# 学位論文

# Replica manipulation of the ground state in one-dimensional quantum spin systems

（1次元量子スピン系における基底状態のレプリカ操作）

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# Abstract

In the last decade it has become evident that information-theoretic quantities, which quantify a quantum entanglement encoded in a quantum state, are extremely useful to analyze a state of mane-body systems. Celebrated examples are the entanglement entropy and the Shannon entropy, which extract universal information of the underlying field theory of the system. Furthermore, a significant concept, the entanglement spectrum, has been proposed recently. It contains more complete information than the entanglement entropy. They have achieved a considerable success in characterizing exotic quantum phases that are beyond conventional descriptions.

Inspired by these developments, we propose a new state, which we name a Rényified state, as follows: for a given quantum state $|\Psi\rangle = \sum_i \psi_i |i\rangle$ and the basis states $|i\rangle$, the Rényified state is defined by raising the wave-function coefficients $\psi_i$ to the power of $n$ (and normalized). The Shannon entropy can be obtained from the Rényified state. Moreover, the latter contains more information than the former.

We study the Rényified state starting from a Tomonaga-Luttinger liquids (TLL), which is an important universality class of one-dimensional quantum systems. A TLL is characterized by a TLL parameter $K$, and described by the free boson field theory with a central charge $c = 1$ in the conformal field theory (CFT) context. We will show, using numerical calculations, that the Rényified state behaves as a TLL described by a modified TLL parameter $\tilde{K} = K/n$ for small $n$. For larger $n$, the Rényified state behaves in a different way: the longitudinal correlation has long-range order while the transverse one remains algebraic. Based on the numerical results, we apply a replica field theory formulation to an analysis of the Rényified state. It suggests that the Rényified state is also a TLL described by a modified TLL parameter $\tilde{K} = K/n$, when $n < 4K$. The field theoretical analysis indicates that the TLL description breaks down at $n = 4K$, which is related to a phase transition in the Rényi-Shannon entropy. Beyond the transition, $n > 4K$, it suggests that the Rényified state is no longer a TLL since the longitudinal correlation has long-range order while the transverse one remains algebraic. This exceptional behavior is unlikely to be realized in the ground state of a Hamiltonian with only short-range interactions. This indicates that the Rényified state beyond the transition belongs to a new class of exotic quantum phase. We explain an origin of the exotic behavior by constructing a particular conformal invariant boundary state of a two-component massless free boson. The relationship between the Rényified state and the Rényi-Shannon entropy is also elucidated in boundary CFT formalism.

# Contents

# Chapter 1

# Introduction

## 1.1 Background

A great variety of phenomena can be found in our world. Such a diversity is caused by an enormity of matters and a complexity of their interactions. Physics has been successful in explaining various discrete phenomena in a simple and a unified way. We believe the universality, and its belief leads us to reach a better understanding of the complicated and cumbersome world. Condensed matter is an ideal playground to discover and investigate the diversity and the universality since it consists of an enormous amount of electrons which interact with each other. It is a profound challenge of condensed matter physics to classify all of phases and/or to categorize transitions between different phases of many-body systems.

An important concept for the classification of the matter is symmetry breaking. The traditional guiding principle is that introducing a local order parameter which characterizes a system and discriminates different phases. The phase transition is interpreted by the spontaneous symmetry breaking of the order parameter. This was developed by Landau and Ginzburg, and grew in sophistication with the advent of the renormalization group theory of Wilson, which gave a unified understanding of the critical phenomena. This scheme has been a great deal of success to understand a variety of phenomena such as crystallization, magnetization, and superconductivity.

It should be noted that the symmetry breaking is not complete to classify phases of matter. An important class of states is the *Tomonaga-Luttinger liquids* (TLL), which is a gapless universality class in one-dimensional quantum systems. No symmetry is spontaneously broken in TLL; yet it represents a distinct phase of matter. In terms of the conformal field theory (CFT), the TLL is described by a $1+1$ dimensional massless free boson field theory with a central charge $c = 1$. This state is generally realized in various situations, such as one-dimensional correlated electron systems, quantum spin chains, carbon nano-tubes and microfabricated systems.

Furthermore, recently, it has been recognized that there are a variety of quantum phases that are beyond the Landau-Ginzburg-Wilson paradigm. They are generally called *topological phases*. Instead of a spontaneous symmetry breaking or a local order parameter, they are characterized by topological features, e.g., the presence of protected edge states and/or a ground state degeneracy which depends on the topology of the system. These type of states, such as quantum Hall states and topological insulators, have been also experimentally realized. An interesting fact is that a TLL emerges as an edge state of quantum Hall states and/or topological insulators. The difficulty of studying the non-symmetry breaking phases is that it cannot be distinguished from a trivial phase by local order parameters. Discriminating such a non-trivial phase is a challenging task in modern condensed matter physics.

In the last decade it has become evident that *information-theoretic quantity* leads to further insight into statistical mechanics and quantum filed theory. Methods developed in information-theoretic theory context is highly useful to analyze a state of many-body systems. However, how to quantify universal information of a quantum state is a vast area of research in its own. A multitude of different measures have been proposed [52]. We do not give an exhaustive review of quantum measures, rather introduce some successful and relevant measures in this thesis.

### 1.1.1 Entanglement measures

#### Entanglement entropy

One of the most prominent measures in physics is the *entanglement entropy*, which quantifies an entanglement between sub-systems of a whole system. For a ground state $|\Psi\rangle$, the von-Neumann entanglement entropy is defined as $S_E = -\mathrm{Tr}_A \rho_A \ln \rho_A$, where the reduced density matrix $\rho_A = \mathrm{Tr}_{\bar{A}} |\Psi\rangle\langle\Psi|$ describes the entanglement between a subsystem $A$ and the rest of the system $\bar{A}$. For one-dimensional quantum systems, celebrated works by Calabrese and Cardy revealed that universal information of the underlying conformal field theory of the system can be read off from its scaling behavior [8, 9, 10, 14]. It has been calculated in many systems including the quantum spin chains which belong to the Ising and the TLL universality class [15]. The entanglement entropy is also useful beyond the one-dimension. For two-dimensional quantum systems, the entanglement entropy obeys an area law in gapped phases with a subleading universal term, which is called the topological entanglement entropy, indicating the topological order [12, 13]. It has been calculated in many systems including fractional quantum Hall systems [24, 25] and quantum spin liquids [26, 27, 28, 29, 30, 31].

### Rényi-entanglement entropy

Calculating a full reduced density matrix $\rho_A$ for a generic interacting Hamiltonian is an astounding challenge. We can avoid the difficulty by taking a different route. This approach is the "replica trick" [8, 10], where we first calculate an alternative quantity, the *Rényi-entanglement entropy*. It is defined as $S_E^{(n)} = \frac{1}{1-n}\ln(\mathrm{Tr}_A \rho_A^n)$. Although calculating $\mathrm{Tr}_A\rho_A^n$ for a generic $n$ in a quantum field theory is still a daunting task, it is hopeful to calculate it for positive integer $n$: the calculation of $\mathrm{Tr}_A\rho_A^n$ is then replaced by a computation of the partition function on a corresponding Riemann surface, where $n$ replicas are connected in a non-trivial way. The analytical continuation reduces the Rényi-entanglement entropy to the entanglement entropy as $\lim_{n\to 1} S_E^{(n)} = -\lim_{n\to 1}\frac{\partial}{\partial n}\mathrm{Tr}_A\rho_A^n = S_E$. The replica formulation has developed not only in analytical methods, also in numerical methods [18, 19, 20, 21, 22].

### Entanglement spectrum

As a generalization of the entanglement entropy, a new idea has been proposed by Li and Haldane [36]. They introduced an *entanglement Hamiltonian* $H_E$ from the reduced density matrix as

$$\rho_A = \mathrm{Tr}_{\bar{A}}|\Psi\rangle\langle\Psi| \equiv \exp(-H_E). \tag{1.1}$$

The entanglement entropy can be interpreted as the thermodynamic entropy of a system described by the "Hamiltonian" $H_E$ at an inverse "temperature" $\beta = 1$. They conjectured universal information is embedded in its low-energy spectrum and more complete information can be read off than the entanglement entropy. Actually, in two-dimensional topological states, such as quantum Hall systems [39, 49, 41], topological insulators [43, 42] and symmetry-protected topological phases [44, 45], its spectrum, which is called the *entanglement spectrum*, has been found to show gapless structures corresponding to the low-energy edge states.

However, there is a rather general problem in this approach, as pointed out in Ref. [50]. We define the canonical ensemble of the entanglement Hamiltonian as

$$\tilde{\rho}_A \equiv \exp(-\beta_E H_E), \tag{1.2}$$

where $\beta_E$ is an inverse temperature for the entanglement Hamiltonian. By definition Eq. (1.1) and Eq. (1.2), physical observables $O_A$ in a sub-system $A$ of the ground state $|\Psi\rangle$ is identical to the thermodynamic expectation value of the canonical ensemble $\tilde{\rho}_A$ at the inverse temperature $\beta_E = 1$, which is described by the identity $\mathrm{Tr}(\rho_A O_A) = \mathrm{Tr}(e^{-H_E} O_A)$. On the other hand, the low-energy spectrum of $H_E$ probes the information in the limit $\beta_E \to \infty$. In general, phases of the entanglement Hamiltonian $H_E$ in these two different temperatures need not be identical. We can expect phase transitions of $H_E$ by varying the entanglement inverse temperature $\beta_E$. This indicates that the low-energy spectrum of $H_E$ does not necessarily reflect the state $|\Psi\rangle$. Even though the low-energy spectrum of $H_E$ changes, it does not always reflect actual phase transitions of $|\Psi\rangle$. This situation is depicted in Fig. 1.1.

It has been also considered that tuning the temperature and investigating the thermodynamics of $H_E$ [46, 47, 48, 49, 51]. These studies teach us a lesson about the entanglement spectrum: Although universal information of a ground state can be read off from the entanglement spectrum, we have to keep in mind that its low-energy spectrum can exhibit a different universal

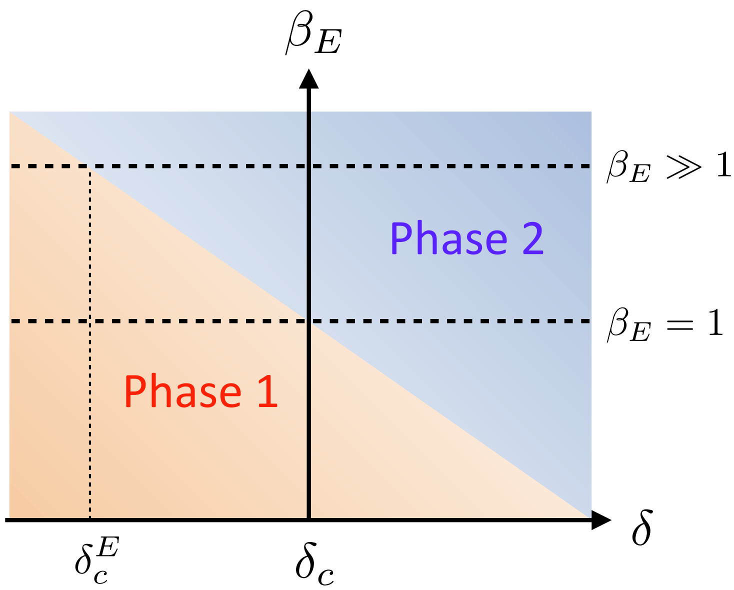


Figure 1.1: Schematic picture of phase diagram of the entanglement Hamiltonian $H_E$. $\delta$ is an external parameter of a parent Hamiltonian of the ground state $|\Psi\rangle$. $\delta_c$ denotes a physical phase transition point of the state $|\Psi\rangle$. The low-energy spectrum of $H_E$ shows a pseudo-phase transition at $\delta_c^E$.

behavior reflecting the phase structure with entanglement temperature. It is indispensable to investigate the properties of the canonical ensemble Eq. (1.2) in the context of entanglement study.

### 1.1.2 Shannon measures

#### (Rényi-)Shannon entropy

Following the great success of the entanglement entropy and the entanglement spectrum, various other information-theoretic quantities have been proposed [52]. As we have seen, the entanglement entropy and the entanglement spectrum are not complete measure, and considerable attention is required. Multifaceted approach from different standpoints is indispensable in the present situation.

Another hopeful measure is the *Shannon entropy*, which quantifies a complexity or localization of a wave function in a given configuration (Hilbert) space. For a ground state $|\Psi\rangle = \sum_i \psi_i |i\rangle$, the Shannon entropy is defined as $S_S = -\sum_i |\psi_i|^2 \ln |\psi_i|^2$. The *Rényi-Shannon entropy* can also be defined as $S_S^{(n)} = \frac{1}{1-n} \ln(\sum_i |\psi_i|^{2n})$ in analogy with the Rényi-entanglement entropy. As taking the limit $n \to 1$, it is identical to the Shannon entropy. For a many-body system where the Hilbert space expands exponentially with the number $N$ of the particles, a volume law $S^{(n)} = a^{(n)}N + b^{(n)} + o(1)$ is naturally expected for the Rényi-Shannon entropy, where $a^{(n)}$ is a non-universal constant and $b^{(n)}$ is a subleading contribution. Although the Rényi-Shannon entropy is obviously basis dependent, recent studies have revealed that the universal properties can be extracted from the subleading term [54, 55, 56, 57, 58, 59, 60, 61, 62].

As an explicit example, we show an application to one-dimensional critical systems. For one dimensional quantum systems, universal information of the underlying conformal field theory of the system can be read off from its scaling behavior [54, 55, 56, 57, 58, 61, 62]. In particular, for a TLL system with periodic boundary condition, the subleading term of the Rényi-Shannon entropy is given as [54, 56]

$$b^{(n)} = -\frac{1}{2}\left(\ln K + \frac{\ln n}{n-1}\right) \quad \text{for } n < n_c = p^2 K \tag{1.3}$$

$$b^{(n)} = \frac{1}{1-n}\left(\frac{n}{2}\ln K + \ln 2\right) \quad \text{for } n > n_c, \tag{1.4}$$

where $p$ is the multiplicity of $|i_{\max}\rangle$, which is the most probable state, and $K$ is a TLL parameter, which characterizes a TLL phase. It is particularly worth noting that the behavior of the subleading term for $n < n_c$ is different from the one above $n_c$. This means that the Rényi-Shannon entropy shows a phase transition, thus it is non-analytic around $n_c$. It is required considerable attention as taking limit $n \to 1$.

It works not only for a one-dimension, but also for several two-dimensional quantum systems. The universal contribution, which is related to symmetry breaking, is contained in the subleading term of the Shannon entropy. The Rényi-Shannon entropy in two-dimensional quantum systems is numerically calculated with the help of powerful quantum Monte Carlo simulations [59, 60].

### Shannon Hamiltonian

We have seen that the entanglement spectrum includes more complete information than the entanglement entropy. It is natural to generalize this picture to the Shannon entropy. For a quantum state $|\Psi\rangle = \sum_i \psi_i |i\rangle$, if the weights of the basis states $\{p_i = |\psi_i|^2\}$ are interpreted as classical Boltzmann weights, we can associate a classical "Hamiltonian" $H_S(i)$ to each configuration $i$ as

$$p_i = |\psi_i|^2 \equiv \exp(-H_S(i)). \tag{1.5}$$

We call it the *Shannon Hamiltonian*. The Shannon entropy can be regarded as the thermodynamic entropy of a system described by the Shannon Hamiltonian $H_S$ at the inverse "temperature" $\beta = 1$. Furthermore, we introduce the canonical ensemble of the Shannon Hamiltonian as

$$\tilde{p}_i \equiv \exp(-\beta_S H_S(i)), \tag{1.6}$$

where $\beta_S$ is an inverse temperature for the Shannon Hamiltonian. In analogy with the entanglement spectrum, it is expected that universal information of the ground state $|\Psi\rangle$ can be accessed from the spectrum of the Shannon Hamiltonian $H_S$. Furthermore, the investigation of the canonical ensemble of the Shannon Hamiltonian is essential to extract veritable information of the ground state $|\Psi\rangle$. The investigation of the Shannon Hamiltonian will hopefully lead us to reach a better understanding of the state $|\Psi\rangle$. However, in the present circumstances, the study of the Shannon Hamiltonian is limited compared with the entanglement spectrum [61]. There is a pressing need to fill in a hole.

## 1.2 Motivations

We have briefly reviewed several information-theoretic quantities as promising tools to detect non-trivial phases. We have seen that the entanglement

Hamiltonian $H_E$ and the Shannon Hamiltonian $H_S$ contain more complete information than the entanglement entropy and the Shannon entropy, respectively. It is crucial to investigate the canonical ensemble of these Hamiltonian.

In this thesis, we attempt to develop a new direction of study, motivated by the Shannon measures. To augment the understanding of the canonical ensemble of the Shannon Hamiltonian Eq. (1.6), we introduce a new state, which we call a *Rényified state*,

$$|\Psi^{(n)}\rangle \equiv \frac{1}{\sqrt{Z^{(n)}}} \sum_i \psi_i^n |i\rangle, \tag{1.7}$$

where

$$Z^{(n)} = \sum_i |\psi_i|^{2n} = \sum_i e^{-nH_S(i)} \tag{1.8}$$

is the normalization factor. This definition means that the quantum expectation value of a physical observable in the Rényified state corresponds to the thermodynamic expectation value under the Shannon Hamiltonian $H_S$ at the inverse temperature $\beta_S = n$ since $\langle\Psi^{(n)}|\hat{A}_{\mathrm{diag.}}|\Psi^{(n)}\rangle = \frac{1}{Z^{(n)}}\sum_i |\psi_i|^{2n} A_{ii} = \frac{1}{Z^{(n)}}\sum_i \frac{1}{\tilde{Z}} e^{-n\tilde{H}_i} A_{ii}$. Thus studying the Rényified state *includes* that of the Shannon Hamiltonian. Actually, the partition function $Z^{(n)}$ has been studied in Refs. [54, 80, 56, 62]. However, the correlation functions in the Rényified state have not been explicitly considered. There are more in the Rényified state than in the classical Shannon Hamiltonian since "non-diagonal physical quantities" with respect to the selected basis, such as $S^x$ correlation function in $S^z$ basis, cannot be interpreted as thermodynamic expectation values under the Shannon Hamiltonian $H_S$. Thus, the study of Rényified states and phase transitions among them could provide us a deeper understanding of the original quantum state. As a first step, we would like to answer the question: *What are the properties of the new quantum state with the change of $n$ ?*

Let us advance the naive consideration about the Rényified state. For an original ground state $|\Psi\rangle = \sum_i \psi_i |i\rangle$, there are the largest amplitudes which correspond to the most probable states $\{|i_{\max}\rangle\}$ in the distribution of amplitudes of the coefficients $\{|\psi_i|^2\}$. By raising the coefficients to the power of $n$, the most probable states are emphasized if $n > 1$. On the other hand, for $n < 1$, they are depressed and the difference among the amplitudes are decreased. Thus we can expect that the Rényified state is in the ordered phase corresponding to the most probable states $\{|i_{\max}\rangle\}$ in the limit $n \to \infty$ while it should be disordered (all amplitudes are identical) as taking the limit $n \to 0$. This indicates that there should be phase transition(s) as changing the Rényi index $n$. It is interesting to see how the Rényified state changes

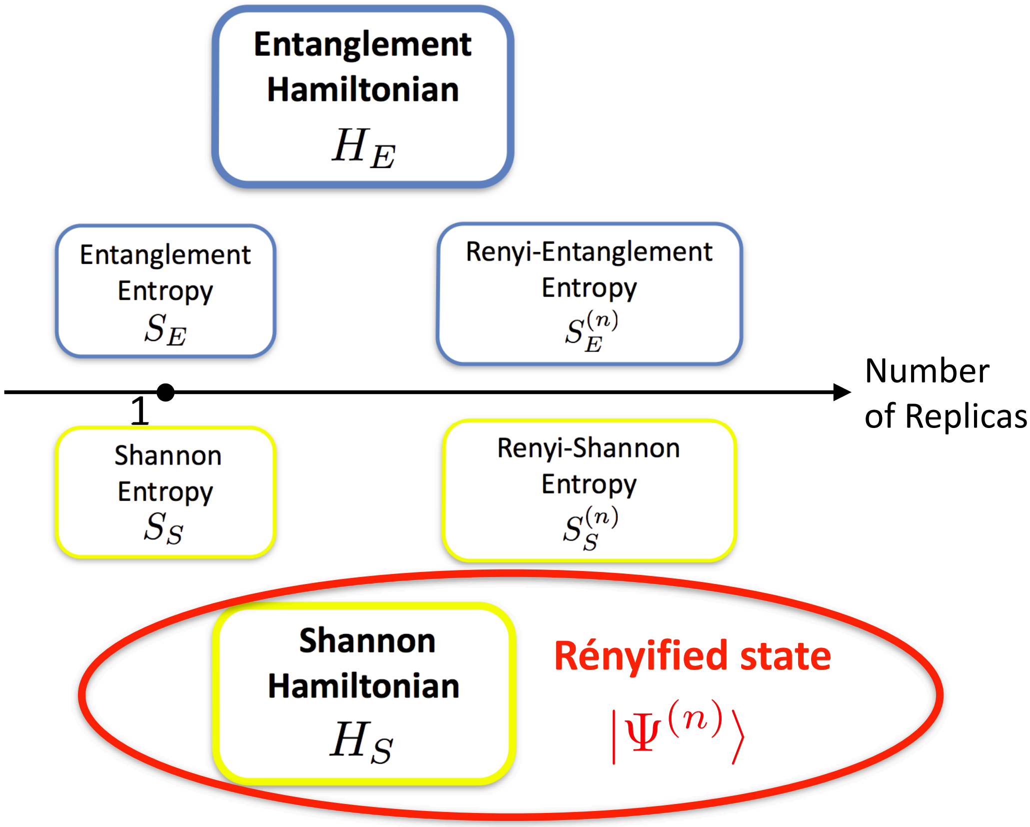


Figure 1.2: Conceptual diagram of an issue in this thesis. The entanglement Hamiltonian contains more information than the (Rényi-)entanglement entropy. The Shannon Hamiltonian includes more information than the (Rényi-)Shannon entropy. We define the Rényified state *beyond* the Shannon Hamiltonian. Investigating the Rényified state is the main issue in this thesis.

with the change of $n$. In addition, its behavior should depend on the initial ground state $|\Psi\rangle$. Thus the introduction of the Rényified state presents a following question: *How is the phase diagram of the Rényified state with the change of $n$ and the initial state ?* We can expect a rich phase diagram in the Rényified state as presented in Fig. 1.3.

Although the manipulation of the "Rényification" is simple, its effect is quite non-trivial since this manipulation acts on a *state*, not on a *Hamiltonian*. Even if we start from a ground state $|\Psi\rangle$ of a Hamiltonian which has short-range interactions, we expect that the Rényified state $|\Psi^{(n)}\rangle$ has a non-trivial parent Hamiltonian which has long-range interactions in general. This

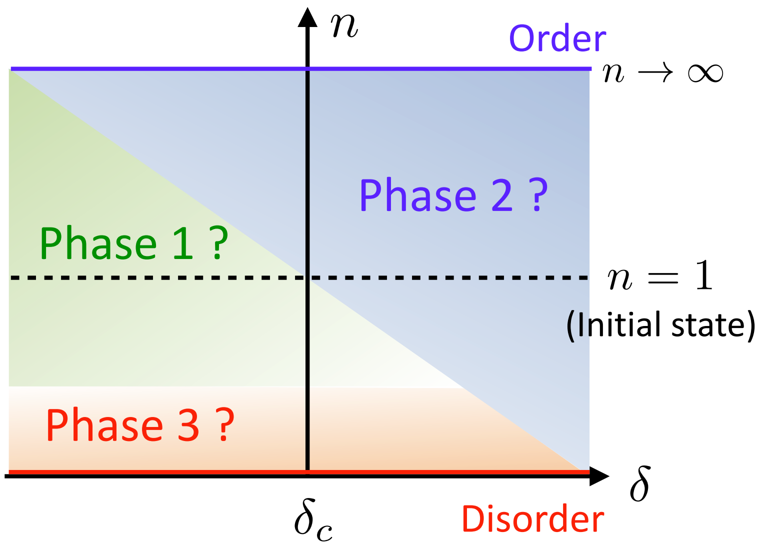


Figure 1.3: Schematic picture of phase diagram of the Rényified state Eq. (1.7). $\delta$ is an external parameter of a parent Hamiltonian of the initial state $|\Psi\rangle$ ($n = 1$). $\delta_c$ denotes a physical phase transition point of the initial state $|\Psi\rangle$. Rich phase diagram can be expected by replica manipulation.

makes us feel that investigating the Rényified state is a daunting challenge in microscopic point of view. However, this problem seems to be handled from a different perspective.

Let us consider an one-dimensional quantum system with periodic boundary condition. We adopt the $1+1$ dimensional point of view by introducing a transfer matrix $\mathcal{T} = e^{-H}$, which connects configurations on neighboring "rings" in imaginary time direction. In terms of the transfer matrix $\mathcal{T}$, a square of a ground state coefficient is expressed as

$$|\psi_i|^2 = |\langle i|\Psi\rangle|^2 = \lim_{\beta\to\infty} \frac{\langle A|\mathcal{T}^{\beta}|i\rangle\langle i|\mathcal{T}^{\beta}|B\rangle}{\langle A|\mathcal{T}^{2\beta}|B\rangle}, \tag{1.9}$$

where $|\Psi\rangle$ is a ground state of $H$, and $|A\rangle$ and $|B\rangle$ are some states which are not orthogonal to the ground state. Thus the numerator and the denominator in Eq. (1.9) are interpreted as partition functions on a cylinder, which extends in imaginary time direction, with or without restriction of configurations at a specific imaginary time (say $\tau = 0$). From this perspective, $|\psi_i|^{2n}$ this can be identified with $n$ replicas of the infinite cylinder (in $\tau$ direction). All replicas are bound at $\tau = 0$, which means that degrees of freedom on each replica has a same configuration at $\tau = 0$. Thus the investigation of the Rényified state can be mapped to a problem in a *replicated system* with a constraint. As we will discuss details in chapter 4, the problem is further mapped to a *boundary problem*, where the boundary is normal to the imaginary time axis, by "folding trick". We note that the boundary is along the imaginary time axis in many condensed-matter applications of the boundary problem, such as an electronic transport through an impurity and Kondo problem. Thus this is, along with the quantum quench, one of exceptional cases in which a boundary perpendicular to the imaginary time axis is relevant.

Although the introduction of the Rényified state, i.e. the replica manipulation is a non-trivial operation, it can be a series of "physical" manipulations under a certain rule as we have stated above. Thus we expect that the replica manipulation can produce a new class of states which cannot be obtained by a simple manipulation of a Hamiltonian. Actually, we will see that this indeed happens later.

In this thesis, we address above questions in the case where $|\Psi\rangle$ belongs to the TLL universality class. Since the present study is, to the best of our knowledge, the first step to investigate the Rényified states, we need a "reliable base" for the study. One-dimensional physics is well understood than higher-dimensional physics with plentiful tools: field theoretical approach, numerical methods and exact solutions. CFT is a very powerful tool to classify one-dimensional critical phenomena. Bosonization deals with a discrete one-dimensional problem as a continuous effective field theory in a unified manner. It gives us a simple viewpoint why many one-dimensional critical systems show a universal behavior described by a free boson filed theory. It is nothing but a TLL, which is described by a $c = 1$ CFT. CFT also well describes a TLL with boundary. The boundary CFT formalism leads a systematic and a precise analysis to boundary problems.

Actually, the TLL is not only tractable but also interesting for the investigation of a Rényified state. Let us consider the $S = 1/2$ XXZ spin chain defined as

$$H = \sum_i \left[ J \left( S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + \Delta S_i^z S_{i+1}^z \right]. \tag{1.10}$$

It is known that the ground state of the model is a gapless TLL state for $-1 < \Delta \leq 1$ in the convention $|J| = 1$. For $\Delta \leq -1$ $(1 < \Delta)$, the system has a gap and the ground state is a ferromagnetic (antiferromagnetic) state. The TLL parameter $K$, which characterizes a TLL phase, is related to the anisotropy as $K^{-1} = 2\left(1 - \frac{1}{\pi}\arccos(\Delta)\right)$. This model is reduced to the spinless fermion chain after the Jordan-Wigner transformation as we will see in the Section 2. In particular, since the anisotropic term $\Delta S_i^z S_{i+1}^z$ in the XXZ model takes a role as an interaction term in a fermion model, the XXZ model at $\Delta = 0$ corresponds to the free fermionic chain. The solution is immediate since it is described as the Slater determinant. We express the ground state in the fermion language as

$$|\Psi(\Delta = 0)\rangle = \sum_{\{x_j\}} \psi_{\{x_j\}} |\{x_j\}\rangle, \tag{1.11}$$

where $\{x_j\} = (x_1, x_2, \cdots, x_N)$ denotes positions of the sites where fermions are located and $N$ is the number of fermions. To translate it into the spin language, it is enough to replace $\{x_j\}$ with positions of the sites where down spins are. The coefficients in Eq. (1.11) are written as [54]

$$\psi_{\{x_j\}} = \frac{1}{\sqrt{L^N}} \prod_{j<k} 2i \sin\left[\frac{\pi}{L}(x_k - x_j)\right]. \tag{1.12}$$

This is a TLL ground state and its TLL parameter is $K = 1$.

Next let us consider another model, the Haldane-Shastry model defined as [92, 93]

$$H = \sum_{j<k} \frac{\vec{S}_j \cdot \vec{S}_k}{\tilde{x}_{j,k}^2}, \tag{1.13}$$

where $\tilde{x}_{j,k} = \frac{L}{\pi} \sin\left[\frac{\pi}{L}(x_k - x_j)\right]$ is the cord distance of the circle (see Fig. 1.4). This model can be regarded as the descendant of the Heisenberg model ($S = 1/2$ XXZ model at $\Delta = 1$) with the long-range interactions. The universality class is the TLL, which has the TLL parameter $K = 1/2$. The ground state is derived exactly as [92, 93]

$$|\Psi\rangle_{\mathrm{HS}} = C \sum_{\{x_j\}} \prod_{j<k} \sin^2\left[\frac{\pi}{L}(x_k - x_j)\right] |\{x_j\}\rangle, \tag{1.14}$$

where $C$ is a normalization factor.

Comparing Eq. (1.11) with Eq. (1.14), a hidden structure of a TLL can be read off. First, they share similarity with the sine form of the ground-state coefficients in the position-basis $\{x_j\}$. Second, the TLL parameter decreases by half by raising the ground-state coefficients to the power of two. Based on these facts, let us introduce a new quantum state as

$$|\Psi^{(n)}(\Delta = 0)\rangle = \frac{1}{\sqrt{Z^{(n)}(\Delta = 0)}} \sum_{\{x_j\}} \prod_{j<k} \sin^n\left[\frac{\pi}{L}(x_k - x_j)\right] |\{x_j\}\rangle, \tag{1.15}$$

where $Z^{(n)}(\Delta = 0) = \sum_{\{x_j\}} \prod_{j<k} \sin^{2n}\left[\frac{\pi}{L}(x_k - x_j)\right]$ is a normalization factor. This is nothing but the Rényified state defined as Eq. (1.7) for the

$S = 1/2$ XXZ spin chain at $\Delta = 0$. We can conjecture that this state belongs to the TLL universality class, which has a TLL parameter $K^{(n)} = 1/n$, if we respect the fact that the distribution of the ground-state coefficients determine the feature of the ground state.

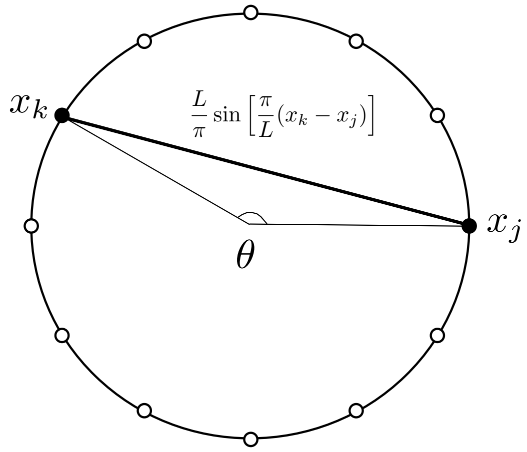


Figure 1.4: The distance of the two charges on a one-dimensional lattice of the size $L$ with periodic boundary condition. The cord distance is given as $\frac{L}{\pi}\sin\left[\frac{\pi}{L}(x_k - x_j)\right]$ since the angle is given by $\theta = \frac{2\pi}{L}(x_k - x_j)$,

Furthermore, let us consider a diagonal quantity $\hat{A}(\{x_j\})$ in the position-basis of the Rényified state as a density-density correlation. It is given by

$$\langle \Psi^{(n)}(\Delta = 0)|\hat{A}(\{x_j\})|\Psi^{(n)}(\Delta = 0)\rangle = \frac{1}{Z^{(n)}} \sum_{\{x_i\}} \prod_{j<k} \sin^{2n}\left[\frac{\pi}{L}(x_k - x_j)\right] A(\{x_i\}) \tag{1.16}$$

As we have seen in the previous section, the quantity Eq. (1.16) can be interpreted as a thermal expectation value in the Boltzmann weight $e^{-\beta_S H_S}$ by defining a corresponding classical Hamiltonian as

$$H_S(\{x_j\}) = -\sum_{j<k} \log\left[\sin\left\{\frac{\pi}{L}(x_k - x_j)\right\}\right], \tag{1.17}$$

and associating the parameter in the Rényified state and the inverse temperature as $2n = \beta_S$. This is known as the Dyson-Gaudin gas model, where charges on a 1D lattice interact via a 2D Coulomb potential [54, 95]. Thus the physics in the Rényified state defined as Eq. (1.15) is governed by the Hamiltonian Eq. (1.17). Changing the index $n$ corresponds to tuning temperature of the system. The phase diagram has been studied in Ref. [95]:

for a rational filling $f = N/L = 1/2, 1/3, \cdots$, a transition point is derived as $n_c = 1/f^2$. At a high temperature, the system is critical and a charge-charge correlation decays algebraically. As the temperature is decreased, the system becomes a crystal and the correlation has a long-range order. If we consider the ground state of the original XXZ model, the total magnetization $\langle \sum_i S_i^z \rangle$ is zero and this corresponds to a half-filling case $f = 1/2$ in the fermionic model. Thus we expect that the transition point is $n_c = 4$. For $n > n_c = 4$, the long-range order of the longitudinal correlation should be seen in the quantum state defined as Eq. (1.15). We note that the classical mapping only works for diagonal quantities. For non-diagonal quantities, as the transverse correlation, we cannot say anything by this analysis.

Based on the consideration stated above, we conjecture a "phase diagram" of the Rényified state $|\Psi^{(n)}(\Delta = 0)\rangle$ as presented in Fig. 1.5. For $n < n_c = 4$, we speculate that the state is a TLL and its TLL parameter changes continuously. On the other hand, for $n > n_c$, we conjecture the state is an antiferromagnetic state which is a counterpart of the charge ordered state in the fermion language. The phase diagram in Fig. 1.5 reminds us of the one for the $S = 1/2$ XXZ spin chain: a TLL phase broadens for $-1 < \Delta \leq 1$ and an antiferromagnetic phase does for $\Delta > 1$. It is necessary to examine whether the phase diagram of the Rényified state for the $S = 1/2$ XXZ spin chain at $\Delta = 0$ in Fig. 1.5 is correct or not. In addition, it is interesting to address a question whether or not the Rényified state starting from a TLL ground state at a general $\Delta$ has a similar structure.

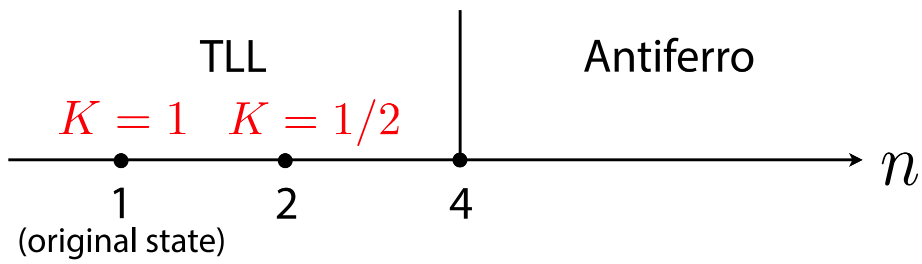


Figure 1.5: A conjecture of a phase diagram of the quantum state $|\Psi^{(n)}(\Delta = 0)\rangle$ defined as Eq. (1.15). At $n = 1$ and $n = 2$, the state is known as the TLL state exactly.

## 1.3 Objectives

We have introduced the Rényified state defined as Eq. (1.7), through the consideration of the Shannon Hamiltonian. The Shannon entropy and the Shannon Hamiltonian have received less attention than the entanglement entropy and the entanglement spectrum although recent works have shown their distinct properties. We also have raised a question about the properties of the Rényified state and its phase diagram starting from a TLL.

The objectives of this thesis are as follows.

- Investigating physical properties and making a phase diagram of the Rényified state starting from a TLL ground state.
- Developing an understanding of the (Rényi-)Shannon entropy and Shannon Hamiltonian through an investigation of the Rényified state.

## 1.4 Outline

The thesis is organized as follows. Two chapters 2 and 3 are review parts of this thesis. Chapter 4 is the main part of this thesis. In chapter 5, we summarize the thesis.

In chapter 2, we review the TLL and the Bosonization technique in one-dimensional systems. We will see the $S = 1/2$ XXZ spin chain can be described by a free boson field theory by bosonization. In chapter 3, we briefly review the Boundary CFT. Chapter 4 consists of a numerical section, two analytical sections, a section about exact solutions and the last section which makes a connection between the Rényified state and Shannon entropy. In the first section, we present numerics for spin correlations in the Rényified state starting from a TLL in the $S = 1/2$ XXZ spin chain. Numerical results indicate a possibility that the Rényified state can be regarded as a TLL with a different TLL parameter. They also suggest an existence of transition and a strange behavior of the spin correlations for large $n$: the longitudinal correlation has long-range order but the transverse correlation decays algebraically. In the second section, we formulate the problem by replica field theory. This can explain why the Rényified state is described by a TLL with a modified TLL parameter $K/n$. A transition mechanism will also be revealed by adding perturbations. In the section 3, we map the problem to the boundary CFT by folding trick. An explicit construction of a conformal invariant boundary state can explain the tricky behavior of the transverse correlation function after the transition. In the last section, we mention a relationship between the Rényified state and the Rényi-Shannon entropy. Boundary CFT approach give a reinterpretation of the behavior of the Rényi-Shannon entropy. In the last chapter, we summarize the thesis.

# Chapter 2

# Tomonaga-Luttinger liquid and bosonization

In this chapter, we present an important class of states in one-dimensional quantum systems, which is called the Tomonaga-Luttinger liquids (TLL). In the context of the conformal field theory (CFT), TLL are described by the free boson field theory, with the central charge $c = 1$. The Luttinger parameter $K$ characterizes a TLL and controls the power-law exponents of the many physical quantities, including spin-spin correlations. We review the low-energy properties of gapless one-dimensional quantum systems, which are generally described in terms of the free boson field theory by bosonization. As a concrete example, we will bosonize the $S = 1/2$ XXZ spin chain and derive the sine-Gordon model, which is the free boson field theory with a vertex operator as a perturbation.

## 2.1 Bosonization of one-dimensional quantum systems

In this section, we introduce a bosonization technique to simply describe the low-energy physical properties of a TLL. Although there are a lot of nice reviews which describe the bosonization technique precisely and in detail [2, 4, 70], we briefly review it here to be self-contained. We choose the "phenomenological" bosonization to give a physical interpretation of the bosonization following Ref. [3], originally proposed by Haldane [5].

We start with a general one-dimensional systems. A density operator is given as

$$\rho(x) = \sum_i \delta(x - x_i), \tag{2.1}$$

where $x_i$ is a position operator of the $i$-th particle of the system. Then we introduce a labeling field operator $\phi_l(x)$, which is continuous function of the position and takes the value $\phi_l(x_i) = 2\pi i$. We can define the field uniquely thanks to the one-dimensional property. The introduction of the labeling field leads the density operator to a tractable form as

$$\begin{aligned} \rho(x) &= \sum_i \partial_x \phi_l(x) \delta(\phi_l(x) - 2\pi i) \\ &= \frac{\partial_x \phi_l(x)}{2\pi} \sum_{n \in \mathbb{Z}} e^{in\phi_l(x)} \end{aligned} \tag{2.2}$$

by using the Poisson summation formula. Let us introduce a field $\phi$ relative to the perfect crystal position as

$$\phi_l(x) = 2\pi\rho_0 x - \frac{\phi}{R}, \tag{2.3}$$

where $\rho_0$ is an average density and $R$ is a constant. As we will explain later, $R$ is interpreted as a periodicity of the filed $\phi$. In terms of the relative field $\phi$, the density is expressed as

$$\rho(x) = \left(\rho_0 - \frac{1}{2\pi R}\partial_x \phi(x)\right) \sum_n e^{in\left(2\pi\rho_0 x - \frac{\phi(x)}{R}\right)}. \tag{2.4}$$

This is a useful expression of the density operator since we can easily see its low-energy property by only taking small $n$ of its expansion.

We furthermore introduce a single-particle creation operator as

$$\psi^\dagger(x) = e^{iR\theta(x)} \rho^{1/2}(x), \tag{2.5}$$

where $\theta$ is a phase operator. It should satisfy the commutation (anticommutation) relation for bosons (fermions) as

$$\left[\psi(x), \psi^\dagger(x')\right] = \delta(x - x'). \tag{2.6}$$

This is satisfied if the bosonic fields $\phi$ and $\theta$ satisfy the commutation relation

$$\left[\phi(x), \frac{1}{2\pi}\partial_{x'}\theta(x')\right] = i\delta(x - x'). \tag{2.7}$$

Thus the field $\phi$ and $\theta$ have a canonical commutation relation. This implies that $\theta$ is a dual field of $\phi$.

A Hamiltonian of the system should contain an interaction term $\int dx \rho^2(x)$ and a kinetic term $\int dx (\partial_x \psi^\dagger(x))(\partial_x \psi(x))$. Due to the form of the operators

Eq. (2.4) and Eq. (2.5), the leading contributions except for constant terms are $(\partial_x\phi(x))^2$ and $(\partial_x\theta(x))^2$. This symmetry consideration allows us to drop the cross term. Thus a general Hamiltonian to describe the low-energy properties of a massless one-dimensional system would be given as

$$H = \frac{1}{4\pi}\int dx \left[ g\left(\partial_x\phi\right)^2 + \frac{1}{g}\left(\partial_x\theta\right)^2 \right], \tag{2.8}$$

where $g$ is determined by a microscopic model, and the spin velocity is set to be 1. The path integral formulation leads to the action

$$S = \frac{g}{4\pi}\int dx d\tau \left(\partial_\mu\phi\right)^2, \tag{2.9}$$

by integrating out the field $\theta$. This is nothing but the free boson field theory. Actually, a TLL is describes by the compactified free boson field theory, where the bosonic fields $\phi$ and $\theta$ are compactified as

$$\phi \sim \phi + 2\pi R \tag{2.10}$$

$$\theta \sim \theta + 2\pi\frac{1}{R}. \tag{2.11}$$

The physics of a TLL is governed by a TLL parameter,

$$K = \left(2gR^2\right)^{-1}. \tag{2.12}$$

Although $g$ and $R$ affects properties of the TLL only through $K$ and hence one of them is redundant, we keep both $g$ and $R$ as parameters in this thesis to compare with other literatures.

Under the compactification Eq. (2.10) and Eq. (2.11), operators which are consistent with their periodicity are only allowed. Thus we expect that operators only have the form as $e^{ip\phi/R}$ and $e^{iqR\theta}$ where $p, q \in \mathbb{Z}$. These are called vertex operators. Long-distance properties of the correlation functions of the vertex operators are given as

$$\left\langle e^{ip\frac{\phi(0)}{R}} e^{-ip\frac{\phi(r)}{R}} \right\rangle \propto e^{\frac{p^2}{R^2}\left(-\frac{1}{g}\ln r\right)} = r^{-\frac{2p^2}{2gR^2}} = r^{-2p^2K} \tag{2.13}$$

$$\left\langle e^{iqR\theta(0)} e^{-iqR\theta(r)} \right\rangle \propto e^{q^2R^2(-g\ln r)} = r^{-q^2\frac{2gR^2}{2}} = r^{-\frac{q^2}{2K}}, \tag{2.14}$$

where we have used Eq. (2.12). These certainly show the power-law decays for the masless theory.

## 2.2 Bosonic Representation of the $S = 1/2$ XXZ spin chain

As a concrete example, we consider the bosonization of the $S = 1/2$ XXZ spin chain defined as

$$H_{\mathrm{XXZ}} = \sum_i \left[ J(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + \Delta S_i^z S_{i+1}^z \right], \tag{2.15}$$

following Ref. [3]. In what follows, we assume the Hamiltonian of the form Eq. (2.15) with negative $J$. By the transformation $S_i^x \to (-1)^i S_i^x, S_i^y \to (-1)^i S_i^y$ and $S_i^z \to S_i^z$, the parameters change as $J \to -J$.

By performing the Jordan-Wigner transformation

$$\begin{aligned} S_i^+ &= c_i^\dagger e^{i\pi \sum_{j=-\infty}^{i-1} c_j^\dagger c_j} && (2.16) \\ S_i^- &= e^{-i\pi \sum_{j=-\infty}^{i-1} c_j^\dagger c_j} c_i && (2.17) \\ S_i^z &= c_i^\dagger c_i - \frac{1}{2}, && (2.18) \end{aligned}$$

the spin Hamiltonian Eq. (2.15) can be mapped to the fermion model as

$$H_{\mathrm{XXZ}} = \sum_i \left[ \frac{J}{2} \left( c_i^\dagger c_{i+1} + \mathrm{h.c.} \right) + \Delta \left( c_i^\dagger c_i - \frac{1}{2} \right) \left( c_{i+1}^\dagger c_{i+1} - \frac{1}{2} \right) \right]. \tag{2.19}$$

Thus the $S = 1/2$ XXZ spin chain is equivalent to a spinless fermion chain with a nearest neighbor interaction. In particular, the system is reduced to the free fermion model for $\Delta = 0$. In this case, the Hamiltonian is described as Eq. (2.8) by bosonization. The interaction part will modify the effective field theory.

We bosonize the interaction part for half-filling case, $\rho_0 = 1/2$. In the continuum limit, a lattice constant $a \to 0$, the interaction part in given as

$$\begin{aligned} H_{\mathrm{int.}} \propto & \int dx \left( -\frac{1}{2\pi R} \partial_x \phi(x+a) + \frac{1}{2\pi\alpha} e^{i\left(\frac{\pi}{a}(x+a) - \frac{\phi(x+a)}{R}\right)} + \mathrm{h.c.} \right) \\ & \times \left( -\frac{1}{2\pi R} \partial_x \phi(x) + \frac{1}{2\pi\alpha} e^{i\left(\frac{\pi}{a}x - \frac{\phi(x)}{R}\right)} + \mathrm{h.c.} \right) \end{aligned} \tag{2.20}$$

to the second-lowest order of the expansion of the density operator Eq. (2.4). $\alpha$ is a cutoff as the lattice constant. A simple calculation gives

$$H_{\mathrm{int.}} \propto \int dx \left[ \frac{2}{(2\pi R^2)^2} \left( \partial_x \phi(x) \right)^2 - \lambda \cos \left( \frac{2\phi(x)}{R} \right) \right]. \tag{2.21}$$

Thus the effective Hamiltonian of the $S = 1/2$ XXZ chain Eq. (2.15) in the masses phase is given by

$$H_{\mathrm{XXZ}} = \frac{1}{4\pi}\int dx \left[\left\{g\left(\partial_x\phi\right)^2 + \frac{1}{g}\left(\partial_x\theta\right)^2\right\} - \lambda\cos\left(\frac{2\phi}{R}\right)\right], \tag{2.22}$$

where the constant $\lambda$ has been determined exactly [85, 87]. We note that the prefactor $g$ in Eq. (2.22) is modified from the original one in Eq. (2.8). The Hamiltonian Eq. (2.22) is known as the sine-Gordon model, which has been studied strenuously [88, 89]. This is the free boson field theory perturbed by a cosine potential. The scaling dimension of the cosine potential is

$$d_\lambda = \frac{4}{2gR^2} = 4K, \tag{2.23}$$

where $K$ is the TLL parameter defined as Eq. (2.12). The TLL parameter $K$ is related to the anisotropy and evaluated analytically from Bethe ansatz as [90, 91]

$$K^{-1} = 2\left(1 - \frac{1}{\pi}\arccos\Delta\right), \tag{2.24}$$

in the convention $|J| = 1$. We have a TLL phase if the most leading perturbation $\cos(2\phi/R)$ becomes irrelevant, $d_\lambda \geq 2$. This indicates that the ground state of the XXZ model is a gapless TLL state for $-1 < \Delta \leq 1$. For $1 < \Delta$, the perturbations becomes relevant and the bosonic field $\phi$ is locked at the minima of the cosine potential. This corresponds to a gapped antiferromagnetic phase. For $\Delta \leq -1$, the ground state is in a ferromagnetic phase.

We furthermore represent the spin operators in a bosonic fields language for the half-filling case ($\rho_0 = 1/2$). The spin operator $S^z$ is just the density as Eq. (2.18), thus it is described as

$$S^z(x) \sim -\frac{1}{2\pi R}\partial_x\phi + a_1(-1)^x\cos\left(\frac{\phi}{R}\right) \tag{2.25}$$

to the second-lowest order of the expansion of the density operator Eq. (2.4). $a_1$ is a non-universal constant, which depends on a microscopic model. The $S^\pm$ operator is more complicated because they contain the string as Eq. (2.16) and Eq. (2.17). Despite its complexity, it can be handled in the continuum limit since the string summation is reduced to the integral of the space-derivative. Finally the $S^\pm$ operator is given as [3]

$$S^\pm(x) \sim e^{\pm iR\theta}\left[b_1 + b_2(-1)^x\cos\left(\frac{\phi}{R}\right)\right], \tag{2.26}$$

in the convention $J$ is negative. $b_1$ and $b_2$ are non-universal coefficients.

The bosonic representation Eq. (2.25) and Eq. (2.26) gives behaviors of spin correlation functions as follows:

$$\langle S^z(0)S^z(r)\rangle = -\frac{K}{2\pi^2}\frac{1}{r^2} + C_1(-1)^r\frac{1}{r^{2K}}, \tag{2.27}$$

$$\langle S^x(0)S^x(r)\rangle = C_2\frac{1}{r^{1/2K}} + C_3(-1)^r\frac{1}{r^{2K+1/2K}}, \tag{2.28}$$

where $C_1$, $C_2$ and $C_3$ are non-universal coefficients. We note that here the prefactor $J$ is defined to be negative in the Hamiltonian Eq. (2.15). For $J > 0$, the uniform part and the staggered part of the $S^{\pm}$ operator Eq. (2.26) are exchanged, thus the uniform part and the staggered part of the transverse correlation Eq. (2.28) will also be exchanged.

# Chapter 3

# Boundary CFT of the free boson field theory

In this chapter, we briefly review the boundary CFT formulation of the multi-component free boson field theory. The study of boundary CFT was greatly made progress by Cardy [63, 64], and in the context of the string theory [65, 66, 67]. It has also been energetically investigated in the context of the condensed matter physics, such as quantum impurity problem [4, 69, 70, 72], defect line [76, 75] and junctions of quantum wires [77, 78, 79]. Furthermore, recent hot topic is the application of the boundary CFT formalism to the study of the entanglement entropy [80] and/or the entanglement spectrum [37, 38]. Here, we summarize relevant formulae in the boundary CFT of the multi-component free boson field theory mainly following Ref. [80].

## 3.1 Boundary condition in CFT

Boundary problems are common in condensed matter physics. An example is a transport problem of one-dimensional electron gas through a point-like barrier. This is a set up of the celebrated work of Kane and Fisher [69, 70]. This type of problems has following features. At first, there are extended gapless quantum degrees of freedom in a bulk. Second, these interact with a localized degrees of freedom, such as an impurity at a point. Generally, these problems can be formulated as a critical bulk filed theory with a boundary. In $1+1$ dimensions, a bulk critical field theory is often described by a CFT. Thus the induced boundary condition is conformal invariant since the bulk is conformal invariant. In other words, conformal transformations should map a boundary onto itself and keep the boundary condition.

Let us consider a CFT on a complex upper half-plane. In this set up, the boundary is regarded as a real axis. The conformal invariance of the boundary implies that there is no flow of energy across the boundary. This gives a condition to energy-momentum tensor as [1]

$$T(z) = \bar{T}(\bar{z}) \quad \text{for} \quad z \in \mathbb{R}, \tag{3.1}$$

where $T(z)$ ($\bar{T}(\bar{z})$) is the holomorphic (anti-holomorphic) component of the energy-momentum tensor on a plane. They are related to infinitesimal local conformal transformations $z \to z' + \epsilon(z)$ and $\bar{z} \to \bar{z}' + \bar{\epsilon}(\bar{z})$ as

$$T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2}, \;\; \bar{T}(\bar{z}) = \sum_{n=-\infty}^{\infty} \bar{L}_n \bar{z}^{-n-2}, \tag{3.2}$$

where $L_n$ and $\bar{L}_n(\bar{z})$ are Virasoro generators of infinitesimal local conformal transformations $\epsilon(z) = \sum_{n=-\infty}^{\infty} \epsilon_n z^{n+1}$ and $\bar{\epsilon}(\bar{z}) = \sum_{n=-\infty}^{\infty} \bar{\epsilon}_n \bar{z}^{n+1}$[1].

Physically realizable boundary conditions in CFT are not only conformal invariant but also stable in a renormalization group (RG) sense. As the CFT classifies bulk field theories, conformally invariant boundary conditions are also classified in the boundary CFT. Let us consider a RG theory of boundary conditions. In the low-energy limit, generically the most stable boundary condition among those permitted by the symmetries of the system will be realized among several kinds of boundary fixed points. Thus the first important problem is to enumerate all possible conformal invariant boundary fixed points for a given critical bulk CFT. We note that this is a classification of the boundary condition in a particular bulk universality class. If the bulk universality is different, the corresponding boundary universality class is changed as in Fig. 3.1.

## 3.2 Conformal invariant boundary state

In 1+1 dimensions, many applications of the boundary CFT can be classified into two groups. First one is a case where a boundary is *along* the imaginary time axis. This includes the electronic transport through an impurity in a quantum wire and Kondo problem. If we consider a half-infinite system, the system can be described as a $1+1$ dimensional field theory on a half-infinite plane by path-integral representation. In this case, a time-evolution Hamiltonian depends on the boundary condition.

Another type of boundary problems is that where a boundary is *normal* to the imaginary time axis. A quantum quench is included in this class of problems [81, 82, 83]. In a quench, an initial state $|\Psi_0\rangle$ is prepared as the ground state of a Hamiltonian $H_0 + \lambda H'$. At $t = 0$, the parameter of the

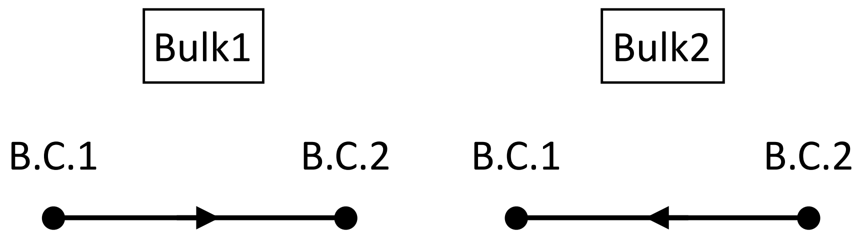


Figure 3.1: Schematic picture of RG flow of a boundary condition. In low-energy theory, a boundary condition renormalizes to a fixed point boundary condition. If the bulk universality is different, the stable fixed boundary condition can be different.

Hamiltonian is quenched as $\lambda \to 0$ and the system evolves unitarily according to the dynamics given by the hamiltonian $H_0$. By regarding the initial state as a *boundary state*, a calculation of physical quantities such as the return amplitude $\langle \Psi_0 | e^{-iHt} | \Psi_0 \rangle$ can be mapped to a boundary problem. This situation is different from the previous examples since the boundary is now *perpendicular* to the imaginary time axis. Here information of the boundary is embedded in a boundary state and time-evolution Hamiltonian is independent of the boundary condition. Of course, a boundary state considering above should be conformal invariant. The conformal invariance gives a condition to a boundary state $|B\rangle$ as

$$\left(T(z) - \bar{T}(\bar{z})\right) |B\rangle = 0, \tag{3.3}$$

at the boundary. Mathematically, these two situations are equivalent under the rotation of the Euclidean space-time. In fact, the equivalence is frequently used in the boundary CFT, as we will see later.

We have considered the boundary problems on the half-infinite plane. We can also consider a boundary theory on a cylinder geometry, which is relevant with a topic of this thesis. We prepare a half-infinite cylinder of the size $L$ in space ($x$) direction and half-infinite in $\tau$-direction. There are a boundary at imaginary time, say $\tau = 0$. This boundary can be regarded as boundary state since this set up is similar to the right case in Fig. 3.2.

As we have seen, physically realizable boundary conditions are invariant under conformal transformations. On a cylinder, this gives a condition to a boundary state $|B\rangle$ as

$$\left(T_{\text{cyl.}}(\tau = 0) - \bar{T}_{\text{cyl.}}(\tau = 0)\right) |B\rangle = 0, \tag{3.4}$$

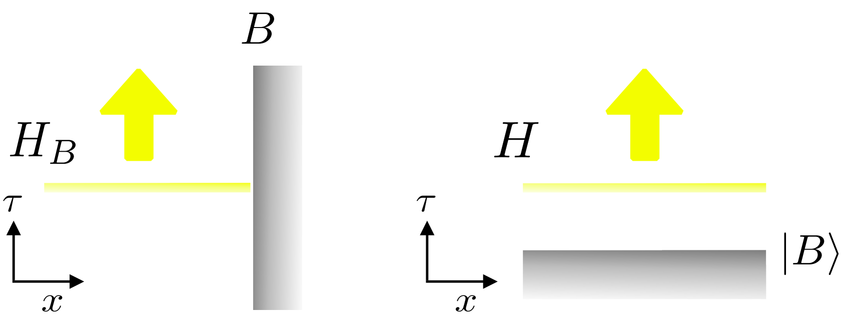


Figure 3.2: Two different set up of boundary problems. In the left panel, a boundary is parallel to the imaginary time axis. In this case, a time-evolution Hamiltonian depends on the boundary condition. On the other hand, a boundary is normal to the imaginary time axis in the right panel. In this situation, a boundary can be viewed as a boundary state. Information of the boundary is embedded in a boundary state and time-evolution Hamiltonian is independent of the boundary condition.

where $\tau = 0$ denotes the position of the boundary. $T_{\text{cyl.}}(\bar{T}_{\text{cyl.}})$ is the holomorphic (anti-holomorphic) component of the energy-momentum tensor on the cylinder. The mode expansion of the energy-momentum tensor is given as [1]

$$T_{\text{cyl.}} = \left(\frac{2\pi}{L}\right)^2 \left(\sum_{m=-\infty}^{\infty} L_m e^{-i\frac{2\pi}{L}m(x+\tau)} - \frac{c}{24}\right) \tag{3.5}$$

$$\bar{T}_{\text{cyl.}} = \left(\frac{2\pi}{L}\right)^2 \left(\sum_{m=-\infty}^{\infty} \bar{L}_m e^{-i\frac{2\pi}{L}m(x-\tau)} - \frac{c}{24}\right) \tag{3.6}$$

In terms of Virasoro generators, the boundary condition Eq. (3.4) becomes

$$\left(L_m - \bar{L}_{-m}\right)|B\rangle = 0, \tag{3.7}$$

for any integer $m$. The solution is derived by Ishibashi, and it is known as the Ishibashi state [73].

We have seen two different pictures of boundary problems, where boundary is along or normal to the $\tau$ direction. In case of cylinder geometry, these situations are connected by performing modular transformation, which exchanges the roles of space and imaginary time directions. This connection gives an additional requirement, *Cardy's consistency condition* [64], on boundary states. In fact, the Ishibashi state, the solution of Eq. (3.7), itself is not physically realizable since it does not satisfy Cardy's condition. In general, we should take a linear combination of the Ishibashi states to satisfy the condition. We will derive this condition in next section.

## 3.3 Cardy's consistency condition

We again consider a boundary theory on a cylinder of the size $L \times \beta$, where $L$ is the circumference in space direction and $\beta$ is the height in $\tau$-direction. There are two boundaries at different imaginary times, say $\tau = 0$ and $\beta$. These two boundaries can be regarded as boundary states as a left panel in Fig. 3.3.

For a given pair of boundary conditions, the partition function is given by

$$Z_{AB} = \langle A|e^{-\beta H}|B\rangle, \tag{3.8}$$

where $|A\rangle$ ($|B\rangle$) is the boundary state at $\tau = 0$ ($\tau = \beta$) and $H$ is a Hamiltonian with periodic boundary condition in the space direction. This is expressed as a function of $q = e^{2\pi i\gamma} = e^{-2\pi\beta/L}$ where $\gamma = i\beta/L$. This scheme is called the *closed string channel* in the string theory terminology.

We can express the partition function on a cylinder in another scheme, in which the time flows *around* the cylinder by performing modular transformation, which exchanges the roles of space and time. Now the system is periodic in the time direction and has the edges in the space direction with boundary conditions $A$ and $B$. This scheme is called the *open string channel*. In this picture, the Hamiltonian depends on the boundary conditions at both ends and its spectrum depends on the boundary conditions. This is to be compared with the scheme in the closed string channel, in which the boundary conditions are embedded in the boundary states and the Hamiltonian is independent of the boundary conditions. We show the schematic picture of these two channels in Fig. 3.3

In the open string channel, the partition function is expressed as a function of $\tilde{q} = e^{-2\pi i/\gamma} = e^{-2\pi L/\beta}$:

$$Z_{AB} = \mathrm{Tr}\left(e^{-LH_{AB}}\right) = \sum_{\tilde{h}} n^{\tilde{h}}_{AB}\chi_{\tilde{h}}(\tilde{q}), \tag{3.9}$$

where $H_{AB}$ is the Hamiltonian with boundary conditions $A$ and $B$ at both edges and $\chi_{\tilde{h}}(\tilde{q})$ is the Virasoro character of the representation $\tilde{h}$ [1]. The coefficients $n^{\tilde{h}}_{AB}$ correspond to the number of primary fields with a conformal

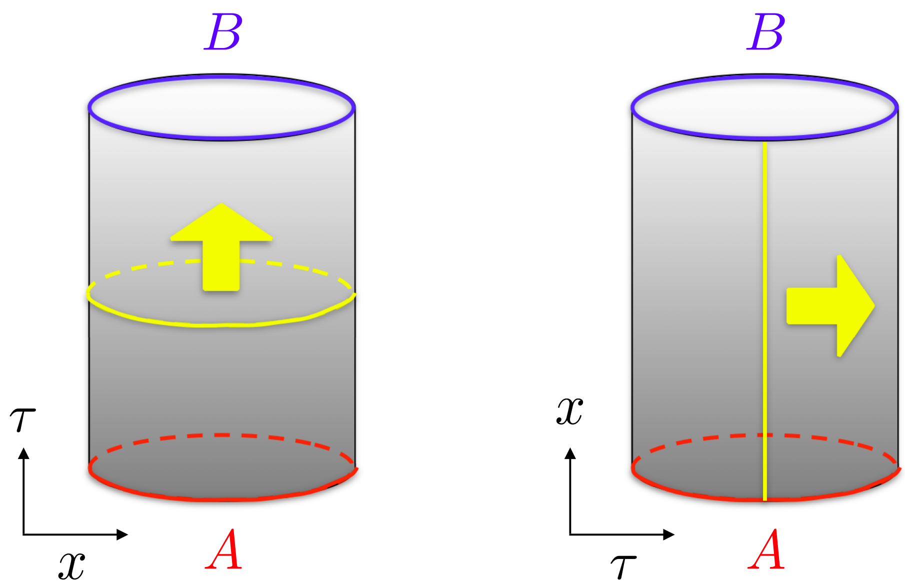


Figure 3.3: Illustration of the closed string channel (left panel) and the open string channel (right panel). In the closed string channel, the time flows *along* the cylinder and the Hamiltonian is independent of the boundary conditions $A$ and $B$. In the open string channel, the time flows *around* the cylinder and the Hamiltonian depends on the boundary conditions.

weight $\tilde{h}$, and they should be non-negative integers [64]. Furthermore, it is usually required that

$$n^0_{AA} = 1, \tag{3.10}$$

since $\tilde{h} = 0$ corresponds to the conformal weight of the identity operator. However, this shall not apply if there are a multiple vacua. While the Virasoro character itself is independent of the boundary conditions, the multiplicities $n^{\tilde{h}}_{AB}$ depend on the boundary conditions. From the partition function in the open string channel Eq. (3.9), we can read off boundary operators which are allowed in boundary conditions $A$ and $B$.

### 3.3.1 Cardy's consistency condition

Cardy's condition is a consistency condition of the partition function between the closed string and open string channel. In the closed string channel, by expanding boundary states in Ishibashi states $\{|h\rangle\rangle\}$, the partition function

Eq. (3.8) is given as

$$Z_{AB}(q) \;=\; \sum_h \langle A|h\rangle\rangle\langle\langle h|B\rangle \chi_h(q), \tag{3.11}$$

where $\chi_h(q)$ is the Virasoro character of the representation $h$ [1]. In the open string channel, the partition function is given by Eq. (3.9). Performing the modular $S$ transformation to Eq. (3.9), we will obtain the partition function in the closed string representation:

$$Z_{AB} \;=\; \sum_{\tilde{h}} n_{AB}^{\tilde{h}} \sum_h S_{\tilde{h}}^h \, \chi_h(q), \tag{3.12}$$

where $S_{\tilde{h}}^h$ is a transformation matrix defined as

$$\sum_h S_{\tilde{h}}^h \chi_h(q) \;=\; \chi_{\tilde{h}}(\tilde{q}). \tag{3.13}$$

The partition functions Eq. (3.11) and Eq. (3.12) should be identical, thus

$$\langle A|h\rangle\rangle\langle\langle h|B\rangle = \sum_{\tilde{h}} n_{AB}^{\tilde{h}} S_{\tilde{h}}^h. \tag{3.14}$$

This is nothing but the Cardy's consistency condition.

## 3.4 Boundary states of the multi-component free boson

In this section, we will construct conformal invariant boundary states of the multi-component free bosons, which is relevant with this thesis. A Lagrangian density of the $N$-dimensional bosonic field theory is given as

$$\mathcal{L} = \frac{g}{4\pi}(\partial_\mu \vec{\phi})^2. \tag{3.15}$$

This is the multi-component version of Eq. (2.9). $\vec{\phi}$ is the $N$-dimensional bosonic field and it is compactified as

$$\vec{\phi} \sim \vec{\phi} + 2\pi \vec{R}, \tag{3.16}$$

where $\vec{R}$ is a vector in the compactification lattice $\Lambda$. When the periodic boundary condition is imposed in the space direction, the bosonic fields is given by the mode expansion as

$$\vec{\phi}(t,x) = \vec{\phi}^{(0)} + \frac{2\pi}{L}\left[\vec{R}x + \vec{P}t\right] + \frac{1}{\sqrt{2g}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left\{\vec{a}_n^L e^{-i\frac{2\pi}{L}nx^+} + \vec{a}_n^R e^{i\frac{2\pi}{L}nx^-} + \text{h.c.}\right\} \tag{3.17}$$

where $L$ is the length of the system in the space direction and $x^{\pm} \equiv x \pm t$. $\vec{P}$ is the conjugate momentum of $\vec{\phi}^{(0)}$. Since the bosonic field $\vec{\phi}$ is compactified as Eq. (3.16), the eigenvalues of $\vec{P}$ is quantized as $\vec{P} = \vec{K}/g$, where $\vec{K}$ belongs to the lattice $\Lambda^*$, which is the dual of $\Lambda$. $\vec{a}^L_n (\vec{a}^{L\dagger}_n)$ is the left-moving annihilation (creation) operators. $\vec{a}^R_n$ and $\vec{a}^{R\dagger}_n$ are the right-moving annihilation and creation operators, respectively. Chiral operators obey the commutation relations

$$[a^L_{m,j}, a^{L\dagger}_{n,k}] = [a^R_{m,j}, a^{R\dagger}_{n,k}] = \delta_{mn}\delta_{jk}, \tag{3.18}$$

and all other commutators vanish.

The bosonic field $\vec{\phi}$ can be decomposed into left-moving and right-moving components as

$$\vec{\phi} = \vec{\phi}^L(x^+) + \vec{\phi}^R(x^-). \tag{3.19}$$

We can define the dual boson field as

$$\vec{\theta} = g(\vec{\phi}^L - \vec{\phi}^R), \tag{3.20}$$

and its mode expansion is given as

$$\vec{\theta}(t,x) = \vec{\theta}^{(0)} + \frac{2\pi}{L}\left[\vec{K}x + g\vec{R}t\right] + \sqrt{\frac{g}{2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left\{\vec{a}^L_n e^{-i\frac{2\pi}{L}nx^+} - \vec{a}^R_n e^{i\frac{2\pi}{L}nx^-} + \text{h.c.}\right\} \tag{3.21}$$

This implies that $g\vec{R}$ is the conjugate momentum of $\vec{\theta}^{(0)}$ and the dual field $\vec{\theta}$ obeys compactification

$$\vec{\theta} \sim \vec{\theta} + 2\pi\vec{K}, \tag{3.22}$$

where $\vec{K} \in \Lambda^*$. The mode expansion of bosonic fields Eq. (3.17) and Eq. (3.21) gives the Hamiltonian for the multi-component free boson as

$$H = \frac{2\pi}{L}\left[\frac{1}{2}\left(g\vec{R}^2 + \frac{1}{g}\vec{K}^2\right) + \sum_{n=1}^{\infty} n\left(\vec{a}^{L\dagger}_n \cdot \vec{a}^L_n + \vec{a}^{R\dagger}_n \cdot \vec{a}^R_n\right) - \frac{c}{12}\right], \tag{3.23}$$

where $c$ is the central charge of the system.

To construct conformally invariant boundary states of the multi-component free boson, we represent the Virasoro generators in terms of oscillation modes. Since the chiral component of the energy− momentum tensor is expressed as $T(x^+) = g : \partial_+\vec{\phi}\ \partial_+\vec{\phi} :$, the Virasoro generators for the multicomponent boson field are given as [80]

$$L_m = \frac{1}{\sqrt{2g}}\sum_l : \vec{\alpha}^L_{m-l}\vec{\alpha}^L_l :, \tag{3.24}$$

$$\bar{L}_m = \frac{1}{\sqrt{2g}}\sum_l : \vec{\alpha}^R_{m-l}\vec{\alpha}^R_l :, \tag{3.25}$$

where

$$\vec{\alpha}_m^L = \begin{cases} -i\sqrt{n}\vec{a}_m^L & (m>0) \\ \frac{1}{\sqrt{2}}\left(\sqrt{g}\vec{R}+\frac{1}{\sqrt{g}}\vec{K}\right) & (m=0) \\ i\sqrt{-m}\vec{a}_{-m}^{L\dagger} & (m<0) \end{cases}, \tag{3.26}$$

and

$$\vec{\alpha}_m^R = \begin{cases} -i\sqrt{m}\vec{a}_m^R & (m>0) \\ \frac{1}{\sqrt{2}}\left(-\sqrt{g}\vec{R}+\frac{1}{\sqrt{g}}\vec{K}\right) & (m=0) \\ i\sqrt{-m}\vec{a}_{-m}^{R\dagger} & (m<0) \end{cases}, \tag{3.27}$$

by using the mode expansion Eq. (3.17).

Although the general solution of the conformal invariant boundary state Eq. (3.7) for the multicomponent free boson is not known, a sufficient condition would be given as

$$\left(\vec{\alpha}_m^L - \mathcal{R}\vec{\alpha}_{-m}^R\right)|B\rangle = 0, \tag{3.28}$$

where $\mathcal{R}$ is an $N \times N$ orthogonal matrix for all integer $m$. For $m \neq 0$, this condition leads the boundary state to the form as

$$\exp\left[-\sum_{n=1}^{\infty}\vec{a}_n^{L\dagger}\mathcal{R}\vec{a}_n^{R\dagger}\right]|\mathrm{vac}\rangle, \tag{3.29}$$

where $|\mathrm{vac}\rangle$ is an oscillator vacuum. Since the Hamiltonian for the multi-component free boson is given as Eq. (3.23), the vacua for oscillator modes are characterized by the zero mode quantum numbers as $|(\vec{R},\vec{K})\rangle$. Thus the solution which satisfies Eq. (3.28) for arbitrary integer $m$ is denoted by

$$|(\vec{R},\vec{K})\rangle\rangle = \exp\left[-\sum_{n=1}^{\infty}\vec{a}_n^{L\dagger}\mathcal{R}\vec{a}_n^{R\dagger}\right]|(\vec{R},\vec{K})\rangle. \tag{3.30}$$

This is nothing but the free boson version of the Ishibashi state [73]. Furthermore, the boundary condition Eq. (3.28) for $m=0$,

$$\left(\sqrt{g}\vec{R}+\frac{1}{\sqrt{g}}\vec{K}\right) = \mathcal{R}\left(-\sqrt{g}\vec{R}+\frac{1}{\sqrt{g}}\vec{K}\right), \tag{3.31}$$

imposes a restriction to a combination of the winding numbers $(\vec{R},\vec{K})$. The linear combination of $|(\vec{R},\vec{K})\rangle\rangle$ which satisfies the condition Eq. (3.31) and the Cardy's consistency condition will be realized for conformal invariant boundary states of the multi-component free boson.

### 3.4.1 Dirichlet boundary state

As an explicit example, we construct a boundary state in a simple case. The simplest choice of the orthogonal matrix is $\mathcal{R} = I$, where $I$ is the identity operator. The solution of Eq. (3.31) for this choice is given by $\vec{R} = 0$. Since $\vec{R}$ represents the winding number of the bosonic field $\vec{\phi}$, $\vec{R} = 0$ indicates that $\vec{\phi}$ is fixed at the boundary. Thus this can be interpreted as the Dirichlet boundary condition (for $\vec{\phi}$ field). The Dirichlet boundary state is given as

$$|D(\vec{\phi}^{(0)})\rangle = g_D \sum_{\vec{K}\in\Lambda^*} e^{i\vec{\phi}^{(0)}\cdot\vec{K}} |(\vec{0},\vec{K})\rangle\rangle, \tag{3.32}$$

where $\Lambda^*$ is the dual compactification lattice and $\vec{\phi}^{(0)}$ is the boundary value of the field $\vec{\phi}$ [80]. The Dirichlet boundary state is a continuous family of boundary states, which are parameterized by the boundary value $\vec{\phi}^{(0)}$. The prefactor $g_D$ is called "$g$-factor" and determined by the Cardy's consistency condition Eq. (3.14). If $\Lambda$ is a hyper cubic lattice with a lattice constant $R$, $g_D$ is given as [80]

$$g_D = \left(2gR^2\right)^{-\frac{N}{4}} = K^{\frac{N}{4}}, \tag{3.33}$$

where $N$ is the number of components of bosons and $K$ is the TLL parameter Eq. (2.12).

### 3.4.2 Neumann boundary state

The next simplest choice of the orthogonal matrix is $\mathcal{R} = -I$. This gives the solution of Eq. (3.31) as $\vec{K} = 0$. This indicates that the dual field $\vec{\theta}$ is fixed at the boundary since $\vec{K}$ represents the winding number of $\vec{\theta}$. By duality, the filed $\vec{\phi}$ can be fluctuated at the boundary. Thus this can be interpreted the Neumann boundary condition (for $\vec{\phi}$ field). The Neumann boundary state is given as

$$|N(\vec{\theta}^{(0)})\rangle = g_N \sum_{\vec{R}\in\Lambda} e^{i\vec{\theta}^{(0)}\cdot\vec{R}} |(\vec{R},\vec{0})\rangle\rangle, \tag{3.34}$$

where $\Lambda$ is the compactification lattice and $\vec{\theta}^{(0)}$ is the boundary value of the field $\vec{\theta}$ [80]. This is again a continuous family of boundary states parameterized by $\vec{\theta}^{(0)}$. The $g$-factor $g_N$ is determined by the Cardy's consistency condition Eq. (3.14). If $\Lambda$ is a hyper cubic lattice with a lattice constant $R$, $g_N$ is given as [80]

$$g_N = \left(\frac{gR^2}{2}\right)^{\frac{N}{4}} = (4K)^{-\frac{N}{4}}, \tag{3.35}$$

where $N$ is the number of component of bosons and $K$ is the TLL parameter Eq. (2.12).

## 3.5 Boundary Entropy and $g$-theorem

We have derived the prefactors of the boundary states, the $g$-factors, by using Cardy's consistency condition. Actually, the $g$-factor represents the "ground-state degeneracy" and its logarithm is interpreted as the "boundary entropy" of the particular boundary condition [68].

This is understood by considering a partition function for a given pair of boundary conditions Eq. (3.8). In the closed string channel, taking a limit $\beta \to \infty$ gives that

$$Z_{AB} = \langle A | e^{-\beta H} | B \rangle \xrightarrow[\beta \to \infty]{} e^{-\beta E_0} \langle A | \Psi \rangle \langle \Psi | B \rangle \tag{3.36}$$

where $|\Psi\rangle$ is the ground state of the Hamiltonian $H$ and $E_0$ is the ground-state energy. Thus $\ln\langle A|\Psi\rangle = \ln g_A$ is interpreted as the entropy due to the boundary condition $A$ (likewise for $B$). Actually, $g$ always decreases under renormalization from a less stable to a more stable boundary critical point in the same bulk universality class. This is known as the $g$-theorem [68]. The $g$-theorem tells us which conformal invariant boundary state is realized in low-energy limit among all conformally invariant boundary states.

Now we apply the $g$-theorem under an assumption that there are only two conformally invariant boundary sates; the Dirichlet and the Neumann boundary state. What we have to do is comparing their $g$-factors in a given TLL parameter $K$. Eq. (3.33) and Eq. (3.35) leads to the following results.

$$g_D < g_N \qquad \text{for} \quad K < \frac{1}{2} \tag{3.37}$$

$$g_D > g_N \qquad \text{for} \quad K > \frac{1}{2} \tag{3.38}$$

This indicates that the Dirichlet boundary condition is realized for $K < 1/2$ and the Neumann boundary condition is realized for $K > 1/2$. At $K = 1/2$, it is known that there are continuous family of "fixed line" boundary conditions [67]. We show in Fig. 3.4 that the renormalization group flow of the conformal invariant boundary condition. Actually, for the one-component free boson field theory, this is a complete picture of the renormalization of the boundary condition. This is related to the famous Kane-Fisher problem [69, 70], which is that a transport of a one-dimensional single-channel interacting electron through a single barrier. They found that electrons with

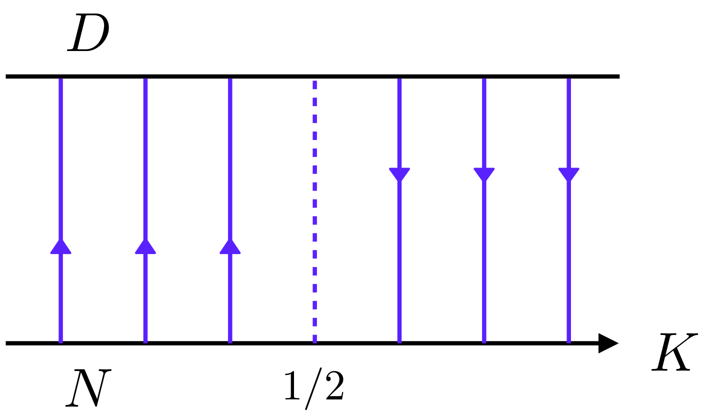


Figure 3.4: Renormalization group flow of the conformal invariant boundary condition of the one-component free boson field theory. $D$ and $N$ denote the Dirichlet and Neumann boundary condition, respectively. $K$ is the TLL parameter of the bulk. The dotted line represent the continuous family of "fixed line" boundary conditions

repulsive interactions are completely reflected by the barrier and electrons with attractive interactions perfectly transmit at zero temperature, while non-interacting electrons are partially transmitted and reflected depending on the strength of the barrier. Since the repulsive (attractive) interaction regime is $K < 1/2$ ($K > 1/2$), the complete reflection (transmission) of electrons are interpreted as the Dirichlet (Neumann) boundary condition in the bosonization context. More than one-component, there are other non-trivial conformal invariant boundary conditions and the renormalization diagram is more complicated [77, 78, 79].

# Chapter 4

# Replica manipulation of the TLL ground states of the XXZ spin chain

In this chapter, we investigate properties of the Rényified state, which is defined as Eq. (1.7), for the TLL ground state of the $S = 1/2$ XXZ spin chain. This is the main part of this thesis. To begin with, we show numerical results for spin correlations. The numerics indicates two interesting features of the Rényified state. First, the Rényified state behaves as a TLL with a different TLL parameter. Second, it shows a "transition" as the Rényi index $n$ is increased. After the transition, a strange behavior of the correlations is seen: long-range order of the longitudinal correlation with a power-law decay of the transverse correlation. To understand this peculiar behavior, we formulate the problem by replica field theory. This will explain why the Rényified state is described by a TLL with the modified TLL parameter $K/n$. A transition at large $n$ will also be explained in terms of perturbations. Finally, we map the problem to the boundary CFT by a folding trick. An explicit construction of a conformal invariant boundary state can explain the peculiar behavior of the transverse correlation function in the Rényified state. The boundary construction will also explain the behavior of the Rényi-Shannon entropy.

## 4.1 Numerics

We first show the numerical results on spin correlations in the Rényified state. We start from the TLL ground state of the $S = 1/2$ XXZ chain, which is defined as Eq. (2.15), for $J = -1$ and $-1 < \Delta \leq 1$. A TLL ground state of the model of the size $L$ is expressed as

$$|\Psi\rangle = \sum_{\{s_i\}} \psi_{\{s_i\}} |\{s_i\}\rangle, \tag{4.1}$$

where $\{s_i\}$ denotes the value of $S^z$ of a $S = 1/2$ spin at each cite. Namely, we choose $S^z$-basis for the expansion.

The numerical procedure is as follows. First, we obtain a TLL ground state with exact diagonalization for a parameter set $(\Delta, L)$. Now we have a set of the coefficients $\{s_i\}$. Next, we make a Rényified state

$$|\Psi^{(n)}\rangle = \frac{1}{\sqrt{Z^{(n)}}} \sum_{\{s_i\}} \psi^n_{\{s_i\}} |\{s_i\}\rangle,$$

which is defined as Eq. (1.7), by raising all the wave-function coefficients $\{\psi_{\{s_i\}}\}$ to the power of $n$. $Z^{(n)}$ is a normalization factor defined as Eq. (1.8). Then we measure physical quantities in the Rényified state with changing the parameter $n$. This is quite simple and straightforward process once we can diagonalize the original Hamiltonian and obtain the set of the coefficients $\{\psi_{\{s_i\}}\}$. We also note that we define $J$ is negative in the Hamiltonian defined as Eq. (2.15). With this convention for the $xy$ part of the interactions, the ground state can be chosen to have real positive coefficients $\{\psi_{\{s_i\}}\}$ with the Perron-Frobenius theorem.

### 4.1.1 Longitudinal correlation

For the XXZ model, the exponents of the algebraic decay of the longitudinal spin correlation (in $S^z$ basis) are related to the TLL parameter $K$ as Eq. (2.27). We reproduce it below as Eq. (4.2).

$$\langle S^z(0)S^z(r)\rangle = -\frac{K}{2\pi^2}\frac{1}{r^2} + C_1(-1)^r \frac{1}{r^{2K}} \tag{4.2}$$

We use Lanczos diagonalization of finite systems up to 32 sites and the results for $|\langle S^z(0)S^z(r)\rangle|$ of the Rényified state are displayed in Fig. 4.1 for $\Delta = -0.5, 0$ and $0.5$ ($K = 3/2, 1$ and $3/4$). In the small $n$ region, the ferromagnetic/uniform term and the antiferromagnetic/staggered term in Eq. (4.2) are comparable. Thus the absolute value of the correlations oscillates. For larger $n$, one of the terms becomes dominant and the data show a smooth power-law decay. In both cases, it is obvious that the correlation exponent changes with $n$. We can also see that the correlation seems to be long-ranged for large $n$.

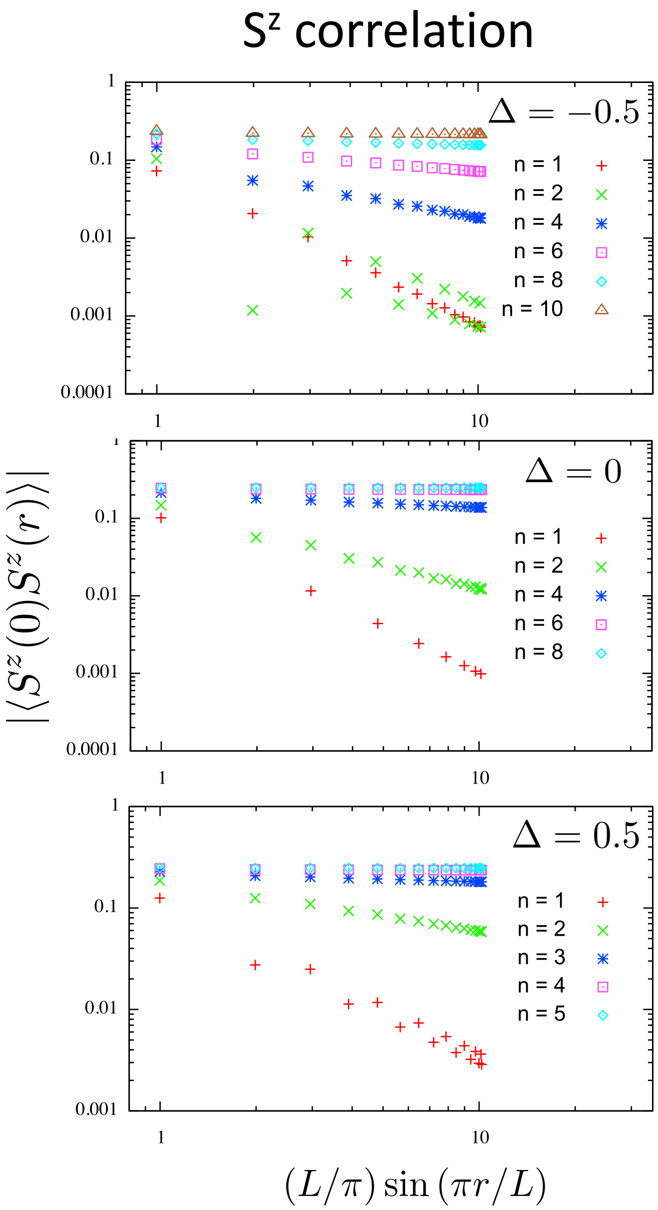


Figure 4.1: Absolute values of longitudinal correlation functions in the Rényified state for $\Delta = -0.5, 0$ and $0.5$ ($K = 3/2, 1$ and $3/4$) based on Lanczos diagonalization of finite systems ($L = 32$). The data are plotted in the cord distance $d(r) = (L/\pi)\sin(\pi r/L)$, which is the standard way to make the power law apparent in conformally invariant systems.

In order to see how the longitudinal correlation behaves clearly, we separate the staggered part from the uniform part of the correlation. We take the staggered part from a difference of the correlation as $\langle S^z(0)S^z(r)\rangle_{\text{stag}} \equiv (-G^z(r-1)+2G^z(r)-G^z(r+1))/4$ where $G^z(r) \equiv \langle S^z(0)S^z(r)\rangle$. The uniform part is defined as $\langle S^z(0)S^z(r)\rangle_{\text{unif}} \equiv G^z(r)-\langle S^z(0)S^z(r)\rangle_{\text{stag}}$. Since the correlation is expected to behave as $G^z(r) = ad(r)^{-\alpha}+b(-1)^r d(r)^{-\beta}$, where $d(r) = (L/\pi)\sin(\pi r/L)$ is the chord-distance across a periodic chain of length $L$, $\langle S^z(0)S^z(r)\rangle_{\text{unif}}$ and $\langle S^z(0)S^z(r)\rangle_{\text{stag}}$ are given as

$$\langle S^z(0)S^z(r)\rangle_{\text{unif}} = ad(r)^{-\alpha}+O(r^{-\alpha-2})+O(r^{-\beta-2}), \tag{4.3}$$

$$\langle S^z(0)S^z(r)\rangle_{\text{stag}} = b(-1)^r d(r)^{-\beta}+O(r^{-\alpha-2})+O(r^{-\beta-2}). \tag{4.4}$$

We show the uniform part and the staggered part of the longitudinal correlations in Fig. 4.2. In the insets, we also show the absolute value of each part in the logarithmic scale. We can immediately see that the antiferromagnetic long-range order is developed when $n$ is increased. For large $n$, the staggered part is dominant thus it is difficult to evaluate the uniform part quantitatively since the error $O(^{-\beta-2})$ is comparable to the term $ad(r)^{-\alpha}$ in Eq. (4.3).

We estimate the exponents from $\langle S^z(0)S^z(r)\rangle$ with the fitting, $ad(r)^{-\alpha}+b(-1)^r d(r)^{-\beta}$, where $a$, $\alpha$, $b$, and $\beta$ are free parameters. In Fig. 4.3, we show the exponents of the uniform (upper panel) and the staggered (lower panel) part of the longitudinal correlations in the Rényified state for $\Delta=-0.5, 0$ and $0.5$ ($K=3/2, 1$ and $3/4$). For low enough $n$, the exponents of the uniform part appears to be close to 2, and that of the staggered part is consistent with $2K/n$. For larger $n$ the data points of the uniform exponent are not shown since the fit quality is inferior. This is the case when the staggered part becomes dominant compared with a rapidly decaying uniform part. For the staggered part in lager $n$ region, the exponents deviate from the curve $2K/n$ and seem to converge 0.

Comparing these results with Eq. (4.2), we conjecture that the Rényified state has a modified TLL parameter $\tilde{K}=K/n$ for low enough $n$. A change of the behavior is observed in lager $n$ region: the longitudinal correlation becomes long-range ordered. We also refer to a relevant previous study. In Ref. [56], this value $\tilde{K}$ has been obtained from the “Gaussian trick”, although this had not been checked on the correlation functions.

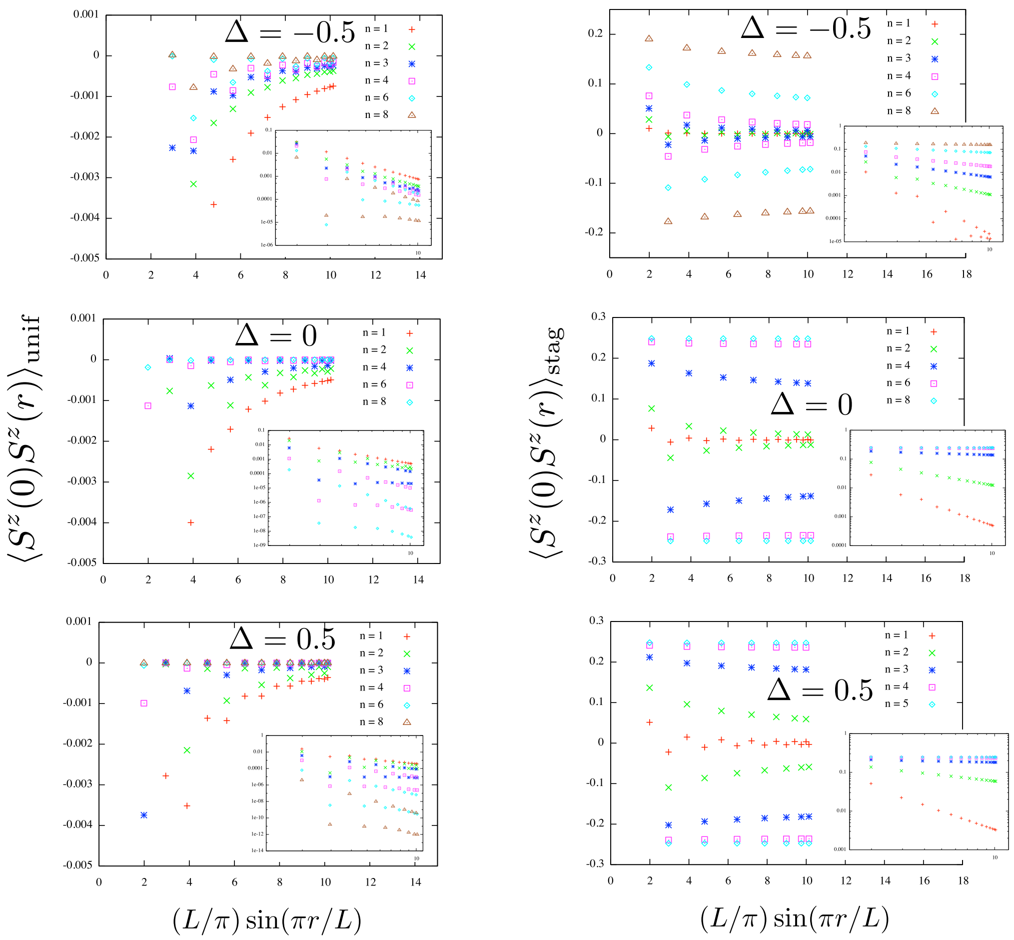


Figure 4.2: Uniform part and staggered part of the longitudinal correlations in the Rényified state for $\Delta = -0.5, 0$ and $0.5$ ($K = 3/2, 1$ and $3/4$). The data are plotted in the cord distance $d(r) = (L/\pi)\sin(\pi r/L)$. In the insets, we show the absolute value of each part in the logarithmic scale.

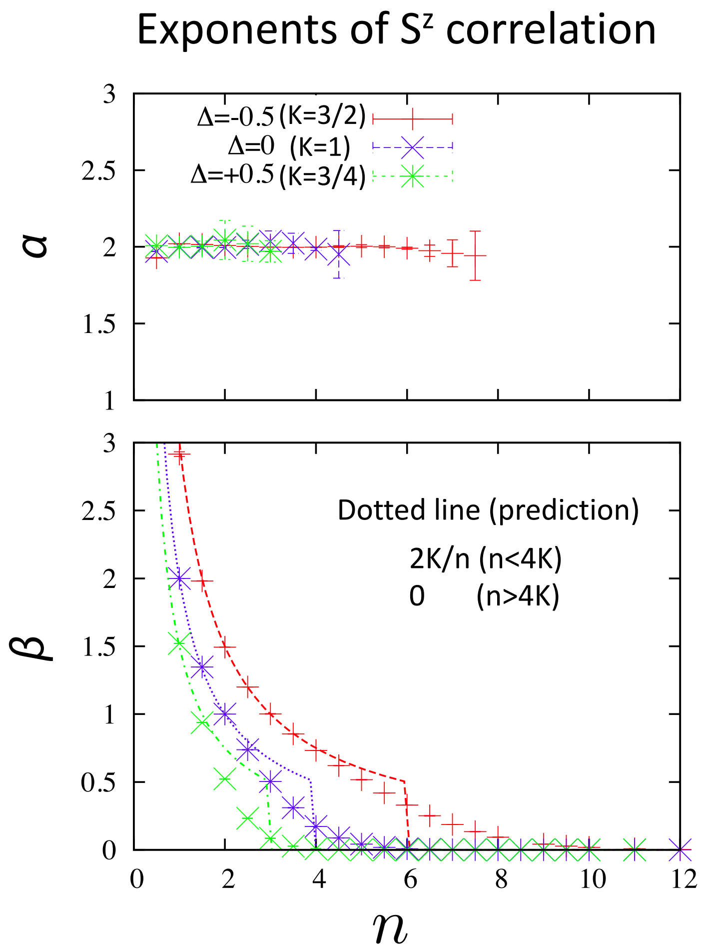


Figure 4.3: $n$ dependence of the exponent of the uniform (upper panel) and the staggered (lower panel) part of the longitudinal correlation in the Rényified state for $\Delta = -0.5, 0$ and $0.5$ ($K = 3/2, 1$ and $3/4$). Exponents are extracted with the fitting, $ad(r)^{-\alpha} + b(-1)^r d(r)^{-\beta}$. Dotted lines: field theoretical predictions obtained in the following sections.

### 4.1.2 Transverse correlation

The transverses spin correlation (in $S^z$ basis) for the XXZ model is described as Eq. (2.28). We reproduce it below as Eq. (4.5).

$$\langle S^x(0)S^x(r)\rangle = C_2\frac{1}{r^{1/2K}} + C_3(-1)^r\frac{1}{r^{2K+1/2K}} \tag{4.5}$$

We note that here the prefactor $J$ is defined to be negative in the Hamiltonian Eq. (2.15). For $J > 0$, the uniform part and the staggered part of the transverse correlation Eq. (4.5) will be exchanged.

We show the numerical results of the transverse correlations for the Rényified state in Fig. 4.4 by Lanczos diagonalization of finite systems up to 32 sites. It is easily seen that the correlation exponents change with $n$. However, contrary to the longitudinal correlation, *the transverse correlation remains algebraic, even at large $n$*. This result contrasts with the naive scenario where the Rényified state for large $n$ would be in a gapped antiferromagnetic phase, analog to that of the ground state of XXZ chain for $\Delta > 1$.

Then we separate the staggered part from the uniform part of the transverse correlation. We take the staggered part from a difference of the correlation as $\langle S^x(0)S^x(r)\rangle_{\mathrm{stag}} \equiv \left(-G^x(r-1) + 2G^x(r) - G^x(r+1)\right)/4$ where $G^x(r) \equiv \langle S^x(0)S^x(r)\rangle$. The uniform part is defined as $\langle S^x(0)S^x(r)\rangle_{\mathrm{unif}} \equiv G^x(r) - \langle S^x(0)S^x(r)\rangle_{\mathrm{stag}}$. Since the correlation is expected to behave as $G^x(r) = ad(r)^{-\alpha} + b(-1)^r d(r)^{-\beta}$, the uniform part and the staggered part of the correlation are given as

$$\langle S^x(0)S^x(r)\rangle_{\mathrm{unif}} = ad(r)^{-\alpha} + O(r^{-\alpha-2}) + O(r^{-\beta-2}), \tag{4.6}$$
$$\langle S^x(0)S^x(r)\rangle_{\mathrm{stag}} = b(-1)^r d(r)^{-\beta} + O(r^{-\alpha-2}) + O(r^{-\beta-2}). \tag{4.7}$$

We show the uniform part and the staggered part of the transverse correlations in Fig. 4.5. We also show the absolute value of each part in the logarithmic scale in the insets. We can see that the transverse correlations decay algebraically even for larger $n$. We can also see that the uniform part is dominant thus it is difficult to evaluate the staggered part quantitatively since the error $O(^{-\alpha-2})$ is rather significant for the term $b(-1)^r d(r)^{-\beta}$ in Eq. (4.7).

In Fig. 4.6, we show the exponents of the uniform (upper panel) and the staggered (lower panel) part of the transverse correlations in the Rényified state. They are extracted using the fitting, $ad(r)^{-\alpha}$ $+b(-1)^r d(r)^{-\beta}$, as done above for the longitudinal correlations. We can see that the exponents of the uniform part obey the prediction $n/2K$, which is guided by dotted curves. The behavior of the staggered part is more tricky: its exponent is compatible

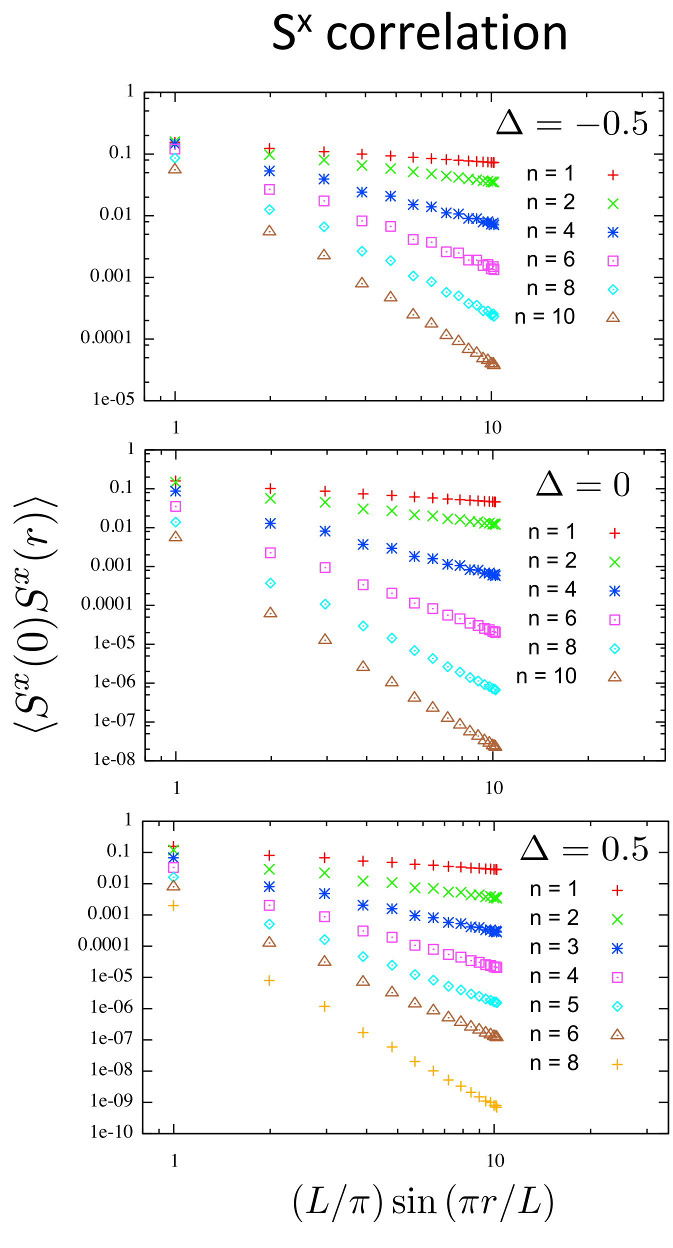


Figure 4.4: Transverse correlation functions in the Rényified state for $\Delta = -0.5, 0$ and $0.5$ ($K = 3/2, 1$ and $3/4$) based on Lanczos diagonalization of finite systems ($L = 32$). The data are plotted in the cord distance $d(r) = (L/\pi)\sin(\pi r/L)$, which is the standard way to make the power law apparent in conformally invariant systems.

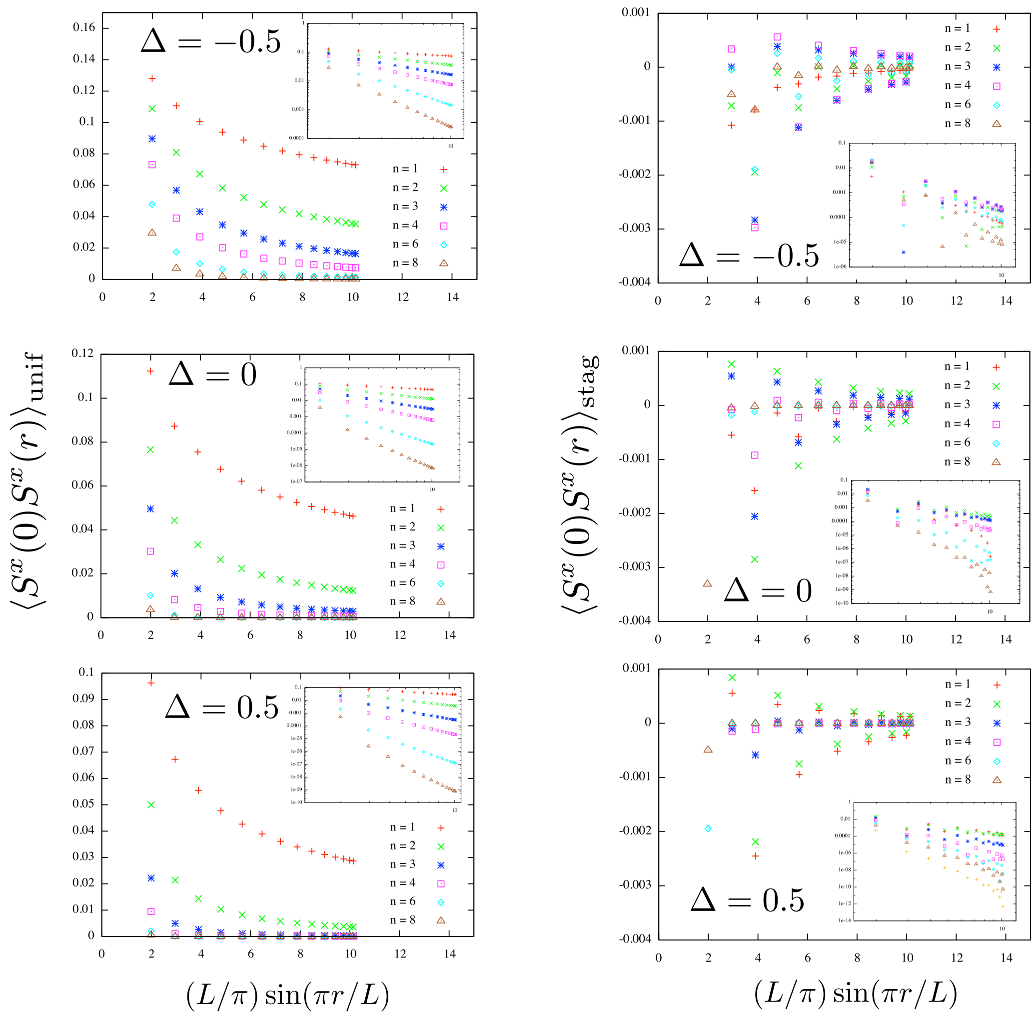


Figure 4.5: Uniform part and staggered part of the transverse correlations in the Rényified state for $\Delta = -0.5, 0$ and $0.5$ ($K = 3/2, 1$ and $3/4$). The data are plotted in the cord distance $d(r) = (L/\pi)\sin(\pi r/L)$. In the insets, we show the absolute value of each part in the logarithmic scale.

with $n/2K + 2K/n$ in small $n$ region, but deviates from it in large $n$ region (we plot its exponents with the theoretical lines which are derived in the following sections). We note that finite-size effects are significant to see its behavior clearly in the numerical calculations.

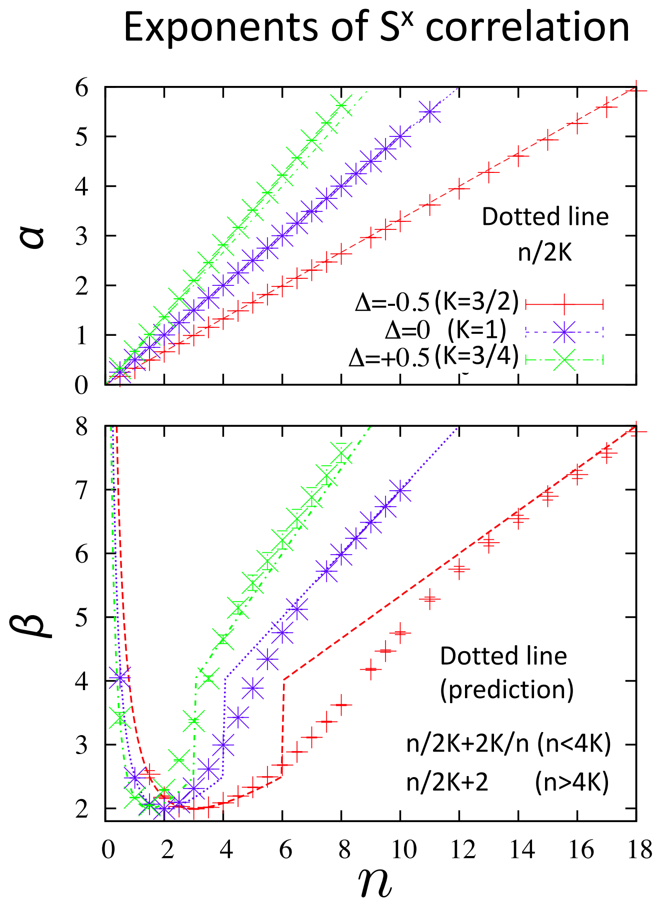


Figure 4.6: $n$ dependence of the exponent of the uniform (upper panel) and the staggered (lower panel) part of the transverse correlation in the Rényified state for $\Delta = -0.5, 0$ and $0.5$ ($K = 3/2, 1$ and $3/4$). Exponents are extracted with the fitting, $ad(r)^{-\alpha} + b(-1)^r d(r)^{-\beta}$. Dotted lines: field theoretical predictions obtained in the following sections.

By comparing with Eq. (4.5), the numerical results indicate that the Rényified state has the modified TLL parameter $\tilde{K} = K/n$ for low enough $n$. For larger $n$, the TLL-like behavior is found in the uniform part while the staggered exponents are no longer that of Eq. (4.5).

We summarize an observation of the longitudinal and transverse correlation function in numerics as follows.

- In the small $n$ region, the Rényified state appears to be a TLL with modified TLL parameter $\tilde{K} = K/n$.
- For larger $n$, it appears to be in a different universality class, which is somewhat unusual: the longitudinal correlation has long-range order while the transverse one decays algebraically.

The numerical results suggest that an existence of a phase transition in the Rényified state at a certain $n$. To explain this more quantitatively, we will formulate the problem by replica field theory in the following section.

## 4.2 Replica field theory formulation

In the previous section, we have seen that the numerical results have indicated that the Rényified ground state behaves as a TLL with a renormalized TLL parameter for low enough $n$. On the other hand, for large enough $n$, it appears to be in a different universality class. In this section, we derive this result by using free-field description of TLL with a replica formulation. The formulation reveals that the behavior of the Rényified ground state could be described by its "center of mass" field with modified TLL parameter. This also leads us to boundary field theory formulation.

### 4.2.1 Longitudinal correlation

Let us start to discuss the longitudinal correlation function $\langle S^z(0)S^z(r)\rangle$ in the Rényified state. We start from a TLL ground state $|\Psi\rangle = \sum_i \psi_i |i\rangle$ (in $S^z$-basis) in a system of the size $L$ in space direction with a periodic boundary condition. Since we are dealing with the diagonal operators with respect to the basis states $|i\rangle$, the longitudinal correlation in the Rényified state Eq. (1.7) is given as

$$\langle \Psi^{(n)}|S^z(0)S^z(r)|\Psi^{(n)}\rangle = \frac{1}{Z^{(n)}}\sum_i \psi_i^{2n}\langle i|S^z(0)S^z(r)|i\rangle. \tag{4.8}$$

For future use, we adopt the $1+1$ dimensional point of view by introducing a transfer matrix $\mathcal{T} = e^{-\mathcal{H}}$, which connects spin configurations on neighboring "rings" in $\tau$ direction. Now we suppose that $\mathcal{H}$ is a Hamiltonian which has a TLL ground state. In terms of the transfer matrix $\mathcal{T}$, a square of a ground state coefficient is expressed as

$$|\psi_i|^2 = |\langle i|\Psi\rangle|^2 = \lim_{\beta\to\infty}\frac{\langle A|\mathcal{T}^\beta|i\rangle\langle i|\mathcal{T}^\beta|B\rangle}{\langle A|\mathcal{T}^{2\beta}|B\rangle}, \tag{4.9}$$

where $|A\rangle$ and $|B\rangle$ are boundary states which are not orthogonal to the ground state. We denote the numerator and the denominator in the right-hand side of Eq. (4.9) as $Z_i = \langle A|\mathcal{T}^\beta|i\rangle\langle i|\mathcal{T}^\beta|B\rangle$ and $Z = \langle A|\mathcal{T}^{2\beta}|B\rangle$, respectively. $Z_i$ and $Z$ are interpreted as partition functions on a cylinder with and without restriction of configurations at a specific imaginary time (say $\tau = 0$) corresponding to an actual system, as depicted in Fig. 4.7. In the continuum limit, a spin configuration $|i\rangle$ in $S^z$-basis is a configuration of a bosonic field $\phi$. Since we are dealing with a TLL ground state, the partition functions are described as

$$Z_i = \int \mathcal{D}\phi\Big|_{\phi(\tau=0)=\phi_i} e^{-S(\phi)}, \quad Z = \int \mathcal{D}\phi\, e^{-S(\phi)}, \tag{4.10}$$

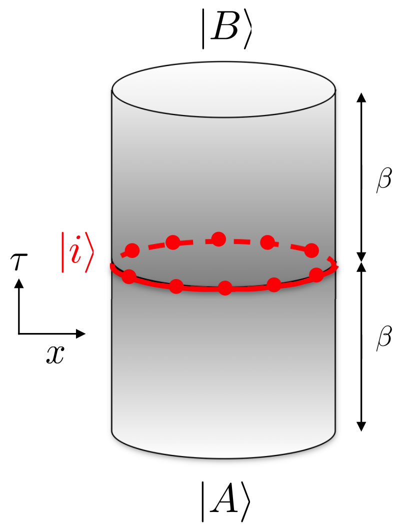


Figure 4.7: $1+1$ dimensional point of view of a square of a ground state coefficient $|\psi_i|^2$. A red ring represents a one-dimensional quantum system with a periodic boundary condition in space direction. For the numerator of Eq. (4.9), $Z_i = \langle A|\mathcal{T}^\beta|i\rangle\langle i|\mathcal{T}^\beta|B\rangle$, a configuration of the system (red ring) is fixed at a particular configuration $|i\rangle$.

where $S(\phi)$ is the free bosonic filed action defined as Eq. (2.9).

Now we express $|\psi_i|^{2n}$ by field theoretical description. For integer $n$, this can be identified with $n$ replicas of TLL systems on the infinite cylinder (in $\tau$ direction). All replicas are bound at $\tau = 0$ as depicted in Fig. 4.8. This means that the bosonic field $\phi$ on each replica has the configuration at $\tau = 0$ corresponding to $|i\rangle$. Hereafter, we represent $\phi(r)$ as $\phi(r, \tau = 0)$. Let us denote the bosonic field on each replica as $\phi_\alpha$. All the replicas having the same configuration which implies that

$$\phi_1 = \phi_2 = \cdots = \phi_n \tag{4.11}$$

at $\tau = 0$. We introduce the new fields as

$$\Phi_\alpha = \sum_{j=1}^{n} u_j^\alpha \phi_j \tag{4.12}$$

with

$$u_j^0 = \frac{1}{\sqrt{n}}, \tag{4.13}$$

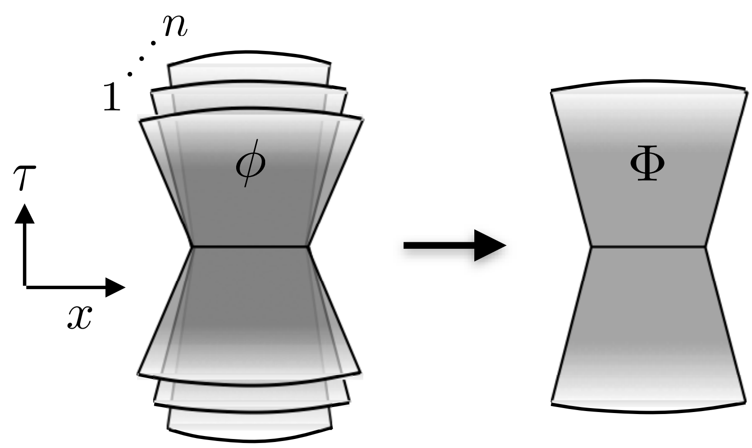


Figure 4.8: Replica picture of the Rényified state. For all replicas, periodic boundary condition is imposed in the $x$ direction and they are glued at the line $\tau = 0$. By changing the basis from $\vec{\phi}$ to $\vec{\Phi}$, the problem is effectively described by the single-component free boson field theory with the modified TLL parameter, $K/n$.

and

$$\sum_{j=1}^{n-1} u_j^{\alpha} u_j^{\alpha'} = \delta_{\alpha,\alpha'} \tag{4.14}$$

$$\sum_{\alpha=0}^{n-1} u_j^{\alpha} u_{j'}^{\alpha} = \delta_{j,j'}. \tag{4.15}$$

Thus the fields $\{\Phi_\alpha\}$ are defined so that they are mutually orthogonal. This orthogonality leads to the "relative coordinate fields" $\Phi_{\alpha\neq 0}$ are fixed as $\Phi_{\alpha\neq 0} = 0$ at $\tau = 0$. In the definition Eq. (4.12) and Eq. (4.13), the "center of mass" field $\Phi_0$ does not vanish at $\tau = 0$. Thus, the problem is reduced to the field theory of the free boson $\Phi_0$ on the infinite cylinder.

The staggered part of the longitudinal spin operator $S^z$ of a TLL is given by $\cos(\phi/R)$, as in Eq. (2.25). In the replica field theory for the Rényified state, thanks to the boundary condition Eq. (4.11), we can measure the longitudinal correlation function on any replica. This should give us the same result for any $\phi_i$ since all the fields have the same configuration. Let us take the first replica $\cos(\phi_1/R)$. We can express the original replica field

$\phi_1$ as a linear combination of the new fields as

$$\phi_1 = \frac{1}{\sqrt{n}}\Phi_0 + \sum_{\alpha=1}^{n-1} u_1^\alpha \Phi_\alpha. \tag{4.16}$$

The relative coordinate fields $\Phi_{\alpha\neq 0}$ are subject to the condition $\Phi_\alpha = 0$ at the line $\tau = 0$. Thus all the terms except for the center of mass field $\Phi_0$ do not have any contributions in the calculation of the longitudinal correlation functions.

Thus the staggered part of the longitudinal correlation function is given by the correlation function of the center of mass filed as

$$\left\langle \cos\left(\frac{\Phi_0(0)}{\sqrt{n}R}\right) \cos\left(\frac{\Phi_0(r)}{\sqrt{n}R}\right)\right\rangle. \tag{4.17}$$

Since the scaling dimension of $\cos(\Phi_0/\sqrt{n}R)$ is $1/2ngR^2 = K/n$, (the staggered part of) the correlation function is given by

$$\langle \Psi^{(n)}|S^z(0)S^z(r)|\Psi^{(n)}\rangle_{\text{stag.}} \sim \frac{1}{r^{2K/n}}. \tag{4.18}$$

By comparing this with Eq. (2.27), we see that the TLL parameter in the Rényified state is modified as $K \to K/n$. The exponent of the uniform part of the longitudinal correlation would not be affected by $n$. This explains the behavior of the longitudinal correlation in the numerics for enough small $n$.

### 4.2.2 Transverse correlation

Now let us consider the transverse correlation function $\langle S^x(0)S^x(r)\rangle$ in the Rényified state Eq. (1.7):

$$\langle \Psi^{(n)}|S^x(0)S^x(r)|\Psi^{(n)}\rangle = \frac{1}{Z^{(n)}} \sum_{i,j} \psi_i^n \psi_j^n \langle i|S^x(0)S^x(r)|j\rangle. \tag{4.19}$$

Although we may still introduce replica fields, we need a different construction from longitudinal one since the $S^x$ operator is not diagonal to the basis states. Let us suppose that the replica fields $\phi_1, \cdots, \phi_n$ are defined on the half-infinite cylinder $\tau < 0$. They are all fixed to a configuration at $\tau = -0$ corresponding to $|i\rangle$. This implies that

$$\phi_1 = \phi_2 = \cdots = \phi_n, \tag{4.20}$$

at $\tau = -0$. We can then regard the other replica fields $\phi_{n+1}, \cdots, \phi_{2n}$ as defined on the half-infinite cylinder $\tau > 0$. Again they obey

$$\phi_{n+1} = \phi_{n+2} = \cdots = \phi_{2n}, \tag{4.21}$$

at $\tau = +0$ corresponding to a configuration $|j\rangle$. Then the new fields can be defined for $\tau < 0$ ($\tau > 0$ ) by the linear combination of $\phi_1, \cdots, \phi_n$ ($\phi_{n+1}, \cdots, \phi_{2n}$) as Eq.(4.12)- (4.15). The relative coordinate fields are subject to be zero at $\tau = 0$. Each center of mass field, however, remains free at $\tau = 0$. Because of the fact that $|i\rangle$ and $|j\rangle$ are almost identical, we require each center of mass fields to be "continuous" at $\tau = 0$. Actually, they should be connected at $\tau = 0$ to have a finite contribution in Eq. (4.19).

Thus we find that the problem is again effectively described by the single-component free boson field theory, $\Phi_0$, defined on the infinite cylinder. Measurements of the correlation functions are done at the intersection $\tau = 0$. Since $S^{\pm}$ operator induces a spin-flip on the basis state $|i\rangle$, which affects all the replicas at the same time, we suppose that the measured spin raising (lowering) operator $S^{+}$ ($S^{-}$) is a product of $S^{+}$ ($S^{-}$) on all the $n$ replicas. Since the $S^{\pm}$ operator of the single-component TLL is given as Eq. (2.26), the uniform part of $S^{\pm}$ operator in the Rényified state would be given by

$$S^{\pm} \sim e^{\pm iR\sum_{\alpha}\theta_{\alpha}} = e^{\pm i\sqrt{n}R\Theta_0}, \tag{4.22}$$

where $\Theta_0$ is the center of mass field for the dual field $\theta$ defined as Eq. (4.12). Thus the uniform part of the transverse correlation function is given by the correlation function

$$\left\langle \cos\left(\sqrt{n}R\Theta_0(0)\right)\cos\left(\sqrt{n}R\Theta_0(r)\right)\right\rangle. \tag{4.23}$$

Since the scaling dimension of $\cos\left(\sqrt{n}R\Theta_0\right)$ is $ngR^2/2 = n/2K$, (the uniform part of) the transverse correlation is given by

$$\langle\Psi^{(n)}|S^x(0)S^x(r)|\Psi^{(n)}\rangle_{\text{unif.}} \sim \frac{1}{r^{n/2K}}. \tag{4.24}$$

We again see that the TLL parameter in the Rényified state is modified as $K \to K/n$ by comparing this with Eq. (2.28). The staggered part of the transverse correlation would be given by

$$\left\langle \cos\left(\sqrt{n}R\Theta_0(0)\right)\cos\left(\sqrt{n}R\Theta_0(r)\right)\cos\left(\frac{\Phi_0(0)}{\sqrt{n}R}\right)\cos\left(\frac{\Phi_0(r)}{\sqrt{n}R}\right)\right\rangle. \tag{4.25}$$

by applying the result in the previous subsection. This indicates the exponent of the staggered part is $n/2K + 2K/n$. This is consistent with the numerical results for small $n$ in the previous section.

We have investigated the correlations of the Rényified state based on the replica field formulation. This explains the behavior of the correlations in numerics for small $n$. So far, we have neglected an effect of perturbations which should be taken into account in a lattice system. In the next section, we consider the effect and see a "transition" of the Rényified state.

### 4.2.3 Effect of relevant perturbations

The replica formulation introduced above indicates that the system can be effectively described by a single-component free boson field $\Phi_0$ and its dual $\Theta_0$ defined on the infinite cylinder. The effective compactification radius of $\Phi_0$ is

$$\tilde{R} = \sqrt{n}R, \tag{4.26}$$

and that of $\Theta_0$ is $1/\tilde{R}$. We can analyze possible perturbations in this formulation. Returning to the original $n$-replica picture, the correlations in the bulk on each replica are unaffected by the number of replicas because replicas are independent of each other in the bulk. Close to the line $\tau = 0$, however, the fluctuations of the scalar fields are no longer that of single sheet if $n \neq 1$. Thus the correlation at $\tau = 0$ will depend on the number of replicas. This implies that the perturbations are renormalized to zero at long distance in the bulk while it can remain at the line $\tau = 0$.

In general, we should expect all the perturbations which are allowed by the symmetry. Since all the vertex operators involving $\Theta_0$ would be forbidden by the U(1) symmetry (rotation about $z$ axis), the only possible perturbations are those of $\Phi_0$. Due to the compactification $\Phi_0 \sim \Phi_0 + 2\pi\sqrt{n}R$, the possible vertex operators are given as $e^{\pm im\Phi_0/\sqrt{n}R}$, where $m$ is an integer. The most relevant perturbation would be $e^{\pm i\Phi_0/\sqrt{n}R}$, which is presumably forbidden by the lattice translation symmetry. Thus the leading boundary perturbation allowed by the symmetries should be $e^{\pm 2i\Phi_0/\sqrt{n}R}$. Its scaling dimension is given as $2/ngR^2 = 4K/n$. Since the perturbation is expected only on the line $\tau = 0$, it is relevant if the scaling dimension is less than 1. Thus the leading perturbation is relevant if

$$n > n_c = 4K. \tag{4.27}$$

We refer to a relevant previous study. In Ref. [94], the longitudinal correlations for $\Delta = 0$ at $n = 2$ and 3 were computed numerically. However, the slow algebraic decay of the staggered part was incorrectly interpreted as long-range order, and the transition point was incorrectly located at $\tilde{n}_c = 2K$ presumably because of some confusion between the bulk and boundary transitions.

When $n > n_c$, the boson field $\Phi_0$ would be pinned to the minima of the potential. In this case, the longitudinal correlation function would have a long-range order. On the other hand, numerical results indicate that the transverse correlation function decays algebraically. Although this behavior seems to be quite unusual, we can understand it as follows.

If the Rényified system were described by a standard TLL, the long-range order in the longitudinal spin component implies a gapped (say, Néel) phase, in which the transverse correlation functions are short-ranged. However, the present problem is mapped to a boundary problem by the "folding trick", as we will see in the following section. In a conformal field theory with perturbations at the boundary, the boundary condition can be renormalized from a conformally invariant one to another conformally invariant one. However, boundary perturbations never opens a mass gap in the bulk. Thus, even if the boundary perturbation becomes relevant, it does not imply that the transverse correlation function falls off exponentially. In fact, it can be shown that the transverse correlation function still decays with the power-law by constructing conformal invariant boundary states explicitly. To make the discussion precise, we need to analyze the problem in terms of boundary conformal field theory.

## 4.3 Boudnary CFT Formalism

In the previous section, we have derived the Rényified state shows TLL behavior with the modified TLL parameter by replica field formulation. By taking into account perturbations, boundary phase transition occurs and the Rényified state is no longer a TLL state. The question is what the Rényified state is after the transition and how its physical quantities behave. In this section, we formulate the problem in boundary conformal field theory language by folding trick. It will make easier to treat with the subtle boundary condition more rigorously. It explains the tricky behavior of the correlation functions, that is, diagonal correlation shows long-range order while transverse correlation shows power-law decay.

### 4.3.1 Folding trick

By using replica formulation, we can effectively describe the Rényified system by a single-component free boson field on the infinite cylinder with perturbations at $\tau = 0$. The bosonic field $\Phi_0$ has effective compactification radius $\tilde{R} = \sqrt{n}R$ derived as Eq. (4.26). Dual field $\Theta_0$ has effective compactification radius $1/\tilde{R}$. For a precise formulation, we first introduce a finite size system of the size $L \times \beta$, then take a limit $\beta \to \infty$.

Let us start from the effective center of mass field theory $\Phi_0$, which corresponds to the right situation in Fig. 4.8. Now we fold the system at $\tau = 0$. Folding reduces the torus to a doubled cylinder of the size $L \times \beta/2$ with boundaries at both ends in the $\tau$-direction (see Fig. 4.9). Thus the problem is mapped to two-component free boson field theory on the cylinder with boundaries at $\tau = 0$ and $\tau = \beta/2$. Since the system is the cylinder which extends in the time direction, the boundary conditions at $\tau = 0$ and $\tau = \beta/2$ can be regarded as the initial state and final state, respectively. These are nothing but boundary states, and lead us a systematic study of boundary conditions.

We define the new fields after the folding as

$$\Phi_0^{(1)}(\tau) = \Phi_0(\tau) \tag{4.28}$$
$$\Phi_0^{(2)}(\tau) = \Phi_0(-\tau) \tag{4.29}$$

for $\forall\tau \geq 0$ and likewise for $\Theta^{(1),(2)}$. The bosonic fields $\vec{\Phi}_0 = (\Phi_0^{(1)}, \Phi_0^{(2)})^T$ and $\vec{\Theta}_0 = (\Theta_0^{(1)}, \Theta_0^{(2)})^T$ obey the campactification

$$\vec{\Phi}_0 = \vec{\Phi}_0 + 2\pi\vec{R} \tag{4.30}$$
$$\vec{\Theta}_0 = \vec{\Theta}_0 + 2\pi\vec{K} \tag{4.31}$$

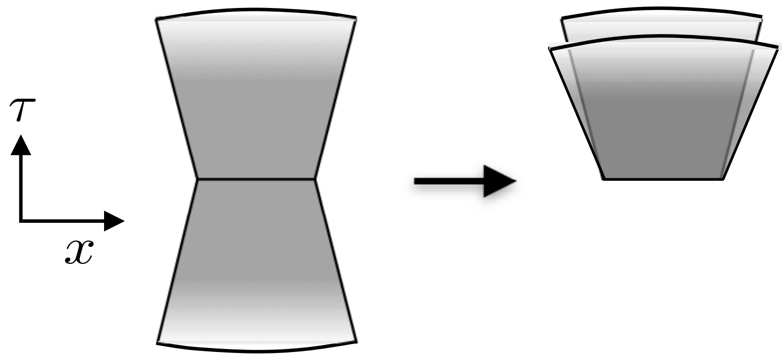


Figure 4.9: Folding of the system of the circumference $L$ in the $x$ direction and the length $\beta$ in the $\tau$ direction. Periodic boundary condition is imposed in both directions. Folding reduces the system to the doubled cylinder of the size $L \times \beta/2$ with boundaries at both ends in the $\tau$-direction.

where $\vec{R} = (N_1\tilde{R}, N_2\tilde{R})^T$ and $\vec{K} = (M_1/\tilde{R}, M_2/\tilde{R})^T$ with $N_1, N_2, M_1, M_2 \in \mathbb{Z}$. Since the original fields $\Phi_0$ and $\Theta_0$ are smooth at $\tau = 0$ before the folding, they have a constraint

$$\Phi_0^{(1)}(\tau = 0) = \Phi_0^{(2)}(\tau = 0) \tag{4.32}$$

$$\Theta_0^{(1)}(\tau = 0) = \Theta_0^{(2)}(\tau = 0). \tag{4.33}$$

Furthermore, let us introduce the new basis $\vec{\tilde{\Phi}} \equiv (\tilde{\Phi}_0, \tilde{\Phi}_1)^T$ by

$$\tilde{\Phi}_0 = \frac{1}{\sqrt{2}}\left(\Phi_0^{(1)} + \Phi_0^{(2)}\right) \tag{4.34}$$

$$\tilde{\Phi}_1 = \frac{1}{\sqrt{2}}\left(\Phi_0^{(1)} - \Phi_0^{(2)}\right) \tag{4.35}$$

and likewise for $\vec{\tilde{\Theta}} \equiv (\tilde{\Theta}_0, \tilde{\Theta}_1)^T$. Due to the boundary condition Eq. (4.32) and Eq. (4.33), $\tilde{\Phi}_0$ and $\tilde{\Theta}_0$ obey the Neumann boundary condition and $\tilde{\Phi}_1$ and $\tilde{\Theta}_1$ obey the Dirichlet boundary condition at $\tau = 0$. Due to the basis transformation, the new fields cannot have the winding numbers independently. In fact, the new fields $\vec{\tilde{\Phi}} = (\tilde{\Phi}_0, \tilde{\Phi}_1)^T$ obey the compactification as

$$\vec{\tilde{R}} = \frac{\tilde{R}}{\sqrt{2}}\begin{pmatrix} n_0 \\ n_1 \end{pmatrix} \tag{4.36}$$

where

$$n_0 \equiv n_1 \bmod 2. \tag{4.37}$$

Likewise, the compactification for $\vec{\tilde{\Theta}} = (\tilde{\Theta}_0, \tilde{\Theta}_1)^T$ is given as

$$\vec{\tilde{K}} = \frac{1}{\sqrt{2}\tilde{R}} \begin{pmatrix} m_0 \\ m_1 \end{pmatrix} \tag{4.38}$$

where

$$m_0 \equiv m_1 \bmod 2. \tag{4.39}$$

We have to keep in mind these "glueing condition" when we construct their boundary states.

### 4.3.2 Boundary state before the transition

In the absence of any boundary perturbation, the system before folding is just a one-component free boson field theory without any defect. After the folding, this corresponds to a particular boundary condition, in which the Neumann boundary condition on $\tilde{\Phi}_0$ and $\tilde{\Theta}_0$ and the Dirichlet boundary condition on $\tilde{\Phi}_1$ and $\tilde{\Theta}_1$ [80]. Following the construction formalism of boundary states for free bosons introduced in the Chapter 3, the corresponding boundary state is given as

$$|P\rangle = g_P \sum_{n_0, m_0 \in \text{even}} |(n_0, n_1 = 0, m_0, m_1 = 0)\rangle\rangle, \tag{4.40}$$

where $|(\vec{\tilde{R}}, \vec{\tilde{K}})\rangle\rangle \equiv |(n_0, n_1, m_0, m_1)\rangle\rangle$ is the bosonic Ishibashi state defined as Eq. (3.30) [73], which is characterized by the winding numbers of the bosonic fields defined as Eq. (4.36) and Eq. (4.38). The restriction on the sum comes from the gluing conditions Eq. (4.37) and Eq. (4.39). The $g$-factor $g_p$ is determined from the Cardy's consistency condition.

By using the Hamiltonian Eq. (3.23), an amplitude between two $|P\rangle$'s is given as

$$\begin{aligned} Z_{PP} &= \langle P| e^{-\frac{\beta}{2}H} |P\rangle \\ &= g_p^2 \left(\frac{1}{\eta(q)}\right)^2 \sum_{n_0, m_0 \in \text{even}} q^{\frac{1}{2}\cdot\frac{1}{2}\left[g\left(\frac{\tilde{R}}{\sqrt{2}}n_0\right)^2 + \frac{1}{g}\left(\frac{m_0}{\sqrt{2}\tilde{R}}\right)^2\right]} \\ &= g_p^2 \left(\frac{1}{\eta(q)}\right)^2 \sum_{\tilde{n}_0, \tilde{m}_0 \in \mathbb{Z}} q^{\frac{1}{2}\left(g\tilde{R}^2\tilde{n}_0^2 + \frac{1}{g\tilde{R}^2}\tilde{m}_0^2\right)}, \end{aligned} \tag{4.41}$$

where $q = e^{2\pi i\gamma} = e^{-2\pi\beta/L}$ where $\gamma = i\beta/L$. $\eta(q)$ is the Dedekind eta function defined as

$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{4.42}$$

The modular $S$ transformation, $\gamma \to -1/\gamma$, exchanges the roles of space and imaginary time. Then the system is periodic in the $\tau$-direction and has the edges in the space direction with boundary conditions. In this picture, the Hamiltonian depends on the boundary conditions at both ends. By performing the modular transformation to Eq. (4.41), we obtain

$$Z_{PP} = g_p^2\left(\frac{1}{\eta(\tilde{q})}\right)^2 \sum_{\tilde{n}_0,\tilde{m}_0\in\mathbb{Z}} \tilde{q}^{\frac{1}{2}\left(\frac{1}{g\tilde{R}^2}\tilde{n}_0^2+g\tilde{R}^2\tilde{m}_0^2\right)}, \tag{4.43}$$

where $\tilde{q} = e^{-2\pi i/\gamma} = e^{-2\pi L/\beta}$. The scaling dimensions of the boundary operators for $P$ boundary condition can be read off from the partition function Eq. (4.43). The smallest scaling dimensions except for the identity operator are $1/2g\tilde{R}^2$ and $g\tilde{R}^2/2$. This implies that the corresponding boundary operators are $\exp(i\tilde{\Phi}_0/\sqrt{2}\tilde{R})$ and $\exp(\pm i\tilde{R}\tilde{\Theta}_0/\sqrt{2})$ since the scaling dimensions of the boundary operators are doubled from the bulk operators. Actually, these are the staggered part of $S^z$ operator and the uniform part of $S^\pm$ operator, respectively. Thus the correlation functions give the power-law decay with the exponent $1/g\tilde{R}^2 = 2K/n$ for the staggered part of the $S^z$ correlation and $g\tilde{R}^2 = n/2K$ for the uniform part of the $S^x$ correlation. This is exactly expected in the picture before the folding.

For the uniform part in the longitudinal correlation, the term $\partial_x\Phi_0$, which gives the exponent 2, obviously does not depend on $n$. In comparison, the staggered part of the transverse correlation is given by the product of operators $\exp(i\tilde{\Phi}_0/\sqrt{2}\tilde{R})\exp(i\tilde{R}\tilde{\Theta}_0/\sqrt{2})$, thus we see the power-law decay with the exponent $2K/n + n/2K$.

We note that Cardy's consistency condition gives

$$g_p = 1. \tag{4.44}$$

The logarithm of the $g$-factor represents the "boundary entropy" [68]. Equation (4.44) means that there is no boundary entropy for $P$ boundary condition. This is again expected since there was no defect before the folding.

### 4.3.3 Boundary state after the transition

Now, let us introduce the relevant perturbation at the boundary. As discussed in the previous section, the leading boundary perturbation allowed by symmetry would be $\cos(2\tilde{\Phi}_0/\sqrt{2}\tilde{R})$. When this boundary perturbation becomes relevant, it is expected that the cosine potential grows to infinity in the low energy limit, and the problem will be described by the Dirichlet boundary condition on $\tilde{\Phi}_0$ at $\tau = 0$. The Dirichlet boundary state with the boundary value $\vec{\tilde{\Phi}} = (\tilde{\Phi}_0, \tilde{\Phi}_1)^T = \vec{\tilde{\Phi}}^{(0)}$ is given as

$$|D(\vec{\tilde{\Phi}}^{(0)})\rangle = g_D \sum_{m_0 \equiv m_1 \mathrm{mod}\ 2} e^{i\vec{\tilde{\Phi}}^{(0)}\cdot\vec{\tilde{K}}}|(n_0 = 0, n_1 = 0, m_0, m_1)\rangle\rangle, \tag{4.45}$$

where $\vec{\tilde{K}}$ is defined as Eq. (4.38), and $|(\vec{\tilde{R}}, \vec{\tilde{K}})\rangle\rangle \equiv |(n_0, n_1, m_0, m_1)\rangle\rangle$ is the bosonic Ishibashi state defined as Eq. (3.30) [73], which is characterized by the winding numbers of the bosonic fields. Since the cosine potential $\cos(2\tilde{\Phi}_0/\sqrt{2}\tilde{R})$ has minima at $\tilde{\Phi}_0 = \tilde{\Phi}^{(0)} + m\pi\sqrt{2}\tilde{R}$, where $m \in \mathbb{Z}$, there are two boundary values $\vec{\tilde{\Phi}}_0 = \vec{\tilde{\Phi}}^{(0)}$ and $\vec{\tilde{\Phi}}^{(0)} + \pi\sqrt{2}\tilde{R}(1,0)^T$ modulo the compactification. The superposition of the Dirichlet boundary states $|D(\vec{\tilde{\Phi}}^{(0)})\rangle$ and $|D(\vec{\tilde{\Phi}}^{(0)} + \pi\sqrt{2}\tilde{R}(1,0)^T)\rangle$ would be realized in the Rényified state. It is simply expressed as

$$|\bar{D}(\vec{\tilde{\Phi}}^{(0)})\rangle = g_{\bar{D}} \sum_{m_0, m_1 \in \mathrm{even}} e^{i\vec{\tilde{\Phi}}^{(0)}\cdot\vec{\tilde{K}}}|(n_0 = 0, n_1 = 0, m_0, m_1)\rangle\rangle. \tag{4.46}$$

Although the expression is similar to Eq. (4.45), it differs in the summation over $m_0$ and $m_1$. Here both $m_0$ and $m_1$ are restricted to even integers. Namely the sum is taken only on the "half" of the set of winding numbers.

A care must be taken in the normalization of such Dirichlet boundary states. In the present case, there are two boundary values (modulo the compactification), which imply the existence of two vacua. This indicates that there are two identity boundary operators. Thus it is required that the multiplicity of the identity operator for the $\bar{D}$ boundary condition is $n^0_{\bar{D}\bar{D}} = 2$ in the diagonal partition function Eq. (3.9) in the open string channel.

The amplitude between two $|\bar{D}\rangle$'s is given as

$$\begin{aligned} Z_{\bar{D}\bar{D}} &= \langle\bar{D}|e^{-\frac{\beta}{2}H}|\bar{D}\rangle \\ &= g_{\bar{D}}^2 \left(\frac{1}{\eta(q)}\right)^2 \sum_{m_0, m_1 \in \mathrm{even}} q^{\frac{1}{2}\cdot\frac{1}{2}\left[\frac{1}{g}\left(\frac{m_0}{\sqrt{2}\tilde{R}}\right)^2 + \frac{1}{g}\left(\frac{m_1}{\sqrt{2}\tilde{R}}\right)^2\right]} \\ &= g_{\bar{D}}^2 \left(\frac{1}{\eta(q)}\right)^2 \sum_{\tilde{m}_0, \tilde{m}_1 \in \mathbb{Z}} q^{\frac{1}{2}\cdot\frac{1}{g\tilde{R}^2}\left(\tilde{m}_0^2 + \tilde{m}_1^2\right)}. \end{aligned} \tag{4.47}$$

The modular transformation gives that

$$Z_{\bar{D}\bar{D}} = g_{\bar{D}}^2 \cdot g\tilde{R}^2 \left(\frac{1}{\eta(\tilde{q})}\right)^2 \sum_{\tilde{m}_0, \tilde{m}_1 \in \mathbb{Z}} \tilde{q}^{\frac{g\tilde{R}^2}{2}\left(\tilde{m}_0^2 + \tilde{m}_1^2\right)}. \tag{4.48}$$

The scaling dimensions of the boundary operators for $\bar{D}$ boundary condition can be read off from the partition function Eq. (4.48). Since it obviously does not contain $n_0$ and $n_1$, there is no spectrum which corresponds to the $S^z$ operator, $\exp(i\tilde{\Phi}_0/\sqrt{2}\tilde{R})$. Thus the longitudinal correlation decays exponentially. In comparison, there is a spectrum which corresponds to the $S^{\pm}$ operator, $\exp(\pm i\tilde{R}\tilde{\Theta}_0/\sqrt{2})$ in the partition function Eq. (4.48). Actually, the scaling dimension of the uniform part of $S^{\pm}$ operator, $\pm g\tilde{R}^2/2$, consists with the smallest scaling dimension of the boundary operators except for the identity operators. This gives the power-law decay of the uniform part of the transverse correlation with the exponent $g\tilde{R}^2 = n/2K$.

For the uniform part of the longitudinal correlation, the term $\partial_x\tilde{\Phi}_0$ has no contribution in the longitudinal correlation since $\vec{\tilde{\Phi}}$ is fixed at the boundary. For the staggered part of the transverse correlation, the situation is more subtle. It is given by the product of operators $\exp(i\tilde{R}\tilde{\Theta}_0/\sqrt{2})\exp(i\tilde{\Phi}_0/\sqrt{2}\tilde{R})$. Since $\vec{\tilde{\Phi}}$ is fixed at the boundary again, the operator $\exp(i\tilde{\Phi}_0/\sqrt{2}\tilde{R})$ would contribute just as a constant. Thus we might expect the power-law decay with the exponent $n/2K$. However, the boundary value is actually given by the superposition of the two values, $\vec{\tilde{\Phi}}^{(0)}$ and $\vec{\tilde{\Phi}}^{(0)} + \pi\sqrt{2}\tilde{R}(1,0)^T$, for $\bar{D}$ boundary condition. This leads $\exp(i\tilde{R}\tilde{\Theta}_0/\sqrt{2})\exp(i\tilde{\Phi}_0/\sqrt{2}\tilde{R})$ has no contribution due to the cancellation. Thus we might expect the next-leading operator would contribute to the transverse correlation. We conjecture this is the operator $\exp(i\tilde{R}\tilde{\Theta}_0/\sqrt{2})\partial_x\tilde{\Theta}_0$, although this term does not appear in the bosonic representation of the $S^x$ operator in general. Actually, the term $\partial_x\tilde{\Theta}_0$ in the $S^x$ operator would break the canonical commutation relation for the spin operators by combining with the term $\partial_x\tilde{\Phi}_0$ in the $S^z$ operator due to the canonical commutation relation Eq. (2.7) for the bosonic operators. However, now $\vec{\tilde{\Phi}}$ is fixed at the boundary then no contribution from $\partial_x\tilde{\Phi}_0$. This might suggest the derivative term $\partial_x\tilde{\Theta}_0$ would be allowed in the expansion of $S^x$ operator. Since the scaling dimension of the operator $\exp(i\tilde{R}\tilde{\Theta}_0/\sqrt{2})\partial_x\tilde{\Theta}_0$ is $g\tilde{R}^2/2 + 1$, we expect the power-law decay of the staggered part of the transverse correlation with the exponent $g\tilde{R}^2 + 2 = n/2K + 2$.

We have seen that the long-range behavior for the longitudinal correlation while the power-law behavior for the transverse correlation in the Rényified state after the transition $n > n_c$. Although this seems strange, there is no paradox since the present problem is mapped to a boundary field theory. In a conformal field theory with perturbations at the boundary, the boundary condition can be renormalized from a conformally invariant one to another conformally invariant one. However, boundary perturbations never opens a mass gap in the bulk. Thus, even if the boundary perturbation becomes relevant, it does not imply that all the correlations falls off exponentially.

We have also discussed the $g$-factor for the $\bar{D}$ boundary state. The Cardy's consistency condition gives

$$g_{\bar{D}} = \sqrt{2/g\tilde{R}^2}. \tag{4.49}$$

The "$g$-theorem" [68], which claims that $g$-factor always decreases under renormalization from a less stable to a more stable fixed point, tells us that the $\bar{D}$ boundary condition realizes if $g_{\bar{D}} < g_P$. This is perfectly consistent with the analysis in the previous section, which predicts that the boundary perturbation becomes relevant when $n > 4K$.

### 4.3.4 Phase diagram of the Rényified state

By the field theoretical analysis above, a phase diagram of the Rényified state has been obtained as presented in Fig. 4.10. Starting from a TLL ground state, which has a TLL parameter $K \geq 1/2$, the Rényified state is still a TLL until $n = 4K$. For $n > 4K$, it is no longer a TLL and belongs to an "exotic phase", where the longitudinal correlation has long-range order while the transverse correlation decays algebraically. In the limit $n \to \infty$, the Rényified state should be an antiferromagnetic state corresponding to the most probable Néel states $|\uparrow\downarrow\uparrow\downarrow\cdots\rangle$ and $|\downarrow\uparrow\downarrow\uparrow\cdots\rangle$. On the other hand, in the limit $n \to 0$, it should be disordered since all amplitudes of coefficients are identical. Furthermore, there is no reason to induce phase transitions for $4K < n < \infty$ and $0 < n < 4K$.

Let us consider a situation starting from a gapped antiferromagnetic state, for $K < 1/2$. It is easily expected that the Rényified state for $n > 1$ is an antiferromagnetic state since the replica manipulation stresses the coefficients which correspond to the Néel states. On the other hand, it is difficult to speculate what the Rényified state is for $n < 1$. Although numerical results suggests that a gapless-like behavior emerges in this region. it is hard to judge whether the state is a veritable gapless state or not. In this region, a field theoretical analysis seems to be daunting since the original state is gapped. For the reasons stated above, we leave a question of a phase in the region for $K < 1/2$ and $n < 1$ an open problem.

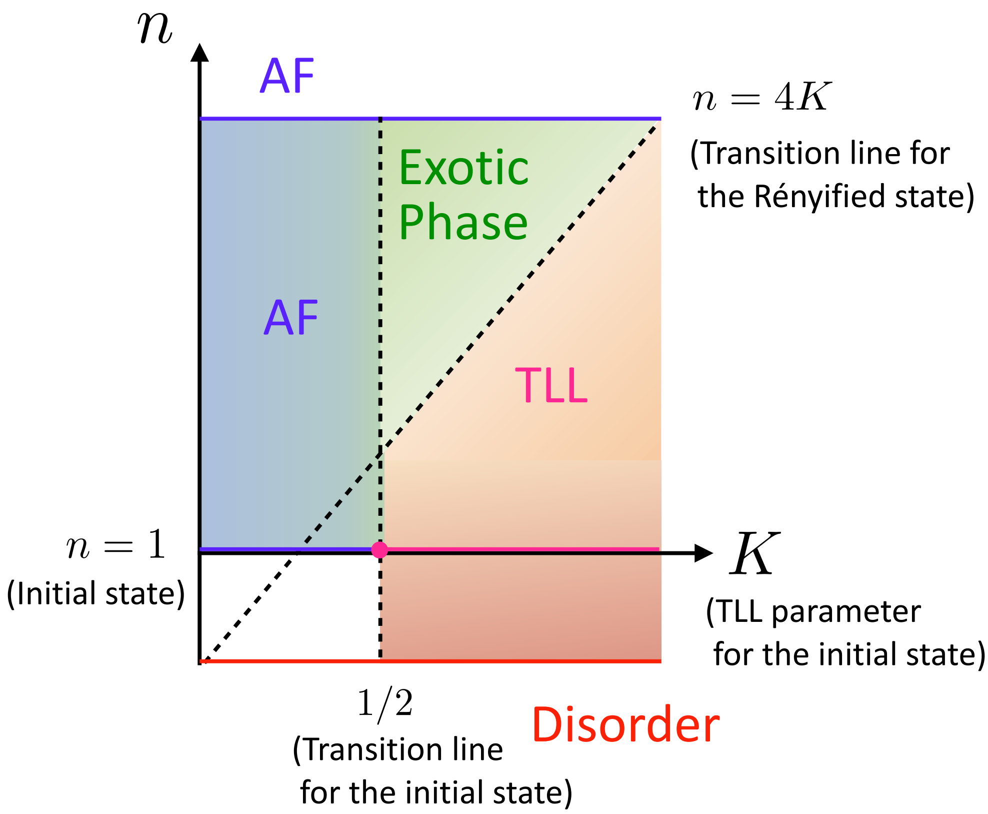


Figure 4.10: Schematic picture of a phase diagram of the Rényified state starting from a TLL derived by the field theoretical analysis. $K$ is a TLL parameter and $n$ is a Rényi index. $K = 1/2$ denotes a physical phase transition point of the original ground state ($n = 1$) of the $S = 1/2$ XXZ model. $n = 4K$ is a transition line for the Rényified state. For $n > 4K$, an exotic states, where the longitudinal correlations have long-range order while the transverse ones remain algebraic, are realized.

## 4.4 Boundary CFT formalism to the Rényi-Shannon entropy

Here we discuss the universal subleading contribution of the Rényi-Shannon entropy

$$S_S^{(n)} = \frac{1}{1-n} \ln \left( \sum_i |\psi_i|^{2n} \right), \tag{4.50}$$

for one-dimensional quantum systems with $c = 1$ by boundary CFT formalism. Although we have already shown the results above and the entropy itself was discussed in Ref. [56], it would be useful to formulate the problem in the boundary CFT language and verify the consistency of the present approach with theirs.

We compute the partition function $Z^{(n)} = \sum_i |\psi_i|^{2n}$ by replica field formulation. $Z^{(n)}$ cay be regarded as the partition function of the $n$-component free boson fields on the infinite cylinder, bound at the line $\tau = 0$. By folding the system at the line, the problem can be formulated in terms of the boundary field theory: $2n$-component free boson fields on the half-infinite cylinder $\tau > 0$ with a boundary.

When all the perturbations are irrelevant, for $n < n_c$, the boundary condition is nothing but the $\mathcal{P}$ boundary condition, derived in the appendix A. Thus the universal ground state degeneracy $g_{\mathcal{P}}$ is given by Eq. (A.13). The corresponding entropy is

$$(O(1) \text{ term of } S_n) \sim \frac{\ln g_{\mathcal{P}}}{1-n} = -\frac{1}{2}\left(\ln K + \frac{\ln n}{n-1}\right). \tag{4.51}$$

This agrees with the result Eq. (1.3) in Ref. [56], after setting $g = 1/2$ to match their convention.

When the leading boundary perturbation becomes relevant, the winding numbers of all the fields do not change except for the center of mass fields, $\tilde{\Phi}_0$ and $\tilde{\Theta}_0$. This implies that the boundary entropy is only affected by the change of the center of mass fields. The boundary condition for them is $P$ for $n < n_c$ and $\bar{D}$ for $n > n_c$. Thus the ground state degeneracy for the $\bar{D}$ boundary condition of $2n$ replica fields is

$$g_{\bar{D}}^{(2n)} = g_{\mathcal{P}} \frac{g_{\bar{D}}}{g_p} = 2\left(\sqrt{2g}R\right)^{-n} = 2K^{\frac{n}{2}} \tag{4.52}$$

where $g_{\bar{D}}$ and $g_p$ are the ground state degeneracy for $\bar{D}$ and $P$ boundary conditions of *two-component* free boson fields. The corresponding entropy is

$$(O(1) \text{ term of } S_n) \sim \frac{\ln g_{\bar{D}}^{(2n)}}{1-n} = \frac{1}{1-n}\left(\frac{n}{2}\ln K + \ln 2\right) \tag{4.53}$$

This again agrees with the result Eq. (1.4) in Ref. [56], after setting $g = 1/2$ (to match their convention) and $d = 2$ (the number of vacua).

# Chapter 5

# Conclusion

We proposed the new quantum state, which we named the Rényified state, which is constructed by taking $n$-th power of wave-function (coefficient of the state vector with respect to a chosen basis). We then discussed its properties starting from a TLL ground state.

In short, we obtained the following results in this thesis.

**(i)** The Rényified state is still a TLL with a modified TLL parameter $K/n$ for small $n$. The field theoretical analysis suggests that the TLL description breaks down at $n = n_c = 4K$. Above $n_c$, the analysis indicates that the Rényified state belongs to an exotic universality class: the longitudinal correlations show long-range order while the transverse ones remain algebraic. A phase diagram has been obtained as presented in Fig. 4.10.

**(ii)** The universal subleading term of the Rényi-Shannon entropy for a TLL has been derived by boundary CFT approach through the investigation of the Rényified state.

Concerning the point (i), we showed that the Rényified state is also a TLL which described by a modified TLL parameter $K/n$ for small $n$ by using numerics and analytical argument. Theoretically, this could be understood as the property of “center of mass field” by the replica field formulation in chapter 4. The filed theoretical analysis suggested that the TLL description breaks down by the effect of perturbations at the “binding” of replicas. A transition of the Rényified state was expected at $n = 4K$ for the $S = 1/2$ XXZ spin chain by the field theoretical analysis. Although we could not see the transition clearly in the numerical calculations of the correlation functions, the behavior of the correlations did not contradict the scenario derived by the field theoretical analysis. We note that the numerical calculations will be able to be performed in larger systems ($L > 32$) for the $S = 1/2$ XXZ spin chain at $\Delta = 0$ since the form of the ground state is exactly known as Eq. (1.11) and Eq. (1.12). This will support our field theoretical analysis. The analysis also suggested that the longitudinal correlations have long-range order while the transverse ones remain algebraic for $n > 4K$. This behavior is quite non-trivial, and such a state is unlikely to be realized as the ground state of a Hamiltonian which has only short-range interactions. This exotic behavior was explained by the construction of a specific conformal invariant boundary state of a two-component free boson.

Concerning the point (ii), the relationship between the Rényified state and the Rényi-Shannon entropy was also elucidated in this thesis. We showed that the phase transition of the Rényified state was related to a phase transition in the Rényi-Shannon Entropy. Furthermore, the boundary CFT formalism gave a re-interpretation of the behavior of the Rényi-Shannon Entropy.

Let us discuss implications of our results on the original ground state. In an original TLL ground state, all perturbations can be neglected, some are prohibited by symmetry, others are irrelevant. We cannot distinguish them as long as we deal with universal large-distance asymptotic behaviors of the original state. In this thesis, we have seen that irrelevant operators hidden in a bulk are stressed and can become relevant at a boundary in the Rényified state. This suggests that we can enhance the effects of the allowed irrelevant operators in given system by replica manipulation. This can be useful in revealing the detailed nature of the given ground state.

The boundary state constructed in the replica field theory formulation, to represent the Rényfied state, would be also interesting by itself. As we have seen above, the replica manipulation amplifies perturbations only at a boundary, then produces a boundary state which depends on the most relevant perturbation. This indicates that exceptional boundary states will be realized if perturbations in the theory are non-trivial. This can be interesting problem when we construct the Rényified state starting from an exotic ground state.

In this thesis, we focused on the investigation of the Rényified state starting from a TLL ground state. The study in higher-dimensional systems is a possible extension of our work. For the analysis, it is necessary to construct a particular boundary state in higher-dimension. It is a challenging but an attractive problem since it is related to the Rényi-Shannon entropy in higher-dimensional systems, which has been energetically investigated recently and found to exhibit interesting properties.

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# Appendix A

# Construction of $\mathcal{P}$ boundary state

In this appendix, we consider the $n$-component free boson field theory on the infinite cylinder with the constraint at $\tau = 0$. By folding the system at $\tau = 0$, the problem is mapped to the $2n$-component free boson field theory on the half-infinite cylinder with the boundary.

By changing the basis, the compactification is tilted and it is not independent in terms of new fields. Although we can construct the boundary condition which satisfies the glueing condition for two-component free boson field theory, it is bothersome to keep track the condition for a large number of fields. Avoiding the formidable task, we handle it with a geometric formulation as we will explain below.

At the boundary, the relative coordinate fields $\Phi_{\alpha\neq 0}$ are fixed as $\Phi_\alpha = 0$. On the other hand, the center of mass field $\Phi_0$ is not fixed at the line. Since the eigenvalue of the orthogonal matrix $\mathcal{R}$ is $1(-1)$ for Dirichlet (Neumann) boundary condition, this implies that the new fields satisfy the following conditions:

$$\mathcal{R}\tilde{\Phi}_0 = -\tilde{\Phi}_0 \tag{A.1}$$

$$\mathcal{R}\tilde{\Phi}_i = \tilde{\Phi}_i \quad \text{for } i \neq 0. \tag{A.2}$$

To satisfy these condition, the matrix $\mathcal{R}$ is given as

$$\mathcal{R} = \mathbf{1} - 2\vec{d}\vec{d}^T \tag{A.3}$$

where $\vec{d} = (1/\sqrt{2n})(1, 1, \cdots, 1)^T$. $\mathcal{R}$ is regarded as the reflection matrix about the plane normal to the $2n$-dimensional vector $\vec{d}$.

Then we have to find a pair of winding numbers to satisfy the condition (3.31) for the orthogonal matrix $\mathcal{R}$ (A.3). Since $\vec{R}$ is reflected by $\mathcal{R}$, $\vec{R}$

should be proportional to $\vec{d}$. Thus it is enough to choose $\vec{R}$ as

$$\vec{R} = n_0 R\sqrt{2n}\vec{d}, \tag{A.4}$$

where $n_0 \in \mathbb{Z}$. On the other hand, $\vec{K}$ is not affected with acting $\mathcal{R}$, thus it is perpendicular to $\vec{d}$. Thus the winding number of $\vec{\Theta}$ will satisfy

$$\vec{K} \in \Xi^*_{\mathcal{P}}, \tag{A.5}$$

where $\Xi^*_{\mathcal{P}}$ is defined as the intersection of $\Lambda^*$, which is the compactification lattice of $\vec{K}$, and the $2n-1$ dimensional hyperplane orthogonal to $\vec{d}$. The corresponding boundary state is given as

$$|\mathcal{P}\rangle = g_{\mathcal{P}} \sum_{\vec{R}=n_o R\sqrt{2n}\vec{d}, \vec{K}\in\Xi^*_{\mathcal{P}}} |(\vec{R}, \vec{K})\rangle\rangle. \tag{A.6}$$

The prefactor $g_{\mathcal{P}}$ is determined by the Cardy's consistency condition. For this purpose, we introduce the amplitude between two $|\mathcal{P}\rangle$'s as

$$\begin{aligned} Z_{\mathcal{P}\mathcal{P}} &= \langle\mathcal{P}|e^{-\frac{\beta}{2}H}|\mathcal{P}\rangle \\ &= g^2_{\mathcal{P}}\left(\frac{1}{\eta(q)}\right)^{2n} \sum_{\vec{R}=n_o R\sqrt{2n}\vec{d}, \vec{K}\in\Xi^*_{\mathcal{P}}} q^{\frac{1}{2}\left(\frac{g}{2}\vec{R}^2+\frac{1}{2g}\vec{K}^2\right)} \\ &= g^2_{\mathcal{P}}\left(\frac{1}{\eta(q)}\right)^{2n} \sum_{n_o\in\mathbb{Z}, \vec{K}\in\Xi^*_{\mathcal{P}}} q^{\frac{1}{2}\left(\frac{2ngR^2}{2}n_0^2+\frac{1}{2g}\vec{K}^2\right)}. \end{aligned} \tag{A.7}$$

Performing the modular $S$ transformation gives that

$$\begin{aligned} Z_{\mathcal{P}\mathcal{P}} &= g^2_{\mathcal{P}}\left(\frac{2}{2ngR^2}\right)^{\frac{1}{2}} (2g)^{\frac{2n-1}{2}} v_0^{-1}\left(\Xi^*_{\mathcal{P}}\right)\left(\frac{1}{\eta(\tilde{q})}\right)^{2n} \\ &\times \sum_{\tilde{n}_o\in\mathbb{Z}, \vec{\tilde{K}}\in\Xi_{\mathcal{P}}} \tilde{q}^{\frac{1}{2}\left(\frac{2}{2ngR^2}\tilde{n}_0^2+2g\vec{\tilde{K}}^2\right)}, \end{aligned} \tag{A.8}$$

where $\Xi_{\mathcal{P}}$ is the dual lattice of $\Xi^*_{\mathcal{P}}$, and $v_0(\Xi^*_{\mathcal{P}})$ is the volume of the unit cell of the Bravais lattice $\Xi^*_{\mathcal{P}}$. The Cardy's consistency condition leads

$$g^2_{\mathcal{P}} = \left(\frac{2}{2ngR^2}\right)^{-\frac{1}{2}} (2g)^{-\frac{2n-1}{2}} v_0(\Xi^*_{\mathcal{P}}). \tag{A.9}$$

Since the $2n-1$ dimensional lattice $\Xi^*_{\mathcal{P}}$ is the intersection of $\Lambda^*$ and the hyperplane perpendicular to $\vec{d}$, the hypercubic lattice $\Lambda^*$ consist of the accumulation of hyperplane $\Xi^*_{\mathcal{P}}$. The distance between the neighboring hyperplane $\Xi^*_{\mathcal{P}}$ is $1/\sqrt{2n}R$, thus the volume of the unit cell of the compactification lattice $\Lambda^*$ is given as

$$v_0(\Lambda^*) = \frac{1}{\sqrt{2n}R} v_0(\Xi^*_{\mathcal{P}}). \tag{A.10}$$

On the other hand, we find

$$v_0(\Lambda^*) = \left(\frac{1}{R}\right)^{2n}, \tag{A.11}$$

since $\Lambda^*$ is the $2n$ dimensional hypercubic lattice with a lattice constant $1/R$. Eq. (A.10) and (A.11) give the volume of the unit cell of the lattice $\Xi^*_{\mathcal{P}}$ as

$$v_0(\Xi^*_{\mathcal{P}}) = \sqrt{2n} R^{-2n+1}. \tag{A.12}$$

Comparing this with Eq. (A.9), the prefactor is obtained as

$$g_{\mathcal{P}} = \sqrt{n} \left(\sqrt{2g}R\right)^{-n+1} = \sqrt{nK^{n-1}} \tag{A.13}$$

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