博士論文（要約）

Essays on Model-Independent Super-Replication of Derivatives

(モデルによらないデリバティブの優複製に関する研究)

東京大学大学院経済学研究科博士課程
金融システム専攻
都築幸宏

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序文

本論文では、デリバティブのモデルに依存しない優劣複製手法および価格の上下限値を研究する。2007 年サブプライム・ローン問題、2008 年リーマンショック、そして 2010 年欧州通貨危機などの相次ぐ金融危機を経て、金融機関のリスク管理が見直されつつある。特に、金融機関が取引しているデリバティブについて、リスクを十分に認識していなかった、適切な評価をしていなかった、などの批判が挙げられている。デリバティブの評価には、一般的に特定のモデルに基づき時価評価されリスク管理が行われているが、モデルを採用することはモデル・リスクやモデル・パラメータの誤推定リスクに晒されていることになる。例えば、複数の資産を参照する証券化商品の評価にガウシアン・コピュラ・モデルを用いて、テイル・リスクを過小評価していたことは記憶に新しい。金融機関が取引しているデリバティブは複雑であり、モデルを使わずに評価やリスク管理を行うことは難しいが、デリバティブのモデル非依存な特性について理解を深めることが、リスク管理の高度化に不可欠である。

0.1　本論文の構成

本論文は、4 部から構成されている。第一部と第二部では、特定のデリバティブ (第一部では複数資産を原資産とするデリバティブ、第二部では経路依存型デリバティブ) の優劣複製手法を考える。第三部では、複製戦略の改善手法を考察する。第四部は、ほかの 3 部とは独立した内容であり、原資産の分布の補正方法を提案する。

第一部と第二部では、特定のデリバティブの優劣複製手法を考える。デリバティブの優 (劣) 複製とは、複製対象のデリバティブのペイオフよりも大きな (小さな) ペイオフを実現するポートフォリオを構築することである。通常、このポートフォリオは流動性の高いデリバティブで構成される。第一部では、複数資産を原資産とするデリバティブに対して、個々の資産を原資産とするバニラ・オプション (コール・オプションとプット・オプション) を使った静的な取引戦略を考える。静的な取引戦略は、初期時点で構築したポートフォリオを変更しない取引戦略のことである。第二部では、ひとつの資産を原資産とする経路依存型デリバティブの優劣複製を研究する。複製取引戦略は、静的または準静的である。ここで、準静的とは初期時点で構築したポートフォリオを高々一度だけ更新することを許す取引手法である。第二部各章の優劣複製の研究では、許容される取引戦略についての前提条件を課し、その条件の下で最良な取引戦略を導出する。この最良な取引戦略に必要な費用が、対象とするデリバティブの価格の上下限値になる。取引戦略が最良であるとは、その取引戦略の費用がデリバティブの価格と一致する (または、限りなく近い) ような同値マルチンゲール測度が存在することである。

このようなモデルに依存しない優劣複製手法が得られることは、デリバティブの流動性を供給する主体で、リスク回避的なリスク選好をもつものにとっては好ましい。なぜならば、デリバティブの売りと買いに応じて優劣複製を行えば、最大損失は初期時点のキャッシュフローに限定されるからである。ただし、一般的にモデル非依存な価格帯は広く、最大損失は限定されるものの、その額は大きく損益の変動も激しいため、取引戦略としては好ましくない。この点を改善する方法を二つ検討する。まず、第一は上記のような優劣複製手法を、個々のデリバティブではなくデリバティブのポートフォリオに適応することを考える。優劣複製手法は、個々の取引に適応するよりも、取引全体のポートフォリオに対して適応した方が複製誤差が小さく (劣加法性)、その結果、価格の上下限値は改善される。ただし、個々のデリバティブに対する優劣複製手法が、そのままポートフォリオに適応できる

とは限らない。この方法は、第一部と第二部で対象とするデリバティブ毎に検討される。第二の方法は、初期時点で構築した優劣複製ポートフォリオを期中に更新する方法である。初期時点で構築したポートフォリオは初期時点では最良であるが、その後は優劣複製するものの最良であるとは限らない。そこで、初期時点以降の任意の時間で、その時の最良な優劣複製ポートフォリオに更新することで、正のキャッシュフローが得られる。この取引を行うことで、取引の満期を待たずに初期時点払った費用の一部を回収できるため、損益の変動を小さくすることが期待できる。この取引戦略は、第三部で研究される。

第四部は、ほかの三部とは独立した内容であり、原資産の分布の補正方法を提案する。第一部と第二部では、優劣複製対象のオプションと同じ満期をもつ、単一資産を原資産とする任意の行使価格のバニラ・オプションが流動的であることを仮定している。これは、原資産の満期時点でのリスク中立測度の下での分布が既知であるという仮定と同値である。ただし、市場で観測されるオプションの価格の数は高々有限であるため、任意のオプションの価格を得るには補間・補外が必要になる。第四部では、オプション価格が無裁定であることが保証されるような補間・補外方法を提案する。

0.2 複数資産を原資産とするデリバティブの優劣複製

0.2.1 背景

複数資産を原資産とするデリバティブは、例えば、バスケット・オプション、スプレッド・オプション、基軸通貨を含まない通貨オプション、クオント・オプション、バリアの参照資産とペイオフの原資産が異なるバリア・オプション、などである。近年の発達したオプション市場では、これらのデリバティブの個々の原資産を参照するバニラ・オプションの流動性は高く、リスク管理に利用できる。バニラ・オプションを使ったモデル非依存な複製手法は、これまでにも多く提案されてきた (例えば、(Laurence and Wang, 2003),(Hobson et al., 2005a),(Hobson et al., 2005b),(Laurence and Wang, 2009),(d'Aspremont and El-Ghaoui, 2006),(Chung and Wang, 2008) を参照)。

これらの理論研究は古くからされ、モデル非依存な価格の上下限値から市場価格がかい離するような裁定機会は、通常期待できない。ただし、個別のオプション市場と複数資産を原資産するデリバティブの市場が必ずしも効率的ではなく、分断されている場合もある。実際、(McCloud, 2011) によれば、2009 年の欧州の CMS(Constant Maturity Swap) 市場において、CMS スプレッド・オプションと CMS オプションの間で裁定機会が存在していたようである。

0.2.2 成果と貢献

1 章では、ペイオフが正の関数の積分であらわされるようなデリバティブに対して優劣複製手法を導出する。このようなデリバティブの例は、バスケット・オプション、通貨オプション、クオント・オプションなどである。先行研究では個々のデリバティブの優劣複製手法は研究されているが、これらのデリバティブを共通の枠組みで統一的に論じたのは本研究が初めてである。優劣複製手法の導出は次のとおりである。まずペイオフに対して Young の不等式を適応することにより優劣複製手法がいくつか得られる。ここで「いくつか」というのは、実数上の単調増加関数と同じだけ自由度を持つという意味である。これらの中で最良の優劣複製手法 (複製ポートフォリオのコストが最小または最大のもの) を与える単調増加関数の特徴を導出する。最良であることは、複製ポートフォリオのコストをデリバティブの価格とするような同値マルチンゲール測度が存在することで示すことができる。この方法により得られる価格の上下限値は、原資産が同時にある方向に変動するような確率空間上での価格と一致する。(Dhaene et al., 2002) では、この挙動を comonotonicity と呼び、プライシングの観点から研究をしているが、複製の観点からこの理論を導出したのが 1 章である。

2 章では、1 章の応用例として、外貨資産と為替レートを参照するクオント・オプションの優劣複製手法を研究する。クオント・オプションのプライシングに関する先行研究は、(Bennett and Kennedy, 2004)、(Jäckel, 2009)、(Jäckel, 2010)、(Giese, 2012) などあるが、優劣複製手法や価格の上下限値を研究したものは、この研究が初めて

である。クオント・オプションのペイオフは外貨建てで表すと、2 つの確率変数の積という形になるため、クオント・オプションのポートフォリオにも 1 章の手法を適応することができる。これはバスケット・オプションなどでは成り立たないクオント・オプションの特徴である。この特徴を利用して、最も流動性のあるクオント・フォワード取引を複製に使うことで複製効率がどのくらい向上できるかを調べる。クオント・フォワード取引をカリブレーションに使う手法は (Giese, 2012) で検討されているが、取引戦略に利用する研究は、本研究が初めてである。

0.3 経路依存型デリバティブの優劣複製

0.3.1 背景

経路依存型デリバティブの代表的な例は、バリア・オプション、ルックバック・オプション、フォワード・スタート・オプション、バリアンス・スワップなどがある。特に、バリア・オプションは流動性が高く、多くの研究がされている。モデルを仮定したり、特定の条件の下での複製、優劣複製については、(Bowie and Carr, 1994)、(Carr and Chou, 1997)、(Carr et al., 1998)、(Fink, 2003)、(Maruhn, 2009)、モデル非依存な優劣複製手法については、(Brown et al., 2001)、(Neuberger and Hodges, 2000)、(Cox and Obłój, 2011a)、(Cox and Obłój, 2011b) などである。ルックバック・オプションは (Hobson, 1998)、フォワード・スタート・オプションは (Hobson and Neuberger, 2012)、バリアンス・スワップは (Hobson and Klimmek, 2012) や (Baldeaux and Rutkowski, 2010) の先行研究がある。これらの理論では、原資産価格がマルチンゲールに従うという仮定を置いている (実際はフォワード価格がマルチンゲールになるため、この仮定は現実には満たされるとは限らない)。近年では、(Labordére et al., 2012) や (Beiglböck et al., 2013) などにより、一般論が整備されてきた。

経路依存型のデリバティブのモデル非依存な優劣複製の重要性について、さらに述べる。成熟したオプション市場ではバニラ・オプションの流動性が高いため、原資産のリスク中立測度の下での周辺分布 (満期時点における分布) についての情報は十分得られる。ところが、モデルをこれらのバニラ・オプションに誤差なくカリブレーションできたとしても、モデルが記述する異なる時点間の原資産の同時分布はモデルごとに異なり、経路依存型のデリバティブはモデル・リスクにさらされる ((Hirsa et al., 2003)、(Lipton and McGhee, 2002)、(Schoutens et al., 2005))。このモデル・リスクを計量するためにも、モデル非依存な価格の上下限値の研究は重要である。

0.3.2 成果と貢献

3 章では、タッチ・オプションを使ってバリア・オプションの静的な優劣複製手法を考える。バリア・オプションのモデルによらない優劣複製の先行研究では、原資産がマルチンゲールであることを仮定し準静的な取引戦略を用いるが、本研究はタッチ・オプションを取引戦略に用いる点と静的な取引戦略に限定している点で、先行研究と異なる新しい研究である。タッチ・オプションとは、原資産価格がバリアに抵触するか否かによって 1 円が支払われるというペイオフをもつバリア・オプションの一種である。この研究はタッチ・オプションが流動的であるという強い条件を仮定するがマルチンゲールの条件を必要とせず、より確実な優劣複製を実現できる。また、満期とバリア条件が同じであれば、バリア・オプションのポートフォリオに対しても同じ手法が適応できる。特に、為替オプション市場において、原資産がバリアに抵触したときに原資産を 1 単位支払うというバリア・オプションはタッチ・オプションと同程度の流動性が期待できる。これを優劣複製に使えば、バリア・オプションの価格の上下限値を改善することができる。数学的な手法は、バリア・オプションのペイオフはバリア判定と満期時点のペイオフの積とみなすことができるため、1 章とほぼ同じである。ただし、バリア判定と満期時点のペイオフの間には依存関係がある点が、1 章と異なる。

4 章では、2 つのタッチ・オプションが同時に取りうる価格の組を求める。タッチ・オプションのモデル非依存な優劣複製は、(Brown et al., 2001)、(Neuberger and Hodges, 2000)、(Cox and Obłój, 2011a)、(Cox and Obłój, 2011b) により、原資産価格がマルチンゲールという条件の下で研究がされてきた。本研究も同様の枠組みを採用するが、2 つのタッチ・オプションを同時に考慮している点が先行研究と異なる。これは、流動性のあるタッチ・

オプションを使って、別のタッチ・オプションを優劣複製する問題と同じである。ただし、タッチ・オプションのバリアはひとつ (シングル・バリア・オプション) とし、2 つのタッチ・オプションのバリアは原資産価格の初期値に対して大小関係が同じであるとする。先行研究では、取引戦略に使う取引の価格を再現するような連続マルチンゲール過程を構築することにより、優劣複製手法の最良性を示している。本研究も、先行研究と同じ手法を使っているが、取引戦略に使う取引が多いため先行研究より計算が複雑になっている。

5 章では、バリア・オプションが満たす無裁定価格の条件についての研究である。バニラ・オプションについて同様の研究は、(Davis and Hobson, 2007)、(Carr and Madan, 2005)、(Cousot, 2007) などですでにされているが、バリア・オプションについての研究は本研究が初めてである。バニラ・オプションの無裁定価格の条件は、コール・オプションの価格が行使価格に関して単調減少凸関数であることと、金利と配当の調整してた価格関数が満期に関して単調増加であることである。数学的には、満期時点の原資産価格の密度関数が正であることと Jensen の不等式が成り立つことに対応する。バニラ・オプションの場合も同様であるが、バリアに抵触した場合とそうでない場合を分けて考える必要があり、バニラ・オプションよりも難しい問題である。これらの結果はバリア・オプションについてではあるが、多くの結果はバリアの参照資産とペイオフの参照資産が異なる場合でも同様の結果が得られる。例えば重要な例としては、Credit Value Adjustment(CVA) がある。CVA は取引相手が倒産した場合に予定されていたキャッシュフローが得られない事象を価格に織り込む調整項である。これは、バリア・オプションのひとつと見なせ、本論文の結果を適応できる。

0.4 優劣複製取引戦略の改善

0.4.1 背景

第一部と第二部で提案した手法は、金融実務の現場では、(McCloud, 2011) のように裁定機会を見つけ、収益を上げる目的で使われることはあるが、リスク管理を目的として使われることは少ない。これには、いくつかの理由が考えられる。例えば、金融機関は多くの取引を保有しているため、個別の取引に注目して、それに特化した戦略を採用するのは、非常に手間がかかり、人的な観点やシステム的な制約で実行できないという理由が挙げられる。ほかの理由としては、金融機関のリスク管理は日々の損益を小さくすることを前提としているため、優劣複製戦略と相性が悪いことが考えられる。優劣複製戦略を行った場合、最大損失を限定することはできるものの日々の損益を時価評価すると損益が大きく変動してしまう。これらは、金融機関のリスク体制と取引戦略との相性という実務的な問題であるが、一方でモデル非依存な優劣複製自体の問題もある。モデル非依存な優劣複製手法をデリバティブのヘッジ取引戦略として使えば、最大損失を確実に限定することができるものの、一般にこの保証される最大損失の額は大きく、ヘッジ取引期間中に損益が大きく変動する。これらの問題は、学術的な問題というよりも実務的な問題ではあるため先行研究は多くないが、モデル非依存な優劣複製を金融機関のリスク管理に用いるためには、非常に重要な問題である。

0.4.2 成果と貢献

第三部 6 章では、最後に挙げた問題の改善手法として、取引期間中の損益の変動を小さくする方法を提案し、その取引戦略の特徴を分析する。この手法は (Neuberger and Hodges, 2000) でバリア・オプションの数値例の一部として考察されている。ただし (Neuberger and Hodges, 2000) では数値実験的な考察にとどまっているのに対し、6 章では、複製対象のデリバティブを限定せず一般的な枠組みで、この戦略の損益を数式を用い表現し、特徴を分析している点で新しい研究である。6 章で考察する取引戦略は、第一部と第二部で検討したような静的優劣複製手法として初期時点で構築したポートフォリオを期中に更新するという手法である。初期時点で構築したポートフォリオはその時点では最良であるが、その後は優劣複製するものの最良であるとは限らない。そこで、初期時点以降の任意の時間で、その時の最良な優劣複製ポートフォリオに更新することで、正のキャッシュフローが得られる。この取引を行うことで、取引の満期を待たずに初期時点払った費用の一部を回収できるため、損益の変動

を小さくすることが期待できる。この応用例として通貨オプションとタッチ・オプションを取り上げ、より具体的に分析する。さらに数値例として通貨オプションのヘッジ・シミュレーションを行い、ダイナミック・ヘッジ、静的な優複製と比較することで、6 章で提案する取引の有効性を確認する。

0.5 密度関数の補正

0.5.1 背景

上記のモデル非依存な複製手法では、個々の原資産を参照資産とするバニラ・オプションは流動的であると仮定している。実際、成熟したオプション市場であれば、多くの種類 (行使価格と満期) のオプションを取引することができる。しかし、実際に観測されるオプションの数は有限であり、任意の行使価格と満期のオプションの価格は補間・補外を用いて代用せざるを得ない。例えば、原資産価格過程にモデルを仮定し観測されているオプション価格にカリブレーションして、このモデルにより補間・補外する方法が一般的である。ところが、オプションのモデル価格に近似解しかない場合は補間・補外の精度が十分でなく、裁定機会を生じるうる。例えば、SABR モデル ((Hagan et al., 2002)) の近似式は行使価格が 0 の近くで、密度が負の値をとることがあり、裁定機会を提供していることになる。この問題に対して (Doust, 2012) は、裁定機会を生じない近似方法を提案している。

0.5.2 成果と貢献

第四部 7 章では、補間・補外されたオプション価格に裁定機会が含まれないように修正する方法を提案する。SABR モデルに関しては (Doust, 2012) が近似式を導出しているが、7 章の手法は特定のモデルや計算方法に依存しない一般的な手法である点で先行研究とは異なり新しい。7 章では、密度関数を適当なヒルベルト空間の元とみなし、密度関数が満たさなければならない条件と満たすことが好ましい条件をヒルベルト空間の内積で記述する。例えば、各点での関数の値が正である、関数の積分値が 1 と等しい、カリブレーション対象のオプション価格を再現する、などである。これらの条件は、ヒルベルト空間の集合として表現したときに凸集合になりさえすればよい。与えられた初期値 (例えば漸近展開で近似した密度関数) に対して、この凸性を利用して最適解を見つけるアルゴリズム ((Deutsch, 2001)) を適応する。設定できる条件の自由度とモデルや計算手法に依存しない点が、この手法の利点である。

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Part I

複数資産を原資産とするデリバティブの優劣複製

Chapter 1

On Optimal Super-Hedging and Sub-Hedging Strategies

This paper proposes optimal super-hedging and sub-hedging strategies for a derivative on two underlying assets without any specification of the underlying processes. Moreover, the strategies are free from any model of the dependency between the underlying asset prices. We derive the optimal pricing bounds by finding a joint distribution under which the derivative price is equal to the hedging portfolio's value; the portfolio consists of liquid derivatives on each of the underlying assets. As examples, we obtain new super-hedging and sub-hedging strategies for several exotic options such as quanto options, exchange options, basket options, forward starting options, and knock-out options.[1]

1.1 Introduction

This paper proposes optimal super-hedging and sub-hedging strategies for a derivative on two underlying assets without any specifications of the underlying processes.

The standard approach to pricing and hedging derivatives is to postulate a particular model for the behavior of the underlying asset prices. Model-parameters are determined by calibration to market prices of liquid derivatives or by estimation from historical data, and hedging is carried out based on the model with only liquid derivatives in the market. For the case of multi-asset derivatives, the dependency structure among the assets is usually estimated and cannot be hedged because there does not exist any derivatives containing information on the dependency. The model with estimated parameters does not necessarily describe the actual behavior of the market, which leads to lack of robustness of the hedging strategy. This is problematic especially in financial turmoils such as the crisis in 2007.

In order to overcome the problem of the standard approach, many researchers have been investigating model-independent super-hedging and sub-hedging strategies for single- or multi-asset derivatives, which is one of the most challenging fields in mathematical finance: currency cross-rate options or spread options by (Chung and Wang, 2008) and (Laurence and Wang, 2009); basket options or asian option by (H. Albrecher and Schoutens, 2005), (d'Aspremont and El-Ghaoui, 2006), (X. Chen and Vanmaele, 2008), (Hobson et al., 2005a), (Laurence

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and Wang, 2003), and (Linders et al., 2012); barrier options by (Brown et al., 2001) and (Neuberger and Hodges, 2000); forward starting options by (Hobson and Neuberger, 2012); lookback options by (Hobson, 1998). They obtain the optimal super-hedging and sub-hedging portfolio for the multi-asset derivatives under an assumption that they have no information on the joint distribution of the assets. The assumption of no information on the joint distribution is reasonable because there is no derivatives including such information in most markets. On the other hand, there exist certain researches (e.g. (Schmutz and Zürcher, 2010) and (Baldeaux and Rutkowski, 2010)) that make use of information embedded in some markets. For single-asset derivatives, it is assumed that marginal distributions of the underlying price at each time are known. Moreover, they add an assumption that the underlying asset price itself is a martingale, which reduces the problem to finding the solution to a Skorohod embedding problem.

We are also in line with the previous works for multi-asset derivatives: our strategies are free from any dependency between two underlying asset prices. The hedging strategy is carried out with a static portfolio which consists of liquid derivatives on each underlying asset. Here, a static portfolio means a portfolio which does not require any transaction after the inception of the contract. A model-independent static hedging strategy with liquid derivatives is effective because it is easy to construct and maintain and never fails to hedge the derivative even in the financial turmoil periods when many models and hedges collapse. We derive the optimal pricing bounds through finding a joint distribution under which the derivative price is equal to the hedging portfolio's value as in the previous works.

On the other hand, we differ from the previous works studying some specific derivatives in that we deal with more general derivatives including existing works such as quanto options, exchange options, basket options, forward starting options and knock-out options. super-hedging and sub-hedging strategies for these apparently different options are derived based on a common well-known inequality, namely Young's inequality. We prove the optimality based on copulas theory which is introduced to mathematical finance by (Cherubini and Luciano, 2002). Under the joint distribution in the optimal case, random variables appearing in a payoff function are co-monotonic or counter-monotonic (see e.g. (Dhaene et al., 2002) for co-monotonicity). In particular, (X. Chen and Vanmaele, 2008) derives optimal super-hedging strategies for basket call options using theory of comonotonicity and (Linders et al., 2012) for basket call and put options. Our approach is more robust than (Brown et al., 2001), (Neuberger and Hodges, 2000) and (Hobson and Neuberger, 2012), which assume a price process of the underlying asset to be a martingale and require a transaction after the inception of the contract. Obviously, as their assumption is violated in the real markets with nonzero interest rates, the result cannot be directly applied in practice. Moreover, it is not necessarily possible to trade during the turmoil periods, which may cause substantial hedging errors. In contrast, we neither impose this assumption nor require any transaction after the inception.

The rest of the paper is as follows. The next sections describes the setup and the problem considered in this paper. Section three proposes our new hedging strategies. In the fourth section, our result is applied to some derivatives including quanto options, exchange options, basket options, forward starting options and knock-out options.

1.2 Setup

We make some assumptions on the market environment. Suppose that two risky assets are traded in the market. Let S_t^X and S_t^Y be the time-t prices of the assets respectively for $t \in [0, T^*]$, where T^* is some arbitrarily determined time horizon. The risk-free interest rate and the dividend yields of the assets are assumed to be zero for simplicity. It is assumed that there exists a risk-neutral probability measure $\mathbb{Q}$, under which the instantaneous expected rate of return on every asset is equal to zero in our settings.

Let X and Y be random variables which are dependent on each asset price S^X and S^Y respectively. Then, a derivative considered in this paper is a product with maturity T whose payoff is expressed for some function $K : \mathbb{R} \times \mathbb{R} \longrightarrow [0, +\infty)$ which is integrable on $\mathbb{R} \times \mathbb{R}$ by

$$\Phi(X,Y) := \int_{\alpha}^{X} \int_{\beta}^{Y} K(x,y) dy dx, \tag{1.2.1}$$

where α and β are some real numbers which are less than the essential infimum of X and Y respectively. Note that the assumption that the function K takes a non-negative value is essential. Concrete examples are shown in Section 1.4.

Remark 1. *A payoff with both long and short is not represented in this form since $K(x,y)$ takes negative values in this case.*

Let us suppose that the marginal distribution functions of the random variable X and Y are known.

Definition 1. *The marginal distribution functions F and G of the random variable X and Y respectively are defined for $x, y \in \mathbb{R}$ by*

$$F(x) := \mathbb{Q}(X \leq x), \tag{1.2.2}$$
$$G(y) := \mathbb{Q}(Y \leq y). \tag{1.2.3}$$

Next, let us introduce some notation to express the joint distribution of the two random variables X and Y with a copula function as in Appendix 1.5.

Definition 2. *A joint distribution function H^C of the random variables X and Y with a copula function C is for $x, y \in \mathbb{R}$ defined by*

$$\begin{aligned} H^C(x,y) &:= C(F(x), G(y)) \\ &= \mathbb{Q}^C(X \leq x, Y \leq y). \end{aligned} \tag{1.2.4}$$

Especially, if the copula function is Maximum copula M,

$$\begin{aligned} H^M(x,y) &:= M(F(x), G(y)) \\ &= \mathbb{Q}^M(X \leq x, Y \leq y), \end{aligned} \tag{1.2.5}$$

where $M(x,y) := \min(x,y)$. In addition, $\mathbb{E}^C$ is defined as the expectation operator under $\mathbb{Q}^C$ with a copula function C.

Remark 2. *The marginal distributions of X and Y under any risk-neutral probability measure $\mathbb{Q}^C$ are independent of choice of the copula function C. We may omit C in $\mathbb{Q}^C$ and $\mathbb{E}^C$ if it is concerned with only the marginal distributions.*

The problem in this paper is stated as follows.

Problem 1. *Suppose that the marginal distribution functions of the random variable X and Y are known, but the joint distribution function is not known. Then, what is the cheapest super-hedging strategy on a derivative with maturity T whose payoff is $\Phi(X,Y)$?*

In order to prove that the super-hedging strategy is the cheapest, one way is to compare the cost of a super-hedging portfolio with the price of the derivative under some measure $\mathbb{Q}^C$ with a copula function C. The upper bound on the price of the derivative with payoff $\Phi(X,Y)$ is

$$\sup_C \mathbb{E}^C(\Phi(X,Y)), \tag{1.2.6}$$

where C is an arbitrary copula function. Since the cost of any super-hedging portfolio is larger than $\mathbb{E}^C(\Phi(X,Y))$ with any copula function C, if we find a particular measure $\mathbb{Q}^C$ and a super-hedging portfolio whose cost is $\mathbb{E}^C(\Phi(X,Y))$, the strategy is the cheapest one.

1.3 Super-hedging and Sub-hedging Strategy

In this section, we first introduce the super-hedging strategy. Then, we derive the sub-hedging strategy using the super-hedging strategy (See (K.C. Cheung and Linders, 2013) for the case where a pay-off function is not twice differentiable). The following lemma is an extended version of Young's inequality (See Theorem 2.3 in (Mitroi and Niculescu, 2011)).

Lemma 1. *Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a non-decreasing function. Then for every Lebesgue locally integrable function $K:\mathbb{R}\times\mathbb{R}\longrightarrow[0,+\infty)$ and real numbers X, Y and α, we have*

$$\begin{aligned}\int_\alpha^X\int_{f(\alpha)}^Y K(x,y)dydx \quad &\leq \quad \int_\alpha^X\left(\int_{f(\alpha)}^{f(x)} K(x,y)dy\right)dx \\ &+ \quad \int_{f(\alpha)}^Y\left(\int_\alpha^{f_{sup}^{-1}(y)} K(x,y)dx\right)dy,\end{aligned} \tag{1.3.1}$$

where f_{sup}^{-1} is the right-continuous inverse function of f:

$$f_{sup}^{-1}(y) := \inf\{x\in\mathbb{R} \mid y < f(x)\}. \tag{1.3.2}$$

If in addition K is strictly positive almost everywhere, then the equality occurs if and only if $y\in[f(x-),f(x+)]$.

Applying Lemma 1 to our problem, the value of $\Phi(X,Y)$ is dominated by the payoff of the following portfolio, if $\beta=f(\alpha)$:

- a derivative with the payoff $\int_\alpha^X\left(\int_{f(\alpha)}^{f(x)} K(x,y)dy\right)dx$
- a derivative with the payoff $\int_{f(\alpha)}^Y\left(\int_\alpha^{f_{sup}^{-1}(y)} K(x,y)dx\right)dy$.

Although these payoffs seem to be complicated, they can be replicated with liquid derivatives. For example, when X and Y are dependent only on S_T^X and S_T^Y respectively, they can be replicated with plain-vanilla options on each asset with maturity T as in (Breeden and Litzenberger, 1978) and (Carr and Madan, 1998) (See (Baldeaux and Rutkowski, 2010) for the case where a payoff function is not twice differentiable).

The question is how to choose the function f in order for the portfolio to be cheap. The answer to the question is that the function f should be f^* in Definition 3, if we assume that Assumption 1 holds.

Assumption 1. *F is continuous.*

Definition 3. *Let $f^* : [\alpha_*, +\infty) \longrightarrow \mathbb{R}$ be a non-decreasing function defined by*

$$f^*(x) := \inf\{y \in \mathbb{R} \mid F(x) < G(y)\}, \tag{1.3.3}$$

where α_* is the essential infimum of X.

The following lemma shows that the function f^* can be viewed as a transform of X to some random variable which has the same distribution of Y.

Lemma 2. *Suppose that Assumption 1 holds. Then, the random variable $f^*(X)$ has the same distribution as the random variable Y:*

$$\mathbb{Q}(f^*(X) \le y) = G(y). \tag{1.3.4}$$

Proof. First, we show $\mathbb{Q}(f^*(X) \le y) = \mathbb{Q}(X \le g^*(y))$, where $g^*(y) := \inf\{x \in \mathbb{R} \mid G(y) < F(x)\}$. $f^*(X) \le y$ means that $f^*(X) < y + \epsilon$ for any $\epsilon > 0$, which is equivalent with $F(X) < G(y+\epsilon)$ for any $\epsilon > 0$. On the other hand, $X \le g^*(y)$ means that $X - \epsilon < g^*(y)$ for any $\epsilon > 0$, which is equivalent with $F(X-\epsilon) \le G(y)$ for any $\epsilon > 0$. By continuity of F, this is equivalent with $F(X) \le G(y)$. Then, we have $\mathbb{Q}(f^*(X) \le y) = \mathbb{Q}(X \le g^*(y))$ because of $\mathbb{Q}(F(X) = G(y)) = 0$.

Next, we have $F(g^*(y)-\epsilon) \le G(y) < F(g^*(y)+\epsilon)$ for any $\epsilon > 0$, when $0 < G(y) < 1$. Then, $F(g^*(y)) = G(y)$ by continuity of F. This is also true when $G(y) = 0$ or $G(y) = 1$. Finally, we obtain $\mathbb{Q}(f^*(X) \le y) = \mathbb{Q}(X \le g^*(y)) = F(g^*(y)) = G(y)$. □

The cheapest super-hedging strategy is obtained by taking a particular probability space such that $Y = f^*(X)$, which means that the random variables X and Y are most "dependent".

Theorem 1. *Suppose that Assumption 1 holds and that f^* is a function defined by Definition 3 which is extended such that $\beta = f^*(\alpha)$ if needed. Then,*

$$\begin{aligned}\Phi(X, f^*(X)) &= \int_\alpha^X \left(\int_{f^*(\alpha)}^{f^*(x)} K(x,y)dy \right) dx \\ &+ \int_{f^*(\alpha)}^{f^*(X)} \left(\int_\alpha^{(f^*)^{-1}_{sup}(y)} K(x,y)dx \right) dy \end{aligned} \tag{1.3.5}$$

and

$$\begin{aligned}\mathbb{E}^M(\Phi(X,Y)) &= \mathbb{E}\left(\int_\alpha^X \left(\int_{f^*(\alpha)}^{f^*(x)} K(x,y)dy \right) dx \right) \\ &+ \mathbb{E}\left(\int_{f^*(\alpha)}^{Y} \left(\int_\alpha^{(f^*)^{-1}_{sup}(y)} K(x,y)dx \right) dy \right), \end{aligned} \tag{1.3.6}$$

where $\mathbb{E}^M$ is the expectation operator of the measure with Maximum copula M defined by Eq.(1.2.5).

Proof. Eq(1.3.5) is followed by Lemma 1. Let x, y be real numbers and $H(x,y) := \mathbb{Q}(X \le x, f^*(X) \le y)$. (i) Suppose that $F(x) < G(y)$. Then we have $f^*(x) \le y$ and $H(x,y) = \mathbb{Q}(X \le x) = F(x)$. (ii) Suppose that $F(x) > G(y)$. Then, we have $f^*(x) \ge y + \epsilon$ for some $\epsilon > 0$ and $H(x,y) = \mathbb{Q}(f^*(X) \le y) = G(y)$. (iii) Suppose that $F(x) = G(y)$. If $f^*(x) = y$, we have $H(x,y) = \mathbb{Q}(X \le x) = F(x)$. Otherwise, we have $H(x,y) = F(x)$ for $f^*(x) < y$ and $H(x,y) = G(y)$ for $f^*(x) > y$. Therefore, the function H is the same as Maximum copula M. □

Remark 3. *Although we have defined α and β as some real numbers which are less than the essential infimum of X and Y respectively, Theorem 1 is also valid for any real numbers α and β such that $\beta = f^*(\alpha)$.*

Remark 4. *Theorem 1 is also valid for functions (e.g. delta function) that can be approximated by series of locally integrable functions.*

Remark 5. *Let $g^*(y) := \inf\{x \in \mathbb{R} \mid G(y) < F(x)\}$. Then, $g^*(Y)$ has the same distribution as X, if G is continuous. In case where both F and G are continuous, we have another hedging strategy where X and f^* are replaced with Y and g^*. However, the hedging strategy dose not depend on the choice, since $f^*(X) = (g^*)^{-1}_{sup}(X)$ holds almost surely.*

Corollary 1. *Suppose that Assumption 1 holds, that $X > 0$ and $Y = 1_A$ for some measurable set A and $\mathbb{Q}(A) > 0$. Then,*

$$-(x_* - X)_+ + x_* 1_A \leq X 1_A \leq (X - x^*)_+ + x^* 1_A, \tag{1.3.7}$$

where x^* and x_* are respectively defined by

$$\begin{aligned} x^* &:= \inf\{x \in \mathbb{R} \mid \mathbb{Q}(A^c) \leq F(x)\} && (1.3.8)\\ x_* &:= \inf\{x \in \mathbb{R} \mid \mathbb{Q}(A) \leq F(x)\}. && (1.3.9) \end{aligned}$$

Proof. Under the assumptions, we have

$$G(y) = \begin{cases} 0 & (y < 0) \\ p & (0 \leq y < 1) \\ 1 & (1 \leq y) \end{cases}, \tag{1.3.10}$$

$$f^*(x) = \begin{cases} 0 & (0 \leq F(x) < p) \\ 1 & (p \leq F(x) \leq 1) \end{cases} \tag{1.3.11}$$

and

$$(f^*)^{-1}(y) = \begin{cases} -\infty & (y < 0) \\ x^* & (0 \leq y < 1) \\ +\infty & (1 \leq y) \end{cases}. \tag{1.3.12}$$

Theorem 1 leads to

$$\begin{aligned} X 1_A &\leq \int_0^X f^*(\xi) d\xi + \int_0^{1_A} (f^*)^{-1}(\xi) d\xi \\ &= (X - x^*)_+ + x^* 1_A. && (1.3.13) \end{aligned}$$

Using the upper bound, we have

$$\begin{aligned} -X 1_{A^c} = X 1_A - X &\leq (X - x^*)_+ - X + x^* 1_A && (1.3.14)\\ &= (x^* - X)_+ - x^* + x^* 1_A \end{aligned}$$

If we view A^c as A, we obtain the lower bound. □

Let α_* and β_* be the essential infimum of X and $-Y$ respectively. Then, we have

$$\begin{aligned}\Phi(X,Y) &= -\int_{\alpha}^{X}\int_{-\beta}^{-Y} K(x,-y)dydx &(1.3.15)\\ &= -\left(\int_{\alpha_*}^{X}\int_{\beta_*}^{-Y} + \int_{\alpha_*}^{X}\int_{-\beta}^{\beta_*} + \int_{\alpha}^{\alpha_*}\int_{-\beta}^{-Y} K(x,-y)dydx\right). &(1.3.16)\end{aligned}$$

We obtain the sub-hedging strategy by applying Theorem 1 to Eq.(1.3.15) or the first integral of Eq.(1.3.16), if all of the tree integrals of Eq.(1.3.16) are finite.

Remark 6. *Note that the sub-hedging strategy is not necessarily determined uniquely, while the lower pricing bound is unique.*

1.4 Examples

In this section, we assume that S_T^X and S_T^Y are positive and their distribution functions are continuous and strictly increasing for simplicity. Moreover, for random variables X and Y and $K : \mathbb{R}\times\mathbb{R} \longrightarrow [0,+\infty)$, let $\Phi_X(x)$ and $\Phi_Y(y)$ be

$$\begin{aligned}\Phi_X(x) &:= \int_{\alpha}^{x}\left(\int_{f^*(\alpha)}^{f^*(x)} K(x,y)dy\right)dx\\ \Phi_Y(y) &:= \int_{f^*(\alpha)}^{y}\left(\int_{\alpha}^{(f^*)^{-1}_{sup}(y)} K(x,y)dx\right)dy, &(1.4.1)\end{aligned}$$

where f^* is defined by Definition 3 for X and Y. Hereafter, we omit sup in f_{sup}^{-1} for simplicity.

1.4.1 Quanto Options

Quanto options are dependent on a price of a foreign asset at maturity and a pre-fixed foreign exchange rate. Let S_T^X be the time-T exchange rate and S_T^Y be the time-T foreign asset price. A quanto call option is a contract which pays the holder a total of

$$(S_T^Y - \kappa)_+ \tag{1.4.2}$$

in the domestic currency, where κ is a positive number. See (Baxter and Rennie, 1996) for more details of the Black formula and (Bennett and Kennedy, 2004) for an application of copulas.

Consider the payoff (1.4.2) denominated in the foreign currency:

$$\frac{1}{S_T^X}(S_T^Y - \kappa)_+. \tag{1.4.3}$$

Then we can directly apply our result to the payoff (2.3.5) with $X = \frac{1}{S_T^X}$, $Y = (S_T^Y - \kappa)_+$ and $K = 1$ and obtain the super-hedging portfolio:

$$XY \le \int_0^X f^*(x)dx + \int_0^Y (f^*)^{-1}(y)dy. \tag{1.4.4}$$

This means that the quanto option is super-hedged by

- an option on the exchange rate whose payoff is $\int_0^{\frac{1}{S_T^X}} f^*(x)dx$
- an option on the foreign asset whose payoff is $\int_0^{(S_T^Y-\kappa)_+}(f^*)^{-1}(y)dy$,

where both of the payoffs are denominated in the foreign currency.

For sub-hedging, we have for any $\alpha > 0$,

$$(X-\alpha)(-Y-\tilde{f}(\alpha)) \leq \int_\alpha^X (\tilde{f}(x)-\tilde{f}(\alpha))dx + \int_{\tilde{f}(\alpha)}^{-Y}(\tilde{f}^{-1}(y)-\alpha)dy. \tag{1.4.5}$$

where $\tilde{f}$ is defined by Definition 3 for X and $-Y$. We obtain the sub-hedging portfolio:

- an option on the exchange rate whose payoff is $-\int_0^{\frac{1}{S_T^X}}(\tilde{f}(x)-\tilde{f}(\alpha))dx - \tilde{f}(\alpha)\frac{1}{S_T^X}$
- an option on the foreign asset whose payoff is $-\int_0^{-(S_T^Y-\kappa)_+}(\tilde{f}^{-1}(y)-\alpha)dy - \alpha((S_T^Y-\kappa)_+ - \tilde{f}(\alpha))$,

where both of the payoffs are denominated in the foreign currency.

Remark 7. *(Tsuzuki, 2011) investigates pricing bounds on quanto options with several numerical examples.*

1.4.2 Exchange Options

Exchange options are options to exchange one risky asset for another (see (Margrabe, 1978)). The options are equivalent to many financial arrangements such as spread options and cross-currency option. (Chung and Wang, 2008) investigates the pricing bounds for a cross-currency option and (Laurence and Wang, 2009) does for spread options. We obtain the same result as theirs.

Let us consider an exchange option whose payoff is:

$$(S_T^X - S_T^Y - \kappa)_+, \tag{1.4.6}$$

where κ is a positive number.

Let $X = S_T^X$, $Y = -S_T^Y$ and $K(x,y) = \delta(x+y-\kappa)$ for deriving the super-hedging portfolio, where $\delta(\cdot)$ is Dirac delta function. Then, the payoff is expressed as:

$$(X+Y-\kappa)_+ = \int_0^X \int_{-\infty}^Y K(x,y)dydx. \tag{1.4.7}$$

Applying Corollary 4, we have

$$\Phi_X(X) = (X-\kappa_X^*)_+ \tag{1.4.8}$$
$$\Phi_Y(Y) = (Y-\kappa_Y^*)_+, \tag{1.4.9}$$

where κ_X^* and κ_Y^* are defined by $\kappa_X^* + f^*(\kappa_X^*) = \kappa$ and by $\kappa_Y^* + (f^*)^{-1}(\kappa_Y^*) = \kappa$ respectively. Note that such κ_X^* and κ_Y^* are uniquely determined by $\mathbb{Q}(S_T^X \leq \kappa_X^*) = \mathbb{Q}(S_T^Y \geq \kappa_Y^*)$ and $\kappa_X^* + \kappa_Y^* = \kappa$. We obtained the hedging portfolio:

- a call option on S_T^X with strike κ_X^*

- a put option on S_T^Y with strike κ_Y^*,

which is the same as (Chung and Wang, 2008) when $\kappa = 0$.

Next, we consider the sub-hedging strategy for the exchange option.

$$\begin{aligned}(X+Y-\kappa)_+ &= -\int_0^X \int_{-\infty}^{-Y} K(x,-y)dydx + \int_0^X \int_{-\infty}^{0} K(x,-y)dydx \\ &\geq -\int_0^X 1_{\{0\le x-\kappa\le \tilde{f}(x)\}}dx - \int_0^{-Y} 1_{\{y+\kappa\le \tilde{f}^{-1}(y)\}}dy + \kappa - (\kappa - X)_+, \end{aligned} \tag{1.4.10}$$

where $\tilde{f}$ is defined by Definition 3 for X and $-Y$. These imply the following sub-hedging portfolio:

- an option on S_T^X whose payoff is $\int_0^{S_T^X} 1_{\{0\le x-\kappa\le \tilde{f}(x)\}}dx - \kappa + (\kappa - S_T^X)_+$
- an option on S_T^Y whose payoff is $\int_0^{S_T^Y} 1_{\{y+\kappa\le \tilde{f}^{-1}(y)\}}dy$.

1.4.3 Basket Options

A basket option is an exotic option whose underlying is a weighted sum of different assets. (Hobson et al., 2005b) derives upper bounds for general n-asset case when prices of call options on each underlying asset with a continuum of strikes or a finite strikes are given. (Hobson et al., 2005a) also investigates lower bounds for 2-asset case under the same circumstance.

We consider a basket option whose underlying is a sum of two assets. Our assumption is the same as the continuum case of (Hobson et al., 2005b) and (Hobson et al., 2005a). Let the payoff of the basket option be

$$(S_T^X + S_T^Y - \kappa)_+, \tag{1.4.11}$$

where κ is a positive number.

In order to derive the super-hedging, let us express the payoff with $X = S_T^X$, $Y = S_T^Y$ and $K(x,y) = \delta(x+y-\kappa)$:

$$(X+Y-\kappa)_+ = \int_{-\infty}^X \int_{-\infty}^Y K(x,y)dydx. \tag{1.4.12}$$

Then, by applying Corollary 4, the payoff of the super-hedging portfolio is

$$\Phi_X(X) = (X-\kappa_X^*)_+ \tag{1.4.13}$$
$$\Phi_Y(Y) = (Y-\kappa_Y^*)_+, \tag{1.4.14}$$

where κ_X^* and κ_Y^* are defined by $\kappa_X^* + f^*(\kappa_X^*) = \kappa$ and by $\kappa_Y^* + \tilde{f}^{-1}(\kappa_Y^*) = \kappa$ respectively. Note that such κ_X^* and κ_Y^* are uniquely determined by $\mathbb{Q}(S_T^X \le \kappa_X^*) = \mathbb{Q}(S_T^Y \le \kappa_Y^*)$ and $\kappa_X^* + \kappa_Y^* = \kappa$. Finally, we obtained the hedging portfolio:

- a call option on S_T^X with strike κ_X^*
- a call option on S_T^Y with strike κ_Y^*.

Let us extend to an N-asset basket option whose underlying assets are S_T^n $(1 \leq n \leq N)$. By appliying mathematical induction, we have the following inequaility:

$$\left(\sum_{n=1}^{N} S_T^n - \kappa\right)_+ \leq \sum_{n=1}^{N} (S_T^n - \kappa_n^*)_+, \tag{1.4.15}$$

where κ_n^* are positive numbers such that $\sum_{n=1}^{N} \kappa_n^* = \kappa$ and $\mathbb{Q}(S_T^n \leq \kappa_n^*)$ is common for all of n.

Next, let us consider the sub-hedging for the basket option:

$$\begin{aligned}(X+Y-\kappa)_+ &= -\int_0^X \int_{-\infty}^{-Y} + \int_0^X \int_{-\infty}^{+\infty} + \int_{-\infty}^{0} \int_{-Y}^{+\infty} K(x,-y)dydx \\ &\geq -\int_0^X 1_{\{x \leq \tilde{f}(x)+\kappa\}} dx - \int_{-\infty}^{-Y} 1_{\{0 \leq y+\kappa \leq \tilde{f}^{-1}(y)\}} dy + X + (Y-\kappa)_+. \end{aligned} \tag{1.4.16}$$

Then, the following portfolio is optimal:

- an option whose payoff is $S_T^X - \int_0^{S_T^X} 1_{\{x \leq \tilde{f}(x)+\kappa\}} dx$
- an option whose payoff is $(S_T^Y - \kappa)_+ - \int_{-\infty}^{-S_T^Y} 1_{\{0 \leq y+\kappa \leq \tilde{f}^{-1}(y)\}} dy$.

Sub-hedging an N-asset basket option$(N > 2)$ is still open to our best knowledge. See (Hobson et al., 2005a).

1.4.4 Forward Starting Options

Forward starting options are options whose strike will be determined at some later date. Let S_t be the underlying asset at time t, T_1 be the date when the strike is determined and T_2 be the date when the payoff is paid. There are two types of payoffs of forward starting options. One is $(S_{T_2} - \kappa S_{T_1})_+$ for a positive number κ. This is equivalent to an exchange option, which has been studied in the previous section and in (Hobson and Neuberger, 2012). Then, in this section, we consider the other type of payoff:

$$\left(\frac{S_{T_2}}{S_{T_1}} - \kappa\right)_+. \tag{1.4.17}$$

First, let us consider super-hedging a payoff XY with $X = S_{T_2}$ and $Y = \frac{1}{S_{T_1}}$. Applying the theorem, we have $XY \leq \Phi_X(X) + \Phi_Y(Y)$, where

$$\Phi_X(X) = \int_0^X f^*(x)dx \tag{1.4.18}$$

$$\Phi_Y(Y) = \int_0^Y (f^*)^{-1}(y)dy. \tag{1.4.19}$$

Then, the payoff of the forward starting option is satisfied with the following inequality as shown in Section 1.4.3:

$$(XY-\kappa)_+ \leq (\Phi_X(X) - \kappa_X)_+ + (\Phi_Y(Y) - \kappa_Y)_+, \tag{1.4.20}$$

where κ_X and κ_Y are any positive number such that $\kappa_X+\kappa_Y=\kappa$. In order to obtain the cheapest super-hedging, let us define x^* by

$$\Phi_X(x^*)+\Phi_Y((f^*)^{-1}(x^*))=\kappa. \tag{1.4.21}$$

Note that x^* is uniquely determined because the left-hand side of (1.4.21) is increasing with respect to x^* from 0 to $+\infty$. If we take $\kappa_X=\Phi_X(x^*)$ and $\kappa_Y=\kappa-\kappa^*$, the right-hand side of (1.4.20) implies the cheapest super-hedging portfolio, because $Y=f^*(X)$ gives an equality of the inequality (1.4.20). Then, the hedging portfolio is as follows:

- an option with maturity T_2 whose payoff is $\left(\int_0^{S_{T_2}} f^*(x)dx-\kappa_X\right)_+$
- an option with maturity T_1 whose payoff is $\left(\int_0^{\frac{1}{S_{T_1}}} (f^*)^{-1}(y)dy-\kappa_Y\right)_+$,

where $\kappa_X=\int_0^{x^*} f^*(x)dx$ and $\kappa_Y=\kappa-\kappa_X$.

Next, let us consider the sub-hedging strategy for the forward starting option. Applying the theorem to XY with $X=S_{T_2}$ and $Y=-\frac{1}{S_{T_1}}$, we have $XY\le\Phi_X(X)+\Phi_Y(Y)$, where

$$\Phi_X(X) = \int_0^X f^*(x)dx \tag{1.4.22}$$

$$\Phi_Y(Y) = \int_{-\infty}^Y (f^*)^{-1}(y)dy. \tag{1.4.23}$$

Then, the payoff of the forward starting option is satisfied with the following inequality:

$$(-XY-\kappa)_+\ge\left(-\Phi_X(X)-\Phi_Y(Y)-\kappa\right)_+. \tag{1.4.24}$$

Since the right-hand side of the above inequality is the same as an exchange option considered in Section 1.4.2, we have

$$\begin{aligned}(-XY-\kappa)_+ &\ge \left(-\Phi_X(X)-\Phi_Y(Y)-\kappa\right)_+\\ &\ge (-\Phi_X(X)-\kappa)_+-\int_0^{-\Phi_X(X)} 1_{\{0\le x-\kappa\le g^*(x)\}}dx\\ &\quad -\int_0^{\Phi_Y(Y)} 1_{\{y+\kappa\le (g^*)^{-1}(y)\}}dy,\end{aligned} \tag{1.4.25}$$

where g^* is defined as Definition 3 for $-\Phi_X(X)$ and $\Phi_Y(Y)$. Since $g^*(-\Phi_X(X))$ and $\Phi_Y(f^*(X))$ have the same distribution as $\Phi_Y(Y)$, we have $g^*\circ(-\Phi_X)=\Phi_Y\circ f^*$ almost surely with respect to F and $g^*(-\Phi_X(X))=\Phi_Y(f^*(X))$ almost surely with respect to $\mathbb{Q}$. If we take $Y=f^*(X)$, then we have $\Phi_Y(Y)=\Phi_Y(f^*(X))$, which means that the two equalities in (1.4.25) occur simultaneously. The hedging portfolio is as follows:

- an option with maturity T_2 whose payoff is the payoff

$$(-\Phi_X(S_{T_2})-\kappa)_+-\int_0^{-\Phi_X(S_{T_2})} 1_{\{0\le x-\kappa\le g^*(x)\}}dx \tag{1.4.26}$$

- an option with maturity T_1 whose payoff is the payoff

$$-\int_0^{\Phi_Y(-\frac{1}{S_{T_1}})} 1_{\{y+\kappa\le (g^*)^{-1}(y)\}} dy. \tag{1.4.27}$$

Remark 8. *Although we assume that the interest rate is zero in this paper, there is a hedging error caused by difference of payment in non-zero interest rate market, because the payoff of the forward starting option is paid at T_2, while one of the components of the hedging portfolio is paid at T_1. It is sufficient that just multiplying a discount factor fills the gap between T_1 and T_2 when interest rate is deterministic.*

Remark 9. *The hedging strategy for an exchange option in Section 1.4.2 can be applied to forward starting options with payoff $(S_{T_2} - \kappa S_{T_1})_+$. It is different from the strategy considered in (Hobson and Neuberger, 2012), which allows to trade forward contracts at time T_1 and requires the martingale condition $\mathbb{E}(S_{T_2} \mid S_{T_1}) = S_{T_1}$.*

1.4.5 Knock-out Options

A knock-out option is a type of an exotic option that provides a payoff only if a certain predetermined event does not occur. Let us first consider an option whose payoff is dependent on an asset price and knock-out event is on another asset:

$$(S_T - \kappa)_+ 1_A, \tag{1.4.28}$$

where S_T is an asset price at maturity and A is an event regarding the other. For example, A is an event that the foreign exchange rate reaches or does not reach a predetermined price, namely "barrier level". An option whose payoff is 1_A is called a one-touch option.

Directly applying Corollary 1 to the payoff with $X = (S_T - \kappa)_+$, we obtain the following inequality:

$$\begin{aligned} & -(x_* - (S_T-\kappa)_+)_+ + x_* 1_A \\ \le\ & (S_T - \kappa)_+ 1_A \\ \le\ & (S_T - (\kappa + x^*))_+ + x^* 1_A, \end{aligned} \tag{1.4.29}$$

where $x^*, x_* > 0$ such that $\mathbb{Q}(A^c) = F(x^*)$ and $\mathbb{Q}(A) = F(x_*)$. This means that the super-hedging portfolio is

- a call option with strike $\kappa + x^*$
- the one touch option

and the sub-hedging portfolio is

- an option whose payoff is $(x_* - (S_T - \kappa)_+)_+$
- the one touch option.

Next, let us consider a knock-out option whose payoff and knock-out event are dependent on a common underlying asset. Suppose that the payoff of the option is:

$$(S_T - \kappa)_+ 1_A, \tag{1.4.30}$$

where $I := [L, U]$ and $A := \{S_t \in I \mid 0 \le \forall t \le T\}$ for some $L, U \in [0, +\infty]$. This is called a single or double barrier call option.

In this case, we cannot obtain the optimal super-hedging and sub-hedging portfolio by directly applying our result. Instead, apply Corollary 1 to $X^* 1_A$ for super-hedging and $X_* 1_A$ for sub-hedging, where $X^* := (S_T - \kappa)_+ 1_{A_T}$ and $X_* := (S_T - \kappa)_+ 1_{A_T} + \infty \cdot 1_{A_T^c}$ (assume $0 \cdot \infty = 0$). Then, we have

$$\begin{aligned} & -(x_* - (S_T - \kappa)_+ 1_{A_T})_+ 1_{A_T} + x_* 1_A \\ \leq \; & (S_T - \kappa)_+ 1_A \\ \leq \; & ((S_T - \kappa)_+ 1_{A_T} - x^*)_+ + x^* 1_A, \end{aligned} \tag{1.4.31}$$

where

$$x^* := \inf\{x \in [0, U - \kappa] \mid \mathbb{Q}(x + \kappa < S_T \leq U) \leq \mathbb{Q}(A)\} \tag{1.4.32}$$

$$x_* := \inf\{x \in [0, U - \kappa] \mid \mathbb{Q}(A) \leq \mathbb{Q}(L \leq S_T \leq x + \kappa)\}. \tag{1.4.33}$$

This means that the super-hedging portfolio is

- an option whose payoff is $((S_T - \kappa)_+ 1_{A_T} - x^*)_+$
- the one touch option

and the sub-hedging portfolio is

- an option whose payoff is $(x_* - (S_T - \kappa)_+ 1_{A_T})_+ 1_{A_T}$
- the one touch option,

which is the same as (Tsuzuki, 2014).

Remark 10. *The reason why the same result as (Tsuzuki, 2014) cannot be obtained by directly applying our result is that the theorem of this paper does not assume any dependency structure between the two random variables, while both of the random variables $(S_T - \kappa)_+$ and 1_A are dependent on S_T.*

1.5 Copula

Some standard notions and well-known results related to the two dimensional Copula are stated in this section (see (Nelson, 1998) for more details).

First of all, let the definition of Copula be introduced.

Definition 4. *A copula is any function $C : [0,1] \times [0,1] \longrightarrow [0,1]$ which has the following properties:*

1. *for every $x, y \in [0,1]$*

$$C(x, 0) = C(0, y) = 0 \tag{1.5.1}$$

and

$$C(x, 1) = x, C(1, y) = y; \tag{1.5.2}$$

2. *for every $x_1, x_2, y_1, y_2 \in [0,1]$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$,*

$$C(x_1, y_1) - C(x_1, y_2) - C(x_2, y_1) + C(x_2, y_2) \geq 0. \tag{1.5.3}$$

Definition 5. *A distribution function F of a random variable X is defined as $F(x) := \mathbb{Q}(X \leq x)$. A distribution function H of two random variables X and Y is defined as $H(x) := \mathbb{Q}(X \leq x, Y \leq y)$. Here, $\mathbb{Q}$ is a probability under which X and Y are defined.*

The following is Sklar's theorem, which elucidates the role that copulas play in the relationship between multivariate distribution functions and their univariate margins.

Lemma 3. *Let H be a joint distribution function with marginal distribution functions F and G. Then there exists a copula function C such that for every $x, y \in \mathbb{R}$*

$$H(x, y) = C(F(x), G(y)). \tag{1.5.4}$$

Moreover, if F and G are continuous, then C is unique. Otherwise, C is uniquely determined on $F(\mathbb{R}) \times G(\mathbb{R})$.

Conversely, if C is a copula and F and G are distribution functions, then the function H defined by Eq.(1.5.4) is a joint distribution function with margins F and G.

Next, we introduce Fréchet-Hoeffding copula boundaries. There are two special copula functions: Minimum copula and Maximum copula, which are defined as follows.

Definition 6. *Minimum copula W is defined by*

$$W(x, y) := \max(x + y - 1, 0). \tag{1.5.5}$$

Maximum copula M is defined by

$$M(x, y) := \min(x, y). \tag{1.5.6}$$

It is easily verified that these functions are copula functions. In addition, these are boundaries in the meanings of the next lemma.

Lemma 4. *For every copula function C and for every $x, y \in [0, 1]$,*

$$W(x, y) \leq C(x, y) \leq M(x, y). \tag{1.5.7}$$

and for every joint distribution function H with marginal distribution functions F and G and for every $x, y \in [0, 1]$,

$$W(F(x), G(y)) \leq H(x, y) \leq M(F(x), G(y)). \tag{1.5.8}$$

Remark 11. *The copula theory holds for $n \in \mathbb{N}$. Note that W is not a copula function for $n > 2$.*

Chapter 2

Pricing Bounds on Quanto Options

This paper proposes model-independent pricing bounds on quanto options and the corresponding replicating strategies, which are static ones whose portfolios consist of plain-vanilla options on the foreign asset and on the FX rate. Since they are derived model-independently, one can make profits with no risk if quanto options are priced outside the bounds. In addition, the pricing bounds can be improved if liquid quanto contracts such as quanto forward contracts are used for replication. Numerical examples show our pricing bounds comparing with the Black pricing formula and that with an ad-hoc adjustment.

2.1 Introduction

This paper proposes model-independent pricing bounds on quanto options and the corresponding replicating strategies.

Quanto options are particular multi-asset options whose payoff are dependent on a price of a foreign asset and a pre-fixed exchange rate. For example, a quanto call option is a contract which pays the holder a total of $\kappa(S_T^A - K)_+$ in the domestic currency, where S_T^A is the foreign asset price denominated in the foreign currency at maturity T, and K and κ are constant. κ is the pre-fixed exchange rate. The fixed exchange rate allows the holder to investment in the foreign asset without carrying the risk of an exchange rate.

In the market, the standard approach to pricing quanto options is based on a Black-type model where correlated lognormal dynamics for the foreign asset prices and the FX rates are assumed. An analytical solution is obtained under the model (see (Baxter and Rennie, 1996)). But, in practice the Black volatility corresponding to a quoted plain-vanilla option price is dependent on the strike of the option, indicating that the assumptions of the Black model do not hold. To incorporate the effect of the volatility smile, practitioners often adopt an ad-hoc approach and modify the Black formula. Note that this ad-hoc modification of the Black formula does not provide a model consistent with the volatility smile observed in the market. This method can lead to a mis-price.

In order to price these options consistently with the volatility smile observed in the market, the distribution of the foreign asset price S_T^A under a domestic risk-neutral measure $\mathbb{Q}^d$ is needed. While the marginal distribution of FX rate S_T^X under $\mathbb{Q}^d$ and that of the foreign asset price S_T^A under a foreign risk-neutral measure $\mathbb{Q}^f$ are known by calibration to each volatility smile respectively, the distribution of the foreign asset price S_T^A under $\mathbb{Q}^d$ cannot be directly known. As shown later, this is equivalent to a joint distribution of the FX rate S_T^X and the foreign asset price S_T^A under $\mathbb{Q}^f$.

A pricing methodology of quanto options is proposed by (Bennett and Kennedy, 2004) based on copulas theory, which is introduced to mathematical finance by (Cherubini and Luciano, 2002). They separate the modeling of the dependence structure of the underlying assets from the modeling of the implied marginal distributions. Another method for quanto options is derived by (Jäckel, 2009), which also assesses the effect of an implied volatility skew for an FX rate on quanto forwards and quanto options of an asset that itself is subject to an implied volatility skew using a simplistic double displaced diffusion model. In an extension to (Jäckel, 2009), (Jäckel, 2010) further investigates the performance of common quanto approximations in a context of stochastic volatility for both the asset and the FX process. While it is well-known that the quanto adjustment in the drift of the underlying has a significant impact on the prices of quanto options, (Giese, 2012) points out that an additional quanto adjustment in the underlying's volatility needs to be considered in the presence of stochastic volatility. In addition, (Giese, 2012) derives closed-form solutions for standard quanto options and uses for calibration quanto forward contracts, which are often liquidly traded.

On the other hand, we provide model-independent pricing bounds on quanto options consistent with volatility smile of the FX rate and the foreign asset as well as the corresponding super-replicating and sub-replicating strategies, whose portfolios consist of plain-vanilla options on the FX rate S_T^X and those on the foreign asset S_T^A. Recently, model-independent super-replication and sub-replication for derivatives on multi-assets have been studied (see (Labordére and Touzi, 2013) and (Tsuzuki, 2013)). Although basket options, spread options and cross-currency options have been well-documented (see (Chung and Wang, 2008), (Hobson et al., 2005b), (Hobson et al., 2005a), (Laurence and Wang, 2003), and (Laurence and Wang, 2009)), this paper is the first attempt to consider model-independent pricing bounds on quanto options to our best knowledge. In addition, we propose another strategy in order to improve our pricing bounds which uses liquid quanto contracts such as quanto forward contracts in the same manner as (Avellaneda et al., 1995).

Pricing bounds on multi-asset derivatives are closely related to theory of comonotonicity (see e.g. (Dhaene et al., 2002) for comonotonicity and (H. Albrecher and Schoutens, 2005), (X. Chen and Vanmaele, 2008) and (K.C. Cheung and Linders, 2013) for the application). Consider a derivative whose payoff is a increasing function on two assets, where marginal distributions of each underlying asset price are known. If a joint distribution on these prices is unknown, a price of the derivative is not uniquely determined in general. An upper bound which a price of the derivative can take is realized when the two asset prices are comonotonic, that is they are perfectly correlated.

The rest of this paper is as followed: In the next section, we describe pricing quanto contracts as well as the Black formula and its ad-hoc modification. The third section introduces our methodology of pricing bounds and the corresponding replicating strategies for quanto contracts as well as its improvement using other quanto contracts. The fourth section provides numerical examples, where we compare our pricing bounds with prices by the Black formula and its ad-hoc modification.

2.2 Quanto Option Pricing

2.2.1 Settings and Notations

For the quanto problem, we need to consider a time-t price of a foreign asset S_t^A and an FX rate S_t^X. Let $D_{t,T}^d$ and $D_{t,T}^f$ be the time-t value in domestic or foreign currency of the zero coupon discount bond with maturity T respectively, $\mathbb{Q}^d$ and $\mathbb{Q}^f$ be the equivalent martingale measures associated with these numeraire and $\mathbb{E}^d[\cdot]$ and $\mathbb{E}^f[\cdot]$ be the expectation operator with respect to these measures. We suppose that each distribution of the time-T prices of a foreign asset S_T^A and an FX rate S_T^X is uniquely determined by prices of call options on each asset.

A quanto call option is a contract which pays the holder a total of

$$\kappa(S_T^A - K)_+ \tag{2.2.1}$$

in the domestic currency. Note that S_T^A is a price denominated in th foreign currency and κ is simply a scaling factor, which represents a pre-determined exchange rate. More generally, a quanto option with the payoff function g is a contract which pays the holder a total of $g\left(S_T^A\right)$ in the domestic currency, which is equivalent to $\frac{1}{S_T^X} g\left(S_T^A\right)$ in the foreign currency.

By standard arbitrage pricing theory, the time-t value in the domestic currency of this quanto option is given by

$$\begin{aligned} G^q(t) &= D_{t,T}^d \mathbb{E}^d\left[g\left(S_T^A\right)\right] \\ &= S_t^X D_{t,T}^f \mathbb{E}^f\left[\frac{1}{S_T^X} g\left(S_T^A\right)\right] \end{aligned} \tag{2.2.2}$$

and that of an options with the payoff function g in the foreign currency:

$$\begin{aligned} G(t) &= D_{t,T}^d \mathbb{E}^d\left[S_T^X g\left(S_T^A\right)\right] \\ &= S_t^X D_{t,T}^f \mathbb{E}^f\left[g\left(S_T^A\right)\right]. \end{aligned} \tag{2.2.3}$$

There is no problem in evaluating Eq.(2.2.3) under our assumption, because the distribution of S_T^A under $\mathbb{Q}^f$ is known. On the other hand, evaluation of Eq.(2.2.2) requires a distribution of S_T^A not under $\mathbb{Q}^d$ but under $\mathbb{Q}^f$, or equivalently a joint distribution of S_T^A and S_T^X under $\mathbb{Q}^f$, which is unknown under our assumption. Uncertainty of a joint distribution implies pricing bounds on the quanto contracts:

$$\inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{S_T^X} g\left(S_T^A\right)\right] \leq \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{S_T^X} g\left(S_T^A\right)\right] \leq \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{S_T^X} g\left(S_T^A\right)\right], \tag{2.2.4}$$

where $\mathcal{P}$ is a set of all equivalent martingale measures associated with the foreign zero coupon bond such that the marginal distribution on S_T^A and S_T^X are equivalent to $\mathbb{Q}^d$ and $\mathbb{Q}^f$ respectively, $\mathbb{Q} \in \mathcal{P}$ and $\mathbb{E}^{\mathbb{Q}}[\cdot]$ is the expectation operator with respect to $\mathbb{Q}$.

2.2.2 Black Formula

We introduce the Black formula for quanto options according to (Baxter and Rennie, 1996) in this subsection. First, an assumption for the formula is as follows.

Assumption 2. *We assume correlated lognormal dynamics for the foreign asset prices and the FX rates under each risk-neutral measure:*

$$dS_t^X = (r^d - r^f) S_t^X dt + \sigma_X S_t^X dW_t^X \tag{2.2.5}$$

$$dS_t^A = (r^f - q) S_t^A dt + \sigma_A S_t^A dW_t^A \tag{2.2.6}$$

$$\rho = \frac{1}{t}\left\langle W^X, W^A \right\rangle_t \tag{2.2.7}$$

where W^X and W^A are Brownian motions under $\mathbb{Q}^d$ and $\mathbb{Q}^f$ respectively, r^d, r^f and q are the domestic interest rate, the foreign interest rate and the dividend rate of the foreign asset S^A, σ_X and σ_A are volatilities of the FX rate S^X and the foreign asset S^A respectively and all of them are constant.

The dynamics of the foreign asset under the domestic risk-neutral measure are derived by (Baxter and Rennie, 1996):

$$dS_t^A = (r^f - q - \rho\sigma_X\sigma_A)S_t^A dt + \sigma_A S_t^A d\tilde{W}_t^A, \tag{2.2.8}$$

where $\tilde{W}^A$ is a Brownian motion under $\mathbb{Q}^d$. Then, the time-t price of a quanto call option under the Black model can be derived:

$$C_B^q(K, \sigma_X, \sigma_A, \rho) := e^{-r^d(T-t)}\left(\tilde{S}_t^A e^{(r^f-q)(T-t)}N(d_1) - KN(d_2)\right), \tag{2.2.9}$$

where

$$\tilde{S}_t^A := S_t^A e^{\rho\sigma_X\sigma_A(T-t)} \tag{2.2.10}$$

$$d_1 := \frac{\log\frac{\tilde{S}_t^A}{K} + (r^f - q + \frac{\sigma_A^2}{2})(T-t)}{\sigma_A\sqrt{T-t}} \tag{2.2.11}$$

$$d_2 := d_1 - \sigma_A\sqrt{T-t} \tag{2.2.12}$$

and $N(\cdot)$ is the standard cumulative Normal distribution function. In particular, the time-t price of a quanto forward contract, which pays the price of the foreign underlying asset S^A at time T converted with a fixed forex rate, is given by:

$$F_B^q(\sigma_X, \sigma_A, \rho) := \tilde{S}_t^A e^{(-r^d+r^f-q)(T-t)}. \tag{2.2.13}$$

We need the following corollary to evaluate pricing bounds on quanto options by the Black formula. It is derived straightforwardly by the Comparison Lemma as it can be found in (Karatzas and Shreve, 1988), Chapter 5, Proposition 2.18.

Corollary 2. *Suppose that Assumption 2 holds. Then, a quanto option price function*

$$G^q(\sigma_X, \sigma_A, \rho) = D_{t,T}^d \mathbb{E}^d\left[g\left(S_T^A\right)\right] \tag{2.2.14}$$

is decreasing with respect to ρ for a non-decreasing function g.

2.2.3 Ad-hoc Adjustment

Although the Black formula admits an analytical solution, the assumptions underlying the Black model do not hold in practice. The Black volatility corresponding to a vanilla option price is dependent on the strike of the option. This well-known feature is termed the volatility smile. The quanto option must be priced consistently with the prices of plain-vanilla options on the foreign asset and those on the FX rate. To incorporate the effect of the volatility smile, practitioners often adopt an ad-hoc approach and modify the Black formula (2.2.9).

Let $\sigma_X(K)$ and $\sigma_A(K)$ be the Black volatility corresponding to the price of the vanilla option on S_T^X and S_T^A respectively. An ad-hoc approximation for the price of the quanto call option is calculated by substituting the actual strike-dependent volatility $\sigma_A(K)$ for σ_A and ATM-volatility σ_X^{ATM} of FX option for σ_X in Eq.(2.2.9):

$$C_A^q(K, \rho) = C_B^q(K, \sigma_X^{ATM}, \sigma_A(K), \rho). \tag{2.2.15}$$

2.3 Pricing Bounds on Quanto Options

This section devotes to deriving pricing bounds and the corresponding super-replication and sub-replication. The domestic interest rate, the foreign interest rate and the dividend yield of the foreign asset are set to be zero for simplicity. The results are still valid when they are not zero.

2.3.1 Review of (Tsuzuki, 2013)

We first review (Tsuzuki, 2013), which provides a general formula of model-independent super-replication and sub-replication for a certain kind of derivatives including quanto options.

Let us consider a derivative which pays $\Phi(X, Y)$ at time T, where X and Y are random variables and $\Phi(X, Y)$ is represented with:

$$\Phi(X, Y) := \int_{\alpha}^{X} \int_{\beta}^{Y} K(x, y) dy dx, \tag{2.3.1}$$

where $K : \mathbb{R} \times \mathbb{R} \longrightarrow [0, +\infty)$, α and β are some real numbers which are less than the essential infimum of X and Y respectively. In addition, we assume that marginal distribution of X and Y are known: $F(x) := \mathbb{Q}(X \leq x)$ and $G(y) := \mathbb{Q}(Y \leq y)$.

Then, an extended version of Young's inequality (See Theorem 2.3 in (Mitroi and Niculescu, 2011)) suggests a super-replicating strategy:

$$\Phi(X, Y) \leq \int_{\alpha}^{x} \left(\int_{f(\alpha)}^{f(x)} K(x, y) dy \right) dx + \int_{f(\alpha)}^{y} \left(\int_{\alpha}^{f^{-1}(y)} K(x, y) dx \right) dy, \tag{2.3.2}$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a non-decreasing function, f^{-1} is the right-continuous inverse function:

$$f^{-1}(y) := \inf\{x \in \mathbb{R} \mid y < f(x)\}. \tag{2.3.3}$$

and $\beta = f(\alpha)$. In particular, if f is chosen as $f^*(x) := \inf\{y \in \mathbb{R} \mid F(x) < G(y)\}$, the value of the super-replicating portfolio is the cheapest, which is also equal to

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}^{\mathbb{Q}} [\Phi(X, Y)] = \mathbb{E}^{\mathbb{M}} [\Phi(X, Y)], \tag{2.3.4}$$

where $\mathcal{P}$ is a set of all equivalent martingale measures such that the marginal distributions on X and Y are equivalent to F and G respectively and $\mathbb{M}$ stands for the joint distribution induced by Maximum copula $\mathbb{Q}^{\mathbb{M}}(X \leq x, Y \leq y) = min(F(x), G(y))$. This shows that the strategy with f^* is the best among possible strategies and its cost is given by an expectation of the payoff under a certain measure.

2.3.2 Pricing Bounds

Let us consider a quanto option whose payoff is $g\left(S_T^A\right)$ at maturity T in the domestic currency. Since the payoff of this option is represented with

$$\frac{1}{S_T^X} \cdot g\left(S_T^A\right) \tag{2.3.5}$$

in the foreign currency, we can directly apply the result of (Tsuzuki, 2013) to the payoff (2.3.5) with $X = \frac{1}{S_T^X}$, $Y = g\left(S_T^A\right)$ and $K = 1$ and obtain the super-replication:

$$XY \leq \int_0^X f^*(x)dx + \int_0^Y (f^*)^{-1}(y)dy. \tag{2.3.6}$$

This means that the quanto option is super-replicated by

- an option on the exchange rate whose payoff is $\int_0^{\frac{1}{S_T^X}} f^*(x)dx$
- an option on the foreign asset whose payoff is $\int_0^{g\left(S_T^A\right)} (f^*)^{-1}(y)dy$,

where both of the payoffs are denominated in the foreign currency. For sub-replication, we have for any $\alpha > 0$,

$$(X-\alpha)(-Y - f_*(\alpha)) \leq \int_\alpha^X (f_*(x) - f_*(\alpha))dx + \int_{f_*(\alpha)}^{-Y} (f_*^{-1}(y) - \alpha)dy, \tag{2.3.7}$$

where f_* is defined for X and $-Y$ in the same manner of f^*. We obtain the sub-replication:

- an option on the exchange rate whose payoff is $-\int_0^{\frac{1}{S_T^X}} (f_*(x) - f_*(\alpha))dx - f_*(\alpha)\frac{1}{S_T^X}$
- an option on the foreign asset whose payoff is $-\int_0^{-g\left(S_T^A\right)} (f_*^{-1}(y) - \alpha)dy - \alpha(g\left(S_T^A\right) - f_*(\alpha))$,

where both of the payoffs are denominated in the foreign currency. Although these payoffs seem to be complicated, they can be replicated with plain-vanilla options on each asset as in (Breeden and Litzenberger, 1978) and (Carr and Madan, 1998).

Remark 12. *The problem of finding the optimal super-replication and sub-replication is a dual problem of finding upper/lower pricing bounds with respect to equivalent martingale measures. While (Tsuzuki, 2013) focuses on the former, (Dhaene et al., 2002) studies the latter using a concept of comonotonicity. Two random variables X and Y are said to be comonotonic (common monotonic), if almost every two outcomes $(X(\omega_1), Y(\omega_1))$ and $(X(\omega_2), Y(\omega_2))$ must be ordered componentwise: $X(\omega_1) \leq X(\omega_2)$, $Y(\omega_1) \leq Y(\omega_2)$ or $X(\omega_2) \leq X(\omega_1)$, $Y(\omega_2) \leq Y(\omega_1)$ (See (Dhaene et al., 2002) for a rigid definition). Eq.(2.3.4) is attained when X and Y are comonotonic. In case of quanto contracts, if a payoff function g is non-decreasing or non-increasing, the upper or lower pricing bound is obtained when $(S_T^X, -S_T^A)$ or (S_T^X, S_T^A) are comonotonic respectively.*

2.3.3 Pricing Bounds Using Liquid Quanto Contracts

There are quanto contracts which are liquidly traded in a market such as quanto forward contracts, which can be used for calibration (see (Giese, 2012)). For the purpose of improving the pricing bounds introduced in the previous subsection, we propose another super-replication and sub-replication using that of (Tsuzuki, 2013) with these quanto contracts. This is in line with (Avellaneda et al., 1995), where they propose super-hedging and sub-hedging strategies based on *uncertain volatility model* and use it to construct hedging portfolios that use other liquid derivatives in addition to the underlying asset.

Let us assume that there are N liquid quanto contracts whose payoff is $h_i(S_T^A)$ and market prices are given by H_i for $i = 1, \cdots, N$. Under no-arbitrage condition, there must exist a probability measure $\mathbb{Q} \in \mathcal{P}$ such that for any $a \in \mathbb{R}^N$

$$\langle a, H \rangle = \mathbb{E}_X^{\mathbb{Q}}\left[\left\langle a, h\left(S_T^A\right)\right\rangle\right], \tag{2.3.8}$$

where $\mathbb{E}_X^{\mathbb{Q}}[\cdot] := \mathbb{E}^{\mathbb{Q}}\left[\cdot \frac{1}{S_T^X}\right]$, $\langle \cdot, \cdot \rangle$ represents an inner product in $\mathbb{R}^N$, $h(s) := (h_i(s)) \in \mathbb{R}^N$ for $s \in \mathbb{R}$ and $H := (H_i) \in \mathbb{R}^N$. A contract that we consider is a quanto contract with payoff $g\left(S_T^A\right)$ whose pricing bounds are finite: $\inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_X^{\mathbb{Q}}\left[g\left(S_T^A\right)\right] > -\infty$ and $\sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_X^{\mathbb{Q}}\left[g\left(S_T^A\right)\right] < +\infty$.

We consider a super-replication where a_i amounts of the i-th quanto contract are used for replication and the result of (Tsuzuki, 2013) is applied with $X = \frac{1}{S_T^X}$, $Y = g\left(S_T^A\right) - \left\langle a, h\left(S_T^A\right)\right\rangle$ and $K = 1$. Then, a price of the quanto contracts based on this strategy is given by

$$\begin{aligned} V_G(a) &:= \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_X^{\mathbb{Q}}\left[g\left(S_T^A\right) - \left\langle a, h\left(S_T^A\right)\right\rangle\right] + \langle a, H\rangle \\ &= \mathbb{E}_X^{a}\left[g\left(S_T^A\right) - \left\langle a, h\left(S_T^A\right)\right\rangle\right] + \langle a, H\rangle, \end{aligned} \tag{2.3.9}$$

where $\mathbb{E}_X^a[\cdot] := \mathbb{E}_X^{\mathbb{Q}^a}[\cdot]$ and $\mathbb{Q}^a$ is defined using the maximum copula for X and $g\left(S_T^A\right) - \left\langle a, h\left(S_T^A\right)\right\rangle$. Note that not every $\mathbb{Q} \in \mathcal{P}$ reproduces the market prices for all of the liquid quanto options. The strategy we propose is the optimal one with respect to $a \in \mathbb{R}^N$ and the pricing upper bound derived from it is:

$$V_G^* := \inf_{a\in\mathbb{R}^N} V_G(a). \tag{2.3.10}$$

Let us investigate properties of the function $V_G : \mathbb{R}^N \longrightarrow \mathbb{R}$. First of all, it is convex because we have for $a = (1-t)a_1 + ta_2$ with $a_1, a_2 \in \mathbb{R}^N, t \in [0,1]$:

$$\begin{aligned} V_G(a) &\leq (1-t)\sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_X^{\mathbb{Q}}\left[g\left(S_T^A\right) - \left\langle a_1, h\left(S_T^A\right)\right\rangle\right] + t\sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_X^{\mathbb{Q}}\left[g\left(S_T^A\right) - \left\langle a_2, h\left(S_T^A\right)\right\rangle\right] + \langle a, H\rangle \\ &= (1-t)V_G(a_1) + tV_G(a_2). \end{aligned} \tag{2.3.11}$$

Next, V_G^* is finite. Indeed, for $a \in \mathbb{R}^N$ and $\mathbb{Q} \in \mathcal{P}$, it holds

$$V_G(a) \geq \inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_X^{\mathbb{Q}}\left[g\left(S_T^A\right)\right] + \langle a, H\rangle - \mathbb{E}_X^{\mathbb{Q}}\left[\left\langle a, h\left(S_T^A\right)\right\rangle\right]. \tag{2.3.12}$$

In particular, if $\mathbb{Q} \in \mathcal{P}$ is satisfied with $\langle a, H\rangle = \mathbb{E}_X^{\mathbb{Q}}\left[\left\langle a, h\left(S_T^A\right)\right\rangle\right]$, it is found that

$$\inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_X^{\mathbb{Q}}\left[g\left(S_T^A\right)\right] \leq V_G^* \leq V_G(0) = \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_X^{\mathbb{Q}}\left[g\left(S_T^A\right)\right], \tag{2.3.13}$$

which implies that V_G^* is finite by our assumption. Moreover, if there exists $\mathbb{Q} \in \mathcal{P}$ for $a \in \mathbb{R}^N$ with $||a|| = 1$ such that $\langle a, H\rangle > \mathbb{E}_X^{\mathbb{Q}}\left[\left\langle a, h\left(S_T^A\right)\right\rangle\right]$, then $V_G(a) \longrightarrow +\infty$ as $||a|| \longrightarrow +\infty$. In this case, the optimization problem (6.3.1) is attained by a certain $a^* \in \mathbb{R}^N$, that is $V_G^* = V_G(a^*)$.

Remark 13. *Suppose that there exists $a \in \mathbb{R}^N$ with $||a|| = 1$ such that*

$$\inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_X^{\mathbb{Q}}\left[\left\langle a, h\left(S_T^A\right)\right\rangle\right] \geq \langle a, H\rangle, \tag{2.3.14}$$

which implies that the market price of a quanto contract with payoff $\left\langle a, h\left(S_T^A\right)\right\rangle$ is equal to the model-independent lower bound using options on each underlying asset, since some measure $\mathbb{Q}_* \in \mathcal{P}$ attains the equality. Let us consider a quanto contract whose model-independent upper bound are given by the same measure $\mathbb{Q}_*$. Then, we have $V_G(ka) = V_G(0) = \mathbb{E}_X^{\mathbb{Q}_*}\left[g\left(S_T^A\right)\right]$ for any $k \geq 0$ and $V_G(ka) \geq V_G(0)$ for $k \in \mathbb{R}$ by convexity of V_G. This means that the quanto contract with payoff $\left\langle a, h\left(S_T^A\right)\right\rangle$ makes no contribution to improving super-replication on the quanto contract with payoff $g\left(S_T^A\right)$. For example, if the quanto forward contract is quoted in a market with its lower bound, the upper bound on a quanto put option, whose payoff is $(K - S_T^A)_+$, is not improved by using the quanto forward contract.

Let $\varphi(x, s; a)$ be a density function of X and S_T^A for the measure $\mathbb{Q}^a$, which is assumed to be differentiable with respect to a. Then, we have

$$\begin{aligned} V_G(a) &= \int\int x((g(s) - \langle a, h(s)\rangle)\, \varphi(x, s; a) dx ds + \langle a, H\rangle \quad (2.3.15)\\ \nabla V_G(a) &= \int\int x((g(s) - \langle a, h(s)\rangle)\nabla\varphi(x, s; a) dx ds - \int\int x h(s)\varphi(x, s; a) dx ds + H, \quad (2.3.16)\end{aligned}$$

where ∇ is a differential operator with respect to a defined by $\nabla := \left(\frac{\partial}{\partial a_i}\right)$. It is reasonable to assume that the first term of the right hand side of Eq.(2.3.16) equals 0, because a function Ψ of $\alpha \in \mathbb{R}^N$ defined by $\Psi(\alpha; a) := \mathbb{E}_X^\alpha \left[g\left(S_T^A\right) - \left\langle a, h\left(S_T^A\right)\right\rangle\right]$ takes the largest value at $\alpha = a$. Then, we have $\mathbb{E}_X^a \left[h_i\left(S_T^A\right)\right] = H_i$ for $a \in \mathbb{R}^N$ with $\nabla V_G(a) = 0$, which means that a probability measure $\mathbb{Q}^a$ reproduces the market price of the liquid quanto contracts. In other words, if $\nabla V_G(a) = 0$, the price V_G^* is the best possible:

$$V_G^* = \sup_{\mathbb{Q}\in\tilde{\mathcal{P}}} \mathbb{E}_X^{\mathbb{Q}} \left[g\left(S_T^A\right)\right], \quad (2.3.17)$$

where $\tilde{\mathcal{P}}$ is a subset of $\mathcal{P}$ such that every element of it reproduces the market prices of the liquid quanto contracts H_i for $i = 1, \cdots, N$.

Finally, let us conclude this section by posing further problems. We have seen that pricing bounds would be tighter if there would be more liquid quanto contract used for replication. For an arbitrary joint distribution $\mathbb{Q}$, a set $\mathcal{H} := \left\{g\left(S_T^A\right) \mid \mathbb{E}_X^{\mathbb{Q}}\left[g\left(S_T^A\right)^2\right] < +\infty\right\}$ is a Hilbert space where the inner product is defined by $\mathbb{E}_X^{\mathbb{Q}}\left[g_1\left(S_T^A\right) g_2\left(S_T^A\right)\right]$ for $g_1\left(S_T^A\right)$ and $g_2\left(S_T^A\right) \in \mathcal{H}$. Any element $g\left(S_T^A\right) \in \mathcal{H}$ can be approximated by a basis in the Hilbert space $\mathcal{H}$, which is interpreted from a financial point of view that any quanto contracts can be replicated with sufficient accuracy using a family of liquid quanto contracts whose payoffs are a basis in the Hilbert space $\mathcal{H}$. Practically, it is desirable for a smaller subset of a basis to span a larger subspace.

The joint distribution of S_T^X and S_T^A is, however, assumed to be unknown in this paper and working on a fixed Hilbert space $\mathcal{H}$ is not allowed. Then, an interesting problem is to find a smaller subset of a basis which spans a large subspace without fixing a Hilbert space. This problem requires formulations: which Banach space is considered, which criterion is adopted to gauge how large a spanned space is and so on. A solution to this problem would be a guideline for market makers of quanto contracts to efficiently provide liquidity to the market. This is our next research topic.

2.3.4 Quanto Call/Put Options

In this subsection, we focus on quanto call and put option, whose payoffs in the domestic currency are defined by $(S_T^A - K)_+$ and $(K - S_T^A)_+$ with a strike price K. First, put-call parity also holds for quanto call and put as well as for plain-vanilla options:

$$(S_T^A - K)_+ = (S_T^A - K) + (K - S_T^A)_+. \quad (2.3.18)$$

Since the payoff functions of call and put are non-decreasing and non-increasing, measures which produce upper bounds and lower bounds are comonotonic ones for (S_T^X, S_T^A) and $(S_T^X, -S_T^A)$. Let $C_L(K)$, $C_G(K)$, $P_L(K)$, $P_G(K)$, F_L and F_G be lower and upper bounds for quanto call and put options and a forward contract.

Then, we have

$$C_L(K) = \mathbb{E}_X^{\mathbb{Q}^*}\left[\left(S_T^A - K\right)_+\right], \quad C_G(K) = \mathbb{E}_X^{\mathbb{Q}_*}\left[\left(S_T^A - K\right)_+\right], \tag{2.3.19}$$

$$P_L(K) = \mathbb{E}_X^{\mathbb{Q}_*}\left[\left(K - S_T^A\right)_+\right], \quad P_G(K) = \mathbb{E}_X^{\mathbb{Q}^*}\left[\left(K - S_T^A\right)_+\right], \tag{2.3.20}$$

$$F_L = \mathbb{E}_X^{\mathbb{Q}^*}\left[S_T^A\right], \quad F_G = \mathbb{E}_X^{\mathbb{Q}_*}\left[S_T^A\right], \tag{2.3.21}$$

where $\mathbb{Q}^*$ and $\mathbb{Q}_*$ are measures under which $(S_T^X, -S_T^A)$ and (S_T^X, S_T^A) are comonotonic respectively.

A distance between an upper bound and a lower bound, $\sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_X^{\mathbb{Q}}\left[g\left(S_T^A\right)\right] - \inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_X^{\mathbb{Q}}\left[g\left(S_T^A\right)\right]$, can be regarded as a measure for uncertainty of price. Using put-call parity, we have

$$(C_G(K) - C_L(K)) + (P_G(K) - P_L(K)) = F_G - F_L, \tag{2.3.22}$$

which means that a sum of uncertainty of price for a quanto call option and that for a put option is that for a quanto forward contract. In addition, if a quanto forward contract is used for replication, price uncertainty for a quanto call option is equal to that for a quanto put option:

$$C_G^*(K) - C_L^*(K) = P_G^*(K) - P_L^*(K), \tag{2.3.23}$$

where $C_G^*(K)$, $C_L^*(K)$, $P_G^*(K)$ and $P_L^*(K)$ are respectively upper and lower bounds with using a quanto forward contract.

2.4 Numerical Examples

In this section, we show numerical examples of our pricing bounds comparing with the Black pricing formula and that with an ad-hoc adjustment.

Suppose that the initial FX rate is $S_0^X = 1.0$, the initial stock price is $S_0^A = 100.0$, and the domestic interest rate, the foreign interest rate and the dividend yield of the foreign asset are set to be zero for simplicity. The maturities of all options in this section are $T = 1.0$. Volatilities of FX volatility and those of the foreign asset volatility are listed in Table 2.1[1] and Table 2.2 respectively. We assume that a quanto forward is liquid and its price is calculated with a joint distribution where S_T^X and S_T^X are independent.

Table 2.1: Implied Volatilities of the FX Rate

strike	0.8	0.9	1.0	1.1	1.2
volatility(%)	15.0	12.0	10.0	11.0	11.0

Table 2.2: Implied Volatilities of the Foreign Asset

strike	80	90	100	110	120
volatility(%)	30.0	25.0	20.0	15.0	15.0

We calculate prices of quanto call and put options with strike prices from 85 to 115:

- C_L, C_G and P_L, P_G are pricing bounds without using a quanto forward contract.

[1]The strike prices are denominated in the foreign currency, namely $\frac{1}{S^X} = 0.8, 0.9, \cdots$.

- C_L^*, C_G^* and P_G^*, P_G^* are pricing bounds using a quanto forward contract.
- C and P are quanto call and put option prices which are also calculated under a joint distribution where S_T^X and S_T^X are independent.
- $C_B^q(\cdot,\rho)$ and $P_B^q(\cdot,\rho)$ are prices of the Black pricing formula with $\sigma_X = 10\%$, $\sigma_A = 20\%$ and correlation ρ (see Eq.(2.2.9)).
- $C_A^q(\cdot,\rho)$ and $P_A^q(\cdot,\rho)$ are prices of the ad-hoc adjusted Black pricing formula with $\sigma_X = 10\%$, $\sigma_A = \sigma_A(K)$ and correlation ρ (see Eq.(2.2.15)).

Table 2.3 reports the numerical results. It is noteworthy that neither of the Black pricing formula nor the ad-hoc adjusted Black pricing formula provides prices for quanto contracts which excludes arbitrage opportunities; some of them are outside our pricing bounds. Largely, the Black pricing formula provides smaller values for lower strikes because it uses smaller volatilities, while the ad-hoc adjusted Black pricing formula does larger values because it uses larger volatilities.

The pricing bounds without using a quanto forward C_L, C_G, P_L and P_G seem wide, which is typical for model-independent pricing bounds. On the other hand, those using a quanto forward C_L^*, C_G^*, P_L^* and P_G^* are actually improved especially for cases of in-the-money. It is not surprising because both of a quanto forward contract and quanto options in the money are significantly exposed to correlation risk between S_T^X and S_T^X and using a quanto forward contract is expected to lead substantial reduction of correlation risk.

Table 2.3: Prices of Quanto Options

strike	85	90	95	100	105	110	115
C_G	20.71	16.49	12.55	8.83	5.67	3.00	1.86
C_G^*	19.95	15.92	12.14	8.58	5.53	2.98	1.84
C	19.40	15.32	11.55	8.00	5.02	2.52	1.53
C_L^*	18.80	14.68	10.93	7.40	4.49	2.10	1.27
C_L	18.03	14.10	10.51	7.16	4.38	2.10	1.27
$C_B^q(\cdot,1)$	18.84	15.11	11.85	9.10	6.84	5.04	3.65
$C_B^q(\cdot,0)$	17.16	13.59	10.52	7.97	5.91	4.29	3.06
$C_B^q(\cdot,-1)$	15.57	12.17	9.29	6.94	5.07	3.63	2.55
$C_A^q(\cdot,1)$	21.54	17.11	13.00	9.10	5.79	2.96	1.84
$C_A^q(\cdot,0)$	19.35	15.27	11.51	7.97	4.99	2.50	1.52
$C_A^q(\cdot,-1)$	17.31	13.57	10.14	6.94	4.28	2.10	1.25
$C_G^* - C_L^*$	1.15	1.23	1.21	1.19	1.04	0.88	0.57
$C_G - C_L$	2.68	2.39	2.04	1.68	1.28	0.90	0.59
$C_B^q(\cdot,1) - C_B^q(\cdot,-1)$	3.28	2.94	2.56	2.16	1.77	1.42	1.10
$C_A^q(\cdot,1) - C_A^q(\cdot,-1)$	4.23	3.54	2.87	2.16	1.51	0.86	0.59
P_G	5.17	6.24	7.64	9.28	11.50	14.21	18.36
P_G^*	4.87	5.83	7.05	8.50	10.45	12.90	16.76
P	4.32	5.23	6.47	7.92	9.94	12.44	16.45
P_L^*	3.72	4.60	5.85	7.31	9.40	12.02	16.19
P_L	3.46	4.23	5.28	6.55	8.37	10.67	14.49
$P_B^q(\cdot,-1)$	2.55	4.15	6.27	8.92	12.05	15.61	19.53
$P_B^q(\cdot,0)$	2.16	3.59	5.52	7.97	10.91	14.29	18.06
$P_B^q(\cdot,1)$	1.82	3.09	4.83	7.08	9.82	13.02	16.63
$P_A^q(\cdot,-1)$	5.03	6.04	7.37	8.92	11.03	13.59	17.74
$P_A^q(\cdot,0)$	4.35	5.27	6.51	7.97	9.99	12.50	16.52
$P_A^q(\cdot,1)$	3.74	4.57	5.71	7.08	9.00	11.45	15.33
$P_G^* - P_L^*$	1.15	1.23	1.21	1.19	1.04	0.88	0.57
$P_G - P_L$	1.71	2.01	2.36	2.73	3.13	3.54	3.87
$P_B^q(\cdot,-1) - P_B^q(\cdot,1)$	0.72	1.06	1.44	1.84	2.23	2.59	2.90
$P_A^q(\cdot,-1) - P_A^q(\cdot,1)$	1.29	1.46	1.66	1.84	2.03	2.14	2.41

Part II

経路依存型デリバティブの優劣複製

Chapter 3

Pricing Bounds on Barrier Options

This paper proposes the optimal pricing bounds on barrier options in an environment where plain-vanilla options and no-touch options can be used as hedging instruments.

Super-hedging and sub-hedging portfolios are derived without specifying any underlying processes, which are static ones consisting of not only plain-vanilla options but also cash-paying no-touch options and/or asset paying no-touch options that pay one cash or one underlying asset respectively if the barrier has not been hit. Moreover, the prices of these portfolios turn out to be the optimal pricing bounds through finding risk-neutral measures under which the barrier option price is equal to the hedging portfolio's value.

The model-independent pricing bounds are useful because a price of a barrier option is significantly dependent on a model. It is demonstrated through numerical examples that prices outside the pricing bounds can be produced by models which are calibrated to market prices of plain-vanilla options, but not to that of a no-touch option.[1]

3.1 Introduction

This paper proposes pricing bounds on barrier options.

Pricing and hedging barrier options have been researched widely so far. In particular, a lot of methods which semi-statically hedge barrier options have been proposed by several researchers (see e.g. (Carr and Chou, 1997), (Carr et al., 1998), (Derman et al., 1995), (Fink, 2003)). Here, semi-static hedging means replication of barrier options by trading plain-vanilla options at no more than one time after inception. Since plain-vanilla options are needed for these hedging strategies, models which price barrier options must be calibrated to plain-vanilla options.

However, model risk on the valuation of barrier options has been pointed out, even if the model is perfectly calibrated to a volatility surface. For instance, it is documented in (Hirsa et al., 2003) and (Lipton and McGhee, 2002) that models may produce similar prices of plain-vanilla options, yet give markedly different prices of barrier options. As a result, they demonstrate that static hedging of barrier options with plain-vanilla options is model dependent. (Schoutens et al., 2005) also reports a general feature of pricing exotic options under several models calibrated nicely to the same volatility surface. Their results about barrier options show that the variation can be significant, especially if the spot price is close to the barrier level.

[1]Electronic version of an article published as Journal of Futures Markets, 34(12):1170–1184, 2014, DOI: 10.1002/fut.21641, ©2013 Wiley Periodicals, Inc., http://onlinelibrary.wiley.com/journal/10.1002/(ISSN)1096-9934

Model risk of barrier options is explained as follows. Since barrier options are path-dependent options, joint distributions of the underlying asset prices at different time points are significantly important in order to price or hedge these options. In contrast, plain-vanilla options determine only the marginal distribution of the risk-neutral measure, but leave considerable freedom in the specification of joint distributions. Joint distributions are determined not by market prices but by models, if the models are calibrated only to plain-vanilla options. Then, what model is used to price barrier options is heavily important. This implies that models must be not only matched to the market prices of plain-vanilla options, but also be consistent with the observable prices of exotic options. (Carr and Crosby, 2010) proposes a model which is calibrated to both the market prices of plain-vanilla options and the observable prices of exotic options. They consider pricing of barrier options in the foreign exchange (FX) options market. In this market, the most actively traded barrier options are double-no-touch (DNT) options. They regard DNT options as instruments which are calibrated to.

In contrast, this study proposes not exact prices for barrier options but pricing bounds, using no-touch options as well as plain-vanilla options, where a no-touch option is a knock-out option which is worthless if the barrier is hit, pays one cash at maturity if the barrier has not been hit. Moreover, the corresponding super-hedging and sub-hedging portfolios are also provided, which are static ones consisting of plain-vanilla options and no-touch options. They are derived without any specification of underlying processes and are the optimal pricing bounds among model-independent bounds. The optimality is proved through finding risk-neutral measures under which the barrier option price is equal to the hedging portfolio's value. In addition, if one can use another type of a no-touch option which pays one underlying asset at maturity if the barrier has not been hit, the pricing bounds are much tighter. In particular, this no-touch option is common in FX options market because it is nothing but a no-touch option which pays one cash for a foreign trader. Hereafter, these no-touch options are called cash-paying no-touch option and asset-paying no-touch option respectively in this article.

The model-independent pricing bounds and the hedging portfolios are useful for checking a barrier option's price. If the price is outside the pricing bounds, it provides an arbitrage opportunity which yields a profit without any risk. It will be demonstrated through numerical examples in this paper that prices outside the pricing bounds may be produced by models which are calibrated to market prices of plain-vanilla options, but not to that of a no-touch option. As (Carr and Crosby, 2010) does, this study also suggests that models for pricing barrier options should be calibrated to the market prices of no-touch options. Otherwise, it is likely that arbitrage opportunities are provided.

The strategy proposed in this article is unique, while model-independent pricing bounds and the hedging portfolios have been studied so far. For example, (Brown et al., 2001) and (Cox and Obłój, 2011a) derive hedging portfolios which consist of only plain-vanilla options and require a certain transaction at the first hitting time, assuming that plain-vanilla options are liquidly traded and an underlying asset price itself is a martingale. The approach of this study is more robust than theirs because their assumption is violated in the real markets with nonzero interest rates and it is not necessarily possible to trade during the turmoil periods, which may cause substantial hedging errors. It should be pointed out that the strategy is crucially dependent on liquidity of no-touch options, while (Brown et al., 2001) is so general that they can be applied to these options as well. Pricing bounds and the corresponding hedging strategies for no-touch options are well documented in these (Brown et al., 2001) and (Cox and Obłój, 2011a). Also, pricing no-touch options under several models is studied by (Lipton and McGhee, 2002). An interesting point in this study is, however, to make use of market prices of no-touch options for the purpose of finding arbitrage opportunities. Actually, these options are likely to be overestimated or underestimated, since they are highly dependent on models and market views of traders.

Numerical examples are used to demonstrate how useful the bounds are for pricing and hedging barrier options. The pricing bounds are compared with exact prices under several models which are calibrated to plain-vanilla options and with pricing bounds proposed by (Brown et al., 2001). In particular, a comparison with the prices in (Schoutens et al., 2005) shows that some models produce prices outside the model-independent bounds. It is suggested that great care should be taken when choosing models to price barrier options and the models should be calibrated to both the market prices of plain-vanilla options and no-touch options.

The rest of this paper is as follows: In the next section, the setup and the problem considered in this paper are described. The third section is devoted to showing pricing bounds on the barrier option and hedging strategies corresponding to them. The fourth section provides numerical examples. Finally, concluding remarks are given in the last section.

3.2 Setup

Consider a problem of pricing and hedging barrier options in an environment where plain-vanilla options and no-touch options are liquid.

In order to state the problem precisely, some assumptions and notations are introduced. First, a barrier option under consideration is assumed to be a single or double knock-out option with maturity T, payoff g and barrier levels l, u where $0 \leq l < u \leq +\infty$. This option is worthless if l or u is hit at any time during its life. If the barrier has not been hit by the expiration date, the terminal payoff is g. Let S_t be the spot price of the underlying asset at time $t \in [0, T]$ and $l < S_0 < u$. Then, the payoff of the barrier option is:

$$g(S_T)1_A, \tag{3.2.1}$$

where $I := [l, u]$ and $A := \{S_t \in I \mid 0 \leq \forall t \leq T\}$. A cash-paying no-touch option with maturity T and barrier levels l, u is a knock-out option which is worthless if the barrier is hit, pays one cash at maturity if the barrier has not been hit. The payoff is 1_A. An asset-paying no-touch option pays one underlying asset instead of one cash. The payoff is $S_T 1_A$.

Next, some assumptions on the market environment are described. The risk-free interest rate r and the dividend yield q of the underlying asset are assumed to be constant for simplicity. Different from other research of barrier options, these two quantities are not required to be equal. In addition, the time-0 prices of all plain-vanilla options with maturity T and a cash-paying no-touch option with the barrier level l, u and the same maturity are assumed to be known. That is, the risk-neutral distribution of S_T is uniquely determined by prices of plain-vanilla options and the risk-neutral probability that the barrier is hit at any time during its life is known. Furthermore, it is assumed that the prices of plain-vanilla options are twice continuously differentiable by strike and the second order derivative is positive. The last assumption ensures that the density of the random variable S_T exists and is a continuous function.

All one knows is the risk-neutral distribution of S_T and the risk-neutral probability that the barrier is hit until time T. In these settings, one has no information about the risk-neutral distribution of S_t $(t < T)$.

The problem is stated as follows: *How can a knock-out call option be priced and hedged in an environment where the underlying asset, all plain-vanilla options with the same maturity and a cash-paying no-touch option with the same maturity and the same barrier levels can be used as hedging instruments?*

Remark 14. *This paper considers a knock-out call option with a single or double barrier. The theorems in this paper introduced in Section 3.3 are however valid for other types of barrier options if it is slightly modified. For example, the knock-out condition A can be replaced with $\{S_t \in I \mid \forall t \in J\}$, where J is a subset of the interval $[0, T]$, and with $\{X_t \in I \mid 0 \leq \forall t \leq T\}$, where X is an another asset price process.*

The approach in this paper to this problem is not to derive an exact price and an exact hedging, but to derive pricing bounds, and super-hedging and sub-hedging strategies corresponding to the bounds. These pricing bounds are the optimal under a certain condition, which means that the pricing bounds cannot be improved without additional condition.

The undiscounted price, or forward price, of a call option and a cash-paying no-touch option at time t are denoted by $C_t(K)$ and $N_t^C(I)$. All one requires of a pricing framework is that they are calibrated to market prices of these options. In order to define this requirement, the following definition is introduced.

Definition 7. *A pair of a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and a stochastic process $\{S_t\}_{t\in[0,T]}$ on it is a calibrated probability space if and only if*

$$C_0(K) = \mathbb{E}((S_T - K)_+), \tag{3.2.2}$$

$$N_0^C(I) = \mathbb{E}(1_A), \tag{3.2.3}$$

where $\forall K > 0$ and $\mathbb{E}$ is the expectation operator under $\mathbb{Q}$. $\mathcal{P}^C$ is a set of all calibrated probability spaces.

In addition, a constraint on trading strategies is imposed: trading is allowed only at the initial time. This kind of strategies is called a *static* strategy in this paper. Limiting the strategy static one means that it is not required that the process $\{e^{-(r-q)t}S_t\}_{t\in[0,T)}$ is a continuous-time martingale[2] and that all of probability one has to consider is $\mathcal{P}^C$.

If a market is incomplete, the price of the barrier option cannot be determined uniquely. The next best thing is to derive the optimal pricing bounds. If there are no arbitrage opportunities, undiscounted values of the barrier option must lie somewhere between the bounds $[W_L^C, W_G^C]$, where

$$W_L^C := \inf_{\mathcal{P}^C} \mathbb{E}(g(S_T)1_A), \tag{3.2.4}$$

$$W_G^C := \sup_{\mathcal{P}^C} \mathbb{E}(g(S_T)1_A). \tag{3.2.5}$$

This study derives pricing bounds which are enforced by a static super-hedging strategy and a static sub-hedging strategy. In particular, the bounds are turned out to be the optimal, which means that they are equal to $[W_L^C, W_G^C]$. This is proved by constructing specific calibrated probability spaces under which a price of the barrier option is equal to the super-hedging and sub-hedging values respectively.

After establishing pricing bounds for this problem, it is considered how the pricing bounds are improved by adding an assumption that one can use an asset-paying no-touch option as a hedging instrument. Similarly to the problem stated above, the following notations are defined:

Definition 8. *$\mathcal{P}^A$ is a subset of $\mathcal{P}^C$ such that its element satisfies*

$$N_0^A(I) = \mathbb{E}(S_T 1_A), \tag{3.2.6}$$

where $N_0^A(I)$ is an undiscounted price of the asset-paying no-touch option.

The pricing bounds with this probability space $\mathcal{P}^A$ are

$$W_L^A := \inf_{\mathcal{P}^A} \mathbb{E}(g(S_T)1_A), \tag{3.2.7}$$

$$W_G^A := \sup_{\mathcal{P}^A} \mathbb{E}(g(S_T)1_A). \tag{3.2.8}$$

Lastly, a measure to be considered is a component of an element of $\mathcal{P}^C or \mathcal{P}^A$ hereafter in this paper. Then, $\mathbb{Q}$ or $\mathbb{E}$ is respectively used to denote the risk-neutral distribution or expectation of S_T and of A for simplicity if it is not ambiguous, because they do not depend on a choice of an element of $\mathcal{P}^C or \mathcal{P}^A$.

[2] It is required that the discrete-time process $\{S_0, e^{-(r-q)T}S_T\}$ is a martingale, but it is imposed from the prices of plain-vanilla options.

3.3 Pricing Bounds

This section derives pricing bounds on barrier options and hedging portfolios corresponding to the bounds. Theorem 2 is the main theorem of this paper.

Theorem 2. *Define* $A_* \subseteq I$ *such that* $\sup_{s\in A_*} g(s) \leq \inf_{s\notin A_*} g(s)$ *and* $A^* \subseteq I$ *such that* $\sup_{s\notin A^*} g(s) \leq \inf_{s\in A^*} g(s)$*. Then,*

$$
\begin{aligned}
& (g(S_T) - g_*)1_{\{S_T\in A_*\}} + g_*1_A && (3.3.1)\\
\leq\ & g(S_T)1_A && (3.3.2)\\
\leq\ & (g(S_T) - g^*)1_{\{S_T\in A^*\}} + g^*1_A, && (3.3.3)
\end{aligned}
$$

where g_* *and* g^* *are arbitrary values such that* $g_* \in [\sup_{s\in A_*} g(s), \inf_{s\notin A_*} g(s)]$ *and* $g^* \in [\sup_{s\notin A^*} g(s), \inf_{s\in A^*} g(s)]$ *respectively. Moreover, suppose that* $N_0^C(I) = \mathbb{Q}(S_T \in A_*)$ *and* $N_0^C(I) = \mathbb{Q}(S_T \in A^*)$*. Then,*

$$
\begin{aligned}
W_L^C &= \mathbb{E}((g(S_T) - g_*)1_{\{S_T\in A_*\}} + g_*1_A), && (3.3.4)\\
W_G^C &= \mathbb{E}((g(S_T) - g^*)1_{\{S_T\in A^*\}} + g^*1_A). && (3.3.5)
\end{aligned}
$$

Remark 15. *The former part of Theorem 2 implies that the barrier option can be dominated by or dominate some portfolios which consist of plain-vanilla options and a cash-paying no-touch option. The latter part of Theorem 2 implies that the portfolios give the optimal pricing bounds respectively.*

Remark 16. *One can find* A_* *and* A^* *because it is assumed that the density of* S_T *is continuous.*

In the following, two propositions which constitute Theorem 2 are provided. The first one shows that there are families of super-hedging and sub-hedging portfolios.

Proposition 1 (super-hedging and sub-hedging portfolio)**.** *Suppose that* g_**,* A_**,* g^* *and* A^* *are defined as in Theorem 2 for* g*. Then,*

$$
\begin{aligned}
& (g(S_T) - g_*)1_{\{S_T\in A_*\}} + g_*1_A && (3.3.6)\\
\leq\ & g(S_T)1_A && (3.3.7)\\
\leq\ & (g(S_T) - g^*)1_{\{S_T\in A^*\}} + g^*1_A. && (3.3.8)
\end{aligned}
$$

Proof. Inequality (5.3.6) is proved by

$$
\begin{aligned}
(g(S_T(\omega)) - g_*)1_{\{S_T(\omega)\in A_*\}} + g_*1_{\{\omega\in A\}} &= \begin{cases} g(S_T(\omega)) & (S_T(\omega) \in A_*, \omega \in A)\\ g_* & (S_T(\omega) \notin A_*, \omega \in A)\\ g(S_T(\omega)) - g_* & (S_T(\omega) \in A_*, \omega \notin A)\\ 0 & (S_T(\omega) \notin A_*, \omega \notin A)\end{cases}\\
&\leq \begin{cases} g(S_T(\omega)) & (S_T(\omega) \in A_*, \omega \in A)\\ g(S_T(\omega)) & (S_T(\omega) \notin A_*, \omega \in A)\\ 0 & (S_T(\omega) \in A_*, \omega \notin A)\\ 0 & (S_T(\omega) \notin A_*, \omega \notin A)\end{cases}\\
&= g(S_T(\omega))1_{\{\omega\in A\}} \qquad (3.3.9)
\end{aligned}
$$

and Inequality (5.3.8) is proved in the same manner. □

Eq.(3.3.6) shows that the payoff of the knock-out option dominates that of the following portfolio:

- g_* unit of the cash-paying no-touch option
- one unit of a European derivative with the payoff $(g(S_T) - g_*)1_{\{S_T \in A_*\}}$.

Similarly, Eq.(3.3.8) shows that the payoff of the knock-out option is dominated by that of the following portfolio:

- g^* unit of the cash-paying no-touch option
- one unit of a European derivative with the payoff $(g(S_T) - g^*)1_{\{S_T \in A^*\}}$.

Remark 17. *Construction of the payoffs $(g(S_T)-g_*)1_{\{S_T\in A_*\}}$ and $(g(S_T)-g^*)1_{\{S_T\in A^*\}}$ is theoretically possible using call and put options (see (Breeden and Litzenberger, 1978)). However, the construction in practice requires high transaction cost. One way to address this problem is to trade these options with an internal option trader not with an external market participant. The internal option trader manages the options as a part of his or her own position and hedging or taking risk is up to him or her.*

Proposition 2 shows the optimality of the pricing bounds by constructing calibrated probability spaces under which a price of the barrier option is equal to W_L^C and W_G^C respectively.

Proposition 2 (the optimality of the bounds)**.** *Suppose that $N_0^C(I) = \mathbb{Q}(S_T \in A_*)$ and $N_0^C(I) = \mathbb{Q}(S_T \in A^*)$.*

(i) *There exists a calibrated probability space $(\Omega^L, \mathcal{F}^L, \mathbb{Q}^L, \{S_t^L\}_{t\in[0,T]}) \in \mathcal{P}^C$ such that*

$$\mathbb{E}^L(g(S_T)1_A) = \mathbb{E}((g(S_T) - g_*)1_{\{S_T\in A_*\}} + g_*1_A). \tag{3.3.10}$$

(ii) *There exists a calibrated probability space $(\Omega^G, \mathcal{F}^G, \mathbb{Q}^G, \{S_t^G\}_{t\in[0,T]}) \in \mathcal{P}^C$ such that*

$$\mathbb{E}^G(g(S_T)1_A) = \mathbb{E}((g(S_T) - g^*)1_{\{S_T\in A^*\}} + g^*1_A). \tag{3.3.11}$$

Proof. First, an arbitrary element $(\Omega, \mathcal{F}, \mathbb{Q}, \{S_t\}_{t\in[0,T]}) \in \mathcal{P}^C$ is chosen.

Let $(\Omega^L, \mathcal{F}^L, \mathbb{Q}^L) = (\Omega, \mathcal{F}, \mathbb{Q})$ and X be a random variable on it such that

$$X := \begin{cases} S_T & (S_T \in A_*) \\ x & (S_T \notin A_*) \end{cases}, \tag{3.3.12}$$

where $x \notin I$ is an arbitrary value. In addition, a stochastic process S^L is defined by

$$S_t^L := S_0 1_{\{t<\frac{1}{3}T\}} + X1_{\{\frac{1}{3}T\le t<\frac{2}{3}T\}} + S_T 1_{\{\frac{2}{3}T\le t\le T\}}. \tag{3.3.13}$$

Then, $(\Omega^L, \mathcal{F}^L, \mathbb{Q}^L, \{S_t^L\}_{t\in[0,T]})$ is a calibrated probability space, because the distribution of S_T^L is coincident with that of S_T and $\mathbb{Q}^L(S_t^L \in I \mid 0 \le t \le T) = \mathbb{Q}^L(S_T^L \in A_*) = \mathbb{Q}(A)$. Then,

$$\begin{aligned} \mathbb{E}^L(g(S_T^L)1_A) &= \mathbb{E}^L(g(S_T^L)1_{\{S_T^L\in A_*\}}) \\ &= \mathbb{E}((g(S_T) - g_*)1_{\{S_T\in A_*\}} + g_*1_{\{S_T\in A_*\}}) \\ &= \mathbb{E}((g(S_T) - g_*)1_{\{S_T\in A_*\}} + g_*1_A). \end{aligned} \tag{3.3.14}$$

Therefore, Eq.(3.3.10) holds.

One can prove Eq.(3.3.11) in the same manner. $\square$

Remark 18. *The calibrated probability space* $(\Omega^G, \mathcal{F}^G, \mathbb{Q}^G, \{S_t^G\}_{t\in[0,T]})$ *is explained intuitively as follows. Consider the worst case scenario for a writer of a knock-out option whose payoff is* $g(S_T)1_A$ *under the condition that the risk-neutral distribution of* S_T *and the risk-neutral probability that the barrier is hit are given. Since the writer has to pay the payoff of the option at maturity if the barrier has not been hit, the worst scenario for the writer is that the payoff is high if the barrier has not been hit and low if the barrier has been hit. Then, the process* $\{S_t^G\}_{t\in[0,T]}$ *should be constructed by distributing* $S_T^G(\omega)$ *to* A^* *for* $\omega \in A$*. The event* A *has the same probability as that of* $\{S_T^G \in A^*\}$*. As a result, the upper bound is an expectation of the plain-vanilla payoff* $g(S_T^G)$ *on the domain* $\{S_T^G \in A^*\}$*.*

In particular, the pricing bounds for a knock-out call option with strike K and an asset-paying no-touch option are obtained:

Corollary 3. *Suppose* $\kappa \in [K, u]$*. Then*

$$\begin{aligned} & (\kappa - K)(1_A - 1_{A_T}) + ((S_T - K)_+ - (S_T - \kappa)_+)1_{A_T} && (3.3.15)\\ \leq\ & (S_T - K)_+ 1_A && (3.3.16)\\ \leq\ & (\kappa - K)1_A + (S_T - \kappa)_+ 1_{A_T}, && (3.3.17)\end{aligned}$$

where $A_T := \{S_T \in I\}$*. Moreover,*

(i) *the expectation of Eq.(3.3.15) takes the supremum value at* $\kappa = \kappa_* \vee K$ *and the value is* W_L^C*,*

(ii) *the expectation of Eq.(3.3.17) takes the infimum value at* $\kappa = \kappa^* \vee K$ *and the value is* W_G^C*,*

where κ_* *and* κ^* *are real numbers in* $[l, u]$ *such that*

$$\begin{aligned} \mathbb{Q}(A) &= \mathbb{Q}(\{l \leq S_T \leq \kappa_*\}) && (3.3.18)\\ \mathbb{Q}(A) &= \mathbb{Q}(\{\kappa^* \leq S_T \leq u\}) && (3.3.19)\end{aligned}$$

and $x \vee y := max(x, y)$*.*

Corollary 4. *The following holds for any element of* $\mathcal{P}^C$

$$\mathbb{E}(S_T 1_{\{l \leq S_T \leq \kappa_*\}}) \leq \mathbb{E}(S_T 1_A) \leq \mathbb{E}(S_T 1_{\{\kappa^* \leq S_T \leq u\}}), \tag{3.3.20}$$

where κ_* *and* κ^* *are defined as in Corollary 3.*

Next, consider how the pricing bounds are improved by adding an assumption that one can use an asset-paying no-touch option as a hedging instrument.

Theorem 3. *Suppose* $\alpha \in \mathbb{R}$*. Then*

$$\begin{aligned} & (h(S_T) - h_*)1_{\{S_T \in A_*\}} + h_* 1_A + \alpha S_T 1_A && (3.3.21)\\ \leq\ & g(S_T)1_A && (3.3.22)\\ \leq\ & (h(S_T) - h^*)1_{\{S_T \in A^*\}} + h^* 1_A + \alpha S_T 1_A, && (3.3.23)\end{aligned}$$

where $h(s) := g(s) - \alpha s$ *and* h_**,* A_**,* h^* *and* A^* *are defined as in Theorem 2 for* h*. Moreover, suppose that*

$$sup_{\lambda \in I - s} \left| \frac{g(s + \lambda) - g(s)}{\lambda} \right| < +\infty \tag{3.3.24}$$

for $s = \kappa_*$ *and* $s = \kappa^*$*, where* κ_* *and* κ^* *are defined as in Corollary 3. Then,*

(i) *the expectation of Eq.(3.3.21) takes the supremum value at* $\alpha = \alpha_*$ *and the value is* W_L^A,

(ii) *the expectation of Eq.(3.3.23) takes the infimum value at* $\alpha = \alpha^*$ *and the value is* W_G^A,

where α_* *and* α^* *are defined by* $N_0^A(I) = \mathbb{E}(S_T 1_{\{S_T \in A_*\}}) = \mathbb{E}(S_T 1_{\{S_T \in A^*\}})$ *respectively.*

Proof. The inequalities are from Theorem 2. The set A_* is close to $\{l \le S_T \le \kappa_*\}$ as $\alpha \longrightarrow -\infty$ and $\{\kappa^* \le S_T \le u\}$ as $\alpha \longrightarrow +\infty$. By Corollary 4, one can find α_* and A_* such that $N_0^A(I) = \mathbb{E}(S_T 1_{\{S_T \in A_*\}})$. One can construct an element of $\mathcal{P}^C$ such that $A = \{S_T \in A_*\}$. Then,

$$\begin{aligned}\mathbb{E}((h(S_T) - h_*)1_{\{S_T \in A_*\}} + h_* 1_A + \alpha_* S_T 1_A) &= \mathbb{E}((h(S_T) - h_*)1_A + h_* 1_A + \alpha_* S_T 1_A) \\ &= \mathbb{E}(g(S_T)1_A). \end{aligned} \tag{3.3.25}$$

□

Eq.(3.3.21) shows that the payoff of the knock-out option dominates that of the following portfolio:

- h_* unit of the cash-paying no-touch option
- α unit of the asset-paying no-touch option
- one unit of a European derivative with the payoff $(g(S_T) - \alpha S_T - h_*)1_{\{S_T \in A_*\}}$.

Similarly, Eq.(3.3.23) shows that the payoff of the knock-out option is dominated by that of the following portfolio:

- h^* unit of the cash-paying no-touch option
- α unit of the asset-paying no-touch option
- one unit of a European derivative with the payoff $(g(S_T) - \alpha S_T - h^*)1_{\{S_T \in A^*\}}$.

3.4 Numerical Examples

This section shows numerical examples. The pricing bounds are compared with exact prices derived by some specific models.

The Heston's stochastic volatility model ((Heston, 1993)) is regarded as the process of the underlying asset, which means all market options are produced by the model. The process of the underlying under the Heston model is as follows:

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma_t dW_t, \tag{3.4.1}$$

$$d\sigma_t^2 = \kappa(\eta - \sigma_t^2)dt + \theta\sigma_t d\tilde{W}_t, \tag{3.4.2}$$

where W and $\tilde{W}$ are Brownian motions with correlation ρ under the risk-neutral measure.

3.4.1 Pricing Bounds on Double Barrier Options

The first example of pricing bounds is for double barrier options. Double barrier options can be priced analytically under the Heston model with $r = q$ and $\rho = 0$ (see (Lipton, 2001)). The barrier option considered in this example is set to be a double knock-out call option with maturity 1-month or 3-month, $K = 0.95$, $l = 0.8$ and $u = 1.1$. The Heston prices, the model-independent pricing bounds and trivial upper bounds for this option are calculated with the spot price varied from l to u under the Heston model with the parameters listed in Table 5.1. Here, a trivial upper bound means a European option whose payoff is $(S_T - K)_+ 1_{A_T}$, where A_T is defined in Corollary 3.

Table 3.1: Parameters of the Heston Model

r	q	σ_0^2	κ	η	θ	ρ
0.03	0.03	0.15^2	3.0	0.2^2	0.4	0.0

The results are Figure 3.1 and Table 3.4 for 1-month and Figure 3.2 and Table 3.5 for 3-month[3], which show that all of the model-independent upper bounds or lower bounds are upper or lower than the exact prices respectively. In addition, the pricing bounds using an asset-paying no-touch option are much closer to the Heston prices than those using only a cash-paying no-touch option.

The upper bound with a cash-paying no-touch option seems much more conservative than the lower bound in these examples. Although this is not true in general, the case of this example is explained as follows: Consider the calibrated probability space $(\Omega^G, \mathcal{F}^G, \mathbb{Q}^G, \{S_t^G\}_{t\in[0,T]}) \in \mathcal{P}^C$, which gives the upper bound for the knock-out call options in the examples. As in Remark 18, $S_T^G(\omega)$ belongs to A^* for a scenario $\omega \in A$ where the barrier is not hit and $S_T^G(\omega)$ does not belongs to A^* for a scenario $\omega \in A^c$. This scenario is, however, far from a reality, because A^* is a subset of I which is close to the barrier levels l and u. On the other hand, the probability space that gives the lower bound is more likely. This is a reason why the upper bound is much more conservative than the lower bound.

3.4.2 Comparison with (Brown et al., 2001)

The second example is comparison with the method proposed by (Brown et al., 2001) for single barrier options, which are up-and-out call options with maturity 1-month or 3-month, $K = 0.95$ and $u = 1.1$. The Heston prices, the model-independent pricing bounds, pricing bounds of (Brown et al., 2001) and trivial upper bounds for this option are calculated with the spot price varied from 0.95 to 1.075 under the Heston model with the parameters listed in Table 3.2.

Table 3.2: Parameters of the Heston Model

r	q	σ_0^2	κ	η	θ	ρ
0.0	0.0	0.15^2	3.0	0.2^2	0.4	0.0

[3] The pricing bounds using asset-paying no-touch options are omitted from the figures since they are very close to the Heston prices.

The results are Table 3.6 for 1-month and Table 3.7 for 3-month. Generally, one cannot claim that either of the two methods is superior to the other. The method proposed by this paper uses a no-touch option and do not assume the underlying asset price is martingale, while (Brown et al., 2001) assume it is martingale. Actually, Table 3.6 and Table 3.7 show that the lower bounds of (Brown et al., 2001) and the upper bounds of this paper are more conservative.

3.4.3 Comparison with (Schoutens et al., 2005)

The third example is a comparison with the results of (Schoutens et al., 2005). They study prices of single barrier options under several models: the Heston model (HEST) and its generalization allowing for jumps in the price process (see e.g. (Bakshi et al., 1997)) (HESJ), the Barndorff-Nielsen-Shephard model introduced in (Barndorff-Nielsen and Shephard, 2001) (BN-S) and Lévy models with stochastic time introduced by (Carr et al., 2001). The Lévy models with stochastic time in their study are NIG Lévy process with CIR Stochastic Clock(NIG-CIR), NIG Lévy process with Gamma-OU Stochastic Clock(NIG-OUΓ), VG Lévy process with CIR Stochastic Clock(VG-CIR), and VG Lévy process with Gamma-OU Stochastic Clock(VG-OUΓ), where NIG is for the Normal Inverse Gaussian distribution and VG for the Variance Gamma distribution.

The barrier options considered in their example are knock-out call options with maturity 3 years, strike equal to the spot S_0 and several barrier levels (ranging from $1.05S_0$ to $1.5S_0$). They price the barrier options under models which are calibrated very well to a set of plain-vanilla options.

The pricing bounds proposed in this paper are compared with their exact prices under the above models. The calculation is based on the Heston model with the parameters listed in Table 3.3. Note that the prices of no-touch options are also calculated by the Heston model, which are different from those by the other models.

Table 3.3: Parameters of the Heston Model in (Schoutens et al., 2005)

S_0	r	q	σ_0^2	κ	η	θ	ρ
2461.44	0.03	0.0	0.0654	0.6067	0.0707	0.2928	-0.7571

The results are listed in Table 3.8.

First, it is noteworthy that there are significant differences in the prices of the barrier options even if the models are calibrated very well to plain-vanilla options, according to (Schoutens et al., 2005). This is due to the different structure in path-behaviour between these models.

Second, whether the prices under the above models are in the model-independent pricing bounds or not is examined. Since the prices of no-touch options in the calculation is based on the Heston model, all prices under the Heston model are in the pricing bounds. On the other hand, some prices are outside the bounds using only cash-paying no-touch options; the prices under NIG-OUΓ, VG-CIR, VG-OUΓ and NIG-CIR take higher values than the upper bounds at $H/S_0 = 0.95, 0.9, 0.85$. The bounds using asset-paying no-touch options are so close to the Heston prices that many prices produced by the other models are outside them.

3.5 Concluding Remarks

This paper provides model-independent pricing bounds on barrier options and corresponding hedging strategies using no-touch options. Moreover, the optimal pricing bounds among them are derived and the optimality is shown through finding risk-neutral measures under which the barrier option price is equal to the hedging portfolio's value. Comparing the pricing bounds proposed by this study with exact prices under several models which are calibrated only to plain-vanilla options, it is demonstrated that some models produce prices outside the model-independent bounds. This implies that great care should be taken when choosing models to price barrier options.

Finally, the next research topic will be to consider a pricing model for barrier options which is calibrated to both the market prices of plain-vanilla options and no-touch options.

Figure 3.1: Pricing bounds and Heston prices (1M)

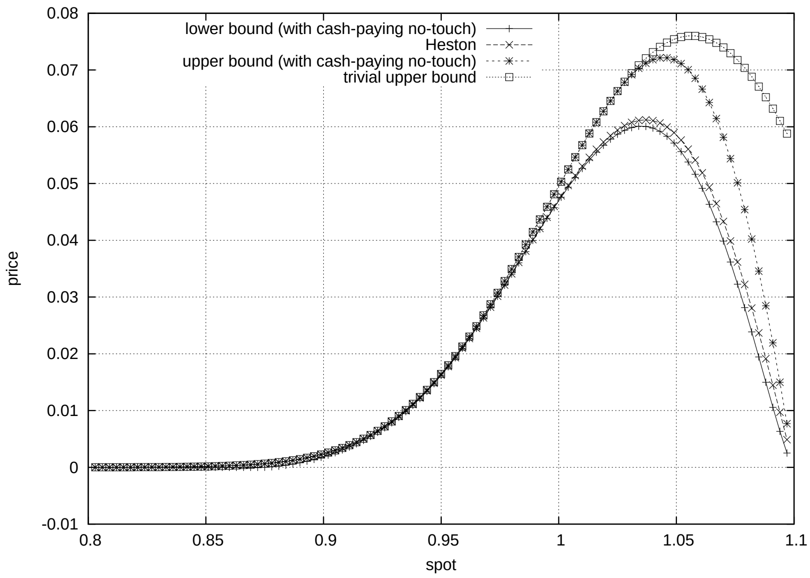


Table 3.4: Pricing bounds and Heston prices (1M)

spot	0.950	0.975	1.000	1.025	1.050	1.075
W_L^C	0.0162	0.0308	0.0470	0.0587	0.0567	0.0336
W_L^A	0.0162	0.0308	0.0472	0.0592	0.0582	0.0373
Heston	0.0162	0.0308	0.0472	0.0594	0.0586	0.0374
W_G^A	0.0163	0.0308	0.0473	0.0595	0.0587	0.0377
W_G^C	0.0165	0.0314	0.0496	0.0663	0.0716	0.0516
trivial upper bound	0.0165	0.0314	0.0496	0.0663	0.0756	0.0722
DNT	0.9940	0.9878	0.9610	0.8886	0.7212	0.4150

Figure 3.2: Pricing bounds and Heston prices (3M)

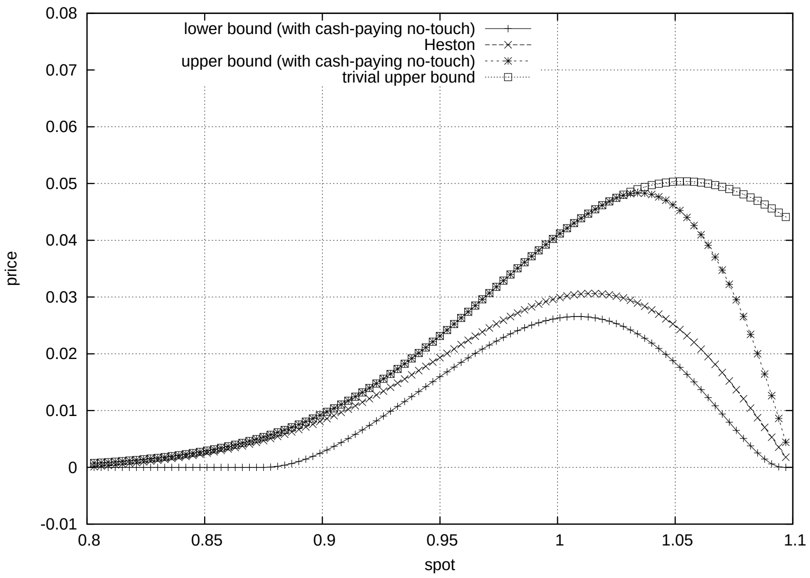


Table 3.5: Pricing bounds and Heston prices (3M)

spot	0.950	0.975	1.000	1.025	1.050	1.075
W_L^C	0.0160	0.0225	0.0263	0.0253	0.0184	0.0070
W_L^A	0.0187	0.0246	0.0284	0.0281	0.0225	0.0126
Heston	0.0193	0.0255	0.0298	0.0302	0.0249	0.0142
W_G^A	0.0220	0.0271	0.0309	0.0310	0.0259	0.0153
W_G^C	0.0231	0.0322	0.0409	0.0475	0.0459	0.0305
trivial upper bound	0.0231	0.0322	0.0409	0.0475	0.0503	0.0487
DNT	0.8739	0.8399	0.7638	0.6400	0.4654	0.2460

Table 3.6: Comparison with (Brown et al., 2001) (1M)

spot	0.95	0.975	1.000	1.025	1.05	1.075
lower bound of (Brown et al., 2001)	0.0162	0.0303	0.0457	0.0549	0.0486	0.0236
W_L^C	0.0163	0.0309	0.0474	0.0595	0.0580	0.0360
Heston	0.0164	0.0310	0.0477	0.0602	0.0598	0.0397
W_G^C	0.0165	0.0315	0.0497	0.0665	0.0723	0.0537
upper bound of (Brown et al., 2001)	0.0165	0.0314	0.0493	0.0657	0.0747	0.0686
trivial upper bound	0.0165	0.0315	0.0497	0.0665	0.0758	0.0724
DNT	0.9976	0.9905	0.9660	0.8961	0.7328	0.4361

Table 3.7: Comparison with (Brown et al., 2001) (3M)

spot	0.95	0.975	1.000	1.025	1.05	1.075
lower bound of (Brown et al., 2001)	0.0161	0.0200	0.0211	0.0179	0.0104	0.0022
W_L^C	0.0187	0.0244	0.0277	0.0266	0.0196	0.0079
Heston	0.0197	0.0262	0.0306	0.0310	0.0260	0.0152
W_G^C	0.0233	0.0324	0.0412	0.0478	0.0469	0.0320
upper bound of (Brown et al., 2001)	0.0223	0.0310	0.0394	0.0459	0.0491	0.0466
trivial upper bound	0.0233	0.0324	0.0412	0.0478	0.0507	0.0491
DNT	0.9295	0.8760	0.7884	0.6591	0.4828	0.2618

Table 3.8: Comparison with the results of (Schoutens et al., 2005)

	H/S_0	NIG-OUΓ	VG-CIR	VG-OUΓ	HEST	HESJ	BN-S	NIG-CIR	W_L^C	W_L^A	W_G^A	W_G^C	trivial upper bound
Call		509.76	511.8	509.33	510.88	510.89	509.89	512.21					
DOB	0.95	300.25	293.28	318.35	173.85	174.64	230.25	284.1	13.77	172.89	177.93	253.80	509.99
	0.9	396.8	391.17	402.24	280.79	282.09	352.14	387.83	38.94	279.38	291.39	369.56	509.99
	0.85	451.61	448.1	452.97	359.05	360.99	423.21	446.52	70.61	355.59	377.78	442.14	509.99
	0.8	481.65	479.83	481.74	414.65	416.63	461.82	479.77	107.38	407.07	439.73	484.96	509.99
	0.75	497	496.95	496.8	452.76	454.33	481.85	496.78	145.63	440.95	480.63	505.74	509.99
	0.7	504.31	505.24	504.05	477.37	479.12	492.62	505.38	185.66	459.08	502.73	509.99	509.99
	0.65	507.53	509.1	507.21	492.76	494.25	498.93	509.34	224.44	470.98	509.02	509.99	509.99
	0.6	508.88	510.75	508.53	501.74	502.84	503.17	511.09	263.95	480.97	509.02	509.99	509.99
	0.55	509.43	511.4	509.06	506.46	507.41	505.93	511.8	301.70	488.72	509.02	509.99	509.99
	0.5	509.64	511.67	509.24	508.91	509.51	507.68	512.08	336.17	495.02	509.02	509.99	509.99
UOB	1.05	0.44	0.27	0.49	0.103	0.08	0.13	0.23	0.00	0.00	2.39	2.39	2.39
	1.1	3.08	2	3.22	0.979	0.89	1.48	1.84	0.00	0.00	9.22	9.55	9.55
	1.15	9.43	6.59	9.77	3.8	3.61	5.58	6.27	0.00	0.00	19.87	22.00	22.00
	1.2	20.71	15.29	21.03	8.96	9.85	13.91	14.8	0.00	0.00	32.66	38.83	38.83
	1.25	37.29	28.95	37.94	20.15	19.96	27.2	28.26	0.00	1.33	47.03	60.88	60.88
	1.3	59.22	48.17	60.1	35.58	36.03	45.38	47.04	4.78	11.58	61.09	86.25	86.25
	1.35	86.14	72.47	87	56.1	58.42	68.39	71.21	20.74	30.62	74.36	116.02	116.02
	1.4	116.75	101.33	117.96	81.93	86.8	94.88	100.04	45.83	57.03	92.24	147.46	147.46
	1.45	149.98	133.74	151.52	111.65	121.33	124.36	132.16	77.84	89.31	117.19	181.78	181.78
	1.5	184.5	168.33	186.53	145.31	160.21	154.96	166.41	115.01	126.01	147.56	215.77	215.77

Chapter 4

No-Arbitrage Bounds on Two One-Touch Options

This paper investigates the pricing bounds of two one-touch options with the same maturity but different barrier levels, where the pricing bound is a range within which a one-touch option can take a price when a price of another one-touch option is given. The upper or lower bounds are the cost of a super-replicating portfolio and a sub-replicating portfolio respectively. These consist of call options, put options, digital options and a one-touch option. We assume that the underlying process is a continuous martingale, but do not postulate a model.[1]

4.1 Introduction

This paper investigates pricing bounds within which a one-touch option can take a price when the price of another one-touch option with the same maturity but a different barrier level is given.

Financial markets trade many barrier option types such as single/double barrier knock-in/-out options. Of these, one-touch and no-touch options are the simplest barrier options and widely are traded. A one-touch option is a barrier option that pays a unit of currency at the maturity if the barrier is hit and is worthless if the barrier has not been hit. In contrast, a no-touch option is worthless if the barrier is hit. These are important instruments for traders of barrier options, because they reflect a market view of the probability of the barrier being hit.

There has been considerable research on pricing and hedging barrier options. In particular, researchers have proposed several methods that semi-statically hedge barrier options (see e.g. (Carr and Chou, 1997), (Carr et al., 1998) and (Derman et al., 1995)). Here, semi-static hedging means the replication of barrier options by trading European puts and calls no more than once after inception. Hedging strategies require options, thus models that price barrier options must be calibrated to these. However, even if the model is perfectly calibrated to a volatility surface there are risks attached to the valuation of barrier options. For instance, (Hirsa et al., 2003), (Lipton and McGhee, 2002) and (Schoutens et al., 2005) all state that although models may produce similar European put and call option prices, they give markedly different barrier option prices. Touch options are recognized as important products because they are used as an instrument to which a model is calibrated (see e.g. (Carr and Crosby, 2010)).

[1] Preprint of an article submitted for consideration in International Journal of Theoretical and Applied Finance, ©2015 World Scientific Publishing Co Pte Ltd, http://www.worldscientific.com/worldscinet/ijtaf

The model-independent approach has also been considered for exotic derivatives including barrier options (see e.g. (Hobson, 1998), (Hobson and Neuberger, 2012), (Labordére et al., 2012) and (Hobson and Klimmek, 2012)). In particular, (Brown et al., 2001) propose robust super-replicating and sub-replicating barrier option strategies including touch-options without assuming any specific models. (Cox and Obłój, 2011a) and (Cox and Obłój, 2011b) focus on touch options with two barrier levels in the same manner as (Brown et al., 2001). They use call options and put options as well as digital options with the same maturity as replicating instruments and trade forward contracts at the first barrier(s) hitting time(s). Generally, pricing bounds derived from model-independent replications tend to be rather wide, which is also the case for touch-options. Hence, it is worth investigating how much these pricing bounds are refined if other instruments are traded.

This paper investigates pricing bounds within which a one-touch option can take a price when a price of another one-touch option with the same maturity but a different barrier level is given and those European options (including call, put and digital options) with the same maturity. Suppose there is a pricing operator on European options with a certain maturity and a touch option with the same maturity and a certain barrier level. The question is how to extend this pricing operator to a space spanned by a touch option with the same maturity but a different barrier level as well as these derivatives. To address this, we propose pricing operators that provide upper and lower bounds for the touch option based on a super-replication and a sub-replication. Our approach is in line with (Brown et al., 2001), (Cox and Obłój, 2011a) and (Cox and Obłój, 2011b), in that we assume the underlying asset process is a continuous martingale and our replications consist of static portfolios and transactions of a forward contract in the first instances of hitting the barrier levels. We differentiate by using a touch option as well as European options for the static portfolios.

Moreover, we provide pricing bounds on a touch/no-touch option that pays one unit of currency if and only if the first barrier is hit but the second is not. In Section 4.4, we consider the pricing bounds on this touch/no-touch option using the one-touch option with the second barrier as well as European options. If we use, instead of the one-touch option, the upper- or lower bounds and the super- or sub-replications on this, we obtain the pricing bounds as well as super- and sub-replications of the touch/no-touch option using only European options.

The next section of this paper describes the settings and notations. The third section reviews the research of (Brown et al., 2001). The super-replications and sub-replications for a one-touch option using another with a different barrier level are derived in the fourth section. The fifth section provides numerical examples.

4.2 Settings and Notations

The settings and notations used in this paper are stated here.

First, we introduce some notations. Let us denote the spot price of the underlying asset at time $t \in [0, T^*]$ by S_t, where T^* is some arbitrary time horizon and the time-t price of a call option and a put option with strike K and maturity $T \in [0, T^*)$ by $C_t(K)$ and $P_t(K)$ respectively. The one-touch option is assumed to be a single knock-in option with maturity T and barrier level $B \in (S_0, +\infty)$. This option is worthless if B has not been hit by the expiration date. If the barrier is hit at any time during the option's life, the terminal payoff is 1. Then, the payoff of the barrier option is $1_{\{\tau_B \leq T\}}$, where τ_B is the first time of hitting B:

$$\begin{aligned} \tau_B &:= \tau_B(S) \\ &:= \inf\{t < T^* \mid S_t \geq B\}. \end{aligned} \tag{4.2.1}$$

A time-t price of this option is denoted as $O_t(B)$. The subscript t may be omitted in case of $t = 0$ such as $C(K)$, $P(K)$ and $O(B)$ for simplicity.

Second, we make some assumptions. The first assumption is that the underlying price process S is a non-negative martingale. The interest rates are also assumed to be zero. This is merely for simplicity, since our results are valid by reading all prices of all options and portfolios as forward T prices in case of a non-zero interest rate. Examples to which our results are applied are that the underlying process is a forward price or that the underlying asset pays continuous dividends equal to the interest rate. We assume that forward transactions are costless and all instruments — such as underlying asset, forward, — are traded without transaction costs. Importantly, we assume that the underlying price process is continuous. This allows us to exchange a call option with strike K, with $(B-K)$ amounts of cash and a put option with the same strike by trading a forward contract with zero cost at the first time of hitting B, since the following parity holds:

$$C_{\tau_B}(K) - P_{\tau_B}(K) = B - K. \tag{4.2.2}$$

This type of trade is used throughout this paper. Moreover, we add an assumption in Section 4.4.1 and 4.4.2 that the distribution of the underlying asset at maturity T under a risk-neutral measure is given. This distribution is centered at S_0. We consider the case where only a finite number of call options are known in Section 4.4.3. Knowledge of the distribution is equivalent to the knowledge of European call option prices without arbitrage opportunities for the continuum of strikes by (Breeden and Litzenberger, 1978). The conditions for no arbitrage are well-documented in (Davis and Hobson, 2007). We assume $C(B) > 0$ to avoid a trivial case. We denote by ν the risk-neutral distribution of the spot price at maturity T determined by prices of these options. It is also assumed that call options, put options and digital call options can be used as replication, where the digital call option with strike K is an option whose payoff is $1_{\{K \le S_T\}}$ in this paper.

Third, we state the aim of this paper: to extend a pricing operator φ that is a linear operator defined on $\mathcal{X} := \mathcal{L}^1([0,+\infty), \nu)$, a set of Lebesgue integrable functions on $[0,+\infty)$ with respect to ν, which associates a payoff of an European option with its initial price such as $\varphi(K) = K$, $\varphi((S_T - K)_+) = C(K)$, $\varphi((K - S_T)_+) = P(K)$. If a price of a one-touch option whose payoff is $1_{\{\tau_B \le T\}}$ is known, we can extend the operator φ to $\mathcal{X} \oplus \mathcal{Y}$, where $\mathcal{Y}$ is a linear space spanned by $1_{\{\tau_B \le T\}}$ and $\oplus$ means a direct sum. This paper examines how to extend the operator φ to $\mathcal{X} \oplus \mathcal{Y} \oplus \tilde{\mathcal{Y}}$, where $\tilde{\mathcal{Y}}$ is a linear space spanned by $1_{\{\tau_{\tilde{B}} \le T\}}$ with another barrier level $\tilde{B}$. To address this, we propose sharp pricing bounds on one-touch options and the corresponding replicating strategies, where *sharpness* means that the pricing bounds can not be improved without adding any other assumption. The lower and upper bounds on the option are defined as follows under our settings:

$$W^L := \inf_{\mathcal{P}} \mathbb{E}\left[1_{\{\tau_{\tilde{B}}(S) \le T\}}\right] \tag{4.2.3}$$

$$W^G := \sup_{\mathcal{P}} \mathbb{E}\left[1_{\{\tau_{\tilde{B}}(S) \le T\}}\right], \tag{4.2.4}$$

where $\mathcal{P}$ is a set of all risk-neutral probability spaces $(\Omega, \mathcal{F}, \mathbb{Q})$ and a continuous martingale process $\{S_t\}_{t \in [0,T^*]}$ on it that satisfies $\varphi(\cdot) = \mathbb{E}[\cdot]$ on $\mathcal{X} \oplus \mathcal{Y}$ and $\mathbb{E}$ is an expectation operator corresponding to the probability space. Prices of super-replicating and sub-replicating portfolios are superior and inferior, but not necessarily equal, to W^G and W^L respectively. To prove the sharpness, we find super-replicating and sub-replicating portfolios whose prices are equal to $\mathbb{E}\left[1_{\{\tau_{\tilde{B}}(S) \le T\}}\right]$ with respect to a certain element of $\mathcal{P}$.

Finally, we introduce some further technical notations. Every function f considered in this paper is a combination of the call price function C. We expand the domain of the function f from $[0,+\infty)$ to $\mathbb{R}$ by $C(K) := C(0) - K$ for $K < 0$ (recall that we assume that the underlying process is non-negative). The function has left- and right-sided directional derivatives as does the function C. In this paper, we denote ∂_K^- as the left-sided derivative operator. Moreover, the derivatives have finite total variations and the derivative ∂_{KK}^- can be defined except for a countable set. The subdifferential of a function f at K can be defined and is denoted by

$$\partial_K f(K) := \{k \in \mathbb{R} \mid f(\kappa) \ge f(K) + k(\kappa - K),\ \forall \kappa \in \mathbb{R}\}. \tag{4.2.5}$$

We introduce the following notation for simplicity:

$$\mathcal{N}(\partial_K f) := \{K \in \mathbb{R} \mid 0 \in \partial_K f(K)\}. \tag{4.2.6}$$

4.3 Review of (Brown et al., 2001)

In this section, we review the replications for a one-touch option with only European options, as proposed by (Brown et al., 2001), because we use these results in Section 4.4. The one-touch option is assumed to have a barrier level B, where $S_0 < B$.

First, we prepare the following lemma:

Lemma 5. *Suppose that there is a measurable set $\Omega_0 \in \mathcal{F}$ such that $S_T \in [B, +\infty)$ on Ω_0 and $\mathbb{E}[S_T : \Omega_0] = B\mathbb{Q}[\Omega_0]$. Then, there exists a continuous martingale $\{S_t^*\}_{t\in[0,T]}$ such that $S_T = S_T^*$ and $\mathbb{Q}[\tau_B(S^*) \le T] = \mathbb{Q}[\Omega_0]$.*

Proof. Let X_0, X_1 and X_2 be random variables defined as $X_0 := S_0$, $X_2 := S_T$ and

$$X_1 := B \cdot 1_{\Omega_0} + \beta \cdot 1_{\Omega_0^c}, \tag{4.3.1}$$

where

$$\beta := B - \frac{B - S_0}{\mathbb{Q}[\Omega_0^c]}. \tag{4.3.2}$$

Note that $\beta < B$ and $\mathbb{E}[S_T : \Omega_0^c] = \beta\mathbb{Q}[\Omega_0^c]$. Then, $\{X_n\}_{n=0,1,2}$ is a discrete martingale with respect to a filtration generated by X. By Dudley's theorem (see, for instance, p.188 of (Karatzas and Shreve, 1988)), the random variables $X_1 - X_0$, $(X_2 - X_1) \cdot 1_{\Omega_0}$ and $(X_2 - X_1) \cdot 1_{\Omega_0^c}$ can be expressed with stochastic integrals with respect to the Winner processes. A continuous martingale process S_t^* such that $\mathbb{Q}[\tau_B(S^*) \le T] = \mathbb{Q}[\Omega_0]$ can be constructed by these stochastic integrals. □

4.3.1 Super-Replication

Consider the following self-financing strategy $\mathcal{G}(K; B)$ for $\forall K \in [0, B)$:

1. At the initial outset
   - Buy $\frac{1}{B-K}$ units of a call option with strike K.
2. At the first time of hitting B
   - Sell $\frac{1}{B-K}$ units of the forward contract.

The strategy $\mathcal{G}(K; B)$ super-replicates the one-touch option with any $K \in [0, B)$. We provide some optimal strategies properties.

Definition 9. *The initial value of strategy $\mathcal{G}(K; B)$ is defined as*

$$G(K; B) := \frac{C(K)}{B - K}, \tag{4.3.3}$$

$G_*(B)$ as the infimum value of $G(K;B)$ with respect to K, $K_G(B)$ as a strike price by which the infimum is attained:

$$\begin{aligned} G_*(B) &:= \inf_{K\in(-\infty,B)} G(K;B) \\ &= G(K_G(B);B) \end{aligned} \tag{4.3.4}$$

and $\mathcal{G}_*(B)$ as the corresponding strategy.

Proposition 3. *The infimum of Eq.(4.3.4) is attained by any element of $\mathcal{N}(\partial_K G(B))$, an interval of $[0,B)$. For all $K_G \in \mathcal{N}(\partial_K G(B))$, the following holds:*

$$\mathbb{Q}\left[K_+ < S_T\right] \le G_*(B) = \mathbb{E}\left[\frac{S_T - K_G}{B - K_G} : K_G \le S_T\right] \le \mathbb{Q}\left[K_- \le S_T\right], \tag{4.3.5}$$

where $K_- := \inf \mathcal{N}(\partial_K G(B))$ and $K_+ := \sup \mathcal{N}(\partial_K G(B))$. In addition, there is a continuous martingale process $\left\{S_t^G\right\}_{t\in[0,T]}$ such that

$$G_*(B) = \mathbb{Q}\left[\tau_B\left(S^G\right) \le T\right]. \tag{4.3.6}$$

Proof. By differentiating G with respect to K, we obtain

$$\begin{aligned} \partial_K^- G(K) &= \frac{1}{B-K}\left(\partial_K^- C(K) + \frac{C(K)}{B-K}\right) \\ &= \frac{1}{B-K}\left(\partial_K^- C(K) + G(K)\right) \end{aligned} \tag{4.3.7}$$

and

$$\begin{aligned} \partial_{KK}^- G(K) &= \frac{1}{B-K}\partial_{KK}^- C(K) + \frac{2}{(B-K)^2}\partial_K^- C(K) + 2\frac{C(K)}{(B-K)^3} \\ &= \frac{1}{B-K}\partial_{KK}^- C(K) + \frac{2}{B-K}\partial_K^- G(K). \end{aligned} \tag{4.3.8}$$

Since $\partial_K^- G(0) = \frac{1}{B}(-1 + \frac{S}{B}) < 0$, $\lim_{K\to B} \partial_K^- G(K) = +\infty$ and because $\partial_{KK}^- G > 0$ if $\partial_K^- G > 0$, the set $\mathcal{N}(\partial_K G(B))$ is an interval of $[0,B)$ and we have Eq.(4.3.5). Apply Lemma 5 with $\Omega_0 \subseteq \Omega$ such that $\{K_G < S_T\} \subseteq \Omega_0 \subseteq \{K_G \le S_T\}$ and $\mathbb{Q}[\Omega_0] = G_*$, then we have a continuous martingale $\left\{S_t^G\right\}_{t\in[0,T]}$. □

4.3.2 Sub-Replication

Consider the following self-financing strategy $\mathcal{L}(K;B)$ for $\forall K \in [0,B)$:

1. At the initial outset
   - Buy $\frac{1}{B-K}$ units of a call option with strike B.
   - Buy 1 unit of a digital call option with strike B.
   - Sell $\frac{1}{B-K}$ units of a put option with strike K.
2. At the first time of hitting B

- Sell $\frac{1}{B-K}$ units of the forward contract.

The strategy $\mathcal{L}(K;B)$ super-replicates the one-touch option with any $K \in [0,B)$. We provide some optimal strategies properties.

Definition 10. *The initial value of the strategy $\mathcal{L}(K;B)$ is defined as*

$$L(K;B) := \frac{C(B)}{B-K} - \frac{P(K)}{B-K} - \partial_K^- C(B), \tag{4.3.9}$$

$L_*(B)$ as the supremum value of $L(K;B)$ with respect to K, $K_L(B)$ as a strike price by which the supremum is attained:

$$\begin{aligned} L_*(B) &:= \sup_{K\in(-\infty,B)} L(K;B) \\ &= L(K_L;B), \end{aligned} \tag{4.3.10}$$

and $\mathcal{L}_*(B)$ as the corresponding strategy.

Proposition 4. *The supremum of Eq.(4.3.10) is attained by any element of $\mathcal{N}(\partial_K L(B))$, an interval of $[0,B)$. For all $K_L \in \mathcal{N}(\partial_K L(B))$, the following holds:*

$$\mathbb{Q}\left[S_T < K_-, B \le S_T\right] \le L_*(B) = \mathbb{E}\left[\frac{S_T - K_L}{B-K_L} : S_T \le K_L, B \le S_T\right] \le \mathbb{Q}\left[S_T \le K_+, B \le S_T\right], \tag{4.3.11}$$

where $K_- := \inf \mathcal{N}(\partial_K L(B))$ and $K_+ := \sup \mathcal{N}(\partial_K L(B))$. In addition, there is a continuous martingale process $\left\{S_t^L\right\}_{t\in[0,T]}$ such that

$$L_*(B) = \mathbb{Q}\left[\tau_B\left(S^L\right) \le T\right]. \tag{4.3.12}$$

Proof. By differentiating L with respect to K, we obtain

$$\begin{aligned} \partial_K^- L(K) &= \frac{1}{B-K}\left(\frac{C(B)}{B-K} - \partial_K^- P(K) - \frac{P(K)}{B-K}\right) \\ &= \frac{1}{B-K}\left(L(K) + \partial_K^- C(B) - \partial_K^- P(K)\right) \end{aligned} \tag{4.3.13}$$

and

$$\begin{aligned} \partial_{KK}^- L(K) &= 2\frac{C(B)}{(B-K)^3} - \frac{1}{B-K}\partial_{KK}^- P(K) - 2\frac{1}{(B-K)^2}\partial_K^- P(K) - 2\frac{P(K)}{(B-K)^3} \\ &= \frac{2}{B-K}\partial_K^- L(K) - \frac{1}{B-K}\partial_{KK}^- P(K). \end{aligned} \tag{4.3.14}$$

Since $\partial_K^- L(0) = \frac{S}{B^2} > 0$, $\lim_{K\to B} \partial_K^- L(K) = -\infty$ and because $\partial_{KK}^- L < 0$ if $\partial_K^- L < 0$, the set $\mathcal{N}(\partial_K L(B))$ is an interval of $[0,B)$ and we have Eq.(4.3.11). Apply Lemma 5 with $\Omega_0 \subseteq \Omega$ such that $\{K_L < S_T, B \le S_T\} \subseteq \Omega_0 \subseteq \{K_L \le S_T, B \le S_T\}$ and $\mathbb{Q}[\Omega_0] = L_*$, then we have a continuous martingale $\left\{S_t^L\right\}_{t\in[0,T]}$. $\square$

4.4 Replication using another One-Touch Option

Here, we consider super-replication and sub-replication for a one-touch option with a barrier level B_1 using European options and a one-touch option with a barrier level B_2, where $S_0 < B_1 < B_2$. Rather than considering the barrier option, we consider a touch/no-touch option whose payoff is $1_{\{\tau_1 \le T < \tau_2\}}$ where τ_1 and τ_2 are the first times of hitting B_1 and B_2 respectively, because of $1_{\{\tau_1 \le T < \tau_2\}} = 1_{\{\tau_1 \le T\}} - 1_{\{\tau_2 \le T\}}$.

For easing expression, we introduce the notation $\pi : [0,1] \longrightarrow \mathcal{F}$: where $\pi(p)$ is an element of $\mathcal{F}$ for $p \in [0,1]$ such that $\mathbb{Q}[\pi(p)] = p$, and $S_T(\omega) \le S_T(\omega^c)$ for $\omega \in \pi(p)$ and $\omega^c \notin \pi(p)$. We also define $\pi([p,q)) := \pi(p)^c \cap \pi(q)$ for $p, q \in [0,1]$ and $\pi(I) := \bigcup_{n=1}^{N} \pi(I_n)$ for $I := \bigcup_{n=1}^{N} I_n$, where I_n are disjoint intervals. The Lebesgue measure on $[0,1]$ is denoted as μ. Then, we have $\mu(I) = \mathbb{Q}\left[\pi(I)\right]$ for any interval $I \subseteq [0,1]$.

4.4.1 Super-Replication

Consider the following self-financing strategy $\mathcal{G}^B(K; B_1, B_2)$ for $\forall K \in [0, B_1)$:

1. At the initial outset
   - Buy $\frac{1}{B_1 - K}$ units of a call option with strike K.
   - Sell $\frac{1}{B_1 - K}$ units of a call option with strike B_2.
   - Buy $\frac{B_2 - B_1}{B_1 - K}$ units of the one-touch option with a barrier level B_2.
   - Sell $\frac{B_2 - K}{B_1 - K}$ units of a digital call option with strike B_2.
2. At the first time of hitting B_1
   - Sell $\frac{1}{B_1 - K}$ units of the forward contract
3. At the first time of hitting B_2
   - Buy $\frac{1}{B_1 - K}$ units of the forward contract.

Fig.4.1 shows that the $\mathcal{G}^B(K; B_1, B_2)$ strategy super-replicates the touch/no-touch option with $K \in [0, B_1)$. We investigate the optimal strategies properties. First, we define the following.

Definition 11. *The initial value of the $\mathcal{G}^B(K; B_1, B_2)$ strategy is defined as*

$$G^B(K; B_1, B_2) := \frac{C(K) - C(B_2)}{B_1 - K} + \frac{B_2 - B_1}{B_1 - K} O(B_2) + \frac{B_2 - K}{B_1 - K} \partial_K^- C(B_2), \tag{4.4.1}$$

$G_*^B(B_1, B_2)$ as the infimum value of $G^B(K; B_1, B_2)$ with respect to K, $K_G^B(B_1, B_2)$ as a strike price by which the infimum is attained:

$$\begin{aligned} G_*^B(B_1, B_2) \quad &:= \quad \inf_{K \in (-\infty, B_1)} G^B(K; B_1, B_2) \\ &= \quad G^B(K_G^B(B_1, B_2); B_1, B_2) \end{aligned} \tag{4.4.2}$$

and $\mathcal{G}_*^B(B_1, B_2)$ as the corresponding strategy.

There is another super-replication: the $\mathcal{G}_*(B_1)$ strategy combined with a short position of the one-touch option with barrier B_2. The following theorem states that the better of the two strategies is the sharp upper bound, because the bound is attained by an expectation of the payoff with respect to a certain martingale.

Theorem 4. *If the set $\mathcal{N}(\partial_K G^B(B_1,B_2))$ is not empty, the infimum of Eq.(4.4.2) is attained by any element of a set $\mathcal{N}(\partial_K G^B(B_1,B_2))$, an interval of $(-\infty,B_1)$. For all $K_G^B \in \mathcal{N}(\partial_K G^B(B_1,B_2))$, the following holds:*

$$\begin{aligned}
& \mathbb{Q}\left[K_+ < S_T < B_2\right] \\
\leq\ & G_*^B(B_1,B_2) = \mathbb{E}\left[\frac{S_T - K_G^B}{B_1 - K_G^B}\ :\ K_G^B < S_T < B_2\right] + \frac{B_2 - B_1}{B_1 - K_G^B}\mathbb{Q}\left[\tau_2 \leq T\right] \\
\leq\ & \mathbb{Q}\left[K_- \leq S_T < B_2\right],
\end{aligned} \tag{4.4.3}$$

where $K_- := \inf \mathcal{N}(\partial_K G^B(B_1,B_2))$ and $K_+ := \sup \mathcal{N}(\partial_K G^B(B_1,B_2))$. If the set $\mathcal{N}(\partial_K G^B(B_1,B_2))$ is empty, the infimum of Eq.(4.4.2) is not attained and $G_*^B(B_1,B_2) = \mathbb{Q}\left[S_T < B_2\right]$.

If $G_*^B(B_1,B_2) < G_*(B_1) - O(B_2)$, then $\mathcal{N}(\partial_K G^B(B_1,B_2))$ is a non-empty interval of $(\sup \mathcal{N}(\partial_K G(B_1)), B_1)$. In addition, there is a continuous martingale process $\left\{S_t^G\right\}_{t\in[0,T]}$ such that

$$\min\left\{G_*^B(B_1,B_2), G_*(B_1) - O(B_2)\right\} = \mathbb{Q}\left[\tau_1\left(S^G\right) \leq T < \tau_2\left(S^G\right)\right]. \tag{4.4.4}$$

Proof. First, by differentiating G^B with respect to K, we obtain

$$\begin{aligned}
\partial_K^- G^B(K) &= \frac{1}{B_1 - K}\partial_K^- C(K) + \frac{C(K) - C(B_2)}{(B_1 - K)^2} + \frac{B_2 - B_1}{(B_1 - K)^2}\left(O(B_2) + \partial_K^- C(B_2)\right) \\
&= \frac{1}{B_1 - K}\left(G^B(K) + \partial_K^- C(K) - \partial_K^- C(B_2)\right)
\end{aligned} \tag{4.4.5}$$

and

$$\begin{aligned}
\partial_{KK}^- G^B(K) &= \frac{1}{B_1 - K}\partial_{KK}^- C(K) + \frac{2}{(B_1 - K)^2}\partial_K^- C(K) + 2\frac{C(K) - C(B_2)}{(B_1 - K)^3} \\
&\quad + 2\frac{B_2 - B_1}{(B_1 - K)^3}\left(O(B_2) + \partial_K^- C(B_2)\right) \\
&= \frac{1}{B_1 - K}\partial_{KK}^- C(K) + \frac{2}{B_1 - K}\partial_K^- G^B(K).
\end{aligned} \tag{4.4.6}$$

Note that $\partial_K^- G^B$ takes at least one positive value, because

$$\lim_{K\to B_1}(B_1 - K)^2\partial_K^- G^B(K) = C(B_1) - C(B_2) + (B_2 - B_1)\left(O(B_2) + \partial_K^- C(B_2)\right) > 0. \tag{4.4.7}$$

Since $\partial_{KK}^- G^B \geq 0$ if $\partial_K^- G^B \geq 0$, $\mathcal{N}(\partial_K G^B(B_1,B_2))$ is empty or a non-empty interval. If $\mathcal{N}(\partial_K G^B(B_1,B_2))$ is empty, we have

$$G^*(B_1,B_2) = \lim_{K\to-\infty} G(K;B_1,B_2) = \mathbb{Q}\left[S_T < B_2\right]. \tag{4.4.8}$$

If $\mathcal{N}(\partial_K G^B(B_1,B_2))$ is not empty, we have Eq.(4.4.3). Moreover, if the following holds:

$$G_*^B(B_1,B_2) < G_*(B_1) - O(B_2), \tag{4.4.9}$$

then $\mathcal{N}(\partial_K G^B(B_1,B_2))$ is a non-empty interval of $(\sup \mathcal{N}(\partial_K G(B_1)), B_1)$, because

$$\mathbb{Q}\left[K_G^B < S_T < B_2\right] \leq G_*^B < \mathbb{Q}\left[K_G(B_1) \leq S_T < B_2\right] - \left(O(B_2) - \mathbb{Q}\left[B_2 \leq S_T\right]\right), \tag{4.4.10}$$

where any $K_G^B \in \mathcal{N}(\partial_K G^B(B_1, B_2))$ and $K_G(B_1) \in \mathcal{N}(\partial_K G(B_1))$. Note that if $\mathcal{N}(\partial_K G^B(B_1, B_2))$ is not empty, we have

$$\begin{aligned}
\mathbb{E}\left[S_T - B_1 : \pi\left(\left[k_G^B, b_2\right)\right)\right] &= \mathbb{E}\left[S_T - K_G^B : \pi\left(\left[k_G^B, b_2\right)\right)\right] + (K_G^B - B_1)\mu\left(\left[k_G^B, b_2\right)\right) \\
&= C(K_G^B) - C(B_2) - (B_2 - K_G^B)\mu\left([b_2, 1]\right) + (K_G^B - B_1)\mu\left(\left[k_G^B, b_2\right)\right) \\
&= C(K_G^B) - C(B_2) - (B_2 - B_1)\mu\left([b_2, 1]\right) - (B_1 - K_G^B)\mu\left(\left[k_G^B, 1\right]\right) \\
&= (B_1 - B_2)\mathbb{Q}\left[\tau_2 \le T\right],
\end{aligned} \tag{4.4.11}$$

where $b_2 = \mathbb{Q}[S_T < B_2]$ and $k_G^B = b_2 - G_*^B$, and if $\mathcal{N}(\partial_K G^B(B_1, B_2))$ is empty, we have

$$\begin{aligned}
\mathbb{E}\left[S_T - B_1 : \pi\left(\left[k_G^B, b_2\right)\right)\right] &= \lim_{K \to -\infty} (B_1 - K)^2 \partial_K^- G^B(K) + (B_1 - B_2)\mathbb{Q}\left[\tau_2 \le T\right] \\
&= (B_1 - B_2)\mathbb{Q}\left[\tau_2 \le T\right].
\end{aligned} \tag{4.4.12}$$

Next, suppose that Eq.(4.4.9) holds. We show that there an interval $[x, y] \subset \left[0, k_G^B\right]$ exists such that

$$\mu\left([x, y) \cup [b_2, 1]\right) = \mathbb{Q}\left[\tau_2 \le T\right] \tag{4.4.13}$$

and

$$\mathbb{E}\left[S_T - B_2 : \pi([x, y) \cup [b_2, 1])\right] = 0. \tag{4.4.14}$$

Let $x = 0$ and y be a real number satisfied with Eq.(4.4.13) with $x = 0$. Then, since $y \ge k_L^{(2)} := L_*(B_2) - (1 - b_2)$, we have

$$\begin{aligned}
\mathbb{E}\left[S_T - B_2 : \pi([0, y) \cup [b_2, 1])\right] &\le \mathbb{E}\left[S_T - B_2 : \pi([0, k_L^{(2)}) \cup [b_2, 1])\right] \\
&= 0.
\end{aligned} \tag{4.4.15}$$

Conversely, let $y = k_G^B$ and x be a real number satisfied with Eq.(4.4.13) with $y = k_G^B$. By Eq.(4.4.9), we have

$$\begin{aligned}
\mu\left(\left[k_G^B, b_2\right)\right) &< G_*(B_1) - \mu\left(\left[x, k_G^B\right) \cup [b_2, 1]\right) \\
&= \mu\left(\left[k_G^{(1)}, 1\right)\right) - \mu\left(\left[x, k_G^B\right) \cup [b_2, 1]\right),
\end{aligned} \tag{4.4.16}$$

where $k_G^{(1)} := 1 - G_*(B_1)$. Then, we have $x > k_G^{(1)}$. In addition, by Eq.(4.4.11), we have

$$\begin{aligned}
\mathbb{E}\left[S_T - B_2 \,:\, \pi\left(\left[x, k_G^B\right) \cup [b_2, 1]\right)\right] &= \mathbb{E}\left[S_T - B_1 \,:\, \pi\left(\left[x, k_G^B\right) \cup [b_2, 1]\right)\right] + (B_1 - B_2)\mathbb{Q}\left[\tau_2 \le T\right] \\
&\ge \mathbb{E}\left[S_T - B_1 \,:\, \pi\left(\left[k_G^{(1)}, k_G^B\right) \cup [b_2, 1]\right)\right] + (B_1 - B_2)\mathbb{Q}\left[\tau_2 \le T\right] \\
&= -\mathbb{E}\left[S_T - B_1 : \pi\left(\left[k_G^B, b_2\right)\right)\right] + (B_1 - B_2)\mathbb{Q}\left[\tau_2 \le T\right] \\
&= 0.
\end{aligned} \tag{4.4.17}$$

Therefore, we can find an interval $[x, y)$ and have

$$\begin{aligned}
\mathbb{E}\left[S_T : \pi\left([x, y) \cup \left[k_G^B, 1\right]\right)\right] &= B_2\mu\left([x, y) \cup [b_2, 1]\right) + \mathbb{E}\left[S_T : \pi\left(\left[k_G^B, b_2\right)\right)\right] \\
&= B_1\mu\left([x, y) \cup \left[k_G^B, 1\right)\right),
\end{aligned} \tag{4.4.18}$$

using Eq.(4.4.11) again. Then, we construct a martingale $\left\{S_t^G\right\}_{t\in[0,T]}$. Let X_1 and X_2 be random variables defined as

$$X_1 := \begin{cases} B_1, & \pi\left([x,y)\cup\left[k_G^B,1\right]\right) \\ \beta_1, & otherwise \end{cases}, \tag{4.4.19}$$

and

$$X_2 := \begin{cases} B_2, & \pi\left([x,y)\cup[b_2,1]\right) \\ \beta_2, & \pi\left(\left[k_G^B,b_2\right)\right) \\ \beta_1, & otherwise \end{cases}, \tag{4.4.20}$$

where $\beta_1 \in [0,B_1)$, $\beta_2 \in [0,B_2)$ are taken as in Lemma 5 and S_t^* is a stochastic process defined as

$$S_t^* := S_0 1_{\{t<\frac{1}{3}T\}} + X_1 1_{\{\frac{1}{3}T\le t<\frac{2}{3}T\}} + X_2 1_{\{\frac{2}{3}T\le t<T\}} + S_T 1_{\{t=T\}}. \tag{4.4.21}$$

Then, applying the same argument from Lemma 5 to $\{S_t^*\}_{t\in[0,T]}$, we obtain a continuous martingale with respect to a certain filtration. We obtain $\mathbb{Q}\left[\tau_1\le T<\tau_2\right] = G_*^B$.

Finally, suppose that Eq.(4.4.9) does not hold. If $O(B_2) = G_*(B_2)$, we have $\mathbb{E}\left[S_T - B_2 \ : \ \pi\left(\left[k_G^{(2)},1\right]\right)\right] = 0$, where $k_G^{(2)} := 1 - G_*(B_2)$. If Eq.(4.4.9) holds with equality, we have

$$\begin{aligned} \mathbb{E}\left[S_T - B_2 \ : \ \pi\left(\left[k_G^{(1)},k_G^B\right]\cup[b_2,1]\right)\right] &= \mathbb{E}\left[S_T - B_1 \ : \ \pi\left(\left[k_G^{(1)},k_G^B\right]\cup[b_2,1]\right)\right] + (B_1-B_2)\mu\left(\left[k_G^{(1)},k_G^B\right]\cup[b_2,1]\right) \\ &= -\mathbb{E}\left[S_T - B_1 \ : \ \pi\left(\left[k_G^B,b_2\right]\right)\right] + (B_1-B_2)\mu\left(\left[k_G^{(1)},k_G^B\right]\cup[b_2,1]\right) \\ &= 0. \end{aligned} \tag{4.4.22}$$

Then, we can take an interval $[x,y) \subseteq \left[k_G^{(1)},b_2\right)$ which is satisfied with Eq.(4.4.13) and Eq.(4.4.14), because of $k_G^{(1)} < k_G^{(2)}$. Similar to the previous case, a continuous martingale can be constructed such that Eq.(4.4.4) holds. □

4.4.2 Sub-Replication

Consider the following self-financing strategy $\mathcal{L}^B(K;B_1,B_2)$ for $\forall K\in[0,B_1)$:

1. At the initial outset
   - Sell $\frac{1}{B_1-K}$ units of a put option with strike K.
   - Buy $\frac{B_2-B_1}{B_1-K}$ units of the one-touch option with a barrier level B_2.
2. At the first time of hitting B_1
   - Sell $\frac{1}{B_1-K}$ units of a forward contract.
3. At the first time of hitting B_2
   - Buy $\frac{1}{B_1-K}$ units of the forward contract.

Fig.4.2 shows that the $\mathcal{L}^B(K;B_1,B_2)$ strategy sub-replicates the touch/no-touch option with $K \in [0, B_1)$. We investigate the optimal strategy properties. First, we define the following.

Definition 12. *The initial value of the strategy $\mathcal{L}^B(K;B_1,B_2)$ is defined as*

$$L^B(K;B_1,B_2) := \frac{-P(K)}{B_1-K} + \frac{B_2-B_1}{B_1-K}O(B_2), \tag{4.4.23}$$

$L^B_*(B_1,B_2)$ as the supremum value of $L^B(K;B_1,B_2)$ with respect to K, $K^B_L(B_1,B_2)$ as a strike price by which the supremum is attained:

$$\begin{aligned} L^B_*(B_1,B_2) &:= \sup_{K\in(-\infty,B_1)} L^B(K;B_1,B_2) \\ &= L^B(K^B_L(B_1,B_2);B_1,B_2) \end{aligned} \tag{4.4.24}$$

and $\mathcal{L}^B_*(B_1,B_2)$ as the corresponding strategy.

There is another sub-replication: the strategy $\mathcal{L}_*(B_1)$ combined with a short position of the one-touch option with barrier B_2. The following theorem states that the better of the two strategies is the sharp lower bound, because the bound is attained by an expectation of the payoff with respect to a certain martingale.

Theorem 5. *The supremum of Eq.(4.4.24) is attained by any element of $\mathcal{N}(\partial_K L^B(B_1,B_2))$, an interval of $(0, \sup\mathcal{N}(\partial_K L(B_1))]$. For all $K^B_L \in \mathcal{N}(\partial_K L^B(B_1,B_2))$, the following holds:*

$$\mathbb{Q}[S_T < K_-] \le L^B_*(B_1,B_2) = \mathbb{E}\left[\frac{S_T-K^B_L}{B_1-K^B_L} : S_T \le K^B_L\right] + \frac{B_2-B_1}{B_1-K^B_L}\mathbb{Q}[\tau_2 \le T] \le \mathbb{Q}[S_T \le K_+], \tag{4.4.25}$$

where $K_- := \inf\mathcal{N}(\partial_K L^B(B_1,B_2))$ and $K_+ := \sup\mathcal{N}(\partial_K L^B(B_1,B_2))$.

In addition, there is a martingale process $\left\{S^L_t\right\}_{t\in[0,T]}$ such that

$$\max\left\{L^B_*(B_1,B_2), L_*(B_1) - O(B_2)\right\} = \mathbb{Q}\left[\tau_1\left(S^L\right) \le T < \tau_2\left(S^L\right)\right]. \tag{4.4.26}$$

Proof. First, by differentiating L^B with respect to K, we obtain

$$\begin{aligned} \partial^-_K L^B(K) &= \frac{-1}{B_1-K}\partial^-_K P(K) - \frac{P(K)}{(B_1-K)^2} + \frac{B_2-B_1}{(B_1-K)^2}O(B_2) \\ &= \frac{1}{B_1-K}\left(-\partial^-_K P(K) + L^B(K)\right) \end{aligned} \tag{4.4.27}$$

and

$$\begin{aligned} \partial^-_{KK} L^B(K) &= \frac{-1}{B_1-K}\partial^-_{KK} P(K) - \frac{2}{(B_1-K)^2}\partial^-_K P(K) - 2\frac{P(K)}{(B_1-K)^3} + 2\frac{B_2-B_1}{(B_1-K)^3}O(B_2) \\ &= \frac{2}{B_1-K}\partial^-_K L^B(K) - \frac{1}{B_1-K}\partial^-_{KK} P(K). \end{aligned} \tag{4.4.28}$$

Note that $\partial^-_K L^B(0) = \frac{B_2-B_1}{B_1^2}O(B_2) > 0$ and by Eq.(4.3.11)

$$\begin{aligned} \partial^+_K L^B(K_+) &= \frac{-1}{B_1-K_+}\partial^+_K P(K_+) - \frac{P(K_+)}{(B_1-K_+)^2} + \frac{B_2-B_1}{(B_1-K_+)^2}O(B_2) \\ &\le -\frac{C(B_1)}{(B_1-K_+)^2} + \frac{B_2-B_1}{(B_1-K_+)^2}O(B_2) \\ &\le 0, \end{aligned} \tag{4.4.29}$$

where $K_+ := \sup \mathcal{N}(\partial_K L(B_1))$ and ∂_K^+ is the right-sided derivative operator. Since $\partial_{KK}^- L^B \le 0$ if $\partial_K^- L^B \le 0$, $\mathcal{N}(\partial_K L^B(B_1, B_2))$ is an interval of $(0, K_+]$ and we have Eq.(4.4.25). Note that

$$\begin{aligned}\mathbb{E}\left[B_1 - S_T : \pi\left(\left[0, k_L^B\right)\right)\right] &= \mathbb{E}\left[K_L^B - S_T : \pi\left(\left[0, k_L^B\right)\right)\right] + \mathbb{E}\left[B_1 - K_L^B : \pi\left(\left[0, k_L^B\right)\right)\right] \\ &= P(K_L^B) + (B_1 - K_L^B)\mu\left(\left[0, k_L^B\right)\right) \\ &= (B_2 - B_1)\mathbb{Q}\left[\tau_2 \le T\right],\end{aligned} \tag{4.4.30}$$

where $k_L^B := L_*^B$.

Next, suppose that the following holds:

$$L_*(B_1) - O(B_2) < L_*^B(B_1, B_2). \tag{4.4.31}$$

We show that there exists an interval $[x, y) \subset \left[k_L^B, b_1\right]$, where $b_1 = \mathbb{Q}[S_T < B_1]$, such that

$$\mu\left([x, y) \cup [b_1, 1]\right) = \mathbb{Q}\left[\tau_2 \le T\right] \tag{4.4.32}$$

and

$$\mathbb{E}\left[S_T - B_2 : \pi\left([x, y) \cup [b_1, 1]\right)\right] = 0. \tag{4.4.33}$$

We can take an interval that satisfies Eq.(4.4.32) because Eq.(4.4.31) implies

$$\begin{aligned}\mathbb{Q}\left[\tau_2 \le T\right] &> \mu\left(\left[0, k_L^{(1)}\right) \cup [b_1, 1]\right) - \mu\left(\left[0, k_L^B\right)\right) \\ &\ge \mu\left([b_1, 1]\right),\end{aligned} \tag{4.4.34}$$

where $k_L^{(1)} := L^*(B_1) - (1 - b_1)$. Let $y = b_1$ and x be a solution of Eq.(4.4.32) with $y = b_1$. We have $x \ge k_G^{(2)} := 1 - G_*(B_2)$ because $\mathbb{Q}\left[\tau_2 \le T\right] \le \mu\left([k_G^{(2)}, 1]\right)$. Then, we have

$$\begin{aligned}\mathbb{E}\left[S_T - B_2 : \pi([x, 1))\right] &\ge \mathbb{E}\left[S_T - B_2 : \pi\left(\left[k_G^{(2)}, 1\right)\right)\right] \\ &= 0.\end{aligned} \tag{4.4.35}$$

Conversely, let $x = k_L^B$ and y be the solution of Eq.(4.4.32) with $x = k_L^B$. We have $y \ge k_L^{(1)}$, because $\mathbb{Q}\left[\tau_2 \le T\right] > \mu\left(\left[k_L^B, k_L^{(1)}\right) \cup [b_1, 1]\right)$ by Eq.(4.4.31). Then, by Eq.(4.4.30), we have

$$\begin{aligned}\mathbb{E}\left[S_T - B_2 : \pi\left(\left[k_L^B, y\right) \cup [b_1, 1]\right)\right] &= \mathbb{E}\left[S_T - B_1 : \pi\left(\left[k_L^B, y\right) \cup [b_1, 1]\right)\right] + (B_1 - B_2)\mathbb{Q}\left[\tau_2 \le T\right] \\ &\le \mathbb{E}\left[S_T - B_1 : \pi\left(\left[k_L^B, k_L^{(1)}\right) \cup [b_1, 1]\right)\right] + (B_1 - B_2)\mathbb{Q}\left[\tau_2 \le T\right] \\ &= \mathbb{E}\left[B_1 - S_T : \pi\left(\left[0, k_L^B\right)\right)\right] + (B_1 - B_2)\mathbb{Q}\left[\tau_2 \le T\right] \\ &= 0.\end{aligned} \tag{4.4.36}$$

Therefore, we can find the interval $[x, y)$ and we have for this interval

$$\begin{aligned}\mathbb{E}\left[S_T : \pi\left(\left[0, k_L^B\right) \cup [x, y) \cup [b_1, 1]\right)\right] &= \mathbb{E}\left[S_T : \pi\left(\left[0, k_L^B\right)\right)\right] + B_2\mu([x, y) \cup [b_1, 1]) \\ &= B_1\mu\left(\left[0, k_L^B\right) \cup [x, y) \cup [b_1, 1]\right),\end{aligned} \tag{4.4.37}$$

using Eq.(4.4.30) again. Then, we construct a martingale $\left\{S_t^L\right\}_{t\in[0,T]}$. Let X_1 and X_2 be random variables defined as

$$X_1 := \begin{cases} B_1, & \pi\left(\left[0, k_L^B\right) \cup [x, y) \cup [b_1, 1]\right) \\ \beta_1, & otherwise \end{cases}, \tag{4.4.38}$$

and

$$X_2 := \begin{cases} B_2, & \pi([x, y) \cup [b_1, 1]) \\ \beta_2, & \pi\left(\left[0, k_L^B\right)\right) \\ \beta_1, & otherwise \end{cases}, \tag{4.4.39}$$

where $\beta_1 \in [0, B_1)$, $\beta_2 \in [0, B_2)$ are taken as in Lemma 5 and S_t^* be a stochastic process defined as

$$S_t^* := S_0 1_{\{t<\frac{1}{3}T\}} + X_1 1_{\{\frac{1}{3}T\le t<\frac{2}{3}T\}} + X_2 1_{\{\frac{2}{3}T\le t<T\}} + S_T 1_{\{t=T\}}. \tag{4.4.40}$$

Then, applying the same argument from Lemma 5 to $\{S_t^*\}_{t\in[0,T]}$, we obtain a continuous martingale with respect to a certain filtration. We obtain $\mathbb{Q}\left[\tau_1 \le T < \tau_2\right] = L_*^B$.

Finally, suppose that Eq.(4.4.31) does not hold. If $O(B_2) = L_*(B_2)$ and let $k_L^{(2)} := L_*(B_2) - (1 - b_2)$, we have $\mathbb{E}\left[S_T - B_2 \ : \ \pi([0, k_L^{(2)}) \cup [b_2, 1])\right] = 0$. If Eq.(4.4.31) holds with equality, we have by Eq.(4.4.30)

$$\begin{aligned} \mathbb{E}\left[S_T - B_2 \ : \ \pi\left(\left[k_L^B, k_L^{(1)}\right) \cup [b_1, 1]\right)\right] &= \mathbb{E}\left[S_T - B_1 \ : \ \pi\left(\left[k_L^B, k_L^{(1)}\right) \cup [b_1, 1]\right)\right] + (B_1 - B_2)\mathbb{Q}\left[\tau_2 \le T\right] \\ &= -\mathbb{E}\left[S_T - B_1 \ : \ \pi\left(\left[0, k_L^B\right)\right)\right] + (B_1 - B_2)\mathbb{Q}\left[\tau_2 \le T\right] \\ &= 0. \end{aligned} \tag{4.4.41}$$

Note that $k_L^{(2)} < k_L^{(1)}$, because $(B - K)^2\partial_K^- L(K; B)$ is decreasing with respect to B and K. Then, we can take a set $D \subseteq [0, k_L^{(1)}] \cup [b_1, b_2)$ which is satisfied with

$$\mu\left(D \cup [b_2, 1]\right) = \mathbb{Q}\left[\tau_2 \le T\right] \tag{4.4.42}$$

and

$$\mathbb{E}\left[S_T - B_2 : \pi\left(D \cup [b_2, 1]\right)\right] = 0. \tag{4.4.43}$$

Similar to the previous case, a continuous martingale can be constructed such that Eq.(4.4.26) holds. □

4.4.3 The Finite Basis Situation

In this section, we consider the case where only a finite number of strikes are given. Suppose that call options with strikes $K_0 < K_1 < \cdots < K_N$, where $K_0 = 0$ and $B_2 \le K_N$, are traded with no-arbitrage prices $\{C_n\}_{n=0}^N$. We consider super-replication and sub-replication for the touch/no-touch option with barrier levels B_1 and B_2 using the one-touch option with a barrier level B_2. We assume that a no-arbitrage price of the digital call option with strike B_2 is given as D_2 in case of the super-replication and that with strike B_1 is given as D_1 in case of the sub-replication. Here, a no-arbitrage price D of digital call option with strike $B \in (K_{n-1}, K_n]$ satisfies

$$-\frac{C_{n+1} - C_n}{K_{n+1} - K_n} \le D \le -\frac{C_{n-1} - C(B)}{K_{n-1} - B}, \tag{4.4.44}$$

where $C(B) := \frac{C_{n+1}-C_n}{K_{n+1}-K_n}(B-K_n)+C_n$. Even if these digital call options are not liquid, we can regard the lower bound as the digital call price with strike B_2 in case of the super-replication and the upper bound as the digital call price with strike B_1 in case of the sub-replication.

First, we consider the super-replication. We suppose that the no-touch option with a barrier level B_2 is traded and the price of this no-touch option satisfies

$$\sup_{K_n<B_2} L(K_n;B_2) \leq O(B_2) \leq \inf_{K_n<B_2} G(K_n;B_2). \tag{4.4.45}$$

The upper bound on the touch/no-touch option derived from the super-replication is

$$\min\left\{G_*^B\left(B_1,B_2;\{K_n\}_{n=0}^N\right), G_*\left(B_1;\{K_n\}_{n=0}^N\right)-O(B_2)\right\}, \tag{4.4.46}$$

where $G_*^B\left(B_1,B_2;\{K_n\}_{n=0}^N\right) := \inf_{K_n<B_1} G^B(K_n;B_1,B_2)$ and $G_*\left(B_1;\{K_n\}_{n=0}^N\right) := \inf_{K_n<B_1} G(K_n;B_1)$. Although the marginal distribution of S_T is not uniquely specified in this case, the following corollary shows that there is a distribution consistent with the given option prices under which we can construct a martingale attaining the upper bound.

Corollary 5. *There is a distribution μ_C of S_T which is consistent with the given call prices, the given digital call option with a strike B_2 and the given no-touch option with a barrier level B_2 satisfying Eq.(4.4.45) such that Eq.(4.4.46) is equal to Eq.(4.4.4) with distribution μ_C.*

Proof. First, we assume $B_2 \in \{K_n\}_{n=0,\cdots,N}$ and $D_2 = -\frac{C_{n-1}-C_n}{K_{n-1}-K_n}$, where $B_2 = K_n$, Let us consider call options prices $\{C(K)\}_{K\in[0,+\infty)}$: $C(K)$ is the linear interpolation of C_n if $K \in [K_0, K_N]$ and an arbitrary extrapolation excluding arbitrage opportunities if $K \in [K_N, +\infty)$. Let μ_C be a distribution implied by the call option prices C. We can apply Proposition 3 and 4 with the distribution μ_C to the no-touch option with a barrier level B_2 and obtain the optimal strikes $K_G(B_2)$ and $K_L(B_2)$. These may not be uniquely determined, but can be taken as one of the given strikes, since the distribution μ_C consists of atoms at K_n on $[0, B_2)$. Hence, the distribution μ_C is consistent with Eq.(4.4.45). By the same reason, Eq.(4.4.4) with distribution μ_C is attained by one of the given strikes. Then, Eq.(4.4.46) is equal to Eq.(4.4.4) with distribution μ_C.

In the general case, two call prices, $C(\tilde{K})$ and $C(B_2)$, with strikes, $\tilde{K} := B_2 - \varepsilon$ and B_2, can be added into the given call price set as: $C(K) := -D_2(K-\hat{K})+\hat{C}$ for $K = \tilde{K}, B_2$, where $B_2 \in (K_{n-1}, K_n]$, $(\hat{K},\hat{C}) = (K_{n-1}, C_{n-1})$ in case of $D_2 > -\frac{C_{n-1}-C_n}{K_{n-1}-K_n}$, $(\hat{K},\hat{C}) = (K_n, C_n)$ in the other case, ε is a sufficiently small positive value such that $\tilde{K}$ is not the optimal strike for $G_*(B_2)$ and $L_*(B_2)$. Then, the same argument from the first case can be applied. □

Next, we consider the sub-replication. This is more involved than the super-replication. The lower bound on the touch/no-touch option derived from the sub-replication is

$$\max\left\{L_*^B\left(B_1,B_2;\{K_n\}_{n=0}^N\right), L_*\left(B_1;\{K_n\}_{n=0}^N\right)-O(B_2)\right\}, \tag{4.4.47}$$

where $L_*^B\left(B_1,B_2;\{K_n\}_{n=0}^N\right) := \sup_{K_n<B_1} L^B(K_n;B_1,B_2)$ and $L_*\left(B_1;\{K_n\}_{n=0}^N\right) := \sup_{K_n<B_1} L(K_n;B_1)$. We assume $B_1 \in \{K_n\}_{n=0,\cdots,N}$ and

$$\sup_{K_n<B_2} L(K_n;B_2) < O(B_2) < \inf_{K_n<B_2} G(K_n;B_2). \tag{4.4.48}$$

Owing to these assumptions, we have the similar result to Corollary 5.

Corollary 6. *There is a distribution μ_C of S_T which is consistent with the given call prices which includes that with strike B_1, the given digital call option with a strike B_1 and the given no-touch option with a barrier level B_2 satisfying Eq.(4.4.48) such that Eq.(4.4.47) is equal to Eq.(4.4.26) with distribution μ_C.*

Proof. Let $n \in \{0, 1, \cdots, N\}$ be such that $B_1 = K_n$. The proof is the same as the first part of Corollary 5, if $D_1 = -\frac{C_{n-1}-C_n}{K_{n-1}-K_n}$. In the general case, let $\tilde{K} := B_1 - \varepsilon > K_{n-1}$ for a sufficiently small positive value ε and $C(\tilde{K}) := -D_1(\tilde{K} - K_n) + C_n$. We can take ε such that

$$\sup_{\tilde{K}, K_i < B_2} L(K_i; B_2) < O(B_2) < \inf_{\tilde{K}, K_i < B_2} G(K_i; B_2). \tag{4.4.49}$$

Let μ_C be a distribution implied by an interpolation of the given call option prices C and $C(\tilde{K})$. Then, we have the conclusion for the distribution μ_C by the same argument from Corollary 5. $\square$

4.5 Numerical Examples

This section provides numerical examples.

We regard Heston's stochastic volatility model ((Heston, 1993)) as the underlying asset process. The process underlying the Heston model is as follows:

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma_t dW_t, \tag{4.5.1}$$

$$d\sigma_t^2 = \kappa(\eta - \sigma_t^2)dt + \theta\sigma_t d\tilde{W}_t, \tag{4.5.2}$$

where W and $\tilde{W}$ are Brownian motions with correlation ρ under a risk-neutral measure. In addition, we assume that the parameters of the Heston model are as shown in Table 5.1.

r	q	σ_0^2	κ	η	θ	ρ
0.0	0.0	0.15^2	3.0	0.2^2	0.4	0.0

Table 4.1: Parameters of the Heston Model

The one-touch option considered has a 3-month maturity and a barrier level of 1.05 USD. We calculate the pricing bounds of our method, those of (Brown et al., 2001) and exact prices by a Monte Carlo simulation with the initial spot price varied from 0.9 USD to 1.04 USD. We calculate pricing bounds derived from $\mathcal{G}_*^B$ and $\mathcal{L}_*^B$ strategies using another one-touch option with $B = 1.06$. This is evaluated by the Heston model with the same parameter set. The results are shown in Fig.4.3 and Table 4.2. Our lower bounds are proved to be higher than those of (Brown et al., 2001) across the entire range and our upper bounds proved lower in the $[0.9, 0.98]$ range.

Additionally, Fig.4.4 shows a relationship between pricing bounds on the two one-touch options with barrier levels 1.05 USD and 1.06 USD, where the market conditions are the same as for the above example and the initial spot price is fixed at 1 USD. The pricing bounds of (Brown et al., 2001) on the two one-touch options are $[0.315, 0.609]$ for the barrier level 1.05 USD and $[0.263, 0.529]$ for barrier level 1.06 USD. However, we established that a condition for no-arbitrage prices of these two options does not lie within these ranges but is within the range indicated in Fig.4.4.

Figure 4.1: Payoff of Strategy $\mathcal{G}^B(K; B_1, B_2)$ with $S_0 = 1$, $K = 0.95$, $B_1 = 1.05$ and $B_2 = 1.06$

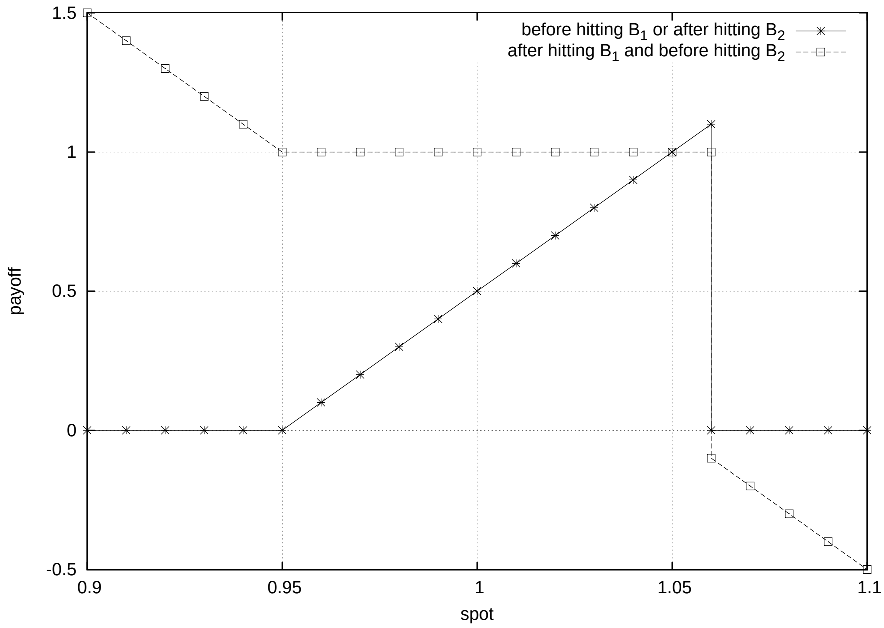


Figure 4.2: Payoff of Strategy $\mathcal{L}^B(K; B_1, B_2)$ with $S_0 = 1$, $K = 0.95$, $B_1 = 1.05$ and $B_2 = 1.06$

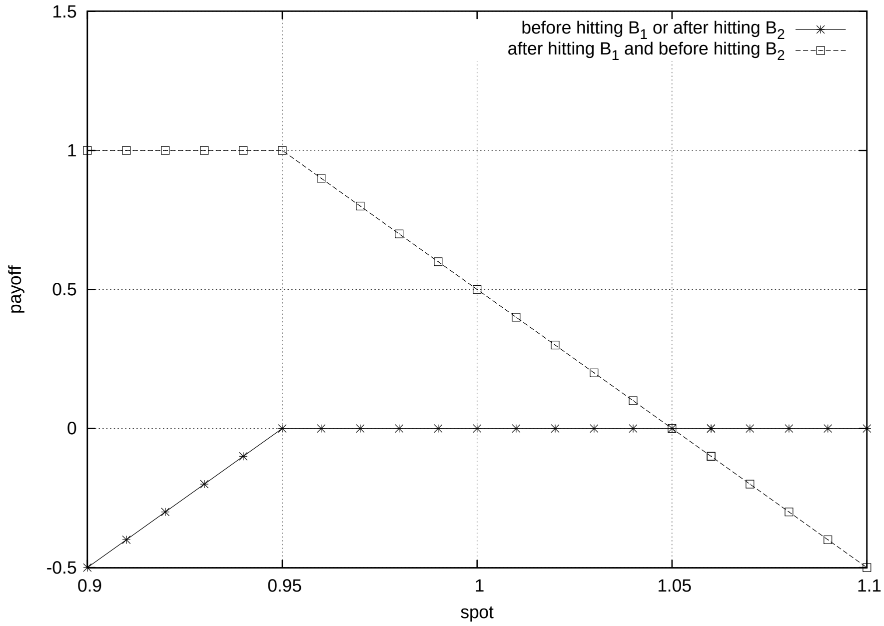


Figure 4.3: Pricing bounds on a one-touch option

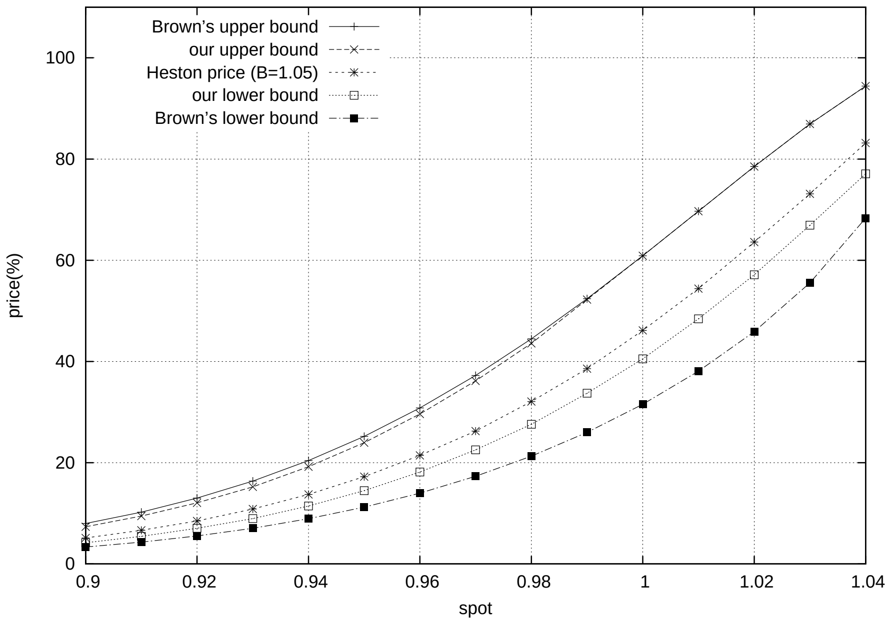
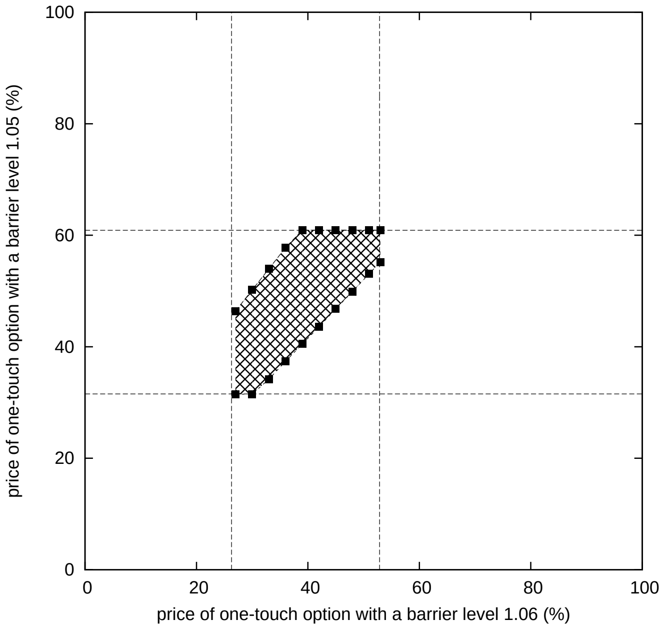


Table 4.2: Pricing bounds on a one-touch option (%)

spot	0.90	0.92	0.94	0.96	0.98	1.00	1.02	1.04
(Brown et al., 2001)'s upper bound	8.0	13.0	20.4	30.8	44.5	60.9	78.5	94.4
Our upper bound W^G	7.3	12.1	19.2	29.6	43.6	61.6	82.8	105.8
Heston price ($B = 1.05$)	5.1	8.5	13.7	21.4	32.1	46.1	63.6	83.2
Our lower bound W^L	4.2	7.0	11.5	18.2	27.6	40.5	57.1	77.1
(Brown et al., 2001)'s lower bound	3.3	5.5	8.9	14.0	21.3	31.5	45.9	68.3
Heston price ($B = 1.06$)	4.1	6.8	11.1	17.6	26.7	39.0	54.7	73.3

Figure 4.4: Pricing bounds on two one-touch options



Chapter 5

No-Arbitrage Conditions for Barrier Options

This paper investigates no-arbitrage conditions for barrier options. No arbitrage conditions for European call options are well-known: absence of calendar and butterfly spread arbitrage, but those for barrier options have not been documented.

The question considered in this paper is whether there are arbitrage opportunities or not for a given price set which consists of barrier options as well as European options. The findings of this study are as follows: a condition that excludes arbitrage opportunities if only static trading strategies are allowed, the pricing bounds on barrier options using other barrier options, how a term structure of touch options improves these bounds in case of single barrier options, and no static arbitrage conditions between barrier options with different maturities.

5.1 Introduction

This paper investigates no-arbitrage conditions for barrier options under various circumstances.

No arbitrage conditions for European call and put options are well-known: absence of calendar and butterfly spread arbitrage (see e.g. (Davis and Hobson, 2007), (Carr and Madan, 2005) and (Cousot, 2007) for details) and many methods of interpolation and extrapolation of volatility smiles/surfaces without arbitrage opportunities have been proposed (e.g. (Fengler, 2009) and (Gatheral and Jacquier, 2013)). These studies on no-arbitrage conditions are quite general and do not rely on modeling an underlying process with a stochastic process. Motivated by the previous research and by the fact that barrier options are most liquidly traded among exotic derivatives, this paper tackles the same kind of problems for barrier options, namely what is no arbitrage condition for a given set of derivative prices and how barrier options are priced in a way of excluding arbitrage opportunities as well as reproducing given derivative prices. Here, the given derivatives considered in this paper are forward contracts, European options and barrier options with the common barrier conditions. Study on barrier options without using specific models for an underlying process is worthy, since barrier option prices are significantly model-dependent even if the model is calibrated to European option prices (see (Hirsa et al., 2003), (Lipton and McGhee, 2002) and (Schoutens et al., 2005)).

This paper considers the following four problems.

The first problem is to establish no-arbitrage conditions for prices of barrier options with the common barrier condition and maturity as well as European options with the same maturity. The characteristics for pricing barrier option in this context are provided by the marginal distribution of the underlying asset, that of the event that the barrier is hit, and a joint distribution of them. The second corresponds to the relevant one-/no-touch option, where the one-/no-touch option is a barrier option which pays a unit of currency if the barrier is/is not hit. The last one is an expectation of the indicator function of the second quantity conditional on the underlying price at maturity. The no-arbitrage condition is reduced to this function and corresponds to absence of butterfly spread arbitrage for European options.

Second, model-independent pricing bounds are considered. Model-independent pricing bounds have been investigated by several authors: (Brown et al., 2001) study when the underlying asset is a martingale, (Cox and Obłój, 2011a), (Cox and Obłój, 2011b) and (Tsuzuki, 2015) focus on several types of touch options under the similar conditions as (Brown et al., 2001). (Tsuzuki, 2014) derives pricing bounds using no-touch options as well as European options. The study in this paper is an extension of (Tsuzuki, 2014) and investigates the bounds on a barrier option if other barrier options can be used in addition to instruments which (Tsuzuki, 2014) uses for replication.

The third problem focuses on no-touch options with a common single barrier but with different maturities. (Tsuzuki, 2014) shows that a knock-out forward contract can be used for replication in addition to a no-touch option, pricing bounds are significantly improved. However, under the condition that the underlying asset is a continuous martingale, the knock-out forward contract is redundant because this can be replicated by the no-touch option. Then, an interesting question is what if the underlying process is not a martingale, but the forward process is and both of them do not necessarily a martingale. Instead of postulating a model, how different the underlying process is from a continuous martingale are given by no-touch options and knock-out forward contracts with different maturities. This paper considers no arbitrage conditions for these options and contracts and whether the pricing bounds are improved or not by them.

The final finding of this paper is no-arbitrage conditions when one trades European options and barrier options with a common barrier conditions but with different two maturities. This corresponds to absence of calendar spread arbitrage for European options.

Moreover, pricing methods incorporating these conditions as well as market prices of relevant derivatives are proposed. An advantage of this method over the standard one which postulates a stochastic process as an underlying process is flexibility for specifying a pricing kernel. Most of the standard method inevitably produce a high probability that the barrier has been hit conditional on that the underlying price at maturity is close to the barrier level. This may be desirable in most cases, but is not necessary.

As a final remark, barrier options considered in this paper may be knock-in/-out options whose knock-in/-out condition is irrelevant to the underlying asset for a terminal payoff, although barrier options are usually dependent on a single asset. An example is an option which pays a payoff dependent on a stock price at maturity if a foreign exchange rate has reached a certain level. CVA, credit value adjustment, is another example and will be studied in this paper as an application.

The next section describes the notations. No-arbitrage conditions mentioned above are investigated in the third section. The fourth section provides a method of how to price barrier options excluding the no-arbitrage opportunities. In the fifth section, some examples that the method is appropriate are given. The sixth section provides numerical examples.

5.2 Notations

The notations used in this paper are stated here.

First, let S_t be an asset price at time $t \in [0, T^*]$ and s_0 be the time-0 price, where $S_t \in \mathbb{R}_+ := [0, +\infty)$ is assumed and T^* is some arbitrary time horizon. This paper focuses on a knock-out option, instead of a knock-in option, whose payoff is an arbitrary function dependent only on the underlying asset price at maturity when it is not knocked out. The results for knock-in options are obtained by in-out parity[1]. The knock-out event is that the underlying asset price has not reached a certain level(s) or some event which is irrelevant to the underlying asset. The former types of barrier options are actively traded in foreign exchange option markets. A payoff of barrier options with maturity T and an arbitrary payoff function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}$ are represented with

$$g(S_T)1_A, \tag{5.2.1}$$

where A is the event that the option pays payoff. For example, the payoff of a double knock-out call option with strike κ and barrier levels L and U, $0 \le L < s_0 < U < +\infty$, is $(S_T - K)_+ 1_A$, where $A := \{S_t \in I \mid 0 \le \forall t \le T\}$ and $I := [L, U]$. Single barrier options can also be expressed in this manner with $L < 0$ or $U = +\infty$. In case where the knock-out event is irrelevant to the underlying asset, the event A is independent of $\{S\}_{t\in[0,T]}$. An example of this kind of barrier options is a knock-out option whose payoff is dependent on a stock price and knock-out event is on a foreign exchange rate. Conventionally, put $I := \mathbb{R}_+$ in this case. In particular, a no-touch option is a knock-out option which is worthless if the knock-out event happens, and pays a unit of currency at the maturity if the knock-out event does not happen. The payoff is 1_A. Let Eq.(5.2.1) be called as *payoff* of a barrier option and $g(S_T)$ be as *terminal payoff* in this paper.

Second, some notations about the market environment are introduced. Let r and q be the risk-free interest rate and the dividend yield of the underlying asset respectively. They are assumed to be deterministic. A time-t_1 price of a domestic zero-coupon bond delivered at time t_2 and that with respect to the dividend yield are denoted by $D^r_{t_1,t_2} := e^{-\int_{t_1}^{t_2} r_s ds}$ and $D^q_{t_1,t_2} := e^{-\int_{t_1}^{t_2} q_s ds}$ respectively and $D^{q/r}_{t_1,t_2} := \frac{D^q_{t_1,t_2}}{D^r_{t_1,t_2}}$, $D^{r/q}_{t_1,t_2} := \frac{D^r_{t_1,t_2}}{D^q_{t_1,t_2}}$. In addition, the time-0 prices of European call options with the same maturity as barrier options under consideration and arbitrary strike prices are assumed to be known at time 0. Knowledge of the distribution of the underlying asset at maturity is equivalent to the knowledge of European call option prices without arbitrage opportunities for the continuum of strikes by (Breeden and Litzenberger, 1978). For a random variable X and a distribution μ, $X \sim \mu$ denotes in this paper that the distribution of X is μ. Let $C_{0,T}(\kappa)$ be the time-0 call price with maturity T and strike κ and μ_T be the distribution of S_T implied by them, that is $S_T \sim \mu_T$. In particular, a forward price of the underlying asset delivered at time T is also given by $\frac{C_{0,T}(0)}{D^r_{0,T}}$, which is assumed to be equal to $D^{q/r}_{0,T} s_0$. Probability measures in this paper are not unique and several notations are used such as $\mathbb{Q}$, $\mathbb{Q}^*$, $\mathbb{Q}_*$ and so on. Expectation operators with respect to these probability measures are denoted as $\mathbb{E}[\cdot]$, $\mathbb{E}^*[\cdot]$ and $\mathbb{E}_*[\cdot]$ respectively and the first time for the knock-out event to happened is denoted as τ, τ^* and τ_* under each probability measure, if there is no ambiguity from the context.

Next, some technical notations are defined. Let $(\mathbb{R}_+, \mathcal{M}, \mu)$ be a measurable space, where $\mathcal{M}$ is the set of all Lebesgue measurable sets on $\mathbb{R}_+$ and μ is a probability measure. Two norms for a function on this space $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ are defined by

$$||f||_1 := \int |f| d\mu, \ ||f||_\infty := \operatorname{ess\,sup} |f|, \tag{5.2.2}$$

$\mathcal{L}^1_\mu$ and $\mathcal{L}^\infty_\mu$ by the Banach space with respect to the norms, and a function $\langle \cdot, \cdot \rangle_\mu$ on $\mathcal{L}^1_\mu \times \mathcal{L}^\infty_\mu$ by

$$\langle f_1, f_\infty \rangle_\mu := \int_0^{+\infty} f_1 \cdot f_\infty d\mu \qquad (f_1 \in \mathcal{L}^1_\mu, f_\infty \in \mathcal{L}^\infty_\mu). \tag{5.2.3}$$

[1]In-out parity is a relationship which states that the payoffs of a knock-in option and a knock-out option sum to the payoff of an option without a barrier condition.

The subscript μ may be omitted if it is obvious.

Finally, some important concepts and results in this paper are introduced. The convex order is one of stochastic orderings between two measures:

Definition 13. *For μ_1 and μ_2 be probability measures on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, where $\mathcal{B}(\mathbb{R}_+)$ is a set of all Borel measurable sets of $\mathbb{R}_+$, μ_2 dominates μ_1 in the convex order, if $\int_{\mathbb{R}_+} \phi d\mu_1 \leq \int_{\mathbb{R}_+} \phi d\mu_2$ for each μ_i-integrable convex function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ for each $i = 1, 2$. This is denoted as $\mu_1 \preceq \mu_2$.*

Note that $\mu_1 \preceq \mu_2$, if $\int_{\mathbb{R}_+} (s-\kappa)_+ d\mu_1(s) \leq \int_{\mathbb{R}_+} (s-\kappa)_+ d\mu_2(s)$ for all $\kappa \geq 0$.

Kellerer's theorem ((Kellerer, 1972)) relates the existence of a martingale to conditions on marginal distributions and is often used in this paper:

Theorem 6 (Kellerer's Theorem)**.** *Let $(\mu_t)_{t\in[0,T]}$ be a family of probability measures on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ with first moment, such that, for $s < t$, μ_t dominates μ_s in the convex order, i.e. for each convex function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}$, μ_t-integrable for each $t \in [0, T]$, one has $\int_{\mathbb{R}_+} \phi d\mu_t \geq \int_{\mathbb{R}_+} \phi d\mu_s$. Then, there exists a Markov process $(M_t)_{t\in[0,T]}$ with these marginal distributions under which it is a submartingale. Furthermore if the means are independent of t then $(M_t)_{t\in[0,T]}$ is a martingale.*

5.3 No Arbitrage Conditions

The following subsections consider no arbitrage conditions under different assumptions about admissible trading strategies and tradable derivatives. All tradable derivatives are assumed to have single cashflows at maturity. Following Definition 4.1 of (Cox and Hobson, 2005), this paper adopts Definition 14 as no-arbitrage.

Definition 14. *There is no arbitrage in the family of derivative prices $\{V_0^\alpha; \alpha \in \mathcal{A}\}$, if there is a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q}, \{\mathcal{F}_t\})$ for which $\{D_{0,t}^{r/q} S_t\}_{t\in[0,T^*)}$ is a non-negative discrete- or continuous-time martingale and $V_0^\alpha = D_{0,T_\alpha}^r \mathbb{E}\left[V_{T_\alpha}\right]$ for all $\alpha \in \mathcal{A}$, where T_α is a maturity, V_{T_α} is a cashflow of the derivative and $\mathbb{E}\left[\cdot\right]$ is an expectation operator of $\mathbb{Q}$.*

The probability space in Definition 14 is called as discrete-time or continuous-time model hereafter.

No static arbitrage conditions on European call option prices are well-known: absence of calendar and butterfly spread arbitrage (see e.g. (Davis and Hobson, 2007), (Carr and Madan, 2005) and (Cousot, 2007) for details). Absence of butterfly spread arbitrage is $C_{0,T}(\kappa - \Delta\kappa) - 2C_{0,T}(\kappa) + C_{0,T}(\kappa + \Delta\kappa) \geq 0$ for all $\kappa, \Delta\kappa \geq 0$ and $T > 0$ and is equivalent to non-negativity of a density function. This kind of condition for barrier options is provided by Section 5.3.1. Absence of calendar spread arbitrage is $D_{T_1,T_2}^q C_{0,T_1}(D_{T_1,T_2}^{r/q}\kappa) \leq C_{0,T_2}(\kappa)$ for all $\kappa \geq 0$ and $T_1 < T_2$ and can be expressed with the convex order as in Definition 13. Section 5.3.4 investigates no arbitrage conditions for barrier options with different two maturities.

5.3.1 No Static Arbitrage Conditions

This section assumes that tradable derivatives are

- European call and put options with maturity T
- Knock-out options with maturity T and with payoff $g(S_T)1_A$ for any $g \in \mathcal{L}^1$.

The distribution of S_T is derived from European option prices: $\mu := \mu_T$. Let $\Phi(g)$ be a forward price of a knock-out option whose payoff is $g(S_T)1_A$, namely $D^r_{0,T}\Phi(g)$ is the knock-out option price. This section investigates a no arbitrage condition on Φ. First, trading is allowed only at inception and then it is relaxed.

Consider the problem in a heuristic way, before no-static arbitrage condition is stated. If a model $(\Omega, \mathcal{F}, \mathbb{Q}, \{\mathcal{F}_t\})$ is given, a forward price of the knock-out option is

$$\begin{aligned}\mathbb{E}\left[g(S_T)1_A\right] &= \mathbb{E}\left[g(S_T)\mathbb{E}\left[1_A|S_T\right]\right] \\ &= \int g\alpha d\mu, \end{aligned} \tag{5.3.1}$$

where $\alpha(s) := \mathbb{E}\left[1_A|S_T = s\right]$. From this expression, the price depends on the three factors: a distribution of S_T, a probability that the knock-out event happens (a distribution of 1_A), and a dependency between the two. The first one is implied by European call prices and the second is by $\Phi(1)$, where 1 is a function that always takes 1. The third one is given by α, a conditional expectation of 1_A conditional on S_T. Let this kink of functions be defined as follows:

Definition 15. *A kernel function for the event A is $\alpha \in \mathcal{L}^\infty_\mu$ such that $0 \le \alpha \le 1$ on $\mathbb{R}_+$ and $\alpha = 0$ on I^c. $\mathcal{A}$ is a set of all kernel functions for the event A.*

Proposition 5 provides no arbitrage condition on Φ under the static trading constraint.

Proposition 5. *The following statements are equivalent:*

(a) *There exists $\alpha \in \mathcal{A}$ such that $\Phi(g) = \langle g, \alpha\rangle$ for any $g \in \mathcal{L}^1$.*

(b) *There is a model with respect to discrete-time $\{0, T\}$.*

Proof. Suppose that (a) holds. Consider two random variables X and Y on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, where the distribution of X is μ and that of Y is given by $\mathbb{Q}[Y = 0] = 1 - \Phi(1)$ and $\mathbb{Q}[Y = 1] = \Phi(1)$. In addition, a joint distribution between the two random variables is assumed to be given by $\mathbb{Q}[X \le s, Y = 1] := \int_{[0,s]} \alpha d\mu$ and $\mathbb{Q}[X \le s, Y = 0] := \int_{[0,s]}(1-\alpha)d\mu$ for $s \in \mathbb{R}_+$. Then, one can define a process $\{S^*_t\}_{t\in[0,T]}$ such that $S^*_T = X$ and the knock-out event happens if and only if $Y = 1$. In case of single or double barrier options, an example is

$$S^*_t := s_0 1_{\{t=0\}} + \left(x1_{\{Y=0\}} + S_T 1_{\{Y=1\}}\right) 1_{\{t\in(0,T)\}} + X1_{\{t=T\}}, \tag{5.3.2}$$

with any $x \notin I$. The similar construction can be applied to a case of a knock-out condition irrelevant to the underlying asset. If $\mathcal{F}_t$ is the filter generated by S^*_t, the probability space $(\Omega, \mathcal{F}_T, \mathbb{Q}, \{\mathcal{F}_t\})$ turns out to be a discrete model.

The converse statement is clear from Eq.(5.3.1). □

Next, it is assumed that continuous trading is allowed. In case of a knock-out condition irrelevant to the underlying asset, the model in Proposition 5 can be made continuous. The other case requires an additional condition under an assumption about $D^{q/r}$ as follows:

Corollary 7. *Suppose that $D^{q/r}_{t,T} = 1$ for all $t \in [0, T]$. The existence of a continuous model is equivalent to (a) in Proposition 5 and*

(in case of a single barrier) *$\beta \notin I$, where*

$$\beta := \frac{\int s(1 - \alpha(s))\mu(ds)}{\int(1 - \alpha(s))\mu(ds)}, \tag{5.3.3}$$

(in case of a double barrier) *there is $c \in \mathbb{R}$ and $\theta \in [0,1]$ such that $\beta_L < L$ and $\beta_U > U$, where*

$$\beta_L := \frac{\int_{[0,c)} s(1-\alpha(s))\mu(ds) + \theta c(1-\alpha(c))\mu(\{c\})}{\int_{[0,c)} (1-\alpha(s))\mu(ds) + \theta(1-\alpha(c))\mu(\{c\})}, \tag{5.3.4}$$

$$\beta_U := \frac{\int_{(c,+\infty)} s(1-\alpha(s))\mu(ds) + (1-\theta)c(1-\alpha(c))\mu(\{c\})}{\int_{(c,+\infty)} (1-\alpha(s))\mu(ds) + (1-\theta)(1-\alpha(c))\mu(\{c\})}. \tag{5.3.5}$$

5.3.2 Pricing Bounds using Liquid Barrier Options

(Tsuzuki, 2014) proposes model-independent pricing bounds using a no-touch option as well as European options. This section devotes to extending results of (Tsuzuki, 2014) to the case where $K+1$ liquid knock-out options are available in a market. The assumptions in this section are that trading is allowed only at inception and that tradable derivatives are

- European call and put options with maturity T
- Knock-out options with maturity T and with terminal payoff $h_i(S_T)1_A$ for $h_i \in \mathcal{L}^1$ $(i = 0, \cdots, K)$, where $h_0 = 1$.

Although (Tsuzuki, 2014) assumes that a distribution of S_T is strictly positive without atom, the same results hold without assuming strict positivity and being atomless. Let $g \in \mathcal{L}^\infty_\mu$ be a terminal payoff of a knock-out option and H_0 be the forward price of the no-touch option. The following is the generalized version of the theorem in (Tsuzuki, 2014).

Proposition 6. *Let $A_* \subseteq I$ be defined such that $\sup_{s\in A_*} g(s) \leq \inf_{s\notin A_*} g(s)$ and $A^* \subseteq I$ be such that $\sup_{s\notin A^*} g(s) \leq \inf_{s\in A^*} g(s)$. Then, one has*

$$(g(S_T) - g_*)1_{\{S_T\in A_*\}} + g_* 1_A \tag{5.3.6}$$

$$\leq g(S_T)1_A \tag{5.3.7}$$

$$\leq (g(S_T) - g^*)1_{\{S_T\in A^*\}} + g^* 1_A, \tag{5.3.8}$$

where g_* and g^* are arbitrary values such that $g_* \in [\sup_{s\in A_*} g(s), \inf_{s\notin A_*} g(s)]$ and $g^* \in [\sup_{s\notin A^*} g(s), \inf_{s\in A^*} g(s)]$ respectively. Moreover, if g_* and g^* are respectively taken as $\mathbb{Q}[g(S_T) < g_*] \leq H_0 \leq \mathbb{Q}[g(S_T) \leq g_*]$ and $\mathbb{Q}[g^* < g(S_T)] \leq H_0 \leq \mathbb{Q}[g^* \leq g(S_T)]$, then there are models $\mathbb{Q}_*$ and $\mathbb{Q}^*$ under which the knock-out option prices are respectively given by

$$\mathbb{E}_*\left[(g(S_T) - g_*)1_{\{S_T\in A_*\}} + g_* 1_A\right], \tag{5.3.9}$$

$$\mathbb{E}^*\left[(g(S_T) - g^*)1_{\{S_T\in A^*\}} + g^* 1_A\right]. \tag{5.3.10}$$

By Proposition 5, there are corresponding kernel functions. Let α^L_g and α^G_g respectively be elements of $\mathcal{A}$ corresponding to the models which provides the lower and upper bounds for a terminal payoff g.

Suppose that in addition to a no-touch option there are K liquid knock-out options in a market whose terminal payoffs are $h_k : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and forward prices are $H_k \in \mathbb{R}$ for $k = 1, \cdots, K$. Let h and H be vectors whose elements are h_k and H_k for $k = 1, \cdots, K$ respectively, $h := (h_k)_{k=1}^K$ and $H := (H_k)_{k=1}^K$ and $\langle \cdot, \cdot \rangle_K$ be an inner product of K-dimensional Euclidean space. This section considers super-replication of a knock-out option whose terminal payoff function is g using the K liquid knock-out options as well as the no-touch option. The same results in this section is obtained for sub-replication by considering $-g$ instead of g.

Apply the super-replication of Proposition 6 to a knock-out option whose terminal payoff is $g - \langle x, h\rangle_K$ for $x \in \mathbb{R}^K$. This is a terminal payoff of the target barrier option and replicating portfolio if $x \in \mathbb{R}^K$ amount of the K liquid knock-out options are used for replication. Then, a kernel function that provides the upper bound for $g - \langle x, h\rangle_K$ is $\Psi_x := \alpha^G_{g-\langle x,h\rangle_K}$ and the upper bound is given by $W^G_{g-\langle x,h\rangle_K} = W^*_g(x) := W_g(x,x)$, where $W_g : \mathbb{R}^K \times \mathbb{R}^K \longrightarrow \mathbb{R}$ is defined by

$$W_g(x,y) := \langle g - \langle x,h\rangle_K , \Psi_y\rangle + \langle x, H\rangle_K = \int_I (g - \langle x,h\rangle_K)\Psi_y d\mu + \langle x, H\rangle_K . \tag{5.3.11}$$

Note that $W_g(x,x) = \sup_{y\in\mathbb{R}^K} W_g(x,y)$.

Since amounts of the K liquid knock-out options for this strategy is arbitrary, the optimal replicating strategy is the one which minimizes the cost of replication $W^*_g(x)$ with respect to $x \in \mathbb{R}^K$. Proposition 7 shows that this strategy is also the best among all static strategies which use European options, the no-touch option and the K liquid knock-out options.

Proposition 7. *Suppose that $\langle x, H\rangle_K > \left\langle \langle x,h\rangle_K , \alpha^L_{\langle x,h\rangle_K}\right\rangle$ for any $x \in \mathbb{R}^K$. Then, there exists $x^* \in \mathbb{R}^K$ such that $W^*_g(x^*) = \inf_{x\in\mathbb{R}^K} W^*_g(x)$ and the kernel function Ψ_{x^*} satisfies $\langle h_k, \Psi_{x^*}\rangle = H_k$ for $k = 1, \cdots, K$.*

Proof. Clearly, the function W_g is linear with respect to x and W^*_g is convex, because for $x, y \in \mathbb{R}^K$ and $a \in [0,1]$,

$$\begin{aligned} W^*_g(ax + (1-a)y) &= aW_g(x, ax + (1-a)y) + (1-a)W_g(y, ax + (1-a)y) \\ &\leq aW^*_g(x) + (1-a)W^*_g(y). \end{aligned} \tag{5.3.12}$$

Then, W_g is continuous. For any $y \in \mathbb{R}^K$ with $\sqrt{\langle y,y\rangle_K} = 1$ and $t > 0$, let $x := ty$. Then, one has

$$\begin{aligned} W^*_g(x) &\geq \left\langle g - \langle x,h\rangle_K , \alpha^L_{\langle y,h\rangle_K}\right\rangle + \langle x, H\rangle_K \\ &= \left\langle g, \alpha^L_{\langle y,h\rangle_K}\right\rangle + t\left(\langle y, H\rangle_K - \left\langle \langle y,h\rangle_K , \alpha^L_{\langle y,h\rangle_K}\right\rangle\right). \end{aligned} \tag{5.3.13}$$

Since $W^*_g(x) \longrightarrow +\infty$ as $t \longrightarrow +\infty$ and W^*_g is a convex function, W^*_g take a finite minimum value.

For fixed $k \leq K$, let $y \in \mathbb{R}^K$ such that $y_n = x^*_n$ for $n \neq k$. Then,

$$\begin{aligned} \frac{\partial W_g}{\partial x_k}(x^*, y) &= \frac{W_g(y,y) - W_g(x^*,y)}{y_k - x^*_k} \\ &= \frac{(W_g(y,y) - W_g(x^*,x^*)) - (W_g(x^*,y) - W_g(x^*,x^*))}{y_k - x^*_k}. \end{aligned} \tag{5.3.14}$$

This derives $\frac{\partial W_g}{\partial x_k}(x^*, y) \geq 0$ for $y_k > x^*_k$ and $\frac{\partial W_g}{\partial x_k}(x^*, y) \leq 0$ for $y_k < x^*_k$. Then, from the continuity of W_g, $\frac{\partial W_g}{\partial x_k}(x^*, x^*) = -\langle h_k, \Psi_{x^*}\rangle + H_k = 0$. □

Remark 19. *One can construct an example to fail the attainability of the infimum. Let $K = 1$, $\kappa > 0$, μ be the uniform distribution on $[0, 2\kappa]$, $g(s) := 1_{\{s>\kappa\}}$ and $h_1(s) := s - \kappa$. In addition, take a knock-out event independent of the underlying process with $\mathbb{Q}[A] = \frac{1}{2}$ and a liquid knock-out option price H_1 as the lower bound: $H_1 = \langle h_1, \alpha^L\rangle$, where $\alpha^L(s) = 1_{\{s<\kappa\}}$. Then, the followings hold for $x > 0$: $\Psi_x = 1_{[0,\kappa-\frac{1}{2x})\cup[\kappa,\kappa+\frac{1}{2x})}$ and*

$$W^*_g(x) = \int_{\kappa-\frac{1}{2x}}^{\kappa} xh_1(s)ds + \int_{\kappa}^{\kappa+\frac{1}{2x}} (1 - xh_1(s))ds = \frac{3}{4x}. \tag{5.3.15}$$

This shows that $\inf_{x\in\mathbb{R}} W^*_g(x)$ is not attainable.

5.3.3 Single Barrier Options with Term-structure of No-touch Options

While this paper has considered barrier options with a general barrier condition so far, this section focuses on a single barrier option with a barrier level $L \in (0, s_0)$. Previous studies of model-independent pricing bounds such as (Brown et al., 2001) consider the problem under the assumption that the underlying asset is a continuous martingale, but (Tsuzuki, 2014) does not. As shown later, if the underlying process is a continuous martingale, the martingale condition is taken into account by adding a knock-out forward contract into liquid knock-out options, where the knock-out forward contract is a knock-out option whose payoff is $g(s) = s$. However, in general, the underlying process S_t is not a martingale, but the forward process $D^{q/r}_{t,T} S_t$ is and both of them do not necessarily a martingale. Then, the study in this section is motivated by how to incorporate the martingale condition of $D^{q/r}_{t,T} S_t$ into the pricing bounds of (Tsuzuki, 2014).

First, consider for an instructive purpose the case where the underlying asset price is continuous and $D^{q/r}_{t,T} = 1$ for all $t \in [0, T]$, namely S_t is a continuous martingale. In this case, if the no-touch option price is given by $N_{0,T}$, the knock-out forward contract is redundant because the contract is replicated as follows:

(1) Buy a forward contract with maturity T at the initial time

(2) Buy $D^{q/r}_{0,T} s_0 - L$ units of the zero-coupon bond with maturity T at the initial time

(3) Buy L units of the no-touch option at the initial time

(4) Sell a forward contract at the first hitting time to the barrier level L.

The price is $D^r_{0,T}(D^{q/r}_{0,T} s_0 - L) + LN_{0,T}$, where τ is the first hitting time to the barrier L. This strategy makes use of two facts: there is zero cost of carry and the underlying asset price is L at the first hitting time. The presence of non-zero cost of carry makes information on when to hit the barrier more important and a possibility of jump introduces uncertainty of the forward price at the first hitting time.

Then, this section relaxes the condition of a continuous martingale and instead adds tradable instruments. Let $T_0 = 0 < T_1 < \cdots < T_N = T$ with $\Delta_n := (T_{n-1}, T_n]$ for $n = 1, \cdots, N$ be a time series. This section assumes that continuous trading is allowed and that time-t prices $D^r_{t,T}$ and $D^q_{t,T}$ are constant on each interval Δ_n and $D^r_{t,T} = D^q_{t,T} = 1$ on Δ_N. Tradable derivatives in this section are as follows:

- European call and put options with maturity T
- no-touch options with a barrier level L and with maturity T_n for $n = 1, \cdots, N$,
- knock-out forward contracts with a barrier level L and with maturity T_n for $n = 1, \cdots, N$.

The above tradable no-touch options and knock-out forward contracts replicate/are replicated by one-touch options which respectively pay a a unit of currency and a unit of the underlying asset at time T if the first hitting time is in Δ_n for $n = 1, \cdots, N$. These one-touch options are used in this section for ease of notation instead of the no-touch options and the knock-out forward contracts, although they are not standard in practice. Let $O^{\Delta_n}_{0,T}$ be prices of the one-touch options which pay a unit of currency and $O^{\Delta_n,A}_{0,T}$ be prices of the one-touch options which pay a unit of the underlying asset. Note that the price of the no-touch option and that of the knock-out forward contract with maturity T are given respectively by $N_{0,T} := D^r_{0,T} - \sum_{n=1}^N O^{\Delta_n}_{0,T}$ and $N^A_{0,T} := D^q_{0,T} s_0 - \sum_{n=1}^N O^{\Delta_n,A}_{0,T}$.

This section first investigates no-arbitrage conditions for these one-touch option prices $\{O^{\Delta_n}_{0,T}\}_{n=1}^N$ and $\{O^{\Delta_n,A}_{0,T}\}_{n=1}^N$ and then investigates whether these one-touch options improve the pricing bounds of Section 5.3.2 with the knock-out forward contract price $N^A_{0,T}$.

For $n = 1, \cdots, N$, define

$$p_n := \frac{O_{0,T}^{\Delta_n}}{D_{0,T}^r},\ q_n := \frac{D_{0,T}^r - \sum_{k=1}^n O_{0,T}^{\Delta_k}}{D_{0,T}^r}, \tag{5.3.16}$$

$$x_n := \frac{O_{0,T}^{\Delta_n,A}}{O_{0,T}^{\Delta_n}},\ y_n := \frac{D_{0,T}^q s_0 - \sum_{k=1}^n O_{0,T}^{\Delta_k,A}}{D_{0,T}^r - \sum_{k=1}^n O_{0,T}^{\Delta_k}}, \tag{5.3.17}$$

$\nu := q_N \delta_{y_N}$ and $\nu^c := \sum_{n=1}^N p_n \delta_{x_n}$, where δ_x is the Dirac delta function located at $x \in \mathbb{R}_+$. Moreover, let μ_N be the smallest restriction of μ whose 0-th and 1-st moments are q_N and $q_N y_N$ respectively, and $\mu_N^c := \mu - \mu_N$. Proposition 8 provides the answer to the first question.

Proposition 8. *Suppose that*

(A1) $\mu_N(I^c) = 0$ *and* $x_n \le D_{T_n,T}^{q/r} L < y_n$ *for* $n = 1, \cdots, N$,

(A2) $\nu^c \preceq \mu_N^c$.

Then, there is a continuous model. Conversely, if there is a continuous model $\mathbb{Q}$*, (A1) and (A2) hold for* $\tilde{\mu}_N$ *and* $\tilde{\mu}_N^c = \mu - \tilde{\mu}_N$ *instead of* μ_N *and* μ_N^c*, where* $\tilde{\mu}_N([a,b)) := \mathbb{Q}(S_T \in [a,b), \tau > T)$ *for* $a, b \in \mathbb{R}_+$.

Proof. Suppose that (A1) and (A2) hold. By (A2) and Theorem 6, there exist subsets $\Omega_n \subseteq \Omega$ for $n = 1, \cdots, N$ such that $O_{0,T}^{\Delta_n} = D_{0,T}^r \mathbb{Q}[\Omega_n]$, $O_{0,T}^{\Delta_n,A} = D_{0,T}^r \mathbb{E}[S_T : \Omega_n]$, $N_{0,T} = D_{0,T}^r \mathbb{Q}\left[\left(\cup_{n=1}^N \Omega_n\right)^c\right]$, and $N_{0,T}^A = D_{0,T}^r \mathbb{E}\left[S_T : \left(\cup_{n=1}^N \Omega_n\right)^c\right]$. Let F_t^* be a process defined by

$$F_t^* := D_{T_0,T}^{q/r} S_0 + \sum_{n=1}^{N-1} X_n 1_{\{t \in \Delta_n\}} + X_N 1_{\{t \in (T_{N-1}, T_N)\}} + S_T 1_{\{t = T\}}, \tag{5.3.18}$$

where

$$X_n(\omega) := \begin{cases} S_T(\omega) & \left(\omega \in \cup_{k=1}^{n-1} \Omega_k\right) \\ x_n & (\omega \in \Omega_n) \\ y_n & (\omega \notin \cup_{k=1}^n \Omega_k) \end{cases}. \tag{5.3.19}$$

The process $S_t^* := \frac{1}{D_{t,T}^{q/r}} F_t^*$ satisfies $S_T^* \sim \mu$. If a filtration $\mathcal{F}_t$ is taken as one generated by Ω_n, $\cup_{k=1}^n \Omega_k$ and $\{B \cap \cup_{k=1}^{n-1} \Omega_k | B \in \mathcal{F}\}$ for $t \in \Delta_n$, then, the process F^* is a martingale with respect to the filtration $\mathcal{F}_t$, because $F_0^* = \mathbb{E}[S_T]$ and for $t \in \Delta_n$

$$\begin{aligned} \mathbb{E}[S_T | \mathcal{F}_t] &= S_T 1_{\cup_{k=1}^{n-1} \Omega_k} + \frac{\mathbb{E}[S_T : \Omega_n]}{\mathbb{Q}[\Omega_n]} 1_{\Omega_n} + \frac{\mathbb{E}[S_T : (\cup_{k=1}^n \Omega_k)^c]}{\mathbb{Q}[(\cup_{k=1}^n \Omega_k)^c]} 1_{(\cup_{k=1}^n \Omega_k)^c} \\ &= S_T 1_{\cup_{k=1}^{n-1} \Omega_k} + x_n 1_{\Omega_n} + y_n 1_{(\cup_{k=1}^n \Omega_k)^c} \\ &= F_t^*. \end{aligned} \tag{5.3.20}$$

Finally, one has $O_{0,T}^{\Delta_n} = D_{0,T}^r \mathbb{E}\left[1_{\{\tau \in \Delta_n\}}\right]$ and $O_{0,T}^{\Delta_n,A} = D_{0,T}^r \mathbb{E}\left[S_T 1_{\{\tau \in \Delta_n\}}\right]$ for $n = 1, \cdots, N$, because of (A4).

The converse statement is obvious. □

Next, consider the second question. Suppose that $K = 1$ and $h_1(s) = s$ and $H_1 = \frac{N^A_{0,T}}{D^r_{0.T}}$. The notations in Section 5.3.2 are $K = 1$ and $h(s) = s$ and $H = \frac{N^A_{0,T}}{D^r_{0.T}}$. The following corollary provides conditions that the pricing bound is not improved by the information of the touch option prices $\{O^{\Delta_n}_{0,T}\}^N_{n=1}$ and $\{O^{\Delta_n,A}_{0,T}\}^N_{n=1}$.

Corollary 8. *Suppose that the assumption of Proposition 7 holds for h and H. Let $g \in \mathcal{L}^1$, λ be the restriction μ whose 0-th and 1-st order moments are respectively q_N and $q_N y_N$, and $x^* \in \mathbb{R}$ be the optimal value so that the optimal upper pricing bound $W^*_g(x^*)$ is represented with*

$$W^*_g(x^*) = \int (g(s) - x^* s) d\lambda + x^* H. \tag{5.3.21}$$

If (A1) and (A2) of Proposition 8 hold for λ and $\lambda^c := \mu - \lambda$, then there is a continuous model for which the price of the knock-out option is $D^r_{0,T} W^*_g$.

From a practical point of view, (A1) and (A2) are likely to hold for λ and λ^c, because $\{x_n\}^N_{n=1}$ are accumulated around L, if the interest rate and the dividend yield are not extremely high and there are no fear about huge jumps in the underlying asset process.

5.3.4 No Arbitrage Condition with Different Maturities

This section devotes to deriving no static arbitrage conditions on two sets of barrier option prices with different maturities, T_1 and T_2 ($T_1 < T_2$). Assumptions in this section are that continuous trading is allowed and that time-t prices D^r_{t,T_2} and D^q_{t,T_2} are constant on Δ_1 and $D^r_{t,T} = D^q_{t,T} = 1$ on Δ_2. Tradable derivatives in this section are as follows:

- European call and put options with maturity T_i for $i = 1, 2$
- Knock-out options with maturity T_i and with payoff $g(S_{T_i}) 1_{A_i}$ for any $g \in \mathcal{L}^1$ for $i = 1, 2$, where A_i is the event for not being knocked out.

Barrier option prices with maturity T_i are individually characterized with (μ_i, α_i) for $i = 1, 2$, where μ_i is a distribution of the underlying asset at time T_i and α_i is a kernel function for A_i. For ease of notation, expressions for S_{T_1} are transformed to those of T_2-forward measure using $\tilde{\alpha}_1(f) := \alpha_1 \left(D^{r/q}_{T_1,T_2} f \right)$ and $\tilde{\mu}_1([0, f)) := \mu_1 \left(\left[0, D^{r/q}_{T_1,T_2} f\right) \right)$. In order to focus on no arbitrage condition on α_2, it is assumed that there is a model up to T_1, namely (μ_1, α_1) satisfies the assumptions of Corollary 7 in case of a single or double barrier. In addition, assume $\tilde{\mu}_1 \preceq \mu_2$ and $n_1 \geq n_2$, where $n_i := \int \alpha_i d\mu_i$.

To exclude static arbitrage opportunities, the following conditions must be satisfied.

Assumption 3. *There is $\alpha_{12} \in \mathcal{L}^\infty_{\mu_2}$ such that*

- *$\alpha_2(s) \leq \alpha_{12}(s)$ for any $s \in \mathbb{R}_+$.*
- *$\int_0^{+\infty} s^m \tilde{\alpha}_1(s) \tilde{\mu}_1(ds) = \int_0^{+\infty} s^m \alpha_{12}(s) \mu_2(ds)$ for $m = 0, 1$,*
- *$\tilde{\alpha}_1 d\tilde{\mu}_1 \preceq \alpha_{12} d\mu_2$ and $(1 - \tilde{\alpha}_1) d\tilde{\mu}_1 \preceq (1 - \alpha_{12}) d\mu_2$.*

As Proposition 9 will show, these assumptions are sufficient if the knock-out condition is irrelevant to the underlying price. However, the cases of a single barrier and a double barrier are more involved and require more than those. Consider the following two distributions μ_{12}^S and μ_{12}^D with $c \in I$ and $\theta \in [0,1]$: for any Borel measurable set B,

$$\mu_{12}^S(B) := \frac{1}{n_1}\int_B \alpha_2 d\mu_2 + \frac{n_1 - n_2}{n_1}\delta_\beta(B), \tag{5.3.22}$$

where $\beta := \frac{1}{n_1 - n_2}\int_0^{+\infty} s(\alpha_{12}(s) - \alpha_2(s))\mu_2(ds)$ and

$$\mu_{12}^D(B) := \frac{1}{n_1}\int_B \alpha_2 d\mu_2 + \frac{n_L}{n_1}\delta_{\beta_L}(B) + \frac{n_U}{n_1}\delta_{\beta_U}(B), \tag{5.3.23}$$

where

$$\beta_L := \frac{1}{n_L}\left(\int_{[0,c)} s(\alpha_{12}(s) - \alpha_2(s))\mu_2(ds) + c\theta(\alpha_{12}(c) - \alpha_2(c))\mu_2(\{c\})\right), \tag{5.3.24}$$

$$\beta_U := \frac{1}{n_U}\left(\int_{(c,+\infty)} s(\alpha_{12}(s) - \alpha_2(s))\mu_2(ds) + c(1-\theta)(\alpha_{12}(c) - \alpha_2(c))\mu_2(\{c\})\right), \tag{5.3.25}$$

$n_L := \int_{[0,c)}(\alpha_{12} - \alpha_2)d\mu_2 + \theta(\alpha_{12}(c) - \alpha_2(c))\mu_2(\{c\})$ and $n_U := n_1 - n_2 - n_L$.

Assumption 4. *There is $\alpha_{12} \in \mathcal{L}_{\mu_2}^\infty$ such that*

(in case of a single barrier) *$\tilde{\alpha}_1 d\tilde{\mu}_1 \preceq d\mu_{12}^S$ and $\beta \notin I$.*

(in case of a double barrier) *$\tilde{\alpha}_1 d\tilde{\mu}_1 \preceq d\mu_{12}^D$, and $\beta_L \leq L$ and $\beta_U \geq U$ for some $c \in I$ and $\theta \in [0,1]$.*

Proposition 9. *Suppose that Assumption 3 holds and that Assumption 4 holds in case of single or double barrier options. Then, there is a continuous model. Conversely, if there is a continuous model $\mathbb{Q}$, Assumption 3 and Assumption 4 hold for $\alpha_{12}(s) = \mathbb{E}[1_{A_1} \mid S_{T_2} = s]$.*

Proof. By Assumption 3 and Theorem 6, one can construct a model such that $\{S_t\}_{t\in[T_1,T_2]}$ is a martingale if the knock-out condition is irrelevant to the underlying asset. In case of a single barrier option, because $\tilde{\alpha}_1 d\tilde{\mu}_1 \preceq d\mu_{12}^S \preceq \alpha_{12}d\mu_2$ and these tree distributions have the same moments up to 1-st order, $\{S_t\}_{t\in[T_1,T_2]}$ can be a model. This is the case for a double barrier case.

Conversely, suppose that there is a continuous model $\mathbb{Q}$. Assumption 3 is clear. In case of a single barrier, one has

$$\beta = \frac{\mathbb{E}[S_{T_2} \,:\, T_1 < \tau \leq T_2]}{\mathbb{Q}[T_1 < \tau \leq T_2]} \notin I. \tag{5.3.26}$$

In case of a double barrier, one has

$$\beta_L \leq \frac{\mathbb{E}\left[S_{T_2} \,:\, \min_{t\in(T_1,T_2]} S_t \leq L\right]}{\mathbb{Q}\left[\min_{t\in(T_1,T_2]} S_t \leq L\right]} \leq L,\ U \leq \frac{\mathbb{E}\left[S_{T_2} \,:\, U \leq \max_{t\in(T_1,T_2]} S_t\right]}{\mathbb{Q}\left[U \leq \max_{t\in(T_1,T_2]} S_t\right]} \leq \beta_U. \tag{5.3.27}$$

□

If the underlying asset is limited to a continuous martingale, Proposition 9 can be simplified in case of a single barrier. Let α^*_{12} be

$$\alpha^*_{12} := \alpha_2 + (1-\alpha_2)\left(1_{(a,b)} + \theta_a 1_{\{a\}} + \theta_b 1_{\{b\}}\right), \tag{5.3.28}$$

where $a, b \in \mathbb{R}_+$ and $\theta_a, \theta_b \in [0,1]$ are taken such that $\int \alpha^*_{12} d\mu_2 = n_1$ and $\frac{1}{n_1 - n_2} \int s(\alpha^*_{12}(s) - \alpha_2(s))\mu_2(ds) = L$.

Corollary 9. *Suppose that $D^{q/r}_{t,T_2} = 1$ for $t \in [0, T_2]$. Then, there is a continuous model for which S_t is continuous, if and only if α^*_{12} is satisfied with $\int_0^{+\infty} s\tilde{\alpha}_1(s)\tilde{\mu}_1(ds) = \int_0^{+\infty} s\alpha^*_{12}(s)\mu_2(ds)$, $(1-\tilde{\alpha}_1)d\tilde{\mu}_1 \preceq (1-\alpha^*_{12})d\mu_2$ and $\tilde{\alpha}_1 d\tilde{\mu}_1 \preceq d\mu^*_{12}$, where μ^*_{12} is μ^S_{12} with α^*_{12}.*

5.4 Construction of Kernel Functions

This section provides some examples to construct kernel functions under several circumstances.

5.4.1 Construction with a No-Touch Option

The first example is to construct a kernel function where a marginal distribution μ of an underlying asset and a no-touch option price $N_{0,T}$ with a maturity T are given. This section focuses on the case where the barrier is single or double: $L \in (-\infty, s_0)$ and $U \in (s_0, +\infty]$, since a copula approach can be applied if a knock-out event is irrelevant to the underlying asset. This method has two steps: the first step is to construct a kernel function which does not necessarily reproduce the given price of the no-touch option and the second step is to adjust the function so as to reproduce it. Let $\mathcal{A}^*$ be a subset of $\mathcal{A}$ such that the no-touch option price is reproduced: $N_{0,T} = D^r_{0,T} \int \alpha d\mu$.

Step 1 Construct a kernel $\tilde{\alpha} \in \mathcal{A}$

Step 2 Adjust the kernel $\tilde{\alpha}$ and obtain $\alpha \in \mathcal{A}^*$

Although a choice of a kernel function in the first step is arbitrary as long as it is in $\mathcal{A}$, it is desirable in ordinary circumstances for a kernel function to take a value close to zero in a neighborhood of the barrier level L or U. In this case, Brownian bridge techniques are useful to construct a kernel function which equips this property. The following lemma is well-known (see e.g. (Revuz and Yor, 1994)).

Lemma 6. *Suppose that $\{S_t\}_{t\in[0,T]}$ is a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$:*

$$dS_t = bS_t dt + \sigma S_t dW_t, \tag{5.4.1}$$

where b and σ are constant and W is a Brownian motion under $\mathbb{Q}$. Let $p(x, y; \sigma, L, U)$ be the probability that the barrier L,U is hit under the condition $S_0 = x$ and $S_T = y$. Then, it holds for $x, y \in [L, U]$:

$$p(x,y;\sigma,L,U) = \begin{cases} 0 & (L=0, U=+\infty) \\ \exp\left(-2\frac{\log(x/L)\log(y/L)}{|\sigma|^2 T}\right) & (L\neq 0, U=+\infty) \\ \exp\left(-2\frac{\log(x/U)\log(y/U)}{|\sigma|^2 T}\right) & (L=0, U\neq +\infty) \\ \sum_{k=-\infty}^{+\infty}(e^{A_k} - e^{B_k}) & (otherwise) \end{cases}, \tag{5.4.2}$$

where

$$A_k := -2\frac{k\log(U/L)(k\log(U/L)+\log(y/x))}{|\sigma|^2T} \tag{5.4.3}$$

and

$$B_k := -2\frac{(k\log(U/L)+\log(x/U))(k\log(U/L)+\log(y/U))}{|\sigma|^2T}. \tag{5.4.4}$$

The assumption of Lemma 6 is not satisfied in most option markets, since they have volatility smiles. A way to incorporate volatility smiles into the Brownian bridge is

$$\tilde{\alpha}_{LV}(s) := 1 - p(S_0, s; \sigma_{iv}(s), L, U), \tag{5.4.5}$$

where $\sigma_{iv}(\cdot) : \mathbb{R}_+ \longrightarrow (0, +\infty)$ is a volatility smile function observed in a market. Let the function (5.4.5) be called as *L.V. kernel.* Numerical examples of these functions are in Section 5.6.

The second step is straightforward: a kernel $\alpha \in \mathcal{A}^*$ is obtained by

$$\alpha(s) = \begin{cases} 1 - \frac{D^r_{0,T} - N_{0,T}}{D^r_{0,T} - \tilde{N}_{0,T}}(1 - \tilde{\alpha}(s)) & (N_{0,T} > \tilde{N}_{0,T}) \\ \frac{N_{0,T}}{\tilde{N}_{0,T}}\tilde{\alpha}(s) & (N_{0,T} \leq \tilde{N}_{0,T}) \end{cases}, \tag{5.4.6}$$

where $\tilde{N}_{0,T} := D^r_{0,T}\int \tilde{\alpha}\,d\mu$.

5.4.2 Construction with Several Knock-out Options

Next, consider how to construct a kernel function in a case as in Section 5.3.2, where there are K liquid knock-out options whose payoffs are $h_n : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and forward prices are H_n for $n = 1, \cdots, K$, in addition to the assumption in Section 5.4.1, namely a marginal distribution μ of an underlying asset and a price of a no-touch option $N_{0,T}$ with a maturity T. For simple notation, let h_0 be the payoff of the no-touch option and H_0 be the forward price. It is assumed that there is no-static arbitrage opportunity among these liquid barrier options. The problem is equivalent to finding $\alpha \in \mathcal{A}^*$ such that $\langle h_k, \alpha\rangle = H_k$ for $k = 0, \cdots, K$.

First, these conditions can be represented in terms of the Banach spaces $\mathcal{L}^1$ and $\mathcal{L}^\infty$:

(1) $\langle \delta_s, \alpha\rangle \in [0, 1]$ for $s \in I$

(2) $\langle \delta_s, \alpha\rangle = 0$ for $s \in I^c$

(3) $\langle h_k, \alpha\rangle = H_k$ for $k = 0, \cdots, K$,

where δ_s is the Dirac delta function located at $s \in \mathbb{R}_+$. Then, the problem is to choose an element of the subset $\mathcal{A}^*_h \subseteq \mathcal{A}$:

$$\mathcal{A}^*_h := \bigcap_{s\in I}\{\alpha \in \mathcal{L}^\infty \mid \langle \delta_s, \alpha\rangle \in [0,1]\} \cap \bigcap_{s\in I^c}\{\alpha \in \mathcal{L}^\infty \mid \langle \delta_s, \alpha\rangle = 0\} \cap \bigcap_{0\leq k\leq K}\{\alpha \in \mathcal{L}^\infty \mid \langle h_k, \alpha\rangle = H_k\}. \tag{5.4.7}$$

Note that $\mathcal{A}^*_h$ is not empty, because no-static arbitrage is assumed. Since every element of the subset $\mathcal{A}^*_h$ is a candidate of a kernel function that reproduces the market prices of liquid knock-out options, one needs some criteria in order to choose one element. One way is to take best approximations of $\mathcal{A}_h^*$ from α_0 which has desirable properties but is not necessarily an element of $\mathcal{A}_h^*$:

$$P_{\mathcal{A}_h^*}(\alpha_0) := \left\{ \alpha^* \in \mathcal{A}_h^* \mid ||\alpha_0 - \alpha^*|| = \inf_{\alpha \in \mathcal{A}_h^*} ||\alpha_0 - \alpha|| \right\}. \tag{5.4.8}$$

The number of elements of $P_{\mathcal{A}_h^*}(\alpha_0)$ is one since the subset $\mathcal{A}_h^*$ is closed and convex in a normed space. For example, the element of $P_{\mathcal{A}_h^*}(\alpha_0)$ is appropriate for those who believe that barrier levels are likely to have been hit when the underlying price S_T at maturity is close to the barrier level L or U, if α_0 is taken as a kernel function using Brownian bridge techniques as in Section 5.4.1.

Finally, a practical procedure for obtaining the element of $P_{\mathcal{A}_h^*}(\alpha_0)$ is proposed. Suppose that h_k for $k = 1, \cdots, K$ are in $\mathcal{L}^2$ instead of $\mathcal{L}^1$. Then, the operator $\langle, \rangle$ can be regarded as an inner product in a Hilbert space $\mathcal{L}^2$. In addition, the problem is discretized with respect to the space I (for example, $I = \cup_{i=1}^{M}[x_i, x_{i+1})$ for some natural number M and an increasing sequence $\{x_i\}_{i=1}^{M+1}$) and reduced to a finite dimensional problem. Then, *Dykstra's algorithm* can be applied to the discretized problem (see (Deutsch, 2001) for more details).

5.4.3 Construction with Different Maturities

The third example is how to construct two kernel functions with different maturities T_1 and T_2 $(T_1 < T_2)$ such that they exclude semi-static arbitrage opportunities as in Section 5.3.4. It is assumed that pairs of a marginal distribution μ_i and a probability n_i that the barrier has not been hit up to maturity T_i are given for $i = 1, 2$, where μ_2 is larger than μ_1 in a sense of convex order and $n_1 > n_2$. In addition, there may be liquid knock-out options for each maturity.

The problem is to construct α_1, α_2 and α_{12} satisfying the conditions in Section 5.3.4. These conditions are represented by convex sets of the triple of Banach space $\mathcal{L}_{\mu_1}^1 \times \mathcal{L}_{\mu_2}^1 \times \mathcal{L}_{\mu_2}^1$. The first step is see if there are no arbitrage opportunity among given market instruments. Suppose that a barrier option with maturity T_2 is quoted in a market with a certain price. Then, calculate the maximum and the minimum of the price using (μ_2, n_2, α_2) over α_1, α_2 and α_{12} which are satisfied with the conditions and compare the price bounds with the market price. If there are no semi-static arbitrage opportunity, there are solutions. Then, *Dykstra's algorithm* can be applied to a discretized problem in the same way as in Section 5.4.2.

5.5 Application

While the approach proposed in this paper can be used to find arbitrage opportunities, it has more applications for pricing barrier options thanks to its flexibility. Its flexibility comes from the fact that it directly describes dependency between a marginal distribution and the event of hitting to barriers. Indeed, standard approaches that postulate a stochastic process as an underlying process may be more appropriate for pricing barrier options under ordinary circumstances. Here, the ordinary circumstance is the case where barriers are likely to have been hit when an underlying price at maturity is close to the barrier levels. However, the approach of this paper is more appropriate under circumstances where an underlying price is not expected to behave in the ordinary circumstance. An example is to price barrier options under potential intervention in a foreign exchange (FX) market. Credit value adjustment (CVA) is another application, because it is exposed to *wrong-way risk* and this risk is difficult to measure. It is helpful for risk management to calculate a range of CVA by modeling a kernel function.

5.5.1 Pricing Barrier Options under Potential Intervention

Central banks sometimes intervene FX markets. This section considers how to price barrier options under potential intervention of a central bank. Suppose a central bank announces that it will intervene when an FX rate S_t reaches a certain level $L \in [0, S_0)$ and it surely does at that time, but it may fail. For simplicity, the intervention in a period $[0, T)$ is regarded as failure if and only if $S_T \leq L$. After the announcement, the market view is incorporated into option prices: the digital put option whose payoff is $1_{[0,L)}$ reflects a probability of failure, the one touch option with barrier L does a probability that the intervention takes place, and so on.

Suppose that there are traders who have the same view about the above two options as the market, but predict a significant change in an FX rate after the intervention. Their view after the intervention is that the intervention fails with the same probability as incorporated in the market, the digital put price, but the intervention is so efficient that the FX rate significantly goes up with some probability. The standard approach that postulates a stochastic process is not appropriate for them, because it is likely that the barrier has not been hit, if the underlying price at T is far from the barrier level. The approach that characterise α is more appropriate for them.

5.5.2 Credit Value Adjustment

Credit value adjustment (CVA) is the difference between the risk-free portfolio value and the true portfolio value that takes into account the possibility of a counterparty's default ((Pykhtin and Zhu, 2007)). Suppose that a bank which is assumed not to default has transactions with a counterparty which may default. Then, CVA for the bank can be defined by[2]

$$CVA := (1-R)\mathbb{E}\left[\int_0^T V_t^+ 1_{\{\tau=t\}} dt\right], \tag{5.5.1}$$

where R is a recovery rate, $\mathbb{E}[\cdot]$ is an expectation operator under a risk-neutral measure, V_t^+ is a positive part of a risk-free portfolio value at time t, and τ is a time when the counterparty defaults. Typically, V_t^+ is dependent on time-t prices of what are observable in markets such as foreign exchange rates, equity prices, interest rates, and so on and the random variable τ is modeled such that market prices of corporate bonds or Credit Default Swap (CDS) of the counterparty are reproduced. If this quantity is discretized with respect to time $\Delta_n := (T_{n-1}, T_n]$ for $n = 1, \cdots, N$ with $T_0 = 0$ and $T_N = T$, Eq.(5.5.1) become

$$\begin{aligned} CVA_\Delta &:= (1-R)\sum_{n=1}^{N} \mathbb{E}\left[V_{T_n}^+ 1_{\{\tau\in\Delta_n\}}\right](T_n - T_{n-1}) \\ &= (1-R)\sum_{n=1}^{N} \left(\mathbb{E}\left[V_{T_n}^+ 1_{\{\tau\in(T_0,T_n]\}}\right] - \mathbb{E}\left[V_{T_n}^+ 1_{\{\tau\in(T_0,T_{n-1}]\}}\right]\right)(T_n - T_{n-1}). \end{aligned} \tag{5.5.2}$$

A bank is said to be exposed to *wrong-way* risk if V_t^+ is expected to be large when the counterparty's probability of default is high. An example of wrong-way risk is an FX forward contract with a company in an emerging country in which the bank pays a fixed amount in an emerging currency and receives a fixed amount in dollar. In this case, it is likely that a strong depreciation of the country currency and deterioration of the credit quality of the counterparty would simultaneously take place when the government in the emerging country declares its default. However, it is impossible to know the actual dependency between the FX rate and

[2] A definition of CVA is not unique. In this paper, the simplest one is adopted.

the counterparty's credit quality, because the counterparty may hedge a depreciation of the currency. To know a range of CVA is important especially in this case.

The approach in Section 5.4 can be applied to this example. In this case, it is necessary to construct a kernel function α_n as well as $\alpha_{n-1,n}$, which is corresponding to α_{12} in Section 5.3.4, for $n = 1, \cdots, N$. The function α_n describes a dependency between the foreign exchange rate at time T_n and the default probability during $(T_0, T_n]$ and the function $\alpha_{n-1,n}$ describes a dependency between the foreign exchange rate at time T_n and the default probability during $(T_0, T_{n-1}]$. In addition, suppose that the counterparty issues its corporate bonds both in dollar and its country currency with maturity T_n, $n = 1, \cdots, N$. This means that information on $\mathbb{E}\left[1_{\{\tau\in\Delta_n\}}\right]$ and $\mathbb{E}\left[S_{T_n}1_{\{\tau\in\Delta_n\}}\right]$ are given, where S_t is the time-t foreign exchange rate. Then, the method proposed in Section 5.4.2 can be applied to constructing kernel functions with $h_0(s) = 1$ and $h_1(s) = s$. This approach can take into account of various dependencies between underlying assets and default probability, and calculate not only CVA with wrong-way risk but also upper and lower bounds of CVA. Since wrong-way risk is difficult to correctly capture, bounds of CVA is useful for risk management.

5.6 Numerical Examples

This section devotes to numerical examples of the method proposed in Section 5.4.1 for pricing barrier options. It is demonstrated that the method is so flexible that it can produce various prices of barrier options inside the model-independent pricing bounds.

It is assumed in this section that the process of the underlying asset price follows the Heston model:

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma_t dW_t \tag{5.6.1}$$

$$d\sigma_t^2 = \kappa(\eta - \sigma_t^2)dt + \theta\sigma_t d\tilde{W}_t, \tag{5.6.2}$$

where W and $\tilde{W}$ are Brownian motions with correlation ρ under a risk-neutral measure. The parameters of the Heston model in the examples are listed in Table 5.1. The Heston model with these parameters produces 6 month implied volatilities as in Table 5.2.

S	r	q	σ_0^2	κ	η	θ	ρ
100	0.03	0.03	0.15^2	1.0	0.2^2	0.4	0.0

Table 5.1: Parameters of the Heston Model

strike	80	90	100	110	120	130
implied volatility (%)	18.28	16.10	15.00	15.93	17.51	18.99

Table 5.2: 6M Volatility Smile

The barrier option considered in the example is double knock-out call options with maturity 6-month, $L = 80$ and $U = 110$. Prices of the double barrier options can be analytically calculated under the Heston model with $r = q$ and $\rho = 0$ (see (Lipton, 2001)). In particular, this model evaluates the double no-touch option as 0.615. Double knock-out call options with strike price varied from L to U are evaluated using the following methods:

(1) the model-independent lower bound of (Tsuzuki, 2014)

(2) the Heston's stochastic volatility model ((Heston, 1993)) with parameters in Table 5.1

(3) the method in Section 5.4.1 using a L.V. kernel function

(4) the method in Section 5.4.1 using a constant kernel function

(5) the model-independent upper bound of (Tsuzuki, 2014),

where the L.V. kernel function in (3) is Eq.(5.4.6) using Eq.(5.4.5) with the volatility smile of Table 5.2 and the constant kernel function in (4) is that using a constant value instead of Eq.(5.4.5).

Figure 5.1 and Table 5.3 are the results. First of all, as is expected, all prices of the methods (3),(4) as well as those of the Heston model (2) are insides the pricing bounds (1) and (5). In addition, prices of the method (3) are close to those of the Heston model (2). This can be interpreted that there remains little freedom in dependency between a distribution of S_T and the knock-out event in ordinary circumstances, where barriers are likely to have been hit when an underlying price at maturity is close to the barrier level. On the other hand, prices of the method (4) are significantly different from those of (2) and (3), which shows that the method in this paper is flexible enough to generate various dependency structure between a distribution of S_T and the knock-out event. The discrepancy between (3) and (4) is due to the difference of kernel functions as described in Fig.5.2 and Fig.5.3.

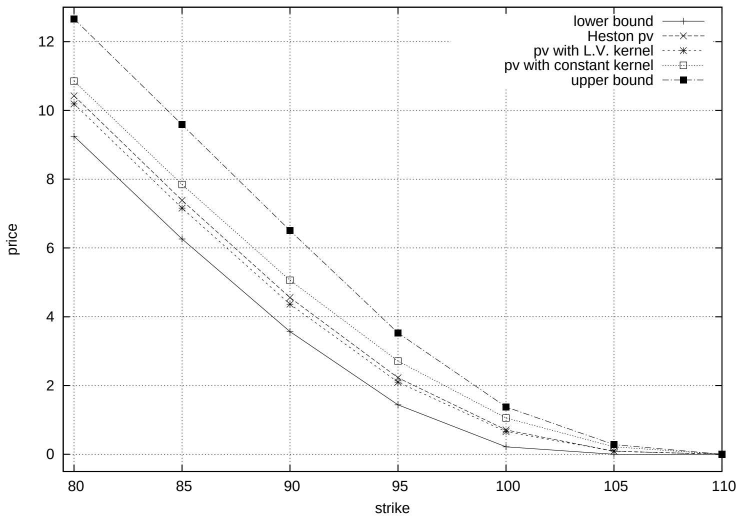


Figure 5.1: Price of double knock-out call options

pricing method \strike	80	85	90	95	100	105	110
(1) lower bound	9.25	6.26	3.57	1.44	0.22	0.00	0.00
(2) Heston model	10.42	7.38	4.56	2.23	0.71	0.09	0.00
(3) L.V. kernel	10.19	7.16	4.37	2.10	0.66	0.09	0.00
(4) constant kernel	10.85	7.84	5.06	2.71	1.06	0.22	0.00
(5) upper bound	12.66	9.59	6.51	3.53	1.38	0.28	0.00

Table 5.3: Price of double knock-out call options

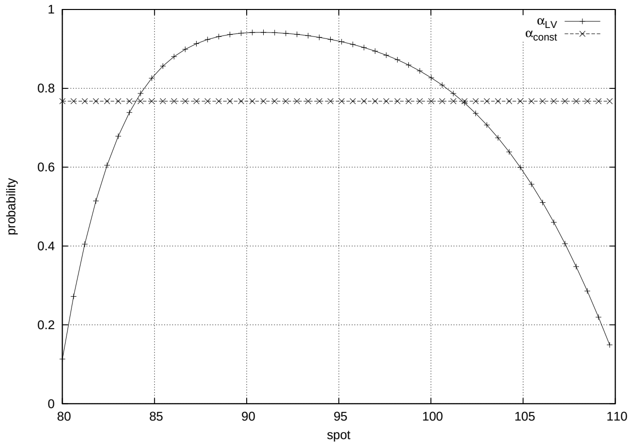


Figure 5.2: Conditional Probability

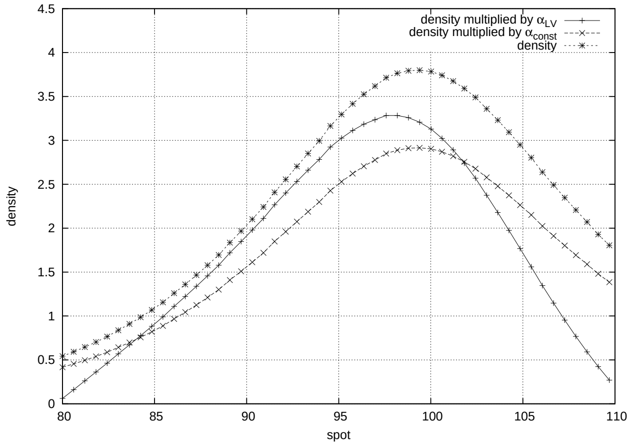


Figure 5.3: Density

Part III

優劣複製取引戦略の改善

Chapter 6

Rebalancing Static Super-Replications

This paper proposes a trading strategy that dynamically rebalances static super-replicating portfolios, which is very useful for both investment and hedging strategies. In order to investigate general properties of the strategy, we derive the Doob-Meyer decomposition for the value process without any specifications of models under the continuous processes of the underlying variables. In particular, we find that the increasing part of the decomposition characterizes the performance of the strategy. Also, we obtain more concrete features for cross-currency and one-touch options based on our general framework. Moreover, numerical examples for cross-currency options demonstrate the effectiveness of our strategy for investment and hedging.

6.1 Introduction

This paper introduces a trading strategy that dynamically rebalances static super-replicating portfolios, which is very attractive for both investment and hedging. Specifically, we derive the Doob-Meyer decomposition for the value process of this strategy without any specifications of models under the continuous processes of the underlying variables: the increasing part of the decomposition is a key element since it characterizes the performance of the strategy.

Super-replications have been more attractive since 2007 after the financial crisis, because they provide a protection against substantial losses. In particular, thanks to the robustness of their model-independent properties, they put a rigid floor on the maximum loss whatever the subsequent paths of the underlying prices.

The problem of finding the cheapest super-replication has first been introduced by (El Karoui and Quenez, 1995) for the case of dynamic trading strategies. Subsequently, various types of super-replicating strategies have been proposed. Among them, several model-independent and static/semi-static super-replications have been investigated by (Chung and Wang, 2008), (Neuberger and Hodges, 2000) and (Tsuzuki, 2013). Here, static replication is a method of replicating a derivative with portfolio whose composition does not change until the maturity of the derivative and semi-static one is a method of replication by trading no more than once after inception. While they are robust, these strategies have a serious drawback that the probability of suffering the maximum loss is extremely high.

In order to overcome this drawback, we propose a dynamically rebalancing strategy of the cheapest super-replication. As an intuitive explanation for the feature of this strategy, consider static super-replications whose portfolios are derived as the cheapest among some family of super-replicating portfolios. First, the super-replicating portfolio is constructed as the cheapest one. After the market conditions have changed, the original portfolio is no longer the cheapest and another one becomes the cheapest. By liquidating the original one and constructing the new cheapest one, an amount of cash is withdrawn from the position:this amount should be positive because the latter is cheaper than the former. The strategy continues this operation until the maturity. Then, thanks to the accumulation of these positive cash flows, the probability of the maximum loss is reduced. We remark that (Neuberger and Hodges, 2000) examines a numerical example of this type of strategy for barrier options. ((Dupire, 2010) called it "roll-down".)

This paper analyzes properties of the strategy without assuming any models under the continuous processes of the underlying variables. In particular, to the best of our knowledge, it is the first work that derives the Doob-Meyer decomposition for the value process, which is a super-martingale process because it is defined as an infimum of a certain family of portfolios. Moreover, we give a financial interpretation to the decomposition and obtain general properties of the strategy through the increasing part of the decomposition;the increasing part is practically important because it characterizes the performance of the strategy. More concrete features become known by applications of our results to specific derivatives such as cross-currency and one-touch options under some additional assumptions that are satisfied for usual cases. Further, numerical examples for cross-currency options demonstrate the effectiveness of our strategy for both investment and hedging.

The organization of this paper is as follows: Section 2 states assumptions and notations. Section 3 is devoted to our main theorem on the Doob-Meyer decomposition. Section 4 applies our result to cross-currency and one-touch options. Numerical examples for cross-currency options are demonstrated in Section 5. The last section gives concluding remarks. Appendix presents analytical results for cross-currency options under the Black-Scholes model.

6.2 Assumptions and Notations

We consider the problem of hedging a derivative by liquid instruments such as bonds, risky assets as well as plain-vanilla options on those assets in a frictionless and no-arbitrage market, which is defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T^*]}, \mathbb{Q})$ for some arbitrary time horizon $T^* > 0$. The no-arbitrage condition ensures the existence of a risk-neutral measure $\mathbb{Q}$ such that the instantaneous expected rate of return on every asset is equal to the instantaneous interest rate. For sake of simplicity, the interest rate and the dividend yields are assumed to be zero.

Let $\mathcal{O}_X$ be a domain of $\mathbb{R}^N$ and $X : \Omega \times [0,T^*] \longrightarrow \mathcal{O}_X$ be an N-dimensional $\{\mathcal{F}_t\}$-adapted continuous process which represents all the underlying random variables such as asset prices and their volatilities:

$$X_t := X_0 + A_t + M_t \tag{6.2.1}$$
$$X_0 := x, \tag{6.2.2}$$

where A_t is an N-dimensional finite variation process, M_t is an N-dimensional continuous local martingale and $x \in \mathcal{O}_X$. The i-th component of the each vector is expressed by $X_t^{(i)}, A_t^{(i)}$ or $M_t^{(i)}$.

Let $Y : \Omega \times [0,T^*] \longrightarrow \mathbb{R}^D$ be an D-dimensional process which denotes time-t prices of all tradable securities, which is $\{\mathcal{F}_t\}$-adapted and continuous. Note that Y is a local martingale under our assumption that the interest rate is zero. We define a trading strategy ϕ by an D-dimensional $\{\mathcal{F}_t\}$-adapted process $\phi : \Omega \times [0,T] \longrightarrow \mathbb{R}^D$, whose i-th component is the number of the security $Y^{(i)}$ held by the strategy, where T is the end of a trading period. In particular, a trading strategy ϕ is a semi-static strategy if and only if ϕ is constant on $[0,\tau)$ and $(\tau,T]$ for some stopping time τ and a strategy is static if and only if $\tau = T$.

Let $Y^* : \Omega \times [0,T^*] \longrightarrow \mathbb{R}$ be a price process of the derivative to be hedged, which is assumed to be $\{\mathcal{F}_t\}$-adapted. Y_t^* may not be measurable with respect to the σ-algebra generated by Y_t for any $t \in [0,T^*]$, which means that the derivative Y^* can not be replicated by the tradable securities. Then, we consider the strategy that super-replicates the derivative. Here, a super-replicating strategy of Y_t^* is a trading strategy such that for any $t \in [0, T^*]$,

$$Y_t^* \leq \sum_{n=1}^{D} \phi_t^{(n)} Y_t^{(n)}. \tag{6.2.3}$$

In particular, suppose that ϕ is a semi-static super-replicating strategy for a stopping time τ. Then, for $t \in [0, \tau)$

$$Y_t^* \leq \sum_{n=1}^{D} \phi_0^{(n)} Y_t^{(n)}. \tag{6.2.4}$$

Assume that there exists a family of static strategies that super-replicate the derivative Y^*. Let us denote the time-t prices of static portfolios in the family by $\{H(t, x, K)\}_{K \in \mathcal{O}_K}$, where H is assumed to depend on a parameter $K \in \mathcal{O}_K$ with some domain $\mathcal{O}_K$ in $\mathbb{R}$ as well as on the time parameter t and market variables $x \in \mathcal{O}_X$.

Remark 20. *Some derivatives can be super-replicated by portfolios consisting of plain-vanilla options whose strike prices are arbitrary. In these cases, the parameter K of $\{H(t, x, K)\}_{K \in \mathcal{O}_K}$ corresponds to the strike price. We will look at those examples in Section 6.4.*

6.3 Rebalancing Super-Replications

This section investigates the strategy which dynamically rebalances super-replications.

The strategy is explained as follows. First, the super-replicating portfolio is constructed at time $t = 0$ as the cheapest one by solving the optimization problem (6.3.1) below. Then, rebalancing the super-replicating portfolio is continuously executed until the maturity T. This is carried out by solving the optimization problem (6.3.1) under the market conditions at time t. It is noteworthy that the strategy is not self-financed because an amount of cash is extracted from the position until the maturity T. The performance of the strategy depends on how much these cash flows are. In order to investigate them, we derive the Doob-Meyer decomposition of the process $\{H_t^*\}_{t \in [0,T]}$, which is the value process of the strategy.

Some assumptions and lemmas are necessary for obtaining the decomposition.

Assumption 5. *$H(t, x, K)$ is assumed to have the unique infimum value with respect to K for all $t \in [0, T)$ and $x \in \mathcal{O}_X$. Let $H^*(t, x)$ be the infimum value and $K^*(t, x)$ be a point where the infimum is attained:*

$$\begin{aligned} H^*(t, x) &:= \inf_{K>0} H(t, x, K) \\ &= H(t, x, K^*(t, x)). \end{aligned} \tag{6.3.1}$$

Hereafter, the following notations will be used for simplicity: $H_t(K) := H(t, X_t, K)$, $K_t^* := K^*(t, X_t)$ and so on.

Assumption 6. *$H(t, x, \cdot)$ and $K^*(t, x)$ are sufficiently smooth with respect to t and x.*

The process $H_t(K)$ for each $K \in \mathcal{O}_K$ is the price process of the static position of a super-replicating portfolio. Under the assumption that the interest rate is assumed to be zero, the process is a local martingale.

Lemma 7. *Suppose $t > 0$ and $K \in \mathcal{O}_K$. Then*

$$\begin{aligned}
&\int_0^t \frac{\partial H}{\partial t}(s, X_s, K)ds + \sum_{i=1}^N \int_0^t \frac{\partial H}{\partial x_i}(s, X_s, K)dA_s^{(i)} \\
&+ \sum_{i,j<N} \int_0^t \frac{1}{2}\frac{\partial^2 H}{\partial x_i \partial x_j}(s, X_s, K)d\left\langle M^{(i)}, M^{(j)}\right\rangle_s = 0
\end{aligned} \tag{6.3.2}$$

and

$$H_t(K) = H_0(K) + \sum_{i=1}^N \int_0^t \frac{\partial H}{\partial x_i}(s, X_s, K)dM_s^{(i)}. \tag{6.3.3}$$

Proof. By Ito's formula, we have for $\forall t > s > 0, \forall K \in \mathcal{O}_K$

$$\begin{aligned}
H_t(K) &= H_s(K) + \int_s^t \frac{\partial H}{\partial t}(u, X_u, K)du \\
&\quad + \sum_{i=1}^N \int_s^t \frac{\partial H}{\partial x_i}(u, X_u, K)dX_u^{(i)} \\
&\quad + \sum_{i,j<N} \int_s^t \frac{1}{2}\frac{\partial^2 H}{\partial x_i \partial x_j}(u, X_u, K)d\left\langle M^{(i)}, M^{(j)}\right\rangle_u .
\end{aligned} \tag{6.3.4}$$

Since the process $\{H_t(K)\}_{t\in[0,T]}$ is a local martingale, Eq.(6.3.2) and Eq.(6.3.3) are obtained. □

Theorem 7 depends on the following assumption, which is satisfied in usual cases where the optimization problem (6.3.1) admits the unique solution.

Assumption 7. *$H(t, x, K)$ is twice continuously differentiable in the neighborhood of $K^*(t, x)$ with respect to K for all $t \in [0, T)$ and $x \in \mathcal{O}_X$. Moreover,*

$$\frac{\partial H}{\partial K}(t, x, K^*(t, x)) = 0 \tag{6.3.5}$$

$$\frac{\partial^2 H}{\partial K^2}(t, x, K^*(t, x)) > 0. \tag{6.3.6}$$

Theorem 7. *Suppose Assumption 7. Then, the process $\{H_t^*\}_{t\in[0,T]}$ is a super-martingale and its Doob-Meyer decomposition is given by*

$$H_t^* = H_0^* - A_t^* + M_t^*, \tag{6.3.7}$$

where the process $\{A_t^*\}_{t\in[0,T]}$ is increasing:

$$A_t^* = \int_0^t \frac{\partial^2 H}{\partial K^2}(s, X_s, K_s^*)\, d\left\langle K^*\right\rangle_s , \tag{6.3.8}$$

and the process $\{M_t^*\}_{t\in[0,T]}$ *is a martingale:*

$$M_t^* = \sum_{i=1}^N \int_0^t \frac{\partial H}{\partial x_i}(s, X_s, K_s^*) dM_s^{(i)}. \tag{6.3.9}$$

Proof. By Assumption 7, we have for any $t \in [0,T]$, $x \in \mathcal{O}_X$,

$$\frac{\partial H}{\partial K}(t, x, K^*(t,x)) \frac{\partial K^*}{\partial t}(t,x) = 0 \tag{6.3.10}$$

$$\frac{\partial H}{\partial K}(t, x, K^*(t,x)) \frac{\partial K^*}{\partial x_i}(t,x) = 0 \tag{6.3.11}$$

and

$$\begin{aligned} & \frac{\partial}{\partial x_i}\left(\frac{\partial H}{\partial K}(t, x, K^*(t,x))\right) \\ = \; & \frac{\partial^2 H}{\partial x_i \partial K}(t, x, K^*(t,x)) + \frac{\partial^2 H}{\partial K^2}(t, x, K^*(t,x)) \frac{\partial K^*}{\partial x_i}(t,x) \\ = \; & 0. \end{aligned} \tag{6.3.12}$$

Ito's formula implies that

$$\begin{aligned} H_t^* &= H_0^* + \int_0^t \frac{\partial H^*}{\partial t}(s, X_s) ds \\ &\quad + \sum_{i=1}^N \int_0^t \frac{\partial H^*}{\partial x_i}(s, X_s) dX_s^{(i)} + \sum_{i,j<N} \int_0^t \frac{1}{2} \frac{\partial^2 H^*}{\partial x_i \partial x_j}(s, X_s) d\left\langle X^{(i)}, X^{(j)} \right\rangle_s \\ &= H_0^* + \int_0^t \left(\frac{\partial H}{\partial t}(s, X_s, K_s^*) + \frac{\partial H}{\partial K}(s, X_s, K_s^*) \frac{\partial K^*}{\partial t}(s, X_s) \right) ds \\ &\quad + \sum_{i=1}^N \int_0^t \left(\frac{\partial H}{\partial x_i}(s, X_s, K_s^*) + \frac{\partial H}{\partial K}(s, X_s, K_s^*) \frac{\partial K^*}{\partial x_i}(s, X_s) \right) dX_s^{(i)} \\ &\quad + \sum_{i,j<N} \int_0^t \frac{1}{2} \left(\frac{\partial^2 H}{\partial x_i \partial x_j}(\cdot, K_s^*) + \frac{\partial^2 H}{\partial x_i \partial K}(\cdot, K_s^*) \frac{\partial K^*}{\partial x_j}(\cdot) \right) (s, X_s) d\left\langle X^{(i)}, X^{(j)} \right\rangle_s \\ &= H_0^* + \sum_{i=1}^N \int_0^t \frac{\partial H}{\partial x_i}(s, X_s, K_s^*) dM_s^{(i)} \\ &\quad + \sum_{i,j<N} \int_0^t \frac{1}{2} \frac{\partial^2 H}{\partial x_i \partial K}(s, X_s, K_s^*) \frac{\partial K^*}{\partial x_j}(s, X_s) d\left\langle X^{(i)}, X^{(j)} \right\rangle_s, \end{aligned} \tag{6.3.13}$$

where we have used Lemma 7.

Let the first integral be defined as M_t^* and the last one be defined as $-A_t^*$. Then, we have

$$\begin{aligned} A_t^* &= -\sum_{i,j<N} \int_0^t \frac{1}{2} \frac{\partial^2 H}{\partial x_i \partial K}(s, X_s, K_s^*) \frac{\partial K^*}{\partial x_j}(s, X_s) d\left\langle X^{(i)}, X^{(j)} \right\rangle_s \\ &= \sum_{i,j<N} \frac{1}{2} \int_0^t \frac{\partial^2 H}{\partial K^2}(s, X_s, K_s^*) \frac{\partial K^*}{\partial x_i}(s, X_s) \frac{\partial K^*}{\partial x_j}(s, X_s) d\left\langle X^{(i)}, X^{(j)} \right\rangle_s \\ &= \frac{1}{2} \int_0^t \frac{\partial^2 H}{\partial K^2}(s, X_s, K_s^*) d\left\langle \sum_{i=1}^N \int_0^\cdot \frac{\partial K^*}{\partial x_i} dX^{(i)} \right\rangle_s . \end{aligned} \tag{6.3.14}$$

It is found that the process $\{M_t^*\}_{t\in[0,T]}$ is a martingale and $\{A_t^*\}_{t\in[0,T]}$ is increasing by Assumption 7. □

Remark 21. *In case where $H^*(t,x)$ takes the infimum value at a boundary point, Eq.(6.3.5) of Assumption 7 may not hold. Nevertheless, Theorem 7 holds under an assumption that $K^*(t,x)$ is constant on the subset of $[0,T) \times \mathcal{O}_X$ that $H^*(t,x)$ takes the infimum value at a boundary point, because Eq.(6.3.10) and (6.3.11) hold and $A_t^* = 0$ on the set.*

A financial interpretation of Theorem 7 is as follows. The variation of the martingale part M is approximated by the difference between the time-t value and the time-s value of the time-s optimal portfolio by Eq.(6.3.3) for all $s < t$, which is a variation of the price of a portfolio held at time s:

$$\begin{aligned} M_t^* - M_s^* &= \sum_{i=1}^N \int_s^t \frac{\partial H}{\partial x_i}(u, X_u, K_u^*) dM_u^{(i)} \\ &\approx \sum_{i=1}^N \frac{\partial H}{\partial x_i}(s, X_s, K_s^*) \Delta M_s^{(i)} \\ &\approx H_t(K_s^*) - H_s(K_s^*), \end{aligned} \tag{6.3.15}$$

where $\Delta M_s^{(i)} := M_t^{(i)} - M_s^{(i)}$ and we have used the approximation

$$H_t(K) - H_s(K) \approx \sum_{i=1}^N \frac{\partial H}{\partial x_i}(s, X_s, K) \Delta M_s^{(i)} \tag{6.3.16}$$

with $K = K_s^*$.

Then, we obtain

$$A_t^* - A_s^* \approx H_t(K_t^*) - H_t(K_s^*), \tag{6.3.17}$$

which is the difference between the time-t value of the time-t optimal portfolio and that of the time-s optimal portfolio. Hence, the increasing part A is regarded as the accumulation of cash flows generated by each rebalancing. Consequently, the value process for a trader shorting the derivative with the optimally rebalancing super-replication strategy is given by

$$H_t^* + A_t^* - Y_t^* = H_0^* + M_t^* - Y_t^*. \tag{6.3.18}$$

This implies that the larger is A_t^*, the more profitable the strategy is.

A general property is derived through this analysis. Eq.(6.3.8) shows that the time-t cash flow is a change in the quadratic variation of the optimal parameter K^* weighted by the second-order derivative of H with respect to the parameter K at $K = K^*$. The former represents the extent of the fluctuation of the optimal parameter K^* while the latter expresses the extent of convexity of the function $H(t, x, \cdot)$ at an optimal point K^*. For further detailed properties, we study some specific options in the next section.

6.4 Applications

This section applies our result to cross-currency and one-touch options, where we assume conditions normally satisfied for plain-vanilla European options;for instance, the prices of these options are sufficiently smooth with respect to every parameter and the delta and the vega [1] of call options are positive.

6.4.1 Cross-Currency Options

This subsection applies Theorem 7 to cross-currency European options. Let a currency exchange rate $X_T^{(i)}$ be the price of the unit amount of Currency i in terms of a base currency such as USD(U.S. dollar). Consider a cross-currency rate representing the price of the unit amount of Currency 1 in terms of Currency 2. Then, the payoff of a call option on the cross-currency with strike 1 and maturity T(in terms of Currency 2) is given by

$$(X_T^{(1)}/X_T^{(2)} - 1)_+ = (1/X_T^{(2)})(X_T^{(1)} - X_T^{(2)})_+.$$

Hence, for pricing this option we need to evaluate an exchange option(see (Margrabe, 1978)) whose payoff is $(X_T^{(1)} - X_T^{(2)})_+$.

Next, note that the following super-/sub-replication is a well-known strategy.

Lemma 8. *For all $K > 0$, the payoff of an exchange option with maturity T must satisfy the following inequalities:*

$$(X_T^{(1)} - X_T^{(2)})_+ \le (X_T^{(1)} - K)_+ + (K - X_T^{(2)})_+, \tag{6.4.1}$$

$$\begin{aligned}(X_T^{(1)} - X_T^{(2)})_+ \quad \ge \quad & max\{(X_T^{(1)} - K)_+ - (X_T^{(2)} - K)_+, \\ & -(K - X_T^{(1)})_+ + (K - X_T^{(2)})_+\}.\end{aligned} \tag{6.4.2}$$

Proof. Suppose $z_1, z_2, k \in \mathbb{R}$. Then

$$\begin{aligned}(z_1 - z_2)_+ &= ((z_1 - k) + (k - z_2))_+ \\ &\le (z_1 - k)_+ + (k - z_2)_+.\end{aligned} \tag{6.4.3}$$

by Jensen's inequality. If we substitute $z_1 = X_T^{(1)}, z_2 = X_T^{(2)}, k = K$ in Inequality (6.4.3), then Inequality (6.4.1) is derived. If we substitute $z_1 = X_T^{(1)}, z_2 = K, k = X_T^{(2)}$ and $z_1 = K, z_2 = X_T^{(2)}, k = X_T^{(1)}$ in Inequality (6.4.3), then Inequality (6.4.2) is derived. □

[1] The delta(vega) is the derivative with respect to the price(volatility) of the underlying asset price.

Let $C^{(i)}(t,x,K)$ and $P^{(i)}(t,x,K)$ be the time-t prices of call and put options on the exchange rate $i (i = 1, 2)$ respectively, where $x \in \mathcal{O}_X$ is a N-dimensional vector consisting of all parameters relevant for option prices, such as the underlying exchange rates and their volatilities. Especially, we assume that the first component x_1 and the second one x_2 stand for the underlying exchange rates. The value process of the super-/sub-replicating portfolio which corresponds to H in the previous section is given as follows:

Definition 16. *Let G and L be the value of super-/sub-replicating portfolios with the strike K:*

$$\begin{aligned} G(t,x,K) &:= C^{(1)}(t,x,K) + P^{(2)}(t,x,K) \quad (6.4.4)\\ L(t,x,K) &:= -max\{L^C(t,x,K), L^P(t,x,K)\}, \quad (6.4.5) \end{aligned}$$

where

$$\begin{aligned} L^C(t,x,K) &:= C^{(1)}(t,x,K) - C^{(2)}(t,x,K), \quad (6.4.6)\\ L^P(t,x,K) &:= -P^{(1)}(t,x,K) + P^{(2)}(t,x,K). \quad (6.4.7) \end{aligned}$$

In addition, let K^G and K^L be the optimal strike prices for super-/sub-replication respectively.

Remark 22. *The optimal strike prices K^G and K^L become the same as those in (Chung and Wang, 2008).*

We see if Assumption 7 is satisfied for these strategies. The optimal super-replicating portfolio is always determined uniquely as in the following lemma. On the other hand, the optimal sub-replicating portfolio is not always determined uniquely. We show a sufficient condition for uniqueness under the Black-Scholes model(see Appendix 6.7).

Lemma 9. *Suppose $H = G$. Then, Assumptions 5 and 7 hold.*

Proof. By differentiating $G(t,x,K)$ with respect to K,

$$\begin{aligned} \frac{\partial G}{\partial K}(t,x,K) &= \frac{\partial C^{(1)}}{\partial K}(t,x,K) + \frac{\partial P^{(2)}}{\partial K}(t,x,K)\\ &= \frac{\partial C^{(1)}}{\partial K}(t,x,K) + \frac{\partial C^{(2)}}{\partial K}(t,x,K) + 1 \quad (6.4.8) \end{aligned}$$

$$\frac{\partial^2 G}{\partial K^2}(t,x,K) = \frac{\partial^2 C^{(1)}}{\partial K^2}(t,x,K) + \frac{\partial^2 C^{(2)}}{\partial K^2}(t,x,K). \quad (6.4.9)$$

The fact that $\frac{\partial^2 C^{(i)}}{\partial K^2}(t,x,K) > 0$ implies $\frac{\partial^2 G}{\partial K^2}(t,x,K) > 0$. Since $\frac{\partial G}{\partial K}(t,x,0) = -1$ and $\frac{\partial G}{\partial K}(t,x,+\infty) = +1$, $G(t,x,K)$ has the infimum value at K which satisfies $\frac{\partial G}{\partial K}(t,x,K) = 0$. □

By investigating the property of K^*, it is found that K^* depends on the correlation of the two underlying exchange rates.

Proposition 10. *Suppose Assumption 7 holds for sub-replication. Then $\frac{\partial K_s^G}{\partial x_1}\frac{\partial K_s^G}{\partial x_2} > 0$ and $\frac{\partial K_s^L}{\partial x_1}\frac{\partial K_s^L}{\partial x_2} < 0$.*

Proof. By Assumption 7,

$$\frac{\partial P^{(1)}}{\partial K}\left(t,x,K^G(t,x)\right) + \frac{\partial P^{(2)}}{\partial K}\left(t,x,K^G(t,x)\right) - 1 = 0 \quad (6.4.10)$$

and

$$\frac{\partial P^{(1)}}{\partial K}\left(t,x,K^L(t,x)\right)=\frac{\partial P^{(2)}}{\partial K}\left(t,x,K^L(t,x)\right). \tag{6.4.11}$$

By differentiating these equations with respect to x_i for $i=1,2$, we obtain

$$\frac{\partial^2 G}{\partial K^2}\left(t,x,K^G(t,x)\right)\frac{\partial K^G}{\partial x_i}=-\frac{\partial^2 P^{(i)}}{\partial K\partial x_i}\left(t,x,K^G(t,x)\right) \tag{6.4.12}$$

and

$$\frac{\partial^2 \tilde{L}}{\partial K^2}\left(t,x,K^L(t,x)\right)\frac{\partial K^L}{\partial x_i}=(-1)^{i+1}\frac{\partial^2 P^{(i)}}{\partial K\partial x_i}\left(t,x,K^L(t,x)\right), \tag{6.4.13}$$

where $\tilde{L}=-L^C$ or L^P. $\frac{\partial^2 P^{(i)}}{\partial K\partial x_i}$ is negative because the probability that the price at the maturity is less than any value goes down if a spot price goes up, and vice versa. This leads to the proposition. □

Remark 23. *Proposition 10 shows that the integrand* $\frac{\partial K_s^*}{\partial x_1}\frac{\partial K_s^*}{\partial x_2}$ *of the last term in the following approximation is positive for* $K^*=K^G$ *and negative for* $K^*=K^L$*:*

$$\begin{aligned}\langle K^*\rangle_t \quad &\approx \quad \int_0^t\left(\frac{\partial K_s^*}{\partial x_1}\right)^2 d\left\langle X^{(1)}\right\rangle_s+\int_0^t\left(\frac{\partial K_s^*}{\partial x_2}\right)^2 d\left\langle X^{(2)}\right\rangle_s \\ &\quad +\int_0^t\frac{\partial K_s^*}{\partial x_1}\frac{\partial K_s^*}{\partial x_2}d\left\langle X^{(1)},X^{(2)}\right\rangle_s.\end{aligned} \tag{6.4.14}$$

This implies the following relations approximately hold:

(i) *The quadratic variation of* K^G *is positively correlated to the quadratic covariation of* $X^{(1)}$ *and* $X^{(2)}$*.*

(ii) *The quadratic variation of* K^L *is negatively correlated to the quadratic covariation of* $X^{(1)}$ *and* $X^{(2)}$*.*

Note that our strategy is useful for an investment in the correlation of the two exchange rates.(See Section 5.1.)

6.4.2 One-Touch Options

We apply Theorem 7 to one-touch options. A one-touch option with maturity T and a barrier level $B\in\mathbb{R}_+$ is an option which is worthless if the barrier has not been hit, and pays one at the maturity if the barrier has been hit. Let the event that the barrier has been hit be

$$A:=\{\omega\in\Omega\,|S_t\notin I\ for\ some\ t\in[0,T)\}, \tag{6.4.15}$$

where $I:=[0,B]$ and S_t is the time-t price of the underlying asset with $S_0<B$. Then, the payoff is 1_A.

(Brown et al., 2001) and (Neuberger and Hodges, 2000) introduced a model-independent static super-replication under the assumption that the risk-free interest rate and the dividend yield of the underlying asset are zero and that the underlying asset process is continuous[2]. They consider the following family of strategies parameterized by a strike price $K\in[0,B)$ for a short position of a one-touch option. The strategy consists of

[2]If the assumption fails, their results are weakened.

at most two steps: the first step is to buy $(B-K)^{-1}$ amounts of a European call option with strike K at the beginning; the second step is to sell the call option and buy $(B-K)^{-1}$ amounts of a European put option with strike K at the first hitting time. By the put-call parity, one amount of cash is left after the operation at the first hitting time.

We apply our theorem to this strategy until the first hitting time. Then, the value of the super-replicating portfolio is given by

$$H(t,x,K)=\frac{C(t,x,K)}{B-K}, \tag{6.4.16}$$

where $C(t,x,K)$ is the time-t price of a European call option and $x\in\mathcal{O}_X$ is a N-dimensional vector consisting of all parameters relevant for option prices such as the underlying price S and its volatility σ.

Lemma 10. *$H(t,x,K)$ satisfies Assumption 5 and 7.*

Proof. By differentiating with respect to K, we obtain

$$\frac{\partial H}{\partial K}(t,x,K) \quad = \quad \frac{g(t,x,K)}{(B-K)^2} \tag{6.4.17}$$

and

$$\frac{\partial^2 H}{\partial K^2}(t,x,K) \quad = \quad \frac{1}{B-K}\frac{\partial^2 C}{\partial K^2}(t,x,K)+\frac{2g(t,x,K)}{(B-K)^3}, \tag{6.4.18}$$

where

$$g(t,x,K):=\frac{\partial C}{\partial K}(t,x,K)(B-K)+C(t,x,K). \tag{6.4.19}$$

$\frac{\partial H}{\partial K}=0$ has solutions by the fact that $\frac{\partial H}{\partial K}(t,x,0)=\frac{1}{B}(-1+\frac{S_t}{B})<0$ and $\lim_{K\to B}\frac{\partial H}{\partial K}(t,x,K)=+\infty$. At these points, we have

$$\frac{\partial^2 H}{\partial K^2}(t,x,K)=\frac{1}{B-K}\frac{\partial^2 C}{\partial K^2}(t,x,K)>0, \tag{6.4.20}$$

which leads to the uniqueness of the solution. □

Let us investigate the properties of the optimal strike K^* which uniquely satisfies $\partial H/\partial K=0$ in Eq.(6.4.17). The optimal strike K^* has a graphical interpretation(see Fig.(6.1)): $(K^*,C(x,K^*))$ is the point at which a line through two points $(B,0)$ and $(K^*,C(x,K^*))$ is tangent to the function $C(x,K)$.

We analyze how the optimal strike K^* is affected by the underlying price S and its volatility σ that are main factors for pricing the options. We assume the following assumption which usually holds true.

Assumption 8. *Suppose that g defined as Eq.(6.4.19) is strictly increasing with respect to S and σ(some components of a N-dimensional vector x).*

Remark 24. *Assumption 8 usually holds true because it is expected that the absolute variation of $\frac{\partial C}{\partial K}$ with respect to S and σ respectively is smaller than that of C.*

Then, we obtain the following property of K^*.

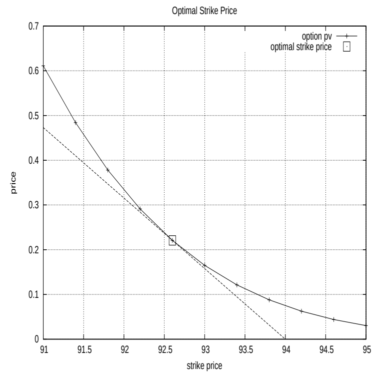


Figure 6.1: The optimal strike ($B = 94$)

Proposition 11. *Suppose Assumption 8 holds. Then* $\frac{\partial K_s^*}{\partial S}\frac{\partial K_s^*}{\partial \sigma} > 0$.

Proof. Let the price of a European call option be $C(\alpha, K)$ with $\alpha = S$ or σ for short, where we ignore other market parameters and K_α^* be the strike price which gives the unique zero for the equation $g(\alpha, K) = 0$. Then, we have $g(\cdot, 0) = S - B < 0$. Let us consider the sign of $g(\beta, K_\alpha^*)$ for $\beta > \alpha$. Assumption 8 implies

$$\begin{aligned} g(\beta, K_\alpha^*) &= \frac{\partial C}{\partial K}(\beta, K_\alpha^*)(B - K_\alpha^*) + C(\beta, K_\alpha^*) \\ &> \frac{\partial C}{\partial K}(\alpha, K_\alpha^*)(B - K_\alpha^*) + C(\alpha, K_\alpha^*) = 0. \end{aligned} \tag{6.4.21}$$

Therefore, K_β^* must be uniquely in the interval $(0, K_\alpha^*)$ by the continuity of g, which leads to the fact that K_α^* is strictly decreasing with respect to α. ☐

Remark 25. *Proposition 11 shows that the integrand* $\frac{\partial K_s^*}{\partial S}\frac{\partial K_s^*}{\partial \sigma}$ *of the last term in the following approximation is positive:*

$$\begin{aligned} \langle K^* \rangle_t &\approx \int_0^t \left(\frac{\partial K_s^*}{\partial S}\right)^2 d\langle S\rangle_s + \int_0^t \left(\frac{\partial K_s^*}{\partial \sigma}\right)^2 d\langle \sigma\rangle_s \\ &\quad + \int_0^t \frac{\partial K_s^*}{\partial S}\frac{\partial K_s^*}{\partial \sigma} d\langle S, \sigma\rangle_s. \end{aligned} \tag{6.4.22}$$

Hence, roughly speaking, the quadratic variation of K^* is positively correlated to the quadratic covariation of S and σ.

6.5 Numerical Examples

In this section, we implement two types of Monte Carlo simulation tests of the dynamically rebalancing super-replication for cross-currency options: the purpose of the first simulation is to confirm that it can be used as an investment strategy on the correlation as stated in Remark 23, and the second is intended to demonstrate the effectiveness of our strategy in hedging through comparing a hedging performance of our strategy with those of other hedging strategies.

Consider a trading strategy where the dynamically rebalancing super-replication is applied against shorting an ATM cross-currency option. For example, let a currency exchange rate $S_t^a(S_t^j)$ represent a time-t price of the unit amount of USD in terms of AUD(JPY)[3]. Consider a cross-currency rate representing the price of the unit amount of AUD in terms of JPY. Then, the payoff of a call option on the cross-currency with strike spot ATM(S_0^j/S_0^a) and maturity T in terms of JPY is given by

$$\left(\frac{S_T^j}{S_T^a} - \frac{S_0^j}{S_0^a}\right)_+ = S_T^j \cdot \frac{1}{S_0^a}\left(\frac{S_0^a}{S_T^a} - \frac{S_0^j}{S_T^j}\right)_+ \tag{6.5.1}$$

In the following simulations, we normalize the processes of the exchange rates so that $S_0^a = S_0^j = 1$.

6.5.1 Investment on Correlation

In order to focus on a correlation investment, we adopt a simple model that is a correlated log-normal model with a constant correlation as in the following assumption.

Assumption 9. *The processes of the exchange rates S_t^j and S_t^a are assumed to be correlated log-normal with constant volatilities and a constant correlation:*

$$\begin{aligned} dS_t^j &= \sigma^j S_t^j dW_t^j && (6.5.2)\\ dS_t^a &= \sigma^a S_t^a dW_t^a, && (6.5.3)\\ \left\langle W^j, W^a \right\rangle_t &= \rho t, && (6.5.4) \end{aligned}$$

where W_t^j and W_t^a are 1 dimensional Brownian motions and σ^j, σ^a and ρ are constant.

Our simulation settings are listed in Table 6.1, where we have two values of the correlation in order to see how the performance of the strategy is affected by the correlation. We sell 100.0 units of a cross-currency option at 14.14% implied volatility, which corresponds to $\rho = -0.25$ and rabalance static portfolios every five days.

Table 6.1: Settings of the simulation

T	S_0^j	S_0^a	σ^j	σ^a	ρ
30(days)	1.0	1.0	10.0(%)	10.0(%)	0.0 / -0.5

Fig.6.2 shows the result of the simulations in terms of JPY; it shows histograms of the performance corresponding to the correlation values. It is found that the higher is the correlation, the more profit is obtained, which is consistent with Remark 23.

6.5.2 Effectiveness as a Hedging Strategy

This subsection considers hedging as an application of the dynamically rebalancing super-replication. Especially, hedging a short position by the strategy seems attractive to risk-averse investors. The strategy has two

[3] AUD and JPY stand for Australian dollar and Japanese yen, respectively.

distinctive features: one is to avoid substantial losses and another is to prevent the worst-case scenario which would often occur if rebalancing would not be carried out.

In order to demonstrate those, a hedging performance of the strategy is compared with those of two other hedging strategies; Black-Scholes dynamic hedging and the static super-replication of (Chung and Wang, 2008), which is a static position introduced in Lemma 8. We implement a simulation test where paths are generated by a realistic model, where the volatilities of both exchange rates are stochastic. The following model is used for generating paths of the simulation.

Assumption 10. *The processes of the exchange rates S_t^j and S_t^a are assumed to follow the model:*

$$\begin{aligned} dS_t^j &= S_t^j \sigma_t^j dW_t^j && (6.5.5) \\ dS_t^a &= S_t^a \sigma_t^a dW_t^a, && (6.5.6) \end{aligned}$$

where $\zeta_t^j := \log \sigma_t^j$ and $\zeta_t^a := \log \sigma_t^a$ follow

$$\begin{aligned} d\zeta_t^j &= \xi^j(\eta^j - \zeta_t^j)dt + \theta^j dZ_t^j, && (6.5.7) \\ d\zeta_t^a &= \xi^a(\eta^a - \zeta_t^a)dt + \theta^a dZ_t^a. && (6.5.8) \end{aligned}$$

$(\xi^j, \eta^j, \theta^j)$ and $(\xi^a, \eta^a, \theta^a)$ are constant and W^j, W^a, Z^j and Z^a are 1-dimensional Brownian motions with

$$d\left\langle W^j, W^a \right\rangle = \rho,\ d\left\langle W^j, Z^j \right\rangle = \rho^j,\ d\left\langle W^a, Z^a \right\rangle = \rho^a, \tag{6.5.9}$$

where the other correlations are zero.

Our simulation settings are listed in Table 6.2. We sell 100.0 units of a cross-currency option at 20.18% implied volatility that is computed by $\sigma_{iv}^j = 14.0\%$, $\sigma_{iv}^a = 16.0\%$ and $\rho = 0.1$. Then, we rebalance static portfolios every five days while rebalancing the delta every day for Black-Scholes hedging, where the delta is evaluated with 20.18% cross-currency rate volatility. The implied volatilities are set to be flat with 14.0% and 16.0%.

Table 6.2: Settings of the simulation

T	S_0^j	S_0^a	ρ	σ_{iv}^j	σ_{iv}^a
30(days)	1.0	1.0	0.1	14.0(%)	16.0(%)

	σ_0^*	ξ^*	η^*	θ^*	ρ^*
j	9.50(%)	347.22	-2.75	23.57	-0.0011
r	11.93(%)	311.08	-2.7	23.3	0.0015

Fig.6.3 shows histograms of the performances of the strategies and Table 6.3 shows their statistics, where C&W stands for (Chung and Wang, 2008). First, while there are substantial losses(over 2.0yen) in the results of B.S., the maximum loss is 1.12 yen in our strategy, which means that out strategy can avoid substantial losses. Second, it is found from Fig.6.3 that almost half scenarios of the strategy of (Chung and Wang, 2008) are the worst, where the maximum loss is 1.12 yen. On the other hand, our strategy mostly avoids the worst case and achieve improvements in VaR over the (Chung and Wang, 2008)(see Table 6.3). Consequently, it is confirmed that our strategy can avoid substantial losses and mostly prevent the worst case scenario of (Chung and Wang, 2008).

Table 6.3: Statistics of Hedging Errors(yen)

Strategy	Mean	Std Err	Mode	Min	1%	5%	10%	25%
B.S.	0.03	1.22	0.43	-23.05	-4.59	-2.03	-1.22	-0.35
C&W	0.00	1.77	-1.12	-1.12	-1.12	-1.12	-1.12	-1.12
out strategy	-0.05	1.39	-0.98	-1.12	-1.11	-1.05	-1.00	-0.84

6.6 Concluding Remarks

We introduced a trading strategy that dynamically rebalances super-replicating portfolios; this strategy is attractive for both investment and hedging. Then, without assuming any models under the continuous processes of the underlying variables, we derived the Doob-Meyer decomposition for the value process of this strategy to obtain the general properties: specifically, we found that the performance of the strategy is characterized by the the increasing part of the decomposition. Also, our general framework was successfully applied to cross-currency and one-touch options, which provides more concrete implications in practice. Moreover, numerical examples for cross-currency options confirmed the property shown in the previous sections, and demonstrated our strategy is useful for hedging under stochastic volatility environment.

Finally, our next research topic will be to analyze properties of the dynamics of the optimal parameter K^* and to evaluate the expectation of the increasing part of the super-martingale process in order to calculate a price of the option based on the strategy. Also an extension of our result to discontinuous processes of the underlying variables is an interesting theme.

6.7 Appendix:Analytical Results for Exchange Options under Black-Scholes Model

In this section, we derive analytical results under the Black-Scholes model of exchange options. For simplicity, we express the price of asset 1 and asset 2 with X_t and Y_t respectively instead of $X_t^{(i)}$. We assume the following assumption.

Assumption 11. *The processes of both asset 1 and asset 2 are assumed to be log-normal with constant volatility:*

$$dX_t = \sigma_X X_t dW_t^X \tag{6.7.1}$$

$$dY_t = \sigma_Y Y_t dW_t^Y, \tag{6.7.2}$$

where W_t^X, W_t^Y are 1 dimensional Brownian motions under the risk neutral measure $\mathbb{Q}$, and σ_X and σ_Y are constant.

In addition, we define some notations:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \tag{6.7.3}$$

$$d_X^{\pm} = \frac{\log \frac{x}{K} \pm \sigma_X^2 \tau}{\sigma_X \sqrt{\tau}} \tag{6.7.4}$$

$$d_Y^{\pm} = \frac{\log \frac{y}{K} \pm \sigma_Y^2 \tau}{\sigma_Y \sqrt{\tau}} \tag{6.7.5}$$

where $\tau = T - t$.

First, we derive the optimal strike price and the value of optimal portfolio for the super-replication and the sub-replication respectively.

Proposition 12. *The optimal strike of the super-replication is:*

$$K^G(t,x,y) = x^{\frac{\sigma_Y}{\sigma_X+\sigma_Y}} y^{\frac{\sigma_X}{\sigma_X+\sigma_Y}} e^{-\frac{1}{2}\sigma_X\sigma_Y\tau}, \tag{6.7.6}$$

and the upper bound of an exchange option is:

$$C_{BS}\left(t,x,y,|\sigma_X+\sigma_Y|\right). \tag{6.7.7}$$

Proof. By Eq.(6.4.8), $K^G(t,x,y)$ must satisfy $\frac{\partial G}{\partial K}(t,x,y,K_t^G(t,x,y)) = 0$. So, $d_X^- = -d_Y^-$. Then, we get (6.7.6).

The value of super-replicating portfolio is:

$$\begin{aligned}
& C^X(t,x,K) + P^Y(t,y,K) \\
= \; & xN(d_X^+) - KN(d_X^-) + KN(-d_Y^-) - yN(-d_Y^+) \\
= \; & xN(d_X^+) - yN(-d_Y^+) \\
= \; & xN(d_1) - yN(d_2) \\
= \; & C_{BS}\left(t,x,y,\sigma_X+\sigma_Y\right),
\end{aligned} \tag{6.7.8}$$

where $K = K^G(t,x,y)$, $d_1 = \frac{\log\frac{x}{y}+\frac{1}{2}(\sigma_X+\sigma_Y)^2\tau}{(\sigma_X+\sigma_Y)\sqrt{\tau}}$, $d_2 = \frac{\log\frac{x}{y}-\frac{1}{2}(\sigma_X+\sigma_Y)^2\tau}{(\sigma_X+\sigma_Y)\sqrt{\tau}}$.

□

Proposition 13. *Assume $\sigma_X < \sigma_Y$. The optimal strike of the sub-replication is:*

$$K^L(t,x,y) = x^{\frac{-\sigma_Y}{\sigma_X-\sigma_Y}} y^{\frac{\sigma_X}{\sigma_X-\sigma_Y}} e^{\frac{1}{2}\sigma_X\sigma_Y\tau}, \tag{6.7.9}$$

and the lower bound of an exchange option is:

$$C_{BS}\left(t,x,y,|\sigma_X-\sigma_Y|\right). \tag{6.7.10}$$

Proof. Put $d_X = d_X^+$, $d_Y = d_Y^+$, we have

$$\frac{\partial^2 C^X}{\partial K^2}(t,K) = \frac{1}{\sqrt{2\pi}}\frac{x}{K^2\sigma_X\sqrt{\tau}}e^{-\frac{1}{2}d_X^2} \tag{6.7.11}$$

$$\frac{\partial^2 C^Y}{\partial K^2}(t,K) = \frac{1}{\sqrt{2\pi}}\frac{y}{K^2\sigma_Y\sqrt{\tau}}e^{-\frac{1}{2}d_Y^2}. \tag{6.7.12}$$

In order to investigate the sign of $\frac{\partial L^C}{\partial K}$ and $\frac{\partial L^P}{\partial K}$, we define the ratio $\psi(K)$:

$$\begin{aligned}
\psi(K) &= \frac{\frac{\partial^2 C^X}{\partial K^2}(K)}{\frac{\partial^2 C^Y}{\partial K^2}(K)} \\
&= \frac{x}{y}\frac{\sigma_Y}{\sigma_X}e^{-\frac{1}{2}(d_X^2-d_Y^2)}.
\end{aligned} \tag{6.7.13}$$

There are at least two roots of the equation $\psi(K)=1$, because $\int_0^{+\infty}\frac{\partial^2 C^X}{\partial K^2}(K)dK=\int_0^{+\infty}\frac{\partial^2 C^Y}{\partial K^2}(K)dK=1$ and $\lim_{K\to 0}\psi(K)=\lim_{K\to+\infty}\psi(K)=0$ by the assumption $\sigma_X<\sigma_Y$. Consider the sign of $\frac{\partial\psi}{\partial K}$:

$$\begin{aligned}\frac{\partial\psi}{\partial K}(K) &= \frac{2\psi(K)}{K}\left(\frac{\log\frac{x}{K}+\frac{1}{2}\sigma_X^2\tau}{\sigma_X^2\tau}-\frac{\log\frac{y}{K}+\frac{1}{2}\sigma_Y^2\tau}{\sigma_Y^2\tau}\right)\\ &= \frac{2\psi(K)}{K\tau}\left(\left(\frac{1}{\sigma_Y^2}-\frac{1}{\sigma_X^2}\right)\log K+\frac{1}{\sigma_X^2}\log x-\frac{1}{\sigma_Y^2}\log y\right).\end{aligned} \tag{6.7.14}$$

Then, the number of the roots of the equation $\psi(K)=1$ is exactly two.

The fact that $\frac{\partial^2 L^C}{\partial K^2}>0$ is equivalent to $\psi(K)>1$ and $\lim_{K\to 0}\frac{\partial L^C}{\partial K}=\lim_{K\to+\infty}\frac{\partial L^C}{\partial K}=0$ implies that the equation $\frac{\partial L^C}{\partial K}=0$ has an unique solution. By the same reason, the equation $\frac{\partial L^P}{\partial K}=0$ has an unique solution.

The definition of $L^C(t,x,y,K)$ and $L^P(t,x,y,K)$ shows that

$$\lim_{K\to 0} L^C(t,x,y,K) = x-y \tag{6.7.15}$$

$$\lim_{K\to 0} L^P(t,x,y,K) = 0, \tag{6.7.16}$$

and

$$\lim_{K\to +\infty} L^C(t,x,y,K) = 0 \tag{6.7.17}$$

$$\lim_{K\to +\infty} L^P(t,x,y,K) = x-y. \tag{6.7.18}$$

As a result, we find that $L^C(t,x,y,K)$ has only one local minimum and $L^P(t,x,y,K)$ has only one local maximum.

In order to compare a maximum of $L^P(t,x,y,K)$ with that of $L^C(t,x,y,K)$, we calculate the maximum of $L^P(t,x,y,K)$. Let K_* be the solution of the equation $\frac{\partial L^P}{\partial K}(t,x,y,K)=0$. By $\frac{\partial P^X}{\partial K}(t,x,y,K)=\frac{\partial P^Y}{\partial K}(t,x,y,K)$, we can derive $d_X^-=d_Y^-$ and then,

$$K_*=x^{\frac{-\sigma_Y}{\sigma_X-\sigma_Y}}\,y^{\frac{\sigma_X}{\sigma_X-\sigma_Y}}\,e^{\frac{1}{2}\sigma_X\sigma_Y\tau}. \tag{6.7.19}$$

The maximum of $L^P(t,x,y,K)$ is:

$$\begin{aligned}&-P^X(t,x,K)+P^Y(t,y,K)\\ =\;&-(KN(-d_X^-)-xN(-d_X^+))+\left(KN(-d_Y^-)-yN(-d_Y^+)\right)\\ =\;& xN(-d_X^+)-yN(-d_Y^+)\\ =\;& xN(d_1)-yN(d_2)\\ =\;& C_{BS}\left(x,y,|\sigma_X-\sigma_Y|\right),\end{aligned} \tag{6.7.20}$$

where $K=K_*$, $d_1=\frac{\log\frac{x}{y}+\frac{1}{2}(\sigma_X-\sigma_Y)^2\tau}{|\sigma_X-\sigma_Y|\sqrt{\tau}}$ and $d_2=\frac{\log\frac{x}{y}-\frac{1}{2}(\sigma_X-\sigma_Y)^2\tau}{|\sigma_X-\sigma_Y|\sqrt{\tau}}$. This maximum is bounded below by $(x-y)_+$.

Finally, we conclude that the optimal strike is:

$$K^L(t,x,y)=K_*, \tag{6.7.21}$$

and the lower bound of an exchange option is:

$$\sup_{K>0} max\{L^C(t,x,y,K), L^P(t,x,y,K)\} = L^P(t,x,y,K^L(x,y)). \tag{6.7.22}$$

□

By the proof of Proposition 13, we have the following proposition, which means that Assumption 7 holds true for sub-replication under a certain condition.

Proposition 14. *Assume $\sigma_X < \sigma_Y$, then the optimal strike price of sub replicating portfolio is determined uniquely. $L(t,x,y,K)$ can be differentiated with respect to x, y and K in the neighborhood of $K = K^L(t,x,y)$ and $K^L(t,x,y)$ can be differentiated with respect to x, y. Moreover, we have*

$$\frac{\partial L}{\partial K}(t,x,y,K^L(t,x,y)) = 0 \tag{6.7.23}$$

$$\frac{\partial^2 L}{\partial K^2}(t,x,y,K^L(t,x,y)) > 0. \tag{6.7.24}$$

We have the Doob–Meyer decomposition explicitly.

Corollary 10. *Assume Assumption 11 and*

$$\langle W^X, W^Y \rangle_t = \rho t, \tag{6.7.25}$$

where $\rho \in [-1,1]$ is constant. Let ψ be the probability density function for the standard normal distribution : $\psi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

Then, in case of the super-replication,

$$A_t^* = \int_0^t \frac{1+\rho}{\sqrt{T-s}} \left(\frac{X_s}{\sigma_X}\psi(d_X^G) + \frac{Y_s}{\sigma_Y}\psi(d_Y^G) \right) \left(\frac{\sigma_X\sigma_Y}{\sigma_X+\sigma_Y} \right)^2 ds \tag{6.7.26}$$

$$M_t^* \quad = \quad \int_0^t \left(\frac{\partial C^X}{\partial x}(s, X_s, K_s^G) dX_s + \frac{\partial P^Y}{\partial y}(s, Y_s, K_s^G) dY_s \right), \tag{6.7.27}$$

where

$$d_X^G \quad := \quad \frac{\log \frac{X_s}{Y_s} + \frac{1}{2}(\sigma_X+\sigma_Y)^2(T-s)}{(\sigma_X+\sigma_Y)\sqrt{T-s}}, \tag{6.7.28}$$

$$d_Y^G \quad := \quad \frac{\log \frac{X_s}{Y_s} - \frac{1}{2}(\sigma_X+\sigma_Y)^2(T-s)}{(\sigma_X+\sigma_Y)\sqrt{T-s}}, \tag{6.7.29}$$

$$K_s^G \quad := \quad X_s^{\frac{\sigma_Y}{\sigma_X+\sigma_Y}} Y_s^{\frac{\sigma_X}{\sigma_X+\sigma_Y}} e^{-\frac{1}{2}\sigma_X\sigma_Y(T-s)}. \tag{6.7.30}$$

In case of the sub-replication with $\sigma_X < \sigma_Y$,

$$A_t^* = \int_0^t \frac{1-\rho}{\sqrt{T-s}} \left(\frac{X_s}{\sigma_X}\psi(d_X^L) - \frac{Y_s}{\sigma_Y}\psi(d_Y^L) \right) \left(\frac{\sigma_X\sigma_Y}{\sigma_X-\sigma_Y} \right)^2 ds \tag{6.7.31}$$

$$M_t^* = \int_0^t \left(\frac{\partial P^X}{\partial x}(s, X_s, K_s^L) dX_s - \frac{\partial P^Y}{\partial y}(s, Y_s, K_s^L) dY_s \right), \tag{6.7.32}$$

where

$$d_X^L := \frac{\log \frac{X_s}{Y_s} + \frac{1}{2}(\sigma_X - \sigma_Y)^2 (T-s)}{(\sigma_X - \sigma_Y)\sqrt{T-s}}, \tag{6.7.33}$$

$$d_Y^L := \frac{\log \frac{X_s}{Y_s} - \frac{1}{2}(\sigma_X - \sigma_Y)^2 (T-s)}{(\sigma_X - \sigma_Y)\sqrt{T-s}}, \tag{6.7.34}$$

$$K_s^L := X_s^{\frac{-\sigma_Y}{\sigma_X - \sigma_Y}} Y_s^{\frac{\sigma_X}{\sigma_X - \sigma_Y}} e^{\frac{1}{2}\sigma_X \sigma_Y (T-s)}. \tag{6.7.35}$$

Proof. Combine Proposition 12, 13 and Theorem 7. $\square$

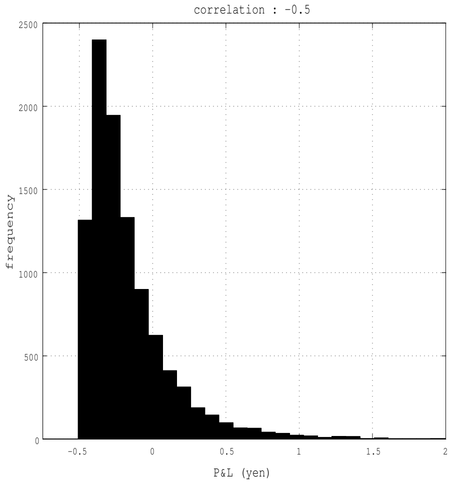


Figure 6.2: Comparison of Correlation

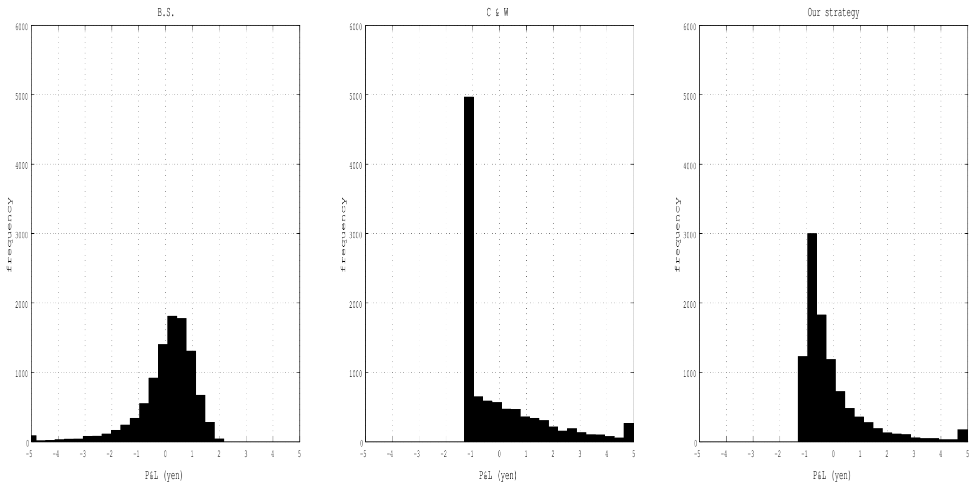


Figure 6.3: Comparison of Hedging Schems

Part IV

密度関数の補正

Chapter 7

A New Improvement Scheme for Approximation Methods of Probability Density Functions

This paper develops a new scheme for improving an approximation method of a probability density function, which is inspired by the idea in *the Hilbert space projection theorem.* Moreover, we apply "Dykstra's cyclic projections algorithm" for its implementation. Numerical examples for application to an asymptotic expansion method in option pricing demonstrate the effectiveness of our scheme under SABR model.

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7.1 Abstract

This paper develops a new scheme for improving density approximation methods, which also provides precise approximations of option values. Specifically, our scheme is inspired by the idea in *the Hilbert space projection theorem* and the so called "Dykstra's cyclic projections algorithm" is applied for its implementation. We also remark that our scheme can be easily implemented in practice, where we need only market data for usual calibration such as option prices with strikes.

We firstly note that our scheme can start with any given approximate density. Then, we improve the density so that it meets a set of conditions such as the non-negativity and the total mass being one that the density function must satisfy. Moreover, based on our method it becomes possible to create a new approximate density to possess certain properties desirable in practice such as calibration to the market forward and option prices. In addition, the method enable a new density to incorporate known information if any, such as the decreasing speed of the tails of the true density. In this manner, we develop a generic scheme which achieves the improvement of the approximation, whatever a starting approximate density is.

[1] http://www.risk.net/type/journal/source/journal-of-computational-finance

[2] This research is supported by JSPS KAKENHI Grant Number 25380389.

Next, let us remark on the criteria on improvement of a density approximation. In general, as the criteria vary such that an improved density provides more accurate ATM option prices, nonnegative prices or butterfly spreads and so on, they are inevitably subjective. For instance, which is the better is not definite between

(1) approximation which excludes negative butterfly spreads

(2) approximation which produces prices close to model prices around ATM, but admits negative butterfly spreads.

For example, Doust (Doust, 2012) is a kind of the first, while (Hagan et al., 2002) is of the second. As for our method, it guarantees the first criterion (non-negativity of butterfly spreads) together with our best effort at the second one (accuracy for model prices). Consequently, the method is robust with respect to the first criterion. In terms of the second criterion, although the accuracy depends on a starting approximation, our method is still robust with a decent initial approximation.

Furthermore, numerical experiments for vanilla option pricing under SABR model demonstrate the validity of our scheme. In fact, with few additional computational costs our scheme improves the third and fifth order asymptotic expansion preserving the required conditions such as nonnegative densities under an appropriate forward measure.

We finally remark that our scheme is general and flexible enough to include a set of conditions and information as one would like to put on an approximate density, and it can be applied to approximation methods other than the asymptotic expansion method. For example, a number of researches have been going on in order to extend SABR model with fixing the problem of the negative densities in the method of (Hagan et al., 2002). (For instance, see (Doust, 2012).) We note that our scheme is also a candidate for handling this issue. Also, the estimate of the absorption probability based on Monte Carlo simulations as in (Doust, 2012) can be consistently reflected in our scheme.

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