博士論文

Essays on Stochastic Macroeconomics
(確率的マクロ経済学に関する論文)

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Introduction

1.1 Modern Macroeconomics and Microfoundations

Over the past four decades, macroeconomic theory has dramatically changed. Once, macroeconomics was clearly distinguished from microeconomics and a synonym to the Keynesian economics. The beginning of this change was arguably the criticism against Keynesian economics by Friedman (1968) in his discussion about the Phillips curve:

> "The 'natural rate of unemployment,' in other words, is the level that would be ground out by the Walrasian system of general equilibrium equations, provided there is imbedded in them the actual structural characteristics of the labor and commodity markets, including market imperfections, stochastic variability in demands and supplies, the cost of gathering information about job vacancies and labor availabilities, the costs of mobility, and so on." (p. 8)

Friedman attempted to understand macroeconomic dynamics by incorporating the various characteristics of economy into the Walrasian system. Since then, the natural rate of employment has been discussed within the framework of *search theory*. Lucas (1972)'s rational expectations model in which money affects real economic activities in the short-run, but is neutral in the long-run provided a foundation for Friedman's "monetarism". Modern macroeconomics from

Friedman, Phelps, Lucas to real business cycle theory (Kydland and Prescott (1982)) is considered to be based on the principles of microeconomic theory, i.e., the optimization of rational economic agents. This recognition most clearly stated by Lucas in his Yrjo Jahnsson lecture (Lucas (1987)) is widely accepted by the profession:

> "The most interesting recent developments in macroeconomic theory seem to me describable as the reincorporation of aggregative problems such as inflation and the business cycle within the general framework of "microeconomic" theory. If these developments succeed, the term 'macroeconomic' will simply disappear from use and the modifier 'micro' will become superfluous. We will simply speak, as did Smith, Ricardo, Marshall and Walras, of economic theory. ... We will be tempted, I am sure, to relieve the discomfort induced by discrepancies between theory and facts by saying that the ill-understood facts are the province of some other, different kind of economic theory. Keynesian "macroeconomics" was, I think, a surrender (under great duress) to this temptation. It led to the abandonment, for a class of problems of great importance, of the use of the only "engine for the discovery of truth" that we have in economics. Now we are once putting this engine of Marshall's to work on the problems of aggregate dynamics." (p. 107–108)

According to his view, there is no boundary between macroeconomics and microeconomics. Macroeconomic theory must be derived from the optimization of rational economic agents such as firms and households. Today's *modern macroeconomics* including real business cycle theory(RBC), endogenous growth theory, and dynamic stochastic general equilibrium theory(DSGE) is all based on this principle, and incorporates optimal behaviors of microeconomic agents into macroeconomic model. It has firmly established itself as the mainstream supported by the majority of economists. They believe that their modern micro-founded macroeconomics outperforms the old macroeconomics on the grounds that it has firm microfoundations similar to the general equilibrium theory represented by Arrow and Debreu (1954). The development of this micro-founded macroeconomics culminated in a declaration of victory by Lucas (2003) in his presidential address to the American Economic Association:

> "[M]acroeconomics ... has succeeded: Its central problem of depression prevention

has been solved, for all practical purposes ..." (p. 1)

Ironically enough, only five years after this presidential address was delivered, the bankruptcy of Lehman Brothers in September, 2008 led to the Great Recession. This was none other than an embarrassing experience for economists, and made them to reconsider what they had achieved and learned from macroeconomic theory. In this section, we shall explain that widely accepted modern macroeconomics, especially DSGE models, do not have any firm microfoundations, and that an alternative theory is urgently needed.

In modern macroeconomics, macroeconomic analysis must be derived from optimization of rational economic agents who maximize their utility or profit function over an infinite horizon under rational expectations. The key ingredient of these models is calculation of dynamic optimization of, e.g., consumers, investors or managers. However, it is not clear at all in what sense these macroeconomic models are microfounded. Do their "general equilibrium" correspond to the real economy? Does macroeconomic analysis reduce to evaluation of dynamic optimization problems? Most macroeconomists seem to believe that DSGE is a tractable and computable version of the well-developed traditional general equilibrium theory as the name "dynamic stochastic *general equilibrium*" suggests. However, the traditional general equilibrium theory actually tells us a completely different story in that we should not use general equilibrium theory as a general basis for the analysis of practical problems.

We briefly review the remarkable results from general equilibrium theory. For simplicity, consider an exchange economy with consumers indexed by i (Of course, the following results can be extended to general cases). General equilibrium in the model is determined by the aggregate excess demand function defined by

$$F(p) \equiv \sum_{i=1}^{N} f_i(p) \tag{1.1}$$

where $f_i(p)$ is the excess demand vector for consumer i, p is a price vector and N is the number of consumers. In elementary textbooks, strong restrictions are imposed on the behavior of consumers; for example, with the identical consumer preference, nice properties such as the uniqueness and/or stability of equilibrium are proven. In such a case, it may be possible to rigorously derive the existence of a representative agent whose behavior characterizes the economy as a whole. However, we cannot expect optimistically that this reasoning holds true in general settings. Sonnenschein (1972), Mantel (1974) and Debreu (1974)(abbreviated as SMD) present the negative answer to this reasoning.

The main message of their results is the following. Under the assumption that consumers are characterized by standard assumptions such as strictly-convex, monotone, continuous and complete preference preorder relations, all the properties we can expect for are as follows:

The aggregate excess demand function F satisfies

$$(i) F \text{ is continuous,} \tag{1.2}$$

$$(ii) F \text{ is homogeneous of degree zero, i.e., } F(\lambda p) = F(p), \tag{1.3}$$

$$(iii) F \text{ satisfies the Walras' law, i.e., } p \cdot F(p) = 0. \tag{1.4}$$

Conversely, an arbitrary function F satisfying $(i) - (iii)$ is given, there exists an economy with consumers whose excess demand functions sum to F.

To our disappointment, these properties are the only properties that general equilibrium theory can impose on the aggregate excess demand function under the general assumptions. The theorem says that unless we impose further restrictions on preferences or endowments, no further restriction can be made on equilibria. Because F can be given arbitrary, the microeconomic structure does not carry over to the aggregate level in general. In other words, general equilibrium theory tells us almost nothing about the properties of aggregate variables.

This result is directly related to the problem of the uniqueness of equilibria. It implies that we cannot expect equilibria to be unique in general, unless we impose further restrictions. It is a common practice in macroeconomics to study how an equilibrium in a model is affected by exogenous shocks, e.g., policy changes. It implicitly assumes that equilibrium is unique. The uniqueness of equilibria is crucial for comparative statics analysis. Thus, the SMD results cast a serious doubt on the relevance of policy analysis based on DSGE models. It is not clear at all what conclusions derived from DSGE of policy analysis really mean when there are more than one solutions to the model. With multiple equilibria, there is always a possibility that shocks move an equilibrium to another one. It must be noted that there is no reason to assume that the multiplicity of equilibrium stands for a small number (e.g. two or three). It may be a million. The bottomline is that we cannot know anything meaningful *a priori* without specifying microeconomic agents. One might argue that the nonuniqueness is largely a theoretical problem, and can be ignored in practical situations. However, this view is illegitimate. For example, Kehoe (1985) considered an economy with production, and showed that the assumptions required for the uniqueness of equilibrium in such economy are too restrictive to be valid in the real economy. In particular, he showed that gross substitutability in demand, which guarantees the uniqueness in pure exchange economy, does not imply the uniqueness in economy with production. He writes,

> "Non-uniqueness of equilibrium does not seem so pathological a situation as to warrant unqualified use of the simple comparative statics method when dealing with general equilibrium models." (p. 145)

(For other cases, see Kehoe (1991) and Mercenier (1995)) Therefore, there is no *a priori* reason to assume that the uniqueness of equilibrium is guaranteed.

What is worse, the SMD results reverberate on *tâtonnement* stability of equilibrium. Economists are prone to focus on the state of equilibrium. However, if a system in disequilibrium does not converge to such equilibrium state the focus on equilibrium becomes meaningless. For equilibrium to have economic meanings, stability is necessary. We briefly review the results relating to tâtonnement stability in the following.

Samuelson (1941) considered a simple price adjustment process,

$$\frac{dp_t}{dt} = F(p_t), \ p_0 \text{ is a given initial condition.} \tag{1.5}$$

This price change mechanism is economically interpretable because the price is adjusted proportionally to the excess demand. However, it is well known that this process is not necessarily globally asymptotically stable, that is, there is no guarantee that the price trajectory converges to some equilibrium price, except for very restrictive cases (see e.g. Scarf (1960)). This result may not be surprising because, as discussed above, the general equilibrium theory does not impose practically any restrictions on the aggregate excess demand function. Distinct from

Samuelson (1941), there have been many efforts to construct globally stable price adjustment mechanism. For example, the Global Newton Method developed by Smale (1976) uses a knowledge of all partial derivatives of the aggregate demand function as well as the aggregate excess demand function itself. This huge increase in the amount of information cannot be avoided, as shown by Saari and Simon (1978). Kamiya (1990) (see also Herings (1997)) succeeded to obtain a globally asymptotically stable price adjustment by combining a simple tâtonnement process and the Global Newton Method. However, it should be noted that these mechanisms also depend on the information on the excess demand function and its partial derivatives to ensure the convergence.

All these results suggest that the stability of equilibria is by no means obvious. In general, information on the prices is sufficient for inducing economic agents to optimal behavior only in equilibrium. In disequilibrium, the amount of informational required for global stablility is prohibitive in every practical sense. All adjustment processes proposed so far as alternatives to Samuelson (1941) have no natural interpretations. In the sense that a huge amount of information is required to guarantee that a market works well, it is far from efficient [1].

They cast doubt on the widespread use of the notion of equilibrium in macroeconomics. Interestingly, it was leading theorist in mathematical economics who took these difficulties seriously. For example, Hildenbrand (1994)

> "When I read in the seventies the publications of Sonnenschein, Mantel and Debreu on the structure of the excess demand function of an exchange economy, I was deeply consternated. Up to that time I had the naive illusion that the microeconomic foundation of the general equilibrium model, which I admired so much, does not only allow us to prove that the model and the concept of equilibrium are logically consistent (existence of equilibria), but also allows us to show that the equilibrium is well determined. This illusion, or should I say rather, this hope, was destroyed, once and for all, at least for the traditional model of exchange economies.
>
> I was tempted to repress this insight and to continue to find satisfaction in proving existence of equilibria for more general models under still weaker assumptions.

[1] As early as the 1950's, K. Arrow pointed out this problem. See, for example, Arrow (1959).

> However, I did not succeed in repressing the newly gained insight because I believe that a theory of economic equilibrium is incomplete if the equilibrium is not well determined." (p. iv)

DSGE apparently solve dynamic optimization problems and obtain definite equilibrium solutions. How does they get around these difficulties discussed above? What the tricks in these models to avoid the difficulties and restore well-behaved properties of equilibria? The answer is simple in that models make additional restrictive assumptions, for example, on a functional form of utility. The assumption of rational behavior of economic agents is by itself so weak that no implications can be obtained in general. Therefore, the additional restrictive assumptions are necessary for DSGE models to obtain well-defined solutions. Arrow (1986) most clearly makes this point,

> "[T]he very large bodies of empirical and theoretical research ... draw powerful implications from utility maximization for, respectively, the behavior of individuals, most especially in the field of labor supply, and the performance of the macroeconomy based on "new classical" or "rational expectations" models. In both domains, this power is obtained by adding strong supplementary assumptions to the general model of rationality." (S388-389)

The strong supplementary assumptions other than rationality are crucial for deriving any clean conclusion. Regarding this point, Arrow (1986) views the homogeneity assumption as especially dangerous because it conceals a fundamental point that in the first place, trading arises from heterogeneity. The supplementary assumptions used in most DSGE models make them tractable, and at the same time throw away fundamental aspects of macroeconomy.

These discussions raise a natural question; *in what sense are the modern macroeconomic models micro-founded?* The DSGE approach apparently works not on the assumption of the rationality of economic agents, but instead said to be on restrictive supplementary assumptions. They are incorporated into macroeconomic models, *for the sake of simplicity*, but are actually not innocuous at all. In summary, general equilibrium theory does not provide any microfoundations for DSGE or modern macroeconomics.

Why do we cling to the DSGE or modern micro-founded macroeconomics in spite of the fact that they are by no means grounded in the principles of microeconomics? Is there any advantage to follow this approach other than that the majority of economists are used to develop models in that framework? Solow (2008) criticizes modern macroeconomics with his witty analogy.

> "It seems to me, therefore, that the claim that "modern macro" somehow has the special virtue of following the principles of economic theory is tendentious and misleading. The analogy that I like to use, and may have overused, is to someone who tells you that his diet consists of carrots and nothing but carrots; when you ask why, he replies grandly that it is because he is a vegetarian. But the principles of vegetarianism offer no support to so extreme a diet. The relevant definition only requires that the diet contain no meat. Carrots-only is at best mere idiosyncrasy and at worst a danger to health." (p. 244-245)

Therefore, we need an alternative approach to study the macroeconomy consisting of many heterogeneous agents.

1.2 Heterogeneity and Stochastic Macroeconomics

So far, we have discussed the difficulties of drawing macroeconomic inferences without imposing restrictions on individual utility functions. Unless we impose unjustifiable restrictions on individual utility functions, we can say nothing about the aggregate behavior. When one attempts to explain the aggregate behavior, there are two alternative views. One is that the all agents behave in a similar manner and such behavior can be observed at the aggregate level. According to this view, the aggregate behavior can be explained by the corresponding property at the microeconomic level. The other view is that microeconomic agents behave very differently and the distribution of agents' characteristics determines the aggregate behavior. The *distributional approach* (e.g. Hildenbrand (1983), Hardle et al. (1991), Grandmont (1992) Hildenbrand (1994), Quah (1997) and Hildenbrand and Kneip (2005)) takes the second view. It infers some regularity of the aggregate behavior by imposing restrictions on the compositions of heterogeneous groups of agents. Here, the individual maximizing behavior plays only a minor role. It is heterogeneity that explains the aggregate behavior. Hildenbrand (1994), Hildenbrand and Kneip (2005) follow this approach; when the distributions of demand and income spreads out enough or an index of behavioral heterogeneity (discussed in Hildenbrand and Kneip (2005)) is large enough, the aggregate demand satisfies some useful properties, e.g., the *Law of Demand*

$$(p - q) \cdot (F(p) - F(q)) < 0. \tag{1.6}$$

Their argument is very suggestive and sheds the new light on micro-foundations for macroeconomics. A higher degree of heterogeneity among microeconomic agents actually gives a solid structure to the aggregate behavior without imposing any restriction on the behavior of microeconomic agents. It is very important to recognize that heterogeneity in Hildenbrand and Kneip (2005) is different from superficial heterogeneity in DSGE models where a specific utility is assumed with a few varying parameters. Heterogeneity discussed in Hildenbrand (1994), Hildenbrand and Kneip (2005) means that it is impossible to derive explicitly the optimal behavior for individuals, and that there is actually no need for such explorations because the regularity of the aggregate behavior does not depend on such microeconomic details.

We follow their lead, and would like to go a step further. As we discussed above, heterogeneity is an important feature of the macroeconomy. Wage (or income) dispersion across households in the economy is a typical example. If a worker's wage is equivalent to the marginal productivity of labor, the wage is uniquely determined in a Walrasian equilibrium because the wage is the price of labor. However, in reality the economy never reaches such an equilibrium state, and heterogeneity or wage dispersion persists. On the other hand, there exists some observed regularity regarding this heterogeneity. For example, Pareto (1896) found that empirical distributions of higher incomes in many countries exhibit power-law behavior (Pareto's law). Champernowne (1953, 1973) tries to answer the question of why such regularity emerges and presents a mathematical model generating a power law distribution. In his model, individuals' incomes are not simply determined by the marginal productivity of labor nor an optimization problem. In his model, the difference between incomes of individuals is explained "in terms of the difference between their various *qualifications* for obtaining income"(Champernowne (1973), p.50). He investigates the qualifications, i.e., factors of production, carefully and exhaustively, and divides them into five classes; income from ability, capital, land, monopoly advantages and windfalls. He then describes how the amounts and relative values of qualifications are affected by various reasons. For example, the main forces that affect the first qualification, ability, are population, eugenics and heredity, the standard of living, educational facilities, the industrial environment, the social mobility between occupations and classes, and so on. Obviously, these forces are so complicated that it is highly unrealistic to assume that we can trace changes of qualifications for all the individuals. Rather, he argues that the observed characteristics of the distribution of incomes can be explained by assuming that an individual income indexed by i, Y_t^i, is described by the following stochastic process,

$$Y_{t+1}^i = r_{t+1}^i Y_t^i, \quad i = 1...N \tag{1.7}$$

where r_{t+1}^i, the growth rate of income, is an i.i.d random variable and N is the number of individuals. It is assumed that r_{t+1}^i does not depend on one's current income Y_t^i. He shows that the stationary distribution generated by this multiplicative process exhibits the observed power law behavior for high incomes.

It is extremely important to recognize that the assumption that an individual income can be described by a stochastic process does not contradict the optimal behavior of individuals. The relative value and amount of qualifications may actually be determined via market transactions as in Walrasian equilibrium, though Champernowne (1973) points out some qualifications are non-marketable. The point is that their changes are too complicated for the eyes of the outside observer, namely economists who analyze the macroeconomy. Thus, from a viewpoint of an observer, even if economic behavior is deterministic, exact information about all the individuals cannot be obtained and, therefore, it is impossible to distinguish between deterministic and stochastic behaviors. Recognizing the complexity of the situation consisting of a large number of individuals who behave randomly, the stochastic modeling is the only way to investigate the characteristics of income distribution.

The most important point is that whereas the individual behavior is stochastic at the microscopic level, some stable regularity can be observed at the macroscopic level. In other words, idiosyncratic shocks cancel out each other, and then the more fundamental, stable relationship emerges at the macroscopic level. This is because the number of microeconomic agents is so large that the averaging effect works. That is, the randomness and the fact that the system is composed of a large number of elements can be considered as a basis for the observed regularity. Pareto's finding about income distributions is one of the typical examples of this situation. Champernowne (1973) clearly recognizes this point;

> Whereas we have found this stability in the characteristics of the distribution, yet individual incomes, we know, are by no means steady in their movements. Our discovery has been that although individual incomes are fluctuating violently about, the net effect of all this movement is a mere reshuffling which leaves little trace on the main characteristics of the *distribution* of these incomes. (Champernowne (1973), p.90)

It is meaningless to trace the random behavior of individuals in detail. The purpose of his analysis is to investigate how the regularity of phenomena at the macroscopic level can arise from a system consisting of stochastically behaved elements. We call this approach *stochastic macroeconomics* in this thesis.

Champernowne's idea is by no means a new one in the macroeconomic literature. Prior to Champernowne, Kondratiev, one of the leading economists in the Russian school and famous for his finding of the very long cycles, discusses the almost the same point mentioned above. In Kondratieff (1998), he states,

> "[F]or example, an individual instance of suicide or criminality occurs relatively randomly in the sphere of social phenomena. However, a study of suicide and criminality as a social, mass phenomenon, however, reveals surprising patterns. The same thing happens in the area of biology, for example, in questions of heredity and mutability. Other areas of knowledge, such as crystallography, astronomy and, finally, physics also give similar results." (p. 228)

According to this view, the pursuit of dynamic optimization problems, the main part of the current macroeconomics, is not important because from a viewpoint of an observer, micro behaviors are stochastic. To specify stochastic process, we usually need some characteristics of individuals' optimal behaviors, but that does not mean that we are required to solve sophisticated optimization problems. It is the regularity of phenomena arising from a large number of stochastic elements that is a basis for macroeconomic analysis [2]. In fact, the reason why probability theory is widely used in various fields lies in this point. For example, in Gnedenko et al. (1968), a monograph of limiting theorems, the applicability of probability theory is described as follows;

> "... all epistemologic value of the theory of probability is based on this: that large-scale random phenomena in their collective action create strict, nonrandom regularity." (p. 1)

That is, microeconomic variables are stochastic whereas the probability distribution of these variables is given deterministically. If a system is comprised of a large number of elements, the probability distribution can be realized as an empirical distribution by the law of large numbers (abbreviated as LLN). It implies that predictions derived from probability theory is testable. Because the macroeconomy is composed of many agents, the argument above can be applied to macroeconomic analysis. In Chapter 2, we investigate the empirically observed characteristics of the distribution of firm growth rates, and derive important implications about firm growth dynamics.

One might expect that by LLN, microeconomic fluctuations would cancel each other out and, therefore, they have no macroeconomic implications. In the end, only shocks affecting all the individuals or firms are important for aggregate economic fluctuations. This view is taken for granted in the literature. The following argument due to Lucas (1977) is typical.

> "These changes [(the changes in technology and taste)] are occurring all the time and, indeed, their importance to individual agents dominates by far the relatively minor movement which constitute the business cycle. Yet these movements should, in general, lead to relative, not general price movements. ... in a complex modern economy, there will be a large number of such shifts in any given period, each small

[2] See also the relevant discussion in Yoshikawa (2010a,b)

> in importance relative to total output. There will be much "averaging out" of such effects across markets.
>
> Cancellation of this sort is, I think, the most important reason why one cannot seek an explanation of the general movements we call business cycles in the mere presence, per se, of unpredictability of conditions in individual markets." (p. 19-20)

One must note that this argument depends heavily on an implicit assumption that microeconomic behaviors of individual agents are independent of each other. This assumption makes analysis considerably simple. However, it is plain that individuals and firms actually interact with each other. Indeed, the difficulties and complexities in general equilibrium theory result from interactions between agents. Interactions between individuals can generate complex phenomena, and, therefore, we cannot resort to the simple LLN argument. In this thesis, especially in Chapter 3 and 4, we explicitly take into account interaction effects among agents.

The idea that the decision or behavior of an agent is influenced by those of others is by no means new in the macroeconomic literature. Externality has been one of the key concepts in economics for a century. In macroeconomics, the existence of this mechanism is shown to generate an origin of multiple equilibria and/or the multiplier effect. For example, Cooper and John (1988) emphasize the importance of *spillovers* (or *strategic complementarities*) in which one player's decision affects the payoffs of other players. When these effects are present, *coordination failure* can occur. That is, Pareto efficient equilibrium may not be achieved because no individual player has incentive to change his/her strategy. In particular, when players are positively interacted, i.e. reaction curves are positively sloped, Cooper and John (1988) show the existence of multiple Pareto-ranked equilibria and a multiplier mechanism associated with exogenous shocks displaying certain features of the Keynesian economics.

There are many examples in the economic literature exhibiting these circumstances. For example, Howitt and McAfee (1988) study an economy with multiple externalities. One is positive trade externalities. In this case, the higher is the fraction of the population employed (denoted by e), the less the transaction cost becomes (see also Diamond (1982)). The other is diseconomy of scale, according to which the marginal adjustment cost to expand its business activity is an increasing function of e. Under these conditions, they investigate the properties (e.g. stability) of equilibria. Murphy and Shleifer (1989) focus on the aggregate demand spillovers. Because of the presence of externality, even if no sector can make profit by industrializing its own technology, simultaneous industrialization of many sectors can make all the sectors profitable. They view industrialization as a transition from a bad to a good equilibrium. Early contributions to this field are summarized in Cooper (1999).

Among these related literature, the importance of Diamond and Fudenberg (1989) (and Diamond (1982)) cannot be overstated. We review the model of Diamond and Fudenberg (1989) in some detail, and discuss the importance from our viewpoint. Diamond and Fudenberg (1989) investigate a model of rational expectations equilibrium with trading externalities. In their model, individuals take two states, referred to as *employed* and *unemployed*. For each unemployed individual, production opportunities arrive as a *Poisson* process, and the production project is undertaken when its cost, which is also a random variable, is less than the reservation cost, c^*. If the project is undertaken, the individual becomes employed. For each employed, the arrival of a trading partner is assumed to be a Poisson process with arrival rate $b(e)$, which depends on e, the fraction of the population employed. The arrival rate $b(e)$ is an increasing function of e, and, in the terms of Cooper and John (1988), represents strategic complementarities. When the transaction is completed, a previously employed individual becomes unemployed and waits for a next production opportunity. The key point is that the reservation cost c^* depends on the expectations of individuals. The higher is the level of employment, e, the better trading opportunities are. Thus, if an agent optimistically anticipates the future of the economy, the reservation cost for accepting project, c^*, becomes higher.

Under the rational expectations assumption that all individuals correctly anticipate the future of the economy and c^* is uniquely determined, they show the existence of multiple equilibria. That is, if everyone is optimistic, the stock of potential trading partners becomes higher and a *good* equilibrium can be achieved.

There are two important features of the model. One is the feedback loop mechanism; the microeconomic behavior of the agents is affected by the macroeconomic condition e, the level of employment in the economy as a whole. That is, the arrival rate $b(e)$ influencing the individuals' decisions, that is, the choice of the reservation cost, c^*, depends on the fraction of employment,

e, which is the aggregation of such individual decisions. The other technically more important assumption is that there are only two states, employment and unemployment, and that the transition between these two states is described by a Poisson process. Because of this assumption, the state of the economy at the macroscopic level is determined by a single variable, e. In addition, because of the simple structure of Poisson processes, the fraction of agents who change their states (from employment to unemployment and vice versa) is given deterministically by the Poisson rate when the number of agents is large. In this framework, the evolution of the state variables can be reduced to a set of deterministic nonlinear differential equations.

The first feature is extraordinarily important not only in search theory, but also in other macroeconomic models. It violates the simple LLN argument as in Lucas (1977), and can generate rich macroeconomic phenomena in the absence of any aggregate exogenous shocks. This concept is very close to the one emphasized by F. Hahn (e.g., in Hahn (2002)), "macro-micro loop". In Chapter 3, assuming "macro-micro loop", we explain the mechanism of the inventory cycle (Kitchin cycle) without assuming the existence of any aggregate exogenous shocks.

The second feature of Diamond and Fudenberg (1989), however, is too restrictive. Although the assumption of binary choice, unemployment and employment, simplifies analysis significantly, real microeconomic situations are more complex. For example, workers' incomes (in Chapter 4) or firms' inventories (in Chapter 3) are distributed continuously and cannot be realistically described by binary choice models. In particular, the state of the macroeconomy cannot be described by a single variable, e.g., the fraction of agents employed as in Diamond and Fudenberg (1989). In general, we must deal with the distribution itself, that is, an infinite dimensional variable. It requires an alternative framework to investigate the time evolution of the economy at the macroscopic level.

A series of studies conducted by Aoki (Aoki (1996, 2004), Aoki and Shirai (2000) and Aoki and Yoshikawa (2007)) wrestle with this problem, and present a more general framework called *jump Markov model*. Suppose that an agent can take n different state $i \in \{1, ..., n\} \equiv S$. The state space S can be easily generalized to a countably infinite set. Agents change their states stochastically, and the time interval between such changes is also a random variable called *holding* or *sojourn time*. If the sojourn time is given by a Poisson process with an arrival rate λ

(for the sake of simplicity, we assume that $\lambda = 1$) and the transition rate from i to j is given by $q(i,j)$, the dynamics of a system, that is, the evolution of the probability distribution at time t, $P(i,t)$, is given by the following partial differential equation,

$$\frac{\partial P(i,t)}{\partial t} = \sum_{j \in S} q(j,i)P(j,t) - P(i,t)\sum_{j \in S} q(i,j) \tag{1.8}$$

This equation is called the *master equation*. The interpretation is very clear. The first term on the right hand side of the master equation represents the probability flow into state i from all other states, wheares the second term represents outflows of probability from state i. Thus, the master equation means that the rate of changes of $P(i,t)$ is given by the difference between inflows and outflows of probability.

$$\frac{\partial P(i,t)}{\partial t} = (\text{Probability inflows into } i) - (\text{Probability outflows out of } i) \tag{1.9}$$

In physics and chemistry, master equation is widely used to describe the time evolution of a system (See e.g. Van Kampen (1992)). In some special cases (for example, when the detail balance condition holds) a stationary distribution can be obtained analytically. In fact, Aoki and Shirai (2000) apply this equation to the model by Diamond and Fudenberg (1989) and investigate it in detail, especially for cases where the number of agents, N, is finite.

Although Aoki's technique expands the scope of macroeconomic analysis and can be applied to various problems, there are some difficulties and situations which his method cannot deal with. First, his framework is not suited for dealing with diffusion. In Chapter 3, we assume that the sale of a firm fluctuates around a constant $\overline{S}$ due to random noise $\xi \equiv \frac{dW}{dt}$ where W is Brownian motion and $\frac{dW}{dt}$ stands for formal derivative with respect to t. In this case, dynamics cannot be easily expressed in the form of the master equation. In addition, in order to investigate dynamics of the probability distribution, we need to specify the transition function $q(i,j)$ and then solve the master equation. However, except for some special cases, it is intractable. In particular, it is difficult to investigate the nonstationary behavior (out of equilibrium) of the probability distribution by solving master equation in most situations. In Chapter 3, by using the *propagation of chaos*, we present an alternative method to investigate how the system, that is, the probability distribution behaves and changes its property when key parameters changed, without directly seeking the probability distribution itself.

To summarize the above discussion, our primary concern in this thesis is to explain how observed regularity at the aggregate level arises from interacting heterogeneous agents that behave stochastically. Because of the presence of interactions, analysis of the aggregate behavior cannot be reduced to that of individual components. Such *collective behaviors* may be observed only at the macroscopic level, and be even qualitatively different from microeconomic behaviors of individual economic agents. This view is cleary expressed cleary by Kirman (1992).

> "... the way to develop appropriate microfoundations for macroeconomics is not to be found by starting from the study of individuals in isolation, but rests in an essential way on studying the aggregate activity resulting from the direct interaction between different individuals." (p. 119)

(See also Kirman (2006, 2010).) This is what we call *stochastic macroeconomics* and the goal of our thesis.

1.3 Review of the related literature

In this section, we briefly review models where agents interact each other stochastically, and discuss the relation with my own study. These models are sometimes referred as *heterogeneous agents models* (HAM); for reviews, see Aoki (2004), Hommes (2006) and references therein.

Follmer (1974) was the first to develop such a model. In his model, he considered random preferences of consumers and the probability depends on the agents' environment. Using ideas in physics (Ising model), he shows that local interactions generate a nontrivial aggregate consequence, a breakdown of price equilibria. The ideas and techniques originally developed in physics, especially statistical physics, are applied to economic models. For applications of statistical physics to concrete economic problems, see, for example, Aoki and Yoshikawa (2007).

One of the most important applications of HAM is found in the literature on financial markets. Here, the existence of heterogeneity of expectations among market participants is emphasized.

They do not share a *rational* identical expectation of asset prices. Needless to say, if market participants share the common rational expectation, no trade can occur; when the price is expected to fall, no one would like to buy stock, and *vice versa*. The huge volume of tradings observed in real financial markets indicates that there exists heterogeneity of expectations. Thus, HAM is an appropriate framework to describe dynamics of financial markets. In Frankel and Froot (1986) and subsequent papers, two different types of agents are considered, namely *chartist* and *fundamentalist*. Fundamentalist bases his/her strategy upon the fundamental values of assets. In contrast, chartists use patterns observed in the past prices to predict future prices. In market where both types of agents exist with their proportions varying over time, interesting market phenomena such as bubbles and crashes can occur.

For example, Follmer and Schweizer (1993) model asset price dynamics with various types of investors. They show that the resulting price process is an Ornstein-Uhlenbeck process with random coefficients. They also investigate its qualitative behavior and stability of the stock price model (see also Follmer et al. (2005)). Kirman (1991) considers a model of foreign exchange market consisting of chartists and fundamentalists. Agents change their opinions as a consequence of stochastic interactions between agents. He investigates how shifts of opinion occur and shows that opinion can be concentrated on one of the extreme states when self-conversion rate at which an agent change his own opinion independently is relatively small. Having developed a model of opinion formation, he studies how shifts of opinion may affect the dynamics of market prices. By way of simulations, he demonstrates bubble like behavior in market.

Heterogeneous expectations among agents are also important in a different context. Brock and Hommes (1997) consider an economy where agents have different expectation formations. Agents can stochastically switch rules of prediction observing the past realized profits. The evolution of expectations and trading strategies is coupled to the equilibrium dynamics of the endogenous variables. In particular, they consider a simple case with two types of expectation, namely rational and naive expectations denoted by H_1 and H_2, respectively. Expectation H_1 is costly, but accurate (perfect foresight) and when all the agents adopt it the price process is stable. With H_2, the expected price in the next period is simply equal to the current actual price. When a large fraction of agents adopts naive expectations the price process become unstable. Suppose that the majority of agents choose H_1 and the price is close to the steady state value. In this situation, the prediction error of H_2 become small, and the advantage of H_1 is lost due to the information costs. Thus, many agents switch their expectations to H_2. As the number of agents who adopt H_2 increases, the prediction error of H_2 become large, and benefits of knowing the accurate price in the next period eventually surpass the information costs. Most agents switch their expectations to H_1, and the price returns to the steady state value. The story repeats itself. The interaction between the evolution of expectations among agents and market dynamics can lead to complicated dynamics. Using bifurcation theory, they demonstrate highly irregular equilibrium price dynamics converging to a strange attractor.

Durlauf (1993) explores the role of complementarities and the resulting stationary probability distribution. He assumes that each individual industry chooses one of two types of production: one denoted by technology 1 is high production with a fixed cost, and the other (technology 2) is low production without fixed costs (nonconvexity assumption). In his model, by complementarities, the relative productivity of technology 1 is enhanced when other industries in the reference group choose technology 1. This positive spillover effect implies that when the number of neighboring industries committing technology 1 increases, the probability of a firm to choose technology 1 becomes large. He shows that these complementarities, when strong enough, lead to multiple equilibria. He also discusses a possibility of *takeoff* to the high-production equilibrium by the growth of leading sectors.

There are other attempts that focus on stochastic behavior and interactions use the term *social interactions* (see, for example, Brock and Durlauf (2001b) and references therein). Social interactions refer to a situation where an individual agent's behavior depends directly on the choices of other individuals and/or expectations that the agent has of the behavior of other individuals in the reference group. The purpose of such an analysis is to analyze the group behavior emerging from the interdependencies across agents: multiple equilibria or big social *multiplier* that transforms a small change in key parameter into a large change at the aggregate level. Suppose, for example, that when an agent's choice is similar to those of his fellow neighbors, he gets higher utility. Then, it reinforces a tendency that agents with similar behavior or preferences gather, that is, *like will to like*. Racial residual segregation investigated by Schelling (1971) is a good example of this type of social interactions. He showed that even weak preferences for neighbors of similar characteristics, i.e., ethnicity can lead to pronounced segregation.

Brock and Durlauf (2001a), following Blume (1993) and Brock (1993), develop a general binary choice model with social interactions which encompasses many of the earlier models of social interactions. They consider a population of N individuals each of whom makes a binary choice $\omega_i \in \{-1, 1\}$. They assume that individual utility function V can be decomposed into three components,

$$V(\omega_i) = u(\omega_i) + S\big(\omega_i, \mu_i^e(\omega_{-i})\big) + \epsilon(\omega_i) \tag{1.10}$$

where $\omega_{-i} = (\omega_1, ..., \omega_{i-1}, \omega_{i+1}, ..., \omega_N)$ denotes the choices of all the agents other than that of i. u represents the private utility that depends only on his/her own choice. $S\big(\omega_i, \mu_i^e(\omega_{-i})\big)$ represents social utility where $\mu_i^e(\omega_{-i})$ denotes the i's belief concerning the choices of other agents, and $\epsilon(\omega_i)$ represents private random utility. They especially consider the case where this social utility term can be described as follows,

$$S\big(\omega_i, \mu_i^e(\omega_{-i})\big) = -E_i \sum_{j \neq i} \frac{J_{i,j}}{2} (\omega_i - \omega_j)^2 \tag{1.11}$$

For the binary choice model, without loss of generality, the private utility can be replaced by a linear function, $\tilde{u} = h\omega_i + k$ where h and k are chosen so that $h + k = u(1)$ and $-h + k = u(-1)$.

$\epsilon(\omega_i)$ are assumed to be independent and extreme-value distributed so that the difference $\epsilon(-1) - \epsilon(1)$ is logistically distributed

$$Prob\{\epsilon(-1) - \epsilon(1) \leq x\} = \frac{1}{1 + \exp(-\beta x)}, \quad \beta \geq 0. \tag{1.12}$$

Then, the expected value of each of the individual choices can be written as

$$E[\omega_i] = \tanh\big(\beta h + \beta J_{i,j} \sum_{j \neq i} E_i[\omega_j]\big). \tag{1.13}$$

Furthermore, the system is assumed to be completely symmetric and globally interacted (i.e., $J_{i,j} = \frac{J}{N-1}$, J is constant). With the self-consistency condition, $m \equiv E[\omega_i] = E_i[\omega_j]$ for all i, j, equation (1.13) reduces to

$$m = \tanh(\beta h + \beta J m). \tag{1.14}$$

This equation shows that how multiple equilibria can emerge in this model. When the choice intensity, β, and/or social interactions, J, are large, multiple equilibria are likely to emerge. This is an example of collective behavior caused by interactions. Note that it cannot be attributed to any individual rational behavior. Equation (1.14) is actually very familiar to physicists and exactly the same as equation describing the *phase transition* for Ising model, where β represents temperature (see, for example, Hashitsume et al. (1991)). One might consider that economic agents have their own will and cannot be described like molecules or atoms in physics. However, as Brock and Durlauf (2001a) show, there is no essential difference between human economic agents and inorganic molecules. Under stochastic formulation, they have the common mathematical structure, and, therefore, conclusions one obtains are the same. This relation is really the same as the well-known fact that probabilistic model to calculate a fire-insurance premium is essentially the same as the calculation of gambling, e.g., coin-tossing devoid of any economic optimal behavior.

Horst and Scheinkman (2006) establish the general results for equilibria in a framework which encompasses many hitherto known theoretical models. They consider an economy consisting of a large number of agents indexed by $a \in \mathbb{A}$. Here, $\mathbb{A}$ is a subset of the d-dimensional lattice $\mathbb{Z}^d$. Actions of all the agents are described by action profile $x \in S \equiv \{x = \{x^b\}_{b \in \mathbb{A}} : x^b \in X\}$, where X is a set of possible actions. The utility of agent a depends on the actions of other agents locally and/or globally (*social interactions*). In addition, the utility is also affected by random variables, θ^a, and $J^a \in \mathbb{R}^{\mathbb{A}/\{0\}}$. θ^a represents taste shock and the realization of $J^a = (J^{a,b})_{b \neq a}$ determines the structure of networks. For example, when agent a interact with his nearest neighbor, $J^{a,b} = 1$ for $\sum_{j=1}^{d} |a^j - b^j| = 1$ and $J^{a,b} = 0$ otherwise. Thus, agent a's utility is defined by

$$U^a(x^a, \{x^b\}_{b \neq a}, \rho, J^a, \theta^a). \tag{1.15}$$

where ρ denotes the empirical distribution of the economy and represents global interactions.

In equilibrium, when global interactions are considered, the optimal choice which takes into account the empirical distribution must result in the same empirical distribution. This is the self-consistency condition corresponding to (1.14) in Brock and Durlauf (2001a). Then, they establish sufficient conditions that the existence and uniqueness of equilibria are guaranteed. As discussed above, the optimal behavior is not essential (compare with Durlauf (1993)). This result complements to previous works, such as Brock and Durlauf (2001a), in the sense that if these conditions are violated, multiple equilibria can emerge.

In line with these researches, we shall attempt to explain some regularity of the aggregate behavior arising from heterogeneous agents. In contrast to DSGE models, rationality of microeconomic agents plays only a minor role in our analyses. Rather, given the fact that microeconomic agents' behavior is heterogeneous and complicated, we shall seek to obtain important implications for macroeconomic phenomena by resorting to stochastic approach.

1.4 Contents

The rest of this thesis is organized as follows.

In Chapter 2, I explore firm growth dynamics. How a firm grows is one of the most important themes in industrial organization literature. Recent empirical studies have demonstrated that the distribution of firm *growth rates* is not Gaussian as predicted by the celebrated Gibrat's law (Gibrat (1931)); rather, it closely follows the Laplace distribution. These findings challenge the existing theoretical models, as well as our understanding of the mechanism of firm growth. To explain the empirical distribution, we consider firm growth dynamics in the framework of the *Lévy processes* and *infinitely divisible distributions*. Our analysis shows that the growth of firm does not result from an accumulation of small shocks as the existing models presume. Instead, it is a handful of large shocks to the firm or jumps that derives firm to marked growth. This result has important implications for our understanding of the nature of innovations.

Chapter 3 investigates the relationship between microeconomic shocks and the aggregate fluctuations. The conventional argument against the relevance of microeconomic shocks for the aggregate fluctuations rests on the law of large numbers. That is, whatever the microeconomic structure is, such shocks would average out at the aggregate level (Lucas (1977)). In order to explain the aggregate fluctuations, aggregate exogenous shocks that affect all the firms in the economy are indispensable. However, Chapter 3 illustrates how aggregate fluctuations can arise from microeconomic stochastic behaviors. At the micro level, empirical data suggest that the standard production-smoothing theory, according to which firm uses its inventories as buffer stock for the cost-minimization, does not hold, and that the firm behavior is characterized by lumpiness (e.g. Blinder and Maccini (1991)). To account for that, a nonconvex cost function is assumed in our model. In addition, we introduce interactions among firms into the model. We then focus on a feedback mechanism. Namely, the behavior of firm is affected by the aggregate economic condition while the economic condition, in the economy as a whole, in turn, is determined by the aggregation of the firms. This is, of course, related to the "macro-micro loop" emphasized by Hahn(2002), where a macro variable acts as externality. Under these assumptions, we show that an endogenous cycle of production emerges at the aggregate level given that the degree of the interaction effects exceeds a certain critical point. It offers an explanation for Kitchin cycles.

Chapter 4 investigates implications of observed statistical regularities of income distributions, by taking into account interactions among individuals. It has been long known that the income distribution is well described by the lognormal distribution (except for the high income range). It is natural to expect there to be the universal structure behind it, and ask what kind of mechanism generates this observed regularity. In fact, the study of the income distribution has long history, and many theoretical investigations have tackled with this problem. However, these previous studies implicitly assume that a worker's income is independent of those of others. In Chapter 4, we explicitly take into account economic interactions among individuals. We then investigate the interaction effects, f, consistent with the observed statistical regularities of income distribution, that is, the lognormal distribution, under two different assumptions about idiosyncratic shocks. Under the assumption of additive shocks, we show that the function $f(x)$ takes its minimum value. Under the assumption of multiplicative shocks, we identify the form of the pairwise interaction between workers. These implications are derived only from the empirical fact that the income distribution is the lognormal distributions. They do not depend on any assumption of preferences and abilities of individuals. In this sense, our findings are robust.

In appendix A, we test the *granular hypothesis* proposed by Gabaix (2011). The granular hypothesis means that a significant fraction of the aggregate fluctuations can be explained by idiosyncratic shocks to very large firms. Focusing on the recent findings showing the similar regularity at both the macro and micro levels, we apply the procedure developed in Chapter 2. Under the granular hypothesis, we calculated the fraction of the aggregate fluctuations attributable to the top 100 largest firms' jumps. We found that the granular hypothesis cannot explain the aggregate fluctuations. Even if the firm size follows power law distribution, the size of large firms is still too small to have significant impacts on the aggregate fluctuations. We showed that macroeconomic fluctuations are not a reflection of microeconomic shocks to large firms.

In Appendix B, we show that the conclusion derived in Chapter 2 holds under more generalized assumptions, and is consistent with the discussions in Chapter 3. Since the seminal work by Gibrat (1931), statistical investigations of empirical distribution have all implicitly presumed the statistical independence of economic agents. In Chapter 2, we also assume that the stochastic processes for the firms' size are independent of each other. This assumption appears to be crucial because the failure of the independence assumption makes the problem complicated and analytically intractable. However, we demonstrate that this assumption can be relaxed in a natural manner in the framework of the nonlinear stochastic differential equations introduced in Chapter 3 as long as the symmetry of the model is preserved. That is, even if each element in the system interacts each other, the conclusion based on the independence assumption still holds true as long as the interactions are formulated as in Chapter 3.

Chapter 2

Firm Growth Dynamics: The Importance of Large Jumps

2.1 Introduction

In this Chapter, we explore the implications of the observed Laplace distribution of growth rates. We show that the growth of a firm is not an accumulaion of a large number of small shocks, but is largely explained by a small number of large jumps. Our findings have profound implications for our understanding of the nature of innovations.

How a firm grows is one of the important themes in the industrial organization literature. It has already been discussed extensively in the literature, and earlier works include Ashton (1926), Gibrat (1931), Simon and Bonini (1958), Little (1962), Steindl (1965), and Ijiri and Simon (1977), to name a few. Among them, in his seminal work, Gibrat (1931) attempts to explain the skewed pattern of firm size distribution. Gibrat's law (Gibrat (1931)), that the growth rate of a firm is independent of its size, continues to receive much attention in the theoretical and empirical literature (for a review, see Sutton (1997)). Gibrat's approach models the size of an existing firm, S_t, as an accumulation of small shocks:

$$S_{t+1} = (1 + \epsilon_t^1)(1 + \epsilon_t^2)(1 + \epsilon_t^3)...(1 + \epsilon_t^n)S_t.$$

Taking the logarithm of both sides, under weak conditions, the Central Limit Theorem (CLT) leads to a log-normal firm size distribution. Note that in this model, when each shock is small enough and the number of shocks n is large, the distribution of growth rates defined by $\log(S_{t+1}) - \log(S_t)$ becomes Gaussian. Following Gibrat (1931), Ijiri and Simon (1977) (and later Sutton (2001)) offer an explanation for skewed firm size distribution, widely known as *the island model*. They assume that firms compete for *business opportunities* in order to grow, and that the size of a firm reflects the number of opportunities it has gained. They show that the firm size distribution converges to the Yule distribution.

A different stream of literature that focus on the R&D investment decision and heterogeneity of firms has proposed theoretical models for firm-level empirical studies. In Ericson and Pakes (1995), firms are exposed to uncertainty arising from investment, leading to variability in their fortunes. Firms maximize the expected present value of their profits based on their beliefs about other firms' behavior. They show that the beliefs of firms are fully consistent with the optimal decisions of all firms at an equilibrium. Klette and Kortum (2004), following Penrose (1959), propose a more parsimonious model. A firm is composed of the goods it produces and can expand into a new market through innovation. When the firm takes over a particular market through innovation, its competitor loses the market (*creative destruction*). They argue that their model can account for firm-level evidence, for example, the persistence of differences in R&D intensity among firms and why productivity growth is not strongly related to firms' R&D.

Recent discoveries have led to interest in firm growth theory again. Since Stanley et al. (1996), subsequent papers have shown that the distribution of firm growth rates is *not* Gaussian; rather, it closely follows the Laplace distribution. This finding is not consistent with the model described above because it predicts a Gaussian distribution. More interestingly, the same characteristics of the distribution of firm growth rates appear at the finer sectoral level. This is a robust feature that holds at various levels of aggregation. On the other hand, the characteristics of the firm *size* distribution differ significantly across industries (see Bottazzi et al. (2007), Bottazzi et al. (2011)). This suggests that firm size distribution is just a consequence of statistical aggregation, and therefore, the explanation of the distribution of firm growth rates is more important for understanding firm growth dynamics [1]. Therefore, we focus on the distribution of firm growth

[1] For this reason, Bottazzi et al. (2011) conclude that "[a]s a result, the distribution of firm size seems to be of

rates, instead of firm size in this Chapter.

The non-normality of growth rates challenges the existing models based on Gibrat's approach because it predicts that the distribution approaches Gaussian. Obviously, it is important to investigate why Gibrat's model fails and which assumption of the model is violated. Several studies have attempted to explain these problems. For example, Bottazzi and Secchi (2006) assume a *success brings success* type of dynamics. A successful firm has a higher probability to achieve another success. They assert that this positive feedback generates a big leap and that the resultant distribution has a fatter tail. A different model, proposed by a group of physicists (Buldyrev et al. (2007a)), assumes that the distribution of firm growth rates is the convolution of two random variables: one is the number of products a firm produces and the other is the size of the market of each product. As we will see later, both models fail to explain the observed Laplace distribution.

In this chapter, we provide an alternative explanation and show how Gibrat's model can be modified to account for the Laplace distribution. We relax one of the assumptions of Gibrat's model, and show that the family of infinitely divisible distributions is appropriate to describe the firm growth dynamics. We further show that the Variance-Gamma process corresponding to the Laplace distribution represents the underlying firm dynamics. We focus on the sample path properties and show that firm growth is not the consequence of an accumulation of many small shocks. Rather, the firm occasionally jumps. That is, the process of growth can be well-described by a handful of large shocks, and therefore, we can identify which events really lead to the prosperity (or decay) of a firm. This is in sharp contrast to the assumptions of existing models such as Klette and Kortum (2004).

The rest of the chapter is organized as follows. Section 2 summarizes empirical evidence related to the distribution of growth rates. In particular, the recent findings that the distribution of firm growth rates closely follows the Laplace distribution have opened a new field of inquiry. Some evidence of the growth rates of Japanese firms is also provided to reconfirm the stylized facts. In Section 3, we briefly review firm growth models proposed thus far, and then develop a general framework to extend Gibrat's model. We investigate the sample path properties of the firm

limited interest to economists."

growth process and show that the firm growth is largely explained by a small number of large jumps. Section 4 discusses the robustness of our analysis developed in Section 3. Introducing subexponential distributions, we show that our conclusion is still valid even in the cases where the distribution of growth rates has a fatter tail than the Laplace distribution. Section 5 discusses the economic implications of our analysis and related literature. Section 6 concludes.

2.2 Empirical Evidence

2.2.1 Size Distributions

One of the important issues attracting many economists in the industrial organization literature is the observed statistical regularity of the firm size distribution. It has long been known that the firm size distribution is positively skewed. The investigation into this observed fact also has a long history and dates back, at least, to a seminal work by Gibrat (1931). Gibrat (1931) considers plant (establishment) sizes in French manufacturing and showed that it can be described by the lognormal distribution quite well (for Gibrat's model generating the log-normal distribution, see section 3). He also presented various data to explain that this is the statistical regularity that can be viewed as a universal law reflecting firm growth dynamics. Hart and Prais (1956) analyze the distribution of quoted companies in the U.K. for the selected years in the period 1885-1950, and presented further evidence that support Gibrat's finding.

Of course, the lognormal distribution is not the unique distribution exhibiting the positive skewness, and there exists other attempts to describe the firm size distribution. For example, Simon and Bonini (1958) and the monograph of their research, Ijiri and Simon (1977) propose Yule (or Yule-Simon) distribution. In particular, they consider entry of new firms, that is, the size distribution is a composition of the cohorts of firms with different ages. The peculiarity of this distribution is that its tail is fatter than that of the lognormal distribution, and exhibits power law behavior in its tail. Today, this is also considered as one of the examples of Zipf's law (see, for example, Saichev et al. (2009)). Zipf's law usually means that the probability distribution $P(s) = \Pr\{S > s\}$ satisfies the following relation, especially in the upper tail,

$$P(s) \sim s^{-1}. \tag{2.1}$$

The above relation represents the slower decay of probablity, so the power law indicates that there exist a few giant firms. Steindl (1965) studies Austrian data and concluded in favor of the Pareto distribution, which exhibits the power law behavior.

Although the skewness of the firm size distribution is widely accepted and various models have been proposed so far, there are some empirical results that lead us to doubt the relevance of the statistical regularity of the skewed nature. For example, Bottazzi and Secchi (2005) and Bottazzi et al. (2011) show that the observed characteristics of the firm size distribution at the aggregate level cannot be observed at the disaggeregated level (see also Dosi et al. (1995), Bottazzi et al. (2007)). They observe significant bimodality in the size distribution at the sector level. Although the characteristics of the firm-size distribution at the aggregated level described by the log-normal or Pareto distribution is robust, it can be considered to be simply a result of a statistical aggregation effect. It is different from what Gibrat (1931) and subsequent papers have envisaged. Examining these results, Bottazzi et al. (2011) conclude that "[a]s a result, the distribution of firm size seems to be of limited interest to economists." On the other hand, they argue that the *growth rates distribution* has robust characteristics observed at various levels of aggregation. That is, the characteristics do not depend on the degree of an aggregation. In the next subsection, we discuss the *stylized facts* of the growth rate distribution.

2.2.2 Distribution of Growth Rates

Although it has long been known that the distribution of firm growth rates is fat-tailed (see, for example, Ashton (1926) and Little (1962)), it was not until recently that the functional form of the distribution is specified. Stanley et al. (1996) analyze publicly traded U.S. manufacturing firms and find that the distribution of growth rates display a tent-shaped density. They propose the Laplace distribution to describe the density:

$$p(x; \gamma, a) = \frac{1}{2a} \exp(-\frac{|x - \gamma|}{a}). \tag{2.2}$$

The goodness of fit that they obtain is striking. This density has a fatter tail than Gaussian; that is, there is a relatively higher probability that a firm experiences more extreme events than predicted by the Gaussian distribution. On the other hand, contrary to power-law distributions, this density has finite moments of all orders, because the tail decays exponentially.

Subsequent papers have confirmed this finding. Amaral et al. (1997) analyze all publicly traded U.S. manufacturing firms and find an exponential tent-shaped form rather than the bell-shaped Gaussian. Bottazzi and Secchi (2003) and Bottazzi and Secchi (2006) analyze the Italian manufacturing industry and Italian firms, respectively, and confirm the Laplace shape. Alfarano and Milakovic (2008) analyze the Forbes Global 2000 list of the world's largest companies and find a similar pattern.

Following these works, we study publicly traded Japanese companies (listed on the first and second section of the Tokyo Stock Exchange, Tokyo Stock Exchange Mothers, and JASDAQ) for the years 2000–2010. The dataset we use is developed by Nikkei Digital Media Inc and contains data based on the companies' annual securities reports. In what follows, we take the number of employees as the definition of firm size [2]. Let S_t be the number of employees of a firm at time t [3]. It is customary to study firm growth on logarithmic scales, and as such, the growth rate is defined as follows:

$$g_t = \log S_{t+1} - \log S_t.$$

The dataset also includes information on whether mergers and acquisitions (M&As) occur in a year. This information can be used to identify whether the growth of a given firm is internal or external, such as via M&A. Large outliers in the dataset are mainly due to these external events. We simply exclude them because we focus on the process of internal growth in the present analysis. This procedure reduces the number of growth rate observations considerably, and we finally have $N = 7549$ observations for 2000–2010.

Figure 2.1 shows a simple histogram of the distribution of growth rates of Japanese firms and

[2] It is interesting that the Laplace distribution emerges across different measures of size, such as sales or value added. See, for example, Bottazzi et al. (2007).

[3] Here, we take consolidated companies as the definition of firms. We exclude banks, insurance, and security companies in our dataset.

the fitted Laplace distribution with the maximum likelihood estimate $a = 0.054$ [4]. It clearly displays a tent-shaped form as previous studies have observed. In Figure 2.2, the histogram is replotted with a logarithmic vertical axis. When plotted in this way, it closely follows a straight line representing the Laplace distribution. Both depict marked departures from Gaussian and are rather Laplace [5].

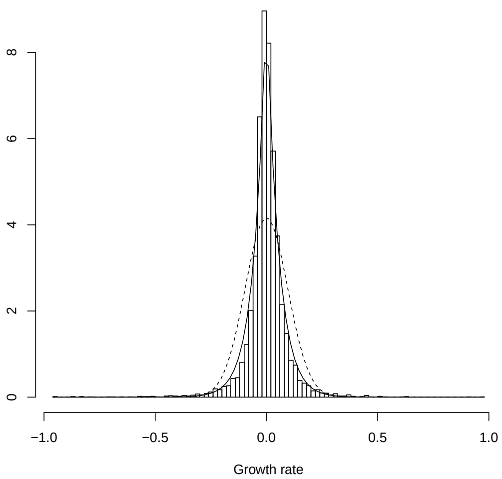

Figure 2.1: Growth rates for Japanese firms, subtracting the sample average (= −0.0049). The solid curve is (2.2) with a maximum likelihood estimate $a = 0.054(0.00062)$ (the value in the parentheses is the Std. Error.). The dotted curve is a Gaussian distribution with standard deviation $\sigma = 0.096$.

2.3 Firm Growth Model

In microeocnomics textbooks following the principle of neoclassical economics, what readers are required to do is to solve profit maximization problems given a cost function. In this theory, firms are considered to go toward the *optimal size*, in which maximization of profits is achieved.

[4] We subtract the average growth rates from the growth rates; that is, we rescale $\gamma = 0$.

[5] Some might notice that the bulk of the empirical distribution is well-fitted by the Laplacian distribution, but it has a fatter tail than the Laplace distribution. On this point, see Section 4.

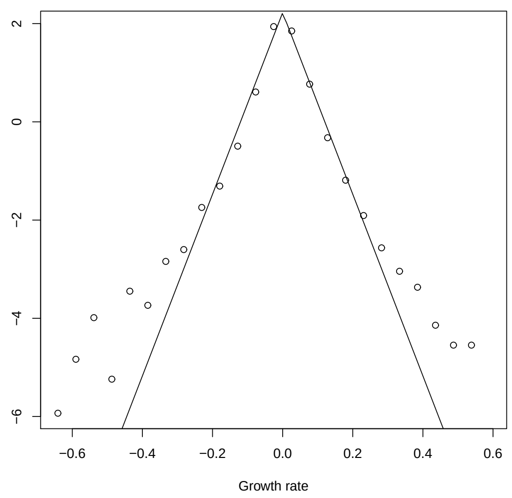

Figure 2.2: The same data in Figure 1 with logarithmic scale. The y-axis is $log(P(g))$.

Firm growth is of no interest to economists because it is just a monotonistically increasing or decreasing path reaching the optimal size. All we need to care is the speed at which the convergence occurs and associated adjustment costs. Especially, in an equilibrium, firms grow no more. Lucas (1978) is one of the examples that follow this principle.

However, though this theory accounts for some part of the undergraduate course, it has no empirical support. In reality, the firm growth exhibits very complex behavior and cannot be described by a monotonistically increasing or decreasing path. Facing these complicated phenomena, for example, Ashton (1926), who studies British textile firms, states,

> "In their growth they obey no one law. A few apparently undergo a steady expansion... With others, increase in size takes place by a sudden leap."

The firm growth dynamics is not such a simple story as considered in neoclassical economics. Examining empirical data, one can easily find that there exists no unique optimal size that firms seek for. In addition, there exists no firm growth at a neoclassical equilibrium. Economists who do not play with desk theories have to investigate empirical data closely. For them, *Gibrat's law* discussed below is a cornerstone of the investigation of firm growth dynamics.

2.3.1 Gibrat's model

In this subsection, we briefly review theoretical and empirical contributions by Gibrat (1931) and related literature. Today, Gibrat's law (also called "law of proportionate effect") refers to an empirical law that the size of a firm and its growth rate are statistically independent. This means that the size of a firm at time $t+1$, denoted by S_{t+1}, can be written by the product of an independent idiosyncratic growth shock, ϵ_t, and the firm size at t, namely,

$$S_{t+1} - S_t = \epsilon_t S_t.$$

This is equivalent to saying that firm size at $t+1$ can be described by multiplicative shocks up to $t+1$ and the initial condition, S_1:

$$S_{t+1} = (1+\epsilon_1)(1+\epsilon_2)...(1+\epsilon_t)S_1.$$

This process is called a multiplicative process. Taking logarithm of both sides of this equation, it can be written as

$$\log S_{t+1} = \log(1+\epsilon_1) + \log(1+\epsilon_2)... + \log(1+\epsilon_t) + \log S_1 \approx \sum_{s=1}^{t} \epsilon_s + \log S_1.$$

where we use an approximation $\log(1+\epsilon) \approx \epsilon$ when ϵ is small, which is equivalent to saying that we consider small interval between $t+1$ and t. When t is large and the effect of the initial condition, $\log S_1$, becomes negligible, the logarithm of the size of a firm can be written as an accumulation of growth shocks,

$$\log S_{t+1} \approx \sum_{s=1}^{t} \epsilon_s.$$

It is clear that if growth shocks are independent of each other and satisfy standard conditions of CLT, the distribution of the logarithm of firm size follow Gaussian distribution. By definition, it implies that the firm size distribution is described by the lognormal distribution. Gibrat first applies this model to income distribution and then plant size distribution.

As described above, his model is very simple, but has some serious limitations. First, the distribution of firm size predicted by his model does not converge to a stationary distribution because the variance of the distribution diverges as t goes to infinity. Of course, this prediction is inconsistent with empirical data. Kalecki (1945) points out this limitation of Gibrat's model and attempts to modify it to have a stationary distribution with a finite variance. He considers a mean revearting process, that is, when the size of a firm is far from the mean, a force pushing the size to the mean comes into play. Although this implies a violation of Gibrat's law, he succeeds in obtaining a stationary distribution with finite variance.

There have been other attempts to modify Gibrat's original model. Simon and Bonini (1958) and Ijiri and Simon (1977) modified Gibrat's model by taking into account the arrival of new firms according to which the number of firms in an economy increases over time. This implies that the size distribution is a composition of the cohorts of firms with different ages. They show that the resulting distribution converges to a Yule (or Yule–Simon) distribution. By considering new entry, their models avoid the conclusion that the variance in firm size distribution increases without limit.

It should be noted that wheares Gibrat's main consern is the explanation of the positive skewness of the firm size distribution, his model is closely related to the distribution of growth rates. The growth rate, g_t, defined in the previous section can also be written as an accumulation of shocks:

$$g_t = \log S_{t+1} - \log S_t = \sum_{i=1}^{n} \log(1 + \epsilon_t^i) \approx \sum_{i=1}^{n} \epsilon_t^i,$$

where the time interval between $t + 1$ and t is divided into n parts, and a growth shock, $\epsilon_t^i, i = 1, ..., n$, corresponds to each one. Using the same arguments as those used for firm size distribution, the distribution of growth rates converges to a Gaussian distribution when n goes to infinity [6]. This property is also shared by Simon and Bonini (1958) and Ijiri and Simon (1977), if we restirict our attention to incumbent firms. Ijiri and Simon (1977) states,

[6] It should be noted that what Gibrat's law (or Gibrat's hypothesis) means differs from author to author. A "weak" version of Gibrat's law merely means that expected growth rate is independent of firm size (for example, see Gabaix (2009)) and does not assume a Gaussian distributon for growth rates. However, once the multiplicative assumption is accpepted on every time scale, "[t]his is indeed a straightforward conjecture"(Bottazzi et al. (2002), p.710). In this sense, Bottazzi et al. (2002) call it "a strong Gibrat hypothesis." Stanley et al. (1996) and Reichstein and Jensen (2005) call this strong version "Gibrat's law" (or "Gibrat's model").

In this Chapter, we simply call this stronger version *Gibrat's model* and a weaker version corresponding to assumptions 1 and 2 (see below) *Gibrat's law.*

> "What distinguishes the Yule distribution from the log-normal distribution is not the first assumption - the law of proportionate effect - but the second - the assumption of a constant "birth rate" for new firms. ... If we assume a random walk, but with a steady introduction of new firms from below, we obtain the Yule distribution"(Ijiri and Simon (1977), p.143).

In short, the basic assumptions of Gibrat's model are summarized as follows:

1. The growth rate of a firm is independent of its size.

2. The successive growth rates of a firm are independent of each other.

3. The growth rate of a firm consists of small shocks that satisfy the conditions of the standard CLT.

In the literature, a large number of studies have been devoted to test the validity of assumptions 1 and 2 (see Mansfield (1962), Singh and Whittington (1975), Evans (1987), Fujiwara et al. (2004), and Santarelli et al. (2006) for reviews). Although their results seem to be inconclusive [7], they suggest that assumptions 1 and 2 are not implausible and can be taken as a good starting point. In particular, it is likely that the goodness of fit of Gibrat's law improves for larger firms. For example, Fujiwara et al. (2004) analyze firms in European countries and show that the conditional distribution $P(R|x_1)$ for large firms is independent of x_1, where x_1 is the size of a firm at a certain time, x_2 is the size at a later time, and $R \equiv x_2/x_1$.

As observed in the previous section, the distribution of growth rates is Laplace, being inconsistent with Gibrat's model. These findings strongly suggest that the observed Laplace distribution arises from the violation of assumption 3. Thus, in the next subsection, we relax assumption 3 and obtain implications of the observed Laplace distribution.

[7] According to Santarelli et al. (2006), "[o]ne cannot conclude that the Law is generally valid nor that it is systematically rejected"(Santarelli et al. (2006), p.43). This is why economists still continue to test Gibrat's law, more than 80 years after Gibrat's publication.

2.3.2 Infinitely Divisible Distributions and Lévy Processes

As discussed above, non-Gaussian distribution of growth rates means the rejection of Gibrat's model. Recent theoretical investigations have tried to explain such empirical distributions by employing different mechanisms. For instance, Bottazzi and Secchi (2006) assume that "the probability that a given firm obtains new opportunities depends on the number of opportunities already caught" (p.251). That is, success leads to further success. This positive feedback generates big leaps and a fat-tailed distribution. Assuming that the Polya Urn statistics describe the feedback, Bottazzi and Secchi (2006) prove that the resulting distribution converges to Laplace. However, their analysis relies crucially on the assumption that "the process of opportunity assignment is repeated anew each year, i.e., that no memory of the previous year's assignment is retained when the new year's opportunities are assigned" (p.251). Otherwise, as time t passes, a single lucky firm would end up gaining all the opportunities and eventually diverging in size. Why is the feedback mechanism cut off abruptly at the end of the year? Why are firms not carrying over their technology or R&D investment to the next year? Their crucial assumptions must be said to be arbitrary.

A different kind of model is proposed by a group of physicists (Buldyrev et al. (2007a)). They model a business firm as a portfolio of products. They assume that the number of products is a random variable whose distribution converges to an exponential distribution. The size of each product is assumed to follow an independent random process. Hence, the distribution of firm growth rates is a combination of these two random variables. In this model, the conditional distribution of growth rates of firms with the same number of products converges to a Gaussian distribution. The Laplace shape is solely due to the aggregation of firms with a different number of products. This implies that if we disaggregate the data by, for example, firm size or sector, the Laplace shape would disappear. However, Stanley et al. (1996) show that the conditional distribution $p(r|s_0)$, where s_0 is firm size (an initial sales value), is well-described by the Laplace distribution. Moreover, Sakai and Watanabe (2010) empirically test this model and conclude that the empirically observed Laplace shape is not explained by their model.

Different from these previous models, we consider a necessary minimum generalization of Gibrat's model to account for the Laplace distribution. As we discussed above, assumption 3 has to be relaxed, and therefore, we assume that growth rate consists of independent random shocks but does not require the conditions of standard CLT. A firm grows for all sorts of reasons, for example, the introduction of new products, quality improvements in existing products, cost reduction induced by technological progress, creation of new markets, and effective advertisement. On the other hand, firm size can also decrease due, for example, to serious conflicts with the labor union, changes in consumer preferences, and emergence of superior substitutes for the firm's product. Because firms are subjected to various shocks stemming from different reasons, it is plausible to assume that these shocks are different in terms of their impacts on the firm's performance, which implies that standard CLT conditions fail. Some shocks are expected to have an disproportional impact on firms. The assumption that shocks are identically distributed cannot be justified. Therefore, we need a broad generalization of standard CLT.

A fundamental limit theorem on sums of independent random variables has been proven by Khintchine (1937) (see also the monograph of limiting theorems, Gnedenko et al. (1968)),

Theorem 2.1 *Let $\{Z_{nk}\}$ be a null array on $\mathbb{R}^d$ with row sums $S_n = \sum_{k=1}^{r_n} Z_{nk}$. If, for some $b_n \in \mathbb{R}^d, n = 1, 2, ...,$ the distribution of $S_n - b_n$ converges to a distribution μ, then μ is infinitely divisible* (Khintchine (1937); see also Sato (1999), p.47).

A double sequence of random variables $\{Z_{nk} : k = 1, 2, ..., r_n; n = 1, 2, ...\}$ on $\mathbb{R}^d$ is called a null array if for each fixed n, $Z_{n1}, Z_{n2}, ..., Z_{nr_n}$ are independent, and for any $\epsilon > 0, \lim_{n\to\infty} \max_{1\le k\le r_n} P[|Z_{nk} > \epsilon] = 0$. The last condition means that no single variable dominates the sum *in probability.*

The definition of infinitely divisible distributions is given as follows.

Definition 2.1 *A probability measure μ on $\mathbb{R}^d$ is infinitely divisible if for any positive integer n, there is a probability measure μ_n on $\mathbb{R}^d$ such that $\mu = \mu_n^n$, where μ^n is the n-fold convolution of the probability measure μ with itself.*

Infinite divisibility means that the random variable with distribution μ can be expressed by the sum of an arbitrary number of independent and identically distributed random variables with distribution μ_n. Gaussian and stable distributions belong to this family.

It is well known that the characteristic function of infinitely divisible distributions, $\hat{\mu}$, can be represented by the Lévy-Khintchin formula:

$$\hat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + i\langle\gamma, z\rangle + \int_{\mathbf{R}^d}(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_D(x))\nu(dx)\right], z \in \mathbb{R}^d \tag{2.3}$$

where $\langle,\rangle$ is an inner product, A is a symmetric nonnegative-definite $d \times d$ matrix, ν is a measure on $\mathbb{R}^d$ satisfying

$$\nu(0) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty,$$

D is the closed unit ball, and $\gamma \in \mathbb{R}^d$.

The Laplace distribution is also included in the family of infinitely divisible distributions. This fact is easily checked by using Lévy-Khintchin formula. The characteristic function of the Laplace distribution with $\gamma = 0$ is written as

$$\hat{p}(z) = \frac{1}{1 + a^2z^2} = \frac{1}{(1 + iaz)(1 - iaz)} \tag{2.4}$$

Factor $\frac{1}{(1-iaz)}$ is the characteristic function of an exponential distribution defined by

$$p(x) = 1_{(0,\infty)}\frac{1}{a}\exp(-\frac{x}{a}).$$

This means that the Laplace distribution is obtained by the convolution of two identical exponential distributions. The characteristic function of an exponential distribution can be written in the form of (2.3):

$$\hat{p}(z) = \frac{1}{(1 - iaz)} = \exp\left[\int_0^\infty (e^{izx} - 1)\frac{e^{-\frac{x}{a}}}{x}dx\right], \quad \text{(Sato (1999), p.45)} \tag{2.5}$$

where $A = 0, \nu(dx) = 1_{(0,\infty)}(x)x^{-1}e^{-\frac{x}{a}}$ in (2.3). It shows that the exponential distribution is infinitely divisible and, therefore, the Laplace distribution is infinitely divisible because the distribution obtained by convolution of two infinitely divisible distribution is also infinitely divisible. When $\gamma \neq 0$, a drift term, $i\langle\gamma, z\rangle$, in (2.3) is added.

Moreover, if an infinitely divisible distribution is given, one can show that there exists a corresponding Lévy process.

Theorem 2.2 *For every infinitely divisible distribution μ on $\mathbb{R}^d$, there is a Lévy process $\{X_t\}$ such that $P_{X_1} = \mu$. It is unique up to identity in law.* (Sato (1999),p.63)

It should be noted that there is the one to one correspondence between Lévy processes and infinitely divisible distributions. It implies that a Lévy process is completely characterized by $\{A, \nu, \gamma\}$ in (2.3) called *generating triplets.*

For convenience, the definition of Lévy process is given as follows,

Definition 2.2 *A stochastic process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ is a Lévy process if the following conditions are satisfied.*

1. $X_0 = 0$ *a.s.*
2. *For any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < ... < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ are independent (independent increments property).*
3. *The distribution of $X_{s+t} - X_s$ does not depend on s (temporal homogeneity or stationary increments property).*
4. *It is stochastically continuous.*
5. *There exists $\Omega_0 \in \mathcal{F}$ with $P[\Omega_0] = 1$ such that for every $\omega \in \Omega_0$, $X_t(\omega)$ is right-continuous in $t \geq 0$ and has left limits in $t > 0$.*

The Condition 1, $X_0 = 0$, is not essential, because, otherwise, we only add a constant. A stochastic process $\{X_t\}$ on $\mathbb{R}^d$ is called stochastically continuous if for every $t \geq 0$ and $\epsilon > 0$,

$$\lim_{s \to t} P[|X_s - X_t| > \epsilon] = 0. \tag{2.6}$$

The stochastic continuity, (2.6), should not be confused with the continuity of sample paths. In fact, the stochastic continuity allows for the discontinuity of sample paths. Suppose, for example, a Poisson process with a rate parameter λ. This process is discontinuous at points where a jump occurs and the value of the process increases by one. However, when such a jump occurs too is a random process and the probability that a particular point on a continuous space, e.g., $[0, T]$ is chosen is 0. Therefore, the Poisson process satisfies the stochastic continuity property. Condition 4 simply means that there exist no predetermined points at which a jump occurs with positive probability.

As easily found, Lévy processes are a natural generalization of Gibrat's model. Indeed, assumptions 1 and 2 of Gibrat's model (in terms of Lévy process, independent increment property) are satisfied. The Lévy processes also include the Brownian motion described by Gibrat's model. In the following, we discuss firm growth dynamics in this framework.

2.3.3 Variance-Gamma Process

As discussed above, there exists a Lévy process corresponding to a given infinitely divisible distribution, especially the Laplace distribution. Accordingly, a process, called the *Variance-Gamma process*, is what we need for our purpose. This was first introduced in option pricing theory by Madan and Seneta (1990), and the mathematical properties of the process and its generalization, pure jump processes, are fully explored by Ferguson and Klass (1972).

Let X_t be the logarithm of the size of a firm, $\log S_t$. Equation (2.4) means that we can represent X_t, that is, the Variance-Gamma process, as the difference between two independent and identical processes Y^1 and Y^2:

$$X_t = Y_t^1 - Y_t^2. \tag{2.7}$$

Here, the characteristic function of Y_t can be written as

$$\hat{P}_{Y_t} = \exp\left[t \int_0^\infty (e^{izx} - 1)\frac{e^{-\frac{x}{a}}}{x} dx\right]. \tag{2.8}$$

Note that Y_t is increasing as a function of t, a.s. This means that firm growth can be divided into two components, a positive growth process and a negative one, both having the same distribution.

To proceed further, we need the Lévy–Itō decomposition to investigate the sample path properties.

Theorem 2.3 *Lévy-Itō decomposition : Let* $\{X_t : t \geq 0\}$ *be a Lévy process on* $\mathbb{R}^d$ *defined on a probability space* $(\Omega, \mathcal{F}, P)$ *with generating triplets* $\{A, \nu, \gamma\}$. *Suppose that the Lévy process satisfies* $\int_0^1 |x| \ \nu(dx) < \infty$ *for all* t. *Let* γ *be the drift of* X_t. *Then, there is* $\Omega_1 \in \mathcal{F}$ *with* $P[\Omega_1] = 1$ *such that, for any* $\omega \in \Omega_1$,

$$X_t^1(\omega) = \int_{(0,t]\times D(0,\infty)} xJ(d(s,x),\omega) \tag{2.9}$$

is defined for all $t \geq 0$. *The process is a Lévy process on* $\mathbb{R}^d$ *such that*

$$E[e^{i<z,X_t^1>}] = \exp[\int_{\mathbb{R}^d}(e^{i<z,x>} - 1)\nu(dx)] \tag{2.10}$$

Define

$$X_t^2(\omega) = X_t(\omega) - X_t^1(\omega), \text{ for } \omega \in \Omega_1 \tag{2.11}$$

There is $\Omega_2 \in \mathcal{F}$ *with* $P[\Omega_2] = 1$ *such that, for any* $\omega \in \Omega_2$, $X_t^2(\omega)$ *is continuous in* t *and* $\{X_t^2\}$ *is a Lévy process on* $\mathbb{R}^d$ *such that*

$$E[e^{i<z,X_t^2>}] = \exp[-\frac{1}{2} < z, Az > +i < \gamma, z >] \tag{2.12}$$

The two processes $\{X_t^1\}$ *and* $\{X_t^2\}$ *are independent* (see Sato (1999), p.121)

Here, $\{J(B) : B \in \mathcal{B}(H)\}$ is a Poisson random measure defined by

$$J(B,\omega) = \#\{s : (s, X_s(\omega) - X_{s-}(\omega)) \in B\}, \text{ for } \omega \in \Omega_0, \text{ and otherwise } 0 \tag{2.13}$$

where H is defined by $H = (0,\infty) \times (\mathbb{R}^d \setminus \{0\}) = (0,\infty) \times D_{0,\infty}$. $\mathcal{B}(H)$ is the Borel σ-algebra of H. $J(B)$ is a counting measure that counts the number of jumps whose size is within B.

This theorem means that we can decompose a Lévy process into the jump part $\{X_t^1\}$ and the continuous part $\{X_t^2\}$, respectively. Because of the characteristic function (2.12), the continuous part is Brownian motion. Thus, the firm growth process described by Gibrat's model corresponds to a continuous Lévy process with $\nu = 0$.

Let us return to our case, the Laplace distribution. From (2.8), that is, $A = 0$, Y_t has no continuous part, and so the process is a pure jump process. Y_t changes its value only by a jump. This is in sharp contrast to the Brownian motion that Gibrat's model predicts. Because the Brownian motion has only a continuous part ($\nu(dx) = 0$), positive growth is assumed to have small continuous movements in Gibrat's model. On the contrary, the positive growth process in our model (i.e., the Laplace distribution), Y_t, has discrete movements, that is, jumps. It should be noted that

$$\nu(\mathbb{R}_+) = \infty,$$

that is, the number of jumps in a finite interval is infinite. This is due to small jumps around 0, and by dropping these small jumps, we can well approximate the process by a compound Poisson process:

$$\hat{P}_{Y_{t,n}} = \exp\left[t\int_{\epsilon_n}^{\infty}(e^{izx}-1)\frac{e^{-\frac{x}{a}}}{x}dx\right], \quad \epsilon_n > 0 \text{ and } \epsilon_n \to 0 \text{ as } n \to \infty. \tag{2.14}$$

$P_{Y_{t,n}}$ converges to the distribution of Y_t (for more details, see Sato (1999) and Madan and Seneta (1990)). When the firm growth process follows a compound Poisson process, jumps occur randomly according to a Poisson process and the size of each jump too is an independent random variable. In the above case, jumps with sizes within the interval $(x, x + dx)$ occur as a Poisson process with intensity

$$\nu(dx) = x^{-1}e^{-\frac{x}{a}}dx.$$

Namely, Y_t is approximated by

$$Y_{t,n} = \sum_{i=1}^{N(t)} J_i, \tag{2.15}$$

where $N(t)$ follows a Poisson distribution with rate

$$\lambda = \int_{\epsilon_n}^{\infty} \frac{e^{-\frac{x}{a}}}{x} dx$$

and J_i is an independent random variable with the probability density defined by

$$\lambda^{-1} \frac{e^{-\frac{x}{a}}}{x} dx \quad \text{on} \quad [\epsilon_n, \infty).$$

Figure 2.3 depicts a typical sample path of $Y_{t,n}$.

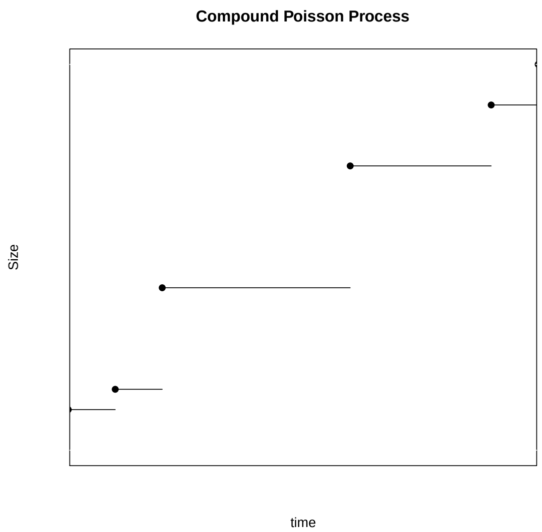

Figure 2.3: A sample path of $Y_{t,n}^1$. $Y_{t,n}^1$ is constant until a shock occurs. When a shock occurs, $Y_{t,n}^1$ jumps by J_i

The properties of the sample path discussed above shed new light on the firm growth dynamics. Suppose that a firm increases its size through innovation as assumed in the existing literature, for example, the introduction of new products (e.g., Klette and Kortum (2004)). An idea for an innovation arrives at the Poisson rate λ. However, the impact of each innovation on firm size too is a random variable, J_i; that is, major innovations contribute greatly to the growth of the firm, while others turn out to be almost useless. A successful innovation would yield rapid expansion. In this way, firm size sometimes jumps. In other words, firm growth is not a consequence of infinitesimally small shocks. If we collect a small number of such large jumps (i.e., successful innovations), they would largely explain the firm's growth. This suggests that we would be able to identify which events lead to the ups and downs of a firm.

This is reminiscent of the concept of *radical innovation* in the literature on innovation management. Radical innovations involve the development or application of something fundamentally new that creates a wholly new industry or causes a complete transformation of the market structure. In our model, radical innovations correspond to large jumps. Radical innovations are crucial to the long-term growth of firms and differ significantly from *incremental innovation* (Ettlie et al. (1984), Chandy and Tellis (2000), Leifer et al. (2000)). Leifer et al. (2000) noted that once a radical innovation was introduced into the existing market,

> "[P]roducts based on one technology were undermined by radically new ones — and incremental improvement to the old technology has done little more than delay the eventual rout." (p.3)

Small firms that engage and succeed in radical innovation usually bring down giants and gain the leading position in the market. Our finding that the growth of firms is determined by large jumps is consistent with these phenomena. A single large jump (successful innovation) or a small number of such jumps (successful innovations) leads to rapid expansion of a firm. Note that while the studies mentioned above are based mainly on interviews and case studies of large firms, our findings are derived explicitly from the distribution of growth rates.

Finally, by using the estimate $a = 0.054$ in (2.2) in the previous section, we can identify the structure of jumps. For example, a positive growth shock larger than 3% has intensity $\nu(0.03, \infty) \simeq 0.50$. This means that such shocks occur, on average, once every two years. In case of an 8% growth shock, $\nu(0.08, \infty) \simeq 0.10$, this figure is once every ten years. Note that these jumps are comparable to annual growth rates (see Figure 2.1). That is, such shocks largely determine the annual growth rate of a firm. The probability of larger jumps can be calculated in a similar manner.

2.4 Further Development

In this section, we discuss the robustness of the implications we have obtained thus far and related empirical results.

An extension of parametric distributions of firm growth rates is conducted by Giulio Bottazzi and coauthors. Bottazzi et al. (2002) propose a wider class of distributions, called the *Subbotin family*, which includes both Laplace and Gaussian distributions as special cases. The Subbotin family is defined by

$$f(x) = \frac{1}{2ab^{1/b}\Gamma(1/b+1)} \exp(-\frac{1}{b}|\frac{x-\gamma}{a}|^b). \tag{2.16}$$

Here, Γ is Gamma function. The case of $b = 1(b = 2)$ corresponds to a Laplace (Gaussian) distribution.

Not only the Subbotin family is useful for parametric estimation, but also has another interpretation, especially called *statistical equilibrium model.* Alfarano and Milakovic (2008) show that the Subbotin family can be considered as a consequence of the *maximum entropy principle.* The maximum entropy principle, proposed by Foley (1994) in the context of economic models, means that the resulting distribution is the combinatorially mostly likely distribution because it maximizes the number of combinations given the constraints. In other words, the maximum entropy principle leads to the least informative distribution under moment constraints. This idea is very familiar to physicists, for example, the distribution of energy of molecules is described by this method. In this case, an allocation of energy is “randomized” via collisions of molecules and the least informative distribution, Gaussian distribution, under the second moment constraint (because the total energy should be conserved) is realized.

Let H denote the entropy function defined by

$$H(p) \equiv -\int_{-\infty}^{\infty} p(x) \log p(x) dx \tag{2.17}$$

where $p(x)$ is a probability density function. Following the maximum entropy principle, we consider the maximization of the entropy function subject to the natural constraint,

$$\int_{-\infty}^{\infty} p(x)dx = 1 \tag{2.18}$$

and the moment constraint,

$$\int_{-\infty}^{\infty} \left| \frac{x-\gamma}{a} \right|^b p(x)dx = 1. \tag{2.19}$$

Alfarano and Milakovic (2008) show that the solution of this variational problem is given by the Subbotin family (2.16). As discussed above, the second moment constraint ($b = 2$) corresponds to a Gaussian distribution. Alfarano and Milakovic (2008) argue that the Laplace distribution of firm growth rates can be considered as a realization of the case of $b = 1$.

From an economical viewpoint, the maximum entropy principle leads to "the growth rate distribution arising from the most decentralized activity of competitive firms that utilize [γ] as a benchmark profit rate in their investment and operating decisions."(Alfarano and Milakovic (2008), p.273) That is, the profit rates for firms are not equalized completely and a capital moves to search for better profits. The complex movements of capital in search of better profits do not converge but generate dispersion around γ. The dispersion is given by the constraint $\int_{-\infty}^{\infty} \left|\frac{x-\gamma}{a}\right| p(x)dx = 1$ ($b = 1$). Under these constraints, the most randomized distribution, i.e., the Laplace distribution, is realized. Although this is not the direction we pursue and the justification of the use of the maximum entropy principle does not seem to be convincing, this approach presents another interpretation of observed statistical regularity of firm growth rate distribution.

Although the Laplace hypothesis is widely accepted and supported empirically, there are some exceptions. Bottazzi et al. (2011) analyze French manufacturing firms and estimate the parameters of the Subbotin distributions. They find that the shape parameter, b, is significantly lower than 1; that is, the distribution of growth rates is more fat-tailed than the Laplace distribution, contrary to previous works on the U.S. and Italian data.

Fu et al. (2005), Buldyrev et al. (2007b) and Buldyrev et al. (2007a) analyze the worldwide pharmaceutical industry by employing the sales figure as the measure of the firm size. They argue that the distribution of growth rates is more leptokurtic than the Laplace distribution. They emphasize the asymptotic behavior of the distribution and show that it can be well fitted by a power law decaying, as $P(x) \sim x^{-3}$, with a slower decay than the exponential. Although they also present a model exhibiting this power law behavior, as discussed earlier, Sakai and Watanabe (2010) show that the observed Laplace shape is not explained by the mechanism described in Buldyrev et al. (2007a).

Both findings (the Subbotin family with $0 < b < 1$ and the power law tail) suggest the possibility that the tail of the distribution of growth rates is fatter than the Laplace distribution. Do these findings invalidate our implication obtained in the previous section? In the next subsection, we briefly review the *tail equivalence* of Lévy processes and consider two fatter-tailed distributions. We show that our main implication is valid even in these cases.

2.4.1 Tail equivalence

Because we consider fatter tails, it is convenient to define the right tail of a measure μ by $\bar{\mu}$ [8],

$$\bar{\mu} \equiv \mu(x, \infty).$$

For the Laplace distribution, it can be written as

$$\bar{\mu}_{Lap} = e^{-cx}.$$

Here, we omit constants irrelevant to our discussion. In this subsection, we consider two distributions fatter-tailed than this tail. One is the distribution exhibiting the power law behavior:

$$\bar{\mu}_{pow} = x^{-\alpha},$$

which is what Fu et al. (2005), Buldyrev et al. (2007b), and Buldyrev et al. (2007a) propose. This tail behavior is included in the class of regularly varying functions denoted by $\mathbf{R}_\alpha$ [9]. The

[8] The analysis developed below can also be applied to the left tail of distributions.
[9] For regularly varying functions $\mathbf{R}_\alpha$, see, for example, Resnick (1987).

other is the so-called *Weibull distributions*:

$$\bar{\mu}_{wei} = e^{-cx^b}, \quad 0 < b < 1.$$

Easy calculations yield that Weibull distributions with $0 < b < 1$ are closely related to the Subbotin family. Comparing the density functions of two classes

$$\exp(-cx^b) \quad \text{for the Subbotin family}$$
$$x^{b-1}\exp(-cx^b) \quad \text{for the Weibull distributions,}$$

because the decay of the distribution is largely determined by e^{-cx^b}, especially when b is close to 1, it is difficult to distinguish these distributions by empirical data. Note that both distributions converge to the Laplace distribution as $b \to 1$. Because the Weibull distributions are more tractable, we consider the Weibull distributions with $0 < b < 1$ as the class having fatter tail than the Laplace distribution, instead of the Subbotin family.

In the remainder of this section, we briefly review a mathematical framework called *tail equivalence* to analyze the sample path properties of a firm growth process given that the process is a Lévy process, and show that our implication obtained in the previous section is valid even in such cases. We denote the η-exponential moment of μ by

$$\tilde{\mu}(\eta) \equiv \int_{-\infty}^{\infty} e^{\eta x}\mu(dx),$$

where $\eta \geq 0$. $f(r) \sim g(r)$ means that

$$\lim_{r\to\infty} f(r)/g(r) = 1.$$

The convolution of distributions μ and ρ is denoted by $\mu * \rho$. We then introduce two classes of distributions, called *exponential class* (denoted by $\mathbf{L}(\eta)$) and *covolution equivalent* ($\mathbf{S}(\eta)$).

Definition 2.3 *Let μ be a distribution on $\mathbb{R}$. Suppose that $\bar{\mu}(r) > 0$ for all $r \in \mathbb{R}$.*

1. *$\mu \in \mathbf{L}(\eta)$ if $\bar{\mu}(r+a) \sim e^{a\eta}\bar{\mu}(r)$ for all $a \in \mathbb{R}$.*

2. *$\mu \in \mathbf{S}(\eta)$ if $\mu \in \mathbf{L}(\eta)$, $\tilde{\mu}(\eta) < \infty$, and $\overline{\mu * \mu}(r) \sim 2\tilde{\mu}(\eta)\bar{\mu}(r)$.*

The case of $\eta = 0$ is particularly important. The distributions in $\mathbf{L}(0)$ and $\mathbf{S}(0)$ are called *long-tailed* and *subexponential*, respectively. The definition of $\mathbf{L}(0)$ means a very slow decay of the tail of the distributions and, in particular, the following property holds;

$$\lim_{x \to \infty} e^{\epsilon x} \bar{\mu}(x) = \infty \quad \text{foreach} \epsilon > 0.$$

That is, the tail of distribution $\in \mathbf{L}(0)$ shows a slower decay than the exponential (see Embrechts et al. (1997), p.41 and Pakes (2004)).

In addition, the distributions in $\mathbf{S}(0)$ have another important property: if $X_i, i = 1...n$ are independent random variables drawn from the same distribution function $F \in \mathbf{S}(0)$, the following property holds:

$$\lim_{r \to \infty} \frac{P(X_1 + ... + X_n > r)}{P(\max(X_1, ..., X_n) > r)} = 1.$$

For the case of random variables on the half line $[0, \infty)$, the proof can be found in Embrechts and Goldie (1982). Pakes (2004) extends this property to the two-sided case, $(-\infty, \infty)$.

This property implies that an extremely large sum is due to a single variable. That is, for such large values, the sum is well-approximated by the largest value. This is in a sharp contrast with Gaussian distributions (Gibrat's model) where an accumulation of small shocks determines the sum and no such variable having a disproportional impact on the sum exists. On the contrary, for $\mathbf{S}(0)$, a single (the largest) variable determines the behavior of the sum and other variables are negligible. As we shall see in the following, this property is closely related to the existence of large jumps when we discuss the sample properties of Lévy processes.

There is a close relationship between $\mathbf{S}(\eta)$ and the Lévy processes. We denote the Lévy measure of an infinitely divisible distribution by ν (see (2.3)). Let $\nu_c(dx)$ denote the jump distribution for $\bar{\nu}(c) > 0$:

$$\nu_c(dx) = \frac{1}{\bar{\nu}(c)} 1_{(c,\infty)}(x) \nu(dx), \quad c > 0. \tag{2.20}$$

This measure simply represents a compound Poisson process (recall the Lévy–Itō decomposition).

Watanabe (2008) proves the following theorem (see also Pakes (2004) and Pakes (2007)).

Theorem 2.4 *Let $\eta \geq 0$. Let μ be an infinitely divisible distribution on $\mathbb{R}$. Then, the following are equivalent:*

1. $\mu \in \mathbf{S}(\eta)$;
2. $\nu_1 \in \mathbf{S}(\eta)$;
3. $\nu_1 \in \mathbf{L}(\eta)$, $\tilde{\mu}(\eta) < \infty$, *and* $\bar{\mu}(x) \sim \tilde{\mu}(\eta)\bar{\nu}(x)$.

This theorem characterizes the class, $\mathbf{S}(\eta)$, of infinitely divisible distributions and shows that the tail is completely determined by ν_1, that is, the distribution of large jumps. It guarantees that we can ignore the effects of small jumps around 0 and the continuous part (Brownian component) on the tail of the distribution.

Let us begin discussing the relation of this theorem and the characteristics of the distribution of firm growth rates. If the tail of the distribution of growth rates is fatter than that of a Laplace, one might expect that the distribution belongs to $\mathbf{S}(\eta)$, or more particularly $\mathbf{S}(0)$. In fact, the following theorem holds.

Theorem 2.5 *Both classes of distributions ($\bar{\mu}_{wei} = e^{-cx^b}$, $0 < b < 1$ and $\bar{\mu}_{pow} = x^{-\alpha}$) are included in $\mathbf{S}(0)$.*

Proof. We begin with the Weibull distributions. Let μ_{wei} be a Weibull distribution with $0 < b < 1$. Define $\mu^+_{wei} = 1_{[0,\infty)}(x)\mu_{wei}(dx) + \mu_{wei}(-\infty, 0)\delta_0(dx)$. Corollary 2.1 in Pakes (2004) implies that for $\eta \geq 0$, $\mu_{wei} \in \mathbf{S}(\eta)$ if and only if $\mu^+_{wei} \in \mathbf{S}(\eta)$. We then choose an absolutely continuous probability distribution ρ satisfying $\lim_{x\to\infty} \bar{\rho}/\bar{\mu}^+_{wei} = c$ where $0 < c < \infty$, so that $\rho = c_1 \exp(-c_2 x^b)$ for $x > c_3$, where c_1, c_2, and c_3 are constants. Because $\rho \in \mathbf{S}(0)$ implies $\mu^+_{wei} \in \mathbf{S}(0)$ (see Embrechts et al. (1979)), all we have to do is to examine whether $\rho \in \mathbf{S}(0)$. From Proposition A.3.16 in Embrechts et al. (1997), which gives a sufficient condition for the membership of $\mathbf{S}(0)$, we can immediately obtain $\rho \in \mathbf{S}(0)$. Therefore, we obtain the desired result, $\mu_{wei} \in \mathbf{S}(0)$.

In the case of distributions with regularly varying tails, using the same argument above and Corollary 1.3.2 in Embrechts et al. (1997), we immediately obtain that they are included in $\mathbf{S}(0)$. Thus, our claims follow. ■

Therefore, in either case (μ_{pow} and μ_{wei}), we can apply Theorem 2.4 with $\eta = 0$. When $\eta = 0$, $\tilde{\mu} = 1$. Hence, Theorem 2.4 states that $\bar{\mu}(x) \sim \bar{\nu}(x)$;

$$\lim_{x\to\infty} \bar{\mu}(x)/\bar{\nu}(x) = 1,$$

that is, μ and ν decay at almost the same rate. In other words, we can estimate the rate at which large jumps occur by estimating the tail of the observed distribution of growth rates. A fatter tail represents the existence of large jumps. In particular, taking into account Theorem 2.5 and the fact that $b = 1$ corresponds to the Laplace distribution, the Laplace hypothesis can be considered as a boundary case. That is, as long as the growth rate distribution can be described by the Laplace distribution or distributions with fatter tails, large jumps determine growth rates. Therefore, in this sense, the recent findings that the distribution of growth rates has a fatter tail than a Laplace distribution reflect the importance of large jumps and support our conclusion.

2.5 Disucussions

As discussed in Section 3 and 4, large jumps are indispensable to firm growth dynamics. However, in the existing literature, a large firm is viewed as the consequence of many small successes. For example, Bottazzi and Secchi (2006) develop a model to generate a Laplace distribution based on the well-known *island model*. What makes their model different from previous studies is the feedback effect; that is, the larger the number of opportunities already obtained, the higher the probability for a given firm to obtain new opportunities. In their model, when the number of shocks per firm (in their notation M/N where M is a finite number representing business opportunities and N the number of firms) goes to infinity, the distribution of growth rates converges to a Laplace distribution. Bottazzi and Secchi (2006) state that

> "...competitive success is seen not as the outcome of a single lucky event granting one firm a persistent, dominant, position, but rather as a firm's ability to build its new success, through a permanent struggle within an extremely volatile environment, on the basis of its past, successful, behavior." (p.252)

The feature of their model is the absence of large shocks. On this point, our analysis presents a sharp contrast to their model.

For another example, Klette and Kortum (2004) view a firm as the portfolio of goods that it produces. The size of a firm is represented by the number of goods it produces. Let n denote the number of the firm's products. A firm grows through innovation and by obtaining new markets, and shrinks when some other firm innovates on a good in its portfolio. The firm that successfully innovates on a particular good takes over the market for that good at the expense of its competitors in the sense of Shumpeterian *creative destruction*. This creative destruction is described by a Poisson process with $\varphi > 0$; that is, the incumbent firm loses one of the markets at the Poisson hazard rate φ. The firm's *innovation production function* I, the arrival rate of innovation, is homogeneous of degree 1 in n and can be written as $I = n\lambda$. Hence, the time evolution of probability density $p(n,t)$ is described as follows

$$\begin{aligned}\frac{dp(n,t)}{dt} &= (n-1)\lambda p(n-1,t) + (n+1)\varphi p(n+1,t) - n(\lambda+\varphi)p(n,t), \quad n \geq 1,\\ \frac{dp(0,t)}{dt} &= \varphi p(1,t).\end{aligned}$$

The meaning of these equations is clear. For example, the first term on the right-hand side of the equation represents that firms whose size is $n-1$ innovate and increase their size by 1. That is, the rate of changes in $p(n,t)$ is given by the difference between inflows and outflows of probability. The reason their model is tractable and simple enough to obtain an analytic solution is that they view a large firm as just a combination of many small firms. As Klette and Kortum (2004) state,

> "A firm of size $n_0 > 1$ at date 0 will evolve as though it consists of n_0 independent

> divisions of size 1. ...the evolution of the entire firm is obtained by summing the evolution of these independent divisions, each behaving as a firm starting with a single product would." (p.995)

For large firms ($n \gg 1$), each shock is very tiny compared to their size ($1/n \ll 1$). Therefore, this model is a typical example showing that the larger firm is the result of successive small innovations, that is, an accumulation of small shocks. Like Bottazzi and Secchi (2006), there are no major shocks that lead to the rapid expansion of firms.

However, the firm growth dynamics is different from what these previous studies envision. In sharp contrast to them, our analysis shows that firm growth is characterized by large jumps. Sizable shocks occasionally hit a firm and the firm grows discontinuously. If we collect a few large jumps of a firm, we can describe the firm's growth path quite well. In other words, we can identify the events (or shocks) that lead to the ups and downs of a firm. There exist particular shocks leading to the rapid expansion of larger firms. This is impossible when shocks are infinitesimally small. This is where our analysis sheds new light and deviates substantially from the existing literature.

2.6 Conclusions

The firm growth dynamics empirically observed shows very complex behavior and predictions drawn from neoclassical economics is quite useless. There seems to be no an attracting "optimal size" derived from a profit maximization procedure nor monotonistically increasing or decreasing growth path converging to it. Even if managers of firms "optimize" their strategies in some sense, their strategies depend on various variables unobservable from observers or economists. These complex and heterogeneous behaviors cannot be explained by arithmetical maximization problems such as economists are able to solve.

At a first glance, there seems to be no law governing the dynamics, as Ashton (1926) said. It is extremely difficult to predict a few years beforehand whether a particular firm extends its business rapidly or shrinks by a business failure. Although we cannot predict the future of a given firm exactly, aggregating firms in an economy, we can find that there are statistical regularities of firms' behavior. Among the most striking is Gibrat's law (Gibrat (1931)). This law holds in various countries and across time, and therefore, suggests some fundamental aspects of firm growth dynamics. In this sense, the importance of Gibrat's law cannot be overstated.

Furthermore, recent findings related to the distribution of growth rates open a new field of inquiry. Since Stanley et al. (1996), subsequent papers have shown that the distribution of firm growth rates closely follows a Laplace distribution. The same characteristics of firm growth rate distributions appear at the finer sectoral level. The aim of this Chapter was to investigate what this robust regularity suggests and clearify the nature of firm growth dynamics.

For that purpose, we generalized Gibrat's model and demonstrated that Lévy processes are an appropriate framework to investigate firm growth dynamics. This is a natural generalization of Gibrat's model. Introducing infinitely divisible distributions, we show that the Variance-Gamma process corresponding to the Laplace distribution represents firm growth dynamics. In particular, this process is a pure jump process, and there exists large jumps that have disproportional impacts on annual growth. Growth rate is largely explained by a few number of these large jumps. The observed Laplace distribution is a reflection of this property.

We also discuss the robustness of our conclusion. There are some studies showing that the distribution of growth rates has a fatter tail than the Laplace distribution, more particularly, the Subbotin family with $0 < b < 1$ (Bottazzi et al. (2011)) and the power law tail (Fu et al. (2005), Buldyrev et al. (2007b), and Buldyrev et al. (2007a)). Introducing the subexponential family, $\mathbf{S}(0)$, and tail equivalence theory, we investigate two fatter-tailed distributions than the Laplace distribution. We show that both families of distributions are included in $\mathbf{S}(0)$. This implies that the growth rate distribution, μ, and the Lévy measure, ν, decay at almost the same rate. That is, a fatter tail implies the existence of large jumps. It guarantees the robustness of our conclusion that a small number of large jumps determine firm growth dynamics.

We also discussed an economic interpretation of our analysis and related literature. In the existing literature (e.g., Bottazzi and Secchi (2006) and Klette and Kortum (2004)), firm growth is explained by an accumulation of small shocks (or innovations). However, the observed Laplace distribution shows that radical innovation (or large jumps) is a key element in the long-term success of a firm. Firms that dominate the existing market, but lag behind in competition for radical innovation, often fail to maintain the leading position. The Laplace distribution of growth rates clearly represents such phenomena. Moreover, radical innovation is an engine of economic growth. It revolutionizes existing markets or opens up whole new industries. As Schumpeter (1942) famously states, “creative destruction is the essential fact about capitalism.” Therefore, investigating the statistical regularities of firm growth rates has important implications for both macroeconomics and firm growth theory.

Chapter 3

Endogenous Business Cycles caused by Nonconvex Costs and Interactions

3.1 Introduction

In this Chapter, we show that endogenous business cycles (inventory cycles) arise from a combination of the nonconvex costs and economic interactions among firms. In particular, we show that the aggregation of randomly behaving microeconomic agents generates deterministic *collective behavior* via interactions.

Economic fluctuations are one of the most important problems in macroeconomics, but it still remains unresolved what causes such aggregate fluctuations? For example, Cochrane (1994) demonstrates that popular economy-wide shocks (e.g., monetary shocks or oil prices) fail to explain the bulk of the aggregate economic fluctuations. He writes,

> "What shocks are responsible for economic fluctuations? Despite at least two hundred years in which economists have observed fluctuations in economic activity, we still are not sure." (p. 295)

Thus, it remains as a challenge to identify the origin of economic fluctuations. We cannot resort to *mysterious* aggregate exogenous shocks to explain them. Rather, because the macroeconomy is composed of many firms, it might be expected that the aggregate fluctuations stem from firm-specific shocks and inherit some their properties.

At the micro level, economic activities are characterized by lumpiness and discreteness. Managers temporary shut down the plants or change the number of shifts for inventory adjustment. This behavior clearly contradicts the well-known production smoothing theory in microeconomics textbooks. In fact, the production smoothing theory is empirically rejected (see Blinder and Maccini (1991)). It is found that, when some fixed costs exist (for example, ordering costs), the cost curve is kinked and nonconvexity emerges. This implies that the cost-minimizing strategy of firms is production bunching (or the bunching of orders). This theory can account for the stylized fact that production is more volatile than sales (e.g. Hall (2000)). The aim of this Chapter is to investigate how these firm-level characteristics are related to the aggregate fluctuations.

There are two different views concerning the effect of microeconomic characteristics on the aggregate fluctuations; one is that microeconomic characteristics disappear at the macroscopic level. Indeed, less attention has been paid to the role of idiosyncratic shocks in the macroeconomic literature simply because these shocks are considered to average out in the aggregate by the law of large numbers (LLN). This view is widely accepted in the literature and Lucas (1977)'s argument is a typical one. Lucas (1977) says,

> "These changes [(the changes in technology and taste)] are occurring all the time and, indeed, their importance to individual agents dominates by far the relatively minor movement with constitute the business cycle. Yet these movements should, in general, lead to relative, not general price movements. ... in a complex modern economy, there will be a large number of such shifts in any given period, each small in importance relative to total output. There will be much "averaging out" of such effects across markets.
>
> Cancellation of this sort is, I think, the most important reason why one cannot seek an explanation of the general movements we call business cycles in the mere presence, per se, of unpredictability of conditions in individual markets." (p. 19)

According to this view, the observed aggregate fluctuations must be explained by the presence of shocks that have a common origin across firms in the economy. By definition, they are *aggregate shocks*.

On the other hand, the second view, which attracts much attention in recent years, emphasizes the effects of interactions between sectors (or firms), especially input–output linkages. Although the LLN argument discussed above depends crucially on the independence assumption, interactions among firms are the fundamental aspects of the macroeconomy. In fact, positive comovement across sectors is a salient feature of the business cycle. In contrast to the LLN argument, it is emphasized that the effects of interactions between sectors (or firms) through input–output linkages, which propagate idiosyncratic shocks throughout the economy, cause the aggregate fluctuations that are unexplained by the usual aggregate shocks (e.g., Long and Plosser (1983), Carvalho (2010), Foerster et al. (2011), Acemoglu et al. (2012), and Carvalho and Gabaix (2013)). The key element of models used in these studies is the existence of sectors that have disproportional impacts on the entire economy (it is discussed in more detail in 3.4.1). This is due to the heterogeneity of input–output linkages; that is, sectors are not equally intense material suppliers. Shocks to general purpose technologies such as oil, electricity, iron and steel propagate to all sectors through the input–output linkages because most sectors rely on them. In this sense, the microeconomic shocks accounting for aggregate fluctuations in these studies can be regarded as *"pseudo–macroeconomic"* shocks. There are other literature following the second view and closely related our analysis, e.g., Bak et al. (1993) and Durlauf (1993), where nonconvex technology and (local) interactions are explicitly considered. Bak et al. (1993) demonstrate that small shocks to final goods can cause an "avalanche" of production increases via the supply chain.

Even though we follow the second view and focus on interactions between firms in order to explain the (regular) business cycle, there exist broad distinctions between our model and previous studies. First, in contrast to Acemoglu et al. (2012) and Carvalho and Gabaix (2013), in our model, each firm can hardly influence the outcome of the entire economy on its own. We assume that each firm is small compared with the economy as a whole. Furthermore, in contrast to Bak et al. (1993), in which shocks to final goods are assumed to be exogenous, we assume that demand for the products depends on the overall economic condition. We assume that the behavior of a firm is affected by the state of the economy as a whole, but at the same time, the economy is composed of the firms themselves. In other words, the macroscopic state of the economy not only is an aggregation of the firms, it also prescribes macroeconomic environment in which the firms engage in business activities. This feedback loop generates many interesting *macro*-phenomena. This idea is closely related to the "macro-micro loop" emphasized by Hahn (2002), where a macro variable acts as an externality. We show that this mechanism can generate *collective behavior* that is different from the motion of an individual firm.

On this point, our approach is close in spirit to *heterogeneous interacting agent models* (see e.g., Delli Gatti et al. (2009) and Stiglitz and Gallegati (2011); for a survey, see Hommes (2006)), especially to Aoki's methods (Aoki (1996, 2004), Aoki and Shirai (2000) and Aoki and Yoshikawa (2007)). Aoki and coauthors develop so-called *jump Markov processes*, where the evolution of the probability distribution is described by *master equations*. Although there is no doubt that Aoki's methods expand the scope of macroeconomic analysis, there exist some difficulties and situations that cannot be dealt with in his framework (see Section 4.1). In particular, in our model, firms' inventories are distributed continuously and affect firms' choice of production. That is, the system is described by an infinite dimensional random variable, that is, the distribution of inventories (and production).

By using the *propagation of chaos* instead of Aoki's methods, we present an alternative method to investigate how the system (that is, the probability distribution) behaves and changes its properties when we change the parameters. On the basis of nonconvexity in the cost function and the above mentioned feedback effect, we show that a regular cyclical movement emerges given that the feedback effect exceeds a certain threshold. This cyclical movement arises endogenously. It offers an explanation for the *Kitchin cycle*.

The rest of this Chapter is organized as follows. Section 2 discusses the firm behavior characterized by nonconvexities, which can explain the empirical puzzle that the volatility of production is larger than that of sales. Section 3 discusses the importance of inventory movement for understanding the business cycle. Section 4 contains our main results and shows that the simple LLN cannot be applied and that an endogenous movement emerges. Section 5 concludes.

3.2 Firm Behavior: Production and Inventory

The standard cost function has been assumed to be convex in output and in the change of output. This means that for cost minimization, the manager of a firm must smooth its production by using inventories as a buffer stock (production-smoothing models; see, e.g., Holt et al. (1960)). This implies that production is less variable than sales.

However, the prediction of the production-smoothing models is known to be inconsistent with the empirical data (see Blinder and Maccini (1991)). In particular, the correlation between sales and inventories is *positive*, not negative as predicted by production-smoothing models. Firms do not use their inventories as a buffer.

Blinder and Maccini (1991) present a well-known (S, s) model in which a firm places an order of size $S - s$ whenever its inventories reach the lower bound, s. They show that it is optimal for the firm to place infrequent large orders when fixed costs of ordering exists, which leads to bunched orders. The inventory series is characterized by a sawtooth pattern. The (S, s) model is strongly supported by empirical data (e.g., Hall and Rust (2000)). Although they emphasize retail and wholesale inventories, that is, the lumpiness of the delivery process, the bunching of orders by the retail sector can induce production bunching in the manufacturing sector even though the latter has the usual increasing marginal costs. Cooper and Haltiwanger (1992) point out this possibility, saying that

> "[D]ownstream bunching of orders by retailers may be the source of upstream production bunching by manufactures." (p. 116)

In relation to these studies, a close examination of data at the micro level (especially for the automobile industry) reveals that changes in production are, in fact, quite lumpy. Managers may shut their plants down for a week or change the number of shifts, varying production. Ramey (1991), Cooper and Haltiwanger (1992), Bresnahan and Ramey (1994), and Hall (2000) focus on the nonconvexity of the cost function to explain these behaviors. In particular, they show that when there are fixed costs associated with opening the plant and adding an additional shift, production bunching becomes the optimal strategy. For example, Cooper and Haltiwanger

(1992) present a simple model and show that a start-up cost for a production run and a constant marginal cost of production lead to production bunching.

To illustrate how the cost function associated with such fixed costs might look, a simple non-convex cost function is depicted in Figure 3.1. If a firm wants to produce, on average, output

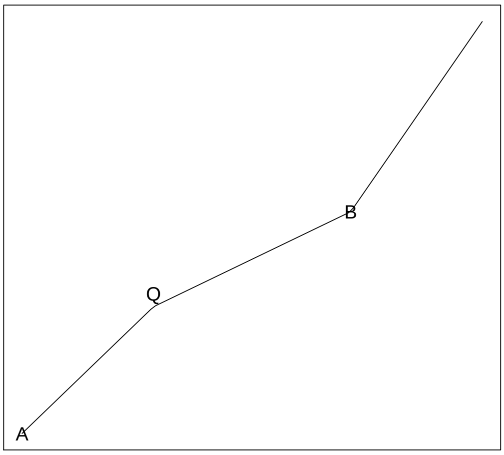

Figure 3.1: A nonconvex cost function. The horizontal (vertical) axis is quantity (costs).

$Q \equiv (A + B)/2$, the average cost can be reduced by alternating between production at A and B rather than keeping the level of production at Q, that is, by production bunching. In this way, the nonconvexity leads to *excess* volatility of production. Furthermore, this nonconvexity is quantitatively important to explain the variation of output. Bresnahan and Ramey (1994) states that

> "[M]ost of the variance of output comes from varying hours over the nonconvex portions of the cost function, rather than from varying hours over the convex portions of the cost function." (p. 610)

The question arises of whether the automobile industry is representative of all manufacturing or is a just special one. On this point, Mattey and Strongin (1997) consider two extremes of technology types. "Pure assemblers" adjust their output through varying plants' work period, that is, through temporary plant shutdowns, adding or dropping shifts, and adding overtime hours (Saturday work). The automobile and transportation industries are typical examples. The other type is "pure continuous processing" operations, where output adjustment is carried out through varying the instantaneous flow rates of production rather than the work period margin. They conclude that

> "[O]n average in all of manufacturing, the use of the plant work period margin is relatively common, so the "pure assembly" type characterization is closer to the truth in the aggregate." (p. 4)

Moreover, among these output adjustment margins, changes in the number of shifts are quantitatively important. Bresnahan and Ramey (1994) show that at a quarterly frequency, changes in the number of shifts account for 40% of plant-level output volatility in the automobile industry and is the most important contributors to the variation of output. Shapiro (1996) shows that close to half of the changes in employment in the U.S. manufacturing take place on late shifts. Thus, we focus on changes in the number of shifts in the following analysis. [1]

The above discussion above suggests that the behavior of firms is as follows. For the sake of simplicity, we assume that the firms choose one of two production states, high and low (the same simplification can be found in the literature; see, for example, Bak et al. (1993) and Durlauf (1993)). Suppose that a manufacturing firm has sufficient inventories (or the wholesale and retail inventories that the firm supplies) and that demand is low. The firm chooses a low production state (e.g., one-shift production) to reduce its inventories. After eliminating the excess inventories, the firm waits for demand to improve. If this happens, the firm adds a new shift to the existing line and increases its output. Even if the sales forecast is overestimated, it is optimal for a manager to maintain the high production for a while because of the fixed costs. After it replenishes its inventories, the firm lays off the workers on the second shift, and returns to the initial state.

Note that the above pattern of behavior is not a deterministic path, but is exposed to various idiosyncratic shocks. Suppose first that the demand (or sales) for the product of a firm indexed

[1] Shapiro (1996) also shows that if the workweek of capital is taken into account, almost all cyclical movement of productivity is eliminated.

by i, s_t^i, fluctuates around $\overline{S}$,

$$s_t^i = \overline{S} + \xi_t^i \tag{3.1}$$

ξ_t^i represents a temporary demand shock with mean 0, that causes unintended changes in inventories. We write $\xi_t^i \equiv -\frac{\sigma_2 dW_t^i}{dt}$, where W_t^i is a standard Brownian motion, σ_2 is a constant, and $\frac{dW_t^i}{dt}$ is the formal derivative with respect to t. We normalize $\overline{S} = 0$. By definition, inventory investment (changes in inventories), dy_t^i, can be written as the difference between production and sales,

$$dy_t^i = (x_t^i - s_t^i)dt \tag{3.2}$$

where x_t^i denotes the production of firm i. We assume that the production is described by the motion in the so-called double-well potential.

$$dx_t^i = (-V'(x_t^i) - ey_t^i)dt + \sigma_1 dW_{1,t}^i, \quad V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 \tag{3.3}$$

where $W_{1,t}^i$ is a standard Brownian motion and $\sigma_1, e > 0$ are constants. The stochastic term represents various idiosyncratic shocks that affects the target level of production—for example, changes in the price of materials. The potential function $V(x)$ is shown in Figure 3.2. The region around $-1(+1)$ corresponds to low(high) production state. This model is a generalization of a two-state Markov chain. Suppose, for example, that $e = 0$. Because -1 and 1 are the local minima, x_t^i stays around there until a large shock occurs, at which point x_t^i goes toward the other local minimum. Thus, the path of x_t^i alternates between low and high production. The second term of equation (3.3) on the right-hand side, ey_t^i, represents the effect of the level of the inventories on the manager's decision. That is, if y_t^i is large, the manager is likely to choose low production around $x_t^i = -1$.

Combining these equations, the behavior of firm i is described by the following two dimensional

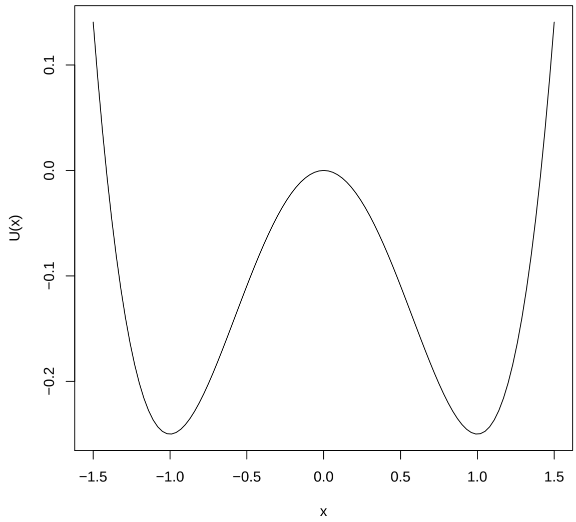

Figure 3.2: The potential function $V(x)$.

stochastic differential equations:

$$\begin{aligned} dx_t^i &= (-V'(x_t^i) - ey_t^i)dt + \sigma_1 dW_{1,t}^i, \quad V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 \\ dy_t^i &= x_t^i dt + \sigma_2 dW_{2,t}^i \end{aligned} \tag{3.4}$$

where $W_{k,t}^i, k = 1, 2$ are independent Brownian motions and $\sigma_1, \sigma_2 > 0$. σ_1 and σ_2 represent the intensities of idiosyncratic shocks.

These equations duplicate the firm behavior discussed above. Suppose that x_t^i is near -1 and $y_t > 0$, that is, the firm has sufficient inventories and chooses low production. Because of the effect of y_t^i, x_t^i stays around -1 until y_t^i is sufficiently reduced. When $y_t^i < 0$, production, x_t^i, is pushed up by the shortage of inventories. Exceeding the top of the curve (around 0), x_t^i goes toward high production (+1), and the inventories are replenished. The stochastic terms $\sigma_1 dW_{1,t}^i$ and $\sigma_2 dW_{2,t}^i$ represent idiosyncratic shocks to firm i. For example, a good market condition $\sigma_2 dW_{2,t}^i < 0$ reduces the inventories beyond expectation and x_t^i might stay around +1 longer. Sample paths of equation (3.4) are depicted in Figures 3.3 and 3.4. Figure 3.3 shows that x_t^i oscillates between +1 and -1 with the stochastic noise. In Figure 3.5, the result of numerical simulations of $N = 20000$ independent copies of equation (3.4) is shown. It clearly shows the bimodality of production.

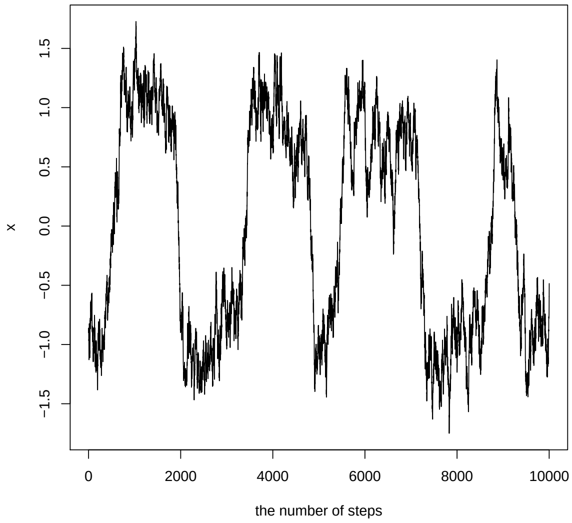

Figure 3.3: A sample path of production x in equation (3.4) with $\sigma_1^2 = \sigma_2^2 = 1/4$ and $e = 0.1$. The interval of a single time step, Δt, is 0.01. The horizontal axis is the number of steps.

3.3 Inventory Investment and Business Cycles

3.3.1 The importance of Inventory Investment

As is well known, the inventory investment (the change in inventories) is a key element in explaining aggregate fluctuations. For example, Blinder and Maccini (1991) demonstrate that the drop in inventory investment accounted for 87% of the drop in the GNP during the average postwar recession in the United States. In addition, a large part of short-run fluctuations (the business cycle frequencies) are explained by the behavior of inventory investment. Blinder (1981) says,

> "Inventory fluctuations are important in business cycles; indeed, to a great extent, business cycles are inventory fluctuations." (p. 500)

Furthermore, there is a consensus in the empirical literature that inventory movements are procyclical and that production is more volatile than sales at the sector and aggregate levels (see

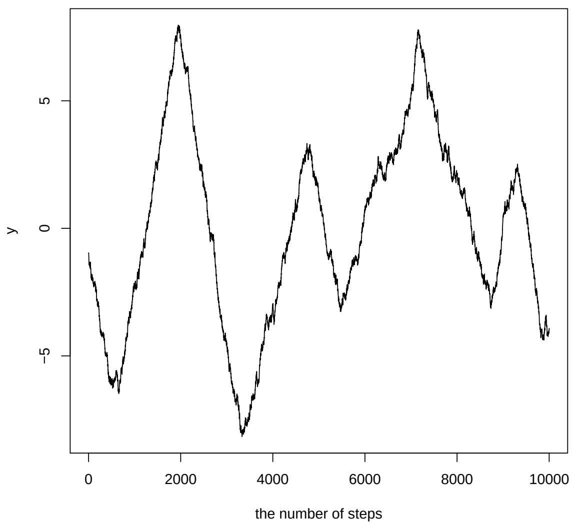

Figure 3.4: A sample path of inventories y in equation (3.4) with $\sigma_1^2 = \sigma_2^2 = 1/4$ and $e = 0.1$.

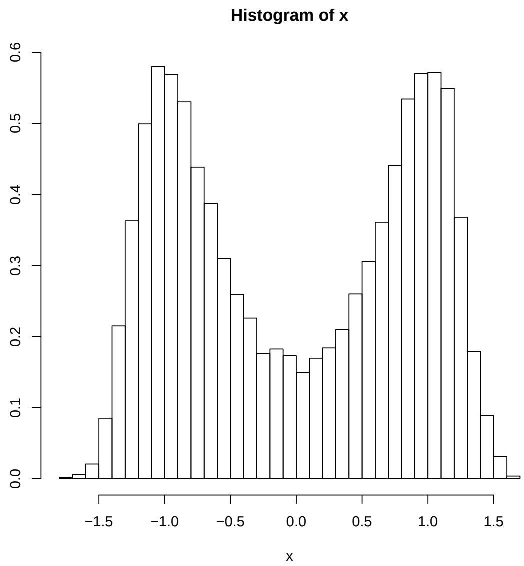

Figure 3.5: Histogram of $N = 20,000$ independent copies of x_t^i.

Ramey and West (1999) and the references cited in Section 2). As discussed in the previous section, these features contradict the production-smoothing theory, which predicts countercyclical inventory movements and smooth production. Thus, from a macroscopic point of view, inventories are considered *destabilizing* factors because they aggravate recessions by the reduction of inventories. In this regard, the inventory accelerator mechanism proposed by Metzler (1941) is a typical example that views inventory movements as destabilizing factors. Without the inventory accelerator, such cycles do not exist.

Interestingly, these "stylized facts" seem to depend on which frequencies we examine. Wen (2005) examines quarterly aggregate data from the U.S. and OECD countries and shows that production and inventories exhibit drastically different behaviors at low and high frequencies. According to his analysis, the procyclicality of inventory investment can be observed only at relatively low cyclical frequencies such as business-cycle frequencies (about 8–40 quarters per cycle). Excess volatility of production can be observed only at these frequencies. On the other hand, at a high frequency (2–3 quarters per cycle), production is less volatile than sales and inventory investment is strongly countercyclical. It can be considered that because of sluggish adjustments in production, managers cannot handle unexpected demand shocks at these high frequencies, and inventories act as buffer stock as production smoothing theory predicts.

As discussed above, which frequencies (or time scales) we examine is important. For example, Hall (2000) examines weekly data for automobile assembly plants and shows that two nonconvex margins (the change in the number of shifts and the temporary shutdown of the plant) play an important role in explaining production behavior. He particularly emphasizes intermittent production such as the weeklong temporary shutdown of plants (for a recent application of this model, see Copeland et al. (2011)). Although weeklong shutdowns used to vary output are relevant to high-frequency (weekly or monthly) production behavior, they are not suitable for explaining business cycles. Bresnahan and Ramey (1994) show that adding or dropping of an additional shift are substantially more important at the quarterly frequency than at the weekly frequency. In fact, they are the most important contributors to the quarter to quarter variation of output. Bresnahan and Ramey (1994) says,

> "While closing the plant temporarily might be important for the week-to-week vari-

ation in output, it might not be as important at the quarterly frequency." (p. 609)

This is why we emphasize changes in the number of shifts to explain the business cycle in Section 2.

3.3.2 DSGE approaches

There is a growing literature that investigates production and inventory behaviors in the framework of DSGE models. It includes Fisher and Hornstein (2000), Thomas (2002), Khan and Thomas (2003, 2007, 2008), Šustek (2011) and Kryvtsov and Midrigan (2012), to name a few.

For example, Khan and Thomas (2007) evaluate two explanations for inventories, the (S,s) and stockout avoidance motives by building DSGE models based on them and comparing the explanation power through calibration. They demonstrate that a DSGE model with (S,s) outperforms a DSGE model with the stockout avoidance motive. Especially, they argue that the average magnitude of inventories in the U.S. and the cyclical regularities can be explained by the (S,s) model.

They are a branch of widely accepted DSGE models and appear to succeed in incorporating inventory behavior (the (S,s) stragety or stockout avoidance motives) into DSGE models with sound *microfoudations*. They criticize the previous studies (e.g. Blinder and Maccini (1991), Ramey (1991) Hall (2000)) as partial equilibrium analyses. However, their arguments are quite misleading. As we mentioned in Chapter 1, general equilibrium theory does not render any microfoundations for DSGE models. In particular, their conclusions depend crucially on how they formulate DSGE models (for example, the functional forms or the choice of parameters), it is unclear that the difference of explanatory power arises from the assumptions about inventory behavior. It might be simply a mirror image of strong supplementary assumptions which can be used to solve a model. In general, we cannot distinguish these effects. Indeed, for example, Wen (2011) propose a tractable DSGE model based on the stockout avoidance motive, and Wang and Wen (2009) use the production-cost-smoothing motives, reaching different conclusions from Khan and Thomas (2007). It suggests that their conclusion depends heavily on the way they build DSGE models, and questions the robustness of their conclusions. Rather, studies that abstract a particular phenomenon and explain it with minimum assumptions, which might be considered (or criticized) as partial equilibrium analyses by DSGE proponents, are more useful and meaningful because it is easier to investigate the mechanism in detail and test such a hypothesis with empirical data. It is not the case that general equilibrium models are always superior to partial equilibrium models. For example, after criticizing the current state of macroeconomics, Solow (2008) says,

> "My general preference is for small, transparent, tailored models, often partial equilibrium, usually aimed at understanding some little piece of the (macro-) economic mechanism." (p. 246)

We follow his idea. In this Chapter, we restrict our attention to a little piece of the macroeconomic mechanism that a large number of randomly behaved firms' behavior generate a regular business cycle observed at the aggregate level. We do not resort to the DSGE approach because it includes many assumptions irrelevant to the explanation of business cycles, especially, inventory cycles (or Kitchin cycle). In the following sections, with minimum assumptions, we explain why such a collective behavior emerges, being contrary to the predictions derived from the law of large numbers.

3.4 Aggregation: Interaction Effects

3.4.1 Interaction

In Section 2, we discussed the importance of lumpiness at the micro level. However, it is unclear whether this lumpiness has any significant impact on business cycles. It might be expected that because the real economy is composed of a large number of firms, lumpiness might be irrelevant at a macroscopic level. In particular, each firm is exposed to idiosyncratic shocks. Indeed, the conventional notion is that microeconomic behaviors would cancel each other out by the law of large numbers and that aggregate exogenous shocks are needed to explain business cycles (e.g., Lucas (1977)). The majority of theoretical models, including Kydland and Prescott (1982), use exogenous shocks to total factor productivity to generate aggregate fluctuations. According to these works, without aggregate shocks, microeconomic fluctuations have no aggregate implications.

However, recent theoretical investigations present another possibility: that aggregate fluctuations can result from microeconomic shocks. The distinctive feature common to these models is the input–output linkages through which sector-specific shocks propagate to other sectors. For example, Long and Plosser (1983) construct a multisector model that generates the comovement of sector outputs even though productivity shocks to each sector are independent of each other. A positive productivity shock to sector i increases not only sector i's output, but also the output of other sectors that use i for materials. Carvalho (2010) shows that the aggregate volatility depends on the structure of intersectoral linkages. In particular, when sectoral outdegrees follow a fat-tailed distribution, the aggregate volatility decays at a lower rate as the size of the economy tends to infinity. Shocks to general purpose technologies such as oil, electricity, iron, and steel propagate to all sectors because most sectors rely on them (see also Acemoglu et al. (2012), Foerster et al. (2011), and Carvalho and Gabaix (2013)).

The literature cited above is closely related to Hulten (1978). For clarity, we briefly review Hulten (1978)'s result (see also Appendix in Acemoglu et al. (2012) and Carvalho and Gabaix (2013)). Suppose that there are N sectors in an economy that produce intermediate and final goods. In order to produce good i, sector i uses capital and labor inputs K_i and L_i, and intermediary inputs $\mathbf{X_i} = (X_{ij})_{j=1,\dots,N}$, where X_{ij} denotes intermediary inputs from sector j. The production function of sector i is assumed to be $Q_i = e^{\pi_i} F^i(K_i, L_i, \mathbf{X_i})$. Here, π_i represents the productivity of sector i. The representative agent consumes a quantity C_i of good i. Therefore, the representative consumer maximizes her utility $U(C_1, ..., C_N)$ subject to

$$C_i + \sum_k X_{ki} \leq e^{\pi_i} F^i(K_i, L_i, \mathbf{X_i})$$
$$\sum_i L_i \leq L, \quad \sum_i K_i \leq K$$

If each sector i has a productivity shock $d\pi_i$, the change of the total factor productivity (TFP) in the aggregate can be written

$$\frac{d\text{TFP}}{\text{TFP}} = \sum_{i=1}^{N} \frac{S_i}{Y} d\pi_i \tag{3.5}$$

where S_i is the value of sales of sector i (i.e., price times production Q_i) and Y is GDP. Equation (3.5) is called Hulten formula. The relation between the change of TFP and that of GDP is given by, for example, $\frac{d\text{Y}}{\text{Y}} = c\frac{d\text{TFP}}{\text{TFP}}$ for some constant $c > 0$ in Carvalho and Gabaix (2013). Note that the sum of the weights $\sum_{i=1}^{N} \frac{S_i}{Y}$ is greater than 1 because $\sum_{i=1}^{N} S_i$ is the sum of *gross output*. Even if the size of *value added* is small, sectors whose sales are large can have a non-negligible effect on aggregate variables. This is because the productivity shock is Hicks-neutral. In this framework, the propagation effect of sector-specific shocks through input–output linkages is summarized by $\frac{S_i}{Y}$.

As the above discussion indicates, significant asymmetry—the existence of sector whose sales are huge compared to other sectors—is necessary to explain aggregate fluctuation by microeconomic (sector-specific) shocks. Otherwise, as Hulten formula (3.5) suggests, these shocks cancel each other out. Namely, the presence of *hubs* leads to aggregate fluctuations. In this sense, their model is closely related to the "granular hypothesis"(Gabaix (2011)) that there exist sectors (or firms) whose sizes are non-negligible compared to an economy as a whole. As Dupor (1999) points out, when the network is more densely connected and the asymmetry disappears, the aggregate fluctuations caused by microeconomic shocks disappear (the "irrelevance theorem"). Dupor (1999) says "the input-output structures in this class provide a poor amplification mechanism for sector shocks". However, in contrast to the literature cited above, we show that by feedback effect, aggregate fluctuations (or regular business cycles) occur even in the absence of asymmetry and aggregate shocks.

Although the literature discussed above does not take into account nonconvex technology, non-convexity and interaction are explicitly considered in Bak et al. (1993) and Durlauf (1993). Bak et al. (1993) assume production bunching: that a firm produces batches of two units of the product or nothing. They assume that positive shocks to final goods leads to orders for final goods. If these orders cannot be filled out of inventories, the final good producer starts production and sends orders to its suppliers for materials. If the inventories of the suppliers are less than the orders, the suppliers start production and send orders, and so on. They call this resulting cascade of production caused by a small shock to final goods an "avalanche." Note that in their model, the shocks to the final goods are exogenous.

However, it is plausible to assume that the shocks to the final goods also depend on economic conditions. That is, if a large fraction of firms expand their production, it would cause increases in the national income (or GDP) and thereby also in the sales of final goods. In general, a firm is affected by the condition of the the macroeconomy, while the macroeconomy consists of the firms themselves. This feedback (or interaction) effect is an important aspect, and it will be shown to be the origin of business cycles.

In this sense, our model is closely related to Durlauf (1993), who explores the role of complementarities and the resulting stationary probability distribution. He assumes that each individual industry chooses one of two types production: one (denoted as technology 1) is high production with a fixed cost and the other (technology 2) is low production without fixed costs (the nonconvexity assumption). In his model, by the complementarities, the relative productivity of technology 1 is enhanced when other industries in the reference group choose technology 1. This positive spillover effect implies that when the number of neighboring industries committing to technology 1 is high, the probability of a firm choosing technology 1 becomes large. He shows that when strong enough, these complementarities lead to multiple equilibria.

Although the assumption of the binary choice (technology 1 and 2) simplifies the analysis significantly, more heterogeneous situations can also be considered. In our model, firms' inventories are distributed continuously and affect firms' choice of production. Inventories act as a state variable of a firm and their behavior cannot be described by the binary choice model. We have to deal with the distribution itself—that is, with an infinite dimensional variable. In this sense, our model is more heterogeneous than that of Durlauf (1993). Of course, there is no *a priori* reason to assume that the distribution is stationary. It requires an alternative framework to investigate the time evolution of an economy at the macroscopic level.

A series of studies conducted by Aoki (Aoki (1996, 2004), Aoki and Shirai (2000) and Aoki and Yoshikawa (2007)) address this problem and present a more general framework called *jump Markov processes*. In this framework, the evolution of the probability distribution $P(i,t)$ is described by the following *master equation*

$$\frac{\partial P(i,t)}{\partial t} = \sum_{j \in S} q(j,i)P(j,t) - P(i,t)\sum_{j \in S} q(i,j) \tag{3.6}$$

Here, $q(i,j)$ is the transition rate from i to j ($i, j \in \{1, ..., n\} \equiv S$). The interpretation is clear. The first term on the right hand side of the master equation represents the probability flow into state i from all other states, whereas the second term represents the outflows of probability from state i. Thus, the master equation means that the rate of change of $P(i,t)$ is given by the difference between the inflows and outflows of probability.

Although there is no doubt that Aoki's methods expand the scope of macroeconomic analysis and can be applied to various problems (for an application to the Diamond search model, see Aoki and Shirai (2000)), there exist some difficulties, and this framework is not suited for our problem. In particular, in our model, firms' inventories are heterogeneous and distributed continuously, and a diffusion process is considered. In such a situation, it is difficult to specify the transition rate and the master equation explicitly. In particular, it is difficult to investigate the nonstationary behavior of the probability distribution by solving master equations. We use the *propagation of chaos* instead of Aoki's methods to present an alternative method to investigate how the probability distribution behaves, without directly seeking the probability distribution itself.

3.4.2 McKean–Vlasov equation

We consider an economy consisting of a large number of firms, and add an interaction term to equation (3.4). We assume that the sales of a firm, i, depend on the conditions of the overall economy,

$$s_t^i = h\langle x \rangle + \xi_t^i, \quad \langle x \rangle \equiv \frac{1}{N}\sum_{j=1}^{N} x_t^j \tag{3.7}$$

where $0 < h < 1$ and N is the number of firms in the economy. The assumption that h is less than 1 means that the sales do not increase as much as an increase in the national income. That is, h is the marginal propensity to consume. In addition, firms are assumed to adjust their production depending on the expectation of the sales. Incorporating these effects into our model, equation (3.4) is modified as

$$\begin{aligned} dx_t^i &= (-V'(x_t^i) - ey_t^i + D(E[s_t^i] - x_t^i))dt + \sigma_1 dW_{1,t}^i, \quad V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 \\ dy_t^i &= (x_t^i - s_t^i)dt \end{aligned} \tag{3.8}$$

where $E[s_t^i]$ refers to the expectation of s_t^i and $D > 0$ represents the strength of the interaction effects. Because $E[s^i] = \frac{h}{N}\sum_{j=1}^N x_t^j$, we finally obtain

$$\begin{aligned} dx_t^i &= (-V'(x_t^i) - ey_t^i + D(h\langle x\rangle - x_t^i))dt + \sigma_1 dW_{1,t}^i, \quad V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2, \quad \langle x\rangle \equiv \frac{1}{N}\sum_{j=1}^N x_t^j \\ dy_t^i &= (x_t^i - h\langle x\rangle)dt + \sigma_2 dW_{2,t}^i \end{aligned} \tag{3.9}$$

The interaction term $D(h\langle x\rangle - x_t^i)$ means that it pushes up the production of a firm, i, if the average of production is higher than the production of the firm. On the other hand, by definition, $\langle x\rangle$ consists of all firms in the economy. This is the interaction and feedback mechanism that we consider in our model.

More generally, using the empirical measure (δ_z denotes the Dirac measure at z) defined by

$$U_t^{(N)} = \frac{1}{N}\sum_{j=1}^N \delta_{z_t^j} \tag{3.10}$$

the system of equations in (3.9) can be written as

$$dz_t^i = b(z_t^i, U_t)dt + a(z_t^i, U_t)dW_t^i \tag{3.11}$$

where z_t^i is a d-dimensional vector. The coefficients depend on the empirical measure. Because the state of the system can be described in terms of the empirical measure, $U_t^{(N)}$ is much more important than the probability distribution of dN dimensional variables. Note that $U_t^{(N)}$ is also a random variable. In particular, we consider the limiting case, $N \to \infty$.

Let u_t denote the limit of the empirical measure (if it exists), that is, $u_t = \lim_{N\to\infty} U_t^{(N)}$ in the weak sense. In probability theory, since the seminal work by McKean (1967), a number of mathematicians have studied the limit of Equation (3.11), the mean-field equation (for reviews, see Sznitman (1991) and Gartner (1988)).

$$dz_t = b(z_t, u_t)dt + a(z_t, u_t)dW_t \tag{3.12}$$

$$u_t(dz) = \text{the law of } \ z_t \tag{3.13}$$

The behavior of z_t is affected by the law of its solution. Furthermore, the law of the solution, u_t, satisfies the following equation.

$$\frac{\partial}{\partial t}u(z,t) = \sum_{k=1}^{d} \frac{\partial}{\partial x_k}(b_k(z,u)u(z,t)) + \sum_{k,l=1}^{d} \frac{\partial^2}{\partial x_k \partial x_l}(\sigma_{kl}(z,u)u(z,t)) \tag{3.14}$$

where $\sigma(z,u) = a(z,u)a^t(z,u)$. This is called the *McKean–Vlasov* equation. This equation is quite similar to the Fokker–Planck equation except that the coefficients depend on the measure u_t. It should be noted that this equation is nonlinear with respect to u_t.

In our case, Equation (3.9) can be written as

$$dz_t^{i,N} = f(z_t^{i,N})dt + g(z_t^{i,N})dW_t^i + \frac{1}{N}\sum_{j=1}^{N} \tilde{b}(z_t^{i,N}, z_t^{j,N})dt \tag{3.15}$$

$$z_t^{i,N} = \begin{pmatrix} x_t^{i,N} \\ y_t^{i,N} \end{pmatrix}, \quad f(z_t^{i,N}) = \begin{pmatrix} -(x_t^{i,N})^3 + x_t^{i,N} - ey_t^{i,N} \\ x_t^{i,N} \end{pmatrix}, \quad g(z_t^{i,N}) = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix},$$

$$dW_t^i = \begin{pmatrix} dW_{1,t}^i \\ dW_{2,t}^i \end{pmatrix}, \quad \tilde{b}(z_t^{i,N}, z_t^{j,N}) = \begin{pmatrix} \tilde{b}_1(z_t^{i,N}, z_t^{j,N}) \\ \tilde{b}_2(z_t^{j,N}) \end{pmatrix},$$

$$\tilde{b}_1(z_t^{i,N}, z_t^{j,N}) = \begin{cases} D(hx_t^{j,N} - x_t^{i,N}) & \text{if } |D(hx_t^{j,N} - x_t^{i,N})| \leq K_1 \\ K_1 & \text{if } D(hx_t^{j,N} - x_t^{i,N}) > K_1 \\ -K_1 & \textit{otherwise} \end{cases},$$

$$\tilde{b}_2(z_t^{j,N}) = \begin{cases} -hx_t^{j,N} & \text{if } |-hx_t^{j,N}| \leq K_2 \\ K_2 & \text{if } -hx_t^{j,N} > K_2 \\ -K_2 & \textit{otherwise} \end{cases}$$

where we have replaced the interaction b with $\tilde{b}$ and $K_1, K_2 > 0$. Although this technical assumption is needed for the following theorem, it is not expected to substantially affect the behavior of z_t^i and $U_t^{(N)}$ given that K_1 and K_2 are very large[2].

The corresponding mean-field equation is given by

$$\begin{aligned} dz_t^i &= f(z_t^i)dt + g(z_t^i)dW_t^i + \int \tilde{b}(z_t^i, v)u_t(dv)dt \\ u_t(dz) &= \text{the law of } \ z_t^i \end{aligned} \tag{3.16}$$

Assuming that the initial condition of $z^i, i = 1, ...N$ is drawn independently from the identical distribution, u_0, we obtain the following results.

Theorem 3.1 *1. The mean-field equation (3.16) is well-posed; that is, there exists a unique solution on* $[0, T]$ *for any* $T > 0$.

2. The process $z_t^{i,N}$ *converges in law to the solution of the mean-field equation (3.16),* z_t^i*, with speed* $1/\sqrt{N}$*; that is,*

$$\sup_N \sqrt{N} E[\sup_{t \leq T} \|z_t^{i,N} - z_t^i\|] < \infty \tag{3.17}$$

3. For any $k \in \mathbb{N}$ *and any* k*-tuple* $(i_1, ..., i_k)$*, the law of the process* $(z_t^{i_1,N}, ..., z_t^{i_k,N}, t \leq T)$ *converges to* $u_t \otimes ... \otimes u_t$.

Proof. Taking into account (3.16), the interaction, $\tilde{b}$, is bounded—that is, $\|\tilde{b}\|^2 \leq K$ for some $K > 0$. Hence, $\tilde{b}$ satisfies the linear growth condition of the interactions (H3) in Baladron et al. (2012). As is easily checked, other conditions about f, g, and $\tilde{b}$ (H1, H2, and H4 in Baladron et al. (2012)) are satisfied. Therefore, applying Theorems 2 and 4 in Baladron et al. (2012), our claim follows. ■

Property 3 in Theorem 1 is called the *propagation of chaos*[3]. It means that the probability distribution of $\{(z_t^1, ..., z_t^m)\}$ evolves as if each element is independent when $N \to \infty$. The

[2] The stability analysis and numerical simulations in the following are performed with b.
[3] This terminology comes from Kac.

motions of k tagged particles approach independent copies of equation (3.16). Moreover, the following theorem is important for our analysis,

Theorem 3.2 *Property 3 in Theorem 3.1 is equivalent to* $U_t^{(N)}$ *(*$M(\mathbb{R}^2)$*-valued random variables, where* $M(\mathbb{R}^2)$ *denotes the set of probability measures on* $\mathbb{R}^2$*), which converges in law to the constant random variable* u_t (Proposition 2.2 in Sznitman (1991)).

This theorem states that the empirical measure tends to concentrate near u_t, the solution of the mean-field equation (3.16). That is, while each element behaves stochastically, the distribution is approximated by the deterministic process as N becomes large. In this sense, this can be considered as a form of the law of large numbers. It should be noted that there is no *a priori* reason to assume that u_t is stationary. In fact, as we will see later, the distribution u_t shows cyclical movement at some parameter values. In the following, we study the behavior of u_t when we change the parameters.

3.4.3 Stability Analysis

In the previous subsection, the McKean–Vlasov equation, which governs the time evolution of the measure, $u(t) = \lim_{N\to\infty} U_t^{(N)}$, was introduced. Because the coefficients depend on $u(t)$, the equation is nonlinear with respect to $u(t)$. Thus, in practice, it is unrealistic to solve it explicitly. Therefore, to investigate the evolution of $u(t)$, an approximation method is needed.

It should be noted that in our model (equation (3.9)), each firm depends on $u(t)$ via mean fields, $\langle x\rangle$ and $\langle y\rangle$, and our primary concern is the behavior of these mean fields which correspond, of course, to macro variables. Instead of investigating $u(t)$ directly, we consider the dynamics of lower moments of $u(t)$ (see, for example, Dawson (1983), Zaks et al. (2005) and Kawai et al. (2004).).

Setting $\varphi = (x-\langle x\rangle)^n(y-\langle y\rangle)^m$ and using Ito's formula, we have

$$\begin{aligned} d\varphi &= n(x-\langle x\rangle)^{n-1}(y-\langle y\rangle)^m dx_t^i + m(x-\langle x\rangle)^n(y-\langle y\rangle)^{m-1}dy_t^i \\ &+ \frac{1}{2}n(n-1)(x-\langle x\rangle)^{n-2}(y-\langle y\rangle)^m\sigma_1^2 dt + \frac{1}{2}m(m-1)(x-\langle x\rangle)^n(y-\langle y\rangle)^{m-2}\sigma_2^2 dt \end{aligned}$$

Then, taking into account equation (3.9) and using the Taylor expansion around $\langle x\rangle$ and $\langle y\rangle$, we obtain the following dynamical system of moments:

$$\begin{aligned}
\dot{\langle x\rangle} &= \langle x\rangle - \langle x\rangle^3 - 3\mu_{2,0}\langle x\rangle - \mu_{3,0} - e\langle y\rangle - D(1-h)\langle x\rangle \\
\dot{\langle y\rangle} &= (1-h)\langle x\rangle \\
\dot{\mu}_{2,0} &= -2D\mu_{2,0} - 2e\mu_{1,1} + 2(1-3\langle x\rangle^2)\mu_{2,0} - 6\langle x\rangle\mu_{3,0} - 2\mu_{4,0} + \sigma_1^2 \\
\dot{\mu}_{1,1} &= -D\mu_{1,1} - e\mu_{0,2} + (1-h)\mu_{2,0} + (1-3\langle x\rangle^2)\mu_{1,1} - 3\langle x\rangle\mu_{2,1} - \mu_{3,1} \\
\dot{\mu}_{0,2} &= 2(1-h)\mu_{1,1} + \sigma_2^2
\end{aligned} \tag{3.18}$$

where $\mu_{n,m} = \langle (x-\langle x\rangle)^n (y-\langle y\rangle)^m\rangle$ and $\langle\rangle$ denotes the expectation with respect to u_t, that is, $\int \varphi(v)u_t(dv)$. $\dot{}$ denotes the time derivative[4].

Now, we focus on a state $\langle x\rangle = \langle y\rangle = 0$ (called the *disordered state*). It corresponds to the situation where idiosyncratic shocks cancel each other out and no aggregate fluctuation appears. Note that there is always a stationary distribution with $\langle x\rangle = \langle y\rangle = 0$ satisfying equation (3.9) because of the symmetry of our model. At this stationary solution, other moments are determined by equation (3.18), that is,

$$0 = -2D\mu^*_{2,0} - 2e\mu^*_{1,1} + 2\mu^*_{2,0} - 2\mu^*_{4,0} + \sigma_1^2 \tag{3.19}$$

$$0 = -D\mu^*_{1,1} - e\mu^*_{0,2} + (1-h)\mu^*_{2,0} + \mu^*_{1,1} - \mu^*_{3,1} \tag{3.20}$$

$$0 = 2(1-h)\mu^*_{1,1} + \sigma_2^2 \tag{3.21}$$

We then apply the Gaussian approximation to investigate the linear stability of the state $\langle x\rangle = \langle y\rangle = 0$. The Gaussian approximation means that we approximate the system by the Gaussian distribution with time-varying parameters (see Zaks et al. (2005) and Kawai et al. (2004)). Because all the moments of Gaussian distributions are determined by the lower moments ($\langle x\rangle$, $\langle y\rangle$, $\mu_{2,0}$, $\mu_{1,1}$, $\mu_{0,2}$,), equation (3.18) becomes a closed-form expression. Specifically, $\mu_{3,0} = 0$, $\mu_{3,1} = 3\mu_{2,0}\mu_{1,1}$, and $\mu_{4,0} = 3\mu_{2,0}^2$ are used in our model.

Next, we carry out the standard linear stability analysis. As is easily checked, the Jacobian of

[4]Equations for the higher moments can be deduced in a similar way, but become irrelevant under the Gaussian approximation. See below.

the five dimensional system of $(\langle x\rangle\ \langle y\rangle\ \mu_{2,0}\ \mu_{0,2}\ \mu_{1,1})$ can be written in a block diagonal form.

$$\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \tag{3.22}$$

A is a 2×2 matrix and B is a 3×3 matrix. Therefore, the behavior of $\langle x\rangle$ and $\langle y\rangle$ around the disordered state can be determined solely by

$$A = \begin{pmatrix} 1-3\mu^*_{2,0}-D(1-h) & -e \\ (1-h) & 0 \end{pmatrix} \tag{3.23}$$

The eigenvalues of A are given by

$$\lambda_\pm = \frac{1}{2}\Big(1-3\mu^*_{2,0}-D(1-h) \pm \sqrt{(1-3\mu^*_{2,0}-D(1-h))^2-4(1-h)e}\Big) \tag{3.24}$$

We examine when the stability of the disordered state gets lost—that is, when the condition that the real parts of the eigenvalues become 0. From (3.24), $\mu^*_{2,0}=\frac{1}{3}(1-D(1-h))$ is implied. From (3.21), $\mu^*_{1,1}=-\frac{\sigma_2^2}{2(1-h)}$. Substituting these values into equation (3.19), we obtain the following condition.

$$f_h(D) \equiv Dh(1-D+Dh) = \frac{3}{2}\big(\sigma_1^2+\frac{e\sigma_2^2}{1-h}\big) \equiv \frac{3}{2}\sigma^2 \tag{3.25}$$

$\sigma^2(\equiv \sigma_1^2+\frac{e\sigma_2^2}{1-h})$ represents the intensity of idiosyncratic shocks. The left-hand side, $f_h(D)$, can be interpreted as the degree of interactions that generate order in the system. In particular,

$$\lim_{h\to 1} f_h(D) = D \tag{3.26}$$

When the two parameters, σ and D, satisfy this relation, bifurcation occurs. The interpretation is clear. When the interaction effect, D, is below the critical point, D^*, the idiosyncratic shocks dominate the system. Any order is destroyed by these shocks, and the system is close to the system with no interaction. Therefore, the simple LLN holds and any no order can be observed. The stationary distribution with $\langle x\rangle = \langle y\rangle = 0$ is stable. The microeconomic structure (e.g.

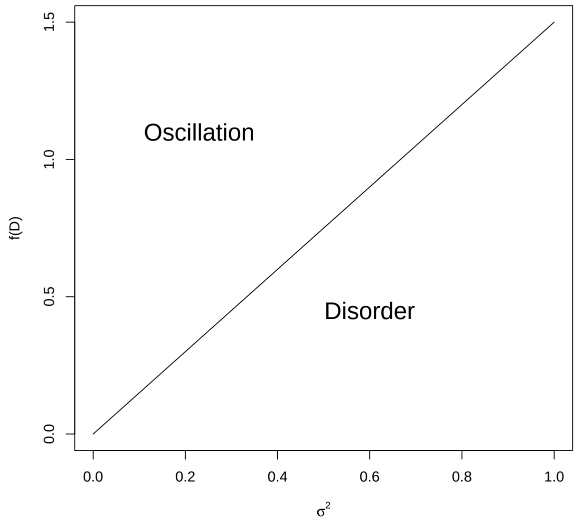

Figure 3.6: Equation (3.25).

lumpiness) is irrelevant to the explanation for the aggregate fluctuations.

However, when D exceeds the critical point, D^*, the situation changes completely. The linear stability analysis above shows that the stationary distribution with $\langle x \rangle = \langle y \rangle = 0$ is no longer stable. That is, the idiosyncratic shocks do not prevent the interaction from generating order in the system. This suggests the possibility of collective behavior in the system. In fact, as we will see in the following, regular cyclical behavior is observed at the aggregate level. In the next subsection, we carry out numerical simulations.

3.4.4 Simulation

Figures 3.7 to 3.17 show the results of simulations for $\langle x \rangle$ and $\langle y \rangle$ of equation (3.9) with different values of D (other parameters are fixed and $N = 20000$). In Figure 3.7 with a small value of D, there is no observable aggregate behavior. Only small variations around $\langle x \rangle = \langle y \rangle = 0$ are observed. This is considered to be the finite number effect of N. It is consistent with our analysis in the previous section. The microeconomic shocks cancel each other out; therefore, lumpiness (or nonconvexity) at the firm level plays no role in the aggregate fluctuations.

On the other hand, Figure 3.10 shows that when the interaction effect is large enough to compen-

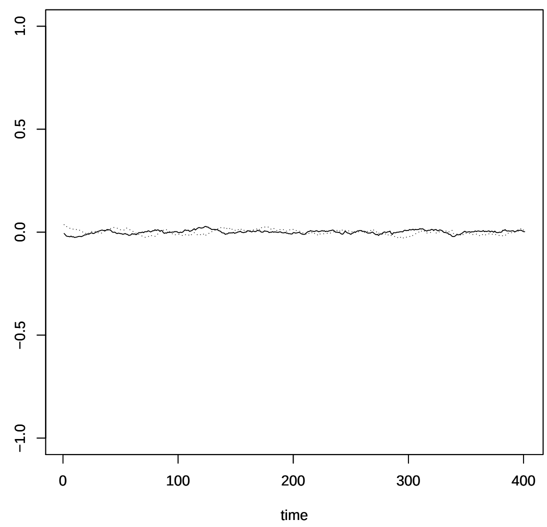

Figure 3.7: Simulation of equation(3.9) with $\sigma_1^2 = \sigma_2^2 = 1/4$, $e = 0.1$, $h = 0.9$ and $D = 0.1$. The solid(dashed) line is $\langle x \rangle (\langle y \rangle)$.

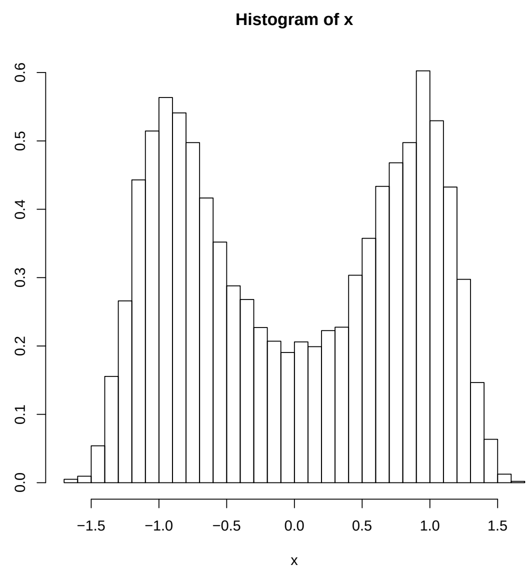

Figure 3.8: The histogram of x_t^i. The parameters are the same as in Figure 3.7.

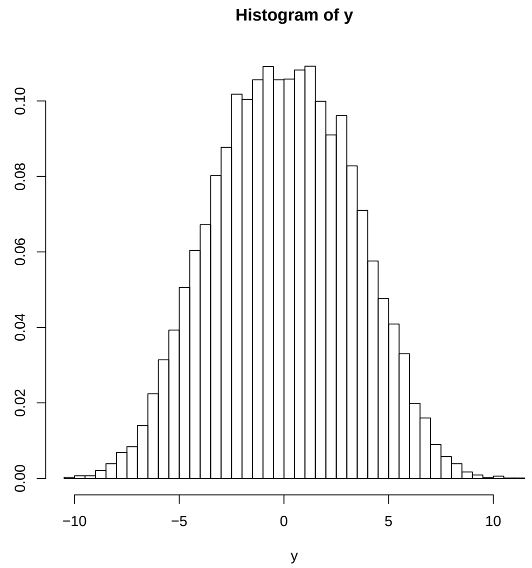

Figure 3.9: The histogram of y_t^i. The parameters are the same as in Figure 3.7.

sate for the disturbances caused by the idiosyncratic shocks, the completely different aggregate behavior appears. That is, an endogenous cyclical movement is observed. This is consistent with the fact that the eigenvalues, (3.24), have an imaginary part different from 0 near the bifurcation point. Interestingly, the movement at the macroscopic level is more regular than at the firm level.

Figure 3.12 shows the histogram of y when $\langle x \rangle = 0.00$ and $\langle y \rangle > 0$—that is, when the economy has excess inventories. It corresponds to a situation in which the economy goes through a phase of contraction to reduce the excess inventories. However, it should be noted that there is heterogeneity among firms and, as Figure 3.12 shows, some firms' inventories are running short. The same argument can be applied to Figure 3.14, where business is good, $\langle x \rangle > 0$. Under this favorable business condition, there exist firms that choose low production depending on their states. The motions of $\langle x \rangle$ and $\langle y \rangle$ are the averaging behaviors of firms in the economy.

The cyclical behavior of $\langle x \rangle$ and $\langle y \rangle$ can be observed to be significantly below the critical value $D^* = 0.92$ predicted by the stability analysis in the previous section. This is related to the fact that the resulting distribution is different from a Gaussian distribution. In particular, the marginal distribution of x_t^i shows clear bimodality. In Figure 3.18, we estimate the spectral density of the cycle of $\langle x \rangle$. This density peaks at 0.014—that is, the period of the cycle is 71.

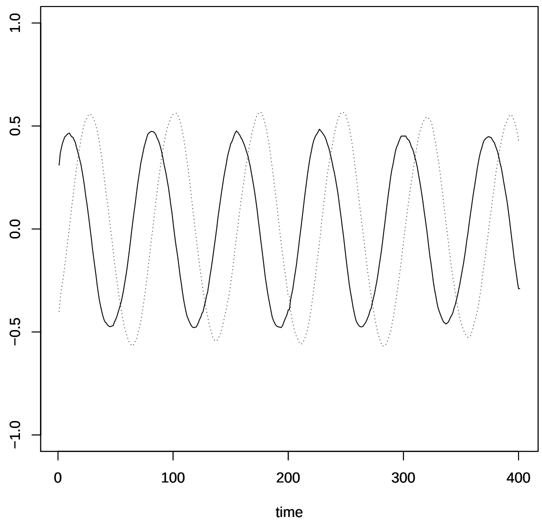

Figure 3.10: Equation (3.9) with $D = 0.32$. The other parameters are the same as in Figure 3.7.

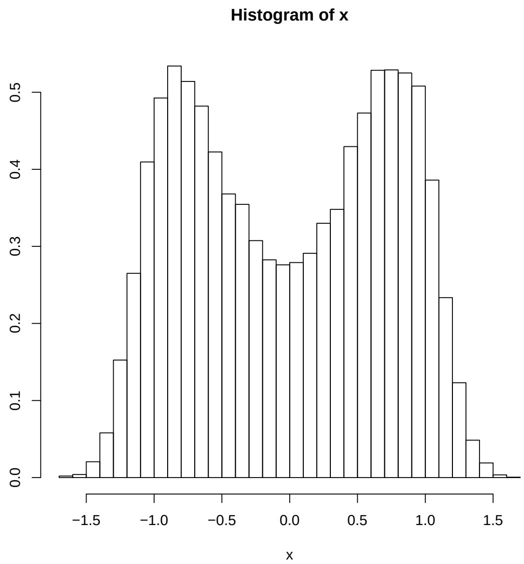

Figure 3.11: The histogram of x_t^i when $\langle x \rangle = 0.00$. The parameters are the same as in Figure 3.10.

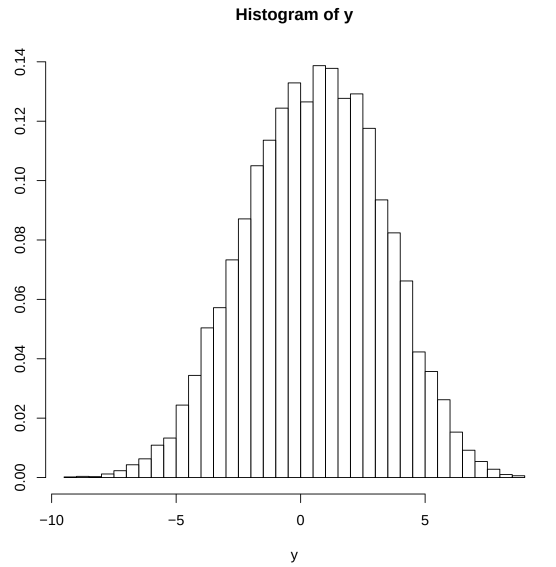

Figure 3.12: The histogram of y_t^i when $\langle x \rangle = 0.00$. It corresponds to an economy going through a phase of contraction due to excess inventories, $\langle y \rangle = 0.56$.

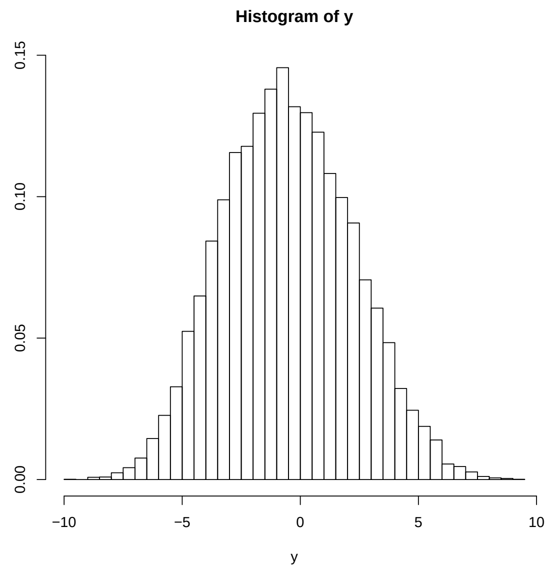

Figure 3.13: The histogram of y_t^i when $\langle x \rangle = 0.00$. It corresponds to an economy going through a phase of expansion, $\langle y \rangle = -0.52$.

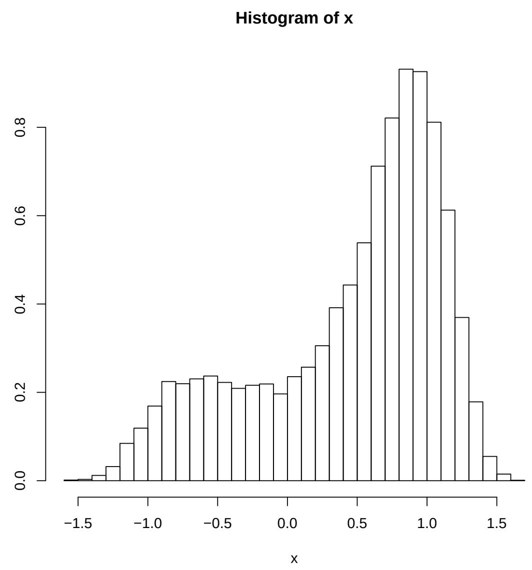

Figure 3.14: The histogram of x_t^i when $\langle x \rangle = 0.45$.

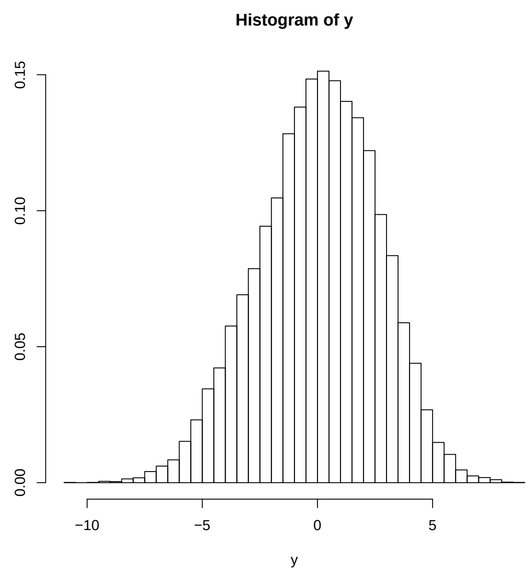

Figure 3.15: The histogram of y_t^i when $\langle x \rangle = 0.45$.

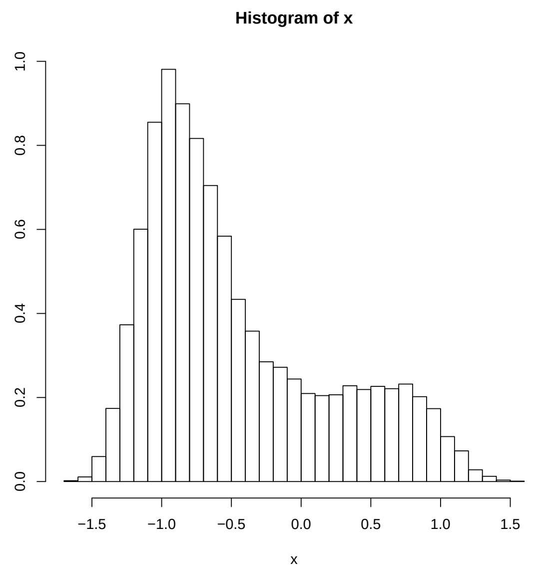

Figure 3.16: The histogram of x_t^i when $\langle x \rangle = -0.46$.

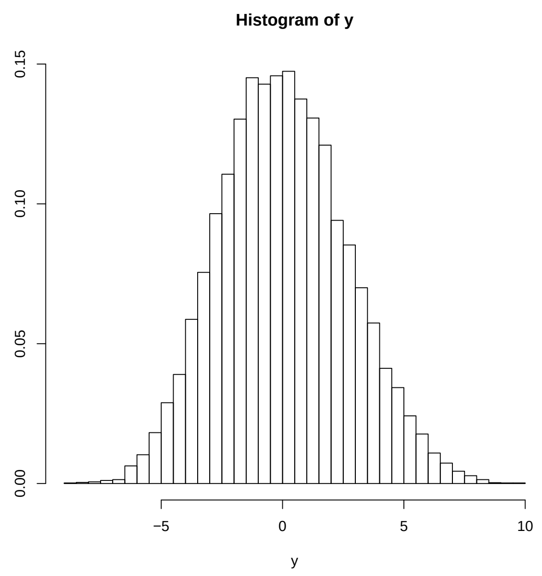

Figure 3.17: The histogram of y_t^i when $\langle x \rangle = -0.46$.

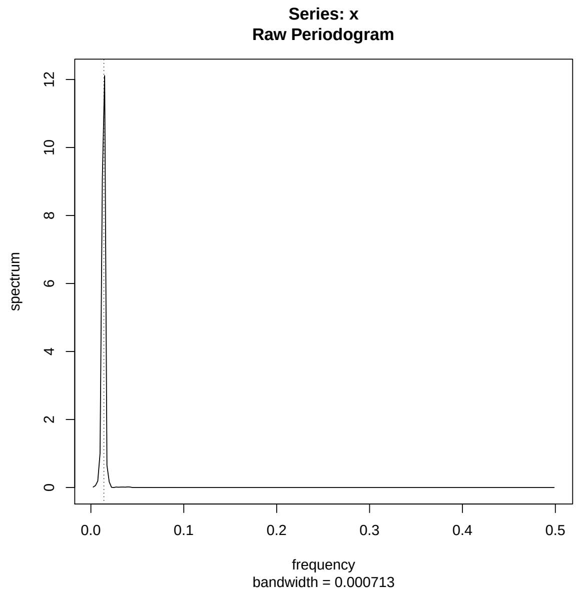

Figure 3.18: Spectral density of $\langle x \rangle$.

On the other hand, from (3.24), the frequency is given approximately by $2\pi/\sqrt{(1-h)e} = 0.016$ near the bifurcation point. The period predicted by the stability analysis is $1/0.016 = 63$, which is relatively close to the estimated value. Therefore, although the critical value of D is overestimated, we conclude that the qualitative feature of our model is captured by the stability analysis[5].

This cyclical behavior of $\langle x \rangle$ and $\langle y \rangle$ is closely related to the well-known *Kitchin cycle*, which is usually explained as follows (see, e.g., Korotayev and Tsirel (2010)). Suppose that firms observe the improvement of their commercial situation. They try to meet an increase in demand by increasing production. The high demand eventually is filled with the increase in supply, but the supply gradually becomes excessive because it takes some time for businesspeople to realize that the supply exceeds the demand. This time lag generates an unexpected increase in inventories, which leads to the reduction of production to decrease the excessive inventories. After the inventories are sufficiently reduced, a new cycle of demand increase is initiated. Accroding to this view, the fundamental factor causing these cycles is the time lags in the information.

At first glance, as shown in Figure 3.10, the behavior of $\langle x \rangle$ and $\langle y \rangle$ appears to be consistent

[5] On this point, Zaks et al. (2005) reach the same conclusion. See Zaks et al. (2005) and the relevant discussion therein.

with the above scenario. An increase in $\langle x\rangle$ is an increase in demand that leads to an increase in $\langle y\rangle$. The cycle of $\langle y\rangle$ lags behind that of $\langle x\rangle$. However, time lags at the firm level are not assumed in our model. Because of nonconvex technology, businesspeople optimally choose the low or high production and increase or decrease their inventories. Furthermore, in contrast to Carvalho (2010), Acemoglu et al. (2012), and Gabaix (2011), each firm has a negligible impact on $\langle x\rangle$ and $\langle y\rangle$ as N is large. $\langle x\rangle$ and $\langle y\rangle$ are the average of firms in the economy. There is no representative firm corresponding to the motion of $\langle x\rangle$ and $\langle y\rangle$. Indeed, as shown in Figures 3.3, 3.4, and 3.10, the behavior of $\langle x\rangle$ and $\langle y\rangle$ is different from that of an individual firm, x^i and y^i. The cyclical behavior of $\langle x\rangle$ and $\langle y\rangle$ is a type of collective behavior that can only be observed at the macroscopic level.

Furthermore, the relation of the two cyclical behavior of $\langle x\rangle$ and $\langle y\rangle$ can be explicitly written. Summing both sides of equation (3.2) over i and dividing them by N, we obtain

$$\frac{1}{N}\sum_{i=1}^{N} dy_t^i = \frac{1}{N}\sum_{i=1}^{N}(x_t^i - s_t^i)dt = \frac{1}{N}\sum_{i=1}^{N}(x_t^i - h\langle x\rangle - \xi_t^i)dt = \Big((1-h)\langle x\rangle - \frac{1}{N}\sum_{i=1}^{N}\xi_t^i\Big)dt \quad (3.27)$$

Taking the limit, $N \to \infty$, we obtain the simple relation $\dot{\langle y\rangle} = (1-h)\langle x\rangle$ by the law of large numbers. This means that the aggregate inventory investment (the change in inventories) comoves with the aggregate production without a time lag. This prediction is consistent with empirical data (see, e.g., Table 2 in Stock and Watson (1999)).

3.5 Concluding Remarks

This Chapter investigated the relationship between microeconomic structures and business cycles. The standard production-smoothing theory has been empirically rejected in the literature; therefore, we focused on the nonconvex cost function. This hypothesis, which has empirical support, can explain the excess volatility of production. The issue is whether this microeconomic structure has nontrivial effects at the aggregate level. If interaction effects are taken into account, this problem becomes very complicated, and the LLN argument (e.g. Lucas (1977)) cannot be applied. In particular, we have to deal with the evolution of the distribution of production and inventories, that is, an infinite-dimensional random variable.

To investigate this problem, the propagation of chaos result is used in our model. Our model explicitly takes into account the feedback loop—that is, the macroscopic state of the economy not only is an aggregation of the firms but, at the same time, prescribes the macroeconomic environment experienced by firms. We have shown that the empirical measure of production and inventories, $U_t^{(N)}$, converges to a $M(\mathbb{R}^2)$-valued constant variable, u_t, as N goes to infinity. This means that whereas each element behaves stochastically, the distribution is approximated by the deterministic process as N becomes large. In this sense, it can be considered a form of the law of large numbers. However, this does not imply that the distribution is stationary. In fact, this feedback loop together with nonconvex technology generates many interesting macrophenomena.

The standard linear stability analysis shows that the disorder state loses its stability, given that the interaction effect exceeds the critical point. This means that the interaction effect generates *order* in the system. With the help of numerical simulations, we have demonstrated that the resulting aggregate behavior shows regular cyclical movement without any aggregate exogenous shocks. This endogenous business cycle is an explanation for the Kitchin cycle. It should be noted that there is no representative firm corresponding to $\langle x \rangle$ and $\langle y \rangle$ and that the behavior of $\langle x \rangle$ and $\langle y \rangle$ is different from that of an individual firm, x^i and y^i. This is one example of the collective behaviors that can be only observed at the aggregate level and are crucial to macroeconomic analysis.

Finally, there exists other microeconomic behavior that is characterized by lumpiness (e.g., Cooper and Haltiwanger (2006)). Investigating how the microeconomic characteristics affect the aggregate fluctuations via interactions is a promising subject for future research.

Chapter 4

Income Distribution among Individuals: The Effects of Economic Interactions

4.1 Introduction

This Chapter investigates the implications of the observed statistical regularities of income distributions, especially taking into account interactions among individuals.

The income distribution among individuals is an important subject in economics. The study of income distribution has a long history and, dating at least to the end of the nineteenth century. Pareto (1896) empirically observes that the distributions of very high incomes exhibit the power-law tail $x^{-\alpha}$, that is, a heavy tail, indicate an unequal distribution. This power law behavior is now known as Pareto's law. For example, Aoyama et al. (2000) analyze the distribution of the income and income tax of individuals in Japan for the fiscal year 1998, and show that it is very well fitted by a power law. Clementi and Gallegati (2005) investigate the shape of the Italian personal income distribution and find a power-law tail.

It should be noted, however, that Pareto's law is applicable only to the tail of the income distribution, that is, for extremely high incomes. Gibrat (1931) (and later Aitchison and Brown

(1957)) shows that the lognormal distribution well describes the income distribution at low and middle income ranges, and proposes a theoretical hypothesis called the *law of proportionate effect* (discussed below). Souma (2001, 2002) examines Japanese income data and shows that the income distribution is well described by the lognormal distribution at low and middle income levels and have power-law decay at high income levels. Montroll and Shlesinger (1983) investigate U.S. personal income data for the years 1935–1936 and show a lognormal distribution with a power-law tail. Willis and Mimkes (2004) study high-quality income data from the United Kingdom and the United States and demonstrate that both distributions are well described by the lognormal distribution[1]. Lopez and Serven (2006) investigate about 800 country-year observations and conclude that the lognormal hypothesis (the null hypothesis) cannot be rejected. The lognormal distribution has been widely used to estimate various measures of poverty and inequality (see, for example, Pinkovskiy and Sala-i Martin (2009) and the references therein). These studies indicate a general consensus that the income distribution obeys a lognormal distribution at low and middle income ranges, while high incomes (for example, the top 1%) exhibit power law behavior[2].

Theoretical investigations have also been conducted. Gibrat (1931) models income as the accumulation of random multiplicative shocks and derives the lognormal distribution (see also Sargan (1957) and Pestieau and Possen (1979), where models leading to the lognormal distribution are proposed with economic interpretations). The main idea of Gibrat's model is simple. Let X_t be an individual income at time t and assume that X_t is described by the following stochastic process:

$$X_{t+1} - X_t = \epsilon_t X_t.$$

Here, ϵ_t is assumed to be independent of X_t (the *law of proportionate effect*). This is equivalent to saying that an individual income at $t+1$ can be described by multiplicative shocks up to $t+1$

[1]Some *econophysicists* (for example, Yakovenko and Rosser (2009)) prefer the Boltzmann distribution (or exponential distribution) to the lognormal distribution. However, it is a monotone decreasing function and cannot describe the hump-shaped pattern of the income distribution. Therefore, we employ the lognormal distribution here.

[2]Although the lognormal distribution can give accurate fits to data, other distributions generalizing the lognormal and Pareto distributions have been discussed in the literature. See, for example, Singh and Maddala (1976) and Bordley et al. (1997).

and the initial condition, X_1:

$$X_{t+1} = (1+\epsilon_1)(1+\epsilon_2)...(1+\epsilon_t)X_1.$$

Taking the logarithm of both sides of this equation, it can be written as

$$\log X_{t+1} = \log(1+\epsilon_1) + \log(1+\epsilon_2)... + \log(1+\epsilon_t) + \log X_1 \approx \sum_{s=1}^{t} \epsilon_s + \log X_1.$$

where we use an approximation $\log(1+\epsilon) \approx \epsilon$ when ϵ is small. When t is large and the effect of the initial condition, $\log X_1$, becomes negligible, the logarithm of income can be written as an accumulation of growth shocks,

$$\log X_{t+1} \approx \sum_{s=1}^{t} \epsilon_s.$$

It is clear that if growth shocks are independent of each other and satisfy standard conditions of the central limit theorem (CLT), the distribution of the logarithm of income follows the Gaussian distribution. By definition, this implies that the income distribution is described by the lognormal distribution.

The straightforward application of CLT to the multiplicative process appears trivial. However, as Kalecki (1945) points out, there is a pitfall in that the variance of $\log X_t$ increases unboundedly as t goes to infinity. Of course, this is inconsistent with the observed data; the distribution of income has finite, fairly stable variance. Kalecki (1945) considers a mean-revearting process; that is, when an worker's income is far from the mean, a force attracting the income to the mean works. Although this violates the law of proportionate effect, he succeeds in obtaining a stationary distribution with a finite variance.

Following these studies, Champernowne (1953, 1973), and Rutherford (1955) examine income distributions and show that slight modification of lognormal models can lead to the stationary distribution exhibiting power law behavior. Closely related to these studies, recent developments with a reflective lower boundary have succeeded in reproducing the observed power law for very high incomes (see, for example, Levy and Solomon (1996), Levy (2003), and Gabaix (1999)).

In the existing literature, theoretical investigations emphasize the power-law behavior of the *tail*; however, not many studies have been done on the distribution of middle and low incomes following the lognormal distribution. In this Chapter, we focus on this middle and low incomes described by the lognormal distribution. As is easily imagined, the main source of these incomes is labor income, that is, wages (see, for example, Souma and Nirei (2005)). Therefore, the stochastic process of low and middle incomes might be different from that of high income earners, a significant proportion of whose incomes are accounted for by capital gains. Nirei and Souma (2007) use a different stochastic process (an additive process) to describe the evolution of labor income. However, because of the lack of an easily available data source for investigating the evolution of income of workers, it is difficult to specify the labor income process[3]. We consider two representative stochastic processes (additive and multiplicative processes) and compare their implications.

This stochastic approach is sometimes criticized on the grounds that *microfoundations* for the models are lacking. Sahota (1978), for example, states,

> "...in Friedman's individual choice theory, chance and luck (including genetic luck) play their roles, but only subsidiary ones, while economic behavior (in the face of uncertainty) is the basic determinant of income inequalities. ... In the face of newly budding theories, the stochastic theory seems to be a 'degenerate problem-shift' " (p. 39).

However, this argument is misleading. First, the assumption that the income process is described by a stochastic process is not inconsistent with an individual optimization. The point is that even if individual workers optimize their decisions and behaviors, we are unable to determine their optimization problems exactly from an observer's viewpoint. This is because they depend on unobservable variables and idiosyncratic shocks, and it is simply beyond our knowledge. Thus, we need *coarse graining*. Champernowne (1953) writes,

> "The forces determining the distribution of incomes in any community are so varied

[3] In contrast to that of lower incomes, the stochastic process of high earners has been examined in detail. See, for example, Fujiwara et al. (2003).

> and complex, and interact and fluctuate so continuously, that any theoretical model must either be unrealistically simplified or hopelessly complicated" (p. 319).

We cannot distinguish a worker's optimal behavior from his/her stochastic behavior. Furthermore, it should be noted that we are not concerned with the behavior of the income of a particular individual, but rather the income distribution observed at the aggregate level, that is, with macroscopic phenomena. This implies that the details of workers are irrelevant. All we know is the information that is indispensable to the macroscopic behavior of the system. This distinction is particularly important; Champernowne (1973) states,

> "Whereas we have found this stability in the characteristics of the distribution, yet individual incomes, we know, are by no means steady in their movements. Our discovery has been that although individual incomes are fluctuating violently about, the net effect of all this movement is a mere reshuffling which leaves little trace on the main characteristics of the *distribution* of these incomes" (p. 90).

In other words, idiosyncratic shocks cancel each other out and a more fundamental, stable relationship appears at the macroscopic level. This is because the number of microeconomic agents is so large that the averaging effect works. That is, the randomness and the fact that the system is composed of a large number of elements can be considered a basis for the observed regularity. The aim of this Chapter is to derive the implications of the observed regularity of income distribution, that is, the lognormal distribution.

In addition, we explicitly focus on economic interactions among individuals in this Chapter. The existing literature cited above implicitly assumes that individual incomes are independent of each other. It is unrealistic to assume that the evolution of an individual income is not affected by others' incomes. It is reasonable to expect that a worker's income increases when his/her customers, employers, or fellow people in his/her community become rich. Otherwise, workers would never migrate from poor countries to seek employment in advanced countries. Obviously, interaction effects (or, in the terms of Cooper and John (1988), *strategic complementarities*), a kind of externality, are important to income distribution. The purpose of the present Chapter is to explicitly investigate the interaction effects in a stochastic model.

The rest of this Chapter is organized as follows. In Section 2, we discuss the interaction effects and introduce a mathematical concept, the *propagation of chaos*. In Section 3, we discuss the two representative candidates to describe the evolution of an individual income, additive, and multiplicative processes. In Section 4, we examine a stationary state of income distribution under the assumption that the underlying process is additive, taking into account the interaction effects. In Section 5, we replace the additive assumption with the multiplicative one and compare the results. Section 6 concludes.

4.2 Economic Interactions and the Propagation of Chaos

The idea that the decision or behavior of an agent is influenced by that of other agents is by no means new in the macroeconomic literature. For example, Cooper and John (1988) emphasize the importance of *spillovers* (or *strategic complementarities*), in which one player's decision affects the payoffs of other players. When these effects are present, a coordination failure can occur. That is, a Pareto-efficient equilibrium might not be achieved because no individual player has an incentive to change his/her strategy. There are many examples in the economic literature exhibiting these circumstances—for example, as cited in Cooper and John (1988), technological complementarities (e.g., Bryant (1983)) or trading externalities (e.g., Diamond (1982)). Especially when players are positively interacted—that is, when reaction curves are positively sloped—Cooper and John (1988) show the existence of multiple Pareto-ranked equilibria and a multiplier mechanism associated with exogenous shocks displaying certain features of Keynesian economics. When we consider the determination of the incomes of workers, positive interaction is a plausible assumption. Namely, a worker's income increases when his/her customers, employers, or fellow people in his/her community become rich. We explicitly take these interactions into account. These interactions can be regarded as attracting force (or centripetal force) preventing the divergence of the variance of incomes because the probability is low that a particular worker's income is high when the average income in the economy is low. It should be noted that, while Nirei and Souma (2007) introduce the lower bound income exogenously to obtain a stationary distribution, the attracting force in our model is endogenously generated by interactions. Then, we build a model in which a worker's income process depends on the state of the economy, that is, the empirical distribution of incomes, in the following.

Consider an economy populated by N workers. Let $X_t^{(i)}$ be the income of the ith worker at time t. We decompose the increment $dX_t^{(i)}$ into two components, an interaction component and a random one representing idiosyncratic shocks. The differential equation of $X_t^{(i)}$ is given by

$$\begin{aligned} dX_t^{(i,N)} &= \frac{1}{N}\sum_{j=1}^{N} b(X_t^{(i,N)}, X_t^{(j,N)})dt + \sigma(X_t^{(i,N)})dW_t^{(i)}, \ i = 1, ..., N, \\ X_0^{(i,N)} &= x_0^i. \end{aligned} \tag{4.1}$$

Here, $W_t^{(i)}$ is an independent Brownian motion. In contrast to the existing models discussed in the Introduction, workers' incomes $X_t^{(i,N)}, i = 1, ..., N$ are not independent of each other because of the function b. This term represents strategic complementarities among workers. Because we consider positive interaction, $b(X_t^{(i,N)}, X_t^{(j,N)}) > 0$ in equation (4.1) if $X_t^{(i,N)}$ is small and $X_t^{(j,N)}$ is large. Because of this interdependence, it is difficult to solve the time evolution of the distribution of X_t in general. However, when N is large, there exists a type of law of large numbers.

Consider a *nonlinear* stochastic differential equation

$$\begin{aligned} dX_t &= \int b(X_t, y)u_t(dy)dt + \sigma(X_t)dW_t, \\ X_0 &= x_0, \quad u_t(dy) \text{ is the law of } X_t. \end{aligned} \tag{4.2}$$

This equation is nonlinear *with respect to* u_t: while the evolution of X_t and its law u_t are determined by the right hand-side of equation (4.2), the right-hand side depends on u_t. The existence and uniqueness of a solution of equation (4.2) are guaranteed under suitable conditions (see Sznitman (1991)). We now introduce independent copies of equation (4.2) corresponding to equation (4.1)

$$\begin{aligned} dX_t^{(i)} &= \int b(X_t^{(i)}, y)u_t(dy)dt + \sigma(X_t^{(i)})dW_t^{(i)}, \ i = 1, ..., N, \\ X_0^{(i)} &= x_0^i, \quad u_t(dy) \text{ is the law of } X_t^{(i)}. \end{aligned} \tag{4.3}$$

Sznitman (1991) shows that as N goes to infinity, $X_t^{(i,N)}$ converges to $X_t^{(i)}$. Especially when the initial condition x_0^i is drawn from a common distribution u_0, one can show that the joint distribution of $(X_t^{(i_1,N)}, ..., X_t^{(i_k,N)})$ converges to $u_t^{\otimes k}$, the k product of u_t. This property is called the *propagation of chaos*. It is surprising that as N goes to infinity, the originally dependent random variables can be described as if they are independent. The interaction effects are represented by the common drift term $\int b(X_t, y)u_t(dy)$.

Moreover, in order to describe the state of an economy, the empirical distribution defined by

$$U_t^{(N)} = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i}$$

is more important than the N-dimensional probability distribution of $X_i,\ i = 1, ..., N$. Here, δ_x denotes the Dirac measure at x. Sznitman (1991) shows that the empirical distribution, which is also a random variable, converges in law to the constant random variable u_t, the solution of equation (4.2) (Proposition 2.2 in Sznitman (1991)). In other words, the macroeconomic environment generated by interaction among workers in the economy is determined by the solution u_t of equation (4.2) as $N \to \infty$. It should be noted that while $\int b(X_t, y)u_t(dy)$ is endogenously determined by the behaviors of workers, it cannot be distinguished from exogenous variables from the standpoint of an individual worker.

In the following section, we especially focus on stationary distribution of workers' income, which is well fitted by the lognormal distribution. That is, we assume that u_t is the stationary lognormal distribution, and set $f(.) = \int b(., y)u_t(dy)$, that is, the interaction effects generated by the lognormal income distribution. Then, we investigate the characteristics of interaction and the implications of the fact that the income distribution is well described by the lognormal distribution.

4.3 The Additive Process vs. the Multiplicative Process

Before proceeding further, we need to specify the characteristic of the underlying idiosyncratic shocks, that is, the function $\sigma(X_t)$.

In the existing literature, multiplicative stochastic processes, that is, $\sigma(X_t) = DX_t$, where D

is a constant, are widely used to explain the shape of the income distribution, especially at the high income range. For example, Champernowne (1973) and Levy and Solomon (1996) use these processes with a reflective boundary condition and generate power-law tails. In the region in which the power law is observed, asset accumulation plays a crucial role in the evolution of individual income (see Souma and Nirei (2005), Atkinson et al. (2011) and references therein). Fujiwara et al. (2003) analyze the personal incomes of about 80,000 high-income taxpayers in Japan for two consecutive years, 1997 and 1998, and show that the assumption of multiplicative processes has good agreement with the data. Apart from the literature cited above, the multiplicative assumption is also used in econometric literature that discusses the heterogeneity and microeconomic structure of individuals (see, for example, Meghir and Pistaferri (2004), Blundell and Stoker (2005), and Guvenen (2007)).

However, the main source of the low and middle income levels on which we focus is labor income, that is, wages and salaries (Souma and Nirei (2005)). Are they described by the same multiplicative process as capital gains? On this point, *model contractual wages* for different age groups, presented in Table 4.1, might provide some insight.

Table 4.1: Model contractual wages by age (2011).

Age	wages for males (1,000 yen)	Differences
22 years old	211	
25	243	32
30	325	82
35	405	80
40	495	90
45	564	69
50	634	70
55	646	12
60	621	-25

Data are drawn from *Japan Statistical Yearbook 2013*, which is edited by the Statistical Research and Training Institute and published by the Statistics Bureau, both of which are under the Ministry of Internal Affairs and Communications, Japan. The *model contractual wage* is defined as “the contractual wage of an employee who joined a company immediately after the graduation, worked continuously for the company, normally promoted and meets the prescribed conditions (sex, occupation, education, age, year of service, and number of dependents) of the model. The contractual wage (wage paid for contractual working hours) includes, in addition to the basic wage, job-related allowances and such living-related allowances as family allowance and housing allowance, but excludes shift allowance and commuting allowance" (Chap. 16-15).

Table 4.1 shows that the increase in the wage is roughly linear with age. The differences in Table 4.1 do not increase as X_t does. If the wages are largely determined by the productivity of labor and the productivity is an accumulation of multiplicative shocks, it is expected that the wages increase exponentially. This linearity suggests that labor income can be described by an additive process, $\sigma(X_t) = D$, where D is a constant, instead of a multiplicative process. Of course, Table 4.1 shows the average growth of wages, not the variance of changes in wages. Although we cannot conclude that idiosyncratic shocks are additive solely by these data, the assumption of additive productivity shocks can be found in the literature. For example, Kydland and Prescott (1982) simply assume that productivity z_t is described by $z_{t+1} = \rho z_t + \epsilon_t$, where $\rho \leq 1$ is a constant and ϵ_t is a random variable.

In addition, related to the observed shape of the income distribution, Nirei and Souma (2007) assume that the labor income evolves as an additive process, where stochastic shocks represent the heterogeneity of labor productivity across workers. Nirei and Souma (2007) also present another interpretation of the process: that it is a wage process of a *job* rather than an individual worker. This interpretation is useful because the overall shape of the distributions of the two processes coincide and we can avoid problems associated with the entry and exit of workers. That is, "[A] job is a vehicle of an individual wage which individuals switch due to demographic or individual risk factors" (p. 452). Silva and Yakovenko (2005) consider the additive assumption as an appropriate one on the grounds that the main source of their incomes is labor income. Silva and Yakovenko (2005) writes,

> "The lower-class income comes from wages and salaries, so the additive process is appropriate, whereas the upper-class income comes from investments, capital gains, etc., where the multiplicative process is applicable" (p. 309).

Which assumption is appropriate should be determined by empirical data and is a task for future research. In the following, we consider these two stochastic shocks (additive and multiplicative) and derive implications under each assumption.

4.4 The Additive Process

In this section, we investigate the properties of the interaction effects, f, given that the idiosyncratic shocks are represented by additive shocks, that is, $\sigma(X_t) = D$. Equation (4.2) then becomes

$$dX_t = f(X_t)dt + DdW_t. \tag{4.4}$$

At first glance, equation (4.4) seems to be a usual one-dimensional stochastic differential equation. However, it should be noted that the first term arises from the interaction term b. As discussed in Section 2, as the number of individuals N goes to infinity, the effects experienced by an individual worker cannot be distinguished from exogenous variables from the worker's point of view; this is reduced to the form shown in equation (4.4) by the propagation of chaos.

Because the empirical distributions of incomes are well described by a lognormal distribution and are stable, the resulting stationary distribution of equation (4.4) must be a lognormal distribution. This can be considered a so-called *self-consistency condition* in our model. Thus, the goal of our analysis is to find the function f for which the stationary distribution derived from equation (4.4) is the lognormal distribution.

4.4.1 The Fokker–Planck Equation

Before identifying the stationary distribution and the corresponding interaction function f, we briefly review the Fokker–Planck equation. In general, the evolution of the probability density derived from equation (4.4) is governed by the Fokker–Planck equation defined by the following (see, for example, Risken (1989) and Hashitsume et al. (1991)):

$$\frac{\partial P(x,t)}{\partial t} = L_{FP} P(x,t), \tag{4.5}$$

$$L_{FP}(x) \equiv -\frac{\partial}{\partial x} f(x) + \frac{D^2}{2}\frac{\partial^2}{\partial^2 x}. \tag{4.6}$$

These can be transformed into

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial S(x,t)}{\partial x}, \tag{4.7}$$

$$S(x,t) \equiv f(x)P(x,t) - \frac{D^2}{2}\frac{\partial}{\partial x}P(x,t). \tag{4.8}$$

This is called a continuity equation in physics. Continuity equations are the local form of conservation laws. In our case, the conserved quantity is the probability because the total of the probability must be 1. S defined in (4.8) is called the *probability current* (see Risken (1989)). Because the probability current must be zero in a stationary state, we thus obtain

$$f(x)P_{st}(x) = \frac{D^2}{2}\frac{\partial}{\partial x}P_{st}(x). \tag{4.9}$$

Suppose that P_{st} can be written as $N \exp Q$ (N is a normalization constant). Substituting it into equation (4.9), we obtain

$$f(x) = \frac{D^2}{2}\frac{dQ(x)}{dx}. \tag{4.10}$$

On the other hand, the lognormal distribution has the following functional form:

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma x}\exp(\frac{-(\ln x - \mu)^2}{2\sigma^2}) \tag{4.11}$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\exp Q, \qquad Q = \frac{-(\ln x - \mu)^2}{2\sigma^2} - \log x, \tag{4.12}$$

where μ is a location parameter and σ is a scale parameter. Substituting equation (4.12) into equation (4.10), we obtain the following proposition.

Proposition

Given the assumptions above, the function f in equation (4.4) consistent with the empirically observed stationary lognormal distribution of X_t is given by

$$f(x) = -\frac{D^2}{2}(\frac{1}{x} - \frac{\mu}{\sigma^2 x} + \frac{\log(x)}{\sigma^2 x}). \tag{4.13}$$

We have identified the interaction effect f consistent with the empirically observed lognormal distribution of individual incomes. In other words, the observed regularity that the income distribution is approximated by the lognormal distribution implies that the interaction effects are given by this formula (4.13) (the self-consistency condition). It should be noted that equation (4.13) is not based on any assumptions on the preferences or abilities of individuals but is derived from the stylized fact that the income distribution is close to the lognormal distribution and the additive assumption. In this sense, condition (4.13) must hold in any country and is expected to reflect a universal structure of the economic system.

4.4.2 Implications

Figure 4.1 shows the graph of the function f (4.13). A positive (negative) value of $f(x)$ means that the expected value of dX_t is positive (negative). For a stationary distribution, the interaction effects f must compensate for the dissipative force caused by DdW_t, and it must hold that $f(x) > (<)0$ at lower (larger) x. A large positive value implies that a worker's income in the next period is more likely to be higher than the current level. Thus, it is reasonable to have a larger positive value of $f(x)$ at a lower x. Figure 4.1 shows that this intuition is correct.

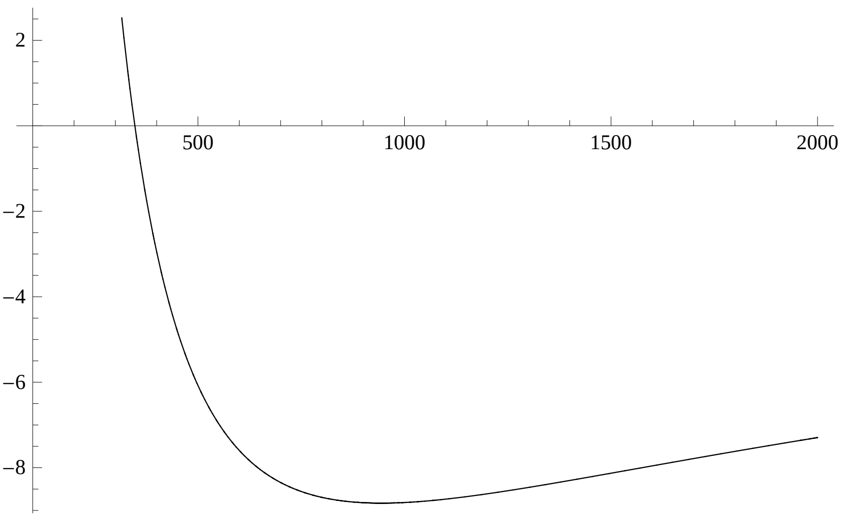

Figure 4.1: The graph of the function f.

However, in the right-side of the graph, for incomes of middle class or relatively high-income earners, a curious property is revealed. Figure 4.1 shows that though f is negative, it has its minimum value and then slightly increases as x becomes large. The negative value of $f(x)$ indicates downward pressure on X_t, so the pressure is weakened as X_t becomes large.

Let us take a closer look at the function f. We calculate the important values of x characterizing f.

Setting $f(x)$ in (4.13) to be zero, the elementary algebra shows that f takes a value of zero at

$$x_{mode} = e^{\mu - \sigma^2}$$

Recall that x_{mode} is the mode of the lognormal distribution with parameters μ and σ. This implies that no pressure on X_t works at x_{mode}.

Next, by differentiating f with respect to x, it can be shown that f takes its minimum value at

$$x_{min} = e^{1+\mu-\sigma^2} = e x_{mode}$$

The existence of x_{min} is significant. Because the downward pressure on X_t is weaker on the right side of x_{min}, the mechanism underlying the income distribution is somewhat "generous" (or favorable) to relatively high-income earners.

The present analysis has other implications for poverty and inequality. Because the income distribution is described by the lognormal distribution, the Gini coefficient is given by

$$2\Phi(\sigma/\sqrt{2}) - 1,$$

where $\Phi()$ is the cumulative distribution function of the standard normal distribution. The degree of inequality is determined by the scale parameter σ only. This simplicity is one of the advantages of the lognormal distributions in describing income distributions (see Pinkovskiy and Sala-i Martin (2009)).

For simplicity, assume that the location parameter μ is fixed. A larger value of σ implies a greater inequality. We can examine the relationship between σ and the interaction effect f.

Differentiating $f(x)$ with respect to σ^2, we have

$$df/d\sigma^2 = -\frac{D^2}{2x\sigma^2}(\mu - \log x), \tag{4.14}$$

Hence,

$$\begin{aligned} df/d\sigma^2 &< 0 \quad \textit{if} \quad x < e^{\mu}, \\ df/d\sigma^2 &> 0 \quad \textit{if} \quad x > e^{\mu}. \end{aligned} \tag{4.15}$$

Note that e^{μ} is the median of the lognormal distribution, separating the upper half of a population from the lower half. What equation (4.15) means is straightforward. Workers whose incomes are in the lower half suffer when σ, that is, the Gini coefficient increases because the interaction effect $f(x) > 0$ decreases, that is, there is a lower expected value of dX_t. This simple relationship is consistent with our interpretation of the function f, which contains all the information about how workers' incomes are affected by macroeconomic conditions.

4.5 The Multiplicative Process

In this section, we investigate the case of multiplicative idiosyncratic shocks, that is, $\sigma(x) = Dx$. It is common procedure to consider $Y_t^i = \log X_t^i$, instead of X_t^i. An advantage of this procedure is that the nonnegativeness of X_t is not important and can consider the stochastic process on the real line, $\mathbb{R}$. Then, the stochastic differential equation that we investigate becomes

$$dY_t^i = f'(Y_t^i)dt + DdW_t^i. \tag{4.16}$$

The empirical findings that the distribution of X_t is well described by the lognormal distribution mean that Y_t^i is normally distributed. Hence, as in the previous section, we seek the conditions under which the interaction term f' leads to a stationary Gaussian distribution of Y_t^i.

We now introduce the Ornstein–Uhlenbeck process, which is given by the following stochastic differential equation:

$$dY_t^i = -\lambda(Y_t^i - \mu)dt + DdW_t^i, \tag{4.17}$$

where λ is a positive constant. This process tends to move back toward its mean, μ, (mean-reverting), with a greater attraction when the process is further from μ. It is well known that the Ornstein–Uhlenbeck process can be explicitly solved,

$$Y_t^i = Y_0^i e^{-\lambda t} + \mu(1 - e^{-\lambda t}) + \int_0^t De^{\lambda(s-t)} dW_s^i, \tag{4.18}$$

and that it leads to a stationary Gaussian distribution with mean μ and variance $\frac{D^2}{2\lambda}$.

In usual cases, the mean μ is given exogenously, but in our case, the mean-reverting effect is considered a consequence of the interaction among workers. That is, the mean (and the distribution of Y_t) of the system is determined endogenously via interaction. Indeed, comparing it with equations (4.1) and (4.3), equation (4.17) can be rewritten as

$$\begin{aligned} dY_t^i &= \Big(\int -\lambda(Y_t^{(i,N)} - y)U^N(dy)\Big)dt + DdW_t^i, \quad U_t^{(N)} = \frac{1}{N}\sum_{j=1}^N \delta_{Y_t^{(j,N)}} \qquad (4.19)\\ &= \frac{1}{N}\sum_{j=1}^N b(Y_t^{(i,N)}, Y_t^{(j,N)})dt + DdW_t^i, \quad b(x,y) = -\lambda(x-y), \end{aligned}$$

where $U_t^{(N)}$ denotes the empirical measure of the system and $N \to \infty$ is considered. Hence, our analysis shows that the interaction between Y_t^i and Y_t^j is given by $-\lambda(Y_t^i - Y_t^j)$. In terms of the level of X_t, the interaction effect can be written as $(\frac{X_t^j}{X_t^i})^\lambda$. The parameter λ characterizing the strength of the interaction is uniquely determined given the variance of the Gaussian distribution.

The interpretation of the interaction effect in the case of the multiplicative process is clear. The larger the difference between two workers, the greater the attraction effect. In contrast to the case of the additive process discussed in the previous section, there is no lower bound. In terms of Y_t, the attractive effect increases linearly. The strength of the interaction is determined by λ, which is inversely related to the variance of the stationary distribution. That is, given that D is fixed, a larger value of λ, that is, a greater attraction, leads to a more equal income distribution.

While the idiosyncratic shocks whose strength is given by D disperses workers' incomes, the interaction represented by λ compensates for that effect. It should be noted that the mean μ is determined endogenously, and therefore, there is no unique stationary distribution to which an arbitrary initial distribution converges in general as $t \to \infty$. Which stationary distribution is realized depends on an initial condition, and, for example, some aggregate shock might result in another stationary distribution different from the original distribution. In this sense, our system is history-dependent (or path-dependent; see Krugman (1991)).

4.6 Concluding Remarks

The income distribution is known to be well described by the lognormal distribution (except for the high income range). It is natural to expect there to be a universal structure behind it and to ask what kind of mechanism generates this observed regularity. In fact, study of income distribution has a long history, and many theoretical investigations have wrestled with this problem. However, these previous studies implicitly assume that the behaviors of individual workers' incomes are independent of each other. Although this assumption simplifies the structure of their models, it is unrealistic to assume that the evolution of an individual income is not affected by others' incomes. It is plausible to expect the labor income of an individual to increase during an economic boom.

In this Chapter, we explicitly took into account economic interactions among individuals. On the one hand, the macroeconomic condition is determined, by definition, by the aggregation of individuals, but on the other hand, an individual income is affected by others' incomes, that is, by the macroeconomic condition. This feedback loop complicates our analysis and makes the system nonlinear with respect to the empirical measure, $U_t^{(N)}$. However, when N goes to infinity, by the propagation of chaos, the interaction effect can be reduced to the function f and an usual one-dimensional stochastic differential equation is obtained.

We investigated the interaction effects, f, under two different assumptions about idiosyncratic shocks. Under the additive assumption, we showed that the function f takes its minimum value. This means that the downward pressure on X_t is weakened given higher values of X_t. That is, the mechanism underlying the income distribution is somewhat “generous” (or favorable) to relatively high-income earners.

Under the multiplicative assumption, we obtained the expression of the pairwise interaction, $b(x, y) = -\lambda(x - y)$. Although the system is described by the Ornstein–Uhlenbeck process, the mean μ, which is usually given exogenously, is determined endogenously in our model as a consequence of the interaction among workers in our system. Therefore, there is no unique stationary distribution to which an arbitrary initial distribution converges. In this sense, our system is history-dependent.

Recall that these implications were derived from the empirical fact that the income distribution is close to the lognormal distributions. They do not depend on any assumptions of the preferences or abilities of individuals. Thus, they can be interpreted as reflecting a universal structure of our economic systems generating the observed regularity. For this reason, our findings are robust and have important meaning for the theory of income distribution as well as the analysis of poverty and inequality.

Appendix A

Aggregate Volatility

In this appendix, we investigate whether aggregate fluctuations can be explained by *independent* microeconomic (especially firm-level) shocks.

As we discussed in Chapter 3, it is difficult to identify the origin of economic fluctuations. According to Cochrane (1994), none of popular candidates of aggregate shocks (e.g. money shocks or oil price) accounts for the bulk of economic fluctuations. If it is the case, what are the shocks that cause economic fluctuations?

Because an economy is composed of many firms, one would expect that the aggregate fluctuation stems from firm-specific shocks and inherits some properties from them. However, this idea has been rejected in macroeconomic literature simply because these shocks would average out in the aggregate by the law of large numbers (e.g. Lucas (1977) or Dupor (1999)). Hence, to explain aggregate fluctuations, existing macroeconomic models (e.g. Real Business Cycle Theory) have used aggregate productivity or technology shocks as the dominant source to cause fluctuations. Since these shocks affect all firms, they do not average out. Idiosyncratic shocks to firms play no role there.

In contrast to this argument, Gabaix (2011) attempts to provide an alternative explanation for the aggregate shocks. He argues that microeconomic shocks do not die out in the aggregate and a significant fraction of aggregate fluctuations can be attributed to idiosyncratic firm-level shocks. This is due to the existence of very large firms. It is well known that the firm size distribution is not thin-tailed, but exhibits power law behavior with exponent $\simeq 1$(e.g. Axtell (2001)). Shocks to a handful of large firms have an impact on aggregate volatility. He calls this view the *granular hypothesis*. According to this hypothesis, macroeconomic events can be examined by looking at the behavior of a handful of large firms. Giovanni and Levchenko (2012) uses a similar idea to explain the fluctuations of exports and investigate the effect of openness to international trade on the aggregate volatility (see also Carvalho (2010)).

In this appendix, we test the granular hypothesis by comparing the dynamics of firm growth developed in Chapter 2 and aggregate fluctuations. Using data of publicly traded Japanese firms, we calculate what fraction of aggregate fluctuations can be explained by firm specific shocks. We show that, on the contrary to Gabaix (2011), the aggregate fluctuations cannot be traced back to microeconomic ones, especially large jumps at the aggregate level. It suggests that there is a common factor (or an incident) affecting some fraction of firms in the economy.

The rest of this appendix is organized as follows. Section 2 presents the empirical facts that the recent studies have revealed. Section 3 develops a framework called Variance-Gamma process. Section 4 shows that *granular hypothesis* cannot explain aggregate fluctuations . Section 5 concludes.

A.1 Empirical evidence

There exists a growing literature studying the statistical properties of the distribution of growth rates of firms, which reveals a new aspect of the firm dynamics (see Chapter 2). Stanley et al. (1996) analyzed the publicly traded U.S manufacturing firm and found that the growth rates display the tent-shape density. They propose the Laplace distribution to describe the distributions:

$$p(x;\gamma,a) = \frac{1}{2a}\exp(-\frac{|x-\gamma|}{a}) \tag{A.1}$$

Subsequent papers have confirmed this regularity. Amaral et al. (1997) analyzes all publicly traded U.S. manufacturing firms exponential “tent-shaped” form rather than the bell-shaped Gaussian. Bottazzi and Secchi (2003) and Bottazzi and Secchi (2006) analyze the Italian manufacturing industry and the Italian firms, respectively, and confirm the Laplace shape. Alfarano and Milakovic (2008) analyze the Forbes Global 2000 list of the world's largest companies and report the similar pattern.

Interestingly, similar statistical regularity can be found at the macroscopic level, i.e., growth rates of aggregate variables. Lee et al. (1998) analyze the fluctuations in GDP of 152 countries for the period 1950-1992 and find that the growth rates of GDP also follow a Laplace distribution. Castaldi and Dosi (2009) find that growth rates of per capita GDP and total GDP for 111 countries for 1960-1996 are "tent-shaped".

To confirm this point, using *the number of employed persons* as the aggregate quantity, S_t, we analyze the growth rates for OECD countries from 1984 to 2012 [1].

As in Chapter 2, *a growth rate*, $s_{t,i}$, is defined as follows,

$$s_{t,i} = \log S_{t,i} - \log S_{t-1,i} \tag{A.2}$$

where $S_{t,i}$ represents the size of the ith country at the period t. We then define *normalized growth rates* as follows,

$$g_{t,i} = \frac{s_{t,i} - \bar{s}_i}{\sigma(s_i)} \tag{A.3}$$

Here, $\bar{s}_i$ and $\sigma(s_i)$ are the average growth rate and the standard deviation of the ith country, respectively.

The simple histogram of the normalized growth rates is shown in Figure A.1,

As can be seen, the Laplacian curve, which is a straight line on a logarithmic scale, describes the observations quiet well. It suggests that a large growth (or shrink) occurs more frequently than expected by Gaussian distributions. We then estimate the parameter a in (A.1) for Japan and obtain the maximum likelihood estimate $a = 0.0075(0.0014)$ for growth rates (not normalized).

[1] The data is drawn from OECD.Stat Extract (http://stats.oecd.org/index.aspx).

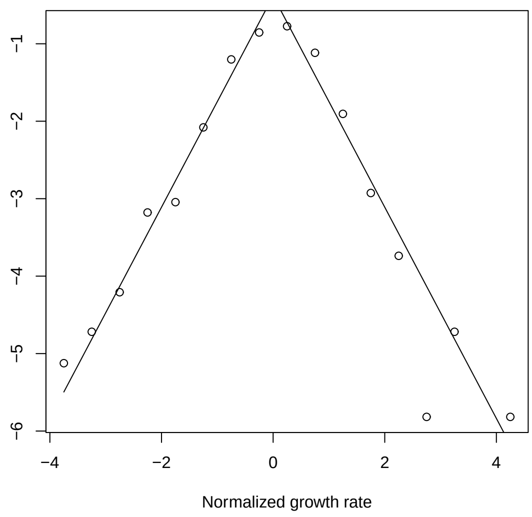

Figure A.1: Normalized growth rate for OECD countries over the period 1984-2012 $a = 0.73(0.028)$. The vertical axis is changed to a logarithmic scale.

A.2 Jumps

In Chapter 2, we show that the firm growth dynamics can be described by the Variance-Gamma process, which is a natural generalization of well-known Gibrat's model and generates the observed Laplace distribution. This process can be seen as a difference between two independent and identical gamma processes, which represent a positive growth process and a negative one. The gamma process is obtained as a limit of a sequence of compound Poisson processes with the distribution P_n defined as

$$\hat{P}_n = \exp\left[t\int_{\epsilon_n}^{\infty}(e^{izx}-1)\frac{e^{-\frac{x}{a}}}{x}dx\right], \quad \epsilon_n > 0 \quad \text{and} \quad \epsilon_n \to 0 \quad \text{as} \quad n \to \infty. \tag{A.4}$$

where $\hat{P}_n$ is the characteristic function of P_n, which converges to the gamma distribution. Hence, we can approximate it as a compound Poisson process.

When the firm growth process follows a compound Poisson process, jumps arrive randomly according to a Poisson process and each jump size is also random variable. In our case, it can be written as

$$\text{The positive growth shocks in a year} = \sum_{i=1}^{N(t)} J_i \tag{A.5}$$

where $N(t)$ follows a Poisson distribution with rate $\lambda = \int_{\epsilon_n}^{\infty} \frac{e^{-\frac{x}{a}}}{x} dx$ and J_i is an independent random variable with distribution $\lambda^{-1} \frac{e^{-\frac{x}{a}}}{x} dx$ on the support $[\epsilon, \infty)$.

The implication is clear. Firm growth is not a consequence of many small shocks, but sometimes a sizable shock hits a firm. Firm growth is characterized by large jumps and its size evolves discontinuously. If we collect a handful of large jumps of a particular firm and ignore other negligible shocks, they can explain almost all the growth path of the firm. In other words, we can identify which event (or shock) leads to the ups and downs of a firm. Hence, there exist particular shocks leading to rapid expansion of larger firms.

The argument developed above can also apply to aggregate variables if the aggregate shocks are just microeconomic shocks to large firms (the granular hypothesis). Note that the key assumption in Chapter 2 is the independence of shocks, and is satisfied in the aggregate case under the granular hypothesis. As observed in the previous section, the aggregate variable (the number of employed persons) follows the Laplace distribution. Thus, the aggregate variable can be described by the same process as the firm growth dynamics, the Variance-Gamma process. That is, the growth of the number of employed persons is not a consequence of infinitesimally small shocks. It is sudden rather than gradual and almost entirely determined by a handful of large shocks, or events. Therefore, it is legitimate to characterize the history of economic growth by several remarkable events or episodes as historians do (e.g. the Volcker shock or the Lehman shock).

How large and how often these large shocks arrive can be calculated by Lévy measure, $\nu(dx) = x^{-1}e^{-\frac{x}{a}}dx$, with $a = 0.0075$. For instance, a positive growth larger than 0.5% (strictly speaking, $log(S_{t+1}) - log(S_t) > 0.005$) has intensity $\nu(0.05, \infty) \simeq 0.398$, so it occurs once in 2.5 years. Other values are reported in the second row in Table A.1.

A.3 Comparison between two shocks

At the first glance, the conclusion that the growth of the aggregate variable is not an accumulation of infinitesimally small shocks appears to support the granular hypothesis because shocks to large firms disproportinally contribute to the aggregate fluctuations. In this section, we distinguish genuine aggregate shocks from microeconomic shocks to large firms. We examine to what extent the shocks to a handful of large firms can explain the aggregate fluctuations.

Recall the stochastic continuity in the definition of Lévy processes (see Chapter 2). It means that jumps of different firms do not occur at the same time. More precisely, if we collect firms $\omega_1, \omega_2, ... \in \Omega_1$ which jump at time t by at least $\epsilon > 0$, the probability $P(\Omega_1)$ must be zero for any ϵ. A jump of the process of the number of employed persons has to correspond to a jump of a single firm. For example, if there is a decrease by 10000 in the aggregate employment, there must be a firm whose size is, for instance, 100,000 and 10% of the total workers are laid off. Therefore, the frequency of jumps derived from the intensity ν^{agg} is just the sum of that of firms derived from ν^{fir}. We can gauge what fraction of the frequency of jumps of the number of employed persons comes from the microeconomic shocks to top 100 largest firms.

Table A.1: Comparison between macro and micro shocks.

jump size	0.05% - 0.075%	0.075% - 0.1%	0.1% - 0.25%	0.25% - 0.5%	0.5% -
ν^{agg}	0.373	0.256	0.738	0.430	0.398
$\sum \nu^{fir}$	0.247	0.0625	0.0301	0.000106	0.000
%	66.2%	24.4%	4.08%	0.0247%	0.00%

Table A.1 presents the results. Though the granular hypothesis can explain the small jumps of the aggregate variable, but it has no explanatory power for larger jumps of the aggregate variable. For example, the intensity ν^{agg} shows that jumps larger than 0.5% occurs once in 2.5 years, shocks to large firms cannot produce them. Thus, the large jumps cannot be attributed to the granular effect [2] [3]. In this way, we can distinguish the genuine aggregate shocks from microeconomic shocks to large firms.

If we use 0.1% as a criteria separating small and large jumps, we can compare the relative importance of these two jumps. We use $\int x\ \nu(dx)$ as a measure because $\nu(dx)$ represents how often shocks arrive. We obtain

$$\int_{0.001}^{\infty} x\ \nu(dx) = 0.00656, \quad \int_{0}^{0.001} x\ \nu(dx) = 0.000936 \tag{A.6}$$

It is clear that an economic growth is determined almost entirely by a small number of the large aggregate shocks which cannot be explained by microeconomic shocks. Therefore, the granular hypothesis is rejected.

As pointed out above, a single firm is unable to generate the aggregate volatility seen in the data, so some fraction of the total firms have to comove to generate a large jump. Notice that the firms jump at the "exact" same time. They are synchronized and generate a large jump observed at the macroscopic level. It strongly suggests that there exists an identifiable origin of the jump (Recall that the number of large jumps within a given interval is finite). Anecdotally, 2009 experiences the largest decrease in the number of employed persons in Japan, which reflects an abrupt drop in exports caused by the recession in the U.S. after the bankruptcy of Lehman Brothers. Facing such a common factor, many firms, which basically behave independently, are suddenly synchronized and cause a large jump of the number of employed persons.

A.4 Concluding Remarks

The granular hypothesis proposed by Gabaix (2011) to provide microfoundation for mysterious aggregate shocks has a valuable meaning because it sheds new light on the role of idiosyncratic

[2] The frequency of very small jumps around 0 under the granular hypothesis can exceed the frequency derived from ν^{agg}, i.e., the percentage ≥ 1. By definition, it is a contradiction. It might be possible to ignore the very small jumps as (A.4) states. It indicates that we need further investigation into the distribution of growth of aggregate employment.

[3] In our analysis, we have ignored an effect of bankruptcy of firms so far. If bankruptcy or liquidation is assumed to occur as a Poisson process with parameter λ, we can immediately add this effect to our model. With a plausible value for λ, 0.003, which is calculated by the number of Delisting on Tokyo Stock Exchange due to bankruptcy, it can be confirmed that the conclusion is not affected.

shocks at the macroscopic level. This appendix investigated the relationship between microeconomic shocks and aggregate fluctuations and test the granular hypothesis. Focusing on the recent findings showing the similar regularity at both macro and micro level, we apply the procedure developed in Chapter 2. Under the granular hypothesis, we calculated what fraction of aggregate fluctuation can be attributed to the top 100 largest firms' jumps.

We found that the granular hypothesis cannot explain aggregate fluctuations. Even if the firm size follows a power law distribution, the firm size in Japan is too small to have an impact on the aggregate fluctuations. Macroeconomic shocks are not a reflection of microeconomic shocks to the large firms.

It strongly suggests that there exists a common factor (or an incident) affecting some fraction of firms in the economy. Facing such a common factor, many firms, which basically behave independently, are suddenly synchronized and cause sizable aggregate shocks. Therefore, the comovement across firms is an indispensable point to explain the aggregate fluctuations.

Appendix B

Nonlinear Stochastic Differential Equations

In this appendix, we show that the conclusion derived in Chapter 2 still holds under more generalized assumptions and is consistent with the discussions in Chapter 3.

Since the seminal work by Gibrat (1931), when statistical features of distributions are investigated, the statistical independence of economic agents has been implicitly assumed (see Sutton (2001) and references cited in Chapter 2). In chapter 2, we also assume that the stochastic processes of the firms' size are independent of each other. This assumption appears to be crucial because the failure of the independence assumption makes the problem unrealistically complicated and analytically intractable. On the other hand, one might argue that the interdependence of economic agents is an indispensable ingredient for economic analysis and the analysis without that is meaningless. Indeed, we demonstrate in chapter 3 that the interaction among firms is the origin of the business cycles. At first glance, these two discussions are appeared to be contradictory to each other. However, we demonstrate that this contradiction can be resolved in a natural manner in the framework of the nonlinear stochastic differential equations introduced in chapter 3 as long as the symmetry of the model is preserved. In other words, the assumptions in chapter 2 is not as restrictive as might be seen.

In chapter 2, we show that the stochastic process of firm size is described by the Variance-

Gamma process. Adding a interaction term in our model in chapter 2, we consider the following equation,

$$X_t^{i,N} = X_0^{i,N} + \int_0^t b(X_t^{i,N}, \mu_s^N)ds \; + \; {}^{V.G.}Z_t^{i,N}, \quad t \in [0,T] \tag{B.1}$$

$$\mu_s^N = \frac{1}{N}\sum_{j=1}^N \delta_{X^{j,N}} \text{ denotes the empirical measure.}$$

where ${}^{V.G.}Z_t^{i,N}$ represents the V.G. process. As in chapter 2, the interaction term can be written as $\int_0^t \frac{1}{N}\sum_{j=1}^N \tilde{b}(X_t^i, X_t^j)$, where we assume that $\tilde{b}$ is Lipschitz continuous. Note that the system is exchangeable, i.e., the system is not changed by any permutation of $i \in \{1, ..., N\}$. Because the integral is not the Brownian case, the argument in chapter 3 is not applicable to this case.

Jourdain et al. (2008) study more general nonlinear stochastic differential equations including the Brownian case as a special one. Let X_0 denote a random variable and Z_t a Lévy process [1]. For the Brownian case, $Z_t = (t, B_t)$. They show well-posedness of the nonlinear stochastic differential equations, i.e., if X_0 and Z_t are square integrable and the mapping σ is Lipschitz continuous, the following equation

$$X_t^i = X_0^i + \int_0^t \sigma(X_{s^-}^i, P_s)dZ_s^i, \quad t \in [0,T] \tag{B.2}$$

$$P_s \text{ denotes the probability distribution of } X_s^i.$$

has a unique solution such that $E(\sup_{t\leq T}|X_t|^2) < +\infty$. Moreover, they show that the propagation of chaos result can be extended to this case. Suppose the following system of n interacting particles corresponding to the above equation,

$$X_t^{i,N} = X_0^{i,N} + \int_0^t \sigma(X_{s^-}^{i,N}, \mu_{s^-}^N)dZ_s^i, \quad t \in [0,T] \tag{B.3}$$

$$\mu_s^N = \frac{1}{N}\sum_{j=1}^N \delta_{X^{j,N}} \text{ denotes the empirical measure.}$$

They show that the $X_t^{i,n}$ converges to X_t^i in law as N tends to infinity, more precisely,

Theorem B.1 *If $\sigma(x,\nu) = \int_{\mathbb{R}} a(x,y)\nu(dy)$, where $a : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{1\times 1}$ is a Lipschitz continuous*

[1] For the sake of simplicity, we restrict ourselves to the one-dimensional case in this appendix.

function, then

$$\sup_{i \le n} E\big(\sup_{t \le T} |X_t^{i,N} - X_t^i|^2\big) \le \frac{C}{N}$$

where the constant C does not depend on N and E denotes the expectation.(Jourdain et al. (2008))

As in Chapter 3, the convergence occurs at the rate N^{-1}.

Let us return to our case. The assumption about Z_t is satisfied because Laplace distribution has the finite second moment. The assumption about σ is clear. Therefore, the discussion above can be applied to our case. As N tends to infinity, the empirical measure, which is a random variable, converges to P_s in equation B.2. Note that P_s evolves deterministically. Thus, the interaction effect $\int_0^t b(X_t^{i,N}, \mu_s^N)ds \to \int_0^t b(X_t^i, P_s)ds$ evolves deterministically and is common to all firms. It means that the interaction effect is experienced by firms via the drift term and not distinguishable from other aggregate shocks as $N \to \infty$. Hence, subtracting these aggregate shocks including the interaction effect, which is performed by subtracting a sample average in practical situations, the independence can be restored because $X_t^i \in \{1, ..., N\}$ in equation (B.2) are independent.

Therefore, even if each element in the system interacts each other, the conclusion based on the independence assumption still holds as long as the interaction is formulated as in equation (B.1).

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