博士論文

Option Pricing under Various Market Restrictions
(種々の市場制約下でのオプション価格付けについて)

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Chapter 1

Introduction

In this thesis, we discuss option pricing under various market restrictions such as liquidity cost of the hedging instrument, market impact on the price of the instrument, and intervention by an authority in the foreign exchange (FX) market. In particular, we consider the following problems: (i) option pricing under a liquidity model with market impacts where the liquidity costs of the hedging instrument and market impacts on the price of the hedging instrument are considered, (ii) continuous time expression of the wealth process of a self-financing strategy under a general shape of a limit order book of the hedging instrument, (iii) pricing FX options under the existence of intervention in the FX market.

In practice, when a trader in a bank deals an option, the trader considers various points which are not assumed in the classical pricing theories. For example, the trader tries to estimate the hedging cost which results from the bid/offer spread in the hedging instrument during the life of the option. The estimation of the cost is difficult in general as it depends on the path of the underlying price process and the market environment. If the price of the hedging instrument fluctuates, the trader has to re-balance the option position often, which results in high hedging costs. If the market environment changes, for instance, the liquidity of the hedging instrument becomes thin, the difference between the executed average price and the mid price becomes large. Therefore it is vital for the trader to take into account the hedging costs when the trader quotes the option price.

In addition, market impact on the price of the hedging instrument by the option hedging activity is another factor which the trader takes into account in pricing. For instance, if the hedging instrument for an option is illiquid, the hedging activity impacts on the limit order book of the instrument and moves the mid price as a result, which also changes the value of the option. Particularly, in the financial crisis, some banks experienced losses in their option positions, which resulted from the market impacts by the option hedging; they had to hedge the same positions from the customer deals in the rapid market movement, which accelerated the further price move, and incurred losses from the revaluation of the option positions. As a result, banks who had aggressively priced and traded options with smaller estimation of hedging costs ran out of the reserve.

Moreover, after the financial crisis, the FX market observed intervention in USDJPY and EURCHF market. The Japanese government intervened in USDJPY market to prevent further JPY strengthening against USD to protect the exporting businesses in the country. Swiss National bank also intervened in EURCHF market to stop strengthening of CHF against EUR caused by the European debt crisis. Considering the fact that options, which have a payoff in the area where the intervention may take place, have been widely traded in the form of structured bonds and forwards, it is important to incorporate the effect of intervention in the model as the values of the options for banks are expected to be affected.

Motivated by the backgrounds, we consider the option pricing problems under the market restrictions. The thesis consists of five chapters and each one discusses the following topics.

In Chapter 2, we investigate an optimal hedging problem of an option in the existence of liquidity cost and market impact in the hedging instrument. In detail, we consider local risk minimization in an incomplete market in a discrete time setting, where a price process of the hedging instruments follows a linear supply curve model with market impacts introduced by Roch [13] and Cetin et al. [3]. In particular, we obtain the following results: (i) in the case of a constant market impact parameter, the unique local risk minimizing strategy exists and is obtained by a recursive manner, (ii) more generally, when the market impact parameter is time dependent, the unique strategy, in the case of cash settlement, and the first order approximation with respect to the market impact parameter, in the case of physical settlement, are obtained by a backward induction. Moreover, we present numerical examples of the local risk minimizing strategies in a finite states model and confirm the effectiveness of the first order expansion of the strategy in the case of physical settlement.

In Chapter 3, we show a continuous time expression of a wealth process of a self-financing strategy in a non-linear supply curve model with market impacts. The result is an extension of the case in Roch [13] where an expression of the process in a linear supply curve with market impacts is given. More in detail, after we show a relationship between maximum/average prices of buying in the limit order book and the supply curve for a general shape of the order density in the order book, we define the size of the market impact by using them and a wealth process of the self-financing in discrete time, and give the limit in probability as the width of the time step tends to zero.

In Chapter 4, we consider FX option pricing in the case where there exists one-sided intervention by an authority. We formulate the effect of intervention by assuming a forward FX rate process stopped by a hitting time of an absorption boundary, which satisfies the arbitrage-free condition. The main results of this chapter are (i) derivation of closed-form pricing formulas for a European put option and a digital option, and Greeks of the European put option under the model, (ii) an extension of the pricing formula to the case where the intervention level by the authority is a random variable which is independent of the Brownian Motion driving the forward FX rate process. We also show numerical experiments in the case of EURCHF options, where the Swiss National Bank has been intervening since September 2011. In the examples, we investigate features of the absorption model and differences between the model prices and the market prices.

Finally, in Chapter 5, we give concluding remarks.

Chapter 2

Local Risk Minimization with Supply Curve Model with Market Impacts

2.1 Introduction

In this chapter, we consider local risk minimization in an incomplete market in discrete time model where liquidity costs and market impacts by the hedger on a hedging instrument are taken into account. Particularly, we adopt a linear supply curve model with market impacts introduced by Cetin et al. [3] and Roch [13]. We show that when the market impact parameter in the model is a constant, the local risk minimizing strategy is obtained uniquely by backward induction.

Moreover, we present even when the market impact parameter is time dependent, the local risk minimizing strategy is obtained recursively in the case of cash settlement for the option payoff. Furthermore, in the case of physical settlement, the first order approximation of the local risk minimizing strategy is also obtained by backward induction by expanding the strategy with respect to the market impact parameter around the strategy in the case of no market impact.

In practice, liquidity cost and market impact are important notions in option pricing. Especially since the financial crisis, due to widening price spreads in hedging instruments and market impacts on the price of the hedging instruments by derivative traders, estimation of the liquidity costs needed for the hedging of contingent claims is important in risk management of banks. In this respect, we introduce a liquidity model with market impacts in order to estimate the liquidity cost and find an optimal hedging strategy.

The linear supply curve model with market impacts introduced by Roch [13] combines a limit order book model, which describes a density of the orders in the order book, and a supply curve model which expresses an average price of a hedging instrument dependent on a volume of a trade (For instance, see [14] for the limit order book model and [3] for the supply curve model). The model incorporates market impacts by trades of a hedger in the linear supply curve model, based on a correspondence between a supply curve and order density of a static limit order book. In the model, it is assumed that the hedger of an option is the only investor whose trades influence on the price of the hedging instrument, and the size of an impact is proportional to the spread between the maximum (or minimum) price the hedger buys (or sells) in the limit order book and the mid price.

Local risk minimization introduced by Schweizer [15] is a criterion that defines an optimal strategy in option pricing in an incomplete market, which is used in various fields in finance such as insurance problem and option pricing with transaction costs (For instance, see [2] for the insurance application and [7] for the option pricing with transaction costs). The idea of this methodology is to find a locally optimal strategy which minimizes an expected quadratic error in the hedging at each time. In the cases where there is no market impact on the price of the hedging instrument, the local risk minimizing exists uniquely and is obtained by backward induction and equivalence of minimization of the remaining risk and the local risk holds (See [7] for instance). However, it is not known if these properties hold in the case where there are market impacts on the price of the hedging instrument.

For this purpose, following Lamberton et al. [7] for the definition of the local risk minimization, we consider the local risk minimization in an incomplete market in discrete time in the case where the price process of the hedging instrument depends on the history of the trades of the hedger. In this paper, we shall observe that, although the equivalence of the minimization of the remaining risk and the local risk does not hold, the local risk minimizing strategy is obtained recursively in the case where the option payoff is cash settled. Moreover, in the case of physical settlement, although the optimal portfolio at each time depends on the positions in the past due to the the dependence on the history of trades, the first order approximation of the strategy is obtained by backward induction.

We remark that we consider the problem in discrete time setting in order to avoid the vanishing liquidity cost which happens in continuous time in supply curve model. In the continuous time setting, it is known that there exists a replicating strategy which nullifies the effect of the liquidity cost of an original strategy by taking limit in dividing the trading amount into small pieces in an instant time (See [3], [13]).

We also remark that our work is different from the one by Abergel et al. [1], which also deals with the local risk minimization in a model with market impacts, in the following points. Their work mainly deals with the problem in continuous time setting using nonequivalent definition of local risk minimization from Lamberton et al. [7] and ours. In their definition, the mean-square error of the local risk instead of the remaining risk is minimized (See Section 2 for the nonequivalence of the definitions), and on account of this, the price impacts cumulated in the price of the hedging instrument are not taken into account in the choice of the optimal strategy. Moreover, our work considers time dependent market impact parameter, which is a complicated case where the cumulative market impact depends on history of the trades, while their work uses constant parameter where the cumulative market impact only depends on the latest position of the hedging instrument by cancellation.

The paper is organized as follows: after the next section introduces local risk minimization for a linear supply curve model with price impacts, Section 3 shows properties of a local risk minimizing strategy, which hold in the case of a constant market impact parameter, and a recursive procedure to obtain the strategy. Section 4 presents our main results which describe backward inductions to obtain the exact strategy in the case of cash settlement and the first order approximation of the strategy in the case of physical settlement when the market impact parameter is time dependent. Section 5 provides numerical examples of the strategies in the case of time dependent market impact parameters. Section 6 concludes. Finally, Appendix provides conditions of the uniqueness of the strategy with the time dependent market impact parameter with physical settlement in a two period case.

2.2 Setup

In this section, we introduce local risk minimization in an incomplete market in a discrete time setting where a hedging instrument follows a linear supply curve model with market impacts. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $\{\mathcal{F}_k\}_{k=0,1,\dots,T}$ be a filtration. Let $L^p(\Omega, \mathcal{F}, \mathbf{P})$ be the totality of p-th integrable random variables for $p \geq 1$. Hereafter we simply denote by $L^p(\Omega)$. We consider an economy which consists of a money market account and a risky asset in discrete times $k = 0, 1, \dots, T$. $\mathcal{F}_k$-measurable real valued random variables $\eta_k \in L^2(\Omega)$ and $\theta_{k+1} \in L^4(\Omega)$, $(k = 0, 1, \dots, T)$ represent positions of the money market account and the risky asset at time k respectively. Here the positive sign of the positions indicates long and the negative sign short. We call the pair $\phi := \{(\eta_k, \theta_{k+1})\}_{k=0,1,\dots,T}$ *a trading strategy.* We assume that the price of the money market account is always 1 for simplicity. Hereafter we denote by ΔX_k $X_k - X_{k-1}$ for any process X_k.

2.2.1 Linear Supply Curve with Market Impacts

First, we introduce a linear supply curve model with price impacts proposed by Roch [13]. We suppose the existence of an unobservable price process $\{\tilde{S}_k\}_{k=0,1,\ldots,T} \subset L^2(\Omega)$ which drives an observable supply curve $S(k, \Delta\theta_{k+1})$. We assume that $\tilde{S}_k \theta_j \in L^2(\Omega)$ for all $k = 0, 1, \ldots, T$ and $j = 1, \ldots, T+1$. Let M and $\{\lambda_j\}_{j=0,1,\ldots,T-1}$ be positive constants.

The supply curve $S(k, \Delta\theta_{k+1})$ is defined by

$$\begin{aligned} S(k, \Delta\theta_{k+1}) &:= S(k,0) + M\Delta\theta_{k+1} \\ &= (\tilde{S}_k + \sum_{j=1}^{k} 2\lambda_{j-1} M \Delta\theta_j) + M\Delta\theta_{k+1}, \ k = 1, \ldots, T. \end{aligned} \tag{2.1}$$

and

$$S(0, \Delta\theta_1) = \tilde{S}_0 + M\Delta\theta_1. \tag{2.2}$$

The first line explains that when one buys $\Delta\theta_{k+1}$ units of the risky asset at time k (sells $|\Delta\theta_{k+1}|$ if the sign of $\Delta\theta_{k+1}$ is negative), the average price $S(k, \Delta\theta_{k+1})$ is the sum of the observable mid price process $S(k, 0)$ and the price spread $M\Delta\theta_{k+1}$, which is proportional to the volume of the trade. This linear supply curve corresponds to the case of a uniform order density of $\frac{1}{2M}$ in a static limit order book. In fact, if one buys $\Delta\theta_{k+1}$ units of the risky asset, one needs to take the offer orders in the prices ranging from $S(k, 0)$ to $S(k, 0) + 2M\Delta\theta_{k+1}$. In this case, one results in buying $2M\Delta\theta_{k+1} \times \frac{1}{2M} = \Delta\theta_{k+1}$ units of the asset with total cost

$$\int_{S(k,0)}^{S(k,0)+2M\Delta\theta_{k+1}} \frac{1}{2M} x dx = S(k,0)\Delta\theta_{k+1} + M(\Delta\theta_{k+1})^2, \tag{2.3}$$

which means the average price is $S(k, 0) + M\Delta\theta_{k+1}$.

The second line explains the market impact of the hedger on the price of the underlying asset. Every time the hedger trades, there is a market impact of $2\lambda M\Delta\theta_{k+1}$ on the underlying price, where λ is a parameter representing the sensitivity of the price impacts. If $\lambda = 1$, this means that the new orders center around the maximum price the hedger has bought, if $\lambda = 0$, around the previous mid price. Here we assume that in the linear supply curve model with price impacts, the hedger is the only market participant whose trades have impacts on the market price.

2.2.2 Local Risk Minimization on a Model with Market Impacts

Second, we define the local risk minimization. We suppose that the payoff of the contingent claim at time T, $(\bar{\eta}_T, \bar{\theta}_{T+1})$, does not depend on the history of the positions of the hedging instrument $\{\theta_{k+1}\}_{k=0,\dots,T-1}$. Hereafter we assume that any trading strategy $\phi = \{(\eta_k, \theta_{k+1})\}_{k=0,1,\dots,T}$ satisfies $\eta_T = \bar{\eta}_T$, and $\theta_{T+1} = \bar{\theta}_{T+1}$. Let us define the value process of a trading strategy.

Definition 1. *The value process of a trading strategy $\phi = \{(\eta_k, \theta_{k+1})\}_{k=0,1,\dots,T}$ is a $\mathcal{F}_k$ adapted process satisfying*

$$V_k(\phi) = \eta_k + \theta_{k+1} S(k, 0). \tag{2.4}$$

Next, we introduce the cost process of a trading strategy.

Definition 2. *The cost process of a trading strategy ϕ is a $\mathcal{F}_k$-adapted process satisfying*

$$\begin{aligned} C_k(\phi) &= V_k(\phi) - \sum_{j=1}^{k} \theta_j \Delta \tilde{S}_j - \sum_{j=1}^{k} 2\lambda M \theta_j \Delta\theta_j + \sum_{j=1}^{k} M(\Delta\theta_{j+1})^2 \\ &= V_k(\phi) - (\sum_{j=1}^{k} \theta_j \Delta S(j,0) - \sum_{j=1}^{k} M(\Delta\theta_{j+1})^2). \end{aligned} \tag{2.5}$$

Note that we can interpret $\sum_{j=1}^{k} \theta_j \Delta S(j,0) - \sum_{j=1}^{k} M(\Delta\theta_{j+1})^2$ as the gain from the risky asset from time 0 to time k including the costs paid for the spread from the mid prices.

Then, we define a risk process and a local risk minimizing strategy.

Definition 3. *The risk process of a trading strategy ϕ is a $\mathcal{F}_k$-adapted process $\{R_k(\phi)\}_{k=0,1,\dots,T}$ satisfying*

$$R_k(\phi) := \mathbf{E}[(C_T(\phi) - C_k(\phi))^2 | \mathcal{F}_k], \text{ for } k = 0, 1, \dots, T. \tag{2.6}$$

Definition 4. *A trading strategy ϕ is called a local risk minimizing strategy if for all $k = 0, 1, \dots, T$, for any trading strategy $\tilde{\phi}$ whose portfolios are the same as ϕ except for at time k,*

$$R_k(\phi) \leq R_k(\tilde{\phi}), \ \mathbf{P} - a.s. \tag{2.7}$$

Remark 1. *The motivation of the definitions is the following. We consider a problem of finding a trading strategy which approximates a derivative payoff at T consisting of $\bar{\theta}_{T+1}$ units of the risky asset and $\bar{\eta}_T$ units of the the* *money market account. We aim to find a trading strategy which is the best approximation at each time k among strategies which have the same portfolio profiles at the other times.*

Let us observe the strategy in detail. Suppose that we start from the position (η_k, θ_{k+1}) at time k. At time $k+1$, the hedger rebalances the position of the risky asset to θ_{k+2} by buying $\Delta\theta_{k+2}$ of the risky asset and borrowing $\theta_{k+2}S(k+1, \Delta\theta_{k+2})$ of the money market account, which results in the position of the portfolio at $k+2$ as $(\eta_k - \Delta\theta_{k+2}S(k+1, \Delta\theta_{k+2}), \theta_{k+2})$. Next, at time $k+2$, the hedger rebalances the position of the risky asset to θ_{k+2} by buying $\Delta\theta_{k+3}$ of the risky asset and borrowing $\theta_{k+3}S(k+2, \Delta\theta_{k+3})$ of the money market account, which results in the position of the portfolio at $k+3$ as $(\eta_k - \Delta\theta_{k+2}S(k+1, \Delta\theta_{k+2}) - \Delta\theta_{k+3}S(k+2, \Delta\theta_{k+3}), \theta_{k+3})$. The hedger continues the rebalancing until $k = T$. At the maturity T, the hedger rebalances the position of the risky asset to $\bar{\theta}_{T+1}$ by buying $\Delta\theta_{T+1}$ of the risky asset and borrowing $\theta_{T+1}S(T, \Delta\theta_T)$ of the money market account, which results in the position of the portfolio at T as

$$(\eta_k - \sum_{j=k+1}^{T} \Delta\theta_{j+1}S(j, \Delta\theta_{j+1}), \bar{\theta}_{T+1}). \tag{2.8}$$

Note that the position of the risky asset is the same as the payoff of the contingent claim, and the difference in the money market account between the portfolio and the contingent claim payoff is $\bar{\eta}_T - (\eta_k - \sum_{j=k+1}^{T} \Delta\theta_{j+1}S(j, \Delta\theta_{j+1}))$. This is rewritten as

$$\begin{aligned}
&\eta_T - (\eta_k - \sum_{j=k+1}^{T} \Delta\theta_{j+1}S(j, \Delta\theta_{j+1})) \\
&= \eta_T - \eta_k + \sum_{j=k+1}^{T} \Delta\theta_{j+1}S(j, 0) + \sum_{j=k+1}^{T} M(\Delta\theta_{j+1})^2 \\
&= (\eta_T + \theta_{T+1}S(T, 0)) - (\eta_k + \theta_{k+1}S(k, 0)) - \sum_{j=k+1}^{T} \theta_j \Delta S(j, 0) + \sum_{j=k+1}^{T} M(\Delta\theta_{j+1})^2 \\
&= (\eta_T + \theta_{T+1}S(T, 0)) - (\eta_k + \theta_{k+1}S(k, 0)) - \sum_{j=k+1}^{T} \theta_j \Delta\tilde{S}_j \\
&\quad - \sum_{j=k+1}^{T} 2\lambda_{j-1}M\theta_j\Delta\theta_j + \sum_{j=k+1}^{T} M(\Delta\theta_{j+1})^2 \\
&= C_T(\phi) - C_k(\phi).
\end{aligned} \tag{2.9}$$

This observation indicates that the local risk minimizing strategy minimizes the quadratic hedging error in the money market account at the maturity date among the strategies which have the same profiles of the portfolios except for at time k.

Remark 2. *It is known that in local risk minimization in the case where there is no market impact, the definition of the local risk minimization is equivalent even if we define the risk function as a conditional variance of $\Delta C_{k+1}(\phi)$ instead of $C_T(\phi) - C_k(\phi)$ and this equivalent definition makes it possible to find the strategy by backward induction. (See [7] for example) In our case, however, because there are price impacts whose sizes depend on the history of the trading strategy of the hedger, the problem does not reduce to the minimization of the quadratic hedging error of $\Delta C_{k+1}(\phi)$. As we shall observe in the following sections, if the parameter for the market impacts λ is time dependent, θ^*_{k+1} in the local minimizing strategy includes $\{\theta^*_i\}_{i=1,\ldots,T+1,\ i\neq k+1}$ in conditional expectations of its expression, which makes it difficult to obtain explicit expressions for $\{\theta^*_{k+1}\}_{k=0,\ldots,T}$. Moreover, as we shall show in the case of two periods, there is a case where there exist multiple local risk minimizing strategies if coefficients in a cubic equation satisfy a certain condition. As such, the local risk minimizing strategy in the case of market impacts is not unique in general. However, we shall observe that we can obtain a first order approximation of a local risk minimizing strategy on $\{\lambda_i\}_{i=0,\ldots,T}$ recursively, if we expand $\{\theta^*_{k+1}\}_{k=0,\ldots,T}$ with respect $\{\lambda_i\}_{i=0,\ldots,T}$ around zero.*

2.3 Multi-Period Case where λ is a Constant

This section investigates the case where the market impact parameter λ is a constant. In this case, we shall observe that the local risk minimizing strategy is determined uniquely by backward induction.

2.3.1 Existence and Uniqueness of the Local Risk Minimizing Strategy

First we introduce the following lemma which is fundamental to local risk minimization.

Lemma 1. *If ϕ is a local risk minimizing strategy, then $\{C_k(\phi)\}_{k=0,1,\ldots,T}$ is a martingale.*

(Proof.) It is enough to show that $\mathbf{E}[(C_T(\phi) - C_k(\phi))|\mathcal{F}_k] = 0,$ for $k =$

$0, 1, \ldots, T$. Consider $\tilde{\phi}$, a perturbed strategy of ϕ, such that

$$\tilde{\eta}_k = \eta_k + \mathbf{E}[(C_T(\phi) - C_k(\phi))|\mathcal{F}_k], \tag{2.10}$$

$$\tilde{\eta}_j = \eta_j \ (j \neq k), \tag{2.11}$$

$$\tilde{\theta}_{j+1} = \theta_{j+1} \ (j = 0, 1, \ldots, T). \tag{2.12}$$

Then, by the definitions of $\tilde{\phi}$ and the cost process,

$$\begin{aligned} R_k(\tilde{\phi}) &= \mathbf{E}[(C_T(\tilde{\phi}) - C_k(\tilde{\phi}))^2|\mathcal{F}_k] \\ &= \mathbf{E}[(C_T(\phi) - C_k(\phi) + C_k(\phi) - C_k(\tilde{\phi}))^2|\mathcal{F}_k] \\ &= \mathbf{E}[((C_T(\phi) - C_k(\phi)) - \mathbf{E}[(C_T(\phi) - C_k(\phi))|\mathcal{F}_k]))^2|\mathcal{F}_k] \\ &= \mathbf{E}[(C_T(\phi) - C_k(\phi))^2|\mathcal{F}_k] - \mathbf{E}[(C_T(\phi) - C_k(\phi)|\mathcal{F}_k]^2 \\ &\leq \mathbf{E}[(C_T(\phi) - C_k(\phi))^2|\mathcal{F}_k] = R_k(\phi). \end{aligned} \tag{2.13}$$

As $R_k(\tilde{\phi}) \geq R_k(\phi)$, $\mathbf{E}[(C_T(\phi) - C_k(\phi)|\mathcal{F}_k] = 0, \ \mathbf{P} - a.s.$ □

Next we show a key proposition in determining a local risk minimizing strategy in the case of a constant market impact parameter.

Proposition 1. *ϕ is a local risk minimizing strategy if and only if*
(i) $\{C_k(\phi)\}_{k=0,1,\ldots,T}$ is a martingale and
(ii) for all $k = 0, 1, \ldots, T-2$,

$$\mathrm{Var}[C_{k+2}(\phi) - C_k(\phi)|\mathcal{F}_k] = \min_{\tilde{\phi} \in \Gamma_k} \mathrm{Var}[C_{k+2}(\tilde{\phi}) - C_k(\tilde{\phi})|\mathcal{F}_k], \tag{2.14}$$

$$\mathrm{Var}[C_T(\phi) - C_{T-1}(\phi)|\mathcal{F}_k] = \min_{\tilde{\phi} \in \Gamma_{T-1}} \mathrm{Var}[C_T(\tilde{\phi}) - C_{T-1}(\tilde{\phi})|\mathcal{F}_{T-1}], \tag{2.15}$$

where Γ_k is a set of perturbations of ϕ at time k.

(Proof.) Let us first show the if part. As $\{C_k(\phi)\}_{k=0,1,\ldots,T}$ is a martingale,

$$\mathrm{Var}[C_T(\phi) - C_k(\phi)|\mathcal{F}_k] = \mathrm{Var}[C_T(\phi) - C_{k+2}(\phi)|\mathcal{F}_k] + \mathrm{Var}[C_{k+2}(\phi) - C_k(\phi)|\mathcal{F}_k]. \tag{2.16}$$

Note that by the definition of $C_k(\phi)$, for any $\tilde{\phi} \in \Gamma_k$,

$$\begin{aligned} &\mathrm{Var}[C_T(\phi) - C_{k+2}(\phi)|\mathcal{F}_k] \\ &= \mathrm{Var}[V_T(\phi) - V_{k+2}(\phi) - \sum_{j=k+3}^{T} \theta_j \Delta \tilde{S}_j - \sum_{j=k+3}^{T} 2\lambda M \theta_j \Delta\theta_j + \sum_{j=k+3}^{T} M(\Delta\theta_{j+1})^2|\mathcal{F}_k] \\ &= \mathrm{Var}[(\eta_T + \theta_{T+1}(\tilde{S}_T + 2\lambda M\theta_T)) - (\eta_{k+2} + \theta_{k+3}(\tilde{S}_{k+3}(\tilde{S}_{k+2} + 2\lambda M\theta_{k+2}))) \\ &\quad - \sum_{j=k+3}^{T} \theta_j \Delta \tilde{S}_j - \sum_{j=k+3}^{T} 2\lambda M \theta_j \Delta\theta_j + \sum_{j=k+3}^{T} M(\Delta\theta_{j+1})^2|\mathcal{F}_k] \\ &= \mathrm{Var}[C_T(\tilde{\phi}) - C_{k+2}(\tilde{\phi})|\mathcal{F}_k]. \end{aligned} \tag{2.17}$$

Therefore

$$\mathrm{Var}[C_T(\phi) - C_k(\phi)|\mathcal{F}_k] = \min_{\tilde{\phi}\in\Gamma_k} \mathrm{Var}[C_T(\tilde{\phi}) - C_k(\tilde{\phi})|\mathcal{F}_k], \text{ for } k = 0, 1, \ldots, T-1. \tag{2.18}$$

Next we show the only if part. (i) is shown in Lemma 1.
For (ii), for $k = 0, 1, \ldots, T$,

$$\begin{aligned}&\mathbf{E}[(C_T(\phi) - C_k(\phi))^2|\mathcal{F}_k] \\ &= \mathrm{Var}[C_T(\phi) - C_{k+2}(\phi)|\mathcal{F}_k] + \mathrm{Var}[C_{k+2}(\phi) - C_k(\phi)|\mathcal{F}_k].\end{aligned} \tag{2.19}$$

For any $\tilde{\phi}$, a perturbation of ϕ at time k,

$$\begin{aligned}\mathbf{E}[(C_T(\tilde{\phi}) - C_k(\tilde{\phi}))^2|\mathcal{F}_k] &= \mathbf{E}[(C_T(\phi) - C_{k+2}(\phi) + C_{k+2}(\tilde{\phi}) - C_k(\tilde{\phi}))^2|\mathcal{F}_k] \\ &= \mathbf{E}[(C_T(\phi) - C_{k+2}(\phi))^2|\mathcal{F}_k] + \mathbf{E}[(C_{k+2}(\tilde{\phi}) - C_k(\tilde{\phi}))^2|\mathcal{F}_k] \\ &\geq \mathrm{Var}[C_T(\phi) - C_{k+2}(\phi)|\mathcal{F}_k] + \mathrm{Var}[C_{k+2}(\tilde{\phi}) - C_k(\tilde{\phi})|\mathcal{F}_k] \\ &= \mathrm{Var}[C_T(\phi) - C_{k+2}(\phi)|\mathcal{F}_k] + \mathrm{Var}[C_{k+2}(\phi) - C_k(\phi)|\mathcal{F}_k] \\ &= \mathbf{E}[(C_T(\phi) - C_k(\phi))^2|\mathcal{F}_k]. \quad \square\end{aligned} \tag{2.20}$$

Then, the next theorem shows that the unique local risk minimizing strategy exists in the case of constant λ.

Theorem 1. *Assume that $\{\theta_j\}_{j=1,\ldots,T}$ recursively defined in (2.26),(2.27) satisfies*

$$\mathrm{Var}[\Delta\tilde{S}_{k+1} - 2\lambda M\theta_{k+2} + 2M\theta_{k+2}|\mathcal{F}_k] \neq 0, \tag{2.21}$$

and the integrability conditions

$$\theta_{k+1} \in L^4(\Omega), (k = 0, 1, \ldots, T), \tag{2.22}$$

$$\tilde{S}_k\theta_j \in L^2(\Omega) \text{ for all } k = 0, 1, \ldots, T \text{ and } j = 1, \ldots, T+1. \tag{2.23}$$

Then there exists the unique local risk minimizing strategy.

(Proof.) The existence part is shown by backward induction in the next subsection. We show the uniqueness part. Let $\phi := \{(\theta_k, \eta_k)\}_{k=0,1,\ldots,T}$ and $\tilde{\phi} := \{(\tilde{\theta}_k, \tilde{\eta}_k)\}_{k=0,1,\ldots,T}$ be local risk minimizing strategies. Then $\tilde{\theta}_k = \theta_k$ for $k = 0, 1, \ldots, T$ as the both are minimizers of $\mathrm{Var}[C_{k+2}(\phi) - C_k(\phi)|\mathcal{F}_k]$, which is a quadratic function with respect to θ_k. Then $\tilde{\eta}_k = \eta_k$ for $k = 0, 1, \ldots, T$ also follows from the fact that $\{C_k(\phi)\}_{k=0,1,\ldots,T}$ is a martingale. $\square$

2.3.2 Backward Induction for the Local Risk Minimizing Strategy

The local risk minimizing strategy is obtained by backward induction as follows. First, we solve the equivalent minimizing problem with respect to θ_j for $j = 1, \ldots, T-1$, which appears in Proposition 1. As we observe in (2.24), $\mathrm{Var}[C_{k+2} - C_k|\mathcal{F}_k]$ is a quadratic function of θ_{k+1}, hence one minimizer exists.

$$
\begin{aligned}
R_k(\phi) &= \mathrm{Var}[C_{k+2}(\phi) - C_k(\phi)|\mathcal{F}_k] \\
&= \mathrm{Var}[V_{k+2} - V_k - \sum_{j=k+1}^{k+2} \theta_j \Delta\tilde{S}_j - \sum_{j=k+1}^{k+2} 2\lambda M\theta_j\Delta\theta_j + \sum_{j=k+1}^{k+2} M(\Delta\theta_{j+1})^2|\mathcal{F}_k] \\
&= \mathrm{Var}[V_{k+2} - V_k - \theta_{k+2}\Delta\tilde{S}_{k+2} - \theta_{k+1}\Delta\tilde{S}_{k+1} - 2\lambda M\theta_{k+2}\Delta\theta_{k+2} - 2\lambda M\theta_{k+1}\Delta\theta_{k+1} \\
&\quad + M(\Delta\theta_{k+3})^2 + M\theta_{k+2}^2 - 2M\theta_{k+2}\theta_{k+1} + M\theta_{k+1}^2|\mathcal{F}_k] \\
&= \mathrm{Var}[\eta_{k+2} + \theta_{k+3}(\tilde{S}_{k+2} + 2\lambda M\theta_{k+2}) - \theta_{k+2}\Delta\tilde{S}_{k+2} - \theta_{k+1}\Delta\tilde{S}_{k+1} - 2\lambda M\theta_{k+2}(\theta_{k+2} - \theta_{k+1}) \\
&\quad + M(\Delta\theta_{k+3})^2 + M\theta_{k+2}^2 - 2M\theta_{k+2}\theta_{k+1}|\mathcal{F}_k]. \qquad (2.24)
\end{aligned}
$$

Taking partial derivative of $R_k(\phi)$ with respect to θ_{k+1}, we have

$$
\begin{aligned}
\frac{\partial}{\partial\theta_{k+1}} R_k(\phi) &= \lim_{h\to 0} \frac{\mathrm{Var}(\theta + h) - \mathrm{Var}(\theta)}{h} \\
&= -2\mathrm{Cov}[\eta_{k+2} + \theta_{k+3}(\tilde{S}_{k+2} + 2\lambda M\theta_{k+2}) - \theta_{k+2}\Delta\tilde{S}_{k+2} - 2\lambda M\theta_{k+2}^2 \\
&\quad + M(\Delta\theta_{k+3})^2 + M\theta_{k+2}^2 - \theta_{k+1}(\Delta\tilde{S}_{k+1} - 2\lambda M\theta_{k+2} + 2M\theta_{k+2}), \\
&\quad \Delta\tilde{S}_{k+1} - 2\lambda M\theta_{k+2} + 2M\theta_{k+2}|\mathcal{F}_k] \\
&= 0. \qquad (2.25)
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\theta_{k+1} \\
&= \mathrm{Cov}[\eta_{k+2} + \theta_{k+3}(\tilde{S}_{k+2} + 2\lambda M\theta_{k+2}) - \theta_{k+2}\Delta\tilde{S}_{k+2} - 2\lambda M\theta_{k+2}^2 + M(\Delta\theta_{k+3})^2 + M\theta_{k+2}^2, \\
&\quad \Delta\tilde{S}_{k+1} - 2\lambda M\theta_{k+2} + 2M\theta_{k+2}|\mathcal{F}_k] \\
&\quad /\mathrm{Var}[\Delta\tilde{S}_{k+1} - 2\lambda M\theta_{k+2} + 2M\theta_{k+2}|\mathcal{F}_k]. \qquad (2.26)
\end{aligned}
$$

Similarly, the minimizer of $R_{T-1}(\phi)$ is

$$
\begin{aligned}
\theta_T =& \mathrm{Cov}[\eta_T + \theta_{T+1}\tilde{S}_T + M\theta_{T+1}^2, \Delta\tilde{S}_T + 2M\theta_{T+1} - 2\lambda M\theta_{T+1}|\mathcal{F}_{T-1}] \\
&/\mathrm{Var}[\Delta\tilde{S}_T + 2M\theta_{T+1} - 2\lambda M\theta_{T+1}|\mathcal{F}_{T-1}]. \qquad (2.27)
\end{aligned}
$$

By (2.26) and (2.27), $\{\theta_k\}_{k=0,1,\ldots,T}$ in the local risk minimizing strategy is obtained recursively. $\{\eta_k\}_{k=0,1,\ldots,T}$ is obtained by the martingale property of $\{C_k\}_{k=0,1,\ldots,T}$.

2.4 Multi-Period Case where λ is Time Dependent

In this section, we observe the local risk minimization in the case where the market impact parameter λ is time dependent. We shall observe that the unique local risk minimizing strategy in the case of cash settlement and the first order approximation in the case of physical settlement, which is expanded around the case of no market impact, are obtained by backward induction.

2.4.1 Minimization at $k = T-1$

First we solve the minimization problem of R_{T-1} with respect to θ_T. By the definition of the cost process,

$$\begin{aligned}
&C_T(\phi) - C_{T-1}(\phi) \\
&= V_T(\phi) - V_{T-1}(\phi) - \theta_T \Delta \tilde{S}_T + M(\Delta\theta_{T+1})^2 - 2\lambda_{T-1} M \theta_T \Delta\theta_T \\
&= \eta_T + \theta_{T+1}(\tilde{S}_T + \sum_{j=1}^{T} 2\lambda_{j-1} M \Delta\theta_j) \\
&\quad - (\eta_{T-1} + \theta_T(\tilde{S}_{T-1} + \sum_{j=1}^{T-1} 2\lambda_{j-1} M \Delta\theta_j)) - \theta_T \Delta\tilde{S}_T + M(\Delta\theta_{T+1})^2 - 2\lambda_{T-1} M\theta_T\Delta\theta_T.
\end{aligned} \tag{2.28}$$

Noting the $\mathcal{F}_{T-1}$-measurable terms in (2.28), we have

$$\begin{aligned}
&\mathrm{Var}[C_T(\phi) - C_{T-1}(\phi)|\mathcal{F}_{T-1}] \\
&= \mathrm{Var}[\eta_T + \theta_{T+1}(\tilde{S}_T + \sum_{j=1}^{T} 2\lambda_{j-1} M\Delta\theta_j) - \theta_T\Delta\tilde{S}_T + M\theta_{T+1}^2 - 2M\theta_{T+1}\theta_T|\mathcal{F}_{T-1}].
\end{aligned} \tag{2.29}$$

Taking partial derivative with respect to θ_{T-1}, we have

$$\begin{aligned}
&\frac{\partial}{\partial\theta_{T-1}}\mathrm{Var}[C_T(\phi) - C_{T-1}(\phi)] \\
&= 2\mathrm{Cov}[\eta_T + \theta_{T+1}(\tilde{S}_T + \sum_{j=1}^{T} 2\lambda_{j-1} M\Delta\theta_j) - \theta_T\Delta\tilde{S}_T + M\theta_{T+1}^2 - 2M\theta_{T+1}\theta_T, \\
&\quad - \Delta\tilde{S}_T - 2M\theta_{T+1} + 2\lambda_{T-1} M\theta_{T+1}|\mathcal{F}_{T-1}] \\
&= 0.
\end{aligned} \tag{2.30}$$

Therefore,

$$\theta_T = \mathrm{Cov}[\eta_T + \theta_{T+1}(\tilde{S}_T - 2\sum_{j=1}^{T-1}\Delta\lambda_j M\theta_j) + M\theta_{T+1}^2, \Delta\tilde{S}_T + 2M\theta_{T+1} - 2\lambda_{T-1}M\theta_{T+1}|\mathcal{F}_{T-1}]$$
$$\Big/\mathrm{Var}[\Delta\tilde{S}_T + 2M\theta_{T+1} - 2\lambda_{T-1}M\theta_{T+1}|\mathcal{F}_{T-1}]. \tag{2.31}$$

2.4.2 Minimization at k

Next we solve the minimization problem of R_k with respect to θ_{k+1} ($0 \le k \le T-2$). By the definition of the cost process, we have

$$\begin{aligned}
&C_T(\phi) - C_k(\phi) \\
&= V_T(\phi) - V_k(\phi) - \sum_{j=k+1}^{T}\theta_j\Delta\tilde{S}_j - \sum_{j=k+1}^{T}2\lambda_{j-1}M\theta_j\Delta\theta_j + \sum_{j=k+1}^{T}M(\Delta\theta_{j+1})^2 \\
&= \eta_T + \theta_{T+1}(\tilde{S}_T + \sum_{j=1}^{T}2\lambda_{j-1}M\Delta\theta_j) - (\eta_k + \theta_{k+1}(\tilde{S}_k + \sum_{j=1}^{k}2\lambda_{j-1}M\Delta\theta_j)) \\
&\quad - \sum_{j=k+1}^{T}\theta_j\Delta S_j - \sum_{j=k+2}^{T}2\lambda_{j-1}M\theta_j\Delta\theta_j - 2\lambda_k M\theta_{k+1}\Delta\theta_{k+1} \\
&\quad + \sum_{j=k+2}^{T}M(\Delta\theta_{j+1})^2 + M\theta_{k+2}^2 - 2M\theta_{k+2}\theta_{k+1} + M\theta_{k+1}^2.
\end{aligned} \tag{2.32}$$

Noting the $\mathcal{F}_k$-measurable random variables in (2.32), we have

$$\begin{aligned}
&\mathrm{Var}[C_T(\phi) - C_k(\phi)|\mathcal{F}_k] \\
&= \mathrm{Var}[\eta_T + \theta_{T+1}(\tilde{S}_T + \sum_{j=1}^{T}2\lambda_{j-1}M\Delta\theta_j) - \sum_{j=k+1}^{T}\theta_j\Delta\tilde{S}_j - \sum_{j=k+2}^{T}2\lambda_{j-1}M\theta_j\Delta\theta_j + \\
&\quad \sum_{j=k+2}^{T}M(\Delta\theta_{j+1})^2 + M\theta_{k+2}^2 - 2M\theta_{k+2}\theta_{k+1}|\mathcal{F}_k].
\end{aligned} \tag{2.33}$$

Taking partial derivative with respect to θ_{k+1}, we have

$$
\begin{aligned}
&\frac{\partial}{\partial \theta_{k+1}} \mathrm{Var}[C_T(\phi) - C_k(\phi)|\mathcal{F}_k] \\
&= 2\mathrm{Cov}[\eta_T + \theta_{T+1}(\tilde{S}_T + \sum_{j=1}^{T} 2\lambda_{j-1} M \Delta\theta_j) - \sum_{j=k+1}^{T} \theta_j \Delta\tilde{S}_j - \sum_{j=k+2}^{T} 2\lambda_{j-1} M \theta_j \Delta\theta_j + \\
&\quad \sum_{j=k+2}^{T} M(\Delta\theta_{j+1})^2 + M\theta_{k+2}^2 - 2M\theta_{k+2}\theta_{k+1}, \\
&\quad - \Delta\tilde{S}_{k+1} - 2\Delta\lambda_{k+1} M\theta_{T+1} + 2\lambda_{k+1} M\theta_{k+2} - 2M\theta_{k+2}|\mathcal{F}_k] \\
&= 0.
\end{aligned} \tag{2.34}
$$

Therefore,

$$
\begin{aligned}
\theta_{k+1} = \mathrm{Cov}[&\eta_T + \theta_{T+1}(\tilde{S}_T + 2\lambda_{T-1} M\theta_T - 2 \sum_{j=1, j\neq k+1}^{T-1} \Delta\lambda_j M\theta_j) - \sum_{j=k+2}^{T} \theta_j \Delta\tilde{S}_j \\
&- \sum_{j=k+3}^{T} 2\lambda_{j-1} M\theta_j \Delta\theta_j - 2\lambda_{k+1} M\theta_{k+2}^2 + \sum_{j=k+2}^{T} M(\Delta\theta_{j+1})^2 + M\theta_{k+2}^2, \\
&\Delta\tilde{S}_{k+1} + 2\Delta\lambda_{k+1} M\theta_{T+1} - 2\lambda_{k+1} M\theta_{k+2} + 2M\theta_{k+2}|\mathcal{F}_k] \\
&\big/ \mathrm{Var}[\Delta\tilde{S}_{k+1} + 2\Delta\lambda_{k+1} M\theta_{T+1} - 2\lambda_{k+1} M\theta_{k+2} + 2M\theta_{k+2}|\mathcal{F}_k].
\end{aligned} \tag{2.35}
$$

2.4.3 The Case of Cash Settlement

In the case of cash settlement, where $\bar{\theta}_{T+1} = \theta_{T+1} = 0$, $\{\theta_{k+1}\}_{k=0,\dots,T}$ can be obtained recursively.

Substituting $\theta_{T+1} = 0$ in (2.31) and (2.35), we have

$$
\theta_T = \frac{\mathrm{Cov}[\eta_T, \Delta\tilde{S}_T|\mathcal{F}_{T-1}]}{\mathrm{Var}[\Delta\tilde{S}_T|\mathcal{F}_{T-1}]}, \tag{2.36}
$$

$$
\begin{aligned}
\theta_{k+1} = \mathrm{Cov}[&\eta_T - \sum_{j=k+2}^{T} \theta_j \Delta\tilde{S}_j - \sum_{j=k+3}^{T} 2\lambda_{j-1} M\theta_j \Delta\theta_j - 2\lambda_{k+1} M\theta_{k+2}^2 + \sum_{j=k+2}^{T} M(\Delta\theta_{j+1})^2 + M\theta_{k+2}^2, \\
&\Delta\tilde{S}_{k+1} - 2\lambda_{k+1} M\theta_{k+2} + 2M\theta_{k+2}|\mathcal{F}_k] \\
&\big/ \mathrm{Var}[\Delta\tilde{S}_{k+1} - 2\lambda_{k+1} M\theta_{k+2} + 2M\theta_{k+2}|\mathcal{F}_k].
\end{aligned} \tag{2.37}
$$

Since θ_{k+1} does not depend on $\theta_1, \dots, \theta_k$, $\{\theta_{k+1}\}_{k=0,\dots,T}$ is obtained by the following manner.

1. Obtain θ_T.

2. Given $\theta_j, j = k+2, \dots, T+1$, obtain θ_{k+1}.

$\{\eta_k\}_{k=0,1,\dots,T}$ is obtained by the martingale property of $\{C_k(\phi)\}_{k=0,1,\dots,T}$.

2.4.4 The Case of Physical Settlement

In the case of physical settlement, where $\theta_{T+1} = \bar{\theta}_{T+1} \neq 0$, the local risk minimizing strategy is a solution of the simultaneous equations on random variables (2.31), (2.35) and cannot be solved explicitly. However, we can obtain the first order approximation of the local risk minimizing strategy in a recursive manner if we expand the strategy with respect to $\{\lambda_i\}_{i=0,1,\dots,T-1}$ around $\lambda_i = 0$. In this subsection, we assume differentiability of each θ_j, $(j = 1, \dots, T)$ with respect to $\{\lambda_i\}_{i=0,1,\dots,T-1}$.

First order approximation of θ_T

Expanding $\{\theta_j\}_{j=1,2,\dots,T}$ with respect to λ_i around 0 $(i = 0, 1, \dots, T-1)$ as

$$\theta_j = \sum_{m_0,\dots,m_{T-1}=0}^{\infty} c^j_{m_0,\dots,m_{T-1}} \lambda_0^{m_0} \dots \lambda_{T-1}^{m_{T-1}}, \tag{2.38}$$

where

$$c^j_{m_0,\dots,m_{T-1}} = \frac{1}{m_0! \dots m_{T-1}!} \frac{\partial^{m_0+\dots+m_{T-1}} \theta_j}{\partial \lambda_0^{m_0} \dots \partial \lambda_{T-1}^{m_{T-1}}} \Big|_{\lambda_0=0,\dots,\lambda_{T-1}=0}, \tag{2.39}$$

we aim to obtain the explicit expression of the first order approximation of θ_T.

The first order expansion of the numerator of θ_T in (2.31) is

$$\begin{aligned} &\mathrm{Cov}[\eta_T + \theta_{T+1}\tilde{S}_T + M\theta_{T+1}^2, \Delta\tilde{S}_T + 2M\theta_{T+1}|\mathcal{F}_{T-1}] \\ &\quad + \mathrm{Cov}[\theta_{T+1}(-2\sum_{j=1}^{T-1} \Delta\lambda_j M \theta_j^{(0)}), \Delta\tilde{S}_T + 2M\theta_{T+1}|\mathcal{F}_{T-1}] \\ &\quad + \mathrm{Cov}[\eta_T + \theta_{T+1}\tilde{S}_T + M\theta_{T+1}^2, -2\lambda_{T-1}M\theta_{T+1}|\mathcal{F}_{T-1}]. \end{aligned} \tag{2.40}$$

Similarly, the first order expansion of the denominator of θ_T is

$$\mathrm{Var}[\Delta\tilde{S}_T + 2M\theta_{T+1}|\mathcal{F}_{T-1}] + 2\mathrm{Cov}[-2\lambda_{T-1}M\theta_{T+1}, \Delta\tilde{S}_T + 2M\theta_{T+1}|\mathcal{F}_{T-1}]. \tag{2.41}$$

Therefore, the first order approximation of θ_T is

$$\begin{aligned}
\theta_T \sim\ & \frac{\mathrm{Cov}[\eta_T + \theta_{T+1}\tilde{S}_T + M\theta_{T+1}^2, \Delta\tilde{S}_T + 2M\theta_{T+1}|\mathcal{F}_{T-1}]}{\mathrm{Var}[\Delta\tilde{S}_T + 2M\theta_{T+1}|\mathcal{F}_{T-1}]} \\
& + \Big\{\mathrm{Cov}[\theta_{T+1}(-2\sum_{j=1}^{T-1}\Delta\lambda_j M\theta_j^{(0)}), \Delta\tilde{S}_T + 2M\theta_{T+1}|\mathcal{F}_{T-1}] \\
& \quad + \mathrm{Cov}[\eta_T + \theta_{T+1}\tilde{S}_T + M\theta_{T+1}^2, -2\lambda_{T-1}M\theta_{T+1}|\mathcal{F}_{T-1}]\Big\} \\
& \quad \Big/ \mathrm{Var}[\Delta\tilde{S}_T + 2M\theta_{T+1}|\mathcal{F}_{T-1}] \\
& - 2\mathrm{Cov}[-2\lambda_{T-1}M\theta_{T+1}, \Delta\tilde{S}_T + 2M\theta_{T+1}|\mathcal{F}_{T-1}] \\
& \quad \times \mathrm{Cov}[\eta_T + \theta_{T+1}\tilde{S}_T + M\theta_{T+1}^2, \Delta\tilde{S}_T + 2M\theta_{T+1}|\mathcal{F}_{T-1}] \\
& \quad \Big/ \mathrm{Var}[\Delta\tilde{S}_T + 2M\theta_{T+1}|\mathcal{F}_{T-1}]^2. \qquad (2.42)
\end{aligned}$$

Note that this expression shows that the first order approximation of θ_T is obtained explicitly if the 0th order approximation of $\{\theta_j\}_{j=1,\ldots,T+1}$ is given. As we have seen in Section 3, the 0 th order approximation, which corresponds to the case of $\lambda = 0$, is determined recursively.

First Order approximation of θ_{k+1}

The 0th order part and the first order part of the numerator of θ_{k+1} in (2.35) are as follows.

(i) The 0th order part

$$\mathrm{Cov}[\eta_T + \theta_{T+1}\tilde{S}_T - \sum_{j=k+2}^{T}\theta_j^{(0)}\Delta\tilde{S}_j + \sum_{j=k+2}^{T}M(\Delta\theta_{j+1}^{(0)})^2 + M\theta_{k+2}^{(0)2}, \Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}|\mathcal{F}_k]. \qquad (2.43)$$

(ii) The first order part

$$\mathrm{Cov}[\theta_{T+1}(2\lambda_{T-1}M\theta_T^{(0)} - 2\sum_{j=1, j\neq k+1}^{T}\Delta\lambda_j M\theta_j^{(0)}) - \sum_{j=k+2}^{T}(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_j^{(0)}\lambda_i) - \sum_{j=k+3}^{T}2\lambda_{j-1}M\theta_j^{(0)}\Delta\theta_j^{(0)}$$
$$- 2\lambda_{k+1}M\theta_{k+2}^{(0)2} + 2\sum_{j=k+2}^{T}M\Delta\theta_{j+1}^{(0)}(\sum_{i=0}^{T-1}(\partial_{\lambda_i}\theta_{j+1}^{(0)} - \partial_{\lambda_i}\theta_j^{(0)})\lambda_i) + 2M\theta_{k+2}^{(0)}(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_{k+2}^{(0)}\lambda_i),$$
$$\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}|\mathcal{F}_k]$$
$$+ \mathrm{Cov}[\eta_T + \theta_{T+1}\tilde{S}_T - \sum_{j=k+2}^{T}\theta_j^{(0)}\Delta\tilde{S}_j + \sum_{j=k+2}^{T}M(\Delta\theta_{j+1}^{(0)2}) + M\theta_{k+2}^{(0)2},$$
$$2\Delta\lambda_{k+1}M\theta_{T+1}^{(0)} - 2\lambda_{k+1}M\theta_{k+2}^{(0)} + 2M(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_{k+2}^{(0)}\lambda_i)|\mathcal{F}_k]. \tag{2.44}$$

The 0th order part and the first order part of the denominator of θ_{k+1} are as follows.

(i) The 0th order part

$$\mathrm{Var}[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}|\mathcal{F}_k]. \tag{2.45}$$

(ii) The first order part

$$2\mathrm{Cov}[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}, 2\Delta\lambda_{k+1}M\theta_{T+1}^{(0)} - 2\lambda_{k+1}M\theta_{k+2}^{(0)} + 2M(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_{k+2}^{(0)}\lambda_i)|\mathcal{F}_k]. \tag{2.46}$$

Therefore, the first order approximation of θ_{k+1} is given by

$$\frac{\mathrm{Cov}[\eta_T + \theta_{T+1}\tilde{S}_T - \sum_{j=k+2}^{T}\theta_j^{(0)}\Delta\tilde{S}_j + \sum_{j=k+2}^{T} M(\Delta\theta_{j+1}^{(0)})^2 + M\theta_{k+2}^{(0)2}, \Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}|\mathcal{F}_k]}{\mathrm{Var}[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}|\mathcal{F}_k]}$$

$$+ \Big\{\mathrm{Cov}[\theta_{T+1}(2\lambda_{T-1}M\theta_T^{(0)} - 2\sum_{j=1, j\neq k+1}^{T}\Delta\lambda_j M\theta_j^{(0)}) - \sum_{j=k+2}^{T}(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_j^{(0)}\lambda_i) - \sum_{j=k+3}^{T} 2\lambda_{j-1}M\theta_j^{(0)}\Delta\theta_j^{(0)}$$

$$- 2\lambda_{k+1}M\theta_{k+2}^{(0)2} + 2\sum_{j=k+2}^{T} M\Delta\theta_{j+1}^{(0)}(\sum_{i=0}^{T-1}(\partial_{\lambda_i}\theta_{j+1}^{(0)} - \partial_{\lambda_i}\theta_j^{(0)})\lambda_i) + 2M\theta_{k+2}^{(0)}(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_{k+2}^{(0)}\lambda_i),$$

$$\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}|\mathcal{F}_k]$$

$$+ \mathrm{Cov}[\eta_T + \theta_{T+1}\tilde{S}_T - \sum_{j=k+2}^{T}\theta_j^{(0)}\Delta\tilde{S}_j + \sum_{j=k+2}^{T} M(\Delta\theta_{j+1}^{(0)2}) + M\theta_{k+2}^{(0)2},$$

$$2\Delta\lambda_{k+1}M\theta_{T+1}^{(0)} - 2\lambda_{k+1}M\theta_{k+2}^{(0)} + 2M(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_{k+2}^{(0)}\lambda_i)|\mathcal{F}_k]\Big\}$$

$$\Big/\mathrm{Var}[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}|\mathcal{F}_k]$$

$$- 2\mathrm{Cov}[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}, 2\Delta\lambda_{k+1}M\theta_{T+1}^{(0)} - 2\lambda_{k+1}M\theta_{k+2}^{(0)} + 2M(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_{k+2}^{(0)}\lambda_i)|\mathcal{F}_k]$$

$$\times \mathrm{Cov}[\eta_T + \theta_{T+1}\tilde{S}_T - \sum_{j=k+2}^{T}\theta_j^{(0)}\Delta\tilde{S}_j + \sum_{j=k+2}^{T} M(\Delta\theta_{j+1}^{(0)})^2 + M\theta_{k+2}^{(0)2}, \Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}|\mathcal{F}_k].$$

$$\Big/\mathrm{Var}[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}|\mathcal{F}_k]^2. \tag{2.47}$$

Backward Induction

Since the first order approximation of θ_{k+1} includes the 0th order part of $\{\theta_j\}_{j=1,\ldots,k,k+2,\ldots,T+1}$ and the first order part of $\{\theta_j\}_{\mathrm{J}=k+2,\ldots,T}$, the first order approximation of $\{\theta_j\}_{j=1,\ldots,T-1}$ is obtained by the following procedure.

1. Obtain the 0th order part of θ_T.
2. Given the 0th order part of $\theta_j, j = k+2, \ldots, T$, obtain the 0th order part of θ_{k+1}.
3. Given the 0th order part of $\theta_j, j = 1, \ldots, T+1$, obtain the first order part of θ_T.
4. Given the first order part of $\theta_j, j = k+2, \ldots, T$, obtain the first order part of θ_{k+1}.

2.5 Numerical Examples

This section considers two period cases ($T = 2$) as numerical experiments.

Let $\Omega = \{\omega_1, \omega_2, \dots, \omega_{16}\}$, $\mathbf{P}(\{\omega_i\}) = 1/16$ for $i = 1, 2, \dots, 16$, $\mathcal{F} = \mathcal{F}_2 = 2^\Omega$, $\mathcal{F}_1 = \sigma(\{\omega_1, \omega_2, \omega_3, \omega_4\}, \dots, \{\omega_{13}, \omega_{14}, \omega_{15}, \omega_{16}\})$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

We assume the following two dimensional binomial model for the unaffected price process of the hedging instrument $\{\tilde{S}_k\}_{k=0,1,2}$ and the price process of the untradable underlying asset in the option payoff $\{A_k\}_{k=0,1,2}$, which is correlated to $\{\tilde{S}_k\}$.

For $k = 0, 1$,

$$\tilde{S}_{k+1} = \tilde{S}_k \exp(-\frac{1}{2}\sigma^{\tilde{S}}\Delta t + \sigma^{\tilde{S}}\sqrt{\Delta t}\xi_{1,k+1}), \tag{2.48}$$

$$A_{k+1} = A_k \exp(-\frac{1}{2}\sigma^A\Delta t + \sigma^A\sqrt{\Delta t}(\rho\xi_{1,k+1} + \sqrt{1-\rho^2}\xi_{2,k+1})), \tag{2.49}$$

where the random variables $\xi_{1,k}$ and $\xi_{2,k}$ take the following value in each ω_i, $i = 1, \dots, 16$,

$$\xi_{1,1}(\omega_i) = \begin{cases} +1, & i \equiv 1, 2, 3, 4, 5, 6, 7, 8 \pmod{16} \\ -1, & i \equiv 9, 10, 11, 12, 13, 14, 15, 0 \pmod{16}, \end{cases} \tag{2.50}$$

$$\xi_{2,1}(\omega_i) = \begin{cases} +1, & i \equiv 1, 2, 3, 4 \pmod{8} \\ -1, & i \equiv 5, 6, 7, 0 \pmod{8}, \end{cases} \tag{2.51}$$

$$\xi_{1,2}(\omega_i) = \begin{cases} +1, & i \equiv 1, 2 \pmod{4} \\ -1, & i \equiv 3, 0 \pmod{4}, \end{cases} \tag{2.52}$$

$$\xi_{2,2}(\omega_i) = \begin{cases} +1, & i \equiv 1 \pmod{2} \\ -1, & i \equiv 0 \pmod{2}. \end{cases} \tag{2.53}$$

In the examples, we consider the local risk minimization problem of a hedger who is short a call option on the untradable asset and aim to replicate the payoff which will be delivered to the buyer at the maturity. We consider the following payoffs in cash settlement, $\bar{\theta}_3 = 0$, $\bar{\eta}_2 = (A_2 - K_A)^+$, and in physical settlement, $\bar{\theta}_3 = 1_{\{A_2 > K_A\}}$, $\bar{\eta}_2 = -K_S 1_{\{A_2 > K_A\}}$. In the cash settlement, the cash amount $A_2 - K_A$ is delivered to the buyer when $A_2 > K_A$. In the physical settlement, one unit of the hedging instrument is delivered to the buyer at price K_S when $A_2 > K_A$. Specifically, the parameters are set to be $\tilde{S}_0 = 100, \sigma^{\tilde{S}} = 10\%, \Delta t = 1, A_0 = 100, \sigma^A = 10\%, K_A = 100, K_S = 100$.

First, Table 1-4 show the position of the local risk minimizing strategy in cash settlement (denoted by θ_1 and θ_2) with time dependent market impact parameters $\lambda_0 = 0.1, \lambda_1 = 0.9$, and Table 5-8 with $\lambda_0 = 0.2, \lambda_1 = 0.4$. R_0 is the value of the risk process at time 0, which measures the mean-square error of the payoff in the cash position at the maturity. V_0 is the value process at time 0, which is the mid value of the initial position. C_0 is the cost to replicate the initial position defined by $V_0 + M\theta_1^2 = \eta_0 + \theta_1 S(0, \theta_1)$. As we observed in Section 4.3, in this case, the unique local risk minimizing strategy is obtained by backward induction.

The correlations ρ are set to be $100\%, 70\%, 50\%$, while the order book density parameters M are $5, 4, 3, 2, 1, 0$ for all correlations. As expected, we observe that the initial cost of the strategy C_0 increases as M increases, which means that when the order book density is thin, the cost for the replication of the option payoff is large. It is also observed that R_0 increases as ρ decreases, which implies that the replication error of the payoff on the untradable asset is large when its correlation with the hedging instrument is low.

Second, Table 9-16 show the first order approximation of the local risk minimizing strategy in the cases of physical settlement (denoted by approx θ_1 and approx θ_2) with time dependent market impact parameters $\lambda_0 = 0.1, \lambda_1 = 0.9$ for Table 9-12, and $\lambda_0 = 0.2, \lambda_1 = 0.4$ for Table 13-16. We compare the approximation results with the exact local risk minimizing strategies, which can be computed solely in the two period case, by solving a cubic equation (denoted by θ_1 and θ_2, see Appendix A) for the correlations $\rho = 100\%, 70\%, 50\%$ and the order book density parameters $M = 5, 4, 3, 2, 1, 0$. Here, approx V_0 and approx C_0 are the initial mid value and the initial replication cost of the approximate strategy.

Then, as expected, we observe that the expansions of θ_1, θ_2 up to the first order closely approximate the exact local risk minimizing strategy computed by solving the cubic equation, in terms of the position of the hedging instrument as well as the initial value and the initial replication cost of the strategy.

Remark 3. *In the cases where* $\rho = 100\%, M = 5, 4, 3, 2, 1$ *in Table 1-12, there are three local risk minimizing strategies, while in all the other cases there is the unique local risk minimizing strategy. We remark that the strategy, which is expanded around the case of no market impact where the unique local risk minimizing strategy exists, approximates the one solution of the three in the cubic equation, which is a sensible strategy taking the value between* 0 *and* 1*. The other two solutions take large values having excessively large initial mid values and replication costs. (See Appendix A for the details of the solutions of the cubic equation. For example, Table 13-14 show coefficients of the cubic equation and its solutions in the case where* $\rho = 100\%, M = 5, \lambda_0 = 0.1, \lambda_1 = 0.9$*. In this case,* $(\theta_1, V_0, C_0) = (0.5379, 6.52, 7.97), (54.00, 12202, 26782), (71.81, 21641, 47424)$*. Similarly in*

Table 15-16, we observe that, in the case where $\rho = 100\%, M = 1, \lambda_0 = 0.2, \lambda_1 = 0.4$, $(\theta_1, V_0, C_0) = (0.5389, 5.55, 5.84), (1007.98, 619663, 1635691), (1024.57, 640240, 16899$

Table 2.1: $\rho = 1$: Local Risk Minimizing Strategy in Cash Settlement, $\lambda_0 = 0.1, \lambda_1 = 0.9$

M	5	4	3	2	1	0
λ_0	0.1	0.1	0.1	0.1	0.1	0.1
λ_1	0.9	0.9	0.9	0.9	0.9	0.9
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.9546	0.9546	0.9546	0.9546	0.9546	0.9546
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.9546	0.9546	0.9546	0.9546	0.9546	0.9546
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
θ_1	0.5446	0.5408	0.5369	0.5330	0.5290	0.5249
R_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
V_0	6.6140	6.3259	6.0436	5.7671	5.4966	5.2321
C_0	8.0967	7.4957	6.9085	6.3353	5.7764	5.2321

Table 2.2: $\rho = 0.7$: Local Risk Minimizing Strategy in Cash Settlement, $\lambda_0 = 0.1, \lambda_1 = 0.9$

M	5	4	3	2	1	0
λ_0	0.1	0.1	0.1	0.1	0.1	0.1
λ_1	0.9	0.9	0.9	0.9	0.9	0.9
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.7081	0.7081	0.7081	0.7081	0.7081	0.7081
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.3167	0.3167	0.3167	0.3167	0.3167	0.3167
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.3958	0.3958	0.3958	0.3958	0.3958	0.3958
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
θ_1	0.3833	0.3809	0.3784	0.3759	0.3734	0.3708
R_0	46.7251	46.5793	46.4331	46.2865	46.1395	45.9921
V_0	6.1134	5.9798	5.8489	5.7206	5.5950	5.4721
C_0	6.8481	6.5600	6.2784	6.0032	5.7344	5.4721

Table 2.3: $\rho = 0.5$: Local Risk Minimizing Strategy in Cash Settlement, $\lambda_0 = 0.1, \lambda_1 = 0.9$

M	5	4	3	2	1	0
λ_0	0.1	0.1	0.1	0.1	0.1	0.1
λ_1	0.9	0.9	0.9	0.9	0.9	0.9
$\theta_2(\omega_i),\ i = 1, 2, 3, 4$	0.4972	0.4972	0.4972	0.4972	0.4972	0.4972
$\theta_2(\omega_i),\ i = 5, 6, 7, 8$	0.2148	0.2148	0.2148	0.2148	0.2148	0.2148
$\theta_2(\omega_i),\ i = 9, 10, 11, 12$	0.3121	0.3121	0.3121	0.3121	0.3121	0.3121
$\theta_2(\omega_i),\ i = 13, 14, 15, 16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
θ_1	0.2748	0.2731	0.2713	0.2696	0.2678	0.2660
R_0	59.9914	59.9136	59.8354	59.7567	59.6777	59.5982
V_0	6.0126	5.9440	5.8768	5.8109	5.7464	5.6832
C_0	6.3902	6.2423	6.0976	5.9563	5.8181	5.6832

Table 2.4: $\rho = 1$: Local Risk Minimizing Strategy in Cash Settlement, $\lambda_0 = 0.2, \lambda_1 = 0.4$

M	5	4	3	2	1	0
λ_0	0.2	0.2	0.2	0.2	0.2	0.2
λ_1	0.4	0.4	0.4	0.4	0.4	0.4
$\theta_2(\omega_i),\ i = 1, 2, 3, 4$	0.9546	0.9546	0.9546	0.9546	0.9546	0.9546
$\theta_2(\omega_i),\ i = 5, 6, 7, 8$	0.9546	0.9546	0.9546	0.9546	0.9546	0.9546
$\theta_2(\omega_i),\ i = 9, 10, 11, 12$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 13, 14, 15, 16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
θ_1	0.6208	0.6052	0.5881	0.5692	0.5483	0.5249
R_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
V_0	7.3444	6.9117	6.4845	6.0624	5.6452	5.2321
C_0	9.2715	8.3769	7.5221	6.7104	5.9458	5.2321

Table 2.5: $\rho = 0.7$: Local Risk Minimizing Strategy in Cash Settlement, $\lambda_0 = 0.2, \lambda_1 = 0.4$

M	5	4	3	2	1	0
λ_0	0.2	0.2	0.2	0.2	0.2	0.2
λ_1	0.4	0.4	0.4	0.4	0.4	0.4
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.7081	0.7081	0.7081	0.7081	0.7081	0.7081
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.3167	0.3167	0.3167	0.3167	0.3167	0.3167
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.3958	0.3958	0.3958	0.3958	0.3958	0.3958
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
θ_1	0.4400	0.4273	0.4141	0.4002	0.3858	0.3708
R_0	50.2396	49.4184	48.5834	47.7342	46.8704	45.9921
V_0	6.2530	6.0920	5.9339	5.7783	5.6245	5.4721
C_0	7.2212	6.8225	6.4482	6.0986	5.7734	5.4721

Table 2.6: $\rho = 0.5$: Local Risk Minimizing Strategy in Cash Settlement, $\lambda_0 = 0.2, \lambda_1 = 0.4$

M	5	4	3	2	1	0
λ_0	0.2	0.2	0.2	0.2	0.2	0.2
λ_1	0.4	0.4	0.4	0.4	0.4	0.4
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.4972	0.4972	0.4972	0.4972	0.4972	0.4972
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.2148	0.2148	0.2148	0.2148	0.2148	0.2148
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.3121	0.3121	0.3121	0.3121	0.3121	0.3121
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
θ_1	0.3154	0.3062	0.2966	0.2867	0.2765	0.2660
R_0	61.8051	61.3920	60.9652	60.5243	60.0687	59.5982
V_0	6.0834	6.0009	5.9199	5.8402	5.7614	5.6832
C_0	6.5808	6.3758	6.1838	6.0046	5.8378	5.6832

Table 2.7: $\rho = 1$: First Order Approximation of Local Risk Minimizing Strategy in Physical Settlement, $\lambda_0 = 0.1, \lambda_1 = 0.9$

M	5	4	3	2	1	0
λ_0	0.1	0.1	0.1	0.1	0.1	0.1
λ_1	0.9	0.9	0.9	0.9	0.9	0.9
approx $\theta_2(\omega_i),\ i = 1, 2, 3, 4$	0.9027	0.9154	0.9280	0.9398	0.9495	0.9546
approx $\theta_2(\omega_i),\ i = 5, 6, 7, 8$	0.9027	0.9154	0.9280	0.9398	0.9495	0.9546
approx $\theta_2(\omega_i),\ i = 9, 10, 11, 12$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
approx $\theta_2(\omega_i),\ i = 13, 14, 15, 16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 1, 2, 3, 4$	0.9434	0.9461	0.9487	0.9510	0.9530	0.9546
$\theta_2(\omega_i),\ i = 5, 6, 7, 8$	0.9434	0.9461	0.9487	0.9510	0.9530	0.9546
$\theta_2(\omega_i),\ i = 9, 10, 11, 12$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 13, 14, 15, 16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
approx θ_1	0.5449	0.5419	0.5380	0.5332	0.5282	0.5249
θ_1	0.5379	0.5358	0.5335	0.5309	0.5281	0.5249
R_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
approx V_0	6.5158	6.2585	6.0006	5.7429	5.4865	5.2321
V_0	6.5191	6.2596	6.0009	5.7432	5.4868	5.2321
approx C_0	8.0003	7.4332	6.8689	6.3116	5.7655	5.2321
C_0	7.9659	7.4080	6.8548	6.3070	5.7657	5.2321

Table 2.8: $\rho = 0.7$: First Order Approximation of Local Risk Minimizing Strategy in Physical Settlement, $\lambda_0 = 0.1, \lambda_1 = 0.9$

M	5	4	3	2	1	0
λ_0	0.1	0.1	0.1	0.1	0.1	0.1
λ_1	0.9	0.9	0.9	0.9	0.9	0.9
approx $\theta_2(\omega_i),\ i = 1,2,3,4$	0.9796	0.9836	0.9851	0.9841	0.9812	0.9773
approx $\theta_2(\omega_i),\ i = 5,6,7,8$	0.5739	0.5516	0.5295	0.5086	0.4907	0.4773
approx $\theta_2(\omega_i),\ i = 9,10,11,12$	0.0270	0.0128	0.0005	-0.0098	-0.0186	-0.0277
approx $\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.9812	0.9823	0.9827	0.9821	0.9804	0.9773
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.5146	0.5092	0.5030	0.4957	0.4872	0.4773
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.0055	0.0011	-0.0042	-0.0106	-0.0184	-0.0277
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
approx θ_1	0.4632	0.4541	0.4430	0.4295	0.4132	0.3937
θ_1	0.4767	0.4635	0.4486	0.4320	0.4137	0.3937
R_0	18.7942	18.5105	18.1795	17.7960	17.3556	16.8563
approx V_0	4.9807	4.7395	4.5000	4.2632	4.0297	3.7997
V_0	4.9910	4.7439	4.5012	4.2632	4.0296	3.7997
approx C_0	6.0535	5.5643	5.0888	4.6321	4.2004	3.7997
C_0	6.1271	5.6032	5.1050	4.6365	4.2008	3.7997

Table 2.9: $\rho = 0.5$: First Order Approximation of Local Risk Minimizing Strategy in Physical Settlement, $\lambda_0 = 0.1, \lambda_1 = 0.9$

M	5	4	3	2	1	0
λ_0	0.1	0.1	0.1	0.1	0.1	0.1
λ_1	0.9	0.9	0.9	0.9	0.9	0.9
approx $\theta_2(\omega_i),\ i = 1,2,3,4$	0.9645	0.9699	0.9738	0.9761	0.9770	0.9773
approx $\theta_2(\omega_i),\ i = 5,6,7,8$	0.5588	0.5380	0.5182	0.5006	0.4865	0.4773
approx $\theta_2(\omega_i),\ i = 9,10,11,12$	0.3848	0.4237	0.4566	0.4798	0.4930	0.5000
approx $\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.9728	0.9740	0.9750	0.9759	0.9767	0.9773
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.5062	0.5009	0.4953	0.4895	0.4835	0.4773
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.4702	0.4760	0.4819	0.4878	0.4939	0.5000
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
approx θ_1	0.5249	0.5214	0.5173	0.5141	0.5126	0.5125
θ_1	0.5217	0.5197	0.5178	0.5158	0.5141	0.5125
R_0	31.3333	31.3059	31.2751	31.2411	31.2044	31.1653
approx V_0	3.8518	3.6004	3.3515	3.1051	2.8602	2.6161
V_0	3.8435	3.5968	3.3507	3.1052	2.8604	2.6161
approx C_0	5.2295	4.6877	4.1544	3.6337	3.1230	2.6161
C_0	5.2042	4.6772	4.1549	3.6374	3.1246	2.6161

Table 2.10: $\rho = 1$: First Order Approximation of Local Risk Minimizing Strategy in Physical Settlement, $\lambda_0 = 0.2, \lambda_1 = 0.4$

M	5	4	3	2	1	0
λ_0	0.2	0.2	0.2	0.2	0.2	0.2
λ_1	0.4	0.4	0.4	0.4	0.4	0.4
approx $\theta_2(\omega_i),\ i = 1,2,3,4$	0.8794	0.8926	0.9069	0.9222	0.9382	0.9546
approx $\theta_2(\omega_i),\ i = 5,6,7,8$	0.8794	0.8926	0.9069	0.9222	0.9382	0.9546
approx $\theta_2(\omega_i),\ i = 9,10,11,12$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
approx $\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.8876	0.8989	0.9112	0.9245	0.9390	0.9546
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.8876	0.8989	0.9112	0.9245	0.9390	0.9546
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
approx θ_1	0.5546	0.5518	0.5478	0.5421	0.5346	0.5249
θ_1	0.5723	0.5663	0.5590	0.5501	0.5389	0.5249
R_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
approx V_0	6.6652	6.4012	6.1279	5.8436	5.5459	5.2321
V_0	6.6866	6.4141	6.1345	5.8459	5.5462	5.2321
approx C_0	8.2030	7.6190	7.0280	6.4315	5.8316	5.2321
C_0	8.3241	7.6969	7.0720	6.4511	5.8366	5.2321

Table 2.11: $\rho = 0.7$: First Order Approximation of Local Risk Minimizing Strategy in Physical Settlement, $\lambda_0 = 0.2, \lambda_1 = 0.4$

M	5	4	3	2	1	0
λ_0	0.2	0.2	0.2	0.2	0.2	0.2
λ_1	0.4	0.4	0.4	0.4	0.4	0.4
approx $\theta_2(\omega_i),\ i = 1,2,3,4$	0.9277	0.9387	0.9492	0.9591	0.9684	0.9773
approx $\theta_2(\omega_i),\ i = 5,6,7,8$	0.6144	0.5917	0.5664	0.5387	0.5089	0.4773
approx $\theta_2(\omega_i),\ i = 9,10,11,12$	0.0949	0.0699	0.0447	0.0197	-0.0046	-0.0277
approx $\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.9296	0.9394	0.9492	0.9589	0.9683	0.9773
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.6051	0.5846	0.5617	0.5363	0.5082	0.4773
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.0911	0.0677	0.0437	0.0195	-0.0045	-0.0277
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
approx θ_1	0.4524	0.4435	0.4334	0.4219	0.4087	0.3937
θ_1	0.4578	0.4469	0.4351	0.4223	0.4085	0.3937
R_0	19.4753	19.1222	18.6798	18.1484	17.5353	16.8563
approx V_0	5.1437	4.8885	4.6274	4.3597	4.0841	3.7997
V_0	5.1394	4.8855	4.6258	4.3590	4.0840	3.7997
approx C_0	6.1669	5.6753	5.1909	4.7156	4.2512	3.7997
C_0	6.1876	5.6844	5.1936	4.7157	4.2509	3.7997

Table 2.12: $\rho = 0.5$: First Order Approximation of Local Risk Minimizing Strategy in Physical Settlement, $\lambda_0 = 0.2, \lambda_1 = 0.4$

M	5	4	3	2	1	0
λ_0	0.2	0.2	0.2	0.2	0.2	0.2
λ_1	0.4	0.4	0.4	0.4	0.4	0.4
approx $\theta_2(\omega_i),\ i = 1, 2, 3, 4$	0.9239	0.9353	0.9464	0.9571	0.9674	0.9773
approx $\theta_2(\omega_i),\ i = 5, 6, 7, 8$	0.6106	0.5883	0.5636	0.5367	0.5078	0.4773
approx $\theta_2(\omega_i),\ i = 9, 10, 11, 12$	0.3365	0.3647	0.3960	0.4293	0.4639	0.5000
approx $\theta_2(\omega_i),\ i = 13, 14, 15, 16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 1, 2, 3, 4$	0.9259	0.9361	0.9465	0.9570	0.9673	0.9773
$\theta_2(\omega_i),\ i = 5, 6, 7, 8$	0.6014	0.5813	0.5591	0.5344	0.5072	0.4773
$\theta_2(\omega_i),\ i = 9, 10, 11, 12$	0.3486	0.3718	0.3990	0.4298	0.4637	0.5000
$\theta_2(\omega_i),\ i = 13, 14, 15, 16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
approx θ_1	0.5234	0.5217	0.5196	0.5172	0.5149	0.5125
θ_1	0.5342	0.5309	0.5270	0.5225	0.5176	0.5125
R_0	30.7850	31.0966	31.3037	31.3892	31.3431	31.1653
approx V_0	3.9402	3.6617	3.3912	3.1281	2.8707	2.6161
V_0	3.9255	3.6534	3.3871	3.1265	2.8703	2.6161
approx C_0	5.3100	4.7505	4.2010	3.6632	3.1358	2.6161
C_0	5.3525	4.7807	4.2202	3.6726	3.1383	2.6161

Table 2.13: $M = 5, \lambda_0 = 0.1, \lambda_1 = 0.9$: Coefficients of Cubic Equation

M	5
λ_0	0.1
λ_1	0.9
a_3	-0.04
a_2	5.19
a_1	-162.11
a_0	85.70

Table 2.14: $M = 5, \lambda_0 = 0.1, \lambda_1 = 0.9$: Solutions of Cubic Equation

θ_1	0.5379	54.00	71.81
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.9434	-17.72	-23.93
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.9434	-17.72	-23.93
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.00	0.00	0.00
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.00	0.00	0.00
$\eta_1(\omega_i),\ i = 1,2,3,4$	-92.90	-7,155	-13,671
$\eta_1(\omega_i),\ i = 5,6,7,8$	-92.90	-7,155	-13,671
$\eta_1(\omega_i),\ i = 9,10,11,12$	0.00	0.00	0.00
$\eta_1(\omega_i),\ i = 13,14,15,16$	0.00	0.00	0.00
V_0	6.52	12,202	21,641
C_0	7.97	26,782	47,424
R_0	0.00	0	89,316

Table 2.15: $M = 1, \lambda_0 = 0.2, \lambda_1 = 0.4$: Coefficients of Cubic Equation

M	1
λ_0	0.2
λ_1	0.4
a_3	-0.00011
a_2	0.22
a_1	-113.33
a_0	61.01

Table 2.16: $M = 1, \lambda_0 = 0.2, \lambda_1 = 0.4$: Solutions of Cubic Equation

θ_1	0.5389	1,007.98	1,024.57
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.9390	-16.49	-16.78
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.9390	-16.49	-16.78
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.00	0.00	0.00
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.00	0.00	0.00
$\eta_1(\omega_i),\ i = 1,2,3,4$	-92.59	-4,962	-5,159
$\eta_1(\omega_i),\ i = 5,6,7,8$	-92.59	-4,962	-5,159
$\eta_1(\omega_i),\ i = 9,10,11,12$	0.00	0.00	0.00
$\eta_1(\omega_i),\ i = 13,14,15,16$	0.00	0.00	0.00
V_0	5.55	619,663	640,240
C_0	5.84	1,635,691	1,689,988
R_0	0.00	0	32,186

2.6 Conclusion

This paper has investigated local risk minimization in an incomplete market in a discrete time setting where the price process of a hedging instrument follows a linear supply curve model with market impacts introduced by Cetin et al. [3] and Roch [13]. We have observed that, in the cases of a constant market impact parameter and a time dependent parameter with cash settlement, the unique local risk minimizing strategy is obtained recursively. Furthermore, in the case of a time dependent market impact parameter with physical settlement, the first order approximation of a local risk minimizing strategy expanded around the no market impact case is also obtained by backward induction.

Investigating the cases of more general types of liquidity models such as a non-linear supply curve or a different shape of dynamic limit order book is our next research topic.

Chapter 3

Representation of Self-Financing Strategy for Non-linear Supply Curve Model

3.1 Introduction

Liquidity is one of the most important issues in the financial market. In the financial crises, illiquid underlying assets increased disastrous derivatives losses. Cross currency basis spreads widened and currency volatilities surged to extreme levels when banks attempted to hedge similar derivatives positions. Consequently, the market lost liquidity and banks incurred huge losses, which led to the further turmoil. Banks should take into account liquidity of underlying assets which affects the cost of hedging in the derivatives position management.

For the sake of the significance of liquidity, various researches on liquidity risk have been done. Among these, there are two widely recognized approaches to incorporate liquidity of an underlying asset in derivatives pricing: the limit order book model and the supply curve model. The limit order book model expresses the state of the order book of the underlying asset, the density of orders at each price at a certain time or instantaneous time interval. Rodgers et al. [14] use the model for option pricing by assuming an absolutely continuous trading strategy for the risky asset. They obtain the option price by minimizing the mean-variance hedging error. On the other hand, the supply curve model expresses the average price of the underlying asset when one trades a certain amount. It was first introduced by Cetin et al. [3] to incorporate liquidity of the underlying asset into pricing (we refer the model as the CJP model hereafter). However, Cetin et al. proved that the total liquidity cost of the hedging can be zero by dividing the amount into infinitely small pieces in their setting where they assume a smooth supply curve at the origin. In order to avoid this counterintuitive result, Roch [13] introduced price impacts by trades of the hedger to the model.

This paper further extends the Roch-CJP supply curve model and provides an expression of the self-financing strategy. We extend the Roch's linear supply curve model with price impacts to include a general supply curve which has discontinuity at the origin. The discontinuity expresses the case where there is a bid-offer spread in the underlying asset. Such a situation is important especially when banks trade derivatives on illiquid underlying assets.

Moreover, in propositions, we present relations between the supply curve and the limit order book density. One of the propositions gives the maximum/minimum price of buying/selling when one trades a certain amount and the density of the limit order book corresponding to a given supply curve. With the maximum/minimum price of buying/selling obtained in the proposition, we extend the Roch model to include the case of a general supply curve.

The paper is organized as follows. In Section 2, we explain the supply curve model and the limit order book model, and show the relations of the two in the propositions. We also give examples of the relations for some supply curves. In Section 3, we show the expression of the self-financing strategy in the extended model in a theorem. We also give examples of the expression for some supply curves. In Section 4, we conclude.

3.2 Connection between Supply Curve and Limit Order Book

In this section, we present two propositions which show relations between the supply curve model and limit order book model.

A supply curve of a risky asset, $S(x)$, is an average price when one buys amount x of it. When x is negative, we interpret that one sells amount $|x|$ of the asset. Let $\rho(y)$ be the density of the order book. If one buys the asset by filling the orders from the price $S(0)$ to y, the total amount he/she buys is $\int_{S(0)}^{y} \rho(\eta)d\eta$, and the total cost he/she pays is $\int_{S(0)}^{y} \eta\rho(\eta)d\eta$. As $S(x)$ is the average price, the total cost he/she pays is also written as $xS(x)$, and the following relations hold.

$$\int_{S(0)}^{y} \rho(\eta)d\eta = x, \tag{3.1}$$

$$\int_{S(0)}^{y} \eta\rho(\eta)d\eta = xS(x). \tag{3.2}$$

Proposition 1 provides the supply curve corresponding to a given density of the limit order book.

Proposition 2. *Let $\rho(y) > 0, \rho(y)$, and $y\rho(y) \in L^1(\mathbf{R})$. Let $S(0) > 0$. Define $f(x)$ as the inverse function of $\int_{S(0)}^{y} \rho(\eta)d\eta$, and assume $f(x) \in L^1(\mathbf{R})$.*

Let $R_1(y)$, $R_2(y)$ be $R_1(y) := \int_{S(0)}^{y} \rho(\eta)d\eta$, $R_2(y) := \int_{S(0)}^{y} \eta\rho(\eta)d\eta$. Then $S(x) := \frac{1}{x}\int_0^x f(s)ds$, $(x \neq 0)$ satisfies

$$R_1(f(x)) = x \tag{3.3}$$

$$R_2(f(x)) = xS(x). \tag{3.4}$$

(Proof.) The first equation follows from the definition of $f(x)$. For the second equation,

$$\begin{aligned}
\frac{d}{dx}\int_{S(0)}^{f(x)} \eta\rho(\eta)d\eta &= (\frac{d}{dx}f(x))f(x)\rho(f(x)) = \frac{1}{\rho(f(x))}f(x)\rho(f(x)) \\
&= f(x) = \frac{d}{dx}\int_0^x f(s)ds, \qquad (3.5)\\
R_2(f(0)) = 0. & \qquad (3.6)
\end{aligned}$$

Therefore $R_2(f(x)) = xS(x)$. $\square$

Proposition 2 provides the density of the limit order book and the maximum price of buying for a given supply curve. The maximum price of buying is a key variable in defining the price impact by trades of the hedger in Section 3. We denote by $S^{(i)}(x)$ the i-th derivative of $S(x)$ with respect to x.

Proposition 3. *Let $S(x)$ be a $\mathbf{R}$-valued function defined on $\mathbf{R}$ which is of class $\mathcal{C}^2$ except for at the origin. Assume that $S(0-)$, $S(0+)$, $S^{(1)}(0-)$ and $S^{(1)}(0+)$ exist, $S(0-) < S(0) < S(0+)$, $S^{(1)}(x) > 0$, and $S^{(2)}(x)x + 2S^{(1)}(x) > 0$ for all $x \in \mathbf{R}\backslash\{0\}$.*

Define $f(x)$ and $\rho(y)$, functions on $\mathbf{R}$, as $f(x) := S(x) + S^{(1)}(x)x$, for all $x \in \mathbf{R}\backslash\{0\}$, $f(0) := S(0)$, $\rho(y) := \frac{d}{dy}f^{-1}(y)$, for all $y \in \mathbf{R}\backslash[S(0-), S(0+)]$, $\rho(y) = 0$, for all $y \in [S(0-), S(0+)]$.

Let $R_1(y)$, $R_2(y)$ be $R_1(y) := \int_{S(0)}^{y} \rho(\eta)d\eta$, $R_2(y) := \int_{S(0)}^{y} \eta\rho(\eta)d\eta$.

Then

$$R_1(f(x)) = x \tag{3.7}$$
$$R_2(f(x)) = xS(x) \tag{3.8}$$

hold.

(Proof.) In the case of $x = 0$, (1) and (2) hold.

First we note that the inverse function $f^{-1}(y)$ is defined for for all $y \in f(\mathbf{R}\backslash\{0\})$ and $\frac{d}{dy}f^{-1}(y)$ exists by the inverse function theorem. We denote the derivative by $\partial f^{-1}(y)$.

For $y \geq S(0+)$,

$$\begin{aligned} R_1(y) &= \int_{S(0)}^{y} \rho(\eta)d\eta \\ &= \int_{S(0)}^{S(0+)} \rho(\eta)d\eta + \int_{S(0+)}^{y} \rho(\eta)d\eta \\ &= \lim_{\epsilon\downarrow 0}[f^{-1}(\eta)]_{S(0+)+\epsilon}^{y} \\ &= f^{-1}(y) - \lim_{\epsilon\downarrow 0} f^{-1}(S(0+)+\epsilon) \\ &= f^{-1}(y) \end{aligned} \tag{3.9}$$

Therefore $R_1(f(x)) = x$, for all $x > 0$.

Next, for $y \geq S(0+)$,

$$\begin{aligned} R_2(y) &= \int_{S(0)}^{y} \eta\rho(\eta)d\eta \\ &= \int_{S(0)}^{S(0+)} \eta\rho(\eta)d\eta + \int_{S(0+)}^{y} \eta\rho(\eta)d\eta \\ &= \int_{S(0+)}^{y} \eta\rho(\eta)d\eta. \end{aligned} \tag{3.10}$$

We show that

$$\lim_{x\to 0+} R_2(f(x)) = \lim_{x\to 0+} xS(x), \tag{3.11}$$
$$\frac{d}{dx}R_2(f(x)) = \frac{d}{dx}(xS(x)). \tag{3.12}$$

For $x > 0$,

$$\begin{aligned} \lim_{x\to 0+} R_2(f(x)) &= R_2(S(0+)) \\ &= 0, \end{aligned} \tag{3.13}$$

$$\lim_{x \to 0+} xS(x) = 0, \tag{3.14}$$

$$\begin{aligned}\frac{d}{dx}R_2(f(x)) &= f(x)\rho(f(x))\frac{d}{dx}f(x) \\ &= f(x)\partial f^{-1}(f(x))\frac{d}{dx}f(x) \\ &= f(x)\frac{d}{dx}(f^{-1}(f(x))) \\ &= f(x),\end{aligned} \tag{3.15}$$

$$\begin{aligned}\frac{d}{dx}(xS(x)) &= S(x) + xS^{(1)}(x) \\ &= f(x).\end{aligned} \tag{3.16}$$

Therefore $R_2(f(x)) = xS(x)$, for all $x > 0$. The same is true for $x < 0$. $\square$

3.2.1 Examples

The above propositions show the links between the supply curve and the limit order book. In the supply curve model with price impacts introduced by Roch [13], the maximum/minimum price of buying/selling obtained in Proposition 2 for the general case plays an important role in defining the price impact. Let us see some examples. Let $\mathrm{sgn}(x)$ be a function defined as $\mathrm{sgn}(x) = +1(x > 0),\ -1(x < 0),\ 0(x = 0)$. Let $M, N > 0$.

Example 1. $S(x) = S(0) + Mx,\ f(x) = S(0) + 2Mx,\ \rho(y) = \frac{1}{2M}$.

Example 1 shows the case of a linear supply curve which appears in Roch [13]. This supply curve corresponds to the constant order book density of $\frac{1}{2M}$. The maximum price $S(0) + 2Mx$ is used to define the size of the price impact in Roch [13].

Example 2. $S(x) = S(0) + \mathrm{sgn}(x)Nx^2,\ f(x) = S(0) + \mathrm{sgn}(x)3Nx^2,\ \rho(y) = \frac{1}{\sqrt{12N|y-S(0)|}}$.

Example 2 shows the case of a non-linear, quadratic supply curve. The corresponding density function is $\frac{1}{\sqrt{12N|y-S(0)|}}$. This is an important example characterizing the case observed in practice, where the orders are centralizing around the mid price.

Example 3. $S(x) = S(0)+\mathrm{sgn}(x)K+Mx,\ f(x) = S(0)+\mathrm{sgn}(x)K+2Mx,\ \rho(y) = 0(y \in [S(0) - K, S(0) + K]),\ \frac{1}{2M}(\text{otherwise})$.

Example 3 shows the case where there is a bid-offer spread of $2K$. In the example, when one buys/sells the underlying asset, there is a spread of K from the mid price $S(0)$ regardless of the size of the trade. The density function indicates that there is no order between the prices $S(0) - K$ and $S(0) + K$, which is the observed in practice for illiquid assets.

3.3 Self-Financing Strategy Expression on Illiquid Underlying Asset

In this section, we show a self-financing strategy expression in the supply curve model with price impacts, which is an extension of the ones obtained in Cetin et al. [3] and Roch [13]. First, we introduce a self-financing strategy in discrete time in the supply curve model with price impacts. Then, in Theorem 1, we obtain the limit of the discretized strategy as the time interval tends to zero, and define it as the self-financing strategy in continuous time. Moreover, we show examples of the strategy, which include an important case where there is a bid-offer spread and the order book density is not uniform.

We consider an economy consisting of a money market account and a risky asset, and a portfolio of them numerated by the money market account. Let Y_t be the position of the money market account, X_t be the position of the risky asset. Let $S(t,x)$ be the supply curve of the risky asset at time t, that is, the average price when one buys amount x of the asset at time t. Hereafter for any process Z_t, we denote by $\Delta^N Z_{t_i}$, $Z_{t_i} - Z_{t_{i-1}}$, $(1 \leq i \leq N)$, and by $\Delta^N Z_{t_0}$, $Z_{t_0} - Z_{t_{0-}}$.

In the setting, the self-financing strategy is characterized by the following equation.

$$Y_{t_k} - Y_{t_{k-1}} + (X_{t_k} - X_{t_{k-1}})S(t_k, \Delta^N X_{t_k}) = 0,$$

This expresses that the cost for the change in the position of the risky asset is compensated by the change in the position of the money market account, and there is no money inflow from outside. By summing from $k = 1$ to N,

$$Y_{t_N} = Y_{t_0} - \sum_{k=1}^{N} (X_{t_k} - X_{t_{k-1}})S(t_k, \Delta^N X_{t_k}). \tag{3.17}$$

Moreover, the changes in the positions at the first trading time t_0 is

$$Y_{t_0} - Y_{t_{0-}} + (X_{t_0} - X_{t_{0-}})S(t_0, \Delta X_{t_0}) = 0.$$

Then we obtain

$$Y_{t_N} = Y_{t_{0-}} - \Delta X_{t_0} S(t_0, \Delta X_{t_0}) - \sum_{k=1}^{N} (X_{t_k} - X_{t_{k-1}})S(t_k, \Delta^N X_{t_k}). \tag{3.18}$$

Next, we introduce the price impact on the risky asset. Let $G(t,x)$ be a spread of the observed price process $S(t,x)$ from the mid price process $S(t,0)$, satisfying

$$S(t,x) = S(t,0) + G(t,x). \tag{3.19}$$

From Proposition 2, the maximum price of buying when one buys the amount $\Delta^N X_{t_k}$ of the risky asset is given by

$$S(t_k, \Delta^N X_{t_k}) + G^{(1)}(t_k, \Delta^N X_{t_k})\Delta^N X_{t_k}, \tag{3.20}$$

where $G^{(1)}(t,x)$ is the first derivative of $G(t,x)$ with respect to x.

We consider the situation where there are price impacts on the observed mid price process $S(t,0)$ after trades by the hedger. We assume that the size of the price impact is proportional to the difference between the observed mid price and the maximum price of buying, which is given by

$$\lambda(G(t_{k-1}, \Delta^N X_{t_{k-1}}) + G^{(1)}(t_{k-1}, \Delta^N X_{t_{k-1}})\Delta^N X_{t_{k-1}}) \tag{3.21}$$

where $0 \leq \lambda \leq 1$ is a constant.

Suppose that there is a price process $\tilde{S}_t$ driving the observed mid price process $S(t, 0)$. We call it the unaffected mid price process. We assume that the change of the observed mid price is the sum of the change of the unaffected mid price and the price impact by the hedges, which is given by

$$S(t_k, 0) = S(t_{k-1}, 0) + \Delta^N \tilde{S}_{t_k} + \lambda(G(t_{k-1}, \Delta^N X_{t_{k-1}}) + G^{(1)}(t_{k-1}, \Delta^N X_{t_{k-1}})\Delta^N X_{t_{k-1}}). \tag{3.22}$$

By summing over k,

$$\begin{aligned} S(t_k, \Delta^N X_{t_k}) =& \tilde{S}_{t_k} + \lambda \sum_{i=1}^{k} (G(t_{i-1}, \Delta^N X_{t_{i-1}}) + G^{(1)}(t_{i-1}, \Delta^N X_{t_{i-1}})\Delta^N X_{t_{i-1}}) \\ &+ G(t_k, \Delta^N X_{t_k}). \end{aligned} \tag{3.23}$$

In the setting of the general supply curve with price impacts, the limit in probability of the self-financing strategy in discrete time is given by Theorem 1, the main result of the paper. We define a self-financing strategy in continuous time as the limit.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions. We denote by $G_t^{(i)}(x)$ the i-th derivative of $G_t(x)$ with respect to x.

Theorem 2. *Let X_t, $\tilde{S}_t$ be semimartingales. Let $\{G_t(x)\}_{x\in\mathbf{R}}$ be a family of continuous semimartingales. We assume that (i) $G_t(x)$ is of class $\mathcal{C}^3$ on x. $G_t(0) = 0$ for all $t \geq 0$, (ii) $G_t^{(1)}(x) > 0$, $xG_t^{(2)}(x) + 2G_t^{(1)}(x) > 0$ for all $x \in \mathbf{R}$, and (iii) $G_t^{(1)}(x)$, $G_t^{(2)}(x)$, $G_t^{(3)}(x)$ are continuous with respect to (t, x). Let $\{\Delta^N\}_{N\in\mathbf{N}}$ be a sequence of partition of $[t_0, t] \subset [0, \infty)$, $\Delta^N : 0 \leq t_0 < t_1 < ... < t_N = t$ satisfying $\lim_{N\to\infty} |\Delta^N| = 0$, where $|\Delta^N|$ is the width of the partition defined by $|\Delta^N| := \max_{1\leq k\leq N} |t_k - t_{k-1}|$. We assume that Δ^N be a finer partition of Δ^{N-1} and $\sum_{k=1}^{N}(\Delta^N X_{t_k})^2$ converges to $[X, X]_t - [X, X]_{t_0}$, $\mathbf{P}-a.s.$. Let $0 \leq \lambda \leq 1$ be a constant. Define $S(t_k, \Delta^N X_{t_k}) := \tilde{S}_{t_k} + \lambda \sum_{i=1}^{k}(G(t_{i-1}, \Delta^N X_{t_{i-1}}) + G^{(1)}(t_{i-1}, \Delta^N X_{t_{i-1}})\Delta^N X_{t_{i-1}}) + G(t_k, \Delta^N X_{t_k})$, for $1 \leq k \leq N$, $S(t_0, 0) := \tilde{S}_{t_0}$.*

Let $S(t, 0)$ be a limit of $S(t_N, 0)$ in probability as $N \to \infty$.
Then $-\sum_{k=1}^{N} \Delta^N X_{t_k} S(t_k, \Delta^N X_{t_k})$ converges to below in probability as $N \to \infty$.

$$
\begin{aligned}
&+ X_{t_0} S(t_0, 0) - X_t S(t, 0) \\
&+ \int_{t_0+}^{t} X_{u-} d\tilde{S}_u \\
&- \int_{t_0+}^{t} G^{(1)}(u, 0) d[X, X]_u \\
&- \sum_{t_0<s\leq t} (G(s, \Delta X_s)\Delta X_s - G^{(1)}(s, 0)(\Delta X_s)^2) \\
&+ 2\lambda \int_{t_0}^{t-} X_{u-} G^{(1)}(u, 0) dX_u \\
&+ 2\lambda \int_{t_0}^{t-} X_{u-} d[G^{(1)}(., 0), X]_u \\
&+ 2\lambda \int_{t_0}^{t-} G^{(1)}(u, 0) d[X, X]_u + \frac{3}{2}\lambda \int_{t_0}^{t-} X_{u-} G^{(2)}(u, 0) d[X, X]_u \\
&+ \lambda \sum_{t_0\leq s<t} X_s (G(s, \Delta X_s) - G^{(1)}(s, 0)\Delta X_s - \frac{1}{2} G^{(2)}(s, 0)(\Delta X_s)^2) \\
&+ \lambda \sum_{t_0\leq s<t} X_s (G^{(1)}(s, \Delta X_s)\Delta X_s - G^{(1)}(s, 0)\Delta X_s - G^{(2)}(s, 0)(\Delta X_s)^2) \\
&+ \frac{3}{2}\lambda \sum_{t_0\leq s<t} G^{(2)}(s, 0)(\Delta X_s)^3. \qquad (3.24)
\end{aligned}
$$

(Proof.) For each $\omega \in \Omega$, for all $\epsilon > 0$, let $A(\epsilon, t)$ and $B(\epsilon, t)$ be subsets of $[0, t]$, satisfying $A(\epsilon, t) \cap B(\epsilon, t) = \emptyset$, $A(\epsilon, t) \cup B(\epsilon, t) = \{s \in [0, t] | \Delta X_s \neq 0\}$, $\# A(\epsilon, t) < \infty$, $\sum_{s \in B(\epsilon,t)} (\Delta X_s)^2 \leq \epsilon^2$. We can take such subsets because the number of jump times in $[0, t]$ is countable as X_s is càdlàg, and $\sum_{s \in A(\epsilon,t) \cup B(\epsilon,t)} (\Delta X_s)^2 \leq [X, X]_t < \infty$.

Hereafter we denote $\sum_{1 \leq k \leq N, (t_{k-1}, t_k] \cap A(\epsilon,t) \neq \emptyset}$ by $\sum_{k, A(\epsilon,t)}$, $\sum_{1 \leq k \leq N, (t_{k-1}, t_k] \cap A(\epsilon,t) = \emptyset}$ by $\sum_{k, B(\epsilon,t)}$, $\sum_{1 \leq k \leq N, (t_{k-2}, t_{k-1}] \cap A(\epsilon,t) \neq \emptyset}$ by $\sum_{k-1, A(\epsilon,t)}$, and $\sum_{1 \leq k \leq N, (t_{k-2}, t_{k-1}] \cap A(\epsilon,t) = \emptyset}$ by $\sum_{k-1, B(\epsilon,t)}$.

First we note that

$$
\begin{aligned}
& - \Delta^N X_{t_k} S(t_k, \Delta^N X_{t_k}) \\
& = -\Delta^N X_{t_k} S(t_k, 0) - \Delta^N X_{t_k} G(t_k, \Delta^N X_{t_k}) \\
& = X_{t_{k-1}} \Delta^N S(t_k, 0) - \Delta^N X_{t_k} G(t_k, \Delta^N X_{t_k}) - \Delta^N (X_{t_k} S(t_k, 0)).
\end{aligned} \tag{3.25}
$$

Then

$$
\begin{aligned}
& - \sum_{k=1}^{N} \Delta^N X_{t_k} S(t_k, \Delta^N X_{t_k}) \\
& = + \sum_{k=1}^{N} X_{t_{k-1}} \Delta^N \tilde{S}(t_k) \\
& + \lambda \sum_{k=1}^{N} X_{t_{k-1}} G(t_{k-1}, \Delta^N X_{t_{k-1}}) + \lambda \sum_{k=1}^{N} X_{t_{k-1}} G^{(1)}(t_{k-1}, \Delta^N X_{t_{k-1}}) \Delta^N X_{t_{k-1}} \\
& - \sum_{k=1}^{N} \Delta^N X_{t_k} G(t_k, \Delta^N X_{t_k}) \\
& - X_{t_N} S_{t_N} + X_{t_0} S_{t_0}.
\end{aligned} \tag{3.26}
$$

Next, by Taylor's theorem and by decomposing $\sum_{k=1}^{N} = \sum_{k, A(\epsilon,t)} + \sum_{k, B(\epsilon,t)}$,

$\sum_{k=1}^N = \sum_{k-1,A(\epsilon,t)} + \sum_{k-1,B(\epsilon,t)}$ for the terms including a remainder.

$$
\begin{aligned}
&-\sum_{k=1}^N \Delta^N X_{t_k} S(t_k, \Delta^N X_{t_k}) \\
&= +\sum_{k=1}^N X_{t_{k-1}} \Delta^N \tilde{S}(t_k) \\
&+ 2\lambda \sum_{k=1}^N X_{t_{k-1}} G^{(1)}(t_{k-1}, 0) \Delta^N X_{t_{k-1}} + \frac{3}{2}\lambda \sum_{k=1}^N X_{t_{k-1}} G^{(2)}(t_{k-1}, 0)(\Delta^N X_{t_{k-1}})^2 \\
&+ \lambda \sum_{k-1,B(\epsilon,t)} X_{t_{k-1}} R_1(t_{k-1}, \Delta^N X_{t_{k-1}}) + \lambda \sum_{k-1,B(\epsilon,t)} X_{t_{k-1}} R_2(t_{k-1}, \Delta^N X_{t_{k-1}}) \Delta^N X_{t_{k-1}} \\
&+ \lambda \sum_{k-1,A(\epsilon,t)} X_{t_{k-1}} (G(t_{k-1}, \Delta^N X_{t_{k-1}}) - G^{(1)}(t_{k-1}, 0) \Delta^N X_{t_{k-1}} - \frac{1}{2} G^{(2)}(t_{k-1}, 0)(\Delta^N X_{t_{k-1}})^2) \\
&+ \lambda \sum_{k-1,A(\epsilon,t)} X_{t_{k-1}} \Delta^N X_{t_{k-1}} (G^{(1)}(t_{k-1}, \Delta^N X_{t_{k-1}}) - G^{(1)}(t_{k-1}, 0) - G^{(2)}(t_{k-1}, 0) \Delta^N X_{t_{k-1}}) \\
&- \sum_{k=1}^N G^{(1)}(t_{k-1}, 0)(\Delta^N X_{t_k})^2 - \sum_{k=1}^N \Delta^N G^{(1)}(t_k, 0)(\Delta^N X_{t_k})^2 \\
&- \sum_{k,B(\epsilon,t)} R_3(t_k, \Delta^N X_{t_k}) \Delta^N X_{t_k} \\
&- \sum_{k,A(\epsilon,t)} \Delta^N X_{t_k} (G(t_k, \Delta^N X_{t_k}) - G^{(1)}(t_k, 0) \Delta^N X_{t_k}) \\
&- X_{t_N} S_{t_N} + X_{t_0} S_{t_0}, \qquad (3.27)
\end{aligned}
$$

where $R_1(t_{k-1}, \Delta^N X_{t_{k-1}})$, $R_2(t_{k-1}, \Delta^N X_{t_{k-1}})$, $R_3(t_k, \Delta^N X_{t_k})$ are the remainders defined by

$$R_1(t_{k-1}, \Delta^N X_{t_{k-1}}) := G(t_{k-1}, \Delta^N X_{t_{k-1}}) - G^{(1)}(t_{k-1}, 0) \Delta^N X_{t_{k-1}} - \frac{1}{2} G^{(2)}(t_{k-1}, 0)(\Delta^N X_{t_{k-1}})^2 \qquad (3.28)$$

$$R_2(t_{k-1}, \Delta^N X_{t_{k-1}}) := G^{(1)}(t_{k-1}, \Delta^N X_{t_{k-1}}) - G^{(1)}(t_{k-1}, 0) - G^{(2)}(t_{k-1}, 0) \Delta^N X_{t_{k-1}} \qquad (3.29)$$

$$R_3(t_k, \Delta^N X_{t_k}) := G(t_k, \Delta^N X_{t_k}) - G^{(1)}(t_k, 0) \Delta^N X_{t_k}, \qquad (3.30)$$

and satisfying the following evaluations.

$$|R_1(t_{k-1}, \Delta^N X_{t_{k-1}})| \leq \frac{1}{6} \sup_{0 \leq |x| \leq |\Delta^N X_{t_{k-1}}|} |G^{(3)}(t_{k-1}, x)||\Delta^N X_{t_{k-1}}|^3 \tag{3.31}$$

$$|R_2(t_{k-1}, \Delta^N X_{t_{k-1}})| \leq \frac{1}{2} \sup_{0 \leq |x| \leq |\Delta^N X_{t_{k-1}}|} |G^{(3)}(t_{k-1}, x)||\Delta^N X_{t_{k-1}}|^2 \tag{3.32}$$

$$|R_3(t_k, \Delta^N X_{t_k})| \leq \frac{1}{2} \sup_{0 \leq |x| \leq |\Delta^N X_{t_k}|} |G^{(2)}(t_k, x)||\Delta^N X_{t_k}|^2. \tag{3.33}$$

Next we evaluate the terms including the remainders. For the term including $R_3(t_k, \Delta^N X_{t_k})$,

$$\begin{aligned} |\sum_{k,B(\epsilon,t)} \Delta^N X_{t_k} R_3(t_k, \Delta^N X_{t_k})| &\leq \frac{1}{2} \sum_{k,B(\epsilon,t)} \sup_{0 \leq |x| \leq |\Delta^N X_{t_k}|} |G^{(2)}(t_k, x)||\Delta^N X_{t_k}|^3 \\ &\leq \frac{1}{2} K_2 \max_{k,B(\epsilon,t)} |\Delta^N X_{t_k}| \sum_{1 \leq k \leq N} |\Delta^N X_{t_k}|^2. \end{aligned} \tag{3.34}$$

As X_t is càdlàg, $\limsup_{N\to\infty} \max_{k,B(\epsilon,t)} |\Delta^N X_{t_k}| \leq \sup_{s \in B(\epsilon,t)} |\Delta X_s| \leq \epsilon$, From the assumption, $\limsup_{N\to\infty} \sum_{k=1}^N |\Delta^N X_{t_k}|^2 \leq [X, X]_t$. By taking $\limsup_{N\to\infty}$ for the both sides,

$$\limsup_{N\to\infty} |\sum_{k,B(\epsilon,t)} \Delta^N X_{t_k} R_3(t_k, \Delta^N X_{t_k})| \leq \frac{1}{2} \epsilon K_2 [X, X]_t. \tag{3.35}$$

Letting $\epsilon \downarrow 0$, we have

$$\lim_{\epsilon\downarrow 0} \limsup_{N\to\infty} |\sum_{k,B(\epsilon,t)} \Delta^N X_{t_k} R_3(t_k, \Delta^N X_{t_k})| = 0. \tag{3.36}$$

By noting $X_s(\omega)$ is càdlàg and bounded in $[0, t]$, for the terms including $R_1(t_{k-1}, \Delta^N X_{t_{k-1}})$, $R_2(t_{k-1}, \Delta^N X_{t_{k-1}})$, we have

$$\lim_{\epsilon\downarrow 0} \limsup_{N\to\infty} |\sum_{k-1,B(\epsilon,t)} X_{t_{k-1}} R_1(t_{k-1}, \Delta^N X_{t_{k-1}})| = 0, \tag{3.37}$$

$$\lim_{\epsilon\downarrow 0} \limsup_{N\to\infty} |\sum_{k-1,B(\epsilon,t)} X_{t_{k-1}} R_2(t_{k-1}, \Delta^N X_{t_{k-1}}) \Delta^N X_{t_{k-1}}| = 0. \tag{3.38}$$

From these and the evaluations of the remainder terms,

$$
\begin{aligned}
&+\lambda \sum_{k-1,B(\epsilon,t)} X_{t_{k-1}} R_1(t_{k-1}, \Delta^N X_{t_{k-1}}) + \lambda \sum_{k-1,B(\epsilon,t)} X_{t_{k-1}} R_2(t_{k-1}, \Delta^N X_{t_{k-1}}) \Delta^N X_{t_{k-1}} \\
&- \sum_{k,B(\epsilon,t)} R_3(t_k, \Delta^N X_{t_k}) \Delta^N X_{t_k} \\
&+\lambda \sum_{k-1,A(\epsilon,t)} X_{t_{k-1}} (G(t_{k-1}, \Delta^N X_{t_{k-1}}) - G^{(1)}(t_{k-1},0)\Delta^N X_{t_{k-1}} - \frac{1}{2} G^{(2)}(t_{k-1},0)(\Delta^N X_{t_{k-1}})^2) \\
&+\lambda \sum_{k-1,A(\epsilon,t)} X_{t_{k-1}} \Delta^N X_{t_{k-1}} (G^{(1)}(t_{k-1}, \Delta^N X_{t_{k-1}}) - G^{(1)}(t_{k-1},0) - G^{(2)}(t_{k-1},0)\Delta^N X_{t_{k-1}}) \\
&- \sum_{k,A(\epsilon,t)} \Delta^N X_{t_k} (G(t_k, \Delta^N X_{t_k}) - G^{(1)}(t_k,0)\Delta^N X_{t_k})
\end{aligned}
\tag{3.39}
$$

converges to

$$
\begin{aligned}
&- \sum_{t_0 < s \le t} (G(s, \Delta X_s)\Delta X_s - G^{(1)}(s,0)(\Delta X_s)^2) \\
&+ \lambda \sum_{t_0 \le s < t} X_s (G(s, \Delta X_s) - G^{(1)}(s,0)\Delta X_s - \frac{1}{2} G^{(2)}(s,0)(\Delta X_s)^2) \\
&+ \lambda \sum_{t_0 \le s < t} X_s (G^{(1)}(s, \Delta X_s)\Delta X_s - G^{(1)}(s,0)\Delta X_s - G^{(2)}(s,0)(\Delta X_s)^2)
\end{aligned}
\tag{3.40}
$$

in absolute convergence as $N \to \infty$.

We note that

$$
\begin{aligned}
&+ 2\lambda \sum_{k=1}^{N} X_{t_{k-1}} G^{(1)}(t_{k-1},0)\Delta^N X_{t_{k-1}} + \frac{3}{2}\lambda \sum_{k=1}^{N} X_{t_{k-1}} G^{(2)}(t_{k-1},0)(\Delta^N X_{t_{k-1}})^2 \\
&= +2\lambda \sum_{k=0}^{N-1} X_{t_{k-1}} \Delta^N G^{(1)}(t_k,0)\Delta^N X_{t_k} + 2\lambda \sum_{k=0}^{N-1} G^{(1)}(t_{k-1},0)(\Delta^N X_{t_k})^2 \\
&+ 2\lambda \sum_{k=0}^{N-1} \Delta^N G^{(1)}(t_k,0)(\Delta^N X_{t_k})^2 + 2\lambda \sum_{k=0}^{N-1} X_{t_{k-1}} G^{(1)}(t_{k-1},0)\Delta^N X_{t_k} \\
&+ \frac{3}{2}\lambda \sum_{k=0}^{N-1} X_{t_{k-1}} \Delta^N G^{(2)}(t_k,0)(\Delta^N X_{t_k})^2 + \frac{3}{2}\lambda \sum_{k=0}^{N-1} G^{(2)}(t_{k-1},0)(\Delta^N X_{t_k})^3 \\
&+ \frac{3}{2}\lambda \sum_{k=0}^{N-1} \Delta^N G^{(2)}(t_k,0)(\Delta^N X_{t_k})^3 + \frac{3}{2}\lambda \sum_{k=0}^{N-1} X_{t_{k-1}} G^{(2)}(t_{k-1},0)(\Delta^N X_{t_k})^2.
\end{aligned}
\tag{3.41}
$$

Then, from the fact that $G^{(1)}(t,0), G^{(2)}(t,0)$ are uniformly continuous in $[0,t]$ and $\limsup_{N\to\infty} \sum_{k=0}^{N-1}(\Delta^N X_{t_k})^2 \leq [X,X]_t < \infty$, $\sum_{k=0}^{N-1} \Delta^N G^{(1)}(t_k,0)(\Delta^N X_{t_k})^2$, $\sum_{k=0}^{N-1} X_{t_{k-1}} \Delta^N G^{(2)}(t_k,0)(\Delta^N X_{t_k})^2$,
and $\sum_{k=0}^{N-1} \Delta^N G^{(2)}(t_k,0)(\Delta^N X_{t_k})^3$ converge to 0 as $N \to \infty$.
We also note that $\sum_{k=0}^{N-1} G^{(2)}(t_{k-1},0)(\Delta^N X_{t_k})^3 \to \sum_{t_0 \leq s < t} G^{(2)}(s,0)(\Delta X_s)^3$.

From these, $-\sum_{k=1}^{N} \Delta^N X_{t_k} S(t_k, \Delta^N X_{t_k})$ converges to

$$
\begin{aligned}
&+ X_{t_0} S(t_0, 0) - X_t S(t, 0) \\
&+ \int_{t_0+}^{t} X_u d\tilde{S}_u \\
&- \int_{t_0+}^{t} G^{(1)}(u, 0) d[X, X]_u \\
&- \sum_{t_0 < s \le t} (G(s, \Delta X_s)\Delta X_s - G^{(1)}(s, 0)(\Delta X_s)^2) \\
&+ 2\lambda \int_{t_0}^{t-} X_{u-} G^{(1)}(u, 0) dX_u \\
&+ 2\lambda \int_{t_0}^{t-} X_{u-} d[G^{(1)}(., 0), X]_u \\
&+ 2\lambda \int_{t_0}^{t-} G^{(1)}(u, 0) d[X, X]_u + \frac{3}{2}\lambda \int_{t_0}^{t-} X_{u-} G^{(2)}(u, 0) d[X, X]_u \\
&+ \lambda \sum_{t_0 \le s < t} X_s (G(s, \Delta^N X_s) - G^{(1)}(s, 0)\Delta X_s - \frac{1}{2} G^{(2)}(s, 0)(\Delta X_s)^2) \\
&+ \lambda \sum_{t_0 \le s < t} X_s (G^{(1)}(s, \Delta^N X_s)\Delta^N X_s - G^{(1)}(s, 0)\Delta X_s - G^{(2)}(s, 0)(\Delta X_s)^2) \\
&+ \frac{3}{2}\lambda \sum_{t_0 \le s < t} G^{(2)}(s, 0)(\Delta X_s)^3
\end{aligned}
\tag{3.42}
$$

in probability as $N \to \infty$. □

The next proposition shows the expression of the observed mid price process $S(t,0)$ which is defined as the limit of the discretized one in Theorem 1.

Proposition 4. $S(t_N,0) := \tilde{S}_{t_N}+\lambda\sum_{i=1}^{N}(G(t_{i-1},\Delta^N X_{t_{i-1}})+G^{(1)}(t_{i-1},\Delta^N X_{t_{i-1}})\Delta^N X_{t_{i-1}})$ *converges to* $\tilde{S}_t+\lambda(2\int_{t_0}^{t-} G^{(1)}(s,0)dX_s+\frac{3}{2}\int_{t_0}^{t-} G^{(2)}(s,0)d[X,X]_s+2\int_{t_0}^{t-} d[G^{(1)}(.,0),X]_s+$ $\sum_{t_0\le s<t}(G(s,\Delta X_s)-G^{(1)}(s,0)\Delta X_s-\frac{1}{2}G^{(2)}(s,0)(\Delta X_s)^2)+\sum_{t_0\le s<t}(G^{(1)}(s,\Delta X_s)-$ $G^{(1)}(s,0)-G^{(2)}(s,0)\Delta X_s)\Delta X_s)$ *in probability as* $N\to\infty$.

(Proof.) First, $S(t_N,0)=\tilde{S}_{t_N}+\lambda\sum_{i=1}^{N}(G(t_{i-1},\Delta^N X_{t_{i-1}})+G^{(1)}(t_{i-1},\Delta^N X_{t_{i-1}})\Delta^N X_{t_{i-1}})$. By Taylor's theorem, $G(t_{i-1},\Delta^N X_{t_{i-1}})=G^{(1)}(t_{i-1},0)\Delta^N X_{t_{i-1}}+\frac{1}{2}G^{(2)}(t_{i-1},0)(\Delta^N X_{t_{i-1}})^2+$ $R^{(1)}(t_{i-1},\Delta^N X_{t_{i-1}})$, $G^{(1)}(t_{i-1},\Delta^N X_{t_{i-1}})=G^{(1)}(t_{i-1},0)+G^{(2)}(t_{i-1},0)\Delta^N X_{t_{i-1}}+$ $R^{(2)}(t_{i-1},\Delta^N X_{t_{i-1}})$.

$$\begin{aligned}\sum_{i=1}^{N}G(t_{i-1},\Delta^N X_{t_{i-1}}) &= \sum_{i=0}^{N-1}(G^{(1)}(t_{i-1},0)\Delta^N X_{t_i}\\ &\quad+\Delta^N G^{(1)}(t_i,0)\Delta^N X_{t_i}+\frac{1}{2}G^{(2)}(t_{i-1},0)(\Delta^N X_{t_i})^2\\ &\quad+\frac{1}{2}\Delta^N G^{(2)}(t_i,0)(\Delta^N X_{t_i})^2+R^{(1)}(t_i,\Delta^N X_{t_i})). \quad (3.43)\end{aligned}$$

converges to $\int_{t_0}^{t-} G^{(1)}(s,0)dX_s+\frac{1}{2}\int_{t_0}^{t-} G^{(2)}(s,0)d[X,X]_s+\int_{t_0}^{t-} d[G^{(1)}(.,0),X]_s$in probability except for the remainder terms.

$$\begin{aligned}\sum_{i=1}^{N}G^{(1)}(t_{i-1},\Delta^N X_{t_{i-1}})\Delta^N X_{t_{i-1}} &= \sum_{i=0}^{N-1}(G^{(1)}(t_{i-1},0)+\Delta^N G^{(1)}(t_i,0)\\ &\quad+G^{(2)}(t_{i-1},0)\Delta^N X_{t_i}+\Delta^N G^{(2)}(t_i,0)\Delta^N X_{t_i}\\ &\quad+R^{(2)}(t_i,\Delta^N X_{t_i}))\Delta^N X_{t_i}. \quad (3.44)\end{aligned}$$

converges to $\int_{t_0}^{t-} G^{(1)}(s,0)dX_s+\int_{t_0}^{t-} G^{(2)}(s,0)d[X,X]_s+\int_{t_0}^{t-} d[G^{(1)}(.,0),X]_s$ except for the remainder terms.

For the remainder terms,

$$\begin{aligned}\sum_{i=0}^{N-1}R^{(1)}(t_i,\Delta^N X_{t_i}) &= \sum_{i,A}R^{(1)}(t_i,\Delta^N X_{t_i})+\sum_{i,B}R^{(1)}(t_i,\Delta^N X_{t_i})\\ &= \sum_{i,A}(G(t_i,\Delta^N X_{t_i})-G^{(1)}(t_i,0)\Delta^N X_{t_i}-\frac{1}{2}G^{(2)}(t_i,0)(\Delta^N X_{t_i})^2)\\ &\quad+\sum_{i,B}R^{(1)}(t_i,\Delta^N X_{t_i}). \quad (3.45)\end{aligned}$$

converges to $\sum_{t_0 \le s < t}(G(s, \Delta X_s) - G^{(1)}(s,0)\Delta X_s - \frac{1}{2}G^{(2)}(s,0)(\Delta X_s)^2)$.

$$\begin{aligned}\sum_{i=0}^{N-1} R^{(2)}(t_i, \Delta^N X_{t_i})\Delta^N X_{t_i} &= \sum_{i,A} R^{(2)}(t_i, \Delta^N X_{t_i})\Delta^N X_{t_i} + \sum_{i,B} R^{(2)}(t_i, \Delta^N X_{t_i})\Delta^N X_{t_i} \\ &= \sum_{i,A}(G^{(1)}(t_i, \Delta^N X_{t_i}) - G^{(1)}(t_i, 0) - G^{(2)}(t_i, 0)(\Delta^N X_{t_i}))\Delta^N X_{t_i} \\ &+ \sum_{i,B} R^{(1)}(t_i, \Delta^N X_{t_i}). \qquad (3.46)\end{aligned}$$

converges to $\sum_{t_0 \le s < t}(G^{(1)}(s, \Delta X_s) - G^{(1)}(s,0) - G^{(2)}(s,0)\Delta X_s)\Delta X_s$, which completes the proof. □

Combining Theorem 1 and the next proposition on the limit of the Stieltjes sum with respect to a finite variation process, we obtain the following corollary which provides the self-financing strategy expression in the case where there is $2K_t$ of a bid-offer spread. Here we assume that the price impact in the case, where there is a bid-offer spread, is proportional to the difference between the maximum/minimum price of buying/selling and the lowest/highest offer/bid price.

Proposition 5. *Let K_t be a nonnegative continuous semimartingale and X_t be a finite variation semimartingale. Then $-\sum_{k=1}^{N} \Delta^N X_{t_k}(K_{t_k}\mathrm{sgn}(\Delta^N X_{t_k}))$ converges to $-\int_{t_0+}^{t} K_u|dX_u|$ in probability as $N \to \infty$.*

(Proof.)

$$\begin{aligned}\sum_{k=1}^{N} -\Delta^N X_{t_k}(K_{t_k}\mathrm{sgn}(\Delta^N X_{t_k})) &= \sum_{k=1}^{N} -K_{t_k}|\Delta^N X_{t_k}| \\ &= \sum_{k=1}^{N} -\Delta^N K_{t_k}|\Delta^N X_{t_k}| + \sum_{k=1}^{N} -K_{t_{k-1}}|\Delta^N X_{t_k}| \qquad (3.47)\end{aligned}$$

converges to $-\int_{t_0+}^{t} K_u|dX_u|$ in probability. □

Corollary 1. *Let K_t be a nonnegative continuous semimartingale, X_t be a finite variation semimartingale, and $\tilde{S}_t$ be a semimartingale. Let $\{G_t(x)\}_{x\in\mathbf{R}}$ be a family of continuous semimartingales. We assume that (i) $G_t(x)$ is of class $\mathcal{C}^3$ on x. $G_t(0) = 0$ for all $t \ge 0$, (ii) $G_t^{(1)}(x) > 0$, $xG_t^{(2)}(x) + 2G_t^{(1)}(x) > 0$ for all $x \in \mathbf{R}$, and (iii) $G_t^{(1)}(x)$, $G_t^{(2)}(x)$, $G_t^{(3)}(x)$ are continuous with respect to (t,x). Let $\{\Delta^N\}_{N\in\mathbf{N}}$ be a sequence of partition of $[t_0, t] \subset [0,\infty)$, $\Delta^N : 0 \le$*

$t_0 < t_1 < ... < t_N = t$ satisfying $\lim_{N\to\infty} |\Delta^N| = 0$, where $|\Delta^N|$ is the width of the partition defined by $|\Delta^N| := \max_{1\le k\le N} |t_k - t_{k-1}|$. We assume that Δ^N be a finer partition of Δ^{N-1} and $\sum_{k=1}^{N}(\Delta^N X_{t_k})^2$ converges to $[X,X]_t - [X,X]_{t_0}$, $\mathbf{P}-a.s.$. Let $0 \le \lambda \le 1$ be a constant. Define $S(t_k, \Delta^N X_{t_k}) := \tilde{S}_{t_k} + \lambda \sum_{i=1}^{k}(G(t_{i-1}, \Delta^N X_{t_{i-1}}) + G^{(1)}(t_{i-1}, \Delta^N X_{t_{i-1}})\Delta^N X_{t_{i-1}}) + G(t_k, \Delta^N X_{t_k}) + K_{t_k}\mathrm{sgn}(\Delta^N X_{t_k})$, for $1 \le k \le N$, $S(t_0, 0) := \tilde{S}_{t_0}$. Let $S(t,0)$ be a limit of $S(t_N, 0)$ in probability as $N \to \infty$. Then $-\sum_{k=1}^{N} \Delta^N X_{t_k} S(t_k, \Delta^N X_{t_k})$ converges to below in probability as $N \to \infty$.

$$
\begin{aligned}
&+ X_{t_0}S(t_0,0) - X_t S(t,0) \\
&+ \int_{t_0+}^{t} X_{u-} d\tilde{S}_u - \int_{t_0+}^{t} K_u |dX_u| \\
&- \sum_{t_0<s\le t} (G(s,\Delta X_s)\Delta X_s) \\
&+ 2\lambda \int_{t_0}^{t-} X_{u-} G^{(1)}(u,0) dX_u \\
&+ 2\lambda \sum_{t_0\le s<t} G^{(1)}(s,0)(\Delta X_s)^2 + \frac{3}{2}\lambda \sum_{t_0\le s<t} X_{s-} G^{(2)}(s,0)(\Delta X_s)^2 \\
&+ \lambda \sum_{t_0\le s<t} X_s (G(s,\Delta X_s) - G^{(1)}(s,0)\Delta X_s - \frac{1}{2} G^{(2)}(s,0)(\Delta X_s)^2) \\
&+ \lambda \sum_{t_0\le s<t} X_s (G^{(1)}(s,\Delta X_s)\Delta X_s - G^{(1)}(s,0)\Delta X_s - G^{(2)}(s,0)(\Delta X_s)^2) \\
&+ \frac{3}{2}\lambda \sum_{t_0\le s<t} G^{(2)}(s,0)(\Delta X_s)^3. \qquad (3.48)
\end{aligned}
$$

3.3.1 Examples

Let us see some examples. Theorem 1 and Corollary 1 include the self-financing strategy expressions which appear in Cetin et al. [3] and Roch [13] and an important case where there is a bid-offer spread and the supply curve is non-linear.

Example 1 shows that when there is no price impact, Theorem 1 derives the case in Cetin et al. [3].

Example 1 (Cetin et al. [3]). In Theorem 1, when $\lambda = 0$ (no price impact) and $X_{t_0-} = 0$,

$$
\begin{aligned}
Y_t =& Y_{t_0-} - X_{t_0} S(t_0, X_{t_0}) + X_{t_0} S(t_0, 0) - X_t S(t, 0) \\
&+ \int_{t_0+}^{t} X_{u-} d\tilde{S}_u - \int_{t_0+}^{t} G^{(1)}(u, 0) d[X, X]_u \\
&- \sum_{t_0 < s \le t} (G(s, \Delta X_s)\Delta X_s - G^{(1)}(s, 0)(\Delta X_s)^2) \\
=& Y_{t_0-} - X_t S(t, 0) \\
&+ \int_{t_0+}^{t} X_{u-} d\tilde{S}_u - \int_{t_0+}^{t} G^{(1)}(u, 0) d[X, X]^c_u \\
&- \sum_{t_0 \le s \le t} G(s, \Delta X_s)\Delta X_s.
\end{aligned} \tag{3.49}
$$

Example 2 shows that when the supply curve is linear, Theorem 1 derives the case in Roch [13]. In the second equality, we use Ito's formula for $X_t^2 M_t$.

Example 2 (Roch [13]). In Theorem 1, when $G(t, x) = M_t x$ (linear supply curve), where M_t is a continuous positive semimartingale, and $\tilde{S}_t$ is a continuous semimartingale,

$$
\begin{aligned}
Y_t =& Y_{t_0-} - \Delta X_{t_0} S(t_0, \Delta X_{t_0}) + X_{t_0} S(t_0, 0) - X_t S(t, 0) \\
&+ \int_{t_0+}^{t} X_{u-} d\tilde{S}_u - \int_{t_0+}^{t} M_u d[X, X]_u \\
&+ 2\lambda \int_{t_0}^{t-} M_u X_{u-} dX_u + 2\lambda \int_{t_0}^{t-} X_{u-} d[M, X]_u + 2\lambda \int_{t_0}^{t-} M_u d[X, X]_u \\
=& Y_{t_0-} + X_{t_0-}(S(t_0, 0) - \lambda X_{t_0-} M_{t_0}) - X_t(S(t+, 0) - \lambda X_t M_t) \\
&+ \int_{t_0}^{t} X_{u-} d\tilde{S}_u - \int_{t_0}^{t} (1 - \lambda) M_u d[X, X]_u - \lambda \int_{t_0}^{t} X_{u-}^2 dM_u.
\end{aligned} \tag{3.50}
$$

Finally, Example 3 shows an important case in practice where there is a bid-offer spread and the supply curve is non-linear.

Example 3. In Corollary 1 where there is $2K_t$ of a bid-offer spread, if X_t is absolutely continuous after t_0 with an expression of $X_t = X_{t_0} + \int_{t_0}^t \dot{X}_u du$ $(t_0 \le t)$, 0 $(0 \le t < t_0)$, and $G(t,x) = N_t x^3$ (cubic supply curve) where N_t is a positive continuous semimartingale,

$$\begin{aligned} Y_t =& Y_{t_0-} - (N_{t_0} X_{t_0}^4 + K_{t_0}|X_{t_0}|) - X_t S(t,0) \\ &+ \int_{t_0+}^t X_u d\tilde{S}_u - \int_{t_0}^t K_u |\dot{X}_u| du + 4\lambda N_{t_0} X_{t_0}^4. \end{aligned} \tag{3.51}$$

3.4 Conclusion

In this paper, we extended Roch's linear supply curve model with price impacts to include the case of a non-linear supply curve which is discontinuous at the origin. We showed an expression of the self-financing strategy of the extended model. Moreover, we presented the relations between the limit order book and the supply curve in the propositions. The extended model is general enough to include the important case where a bid-offer spread exists for the underlying asset and the order book density is not uniform. Such a situation has always been a key issue in practice.

Chapter 4

Foreign Exchange Option Pricing under Intervention

4.1 Introduction

In this study, we consider option pricing for a foreign exchange (FX) rate where interventions by an authority may take place when the rate approaches to a certain level at the down side. We formulate the forward FX model by a diffusion process which is stopped by a hitting time of an absorption boundary. Moreover, for a deterministic volatility case with a moving absorption whose level is described by an ordinary differential equation (ODE), we obtain closed-form pricing formulas for a European put option, a digital option, and Greeks of the put option. Furthermore, we show an extension of the pricing formula to the case where the intervention level is unknown. In numerical examples, we show option prices for different strikes under the absorption model and observe features of the model. We also investigate differences between the model prices and market prices of EURCHF options, where the options which have a low strike show lower present values with the model than the market.

Since the recent financial crisis, increasing intervention has been observed by authorities in FX markets such as USDJPY and EURCHF. For example, the Japanese government intervened in the USDJPY market to prevent an excessive appreciation of JPY against USD in order to protect exporting industries in the country. Similarly in September 2011, in response to the CHF appreciation against EUR after the Eurozone crisis, the Swiss National Bank intervened in the EURCHF market by announcing unlimited interventions to defend the spot rate of 1.20.

While we observe interventions in FX markets, FX options with large payoffs in a low strike area have been actively traded. For instance, structured notes which embed a short position of a USDJPY low strike option have been sold to global investors since the late 1990s. In particular, after the financial crisis in 2008, the investors incurred large mark-to-market losses on the notes due to the drop in USDJPY. As a result, there have been many disputes on the mark-to-market values of the notes. Considering the effect of interventions, it is imagined that the present values of such skewed payoffs differ critically. For this reason, we consider option pricing with a model which incorporates the effect of intervention.

We formulate the forward FX rate model by a diffusion process stopped by an absorption boundary, which satisfies the arbitrage-free condition. We remark that if the boundary is reflection, it is not arbitrage-free and the option price cannot be defined uniquely. In detail, there exists an arbitrage strategy where one buys the FX when it hits the reflection boundary. We also show an extension to the case of an unknown boundary, which corresponds to a situation where the market does not know the target level of intervention such as USDJPY.

Related researches such as Lipton (2001), De Jong et al. (2001), Larsen et al. (2007) and Veestraeten (2013) deal with bounds of a FX rate. Particularly, they were motivated by the transition period of European currencies into Euro when the movement of the exchange rates were regulated in a band. The first three formulate a FX rate in a band whose diffusion part converges to zero if the rate approaches an upper or a lower boundary. Lipton (2001) gives a closed-form solution for a hyperbolic form of the local volatility function for the forward FX rate. De Jong et al. (2001) and Larsen et al. (2007) consider a mean reverting process in a band for a European option, but do not give a closed-form solution. They show the option prices in numerical examples by simulation. Veestraeten (2013) models a FX rate in a band by using a reflecting Brownian motion and shows closed-form solution for the case of a constant volatility and fixed intervention levels, however the model does not satisfy the arbitrage-free condition in the usual sense. Our study considers option pricing under one-sided intervention by a diffusion model with a time dependent or an unknown absorption boundary, which satisfies the arbitrage-free condition. It is motivated by the interventions in the currency market such as USDJPY and EURCHF after the financial crisis. We give closed-form solutions for the option prices and Greeks for a deterministic volatility and numerical examples.

From a mathematical view point, our study deals with expectations of a functional of a stopped process by a hitting time of a deterministic boundary. Kunitomo and Ikeda (1992), Omberg (1987) and Takahashi et al. (2001) also study the similar problems, but in different topics: Kunitomo and Ikeda

(1992) for double knock-out options, Omberg (1987) for exercise boundary in American put option, and Takahashi et al. (2001) for the structural approach for convertible bond pricing. While they consider a case of a constant volatility and a shape of the boundary which does not cross the strike level, our study deals with a deterministic volatility process and allows the boundary to cross the strike level. Moreover, we express the stopped process in a form of SDE and prove the pathwise uniqueness of the strong solution, although the SDE does not satisfy the Lipshitz condition.

The paper is organized as follows: after the next section describes the formulation of the absorption model, Section 3 derives the closed-form formulas for the European and digital put option prices, and Greeks of the European option. Section 4 presents an extension of the absorption model to the case of an unknown boundary. Section 5 provides numerical examples of the European option prices with the model. Specifically, we investigate the features of the absorption model and differences between the model prices and the market prices in the case of EURCHF options. Finally, Section 6 concludes. Appendix 1 presents the derivation of the closed-form formulas. Appendix 2 proves the pathwise uniqueness of the stochastic differential equation (SDE) for the stopped forward FX rate process by the absorption boundary. Appendix 3 shows the derivation of the Greeks of the European put option.

4.2 Settings of the Absorption Model

In this section, we formulate the absorption model as a forward FX rate process under the domestic forward martingale measure. We define the forward FX rate process as a stopped process by a hitting time of an absorption boundary set at the downside of the initial forward FX rate.

4.2.1 Formulation of the Absorption Model

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions. Let $\{W_t\}_{0 \leq t \leq T}$ be a $\mathcal{F}_t$-Brownian motion. $\mathbf{P}$ is the domestic forward martingale measure. $F_{0,T} > 0$ is an initial value of the forward FX rate for settlement T. σ_t is the volatility process of the forward FX rate, which is deterministic, bounded and strictly positive on $[0, T]$.

First, we define the absorption boundary in terms of the log return of the forward FX rate. Let $a_0 < 0$ be the initial value of the absorption. We assume that the absorption moves downward in proportion to the variance of the log return, that is,

$$a_t := a_0 - \alpha \int_0^t \sigma_s^2 ds, \tag{4.1}$$

where $\alpha \geq 0$.

Next, we set X_t as the log return of the forward FX rate, and τ as its hitting time of the absorption a_t as follows:

$$X_t := -\frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s, \tag{4.2}$$

$$\tau := \inf\{t \geq 0 | X_t = a_t\}, \quad \inf \emptyset = \infty. \tag{4.3}$$

Then, we define the forward FX rate $F_{t,T}$ as the stopped process of $F_{0,T}\exp(X_t)$ by the hitting time τ,

$$F_{t,T} := F_{0,T}\exp(X_{\tau \wedge t}). \tag{4.4}$$

Here, we note that $\{F_{t,T}\}_{0 \leq t \leq T}$ is a $\mathcal{F}_t$-martingale under $\mathbf{P}$ by the optional sampling theorem; therefore, this model is arbitrage-free.

4.2.2 Equivalent Expression of the Forward FX Rate Process by Stochastic Differential Equation

In this subsection, we give an equivalent expression of the stopped forward FX rate process by SDE. First, the corresponding absorption A_t in terms of the forward FX rate is defined as follows:

$$A_t := F_{0,T}\exp(a_t). \tag{4.5}$$

This satisfies the ordinary differential equation (ODE):

$$\frac{d}{dt}A_t = -\alpha\sigma_t^2 A_t, \tag{4.6}$$

and the hitting time τ is rewritten in terms of $F_{t,T}$ and A_t:

$$\tau = \inf\{t \geq 0 | F_{t,T} = A_t\}. \tag{4.7}$$

Here, we note that $A_0 < F_{0,T}$ and α is interpreted as a speed parameter of the movement of A_t. It is easy to see that $\{F_{t,T}\}_{0 \leq t \leq T}$ is a strong solution of the SDE with the initial value $F_{0,T}$:

$$\begin{aligned} dF_{t,T} &= \sigma_t 1_{\{\tau^F > t\}} F_{t,T} dW_t, \\ \tau^F &= \inf\{t \geq 0 | F_{t,T} = A_t\}. \end{aligned} \tag{4.8}$$

The pathwise uniqueness of the SDE also holds. We give the proof in Appendix 2. In the above SDE, we observe that the forward FX rate $F_{t,T}$ stops once it hits the absorption boundary A_t. By letting a_0 to $-\infty$, we can regard the absorption model as a Black–Sholes model for deterministic volatility.

Remark 4. *In the case where the domestic and foreign instantaneous interest rate processes are deterministic, we can rewrite the absorption model in the spot FX rate term. Let r^d_t and r^f_t be the domestic and foreign instantaneous interest rate processes, respectively. Let S_t be the spot FX rate process defined by $S_t := F_{t,T}\exp(\int_t^T(r^f_s - r^d_s)ds)$. Let A^S_t be the absorption boundary corresponding to the spot rate process defined by $A^S_t := A_t\exp(\int_t^T(r^f_s - r^d_s)ds)$. Then, according to Ito's formula, the SDE for S_t is as follows:*

$$\begin{aligned} dS_t &= (r^d_t - r^f_t)S_t dt + \sigma_t S_t 1_{\{\tau^S > t\}} dW_t, \\ \tau^S &= \inf\{t \geq 0 | S_t = A^S_t\}. \end{aligned} \tag{4.9}$$

Here, note that the domestic forward martingale measure $\mathbf{P}$ coincides with the domestic risk neutral measure when the instantaneous rates are deterministic. From the SDE, we see that after the spot FX rate hits the boundary A^S_t, the spot rate behaves as a deterministic process which evolves at the rate of the interest rates differential so that S_T realises the forward FX rate at time τ such that

$$\begin{aligned} S_T &= S_\tau \exp(\int_\tau^T (r^d_s - r^f_s)ds) \\ &= F_{\tau,T}. \end{aligned} \tag{4.10}$$

Remark 5. *When domestic and foreign interest rate processes are deterministic, the SDE for the spot FX rate process under the physical measure $\tilde{\mathbf{P}}$ with the initial value S_0 defined by*

$$\begin{aligned} dS_t &= (r^d_t - r^f_t - \sigma_t\lambda_t 1_{\{\tau^S > t\}})S_t dt + \sigma_t 1_{\{\tau^S > t\}} S_t d\tilde{W}_t, \\ \tau^S &= \inf\{t \geq 0 | S_t = A^S_t\} \end{aligned} \tag{4.11}$$

corresponds to the SDE (2.9) under the domestic risk neutral measure $\mathbf{P}$, where λ_t is a $\{\mathcal{F}_t\}$-adapted process satisfying $\mathbf{E}\left[\exp\left(\frac{1}{2}\int_0^T \lambda_t^2 dt\right)\right] < \infty$. Note that we can interpret λ_t as the market price of risk.

In fact, we can derive SDE (2.9) under the domestic risk neutral measure $\mathbf{P}$ *from the SDE (2.11) under the physical measure* $\tilde{\mathbf{P}}$. *If we define* $\mathbf{P}$ *by*

$$\frac{d\mathbf{P}}{d\tilde{\mathbf{P}}} = \exp\left(-\frac{1}{2}\int_0^T \lambda_t^2 dt + \int_0^T \lambda_s d\tilde{W}_s\right),$$

then

$$W_t = \tilde{W}_t - \int_0^t \lambda_s ds$$

is a $\{\mathcal{F}_t\}$-Brownian motion under $\mathbf{P}$, and

$$\begin{aligned}
dS_t &= (r_t^d - r_t^f - \sigma_t\lambda_t 1_{\{\tau^S>t\}})S_t dt + \sigma_t 1_{\{\tau^S>t\}} S_t d\tilde{W}_t,\\
&= (r_t^d - r_t^f - \sigma_t\lambda_t 1_{\{\tau^S>t\}})S_t dt + \sigma_t 1_{\{\tau^S>t\}} S_t (dW_t + \lambda_t dt),\\
&= (r_t^d - r_t^f)S_t dt + \sigma_t 1_{\{\tau^S>t\}} S_t dW_t,\\
\tau^S &= \inf\{t \geq 0 | S_t = A_t^S\}.
\end{aligned}$$

4.3 Closed-form formulas for European Put and Digital Options under the Absorption Model

In this section, we show closed-form pricing formulas for a European put option and a digital put option under the absorption model introduced in Section 2. We also show Greeks of the European put option in the corollary. Hereafter, we fix $T > 0$, and denote the cumulative distribution function of the standard normal distribution by $N(x)$.

Theorem 3. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions. Let $\{W_t^{\mathbf{P}}\}_{0\leq t\leq T}$ be a $\mathcal{F}_t$-Brownian motion. Let $\{\sigma_t\}_{0\leq t\leq T}$ be a bounded deterministic process on $[0,T]$ which is strictly positive on the interval. Let $F_{0,T} > 0$, $\alpha \geq 0$, $a_0 < 0$ and $K > 0$.*

We define an absorption boundary $\{A_t\}_{0\leq t\leq T}$ as the unique solution of an ODE,

$$\begin{aligned}
\frac{dA_t}{dt} &= -\alpha\sigma_t^2 A_t,\\
A_0 &= F_{0,T}\exp(a_0),
\end{aligned} \tag{4.12}$$

and a forward FX rate process $\{F_{t,T}\}_{0\leq t\leq T}$ as the pathwise unique strong solution of a SDE with the initial value $F_{0,T}$,

$$\begin{aligned}
dF_{t,T} &= \sigma_t 1_{\{\tau>t\}} F_{t,T} dW_t,\\
\tau &= \inf\{t \geq 0 | F_{t,T} = A_t\}. \ (\inf\emptyset = \infty)
\end{aligned} \tag{4.13}$$

Define t_0 as

$$t_0 := \inf\{t \geq 0 | A_t \leq K\}. \ (\inf\emptyset = \infty) \tag{4.14}$$

If (i) $\alpha = 0,\ a_0 < k$, or (ii) $\alpha > 0,\ 0 \le t_0 < T$, then we have
(1) the European put option price

$$\begin{aligned}
\mathbf{E}^{\mathbf{P}}[(K - F_{T,T})^+] = K\Bigg[& N\left(\frac{-a_0 + m(t_0)\theta}{\sqrt{m(t_0)}}\right) - \exp(2a_0\theta) N\left(\frac{a_0 + m(t_0)\theta}{\sqrt{m(t_0)}}\right) \\
& - N\left(\frac{-\max(\tilde{k}, a_0) + m(T)\theta}{\sqrt{m(T)}}\right) \\
& + \exp(2a_0\theta) N\left(\frac{-\max(\tilde{k}, a_0) + 2a_0 + m(T)\theta}{\sqrt{m(T)}}\right)\Bigg] \\
& - F_{0,T}\Bigg[N\left(\frac{-a_0 + m(t_0)\tilde{\theta}}{\sqrt{m(t_0)}}\right) - \exp(2a_0\tilde{\theta}) N\left(\frac{a_0 + m(t_0)\tilde{\theta}}{\sqrt{m(t_0)}}\right) \\
& - N\left(\frac{-\max(\tilde{k}, a_0) + m(T)\tilde{\theta}}{\sqrt{m(T)}}\right) \\
& + \exp(2a_0\tilde{\theta}) N\left(\frac{-\max(\tilde{k}, a_0) + 2a_0 + m(T)\tilde{\theta}}{\sqrt{m(T)}}\right)\Bigg],
\end{aligned} \tag{4.15}$$

(2) the digital put option price

$$\begin{aligned}
\mathbf{E}^{\mathbf{P}}[1_{\{F_{T,T} < K\}}] = & N\left(\frac{-a_0 + m(t_0)\theta}{\sqrt{m(t_0)}}\right) - \exp(2a_0\theta) N\left(\frac{a_0 + m(t_0)\theta}{\sqrt{m(t_0)}}\right) \\
& - N\left(\frac{-\max(\tilde{k}, a_0) + m(T)\theta}{\sqrt{m(T)}}\right) \\
& + \exp(2a_0\theta) N\left(\frac{-\max(\tilde{k}, a_0) + 2a_0 + m(T)\theta}{\sqrt{m(T)}}\right),
\end{aligned} \tag{4.16}$$

(3) Distribution function of τ

$$\mathbf{P}(t < \tau) = N\left(\frac{-a_0 + m(t)\theta}{\sqrt{m(t)}}\right) - \exp(2a_0\theta) N\left(\frac{a_0 + m(t)\theta}{\sqrt{m(t)}}\right), \tag{4.17}$$

where

$$
\begin{aligned}
t &> 0, \\
m(t) &:= \int_0^t \sigma_s^2 ds, \\
k &:= \log\left(\frac{K}{F_{0,T}}\right), \\
\tilde{k} &:= \log\left(\frac{K}{F_{0,T}}\right) + \alpha m(T), \\
\theta &:= -\frac{1}{2} + \alpha, \\
\tilde{\theta} &:= \frac{1}{2} + \alpha.
\end{aligned}
$$

Proof. See Appendix 1.

Corollary 2. *Let $n(x) := \exp(-\frac{1}{2}x^2)$. Let $F_{t,T}^{\epsilon}$ be the pathwise unique strong solution of the SDE substituting $\sigma_t^{\epsilon} := \sigma_t + \epsilon$ for σ_t in the ODE (3.1) and the SDE (3.2). Let $p(t) := \int_0^t \sigma_s ds$. Then the Greeks of the European put option are given as follows.*

(1)Delta

$$
\begin{aligned}
&\frac{\partial}{\partial F_{0,T}} \mathbf{E}^{\mathbf{P}}\left[(K - F_{T,T})^+\right] \\
&= -N\left(\frac{-a_0 + m(t_0)\tilde{\theta}}{\sqrt{m(t_0)}}\right) + \exp(2a_0\tilde{\theta})N\left(\frac{a_0 + m(t_0)\tilde{\theta}}{\sqrt{m(t_0)}}\right) \\
&\quad + N\left(\frac{-\max(a_0, \tilde{k}) + m(T)\tilde{\theta}}{\sqrt{m(T)}}\right) \\
&\quad - \exp(2a_0\tilde{\theta})N\left(\frac{-\max(a_0, \tilde{k}) + 2a_0 + m(T)\tilde{\theta}}{\sqrt{m(T)}}\right),
\end{aligned}
\tag{4.18}
$$

(2) Gamma

$$
\begin{aligned}
&\frac{\partial^2}{\partial F_{0,T}^2} \mathbf{E}\left[(K - F_{T,T})^+\right] \\
&= \frac{1}{F_{0,T}\sqrt{m(T)}}\left[n\left(\frac{-\tilde{k} + m(T)\tilde{\theta}}{\sqrt{m(T)}}\right)\right. \\
&\quad \left. - \exp(2a_0\tilde{\theta})n\left(\frac{-\tilde{k} + 2a_0 + m(T)\tilde{\theta}}{\sqrt{m(T)}}\right)\right] 1_{\{\tilde{k} > a_0\}},
\end{aligned}
\tag{4.19}
$$

(3) Theta

$$
\begin{aligned}
&\frac{\partial}{\partial \epsilon}\mathbf{E}\left[(K - F_{T-\epsilon,T})^{+}\right]\Big|_{\epsilon=0} \\
&= \frac{\sigma_T^2 K}{2\sqrt{m(T)^3}}\left[\left(\max(a_0,\tilde{k}) + m(T)\theta\right) n\left(\frac{-\max(a_0,\tilde{k}) + m(T)\theta}{\sqrt{m(T)}}\right)\right. \\
&\quad \left. + \exp(2a_0\theta)\left(-\max(a_0,\tilde{k}) + 2a_0 - m(T)\theta\right) n\left(\frac{-\max(a_0,\tilde{k}) + 2a_0 + m(T)\theta}{\sqrt{m(T)}}\right)\right] \\
&\quad - \frac{\sigma_T^2 F_{0,T}}{2\sqrt{m(T)^3}}\left[\left(\max(a_0,\tilde{k}) + m(T)\tilde{\theta}\right) n\left(\frac{-\max(a_0,\tilde{k}) + m(T)\tilde{\theta}}{\sqrt{m(T)}}\right)\right. \\
&\quad \left. + \exp(2a_0\tilde{\theta})\left(-\max(a_0,\tilde{k}) + 2a_0 - m(T)\tilde{\theta}\right) n\left(\frac{-\max(a_0,\tilde{k}) + 2a_0 + m(T)\tilde{\theta}}{\sqrt{m(T)}}\right)\right],
\end{aligned}
\tag{4.20}
$$

(4) Vega

$$
\begin{aligned}
&\frac{\partial}{\partial\epsilon}\mathbf{E}[(K - F^{\epsilon}_{T,T})^{+}]\big|_{\epsilon=0} \\
&= K\Bigg[\frac{p(t_0)}{\sqrt{m(t_0)^3}}\left(a_0 + m(t_0)\theta\right) n\left(\frac{-a_0 + m(t_0)\theta}{\sqrt{m(t_0)}}\right) \\
&- \exp(2a_0\theta)\frac{p(t_0)}{\sqrt{m(t_0)^3}}\left(-a_0 + m(t_0)\theta\right) n\left(\frac{a_0 + m(t_0)\theta}{\sqrt{m(t_0)}}\right) \\
&- \frac{p(T)}{\sqrt{m(T)^3}}\left(\max(a_0, \tilde{k}) + m(T)\theta\right) n\left(\frac{-\max(a_0, \tilde{k}) + m(T)\theta}{\sqrt{m(T)}}\right) \\
&+ \exp(2a_0\theta)\frac{p(T)}{\sqrt{m(T)^3}}\left(\max(a_0, \tilde{k}) - 2a_0 + m(T)\theta\right) n\left(\frac{-\max(a_0, \tilde{k}) + 2a_0 + m(T)\theta}{\sqrt{m(T)}}\right)\Bigg] \\
&- F_{0,T}\Bigg[\frac{p(t_0)}{\sqrt{m(t_0)^3}}\left(a_0 + m(t_0)\tilde{\theta}\right) n\left(\frac{-a_0 + m(t_0)\tilde{\theta}}{\sqrt{m(t_0)}}\right) \\
&- \exp(2a_0\tilde{\theta})\frac{p(t_0)}{\sqrt{m(t_0)^3}}\left(-a_0 + m(t_0)\tilde{\theta}\right) n\left(\frac{a_0 + m(t_0)\tilde{\theta}}{\sqrt{m(t_0)}}\right) \\
&- \frac{p(T)}{\sqrt{m(T)^3}}\left(\max(a_0, \tilde{k}) + m(T)\tilde{\theta}\right) n\left(\frac{-\max(a_0, \tilde{k}) + m(T)\tilde{\theta}}{\sqrt{m(T)}}\right) \\
&+ \exp(2a_0\tilde{\theta})\frac{p(T)}{\sqrt{m(T)^3}}\left(\max(a_0, \tilde{k}) - 2a_0 + m(T)\tilde{\theta}\right) n\left(\frac{-\max(a_0, \tilde{k}) + 2a_0 + m(T)\tilde{\theta}}{\sqrt{m(T)}}\right)\Bigg].
\end{aligned} \tag{4.21}
$$

Proof. See Appendix 3.

4.4 Extension to the Case of Unknown Boundary

In this section, we extend the option pricing formula for the European put option to the case where there are some possible levels for the absorption boundary. We assume that the random level of the absorption boundary is independent of the filtration of the FX rate process. In particular, this corresponds to the situation in which the market does not know at which level the authority intervenes.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $\mathbf{P}$ be the domestic forward martingale measure. Let $\{\mathcal{F}^W_t\}_{0\le t\le T}$ an augmented filtration generated by a one-dimensional Brownian motion W_t, that is, $\mathcal{F}_t^W := \sigma(W_s; 0 \leq s \leq t) \vee \mathcal{N}$ where $\mathcal{N}$ is a collection of sets of measure zero.

First, let X be a random variable which takes values in $\{1, 2, .., N\}$, and suppose that X and $\mathcal{F}_T^W$ are independent. Let $\{\mathcal{G}_t\}$ be a filtration generated by $\{\mathcal{F}_t^W\}_{0 \leq t \leq T}$ and X, that is, $\mathcal{G}_t := \mathcal{F}_t^W \vee \sigma(X)$.

Next, let $F_{t,T}$ be the unique strong solution of the SDE

$$dF_{t,T} = \sigma_t F_{t,T} dW_t, \tag{4.22}$$

with an initial value $F_{0,T}$. Let $\{\tau_i\}_{1 \leq k \leq N}$ be a set of hitting times defined by

$$\tau_i := \inf\{t \geq 0 | F_{t,T} = A_t^i\}, \ (1 \leq i \leq N). \tag{4.23}$$

where we set $\inf \emptyset = \infty$.

Furthermore, we define a $\mathcal{G}_t$- stopping time $\tilde{\tau}$ as follows:

$$\tilde{\tau} := \sum_{i=1}^{N} \tau_i 1_{\{X=i\}}, \tag{4.24}$$

and the forward FX rate process as $\{F_{\tilde{\tau} \wedge t,T}\}_{0 \leq t \leq T}$.

Then the European put option price becomes

$$\begin{aligned} \mathbf{E}\left[(K - F_{\tilde{\tau} \wedge T,T})^+\right] &= \mathbf{E}\left[\sum_{i=1}^{N} (K - F_{\tau_i \wedge T})^+ 1_{\{X=i\}}\right] \\ &= \sum_{i=1}^{N} \mathbf{E}\left[(K - F_{\tau_i \wedge T})^+\right] \mathbf{P}(X = i). \end{aligned} \tag{4.25}$$

We note that the model satisfies the arbitrage-free condition, since the stopped process $\{F_{\tilde{\tau} \wedge t,T}\}_{0 \leq t \leq T}$ is a $\mathcal{G}_t$-martingale by the optional sampling theorem.

We can interpret the case as the situation in which there are N possible intervention boundaries A_t^i $(1 \leq i \leq N)$ and the boundary is determined independently from the information of the FX rate behavior. $\mathbf{P}(X = i)$ is the probability of the event where the authority intervenes at the boundary A_t^i. As we can observe in (3.14), the European option price for the random absorption boundary is the weighted sum of the expectations whose closed-form formula we obtained in the previous section.

4.5 Numerical Examples

This section provides numerical examples of the European put option prices under the absorption model. Moreover, we observe the features of the absorption model and investigate differences between the model prices and the market prices of EURCHF options, in which spot market the Swiss National Bank intervenes to defend the 1.20 level.

First, Table 1 shows European put option prices under the absorption model (denoted by Model Put) and market prices of EURCHF European put options as of 14 May 2013 (denoted by Market Put). Market Vol and Model Vol represent Black–Sholes implied volatility of the model price and the market price respectively. Specifically, the parameters of the absorption model are set to be $F_{0,T} = 1.238, A_0 = 1.198, \sigma_t = 6.69\%, A_0^S = 1.200, A_T = A_T^S = 1.168, \alpha = 5.50$. The option maturity is set to be 1 year, while the strike prices are 1.134, 1.189, 1.238, 1.293, 1.368 which correspond to 10 Delta Put (10DP), 25 Delta Put (25DP), Delta Neutral (DN), 25 Delta Call (25DC), 10 Delta Call (10DC) respectively where the each moneyness is calculated by the Black–Sholes delta.

Then, as expected, we observe that the absorption model shows the upward sloping smile curve in the high strike area without assuming any specific volatility process such as local or stochastic volatility. The market also presents an upward sloping smile curve because of the market expectation that the FX movement in the downside area is limited. Moreover, the model shows zero or marginal prices for the low strike options, such as 25DP and 10DP, due to the absorption boundary. In contrast, the market shows larger present values for these options, on account of the market expectation that the intervention by the central bank to support the rate of 1.20 may fail.

Table 1: 1 Year EURCHF European put option prices by the absorption model and the market as of 14 May 2013 for 5 strikes (10DP, 25DP, DN, 25DC, and 10DC).

Strike	Strike Level	Market Put	Model Put	Market Vol	Model Vol
10DP	1.134	0.0039	0.0000	6.89%	0.00%
25DP	1.189	0.0106	0.0017	5.92%	3.10%
DN	1.238	0.0282	0.0282	5.72%	5.72%
25DC	1.293	0.0669	0.0669	6.39%	6.39%
10DC	1.368	0.1345	0.1323	7.67%	6.65%

Second, Table 2 presents European put option prices under the model (denoted by Model Put) with parameters $F_{0,T} = 1.190, \sigma_t = 12.80\%, A_0 = 1.185, A_T = 0.111, \alpha = 145$, and the EURCHF put option prices in the market as of 13 September 2011 (denoted by Market Put), which is one week after the announcement of the unlimited interventions by the Swiss National Bank. The option maturity is set to be 1 year, while the strikes are 0.915,1.071,1.190,1.279,1.384 which correspond to 10DP, 25DP, DN, 25DC, 10DC respectively. In this case where α is set to be high compared to the case in Table 1, it is observed that the model shows high implied volatilities in the low strike area because of the fast moving boundary.

On the other hand, the market prices show that the two low strike options, 25DP and 10DP, were traded at significantly higher implied volatilities than the absorption model, which was due to the market expectation that the unlimited intervention to support the level was impossible. It is notable that, as a consequence, from the trade date to the expiry date, the Swiss National Bank succeeded in keeping the spot rate above the 1.20 level by intervention. In such a case, buying the low strike options resulted in losing the premiums of the options without earning from gamma trading. As seen from this example, as long as intervention is effective, it is important to incorporate intervention into modeling to evaluate options which have a payoff in a low strike area.

Table 2: 1 Year EURCHF European put option prices by the absorption model and the market as of 13 September 2011 for 5 strikes (10DP, 25DP, DN, 25DC, and 10DC).

Strike	Strike Level	Market put	Model put	Market vol	Model vol
10DP	0.915	0.0115	0.0006	21.24%	12.00%
25DP	1.071	0.0266	0.0105	15.72%	10.75%
DN	1.190	0.0568	0.0419	11.98%	8.83%
25DC	1.279	0.1091	0.1091	10.70%	10.70%
10DC	1.384	0.2012	0.2012	11.61%	11.61%

4.6 Conclusion

This paper has proposed FX options pricing under the existence of intervention by absorption modeling. We have shown closed-form formulas for European put option and digital put option prices and Greeks of the European put option under the forward FX rate model which assumes a diffusion process stopped by a hitting time of the deterministic absorption boundary moving downward. Moreover, we have presented an extension of the model to the case where the absorption level is unknown which corresponds to the case of USDJPY for example. In the numerical examples, we have observed the features of the absorption model, and investigated the differences between the model prices and the market prices in the case of EURCHF options, in which spot market the central bank has been intervening to support the spot rate above 1.20. Extending the absorption model to the case of a stochastic volatility with a stochastic absorption boundary which moves correlated with the FX rate is our next research topic.

Chapter 5

Concluding Remarks

In the thesis, we have investigated option pricing under market restrictions such as liquidity cost, market impact, and intervention in FX market. In the presence of liquidity cost and market impact in a hedging instrument in an incomplete market, we have derived an explicit expression of a local risk minimizing strategy. Furthermore, in continuous time setting, we have derived an expression of a wealth process when the price process of a risky asset follows non-linear version of supply curve with market impacts. Finally, under the setting of existence of intervention in FX market, which is formulated by absorption, we have derived closed-form formulas for price and Greeks of FX European options.

Although we have formulated the restrictions in rather simple ways, the restrictions in practice are more complicated. For example, the density of the order book is stochastic and related to the economic situation. Moreover, on the market impact, there are multiple traders who may affect on the market by their hedging. Intervention levels in the FX market is stochastic which may change depending on the FX level and economic status. These are important and interesting topics, and mathematical formulation and solution will remain for future studies.

Appendix A

Proof of Theorem 3.1

In this appendix, we provide the proof of Theorem 3.1.

First note that

$$A_t = F_{0,T} \exp\left(a_0 - \alpha \int_0^t \sigma_s^2 ds\right). \tag{A.1}$$

Define

$$\begin{gathered} X_t := -\frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s^{\mathbf{P}}, \\ a_t := \log\left(\frac{A_t}{F_{0,T}}\right) = a_0 - \alpha \int_0^t \sigma_s^2 ds, \\ k := \log\left(\frac{K}{F_{0,T}}\right), \end{gathered} \tag{A.2}$$

Then,

$$\tau = \inf\{t \geq 0 | X_t = a_t\},$$

and

$$F_{t,T} = F_{0,T} \exp\left(-\frac{1}{2}\int_0^{t\wedge\tau} \sigma_s^2 ds + \int_0^{t\wedge\tau} \sigma_s dW_s^{\mathbf{P}}\right) \tag{A.3}$$

by Appendix 2.

$$\begin{aligned} \mathbf{E}^{\mathbf{P}}[(K - F_{T,T})1_{\{F_{T,T}<K\}}] &= K\mathbf{E}^{\mathbf{P}}[1_{\{F_{T,T}<K\}}] - F_{0,T}\mathbf{E}^{\mathbf{P}}[\exp(X_{T\wedge\tau})1_{\{F_{T,T}<K\}}] \\ &= K\mathbf{E}^{\mathbf{P}}[1_{\{F_{T,T}<K\}}] - F_{0,T}\mathbf{E}^{\mathbf{Q}}[1_{\{F_{T,T}<K\}}], \end{aligned} \tag{A.4}$$

where we set

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = \exp\left(-\frac{1}{2}\int_0^T \sigma_s^2 1_{\{s<\tau\}} ds + \int_0^T \sigma_s 1_{\{s<\tau\}} dW_s^{\mathbf{P}}\right).$$

$$1_{\{\tau\le T\}} 1_{\{X_\tau<k\}} = 1_{\{t_0<\tau\le T\}}. \tag{A.5}$$

Proof of Lemma 1. In the case (i), noting that $t_0 = 0$ and $\tau > 0$, if $\tau \le T$ and $X_\tau < k$, then $0 < \tau \le T$. Conversely, if $0 < \tau \le T$, then $X_\tau = a_\tau = a_0 < k$.

In the case (ii), if $\tau \le T$ and $X_\tau < k$, then $X_\tau = a_\tau < k$. If $t_0 = 0$, then $t_0 = 0 < \tau$. If $t_0 > 0$, then $t_0 < \tau$ since $a_0 > k = a_{t_0} > a_\tau$ and a_t is strictly decreasing. Conversely, if $t_0 < \tau \le T$, then $X_\tau = a_\tau < a_{t_0} \le k$. $\square$

Note that by Girsanov's theorem,

$$W_t^{\mathbf{Q}} = W_t^{\mathbf{P}} - \int_0^t \sigma_s 1_{\{s<\tau\}} ds$$

is a $\mathcal{F}_t$-Brownian motion under $\mathbf{Q}$.

By Lemma 1,

$$\begin{aligned}
1_{\{F_{T,T}<K\}} &= 1_{\{X_{T\wedge\tau}<k\}} \\
&= 1_{\{\tau\le T\}} 1_{\{X_\tau<k\}} + 1_{\{\tau>T\}} 1_{\{X_T<k\}} \\
&= 1_{\{t_0<\tau\le T\}} + 1_{\{\tau>T\}} 1_{\{X_T<k\}} \\
&= 1_{\{t_0<\tau\}} - 1_{\{\tau>T\}} 1_{\{X_T\ge k\}} \\
&= 1_{\{\min_{0\le t\le t_0}(X_t-a_t)>0\}} - 1_{\{\min_{0\le t\le T}(X_t-a_t)>0\}} 1_{\{X_T\ge k\}} \\
&= 1_{\{\min_{0\le t\le t_0}(\theta\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s^{\mathbf{P}})>a_0\}} \\
&\quad - 1_{\{\min_{0\le t\le T}(\theta\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s^{\mathbf{P}})>a_0\}} 1_{\{\theta\int_0^T \sigma_s^2 ds + \int_0^T \sigma_s dW_s^{\mathbf{P}}\ge\tilde{k}\}} \\
&= 1_{\{\min_{0\le t\le t_0}(\tilde\theta\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s^{\mathbf{Q}})>a_0\}} \\
&\quad - 1_{\{\min_{0\le t\le T}(\tilde\theta\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s^{\mathbf{Q}})>a_0\}} 1_{\{\tilde\theta\int_0^T \sigma_s^2 ds + \int_0^T \sigma_s dW_s^{\mathbf{Q}}\ge\tilde{k}\}}.
\end{aligned} \tag{A.6}$$

The last equality holds as $t_0 < \tau$ and $T < \tau$ in each term.

Since $m(t)$ is strictly increasing,

$$m : [0, T] \to [0, m(T)]$$

is bijective.

For $0 \le t \le m(T)$, define

$$\begin{aligned}
\hat{W}_t^{\mathbf{P}} &:= \int_0^{m^{-1}(t)} \sigma_s dW_t^{\mathbf{P}}, \\
\hat{W}_t^{\mathbf{Q}} &:= \int_0^{m^{-1}(t)} \sigma_s dW_t^{\mathbf{Q}}, \\
\hat{\mathcal{F}}_t &:= \mathcal{F}_{m^{-1}(t)}.
\end{aligned}$$

Then, $\hat{W}_t^{\mathbf{P}}$ and $\hat{W}_t^{\mathbf{Q}}$ are $\hat{\mathcal{F}}_t$-Brownian motions under $\mathbf{P}$ and $\mathbf{Q}$, respectively.

We define new probability measures $\mathbf{P}^*$ and $\mathbf{Q}^*$by

$$\frac{d\mathbf{P}^*}{d\mathbf{P}} = \exp\left(-\frac{1}{2}\theta^2 m(T) - \theta \hat{W}^{\mathbf{P}}_{m(T)}\right),$$
$$\frac{d\mathbf{Q}^*}{d\mathbf{Q}} = \exp\left(-\frac{1}{2}\tilde{\theta}^2 m(T) - \tilde{\theta} \hat{W}^{\mathbf{Q}}_{m(T)}\right).$$

Then, by Girsanov's theorem, for $0 \le t \le m(T)$,

$$\hat{W}_t^{\mathbf{P}^*} := \hat{W}_t^{\mathbf{P}} + \theta t,$$
$$\hat{W}_t^{\mathbf{Q}^*} := \hat{W}_t^{\mathbf{Q}} + \tilde{\theta} t$$

are $\hat{\mathcal{F}}_t$-Brownian motions under $\mathbf{P}^*$ and $\mathbf{Q}^*$, respectively.

Then,

$$\begin{aligned}
&\mathbf{E}^{\mathbf{P}}[1_{\{F_{T,T}<K\}}] \\
&= \mathbf{E}^{\mathbf{P}^*}\left[1_{\{\min_{0\le t\le m(t_0)} \hat{W}_t^{\mathbf{P}^*} > a_0\}} \exp\left(-\frac{1}{2}\theta^2 m(t_0) + \theta \hat{W}^{\mathbf{P}^*}_{m(t_0)}\right)\right] \\
&\quad - \mathbf{E}^{\mathbf{P}^*}\left[1_{\{\min_{0\le t\le m(T)} \hat{W}_t^{\mathbf{P}^*} > a_0\}} 1_{\{\hat{W}^{\mathbf{P}^*}_{m(T)} \ge \tilde{k}\}} \exp\left(-\frac{1}{2}\theta^2 m(T) + \theta \hat{W}^{\mathbf{P}^*}_{m(T)}\right)\right],
\end{aligned} \tag{A.7}$$

and

$$\begin{aligned}
&\mathbf{E}^{\mathbf{Q}}[1_{\{F_{T,T}<K\}}] \\
&= \mathbf{E}^{\mathbf{Q}^*}\left[1_{\{\min_{0\le t\le m(t_0)} \hat{W}_t^{\mathbf{Q}^*} > a_0\}} \exp\left(-\frac{1}{2}\tilde{\theta}^2 m(t_0) + \tilde{\theta} \hat{W}^{\mathbf{Q}^*}_{m(t_0)}\right)\right] \\
&\quad - \mathbf{E}^{\mathbf{Q}^*}\left[1_{\{\min_{0\le t\le m(T)} \hat{W}_t^{\mathbf{Q}^*} > a_0\}} 1_{\{\hat{W}^{\mathbf{Q}^*}_{m(T)} \ge \tilde{k}\}} \exp\left(-\frac{1}{2}\tilde{\theta}^2 m(T) + \tilde{\theta} \hat{W}^{\mathbf{Q}^*}_{m(T)}\right)\right].
\end{aligned} \tag{A.8}$$

By the reflection principle of the Brownian motion, for a Brownian motion W_t starting from zero, the joint probability density function for $(W_{m(T)}, \min_{0\le s\le m(T)} W_s)$ is as follows:

$$f(x,y) = \frac{2(x-2y)}{\sqrt{2\pi m(T)^3}} \exp\left(-\frac{(x-2y)^2}{2m(T)}\right) \tag{A.9}$$

for $y \le 0, \ y \le x$.

Set

$$F(x,y) := \frac{1}{\sqrt{2\pi m(T)}} \exp\left(-\frac{(x-2y)^2}{2m(T)}\right), \tag{A.10}$$

then

$$\frac{\partial}{\partial y} F(x,y) = f(x,y),\ F(x,x) = F(x,0).$$

Hence

$$\begin{aligned}
&\mathbf{E}^{\mathbf{P}^*}\left[1_{\{\min_{0\le t\le m(T)} \hat{W}^{\mathbf{P}^*} > a_0\}} 1_{\{\hat{W}^{\mathbf{P}^*}_{m(T)} \ge \tilde{k}\}} \exp\left(-\frac{1}{2}\theta^2 m(T) + \theta \hat{W}^{\mathbf{P}^*}_{m(T)}\right)\right] \\
&= \int_{\min(0,\max(a_0,\tilde{k}))}^{0} \left(\int_{a_0}^{x} f(x,y) \exp\left(-\frac{1}{2}\theta^2 m(T) + \theta x\right) dy\right) dx \\
&\quad + \int_{\max(0,\tilde{k})}^{\infty} \left(\int_{a_0}^{0} f(x,y) \exp\left(-\frac{1}{2}\theta^2 m(T) + \theta x\right) dy\right) dx \\
&= \int_{\min(0,\max(a_0,\tilde{k}))}^{0} \exp\left(-\frac{1}{2}\theta^2 m(T) + \theta x\right) (F(x,x) - F(x,a_0)) dx \\
&\quad + \int_{\max(0,\tilde{k})}^{\infty} \exp\left(-\frac{1}{2}\theta^2 m(T) + \theta x\right) (F(x,0) - F(x,a_0)) dx \\
&= \int_{\max(a_0,\tilde{k})}^{\infty} \exp\left(-\frac{1}{2}\theta^2 m(T) + \theta x\right) (F(x,0) - F(x,a_0)) dx \\
&= N\left(\frac{-\max(a_0,\tilde{k}) + m(T)\theta}{\sqrt{m(T)}}\right) - \exp(2a_0\theta) N\left(\frac{-\max(a_0,\tilde{k}) + 2a_0 + m(T)\theta}{\sqrt{m(T)}}\right).
\end{aligned} \tag{A.11}$$

From this, it is easy to see that

$$\begin{aligned}
&\mathbf{E}^{\mathbf{P}^*}\left[1_{\{\min_{0\le t\le m(t_0)} \hat{W}^{\mathbf{P}^*}_t > a_0\}} \exp\left(-\frac{1}{2}\theta^2 m(t_0) + \theta \hat{W}^{\mathbf{P}^*}_{m(t_0)}\right)\right] \\
&= N\left(\frac{-a_0 + m(t_0)\theta}{\sqrt{m(t_0)}}\right) - \exp(2a_0\theta) N\left(\frac{a_0 + m(t_0)\theta}{\sqrt{m(t_0)}}\right),
\end{aligned} \tag{A.12}$$

and (3) also follows.

Therefore,

$$
\begin{aligned}
&\mathbf{E}^{\mathbf{P}}[(K-F_{T,T})^+] \\
&= K\mathbf{E}^{\mathbf{P}}[1_{\{F_{T,T}<K\}}] - F_{0,T}\mathbf{E}^{\mathbf{Q}}[1_{\{F_{T,T}<K\}}] \\
&= K\Bigg[N\left(\frac{-a_0+m(t_0)\theta}{\sqrt{m(t_0)}}\right) - \exp(2a_0\theta)N\left(\frac{a_0+m(t_0)\theta}{\sqrt{m(t_0)}}\right) \\
&\quad - N\left(\frac{-\max(a_0,\tilde{k})+m(T)\theta}{\sqrt{m(T)}}\right) + \exp(2a_0\theta)N\left(\frac{-\max(a_0,\tilde{k})+2a_0+m(T)\theta}{\sqrt{m(T)}}\right)\Bigg] \\
&\quad - F_{0,T}\Bigg[N\left(\frac{-a_0+m(t_0)(\tilde{\theta})}{\sqrt{m(t_0)}}\right) - \exp(2a_0\tilde{\theta})N\left(\frac{a_0+m(t_0)\tilde{\theta}}{\sqrt{m(t_0)}}\right) \\
&\quad - N\left(\frac{-\max(a_0,\tilde{k})+m(T)\tilde{\theta}}{\sqrt{m(T)}}\right) + \exp(2a_0\tilde{\theta})N\left(\frac{-\max(a_0,\tilde{k})+2a_0+m(T)\tilde{\theta}}{\sqrt{m(T)}}\right)\Bigg],
\end{aligned}
\tag{A.13}
$$

and

$$
\begin{aligned}
&\mathbf{E}^{\mathbf{P}}[1_{\{F_{T,T}<K\}}] \\
&= N\left(\frac{-a_0+m(t_0)\theta}{\sqrt{m(t_0)}}\right) - \exp(2a_0\theta)N\left(\frac{a_0+m(t_0)\theta}{\sqrt{m(t_0)}}\right) \\
&\quad - N\left(\frac{-\max(a_0,\tilde{k})+m(T)\theta}{\sqrt{m(T)}}\right) + \exp(2a_0\theta)N\left(\frac{-\max(a_0,\tilde{k})+2a_0+m(T)\theta}{\sqrt{m(T)}}\right),
\end{aligned}
\tag{A.14}
$$

which completes the proof. $\square$

Appendix B

Pathwise Uniqueness of the Strong Solution

In this appendix, we show the uniqueness of the strong solution for the SDE,

$$\begin{aligned} dX_t &= \sigma_s X_t 1_{\{\tau^X > t\}} dW_t, \\ \tau^X &:= \inf\{t \geq 0 | X_t = A_t\}, \ (\inf \emptyset = \infty) \end{aligned} \tag{B.1}$$

with the initial condition $X_0 = x$, where x is a positive constant, σ_t is bounded, $A_0 \neq x$, and A_t is continuous.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions. Let W_t be a one-dimensional $\{\mathcal{F}_t\}$-Brownian motion.

We first see that

$$\begin{aligned} X_t &= x \exp\left(-\frac{1}{2}\int_0^{t\wedge\tau} \sigma_s^2 ds + \int_0^{t\wedge\tau} \sigma_s dW_s\right), \\ \tau &:= \inf\left\{t \geq 0 \middle| -\frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s = \log\left(\frac{A_t}{x}\right)\right\}, \ (\inf \emptyset = \infty) \end{aligned} \tag{B.2}$$

is a strong solution of the SDE since

$$\tau = \inf\{t \geq 0 | X_t = A_t\} = \tau^X,$$

and by Ito's formula,

$$
\begin{aligned}
&X_0 + \int_0^t \sigma_s X_s 1_{\{\tau^X > s\}} dW_s \\
&= x + \int_0^{t\wedge\tau^X} \sigma_s X_s dW_s \\
&= x\left(1 + \int_0^{t\wedge\tau^X} \sigma_s \exp\left(-\frac{1}{2}\int_0^{s\wedge\tau} \sigma_u^2 du + \int_0^{s\wedge\tau} \sigma_u dW_u\right) dW_s\right) \\
&= x\exp\left(-\frac{1}{2}\int_0^{t\wedge\tau} \sigma_s^2 ds + \int_0^{t\wedge\tau} \sigma_s dW_s\right) \\
&= X_t.
\end{aligned}
$$

Next, we show the pathwise uniqueness of the SDE. Let X_t, Y_t be solutions of the SDE on the same probability space and for the same Brownian motion W_t.

Suppose that $X_0 = Y_0 = x$ and

$$
\begin{aligned}
X_t &= X_0 + \int_0^t \sigma_s X_s 1_{\{\tau^X > s\}} dW_s,\ 0 \le \forall t < \infty,\ \mathbf{P} - a.s. \\
Y_t &= Y_0 + \int_0^t \sigma_s Y_s 1_{\{\tau^Y > s\}} dW_s,\ 0 \le \forall t < \infty,\ \mathbf{P} - a.s.
\end{aligned}
\tag{B.3}
$$

where

$$
\begin{aligned}
\tau^X &:= \inf\{t \ge 0 | X_t = A_t\}\ (\inf\emptyset = \infty) \\
\tau^Y &:= \inf\{t \ge 0 | Y_t = A_t\}\ (\inf\emptyset = \infty).
\end{aligned}
\tag{B.4}
$$

We first define

$$
\begin{aligned}
\tilde{X}_t &:= X_{t\wedge\tau^Y}, \\
\tilde{Y}_t &:= Y_{t\wedge\tau^X},
\end{aligned}
$$

and show

$$
\tilde{X}_t = \tilde{Y}_t, 0 \le t < \infty, \mathbf{P} - a.s. \tag{B.5}
$$

Note that

$$
\begin{aligned}
\tilde{X}_t &= X_0 + \int_0^{t\wedge\tau^X\wedge\tau^Y} \sigma_s X_s dW_s \\
&= \tilde{X}_0 + \int_0^{t\wedge\tau^X\wedge\tau^Y} \sigma_s \tilde{X}_s dW_s,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{Y}_t &= Y_0 + \int_0^{t\wedge\tau^X\wedge\tau^Y} \sigma_s Y_s dW_s \\
&= \tilde{Y}_0 + \int_0^{t\wedge\tau^X\wedge\tau^Y} \sigma_s \tilde{Y}_s dW_s.
\end{aligned}
$$

For any $N > 0$, set a hitting time τ_N and its stopped processes as

$$
\begin{aligned}
\tau_N &:= \inf\{t \geq 0 || \tilde{X}_t| \vee |\tilde{Y}_t| \geq N\} \ (\inf \emptyset = \infty), \\
\tilde{X}_t{}^N &:= \tilde{X}_{t\wedge\tau_N}, \\
\tilde{Y}_t{}^N &:= \tilde{Y}_{t\wedge\tau_N}.
\end{aligned}
$$

Then, there exists some constant $K > 0$ and

$$
\begin{aligned}
\mathbf{E}[(\tilde{X}_t{}^N - \tilde{Y}_t{}^N)^2] &= \mathbf{E}\left[\left(\int_0^{t\wedge\tau_X\wedge\tau_Y\wedge\tau_N} \sigma_s(\tilde{X}_s^N - \tilde{Y}_s^N)dW_s\right)^2\right] \\
&= \mathbf{E}\left[\int_0^{t\wedge\tau_X\wedge\tau_Y\wedge\tau_N} \sigma_s^2(\tilde{X}_s^N - \tilde{Y}_s^N)^2 ds\right] \\
&\leq K\mathbf{E}\left[\int_0^t (\tilde{X}_s^N - \tilde{Y}_s^N)^2 ds\right] \\
&= K\int_0^t \mathbf{E}[(\tilde{X}_s^N - \tilde{Y}_s^N)]^2 ds.
\end{aligned}
$$

By Gronwall's inequality, we see

$$
\tilde{X}_t^N = \tilde{Y}_t^N, \ \mathbf{P} - a.s.
$$

Since $\lim_{N\to\infty} \tau_N = \infty$, we have

$$
\tilde{X}_t = \tilde{Y}_t, \ \mathbf{P} - a.s.
$$

By continuity of $\tilde{X}_t$, $\tilde{Y}_t$,

$$
\tilde{X}_t = \tilde{Y}_t, \ 0 \leq \forall t < \infty, \ \mathbf{P} - a.s. \tag{B.6}
$$

Next, we would like to show

$$
\tau^X = \tau^Y, \ \mathbf{P} - a.s. \tag{B.7}
$$

For ω satisfying

$$
\tau^X(\omega) = \tau^Y(\omega) = \infty, \tag{B.8}
$$

we see

$$X_t = \tilde{X}_t = \tilde{Y}_t = Y_t, \ 0 \le \forall t < \infty. \tag{B.9}$$

For ω satisfying

$$\tau^X(\omega) \ge \tau^Y(\omega), \ \tau^Y(\omega) < \infty, \tag{B.10}$$

noting that

$$\tilde{X}_{\tau^Y} = \tilde{Y}_{\tau^Y},$$

and

$$\tilde{X}_{\tau^Y} = X_{\tau^Y \wedge \tau^Y} = X_{\tau^Y}, \quad \tilde{Y}_{\tau^Y} = Y_{\tau^Y \wedge \tau^X} = Y_{\tau^Y},$$

we see

$$X_{\tau^Y} = Y_{\tau^Y} = A_{\tau^Y}.$$

Since

$$\tau^X := \inf \{t \ge 0 | X_t = A_t\},$$

we have

$$\tau^X = \tau^Y.$$

Hence

$$\tilde{X}_t = X_t, \ \tilde{Y}_t = Y_t,$$

and

$$X_t = Y_t, \ 0 \le \forall t < \infty. \tag{B.11}$$

The same is true for ω satisfying

$$\tau^X(\omega) \le \tau^Y(\omega), \ \tau^X(\omega) < \infty. \tag{B.12}$$

Therefore,

$$X_t = Y_t, \ 0 \le \forall t < \infty, \ \mathbf{P} - a.s. \quad \square$$

Appendix C

Greeks of the European Put Option

In this appendix, we show derivation of the Greeks. For Delta, by differentiating the put option price with respect to the initial value $F_{0,T}$, we have

$$\begin{aligned}
\frac{\partial}{\partial F_{0,T}} \mathbf{E}^{\mathbf{P}} \left[(K - F_{T,T})^+ \right] &= \mathbf{E}^{\mathbf{P}} \left[-1_{\{F_{T,T} < K\}} \frac{\partial}{\partial F_{0,T}} F_{T,T} \right] \\
&= -\mathbf{E}^{\mathbf{P}} \left[1_{\{F_{T,T} < K\}} X_{T \wedge \tau} \right] \\
&= -\mathbf{E}^{\mathbf{Q}} \left[1_{\{F_{T,T} < K\}} \right] \\
&= -N \left(\frac{-a_0 + m(t_0)\tilde{\theta}}{\sqrt{m(t_0)}} \right) + \exp(2a_0\tilde{\theta}) N \left(\frac{a_0 + m(t_0)\tilde{\theta}}{\sqrt{m(t_0)}} \right) \\
&\quad + N \left(\frac{-\max(a_0, \tilde{k}) + m(T)\tilde{\theta}}{\sqrt{m(T)}} \right) \\
&\quad - \exp(2a_0\tilde{\theta}) N \left(\frac{-\max(a_0, \tilde{k}) + 2a_0 + m(T)\tilde{\theta}}{\sqrt{m(T)}} \right).
\end{aligned} \tag{C.1}$$

For Gamma, by differentiating the delta with respect to the initial value $F_{0,T}$, we have

$$\begin{aligned}
\frac{\partial^2}{\partial F_{0,T}^2}\mathbf{E}\left[(K-F_{T,T})^+\right] &= \frac{\partial}{\partial F_{0,T}}N\left(\frac{-\max(a_0,\tilde{k})+m(T)\tilde{\theta}}{\sqrt{m(T)}}\right) \\
&\quad - \frac{\partial}{\partial F_{0,T}}\exp(2a_0\tilde{\theta})N\left(\frac{-\max(a_0,\tilde{k})+2a_0+m(T)\tilde{\theta}}{\sqrt{m(T)}}\right) \\
&= \frac{1}{F_{0,T}\sqrt{m(T)}}\left[n\left(\frac{-\tilde{k}+m(T)\tilde{\theta}}{\sqrt{m(T)}}\right)\right. \\
&\quad \left. - \exp(2a_0\tilde{\theta})n\left(\frac{-\tilde{k}+2a_0+m(T)\tilde{\theta}}{\sqrt{m(T)}}\right)\right]1_{\{\tilde{k}>a_0\}}.
\end{aligned} \tag{C.2}$$

For Theta, by differentiating the price with respect to ϵ in $F_{T-\epsilon,T}$,

$$\begin{aligned}
&\frac{\partial}{\partial \epsilon}\mathbf{E}\left[(K-F_{T-\epsilon,T})^+\right]\Big|_{\epsilon=0} \\
&= K\left[-\frac{\partial}{\partial\epsilon}N\left(\frac{-\max(a_0,\tilde{k})+m(T-\epsilon)\theta}{\sqrt{m(T-\epsilon)}}\right)\Big|_{\epsilon=0}\right. \\
&\quad \left. + \exp(2a_0\theta)\frac{\partial}{\partial\epsilon}N\left(\frac{-\max(a_0,\tilde{k})+2a_0+m(T-\epsilon)\theta}{\sqrt{m(T-\epsilon)}}\right)\Big|_{\epsilon=0}\right] \\
&\quad - F_{0,T}\left[-\frac{\partial}{\partial\epsilon}N\left(\frac{-\max(a_0,\tilde{k})+m(T-\epsilon)\tilde{\theta}}{\sqrt{m(T-\epsilon)}}\right)\Big|_{\epsilon=0}\right. \\
&\quad \left. + \exp(2a_0\tilde{\theta})\frac{\partial}{\partial\epsilon}N\left(\frac{-\max(a_0,\tilde{k})+2a_0+m(T-\epsilon)\tilde{\theta}}{\sqrt{m(T-\epsilon)}}\right)\Big|_{\epsilon=0}\right], \\
&= \frac{\sigma_T^2 K}{2\sqrt{m(T)^3}}\left[\Big(\max(a_0,\tilde{k})+m(T)\theta\Big)\, n\left(\frac{-\max(a_0,\tilde{k})+m(T)\theta}{\sqrt{m(T)}}\right)\right. \\
&\quad \left. + \exp(2a_0\theta)\Big(-\max(a_0,\tilde{k})+2a_0-m(T)\theta\Big)\, n\left(\frac{-\max(a_0,\tilde{k})+2a_0+m(T)\theta}{\sqrt{m(T)}}\right)\right] \\
&\quad - \frac{\sigma_T^2 F_{0,T}}{2\sqrt{m(T)^3}}\left[\Big(\max(a_0,\tilde{k})+m(T)\tilde{\theta}\Big)\, n\left(\frac{-\max(a_0,\tilde{k})+m(T)\tilde{\theta}}{\sqrt{m(T)}}\right)\right. \\
&\quad \left. + \exp(2a_0\tilde{\theta})\Big(-\max(a_0,\tilde{k})+2a_0-m(T)\tilde{\theta}\Big)\, n\left(\frac{-\max(a_0,\tilde{k})+2a_0+m(T)\tilde{\theta}}{\sqrt{m(T)}}\right)\right].
\end{aligned} \tag{C.3}$$

For Vega, let $F^{\epsilon}_{t,T}$ be a strong solution of the SDE substituting $\sigma^{\epsilon}_t := \sigma_t + \epsilon$ for σ_t in the SDE. Let $m^{\epsilon}(t) := \int_0^t (\sigma^{\epsilon})^2 ds$.

Then Vega is given by

$$
\begin{aligned}
&\frac{\partial}{\partial \epsilon}\mathbf{E}[(K - F^{\epsilon}_{T,T})^{+}]\Big|_{\epsilon=0} \\
&= K\Bigg[\frac{\partial}{\partial \epsilon}N\left(\frac{-a_0 + m^{\epsilon}(t_0)\theta}{\sqrt{m^{\epsilon}(t_0)}}\right)\Bigg|_{\epsilon=0} - \exp(2a_0\theta)\frac{\partial}{\partial \epsilon}N\left(\frac{a_0 + m^{\epsilon}(t_0)\theta}{\sqrt{m^{\epsilon}(t_0)}}\right)\Bigg|_{\epsilon=0} \\
&\quad - \frac{\partial}{\partial \epsilon}N\left(\frac{-\max(a_0,\tilde{k}) + m^{\epsilon}(T)\theta}{\sqrt{m^{\epsilon}(T)}}\right)\Bigg|_{\epsilon=0} \\
&\quad + \exp(2a_0\theta)\frac{\partial}{\partial \epsilon}N\left(\frac{-\max(a_0,\tilde{k}) + 2a_0 + m^{\epsilon}(T)\theta}{\sqrt{m^{\epsilon}(T)}}\right)\Bigg|_{\epsilon=0}\Bigg] \\
&\quad - F_{0,T}\Bigg[\frac{\partial}{\partial \epsilon}N\left(\frac{-a_0 + m^{\epsilon}(t_0)\tilde{\theta}}{\sqrt{m^{\epsilon}(t_0)}}\right)\Bigg|_{\epsilon=0} - \exp(2a_0\tilde{\theta})\frac{\partial}{\partial \epsilon}N\left(\frac{a_0 + m^{\epsilon}(t_0)\tilde{\theta}}{\sqrt{m^{\epsilon}(t_0)}}\right)\Bigg|_{\epsilon=0} \\
&\quad - \frac{\partial}{\partial \epsilon}N\left(\frac{-\max(a_0,\tilde{k}) + m^{\epsilon}(T)\tilde{\theta}}{\sqrt{m^{\epsilon}(T)}}\right)\Bigg|_{\epsilon=0} \\
&\quad + \exp(2a_0\tilde{\theta})\frac{\partial}{\partial \epsilon}N\left(\frac{-\max(a_0,\tilde{k}) + 2a_0 + m^{\epsilon}(T)\tilde{\theta}}{\sqrt{m^{\epsilon}(T)}}\right)\Bigg|_{\epsilon=0}\Bigg] \\
&= K\Bigg[\frac{p(t_0)}{\sqrt{m(t_0)^3}}\left(a_0 + m(t_0)\theta\right) n\left(\frac{-a_0 + m(t_0)\theta}{\sqrt{m(t_0)}}\right) \\
&\quad - \exp(2a_0\theta)\frac{p(t_0)}{\sqrt{m(t_0)^3}}\left(-a_0 + m(t_0)\theta\right) n\left(\frac{a_0 + m(t_0)\theta}{\sqrt{m(t_0)}}\right) \\
&\quad - \frac{p(T)}{\sqrt{m(T)^3}}\left(\max(a_0,\tilde{k}) + m(T)\theta\right) n\left(\frac{-\max(a_0,\tilde{k}) + m(T)\theta}{\sqrt{m(T)}}\right) \\
&\quad + \exp(2a_0\theta)\frac{p(T)}{\sqrt{m(T)^3}}\left(\max(a_0,\tilde{k}) - 2a_0 + m(T)\theta\right) n\left(\frac{-\max(a_0,\tilde{k}) + 2a_0 + m(T)\theta}{\sqrt{m(T)}}\right)\Bigg] \\
&\quad - F_{0,T}\Bigg[\frac{p(t_0)}{\sqrt{m(t_0)^3}}\left(a_0 + m(t_0)\tilde{\theta}\right) n\left(\frac{-a_0 + m(t_0)\tilde{\theta}}{\sqrt{m(t_0)}}\right) \\
&\quad - \exp(2a_0\tilde{\theta})\frac{p(t_0)}{\sqrt{m(t_0)^3}}\left(-a_0 + m(t_0)\tilde{\theta}\right) n\left(\frac{a_0 + m(t_0)\tilde{\theta}}{\sqrt{m(t_0)}}\right) \\
&\quad - \frac{p(T)}{\sqrt{m(T)^3}}\left(\max(a_0,\tilde{k}) + m(T)\tilde{\theta}\right) n\left(\frac{-\max(a_0,\tilde{k}) + m(T)\tilde{\theta}}{\sqrt{m(T)}}\right) \\
&\quad + \exp(2a_0\tilde{\theta})\frac{p(T)}{\sqrt{m(T)^3}}\left(\max(a_0,\tilde{k}) - 2a_0 + m(T)\tilde{\theta}\right) n\left(\frac{-\max(a_0,\tilde{k}) + 2a_0 + m(T)\tilde{\theta}}{\sqrt{m(T)}}\right)\Bigg].
\end{aligned} \tag{C.4}
$$

Appendix D

Physical Settlement in Two Period Case with Time Dependent λ

We aim to obtain explicit expressions for $\theta_2 \in \mathcal{F}_1$, $\theta_1 \in \mathcal{F}_0$ which satisfy the both equalities.

$$\theta_2 = \frac{\mathrm{Cov}[\eta_2 + \theta_3(\tilde{S}_2 - 2(\lambda_1 - \lambda_0)M\theta_1) + M\theta_3^2, -2\lambda_1 M\theta_3 + \Delta\tilde{S}_2 + 2M\theta_3 | \mathcal{F}_1]}{\mathrm{Var}[-2\lambda_1 M\theta_3 + \Delta\tilde{S}_2 + 2M\theta_3 | \mathcal{F}_1]}. \tag{D.1}$$

$$\begin{aligned}\theta_1 =& \mathrm{Cov}[\eta_2 + \theta_3(\tilde{S}_2 + 2\lambda_1 M\theta_2) - \theta_2\Delta\tilde{S}_2 + M(\Delta\theta_3)^2 + M\theta_2^2 - 2\lambda_1 M\theta_2^2, \\ & \Delta\tilde{S}_1 + 2M\theta_2 - 2\lambda_1 M\theta_2 + 2(\lambda_1 - \lambda_0)M\theta_3)] \\ & /\mathrm{Var}[\Delta\tilde{S}_1 + 2M\theta_2 - 2\lambda_1 M\theta_2 + 2(\lambda_1 - \lambda_0)M\theta_3].\end{aligned} \tag{D.2}$$

Rewrite θ_2 as

$$\begin{aligned}\theta_2 =& -2(\lambda_1 - \lambda_0)M\frac{\mathrm{Cov}[\theta_3, -2\lambda_1 M\theta_3 + \Delta\tilde{S}_2 + 2M\theta_3 | \mathcal{F}_1]}{\mathrm{Var}[-2\lambda_1 M\theta_3 + \Delta\tilde{S}_2 + 2M\theta_3 | \mathcal{F}_1]}\theta_1 \\ & + \frac{\mathrm{Cov}[\eta_2 + \theta_3\tilde{S}_2 + M\theta_3^2, -2\lambda_1 M\theta_3 + \Delta\tilde{S}_2 + 2M\theta_3 | \mathcal{F}_1]}{\mathrm{Var}[-2\lambda_1 M\theta_3 + \Delta\tilde{S}_2 + 2M\theta_3 | \mathcal{F}_1]} \\ :=& \alpha_1\theta_1 + \beta_1.\end{aligned} \tag{D.3}$$

Substituting this into the equation for θ_1, we obtain

$$\begin{aligned}
&\theta_1 \mathrm{Var}[\Delta\tilde{S}_1 + 2(1-\lambda_1)M(\alpha_1\theta_1+\beta_1) + 2(\lambda_1-\lambda_0)M\theta_3] \\
&= \mathrm{Cov}[\eta_2 + \theta_3(\tilde{S}_2 + 2\lambda_1 M(\alpha_1\theta_1+\beta_1)) - (\alpha_1\theta_1+\beta_1)\Delta\tilde{S}_2 \\
&\quad + M(\theta_3 - (\alpha_1\theta_1+\beta_1))^2 + (1-2\lambda_1)M(\alpha_1\theta_1+\beta_1)^2, \\
&\quad \Delta\tilde{S}_1 + 2M\theta_2 - 2\lambda_1 M\theta_2 + 2(\lambda_1-\lambda_0)M\theta_3)].
\end{aligned} \tag{D.4}$$

The left hand side of the equation is

$$\begin{aligned}
&\theta_1(\mathrm{Var}[2(1-\lambda_1)M\alpha_1\theta_1] + \mathrm{Var}[\Delta\tilde{S}_1 + 2(1-\lambda_1)M\beta_1 + 2(\lambda_1-\lambda_0)M\theta_3] \\
&\quad + 2\mathrm{Cov}[2(1-\lambda_1)M\alpha_1\theta_1, \Delta\tilde{S}_1 + 2(1-\lambda_1)M\beta_1 + 2(\lambda_1-\lambda_0)M\theta_3]) \\
&= \theta_1^3(\mathrm{Var}[2(1-\lambda_1)M\alpha_1]) \\
&\quad + \theta_1^2(2\mathrm{Cov}[2(1-\lambda_1)M\alpha_1\theta_1, \Delta\tilde{S}_1 + 2(1-\lambda_1)M\beta_1 + 2(\lambda_1-\lambda_0)M\theta_3]) \\
&\quad + \theta_1\mathrm{Var}[\Delta\tilde{S}_1 + 2(1-\lambda_1)M\beta_1 + 2(\lambda_1-\lambda_0)M\theta_3].
\end{aligned} \tag{D.5}$$

The right hand side of the equation is

$$\begin{aligned}
&\mathrm{Cov}[\theta_1^2(\alpha_1^2 M + \alpha_1^2 M(1-2\lambda_1)) \\
&\quad + \theta_1(2\lambda_1 M\alpha_1\theta_3 - \alpha_1\Delta\tilde{S}_2 - 2\alpha_1 M(\theta_3-\beta_1) + 2M(1-2\lambda_1)\alpha_1\beta_1) \\
&\quad + (\eta_2 + \theta_3(\tilde{S}_2 + 2\lambda_1 M\beta_1) - \beta_1\Delta\tilde{S}_2 + M(\theta_3-\beta_1)^2 + M(1-2\lambda_1)\beta_1^2), \\
&\quad \theta_1(2(1-\lambda_1)M\alpha_1) \\
&\quad + (\Delta\tilde{S}_1 + 2(1-\lambda_1)M\beta_1 + 2(\lambda_1-\lambda_0)M\theta_3))] \\
&= \theta_1^3\mathrm{Cov}[2(1-\lambda_1)\alpha_1^2 M, 2(1-\lambda_1)M\alpha_1] \\
&\quad + \theta_1^2(\mathrm{Cov}[2(1-\lambda_1)\alpha_1^2 M, \Delta\tilde{S}_1 + 2(1-\lambda_1)M\beta_1 + 2(\lambda_1-\lambda_0)M\theta_3)] \\
&\quad + \mathrm{Cov}[2\lambda_1 M\alpha_1\theta_3 - \alpha_1\Delta\tilde{S}_2 - 2\alpha_1 M(\theta_3-\beta_1) + 2M(1-2\lambda_1)\alpha_1\beta_1, 2(1-\lambda_1)M\alpha_1]) \\
&\quad + \theta_1(\mathrm{Cov}[2\lambda_1 M\alpha_1\theta_3 - \alpha_1\Delta\tilde{S}_2 - 2\alpha_1 M(\theta_3-\beta_1) + 2M(1-2\lambda_1)\alpha_1\beta_1, \\
&\quad \Delta\tilde{S}_1 + 2(1-\lambda_1)M\beta_1 + 2(\lambda_1-\lambda_0)M\theta_3)] \\
&\quad + \mathrm{Cov}[\eta_2 + \theta_3(\tilde{S}_2 + 2\lambda_1 M\beta_1) - \beta_1\Delta\tilde{S}_2 + M(\theta_3-\beta_1)^2 + M(1-2\lambda_1)\beta_1^2, 2(1-\lambda_1)M\alpha_1]) \\
&\quad + \mathrm{Cov}[\eta_2 + \theta_3(\tilde{S}_2 + 2\lambda_1 M\beta_1) - \beta_1\Delta\tilde{S}_2 + M(\theta_3-\beta_1)^2 + M(1-2\lambda_1)\beta_1^2, \\
&\quad \Delta\tilde{S}_1 + 2(1-\lambda_1)M\beta_1 + 2(\lambda_1-\lambda_0)M\theta_3].
\end{aligned} \tag{D.6}$$

Setting the equation as

$$f(\theta_1) = a_3\theta_1^3 + a_2\theta_1^2 + a_1\theta_1^1 + a_0 = 0, \tag{D.7}$$

we reduce the problem of finding the minimizer of θ_1 to solving the cubic equation. It is easy to see that the conditions for the equation has 3 solutions are

$$a_3 \neq 0, \tag{D.8}$$
$$a_2^2 - 3a_3a_1 > 0, \tag{D.9}$$

and

$$f\left(\frac{-a_2 + \sqrt{a_2^2 - 3a_3a_1}}{3a_3}\right) f\left(\frac{-a_2 - \sqrt{a_2^2 - 3a_3a_1}}{3a_3}\right) < 0. \tag{D.10}$$

Substituting θ_1 into the equation for θ_2, we obtain θ_2.

Bibliography

[1] F.Abergel, G.Loeper (2013): Pricing and hedging contingent claims with liquidity costs and market impact. *Working Paper*

[2] J.Barbarin (2007): Local risk-minimization for payment processes. Application to insurance contracts with surrender option or to default sensitive contingent claims. *working paper, Universite Catholique de Louvain.*

[3] U. Cetin, R. Jarrow, P. Protter (2004): Liquidity risk and arbitrage pricing theory. *Finance Stoch*, 8: 311-341.

[4] F.De Jong, F.Drost, B.Werker (2001): A Jump-Diffusion Model for Exchange Rates in a Target Zone. *Statistica Neerlandica*, 3: 270-300.

[5] N.Ikeda, S.Watanabe (1989): Stochastic Differential Equations and Diffusion Processes Second Edition. North-Holland, Kodansha, 1989.

[6] N.Kunitomo, M.Ikeda (1992): Pricing Options with Curved Boundaries. *Mathematical Finance*, 2: 275-298.

[7] D.Lamberton, H.Pham, M.Schweizer (1998): Local risk-minimization under transaction costs. *Mathematics of Operations Research*, 23: 585-612.

[8] K.S.Larsen, M.Sørensen (2007): Diffusion Models For Exchange Rates in a Target Zone. *Mathematical Finance*, 17: 285-306.

[9] A.Lipton (2001): Mathematical Methods for Foreign Exchange: A Financial Engineers Approach. World Scientific Pub Co Inc.

[10] P. Medvegyev (2007): Stochastic Integration Theory. *Oxford.*

[11] E.Omberg (1987): The Valuation of American Put Options with Exponential Exercise Policies. *Advances in Futures and Options research*, 2: 117-142.

[12] P. Protter (2003): Stochastic Integration and Differential Equations Second Edition. *Springer.*

[13] A. Roch (2011): Liquidity risk, price impacts and the replication problem. *Finance Stoch*, 15: 399-419.

[14] L.C.G. Rogers, S.Singh (2010): The cost of liquidity and its effects on hedging. *Mathematical Finance*, 20:

[15] M.Schweizer (1988): Hedging of options in a general semimartingale model. *Diss. ETHZ*, No.8615.

[16] M.Shirakawa (2008): Modern Monetary Policy in Theory and Practice. Nikkei.

[17] A.Takahashi, T.Kobayashi, N.Nakagawa (2001): Pricing Convertible Bonds with Default Risk: A Duffie-Singleton Approach. *The Journal of Fixed Income*, 11: 20-29.

[18] D.Veestraeten (2013): Currency Option Pricing in a Credible Exchange Rate Target Zone. *Applied Financial Economics*, 23: 951-962.

[19] M.Yor (2009): Mathematical Methods for Financial Markets. Springer.