論文の内容の要旨

論文題目　Relative-entropy conservation law in quantum measurement and its applications to continuous measurements
（量子測定における相対エントロピーの保存則とその連続測定への応用）

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In a quantum measurement process, there is a back-action on the system and the information about the measured observable is in general disturbed by the measurement back-action. Still in some quantum measurements such as a quantum non-demolition measurement or a photon-counting measurement, the information about the measured observable is conserved. Ban discussed these types of measurements quantitatively based on the Shannon entropy and the mutual information [1, 2, 3]. In his formulation, the obtained information is quantified in terms of the mutual information between the measured observable X and the measurement outcome Y. He established a condition for the Shannon entropy conservation which states that the amount of the obtained information is equal to the decrease in the system's Shannon entropy of X due to the measurement. However, the Shannon entropy for a continuous variable cannot be interpreted as the proper information content in general since its value changes by a one-to-one transformation, which is just a relabelling and should not increase or decrease the information. Furthermore, the physical meaning of the established condition for the Shannon entropy conservation is not clear. The aim of this thesis is to study these information-conserving measurement processes based on the relative entropy [4] and establish the conservation relation for this kind of information content.

This thesis consists of eight chapters and two appendices. Chapter 1 is the introduction. Chapters 2, 3 and 4 are review parts. Chapters 5, 6 and 7 contain the new results obtained by the present author. Chapter 8 summarizes this thesis.

Chapter 2 reviews the quantum measurement theory and in this chapter we introduce the positive operator-valued measure (POVM) and the completely positive (CP) instrument. The POVM is a generalization of the projection-valued measure (PVM) associated with a measurement on a self-conjugate operator and it determines the statistics of the measurement outcome for a non-ideal quantum measurement. The CP instrument describes both the measurement outcome and the conditional state change due to the measurement back-action. In the introduction of these concepts, we consider the measurement outcome based on the general theory of measure and measurable sets which can handle discrete and continuous sample spaces of the measurement outcome in a coherent manner.

Chapter 3 reviews classical information theory. We first introduce three entropic information contents, namely the Shannon entropy, the mutual information and the relative entropy. We point out that the Shannon entropy for a continuous variable changes its value by a one-to-one transformation while the other two contents do not. We also introduce the concept of a sufficient statistic for the statistical model introduced by Halmos and Savage [5] and review the result by Kullback and Leibler [4] that the sufficiency of a statistic can be characterized by the conservation of the relative entropy.

Chapter 4 reviews the Shannon entropy conservation for quantum measurements derived by Ban [2, 3]. We consider a measurement process Y described by a CP instrument and a system's observable X described by a POVM. We prove the Shannon entropy conservation under some conditions for X and Y. We also consider a case in which X is a PVM. As an example of the Shannon-entropy-conserving measurement, we discuss a quantum non-demolition measurement which does not alter the probability distribution of the measured observable by the measurement back-action.

Chapter 5 establishes the relative entropy conservation. As in the previous chapter we consider an observable X described by a POVM $\hat{E}_x^X$ and a measurement process Y corresponding to a CP instrument $\mathcal{E}_y^Y$. We quantify the obtained information as the relative entropy $D(p_{\hat{\rho}}^Y||p_{\hat{\sigma}}^Y)$, where $\hat{\rho}$ and $\hat{\sigma}$ are candidate states of the system and $p_{\hat{\rho}}^Y(y)$ is the probability distribution for the measurement outcome of Y. Under some conditions for X and Y, we prove the following relative-entropy conservation law:

$$D(p_{\hat{\rho}}^Y||p_{\hat{\sigma}}^Y) = D(p_{\hat{\rho}}^X||p_{\hat{\sigma}}^X) - E_{\hat{\rho}}[D(p_{\hat{\rho}_y}^X||p_{\hat{\sigma}_y}^X)], \tag{1}$$

where $p_{\hat{\rho}}^X$ is the probability distribution of X for a quantum state $\hat{\rho}$, $\hat{\rho}_y$ is the post-measurement state for a given measurement outcome y, and $E_{\hat{\rho}}[\cdot]$ denotes the ensemble average over the measurement outcome y for a given pre-measurement state $\hat{\rho}$. The left-hand side of Eq. (1) is the amount of the obtained information from the measurement outcome as to which state $\hat{\rho}$ or $\hat{\sigma}$ is actually prepared. The right-hand side of the relative-entropy conservation law is the average decrease in the system's relative entropy of X. To understand the meaning of the relative-entropy conservation law, we consider a joint successive measurement process of X after Y. Then we show that the relative-entropy conservation law (1) is equivalent to another relative entropy conservation law

$$D(\tilde{p}_{\hat{\rho}}^{XY}||\tilde{p}_{\hat{\sigma}}^{XY}) = D(p_{\hat{\rho}}^X||p_{\hat{\sigma}}^X), \tag{2}$$

where $\tilde{p}_{\hat{\rho}}^{XY}(x,y)$ is the probability distribution for the joint successive measurement. The left-hand side of Eq. (2) is the obtained information for the joint measurement process. The established condition for the relative entropy conservation is interpreted as the existence of a sufficient statistic $\tilde{x}(x;y)$ for the joint successive measurement whose probability distribution coincides with that of X. For the case X is a discrete PVM and Y is discrete, the established condition is shown to be equivalent to the relative-entropy conservation law (1) or (2) for arbitrary quantum states $\hat{\rho}$ and $\hat{\sigma}$. We also show that the condition for the relative-entropy conservation law is less restrictive than that for the Shannon entropy conservation by Ban, i.e. our condition applies to a wider class of quantum measurements. For a case in which X is a discrete non-degenerate PVM and Y is discrete, Ban's condition for the Shannon entropy conservation is shown to be equivalent to a condition that the post-measurement state is an eigenstate of X if the pre-measurement state is also an eigenstate of X. As an example of a measurement process which does not satisfy Ban's condition but do satisfy our condition for the relative-entropy conservation law, we discuss a destructive measurement Y of

X in which measurement outcome of Y is equivalent to the projective measurement of X while the state changes to a completely mixed state due to the measurement back-action.

In Chapter 6, we apply the general theorem for the relative-entropy conservation law established in Chapter 5 to continuous destructive measurements on a single-mode photon field, namely, photon-counting, quantum counting, homodyne, and heterodyne measurements. We prove that all of these measurement satisfy the established condition for the relative-entropy conservation law, whereas the Shannon entropy conservation does not hold except for the photon-counting measurement. The common reason in these examples for the non-conservation of the Shannon entropy is that the Jacobian of the transformation $x \to \tilde{x}(x; y)$ is not unit, which is due to the strong dependence of the Shannon entropy for a continuous variable on a reference measure of the variable.

In Chapter 7 we consider a problem of whether or not there exists a relative-entropy-conserving observable X for a given measurement process Y described by a CP instrument. We show that the answer is affirmative when the sample space of Y is a standard Borel space. The typical standard Borel spaces include finite space, countable discrete space and all the Borel subsets of the Euclidean space $\mathbb{R}^d$. Thus the assumption of a standard Borel sample space is as general as to include usual examples of the measurement processes encountered in the physical problems. The constructed observable X corresponds to the measurement outcome of the infinite successive measurements of the same measurement process Y. Since the sample space of X becomes continuous even when Y's sample space is finite, the measure theoretic consideration takes part in the construction, which includes the Kolmogorov extension theorem for POVM [6].

References

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