\prod_{l=0}^{n-1} A_{l,0} = Dg^n(q)$, might lose its hyperbolicity before t arrives 1, which will destroy the established plan. To overcome this obstacle, our strategy is to introduce another perturbation D_t which is used to ensure the hyperbolicity of the perturbed derivatives by recovering its expanding eigenvector for every $t \in [0,1]$. More precisely, such D_t should satisfy the following conditions:

(D1) For every $t \in [0,1]$,

$$D_t \begin{pmatrix} v \\ Lv \end{pmatrix} = \begin{pmatrix} v - tCLv \\ Lv \end{pmatrix};$$

(D2)

$$D_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

where $\begin{pmatrix} x \\ y \end{pmatrix}$ is the eigenvector with weak eigenvalue λ^n obtained before;

(D3) D_t is sufficiently near the identity, that is, for every $t \in [0,1]$,

$$\|D_t - \mathrm{id}\| < \frac{\epsilon}{8D}.$$

First, let us show the existence of such D_t. Denote by G the 2-dimensional plane spanned by $(v, 0)$ and $(0, Lv)$. Recall that L is the linear map from $(E^s(q))^{\perp}$ to $E^s(q)$ whose graph is $E^u(q)$.

Claim. $(x, y) \notin G$.

In fact, otherwise, there are $b_1, b_2 \in \mathbb{R}$, satisfying

$$\begin{pmatrix} v \\ \lambda^n(\lambda^n - B)^{-1}PA^{-1}v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = b_1 \begin{pmatrix} v \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ Lv \end{pmatrix}$$

which implies

$$\lambda^n(\lambda^n - B)^{-1}PA^{-1}v = b_2 Lv.$$

combining $Lv = PA^{-1}v + BLA^{-1}v$, and noticing that A is actually a real number, we obtain

$$\lambda^n Lv - \lambda^n A^{-1}BLv = b_2(\lambda^n - B)Lv,$$
$$(1-b_2)\lambda^n Lv = (\lambda^n A^{-1} - b_2)BLv.$$

But according to (K2), Lv and BLv are linearly independent, we conclude $1 = b_2 = \lambda^n A^{-1} < 1$, which is absurd. The claim is proved.

Continue the construction of D_t. We will take $F := \mathrm{span}\{(x,y)\}$ and G as two invariant subspaces of D_t and give the definition of D_t $(1 \le t \le 1)$ on them separately.

- Define $D_t|_F$ as the identity map;
- Define $D_t|_G$ as a rotation of the form (under some 2-dimensional standard orthogonal coordinate chart of G)
$$D_t|_G = \rho_t \begin{pmatrix} \cos\omega_t & -\sin\omega_t \\ \sin\omega_t & \cos\omega_t \end{pmatrix}$$
such that
$$D_t \begin{pmatrix} v \\ Lv \end{pmatrix} = \begin{pmatrix} v - tCLv \\ Lv \end{pmatrix},$$
where
$$\rho_t = \left\| \begin{pmatrix} v - tCLv \\ Lv \end{pmatrix} \right\| \Big/ \left\| \begin{pmatrix} v \\ Lv \end{pmatrix} \right\| \text{ and } \omega_t = \mathrm{Angle}\left(\begin{pmatrix} v \\ Lv \end{pmatrix}, \begin{pmatrix} v - tCLv \\ Lv \end{pmatrix} \right).$$

Obviously, (D1) and (D2) follow directly from this definition, it remains to check (D3). In fact,

$$\omega_t \le \omega_1 \le \frac{2\|CLv\|}{\|Lv\|} \le 2\|C\| \le 12\theta_q$$
$$\rho_t - 1 \le \rho_1 - 1 \le \theta_q$$

whenever θ_q is sufficiently small. Just notice $\rho_1 - 1$ is a higher order infinitesimal of θ_q. Thus,

$$\begin{aligned}\|(D_t - \mathrm{id})|_G\| \le \|(D_1 - \mathrm{id})|_G\| &= \left\| \rho_1 \begin{pmatrix} \cos\omega_1 & -\sin\omega_1 \\ \sin\omega_1 & \cos\omega_1 \end{pmatrix} - \rho_1 \cdot \rho_1^{-1} \right\| \\ &\le \rho_1 \left(\left\| \begin{pmatrix} \cos\omega_1 & -\sin\omega_1 \\ \sin\omega_1 & \cos\omega_1 \end{pmatrix} - \mathrm{id} \right\| + \left\| \mathrm{id} - \rho_1^{-1} \right\| \right) \\ &\le 2\rho_1\omega_1 + (\rho_1 - 1) \le 24\theta_q(1+\theta_q) + \theta_q < 50\theta_q.\end{aligned}$$

Let $\beta = \angle(F,G)$, we will estimate this angle in the triangle spanned by (v, Lv) and (x,y). Let

$$\begin{aligned}\vec{n} := \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} v \\ Lv \end{pmatrix} &= \begin{pmatrix} v \\ \lambda^n(\lambda^n - B)^{-1}PA^{-1}v \end{pmatrix} - \begin{pmatrix} v \\ Lv \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \lambda^n(\lambda^n - B)^{-1}PA^{-1}v - Lv \end{pmatrix}\end{aligned}$$

which is parallel to $E^s(q)$. Moreover, since

$$\begin{aligned}\lambda^n(\lambda^n - B)^{-1}PA^{-1}v - Lv &= \lambda^n(\lambda^n - B)^{-1}(Lv - BLA^{-1}v) - (\lambda^n - B)^{-1}(\lambda^n - B)Lv \\ &= (\lambda^n - B)^{-1}(\lambda^n Lv - \lambda^n A^{-1}BLv - \lambda^n Lv + BLv) \\ &= \lambda^{-n}(I - \lambda^{-n}B)^{-1}(I - \lambda^n A^{-1})BLv.\end{aligned}$$

By assumption (K2) and (K6),

$$\mathrm{Angle}(\vec{n}, G) \ge \sigma - \frac{\sigma}{2} = \frac{\sigma}{2}.$$

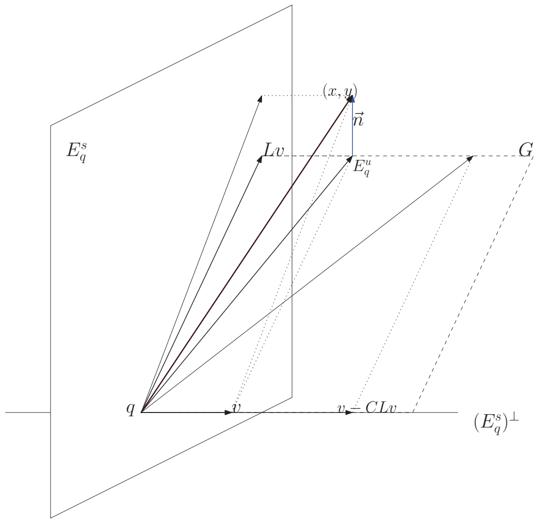

Figure 1

Intuitively, we see that $\vec{n}$ has stood in G (see Figure 1). Since σ is independent of $k \geq 1$ which have been omitted after (K6), we are allowed to estimate β using $\dfrac{\|\vec{n}\|}{\|Lv\|}$, up to a constant multiple, which can be assumed equal to 1 for simplicity. By (K5) and (K6), we have

$$\|\vec{n}\| = \|\lambda^{-n}(I - \lambda^{-n}B)^{-1}(I - \lambda^n A^{-1})BLv\| \in \left[\frac{\|BLv\|}{4\lambda^n}, \frac{4\|BLv\|}{\lambda^n}\right].$$

As a result, combining (K3),

$$\frac{\|BLv\|}{4\lambda^n\|Lv\|} \leq \beta \leq \frac{4\|BLv\|}{\lambda^n\|Lv\|} \leq \frac{4\|B\|}{\lambda^n} \leq 2.$$

Then, by Lemma 2.10,

$$\begin{aligned} \|D_t - \mathrm{id}\| \leq \|D_1 - \mathrm{id}\| &\leq \frac{1+\beta}{\beta}\big(\,\|(D_1 - \mathrm{id})|_F\| + \|(D_1 - \mathrm{id})|_G\|\,\big) \\ &< \frac{1+\beta}{\beta}\big(0 + 50\theta_q\big) \leq \frac{150\theta_q}{\beta} \leq 600\lambda^n\theta_q\frac{\|Lv\|}{\|BLv\|} \leq \frac{\epsilon}{8D}, \end{aligned}$$

where the last inequality comes from (K4). Recall that ξ is the unit vector in Lv direction.

Now, we complete showing the existence of D_t which satisfies (D1)-(D3). Using this D_t, let us finish the proof of Lemma 3.1.

Consider the following isotopic perturbation $(A_{l,t})_{\substack{l=0,\ldots,n-1 \\ t\in[0,1]}}$:

- $A_{0,t} = Dg(q) \circ C_t \circ D_t$;
- $A_{l,t} = Dg(g^l(q))$ $(l = 1, \ldots, n-1)$.

Clearly, $A_{l,0} = Dg(g^l(q))$ for all $l = 0, \ldots, n-1$. To estimate the radius of the path, since

$$\|C_1^{\pm 1} - \mathrm{id}\| \leq \|C\| \leq \frac{\epsilon}{8D} \leq 1$$

gives $\|C_1\| \le 2$, as a consequence,

$$\begin{aligned}\max_{\substack{l=0,\dots,n-1\\ t\in[0,1]}}\left\{\|A_{l,t}-A_{l,0}\|\right\} &\le \max_{t\in[0,1]}\|A_{0,t}-A_{0,0}\| = \|A_{0,1}-A_{0,0}\| \\ &\le \|Dg(q)\circ C_1\circ D_1 - Dg(q)\| \le \|Dg(q)\|\cdot\|C_1\circ D_1-\mathrm{id}\| \\ &\le D\|C_1\|\big(\|D_1-\mathrm{id}\|+\|\mathrm{id}-C_1^{-1}\|\big) < 2D\left(\frac{\epsilon}{8D}+\frac{\epsilon}{8D}\right)=\frac{\epsilon}{2}.\end{aligned}$$

Similar calculation shows that

$$\max_{\substack{l=0,\dots,n-1\\ t\in[0,1]}}\left\{\|A_{l,t}^{-1}-A_{l,0}^{-1}\|\right\} < \frac{\epsilon}{2}.$$

Hence, we conclude that the radius of the path is less than $\dfrac{\epsilon}{2}$.

Immediately, we have two cases: *either*

(I) $\prod_{l=0}^{n-1} A_{l,t}$ keeps its hyperbolicity during all the time when t varies from 0 to 1; *or*

(II) $\prod_{l=0}^{n-1} A_{l,t}$ lose its hyperbolicity for the first time at some $t_0\in(0,1]$.

If case (I) occurs, applying Lemma 2.9 to $\mathrm{orb}_g(q)$ and $(A_{l,t})_{\substack{l=0,\dots,n-1\\ t\in[0,1]}}$, we obtain an $\dfrac{\epsilon}{2}$-perturbation h of g (as a consequence, $\mathrm{dist}_{C^1}(h,f)\le \mathrm{dist}_{C^1}(h,g)+\mathrm{dist}_{C^1}(g,f)<\epsilon$ by (K1)), satisfying

- $h^l(q)=g^l(q)$ for $l=0,\dots,n-1$;
- q is homoclinically related to p_h;
- $Dh(h^l(q))=A_{l,1}$ for $l=0,\dots,n-1$.

Since we also have, by (D2),

$$\begin{aligned}Dh^n(q)\begin{pmatrix}x\\y\end{pmatrix} &= Dg^n(q)\circ C_1\circ D_1\begin{pmatrix}x\\y\end{pmatrix} = Dg^n(q)\circ C_1\begin{pmatrix}x\\y\end{pmatrix}\\ &= \begin{pmatrix}A&0\\P&B\end{pmatrix}\begin{pmatrix}I&C\\0&I\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \lambda^n\begin{pmatrix}x\\y\end{pmatrix},\end{aligned}$$

$Dh^n(q)$ admits a contracting eigenvalue λ^n. In other words, we have found ϵ-perturbation h of f, having $(1-\lambda)$-weak contracting eigenvalues associated to $q\in \pitchfork(p_h)$.

On the other hand, if case (II) occurs, $\prod_{l=0}^{n-1} A_{l,t}$ is hyperbolic for all $t\in[0,t_0)$. Then we cut the path $(A_{l,t})$ just before t arrives t_0 such that its end point admits an eigenvalue as weak as we desired (in particular, weaker than $1-\lambda$). Applying Lemma 2.9 to the tail-cut curve, we also obtain ϵ-perturbation h of f, having $(1-\lambda)$-weak eigenvalues associated to $q\in\pitchfork(p_h)$. By our construction, this weak eigenvalue must be contracting. In fact, recalling (D1), for every $t\in[0,1]$, we have

$$\begin{aligned}\left(\prod_{l=0}^{n-1}A_{l,t}\right)\begin{pmatrix}v\\Lv\end{pmatrix} &= Dg^n(q)\circ C_t\circ D_t\begin{pmatrix}v\\Lv\end{pmatrix} = Dg^n(q)\circ C_t\begin{pmatrix}v-tCLv\\Lv\end{pmatrix}\\ &= Dg^n(q)\begin{pmatrix}I&tC\\0&I\end{pmatrix}\begin{pmatrix}v-tCLv\\Lv\end{pmatrix} = Dg^n(q)\begin{pmatrix}v\\Lv\end{pmatrix}=\lambda^u\begin{pmatrix}v\\Lv\end{pmatrix}\end{aligned}$$

where λ^u is the expanding eigenvalue of $Dg^n(q)$ associated to $E^u(q)$. In other words, (v,Lv) is an expanding eigenvector of $\prod_{l=0}^{n-1}A_{l,t}$ for all $t\in[0,1]$, which indicates, when t increases from 0 to 1, that $\prod_{l=0}^{n-1}A_{l,t}$ lose its hyperbolicity for the first time by the absolute value of one of its contracting eigenvalues passes through 1 from the left to the right. Therefore, the weak eigenvalue obtained in case (II) should be contracting. Now, the proof of Lemma 3.1 is complete.

4 A horseshoe model: Proof of Theorem C

Theorem C is a straightforward consequence of Lemma 3.1 and the following

Lemma 4.1. Given $\delta > 0$. Suppose p is a hyperbolic periodic saddle of $f \in \mathrm{Diff}^1(M^3)$ satisfying:

- p has non-real contracting eigenvalues and $\chi_2(p) + \chi_3(p) > \log(1-\delta)$; and
- f exhibits a homoclinic tangency associated to p.

Then, there exist sequences λ_k, $g_k \to f$ and $q_k \in \pitchfork(p_{g_k})$ of period $n_k \to \infty$ with $\lambda_k^{n_k} \to 0$ such that: λ_k, g_k and q_k satisfy the hypothesis of Lemma 3.1. Moreover,

- If $\chi_2(p) + \chi_3(p) \geq 0$, we have $\lambda_k \to 1-$;
- If $0 > \chi_2(p) + \chi_3(p) > \log(1-\delta)$, we have $\lambda_k = \lambda = 1-\delta$.

By using the diagonal argument, we see immediately from the following lemma that the above assumption of homoclinic tangency can be replaced by assuming that $H(p)$ does not admit dominated splittings of dimension $\mathrm{ind}(p)$.

Lemma 4.2. Let $f \in \mathrm{Diff}^1(M)$ and p be a hyperbolic saddle of f. If $H(p)$ does not admit dominated splittings of dimension $\mathrm{ind}(p)$, then, there exists arbitrarily small perturbation g of f such that p_g has a homoclinic tangency.

This result actually follows from [SV, Section 2], although the above statement is not explicitly given in it. We outline the proof for readers' convenience. First, under the hypothesis that $H(p)$ does not have dominated splittings of dimension $\mathrm{ind}(p)$, [SV, Proposition 2] provides sequences $g_k \to f$ and homoclinic point x_k of p_{g_k} satisfying $\angle(T_{x_k}W^u(p_{g_k}), T_{x_k}W^s(p_{g_k})) < \frac{1}{k}$. Once such sequences are obtained, [SV, Proposition 1] gives $\frac{1}{k}$-perturbation $\tilde{g}_k$ of g_k such that $\tilde{g}_k$ has a homoclinic tangency associated to $p_{\tilde{g}_k}$. We remind the readers that in [SV, Proposition 1], the homoclinic tangency is created as a contradiction to the assumption of persistent expansiveness, which we do not need to care. This completes the outline of the proof.

Let us turn to the proof of Lemma 4.1. Without loss of generality, we can always assume that p is a fixed point of f. Otherwise, it is enough to consider $f^{\pi(p)}$ instead of f.

Under the assumption of Lemma 4.1, by an arbitrarily small perturbation (the readers can refer the proof of [F, Lemma 1.1]), there exists a local coordinate chart $\phi : U_p \to \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$ defined in a small neighborhood $U_p \subset M$ of p satisfying the following:

(A1) $\phi(U_p) = \{\mathbf{z} \in \mathbb{C} : |\mathbf{z}| < 1\} \times (-1,1) =: B^s \times B^u =: B$;

(A2) $\phi(p) = (\mathbf{0}, 0) \in B^s \times B^u$;

(A3) $B^s \subset W^s_{loc}(p)$, $B^u \subset W^u_{loc}(p)$;

(A4) There exists $x = (\mathbf{z}, 0)$ at which $W^s(p)$ intersects $W^u(p)$ non-transversally;

(A5) $\phi \circ f \circ \phi^{-1}$ defined on B act as the linear transformation $\phi \circ Df(p) \circ \phi^{-1}$, that is, for any $(\mathbf{z}, y) \in B$, we have $\phi \circ f \circ \phi^{-1}(\mathbf{z}, y) = (\mu^s \mathbf{z}, \mu^u y)$. In the following, if no confusion arises, in this coordinate chart, we identify f with its conjugation $\phi \circ f \circ \phi^{-1}$;

(A6) There exists $T \in \mathbb{N}$ such that $\phi^{-1}(x) \notin f^{T-1}(U_p)$ but $\phi^{-1}(x) \in f^T(U_p)$, we call f^T the *transition map*. Moreover, replace x if necessary, we can also assume $\phi^{-1}(x) \notin f^j(U_p)$ for $j = 1, \dots, T-1$.

(From now on, points and vectors in B^s will be identified as complex numbers.) It is worth pointing out that (A5) can be obtained by an arbitrarily small perturbation as long as U_p is taken small enough. For any $\epsilon > 0$ fixed, we are going to construct a 2ϵ-perturbation g of f and a periodic point $q \in \pitchfork(p_g)$. Finally the sequence g_k and q_k will be obtained by letting ϵ tend to zero.

We divide the following proof into three parts. In the first part, a hyperbolic horseshoe is built along the orbit of a homoclinic point with small angle. Cone fields are constructed to prove the hyperbolicity. This part is relatively standard, readers can refer [PT, Section 2.3] for a two dimensional model although we are dealing with a three dimensional one. In the second part, we select a periodic point in the horseshoe, consider its iterations and give appropriate perturbations in its orbit at which E^s and E^u have large angle. In the last part, we verify conditions (i)-(iii) of Lemma 3.1 for the sequences.

Before constructing the horseshoe, as preparations, let us fix some important constants. For convenience, two cases are considered in separated ways: (note that the case of $\chi_2(p)+\chi_3(p)=0$ can be dealt using a limit process)

Case (I): $\chi_2(p)+\chi_3(p)>0$**.** For $\epsilon>0$ fixed before, choose $\lambda=\lambda(\epsilon)\in(0,1)$ sufficiently close to 1, such that

$$\frac{\log|\mu^s|-\log|\mu^u|+2\log(1-\frac{\epsilon}{D})}{\log|\mu^s|+\log(1-\frac{\epsilon}{D})-\log\lambda}<1-\frac{\log|\mu^u|}{\log|\mu^s|},$$

where $D=\sup\{\|Dg\|+\|Dg^{-1}\|:\mathrm{dist}_{C^1}(g,f)\le 1\}$. This is possible because the above inequality is equivalent to

$$\begin{aligned}\log|\mu^s|\left(\log|\mu^s|-\log|\mu^u|+2\log\left(1-\frac{\epsilon}{D}\right)\right)&<(\log|\mu^s|-\log|\mu^u|)\left(\log|\mu^s|+\log\left(1-\frac{\epsilon}{D}\right)-\log\lambda\right)\\ 2\log|\mu^s|\log\left(1-\frac{\epsilon}{D}\right)&<(\log|\mu^s|-\log|\mu^u|)\left(\log\left(1-\frac{\epsilon}{D}\right)-\log\lambda\right)\\ \log|\mu^s\mu^u|\log\left(1-\frac{\epsilon}{D}\right)&<\log\lambda\log\left|\frac{\mu^u}{\mu^s}\right|. \qquad (4)\end{aligned}$$

By assumption $|\mu^s\mu^u|>1$, it suffice to choose $\lambda\in(0,1)$ sufficiently close to 1. Notice that $\lambda(\epsilon)\to 1-$ when $\epsilon\to 0$. Next, select $\kappa\in\mathbb{R}$ such that

$$1-\frac{\log|\mu^u|}{\log|\mu^s|}<\kappa<\frac{\log|\mu^s|-\log|\mu^u|}{\log|\mu^s|-\log\lambda}.$$

This definition does make sense because it is easy to see

$$1-\frac{\log|\mu^u|}{\log|\mu^s|}<\frac{\log|\mu^s|-\log|\mu^u|}{\log|\mu^s|-\log\lambda}\iff 0<\lambda<1.$$

The choice of κ provides us two inequalities:

$$|\mu^u(\mu^s)^{\kappa-1}|<1 \text{ and} \qquad (5)$$

$$0<\lambda^\kappa<|\mu^u(\mu^s)^{\kappa-1}|. \qquad (6)$$

Moreover, since $|\mu^s\mu^u|>1$ gives $-\dfrac{\log|\mu^u|}{\log|\mu^s|}>1$, we know $\kappa>2$. Thus

$$|\mu^s(\mu^u)^{\kappa-1}|>|\mu^s\mu^u|>1. \qquad (7)$$

Beside, since

$$\kappa>1-\frac{\log|\mu^u|}{\log|\mu^s|}>\frac{\log|\mu^s|-\log|\mu^u|+2\log(1-\frac{\epsilon}{D})}{\log|\mu^s|+\log(1-\frac{\epsilon}{D})-\log\lambda},$$

we have

$$\lambda^\kappa>\left(1-\frac{\epsilon}{D}\right)^{\kappa-2}|\mu^u(\mu^s)^{\kappa-1}|. \qquad (8)$$

Case (II): $0 > \chi_2(p) + \chi_3(p) > \log(1-\delta)$**.** For $\epsilon > 0$ fixed before, let $\lambda = 1-\delta$. Notice that in contrast with the previous case, here, λ does not depend on ϵ. Thus, $0 > \chi_2(p) + \chi_3(p) > \log(1-\delta)$ can be rewritten as $1 > |\mu^s\mu^u| > \lambda$. Moreover, we can always assume $|\mu^s| < \lambda$, for otherwise, p itself has δ-weak contacting eigenvalues and there is nothing to prove. By the choice of λ, we have

$$1 - \frac{\log|\mu^s|}{\log|\mu^u|} < \frac{\log|\mu^s| - \log|\mu^u| + 2\log(1-\frac{\epsilon}{D})}{\log|\mu^s| + \log(1-\frac{\epsilon}{D}) - \log\lambda}.$$

This is because the above inequality is equivalent to

$$\log|\mu^s\mu^u| \log\left|\frac{\mu^u}{\mu^s}\right| > \log\lambda \log\left|\frac{\mu^u}{\mu^s}\right| + \log|\mu^s\mu^u| \log\left(1-\frac{\epsilon}{D}\right),$$

but $|\mu^s\mu^u| > \lambda$, thus it suffice to shrink ϵ if necessary. Next, select $\kappa \in \mathbb{R}$ satisfying

$$\frac{\log|\mu^s| - \log|\mu^u| + 2\log(1-\frac{\epsilon}{D})}{\log|\mu^s| + \log(1-\frac{\epsilon}{D}) - \log\lambda} < \kappa < \frac{\log|\mu^u| - \log|\mu^s|}{\log\lambda - \log|\mu^s|}.$$

κ is well-defined since

$$\frac{\log|\mu^s| - \log|\mu^u| + 2\log(1-\frac{\epsilon}{D})}{\log|\mu^s| + \log(1-\frac{\epsilon}{D}) - \log\lambda} < \frac{\log|\mu^u| - \log|\mu^s|}{\log\lambda - \log|\mu^s|} \iff |\mu^s\mu^u| > \lambda^2.$$

The choice of κ provides two inequalities:

$$0 < \lambda^\kappa < |\mu^u(\mu^s)^{\kappa-1}|;$$

$$\lambda^\kappa > \left(1-\frac{\epsilon}{D}\right)^{\kappa-2} |\mu^u(\mu^s)^{\kappa-1}|.$$

Moreover,

$$\kappa > \frac{\log|\mu^s| - \log|\mu^u| + 2\log(1-\frac{\epsilon}{D})}{\log|\mu^s| + \log(1-\frac{\epsilon}{D}) - \log\lambda} > 1 - \frac{\log|\mu^s|}{\log|\mu^u|}$$

gives

$$|\mu^s(\mu^u)^{\kappa-1}| > 1.$$

Since $|\mu^s\mu^u| < 1$ as we assumed in this case, $\kappa > 1 - \dfrac{\log|\mu^s|}{\log|\mu^u|} > 2$, which implies

$$|\mu^u(\mu^s)^{\kappa-1}| < |\mu^u\mu^s| < 1.$$

Therefore, we still have a constant $\kappa > 2$ satisfying completely the same inequalities (5)-(8) as in Case (I). It should be pointed out that as long as these inequalities are obtained, the following constructions are fit for both Case (I) and Case (II). The only difference is, $\lambda(\epsilon) \to 1-$ $(\epsilon \to 0)$ when $|\mu^s\mu^u| > 1$ while $\lambda = 1-\delta$ independent of ϵ when $1 > |\mu^s\mu^u| > 1-\delta$.

Now, let us construct the horseshoe. For any $\theta \in (0, \epsilon)$, by a θ-small perturbation of f near $f^{-1}(x)$, we get g such that the above facts (A1)-(A6) still hold for g except (A4), which becomes

(A4') $x = (\mathbf{z}, 0) \in B$ is a transversal homoclinic point of $p_g = p$ with $\mathrm{Angle}(T_xW^s(p), T_xW^u(p)) = \theta$.

Define $N = N(\theta) \in \mathbb{R}$ by

$$\left|\frac{\mu^u}{\mu^s}\right|^N \theta = 1.$$

Clearly, $N \to \infty$ when $\theta \to 0$. For some iteration of g, we are going to build a horseshoe inside B, but notice that the above perturbation from f to g only affect the angle between $T_xW^s(p)$ and $T_xW^u(p)$

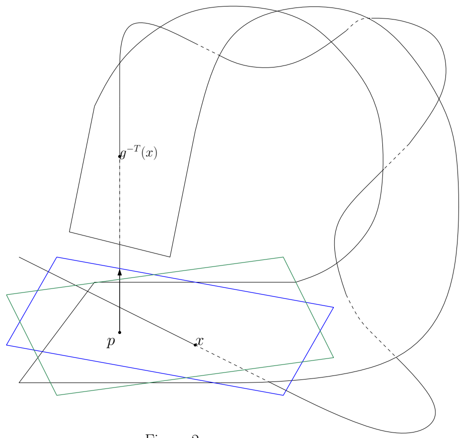

Figure 2

which is in general not sufficient to guarantee $g^T(B)$ passing through B from its top to the bottom. Therefore, we need to cut B to get some subset with smaller height. To be more precise, define

$$B_H^\theta = B^s \times \left(\frac{1}{|\mu^u|^{[\kappa N]}} B^u\right) = \{\mathbf{z} \in \mathbb{C} : |\mathbf{z}| < 1\} \times \left(-\frac{1}{|\mu^u|^{[\kappa N]}}, \frac{1}{|\mu^u|^{[\kappa N]}}\right).$$

Here and subsequently, for $s \in \mathbb{R}$, let $[s]$ denote the smallest integer that is larger than s. (Note that this notation is different from the usual one.) When $\theta \to 0$, the height of B_H^θ, denoted by $h(B_H^\theta)$, decreases to zero, but by inequality (7), we have

$$\lim_{\theta\to 0} \frac{\theta}{h(B_H^\theta)} = \lim_{N\to\infty} \left(\left|\frac{\mu^s}{\mu^u}\right|^N |\mu^u|^{[\kappa N]} \right) \geq \lim_{N\to\infty} \left|\mu^s (\mu^u)^{\kappa-1}\right|^N = \infty,$$

which shows that $h(B_H^\theta)$ decrease more rapidly than θ. Therefore, by local linearization property of the derivatives, we can always assume that the connected component of $g^T(\{\mathbf{0}\} \times B^u) \cap B_H^\theta$ containing x is a 1-dimensional straight segment whose boundary is contained in $B^s \times \partial\left(\frac{1}{|\mu^u|^{[\kappa N]}} B^u\right)$. Briefly, we say that $g^T(\{\mathbf{0}\} \times B^u)$ passes through B_H^θ along B^u-direction. By continuity, we have

Fact 4.3. If $D^s \subset B^s$ is a disk centered at $\mathbf{0} \in B^s$ whose radius is sufficiently small, then $g^T(D^s \times B^u)$ also passes through B_H^θ along B^u-direction.

Symmetrically, let

$$B_V^\theta = \left(|\mu^s|^{[\kappa N]} B^s\right) \times B^u = \left\{\mathbf{z} \in \mathbb{C} : |\mathbf{z}| < |\mu^s|^{[\kappa N]}\right\} \times (-1, 1).$$

When $\theta \to 0$, the width of B_V^θ, denoted by $v(B_V^\theta)$, decrease to zero. Moreover, since T is independent of θ, there exists a constant $c_T \geq 1$ which only depends on T and a fixed neighborhood of f, such that

$$\mathrm{Angle}(T_{g^{-T}(x)} W^s(p), T_{g^{-T}(x)} W^u(p)) =: \Theta \in [c_T^{-1}\theta, c_T\theta]. \tag{9}$$

As above, by the inequality (5), we have

$$\lim_{\theta\to 0} \frac{\Theta}{v(B_V^\theta)} \geq \lim_{\theta\to 0} \frac{c_T^{-1}\theta}{v(B_V^\theta)} = \lim_{N\to\infty} \left(\frac{c_T^{-1}}{|\mu^s|^{[\kappa N]}} \left|\frac{\mu^s}{\mu^u}\right|^N \right) \geq \lim_{N\to\infty} c_T^{-1} \left(\frac{1}{|\mu^u (\mu^s)^{\kappa-1}|}\right)^N = \infty,$$

which shows that $v(B_V^\theta)$ decrease to zero faster than $\mathrm{Angle}(T_{g^{-T}(x)} W^s(p), T_{g^{-T}(x)} W^u(p))$. Hence by an arbitrarily small perturbation, we can assume the connected component of $g^{-T}(B^s \times \{0\}) \cap B_V^\theta$ containing $g^{-T}(x)$ is a 2-dimensional disk whose boundary is contained in $\partial(|\mu^s|^{[\kappa N]} B^s) \times B^u$. Briefly, we say that $g^{-T}(B^s \times \{0\})$ passes through B_V^θ along B^s-direction.

The above construction shows the existence of a topological horseshoe of $g^{[\kappa N]+T}$ inside B_H^θ. In fact,

$$\Lambda_H^\theta := \bigcap_{n=-\infty}^{+\infty} g^{([\kappa N]+T)n}(B_H^\theta)$$

is a $g^{[\kappa N]+T}$-invariant subset, on which $g^{[\kappa N]+T}$ conjugate to a full shift of two symbols. By using the cone field criterion, we will prove that Λ_H^θ is actually a hyperbolic horseshoe. To see this, notice that $g^{[\kappa N]+T}(B_H^\theta) = g^T(B_V^\theta)$. Thus, when θ is small, according to Fact 4.3, there are two connected components of $g^{[\kappa N]+T}(B_H^\theta) \cap B_H^\theta$, both of them pass through B_H^θ along B^u-direction. As a result, $g^{-([\kappa N]+T)}(B_H^\theta) \cap B_H^\theta$ consists of another two components which pass through B_H^θ along B^s-direction. Therefore, $B_H^\theta \cap g^{[\kappa N]+T}(B_H^\theta) \cap g^{-([\kappa N]+T)}(B_H^\theta)$ has totally four components, three of which contain p, x, $g^{-([\kappa N]+T)}(x)$, respectively. We denote these three components by $\mathrm{comp}(p)$, $\mathrm{comp}(x)$, $\mathrm{comp}(g^{-([\kappa N]+T)}(x))$ and the rest one by $\mathrm{comp}(\star)$. Let us define a unstable cone field on Λ_H^θ as follows:

(UC1) For every $w \in \mathrm{comp}(p) \cup \mathrm{comp}(g^{-([\kappa N]+T)}(x))$, let $C_w^u(1) = \{(\mathbf{z}, y) \in T_w M : |\mathbf{z}| \le |y|\}$;

(UC2) For every $w \in \mathrm{comp}(x) \cup \mathrm{comp}(\star)$, let $C_w^u\left(\frac{4}{3\theta}\right) = \left\{(\mathbf{z}, y) \in T_w M : |\mathbf{z}| \le \frac{4}{3\theta}|y|\right\}$.

In order to define a stable cone field, for convenience, we consider B_V^θ instead of B_H^θ. Similarly as before, since $g^{-([\kappa N]+T)}(B_V^\theta) = g^{-T}(B_H^\theta)$, when θ is small, there are two connected components of $g^{-([\kappa N]+T)}(B_V^\theta) \cap B_V^\theta$, both of them pass through B_V^θ along B^s-direction. As a result, $g^{[\kappa N]+T}(B_V^\theta) \cap B_V^\theta$ consists of another two components which pass through B_V^θ along B^u-direction. We denote the four connected components of $B_V^\theta \cap g^{[\kappa N]+T}(B_V^\theta) \cap g^{-([\kappa N]+T)}(B_V^\theta)$ by $\mathrm{Comp}(p)$, $\mathrm{Comp}(g^{-T}(x))$, $\mathrm{Comp}(g^{[\kappa N]}(x))$ and $\mathrm{Comp}(\star)$, where the first three contains p, $g^{-T}(x)$ and $g^{[\kappa N]}(x)$, respectively. Define a stable cone field on Λ_V^θ as the following:

(SC1) For every $w \in \mathrm{Comp}(p) \cup \mathrm{Comp}(g^{[\kappa N]}(x))$, let $C_w^s(1) = \{(\mathbf{z}, y) \in T_w M : |y| \le |\mathbf{z}|\}$;

(SC2) For every $w \in \mathrm{Comp}(g^{-T}(x)) \cup \mathrm{Comp}(\star)$, let $C_w^s\left(\frac{4}{3\Theta}\right) = \left\{(\mathbf{z}, y) \in T_w M : |y| \le \frac{4}{3\Theta}|\mathbf{z}|\right\}$.

Here, we gave the definition of unstable cone field on B_H^θ while stable cone field on B_V^θ. Indeed, it does not matter because $g^{[\kappa N]}(B_H^\theta) = B_V^\theta$, which implies

$$\begin{aligned}\Lambda_V^\theta &= \bigcap_{n=-\infty}^{+\infty} g^{([\kappa N]+T)n}(B_V^\theta) = \bigcap_{n=-\infty}^{+\infty} g^{([\kappa N]+T)n+[\kappa N]}(B_H^\theta) \\ &= \bigcap_{n=-\infty}^{+\infty} g^{([\kappa N]+T)n-T}(B_H^\theta) = g^{-T}\left(\bigcap_{n=-\infty}^{+\infty} g^{([\kappa N]+T)n}(B_H^\theta)\right) = g^{-T}(\Lambda_H^\theta).\end{aligned}$$

Thus Λ_V^θ and Λ_H^θ only differ at some fixed number of iterations. If we can show C_w^s is a stable cone field on Λ_V^θ, define $E^s(w) = \cap_{n=0}^{\infty} Dg^{-([\kappa N]+T)n}(C^s_{g^{([\kappa N]+T)n}(w)})$ for every $w \in \Lambda_V^\theta$. Then $g^T(E^s(g^{-T}(w)))$ is the stable bundle for every $w \in \Lambda_H^\theta$. In other words, to prove the hyperbolicity of Λ_H^θ (or Λ_V^θ), it suffice to prove that C_w^u is a unstable cone field on Λ_H^θ and C_w^s is a stable cone field on Λ_V^θ.

Lemma 4.4. (Uniform invariance) For sufficiently small $\theta > 0$,

(1) $Dg^{[\kappa N]+T}(w)(C_w^u(\cdot)) \subset \mathrm{int} C^u_{g^{[\kappa N]+T}(w)}(\frac{6}{7}\cdot) \cup \{0\}$ for every $w \in \Lambda_H^\theta$;

(2) $Dg^{-([\kappa N]+T)}(w)(C_w^s(\cdot)) \subset \mathrm{int} C^s_{g^{-([\kappa N]+T)}(w)}(\frac{6}{7}\cdot) \cup \{0\}$ for every $w \in \Lambda_V^\theta$.

Proof. We define the H-slope (V-slope) of a vector $v = (\mathbf{z}, y) \in B^s \times B^u$ as $|\mathbf{z}|/|y|$ (resp. $|y|/|\mathbf{z}|$). Thus the definition of unstable (stable) cone field can be easily rewritten using the notion of H-slope (resp. V-slope).

(1a) For any $w \in \Lambda_H^\theta \cap (\mathrm{comp}(p) \cup \mathrm{comp}(x))$, we have

$$g^{[\kappa N]+T}(w) \in \mathrm{comp}(p) \cup \mathrm{comp}(g^{-([\kappa N]+T)}(x)).$$

Take any vector $(\mathbf{z}, y) \in C_w^u(*)$, where $* = 1$ or $\frac{4}{3\theta}$, depending on the component that w belongs to. Then $Dg^{[\kappa N]+T}(w)(\mathbf{z}, y) = \big((\mu^s)^{[\kappa N]+T}\mathbf{z}, (\mu^u)^{[\kappa N]+T}y\big)$ whose H-slope is

$$\frac{|\mathbf{z}|}{|y|}\left|\frac{\mu^s}{\mu^u}\right|^{[\kappa N]+T} \leq \max\left\{1, \frac{4}{3\theta}\right\}\left|\frac{\mu^s}{\mu^u}\right|^{\kappa N+T} = \frac{4}{3}\left|\frac{\mu^s}{\mu^u}\right|^{T}\theta^{\kappa-1} < \frac{6}{7}$$

whenever θ is sufficiently small. Recall that $\kappa > 2$.

(1b) For any $w \in \Lambda_H^\theta \cap (\mathrm{comp}(g^{-([\kappa N]+T)}(x)) \cup \mathrm{comp}(\star))$, we have

$$g^{[\kappa N]+T}(w) \in \mathrm{comp}(x) \cup \mathrm{comp}(\star).$$

Take any vector $(\mathbf{z}, y) \in C_w^u(*)$, where $* = 1$ or $\frac{4}{3\theta}$, depending on the component that w belongs to. then $Dg^{[\kappa N]}(w)(\mathbf{z}, y) = \big((\mu^s)^{[\kappa N]}\mathbf{z}, (\mu^u)^{[\kappa N]}y\big)$ whose H-slope is

$$\frac{|\mathbf{z}|}{|y|}\left|\frac{\mu^s}{\mu^u}\right|^{[\kappa N]} \leq \max\left\{1, \frac{4}{3\theta}\right\}\left|\frac{\mu^s}{\mu^u}\right|^{\kappa N} = \frac{4}{3}\theta^{\kappa-1}.$$

Thus,

$$\mathrm{Angle}\left(Dg^{[\kappa N]}(w)(\mathbf{z}, y), \{\mathbf{0}\} \times B^u\right) \leq \arctan\left(\frac{4}{3}\theta^{\kappa-1}\right) \tag{10}$$

which is a higher order infinitesimal of θ. Since $g^T(\{\mathbf{0}\} \times B^u)$ intersects $B^s \times \{0\}$ at x with angle θ (recall (A4') before), we obtain

$$\mathrm{Angle}\left(Dg^{[\kappa N]+T}(w)(\mathbf{z}, y), B^s \times \{0\}\right) \in \left[\theta - c_T \arctan\left(\frac{4}{3}\theta^{\kappa-1}\right), \theta + c_T \arctan\left(\frac{4}{3}\theta^{\kappa-1}\right)\right].$$

The H-slope of $Dg^{[\kappa N]+T}(w)(\mathbf{z}, y)$ is smaller than

$$\cot\left(\theta - c_T \arctan\left(\frac{4}{3}\theta^{\kappa-1}\right)\right) < \frac{8}{7\theta} = \frac{6}{7} \cdot \frac{4}{3\theta}$$

whenever $\theta > 0$ is sufficiently small. The last inequality holds because

$$\lim_{\theta \to 0} \frac{7\theta}{8} \cot\left(\theta - c_T \arctan\left(\frac{4}{3}\theta^{\kappa-1}\right)\right) = \frac{7}{8} < 1.$$

Now, (1a) and (1b) together imply (1).

(2a) For any $w \in \Lambda_V^\theta \cap (\mathrm{Comp}(p) \cup \mathrm{Comp}(g^{-T}(x)))$, we have

$$g^{-([\kappa N]+T)}(w) \in \mathrm{Comp}(p) \cup \mathrm{Comp}(g^{[\kappa N]}(x)).$$

Take any vector $(\mathbf{z}, y) \in C_w^s(*)$, where $* = 1$ or $\frac{4}{3\Theta}$, depending on the component that w belongs to. Then $Dg^{-([\kappa N]+T)}(w)(\mathbf{z}, y) = \big((\mu^s)^{-([\kappa N]+T)}\mathbf{z}, (\mu^u)^{-([\kappa N]+T)}y\big)$ whose V-slope is, recalling that $c_T^{-1}\theta \leq \Theta \leq c_T\theta$,

$$\frac{|y|}{|\mathbf{z}|}\left|\frac{\mu^u}{\mu^s}\right|^{-([\kappa N]+T)} \leq \max\left\{1, \frac{4}{3\Theta}\right\}\left|\frac{\mu^s}{\mu^u}\right|^{\kappa N+T} \leq \frac{4}{3}\left|\frac{\mu^s}{\mu^u}\right|^{T} c_T^\kappa \Theta^{\kappa-1} < \frac{6}{7}$$

whenever θ (hence Θ) is sufficiently small.

(2b) For any $w \in \Lambda_V^\theta \cap (\mathrm{Comp}(g^{[\kappa N]}(x)) \cup \mathrm{Comp}(\star))$, we have

$$g^{-([\kappa N]+T)}(w) \in \mathrm{Comp}(g^{-T}(x)) \cup \mathrm{Comp}(\star).$$

Take any vector $(\mathbf{z}, y) \in C_w^s(*)$ where $* = 1$ or $\frac{4}{3\Theta}$, depending on the component that w belongs to. Then $Dg^{-[\kappa N]}(w)(\mathbf{z}, y) = \big((\mu^s)^{-[\kappa N]}\mathbf{z}, (\mu^u)^{-[\kappa N]}y\big)$ whose V-slope is

$$\frac{|y|}{|\mathbf{z}|}\left|\frac{\mu^u}{\mu^s}\right|^{-[\kappa N]} \le \max\left\{1, \frac{4}{3\Theta}\right\}\left|\frac{\mu^s}{\mu^u}\right|^{\kappa N} \le \frac{4}{3}c_T^\kappa \Theta^{\kappa-1}.$$

Thus,

$$\mathrm{Angle}\left(Dg^{-[\kappa N]}(w)(\mathbf{z}, y), B^s \times \{0\}\right) \le \arctan\left(\frac{4}{3}c_T^\kappa \Theta^{\kappa-1}\right) \tag{11}$$

which is a higher order infinitesimal of Θ. Since $g^{-T}(B^s \times \{0\})$ intersects $\{\mathbf{0}\} \times B^u$ at $f^{-T}(x)$ with angle Θ (recall (9)), we get

$$\mathrm{Angle}(Dg^{-([\kappa N]+T)}(w)(\mathbf{z}, y), \{\mathbf{0}\} \times B^u) \in \left[\Theta - c_T \arctan\left(\frac{4}{3}c_T^\kappa \Theta^{\kappa-1}\right), \Theta + c_T \arctan\left(\frac{4}{3}c_T^\kappa \Theta^{\kappa-1}\right)\right]$$

The V-slope of $Dg^{-([\kappa N]+T)}(w)(\mathbf{z}, y)$ is smaller than

$$\cot\left(\Theta - c_T \arctan\left(\frac{4}{3}c_T^\kappa \Theta^{\kappa-1}\right)\right) < \frac{8}{7\Theta} = \frac{6}{7}\cdot\frac{4}{3\Theta}$$

whenever θ (hence Θ) is sufficiently small. The last inequality holds because

$$\lim_{\Theta\to 0}\frac{7\Theta}{8}\cot\left(\Theta - c_T \arctan\left(\frac{4}{3}c_T^\kappa \Theta^{\kappa-1}\right)\right) = \frac{7}{8} < 1.$$

Now, (2a) and (2b) together imply (2). Lemma 4.4 is proved.

Remark 4.5. In the above proof, we used the definition of unstable cone field given by (UC). However, in the proof of (1b) above, since $c_T \arctan(\frac{4}{3}\theta^{\kappa-1})$ is a higher order infinitesimal of θ, as a result, for every $w \in \Lambda_H^\theta \cap \big(\mathrm{comp}(g^{-([\kappa N]+T)}(x)) \cup \mathrm{comp}(\star)\big)$, we see that $Dg^{[\kappa N]+T}(C_w^u)$ is contained in $\tilde{C}_{w'}^u := \{v \in T_{w'}M : \mathrm{Angle}(v, T_x W^u(p)) \le \frac{\theta}{4}\}$ where $w' = g^{[\kappa N]+T}(w) \in \Lambda_H^\theta \cap (\mathrm{comp}(x) \cup \mathrm{comp}(\star))$. Thus we can modify the definition of the unstable cone field as follows:

- For every $w \in \mathrm{comp}(p) \cup \mathrm{comp}(g^{-([\kappa N]+T)}(x))$, let $\tilde{C}_w^u = \{v \in T_w M : \mathrm{Angle}(v, \{\mathbf{0}\} \times B^u) \le \frac{\pi}{4}\}$;
- For every $w \in \mathrm{comp}(x) \cup \mathrm{comp}(\star)$, let $\tilde{C}_w^u = \{v \in T_w M : \mathrm{Angle}(v, T_x W^u(p)) \le \frac{\theta}{4}\}$.

Similarly, the definition of the stable cone field can be replaced by:

- $\tilde{C}_w^s = \{v \in T_w M : \mathrm{Angle}(v, B^s \times \{0\}) \le \frac{\pi}{4}\}$ for every $w \in \mathrm{Comp}(p) \cup \mathrm{Comp}(g^{[\kappa N]}(x))$;
- $\tilde{C}_w^s = \{v \in T_w M : \mathrm{Angle}(v, T_{g^{-T}(x)} W^s(p)) \le \frac{\theta}{4}\}$ for every $w \in \mathrm{Comp}(g^{-T}(x)) \cup \mathrm{Comp}(\star)$.

From now on, to simplify the notation, we write $\tilde{C}_w^u$ and $\tilde{C}_w^s$ again by C_w^u and C_w^s. One can easily verify that Lemma 4.4 still holds for these newly defined C_w^u and C_w^s.

Lemma 4.6. (Uniform expansion) For sufficiently small $\theta > 0$,

(1) $\|Dg^{[\kappa N]+T}(w)v\| \ge 2\|v\|$ for all $w \in \Lambda_H^\theta$ and all $v \in C_w^u$;

(2) $\|Dg^{-([\kappa N]+T)}(w)v\| \ge 2\|v\|$ for all $w \in \Lambda_V^\theta$ and all $v \in C_w^s$.

Proof. By the definition of unstable cone field in the previous remark, vectors in C^u_w ($w \in \mathrm{comp}(p) \cup \mathrm{comp}(g^{-([\kappa N]+T)}(x))$) expands much more than vectors in C^u_w ($w \in \mathrm{comp}(x) \cup \mathrm{comp}(\star)$). Hence we only need to verify the expanding rate of the latter one. Take any $w \in \mathrm{comp}(x) \cup \mathrm{comp}(\star)$ and any unit vector $v = (\mathbf{z}, y) \in C^u_w$. Obviously, whenever θ is small, $|\mathbf{z}| > \frac{1}{2}$ and $\frac{|y|}{|\mathbf{z}|} \geq \frac{3}{4}\tan\theta \geq \frac{\theta}{2}$. we have $Dg^{[\kappa N]+T}(w)v = Dg^T(g^{[\kappa N]}(w))((\mu^s)^{[\kappa N]}\mathbf{z}, (\mu^u)^{[\kappa N]}y)$ whose norm is larger than

$$\begin{aligned}\frac{1}{D^T}\left\|((\mu^s)^{[\kappa N]}\mathbf{z}, (\mu^u)^{[\kappa N]}y)\right\| &\geq \frac{1}{D^T}|(\mu^u)^{[\kappa N]}y| = \frac{|\mathbf{z}|}{D^T}\left|\frac{y}{\mathbf{z}}\right||\mu^u|^{[\kappa N]} \\ &\geq \frac{\theta|\mathbf{z}|}{2D^T}|\mu^u|^{\kappa N} \geq \frac{1}{4D^T}|\mu^s(\mu^u)^{\kappa-1}|^N > 2.\end{aligned}$$

Recall that g is a θ-perturbation of f hence $\|Dg\| \leq D$. The last inequality holds because $4D^T$ is a constant which is independent of θ while (7) tell us $|\mu^s(\mu^u)^{\kappa-1}|^N$ goes to infinity as $\theta \to 0$.

Symmetrically, by the definition of stable cone field in Remark 4.5, under negative iterations, vectors in C^s_w ($w \in \mathrm{Comp}(p) \cup \mathrm{Comp}(g^{[\kappa N]}(x))$) expand much more than vectors in C^s_w ($w \in \mathrm{Comp}(g^{-T}(x)) \cup \mathrm{Comp}(\star)$). Thus it suffice to verify the latter one. Take any $w \in \mathrm{Comp}(g^{-T}(x)) \cup \mathrm{Comp}(\star)$ and any unit vector $v = (\mathbf{z}, y) \in C^s_w$, we have $|\mathbf{z}| \geq \frac{3}{4}\tan\Theta \geq \frac{\Theta}{2}$. Thus

$$Dg^{-([\kappa N]+T)}(w)v = Dg^{-T}(g^{-[\kappa N]}(w))((\mu^s)^{-[\kappa N]}\mathbf{z}, (\mu^u)^{-[\kappa N]}y)$$

whose norm is larger than

$$\frac{1}{D^T}|(\mu^s)^{-[\kappa N]}\mathbf{z}| \geq \frac{\Theta}{2D^T|\mu^s|^{\kappa N}} > \frac{1}{2c_T D^T|\mu^u(\mu^s)^{\kappa-1}|^N} > 2 \tag{12}$$

whenever θ is small. The last inequality follows from (5). The proof of Lemma 4.6 is complete.

To summarize, for any $\theta \in (0, \epsilon)$ sufficiently small, Lemmas 4.4 and 4.6 together imply that Λ^θ_H (also Λ^θ_V) is a hyperbolic horseshoe of $g^{[\kappa N]+T}$. In fact, for every $w \in \Lambda^\theta_H$,

$$\begin{aligned}E^u(w) &= \bigcap_{n=0}^{\infty} g^{([\kappa N]+T)n}(C^u_{g^{-([\kappa N]+T)n}(w)}) \\ E^s(w) &= g^T\left(\bigcap_{n=0}^{\infty} g^{-([\kappa N]+T)n}(C^s_{g^{([\kappa N]+T)n-T}(w)})\right)\end{aligned}$$

are the expanding and contracting bundles of Λ^θ_H. As a consequence, there are exactly two fixed points of $g^{[\kappa N]+T}$ in Λ^θ_H: one is p, which is also a fixed point of g, and the other one is denoted by $q = q(\theta)$. Recall that g also depends on θ, where we have been omitting the symbol θ for simplicity. By construction, q is homoclinically related to p_g.

Remark 4.7. Lemmas 4.4 and 4.6 are proved under the hypothesis that (recall (9))

$$\mathrm{Angle}(T_{g^{-T}(x)}W^s(p), T_{g^{-T}(x)}W^u(p)) = \Theta \in [c_T^{-1}\theta, c_T\theta].$$

But from the above calculations, we see that actually, c_T does not bring any essential affection, just appearing as a constant which is independent of ϵ and θ. Thus, in what follows, let us set $c_T = 1$ for simplicity. In other words, take $\Theta = \theta$ directly. The readers can verify step by step as in the proof of Lemmas 4.4 and 4.6 that such a simplification involves no loss of generality.

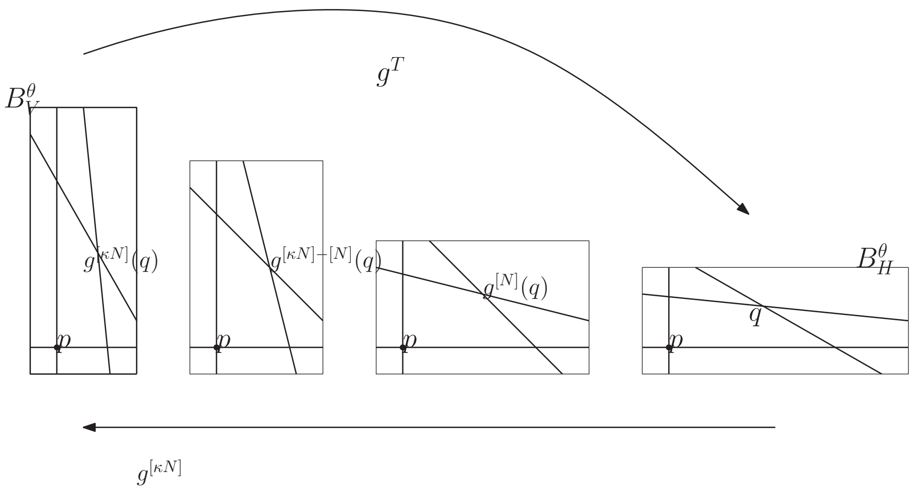

Figure 3

Remark 4.8. Combining Lemma 4.4, Remarks 4.5 and 4.7, we summarize the following facts which will be convenient for later use.

(1) $\mathrm{Angle}(E^u(q), T_xW^u(p)) \in (\frac{4}{5}\theta^{\kappa-1}, \frac{4}{3}\theta^{\kappa-1})$;

(2) $\mathrm{Angle}(E^s(q), B^s \times \{0\}) \in (\frac{4}{5}\theta^{\kappa-1}, \frac{4}{3}\theta^{\kappa-1})$;

(3) $\mathrm{Angle}(E^s(q), E^u(q)) < 2\theta$;

In fact, (1) and (2) comes from (10) and (11), respectively, and (3) is straightforward from (1) and (2) since $\theta^{\kappa-1}$ is a higher order infinitesimal of θ.

Let us divide the iteration from q to $g^{[\kappa N]}(q)$ into three parts: $[\kappa N] = [N] + ([\kappa N] - 2[N]) + [N]$ (see Figure 3 for a conceptional picture), recalling that $\kappa > 2$. The following lemma tells us that, in the middle part, E^s and E^u exhibit large angles.

Lemma 4.9. For $\theta > 0$ small enough, $\angle(E^s(g^i(q)), E^u(g^i(q))) > \dfrac{1}{2}$ for every $i = [N], \ldots, [\kappa N] - [N]$.

Proof. By Remark 4.8, using similar estimations as in the proof of Lemma 4.4, for any $a \in [0, \kappa]$,

$$\inf\{\text{V-slope}(u) : u \in C^u_{g^{[aN]}(q)}\} \geq \frac{3}{4}\theta \left|\frac{\mu^u}{\mu^s}\right|^{aN} = \frac{3}{4}\theta^{1-a},$$
$$\sup\{\text{V-slope}(u) : u \in C^s_{g^{[aN]}(q)}\} \leq \frac{4}{3\theta} \left|\frac{\mu^s}{\mu^u}\right|^{(\kappa-a)N} = \frac{4}{3}\theta^{\kappa-1-a}.$$

Thus, letting $\sigma_a = \mathrm{Angle}(E^s(g^{[aN]}(q)), E^u(g^{[aN]}(q)))$,

$$\begin{aligned}\angle(E^s(g^{[aN]}(q)), E^u(g^{[aN]}(q))) = \tan\sigma_a &\geq \tan\left(\arctan\left(\frac{3}{4}\theta^{1-a}\right) - \arctan\left(\frac{4}{3}\theta^{\kappa-1-a}\right)\right) \\ &= \frac{\frac{3}{4}\theta^{1-a} - \frac{4}{3}\theta^{\kappa-1-a}}{1 + \theta^{\kappa-2a}} =: \Gamma_a. \end{aligned} \tag{13}$$

As a lower bound of $\tan\sigma_a$, let us analysis how Γ_a varies with a.

Claim. $\Gamma_a \geq \Gamma_1 = \Gamma_{\kappa-1}$ for $a \in [1, \kappa-1]$.

In fact, take $a = 1$ and $a = \kappa - 1$ in (13), we see that

$$\Gamma_1 = \Gamma_{\kappa-1} = \frac{\frac{3}{4} - \frac{4}{3}\theta^{\kappa-2}}{1 + \theta^{\kappa-2}}.$$

On the other hand, $\Gamma_a \geq \Gamma_1$ if and only if

$$\frac{\frac{3}{4} - \frac{4}{3}\theta^{\kappa-2}}{\theta^{a-1} + \theta^{\kappa-1-a}} = \frac{\frac{3}{4}\theta^{1-a} - \frac{4}{3}\theta^{\kappa-1-a}}{1 + \theta^{\kappa-2a}} \geq \frac{\frac{3}{4} - \frac{4}{3}\theta^{\kappa-2}}{1 + \theta^{\kappa-2}},$$

implying that

$$1 + \theta^{\kappa-2} \geq \theta^{a-1} + \theta^{\kappa-1-a}.$$

By analyzing the derivatives, when a increasing from 1 to $\kappa - 1$, the right hand term first decreases, and then increases after reaching its minimum at $a = \kappa/2$, which implies that

$$\max_{a\in[1,\kappa-1]} \left\{\theta^{a-1} + \theta^{\kappa-1-a}\right\} = 1 + \theta^{\kappa-2}.$$

The claim is proved. As a result, when θ is sufficiently small, $\tan\sigma_a \geq \tan\sigma_1 \geq \frac{1}{2}$ which completes the proof of Lemma 4.9.

Remark 4.10. Since the angle is larger than a universal constant, when giving perturbations of the derivatives at these points keeping E^s and E^u invariant, we are allowed to estimate the size of the perturbation by considering the sum of its norms on E^s and E^u. In other words, up to a constant multiple, we can assume that the perturbation size comes from only perturbations over individual subbundle (recall Lemma 2.10).

In the coordinate chart $B^s \times B^u$, we set $B^s \times \{0\}$ as the horizontal plane and $\{\mathbf{0}\} \times B^u$ as the vertical axis. Given any 2-dimensional plan G which is not parallel to $B^s \times \{0\}$, there are two uniquely defined unit vectors in G, denoted by u_h and u_l, where u_h is called the *horizontal direction*, which is parallel to $B^s \times \{0\}$ and u_l is called the *slope direction*, which is perpendicular to u_h (see Figure 4). Obviously, u_l have the largest angle with $B^s \times \{0\}$ among all vectors in G. It is very easy to check that under the iterations of g, the horizontal (slope) direction is still sent to the horizontal (slope) direction.

Fact 4.11. Suppose $\mathrm{Angle}(E, B^s \times \{0\}) = \gamma$, let u be any vector in E with $\mathrm{Angle}(u, u_h) = \alpha$. Then $\mathrm{Angle}(u, B^s \times \{0\}) = \arcsin(\sin\gamma \sin\alpha)$.

Indeed, set $G = \mathrm{span}\{AB, AC\}$, $B^s \times \{0\} = \mathrm{span}\{AD, AE\}$, $u_h = BC$, $u_l = AB$, $u = AC$ and $|AB| = 1$ as in Figure 4 and 5. Then, by assumption, $|CD| = |BE| = \sin\gamma$ and $|AC| = \dfrac{1}{\sin\alpha}$. Hence in $\triangle ACD$,

$$\sin \mathrm{Angle}(u, B^s \times \{0\}) = \sin \mathrm{Angle}(AC, AD) = \frac{|CD|}{|AC|} = \sin\gamma \sin\alpha,$$

which gives the conclusion immediately.

Observe that when θ decreasing to zero in $\mathbb{R}$ continuously, $\pi(q) = \pi(q(\theta)) = [\kappa N(\theta)] + T$ increases to infinity in $\mathbb{N}$ continuously. According to Remark 4.8, when θ is small enough, both $\mathrm{Angle}(E^s(q), B^s \times \{0\})$ and $\mathrm{Angle}(E^u(q), T_x W^u(p))$ are higher order infinitesimal of θ. Hence, if we use the same notation $\xi(q)$ as in Lemma 3.1 to denote the orthogonal projection of $E^u(q)$ into $E^s(q)$, then $\mathrm{Angle}(u_h(q), \xi(q))$ varies on the unit circle as a rotation with angle ϕ where ϕ is the argument of the complex contracting eigenvalue of p. With arbitrarily small perturbation if necessary, we can take ϕ to be irrational. As a result, for the $\epsilon > 0$ which has been fixed at the very beginning, by decreasing θ if necessary, we are allowed to take θ such that $\mathrm{Angle}(u_h(q), \xi(q)) < \dfrac{\epsilon}{32cD}$, where $c = \left|\dfrac{\mu^u}{\mu^s}\right| > 1$, only depending on p.

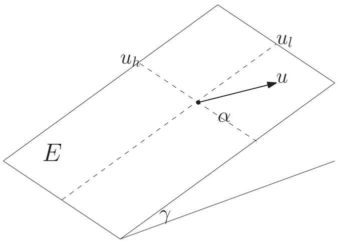

Figure 4

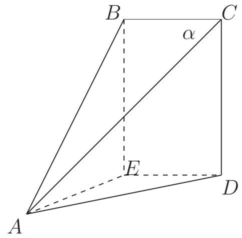

Figure 5

Lemma 4.12. $\mathrm{Angle}(Dg^{[N]}(q)\xi(q), u_h(g^{[N]}(q))) \le \dfrac{\epsilon}{2D}$.

Proof. Write $\xi(q) = (\mathbf{z}, y) \in E^s(q)$. By item (2) of Remark 4.8, $\mathrm{Angle}(E^s(q), B^s \times \{0\}) \le 2\theta^{\kappa-1}$. Apply Fact 4.11 to get $\mathrm{Angle}(\xi(q), B^s \times \{0\}) \le \arcsin\left(\sin(2\theta^{\kappa-1})\sin\left(\dfrac{\epsilon}{32cD}\right)\right)$. Thus,

$$\left|\frac{y}{\mathbf{z}}\right| = \tan \mathrm{Angle}(\xi(q), B^s \times \{0\}) \le \frac{\epsilon\theta^{\kappa-1}}{8cD}.$$

Then $Dg^{[N]}(q)\xi(q) = ((\mu^s)^{[N]}\mathbf{z}, (\mu^u)^{[N]}y)$ whose V-slope is

$$\left|\frac{(\mu^u)^{[N]}y}{(\mu^s)^{[N]}\mathbf{z}}\right| \le c\left|\frac{(\mu^u)^N y}{(\mu^s)^N \mathbf{z}}\right| = \frac{c}{\theta}\left|\frac{y}{\mathbf{z}}\right| \le \frac{\epsilon\theta^{\kappa-2}}{8D}.$$

On the other hand, it follows easily by Remark 4.8 (2) that

$$\mathrm{Angle}(E^s(g^{[N]}(q)), B^s \times \{0\}) \ge \left|\frac{\mu^u}{\mu^s}\right|^N \frac{\theta^{\kappa-1}}{2} = \frac{\theta^{\kappa-2}}{2}.$$

Applying Fact 4.11 again, we obtain

$$\begin{aligned}\frac{\epsilon\theta^{\kappa-2}}{8D} &\ge \tan \mathrm{Angle}(Dg^{[N]}(q)\xi(q), B^s \times \{0\}) \\ &\ge \tan\circ\arcsin\left(\sin\left(\frac{\theta^{\kappa-2}}{2}\right)\sin \mathrm{Angle}\left(Dg^{[N]}(q)\xi(q), u_h(g^{[N]}(q))\right)\right) \\ &\ge \frac{\theta^{\kappa-2}}{4}\mathrm{Angle}(Dg^{[N]}(q)\xi(q), u_h(g^{[N]}(q))).\end{aligned}$$

That is, $\mathrm{Angle}(Dg^{[N]}(q)\xi(q), u_h(g^{[N]}(q))) \le \dfrac{\epsilon}{2D}$ as desired.

In the following, we will give perturbations at the central part of $\mathrm{orb}_g(q)$. All of them take place in E^s while E^u will not be changed all the time. We point out that in this process, the angle between E^s and E^u need not to be considered, see Remark 4.10.

Step 1. Write $\omega = \mathrm{Angle}(Dg^{[N]}(q)\xi(q), u_h(g^{[N]}(q)))$. Under some standard orthogonal coordinate chart of $E^s(g^{[N]}(q))$, the isotopic perturbation $\begin{pmatrix} \cos t\omega & -\sin t\omega \\ \sin t\omega & \cos t\omega \end{pmatrix} \circ Dg|_{E^s}(g^{[N]}(q))$ of $Dg|_{E^s}(g^{[N]}(q))$ sends $Dg^{[N]}(q)\xi(q)$ into $u_h(g^{[N]}(q))$, where $t \in [0,1]$. It can be easily verified as we did in the proof of Lemma 4.6 that the corresponding path of the first return map keeps being hyperbolic for all $t \in [0,1]$. Thus, by the estimation in the previous lemma, there exists ϵ-perturbation G_1 of g, satisfying $DG_1^{[N]}(q)\xi(q) = u_h(G_1^{[N]}(q))$ and $q \in \pitchfork(p_{G_1})$. That is, $DG_1^{[N]}\xi(q)$ is exactly the horizontal direction of $E^s(G_1^{[N]}(q))$.

Step 2. In the following $([\kappa N]-2[N])$ times iterations, we will contract the slope direction of E^s by a factor $(1-\frac{\epsilon}{D})$. More precisely, for $i=[N]+1,\ldots,[\kappa N]-[N]$, under the coordinate chart $\{u_h(G_1^i(q)),u_l(G_1^i(q))\}$ of $E^s(G_1^i(q))$, using

$$\begin{pmatrix} 1 & 0 \\ 0 & 1-\frac{\epsilon}{D} \end{pmatrix} \circ DG_1|_{E^s}(G_1^i(q))$$

to replace $DG_1|_{E^s}(G_1^i(q))$, and leave $DG_1|_{E^u}(G_1^i(q))$ unchanged. By the isotopic Franks Lemma, we get an ϵ-perturbation G_2 of G_1 such that from $G_2^{[N]}(q)$ to $G_2^{[\kappa N]-[N]}(q)$, the slope direction is contracted by $(1-\frac{\epsilon}{D})^{[\kappa N]-[N]}$. Recall that the slope direction is mapped to the slope direction. Thus the above $(1-\frac{\epsilon}{D})$-contraction in each iteration can be accumulated. Clearly, q is still homoclinically related to p_{G_2}. Finally, since $DG_1^{[N]}(q)\xi(q)$ is exactly the horizontal direction of $E^s(G_1^{[N]}(q))$ and the above contraction in the slope direction does not affect the horizontal direction, we see that $DG_2^{[\kappa N]-[N]}\xi(q)$ is still the horizontal direction of $E^s(G_2^{[\kappa N]-[N]}(q))$.

In what follows, we replace the notation G_2 by g again. To summarize, starting from f with a homoclinic tangency, first, by θ-small perturbation, a hyperbolic horseshoe Λ_H^θ was constructed. Then, we selected $q\in\Lambda_H^\theta$. Finally, after the above two steps, we complete all the perturbations, obtaining g with $\mathrm{dist}_{C^1}(f,g)<2\epsilon$.

Proof of Lemma 4.1. For sufficiently small $\epsilon>0$ fixed in advance, the above constructions provide us a 2ϵ-perturbation g of f. Letting $\epsilon=\epsilon_k\to 0$, we obtain sequences g_k and $q_k\in \pitchfork(p_{g_k})$. Obviously, $\pi(q_k)\to\infty$. To finish the proof, it remains to verify conditions (i)-(iii) of Lemma 3.1 for these sequences.

(i) $\limsup_{k\to\infty}\angle(Dg_k^{n_k}(q_k)\xi_k,\xi_k)>0$.

In fact, we can prove a stronger result that $\liminf_{k\to\infty}\angle(Dg_k^{n_k}(q_k)\xi_k,\xi_k)>0$. It suffice to show that $\mathrm{Angle}(Dg_k^{n_k}(q_k)\xi_k,\xi_k)$ is bounded away from zero by a constant which is independent of k. Indeed, for every $k\in\mathbb{N}$ large enough, using the previous notations (k has been fixed, thus we can omit it for a while). By Step 2 above, $Dg^{[\kappa N]-[N]}(q)\xi(q)$ is exactly the horizontal direction of $E^s(g^{[\kappa N]-[N]}(q))$, hence $Dg^{[\kappa N]}(q)\xi(q)$ remains in the horizontal direction of $E^s(g^{[\kappa N]}(q))$. Next, since the transition map $Dg^T:T_{g^{[\kappa N]}(q)}M\to T_qM$ does not depend on ϵ and Dg^T sends the slope direction of $E^s(g^{[\kappa N]}(q))$ close to $\xi(q)\in E^s(q)$ (this is because before the perturbation, the tangent direction of $f^{-T}(x)$ is sent to the tangent direction of x but $\theta>0$ will be selected very small), we obtain

$$\mathrm{Angle}(Dg^{\pi(q)}(q)\xi(q),\xi(q))\geq\frac{\pi}{4}C_T>0,$$

where $C_T>0$ is a constant only depending on T and a fixed neighborhood of f. This estimation holds for every $\epsilon=\epsilon_k$, as a consequence, $\liminf_{k\in\mathbb{N}}\angle(Dg_k^{n_k}(q_k)\xi_k,\xi_k)>0$.

(ii) $\lim_{k\to\infty}\dfrac{\|Dg_k^{n_k}|_{E^s}(q_k)\|}{\lambda_k^{n_k}}=0;$

Let us first verify that $\lambda_k^{n_k}\to 0$. Indeed, for every $k\in\mathbb{N}$ fixed, in the previous calculations, we see that θ_k is selected just before Lemma 4.12 to get $\mathrm{Angle}(\xi(q_k),u_h(q_k))<\dfrac{\epsilon_k}{32cD}$. But in fact, for fixed λ_k, this θ_k can be taken arbitrarily small in $(0,\epsilon_k)$, in particular, satisfying $\lambda_k^{n_k}<1/k$. Recall that $n_k=\pi(q_k)=[\kappa N_k]+T$ where $N_k=N(\theta_k)$ can be arbitrarily large as long as θ_k is arbitrarily small.

Now, we prove (ii). It is easy to see that among all directions of $E^s(q_k)$, the slope direction have the weakest contracting rate. Therefore, letting u_l denote the unit vector in the slope direction of $E^s(q_k)$, we observe (refer (12))

$$\|Dg_k^{n_k}(q_k)u_l\| \leq 2D^T|\mu^s|^{[\kappa N_k]-[N_k]}\left(1-\frac{\epsilon}{D}\right)^{[\kappa N_k]-2[N_k]}|\mu^u|^{[N_k]},$$

where $(1-\frac{\epsilon}{D})^{[\kappa N_k]-2[N_k]}$ comes from the additional contraction given by perturbations on the center $[\kappa N_k]-2[N_k]$ times iterations in Step 2. Thus

$$\frac{\|Dg_k^{n_k}|_{E^s}(q_k)\|}{\lambda_k^{n_k}} \leq (2D)^T\frac{4}{|\mu^s|}\left|\frac{\mu^u(\mu^s)^{\kappa-1}(1-\frac{\epsilon}{D})^{\kappa-2}}{\lambda_k^\kappa}\right|^{N_k} \to 0 \quad (k\to\infty)$$

The last limit holds because on the one hand, we have (8), on the other hand, choose θ_k sufficiently small in $(0,\epsilon_k)$ such that $\left|\frac{\mu^u(\mu^s)^{\kappa-1}(1-\frac{\epsilon}{D})^{\kappa-2}}{\lambda_k^\kappa}\right|^{N_k} < \frac{1}{k}$ for every k.

(iii) $\lim_{k\to\infty}\dfrac{\lambda_k^{n_k}\angle(E^s(q_k),E^u(q_k))}{\|Dg_k^{n_k}(q_k)\xi_k\|}=0.$

By item (3) of Remark 4.8,

$$\angle(E^u(q_k),E^s(q_k)) = \tan \mathrm{Angle}(E^u(q_k),E^s(q_k)) \leq 4\theta_k.$$

On the other hand, since $Dg_k^{[\kappa N_k]}|_{E^s}(q_k)$ contracts the most in the horizontal direction, we have

$$\|Dg^{n_k}(q_k)\xi_k\| \geq D^{-T}|\mu^s|^{[\kappa N_k]}.$$

Therefore,

$$\frac{\lambda_k^{n_k}\angle(E^s(q_k),E^u(q_k))}{\|Dg_k^{n_k}\xi_k\|} \leq 4D^T\frac{\lambda_k^{[\kappa N_k]}\theta_k}{|\mu^s|^{[\kappa N_k]}} \leq \frac{4D^T}{|\mu^s|}\left|\frac{\lambda_k^\kappa}{\mu^u(\mu^s)^{\kappa-1}}\right|^{N_k} \to 0 \quad (k\to\infty)$$

The last convergence is similar as above. Using (6) and choosing θ_k small enough in $(0,\epsilon_k)$ (hence N_k sufficiently large) such that $\left|\frac{\lambda_k^\kappa}{\mu^u(\mu^s)^{\kappa-1}}\right|^{N_k} < \frac{1}{k}$ for every k.

Remark 4.13. By investigating the above proof carefully, it is easy to see that we can require $\mathrm{orb}(q)$ spends a large proportion of its iterations in a small neighborhood of p, where the norms of its derivatives and the inverse are close to that of f at p. More precisely, using the previous notations, for any neighborhood $U_p \subset M$ of p, by decreasing θ if necessary, $\dfrac{[\kappa N]}{[\kappa N]+T}$ can be taken close to one as much as we want. This fact will be useful in the proof of Theorem A.

5 Index change: Proof of Theorem A and Corollary B

To prove Theorem A, firstly, under the assumption of non-existence of dominated splittings of dimension $(i-1)$, we construct a strong homoclinic intersection using Lemma 5.1 and transport this strong homoclinic intersection to periodic points with weak contracting eigenvalues using Lemma 5.2. Secondly, applying the Connecting Lemma (Lemma 5.3) to create a strong heteroclinic intersection. Thirdly, perturb the heteroclinic cycle to a heterodimensional cycle. Finally, by stabilizing this heterodimensional cycle, we obtain $H(p_g)$ containing periodic points of different indices, by which robust homoclinic tangency follows immediately.

Lemma 5.1. ([BCDG, Proposition 7.1]) For every $D > 1$, $\epsilon > 0$, and $d \geq 2$, there exists a constant $k = k(D, \epsilon, d)$ with the following property. Consider $f \in \mathrm{Diff}^1(M)$, $\dim M = d$, such that the norms of Df and Df^{-1} are bounded by D, if p is a periodic point of f with index $2 \leq i \leq d$ such that $H(p)$ is non-trivial and has no k-dominated splitting of dimension $(i-1)$, then there exists an ϵ-perturbation g of f and $q(g) \in \pitchfork(p_g)$ such that $q(g)$ has a center contracting eigenvalue with multiplicity one and $W^{ss}_{i-1}(q(g)) \cap W^u(q(g)) \setminus \{q(g)\} \neq \emptyset$.

Lemma 5.2. ([PPV, Proposition 4.3] or [BCDG, Claim 8.3]) Let $\delta > 0$, $f \in \mathrm{Diff}^1(M)$ and p be a periodic point of f with $\mathrm{ind}(p) \geq 2$. Suppose

- there exists $q_1 \in \pitchfork(p)$ satisfying $|\lambda^{cs}(q_1)| > (1-\delta)^{\pi(q_1)}$;
- there exists $q_2 \in \pitchfork(p)$ satisfying $W^{ss}_{i-1}(q_2) \cap W^u(q_2) \setminus \{q_2\} \neq \emptyset$.

Then, there is an arbitrarily small perturbation g of f which has a periodic point $q(g) \in \pitchfork(p_g)$ such that $q(g)$ inherits both the above properties of q_1 and q_2. That is,

- $|\lambda^{cs}(q(g))| > (1-\delta)^{\pi(q(g))}$;
- $W^{ss}_{i-1}(q(g)) \cap W^u(q(g)) \setminus \{q(g)\} \neq \emptyset$.

Lemma 5.3. (Connecting Lemma ([H, Theorem A])) Let a_f and b_f be a pair of saddles of $f \in \mathrm{Diff}^1(M)$ such that there are sequences of points y_n and of natural numbers k_n satisfying:

- $y_n \to y \in W^u_{loc}(a_f)$ $(n \to \infty)$, $y \neq a_f$; and
- $f^{k_n}(y_n) \to z \in W^s_{loc}(b_f)$ $(n \to \infty)$, $z \neq b_f$.

Then there is a diffeomorphism g arbitrarily C^1-close to f such that $W^u(a_g)$ and $W^s(b_g)$ have a non-empty intersection arbitrarily close to y. In particular, $W^s_{loc}(b_f)$ and $W^s_{loc}(b_g)$ can be replaced by $W^{ss}_{loc}(b_f)$ and $W^{ss}_{loc}(b_g)$, respectively.

Proof of Theorem A. Given any $a > 1$, fix constants b and δ_0 such that

$$1 < b < a \text{ and } 0 < \delta_0 < 1 - \tfrac{b}{a}.$$

Obviously, $\delta_0 \to 0$ as $a \to 1$. For any $\delta \in (1, \delta_0)$, take $\epsilon > 0$ satisfying $3\epsilon < \delta(a - \frac{b}{1-\delta})\|Df^{\pm}(p)\|$. Fix neighborhoods $\mathcal{U} \subset \mathrm{Diff}^1(M)$ of f and $U \subset M$ of $\mathrm{orb}(p)$ such that $\|Dg^{\beta}(x)\| \leq b\|Df^{\pm}(p)\|$ for all $g \in \mathcal{U}$ and $x \in U$ where $\beta = \pm 1$. By Theorem C, there exists an ϵ-perturbation $g_1 \in \mathcal{U}$ of f and $r \in \pitchfork(p_{g_1})$ admitting contracting eigenvalue $\lambda^2(r)$ satisfying $|\lambda^2(r)| > (1-\delta)^{\pi(r)}$. With additional arbitrarily small perturbation and replace r if necessary, we can assume this $\lambda^2(r)$ is central contracting, having multiplicity one (see Remark 3.2). By continuity, there is a neighborhood $\mathcal{W}_1 \subset \mathcal{U}$ of g_1 such that

(F1) For every $g \in \mathcal{W}_1$, we have $r_g \in \pitchfork(p_g)$ and $|\lambda^2(r_g)| > (1-\delta)^{\pi(r_g)}$.

Shrinking $\mathcal{W}_1$ if necessary, we can always assume $\mathrm{dist}_{C^1}(f, g) < 2\epsilon$ whenever $g \in \mathcal{W}_1$. On the other hand, since p_{g_1} has non-real contracting eigenvalue, $H(p_{g_1})$ does not have dominated splitting of dimension one. Applying Lemma 5.1 to $H(p_{g_1})$, we get a perturbation $g_2 \in \mathcal{W}_1$ of g_1 which admits a strong homoclinic intersection associated to some $s(g_2) \in \pitchfork(p_{g_2})$, that is

(F2) $W^{ss}_1(s(g_2)) \cap W^u(s(g_2)) \setminus \{s(g_2)\} \neq \emptyset$.

Since $g_2 \in \mathcal{W}_1$, combining facts (F1) and (F2) above, we conclude by Lemma 5.2 that there exist $g_3 \in \mathcal{W}_1$ arbitrarily close to g_2 and $q(g_3) \in \pitchfork(p_{g_3})$ satisfying

- $|\lambda^2(q(g_3))| > (1-\delta)^{\pi(q(g_3))}$;
- $\exists x \in W^{ss}_1(q(g_3)) \cap W^u(q(g_3)) \setminus \{q(g_3)\}$.

Moreover, note that $q(g_3)$ can be taken such that its orbit spend a large proportion (close to one as much as we want) in U. Since $q(g_3) \neq x \in H(p_{g_3})$ and $H(p_{g_3})$ is transitive, using the Connecting Lemma, we obtain an arbitrarily small perturbation $g_4 \in \mathcal{W}_1$ of g_3 satisfying $W_1^{ss}(q_{g_4}) \cap W^u(p_{g_4}) \neq \emptyset$. Moreover, since $q(g_3)$ is homoclinically related to p_{g_3}, by robustness of transversal intersections, we can also assume $W^s(p_{g_4}) \cap W^u(q_{g_4})$ remains non-empty. Now, we apply Lemma 2.9 to $\mathrm{orb}(q_{g_4})$. For $l = 0, \dots, \pi(q_{g_4}) - 1$ and $t \in [0,1]$, let

- $A_{l,t} = ((1-t) + t\lambda^{-1}) \circ Dg_4(g_4^l(q_{g_4}))$ if $g_4^l(q_{g_4}) \in U$;
- $A_{l,t} = Dg_4(g_4^l(q_{g_4}))$ if $g_4^l(q_{g_4}) \notin U$,

where $\lambda \in (0,1)$ is selected satisfying

$$\max\left\{|\lambda^1(q_{g_4})|^{\frac{1}{\pi(q_{g_4})}}, (1-\delta)\right\} < \lambda < |\lambda^2(q_{g_4})|^{\frac{1}{\pi(q_{g_4})}}.$$

Here, λ^1 denote the other contracting eigenvalue beside λ^2. Slightly different from before, this time we will pay attention to the behavior of one dimensional strong stable manifold under the perturbation. By the choice of λ and Remark 4.13, one can easily verify that $\prod_{l=1}^{\pi(q_{g_4})-1} A_{l,t}$ keeps having an 1-dimensional strong stable direction for all $t \in [0,1]$ and its endpoint $\prod_{l=1}^{\pi(q_{g_4})-1} A_{l,1}$ is a hyperbolic matrix with index one. Moreover, we have:

$$\max_{\substack{l=0,\dots,\pi(q_{g_4})-1 \\ t\in[0,1]}} \{\|A_{l,t} - A_{l,0}\|\} < b\|Df^{\pm}(p)\|\left(\frac{1}{\lambda} - 1\right) < \frac{b\delta}{1-\delta}\|Df^{\pm}(p)\|.$$

Similar estimation also works for the inverse. Thus, by Lemma 2.9 we get a perturbation g_5 of g_4 such that:

- $\mathrm{dist}_{C^1}(g_5, g_4) < \dfrac{b\delta}{1-\delta}\|Df^{\pm}(p)\|$;
- $q_{g_5} = q_{g_4}$ is periodic with $\mathrm{ind}(q_{g_5}) = \mathrm{ind}(q_{g_4}) - 1 = 2 - 1 = 1$;
- $W^u(p_{g_5}) \cap W^s(q_{g_5}) \neq \emptyset$ and $W^u(q_{g_5}) \cap W^s(p_{g_5}) \neq \emptyset$.

In particular, g_5 has a co-index one heterodimensional cycle associated to p_{g_5} and q_{g_5}. Noticing that $H(p_{g_5})$ is non-trivial, by Lemma 2.7 there exists g arbitrarily close to g_5, admitting robust heterodimensional cycle associated to transitive hyperbolic sets $\Gamma_g \ni p_g$ and $\Lambda_g \ni q_g$. By robustness, we are allowed to select g in the residual set $\mathcal{R}$ of Lemma 2.6 and satisfying $\mathrm{dist}(g_5, g) < \epsilon$. Therefore, by Lemma 2.3, $H(p_g) = C(p_g)$ which contains periodic points of index one and two. Moreover, note that p_g has complex contracting eigenvalues, $H(p_g)$ does not have dominated splittings of dimension one. Now, apply Lemma 2.6 to $H(q_g)$, it follows that g exhibits a robust homoclinic tangency. Finally,

$$\begin{aligned}\mathrm{dist}_{C^1}(g, f) &\leq \mathrm{dist}_{C^1}(g, g_5) + \mathrm{dist}_{C^1}(g_5, g_4) + \mathrm{dist}_{C^1}(g_4, f) \\ &\leq \epsilon + \frac{b\delta}{1-\delta}\|Df^{\pm}(p)\| + 2\epsilon < a\delta\|Df^{\pm}(p)\|\end{aligned}$$

as desired. This complete the proof of Theorem A.

Proof of Corollary B. It suffice to notice that, in the proof of Theorem A, at the last moment, we obtain an $a\delta\|Df^{\pm}(p)\|$-perturbation g of f such that $H(p_g)$ contains periodic points of index one and two. Thus, under the additional assumption, we are allowed to apply Lemma 2.6 to $H(p_g)$, obtaining a robust homoclinic tangency associated to a hyperbolic set containing p_g.

6 Proof of Theorem D

We begin with a general result on R-robustly entropy-expansive diffeomorphisms. Detailed definitions and background can be found in [L]. Note that different from before, the following two propositions are also valid in higher dimensional case. Let min-ind(Λ) and max-ind(Λ) denote the minimal index and maximal indices of periodic points in Λ.

Proposition 6.1. Let $f \in \mathrm{Diff}^1(M)$ and let $H(p)$ be a non-trivial homoclinic class associated to a hyperbolic periodic saddle p of f. If $H(p)$ is R-robustly entropy-expansive in a neighborhood $\mathcal{U}_f$ of f, then, there is an open and dense subset $\mathcal{O}_f$ of $\mathcal{U}_f$ such that for any $g \in \mathcal{O}_f$, the chain recurrent class $C(p_g)$ admits a dominated splitting of the form

$$E \oplus F_1 \oplus \cdots \oplus F_k \oplus G \qquad (k \in \mathbb{N})$$

where all of F_l $(l = 1, \ldots, k)$ are one dimensional and non-hyperbolic. Moreover, the splitting is *index-adapted*. That is,

$$\dim E = \text{min-ind}(C(p_g)),$$
$$\dim M - \dim G = \text{max-ind}(C(p_g)).$$

This is an alternative statement of [L, Theorem A] but slightly stronger. To prove it, we need to know how the index-interval of a chain recurrent classes vary with f.

Lemma 6.2. For generic f in $\mathrm{Diff}^1(M)$, given any periodic point p of f and its chain recurrent class $C(p)$, there exists a neighborhood $\mathcal{V}_f$ of f in $\mathrm{Diff}^1(M)$, such that for any g in $\mathcal{V}_f$, the index-set of $C(p_g)$ coincide with that of $C(p)$, which is an interval of $\mathbb{N}$.

Proof. Let us fix $f \in \mathcal{G} := \mathcal{G}_1 \cap \mathcal{G}_2$ where $\mathcal{G}_1$ and $\mathcal{G}_2$ are the residual subsets obtained in Lemmas 2.1 and 2.2, respectively. For any chain recurrent class $C(p)$ of f, we have $C(p) = H(p)$ by Lemma 2.3, hence $\mathrm{ind}(C(p)) = \mathrm{ind}(H(p))$ which is an interval of $\mathbb{N}$, denoted by $[i, j]$. For every $k \in [i, j]$, take $q^k \in C(p)$ with $\mathrm{ind}(q^k) = k$. Then, apply Lemma 2.1 to q^k and p to get a neighborhood $\mathcal{U}_k$ of f in $\mathrm{Diff}^1(M)$ such that $H(p_g) = H(q_g^k)$ for all $g \in \mathcal{U}_k \cap \mathcal{G}$. We claim that $q_g^k \in C(p_g)$ for all $g \in \mathcal{U}_k$. In fact, otherwise, we can find some $g \in \mathcal{U}_k$ satisfying $q_g^k \notin C(p_g)$. By Conley's Fundamental Theorem (see [BDV, Theorem 10.3] for instance), there exists a neighborhood $\mathcal{W}_g$ of g in $\mathrm{Diff}^1(M)$ such that $q_h^k \notin C(p_h)$ for all $h \in \mathcal{W}_g$. This is a contradiction because if we take $h \in \mathcal{W}_g \cap \mathcal{U}_k \cap \mathcal{G}$, it follows that $q_h^k \in H(q_h^k) = H(p_h) = C(p_h)$, where the last equality comes from Lemma 2.3 again. This claim shows that for every $g \in \mathcal{V}_f := \cap_{k=i}^{j} \mathcal{U}_k$, we have $\mathrm{ind}(C(p_g)) \supset [i, j]$. In particular, $\text{max-ind}(C(p_g)) \geq \text{max-ind}(C(p))$ and $\text{min-ind}(C(p_g)) \leq \text{min-ind}(C(p))$. That is, when restricted to $\mathcal{G}$, max-ind($C(p)$) (resp. min-ind($C(p)$)) depends lower semi-continuously (resp. upper semi-continuously) on f. As a result, both of them vary continuously on a residual subset $\mathcal{R}$ of $\mathcal{G}$. It follows immediately that $\mathcal{R}$ is also residual in $\mathrm{Diff}^1(M)$. On the other hand, since both of these two functions are integer-valued, they must be constant in a sufficiently small neighborhood of f. Thus, shrink $\mathcal{V}_f$ if necessary, we have $\mathrm{ind}(C(p_g)) = [i, j]$ for all $g \in \mathcal{V}_f$ which completes the proof.

Proof of Proposition 6.1. Under the hypothesis of R-robust entropy-expansiveness, the proof of [L, Theorem A] gives a residual subset $\mathcal{R}_f$ of $\mathcal{U}_f$ such that for all $g \in \mathcal{R}_f$, $H(p_g)$ (which coincide with $C(p_g)$ by Lemma 2.3) admit a dominated splitting

$$E \oplus F_1 \oplus \cdots \oplus F_k \oplus G \qquad (k \in \mathbb{N}) \qquad (\star)$$

where all of F_l $(l = 1, \ldots, k)$ are one dimensional and non-hyperbolic. The remark of [L, Proposition 3.1] said, the dominated splitting $(\star)$ is index-adapted for $g \in \mathcal{R}_f$.

On the other hand, since $\mathcal{R}_f$ is residual in $\mathcal{U}_f$, for any $g \in \mathcal{R}_f$ fixed, there is a neighborhood $\mathcal{V}_g$ of g such that for any $h \in \mathcal{V}_g$,

(1) $\mathrm{ind}(C(p_h)) = \mathrm{ind}(C(p_g))$;

(2) $C(p_h)$ admits a dominated splitting of the same form as that of $C(p_g)$.

Indeed, (1) comes from Lemma 6.2 and (2) is obtained by Lemma 2.5. To see this, it suffice to notice that as a set-valued function, $f \mapsto C(p_f)$ depend upper semi-continuously on f, thus, restricted to a residual subset, $C(p_f)$ moves continuously.

Now, define $\mathcal{O}_f = \bigcup_{g \in \mathcal{R}_f} \mathcal{V}_g$, which is an open and dense subset of $\mathcal{U}_f$. Combining the above observations, we conclude that for every $h \in \mathcal{O}_f$, $C(p_h)$ admits an index-adapted dominated splitting and finish the proof of Proposition 6.1.

Proof of Theorem D. Let $a = 1 + \dfrac{\rho}{\|Df^{\pm}(p)\|}$. In the beginning of the proof of Theorem A, take $b \in (1, a)$ sufficiently close to 1 such that $\delta \in (0, 1 - \frac{b}{a})$ is sufficiently close to $1 - \dfrac{1}{a} = \dfrac{\rho}{\rho + \|Df^{\pm}(p)\|} = \sigma$, in particular, satisfying $\chi_2(p) + \chi_3(p) > \log(1 - \delta)$. To prove Theorem D, suppose by contradiction that $H(p_f)$ does not have dominated splittings of dimension $\mathrm{ind}(p) = 2$, combining with Lemma 4.2, Theorem A provides a perturbation g of f, having a co-index one heterodimensional cycle associated to p_g and some $q(g)$ with index 1. In addition,

$$\mathrm{dist}_{C^1}(f, g) < a\delta \|Df^{\pm}(p)\| < a\sigma \|Df^{\pm}(p)\| = \rho.$$

This heterodimensional cycle can be stabilized by Lemma 2.7. As a result, $\mathrm{ind}(C(p_h)) = \{1, 2\}$ for every h near g. In particular, we can always select $h \in \mathcal{O}_f$ where $\mathcal{O}_f$ is the open and dense subset in Proposition 6.1. Since the dominated splitting provided by Proposition 6.1 is index-adapted, it follows that $C(p_h)$ admit dominated splittings of dimension 1. But $C(p_h)$ contains p_h which has non-real contracting eigenvalues, this contradiction completes the proof of Theorem D.

Acknowledgments

The author would like to express my sincere gratitude to Prof. S. Hayashi, my supervisor, for his constant encouragement and guidance, who give me many valuable suggestions, especially in the preparation of this paper. Also the author would like to thank Prof. L. J. Díaz for his helpful comments. Special thanks should also go to Dr. K. Shinohara for many useful discussions. Finally, the author thanks the financial support of Japanese Government Scholarship.

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