# 博士論文

# A remark on default risks in financial models: a filtering model and a remark on copula

$$\left(\begin{array}{c}\text{デフォルトリスクに対する}\\ \text{ファイナンスモデルに関する考察：}\\ \text{フィルタリングモデルとコピュラモデルについて}\end{array}\right)$$

中島 武信

平成 26 年 12 月 26 日

# 目 次

# Notation

A filtering model

- $(\Omega, \mathcal{B}, P, \{\mathcal{B}_t\}_{t\in[0,\infty)})$; A complete filtrated probability space.
- $\{\mathcal{B}_t\}_{t\geq 0}$: A filtration satisfies usual conditions.
- $(\mathcal{G}_t^X)$: The right continuous filtration generated by continuous stochastic process $X$.
- $\tau = \inf\{t > 0;\ X_t = 0\}$
- $N_t = 1_{\{\tau \leq t\}}$
- $\mathcal{F}_t = \bigcap_{u>t}(\mathcal{G}_u^Y \vee \sigma\{\tau \wedge u\})$
- $a > 0$
- $B_t^a = a + B_t$
- $\tau^a = \inf\{t > 0;\ B_t^a = 0\}$
- $N_t^a = 1_{\{\tau^a \leq t\}}$
- $\mathcal{F}_t^W = \bigcap_{u>t}(\mathcal{G}_u^W \vee \sigma\{\tau^a \wedge u\})$.
- $q_a(t) = \int_t^\infty \frac{a}{\sqrt{2\pi s^3}} \exp(-\frac{a^2}{2s})ds$
- $\lambda_a(t) = -\frac{d}{dt}\log q_a(t) = \frac{a}{\sqrt{2\pi t^3}} q_a(t)^{-1} \exp(-\frac{a^2}{2t})$.
- $P[\tau^a > t] = q_a(t) = e^{-\int_0^t \lambda_a(u)du}$
- $P[\tau^a \in dt] = \lambda_a(t)e^{-\int_0^t \lambda_a(u)du}dt = \gamma_a(t)dt$

- $M_t = N_t^a - \int_0^t (1 - N_s^a)\lambda_a(s)ds$
- $\mathcal{L}^p$, $\quad p \in (1,\infty)$: Space of $\{\mathcal{B}_t\}$-progressively measurable functions $\varphi$ such that $E[\int_0^T |\varphi|_s^p ds] < \infty$ for any $T > 0$.
- $\mathcal{L}^{p+} = \bigcup_{q>p} \mathcal{L}^q, p \geq 1$.
- $g(t,x) = \frac{1}{\sqrt{2\pi t}} \exp(-\frac{x^2}{2t})$
- $\Phi(t,x) = \int_{-\infty}^x g(t,y)dy$
- $g_0(t,x,y) = g(t,y-x) - g(t,y+x) = g(t,y-x)(1 - e^{-2xy/t}), \qquad x,y \geq 0, \quad t > 0.$
- $H_a^{(k)}(t,s;f) = E[1_{\{\tau^a > s\}} f_s \frac{\partial^k g}{\partial x^k}(t-s, B_s^a)|\mathcal{G}_s^W], \quad t > s, \quad f \in \mathcal{L}^{1+}, \quad k = 0,1,2.$
- $\hat{H}_a^{(k)}(t;f) = \int_0^t H_a^{(k)}(t,u;f)du, \quad f \in \mathcal{L}^{\frac{4}{3-k}+}, \quad k = 0,1,2.$
- $\bar{H}_a(t;f) = e^{\int_0^t \lambda_a(r)dr}(\hat{H}_a^{(2)}(t;f) + 2\lambda_a(t)\hat{H}_a^{(0)}(t;f)), \quad f \in \mathcal{L}^{4+}.$
- $U_a(t,s;f) = E[1_{\{\tau^a > s\}} f_s (2\Phi(t-s, B_s^a) - 1)|\mathcal{G}_s^W], \quad t > s, \quad f \in \mathcal{L}^{1+}.$
- $\bar{U}_a(t,s;f) = e^{\int_0^t \lambda_a(r)dr}(H_a^{(1)}(t,s;f) + \lambda_a(t)U_a(t,s;f)), \quad t > s, \quad f \in \mathcal{L}^{1+}.$
- $dX_t = dB_t + b_0(t, X_t, Z_t)dt, \qquad X_0 = x_0 > 0.$
- $dZ_t = \sigma_1(t, X_t, Z_t)d\hat{B}_t + b_1(t, X_t, Z_t)dt, \qquad Z_0 = z_0 \in \mathbf{R}^N.$
- $dY_t = \sigma_2(t, Y_t)dW_t + b_2(t, X_{t\wedge\tau}, Y_t)dt, \qquad Y_0 = y_0 \in \mathbf{R}.$
- $\rho_t = \exp(\int_0^t b_0(s, X_s, Z_s)dB_s + \int_0^t \beta(s, X_{s\wedge\tau}, Y_s)dW_s + \frac{1}{2}\int_0^t (b_0(s, X_s, Z_s)^2 + \beta(s, X_{s\wedge\tau}, Y_s)^2)ds)$
- $\beta(t,x,y) = \sigma_2(t,y)^{-1} b_2(t,x,y)$
- $d\widetilde{P} = \rho_t^{-1} dP$
- $\widetilde{B}_t = B_t + \int_0^t b_0(s, X_s, Z_s)ds$
- $\widetilde{W}_t = W_t + \int_0^t \beta(s, X_{s\wedge\tau}, Y_s)ds$
- $\widetilde{M}_t = N_t - \int_0^t (1 - N_{s-})\lambda_{x_0}(s)ds$

- $I^{(k)}(t,s;f) = \widetilde{E}[1_{\{\tau>s\}}\frac{\partial^k g}{\partial x^k}(t-s,X_s)\rho_{s-}f_s|\mathcal{G}_s^Y], \quad t>s, \quad k=0,1,2.$
- $(\widetilde{D_0}F)_t = \beta(t,X_{t\wedge\tau},Y_t)F_{t\wedge\tau} + 1_{\{\tau>t\}}f_2(t)$
- $(\widetilde{D_1}F)_t = b_0(t,X_t,Z_t)F_{t\wedge\tau} + 1_{\{\tau>t\}}f_3(t)$
- $(\widetilde{D_2}F)_t = 1_{\{\tau>t\}}f_4(t)$
- $(\widetilde{L}F)_t = 1_{\{\tau>t\}}f_1(t)$
- $\hat{V}(r;F) = \widetilde{\rho}_{r-}^{-1}e^{\int_0^r \lambda_{x_0}(u)du}(\hat{V}_1(r,r;F)+\lambda_{x_0}(r)(-(2\Phi(r,x_0)-1)F_0+\widetilde{E}[1_{\{\tau>r\}}\rho_r F_r|\mathcal{G}_r^Y]))$
- $\hat{V}_1(r,s;F) = \int_0^s I^{(1)}(r,u;\widetilde{D_0}F)d\widetilde{W}_u + \int_0^s(I^{(2)}(r,u;\widetilde{D_1}F) + I^{(1)}(r,u;\widetilde{L}F))du$
- $\tilde{\tilde{\lambda}}(s) = \lambda_{x_0}(s) + \hat{V}(s;1)$
- $\widetilde{\widetilde{M}}_t = N_t - \int_0^t(1-N_s)\tilde{\tilde{\lambda}}(s)ds$
- $\widetilde{\widetilde{W}}_t = \widetilde{W}_t - \int_0^t E[\beta(r,X_{r\wedge\tau},Y_r)|\mathcal{F}_r]dr$
- $\widetilde{H}_a^{(k)}(t,s;f) = E[1_{\{\tau^a>s\}}|f_s\frac{\partial^k g}{\partial x^k}(t-s,B_s^a)||\mathcal{G}_s^W], \quad t>s>0, \quad k=0,1,2.$
- $V(t,s;f) = \widetilde{E}[\rho_{s-}1_{\{\tau>s\}}f_s(2\Phi(t-s,X_s)-1)|\mathcal{G}_s^Y]$
- $\bar{V}(t,s;f) = e^{\int_0^t \lambda_{x_0}(r)dr}(I^{(1)}(t,s;f) + \lambda_{x_0}(t)V(t,s;f))$
- $\bar{I}(t,s;f) = e^{\int_0^t \lambda_{x_0}(r)dr}(\int_0^s I^{(2)}(t,u;f)du + 2\lambda_{x_0}(t)\int_0^s I^{(0)}(t,u;f)du)$
- $\hat{V}(r,s;F) = \widetilde{\rho}_{r-}^{-1}e^{\int_0^r \lambda_{x_0}(u)du}(\hat{V}_1(r,s;F) + \lambda_{x_0}(r)\hat{V}_2(r,s;F)), \quad s\le r.$
- $\hat{V}_2(r,s;F) = \int_0^s V(r,u;\widetilde{D_0}F)d\widetilde{W}_u + \int_0^s(V(r,u;\widetilde{L}F) + 2I^{(0)}(r,u;\widetilde{D_1}F))du$
- $S_t = E[X_{t\wedge\tau}|\mathcal{F}_{t\wedge\tau}]$
- $R_t = e^{-\alpha(T-t)}E[1_{\{\tau>T\}}|\mathcal{F}_t]$
- $\tilde{R}_t = E[1_{\{\tau>T\}}|\mathcal{B}_t]$

A remark on credit risk and copula

- $(\Omega, \mathcal{F}, P)$: A complete probability space.
- $W(t) = (W^k(t))_{k=1,\ldots,d}$, $t \geqq 0$: A $d$-dimensinal standard Wiener process.
- $\mathcal{G}_t = \sigma\{W(s),\ s \in [0,t]\} \vee \mathcal{N}$
- $\mathcal{N} = \{B \in \mathcal{F};\ P(B) = 0 \text{ or } 1\}$
- $N \geqq 2$, $\tau_i : \Omega \to [0, \infty)$, $i = 1, \ldots, N$: Random variables
- $\mathcal{F}_t = \mathcal{G}_t \vee \sigma\{\tau_i \wedge t, i = 1, \ldots, N\}$
- $\xi_i : [0, \infty) \times \Omega \to [0, \infty)$, $i = 1, \ldots, N$, $\mathcal{G}$-progressively measurable processes
- $\Theta$: An open subset in $\mathbf{R}^M$
- $\theta$: $[0, \infty) \times \Omega \to \mathbf{R}^M$, $\mathcal{G}$-Ito process, i.e., $\theta$ is $\mathcal{G}$-progressively measurable, $\theta(t, \omega)$ is continuous in $t$ for all $\omega \in \Omega$, and there are $\mathbf{R}^M$-valued $\mathcal{G}$-progressively measurable processes $\eta_k$, $k = 1, \ldots, d$, and $b$ satisfying

$$P(\sum_{k=1}^{d} \int_0^T |\eta_k(t)|^2 dt + \int_0^T |b(t)| dt < \infty) = 1, \text{ for any } T > 0,$$

and

$$\theta(t) = \theta(0) + \sum_{k=1}^{d} \int_0^t \eta_k(s) dW^k(s) + \int_0^t b(s) ds.$$

# 第1章　論文の概要と背景

本論文では、デフォルトに関する 2 つの話題、具体的には「デフォルトを含むフィルタリングモデル」および「コピュラ」を取り扱う。前者は学術的観点から、後者は実務的観点から興味を持たれている。これらを確率論の枠組みで定式化し、主にマルチンゲール理論を用いて考察していく。

## 1.1　デフォルトを含むフィルタリングモデルの概要

デフォルトとは債務不履行を表すファイナンス用語である。社債や CDS(Credit Default Swap) といった、個別企業の信用リスクが絡む金融商品を、理論的であれ、実務的であれ、取り扱う場合には必ずと言っていいほど良く用いられる概念である。

例えば社債の場合、社債の満期まで発行体（社債を発行した企業) がデフォルトしなければ、期中の利払い日には定められたクーポンが、満期日には元本が、社債の購入者に支払われる。その一方で、満期前に発行体がデフォルトしてしまうと、クーポンの支払いは止まり、元本部分は一部しか、あるいは、全く支払われなくなるケースが多い。

また、株式に関しても、TOPIX といった株価指数ではなく、個別株を扱う場合は、デフォルトを意識する必要が出てくる。それは、デフォルトが発生すると、通常、株式の発行体の株価はゼロになるためである。

ここで、簡単なモデルを考えてみよう。$(\Omega, \mathcal{B}, P, \{B\}_{t\in[0,\infty)})$ をフィルトレーション付きの確率空間とする。ある社債の発行体のデフォルト時刻を $\{B\}_{t\in[0,\infty)}$-停止時刻とする。簡単のためにこの社債は割引債であるとしよう。満期まで発行体がデフォルトしなければ元本が回収できるが、満期前のデフォルトすると、元本は一切回収できず、社債の価値はゼロになるとする。この社債の満期を $T$ としよう。$r$ を無リスク金利、$Q$ は $P$ と同値なリスク中立確率とする。このとき、時刻 $t$ における社債の価格 $p(t)$ は次で与え

られる。

$$p(t) = 1_{\{\tau>t\}} E^Q[e^{-\int_t^T r(s)ds} 1_{\{\tau>t\}} | \mathcal{B}_t]$$

この例から分かる通り、デフォルトリスクをモデル化する場合、デフォルト時刻 $\tau$ の特徴づけや、リスク中立確率の下での条件付き期待値 $E^Q[\cdot|\mathcal{B}_t]$ に関する考察が必要となってくる。デフォルト時刻 $\tau$ は、Structual アプローチ、あるいは、Reduced form アプローチにより特徴づけられることが多い。Structual アプローチでは、企業価値がある一定水準を下回った時刻をデフォルト時刻と定義している。一方で Reduced form アプローチでは、デフォルト時刻を外生的に与えている。これらの詳細は楠岡・青沼・中川 [1] を参照されたい。本稿では、概要だけ述べることとする。

(1)Structual アプローチ

Structual アプローチは Merton[14] において初めて用いられるようになったといわれている。さて、企業価値 $V_t$ は以下の確率微分方程式を満たすとする。

$$dV_t = \mu(V_t, t)dt + \sigma(V_t, t)dW_t. \tag{1.1.1}$$

そして、企業価値がゼロを下回った時刻をデフォルト時刻 $\tau$ とする。

$$\tau = \inf\{t > 0 : V_t \leq 0\}.$$

このモデルに関して、主に 2 つの問題点が指摘されている。一つは、一般的にデフォルトは突発的に発生することが多いにもかかわらず、このモデルでは、デフォルトがある程度予測可能ということである。あと一つは、企業価値を常に観測できるのは、インサイダー以外はほぼ不可能であるという点である。例えば社債の価格プロセスを考えるときに、社債の価格付けを行う社債市場参加者はインサイダーではないため、本モデルをそのまま社債価格のプロセスに用いるのは現実的ではないと考えられる。実際には、企業価値が観測できるのは、半年毎の決算時のみであることが多いため、常に企業価値を観測できるという条件は外す必要があるかもしれない。また、その企業の価値に関する全ての情報が公表されるわけではないため、取得できる情報はノイズを含んだものと考えた方が現実的であろう。

(2)Reduced form アプローチ

Reduced form アプローチは Duffie-Singleton[10] や Duffie and Land[9] に代表されるアプローチである。デフォルト時刻 $\tau$ はあるフィルトレーション $\{\mathcal{F}_t\}$ に関する停止時刻

とし、$N_t = 1_{\{\tau \leq t\}}$ とする。ハザードレート $h_t$ は

$$N_t - \int_0^t (1 - N_s) h_s ds \tag{1.1.2}$$

が $\{\mathcal{F}_t\}$-マルチンゲールになるような、$\{\mathcal{F}_t\}$-発展的可測過程であるとする。このとき、$s < t$ における、条件付きデフォルト確率は以下で与えられることが知られている。

$$P(\tau > t | \mathcal{F}_s) = 1_{\{\tau > s\}} E[e^{-\int_s^t h_u du} | \mathcal{F}_s].$$

この式から、デフォルト時刻 $\tau$ を定義することは、ハザードレート $h_t$ を定義することと本質的に同じであることが分かる。

以上から、デフォルトリスクをモデル化するにあたって、デフォルト時刻とフィルトレーションの特徴付けが重要であることが分かる。

ここで、フィルタリングをファイナンスに用いた最初の論文である、Duffie and Land[9] について言及しておきたい。楠岡・青沼・中川 [1] でも纏められているとおり、Duffie and Land[9] は、ある企業が株と社債を発行しているケースを考え、株主は企業価値について完全な情報を保有している一方、社債保有者は不完全な情報しか保有していないとした。そして株主は、当初株価が最大化するように、企業の清算時点を決めることができるとした。これは、株主が経営権を持つケースであるから、オーナー企業を想定していると考えられる。

当初株価の最大化に当たって株主は、株の配当、社債の発行利回り、税効果、清算によるコストを考慮するとしたが、これらに関しては株主は変更を行うことはできず、唯一、清算時点のみをコントロールできるとした。このとき、清算時点は、企業価値がある一定水準を下回った時点として与えられることを、Duffie and Land[9] は証明している。

さらに、ある技術的な条件のもとで、Structual アプローチにおける変数と、Reduced アプローチにおける変数がある等式で結ばれることを示している。その点について簡単に触れておきたい。株主が保有している完全情報をフィルトレーション $\{\mathcal{F}_t\}$ とし、社債保有者が保有している不完全情報をフィルトレーション $\{\mathcal{H}_t\}$ とする。ここで、$\{\mathcal{H}_t\}$ は $\{\mathcal{F}_t\}$ の部分フィルトレーションとする。そして、(1)Structual アプローチのように、企業価値 $V_t$ が式 (1.1.1) で与えられるとする。清算時刻を $\{\mathcal{H}_t\}$-停止時刻 $\tau$、$N_t = 1_{\{\tau \leq t\}}$

とする。そして (2)Reduced アプローチのように、ハザードレート $h_t$ は、式 (1.1.2) が $\{\mathcal{H}_t\}$-マルチンゲールになるものとする。

上述の通り、Duffie and Land[9] では清算時刻 $\tau$ は、企業価値 $V_t$ がある水準を下回った時刻ということを示しているが、ここでは説明を、Structual アプローチと Reduced form アプローチの関係性に集中させるために、清算時刻 $\tau$ は、企業価値 $V_t$ がゼロ以下になった時としよう。このとき、

$$c(t) = \lim_{x \downarrow 0} \frac{1}{x} \frac{P(V_t \in dx|\mathcal{H}_t)}{dx}$$

が存在し、他の技術的条件が満たされるのであれば、$t < \tau$ において、

$$h_t = \frac{1}{2}\sigma(0,t)^2 c(t)$$

が成立することが示されている。Duffie and Land[9] では、社債保有者は決算期ごとに、つまり離散時間において不完全情報が与えられるとしているが、Kusuoka[13] では、デフォルトリスクに関する様々な数理モデルについて考察を行う中で、Duffie and Land[9] の連続版に相当するモデルについても言及している。

さらに Nakagawa[15] は、デフォルトリスクに関するフィルタリングモデルを構築し、不完全情報の下での表現定理を示したほか、Kusuoka[13] の結果と併せて、$P$ と同値な確率測度 $Q$ の下でのハザードレート $\lambda_t$ と、$P$ の下でのハザードレート $h_t$ の関係性を示した。ただし、Nakagawa[15] ではピン止めされたブラウン運動の確率測度という扱いにくい測度を用いているため、本論文では、Nakagawa[15] の方法を踏襲しながら、より扱いやすい測度の下での表現定理を示した。以下で、本論文の概要を纏めることとする。

$(\Omega, \mathcal{B}, P, \{\mathcal{B}_t\}_{t\in[0,\infty)})$ を完備な確率空間とする。$B_t,\ \hat{B}_t,\ W_t$ は独立な $\mathcal{B}_{t\in[0,\infty)}$-ブラウン運動とする。$(\mathcal{G}_t^X)$ をある確率過程 $X$ から生成される右連続なフィルトレーションとする。例えば $\mathcal{G}_t^B = \bigcap_{u>t}\sigma\{B_s, s \le u\}$. である。次のような停止時刻 $\tau$ を考えよう $\tau = \inf\{t > 0;\ X_t = 0\}$。そして $N_t = 1_{\{\tau \le t\}}$, $\mathcal{F}_t = \bigcap_{u>t}(\mathcal{G}_u^Y \vee \sigma\{\tau \wedge u\})$ とする。$a > 0$ とし、$B_t^a = a + B_t$, $\tau^a = \inf\{t > 0;\ B_t^a = 0\}$, $N_t^a = 1_{\{\tau^a \le t\}}$, $\mathcal{F}_t^W = \bigcap_{u>t}(\mathcal{G}_u^W \vee \sigma\{\tau^a \wedge u\})$ としよう。ここで $q_a(t) = \int_t^\infty \frac{a}{\sqrt{2\pi s^3}} \exp(-\frac{a^2}{2s})ds$ とし、次のようなハザードレートを考える $\lambda_a(t) = -\frac{d}{dt}\log q_a(t) = \frac{a}{\sqrt{2\pi t^3}} q_a(t)^{-1} \exp(-\frac{a^2}{2t})$. このとき、$P[\tau^a > t] = q_a(t) =$

$e^{-\int_0^t \lambda_a(u)du}$ であり $P[\tau^a \in dt] = \lambda_a(t)e^{-\int_0^t \lambda_a(u)du}dt = \gamma_a(t)dt$ となる。また、

$$M_t = N_t^a - \int_0^t (1 - N_s^a)\lambda_a(s)ds$$

が $\mathcal{F}_t^W$ マルチンゲールであることが分かる。ここで $\mathcal{L}^p$, $p \in (1,\infty)$ を $\{\mathcal{B}_t\}$-発展的可測な関数 $\varphi$ の空間とし、$\varphi$ はどのような $T > 0$ に対しても $E[\int_0^T |\varphi|_s^p ds] < \infty$ を満たすものとする。また、$\mathcal{L}^{p+} = \bigcup_{q>p} \mathcal{L}^q, p \geq 1$ と表記する。この条件下で、次の表現定理を示した。

**Theorem 1.1.1.** (1) 任意の $t, T \geq 0$ と $f \in \mathcal{L}^{4+}$ に対して、以下が成立する。

$$E[\int_0^T f_s dB_s|\mathcal{F}_t^W] = -\int_0^t \bar{H}_a(s; f1_{(0,T]}(\cdot))\lambda_a(s)^{-1}dM_s.$$

(2) 任意の $t \geq 0$ と $f \in \mathcal{L}^{4+}$ に対して、以下が成立する。

$$E[\int_0^t f_s ds|\mathcal{F}_t^W] = \int_0^t E[f_s|\mathcal{F}_s^W]ds - \int_0^t (\int_0^s \bar{U}_a(s,r;f)dr)\lambda_a(s)^{-1}dM_s.$$

(3) 任意の $t, T \geq 0$ と $f \in \mathcal{L}^{6+}$、以下が成立する。

$$E[\int_0^T f_s dW_s|\mathcal{F}_t^W] = \int_0^{T\wedge t} E[f_s|\mathcal{F}_s^W]dW_s - \int_0^t (\int_0^s \bar{U}_a(s,r;f1_{[0,T]}(\cdot))dW_r)\lambda_a(s)^{-1}dM_s.$$

(4) 任意の $t \geq 0, \hat{f}_i, \in \mathcal{L}^{2+}, i = 1, \cdots, d$ に対して、以下が成立する。

$$E[\sum_{i=1}^d \int_0^t \hat{f}_s^i d\hat{B}_s^i|\mathcal{F}_t^W] = 0.$$

ここで $X_t, Z_t, Y_t$ なる 3 つの確率過程を考える。$X_t, Z_t, Y_t$ は次の確率微分方程式の解とする。

$$dX_t = dB_t + b_0(t, X_t, Z_t)dt, \qquad X_0 = x_0 > 0,$$

$$dZ_t = \sigma_1(t, X_t, Z_t)d\hat{B}_t + b_1(t, X_t, Z_t)dt, \qquad Z_0 = z_0 \in \mathbf{R}^N,$$

$$dY_t = \sigma_2(t, Y_t)dW_t + b_2(t, X_{t\wedge\tau}, Y_t)dt, \qquad Y_0 = y_0 \in \mathbf{R}.$$

ここで、次の確率微分方程式の解である $\rho_t$ を考える。

$$\rho_t =$$

$$\exp(\int_0^t b_0(s, X_s, Z_s)dB_s + \int_0^t \beta(s, X_{s\wedge\tau}, Y_s)dW_s$$

$$+\frac{1}{2}\int_0^t (b_0(s, X_s, Z_s)^2 + \beta(s, X_{s\wedge\tau}, Y_s)^2)ds),$$

そして、$\widetilde{P}$ を $d\widetilde{P} = \rho_t^{-1}dP$ で与えられる、 $(\Omega, \mathcal{F})$ 上の確率測度とする。このとき

$$\widetilde{M}_t = N_t - \int_0^t (1 - N_{s-})\lambda_{x_0}(s)ds$$

は $\widetilde{P}$-$\mathcal{F}_t$-マルチンゲールとなる。$\Sigma$ を $\mathcal{B}$ -適合な連続過程 $F$ の集合とし、ここで $F$ は、$f_i,\quad i = 1, 2, 3 \in \mathcal{L}^{6+}$, $(f_4^j), j = 1, \cdots, d \in \mathcal{L}^{6+}$ があって、

$$F_t = F_0 + \int_0^t f_1(s)ds + \int_0^t f_2(s)dW_s + \int_0^t f_3(s)dB_s + \sum_{j=1}^d \int_0^t f_4^j(s)d\hat{B}_s^j$$

と表現されるとする。ここで、

$$\tilde{\tilde{\lambda}}(s) = \lambda_{x_0}(s) + \hat{V}(s; 1), \quad \widetilde{\widetilde{M}}_t = N_t - \int_0^t (1 - N_s)\tilde{\tilde{\lambda}}(s)ds,$$

$$\widetilde{\widetilde{W}}_t = \widetilde{W}_t - \int_0^t E[\beta(r, X_{r\wedge\tau}, Y_r)|\mathcal{F}_r]dr$$

とすると、$\widetilde{\widetilde{M}}_t$ が $P$-$\mathcal{F}_t$-マルチンゲールであり、$\widetilde{\widetilde{W}}_t$ が $P$-$\mathcal{F}_{t\in[0,\infty)}$-ブラウン運動となることを本稿で示した。また、本稿では次の表現定理を得た。

**Theorem 1.1.2.** $F \in \Sigma$ とし $\bar{F}_t = E[F_{t\wedge\tau}|\mathcal{F}_t]$ とする。このとき以下を得る。

(1)

$$E[F_{t\wedge\tau}|\mathcal{F}_t] = F_0 + \int_0^t \bar{f}_0(r; F)d\widetilde{\widetilde{M}}_r + \int_0^t \bar{f}_1(r; F)dr + \int_0^t \bar{f}_2(r; F)d\widetilde{\widetilde{W}}_r.$$

ここで、

$$\bar{f}_0(r; F) = -1_{\{\tau>r\}}(\hat{V}(r; F) + \hat{V}(r; 1)\bar{F}_{r-})\tilde{\tilde{\lambda}}(r)^{-1},$$

$$\bar{f}_1(r; F) = 1_{\{\tau>r\}}E[1_{\{\tau>r\}}(\widetilde{L}F)_r|\mathcal{F}_r],$$

$$\bar{f}_2(r; F) = E[(\widetilde{D_0}F)_r|\mathcal{F}_r] - E[\beta(r, X_r, Y_r)|\mathcal{F}_r]\bar{F}_{r-}$$

である。
(2) さらに、$t > 0$ に対して $1_{\{|X_t| \le 1\}} 1_{\{\tau > t\}} |F_t| \le C|X_t|^\alpha$ となるような $C > 0$ と $\alpha \in (0, 1)$ が存在するのであれば、以下が成立する。$\bar{f}_0(r; F) = -1_{\{\tau > r\}} \bar{F}_{r-}$.

この表現定理の応用として、ある企業の株価と社債の価格過程を与えた。これらの結果の詳しい証明等は第 2 章で与える。

## 1.2　クレジットモデルとコピュラの考察の概要

前章ではデフォルト・リスクの数理モデル化に関して議論を行ったが、そこでの議論は、基本的に、単一企業のデフォルトを想定している。社債など、発行体のデフォルトのみが影響する商品を取り扱う場合であれば、前章の設定で十分であるが、CDO(Collateralized Debt Obligation) に代表されるような、複数の企業群を参照する金融商品を取り扱う場合は、各々の債務の分布ではなく、それらの同時分布を考える必要がある。

ここで CDO を含め、金融市場においてクレジットに分類される商品について簡単に触れておきたい。金融商品は現物とデリバティブに分類されるが、クレジットにおける現物は社債とローンである。社債は 1 社の債務者 (社債の発行体) に対し、複数社の債権者 (社債の購入者) が存在する。また、社債は様々な業態が発行している。一方でローンは、通常、債権者 (ローンの貸し手) は銀行 1 行であり、債務者 (ローン借り手) も 1 社であることが多い。ローンの債権者の業態は既述の通り、ほぼ銀行に限られる。

クレジットの代表的なデリバティブは、CDS(Credit Default Swap) と CDO である。CDS にも個別企業を参照したり、特定のローンや証券化商品を参照する CDS 等があるが、ここでは個別 CDS(以下、単に CDS と呼ぶ時は個別 CDS を指すこととする) に絞って解説を行う。

CDS は保険契約に似ており、CDS の買い手は、契約期間中、CDS の売り手にプレミアムを支払う代わりに、契約期間中に CDS の参照企業がデフォルトすれば、CDS の売り手から保証金を受け取ることができる。CDS の売りは、社債の買いと良く似ている部分がある。どちらも、発行体、ないし、参照体のクレジットが改善すれば、価値が上昇するし、デフォルトが発生しない限りは、プレミアムやクーポンを受け取ることが

できる。ただし、社債と CDS は次の点で事なる。まず、社債は現物のため、購入に際し現金が必要であるが、CDS はデリバティブのため、ほぼ現金は不要である (証拠金のみでよい)。その代わりに社債は満期の元本払いがある一方で、CDS は満期において元本に対応するキャッシュフローは発生しない。また、社債は通常、買いからしか入れないが (社債の売り手は発行体のみ)、CDS は売りからも買いからも入ることができる。さらに、社債でいうところの狭義のデフォルトと、CDS のデフォルト条項は異なる。前者は、利払いも元本の支払いも行われない状態を指すが、後者は、利子の減額や利払いの遅れもデフォルトとなることが多い (何をもってデフォルトとするかは、ISDA と呼ばれる契約書により定義される)。

次に CDO についてであるが、CDO については小宮 [3] に詳細な記述があるため、概要を述べるに留めたい。CDO はクレジット商品のポートフォリオ (複数のクレジット商品) を参照するデリバティブである。参照商品がクレジットの現物ならキャッシュ CDO(社債なら CBO、ローンなら CLO)、クレジットデリバティブ (CDS 等) を参照する場合は、シンセティック CDO と呼ばれる。CDO を参照する CDO も存在し、それは $\mathrm{CDO}^2$(CDO スクエアード) と呼ばれる。

さて、CDO を考える場合、参照ポートフォリオの中に入っているクレジット商品 (各社の社債やローン等) 同士の相関を考える必要がある。ここでの相関とは、連鎖倒産に代表されるような、デフォルトの意味での相関である。この相関を考える上で良く用いられるのが、次に述べるコピュラである。

コピュラとは、複数の確率変数 $\{X_i\}_{i=1,\cdot,n}$ があったときに、各々の周辺分布 $F_i(x_i)_{i=1,\cdots,n}$ と、それらの同時分布関数 $F(x_1,\cdots,x_n)$ を繋ぐ関数を指し、具体的には、

$$F(x_1,\cdot,x_n)=C(F_1(x_1),\cdots,F_n(x_n))$$

なる関数 $C$ がコピュラである。また、Sklar の定理 [16] により、同時分布関数 $F$ と周辺分布関数 $\{F_i\}_{i=1,\cdots,n}$ をつなぐコピュラ $C$ が一意に存在することが証明されている。ただし、周辺分布関数 $\{F_i\}_{i=1,\cdots,n}$ を決めればコピュラ $C$ が一意に決まる訳ではないので、参照する債務のポートフォリオの特徴に応じて、適切と思われるコピュラを選択する必要が出てくる。戸坂・吉羽 [2] ではコピュラを、変数間の相関関係を行列で表現するコピュラと、1 種類のパラメータで表現するコピュラに分類しており、前者の例として、正規コピュラ、$t$ コピュラを、後者の例として、1 パラメータ・アルキメディアン・コピュ

ラ (ガンベル (Gumbel)・コピュラ、クレイトン (Clayton)・コピュラ、フランク (Frank)・コピュラ等) を挙げている。

(1) 変数間の相関関係を行列を用いて表現するコピュラ

(1-1) 正規コピュラ (Gaussian コピュラ)

確率変数 $\{X_i\}_{i=1,\cdots,n}$ が $n$ 変数標準正規分布関数 $\Phi_n(x_1,\cdots,x_n;\Sigma)$ に従うとする。ここで $\Sigma$ は $\{X_i\}_{i=1,\cdots,n}$ の相関行列である。このときコピュラ $C$ を

$$C(x_1\cdots,x_n)=\Phi_n(\Phi^{-1}(x_1),\cdots,\Phi^{-1}(x_n);\Sigma)$$

とおけば、次のように同時分布関数と周辺分布関数を結びつけることができる。

$$\Phi_n(x_1,\cdots,x_n;\Sigma)=C(\Phi_1(x_1),\cdots,\Phi_1(x_n)).$$

このコピュラ $C$ を正規コピュラ (Gaussian コピュラ) と呼ぶ。

(1-2)t コピュラ

自由度 $\nu$、相関行列 $\Sigma$ の $n$ 変量分布関数を $t_\nu^n(x_1\cdots,x_n;\Sigma)$、自分度 $\nu$ の 1 変量 $t$ 分布関数を $t_\nu$ とすると、次で定義される $C$ を $t$ コピュラと呼ぶ。

$$C_\nu^n(x_1\cdots,x_n;\Sigma)=t_\nu^n(t_\nu^{-1}(x_1),\cdots,t_\nu^{-1}(x_n);\Sigma).$$

(2) アルキメディアン・コピュラ

$\phi(1)=0$, $\phi'(u)<0$, $\phi''(u)\geq 0$ for any $u\in(0,1)$ を満たす関数 $\phi(\cdot)$ を用いて、次で表現されるコピュラをアルキメディアン・コピュラと呼ぶ。

$$C(x_1,\cdots,x_n)=\phi^{-1}(\phi(x_1)+\cdots+\phi(x_n)).$$

そして 1 パラメータで表現可能なコピュラにはガンベル (Gumbel)・コピュラ、クレイトン (Clayton)・コピュラ、フランク (Frank)・コピュラ等がある。

(2-1) ガンベル・コピュラ

ガンベル・コピュラは次で与えられる。

$$C(x_1,\cdots,x_n)=\exp(-(\sum_{i=1}^{n}(-\ln x_i)^\alpha)^{1/\alpha}).$$

(2-2) 反転ガンベル・コピュラ

反転ガンベル・コピュラは次で与えられる。

$$C(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i - 1 + \exp(-(\sum_{i=1}^{n}(-\ln(1-x_i))^{\alpha})^{1/\alpha}).$$

(2-3) クレイトン・コピュラ

クレイトン・コピュラはパラメータ $\alpha > 0$ に対し、次で与えられる。

$$C(x_1, \cdots, x_n) = (\sum_{i=1}^{n} x_i^{-\alpha} - n + 1)^{-1/\alpha}.$$

(2-4) フランク・コピュラ

フランク・コピュラはパラメータ $\alpha > 0$ に対し、次で与えられる。

$$C(x_1, \cdots, x_n) = -\frac{1}{\alpha}\left(1 + \frac{\prod_{i=1}^{n}(e^{-\alpha x_i} - 1)}{(e^{-\alpha} - 1)^{n-1}}\right).$$

さて本稿では、このコピュラに焦点を当てている。コピュラは、静的な条件下で、クレジット商品のポートフォリオに関するパラメーターの推計、リスク量の算出に用いられている。そして、クレジット商品のポートフォリオを原資産とする CDO の価格付けにもコピュラが用いられているが、価格付けにコピュラを用いる場合、コピュラが動的なデフォルト時刻モデルと整合的であるか、という点が重要となってくる。コピュラの性質や、実務的にどのように用いられているかについては、Li[8]、小宮 [3]、戸坂・吉羽 [2] 等に記載されているが、この整合性について分析を行った論文は筆者の調べた限りでは、存在しない。しかしコピュラが整合性を満たさないということになれば、そもそも実務で行われているような、CDO の価格付けにコピュラを用いること自体、必ずしも妥当ではない (少なくともオプションプライシングに Black Scholes が用いられる時程の妥当性はない) ということになろう。

Björk-Christernsen [4] は無裁定な金利モデルとフォワードレート・カーブが整合的となる、つまり、ある金利モデルによって、別のあるフォワードカーブの族が生成される必要十分条件を導出した。また上記論文では、例として、Nelson-Sigel 型のフォワー

ドレート・カーブは Ho-Lee モデルをはじめとする、拡散過程モデル全般と整合的でないことを指摘している。

本稿では、有限次元のパラメータを持つコピュラの族と、動的なデフォルト時刻の関係を分析し、動的なデフォルト時刻と整合的となるコピュラは、(Baire の第一類の意味で) 稀であるということを示した。つまり、ほとんどすべてのコピュラは本稿でいうところの整合性を満たしておらず、CDO のプライシングという動的な設定においてを用いる場合は、整合性を確認する必要があるというのが本稿の結論である。なお本稿では、$n=3$ の Gumbel コピュラと逆 Gumbel を取り上げ、数値計算によりこれらが整合性を満たさないことを示した。

コピュラの代わりに CDO のプライシングに何を用いればよいのか、つまり、動的なデフォルト時刻モデルと整合的な関数としてどのようなものが存在するのか、については今後の研究課題としたい。

最後に本稿の設定と、結論の概要について述べることとする。$(\Omega, \mathcal{F}, P)$ を完備な確率空間とする。$W(t) = (W^k(t))_{k=1,\dots,d}$, $t \geqq 0$, は $d$-次元の標準ブラウン運動、$\mathcal{N} = \{B \in \mathcal{F};\ P(B) = 0 \text{ or } 1\}$,$\mathcal{G}_t = \sigma\{W(s),\ s \in [0,t]\} \vee \mathcal{N}$ とする。

また、$N \geqq 2,\ \tau_i : \Omega \to [0,\infty),\ i = 1,\dots,N,$ を random variable とし $\mathcal{F}_t = \mathcal{G}_t \vee \sigma\{\tau_i \wedge t, i = 1,\dots,N\}$ とした。さらに、$\xi_i : [0,\infty) \times \Omega \to [0,\infty),\ i = 1,\dots,N,$ は $\mathcal{G}$-発展的可測過程とした。

まず、次のような仮定をおいた。

(SC)　任意の $I \subset \{1,\dots,N\}$ と $t, t_i \in [0,\infty), i \in I$、ただし、$t \leqq \min_{i \in I} t_i$ に対し、以下が成立する。

$$(\prod_{i \in I} 1_{\{\tau_i > t\}}) P(\tau_i > t_i, i \in I | \mathcal{F}_t) = (\prod_{i \in I} 1_{\{\tau_i > t\}}) E[\exp(-\sum_{i \in I} \int_t^{t_i} \xi_i(s) ds) | \mathcal{G}_t]\ a.s.$$

(PO) 任意の $t \geqq 0$ に対し、以下が成立する。

$$P(\bigcap_{i=1}^{N} \{\tau_i > t\}) | \mathcal{G}_t] > 0\ a.s.$$

さらに、技術的な理由で、次の仮定をおいた。

(A-1) 任意の $T > 0$ に対し、以下が成立する。

$$\sum_{i=1}^{N} \int_0^T E[\xi_i(t)^4]dt < \infty.$$

(A-2) $b > a$ とする。任意の $a, b > 0$ と任意の $i = 1, \ldots, N$ に対し、以下が成立する。

$$\int_0^\infty \xi_i(t) = \infty \quad a.s. \text{ かつ } \int_a^b \xi_i(t) > 0 \quad a.s.$$

(A-3) $\displaystyle\sum_{i=1}^{N} \int_0^\infty (1+t)^2 E[\xi_i(t)^2 \exp(-2\int_0^t \xi_i(s)ds)]dt < \infty.$

ここで $\theta : [0, \infty) \times \Omega \to \mathbf{R}^M$ を $\mathcal{G}$-伊藤過程とし、$\theta(t, \omega)$ は全ての $\omega \in \Omega$ に対して $t$ 連続であり、さらに、発展的可測過程 $\eta_k \in \mathbf{R}^M$, $k = 1, \ldots, d,$ と $b$ があって、任意の $T > 0$ に対して、

$$P(\sum_{k=1}^{d} \int_0^T |\eta_k(t)|^2 dt + \int_0^T |b(t)|dt < \infty) = 1$$

と

$$\theta(t) = \theta(0) + \sum_{k=1}^{d} \int_0^t \eta_k(s)dW^k(s) + \int_0^t b(s)ds.$$

を満たすとした。

$\Theta$ を $\mathbf{R}^M$ の開部分集合とし、$K \in C([0,1]^N \times \Theta; [0,1])$ とした。さらに、以下を仮定した。

(A-4) $P($任意の $t \geq 0$ に対し,$\theta(t) \in \bar{\Theta}) = 1$ が成立するとした。ここで $\bar{\Theta}$ は $\Theta \in \mathbf{R}^M$ の閉包であるとした。

(A-5) $e^{-t}dt \otimes P(d\omega)$ における $\theta(t, \omega)$ の確率密度の台は、$\Theta$ において空でない開集合を含む。つまり、$U_0 \in \Theta$ に空でない開集合があって、任意の $\theta_0 \in U_0$ と $\varepsilon > 0$ に対して以下を満たす。

$$\int_0^\infty P(|\theta(t) - \theta_0| < \varepsilon)\, e^{-t}dt > 0$$

(CP) $K(\cdot, \theta) : [0,1]^N \to [0,1]$ は任意の $\theta \in \Theta$ に対してコピュラ関数であり、任意の $t, t_1, \ldots, t_N > 0$, $t < \min_{i=1,\ldots,N} t_i$ に対して以下を満たす。

$$(\prod_{i=1}^{N} 1_{\{\tau_i > t\}}) 1_\Theta(\theta(t)) P(\tau_i > t_i, i = 1, \ldots, N | \mathcal{F}_t)$$

$$= (\prod_{i=1}^{N} 1_{\{\tau_i > t\}}) 1_\Theta(\theta(t)) K(P(\tau_1 > t_1 | \mathcal{F}_t), \ldots, (P(\tau_N > t_N | \mathcal{F}_t), \theta(t)) \ a.s.$$

ここで上記の仮定を満たす $((\Omega, \mathcal{F}, P), (W_t^k)_{k=1,\ldots,d}, (\tau_i)_{i=1,\ldots,N}, (\xi_i(t))_{i=1,\ldots,N}, \theta(t), \Theta, K)$ を動的デフォルト時刻コピュラモデル呼ぶことにし、このような $K$ をこのモデルに関連付けられたコピュラ関数の族と呼ぶことにする。

**Definition 1.2.1.** $\Theta$ を $\mathbf{R}^M$ の開部分集合とする。動的デフォルト時刻コピュラモデルがあって、$K \in C([0,1]^N \times \Theta; [0,1])$ がそのモデルに関連付けられたコピュラの族であるとき、$K$ を許容可能なコピュラの族と呼ぶことにする。

本稿の目的は許容可能なコピュラの族に対する解析的な条件を示すことである。例えば、以下のようなことを示した。

$N, M \geqq 1, \Theta$ を空でない $\mathbf{R}^M$ の開部分集合であるとする。そして $\mathcal{C}_{(N)}(\Theta)$ を $C([0,1]^N \times \Theta; [0,1])$ の部分集合とし、次のような $K$ を含むものとする。ここでの $K(\cdot, \theta) : [0,1]^N \to [0,1]$ とはどの $\theta \in \Theta$ に対してもコピュラであり、$K|_{(0,1)^N \times \Theta}$ が $C^\infty$ 関数であるものである。

$\Theta$ の部分集合 $D_n$ をは増加列でコンパクトであるとするとし、$\bigcup_{n=1}^\infty D_n = \Theta$ を満たすとする。このとき $\mathcal{C}_{(N)}(\Theta)$ を、次のような距離関数 $dis$ を持つポーランド空間とみなすことができる。

$$\begin{aligned} & dis(K_1, K_2) \\ &= \sum_{n=1}^\infty 2^{-n} \wedge \sup\{|K_1(x,\theta) - K_2(x,\theta)|;\ x \in [0,1]^N,\ \theta \in D_n\} \\ &+ \sum_{n=1}^\infty 2^{-n} \wedge (\sum_{\alpha_1, \ldots, \alpha_{N+M}=0}^{n} \sup\{|\frac{\partial^{\alpha_1 + \cdots \alpha_{N+M}} (K_1 - K_2)}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N} \partial \theta_1^{\alpha_{N+1}} \partial \ldots \theta_M^{\alpha_{N+M}}}(x,\theta); \\ & \qquad x \in [1/4n, 1 - 1/4n]^N,\ \theta \in D_n\}). \end{aligned}$$

本稿の主定理は次のとおりである。

**Theorem 1.2.1.** $N \geqq 3,\ M \geqq 1,$ とし、$\Theta$ は空でない $\mathbf{R}^M$ の開部分集合とする。このとき、$\mathcal{C}_{(N)}(\Theta)$ の部分集合の中で、許容可能なコピュラ関数からなるものは、$Baire$ の第一類、つまり、加算個の疎集合の和になっている。

またコピュラの例として、Gumbel コピュラと逆 Gumbel を取り上げ、数値計算によりこれらが許容可能でないということを示した。正規コピュラ (Gaussian コピュラ) も許容可能ではないと思われるが、計算精度が足りず、示すことができなかった。計算ツールとしては C 言語と統計ソフト R をそれぞれ試したが、同様の結果であり、上記の部分を改善することはできなかった。これらの結果の詳しい証明は第 3 章で与える。

共著して頂いた A remark on credit risk models and copula はもとより、A filtering model や本博士論文を纏めるにあたり、楠岡　成雄教授には数多くのご助言を頂戴致しました。心より感謝いたします。

# 第2章　A filtering model

## 2.1 Introduction

Duffie and Land [9] studied the implications of imperfect information for term structures of credit spreads on corporate bonds. They supposed that the bond investor cannot observe the issuer's assets directly and they can receive only periodic and imperfect accounting information. Then they delivered a relationship between the volatility of the issuer's asset price and its hazard rate. Their model is a kind of a filtering model. Jarrow, Protter and Deniz [11] provided an alternative credit risk model based on information reduction where the market only observes the company's asset value when it crosses certain levels, interpreted as changes significant enough for the company's management to make a public announcement. Jeanblanc and Valchev [12] examined three types of information for a company's unlevered asset value to the secondary bond market, the classical case of continuous and perfect information, observation of past and contemporaneous asset values at selected discrete times, and observation of contemporaneous asset value at discrete times. Nakagawa[15] constructed a filtering model on a default risk and showed representation formulas under imperfect information. He analyzed properties of the processes under $\nu_{0,x_1}^{u,x_2}$ which is a probability measure on $C([0,u];\mathbf{R})$, and the law of Brownian motion $(B_s)_{s\in[0,u]}$ conditioned to start from $x_1>0$, stay in $(0,\infty)$ for $s\le u$ and reach $x_2>0$ at time $u$ under $P$. However since this measure is hard to deal with, we give representation formulas without using the measure $\nu$. Let $(\Omega,\mathcal{B},P,\{\mathcal{B}_t\}_{t\in[0,\infty)})$ be a complete filtrated probability space and the filtration $\{\mathcal{B}_t\}_{t\ge 0}$ satisfies usual conditions. Let $B_t$, $\hat{B}_t$ and $W_t$ be independent $\mathcal{B}_{t\in[0,\infty)}$-Brownian motions with values in $\mathbf{R}$,$\mathbf{R}^d$ and $\mathbf{R}$ respectively. We denote by $(\mathcal{G}_t^X)$ the right continuous filtration generated by continuous stochastic process $X$. For example, $\mathcal{G}_t^B=\bigcap_{u>t}\sigma\{B_s,s\le u\}$. Let

$\tau = \inf\{t > 0;\ X_t = 0\}$, $N_t = 1_{\{\tau \le t\}}$ and $\mathcal{F}_t = \bigcap_{u>t}(\mathcal{G}_u^Y \vee \sigma\{\tau \wedge u\})$. Let $a > 0$, $B_t^a = a + B_t$, $\tau^a = \inf\{t > 0;\ B_t^a = 0\}$, $N_t^a = 1_{\{\tau^a \le t\}}$ and $\mathcal{F}_t^W = \bigcap_{u>t}(\mathcal{G}_u^W \vee \sigma\{\tau^a \wedge u\})$. Let $q_a(t) = \int_t^\infty \frac{a}{\sqrt{2\pi s^3}} \exp(-\frac{a^2}{2s})ds$ and $\lambda_a(t) = -\frac{d}{dt}\log q_a(t) = \frac{a}{\sqrt{2\pi t^3}} q_a(t)^{-1}\exp(-\frac{a^2}{2t})$. Then $P[\tau^a > t] = q_a(t) = e^{-\int_0^t \lambda_a(u)du}$ and $P[\tau^a \in dt] = \lambda_a(t)e^{-\int_0^t \lambda_a(u)du}dt = \gamma_a(t)dt$ . We also see that

$$M_t = N_t^a - \int_0^t (1 - N_s^a)\lambda_a(s)ds$$

is $\mathcal{F}_t^W$-martingale. Let

$$g(t,x) = \frac{1}{\sqrt{2\pi t}}\exp(-\frac{x^2}{2t}), \qquad \Phi(t,x) = \int_{-\infty}^x g(t,y)dy,$$

$$g_0(t,x,y) = g(t,y-x) - g(t,y+x) = g(t,y-x)(1 - e^{-2xy/t}), \qquad x,y \ge 0, \quad t > 0.$$

Here we note that

$$\frac{\partial g}{\partial x}(t,x) = -\frac{x}{t}g(t,x), \quad \frac{\partial^2 g}{\partial x^2}(t,x) = 2\frac{\partial g}{\partial t}(t,x) = -\frac{x^2 - t}{t^2}g(t,x).$$

We denote by $\mathcal{L}^p$, $p \in (1,\infty)$, the space of $\{\mathcal{B}_t\}$-progressively measurable functions $\varphi$ such that $E[\int_0^T |\varphi|_s^p ds] < \infty$ for any $T > 0$ and denote $\mathcal{L}^{p+} = \bigcup_{q>p}\mathcal{L}^q, p \ge 1$. Let

$$H_a^{(k)}(t,s;f) = E[1_{\{\tau^a > s\}} f_s \frac{\partial^k g}{\partial x^k}(t-s, B_s^a)|\mathcal{G}_s^W], \quad t > s, \quad f \in \mathcal{L}^{1+}, \quad k = 0,1,2,$$

$$\hat{H}_a^{(k)}(t;f) = \int_0^t H_a^{(k)}(t,u;f)du, \quad f \in \mathcal{L}^{\frac{4}{3-k}+}, \quad k = 0,1,2,$$

$$\bar{H}_a(t;f) = e^{\int_0^t \lambda_a(r)dr}(\hat{H}_a^{(2)}(t;f) + 2\lambda_a(t)\hat{H}_a^{(0)}(t;f)), \quad f \in \mathcal{L}^{4+},$$

$$U_a(t,s;f) = E[1_{\{\tau^a > s\}} f_s (2\Phi(t-s, B_s^a) - 1)|\mathcal{G}_s^W], \quad t > s, \quad f \in \mathcal{L}^{1+}$$

and

$$\bar{U}_a(t,s;f) = e^{\int_0^t \lambda_a(r)dr}(H_a^{(1)}(t,s;f) + \lambda_a(t)U_a(t,s;f)), \quad t > s, \quad f \in \mathcal{L}^{1+}.$$

We will show that these are well-defined in Section 2.2. Then we have the following Theorem.

**Theorem 2.1.1.** (1) *For any* $t, T \geq 0$ *and* $f \in \mathcal{L}^{4+}$,

$$E[\int_0^T f_s dB_s|\mathcal{F}_t^W] = -\int_0^t \bar{H}_a(s; f1_{(0,T]}(\cdot))\lambda_a(s)^{-1}dM_s.$$

(2) *For any* $t \geq 0$ *and* $f \in \mathcal{L}^{4+}$,

$$E[\int_0^t f_s ds|\mathcal{F}_t^W] = \int_0^t E[f_s|\mathcal{F}_s^W]ds - \int_0^t (\int_0^s \bar{U}_a(s, r; f)dr)\lambda_a(s)^{-1}dM_s.$$

(3) *For any* $t, T \geq 0$ *and* $f \in \mathcal{L}^{6+}$,

$$E[\int_0^T f_s dW_s|\mathcal{F}_t^W] = \int_0^{T\wedge t} E[f_s|\mathcal{F}_s^W]dW_s - \int_0^t (\int_0^s \bar{U}_a(s, r; f1_{[0,T]}(\cdot))dW_r)\lambda_a(s)^{-1}dM_s.$$

(4) *For any* $t \geq 0, \hat{f}_i, \in \mathcal{L}^{2+}, i = 1, \cdots, d$,

$$E[\sum_{i=1}^d \int_0^t \hat{f}_s^i d\hat{B}_s^i|\mathcal{F}_t^W] = 0.$$

Let $X$ and $Z$ be solutions of the following stochastic differential equations under $P$,

$$dX_t = dB_t + b_0(t, X_t, Z_t)dt, \qquad X_0 = x_0 > 0,$$

$$dZ_t = \sigma_1(t, X_t, Z_t)d\hat{B}_t + b_1(t, X_t, Z_t)dt, \qquad Z_0 = z_0 \in \mathbf{R}^N,$$

where $b_0 : [0, \infty) \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$, $\sigma_1 : [0, \infty) \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^{N\times d}$ and $b_1 : [0, \infty) \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ are bounded and continuously differentiable functions. Let $Y$ be a solution of the following stochastic differential equations,

$$dY_t = \sigma_2(t, Y_t)dW_t + b_2(t, X_{t\wedge\tau}, Y_t)dt, \qquad Y_0 = y_0 \in \mathbf{R},$$

where $\sigma_2 : [0, \infty) \times \mathbf{R} \to \mathbf{R}$ and $b_2 : [0, \infty) \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are bounded and continuously differentiable functions. We assume that there exists $\epsilon > 0$ and $\sigma_2(t, y)$ satisfies $\sigma_2(t, y) \geq \epsilon$ for any $t \in [0, \infty)$, $y \in \mathbf{R}$. Then $\sigma_2(t, y)^{-1}$ exists and satisfies

$$\sigma_2(t, y)^{-1} \leq \epsilon^{-1}, \quad y \in \mathbf{R}, \quad t \in [0, \infty).$$

Let $\rho_t$, $t \in [0, T]$, be given by

$$\rho_t = \\ \exp(\int_0^t b_0(s, X_s, Z_s)dB_s + \int_0^t \beta(s, X_{s\wedge\tau}, Y_s)dW_s \\ +\frac{1}{2}\int_0^t (b_0(s, X_s, Z_s)^2 + \beta(s, X_{s\wedge\tau}, Y_s)^2)ds),$$

where $\beta(t, x, y) = \sigma_2(t, y)^{-1} b_2(t, x, y)$. Let $\widetilde{P}$ be a probability measure on $(\Omega, \mathcal{F})$ given by $d\widetilde{P} = \rho_t^{-1} dP$. We see that $\rho, \rho^{-1} \in \bigcap_{p\geq 1} \mathcal{L}^p$ by Novikov's Theorem. Let $\widetilde{\rho}_t = \widetilde{E}[\rho_t|\mathcal{F}_t]$. Here we will denote by $\tilde{E}[\cdot]$ denote an expectation under the probability measure $\tilde{P}$. Let

$$\widetilde{B}_t = B_t + \int_0^t b_0(s, X_s, Z_s)ds$$

and

$$\widetilde{W}_t = W_t + \int_0^t \beta(s, X_{s\wedge\tau}, Y_s)ds.$$

Then $\widetilde{B}_t$, $\hat{B}_t$ and $\widetilde{W}_t$ are independent $\widetilde{P}$-$\{\mathcal{B}_t\}_{t\in[0,\infty)}$ -Brownian motions. Stochastic processes $X$, $Z$ and $Y$ are described in the following

$$dX_t = d\widetilde{B}_t,$$

$$dZ_t = \sigma_1(t, X_t, Z_t)d\hat{B}_t + b_1(t, X_t, Z_t)dt,$$

$$dY_t = \sigma_2(t, Y_t)d\widetilde{W}_t.$$

By the above equations, we see that $\{\mathcal{G}_t^X\}_{t\in[0,\infty)}$ coincides with the natural filtration generated by $\{\widetilde{B}_t\}_{t\in[0,\infty)}$. Since $d\widetilde{W}_t = \sigma(t, Y_t)^{-1} dY_t$, we can see that $\mathcal{G}_t^Y = \mathcal{G}_t^{\widetilde{W}}$ and $\mathcal{F}_t = \bigcap_{u>t}(\mathcal{G}_u^{\widetilde{W}} \vee \sigma\{\tau \wedge u\})$. In addition, we can see that

$$\widetilde{M}_t = N_t - \int_0^t (1 - N_{s-})\lambda_{x_0}(s)ds$$

is $\widetilde{P}$-$\mathcal{F}_t$-martingale. Let

$$I^{(k)}(t, s; f) = \widetilde{E}[1_{\{\tau>s\}} \frac{\partial^k g}{\partial x^k}(t-s, X_s)\rho_{s-} f_s | \mathcal{G}_s^Y], \quad t > s, \quad k = 0, 1, 2$$

for $f \in \mathcal{L}^{2+}$. Let $\Sigma$ denote the set of $\mathcal{B}$ -adapted continuous processes $F$ for which there exist $f_i, \quad i = 1, 2, 3 \in \mathcal{L}^{6+}$ and $(f_4^j), j = 1, \cdots, d \in \mathcal{L}^{6+}$such that

$$F_t = F_0 + \int_0^t f_1(s)ds + \int_0^t f_2(s)dW_s + \int_0^t f_3(s)dB_s + \sum_{j=1}^d \int_0^t f_4^j(s)d\hat{B}_s^j. \qquad (2.1.1)$$

For $F \in \Sigma$, let

$$(\widetilde{D_0}F)_t = \beta(t, X_{t\wedge\tau}, Y_t)F_{t\wedge\tau} + 1_{\{\tau>t\}}f_2(t), \quad (\widetilde{D_1}F)_t = b_0(t, X_t, Z_t)F_{t\wedge\tau} + 1_{\{\tau>t\}}f_3(t),$$

$$(\widetilde{D_2}F)_t = 1_{\{\tau>t\}}f_4(t), \quad (\widetilde{L}F)_t = 1_{\{\tau>t\}}f_1(t)$$

and

$$\hat{V}(r; F) =$$

$$\widetilde{\rho}_{r-}^{-1} e^{\int_0^r \lambda_{x_0}(u)du}(\hat{V}_1(r, r; F) + \lambda_{x_0}(r)(-(2\Phi(r, x_0) - 1)F_0 + \widetilde{E}[1_{\{\tau>r\}}\rho_r F_r|\mathcal{G}_r^Y])), \quad r > 0,$$

where

$$\hat{V}_1(r, s; F)$$

$$= \int_0^s I^{(1)}(r, u; \widetilde{D_0}F)d\widetilde{W}_u + \int_0^s (I^{(2)}(r, u; \widetilde{D_1}F) + I^{(1)}(r, u; \widetilde{L}F))du, \quad r > s > 0.$$

Let

$$\tilde{\tilde{\lambda}}(s) = \lambda_{x_0}(s) + \hat{V}(s; 1), \quad \widetilde{\widetilde{M}}_t = N_t - \int_0^t (1 - N_s)\tilde{\tilde{\lambda}}(s)ds$$

and

$$\widetilde{\widetilde{W}}_t = \widetilde{W}_t - \int_0^t E[\beta(r, X_{r\wedge\tau}, Y_r)|\mathcal{F}_r]dr.$$

Then we will show that $\widetilde{\widetilde{M}}_t$ is $P$-$\mathcal{F}_t$-martingale and $\widetilde{\widetilde{W}}_t$ is $P$-$\mathcal{F}_{t\in[0,\infty)}$-Brownian motion. Then we have the following theorem.

**Theorem 2.1.2.** *Let $F \in \Sigma$ and $\bar{F}_t = E[F_{t\wedge\tau}|\mathcal{F}_t]$. Then we have the following.*

(1)

$$E[F_{t\wedge\tau}|\mathcal{F}_t] = F_0 + \int_0^t \bar{f}_0(r; F)d\widetilde{\widetilde{M}}_r + \int_0^t \bar{f}_1(r; F)dr + \int_0^t \bar{f}_2(r; F)d\widetilde{\widetilde{W}}_r,$$

*where*

$$\bar{f}_0(r; F) = -1_{\{\tau>r\}}(\hat{V}(r; F) + \hat{V}(r; 1)\bar{F}_{r-})\tilde{\tilde{\lambda}}(r)^{-1},$$

$$\bar{f}_1(r; F) = 1_{\{\tau>r\}}E[1_{\{\tau>r\}}(\widetilde{L}F)_r|\mathcal{F}_r],$$

$$\bar{f}_2(r;F) = E[(\widetilde{D_0}F)_r|\mathcal{F}_r] - E[\beta(r,X_r,Y_r)|\mathcal{F}_r]\bar{F}_{r-}.$$

(2) *Moreover, if there exist* $C > 0$ *and* $\alpha \in (0,1)$ *such that* $1_{\{|X_t|\le 1\}}1_{\{\tau>t\}}|F_t| \le C|X_t|^\alpha$ *for* $t > 0$*, we have* $\bar{f}_0(r;F) = -1_{\{\tau>r\}}\bar{F}_{r-}$.

As an application of aforementioned theorems, the Proposition 2.6.1 and Proposition 2.6.3 provide the stochastic equations which the stock and bond price follow. In Section 2.6 , the stock price $S_t$ is defined as a value of company looked over the filtration $\mathcal{F}_t$ constructed of the observable process $Y_t$ and the information about whether the default takes place or not. Let $T$ be the maturity of a bond issued by this company, the recovery rate of this bond be zero, the risk free rate be also zero and $\alpha > 0$ be constant risk premium.

The author appreciates Prof. Kusuoka for his helpful advice.

## 2.2 Evaluation of integrands

For $f \in \mathcal{L}^1$, let

$$\widetilde{H}_a^{(k)}(t,s;f) = E[1_{\{\tau^a>s\}}|f_s\frac{\partial^k g}{\partial x^k}(t-s,B_s^a)||\mathcal{G}_s^W], \quad t>s>0, \quad k=0,1,2.$$

**Proposition 2.2.1.** *For* $q > 1$ *and* $k = 0,1,2$*, we have*

$$E[1_{\{\tau^a>u\}}|\frac{\partial^k g}{\partial x^k}(t-u,B_u^a)|^q|\mathcal{G}_u^W] \le C_1^{(k)}(a) + C_2^{(k)}(q,a)(t-u)^{\frac{-kq-q+2}{2}}$$

*for any* $t > u \ge 0$ *with* $t-u \le 1$*. Here*

$$C_1^{(k)}(a) = \sup_{x\ge a/2, t>0} |\frac{\partial^k g}{\partial x^k}(t,x)| < \infty$$

*and*

$$C_2^{(k)}(q,a) = 2a(\int_0^\infty y|\frac{\partial^k g}{\partial y^k}(1,y)|^q dy) \sup_{u>0}\frac{g(u,\frac{a}{2})}{u} < \infty.$$

*Proof.* We have

$$\frac{\partial^k g}{\partial x^k}(t,x) = \frac{\partial^k}{\partial x^k}(t^{-\frac{1}{2}}g(1,t^{-\frac{1}{2}}x)) = t^{-\frac{k+1}{2}}\frac{\partial^k g}{\partial x^k}(1,t^{-\frac{1}{2}}x).$$

Since $\{B_t^a\}$ and $\{W_t\}$ are independent,

$$\begin{aligned} & E[1_{\{\tau^a>u\}}|\frac{\partial^k g}{\partial x^k}(t-u,B_u^a)|^q|\mathcal{G}_u^W] \\ &= \int_0^\infty (g(u,x-a)-g(u,x+a))|\frac{\partial^k g}{\partial x^k}(t-u,x)|^q dx \\ &= \int_0^\infty g(u,x-a)(1-\exp(-\frac{2ax}{u}))|\frac{\partial^k g}{\partial x^k}(t-u,x)|^q dx \\ &\leq \int_{a/2}^\infty g(u,x-a)|\frac{\partial^k g}{\partial x^k}(t-u,x)|^q dx + \frac{2a}{u}\int_0^{a/2} g(u,x-a)x|\frac{\partial^k g}{\partial x^k}(t-u,x)|^q dx. \quad (2.2.1) \end{aligned}$$

For the first term, we have

$$\int_{a/2}^\infty g(u,x-a)|\frac{\partial^k g}{\partial x^k}(t-u,x)|^q dx \leq C_1^{(k)}(a)\int_{a/2}^\infty g(u,x-a)dx \leq C_1^{(k)}(a).$$

For the second term, we have

$$\begin{aligned} \frac{2a}{u}\int_0^{a/2} g(u,x-a)x|\frac{\partial^k g}{\partial x^k}(t-u,x)|^q dx &\leq g(u,\frac{a}{2})\frac{2a}{u}\int_0^\infty x|\frac{\partial^k g}{\partial x^k}(t-u,x)|^q dx \\ &= g(u,\frac{a}{2})\frac{2a}{u}\int_0^\infty x|(t-u)^{-\frac{k+1}{2}}\frac{\partial^k g}{\partial x^k}(1,(t-u)^{-\frac{1}{2}}x)|^q dx \\ &= g(u,\frac{a}{2})\frac{2a}{u}(\int_0^\infty y|\frac{\partial^k g}{\partial y^k}(1,y)|^q dy)(t-u)^{\frac{-kq-q+2}{2}} \leq C_2^{(k)}(q,a)(t-u)^{\frac{-kq-q+2}{2}}. \end{aligned}$$

Then we have our assertion. □To give a representation of conditional expectation under $P$ with respect to $\{\mathcal{G}_t^W\}$and $\{\mathcal{F}_t^W\}$, we need to show some inequalities to define stochastic integrals. Proposition 2.2.2 and Proposition 2.2.3 are supposed to valuate $\bar{H}_a$ and $\bar{U}_a$ in Theorem 2.1.1, respectively.

**Proposition 2.2.2.** *Let* $p \in (1,\infty)$ *and* $q = \frac{p}{p-1}$.
(1) *For* $k = 0,1,2$, *there is* $C_3^{(k)}(q,a)$ *and* $C_4^{(k)}(q,a) \in (0,\infty)$ *such that*

$$\widetilde{H}_a^{(k)}(t,u;f) \leq (C_3^{(k)}(q,a) + C_4^{(k)}(q,a)(t-u)^{\frac{-kq-q+2}{2q}})E[|f_u|^p|\mathcal{G}_u^W]^{\frac{1}{p}},$$

*for any* $f \in \mathcal{L}^p$, $t > u > 0$.

(2) *Let* $k = 0, 1, 2$ *and* $p > \frac{4}{3-k}$. *Then there are* $C_{5,1}^{(k)}(q,a)$ *and* $C_{6,1}^{(k)}(q,a) \in (0,\infty)$ *such that*

$$\int_0^t \widetilde{H}_a^{(k)}(t,u;f)du \le (C_{5,1}^{(k)}(q,a)t^{\frac{1}{q}} + C_{6,1}^{(k)}(q,a)t^{\frac{-kq-q+4}{2q}})(\int_0^t E[|f_u|^p]du)^{\frac{1}{p}},$$

*for any* $t > 0$, $f \in \mathcal{L}^p$.

(3) *Let* $k = 0, 1$, $p > \frac{3}{2-k}$. *There are* $C_{5,2}^{(k)}(q,a)$ *and* $C_{6,2}^{(k)}(q,a) \in (0,\infty)$ *such that*

$$\int_0^t \widetilde{H}_a^{(k)}(t,u;f)^2 du \le (C_{5,2}^{(k)}(q,a)t^{\frac{1}{q}} + C_{6,2}^{(k)}(q,a)t^{\frac{-kq-q+3}{q}})(\int_0^t E[|f_u|^{2p}]du)^{\frac{1}{p}},$$

*for any* $t > 0$, $f \in \mathcal{L}^{2p}$.

(4) *Let* $s \in [0,T]$. *There is* $\hat{C}_1(T,q,a) \in (0,\infty)$ *such that*

$$E[\int_0^s |\bar{H}_a(t;f)|dt] \le \hat{C}_1(T,q,a)(\int_0^s E[|f_u|^p]du)^{\frac{1}{p}},$$

*for any* $f \in \mathcal{L}^p$, $p > 4$.

(5) *Let* $0 \le s_0 < s_1$ *and* $\xi$ *be a bounded* $\mathcal{F}_{s_0}$*-measurable random variable. Then we have*

$$\int_0^s \hat{H}_a^{(2)}(r;\xi 1_{(s_0,s_1]}(\cdot))dr = 2\hat{H}_a^{(0)}(s;\xi 1_{(s_0,s_1]}(\cdot)).$$

*Proof.* (1) By Proposition 2.2.1, we have

$$\begin{aligned}
&|\widetilde{H}_a^{(k)}(t,u;f)| \\
&\le E[1_{\{\tau^a > u\}}|\frac{\partial^k g}{\partial x^k}(t-u,B_u^a)|^q|\mathcal{G}_u^W]^{\frac{1}{q}}E[|f_u|^p|\mathcal{G}_u^W]^{\frac{1}{p}} \\
&\le (C_1^{(k)}(a) + C_2^{(k)}(q,a)(t-u)^{\frac{-kq-q+2}{2}})^{\frac{1}{q}}.E[|f_u|^p|\mathcal{G}_u^W]^{\frac{1}{p}} \\
&\le (C_3^{(k)}(q,a) + C_4^{(k)}(q,a)(t-u)^{\frac{-kq-q+2}{2q}})E[|f_u|^p|\mathcal{G}_u^W]^{\frac{1}{p}},
\end{aligned}$$

where

$$C_3^{(k)}(q,a) = 2^{\frac{1}{q}}C_1^{(k)}(a)^{\frac{1}{q}}, \quad C_4^{(k)}(q,a) = 2^{\frac{1}{q}}C_2^{(k)}(q,a)^{\frac{1}{q}}.$$

Next, we will show assertion (2) and (3). Let $m = 1, 2$.

$$\int_0^t \widetilde{H}_a^{(k)}(t,u;f)^m du$$

$$\leq \int_0^t (C_3^{(k)}(q,a) + C_4^{(k)}(q,a)(t-u)^{\frac{-kq-q+2}{2q}})^{mq} E[|f_u|^p|\mathcal{G}_u^W]^{\frac{m}{p}} du$$

$$\leq (\int_0^t (C_3^{(k)}(q,a) + C_4^{(k)}(q,a)(t-u)^{\frac{-kq-q+2}{2q}})^{mq} du)^{\frac{1}{q}} (\int_0^t E[|f_u|^p|\mathcal{G}_u^W]^m du)^{\frac{1}{p}}$$

$$\leq 2^m (\int_0^t (C_3^{(k)}(q,a)^{mq} + C_4^{(k)}(q,a)^{mq}(t-u)^{\frac{(-kq-q+2)m}{2}}) du)^{\frac{1}{q}} (\int_0^t E[|f_u|^p|\mathcal{G}_u^W]^m du)^{\frac{1}{p}}.$$

If $m=1$ and $p > \frac{4}{3-k}$, or if $m=2$ and $p > \frac{3}{2-k}$, we have $p > \frac{2+2m}{2+m-mk}$ and $\frac{(-kq-q+2)m}{2} > -1$. Then we have

$$\int_0^t (C_3^{(k)}(q,a)^{mq} + C_4^{(k)}(q,a)^{mq}(t-u)^{\frac{(-kq-q+2)m}{2}}) du$$

$$\leq C_3^{(k)}(q,a)^{mq} t + C_4^{(k)}(q,a)^{mq} \frac{2}{|(-kq-q+2)m+2|} t^{\frac{(-kq-q+2)m+2}{2}}.$$

Then we have the following for $f \in \mathcal{L}^{mp}$.

$$\int_0^t \widetilde{H}_a^{(k)}(t,u;f)^m du$$

$$\leq (C_{5,m}^{(k)}(q,a) t^{\frac{1}{q}} + C_{6,m}^{(k)}(q,a) t^{\frac{(-kq-q+2)m+2}{2q}}) (\int_0^t E[|f_u|^{mp}] du)^{\frac{1}{p}},$$

where

$$C_{5,m}^{(k)}(q,a) = 2^{\frac{mq+m}{q}} C_1^{(k)}(a)^{\frac{m}{q}}, \quad C_{6,m}^{(k)}(q,a) = \frac{2^{\frac{mq+m}{q}}}{|(-kq-q+2)m+2|^{\frac{1}{q}}} C_2^{(k)}(q,a)^{\frac{m}{q}}.$$

(4) We can see that $H_a^{(k)}(t,f)$, $k=0,1,2$, are well-defined for $f \in \mathcal{L}^{\frac{4}{3-k}+}$ by Assertion (2). Then $\bar{H}_a(t;f)$ is well-defined for $p \in \mathcal{L}^{4+}$. Since $p > 4$ and $\frac{-3q+4}{2q} > 0$, Assertion (1) implies

$$E[\int_0^s |\bar{H}_a(t;f)| dt] \leq E[e^{\int_0^s \lambda_a(r) dr} \int_0^T (\hat{H}_a^{(2)}(t;f) + 2\lambda_a(t)\hat{H}_a^{(0)}(t;f)) dt]$$

$$\times (\int_0^s E[|f_u|^p] du)^{\frac{1}{p}} \leq \hat{C}_1(T,q,a) (\int_0^s E[|f_u|^p] du)^{\frac{1}{p}},$$

where

$$\hat{C}_1(s,q,a)$$

$$= e^{\int_0^s \lambda_a(r)dr}(\int_0^s ((C_3^{(2)}(a)^{\frac{1}{q}} t^{\frac{1}{q}} + C_4^{(2)}(q,a)^{\frac{1}{q}} t^{\frac{-3q+4}{2q}}) + 2\lambda_a(t)(C_3^{(0)}(a)^{\frac{1}{q}} t^{\frac{1}{q}} + C_4^{(0)}(q,a)^{\frac{1}{q}} t^{\frac{-q+4}{2q}})dt).$$

(5) Since $1_{\{\tau^a>u\}} \int_u^s \frac{\partial g}{\partial r}(r-u, B_u^a)dr = 1_{\{\tau^a>u\}} g(s-u, B_u^a)$, we have the following by Assertion (2).

$$\int_0^s \hat{H}_a^{(2)}(r; \xi 1_{(s_0,s_1]}(\cdot))dr$$

$$= \int_0^s (\int_0^r H_a^{(2)}(r,u;\xi 1_{(s_0,s_1]}(\cdot))du)dr = \int_0^s (\int_u^s H_a^{(2)}(r,u;\xi 1_{(s_0,s_1]}(\cdot))dr)du$$

$$= \int_0^s (\int_u^s E[1_{\{\tau^a>u\}} \xi 1_{(s_0,s_1]}(u) \frac{\partial^2 g}{\partial x^2}(r-u, B_u^a) | \mathcal{G}_u^W]dr)du$$

$$= 2\int_0^s (\int_u^s E[1_{\{\tau^a>u\}} \xi 1_{(s_0,s_1]}(u) \frac{\partial g}{\partial r}(r-u, B_u^a) | \mathcal{G}_u^W]dr)du$$

$$= 2\hat{H}_a^{(0)}(s; \xi 1_{(s_0,s_1]}(\cdot)).$$

□

**Proposition 2.2.3.** *Let $T>0$, $p>3$, $q=\frac{p}{p-1}$. Then $\bar{U}_a$ is well-defined, for any $f \in \mathcal{L}^{6+}$ and there are $\widetilde{C}_1(q,a,T), \widetilde{C}_2(q,T) \in (0,\infty)$ such that*

$$E[\int_0^T (\int_0^t \bar{U}_a(t,u;f)^2 du)dt] \le \widetilde{C}_1(q,a,T)(\int_0^t E[|f_u|^{2p}]du)dt)^{\frac{1}{p}} + \widetilde{C}_2(T)E[\int_0^t f_u^2 du]$$

*for any $f \in \mathcal{L}^{6+}$.*

*Proof.* Because $0 \le \Phi(t-s, B_s^a) \le 1$, for any $f \in \mathcal{L}^{6+}$ ,we have

$$\int_0^t E[U_a(t,u;f)^2]du \le \int_0^t E[E[1_{\{\tau^a>s\}} f_u(2\Phi(t-u,B_u^a)-1)|\mathcal{G}_u^W]^2]du$$

$$\le \int_0^t E[f_u^2(2\Phi(t-u,B_u^a)-1)^2]du \le \int_0^t f_u^2 du.$$

By the above evaluation and Proposition 2.2.2 (2), we have

$$E[\int_0^T (\int_0^t \bar{U}_a(t,u;f)^2 du)dt]$$

$$= E[\int_0^T (\int_0^t e^{2\int_0^t \lambda_a(r)dr}(H_a^{(1)}(t,u;f) + \lambda_a(t)^2 U_a(t,u;f))^2 du)dt]$$

$$\leq 2e^{2\int_0^T \lambda_a(r)dr} E[\int_0^T \int_0^t (\tilde{H}_a^{(1)}(t,u;f)^2 du)dt + \int_0^T \lambda_a(t)^2 (\int_0^t |U_a(t,u;f)|du)^2 dt]$$

$$\leq 2e^{2\int_0^T \lambda_a(r)dr} \int_0^T ((C_{5,2}^{(1)}(q,a)t + C_{6,2}^{(1)}(q,a)t^{-2q+3})^{\frac{1}{q}} \lambda_a(t)^2 (\int_0^t E[|f_u|^p]^2 du)^{\frac{1}{p}})dt$$

$$+2Te^{2\int_0^T \lambda_a(r)dr} (\sup_{0\leq t\leq T} \lambda_a(t)^2) E[\int_0^T f_u^2 du].$$

For a part of first term, we have

$$\int_0^T ((C_{5,2}^{(1)}(q,a)t + C_{6,2}^{(1)}(q,a)t^{-2q+3})^{\frac{1}{q}} (\int_0^t E[|f_u|^{2p}]du)^{\frac{1}{p}})dt$$

$$\leq (\int_0^T (C_{5,2}^{(1)}(q,a)t + C_{6,2}^{(1)}(q,a)t^{-2q+3})dt)^{\frac{1}{q}} (\int_0^T (\int_0^t E[|f_u|^{2p}]du)dt)^{\frac{1}{p}}.$$

Then we have the assertion where

$$\widetilde{C}_1(q,a,T) = 2e^{2\int_0^T \lambda_a(r)dr} (\int_0^T (C_{5,2}^{(1)}(q,a)t + C_{6,2}^{(1)}(q,a)t^{-2q+3})dt)^{\frac{1}{q}} (\int_0^T (\lambda_a(t)^2 \int_0^t E[|f_u|^{2p}]du)dt)^{\frac{1}{p}}$$

and

$$\widetilde{C}_2(T) = 2Te^{2\int_0^T \lambda_a(r)dr} \sup_{0\leq t\leq T} \lambda_a(t)^2.$$

□

## 2.3 Representation theorem

We saw some integrals are well-defined under corresponding conditions in Section 2.2. Then Theorem 2.1.1 which is the representation theorem under $\mathcal{F}_t^W$ is proved in this section.

**Proposition 2.3.1.** *Let $t > u > 0$ and $\xi$ be a bounded $\mathcal{B}_u$-measurable random variable. Then we have*

$$E[\xi|\mathcal{G}_\infty^W] = E[\xi|\mathcal{G}_u^W] \ \text{ and } \ E[\xi(B_t - B_u)|\mathcal{G}_\infty^W] = 0.$$

*Proof.* Let $h_0$ be a bounded $\mathcal{G}_u^W$ -measurable random variable and $h_1$ be a bounded $\sigma\{W(s) - W(u); s \geq u\}$ measurable random variable.Then

$$E[\xi h_0 h_1] = E[\xi h_0 E[h_1|\mathcal{B}_u]] = E[\xi h_0]E[h_1]$$

$$= E[E[\xi|\mathcal{G}_u^W]h_0]E[h_1] = E[E[\xi|\mathcal{G}_u^W]h_0h_1]$$

and

$$E[\xi(B_t - B_u)h_0h_1] = E[\xi h_0]E[(B_t - B_u)h_1] = 0.$$

So we have our assertion. □

**Proposition 2.3.2.** *Let $0 \leq s_0 < s_1$ and $\xi$ be a bounded $\mathcal{B}_{s_0}$-measurable random variable. Then we have the following for $t \geq 0$ ,*

$$E[\xi 1_{\{\tau^a > t\}}(B^a_{s_1} - B^a_{s_0})]$$
$$= -\int_t^\infty (\int_{s_0}^{s_1} 1_{\{u<r\}} E[\xi 1_{\{\tau^a > u\}} \frac{\partial^2 g}{\partial x^2}(r-u, B^a_u)]du)dr. \tag{2.3.1}$$

*Proof.* Let

$$\varphi(s,x,t) = \int_0^\infty \int_0^\infty (y-x)g_0(s,x,y)g_0(t,y,z)dydz, \qquad x > 0,\ s,t > 0.$$

At first, let us think about the case $t > s_1$. Then we have

$$1_{\{\tau^a > s_0\}} E[1_{\{\tau^a > t\}}(B^a_{s_1} - B^a_{s_0})|\mathcal{B}_{s_0}] = 1_{\{\tau^a > s_0\}} \varphi(s_1 - s_0, B^a_{s_0}, t - s_1).$$

Then

$$E[\xi 1_{\{\tau^a > t\}}(B^a_{s_1} - B^a_{s_0})] = E[\xi 1_{\{\tau^a > s_0\}} \varphi(s_1 - s_0, B^a_{s_0}, t - s_1)].$$

Note that

$$|\varphi(s,x,t)| \leq \int_{-\infty}^\infty \int_{-\infty}^\infty |y-x| g(s, x-y)g(t, y-z)dzdy$$
$$= \int_{-\infty}^\infty |y-x|g(s,x-y)dy = E[|B_s|] = s^{\frac{1}{2}}. \tag{2.3.2}$$

Since

$$d_s\varphi(s_1 - s, B^a_s, r) = (-\frac{\partial}{\partial s} + \frac{1}{2}\frac{\partial^2}{\partial x^2})\varphi(s_1 - s, B^a_s, r)ds + \frac{\partial \varphi}{\partial x}(s_1 - s, B^a_s, r)dB^a_s$$

and

$$(-\frac{\partial}{\partial s} + \frac{1}{2}\frac{\partial^2}{\partial x^2})\varphi(s,x,r)$$
$$= -\int_0^\infty \int_0^\infty \frac{\partial g_0}{\partial x}(s,x,y)g_0(r,y,z)dydz = -\int_0^\infty \frac{\partial g_0}{\partial x}(s+r,x,z)dz$$

$$= -\int_0^\infty (\frac{\partial g}{\partial x}(s+r, x-z) - \frac{\partial g}{\partial x}(s+r, x+z))dz = -2g(s+r, x), \quad x > 0, \ \ s, r > 0,$$

we have

$$1_{\{\tau^a > s_0\}}(\varphi(s_1 - s \wedge \tau^a, B^a_{s\wedge\tau^a}, t - s_1) - \varphi(s_1 - s_0, B^a_{s_0}, t - s_1))$$
$$= -2\int_{s_0}^{s\wedge\tau^a} g(t-u, B^a_u)du + \int_{s_0}^{s\wedge\tau^a} \frac{\partial \varphi}{\partial x}(s_1 - u, B^a_u, t - s_1)dB^a_u, \qquad s \in [s_0, s_1).$$

Then we have

$$E[\xi 1_{\{\tau^a > t\}}(B^a_{s_1} - B^a_{s_0})]$$
$$= E[\xi 1_{\{\tau^a > s_0\}}\varphi(s_1 - s\wedge\tau^a, B^a_{s\wedge\tau^a}, t - s_1)]$$
$$+2E[\xi 1_{\{\tau^a > s_0\}}(\int_{s_0}^{s} 1_{\{\tau^a > u\}} g(t-u, B^a_u)du)], \quad s \in [s_0, s_1).$$

Since $\varphi(s, 0, t) = 0$ and $\varphi(s, x, t) \to 0$, $s \downarrow 0$, we have

$$\lim_{s\to s_1} E[\xi 1_{\{\tau^a > s_0\}}\varphi(s_1 - s\wedge\tau^a, B^a_{s\wedge\tau^a}, t - s_1)] \to 0$$

by Equation (2.3.2) and the bounded convergence theorem. Then we have

$$E[\xi 1_{\{\tau^a > t\}}(B^a_{s_1} - B^a_{s_0})] = -2\int_{s_0}^{s_1} E[\xi 1_{\{\tau^a > u\}} g(t-u, B^a_u)]drdu$$
$$= -2\int_{s_0}^{s_1} E[\xi \int_t^\infty \frac{\partial g}{\partial r}(r-u, B^a_u)dr]du$$
$$= -\int_t^\infty (\int_{s_0}^{s_1} 1_{\{u<r\}} E[\xi 1_{\{\tau^a > u\}} \frac{\partial^2 g}{\partial x^2}(r-u, B^a_u)]du)dr \tag{2.3.3}$$

for any $t > s_1$. By taking $t \downarrow s_1$, we also have our assertion for $t = s_1$.

Second, let us think of the case $t \in (s_0, s_1]$.

$$E[\xi 1_{\{\tau^a > t\}}(B^a_{s_1} - B^a_{s_0})] = E[\xi 1_{\{\tau^a > t\}} E[(B^a_{s_1} - B^a_{s_0})|\mathcal{B}_t]]$$
$$= E[\xi 1_{\{\tau^a > t\}}(B^a_t - B^a_{s_0})] = -\int_t^\infty (\int_{s_0}^{t} 1_{\{u<r\}} E[\xi 1_{\{\tau^a > u\}} \frac{\partial^2 g}{\partial x^2}(r-u, B^a_u)]du)dr.$$

Let $p > 4$ and $q = \frac{p}{p-1}, r > u \geq 0$. Then we have

$$E[|1_{\{\tau^a > u\}}\xi \frac{\partial^2 g}{\partial x^2}(r-u, B^a_u)|]$$

$$\leq E[|1_{\{\tau^a>u\}}|\xi|^p]^{\frac{1}{p}} E[|1_{\{\tau^a>u\}}|\frac{\partial^2 g}{\partial x^2}(r-u, B^a_u)|^q]^{\frac{1}{q}}$$
$$\leq E[|1_{\{\tau^a>u\}}|\xi|^p]^{\frac{1}{p}} E[(C_1^{(2)}(a) + C_2^{(2)}(2,a)(r-u)^{\frac{-3q+2}{2}}]^{\frac{1}{q}}$$

by Proposition 2.2.1. Since $\frac{-3q+2}{2} > -1$, we can use Fubini's Theorem in the following equation.

$$\int_t^\infty (\int_t^{s_1} 1_{\{u<r\}} E[1_{\{\tau^a>u\}}\xi \frac{\partial^2 g}{\partial x^2}(r-u, B^a_u)]du)dr$$
$$= \int_t^{s_1} (\int_t^\infty 1_{\{u<r\}} E[1_{\{\tau^a>u\}}\xi \frac{\partial g}{\partial r}(r-u, B^a_u)]dr)du$$
$$= -2\int_t^{s_1} (E[1_{\{\tau^a>u\}}\xi \int_u^\infty \frac{\partial g}{\partial r}(r-u, B^a_u)]dr)du = 0.$$

So we have Equation (2.3.3) for $t \in (s_0, s_1]$.
When $t \in [0, s_0]$,

$$E[\xi 1_{\{\tau^a>t\}}(B^a_{s_1} - B^a_{s_0})] = E[\xi 1_{\{\tau^a>t\}} E[B^a_{s_1} - B^a_{s_0} | \mathcal{B}_{s_0}]] = 0.$$

So we see Equation (2.3.3) is valid for $t \geq 0$. □

**Proposition 2.3.3.** *Let $0 \leq s_0 < s_1$, $t > 0$ and $\xi$ be a bounded $\mathcal{F}_{s_0}$-measurable random variable. Then we have*

$$E[\xi 1_{\{\tau^a>s_0\}} 1_{\{\tau^a>t\}}] = -\int_{s_0\vee t}^\infty E[1_{\{\tau^a>s_0\}}\xi \frac{\partial g}{\partial x}(r-s_0, B^a_{s_0})]dr.$$

*Proof.* We assume that $t > s_0$, then we have

$$E[\xi 1_{\{\tau^a>t\}}] = E[\xi 1_{\{\tau^a>s_0\}} E[1_{\{\tau^a>t\}} | \mathcal{B}_{s_0}]] = E[\xi 1_{\{\tau^a>s_0\}} (\int_0^\infty g_0(t-s_0, B^a_{s_0}, y)dy)].$$

For $x > 0$ and $t > 0$, we have

$$\int_0^\infty g_0(t,x,y)dy = -\int_0^\infty (\int_t^\infty \frac{\partial g_0}{\partial s}(s,x,y)ds)dy$$
$$= -\frac{1}{2}\int_t^\infty (\int_0^\infty \frac{\partial^2 g_0}{\partial y^2}(s,x,y)dy)ds = \frac{1}{2}\int_t^\infty \frac{\partial g_0}{\partial y}(s,x,0)ds = -\int_t^\infty \frac{\partial g}{\partial x}(s,x)ds.$$

Considering Equation (2.2.1) in Proposition 2.2.1, we have

$$E[\xi 1_{\{\tau^a>s_0\}} 1_{\{\tau^a>t\}}] = -\int_{s_0\vee t}^\infty E[\xi 1_{\{\tau^a>s_0\}} \frac{\partial g}{\partial x}(r-s_0, B^a_{s_0})]dr.$$

□

**Proposition 2.3.4.** *Let* $0 \le s_0 < s_1$ *and* $\xi$ *be a bounded* $\mathcal{F}_{s_0}$*-measurable random variable,* $v : [0,\infty) \to \mathbf{R}$ *be a bounded Borel measurable function. Then we have the following.*

(1)

$$E[\xi(B^a_{s_1} - B^a_{s_0})v(\tau^a)] = -\int_0^\infty v(r)(\int_{s_0}^{s_1} 1_{\{u<r\}}E[\xi 1_{\{\tau^a>u\}}\frac{\partial^2 g}{\partial x^2}(r-u, B^a_u)]du)dr.$$

(2)

$$E[\xi 1_{\{\tau^a>s_0\}}v(\tau^a)] = -\int_{s_0}^\infty v(r)E[1_{\{\tau^a>s_0\}}\xi\frac{\partial g}{\partial x}(r-s_0, B^a_{s_0})]dr.$$

(3)

$$E[\xi(B^a_{s_1} - B^a_{s_0})|\mathcal{F}^W_\infty] = -\int_0^\infty \gamma_a^{-1}(r)\hat{H}^{(2)}_a(r;\xi 1_{(s_0,s_1]}(\cdot))dN^a_r.$$

(4)

$$E[\xi(B^a_{s_1} - B^a_{s_0})|\mathcal{F}^W_t] = -\int_0^t \bar{H}_a(s;\xi 1_{(s_0,s_1]}(\cdot))\lambda_a(s)^{-1}dM_s, \quad t>0.$$

*Proof.* (1) For $v = 1_{[t,\infty)}$, Assertion (1) is valid by Proposition 2.3.2. Let $\mathcal{V}$ be the collection of bounded measurable functions $v$ which satisfy Assertion (1). Then $\mathcal{V}$ is a vector space. In addition, if $\{v_n\}_{n\in\mathbf{N}}$ is an increasing sequence of non-negative functions in $\mathcal{V}$ and if $\lim_{n\to\infty} v_n$ exists and bounded then $\lim_{n\to\infty} v_n \in \mathcal{V}$. Let $\mathcal{A} = \{A \subset \mathbf{R}; 1_A \in \mathcal{V}\}$ then $(t,\infty) \in A$ for each $t > 0$. $\mathcal{A}$ is d-system by the monotone convergence Theorem and $\mathcal{A}' = \{(t,\infty); t > 0\} \in \mathcal{A}$ is $\pi$-system. Then we have our assertion by the monotone class theorem.

(2) By the same way with Assertion (1), we see that this assertion is valid for any bounded Borel measurable function $v : (0,\infty) \to \mathbf{R}$. This completes the proof of Assertion.

(3) Let $v : [0,\infty) \to \mathbf{R}$ be a bounded Borel measurable function, $h_0$ be a bounded $\mathcal{G}^W_{s_0}$-measurable Borel function and $h_1$ be a bounded $\sigma\{W_t - W_{s_0};\ t > s_0\}$-measurable function. Note that $\mathcal{B}_{s_0} \vee \mathcal{G}^B_\infty$ and $\sigma\{W_t - W_{s_0}; t > s_0\}$ are independent. By Proposition 2.3.1 and Proposition 2.3.4 (1), we have

$$E[\xi(B^a_{s_1} - B^a_{s_0})v(\tau^a)h_0h_1] = -E[h_0\xi(B^a_{s_1} - B^a_{s_0})v(\tau^a)]E[h_1]$$

$$= -(\int_0^\infty v(r)(\int_{s_0}^{s_1} 1_{\{u<r\}} E[h_0 \xi 1_{\{\tau^a>u\}} \frac{\partial^2 g}{\partial x^2}(r-u, B_u^a)]du)dr)E[h_1]$$
$$= -(\int_0^\infty v(r)(\int_{s_0}^{s_1} 1_{\{u<r\}} E[h_0 h_1 E[\xi 1_{\{\tau^a>u\}} \frac{\partial^2 g}{\partial x^2}(r-u, B_u^a)|\mathcal{G}_\infty^W]]du)dr)$$
$$= -E[h_0 h_1(\int_0^\infty v(r)(\int_0^\infty 1_{\{u<r\}} E[\xi 1_{(s_0,s_1]}(u) 1_{\{\tau^a>u\}} \frac{\partial^2 g}{\partial x^2}(r-u, B_u^a)|\mathcal{G}_\infty^W]du])dr)$$
$$= -E[h_0 h_1 v(\tau^a)\gamma_a(\tau^a)^{-1} \int_0^\infty 1_{\{\tau^a>u\}} E[\xi 1_{(s_0,s_1]}(u) 1_{\{\tau^a>u\}} \frac{\partial^2 g}{\partial x^2}(\tau^a-u, B_u^a)|\mathcal{G}_u^W]du]$$
$$= -E[h_0 h_1 v(\tau^a)\gamma_a(\tau^a)^{-1} \hat{H}_a^{(2)}(\tau^a; \xi 1_{(s_0,s_1]}(\cdot))]$$
$$= -E[h_0 h_1 v(\tau^a) \int_0^\infty \gamma_a(r)^{-1} \hat{H}_a^{(2)}(r; \xi 1_{(s_0,s_1]}(\cdot)) dN_r^a].$$

Then we have the assertion.

(4) We note that $\hat{H}_a^{(2)}(t; \xi 1_{(s_0,s_1]}(\cdot)) = 0$ for $t \le s_0$ and

$$E[\xi(B_{s_1}^a - B_{s_0}^a)|\mathcal{F}_{s_0}^W] = E[E[\xi(B_{s_1}^a - B_{s_0}^a)|\mathcal{B}_{s_0}]|\mathcal{F}_{s_0}^W] = 0.$$

Then we have

$$E[\xi(B_{s_1}^a - B_{s_0}^a)|\mathcal{F}_t^W] = 0 = \int_0^t \bar{H}_a(s; \xi 1_{(s_0,s_1]}(\cdot))\lambda_a(s)^{-1} dM_s, \quad t \le s_0.$$

By Proposition 2.3.1, we have

$$\int_0^\infty \hat{H}_a^{(2)}(s; \xi 1_{(s_0,s_1]}(\cdot)) ds = E[\xi(B_{s_1}^a - B_{s_0}^a)|\mathcal{G}_\infty^W] = 0$$

and then

$$E[\int_t^\infty \hat{H}_a^{(2)}(s; \xi 1_{(s_0,s_1]}(\cdot)) ds|\mathcal{G}_t^W] = -\int_0^t \hat{H}_a^{(2)}(s; \xi 1_{(s_0,s_1]}(\cdot)) ds. \tag{2.3.4}$$

By Assertion (3) and Equation (2.3.4), we see that

$$E[\xi(B_{s_1}^a - B_{s_0}^a)|\mathcal{F}_t^W] = -E[\int_0^\infty \gamma_a^{-1}(r) \hat{H}_a^{(2)}(r; \xi 1_{(s_0,s_1]}(\cdot)) dN_r^a|\mathcal{F}_t^W]$$
$$= -\int_0^t \gamma_a^{-1}(r) \hat{H}_a^{(2)}(r; \xi 1_{(s_0,s_1]}(\cdot)) dN_r^a - E[\int_t^\infty \gamma_a^{-1}(r) \hat{H}_a^{(2)}(r; \xi 1_{(s_0,s_1]}(\cdot)) dN_r^a|\mathcal{F}_t^W]$$
$$= -\int_0^t e^{\int_0^s \lambda_a(r)dr} \hat{H}_a^{(2)}(s; \xi 1_{(s_0,s_1]}(\cdot))\lambda_a(s)^{-1} dM_s - \int_0^t e^{\int_0^s \lambda_a(r)dr} \hat{H}_a^{(2)}(s; \xi 1_{(s_0,s_1]}(\cdot))(1-N_s^a) ds$$

$$+e^{\int_0^t \lambda_a(r)dr}(1-N_t^a)\int_0^t \hat{H}_a^{(2)}(s;\xi 1_{(s_0,s_1]}(\cdot))ds. \qquad (2.3.5)$$

Here we note that $e^{\int_0^t \lambda_a(r)dr}(1-N_t^a) = 1 - \int_0^t e^{\int_0^s \lambda_a(r)dr}dM_s$ and

$$e^{\int_0^t \lambda_a(r)dr}(1-N_t^a)\int_0^t \hat{H}_a^{(2)}(s;\xi 1_{(s_0,s_1]}(\cdot))ds$$

$$= \int_0^t \hat{H}_a^{(2)}(s;\xi 1_{(s_0,s_1]}(\cdot))ds - (\int_0^t e^{\int_0^s \lambda_a(r)dr}dM_s)(\int_0^t \hat{H}_a^{(2)}(s;\xi 1_{(s_0,s_1]}(\cdot))ds)$$

$$= -\int_0^t e^{\int_0^s \lambda_a(r)dr}(\int_0^s \hat{H}_a^{(2)}(r;\xi 1_{(s_0,s_1]}(\cdot))dr)dM_s$$

$$+\int_0^t \hat{H}_a^{(2)}(s;\xi 1_{(s_0,s_1]}(\cdot))ds + \int_0^t (\hat{H}_a^{(2)}(s;\xi 1_{(s_0,s_1]}(\cdot))\int_0^s e^{\int_0^r \lambda_a(u)du}dM_r)ds.$$

Then we have the following for $t \geq s_0$,

$$E[\xi(B_{s_1}^a - B_{s_0}^a)|\mathcal{F}_t^W] = -\int_0^t e^{\int_0^s \lambda_a(r)dr}\hat{H}_a^{(2)}(s;\xi 1_{(s_0,s_1]}(\cdot))\lambda_a(s)^{-1}dM_s$$

$$+\int_0^t(\int_0^s \hat{H}_a^{(2)}(r;\xi 1_{(s_0,s_1]}(\cdot))dr)d(e^{\int_0^s \lambda_a(r)dr}(1-N_{s-}^a))$$

$$= -\int_0^t (e^{\int_0^s \lambda_a(r)dr}(\hat{H}_a^{(2)}(s;\xi 1_{(s_0,s_1]}(\cdot)) + \lambda_a(s)\int_0^s \hat{H}_a^{(2)}(r;\xi 1_{(s_0,s_1]}(\cdot))dr))\lambda_a(s)^{-1}dM_s.$$

Finally, we have Assertion by Proposition 2.2.2 (4). $\square$Let $\widetilde{\mathcal{L}}_0$ be the space of progressively measurable processes $\varphi_t$ for which there exist $\mathcal{B}_{s_k}$- measurable bounded random variables $\xi_{s_k}$ such that

$$\varphi_t = \sum_{k=0}^{n-1}\xi_k 1_{(s_k,s_{k+1}]}(t), \quad t \geq 0,$$

where $0 \leq s_0 < s_1 < \cdots < s_n \leq T$. For any $p \geq 1$ and $f \in \mathcal{L}^p$, there exist $f_n \in \widetilde{\mathcal{L}}^0$, $n = 1, 2, \cdots$, such that

$$\lim_{n\to\infty} E[\int_0^T |f_n(s,\omega) - f(s,\omega)|^p ds] = 0 \quad \textit{for any } T > 0.$$

**Corollary 2.3.5.** *Let $T > 0$. Then we have*

$$E[\int_0^T f_s dB_s|\mathcal{F}_\infty^W] = -\int_0^\infty \gamma_a(s)^{-1}(\int_0^\infty H_a^{(2)}(s,u;f1_{[0,T]}(\cdot))du)dN_s^a$$

*for any* $f \in \mathcal{L}^{4+}$ *and*

$$E[\int_0^T f_s dB_s|\mathcal{F}_t^W] = -\int_0^t \bar{H}_a(s, f1_{[0,T]}(\cdot))\lambda_a(s)^{-1}dM_s, \quad t > 0$$

*for any* $f \in \mathcal{L}^{4+}$.

*Proof.* Let $s_1 > s_0 \geq 0$ and $\widetilde{f}$ be a bounded $\mathcal{B}_{s_0}$ -measurable function and $f_t = \widetilde{f}1_{(s_0,s_1]}(t)$. Then we see that the first and second assertion are valid for $f \in \widetilde{\mathcal{L}}^{(0)}$ by Proposition 2.3.4 (3) and (4), respectively. We can see that $\int_0^\infty H_a^{(2)}(s, u; f1_{[0,T]}(\cdot))du$ in the first assertion is well-defined for any $f \in \mathcal{L}^{4+}$ by Proposition 2.2.2 (2). As for the second assertion, let us take $\{\widetilde{\xi}_n\} \in \widetilde{\mathcal{L}}_0$ such that

$$\lim_{n\to\infty} E[|\widetilde{\xi}_n(r) - f_r|] = 0 \; for \; all \; r > 0.$$

Then we have

$$E[\int_0^T \widetilde{\xi}_n(s)dB_s|\mathcal{F}_t^W] = -\int_0^t \bar{H}_a(s, \widetilde{\xi}_n 1_{[0,T]}(\cdot))\lambda_a(s)^{-1}dM_s, \quad t > 0,$$

by Proposition 2.3.4 (2). Since $\sigma\{W_t; t \geq 0\}$ and $\sigma\{N_t; t \geq 0\}$ are independent, we have

$$E[\int_0^T |(\bar{H}_a(s;\widetilde{\xi}_n) - \bar{H}_a(s;f))|\lambda_a(s)^{-1}dN_s^a] = E[\int_0^T E[|(\bar{H}_a(s;\widetilde{\xi}_n) - \bar{H}_a(s;f))|]\lambda_a(s)^{-1}dN_s^a]$$

$$= \int_0^T E[|(\bar{H}_a(s;\widetilde{\xi}_n - f)|]e^{-\int_0^s \lambda_a(u)du}ds \to 0, \quad as \quad n \to \infty, \; for \; all \; T > 0$$

by Proposition 2.2.2 (2) . So $\int_0^t \bar{H}_a(s, f1_{[0,T]}(\cdot))\lambda_a(s)^{-1}dM_s$ is well-defined, and we have the assertion. □

**Proposition 2.3.6.** *Let* $s > 0$ *and* $f$ *be a bounded,* $\mathcal{F}$*-progressively measurable process. Then we have the following.*

(1)
$$E[f_s|\mathcal{F}_\infty^W] = E[f_s|\mathcal{F}_s^W]1_{\{\tau^a \leq s\}} - \int_s^\infty \gamma_a(r)^{-1}H_a^{(1)}(r, s; f)dN_r^a.$$

(2)
$$E[f_s|\mathcal{F}_t^W] = E[f_s|\mathcal{F}_s^W] - \int_s^t \gamma_a(r)^{-1}\bar{U}_a(r, s; f)\lambda_a(r)^{-1}dM_r, \quad t > s.$$

*Proof.*

(1) Let $s > 0$, $h_0$ be a bounded $\mathcal{G}_s^W$ -measurable Borel function, $h_1$ be a bounded $\sigma\{W_t - W_s;\ t > s\}$-measurable function and $v : [0,\infty) \to \mathbf{R}$ be a bounded Borel measurable function. Then we have

$$E[f_s v(\tau^a)h_0h_1] = E[f_s 1_{\{\tau^a \le s\}} v(\tau^a)h_0h_1] + E[h_1]E[f_s 1_{\{\tau^a > s\}} v(\tau^a)h_0]$$

and

$$E[f_s 1_{\{\tau^a \le s\}} v(\tau^a)h_0h_1] = E[h_1]E[f_s 1_{\{\tau^a \le s\}} v(\tau^a)h_0]$$
$$= E[h_1]E[E[f_s|\mathcal{F}_s^W]1_{\{\tau^a \le s\}} v(\tau^a)h_0] = E[E[f_s|\mathcal{F}_s^W]1_{\{\tau^a \le s\}} v(\tau^a)h_0h_1].$$

Since $\sigma\{W_t; t \ge 0\}$ and $\sigma\{N_t; t \ge 0\}$ are independent, we have the following by Proposition 2.3.4 (2),

$$E[h_1]E[f_s 1_{\{\tau^a > s\}} v(\tau^a)h_0]$$
$$= E[h_1]\int_s^\infty v(r)E[1_{\{\tau^a > s\}} f_s \frac{\partial g}{\partial x}(r - s, B_s^a)h_0]dr$$
$$= -E[h_1]\int_s^\infty v(r)\gamma_a(r)E[\gamma_a(r)^{-1}H_a^{(1)}(r, s; f)h_0]dr$$
$$= -\int_s^\infty v(r)\gamma_a(r)E[\gamma_a(r)^{-1}H_a^{(1)}(r, s; f)h_0h_1]dr$$
$$= -E[E[v(r)\gamma_a(r)^{-1}H_a^{(1)}(r, s; f)h_0h_1]|_{r=\tau^a}1_{\{\tau^a > s\}}]$$
$$= -E[\gamma_a(\tau^a)^{-1}H_a^{(1)}(\tau^a, s; f)1_{\{\tau^a > s\}} v(\tau^a)h_0h_1]$$
$$= -E[(\int_s^\infty \gamma_a(r)^{-1}H_a^{(1)}(r, s; f)dN_r^a)v(\tau^a)h_0h_1].$$

So we have

$$E[f_s 1_{\{\tau^a > s\}}|\mathcal{F}_\infty^W] = -\int_s^\infty \gamma_a(r)^{-1}H_a^{(1)}(r, s; f)1_{\{\tau^a > s\}}dN_r^a.$$

Thus we have Assertion.

(2) Note that

$$\frac{\partial}{\partial t}(2\Phi(t - s, x) - 1)$$
$$= 2\int_{-\infty}^x \frac{\partial g}{\partial t}(t - s, y)dy = \int_{-\infty}^x \frac{\partial^2 g}{\partial y^2}(t - s, y)dy = \frac{\partial g}{\partial x}(t - s, x)$$

and that

$$2\Phi(t-s,x)-1=-\int_t^\infty \frac{\partial}{\partial r}(2\Phi(r-s,x)-1)dr=-\int_t^\infty \frac{\partial g}{\partial x}(r-s,x)dr.$$

Here we note that

$$\lim_{t\to\infty}\Phi(t-s,x)=\frac{1}{2},\quad x>0$$

and

$$\lim_{t\downarrow s}\Phi(t-s,x)=1,\quad x>0.$$

Let

$$L_t=1-\exp(\int_0^t \lambda_a(s)ds)(1-N_t^a).$$

Then we have

$$dL_t=\exp(\int_0^t \lambda_a(s)ds)(dN_t^a-\lambda_a(t)(1-N_t^a)dt)=\exp(\int_0^t \lambda_a(s)ds)dM_t.$$

We note that

$$dN_t^a=\exp(-\int_0^t \lambda_a(s)ds)dL_t+\lambda_a(t)(1-N_t^a)dt$$

and

$$\gamma_a(t)^{-1}dN_t^a=\lambda_a(t)^{-1}dL_t+\exp(\int_0^t \lambda_a(s)ds)(1-N_t^a)dt$$

$$=\lambda_a(t)^{-1}dL_t-L_tdt+\exp(\int_0^t \lambda_a(s)ds)dt.$$

Then we have

$$U_a(t,s,f)$$

$$=E[1_{\{\tau^a>s\}}f_s(2\Phi(t-s,B_s^a)-1)|\mathcal{G}_s^W]$$

$$=-E[\int_t^\infty 1_{\{\tau^a>s\}}f_s\frac{\partial g}{\partial x}(r-s,B_s^a)|\mathcal{G}_s^W]dr=-\int_t^\infty H_a^{(1)}(r,s;f)dr.$$

It is obvious that

$$\int_s^\infty H_a^{(1)}(r,s;f)\gamma_a(r)^{-1}dN_r^a$$

$$=\int_s^\infty H_a^{(1)}(r,s;f)\lambda_a(r)^{-1}dL_r-\int_s^\infty H_a^{(1)}(r,s;f)L_rdr+\int_s^\infty H_a^{(1)}(r,s;f)e^{\int_0^r \lambda_a(u)}dr.$$

Here we note that the third term at the last equation is $\mathcal{F}_s$-measurable. And the second term of the above can be described in the following.

$$
\begin{aligned}
-\int_s^\infty H_a^{(1)}(r,s;f)L_r dr &= -\int_s^\infty H_a^{(1)}(r,s;f)(\int_s^r dL_u + L_s)dr \\
&= -\int_s^\infty (\int_0^\infty H_a^{(1)}(r,s;f)dr)dL_u - \int_s^\infty H_a^{(1)}(r,s;f)L_s dr \\
&= \int_s^\infty U_a(r,s;f)dL_r + L_s U_a(s+,s,;f).
\end{aligned}
$$

Here we note that the second term at the last equation is $\mathcal{F}_s$-measurable. Then we have

$$
\begin{aligned}
&E[f_s|\mathcal{F}_\infty^W] \\
&= (E[f_s|\mathcal{F}_s^W]1_{\{\tau^a \leq s\}} - L_s U_a(s+,s;f) - \int_s^\infty H_a^{(1)}(r,s;f)e^{\int_0^r \lambda_a(s)ds}dr) \\
&\quad - \int_s^\infty (H_a^{(1)}(r,s;f)\lambda_a(r)^{-1} + U_a(r,s;f))dL_r.
\end{aligned}
$$

The first three terms are $\mathcal{F}_s^W$-measurable and the summation should be equal to $E[f_s|\mathcal{F}_s^W]$. The last term is equal to

$$
\begin{aligned}
&\int_s^\infty \exp(\int_0^r \lambda_a(u)du)(H_a^{(1)}(r,s;f) + \lambda_a(r)U_a(r,s;f))\lambda_a(r)^{-1}dM_r \\
&= \int_s^\infty \bar{U}_a(r,s;f)\lambda_a(r)^{-1}dM_r.
\end{aligned}
$$

Then we have our assertion. $\square$

**Proposition 2.3.7.** *Let $s_1 > s_0 \geq 0$, and $\xi$ be a bounded $\mathcal{F}$ -measurable process. Then we have*

$$
\begin{aligned}
&E[\int_0^\infty \xi 1_{(s_0,s_1]}(r)dW_r|\mathcal{F}_\infty^W] \\
&= \int_0^\infty E[\xi 1_{(s_0,s_1]}(r)|\mathcal{F}_r^W]dW_r - \int_0^\infty (\int_0^r \bar{U}_a(r,u;\xi 1_{(s_0,s_1]}(\cdot))dW_u)\lambda_a(r)^{-1}dM_r.
\end{aligned}
$$

*In particular, for any $T > 0$, $f \in \mathcal{L}^{6+}$,*

$$
\begin{aligned}
&E[\int_0^\infty f_r 1_{[0,T]}(r)dW_r|\mathcal{F}_t^W] \\
&= \int_0^t E[f_r 1_{[0,T]}(r)|\mathcal{F}_r^W]dW_r - \int_0^t (\int_0^r \bar{U}_a(r,u;f1_{[0,T]}(\cdot))dW_u)\lambda_a(r)^{-1}dM_r.
\end{aligned}
$$

*Proof.*

Note that

$$E[\int_0^\infty \xi 1_{(s_0,s_1]}(r)dW_r|\mathcal{F}_\infty^W] = E[\xi|\mathcal{F}_\infty^W](W_{s_1} - W_{s_0}).$$

By Proposition2.3.6, we have

$$E[\xi|\mathcal{F}_\infty^W] = E[\xi|\mathcal{F}_{s_0}^W] - \int_{s_0}^\infty \bar{U}_a(r, s_0; \xi 1_{[s_0,\infty)}(\cdot))\lambda_a(r)^{-1}dM_r$$

and then

$$E[\int_0^\infty \xi 1_{(s_0,s_1]}(r)dW_r|\mathcal{F}_\infty^W]$$
$$= \int_{s_0}^{s_1} E[\xi|\mathcal{F}_r^W]dW_r - \int_{s_0}^\infty \bar{U}_a(r, s_0; \xi 1_{[s_0,s_1)}(\cdot))(W_{r\wedge s_1} - W_{s_0})\lambda_a(r)^{-1}dM_r$$
$$= \int_0^\infty E[\xi 1_{(s_0,s_1]}(r)|\mathcal{F}_r^W]dW_r - \int_{s_0}^\infty (\int_0^{r\wedge s_1} \bar{U}_a(r, s_0; \xi 1_{(s_0,s_1]}(\cdot))dW_u)\lambda_a(r)^{-1}dM_r.$$

Here we note that

$$1_{\{\tau^a > s_0\}}(\frac{\partial g}{\partial x}(t - s\wedge\tau^a, B^a_{s\wedge\tau^a}) - \frac{\partial g}{\partial x}(t - s_0, B^a_{s_0}))$$
$$= 1_{\{\tau^a > s_0\}} \int_{s_0}^{s\wedge\tau^a} \frac{\partial^2 g}{\partial x^2}(t - r, B^a_r)dB^a_r, \qquad s \in (s_0, t).$$

Then we have

$$H_a^{(1)}(t, s_0; \xi 1_{[s_0,s_1)}(\cdot)) = E[1_{\{\tau^a > s_0\}}\xi 1_{[s_0,s_1)}(\cdot)(\frac{\partial g}{\partial x}(t - s\wedge\tau^a, B^a_{s\wedge\tau^a}))|\mathcal{G}_{s_0}^W]$$
$$= H_a^{(1)}(t, s; \xi 1_{[s_0,s_1)}(\cdot)), \quad s \in (s_0, t\wedge s_1).$$

Also we have

$$1_{\{\tau^a > s_0\}}(\Phi(t - s\wedge\tau^a, B^a_{s\wedge\tau^a}) - \Phi(t - s_0, B^a_{s_0}))$$
$$= 1_{\{\tau^a > s_0\}} \int_{s_0}^{s\wedge\tau^a} g(t - r, B^a_r)dB^a_r, \quad s \in (s_0, t\wedge s_1).$$

Thus we have

$$U_a(t, s_0; \xi 1_{[s_0,t\wedge s_1)}(\cdot)) = U_a(t, s; \xi 1_{[s_0,t\wedge s_1)}(\cdot)), \quad s \in (s_0, t\wedge s_1).$$

Since

$$\bar{U}_a(r, u; \xi 1_{(s_0,s_1]}(\cdot)) = 0, \quad r \in [0, s_0],$$

we can see that $\int_0^\infty(\int_0^r \bar{U}_a(r,u;\xi 1_{(s_0,s_1]}(\cdot))dW_u)\lambda_a(r)^{-1}dM_r$ is well-defined. Then we have the first assertion. For $\widetilde{\xi} \in \widetilde{\mathcal{L}}_0$, we have the following by the first assertion,

$$E[\int_0^T \widetilde{\xi}_r dW_r|\mathcal{F}_\infty^W] = \int_0^T E[\widetilde{\xi}_r|\mathcal{F}_r^W]dW_r - \int_0^T(\int_0^r \bar{U}_a(r,u;\widetilde{\xi})dW_u)\lambda_a(r)^{-1}dM_r.$$

Let us take $\{\widetilde{\xi}_n\} \in \widetilde{\mathcal{L}}_0$ such that

$$\lim_{n\to\infty} E[\int_0^T |\widetilde{\xi}_n(r) - f_r|dr] = 0 \; for\ all\ T > 0.$$

Since $\sigma\{W_t; t \geq 0\}$ and $\sigma\{N_t; t \geq 0\}$ are independent, we have

$$\begin{aligned} &E[\int_0^T |\int_0^r (\bar{U}_a(r,u;\widetilde{\xi}_n) - \bar{U}_a(r,u;f))dW_u|\lambda_a(r)^{-1}dN_r^a] \\ &= E[\int_0^T E[|\int_0^r (\bar{U}_a(r,u;\widetilde{\xi}_n) - \bar{U}_a(r,u;f))dW_u|]\lambda_a(r)^{-1}dN_r^a] \\ &= \int_0^T E[|\int_0^r (\bar{U}_a(r,u;\widetilde{\xi}_n - f)dW_u|]q_a(r)dr \\ &\leq \int_0^T E[\int_0^r (\bar{U}_a(r,u;\widetilde{\xi}_n - f)^2du]^{1/2}dr \\ &\to 0, \quad as \quad n \to \infty, \; for\ all\ T > 0 \end{aligned}$$

by Proposition 2.2.3 for $f \in \mathcal{L}^{6+}$. So we have Assertion. □

**Proposition 2.3.8.** *Let $T, t > 0$, $f \in \mathcal{L}^{2+}$. Then we have*

$$E[\int_0^t f_s ds|\mathcal{F}_t^W] = \int_0^t E[f_s|\mathcal{F}_t^W]ds$$

$$= \int_0^t E[f_s 1_{[0,T](s)}|\mathcal{F}_s^W]ds - \int_0^t(\int_0^r \bar{U}_a(s,r;f1_{[0,T]})ds)\lambda_a(s)^{-1}dM_r.$$

*Proof.* Remember that $U_a(t,s;f) = E[1_{\{\tau^a > s\}}f_s(2\Phi(t-s, B_s^a) - 1)|\mathcal{G}_s^W]$. We can see that $\int_0^t(\int_0^r \bar{U}_a(s,r;f1_{[0,T]})ds)\lambda_a(s)^{-1}dM_r$ is well-defined for any $f \in \mathcal{L}^{2+}$ by Proposition 2.2.2 (2) . Then we have the assertion by Proposition 2.2.3 and Proposition 2.3.6. □

**Proposition 2.3.9.** *Let $T, t > 0$ and $\hat{f}^j \in \mathcal{L}^{2+}, j = 1, \cdots, d$. Then we have*

$$E[\sum_{j=1}^d \int_0^t \hat{f}_s^j d\hat{B}_s^j|\mathcal{F}_t^W] = 0.$$

*Proof.* Because $B$, $\hat{B}$ and $W$ are independent and $\mathcal{F}_t^W \subset \sigma\{B_s, W_s; s \in [0,\infty)\}$,

$$E[\sum_{j=1}^d \int_0^t \hat{f}_s^j d\hat{B}_s^j | \mathcal{F}_t^W] = E[\sum_{j=1}^d \int_0^t \hat{f}_s^j d\hat{B}_s^j] = 0.$$

□

Now Theorem2.1.1 is proved by Corollary 2.3.5, Proposition 2.3.8 and Proposition 2.3.9.

**Proposition 2.3.10.** *Let $f \in \mathcal{L}^{4+}$. Then we have*

$$\hat{H}_a^{(2)}(t; f) = \lim_{u \uparrow t} E[(\int_0^u 1_{\{\tau^a > s\}} f_s dB_s^a) \frac{\partial g}{\partial x}(t-u, B_u^a) | \mathcal{G}_t^W].$$

*The right hand side of the above corresponds to a representation Theorem in Nakagawa [15] .*

*Proof.* Note that

$$(-\frac{\partial}{\partial u} + \frac{1}{2}\frac{\partial^2}{\partial x^2})\frac{\partial g}{\partial x}(t-u, x) = 0$$

and so

$$d_u \frac{\partial g}{\partial x}(t-u, B_u^a) = \frac{\partial^2 g}{\partial x^2}(t-u, B_u^a) dB_u^a, \quad u < t.$$

By Ito lemma, we have

$$d_u((\int_0^u 1_{\{\tau^a > s\}} f_s dB_s^a)\frac{\partial g}{\partial x}(t-u, B_u^a))$$

$$= 1_{\{\tau^a > u\}} f_u \frac{\partial g}{\partial x}(t-u, B_u^a) dB_u^a + (\int_0^u 1_{\{\tau^a > s\}} f_s dB_s^a)\frac{\partial^2 g}{\partial x^2}(t-u, B_u^a) dB_u^a$$

$$+ 1_{\{\tau^a > u\}} f_u \frac{\partial^2 g}{\partial x^2}(t-u, B_u^a) du, \quad u < t.$$

And then

$$E[(\int_0^u 1_{\{\tau^a > s\}} f_s dB_s^a)\frac{\partial g}{\partial x}(t-u, B_u^a) | \mathcal{G}_t^W]$$

$$= \int_0^u E[1_{\{\tau^a > s\}} f_s \frac{\partial^2 g}{\partial x^2}(t-s, B_s^a) | \mathcal{G}_t^W] ds = \int_0^u E[1_{\{\tau^a > s\}} f_s \frac{\partial^2 g}{\partial x^2}(t-s, B_s^a) | \mathcal{G}_s^W] ds$$

$$= \int_0^u H^{(2)}(t, s; f) ds, \quad u < t.$$

So we have

$$\hat{H}_a^{(2)}(t; f) = \lim_{u \uparrow t} E[(\int_0^u 1_{\{\tau^a > s\}} f_s dB_s^a) \frac{\partial g}{\partial x}(t-u, B_u^a) | \mathcal{G}_t^W].$$

□

## 2.4 Equivalent probability measures

Note that

$$\rho_t = 1 + \int_{0+}^t \rho_{s-}(b_0(s, X_s, Z_s)d\widetilde{B}_s + \beta(s, X_{s\wedge\tau}, Y_s)d\widetilde{W}_s).$$

Let $F \in \Sigma$ be given by Equation (2.1.1) in the Introduction. Then we have

$$F_t = F_0 + \int_0^t (f_1(s) - \beta(s, X_s, Y_s)f_2(s) - b_0(s, X_s, Z_s)f_3(s))ds$$

$$+ \int_0^t f_2(s)d\widetilde{W}_s + \int_0^t f_3(s)d\widetilde{B}_s + \int_0^t f_4(s)d\hat{B}_s$$

and so

$$\begin{aligned} &\rho_t F_{t\wedge\tau} \\ &= \rho_0 F_0 + \int_0^t F_{s\wedge\tau} d\rho_s + \int_0^{t\wedge\tau} \rho_{s-} dF_s + [\rho, F]_{t\wedge\tau} \\ &= F_0 + \int_{0+}^t \rho_{s-}(\widetilde{D_1}F)_s d\widetilde{B}_s + \int_{0+}^t \rho_{s-}(\widetilde{D_2}F)_s d\hat{B}_s \\ &\quad + \int_{0+}^t \rho_{s-}(\widetilde{D_0}F)_s d\widetilde{W}_s + \int_{0+}^t \rho_{s-}(\widetilde{L}F)_s ds. \end{aligned}$$

Let

$$V(t, s; f) = \widetilde{E}[\rho_{s-}1_{\{\tau>s\}} f_s(2\Phi(t-s, X_s) - 1)|\mathcal{G}_s^Y],$$

$$\bar{V}(t, s; f) = e^{\int_0^t \lambda_{x_0}(r)dr}(I^{(1)}(t, s; f) + \lambda_{x_0}(t)V(t, s; f)),$$

$$\bar{I}(t, s; f) = e^{\int_0^t \lambda_{x_0}(r)dr}(\int_0^s I^{(2)}(t, u; f)du + 2\lambda_{x_0}(t)\int_0^s I^{(0)}(t, u; f)du),$$

$$\hat{V}(r, s; F) = \widetilde{\rho}_{r-}^{-1} e^{\int_0^r \lambda_{x_0}(u)du}(\hat{V}_1(r, s; F) + \lambda_{x_0}(r)\hat{V}_2(r, s; F)), \quad s \le r$$

where

$$\hat{V}_2(r, s; F) = \int_0^s V(r, u; \widetilde{D_0}F)d\widetilde{W}_u + \int_0^s (V(r, u; \widetilde{L}F) + 2I^{(0)}(r, u; \widetilde{D_1}F))du$$

for $f \in \mathcal{L}^{6+}$ and $F \in \Sigma$. Then we have the following by Theorem 2.1.1.

$$\widetilde{E}[\rho_t F_{t\wedge\tau}|\mathcal{F}_t]$$

$$= F_0 - \int_0^{t\wedge\tau} (\bar{I}(r,r;\widetilde{D_1}F) + (\int_0^r \bar{V}(r,u;\widetilde{D}_0 F)d\widetilde{W}_u) + (\int_0^r \bar{V}(r,u;\widetilde{L}F)du))\lambda_{x_0}(r)^{-1}d\widetilde{M}_r$$

$$+ \int_0^{t\wedge\tau} \widetilde{E}[\rho_{r-}(\widetilde{L}F)_r|\mathcal{F}_r]dr + \int_0^{t\wedge\tau} \widetilde{E}[\rho_{r-}(\widetilde{D_1}F)_r|\mathcal{F}_r]d\widetilde{W}_r$$

$$= F_0 - \int_0^t \widetilde{\rho}_{r-}\hat{V}(r,r;F)\lambda_{x_0}(r)^{-1}d\widetilde{M}_r$$

$$+ \int_0^t \widetilde{E}[\rho_{r-}(\widetilde{L}F)_r|\mathcal{F}_r]dr + \int_0^t \widetilde{E}[\rho_{r-}(\widetilde{D_1}F)_r|\mathcal{F}_r]d\widetilde{W}_r. \tag{2.4.1}$$

Here we note that

$$\bar{I}(r;\rho(\widetilde{D_1}F)) + \int_0^r \bar{V}(r,u;\widetilde{D_0}F)d\widetilde{W}_u + \int_0^r \bar{V}(r,u;\widetilde{L}F)du$$

$$= e^{\int_0^r \lambda_{x_0}(u)du}((\int_0^r I^{(1)}(r,u;\widetilde{D_0}F)d\widetilde{W}_u + \int_0^r (I^{(2)}(r,u;\widetilde{D_1}F) + I^{(1)}(r,u;\widetilde{L}F))du)$$

$$+\lambda_{x_0}(r)(\int_0^r V(r,u;\widetilde{D_0}F)d\widetilde{W}_u + \int_0^r (V(r,u;\widetilde{L}F) + 2\int_0^r I^{(0)}(r,u;\widetilde{D_1}F))du))$$

$$= \widetilde{\rho}_{r-}\hat{V}(r,r;F).$$

**Proposition 2.4.1.** *Let $T > 0$ and $F \in \Sigma$. Then we have*

$$\hat{V}_1(r,s;F) = -\frac{\partial g}{\partial x}(r,x_0)F_0 + I^{(1)}(r,s;F), \quad 0 < s < r \le T.$$

*Proof.* Because $\frac{\partial g}{\partial r}(r,x) - \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(r,x) = 0$, we have

$$\frac{\partial g}{\partial x}(r-s,X_s) = \int_0^s \frac{\partial^2 g}{\partial x^2}(r-u,X_u)d\widetilde{B}_u.$$

So we have

$$d(\frac{\partial g}{\partial x}(r-u,X_u)\rho_u F_u)$$

$$= \rho_u((\frac{\partial^2 g}{\partial x^2}(r-u,X_u)F_s + \frac{\partial g}{\partial x}(r-u,X_u)(\widetilde{D_1}F)_u)d\widetilde{B}_u$$

$$+\frac{\partial g}{\partial x}(r-u,X_u)\rho_u(\widetilde{D_2}F)_u d\hat{B}_u + \frac{\partial g}{\partial x}(r-u,X_u)\rho_u(\widetilde{D_0}F)_u d\widetilde{W}_u$$

$$+(\frac{\partial g}{\partial x}(r-u,X_u)(\widetilde{L}F)_u + \frac{\partial^2 g}{\partial x^2}(r-u,X_u)\rho_u(\widetilde{D_1}F)_u)du).$$

Since $\mathcal{G}_s^Y$, $\sigma\{\tilde{B}_u, \hat{B}_u; u \leq s\}$ and $\sigma\{\widetilde{M}_u; u \leq s\}$ are independent, we have the following for $r > s$.

$$\widetilde{E}[\frac{\partial g}{\partial x}(r-s, X_s)1_{\{\tau>s\}}\rho_s F_s|\mathcal{G}_s^Y]$$

$$= \frac{\partial g}{\partial x}(r, x_0)F_0 + \int_0^s I^{(1)}(r, u; \widetilde{D_0}F)d\widetilde{W}_u + \int_0^s (I^{(2)}(r, u; \widetilde{D_1}F) + I^{(1)}(r, u; \widetilde{L}F))du$$

$$= \frac{\partial g}{\partial x}(r, x_0)F_0 + \hat{V}_1(r, s; F).$$

Then we have our assertion. □

**Proposition 2.4.2.** *Let $T > 0$ and $F \in \Sigma$. Then we have*

$$\hat{V}_2(r, s; F) = -(2\Phi(r, x_0) - 1)F_0 + V(r, s; F), \quad 0 < s < r \leq T$$

*and*

$$\lim_{s\uparrow r} \hat{V}_2(r, s; F) = -(2\Phi(r, x_0) - 1)F_0 + \widetilde{E}[1_{\{\tau>r\}}\rho_r F_r|\mathcal{G}_r^Y].$$

*In particular, $\hat{V}(r, r; F) = \hat{V}(r; F)$ .*

*Proof.* Because $\frac{\partial \Phi}{\partial r}(r, x) - \frac{1}{2}\frac{\partial^2 \Phi}{\partial x^2}(r, x) = 0$ and $\frac{\partial \Phi}{\partial x}(r, x) = g(r, x)$,

$$\Phi(r-s, X_s) = \int_0^s g(r-u, X_u)d\widetilde{B}_u.$$

So we have

$$(2\Phi(r-s, X_s) - 1)\rho_s F_s = (2\Phi(r, x_0) - 1)F_0 + 2\int_0^s \rho_{u-}F_u g(r-u, X_u)d\widetilde{B}_u$$

$$+ \int_0^s (2\Phi(r-u, X_u)-1)(\rho_{u-}(\widetilde{D_1}F)_u d\widetilde{B}_u + \rho_{u-}(\widetilde{D_2}F)_u d\hat{B}_u + \rho_{u-}(\widetilde{D_0}F)_u d\widetilde{W}_u + \rho_{u-}(\widetilde{L}F)_u du)$$

$$+2\int_0^s \rho_{u-}g(r-u, X_u)(\widetilde{D_1}F)_u du$$

and then we have the following by Proposition 2.3.1.

$$V(r, s; F) = \widetilde{E}[(2\Phi(r, x_0) - 1)F_0 + \int_0^s 1_{\{\tau>u\}}(2\Phi(r-u, X_u) - 1)\rho_{u-}(\widetilde{D_0}F)_u d\widetilde{W}_u$$

$$+ \int_0^s 1_{\{\tau>u\}}((2\Phi(r-u, X_u) - 1)\rho_{u-}(\widetilde{L}F)_u + 2g(r-u, X_u)\rho_{u-}(\widetilde{D_1}F)_u)du|\mathcal{G}_r^Y]$$

$$= (2\Phi(r,x_0)-1)F_0 + \int_0^s V(r,u;\widetilde{D_0}F)d\widetilde{W}_u + \int_0^s (V(r,u;\widetilde{L}F) + 2I^{(0)}(r,u,\widetilde{D_1}F))du$$
$$= (2\Phi(r,x_0)-1)F_0 + \hat{V}_2(r,s;F).$$

Then we have

$$(2\Phi(r,x_0)-1)F_0 + \lim_{s\uparrow r}\hat{V}_2(r,s;F) = \lim_{s\uparrow r} V(r,s;F)$$
$$= \widetilde{E}[(2\Phi(r-r\wedge\tau, X_{r\wedge\tau})-1)\rho_{r\wedge\tau}F_{r\wedge\tau})|\mathcal{G}_r^Y]$$
$$= \widetilde{E}[1_{\{\tau>r\}}(2\Phi(0,X_r)-1)\rho_r F_r)|\mathcal{G}_r^Y] - \widetilde{E}[1_{\{\tau\le r\}}(2\Phi(r-\tau,0)-1)\rho_\tau F_\tau)|\mathcal{G}_r^Y]$$
$$= \widetilde{E}[1_{\{\tau>r\}}\rho_r F_r|\mathcal{G}_r^Y].$$

□

So we have $\hat{V}(r;F)$ introduced in Introduction can be written as

$$\hat{V}(r;F) = \widetilde{\rho}_{r-}^{-1}e^{\int_0^r \lambda_{x_0}(u)du}(\hat{V}_1(r,r;F) + \lambda_{x_0}(r)V(r,r;F))$$
$$= \widetilde{\rho}_{r-}^{-1}e^{\int_0^r \lambda_{x_0}(u)du}(\hat{V}_1(r,r;F) + \lambda_{x_0}(r)(-(2\Phi(r,x_0)-1)F_0 + \widetilde{E}[1_{\{\tau>r\}}\rho_r F_r|\mathcal{G}_r^Y]).$$

**Proposition 2.4.3.**

$$\lambda_a(t)(2\Phi(t,a)-1) + \frac{\partial g}{\partial x}(t,a) = 0.$$

*In particular,*

$$\hat{V}(r;F) =$$
$$\widetilde{\rho}_{r-}^{-1}e^{\int_0^r \lambda_{x_0}(u)du}(\hat{V}_1(r,r;F) - \frac{\partial g}{\partial x}(r,x_0)F_0 + \widetilde{E}[1_{\{\tau>r\}}\rho_r F_r|\mathcal{G}_r^Y])).$$

*Proof.* Since $P[\tau^a<t] = P[\tau^a<t, B_t^a>0] + P[\tau^a<t, B_t^a<0]$ and $P[\tau^a<t, B_t^a>0] = P[B_t^a>0]$, we have $q_a(t) = P[\tau^a>t] = 1-P[\tau^a<t] = 1-\frac{2}{\sqrt{2\pi t}}\int_0^\infty e^{-\frac{(x-a)^2}{2t}}dx = 2\Phi(t,a)-1$. So we have

$$\frac{\partial}{\partial t}q_a(t) = 2\frac{\partial\Phi}{\partial x}(t,x) = 2\int_{-\infty}^x \frac{\partial g}{\partial t}(t,y)dy = \frac{\partial g}{\partial x}(t,x).$$

Since $\lambda_a(t) = -\frac{d}{dt}\log q_a(t)$, we have the assertion. □

**Proposition 2.4.4.** *Let $Z$ be a random variable and $r>0$. Then we have*

$$\tilde{E}[Z1_{\{\tau>r\}}|\mathcal{G}_r^Y]1_{\{\tau>r\}} = e^{-\int_0^r \lambda_{x_0}(u)du}\tilde{E}[Z|\mathcal{F}_r]1_{\{\tau>r\}}.$$

*Proof.* Let $A \in \mathcal{F}_r$. Then there exists $B \in \mathcal{G}_r^Y$ such that $A \cap \{\tau > r\} = B \cap \{\tau > r\}$. Since $\mathcal{G}_r^Y$ and $1_{\{\tau>r\}}$ are independent, we have the following.

$$\tilde{E}[\tilde{E}[Z1_{\{\tau>r\}}|\mathcal{G}_r^Y]1_{\{\tau>r\}}, A] = \tilde{E}[\tilde{E}[Z1_{\{\tau>r\}}|\mathcal{G}_r^Y]1_{\{\tau>r\}}1_B]$$

$$= \tilde{E}[\tilde{E}[Z1_{\{\tau>r\}}1_B|\mathcal{G}_r^Y]1_{\{\tau>r\}}] = \tilde{E}[\tilde{E}[Z1_B|\mathcal{G}_r^Y]\tilde{E}[1_{\{\tau>r\}}]]$$

$$= \tilde{E}[Z, A]\tilde{P}[\tau > r] = \tilde{E}[e^{-\int_0^r \lambda_{x_0}(u)du} Z, A] = \tilde{E}[e^{-\int_0^r \lambda_{x_0}(u)du}\tilde{E}[Z|\mathcal{F}_r], A].$$

Then we have Assertion. □

**Proposition 2.4.5.** *Let $F \in \Sigma$. Assume that there exist $C > 0$ and $\alpha \in (0,1)$ such that $1_{\{|X_r|\le 1\}}1_{\{\tau>r\}}|F_r| \le C|X_r|^\alpha$ for $r > 0$. Then we have $\hat{V}_1(r, r; F) = -\frac{\partial g}{\partial x}(r, x_0)F_0$ and*

$$\hat{V}(r; F) = \widetilde{\rho}_{r-}^{-1} e^{\int_0^r \lambda_{x_0}(u)du}\lambda_{x_0}(r)\widetilde{E}[1_{\{\tau>r\}}\rho_r F_r|\mathcal{G}_r^Y].$$

*In particular,*

$$1_{\{\tau>r\}}\hat{V}(r; F) = 1_{\{\tau>r\}}\widetilde{\rho}_{r-}^{-1}\lambda_{x_0}(r)\widetilde{E}[1_{\{\tau>r\}}\rho_r F_r|\mathcal{F}_r].$$

*Proof.* Let $1 < p < \frac{2}{2-\alpha}, q = \frac{p}{p-1}$ and $r > s > 0$. Then we have

$$\widetilde{E}[|I^{(1)}(r, s; \widetilde{F})|] \le \widetilde{E}[|\frac{\partial g}{\partial x}(r-s, X_s)|1_{\{\tau>s\}}\rho_s|F_s|] \le \widetilde{E}[1_{\{\tau>r\}}|\frac{\partial g}{\partial x}(r-s, X_s)|^p|F_s|^p]^{\frac{1}{p}}\widetilde{E}[\rho_s^q]^{\frac{1}{q}}.$$

Note that $x_0 > 0$. We have the following by Mean-Value Theorem.

$$e^{-\frac{(x-x_0)^2}{2s}} - e^{-\frac{(x+x_0)^2}{2s}} \le \frac{2x_0}{s}x, \quad x \in (0,\infty), \quad s \in (0, r).$$

Since $\frac{2-p(2-\alpha)}{2} > 0$, we have

$$\widetilde{E}[1_{\{|X_s|\le 1\}}1_{\{\tau>r\}}|\frac{\partial g}{\partial x}(r-s, X_s)|^p|F_s|^p]$$

$$\le \int_0^1 (\frac{1}{\sqrt{2\pi(r-s)}}\frac{x}{r-s}e^{-\frac{x^2}{2(r-s)}})^p(Cx^\alpha)^p\frac{e^{-\frac{(x-x_0)^2}{2s}} - e^{-\frac{(x+x_0)^2}{2s}}}{\sqrt{2\pi s}}dx$$

$$\le 2x_0C^p(r-s)^{-\frac{3p}{2}}s^{-\frac{3}{2}}\int_0^\infty x^{(1+\alpha)p+1}e^{-\frac{px^2}{2(r-s)}}dx$$

$$= 2x_0C^p(r-s)^{\frac{2-p(2-\alpha)}{2}}s^{-\frac{3}{2}}\int_0^\infty y^{(1+\alpha)p+1}e^{-\frac{py^2}{2}}dy \to 0 \quad as \quad s \uparrow r.$$

Let $p' > 1$, $q' = \frac{p'}{p'-1}$. For $r > s > 0$, we have

$$\widetilde{E}[1_{\{|X_s|>1\}}1_{\{\tau>r\}}|\frac{\partial g}{\partial x}(r-s, X_s)|^p|F_s|^p]$$

$$\leq \widetilde{E}[1_{\{|X_s|>1\}}1_{\{\tau>r\}}|\frac{\partial g}{\partial x}(r-s, X_s)|^{pp'}]^{\frac{1}{p'}}E[|F_s|^{pq'}]^{\frac{1}{q'}}$$

$$= (\int_1^\infty (\frac{1}{\sqrt{2\pi(r-s)}}\frac{x}{r-s}e^{-\frac{x^2}{2(r-s)}})^{pp'}\frac{e^{-\frac{(x-x_0)^2}{2s}} - e^{-\frac{(x+x_0)^2}{2s}}}{\sqrt{2\pi s}}dx)^{\frac{1}{p'}}E[|F_s|^{pq'}]^{\frac{1}{q'}}$$

$$\to 0 \quad as \quad s \uparrow r.$$

Then we have $\hat{V}_1(r, r; F) = -\frac{\partial g}{\partial x}(r, x_0)F_0 + \lim_{s\uparrow r} I^{(1)}(r, s; F) = -\frac{\partial g}{\partial x}(r, x_0)F_0$. By Proposition 2.4.3, we have the first assertion. We have the second assertion by Proposition 2.4.4. □

**Proposition 2.4.6.** *Let $\xi$ be $\mathcal{B}$ - measurable process. Then we have*

$$E[\xi_{t\wedge\tau}|\mathcal{H}] = \frac{\widetilde{E}[\rho_t\xi_{t\wedge\tau}|\mathcal{H}]}{\widetilde{E}[\rho_t|\mathcal{H}]}, \quad \mathcal{H} \subset \mathcal{B}_t.$$

*Proof.* For $A \in \mathcal{H} \subset \mathcal{B}_t$, we have

$$E[\xi_{t\wedge\tau}, A] = E[E[\xi_{t\wedge\tau}|\mathcal{F}_t], A]$$

$$= \widetilde{E}[\rho_T E[\xi_{t\wedge\tau}|\mathcal{H}], A] = \widetilde{E}[\widetilde{E}[\rho_T|\mathcal{B}_t]E[\xi_{t\wedge\tau}|\mathcal{H}], A] = \widetilde{E}[\widetilde{E}[\rho_t|\mathcal{H}]E[\xi_{t\wedge\tau}|\mathcal{H}], A].$$

At the same time,

$$E[\xi_{t\wedge\tau}, A] = \widetilde{E}[\rho_T\xi_{t\wedge\tau}, A] = \widetilde{E}[\widetilde{E}[\rho_T|\mathcal{B}_t]\xi_{t\wedge\tau}, A] = \widetilde{E}[\rho_t\xi_{t\wedge\tau}, A] = \widetilde{E}[\widetilde{E}[\rho_t\xi_{t\wedge\tau}|\mathcal{H}], A].$$

□

**Proposition 2.4.7.**

$$\widetilde{E}[\rho_t F_{t\wedge\tau}|\mathcal{F}_t] = F_0$$

$$-\int_0^{t\wedge\tau}\widetilde{\rho}_{r-}\hat{V}(r; F)\lambda_{x_0}(r)^{-1}d\widetilde{M}_r + \int_0^{t\wedge\tau}\widetilde{E}[\rho_{r-}(\widetilde{L}F)_r|\mathcal{F}_r]dr + \int_0^{t\wedge\tau}\widetilde{E}[\rho_{r-}(\widetilde{D_0}F)_r|\mathcal{F}_r]d\widetilde{W}_r.$$

*Proof.* Since $\hat{V}(r,r;F) = \hat{V}(r;F)$ by Proposition 2.4.2, we have our assertion by Equation (2.4.1). □

Let $\widetilde{\rho}_t$ be $\widetilde{\rho}_t = \widetilde{E}[\rho_t|\mathcal{F}_t]$.

**Proposition 2.4.8.**

$$\widetilde{\rho}_t = 1 + \int_0^t \widetilde{\rho}_{r-}\hat{V}(r;1)\lambda_{x_0}(r)^{-1}d\widetilde{M}_r + \int_0^t \widetilde{\rho}_{r-}E[\beta(r,X_r,Y_r)|\mathcal{F}_r]d\widetilde{W}_r$$

*and*

$$\widetilde{\rho}_t^{-1} = 1 - \int_0^t \widetilde{\rho}_{r-}^{-1}\frac{\hat{V}(r;1)}{\lambda_{x_0}(r)+\hat{V}(r;1)}d\widetilde{M}_r$$

$$+\int_0^t \widetilde{\rho}_r^{-1}(E[\beta(r,X_r,Y_r)|\mathcal{F}_r]^2 + \frac{\hat{V}(r;1)^2}{\lambda_{x_0}(r)+\hat{V}(r;1)}1_{\{\tau>r\}})dr - \int_0^t \widetilde{\rho}_{r-}^{-1}E[\beta(r,X_r,Y_r)|\mathcal{F}_r]d\widetilde{W}_r.$$

*Proof.* Letting $F_t = 1$ in Proposition 2.4.7, we have the first assertion. Then we have

$$\widetilde{\rho}_t^{-1} = 1 - \int_0^t \widetilde{\rho}_{r-}^{-2}d\widetilde{\rho}_r + \int_0^t \widetilde{\rho}_{r-}^{-3}d[\widetilde{\rho},\widetilde{\rho}]_r^c + \sum_{0<r\le t}(\widetilde{\rho}_r^{-1} - \widetilde{\rho}_{r-}^{-1} + \widetilde{\rho}_{r-}^{-2}(\widetilde{\rho}_r - \widetilde{\rho}_{r-}))$$

$$= 1 - \int_0^t \widetilde{\rho}_{r-}^{-1}\hat{V}(r;1)\lambda_{x_0}(r)^{-1}d\widetilde{M}_r - \int_0^t \widetilde{\rho}_{r-}^{-1}\widetilde{E}[\beta(r,X_r,Y_r)|\mathcal{F}_r]d\widetilde{W}_r$$

$$+\int_0^t \widetilde{\rho}_{r-}^{-1}\widetilde{E}[\beta(r,X_r,Y_r)|\mathcal{F}_r]^2dr + \int_0^t \widetilde{\rho}_{r-}^{-1}\frac{\hat{V}(r;1)^2}{\lambda_{x_0}(r)+\hat{V}(r;1)}\lambda_{x_0}(r)^{-1}dN_r.$$

Here we use the fact that

$$\sum_{0<r\le t}(\widetilde{\rho}_r^{-1} - \widetilde{\rho}_{r-}^{-1} + \widetilde{\rho}_{r-}^{-2}(\widetilde{\rho}_r - \widetilde{\rho}_{r-}))$$

$$= \sum_{0<r\le t}\frac{(\widetilde{\rho}_r - \widetilde{\rho}_{r-})^2}{\widetilde{\rho}_{r-}^2\widetilde{\rho}_r} = \int_0^t \widetilde{\rho}_{r-}^{-1}\frac{\hat{V}(r;1)^2}{\lambda_{x_0}(r)+\hat{V}(r;1)}\lambda_{x_0}(r)^{-1}dN_r.$$

Then we have the assertion. □

**Proposition 2.4.9.** $\widetilde{\widetilde{M}}_t$ *is* $P$-$\mathcal{F}_t$-*martingale and* $\widetilde{\widetilde{W}}_t$ *is* $P$-$\mathcal{B}_t$-*Brownian motion.*

*Proof.* By Proposition 2.4.8,

$$d[\widetilde{\rho}^{-1}, \widetilde{M}]_t = -\widetilde{\rho}_{t-}^{-1} \frac{\hat{V}(t;1)}{\lambda_{x_0}(t) + \hat{V}(t;1)} dN_t$$

and then we have

$$d(\widetilde{\rho}_t^{-1}\widetilde{M}_t) = \widetilde{\rho}_{t-}^{-1} d\widetilde{M}_t + \widetilde{M}_{t-} d(\widetilde{\rho}^{-1})_t + d[\widetilde{\rho}^{-1}, \widetilde{M}]_t = \frac{\widetilde{\rho}_{t-}^{-1}\lambda_{x_0}(t)}{\lambda_{x_0}(t) + \hat{V}(t;1)} d\widetilde{\widetilde{M}}_t + \widetilde{M}_{t-} d(\widetilde{\rho}^{-1})_t.$$

Since $\widetilde{\rho}_t^{-1}\widetilde{M}_t$ and $\widetilde{\rho}_t^{-1}$ are $P$-$\mathcal{F}_t$-martingale, we can see $\widetilde{\widetilde{M}}_t$ is also $P$-$\mathcal{F}_t$-martingale. We can see that $\widetilde{\widetilde{W}}_t$ is $P$-$\mathcal{B}_t$ -Brownian motion by the following.

$$d(\widetilde{\rho}_t^{-1}\widetilde{W}_t) = \widetilde{\rho}_{t-}^{-1} d\widetilde{W}_t + \widetilde{W}_t d(\widetilde{\rho}^{-1})_t + d[\widetilde{\rho}^{-1}, \widetilde{W}]_t = \widetilde{\rho}_{t-}^{-1} d\widetilde{\widetilde{W}}_t + \widetilde{W}_t d(\widetilde{\rho}^{-1})_t.$$

□

Now let us prove Theorem 2.1.2. Let $\hat{F}_t = \widetilde{E}[\rho_t F_{t\wedge\tau}|\mathcal{F}_t]$, then we have the following by Proposition 2.4.7 and Proposition 2.4.8 .

$$\hat{F}_t = F_0 + \int_0^t \hat{f}_0(r;F) d\widetilde{\widetilde{M}}_r + \int_0^t \hat{f}_1(r;F) dr + \int_0^t \hat{f}_2(r;F) d\widetilde{\widetilde{W}}_r,$$

$$\widetilde{\rho}_t^{-1} = 1 + \int_0^t \widetilde{\rho}_{r-}^{-1} \tilde{f}_0(r) d\widetilde{\widetilde{M}}_r + \int_0^t \widetilde{\rho}_{r-}^{-1} \tilde{f}_2(r) d\widetilde{\widetilde{W}}_r$$

where

$$\hat{f}_0(r;F) = -\frac{\hat{V}(r;F)}{\tilde{\tilde{\lambda}}(r) - \hat{V}(r;1)} \widetilde{\rho}_{r-}, \quad \hat{f}_2(r;F) = \widetilde{E}[\rho_{r-}(\widetilde{D_0}F)_r|\mathcal{F}_r],$$

$$\hat{f}_1(r;F) =$$

$$\widetilde{E}[\rho_{r-}(\widetilde{L}F)_r|\mathcal{F}_r] + 1_{\{\tau>r\}} \frac{\hat{V}(r;1)\hat{V}(r;F)}{\tilde{\tilde{\lambda}}(r) - \hat{V}(r;1)} \widetilde{\rho}_{r-} + E[\beta(r, X_{r\wedge\tau}, Y_r)|\mathcal{F}_r] \widetilde{E}[\rho_{r-}(\widetilde{D_0}F)_r|\mathcal{F}_r],$$

$$\tilde{f}_0(r) = -\hat{V}(r;1)\tilde{\tilde{\lambda}}(r)^{-1}, \quad \tilde{f}_2(r) = -E[\beta(r, X_r, Y_r)|\mathcal{F}_r].$$

Note that $d\widetilde{M}_t = d\widetilde{\widetilde{M}}_t + (1 - N_{t-})\hat{V}(t;1)dt$ and $d\widetilde{W}_t = d\widetilde{\widetilde{W}}_t + E[\beta(t, X_{t\wedge\tau}, Y_t)|\mathcal{F}_t]dt$. Then $E[F_{t\wedge\tau}|\mathcal{F}_t] = \widetilde{\rho}_t^{-1}\hat{F}_t$. Let $\bar{F}_t = E[F_{t\wedge\tau}|\mathcal{F}_t]$ and we have the following.

$$\widetilde{\rho}_t^{-1}\hat{F}_t$$

$$= F_0 + \int_0^{t\wedge\tau} \widetilde{\rho}_{r-}^{-1} d\hat{F}_r + \int_0^{t\wedge\tau} \hat{F}_{r-} d\widetilde{\rho}_r^{-1} + [\hat{F}, \widetilde{\rho}^{-1}]_{t\wedge\tau}$$

$$
\begin{aligned}
&= F_0 + \int_0^{t\wedge\tau} \widetilde{\rho}_{r-}^{-1}(\hat{f}_0(r;F) + \tilde{f}_0(r)\hat{F}_{r-})d\widetilde{\widetilde{M}}_r + \int_0^{t\wedge\tau} \widetilde{\rho}_{r-}^{-1}(\hat{f}_1(r;F) + \tilde{f}_2(r)\hat{f}_2(r;F))dr \\
&\quad + \int_0^{t\wedge\tau} \widetilde{\rho}_{r-}^{-1}(\hat{f}_2(r;F) + \tilde{f}_2(r)\hat{F}_{r-})d\widetilde{\widetilde{W}}_r + \sum_{0<r\le t\wedge\tau} (\widetilde{\rho}_r^{-1} - \widetilde{\rho}_{r-}^{-1})(\hat{F}_r - \hat{F}_{r-}) \\
&= F_0 + \int_0^{t\wedge\tau} \widetilde{\rho}_{r-}^{-1}(\hat{f}_0(r;F) + \tilde{f}_0(r)\hat{F}_{r-} + \tilde{f}_0(r)\hat{f}_0(r;F))d\widetilde{\widetilde{M}}_r \\
&\quad + \int_0^{t\wedge\tau} \widetilde{\rho}_{r-}^{-1}(\hat{f}_1(r;F) + \tilde{f}_2(r)\hat{f}_2(r;F) + 1_{\{\tau>r\}}\tilde{f}_0(r)\hat{f}_0(r;F)\tilde{\tilde{\lambda}}(r))dr \\
&\quad + \int_0^{t\wedge\tau} \widetilde{\rho}_{r-}^{-1}(\hat{f}_2(r;F) + \tilde{f}_2(r)\hat{F}_{r-})d\widetilde{\widetilde{W}}_r
\end{aligned}
$$

Here we note that

$$
\sum_{0<r\le t} (\widetilde{\rho}_r^{-1} - \widetilde{\rho}_{r-}^{-1})(\hat{F}_r - \hat{F}_{r-}) = \int_0^t \widetilde{\rho}_{r-}^{-1}\tilde{f}_0(r)\hat{f}_0(r;F)dN_r.
$$

Then we have Theorem 2.1.2(1) as the following.

$$
E[F_{t\wedge\tau}|\mathcal{F}_t] = F_0 - \int_0^t 1_{\{\tau>r\}}(\hat{V}(r;F) + \hat{V}(r;1)\bar{F}_{r-})\tilde{\tilde{\lambda}}(r)^{-1}d\widetilde{\widetilde{M}}_r
$$

$$
+ \int_0^t 1_{\{\tau>r\}}E[1_{\{\tau>r\}}(\widetilde{L}F)_r|\mathcal{F}_r]dr + \int_0^t (E[(\widetilde{D_0}F)_r|\mathcal{F}_r] - E[\beta(r,X_r,Y_r)|\mathcal{F}_r]\bar{F}_{r-})d\widetilde{\widetilde{W}}_r.
$$

If there exist $C > 0$ and $\alpha \in (0,1)$ such that $1_{\{|X_t|\le 1\}}1_{\{\tau>t\}}|F_t| \le C|X_t|^\alpha$ for $t > 0$, we have

$$
\begin{aligned}
1_{\{\tau>r\}}\hat{V}(r;F) &= 1_{\{\tau>r\}}\widetilde{\rho}_{r-}^{-1}\lambda_{x_0}(r)\widetilde{E}[1_{\{\tau>r\}}\rho_r F_r|\mathcal{F}_r] \\
&= 1_{\{\tau>r\}}\widetilde{\rho}_{r-}^{-1}\lambda_{x_0}(r)e^{\int_0^r \lambda_{x_0}(u)du}\widetilde{E}[1_{\{\tau>r\}}\rho_r F_r|\mathcal{G}_r].
\end{aligned}
$$

by Proposition 2.4.4 and Proposition 2.4.5. Then we have

$$
1_{\{\tau>r\}}(\hat{V}(r;F) + \hat{V}(r;1)\bar{F}_{r-})\tilde{\tilde{\lambda}}(r)^{-1} = 1_{\{\tau>r\}}\bar{F}_{r-}
$$

which gives Theorem 2.1.2(2).

## 2.5 Examples

**Example 2.5.1.** $K(t,x,z,y) \in \mathcal{C}^2([0,\infty)\times \mathbf{R}\times \mathbf{R}^N \times \mathbf{R};\mathbf{R})$ *is a bounded and continuously differentiable function which satisfies* $K(t,0,z,y)=0$*. Let constants* $m, C' > 0$ *exist such that* $|K(t,x,y,z)|+|\frac{\partial K}{\partial t}(t,x,z,y)|+|\frac{\partial K}{\partial x}(t,x,z,y)|+|\frac{\partial K}{\partial z}(t,x,z,y)|+|\frac{\partial K}{\partial y}(t,x,z,y)| \le C'(1+|x|+|z|+|y|)^m$ *. Let us think of the case* $K_t = K(t,X_t,Z_t,Y_t)$ *in Theorem 2.1.2. Then we have*

$$K_t = K_0 + \int_0^t (\widetilde{L_0}K)_u du + \int_0^t \sigma_2(u,Y_u)\frac{\partial K}{\partial y}(u,X_u,Z_u,Y_u)dW_u$$

$$+\int_0^t \frac{\partial K}{\partial x}(u,X_u,Z_u,Y_u)dB_u + \int_0^t \sigma_1(u,X_u,Z_u)\bigtriangledown_z K(u,X_u,Z_u,Y_u)d\hat{B}_u$$

*where*

$$(\widetilde{L_0}K)(t,x,z,y) = (\frac{\partial}{\partial t} + b_0(t,x,z)\frac{\partial}{\partial x} + b_1(t,x,z)\bigtriangledown_z + b_2(t,x,y)\frac{\partial}{\partial y}$$

$$+\frac{1}{2}(\frac{\partial^2}{\partial x^2} + \sigma_1(t,x,z)^2\bigtriangledown_z^2 + \sigma_2(t,y)^2\frac{\partial^2}{\partial y^2}))K(t,x,z,y).$$

*Then we have*

$$(\widetilde{D_0}K)_t = \beta(t,X_t,Y_t)K_{t\wedge\tau} + 1_{\{\tau>t\}}\sigma_2(u,Y_u)\frac{\partial K}{\partial y}(u,X_u,Z_u,Y_u),$$

$$(\widetilde{L}K)_t = 1_{\{\tau>t\}}(\widetilde{L_0}K)_t$$

*and*

$$E[K_{t\wedge\tau}|\mathcal{F}_t]$$

$$= K_0 - \int_0^t (\hat{V}(r;K)+\hat{V}(r;1)\bar{K}_{r-})\tilde{\tilde{\lambda}}(r)^{-1}d\widetilde{\widetilde{M}}_r + \int_0^t E[1_{\{\tau>r\}}(\widetilde{L_0}K)_r|\mathcal{F}_r]dr$$

$$+\int_0^t (E[\beta(r,X_r,Y_r)K_{r\wedge\tau} + 1_{\{\tau>r\}}\sigma_2(r,Y_r)\frac{\partial K}{\partial y}(r,X_r,Z_r,Y_r)|\mathcal{F}_r]$$

$$-E[\beta(r,X_r,Y_r)|\mathcal{F}_r]\bar{K}_{r-})d\widetilde{\widetilde{W}}_r$$

*by Theorem 2.1.2(1).*

**Example 2.5.2.** *In the case that* $F_t = X_t = B_t + \int_0^t b_0(r, X_r)dr$*, we have* $(\widetilde{D_0}X)_t = \beta(t, X_t, Y_t)X_t$ *and* $(\widetilde{L}X)_t = 1_{\{\tau>r\}}b_0(t, X_t)$*. Then we have*

$$E[X_{t\wedge\tau}|\mathcal{F}_t]$$

$$= x_0 - \int_0^t \bar{X}_{r-}d\widetilde{\widetilde{M}}_r + \int_0^t E[1_{\{\tau>r\}}b_0(r, X_r, Z_r)|\mathcal{F}_r]dr$$

$$+ \int_0^t 1_{\{\tau>r\}}(E[\beta(r, X_r, Y_r)X_r|\mathcal{F}_r] - E[\beta(r, X_r, Y_r)|\mathcal{F}_r]\bar{X}_{r-})d\widetilde{\widetilde{W}}_r$$

*by Theorem 2.1.2(2).*

**Example 2.5.3.** *Let* $\hat{f}_i \in \mathcal{L}^{6+}$*,* $i = 0, 1, \cdots, d$ *and*

$$F_t = F_0 + \int_0^t \hat{f}_0(s)dB_s + \sum_{j=1}^d \int_0^t \hat{f}_j(s)d\hat{B}_s^j,$$

*then* $(\widetilde{D_0}F)_t = \beta(t, X_t, Y_t)F_t$*,* $(\widetilde{L}F)_t = 0$ *and*

$$E[F_{t\wedge\tau}|\mathcal{F}_t]$$

$$= F_0 - \int_0^t (\hat{V}(r; F) + \hat{V}(r; 1)\bar{F}_{r-})\tilde{\tilde{\lambda}}(r)^{-1}d\widetilde{\widetilde{M}}_r$$

$$+ \int_0^t 1_{\{\tau>r\}}(E[\beta(t, X_t, Y_t)F_{r\wedge\tau}|\mathcal{F}_r] - E[\beta(r, X_r, Y_r)|\mathcal{F}_r]\bar{F}_{r-})d\widetilde{\widetilde{W}}_r.$$

## 2.6 A Financial market model

We will give an application to the financial market. Let $X$ and $Y$ be solutions of the following stochastic differential equations under $P$.

$$dX_t = dB_t + b_0(t, X_t)dt, \qquad X_0 = x_0 > 0,$$

$$dY_t = \sigma_1(t, Y_t)dW_t + b_1(t, X_{t\wedge\tau}, Y_t)dt, \qquad Y_0 = y_0 \in \mathbf{R}.$$

Let $S_t = E[X_{t\wedge\tau}|\mathcal{F}_{t\wedge\tau}]$ and we regard $S_t$ as a stock price process. We consider the stock price $S_t$ to be the same with a value of company looked over the filtration $\mathcal{F}_t$ which constructed of the observable process $Y_t$ and the fact that whether the default takes place or not. Let $T$ be the maturity of a bond issued by this company, the recovery rate of this bond be zero, the risk free rate be also zero and $\alpha > 0$ be constant risk premium. Then we will define the bond price $R_t$ as $R_t = e^{-\alpha(T-t)}E[1_{\{\tau>T\}}|\mathcal{F}_t]$. By Example 2.5.2, we have the following.

**Proposition 2.6.1.**

$$S_t = x_0 + \int_0^t \bar{f}_0(r;X)d\widetilde{\widetilde{M}}_r + \int_0^t \bar{f}_1(r;X,Z)dr + \int_0^t \bar{f}_2(r;X,Y)d\widetilde{\widetilde{W}}_r$$

*where*

$$\bar{f}_0(r;X) = -S_{r-}, \quad \bar{f}_1(r;X,Z) = E[1_{\{\tau>r\}}b_0(r,X_r,Z_r)|\mathcal{F}_r]$$

*and*

$$\bar{f}_2(r;X,Y) = 1_{\{\tau>r\}}(E[\beta(r,X_r,Y_r)X_r|\mathcal{F}_r] - E[\beta(r,X_r,Y_r)|\mathcal{F}_r]S_{r-}).$$

**Proposition 2.6.2.** *If*

$$\frac{\partial\beta}{\partial x}(r,x,y) > 0 \qquad for\ all\ r,x,y \in \mathbf{R},$$

*then* $\bar{f}_2(t;X) > 0$ *a.e.* $t > 0$ *in Proposition 2.6.1 for* $F \in \tilde{\Sigma}$.

*Proof.*

$$\bar{f}_2(r;X)$$

$$E[1_{\{\tau>t\}}\beta(r,X_r,Y_r)X_r|\mathcal{F}_r] - E[1_{\{\tau>r\}}\beta(r,X_r,Y_r)|\mathcal{F}_r]E[1_{\{\tau>r\}}X_r|\mathcal{F}_r]$$

$$= 1_{\{\tau>r\}}E[(\beta(r,X_r,Y_r) - \beta(r,E[X_r|\mathcal{F}_r],Y_r))(X_r - E[X_r|\mathcal{F}_r])|\mathcal{F}_r]$$

$$= 1_{\{\tau>r\}}E[(\int_{E[X_r|\mathcal{F}_r]}^{X_r} \frac{\partial\beta}{\partial x}(r,x,Y_r)dx)(X_r - E[X_r|\mathcal{F}_r])|\mathcal{F}_r] > 0 \quad a.s. \tag{2.6.1}$$

Since

$$\widetilde{P}(X_r - E[X_r|\mathcal{F}_r] = 0, r < \tau) = \widetilde{P}(X_r - E[X_r|\mathcal{G}_r^Y] = 0, r < \tau)$$

$$\leq \widetilde{P}(X_r - E[X_r] = 0) = 0,$$

we have our assertion. $\square$

**Proposition 2.6.3.** *Let* $\tilde{R}_t = E[1_{\{\tau>T\}}|\mathcal{B}_t]$. *Then* $R_t = e^{-\alpha(T-t)}E[\tilde{R}_t|\mathcal{F}_t]$ *and we have*

$$R_t$$

$$= R_0 - \int_0^t e^{-\alpha(T-r)}(\hat{V}(r;\tilde{R}) + \hat{V}(r;1)R_{r-}|\mathcal{F}_r])\tilde{\tilde{\lambda}}(r)^{-1}d\widetilde{\widetilde{M}}_r$$

$$+\int_0^t \alpha e^{-\alpha(T-r)}\tilde{R}_r dr$$

$$+\int_0^t e^{-\alpha(T-r)}1_{\{\tau>r\}}(E[\beta(t,X_t,Y_t)\tilde{R}_{r\wedge\tau}|\mathcal{F}_r] - E[\beta(r,X_r,Y_r)|\mathcal{F}_r]R_{r-})d\widetilde{\widetilde{W}}_r$$

*which means, in case* $\alpha = 0$, $R_t$ *is* $P$-$\mathcal{F}_t$*-martingale.*

*Proof.* It is well known that there is $u : [0,T]\times[0,\infty) \to \mathbf{R}$ such that $u(t,x)$ is bounded, is continuous in $(t,x) \in [0,T]\times[0,\infty)$, $C^1$ in $t \in (0,T)$ and is $C^2$ in $x \in (0,\infty)$, which satisfies

$$\begin{cases} \frac{\partial u}{\partial t} + (Lu)(t,x) = 0, & x > 0, \\ u(t,0) = 0, & t \in [0,T), \\ u(T,x) = 1, & x > 0 \end{cases}$$

where

$$(Lu)(t,x) = (b_0(t,x)\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2})u(t,x).$$

Then we have

$$1_{\{t<\tau\}}u(t,X_t) = u(t\wedge\tau, X_{t\wedge\tau}) = u(0,x_0) + \int_0^{t\wedge\tau}\frac{\partial u}{\partial x}(r,X_r)dB_r, \quad t < T.$$

Letting $t \uparrow T$,

$$1_{\{T<\tau\}} = u(0,x_0) + \int_0^{T\wedge\tau}\frac{\partial u}{\partial x}(r,X_r)dB_r$$

Then we have the following by Example 2.5.3.

$$E[\tilde{R}_t|\mathcal{F}_t] = E[1_{\{T<\tau\}}|\mathcal{F}_t]$$

$$= \tilde{R}_0 - \int_0^t (\hat{V}(r;\tilde{R}) + \hat{V}(r;1)R_{r-}])\tilde{\tilde{\lambda}}(r)^{-1}d\widetilde{\widetilde{M}}_r$$

$$+\int_0^t 1_{\{\tau>r\}}(E[\beta(t,X_t,Y_t)\tilde{R}_{r\wedge\tau}|\mathcal{F}_r] - E[\beta(r,X_r,Y_r)|\mathcal{F}_r]R_{r-})d\widetilde{\widetilde{W}}_r.$$

Here we note that

$$
\begin{aligned}
&e^{\alpha T} R_t = e^{\alpha t} \tilde{R}_t \\
&= \tilde{R}_0 - \int_0^t e^{\alpha r} (\hat{V}(r; \tilde{R}) + \hat{V}(r; 1) R_{r-}) \tilde{\tilde{\lambda}}(r)^{-1} d\widetilde{\widetilde{M}}_r \\
&+ \int_0^t \alpha e^{\alpha r} \tilde{R}_r dr \\
&+ \int_0^t e^{\alpha r} 1_{\{\tau > r\}} (E[\beta(t, X_t, Y_t) \tilde{R}_{r \wedge \tau} | \mathcal{F}_r] - E[\beta(r, X_r, Y_r) | \mathcal{F}_r] R_{r-}) d\widetilde{\widetilde{W}}_r.
\end{aligned}
$$

Then we have the assertion. □

# 第3章 A remark on credit risk models and copula

## 3.1 Introduction

Recently, copula models are widely used to describe the joint distribution of stopping times (such as borrower default times). But it is not clear whether these copula models are dynamically consistent. For consistency, there is an enormous family of corresponding martingale restrictions. Specifically, the process describing the conditional probability, as time passes and new information is gathered, of any specific joint default scenario must be a martingale, by the law of iterated expectations. The scenario could be, for example, that borrowers A and B default by January 1, 2015, while borrower C survives until at least June 1, 2015.

In a static setting, the copula has been conveniently used to parameterize the likelihoods of such scenarios, particularly for purposes of analyzing collateralized debt obligations and for capital-sufficiency analysis of credit portfolios. But the same copulas have been applied in dynamic settings, routinely, and nobody to this point has ever checked whether this can be done consistently in a dynamic setting. This is critical for dynamic risk management purposes and obviously for checking the validity of the chosen copula function.

Björk-Christernsen [4] considered the relationship between a family of forwardrate curves parameterized by finite factors and a dynamical interest rate model free of arbitrage, and showed that there are some analytic constraint conditions for a family of forwardrate curves which comes from a dynamical interest rate model free of arbitrage.

In the present paper, we study the relationship between a family of copula functions parameterized by finite dimensional parameters and dynamical default time models. Then we show that the set of copula models that are consistent in this dynamic sense and satisfy some technical regularity conditions, is extremely rare (a set of the first category in the Baire sense). In economic theory, this notion of rareness is described by saying that the consistency condition is "generically" not satisfied. The paper also analyzes consistency of the family of Gumbel copulas and concludes that it is not consistent.

The setup in this paper is the following.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $W(t) = (W^k(t))_{k=1,\dots,d}$, $t \geqq 0$, be a $d$-dimensinal standard Wiener process, and $\mathcal{G}_t = \sigma\{W(s),\ s \in [0,t]\} \vee \mathcal{N}$, where $\mathcal{N} = \{B \in \mathcal{F};\ P(B) = 0 \text{ or } 1\}$. Let $N \geqq 2$, $\tau_i : \Omega \to [0,\infty)$, $i = 1,\dots,N$, be random variables, and let $\mathcal{F}_t = \mathcal{G}_t \vee \sigma\{\tau_i \wedge t, i = 1,\dots,N\}$. Let $\xi_i : [0,\infty) \times \Omega \to [0,\infty)$, $i = 1,\dots,N$, be $\mathcal{G}$-progressively measurable processes.

First, we assume the following conditions.

(SC) $\quad (\prod_{i\in I} 1_{\{\tau_i > t\}}) P(\tau_i > t_i, i \in I | \mathcal{F}_t) = (\prod_{i\in I} 1_{\{\tau_i > t\}}) E[\exp(-\sum_{i\in I}\int_t^{t_i} \xi_i(s)ds)|\mathcal{G}_t]\ a.s.$
for any $I \subset \{1,\dots,N\}$ and $t, t_i \in [0,\infty), i \in I$ with $t \leqq \min_{i\in I} t_i$.

(PO) For any $t \geqq 0$,

$$P(\bigcap_{i=1}^N \{\tau_i > t\})|\mathcal{G}_t] > 0\ a.s.$$

We also assume the following technical assumptions.

(A-1) For any $T > 0$,

$$\sum_{i=1}^N \int_0^T E[\xi_i(t)^4]dt < \infty.$$

(A-2) For any $i = 1,\dots,N$,

$$\int_0^\infty \xi_i(t) = \infty \quad a.s. \text{ and } \int_a^b \xi_i(t) > 0 \quad a.s. \text{ for any } a, b > 0 \text{ with } b > a.$$

(A-3) $\quad \sum_{i=1}^N \int_0^\infty (1+t)^2 E[\xi_i(t)^2 \exp(-2\int_0^t \xi_i(s)ds)]dt < \infty.$

Let $\theta : [0,\infty) \times \Omega \to \mathbf{R}^M$ be a $\mathcal{G}$-Ito process, i.e., $\theta$ is $\mathcal{G}$-progressively measurable, $\theta(t,\omega)$ is continuous in $t$ for all $\omega \in \Omega$, and there are $\mathbf{R}^M$-valued $\mathcal{G}$-progressively measurable processes $\eta_k$, $k = 1, \dots, d$, and $b$ satisfying

$$P(\sum_{k=1}^{d} \int_0^T |\eta_k(t)|^2 dt + \int_0^T |b(t)| dt < \infty) = 1, \text{ for any } T > 0,$$

and

$$\theta(t) = \theta(0) + \sum_{k=1}^{d} \int_0^t \eta_k(s) dW^k(s) + \int_0^t b(s) ds. \tag{3.1.1}$$

Let $\Theta$ be an open subset in $\mathbf{R}^M$ and $K \in C([0,1]^N \times \Theta; [0,1])$. We assume the following, moreover.

(A-4) $P(\theta(t) \in \bar{\Theta}$ for all $t \geqq 0) = 1$, where $\bar{\Theta}$ is the closure of $\Theta$ in $\mathbf{R}^M$.

(A-5) the support of probability law of $\theta(t, \omega)$ under $e^{-t} dt \otimes P(d\omega)$ contains a non-empty open set in $\Theta$, i.e., there is a non-empty open set $U_0$ in $\Theta$ such that for any $\theta_0 \in U_0$ and $\varepsilon > 0$

$$\int_0^\infty P(|\theta(t) - \theta_0| < \varepsilon) \, e^{-t} dt > 0.$$

(CP) $K(\cdot, \theta) : [0,1]^N \to [0,1]$ is a copula function for any $\theta \in \Theta$, and

$$(\prod_{i=1}^{N} 1_{\{\tau_i > t\}}) 1_\Theta(\theta(t)) P(\tau_i > t_i, i = 1, \dots, N | \mathcal{F}_t)$$

$$= (\prod_{i=1}^{N} 1_{\{\tau_i > t\}}) 1_\Theta(\theta(t)) K(P(\tau_1 > t_1 | \mathcal{F}_t), \dots, (P(\tau_N > t_N | \mathcal{F}_t), \theta(t)) \ a.s.$$

for any $t, t_1, \dots, t_N > 0$ with $t < \min_{i=1,\dots,N} t_i$.

We call a family $((\Omega, \mathcal{F}, P), (W_t^k)_{k=1,\dots,d}, (\tau_i)_{i=1,\dots,N}, (\xi_i(t))_{i=1,\dots,N}, \theta(t), \Theta, K)$ satisfying the above assumptions a dynamical default time copula model, and we call $K$ the associated family of copula functions to this model.

**Definition 3.1.1.** *Let $\Theta$ be an open subset in $\mathbf{R}^M$. We say that $K \in C([0,1]^N \times \Theta; [0,1])$ is an admissible family of copula functions, if there is a dynamical default time copula model and $K$ is the associated family of copula functions to the model.*

The purpose of the present paper is to show that there are some analytic constraint conditions for an admissible family of copula functions. For example we will prove the following.

Let $N, M \geqq 1$, $\Theta$ be a non-void open subset in $\mathbf{R}^M$. Let $\mathcal{C}_{(N)}(\Theta)$ denote the subset of $C([0,1]^N \times \Theta; [0,1])$ consisting of elements $K$ such that $K(\cdot,\theta) : [0,1]^N \to [0,1]$ is a copula function for any $\theta \in \Theta$, and $K|_{(0,1)^N \times \Theta}$ is a $C^\infty$ function.

Let $D_n$ be an increasing sequence of compact subsets in $\Theta$ such that $\bigcup_{n=1}^\infty D_n = \Theta$. Then we can regard $\mathcal{C}_{(N)}(\Theta)$ as a Polish space with a metric function $dis$ given by

$$dis(K_1, K_2)$$
$$= \sum_{n=1}^\infty 2^{-n} \wedge \sup\{|K_1(x,\theta) - K_2(x,\theta)|;\ x \in [0,1]^N,\ \theta \in D_n\}$$
$$+ \sum_{n=1}^\infty 2^{-n} \wedge \Big( \sum_{\alpha_1,\ldots,\alpha_{N+M}=0}^{n} \sup\{| \frac{\partial^{\alpha_1+\cdots\alpha_{N+M}}(K_1 - K_2)}{\partial x_1^{\alpha_1}\ldots\partial x_N^{\alpha_N}\partial\theta_1^{\alpha_{N+1}}\partial\ldots\theta_M^{\alpha_{N+M}}}(x,\theta);$$
$$x \in [1/4n, 1-1/4n]^N,\ \theta \in D_n\}\Big).$$

Our main result is the following.

**Theorem 3.1.1.** *Let $N \geqq 3$, $M \geqq 1$, and $\Theta$ be a non-void open subset in $\mathbf{R}^M$. Then the subset of $\mathcal{C}_{(N)}(\Theta)$ whose elements are admissible families of copula functions is a set of the first category in Baire's sense.*

We also show that a family of Gumbel copula functions of 3 variables is not admissible by relying on numerical computation in Section 7.

**Acknowledgement**

The authors thank the referee for useful suggestions and comments.

## 3.2 Preliminary results

Let $\xi : [0,\infty) \times \Omega \to [0,\infty)$ be a $\mathcal{G}$-progressively measurable satisfying the following three conditions.

(B-1) $\displaystyle\int_0^T E[\xi(t)^4]dt < \infty$ for any $T > 0$.

(B-2) $\int_0^\infty \xi(t)dt = \infty$ *a.s.*, and $\int_a^b \xi(t)dt > 0$ *a.s.* for any $b > a \geqq 0$.
(B-3) $E[\int_0^\infty (1+t)^2\xi(t)^2 \exp(-2\int_0^t \xi(r)dr)dt] < \infty$.

For each $s \geqq 0$, let $\{M(t,s); t \geqq 0\}$ is a continuous martingale given by

$$M(t,s) = E[\exp(-\int_0^s \xi(r)dr)|\mathcal{G}_t], \qquad t \geqq 0.$$

**Proposition 3.2.1.** *There is $f : [0,\infty) \times [0,\infty) \times \Omega \to (0,\infty)$ satisfying the following.*
(1) *For any $t, s \geqq 0$,*

$$f(t,s) = E[\exp(-\int_{t\wedge s}^s \xi(r)dr)|\mathcal{G}_t] = \exp(\int_0^{t\wedge s} \xi(r)dr)M(t,s) \qquad a.s.$$

(2) *For any $\omega \in \Omega$ $f(\cdot, * : \omega) : [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous.*
(3) *If $s_2 > s_1 > t$, then $f(t,s_1,\omega) > f(t,s_2,\omega) > 0$, $\omega \in \Omega$.*
(4) *For any $t \geqq 0$*

$$f(t,t) = 1, \quad \lim_{s\uparrow\infty} f(t,s) = 0.$$

*Proof.* Note that for $0 \leqq s_1 \leqq s_2$

$$E[\sup_{t\in[0,\infty)} |M(t,s_1) - M(t,s_2)|^4] \leqq 4E[|M(s_1,s_1) - M(s_2,s_2)|^4]$$

$$\leqq 4E[|\int_{s_1}^{s_2} \xi(r)dr|^4] \leqq 4(s_2-s_1)^3(\int_{s_1}^{s_2} E[\xi(r)^4]dr).$$

So by Kolmogorov's continuity theorem and the assumption (B-1), we see that there is a $\tilde{M} : [0,\infty) \times [0,\infty) \times \Omega \to [0,\infty)$ such that $\tilde{M}(\cdot, *, \omega) \to [0,\infty)$ is continuous and $P(\tilde{M}(t,s) = M(t,s)) = 1$, $t, s \geqq 0$. Let

$$\tilde{f}(t,s) = \exp(\int_0^{t\wedge s} \xi(r)dr)\tilde{M}(t,s) \qquad t, s \geqq 0.$$

Then $\tilde{f}(t,s)$ is continuous in $(t,s)$. Let $0 \leqq s_1 < s_2$. Then

$$\tilde{f}(t,s_1) - \tilde{f}(t,s_2) = \exp(\int_0^t \xi(r)dr)(\tilde{M}(t,s_1) - \tilde{M}(t,s_2)), \qquad t \in [0,s_1].$$

By the assumption (B-2), we have

$$\tilde{M}(s_1,s_1) - \tilde{M}(s_1,s_2) = \exp(-\int_0^{s_1} \xi(r)dr)E[1 - \exp(-\int_{s_1}^{s_2} \xi(r)dr))|\mathcal{G}_{s_1}] > 0 \ a.s.$$

Let $\tau = \inf\{t \geqq 0;\ \tilde{M}(t, s_1) - \tilde{M}(t, s_2) = 0\} \wedge s_1$. Then we see that

$$\tilde{M}(\tau, s_1) - \tilde{M}(\tau, s_2) = E[\tilde{M}(s_1, s_1) - \tilde{M}(s_1, s_2)|\mathcal{G}_\tau] > 0, \qquad a.s.$$

So we see that

$$\inf_{t \in [0, s_1]} (\tilde{M}(t, s_1) - \tilde{M}(t, s_2)) > 0 \ a.s.,$$

and so we have

$$\inf_{t \in [0, s_1]} (\tilde{f}(t, s_1) - \tilde{f}(t, s_2)) > 0 \ a.s.$$

for any $s_1, s_2 \in \mathbf{Q}$ with $s_2 > s_1 \geqq 0$. So there is an $\Omega_1 \in \mathcal{F}$ with $P(\Omega_1) = 1$ such that $\tilde{f}(t, s_1, \omega) > \tilde{f}(t, s_2, \omega)$ for any $\omega \in \Omega_1$, $s_1, s_2 \in \mathbf{Q}$ with $s_2 > s_1 > 0$ and $t \in [0, s_1)$. Since $\tilde{f}(t, s)$ is continuous in $(t, s)$, we see that $\tilde{f}(t, s)$ is non-increasing in $s$. So we see that $\tilde{f}(t, s_1, \omega) > \tilde{f}(t, s_2, \omega)$ for any $\omega \in \Omega_1$, $t, s_1, s_2 \in [0, \infty)$ with $s_2 > s_1 > t$. Similarly we can show that there is an $\Omega_2 \in \mathcal{F}$ with $P(\Omega_2) = 1$ such that $\tilde{f}(t, s, \omega) > 0$ for any $\omega \in \Omega_2$, $t, s \in [0, \infty)$.

We see that

$$E[\varliminf_{s \to \infty} \tilde{f}(t, s)] \leqq \varliminf_{s \to \infty} E[\tilde{f}(t, s)] = \varliminf_{s \to \infty} E[\exp(-\int_t^s \xi(r) dr] = 0.$$

Since $\lim_{s \to \infty} \tilde{f}(t, s, \omega)$ exists for $\omega \in \Omega_1$, we see that $\lim_{s \to \infty} \tilde{f}(t, s, \omega) = 0$ a.s. Also, it is easy to see $\tilde{f}(t, t) = 1$ $a.s.$ Therefore we can take a good version $f$ of $\tilde{f}$ satisfying the assertion. □

**Proposition 3.2.2.** *There exist* $\hat{\sigma}_k : [0, \infty] \times [0, \infty) \times \Omega \to \mathbf{R}$, $k = 1, \dots, d$, *satisfying the following.*
(1) $\hat{\sigma}_k(t, \cdot, \omega) : [0, \infty] \to \mathbf{R}$, $k = 1, \dots, d$, *is continuous for any* $t \in [0, \infty)$ *and* $\omega \in \Omega$. *Moreover,* $\hat{\sigma}_k(t, s, \omega) = 0$, $t \geqq s$, *and* $\lim_{s \to \infty} \hat{\sigma}_k(t, s, \omega) = 0$ *for any* $t \in [0, \infty)$ *and* $\omega \in \Omega$.
(2) $\hat{\sigma}_k(\cdot, s) : [0, \infty) \times \Omega \to \mathbf{R}$, $k = 1, \dots, N$, *is* $\mathcal{G}$*-progressively measurable for any* $s \geqq 0$ *and*

$$M(t, s) = M(0, s) + \sum_{k=1}^d \int_0^t \hat{\sigma}_k(r, s) dW^k(r), \quad t \geqq 0, \ a.s.$$

*for any* $s > 0$.

*Proof.* For each $s \geqq 0$, let $N(t,s), t \in [0,\infty)$ be a continuous martingale given by

$$N(t,s) = E[\xi(s)\exp(-\int_0^s \xi(r)dr)|\mathcal{G}_t].$$

By Ito's representation theorem, we see that for any $s \geqq 0$ there exist $\mathcal{G}$-progressively measurable processes $c_k(\cdot,s) : [0,\infty)\times\Omega \to \mathbf{R}$, $k = 1,\ldots,d$, such that

$$N(t,s) = N(0,s) + \sum_{k=1}^d \int_0^t c_k(r,s)dW^k(r), \quad t \geqq 0.$$

Since the map from $[0,\infty)$ to $L^2(\Omega,\mathcal{F},P)$ coresponding $s$ to $N(t,s)$ is measurable, we may assume that $c_k : [0,\infty)\times[0,\infty)\times\Omega \to \mathbf{R}$ is measurable. Note that

$$N(0,s)^2 + \sum_{k=1}^d \int_0^\infty E[c_k(r,s)^2]dr$$

$$= \lim_{t\to\infty} E[N(t,s)^2] \leqq E[\xi(s)^2\exp(-2\int_0^s \xi(r)dr)].$$

Therefore by the assumption (B-3), we see that

$$E[\int_{[0,\infty)\times[0,\infty)} (1+s)^2 c_k(r,s)^2 drds] < \infty, \qquad k = 1,\ldots,d.$$

Let us define $\tilde{\sigma}_k : [0,\infty)\times[0,\infty)\times\Omega \to \mathbf{R}$, $k = 1,\ldots,N$, by

$$\tilde{\sigma}_k(t,s) = \begin{cases} -\int_0^s c_k(t,u)du, & \text{if } \int_{[0,\infty)}(1+u)^2 c_k(t,u)^2 ds < \infty, \\ 0, & \text{otherwise .} \end{cases}$$

Then we see that $\tilde{\sigma}_k(\cdot,s) : [0,\infty)\times\Omega \to \mathbf{R}$, is $\mathcal{G}$-progressively measurable for any $s \geqq 0$ and $\tilde{\sigma}_k(t,\cdot) : [0,\infty) \to \mathbf{R}$ is continuous. Also, by stochastic Fubini's theorem, we have

$$-\int_0^s N(0,u)du + \sum_{k=1}^d \int_0^t \tilde{\sigma}_k(r,s)dW^k(r)$$

$$= -\int_0^s N(0,u)du - \sum_{k=1}^d \int_0^t (\int_0^s c_k(r,u)du)dW^k(r)$$

$$= -\int_0^s N(t,u)du = E[\exp(-\int_0^s \xi(u)du) - 1|\mathcal{G}_t] = M(t,s) - 1 \qquad a.s.$$

So we see that

$$M(t,s) = M(0,s) + \sum_{k=1}^{d} \int_0^t \tilde{\sigma}_k(r,s) dW^k(r).$$

Note that for $0 < s_1 < s_2 < \infty$, we have

$$|\tilde{\sigma}_k(t,s_1) - \tilde{\sigma}_k(t,s_2)|^2 \leqq (\int_{s_1}^{s_2} |c_k(t,u)| du)^2$$

$$\leqq (\int_{s_1}^{s_2} (1+u)^{-2} du)(\int_{s_1}^{s_2} (1+u)^2 c_k(t,u)^2 du) \leqq \int_{s_1}^{\infty} (1+u)^2 c_k(t,u)^2 du.$$

So we have

$$E[\int_{[0,\infty)} dt (\sup_{s_1,s_2>s} |\tilde{\sigma}_k(t,s_1) - \tilde{\sigma}_k(t,s_2)|^2)]$$

$$\leqq E[\int_{[0,\infty)\times[s,\infty)} (1+u)^2 c_k(r,u)^2 dr du] \to 0, \qquad s \to \infty.$$

Therefore we see that

$$\sup_{s_1,s_2>s} |\tilde{\sigma}_k(t,s_1) - \tilde{\sigma}_k(t,s_2)|^2 \to 0, \qquad s \to \infty \qquad dt \otimes P(d\omega) - a.e.(t,\omega).$$

This implies that $\tilde{\sigma}_k(t,s)$ converges as $s \to \infty$ for $dt \otimes P(d\omega) - a.e.(t,\omega)$.

Also, we see by (B-2) that

$$E[\int_0^{\infty} (\varliminf_{s\to\infty} \tilde{\sigma}_k(t,s)^2) dt] \leqq \varliminf_{s\to\infty} E[\int_0^{\infty} \tilde{\sigma}_k(t,s)^2 dt] \leqq \varliminf_{s\to\infty} E[\exp(-2\int_0^s \xi_i(u) du)] = 0.$$

Thus we see that $\tilde{\sigma}_k(t,s) \to 0$, $s \to \infty$ for $dt \otimes P(d\omega) - a.e.(t,\omega)$.

Let $\hat{\sigma}_k$, $k = 1, \ldots, d$, be given by

$$\hat{\sigma}_k(t,s) = \begin{cases} \tilde{\sigma}_k(t,s), & \text{if } \tilde{\sigma}_k(t,s) \to 0, \text{ as } s \to \infty, \\ 0, & \text{otherwise ,} \end{cases}$$

Then we have our assertion. □

By Ito's formula, we have

$$f(t,s)$$

$$= f(0,s) + \int_0^{t\wedge s} \xi(r) f(r,s) dr + \sum_{k=1}^{d} \int_0^t \exp(\int_0^{r\wedge s} \xi(u) du) \hat{\sigma}_k(r,s) dW^k(r), \quad t \geqq 0,$$

for any $s \geqq 0$. So we have the following as a corollary to Proposition 3.2.2.

**Corollary 3.2.3.** *There exist* $\tilde{\sigma}_k : [0,\infty] \times [0,\infty) \times \Omega \to \mathbf{R}$, $k = 1,\dots,d$, *such that*
(1) $\tilde{\sigma}_k(t,\cdot,\omega) : [0,\infty] \to \mathbf{R}$, $k = 1,\dots,d$, *is continuous for any* $t \in [0,\infty)$ *and* $\omega \in \Omega$. *Moreover,* $\tilde{\sigma}_k(t,s,\omega) = 0$, $t \geqq s$, *and* $\lim_{s\to\infty} \tilde{\sigma}_k(t,s,\omega) = 0$ *for any* $t \in [0,\infty)$ *and* $\omega \in \Omega$.
(2) $\tilde{\sigma}_{i,k}(\cdot,s) : [0,\infty) \times \Omega \to \mathbf{R}$, $k = 1,\dots,N$, *is* $\mathcal{G}$*-progressively measurable for any* $s \geqq 0$ *and*

$$f(t,s) = f(0,s) + \int_0^{t\wedge s} \xi(r) f(r,s) dr + \sum_{k=1}^d \int_0^t \tilde{\sigma}_k(r,s) dW^k(r), \quad t \geqq 0, \ a.s.$$

*for any* $s > 0$.

By Proposition 3.2.1, we have the following immediately.

**Proposition 3.2.4.** *Let* $T : [0,\infty) \times (0,1] \times \Omega \to [0,\infty]$ *be given by*

$$T(t,x) = \inf\{s \geqq t;\ f(t,s) < x\}, \qquad x \in (0,1].$$

*Then* $T(t,\cdot,\omega) : (0,1] \to [0,\infty)$ *is continuous and strictly decreasing and* $\lim_{x\downarrow 0} T(t,x,\omega) = \infty$ *for any* $t \geqq 0$ *and* $\omega \in \Omega$.

Now let $X : [0,\infty) \times [0,\infty) \times (0,1] \times \Omega \to (0,1]$ be given by

$$X(t,s,x) = f(t \vee s, T(s,x)), \qquad t,s \geqq 0,\ x \in (0,1].$$

Then we see that $\lim_{x\to 0} X(t,s,x,\omega) = 0$. So by defining $X(t,s,0) = 0$, we can define $X : [0,\infty) \times [0,\infty) \times [0,1] \times \Omega \to [0,1]$ such that $X(\cdot,*,**,\omega) : [0,\infty) \times [0,\infty) \times [0,1] \to [0,1]$ is continuous for any $\omega \in \Omega$, $X(t,s,\cdot,\omega) : [0,1] \to [0,1]$ is continuous and non-decreasing, and

$$f(t \vee s, r) = X(t,s,f(s,r)), \qquad r \geqq s \geqq 0,\ t \geqq 0.$$

Then we see that $X(t,t,x) = x$, and for $t \geqq s \geqq r \geqq 0$,

$$X(t,s,X(s,r,x)) = f(t,T(s,f(s,T(r,x)))) = f(t,T(r,x) \vee s) = f(t,T(r,x)) = X(t,r,x).$$

Let $Y(t,s) = \inf\{x \in [0,1]; X(t,s,x) = 1\}$. Then we see that $T(s,x) \geqq t$ iff $x \geqq Y(t,s)$, and that $Y(t,s)$ is $\mathcal{G}_s$-measurable.

## 3.3 A remark on support

Let $(\Omega, \mathcal{F}, P)$ be a probability measure, $\Theta$ be a non-empty open set in $\mathbf{R}^M$, and $\mathcal{M}_0$ be a Polish space. Also, let $\xi : [0,\infty) \times \Omega \to [0,\infty)$, $\theta : [0,\infty) \times \Omega \to \bar{\Theta}$, and $Y : [0,\infty) \times \Omega \to \mathcal{M}_0$ be measurable processes. Remind that $\bar{\Theta}$ is the closure of $\Theta$ in $\mathbf{R}^M$. We assume that $\theta(\cdot,\omega) \to \bar{\Theta}$ is continuous for all $\omega \in \Omega$ and that $P(\int_a^b \xi(t)dt > 0) = 1$ for any $a, b \geqq 0$ with $a < b$.

Let $\tilde{\Omega} = [0,\infty) \times \Omega$. Let $\nu_0$ be a probability measure on $[0,\infty)$ given by $\nu_0(dt) = e^{-t}dt$, and $\nu$ be a probability measure on $(\tilde{\Omega}, \mathcal{B}([0,\infty)) \times \mathcal{F})$ given by $\nu = \nu_0 \otimes P$. Then $\xi$, (resp. $\theta$, $Y$) can be regarded as a $[0,\infty)$ (resp. $\Theta$, $\mathcal{M}_0$)-valued random variable defined in a probability space $(\tilde{\Omega}, \mathcal{B}([0,\infty)) \times \mathcal{F}, \nu)$.

Let $\mu$ be a probability law of $(\xi, Y, \theta)$ and $\mu_\theta$ be a probability law of $\theta$ unde $\nu$. Then $\mu$ and $\mu_\theta$ be probability measures on $[0,\infty) \times \mathcal{M}_0 \times \bar{\Theta}$ and $\bar{\Theta}$ respectively. Let $\Gamma$ and $\Gamma_\theta$ be the support of probability measures $\mu$ and $\mu_\theta$ respectively. Then $\Gamma$ and $\Gamma_\theta$ are closed subsets of $[0,\infty) \times \mathcal{M}_0 \times \bar{\Theta}$ and $\bar{\Theta}$ respectively. Let $\pi : [0,\infty) \times \mathcal{M}_0 \times \bar{\Theta} \to \bar{\Theta}$ be a natural projection and let $\Gamma_0 = \pi((0,\infty) \times \mathcal{M}_0 \times \Theta)$.

Then we have the following.

**Proposition 3.3.1.** *The closure of* $\Gamma_0$ *contains* $\Gamma_\theta \cap \Theta$.

*Proof.* Let $\Phi : \tilde{\Omega} \to [0,\infty) \times \mathcal{M}_0 \times \bar{\Theta}$ be given by $\Phi(t,\omega) = (\xi(t,\omega), Y(t,\omega), \theta(t,\omega))$. Let $A = \Phi^{-1}(\Gamma)$. Then we have

$$1 = \nu(A) = \int_\Omega \nu_0(A_\omega) P(d\omega),$$

where $A_\omega = \{t \in [0,\infty);\ (t,\omega) \in A\}$. Let

$$B = \{\omega \in \Omega;\ \nu_0(A_\omega) = 1,\ \int_r^{r'} \xi(t,\omega)dt > 0 \text{ for any } r.r' \in \mathbf{Q} \text{ with } r < r'\ \}.$$

Then we see that $P(B) = 1$. Let $A' = A \cap ([0,\infty) \times B)$. Then we see that $\nu(A') = 1$. Let $\theta_0 \in \Gamma_\theta \cap \Theta$. Then for any $n \geqq 1$,

$$\nu(\{(t,\omega) \in A';\ |\theta(t,\omega) - \theta_0| < \frac{1}{2n}\}) > 0.$$

Therefore there is a $(t_n, \omega_n) \in A'$ such that $|\theta(t_n, \omega_n) - \theta_0| < 1/2n$. For any $m \geqq 1$, we see that $\int_{t_n}^{t_n+1/m} \xi(t, \omega_n)dt > 0$, and so there is a $s_{n,m} \in (t_n, t_n + 1/m) \cap A_{\omega_n}$ such that $\xi(s_{n,m}, \omega_n) > 0$. Since $\theta(t, \omega_n)$ is continuous in $t$, we see that there is a $m(n) \geqq 1$ such that $|\theta(s_{n,m(n)}, \omega_n) - \theta(t_n, \omega_n)| < 1/2n$. Now let $\xi_n = \xi(s_{n,m(n)}, \omega_n)$, $\theta_n = \theta(s_{n,m(n)}, \omega_n)$, and $y_n = Y(s_{n,m(n)}, \omega_n)$. Then we see that $(\xi_n, y_n, \theta_n) \in \Gamma$, $\xi_n > 0$, and $|\theta_n - \theta_0| < 1/n$. Since $\Theta$ is open, $\theta_n \in \Theta$ for sufficiently large $n$. So we have our assertion. □

## 3.4 Fundamental Relations

Let $(\Omega, \mathcal{F}, P, (W_t^k)_{k=1,\ldots,d}, (\tau_i)_{i=1,\ldots,N}, (\xi_i(t))_{i=1,\ldots,N}, \theta(t), \Theta, K)$ be a dynamical default time copula model as in Introduction. We also assume that $K|_{(0,1)^N \times \Theta}$ is $C^2$. We think about conditions which $K$ must satisfy.

By Proposition 3.2.1, we see that there are $f_i : [0,\infty) \times [0,\infty) \times \Omega \to (0,\infty)$, $i = 1, \ldots, N$, such that

$$f_i(t,s) = E[\exp(-\int_{t\wedge s}^{s} \xi_i(r)dr)|\mathcal{G}_t] \quad a.s. \qquad t,s \geqq 0,$$

$f_i(\cdot, *, \omega) : [0,\infty) \times [0,\omega)$, are continuous for any $\omega \in \Omega$, $f_i(t, s_1, \omega) > f_i(t, s_2, \omega) > 0$ for $s_2 > s_1 > t$, $\omega \in \Omega$, and

$$f(t,t,\omega) = 1, \quad \lim_{s\uparrow\infty} f(t,s,\omega) = 0, \qquad t \geqq 0, \ \omega \in \Omega.$$

Also by Corollary 3.2.3, we see that there are $\tilde{\sigma}_{i,k} : [0,\infty] \times [0,\infty) \times \Omega \to \mathbf{R}$, $k = 1, \ldots, d$, $i = 1, \ldots, N$, satisfying the following.

(1) $\tilde{\sigma}_{i,k}(t, \cdot, \omega) : [0,\infty] \to \mathbf{R}$, $k = 1, \ldots, d$, is continuous for any $t \in [0,\infty)$ and $\omega \in \Omega$.

(2) $\tilde{\sigma}_{i,k}(t,s,\omega) = 0$, $t \geqq s$, and $\lim_{s\to\infty} \tilde{\sigma}_{i,k}(t,s,\omega) = 0$ for any $t \in [0,\infty)$ and $\omega \in \Omega$.

(3) $\sigma_{i,k}(\cdot, s) : [0,\infty) \times \Omega \to \mathbf{R}$, $k = 1, \ldots, N$, is $\mathcal{G}$-progressively measurable for any $s \geqq 0$.

(4) For any $s > 0$

$$f_i(t,s) = f_i(0,s) + \int_0^{t\wedge s} \xi_i(r) f_i(r,s)dr + \sum_{k=1}^{d} \int_0^t \tilde{\sigma}_{i,k}(r,s)dW^k(r).$$

Let $T_i : [0,\infty) \times (0,1) \times \Omega \to (0,\infty)$, $i = 1,\dots,N$, be given by

$$T_i(t,x) = \inf\{s \geqq t, f_i(t,s) \leqq x\}, \qquad x \in [0,1).$$

Then by Proposition 3.2.4 we see that $T_i(t,\cdot,\omega) : (0,1) \to (0,\infty)$ is continuous and strictly decreasing, $\lim_{x\downarrow 0} T_i(t,x,\omega) = \infty$ and $\lim_{x\uparrow 1} T_i(t,x,\omega) = 0$ for any $t \geqq 0$ and $\omega \in \Omega$. Let $\sigma_{i,k} : [0,\infty) \times (0,1) \times \Omega \to \mathbf{R}$, $i = 1,\dots,N$, $k = 1,\dots,d$, be given by

$$\sigma_{i,k}(t,x) = \tilde{\sigma}_{i,k}(t, T_i(t,x)) \qquad t \geqq 0,\ x \in (0,1).$$

Then we see that

$$\lim_{x\downarrow 0} \sigma_{i,k}(t,x) = 0, \qquad \lim_{x\uparrow 1} \sigma_{i,k}(t,x) = 0.$$

So we can extend this $\sigma_{i,k}$ as a function $\sigma_{i,k} : [0,\infty) \times [0,1] \times \Omega \to \mathbf{R}$, for which $\sigma_{i,k}(t,\cdot,\omega) : [0,1] \to \mathbf{R}$ is continuous for any $t \geqq 0$, $\omega \in \Omega$, and $\sigma_{i,k}(t,0) = \sigma_{i,k}(t,1) = 0$.

Let $\nu$ be a probability measure on $(0,\infty) \times \Omega$ given by $\nu(dt, d\omega) = e^{-t} dt \otimes P(d\omega)$. By the assumption (SC), we see that

$$1_{\{\tau_i > t\}} f_i(t,s) = 1_{\{\tau_i > t\}} P(\tau_i > s | \mathcal{F}_t) \quad a.s. \quad s \geqq t,\ i = 1,\dots,N,$$

and

$$\prod_{i=1}^N 1_{\{\tau_i > t\}} \exp(-\sum_{i=1}^N \int_0^t \xi_i(r) dr) P(\tau_1 > s_1, \dots, \tau_N > s_n | \mathcal{F}_t)$$
$$= \prod_{i=1}^N 1_{\{\tau_i > t\}} E[\exp(-\sum_{i=1}^N \int_0^{s_i} \xi_i(r) dr) | \mathcal{G}_t] \ a.s.$$

for $t \in [0, \min_{i=1,\dots,N} s_i]$. So by the assumption (CP) we have

$$\prod_{i=1}^N 1_{\{\tau_i > t\}} 1_\Theta(\theta(t)) \exp(-\sum_{i=1}^N \int_0^t \xi_i(r) dr) K(f_1(t,s_1), \dots, f_N(t,s_N), \theta(t))$$

$$= \prod_{i=1}^N 1_{\{\tau_i > t\}} 1_\Theta(\theta(t)) E[\exp(-\sum_{i=1}^N \int_0^{s_i} \xi_i(r) dr) | \mathcal{G}_t] \ a.s.$$

for $t \in [0, \min_{i=1,\dots,N} s_i]$. Therefore by the assumption (PO), we have

$$1_\Theta(\theta(t)) \exp(-\sum_{i=1}^N \int_0^t \xi_i(r) dr) K(f_1(t,s_1), \dots, f_N(t,s_N), \theta(t))$$

$$= 1_\Theta(\theta(t))E[\exp(-\sum_{i=1}^N \int_0^{s_i} \xi_i(r)dr)|\mathcal{G}_t] \ a.s.$$

for $t \in [0, \min_{i=1,\dots,N} s_i]$.

Now let us take a non-empty open set $U$ in $\mathbf{R}^M$ such that $\bar{U} \subset \Theta$ and fix it for a while. For $T \geqq 0$, let $\tau_T^U : \Omega \to [0,\infty)$ be given by

$$\tau_T^U = \inf\{t \geqq T;\ \theta(t) \notin \bar{U}\} \wedge (T+1).$$

Then by the assumption (CP), we see that for any $s_1, \dots, s_N \geqq T$

$$1_U(\theta(T)) \exp(-\sum_{i=1}^N \int_0^{t\wedge\tilde{\tau}} \xi_i(r)dr) K(f_1(t\wedge\tilde{\tau}, s_1), \dots, f_N(t\wedge\tilde{\tau}, s_N), \theta(t\wedge\tilde{\tau})),\ t \in [T, T+1],$$

is a $\{\mathcal{G}_t\}_{t\in[T,T+1]}$-maringale, where $\tilde{\tau} = \tau_T^U \wedge \min_{i=1,\dots,N} s_i$.

Note that $\theta(t)$ is an Ito process satisfying Equation (3.1.1). Therefore, applying Ito's formula and comparing finite total variation process, we have for any $s_1, \dots, s_N \geqq T$

$$\begin{aligned}
1_U(\theta(T))1_{[t,T+1]}(\tau_T^U \wedge \min_{i=1,\dots,N} s_i)\{-(\sum_{i=1}^N \xi_i(t))K(f_1(t,s_1), \dots, f_N(t,s_N), \theta(t)) \\
+\sum_{i=1}^N \xi_i(t) f_i(t,s_i)\frac{\partial K}{\partial x_i}(f_1(t,s_1), \dots, f_N(t,s_N), \theta(t)) \\
+\sum_{j=1}^M b^j(t)\frac{\partial K}{\partial \theta^j}(f_1(t,s_1), \dots, f_N(t,s_N), \theta(t)) \\
+\frac{1}{2}\sum_{i,i'=1}^N \sum_{k=1}^d \tilde{\sigma}_{i,k}(t,s_i)\tilde{\sigma}_{i',k}(t,s_{i'})\frac{\partial^2 K}{\partial x_i \partial x_{i'}}(f_1(t,s_1), \dots, f_N(t,s_N), \theta(t)) \\
+\frac{1}{2}\sum_{j,j'=1}^M \sum_{k=1}^d \eta_k^j(t)\eta_k^{j'}(t)\frac{\partial^2 K}{\partial \theta_j \theta_{j'}}(f_1(t,s_1), \dots, f_N(t,s_N), \theta(t)) \\
+\sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^d \tilde{\sigma}_{i,k}(t,s_i)\eta_k^j(t)\frac{\partial^2 K}{\partial x_i \theta_j}(f_1(t,s_1), \dots, f_N(t,s_N), \theta(t))\} = 0
\end{aligned} \tag{3.4.1}$$

for $\nu - a.e.(t,\omega) \in (T, T+1) \times \Omega$.

Note that the left hand side of Equation (3.4.1) is right continuous in $s_1, \ldots, s_N$. So we see that there is an $B_T^U \in \mathcal{B}((T, T+1)) \times \mathcal{F}$ such that $\nu(((T, T+1) \times \Omega) \setminus B_T^U) = 0$ and Equation (3.4.1) holds for all $(t, \omega) \in B_T^U$ and $s_1, \ldots, s_N \in [T, \infty)$.

Also, substituting $s_i = T_i(t, x_i)$, $i = 1, \ldots, N$, to Equation (3.4.1), we see that for all $(t, \omega) \in B_T^U$

$$
\begin{aligned}
&1_U(\theta(T)) 1_{[t,T+1]}(\tau_T^U)\{-(\sum_{i=1}^N \xi_i(t)) K(x_1, \ldots, x_N, \theta(t)) \\
&+ \sum_{i=1}^N \xi_i(t) x_i \frac{\partial K}{\partial x_i}(x_1, \ldots, x_N, \theta(t)) + \sum_{j=1}^M b^j(t) \frac{\partial K}{\partial \theta^j}(x_1, \ldots, x_N, \theta(t)) \\
&+ \frac{1}{2} \sum_{i,i'=1}^N \sum_{k=1}^d \sigma_{i,k}(t, x_i) \sigma_{i',k}(t, x_{i'}) \frac{\partial^2 K}{\partial x_i \partial x_{i'}}(x_1, \ldots, x_N, \theta(t)) \\
&+ \frac{1}{2} \sum_{j,j'=1}^M \sum_{k=1}^d \eta_k^j(t) \eta_k^{j'}(t) \frac{\partial^2 K}{\partial \theta_j \theta_{j'}}(x_1, \ldots, x_N, \theta(t)) \\
&+ \sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^d \sigma_{i,k}(t, x_i) \eta_k^j(t) \frac{\partial^2 K}{\partial x_i \theta_j}(x_1, \ldots, x_N, \theta(t))\} = 0
\end{aligned}
\tag{3.4.2}
$$

for any $x_1, \ldots, x_N \in (0, 1)$.

Let $J_2 = \{(j, j') \in \{0, 1, \ldots, M\} \times \{1, \ldots, M\};\ j \leqq j'\}$. We define linear operators $S_{ii'}^{(2)}$, $i, i' = 1, \ldots, N$, $i < i'$, $S_{ij}^{(1)}$, $i = 1, \ldots, N$, $j = 0, 1, \ldots, M$, and $S_{jj'}^{(0)}$, $(j, j') \in J_2$, from $C^2((0,1)^N \times \Theta)$ to $C((0,1)^N \times \Theta)$ by

$$(S_{ii'}^{(2)} F)(x, \theta) = \frac{\partial^2 F}{\partial x_i \partial x_{i'}}(x, \theta), \qquad 1 \leqq i < i' \leqq N,$$

$$(S_{i0}^{(1)} F)(x, \theta) = \frac{\partial^2 F}{(\partial x_i)^2}(x, \theta), \qquad i = 1, \ldots, N,$$

$$(S_{ij}^{(1)} F)(x, \theta) = \frac{\partial^2 F}{\partial x_i \partial \theta_j}(x, \theta), \qquad i = 1, \ldots, N,\ j = 1, \ldots, M,$$

$$(S_{jj'}^{(0)} F)(x, \theta) = \frac{\partial^2 F}{\partial \theta_j \partial \theta_{j'}}(x, \theta), \qquad 1 \leqq j \leqq j' \leqq N,$$

$$(S_{0j'}^{(0)} F)(x, \theta) = \frac{\partial F}{\partial \theta_{j'}}(x, \theta), \qquad 1 \leqq j' \leqq N,$$

for any $F \in C^2((0,1)^N \times \Theta)$.

Also, let us define $a^{(2)}_{ii'} : [0,\infty) \times [0,1] \times [0,1] \times \Omega \to \mathbf{R}$, $i, i' = 1, \dots, N$, $i < i'$, $a^{(1)}_{ij} : [0,\infty) \times [0,1] \times \Omega \to \mathbf{R}$, $i = 1, \dots, N$, $j = 0, 1, \dots, M$, and $a^{(0)}_{jj'} : [0,\infty) \times [0,1] \times \Omega \to \mathbf{R}$, $(j, j') \in J_2$, by the following.

$$a^{(2)}_{ii'}(t, x_i, x_{i'}) = \sum_{k=1}^{d} \sigma_{i,k}(t, x_i)\sigma_{i',k}(t, x_{i'}), \qquad 1 \leqq i < i' \leqq N,$$

$$a^{(1)}_{i0}(t, x_i) = \frac{1}{2}\sum_{k=1}^{d} \sigma_{i,k}(t, x_i)^2 \qquad i = 1, \dots, N,$$

$$a^{(1)}_{ij}(t, x_i) = \sum_{k=1}^{d} \hat{\sigma}_{i,k}(t, x_i)\eta_{j,k}(t), \qquad i = 1, \dots, N, \ j = 1, \dots, M,$$

$$a^{(0)}_{jj'}(t) = \sum_{k=1}^{d} \eta_{j,k}(t)\eta_{j',k}(t), \qquad j, j' = 1, \dots, M, \ \text{with } j < j',$$

$$a^{(0)}_{jj}(t) = \frac{1}{2}\sum_{k=1}^{d} \eta_{j,k}(t)^2, \qquad j = 1, \dots, M,$$

and

$$a^{(0)}_{0j'}(t) = b_{j'}(t), \qquad j' = 1, \dots, M.$$

Then we have for all $(t, \omega) \in B^U_T$

$$\begin{aligned} &1_U(\theta(T))1_{[t,T+1]}(\tau^U_T)\{\sum_{i=1}^{N} \xi_i(t)(x_i \frac{\partial K}{\partial x_i}(x, \theta(t)) - K(x, \theta(t))) \\ &+ \sum_{1 \leqq i < i' \leqq N} a^{(2)}_{ii'}(t, x_i, x_{i'})(S^{(2)}_{ii'}K)(x, \theta(t)) + \sum_{i=1}^{N}\sum_{j=0}^{d} a^{(1)}_{ij}(t, x_i)(S^{(1)}_{ij}K)(x, \theta(t)) \\ &+ \sum_{(j,j') \in J_2} a^{(0)}_{jj'}(t)(S^{(0)}_{jj'}K)(x, \theta(t))\} = 0, \qquad x_1, \dots, x_N \in (0,1). \end{aligned} \tag{3.4.3}$$

Now let $U_n$, $n = 1, 2, \dots$, be non-empty open sets in $\mathbf{R}^M$ such that the closure of $U_n$ is contained in $\Theta$ for each $n$, and $\bigcup_{n=1}^{\infty} U_n = \Theta$. Since $\theta(t)$ is continuous in $t$, we see that

$$\{(t, \omega) \in (0, \infty) \times \Omega; \ \theta(t, \omega) \in \Theta\}$$

$$= \bigcup_{T\in\mathbf{Q}_{\geqq 0}} \bigcup_{n=1}^{\infty} \{(t,\omega)\in (T,T+1)\times\Omega;\ \theta(T,\omega)\in U_n,\ t\leqq \tau_T^{U_n}(\omega)\}.$$

So let

$$B_0 = \bigcup_{T\in\mathbf{Q}_{\geqq 0}} \bigcup_{n=1}^{\infty} (B_T^{U_n}\cap\{(t,\omega)\in (T,T+1)\times\Omega;\ \theta(T,\omega)\in U_n,\ t\leqq \tau_T^{U_n}(\omega)\})$$

and $B_1 = B_0\cup\{(t,\omega)\in(0,\infty)\times\Omega;\ \theta(t,\omega)\notin\Theta\}$. Then we see that $\nu(B_1)=1$. Also, we see that for all $(t,\omega)\in B_1$

$$1_\Theta(\theta(t))\{\sum_{i=1}^N \xi_i(t)(x_i\frac{\partial K}{\partial x_i}(x,\theta(t)) - K(x,\theta(t)))$$

$$+\sum_{1\leqq i<i'\leqq N} a_{ii'}^{(2)}(t,x_i,x_{i'})(S_{ii'}^{(2)}K)(x,\theta(t)) + \sum_{i=1}^N\sum_{j=0}^d a_{ij}^{(1)}(t,x_i)(S_{ij}^{(1)}K)(x,\theta(t))$$

$$+\sum_{(j,j')\in J_2} a_{jj'}^{(0)}(t)(S_{jj'}^{(0)}K)(x,\theta(t))\} = 0,\qquad x_1,\dots,x_N\in(0,1). \tag{3.4.4}$$

Let $\mathcal{C}_2$ be the set of continuous functions $a:[0,1]\times[0,1]\to\mathbf{R}$ with $a(0,x)=a(1,x)$ $=a(1,x)=a(x,1)=0$, $x\in[0,1]$, and $\mathcal{C}_1$ be the set of continuous functions $\tilde{a}:[0,1]\to\mathbf{R}$ with $a(0)=a(1)=0$. Then we see that $a_{ii'}^{(2)}(t,\cdot,*)\in\mathcal{C}^2$, $1\leqq i<i'\leqq N$, $a_{ij}^{(1)}\in\mathcal{C}^1$, $i=1,\dots,N$, $j=0,1,\dots,M$, for $\nu-a.e.$ $(t,\omega)$.

Also, let $\mathcal{M}=(\mathcal{C}_2)^{N(N-1)/2}\times(\mathcal{C}_1)^{N(1+M)}\times\mathbf{R}^{J_2}$. Then $\mathcal{M}$ is a Poilish space. Let $Y(t,\omega)$
$=((a_{ii'}^{(2)}(t,\cdot,*,\omega))_{1\leqq i<i'\leqq N}, (a_{ij}^{(1)}(t,\cdot,\omega))_{i=1,\dots,N,j=0,1,\dots,M}, (a_{jj'}^{(0)}(t,\omega))_{(j,j')\in J_2})$. Then we see that $Y(t,\omega)\in\mathcal{M}$ for $\nu-a.e.$ $(t,\omega)$. Therefore under the probability measure $\nu$ on $[0,\infty)\times\Omega$, $((\xi_i(t,\omega))_{i=1,\dots,N}, Y(t,\omega),\theta(t,\omega))$ is a $[0,\infty)^N\times\mathcal{M}\times\bar{\Theta}$-valued ranndom variable. Let $\Gamma$ be the support of the probability law of this random variable.

Then we have the following.

**Proposition 3.4.1.** *If* $((\xi_i)_{i=1,\dots,N}, (\tilde{a}_{ii'}^{(2)})_{1\leqq i<i'\leqq N}, (\tilde{a}_{ij}^{(1)})_{i=1,\dots,N,j=0,1,\dots,M}, (\tilde{a}_{jj'}^{(0)})_{jj'})_{(j,j')\in J_2},\theta)$ *belongs to* $\Gamma$, *and* $\theta\in\Theta$,*then*

$$\sum_{i=1}^N \xi_i(x_i\frac{\partial K}{\partial x_i}(x,\theta) - K(x,\theta))$$

$$+\sum_{1\leqq i<i'\leqq N}\tilde{a}^{(2)}_{ii'}(x_i,x_{i'})(S^{(2)}_{ii'}K)(x,\theta)+\sum_{i=1}^{N}\sum_{j=0}^{d}\tilde{a}^{(1)}_{ij}(x_i)(S^{(1)}_{ij}K)(x,\theta)$$

$$+\sum_{(j,j')\in J_2}\tilde{a}^{(0)}_{jj'}(S^{(0)}_{jj'}K)(x,\theta)=0,\ \textit{for all } x=(x_1,\ldots,x_N)\in(0,1)^N$$

*and*

$$\sum_{1\leqq i<i'\leqq N}\tilde{a}^{(2)}_{ii'}(x_i,x_{i'})z_iz_{i'}+\sum_{i=1}^{N}\tilde{a}^{(1)}_{i0}(x_i)z_i^2+\sum_{i=1}^{N}\sum_{j=1}^{d}\tilde{a}^{(1)}_{ij}(x_i)z_iy_j+\sum_{(j,j')\in J_2}\tilde{a}^{(0)}_{jj'}y_jy_{j'}\geqq 0$$

*for all* $x=(x_1,\ldots,x_N)\in(0,1)^N$ *and* $z_1,\ldots,z_N,y_1,\ldots,y_M\in\mathbf{R}$.

Let $\Gamma_\theta$ be the support of $\theta(t,\omega)$ under $\nu$. Let $\pi:[0,\infty)^N\times\mathcal{M}\times\bar{\Theta}\to\bar{\Theta}$ be the natural projection, and let $\Gamma_0=\pi(\Gamma\cap((0,\infty)\times[0,\infty)^{N-1}\times\mathcal{M}\times\Theta))$. Then Proposition 3.3.1 implies that the closure of $\Gamma_0$ contains $\Gamma_\theta\cap\Theta$.

Then we have the following from the previous Proposition.

**Lemma 3.4.2.** *Let $N\geqq 2$, $M\geqq 1$, $\Theta$ be an open set in $\mathbf{R}^M$, and $K\in C([0,1]^N\times\Theta;[0,1])$. Assume that $K$ is an admissible family of copula functions and that $K|_{(0,1)^N\times\Theta}$ is $C^2$. Then there is a subset $A$ of $\Theta$ such that the closure of $A$ contains non-void open set in $\Theta$, and for any $\theta\in A$, there are $\xi_1>0$, $\xi_i\geqq 0$, $i=2,\ldots,N$, $\tilde{a}^{(2)}_{ii'}\in\mathcal{C}_1$, $1\leqq i<i'\leqq N$, $\tilde{a}^{(1)}_{ij}\in\mathcal{C}_1$, $i=1,\ldots,N$, $j=0,1,\ldots,M$, and $\tilde{a}^{(0)}_{jj'}\in\mathbf{R}$, $(j,j')\in J_2$, such that*

$$\sum_{i=1}^{N}\xi_i(x_i\frac{\partial K}{\partial x_i}(x,\theta)-K(x,\theta))$$

$$+\sum_{1\leqq i<i'\leqq N}\tilde{a}^{(2)}_{ii'}(x_i,x_{i'})(S^{(2)}_{ii'}K)(x,\theta)+\sum_{i=1}^{N}\sum_{j=0}^{d}\tilde{a}^{(1)}_{ij}(x_i)(S^{(1)}_{ij}K)(x,\theta)$$

$$+\sum_{(j,j')\in J_2}\tilde{a}^{(0)}_{jj'}(S^{(0)}_{jj'}K)(x,\theta)=0,\ \textit{for all } x=(x_1,\ldots,x_N)\in(0,1)^N$$

*and*

$$\sum_{1\leqq i<i'\leqq N}\tilde{a}^{(2)}_{ii'}(x_i,x_{i'})z_iz_{i'}+\sum_{i=1}^{N}\tilde{a}^{(1)}_{i0}(x_i)z_i^2+\sum_{i=1}^{N}\sum_{j=1}^{d}\tilde{a}^{(1)}_{ij}(x_i)z_iy_j+\sum_{(j,j')\in J_2}\tilde{a}^{(0)}_{jj'}y_jy_{j'}\geqq 0$$

*for all* $x=(x_1,\ldots,x_N)\in(0,1)^N$ *and* $z_1,\ldots,z_N,y_1,\ldots,y_M\in\mathbf{R}$.

## 3.5 Verification

Let $N, M \geqq 1$, and $\Theta$ be an open set in $\mathbf{R}^M$. Let $n \geqq 1$, and $\vec{z} = (z_{ik})_{i=1,\dots,N,k=1,\dots,n} \in (0,1)^{nN}$.

For $\vec{k} = (k_1, \dots, k_N) \in \{1, \dots, n\}^N$, and $\vec{z} \in (0,1)^{nN}$, let $Z_i(\vec{z}, \vec{k}) = z_{ik_i}$, $i = 1, \dots, N$, and $\vec{Z}(\vec{z}, \vec{k}) = (z_{1k_1}, \dots, z_{Nk_N}) \in (0,1)^N$.

Let $K \in C([0,1]^N \times \Theta; [0,1])$ be an admissible family of copula functions, and assume that $K|_{(0,1)^N \times \Theta}$ is $C^2$. Now let $A$ be a subset in $\Theta$ as in Lemma 3.4.2 Then for any $\theta \in A$, there are $\xi_i$, $i = 1, \dots, N$, $\tilde{a}^{(2)}_{ii'}$, $1 \leqq i < i' \leqq N$, $\tilde{a}^{(1)}_{ij}$, $i = 1, \dots, N, j = 0, 1, \dots, M$, and $\tilde{a}^{(0)}_{jj'}$, $(j, j') \in J_2$, be as in Lemma 3.4.2. Then we have

$$\sum_{i=1}^N \xi_i(Z_i(\vec{z}, \vec{k})) \frac{\partial K}{\partial x_i}(\vec{Z}(\vec{z}, \vec{k}), \theta) - K(\vec{Z}(\vec{z}, \vec{k}), \theta))$$

$$+ \sum_{1 \leqq i < i' \leqq N} \tilde{a}^{(2)}_{ii'}(Z_i(\vec{z}, \vec{k}), Z_{i'}(\vec{z}, \vec{k}))(S^{(2)}_{ii'}K)(\vec{Z}(\vec{z}, \vec{k}), \theta) + \sum_{i=1}^N \sum_{j=0}^d \tilde{a}^{(1)}_{ij}(Z_i(\vec{z}, \vec{k}))(S^{(1)}_{ij}K)(\vec{Z}(\vec{z}, \vec{k}), \theta)$$

$$+ \sum_{(j,j') \in J_2} \tilde{a}^{(0)}_{jj'}(S^{(0)}_{jj'}K)(\vec{Z}(\vec{z}, \vec{k}), \theta) = 0.$$

That is

$$\begin{aligned} &\sum_{i=1}^N \xi_i(Z_i(\vec{z}, \vec{k}) \frac{\partial K}{\partial x_i}(\vec{Z}(\vec{z}, \vec{k}), \theta) - K(\vec{Z}(\vec{z}, \vec{k}), \theta)) \\ &+ \sum_{p,q=1}^n \sum_{1 \leqq i < i' \leqq N} \tilde{a}^{(2)}_{ii'}(z_{ip}, z_{i'q})\delta_{p,k_i}\delta_{q,k_{i'}}(S^{(2)}_{ii'}K)(\vec{Z}(\vec{z}, \vec{k}), \theta) \\ &+ \sum_{p=1}^n \sum_{i=1}^N \sum_{j=0}^d \tilde{a}^{(1)}_{ij}(z_{ip})\delta_{p,k_i}(S^{(1)}_{ij}K)(\vec{Z}(\vec{z}, \vec{k}), \theta) \\ &+ \sum_{(j,j') \in J_2} \tilde{a}^{(0)}_{jj'}(S^{(0)}_{jj'}K)(\vec{Z}(\vec{z}, \vec{k}), \theta) = 0, \qquad \vec{k} \in \{1, \dots, n\}^N. \end{aligned} \tag{3.5.1}$$

Let

$$C^{(2)}_n = \{(i, i') \in \{1, 2, \dots, N\}^2;\ i < i'\} \times \{1, 2, \dots, n\}^2$$

and

$$C^{(1)}_n = \{1, 2, \dots, N\} \times \{0, 1, \dots, M\} \times \{1, 2, \dots, n\}.$$

For any $G \in C^2((0,1)^N \times \Theta)$, $n \geqq 1$, and $\vec{k} \in \{1,\dots,n\}^N$, we define continuous functions defined in $(0,1)^{nN} \times \Theta$, $(M_i^{(n)I}G)(\cdot,\vec{k})$, $i = 1,\dots,N$, $(M_{ii'pq}^{(n)(2)}G)(\cdot,\vec{k})$, $(i,i',p,q) \in C_n^{(2)}$, $(M_{ijp}^{(n)(1)}G)(\cdot,\vec{k})$, $(i,j,p) \in C_n^{(1)}$, $(M_{jj'}^{(n)(0)}G)(\cdot,\vec{k})$, $(j,j') \in J_2$, by

$$(M_i^{(n)I}G)(\vec{z},\theta,\vec{k}) = Z_i(\vec{z},\vec{k})\frac{\partial G}{\partial x_i}(\vec{Z}(\vec{z},\vec{k}),\theta) - G(\vec{Z}(\vec{z},\vec{k}),\theta) \quad i = 1,\dots,N,$$

$$(M_{ii'pq}^{(n)(2)}G)(\vec{z},\theta,\vec{k}) = \delta_{p,k_i}\delta_{q,k_{i'}}(S_{ii'}^{(2)}G)(\vec{Z}(\vec{z},\vec{k}),\theta) \quad (i,i',p,q) \in C_n^{(2)},$$

$$(M_{ijp}^{(n)(1)}G)(\vec{z},\theta,\vec{k}) = \delta_{p,k_i}(S_{ij}^{(1)}G)(\vec{Z}(\vec{z},\vec{k}),\theta), \quad (i,j,p) \in C_n^{(1)},$$

$$(M_{jj'}^{(n)(0)}G)(\vec{z},\theta,\vec{k}) = (S_{jj'}^{(0)}G)(\vec{Z}(\vec{z},\vec{k}),\theta), \quad (j,j') \in J_2,$$

for any $\vec{z} \in (0,1)^{nN}$ and $\theta \in \Theta$.

Let $C_{n0} = C_n^{(2)} \cup C_n^{(1)} \cup J_2$, and $C_n = \{1,\dots,N\} \cup C_{n0}$. Note that the cardinal $\#(C_{n0})$ of $C_{n0}$ is equal to $n^2N(N-1)/2 + nN(M+1) + M(M+3)/2$, and $\#(C_n) = N + \#(C_{n0})$.

For any $G \in C^2((0,1)^N \times \Theta)$, $n \geqq 1$, and $\gamma \in C_n$ we define a continuous function $(\vec{M}^{(n)}G)_\gamma : (0,1)^{nN} \times \Theta \to \mathbf{R}^{\{1,\dots,n\}^N}$ by

$$(\vec{M}^{(n)}G)_\gamma(\vec{z},\theta)$$

$$= \begin{cases} ((M_{ii'pq}^{(n)(2)}G)(\vec{z},\theta,\vec{k}))_{\vec{k}\in\{1,\dots,n\}^N} & \text{if } \gamma = (i,i',p,q) \in C_n^{(2)}, \\ (M_{ijp}^{(n)(1)}G)(\vec{z},\theta,\vec{k}))_{\vec{k}\in\{1,\dots,n\}^N} & \text{if } \gamma = (i,j,p) \in C_n^{(1)}, \\ (M_{jj'}^{(n)(0)}G)(\vec{z},\theta,\vec{k}))_{\vec{k}\in\{1,\dots,n\}^N} & \text{if } \gamma = (j,j') \in J_2, \\ ((M_i^{(n)I}G))(\vec{z},\theta,\vec{k}))_{\vec{k}\in\{1,\dots,n\}^N} & \text{if } \gamma = i \in \{1,\dots,N\}. \end{cases}$$

For any $G \in C^2((0,1)^N \times \Theta)$, $n \geqq 1$, $\vec{z} \in (0,1)^{nN}$ and $\theta \in \Theta$, let $V_n(G,\vec{z},\theta)$ (resp. $V_{n0}(G,\vec{z},\theta)$) be the vector subspace of $\mathbf{R}^{\{1,\dots,n\}^N}$ spaned by $\{(\vec{M}^{(n)}G)_\gamma(\vec{z},\theta);\ \gamma \in C_n\}$ (resp. $\{(\vec{M}^{(n)}G)_\gamma(\vec{z},\theta);\ \gamma \in C_{n0}\}$). Also, let $N_{(n)}(G,\vec{z},\theta)$ be a vector space in $\mathbf{R}^N$ given by

$$N_{(n)}(G,\vec{z},\theta) = \{(v_1,\dots,v_N) \in \mathbf{R}^N;\ \sum_{i=1}^n v_i(\vec{M}^{(n)}G)_i(\vec{z},\theta) \in V_{n0}(G,\vec{z},\theta)\}.$$

Then we have $N_{(n)}(K,\vec{z},\theta) \cap [0,\infty)^N \neq \{0\}$, for any $\theta \in A$.

Therefore we have the following.

**Lemma 3.5.1.** *Let $N \geqq 2$, $M \geqq 1$, and $\Theta$ be an open subset of $\mathbf{R}^M$ Let $K \in C([0,1]^N \times \Theta; [0,1])$. Assume that $K$ is an admissible family of copula functions, and that $K|_{(0,1)^N \times \Theta}$ is $C^2$. Then there is a subset $A$ of $\Theta$ such that the closure of $A$ contains non-void open set in $\Theta$, and for any $\theta \in A$ and $\vec{z} \in (0,1)^{nN}$, $N_{(n)}(K, \vec{z}, \theta) \cap [0,\infty)^N \neq \{0\}$.*

As a corollary we have the following.

**Corollary 3.5.2.** *Let $N \geqq 2$, $M \geqq 1$, and $\Theta$ be an open subset of $\mathbf{R}^M$. Let $K \in C([0,1]^N \times \Theta; [0,1])$. Assume that $K$ is an admissible family of copula functions, and that $K|_{(0,1)^N \times \Theta}$ is $C^2$. Then for any $n \geqq 1$ and $\vec{z} \in (0,1)^{nN}$, there is a non-void open subset $U$ of $\Theta$ such that*

$$\dim V_n(K, \vec{z}, \theta) \leqq \#(C_n) - 1, \qquad \theta \in U.$$

*Proof.* Since $\dim V_n(K, \vec{z}, \theta) \leqq n^N$, the assertion is obvious in the case that $n^N \leqq \#(C_n) - 1$. So assume that $n^N \geqq \#(C_n)$.

Let $A$ be a subset in $\Theta$ as in Lemma 3.5.1. It is easy to see that for any $\theta \in A$

$$\dim V_n(K, \vec{z}, \theta) = \dim V_{n0}(K, \vec{z}, \theta) + N - \dim N_{(n)}(G, \vec{z}, \theta) \leqq \#(C_n) - 1.$$

Let $H$ be the set of injections from $C_n$ to $\{1, \dots, n\}^N$ and let

$$\varphi(\theta) = \sum_{h \in H} \det((\vec{M}^{(n)} G)_{\gamma_1}(\vec{z}, \theta, h(\gamma_2)))_{\gamma_1, \gamma_2 \in C_n})^2, \quad \theta \in \Theta.$$

Then we see $\varphi(\theta) = 0$ for $\theta \in A$. Since $\varphi : \theta \to \mathbf{R}$ is continuous, we see that $\varphi(\theta) = 0$ for $\theta \in \bar{A}$. So we see that

$$\dim V_n(K, \vec{z}, \theta) \leqq \#(C_n) - 1, \qquad \theta \in \bar{A}.$$

This implies our assertion. □

## 3.6 Proof of Theorem 3.1.1

Now let $N \geqq 2$, $M \geqq 1$ and $n \geqq 1$. We say that $h : C_n^{(2)} \cup C_n^{(1)} \to \{1, \dots, n\}^N$ is a matching map, if $h$ is injective and satisfying the following.

$$h((i, i', p, q))_i = p, \quad h((i, i', p, q))_{i'} = q \quad \text{for any } (i, i', p, q) \in C_n^{(2)},$$

and

$$h((i,j,p))_i = p \quad \text{for any } (i,j,p) \in C_n^{(1)}.$$

**Proposition 3.6.1.** *Let $N \geqq 3$. Assume that there is a matching map $h_0 : C_n^{(2)} \cup C_n^{(1)} \to \{1,\dots,n\}^N$, and that $\#(C_n) \leqq n^N$. Let $0 < c_{i1} < c_{i2} < \dots < c_{in} < 1$, $i = 1,\dots,N$, $\vec{c} = (c_{ik})_{i=1,\dots,N,k=1,\dots,n} \in (0,1)^{nN}$, and $\theta_0 \in \mathbf{R}^M$. Then there is a $K \in \mathcal{C}_{(N)}(\mathbf{R}^M)$ such that* $\dim V_n(K,\vec{c},\theta_0) = \#(C_n)$.

*Proof.* From the assumption, there is an injective map $h : C_n \to \{1,\dots,n\}^N$ such that the restriction of $h$ to $C_n^{(2)} \cup C_n^{(1)}$ is equal to $h_0$. Note that $\vec{Z}(\vec{c},\vec{k})$, $\vec{k} \in \{1,\dots,n\}^N$, are distinct points. Let

$$\varepsilon_0 = \min\{|\vec{Z}(\vec{c},\vec{k}) - \vec{Z}(\vec{c},\vec{k}')|;\ \vec{k},\vec{k}' \in \{1,\dots,n\}^N, \vec{k} \neq \vec{k}'\},$$

$$\varepsilon_1 = \min\{c_{i1};\ i = 1,\dots,N\} \wedge \min\{1 - c_{in};\ i = 1,\dots,N\},$$

and $\varepsilon = \varepsilon_0 \wedge \varepsilon_1$. Let $\varphi_0 \in C_0^\infty(\mathbf{R}^N)$ and $\varphi_1 \in C_0^\infty(\mathbf{R}^M)$ such that $\varphi_0(x) = 1$, $|x| < \varepsilon/3$, $\varphi_0(x) = 0$, $|x| > 2\varepsilon/3$, and $\varphi_1(\theta) = 1$, $|\theta| < 1$.

Let $F : \mathbf{R}^N \times \mathbf{R}^M \times C_n \to \mathbf{R}$ be given by the following.

$$F(x,\theta,i) = -\varphi_0(x - \vec{Z}(\vec{c},h(i))), \qquad i \in I,$$

$$F(x,\theta,(0,j)) = \varphi_0(x - \vec{Z}(\vec{c},h((0,j)))(\theta_j - \theta_{0j})\varphi_1(\theta - \theta_0), \qquad j = 1,\dots,M,$$

$$F(x,\theta,(j,j)) = \frac{1}{2}\varphi_0(x - \vec{Z}(\vec{c},h((j,j)))(\theta_j - \theta_{0j})^2\varphi_1(\theta - \theta_0), \qquad j = 1,\dots,M,$$

$$F(x,\theta,(j,j')) = \varphi_0(x - \vec{Z}(\vec{c},h((j,j')))(\theta_j - \theta_{0j})(\theta_{j'} - \theta_{0j'})\varphi_1(\theta - \theta_0), \qquad 1 \leqq j < j' \leqq M,$$

$$F(x,\theta,(i,i',p,q))$$

$$= \varphi_0(x - \vec{Z}(\vec{c},h((i,i',p,q)))(x_i - \vec{Z}(\vec{c},h((i,i',p,q))_i)(x_{i'} - \vec{Z}(\vec{c},h((i,i',p,q))_{i'}),$$

$(i,i',p,q) \in C_n^{(2)}$,

$$F(x,\theta,(i,0,p)) = \frac{1}{2}\varphi_0(x - \vec{Z}(\vec{c},h((i,0,p)))(x_i - \vec{Z}(\vec{c},h((i,0,p))_i)^2, \qquad (i,0,p) \in C_n^{(1)},$$

$$F(x,\theta,(i,j,p))$$

$$= \varphi_0(x - \vec{Z}(\vec{c},h((i,j,p)))(x_i - \vec{Z}(\vec{c},h((i,j,p,q))_i)(\theta_j - \theta_{0j})\varphi_1(\theta - \theta_0), \quad (i,j,p) \in C_n^{(1)}, j \geqq 1.$$

Then we see that $(\vec{M}^{(n)}F(\cdot,u))_\gamma(\vec{c},\theta_0,h(\alpha)) = \delta_{\gamma\alpha}$, $\gamma,\alpha \in C_n$, and that $(\vec{M}^{(n)}F(\cdot,u))_\gamma(\vec{c},\theta_0,\vec{k})$ $= 0$, $\gamma \in C_n$, $\vec{k} \in \{1,\dots,n\}^N$, with $\vec{k} \neq h(\gamma)$. Now let $F_0 \in C_0^\infty(\mathbf{R}^N \times \mathbf{R}^M)$ be given by

$$F_0(\cdot,*) = \sum_{\gamma \in C_n} F(\cdot,*,\gamma).$$

Then we see that derivatives of $F_0$ of any order are bounded functions defined in $\mathbf{R}^N \times \mathbf{R}^M$,

$$\det((\vec{M}^{(n)}F_0)_\gamma(\vec{c},\theta_0,h(\alpha))_{\gamma,\alpha\in C_n}) = 1,$$

and $F_0(x_1,\dots,x_N,\theta) = 0$, if $x_i < \varepsilon/3$ or $x_i > 1-(\varepsilon/3)$ for some $i = 1,\dots,N$.

Let

$$f_0(x,\theta) = \frac{\partial^N F_0}{\partial x_1 \dots \partial x_N}(x,\theta)$$

Then we have

$$F_0(x,\theta) = \int_0^{x_1} \cdots \int_0^{x_N} f_0(y_1,\dots,y_N,\theta)dy_1 \dots dy_N.$$

Let $c = \sup\{|f_0(x,\theta)|;\ (x,\theta) \in \mathbf{R}^N \times \mathbf{R}^M\} < \infty$, and let

$$\begin{aligned} G_s(x,\theta) &= \int_0^{x_1} \cdots \int_0^{x_N} (1 + sf_0(y_1,\dots,y_N,\theta))dy_1 \dots dy_N \\ &= x_1 \cdots x_N + sF_0(x_1,\dots,x_N.\theta) \end{aligned}$$

for $s \in \mathbf{R}$, $x \in [0,1]^N$, $\theta \in \mathbf{R}^M$. Then

$$p(s) = \det(((\vec{M}^{(n)}G_s)_\gamma(\vec{c},\theta_0,h(\alpha)))_{\gamma,\alpha\in C_n})$$

is a polynomial in $s$ and

$$\lim_{s\to\infty} s^{-\#(C_n)}p(s) = 1.$$

Therefore there is a $\tilde{s}$ with $0 < \tilde{s} < 1/(2c+1)$ such that

$$\det(((\vec{M}^{(n)}G_{\tilde{s}})_\gamma(\vec{c},\theta_0,h(\alpha)))_{\gamma,\alpha\in C_n}) \neq 0.$$

Note that $1 + \tilde{s}f_0(y_1,\dots,y_N,\theta) > 1/(2c+1)$. So it is easy to see that $G_{\tilde{s}}(\cdots,\theta)$ is a copula function for all $\theta \in \mathbf{R}^M$.

This shows our assertion. $\square$

**Proposition 3.6.2.** *Let $N = 3$, $M \geqq 1$ and $n \geqq 1$. Suppose that $n \geqq M + 3$, and $n \equiv 1$ or 5 mod 6. Then there is a matching map $h : C_n^{(2)} \cup C_n^{(1)} \to \{1, \ldots, n\}^N$.*

*Proof.* First we prove the following.

**Claim.** Let $p, q, r = 1, \ldots, n$. If $p \neq q$ and $r \equiv 2q - p \mod n$, then $r \neq p, q$, $p \not\equiv 2r - q \mod n$, and $q \not\equiv 2p - r \mod n$.

Actually if $r = p$, we have $2p \equiv 2q \mod n$, which implies $p = q$. If $r = q$, we have $q \equiv p \mod n$. which implies $p = q$. If $p \equiv 2r - q \mod n$, then $3p \equiv 3q \mod n$, which implies $p = q$. If $q \equiv 2p - r \mod n$, then $3q \equiv 3p \mod n$, which implies $p = q$. Therefore we have our Claim.

Let us define $h : C_n^{(2)} \cup C_n^{(1)} \to \{1, \ldots, n\}^N$ by the following.

$h|_{C_n^{(2)}}$ is given by the following. For $p, q = 1, \ldots, n$, with $p \neq q$

$$h(1, 2, p, q) = (p, q, r), \quad h(1, 3, q, p) = (q, r, p) \quad h(2, 3, p, q) = (r, p, q),$$

where $r = 1, \ldots, n$, with $r \equiv 2q - p \mod n$. For $p = 1, \ldots, n$,

$$h(1, 2, p, p) = (p, p, r), \quad h(1, 3, p, p) = (p, r, p) \quad h(2, 3, p, p) = (r, p, p),$$

where $r = 1, \ldots, n$, with $r \equiv p - 1 \mod n$.

$h|_{C_n^{(1)}}$ is given by the following. For $p = 1, \ldots, n$, and $j = 0, 1, \ldots, M$,

$$h(1, j, p) = (p, r, r), \quad h(2, j, p) = (r, p, r) \quad h(3, j, p) = (r, r, p),$$

where $r = 1, \ldots, n$, with $r \equiv p + j + 2 \mod n$.

By the above Claim, we can easily check $h$ is a matching map. $\square$

**Proposition 3.6.3.** *Let $N \geqq 4$, $M \geqq 1$ and $n \geqq 1$. Suppose that $n \geqq M + 2$. Then there is a matching map $h : C_n^{(2)} \cup C_n^{(1)} \to \{1, \ldots, n\}^N$.*

*Proof.* First take a map $R : \{i, \ldots, n\}^2 \to \{1, \ldots, n\}$ such that $R(p, q) \neq p, q$ and $R(p, p) \equiv p + 1 \mod n$. Since $n \geqq 3$, we can take such a map. Now let us define $h : C_n^{(2)} \cup C_n^{(1)} \to \{1, \ldots, n\}^N$ by the following.

$h(i,i',p,q) = (k_1,\ldots,k_N)$, $(i,i',p,q) \in C_n^{(2)}$, where $k_i = p$, $k_{i'} = q$, $k_r = R(p,q)$, $r \neq i,i'$. $h(i,j,p) = (k_1,\ldots,k_N)$, $(i,j,p) \in C_n^{(1)}$, where $k_i = p$, $k_r = 1,\ldots,n$, with $k_r \equiv p+j+1 \bmod n$.

Since $p+2 \not\equiv p \bmod n$, we can easily check $h$ is a matching map. $\square$

Now let us prove Theorem 3.1.1.

Let $N \geqq 3$, $M \geqq 1$, and $\Theta$ is a non-empty open set in $\mathbf{R}$. By Propositions 3.6.2 and 3.6.3, there are $n \geqq 1$ and an injective map $h : C_n \to \{1,\ldots,n\}^N$ for which $h|_{C_n^{(2)} \cup C_n^{(1)}}$ is a matching map. Fix such an $n$ and let us take $\vec{c} \in \mathbf{R}^{Nn}$ such that $0 < c_{i1} < c_{i2} < \ldots < c_{in} < 1$, $i = 1,\ldots,N$, $i = 1,\ldots,N$. Let $D(\theta)$, $\theta \in \Theta$, be a set given by

$$D(\theta) = \{K \in \mathcal{C}_{(N)}(\Theta);\ \dim V_n(K,\vec{c},\theta_0) = \#(C_n)\}.$$

Then by Propositions 3.6.1, we see that $D(\theta) \neq$ for all $\theta \in \Theta$. Let $H$ be the set of injections from $C_n$ to $\{1,\ldots,n\}^N$. Let $\varphi : \mathcal{C}_{(N)}(\Theta) \to \mathbf{R}$ be given by

$$\varphi(K) = \sum_{h\in H} \det((\vec{M}^{(n)}K)_{\gamma_1}(\vec{z},\theta,h(\gamma_2)))_{\gamma_1,\gamma_2\in C_n})^2.$$

Then we see that $\varphi : \mathcal{C}_{(N)}(\Theta) \to \mathbf{R}$ is continuous and $D(\theta) = \{K \in \mathcal{C}_{(N)}(\Theta);\ \varphi(K) > 0\}$. So we see that $D(\theta)$ is an open subset $\mathcal{C}_{(N)}(\Theta)$.

Let $G \in D(\theta)$. For any $K \in \mathcal{C}_{(N)}(\Theta)$ and $s \in [0,1]$, $(1-s)K + sG_0 \in \mathcal{C}_{(N)}(\Theta)$. Also, $\varphi((1-s)K+sG)$ is a polynomial in $s$, and so is not equal to 0 except finite $s$'s. Therefore there is a $\{s_\ell\}_{\ell=1}^\infty \subset [0,1]$ such that $s_\ell \downarrow 0$, $\ell \to \infty$, and $(1-s_\ell)K + s_\ell)G) \in D(\theta)$, $\ell \geqq 1$. This observation shows that $D(\theta)$ is dense in $\mathcal{C}_{(N)}(\Theta)$ for all $\theta \in \Theta$.

Now let $\{\theta_m\}_{m=1}^\infty$ be a dense set in $\Theta$, and let

$$D = \bigcap_{m=1}^\infty D(\theta_m).$$

Then by Corollary 3.5.2, we see that any element of $D$ is not admissible family of copula functions. This proves Theorem 3.1.1.

## 3.7 Remarks

Let $N \geqq 3$, $M \geqq 1$, and $\Theta$ be an open set in $\mathbf{R}^M$. Let $K \in \mathcal{C}_{(N)}(\Theta)$. Assume that $n \geqq N$.

Let

$$A^{(2)}$$
$$= \{(1,2,p,q);\ p,q = 1,\dots,n\} \cup \{(1,i,p,q);\ i = 3,\dots,N,\ p = 1,\dots,n,\ q = 2,\dots,n\}$$
$$\cup \{(i,i',p,q);\ 2 \leqq i < i' \leqq N,\ p,q = 2,\dots,n\} \subset C_n^{(2)}.$$

Also, let $\vec{k}_{ii'pq} \in \mathbf{R}^{\{1,\dots,n\}^N}$, $(i,i',p,q) \in C_n^{(2)}$, be given by

$$\vec{k}_{ii'pq} = (1,\dots,1,\underset{i}{p},1,\dots,1,\underset{i'}{q},1,\dots,1).$$

Then we have the following.

**Proposition 3.7.1.** *Let $\theta_0 \in \Theta$, and assume that*

$$\frac{\partial^2 K}{\partial x_i \partial x_{i'}}(x,\theta_0) > 0, \qquad x \in (0,1)^N,\ 1 \leqq i < i' \leqq N.$$

*Then for any $n \geqq N$, and $\vec{z} \in (0,1)^{3n}$,*

$$\dim V_{n0}(K,\vec{c},\theta_0) \geqq \#(A^{(2)}) = \frac{N(N-1)}{2}n^2 - nN(N-2) + \frac{(N-1)(N-2)}{2}.$$

*Proof.* Remind that

$$(M^{(n)(2)}_{ii'pq}K)(\vec{z},\theta_0,\vec{k}) = \delta_{p,k_i}\delta_{q,k_{i'}}\frac{\partial^2 K}{\partial x_i \partial x_{i'}}(z_{1k_1},\dots,z_{Nk_N},\theta_0), \quad p,q = 1,\dots,n.$$

So for $(i,i',p,q),(j,j',r,\ell) \in A^{(2)}$, we see that $(M^{(n)(2)}_{ii'pq}K)(\vec{z},\theta_0,\vec{k}_{jj'r\ell}) = 0$ if $i > j$, or if $i = j$ and $i' > j$, and that $(M^{(n)(2)}_{ii'pq}K)(\vec{z},\theta_0,\vec{k}_{ii'r\ell}) = \delta_{p,r}\delta_{q,\ell}c_{ii'pq}$, for some positive numbers $c_{ii'pq}$. So we see that $\{(\vec{M}^{(n)}K)_\gamma(\vec{z},\theta_0);\ \gamma \in A^{(2)}\}$ is linearly independent. So we have our assertion. □

From now on we think of a special case. We assume that $K$ is a family of Archimedian copula functions, i.e., there are smooth functions $\varphi : (0,1)\times\Theta \to (0,\infty)$ and $\rho : (0,\infty)\times\Theta \to (0,1)$ such that

$$K(x_1,\dots,x_N,\theta) = \rho(\sum_{k=1}^N \varphi(x_k,\theta),\theta), \qquad x_1,\dots,x_N \in (0,1),\ \theta \in \Theta.$$

Then $\rho(\cdot,\theta)$ must be the inverse function of $\varphi(\cdot,\theta)$.

Then we have the following.

**Proposition 3.7.2.** *Let*

$$m_0 = \frac{N(N-1)}{2}n^2 + N(M+3-N)n - \frac{(N-1)(2M+4-N)}{2} + \frac{M(M+3)}{2}.$$

*Then we have the following.*
(1) $\dim V_{n0}(K,\vec{z},\theta) \leqq m_0$. *and* $\dim V_n(K,\vec{z},\theta) \leqq m_0+1$ *for any* $\vec{z} \in (0,1)^{Nn}$, *and* $\theta \in \Theta$.
(2) *Assume that* $\Theta$ *is connected and that* $\varphi : (0,1)\times\Theta \to (0,\infty)$ *is real analytic. If there exists a* $\theta_0 \in \Theta$ *such that* $\dim V_n(K,\vec{z},\theta_0) = m_0+1$, *then* $K$ *is not an admissible family of copula functions.*

Since the proof is rather long, we will give it in the next section.

Now let us think of a family of Gumbel copula functions. Let $N=3$, $M=1$, and $\Theta=(0,1)$. Let $K \in \mathcal{C}_{(3)}((0,1))$ be given by

$$K(x_1,x_2,x_3,\theta) = \exp(-(\sum_{i=1}^{3}(-\log x_i)^{\theta})^{1/\theta}), \qquad x_1,x_2,x_3 \in (0,1),\ \theta \in (0,1).$$

Then letting $\varphi(x,\theta) = (-\log x)^{\theta}$, $\rho(y,\theta) = \exp(-y^{1/\theta})$, we see that

$$K(x_1,x_2,x_3,\theta) = \rho(\sum_{i=1}^{3}\varphi(x_i,\theta),\theta), \qquad x_1,x_2,x_3 \in (0,1),\ \theta \in (0,1).$$

Let $n=5$. Then we have $m_0 = 89$. So by Proposition 3.7.2 we see that if there exist $\theta_0 \in (0,1)$ and $\vec{z} = (z_{ip})_{i=1,2,3,p=1,\ldots,5} \in (0,1)^{15}$ such that $\dim V_n(K,\vec{z},\theta_0) = 90$, we see that $K$ is not admissible family of copula functions.

By using numerical computation, we check that $\dim V_n(K,\vec{z},\theta_0) = 90$ for $(z_{i1},\ldots,z_{i5}) = (0.55, 0.65, 0.75, 0.85, 0.95)$, $i=1,2,3$, and $\theta_0 = 0.4$ or $0.6$. Actuary, we compute the dimension of the vector subspace in $\mathbf{R}^{\{1,2,3,4,5\}^3}$ spanned by $e^{(2)}_{ii'pq}(\vec{z},\theta_0)$, $(i,i',p,q) \in A^{(2)}$, $e^{(1)}_{ijp}(\vec{z},\theta_0)$, $(i,j,p) \in A^{(1)}$, $e^{(0)}_{jj'}(\vec{z},\theta_0)$, $(j,j') \in J_2$, and $e_0(\vec{z},\theta_0)$ given in the next section by applying Householder transformation for the associated matrix, and we are convinced that it is 90. As we show in the next section, the dimension of this vector subspace is the same as $\dim V_n(K,\vec{z},\theta_0)$ in this case.

## 3.8 Proof of Proposition 3.7.2

For $\vec{z} \in (0,1)^{3n}$, $\theta \in \Theta$ and $\vec{k} \in \{1,\ldots,n\}^3$, let

$$\Phi(\vec{z},\theta,\vec{k}) = \sum_{i=1}^{3} \varphi(z_{ik_i},\theta),$$

$$e^{(2)}_{ii'pq}(\vec{z},\theta,\vec{k}) = \delta_{pk_i}\delta_{qk_{i'}}\frac{\partial^2\rho}{\partial y^2}(\Phi(\vec{z},\theta,\vec{k}),\theta), \quad (i,i',p,q) \in C^{(2)}_n,$$

$$e^{(1)}_{i0p}(\vec{z},\theta,\vec{k}) = \delta_{pk_i}\frac{\partial\rho}{\partial y}(\Phi(\vec{z},\theta,\vec{k}),\theta), \quad (i,0,p) \in C^{(1)}_n,$$

$$e^{(1)}_{ijp}(\vec{z},\theta,\vec{k}) = \delta_{pk_i}\frac{\partial^2\rho}{\partial\theta_j\partial y}(\Phi(\vec{z},\theta,\vec{k}),\theta) \quad (i,j,p) \in C^{(1)}_n,\ j \geqq 1,$$

$$e^{(0)}_{jj'}(\vec{z},\theta,\vec{k}) = \frac{\partial^2\rho}{\partial\theta_j\partial\theta_{j'}}(\Phi(\vec{z},\theta,\vec{k}),\theta), \quad (j,j') \in J_2,\ j \geqq 1,$$

$$e^{(0)}_{0j}(\vec{z},\theta,\vec{k}) = \frac{\partial\rho}{\partial\theta_j}(\Phi(\vec{z},\theta,\vec{k}),\theta). \quad (0,j) \in J_2,$$

and

$$e_0(\vec{z},\theta,\vec{k}) = -\rho(\Phi(\vec{z},\theta,\vec{k}),\theta).$$

Let $e^{(2)}_{ii'pq}(\vec{z},\theta)$, $(i,i',p,q) \in C^{(2)}_n$, $e^{(1)}_{ijp}(\vec{z},\theta)$, $(i,j,p) \in C^{(1)}_n$, $e^{(0)}_{jj'}(\vec{z},\theta)$, $(j,j') \in J_2$, and $e_0(\vec{z},\theta)$ be elements of $\mathbf{R}^{\{1,\ldots,n\}^3}$ given by $e^{(2)}_{ii'pq}(\vec{z},\theta) = (e^{(2)}_{ii'pq}(\vec{z},\theta,\vec{k}))_{\vec{k}\in\{1,\ldots,n\}^3}$ etc.

Let $A^{(1)}$ be the subset of $C^{(1)}_n$ given by

$$\begin{aligned} &A^{(1)} \\ &= \{(1,j,p);\ j=0,1,\ldots,M,\ p=1,\ldots,n\} \\ &\quad \cup\{(i,j,p);\ i=2,\ldots,N,\ j=0,1,\ldots,M,\ p=2,\ldots,n\}, \end{aligned}$$

and $A^{(2)}$ be the subset of $C^{(2)}_n$ given in the previous section.

Let $U_0(\vec{z},\theta)$ be a vector subspace in $\mathbf{R}^{\{1,\ldots,n\}^N}$ spaned by $\{e^{(2)}_{ii'pq}(\vec{z},\theta);\ (i,i',p,q) \in A^{(2)}\}$, $\{e^{(1)}_{ijp}(\vec{z},\theta);\ (i,j,p) \in A^{(1)}\}$ and $\{e^{(0)}_{jj'}(\vec{z},\theta);\ (j,j') \in J_2\}$. Since $\#(A^{(2)}) + \#(A^{(1)}) + \#(J_2) = m_0$, we see that $\dim U_0(\vec{z},\theta) \leqq m_0$.

First, we prove the following.

**Proposition 3.8.1.** (1) $e^{(2)}_{ii'pq}(\vec{z},\theta) \in U_0(\vec{z},\theta)$ *for all* $(i,i',p,q) \in C^{(2)}_n$.
(2) $e^{(1)}_{ijp}(\vec{z},\theta) \in U_0(\vec{z},\theta)$ *for all* $(i,j,p) \in C^{(1)}_n$.

*Proof.* Let

$$\tilde{e}^{(2)}_{1p}(\vec{z},\theta) = \sum_{q=1}^{n} e^{(2)}_{12pq}(\vec{z},\theta) \in U_0(\vec{z},\theta), \quad p = 1,\dots,n,$$

and

$$\tilde{e}^{(2)}_{ip}(\vec{z},\theta) = \sum_{q=1}^{n} e^{(2)}_{1iqp}(\vec{z},\theta) \in U_0(\vec{z},\theta), \quad i = 2,\dots,N-1,\ p = 2,\dots,n.$$

Then we see that

$$e^{(2)}_{1ip1}(\vec{z},\theta) = \hat{e}^{(2)}_{1p}(\vec{z},\theta) - \sum_{q=2}^{n} e^{(2)}_{1ipq}(\vec{z},\theta) \in U_0(\vec{z},\theta), \quad i = 3,\dots,N,\ p = 1,\dots,n.$$

So we see that $e^{(2)}_{1ipq}(\vec{z},\theta) \in U_0(\vec{z},\theta)$, $i = 2,\dots,N$, $p,q = 1,\dots,n$.

Also, we see taht

$$e^{(2)}_{ii'1q}(\vec{z},\theta) = \hat{e}^{(2)}_{i'q}(\vec{z},\theta) - \sum_{p=2}^{n} e^{(2)}_{ii'pq}(\vec{z},\theta) \in U_0(\vec{z},\theta), \quad i = 2,\dots,N,\ q = 2,\dots,n,$$

and so

$$e^{(2)}_{ii'11}(\vec{z},\theta) = \hat{e}^{(2)}_{i1}(\vec{z},\theta) - \sum_{q=2}^{n} e^{(2)}_{ii'1q}(\vec{z},\theta) \in U_0(\vec{z},\theta), \quad i = 2,\dots,N.$$

These show that the assetion (1).

Let

$$\tilde{e}^{(1)}_{j}(\vec{z},\theta) = \sum_{p=1}^{n} e^{(1)}_{1jp}(\vec{z},\theta) \in U_0(\vec{z},\theta), \quad j = 0,\dots,M.$$

Then we see that

$$e^{(1)}_{ij1}(\vec{z},\theta) = \hat{e}^{(1)}_{i}(\vec{z},\theta) - \sum_{p=2}^{n} e^{(1)}_{ijp}(\vec{z},\theta) \in U_0(\vec{z},\theta), \quad j = 0,\dots,M.$$

This proves the assertion (2). □

Now note that

$$(M^{(n)(2)}_{ii'pq}K)(\vec{z},\theta,\vec{k}) = \frac{\partial \varphi}{\partial x}(z_{ip},\theta)\frac{\partial \varphi}{\partial x}(z_{i'q},\theta)e^{(2)}_{ii'pq}(\vec{z},\theta,\vec{k}), \quad (i,i',p,q) \in C^{(2)}_n,$$

$$(M_{i0p}^{(n)(1)}K)(\vec{z},\theta,\vec{k}) = \frac{\partial^2\varphi}{\partial x^2}(z_{ip},\theta)e_{i0p}^{(1)}(\vec{z},\theta,\vec{k}) + (\frac{\partial\varphi}{\partial x}(z_{ip},\theta))^2\tilde{e}_{ip}^{(2)}(\vec{z},\theta,\vec{k})), \quad (i,0,p) \in C_n^{(1)},$$

$$(M_{ijp}^{(n)(1)}K)(\vec{z},\theta,\vec{k}) = \frac{\partial\varphi}{\partial x}(z_{ip},\theta)e_{ijp}^{(1)}(\vec{z},\theta,\vec{k})$$

$$+\frac{\partial\varphi}{\partial x}(z_{ip},\theta)(\sum_{q=1}^{n}\frac{\partial\varphi}{\partial\theta_j}(z_{iq},\theta)e_{iq}^{(2)}(\vec{z},\theta,\vec{k})))+\frac{\partial^2\varphi}{\partial\theta_j\partial x}(z_{ip},\theta)e_{i0p}^{(1)}(\vec{z},\theta,\vec{k}), \quad (i,j,p) \in C_n^{(1)},\ j \geqq 1,$$

$$(M_{0j}^{(n)(0)}K)(\vec{z},\theta,\vec{k}) = \sum_{i=1}^{N}\sum_{p=1}^{n}\frac{\partial\varphi}{\partial\theta}(z_{ip},\theta)e_{i0p}^{(1)}(\vec{z},\theta,\vec{k}) + e_{0j}^{(0)}(\vec{z},\theta,\vec{k}), \quad (0,j) \in J_2,$$

$$(M_{jj'}^{(n)(0)}K)(\vec{z},\theta,\vec{k})$$

$$= \sum_{1\leqq i<i'\leqq N}\sum_{p,q=1}^{n}(\frac{\partial\varphi}{\partial\theta_j}(z_{ip},\theta)\frac{\partial\varphi}{\partial\theta_{j'}}(z_{i'q},\theta) + \frac{\partial\varphi}{\partial\theta_j}(z_{ip},\theta)\frac{\partial\varphi}{\partial\theta_{j'}}(z_{i'q},\theta))e_{ii'pq}^{(2)}(\vec{z},\theta,\vec{k})$$

$$+\sum_{i=1}^{N}\sum_{p=1}^{n}\frac{\partial\varphi_j}{\partial\theta_j}(z_{ip},\theta)\frac{\partial\varphi_{j'}}{\partial\theta_{j'}}(z_{ip},\theta)\tilde{e}_{ip}^{(2)}(\vec{z},\theta,\vec{k})$$

$$+\sum_{i=1}^{N}\sum_{p=1}^{n}\frac{\partial\varphi}{\partial\theta_j}(z_{ip},\theta)e_{ij'p}^{(1)}(\vec{z},\theta,\vec{k}) + \frac{\partial\varphi}{\partial\theta_{j'}}(z_{ip},\theta)e_{ijp}^{(1)}(\vec{z},\theta,\vec{k})$$

$$+\sum_{i=1}^{N}\sum_{p=1}^{n}\frac{\partial^2\varphi}{\partial\theta_j\partial\theta_{j'}}(z_{ip},\theta)e_{i0p}^{(2)}(\vec{z},\theta,\vec{k}) + e_{jj'}(\vec{z},\theta,\vec{k}).$$

Therefore from the assumption, we see that

$$V_{n0}(K,\vec{z},\theta) \subset U_0(\vec{z},\theta).$$

So we have the first assertion of Proposition 3.7.2 (1).

We remark that if

$$\frac{\partial\varphi}{\partial x}(z_{ip},\theta) > 0, \qquad \frac{\partial^2\varphi}{(\partial x)^2}(z_{ip},\theta) \neq 0,$$

for any $i = 1,\ldots,N$ and $p = 1,\ldots,n$, then

$$U_0(\vec{z},\theta) = V_{n0}(K,\vec{z},\theta).$$

Now note that

$$(M_i^{(n)I}K)(\vec{z},\theta,\vec{k}) = e_0(\vec{z},\theta,\vec{k}) + \sum_{p=1}^{n} z_{ip}\frac{\partial\varphi}{\partial x}(z_{ip},\theta)e_{i0p}^{(1)}(\vec{z},\theta,\vec{k}).$$

So we have $(\vec{M}^{(n)}K)_i(\vec{z},\theta) - e_0(\vec{z},\theta) \in V_{n0}(K,\vec{z},\theta)$, $i \in I$ This implies that the second assertion of Proposition 3.7.2 (1).

Now let us prove the assertion (2) of Proposition 3.7.2. Suppose that $K$ is an admissible family of copula functions. Then by Lemma 3.4.2, we see that there is a subset $A$ of $\Theta$ such that the closure of $A$ contains a non-void open subset of $\Theta$ and for any $\theta \in A$ there are $\xi_i \geqq 0$, $i = 1,\ldots,N$, such that $\sum_{i\in I}\xi_i > 0$ and $\sum_{i\in I}\xi_i(\vec{M}^{(n)}K)_i(\vec{z},\theta)$ $\in V_{n0}(K,\vec{z},\theta)$. Then we see that $e_0(\vec{z},\theta) \in V_{n0}(K,\vec{z},\theta)$. This implies that $V_n(K,\vec{z},\theta) \subset U_0(\vec{z},\theta)$.

Then by the assertion (1), we see that $\dim V_n(K,\vec{z},\theta) \leqq m_0$, $\theta \in A$. Let $H_1$ (resp. $H_2$) be the set of injective maps from $\{1,\ldots,m_0\}$ to $C_n$ (resp. $\{1,\ldots,n\}^N$. ) Now let

$$f(\theta) = \sum_{h_1\in H_1}\sum_{h_2\in H_2} \det(((\vec{M}^{(n)}K)_{h_1(r)})(\vec{z},\theta,h_2(\ell)))_{r,\ell=1,\ldots,m_0})^2,\ \theta\in\Theta.$$

Then we see that $f(\theta) = 0$, $\theta \in A$. From the assumption, we see that $f : \Theta \to \mathbf{R}$ is real analytic. So we see that $f(\theta) = 0$, $\theta \in \bar{A}\cap\Theta$. Since $\bar{A}\cap\Theta$ contains a non-void open set and $\Theta$ is connected, we see that $f = 0$ on $\Theta$. In particular, $f(\theta_0) = 0$. But this implies that $\dim V_n(K,\vec{z},\theta_0) \leqq m_0 - 1$. This contradicts to the assumption. Therefore $K$ is not admissible.

This completes the proof of Proposition 3.7.2.

## 3.9 Examples of dynamical default time copula models

Let $(\Omega,\mathcal{F},P)$ be a complete probability space, $W(t) = (W^k(t))_{k=1,\ldots,d}$, $t \geqq 0$, be a $d$-dimensinal standard Wiener process. Let $N \geqq 2$, and $Z_1,\ldots,Z_N$ be a independent identically distributed random variables whose distributions are uniform distribution on $(0,1)$. We assume that $\sigma\{Z_1,\ldots,Z_N\}$ and $\sigma\{W(t), t \geqq 0\}$ are independent. Let $M \geqq 1$. Let $\sigma_k : \mathbf{R}^M \to \mathbf{R}^M$, $k = 0,1,\ldots,d$, be Lipschitz continuous functions and $h_i : \mathbf{R}^M \to (0,\infty)$, $i = 1,\ldots,N$, be continuous functions.

Let $Y$ be the unique solution to the following stochastic differential equation on $\mathbf{R}^M$.

$$dY(t,y) = \sum_{k=1}^{d} \sigma_k(t, Y(t,y))dW^k(t) + \sigma_0(t, Y(t,y))dt,$$
$$Y(0,y) = y \in \mathbf{R}^M.$$

Let $y_0 \in \mathbf{R}^M$. We also assume that

$$P(\int_0^\infty h_i(Y(t,y_0))dt = \infty) = 1, \qquad i = 1,\ldots,N,$$

and the support of the distribution of $Y(t,y_0)$ under $e^{-t} \otimes P(d\omega)$ contains non-empty open set.

Now let us define random times $\tau_1,\ldots,\tau_N$ by

$$\tau_i = \inf\{t > 0;\ \exp(-\int_0^t h_i(Y(s,y_0))ds) < Z_i\}, \qquad i = 1,\ldots,N.$$

Then we see that

$$(\prod_{i\in I} 1_{\{\tau_i > t\}})P(\tau_i > t_i,\ i \in I|\mathcal{F}_t)$$
$$= (\prod_{i\in I} 1_{\{\tau_i > t\}})E[\exp(-\sum_{i\in I}\int_t^{t_i} h_i(Y(s,y_0)ds)|Y(t,y_0)]$$

for $t, t_1, \ldots .t_N \geqq 0$ with $t < \min\{t_i;\ i \in I\}$ (c.f. [6],[5],[7]). Let

$$H(s_1,\ldots,s_N,y) = E[\exp(-\sum_{i=1}^{N}\int_0^{s_i} h_i(Y(r,y)dr)] \qquad s_1,\ldots,s_N \geqq 0,\ y \in \mathbf{R}^M,$$

and

$$H_i(s,y) = E[\exp(-\int_0^s h_i(Y(r,y)dr)]. \qquad i = 1,\ldots,N,\ s \geqq 0,\ y \in \mathbf{R}^M.$$

Then $H_i(\cdot,y) : [0,\infty) \to (0,1]$, $i = 1,\ldots,N$, is strictly decreasing surjective function. So the inverse functions $H_i^{-1}(\cdot,y) : (0,1] \to [0,\infty)$, $i = 1,\ldots,N$, exist. Let $K : [0,1]^N \times \mathbf{R}^M \to [0,1]$ be given by

$$K(x_1,\ldots,x_N,y) = \begin{cases} H(H_1^{-1}(\cdot,y)(x_1),\ldots,H_N^{-1}(\cdot,y),y), & \text{if } x_1,\ldots,x_N \in (0,1], \\ 0, & \text{if one of } x_1,\ldots,x_N = 0. \end{cases}$$

Then we have

$$(\prod_{i=1}^{N} 1_{\{\tau_i > t\}}) P(\tau_i > t_i,\ i = 1, \ldots, N | \mathcal{F}_t)$$

$$= (\prod_{i=1}^{N} 1_{\{\tau_i > t\}}) K(P(\tau_1 > t_1 | \mathcal{F}_t), \ldots, P(\tau_N > t_N, Y(t, y_0))\ a.s.$$

for any $t \geqq 0$, and $t_1, \ldots, t_N \in [t, \infty)$.

So we see that $K$ is an admissible family of copula functions.

# 関連図書


[1] 楠岡成雄, 青沼君明, 中川秀敏, クレジット・リスク・モデル, きんざい, 2001.

[2] 戸坂凡展, 吉羽要直, CDO のプライシング・モデルとそれを用いた CDO の特性等の考察：CDO の商品性、国内市場の概説とともに, 日本銀行, 金融研究, 2003.11.

[3] 小宮清孝, コピュラの金融実務での具体的な活用方法の解説, 日本銀行, 金融研究第 24 巻別冊第 2 号, 2005.

[4] Björk, T. and B. Christernsen, Interestrate Dynamics and Consistent Forwardrate Curves, Mathematical Finance, 9(1999), 323-348.

[5] Bielecki, T. R., and M. Rutkowski, Credit risk: modelling, valuation and hedging, Springer Finance. Springer-Verlag, Berlin, 2002.

[6] Bielecki, T. R., M. Jeanblanc, and M. Rutkowski, Modeling and valuation of credit risk. Stochastic methods in finance, Lecture Notes in Math., 1856, pp. 27?126, Springer, Berlin, 2004.

[7] Chesney, M., M. Jeanblanc, and M. Yor, Mathematical methods for financial markets, Springer Finance. Springer-Verlag London, Ltd., London, 2009.

[8] David X. Li., On Default Correlation: A copula Function Approach, Journal of Fixed Income 9: 43-54, 2000.

[9] Duffie, D., and Lando, D., Term structures of credit spreads with incomplete accounting information, Econometrica 69.3: 633-664, 2001.

[10] Duffie, D., Singleton, K., Modeling Term Structures of Defaulttable Bonds, Working paper, Stanford University, 1994.

[11] Jarrow, Robert A., Philip Protter, and A. Deniz Sezer, Information Reduction via Level Crossing in a Credit Riks Model, Finance and Stochastics 11.2: 195-212, 2007.

[12] Jeanblanc, Monique, and Stoyan Valchev, Partial information and hazard process, International Journal of Theoretical and Applied Finance 8.06: 807-838, 2005.

[13] Kusuoka, S., A remark on default risk models, in Advances in Mathematical Economics vol. 1, ed. S.Kusuoka, M.Maruyama , pp.69-82, Springer 1999.

[14] Merton, R., On the pricing of corporate debt: The risk structure of interest rate, Journal of Finance 29: 449-470, 1974.

[15] Nakagawa, H., A Filtering Model on Default Risk, J. Math. Sci. Univ. Tokyo 8: 107-142, 2001.

[16] Sklar, M., Fonctions de rèpartition à n dimensions et leurs marges, Publ. Inst. Statist. Univ. Paris 8: 229-231, 1959.