Measurements in Time Symmetric Quantum Mechanics

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Abstract

In this thesis, we develop a probabilistic description of the time-symmetric formulation of quantum mechanics. We derive a generalized form of the probability distribution of a quantum process which is determined by performing pre- and post-selections on a system. It gives the well-known value for an observable associated to a quantum process, called weak value, as an expectation value. We show that the generalized probability satisfies the law of total probability and admits the transitive form by means of conditional probabilities. This motivates us to step forward to a classical ensemble picture in quantum mechanics. In addition, we present the description of quantum measurement in terms of the generalized probability. It is suggested that the difference between the projective measurement and the weak measurement corresponds to the choice of selections of the forward and backward propagating processes.

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Chapter 1

Introduction

Quantum mechanics is the most successful theory for describing microscopic systems with an incredibly accurate expression of the microscopic nature. Although quantum mechanics has been used extensively for about a century, we have not quite succeeded in obtaining its underlying picture and grasping its implications in a coherent framework. In particular, no one has succeeded in describing the dynamics of measurements on quantum system without classical intervention.

By now, we are more or less forced to accept the idea that one has to be accustomed to the characteristics of quantum mechanics which are beyond our intuitive picture obtained by classical mechanics, because it is difficult to find a description of quantum mechanics with the formalism in which our intuition can work properly. One of the reasons comes from the absence of values assigned to observables on a quantum system, whereas definite values are assigned to observables on a classical system. In fact, the absence of values has been seriously discussed since the first proposal of Einstein, Podolsky and Rosen argument on local hidden variables [3]. After three decades, Kochen and Specker proved that it is actually impossible to assign values to all observables simultaneously [4]. We can perform assignment of values at a time not for all observables but for the partial set of observables, which is a set of mutually commutative observables. The choice of the set of mutually commutative observables is called the context, and the dependence of the assignment of values of observables on the context is called the contextuality of value assignment. Since the contextuality does not appear in classical mechanics, it seems that we have to abandon something which exists in classical mechanics to describe quantum mechanics. Although it may be possible to construct a completely new dynamics in microscopic world, e.g., the Bohmian theory [11] or the continuous spontaneous localization models [12], none of the formulations has been completed yet.

Meanwhile, a theory which can explain the assignment of values of observables without the problem of contextuality has been proposed based on the time-symmetric two-state vector formulation of quantum mechanics [30, 31, 34]. The motivation of the formalism is to obtain an intuitive picture for quantum mechanics by considering ensembles of the process determined by the initial and final conditions of the process. Aharnov et al. introduced the forward and backward propagating quantum states whose counterparts in classical mechanics may be the

advanced and retarded waves in the absorber theory of electromagnetic radiation [13]. They solved the contextual assignment of values of observables under the condition that we require all boundary conditions which are determined by the results of projective measurements performed on a system. One of the novel point of their formalism is that we have only to implement a factual assignment for a part of observables, not all observables. Since there are no other projective measurements except at the two boundaries in a process, we need the contextual assignments only for the set of compatible observables with both of the boundaries.

On the other hand, there is another approach based on the probability theory which plays a key role in quantum mechanics. Indeed, recently some modifications of the interpretation of probability have been discussed [15, 16, 17], where it is argued that the characteristics of quantum mechanics can be described by an epistemic theory with subjective probability. If the probability described in quantum mechanics is subjective, we do not need to consider the ensembles underlying the probability. It requires no more explanation for the physical dynamics of the transition of probability distributions of a system.

In the present thesis, however we consider an alternative probabilistic approach in the time-symmetric formulation of quantum mechanics. The main aim of this paper is to recover a probabilistic description with underlying ensembles by introducing a probability distribution of quantum processes. In classical mechanics, a system admits ensemble with ordinary real probability. Each of the ensembles has a definite value assigned to an observable. If one obtains knowledge about the system, the probability distribution changes to the conditional probability distribution, which we call 'transition of probability'. In the standard formulation of quantum mechanics, the probability to obtain a certain result is given by Born's rule. However, since we cannot define ensembles on a quantum system, the transition probability in quantum system cannot be given by conditional probability. Here, we try to associate the probability distribution to quantum processes for which a solution for the contextuality can be expected. An ensemble picture in quantum mechanics as classical mechanics may then be expected to be found.

To this end, specifically we consider a general form of probability in a quantum process defined by two states which correspond to the initial and final states of the process. If we allow the probability to extend from the real valued interval $[0, 1]$ to the whole complex plane, the probability is generalized to the same extent that has been used for the weak value of projective operators.

Once our generalized complex-valued probability is adapted, we can understand the weak value of an observable in a coherent manner. In fact, the weak value, which appears in the two-state vector formalism, is supposed to give a noncontextual assignment of the values of observables in the formalism, which is also a requirement for probability. The weak value is interpreted as the expectation value of the observable with the generalized probability, whose derivation and discussions have been presented in Ref. [1]. We can also define the conditional probability for the generalized probability in such a way that it describes the transition of the generalized probability to a process. We find that our generalized probability sat-

isfies the law of total probability for conditional probabilities. This feature does not exist in the ordinary probability in quantum mechanics. The law of total probability is necessary for constructing the classical ensemble picture from probability distribution, though it is still difficult to restore the complete ensemble from the generalized probability. Indeed, the transitive form of probability, which we derive as the conditional probability, is found to be consistent with the transition of a process.

Regarding the description of measurement in a process, although the two-state vector formalism requires distinct treatments for describing projective measurement and weak measurement, we can also see the connection between the two measurements as two different ways of selections of processes based on the generalized probability. Moreover, in our treatment, the forward and backward propagating processes are understood in a natural manner.

The plan of the present thesis is the following. First, we give a preliminary account on the classical difficulty in describing outcomes of quantum mechanics in Chapter 2. Although there are arguments for the locality of the assignment of values of dynamical variables, we focus on the contextual dependence of the assignment of the values, as it is one of the most critical differences between classical mechanics and quantum mechanics. Then in Chapter 3, we introduce the time-symmetric formulation of quantum mechanics. The formalism avoids the contextuality problem by the description of only factual processes. However, we mention that there are insufficient elements in their formalism. One of the problems lies in the notion of the forward and backward propagating states, in that the distribution of values of observables is not defined in each propagating state. Chapter 4 and Chapter 5 are devoted to our main works, of which one is the derivation of the generalized probability on a quantum system [1] and the other is the acquisition of the descriptions of a quantum system and quantum measurement through the generalized probability. In Chapter 4, we introduce the generalized probability in a quantum process. the distribution of processes is defined as probability extended to complex plane. We present how the system with imaginary generalized probability behave in oprical paths. We show that the probability satisfies the law of total probability and that a consistent description of its transition becomes possible as a part of the requirements for underling ensembles. Finally, we present the description of quantum measurement in terms of the generalized probability in Chapter 5. It is suggested that the difference between the projective measurement and the weak measurement corresponds to the choice of selections of the forward and backward propagating processes. We conclude this thesis with discussions on our results in Chapter 6.

Chapter 2

Preliminaries

2.1 Classical Ensemble Picture

First of all, we explain the classical ensemble picture which forms a basis of the quantum ensemble we aim to obtain. A classical system is represented by an object which has definite values of dynamical variables, e.g., a massive particle which has values of position and momentum simultaneously. The dynamical variables evolve according to the law of given dynamics. The state of a system at a time is determined by all values of dynamical variables at the time. Since an observable corresponds to a function on these dynamical variables, the value of an observable on a state at a time is evaluated as the value of the function associated to the observable on the state at the time. Multiple objects in a system give the distribution of dynamical variables. The proportion of dynamical variables can be represented by a probability distribution. The value of an observable on multiple objects is given by the weighted sum of the value of the function associated to the observable with the probability distribution. In this sense, the probability distribution in classical mechanics is based on objects having definite values of dynamical variables.

Accordingly we here define the classical ensemble picture as follows.

- A system is represented by objects having definite values which determine the value of an arbitrary observable.

- A system is specified by a probability distribution of the values of observables.

The probability distribution of observables is given by the probability distribution of dynamical variables through the function associated to the observable. In the next section, we see the reason why it is difficult to obtain the classical ensemble picture in quantum system.

2.2 Problems on Classical Ensemble Picture in Quantum Mechanics

Measurement is an operation to find the value of an observable in a system. We can obtain the value of an observable through a measurement on a macroscopic system. It is assumed that we can obtain the value of any dynamical variable of a state by an appropriate measurement in classical mechanics. However, we cannot obtain the value of a dynamical variable on a microscopic system without disturbing other dynamical variables, according to the error-disturbance relation of quantum mechanics. If we measure observables A and B on a microscopic system, the square roots of the variances of obtained results $\sigma(A)$ and $\sigma(B)$ must satisfy the Robertson-Kennard uncertainty relation [9, 10].

$$\sigma(A)\sigma(B) \geq \frac{1}{2} \left| \langle [A, B] \rangle \right|, \tag{2.1}$$

where $\sigma(A) := \sqrt{\langle A^2 - \langle A \rangle^2 \rangle}$ and $\langle \cdot \rangle := \langle \psi | \cdot | \psi \rangle$ in the case that the state of the system prepared in $| \psi \rangle$. It is interpreted that the measurement disturbs the system and randomize the values of observables. Thus we cannot determine all the values of observables sharply at a time. It implies that the classical ensemble picture in quantum mechanics is impossible. However, it was nevertheless suggested that if we could precisely understand the dynamics of the disturbance in the measurement, we could still trace the values of observables. This expectation was defeated by the famous paper [3] in an elegant way.

2.2.1 EPR Paradox

The key to the statement of that paper is the measurement of an entangled state which is called "EPR state". If we measure one of the entangled pair, we can know the value of dynamical variables on the system. We can perform a measurement without disturbing its system under the assumption of the cluster decomposition property of these variables.

In Ref. [3], the authors consider the entangled state of the positions, which is given by

$$|\Psi\rangle := \int_{-\infty}^{\infty} dx \left| x_1 = x + \frac{x_0}{2} \right\rangle \left| x_2 = x - \frac{x_0}{2} \right\rangle, \tag{2.2}$$

where $|x_1 = a\rangle$ represents the localized state of system 1 around a (Fig. 2.1). If we measure the position of system 1 and find it around a, we find that the position of system 2 gets localized around $a - x_0$ according to the calculation

$$
\begin{aligned}
\langle x_1 = a | \Psi \rangle &= \int_{-\infty}^{\infty} dx \delta \left(x + \frac{x_0}{2} - a \right) \left| x_2 = x - \frac{x_0}{2} \right\rangle \\
&= |x_2 = a - x_0\rangle.
\end{aligned}
\tag{2.3}
$$

Similarly, if we measure the momentum of system 1 as $p_1 = b$, we find that the

EPR state
(Entangled state)

System 1   System 2

$$x_1 = a \longrightarrow x_2 = a - x_0$$
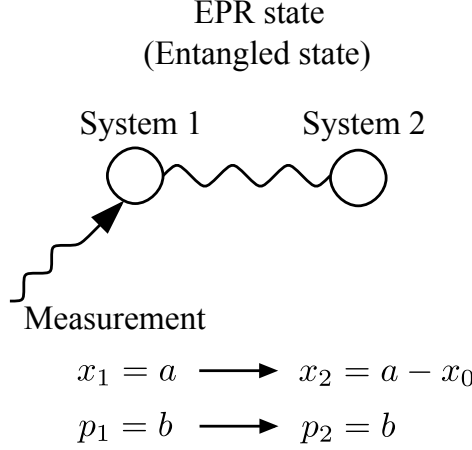$$p_1 = b \longrightarrow p_2 = b$$

Figure 2.1: If we measure the position or momentum of system 1 of the EPR state, we obtain the position or momentum of system 2 without disturbing the system 2.

momentum of system 2 is the same $p_2 = b$.

$$
\begin{aligned}
\langle p_1 = b | \Psi \rangle &= \int_{-\infty}^{\infty} dx e^{-ib(x+\frac{x_0}{2})} \left| x_2 = x - \frac{x_0}{2} \right\rangle \\
&= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, e^{-ib(x+\frac{x_0}{2})} e^{-ip(x-\frac{x_0}{2})} | p_2 = p \rangle \\
&= \int_{-\infty}^{\infty} dp\, \delta(b+p) e^{-i(b-p)\frac{x_0}{2}} | p_2 = p \rangle \\
&= e^{-ibx_0} | p_2 = b \rangle ,
\end{aligned}
\tag{2.4}
$$

where we put $\hbar = 1$.

In the original arguments, they define the element of reality.

> *"If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."*

We argue above that the position and momentum of system 2 can be measured without disturbing the system 2 by the measurements on system 1. According to their definition of the element of reality, the position and momentum of system 2 are the elements of reality according to their definition. Such elements of reality, which are the position and momentum of system 2, have no simultaneous eigenstates in quantum mechanics. This means that we cannot obtain the values of observables with probability equal to unity. Consequently, they claimed that quantum mechanics is not a complete theory of physics. It is important to recognize that they supposed the cluster decomposition property of dynamical variables, which means that the values of dynamical variables are locally determined. In terms of dynamical variables, the position and momentum of system 2 are precisely determined against the uncertainty relation in quantum mechanics. The

question was experimentally examined in 1982 [7] and the results contradicted the explanation by the dynamical variables of local systems. If we can obtain values of observables in some way, there must be a local operation extracting the values of observables determined by local dynamical variables. The above example shows that unlike classical mechanics, the microscopic system cannot be described by dynamical variables on a local system.

Even though if we remove the assumption of locality, it is known that we cannot define the value of dynamical variable on a system at a time. The next theorem shows that the assignments of values must depend on the context, which is determined by the ordering of measurements.

2.2.2 Kochen-Specker Theorem

The Kochen-Specker theorem argues the impossibility of assignments of values to observables independently of the context [4]. In quantum mechanics, the value assigned to an observable is supposed to be an eigenvalue of the observable, because we obtain one of the eigenvalues of the observable through the measurement with probability unity when we prepare the system in the eigenstate of the observable. Thus the value assigned to an observable in a system is represented by its eigenstate. Though the simultaneous eigenstate for observables assigns the value of them, there is no simultaneous eigenstate for non-commutative observables. However, we cannot say that we are not able to assign values of observables on a system due to absence of simultaneous eigenstate. If we measure non-commutative observables, the measurement disturb the system and change the values of them. The values of non-commutative observables before the measurement cannot be obtained through the measurement due to the disturbance, hence we can expect that the values assigned to observables are determined before the measurement on the system, not as the result of measurement. Since there is an object having definite values which determines a value of an observable in classical mechanics, we expect that there are values to be assigned to observables on a system in quantum system similarly as classical mechanics. Although the value of an observable is determined by underlying variables through the function corresponding to the observable, it is enough to consider the assignment of the value of the observable determined by underlying variables.

This theorem states the impossibility of assignment of values of observables which is compatible with consistency conditions in quantum mechanics. This means that the classical ensemble picture is not achievable in quantum mechanics since there is no object which determines the values of observables consistently. We here introduce the simple proof by Peres [5, 6] instead of the original complicated discussion, where one introduced a set of observables which partially commute with each other. In a spin-1 particle system, the squares of spin components S_x^2, S_y^2 and S_z^2 are mutually commutative and satisfy the sum condition

$$S_x^2 + S_y^2 + S_z^2 = 2, \tag{2.5}$$

where, for simplicity, we set $\hbar = 1$. Let us consider the assignment of values of the spin components in various directions. We use Miller index for the description of
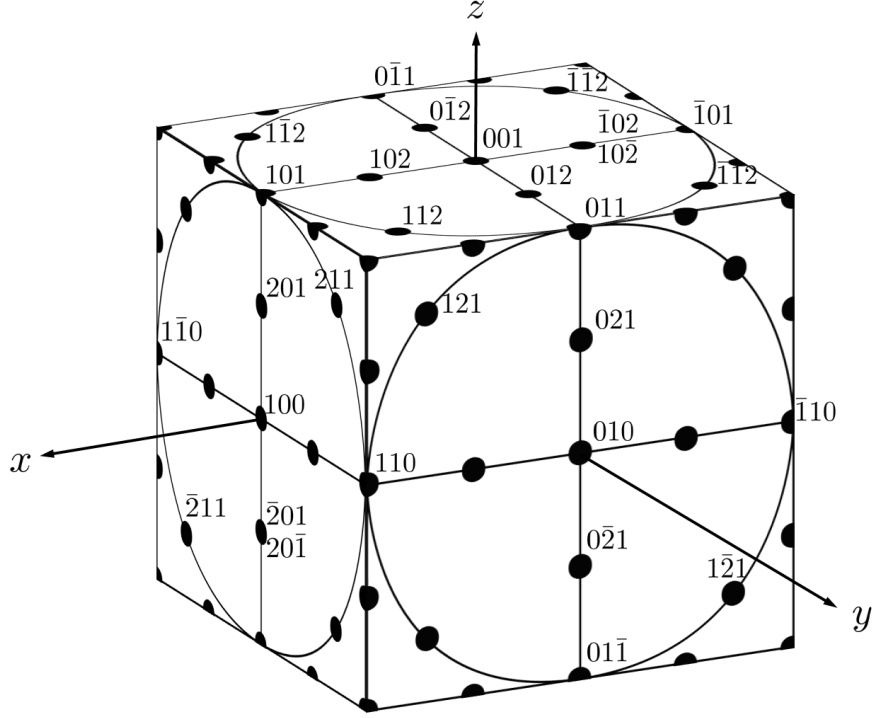
Figure 2.2: The directions of spin components in the proof of Kochen Specker theorem. The direction indicated by the Miller index is that from the origin to its point.

the direction of spin components (see Fig. 2.2). First, we assign 1 in the directions 100 and 010 and assign 0 to 001 to satisfy the sum condition (2.5). Then, we continue to assign eigenvalues to spin components in 33 directions in the following table:

001 100 010	110 $\bar{1}$10
101 $\bar{1}$01 010	
011 0$\bar{1}$1 100	
1$\bar{1}$2 $\bar{1}$12 110	021 $\bar{2}$01
102 20$\bar{1}$ 010	$\bar{2}$11
211 01$\bar{1}$ $\bar{2}$11	$\bar{1}$02
201 010 10$\bar{2}$	$\bar{1}\bar{1}$2
112 1$\bar{1}$0 $\bar{1}\bar{1}$2	0$\bar{2}$1
012 100 0$\bar{2}$1	1$\bar{2}$1
121 $\bar{1}$01 1$\bar{2}$1	0$\bar{1}$2

We assign 0 to the direction in the left-hand side of the left column of table from top

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to bottom. The other directions are assigned 1. In the table, we gave 0 assignments for $001, 101, 011$ and $1\bar{1}2$, and the other assignment is automatically determined. If we assign 0 in the direction 001, 101 must have the different assignment against $\bar{1}01$ because 010 must be assigned 1. Similarly, each of $011, 112$ and $1\bar{1}2$ has different assignments against $0\bar{1}1, \bar{1}\bar{1}2$ and $\bar{1}12$. Since we find the assignments in which some neighbor pairs of directions $101, 1\bar{1}2, 0\bar{1}1, \bar{1}\bar{1}2, \bar{1}01, \bar{1}12, 011$ and 112 are assigned the same value, we can obtain the above assumed assignments by interchanging the x, y and z axes. Finally, we find that mutually orthogonal directions $100, 021$ and $0\bar{1}2$ have the assigned value 1, which contradicts the sum condition (2.5) which is satisfied in the three orthogonal directions.

Here is another simple proof of this theorem. Consider a composite system of two spin-1/2 particles and spin components on the system:

σ_x^1	σ_x^2	$\sigma_x^1\sigma_x^2$
σ_y^2	σ_y^1	$\sigma_y^1\sigma_y^2$
$\sigma_x^1\sigma_y^2$	$\sigma_x^2\sigma_y^1$	$\sigma_z^1\sigma_z^2$

where σ_a^i represents the spin component in the a direction of system i for $a = x, y, z$ and $i = 1, 2$. The row and column of this table consist of mutually-commutative observables. Since we can measure these mutually-commutative observables without disturbance for the system by appropriate measurements, the assigned value of $\sigma_x^1\sigma_y^2$ must be 1 if we assigned 1 to σ_x^1 and 1 to σ_y^2. The commutative set of observables is called "context". As long as we treat the observables in one context, we can assign their values by a simultaneous eigenstate of them. Now consider the assignment of values over contexts. The simultaneous assignment for the observables in different contexts is called the counter-factual assignment. This comes from the impossibility of obtaining the assigned values in actual simultaneous measurements.

If we assign 1 for four observables $\sigma_x^1, \sigma_x^2, \sigma_y^2, \sigma_y^1$, the products of the pair of them, which are $\sigma_x^1\sigma_x^2, \sigma_y^1\sigma_y^2, \sigma_x^1\sigma_y^2, \sigma_x^2\sigma_y^1$, must be 1. Then, the product of $\sigma_x^1\sigma_x^2$ and $\sigma_y^1\sigma_y^2$ and the product of $\sigma_x^1\sigma_y^2$ and $\sigma_x^2\sigma_y^1$ must be 1. However, we have the relation of the product of operators of spin components as

$$\sigma_x^1\sigma_y^2\sigma_x^2\sigma_y^1 = \sigma_z^1\sigma_z^2$$
$$\sigma_x^1\sigma_x^2\sigma_y^1\sigma_y^2 = -\sigma_z^1\sigma_z^2,$$

which have different signs. It is a contradiction. Furthermore, any other assignments do not satisfy the relation. We can easily see the impossibility of assignment values by the fact that there is no 3x3 magic square satisfying that the product of the three operators in any row or any column must be 1 except the rightmost column in which the product of three operators is -1.

This theorem shows that we cannot define values of observables which are given by functions of dynamical variables independently from the context. This is called the contextuality of the value of observables. Therefore we cannot explain the quantum system by the statistical ensembles of the objects which have

dynamical variables determining the values of all observables. Although the selection of the context is mathematically defined by the selection of a set of mutually commutative observables, it means physically the choice of the observables which are measured not necessarily simultaneously. Consider that we first measure the square of total angular momentum $\sigma^2 := \sigma_x^2 + \sigma_y^2 + \sigma_z^2$ on a spin-1/2 system and obtain its value 3/4. Then we can perform the second measurement of either σ_x or σ_y, not the both at a time, to obtain its value. We cannot measure them at a time due to the non-commutativity of observables $[\sigma_x, \sigma_y] \neq 0$. This choice of measurements is equivalent to the selection of context. If we can choose freely the second measurement, the contextuality of the dynamical variables is a fatal problem for the classical ensemble picture. Conway et al. discussed this point in terms of the free will of observers [8] in choosing the second measurements on the entangled systems.

We note here that the contextuality of the value of an observable cannot be explained by the disturbance model which is independent from the context. The measurement causes the disturbance which prevents measurements of observables in other contexts. We expect that an appropriate disturbance model enables us to construct a context free variable model which is equal to the classical ensemble picture. We may assume that we can measure the system without disturbance only when we measure the observables in one context, since we can assign the values and obtain them with probability unity only for these observables. However, the disturbance model must depend on the context of observables. Consider the disturbance in measuring the spin component of y. If we first measure σ^2, the disturbance in measuring σ_y does not occur since both of observables being to the same context. On the other hand, if we first measure σ_x before measuring σ^2, a disturbance occurs in the second measurement of σ_y. This means that the state at present is not enough to evaluate the disturbance. It is difficult to obtain the explanation of the contextuality by such disturbance model. One of the reasons for the difficulty is that the model loses the Markov property which quantum mechanics assumes. In addition, we have to explain the impossibility of an appropriate measurement of the observables in different contexts.

These are the reasons why it is difficult to construct a physical theory determined by the dynamical variables, which is called the hidden-variable theory. This means that building a classical ensemble picture in quantum mechanics has a considerable obstacle to overcome.

2.3 Discussion on Assignment of Values to Observables

We here see how quantum mechanics avoids the problem of assignment of values to observables together with possible explanations on the problem. The connection between the observables and their values must be contextual. Quantum mechanics performs a contextual assignment of the results of measurements at each time of

the measurements. For a quantum state $|\psi\rangle$, we can consistently assign values to observables when one of its eigenstates is $|\psi\rangle$. That is, we can perform assignment for observables which commute with the projective operator $P_\psi := |\psi\rangle\langle\psi|$. The assignment for the other observables is given at random according to the frequency given by Born's rule in quantum mechanics. The contextuality is consistent with the fact that quantum mechanics does not explain the mechanism of determining values of observables. In quantum mechanics, the transition of the complete set of assigned values, which corresponds to the context, is described by the transition of a quantum state. One of the transitions is the time evolution according to the Schrödinger equation. The equation of motion in classical mechanics is its counterpart. The other is the transition enforced by the measurement on the system, which causes the contextuality.

To explain the contextuality caused by the ordering of the measurements, some researches recognize that the transition by measurements should be described by the subjective probability [15, 16], which appears in the epistemic theory. In the epistemic theory, the probability distribution is modified by the acquisition of knowledge of a system. The transition of the probability distribution also occurs in classical mechanics when the system has ambiguity. For example, in the Bayesian estimation theory, we suppose that a distribution of parameters represents incomplete knowledge. We update the distribution by the results of sequential measurements about the related parameters since the distribution in the Bayesian theory is a subjective probability distribution of the parameters. The epistemic theory supposes that the probability distribution of observables is a subjective probability, not an objective one. The subjective probability seems to be suitable for describing quantum mechanics since it can be defined even if there is no underlying object. According to this view, there is no object which determines the values of observables in quantum mechanics. This means that there is no objective probability distribution for definite value of observables. Moreover, the subjective probability provides a natural explanation for the discrete transition of the distribution which is one of the problems on the locality in quantum mechanics. In this respect, The epistemic theory provides a possible way to explain the quantum mechanics consistently.

However, there is another solution for the contextuality. It has been introduced by Aharonov et al. [31], called the two-state vector formalism, which is the time-symmetric formalism of quantum mechanics. In this formalism, past and future boundary conditions determine the assignment of the contextual values of observables. Furthermore, the theory is capable of eliminating the jump of distributions by a unique modification of the definition of a system. We introduce the formulation in the next chapter.

Chapter 3

Two-State Vector Formalism

We review one of the time-symmetric formulations of quantum mechanics, called two-state vector formalism introduced by Aharonov et al. [31, 34]. Their formulation leads to the discovery of the weak value and the weak measurement in a quantum process. Our main discussion which starts from the next chapter is based on this two-state vector formalism.

3.1 Time Symmetry in Quantum Mechanics

The two-state vector formalism (TSVF) is a new time-symmetric formulation of quantum mechanics developed from the paper by Aharonov, Bergman, and Lebowitz [31], where they consider time symmetry in quantum mechanics. **The time symmetric description of quantum mechanics means here that the description of a quantum system is invariant under interchanging the arrow of time**. Their aim is to recover the time symmetric description of quantum mechanics which is broken by measurements in a quantum system.

In the standard description of quantum mechanics, the time symmetry is broken. Consider a system which evolves with a time independent Hamiltonian H. First, we prepare the system in the quantum state $|\psi\rangle$ at time t_i. After a while, we measure the system and find the quantum state $|\phi\rangle$ at time t_f. We assume that the quantum state $|\psi\rangle$ evolves forward in time according to the Schrödinger equation and the state of the system becomes $|\phi\rangle$ by the measurement. Now, let us consider the same system backward in time. The preparation of a quantum state and the measurement to obtain a result are equivalent operations since they are ideally projective measurements. Thus, we can may also assume that the quantum state $|\phi\rangle$ is prepared at time t_f and evolves backward in time, and then we find $|\psi\rangle$ by the measurement at time t_i. These two, forward and backward, descriptions of the system are not equivalent owing to the difference of the quantum state of the system at any time other than t_i and t_f. For example, we can interpret the system between the preparation and measurement as two different states $|\psi\rangle$ and $|\phi\rangle$ when we are allowed to ignore the time evolution of the system.

Aharonov et al. found that if we introduce the description which treats equiv-

alently the two states corresponding to the preparation and the measurement, we can obtain a time-symmetric formulation of quantum mechanics. In fact, the equivalent treatment of the two states corresponding to the preparation and the measurement has already appeared in the standard description of quantum mechanics. The quantum theory predicts the probability distribution which is obtained from the given quantum state. In the above system, the probability of obtaining the quantum state $|\phi\rangle$ at time t_f is given by

$$\left| \langle\phi| \exp\left(-\frac{i}{\hbar} \int_{t_i}^{t_f} Hdt \right) |\psi\rangle \right|^2. \tag{3.1}$$

We regard this expression such that the quantum state $|\psi\rangle$ evolves forward in time according to the Schrödinger equation. However, it is also possible to interpret it as the quantum state $|\phi\rangle$ evolving backward in time. The backward propagating picture is also obtained according to the Schrödinger equation for the bra state (in other words, the complex conjugate of the wave function).

This picture is consistent even when we consider an intermediate measurement in the system where we prepare the state $|\psi\rangle$ at time t_i and find the state $|\phi\rangle$ by the measurement at time t_f. We hereafter call the system between the two selections "a quantum process". If we measure an observable which is the projector P_a of state $|a\rangle$ at an intermediate time t_m in the quantum process, the probability $p(a, t_m)$ of obtaining $|a\rangle$ at t_m and then $|\phi\rangle$ at t_f is given by

$$p(a, t_m) := \frac{1}{N} \left| \langle\phi| \exp\left(-\frac{i}{\hbar} \int_{t_m}^{t_f} Hdt \right) P_a \exp\left(-\frac{i}{\hbar} \int_{t_i}^{t_m} Hdt \right) |\psi\rangle \right|^2, \tag{3.2}$$

where $N := \sum_a p(a, t_m)$ is a normalization constant. This expression is also time symmetric. Similarly, the time symmetry holds when we consider an imperfect measurement which is described by using a positive operator valued measure (POVM). The imperfect measurement is a measurement which does not change the state of a system into the eigenstate of the observable according to the measurement outcome. The state of the system is turned into another state which is represented by a density operator by the imperfect measurement. The transition of quantum state by the imperfect measurement is represented as the map from a quantum state to another quantum state. For example, if we obtain the result a of the imperfect measurement on a quantum state ρ, the quantum state after the measurement ρ_a is represented by using Kraus operators $\{M_a\}$ as

$$\rho_a := \frac{M_a \rho M_a^\dagger}{\mathrm{Tr}\left(M_a^\dagger M_a \rho \right)}, \tag{3.3}$$

where we ignore the time evolution of a quantum state for simplicity. The POVM Π_a is defined as $\Pi_a := M_a^\dagger M_a$ and satisfies $\sum_a \Pi_a = \mathbb{1}$. If we perform the imperfect measurement in the above process, the probability $p'(a, t_m)$ of obtaining the result a in the measurement at the intermediate time t_m is given by

$$p'(a, t_m) := \frac{1}{N'} \left| \langle\phi| \exp\left(-\frac{i}{\hbar} \int_{t_m}^{t_f} Hdt \right) M_a \exp\left(-\frac{i}{\hbar} \int_{t_i}^{t_m} Hdt \right) |\psi\rangle \right|^2, \tag{3.4}$$

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where $N' := \sum_a p'(a, t_m)$ is a normalization constant. This is also time-symmetric as in the case of a projective measurement (3.2).

Although we can describe the prediction by the probability time-symmetrically, one may say that the transition of quantum states by measurements seems to be time asymmetrical. However, we do not need to describe the transition of the quantum state, called the jump of a quantum state, in the two-state vector formalism. We can predict the probability without considering the jump if we give all boundary conditions of the system. In the time-symmetric formulation of the quantum mechanics, we give all boundary conditions as the initial state and the final state of the system to obtain the predictions for the system. We regard that each state evolves forward and backward in the quantum process.

The most important point in the time-symmetric formulation is that it provides a reasonable interpretation for the contextuality. We see this point later in this chapter.

3.2 Formulation

The time-symmetric formulation of quantum mechanics must have a time-symmetric definition of states. Aharonov et al. [34] defined such states, named two-states, by two vectors of the Hilbert space of a system. This is why the time-symmetric formulation is called "two-states vector" formalism. Each vector represents the time boundary condition of the system. We can select the initial state of the system by performing a projective measurement by a certain state $|\psi\rangle$. This is called "pre-selection" and the state $|\psi\rangle$ is interpreted as the initial boundary condition of the system. Then we can perform another projective measurement at a later time to examine whether the state is $|\phi\rangle$ or not. Since this projective measurement can be treated on the same footing as the first projective measurement in the time-symmetric view, this projective measurement is called "post-selection" and the state $|\phi\rangle$ is interpreted as the final condition of the system. They call the initial state $|\psi\rangle$ "pre-selected state" and the final state $|\phi\rangle$ "post-selected state". The two-state of the system is defined by using the two states, $|\psi\rangle$ and $|\phi\rangle$.

Definition 1 (Two-states: the states in the two-state vector formalism)
The two-state of a quantum process, which is pre-selected by $|\psi\rangle$ and post-selected by $|\phi\rangle$, is defined as

$$\varrho := |\psi\rangle \langle\phi| , \tag{3.5}$$

if we ignore the time evolution of the system.

The two-state $\varrho(t)$ evolves according to the Liouville-von-Neumann equation just like the standard quantum states:

$$i\hbar \frac{\partial \varrho(t)}{\partial t} = [H, \varrho(t)], \tag{3.6}$$

t_f — $|\phi\rangle$ final state

$U(t - t_f)$

t — $\varrho(t)$ two-state

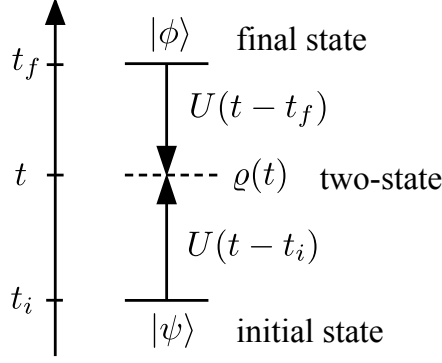$U(t - t_i)$

t_i — $|\psi\rangle$ initial state

Figure 3.1: The two-state, which is a state in the two-state vector formalism, is defined by two states specifying the time boundary. The initial state does not have to evolve into the final state. Each of the boundary states is determined by projective measurements at each time, that is, it is given by the eigenstate which corresponds to the result of the projective measurement at the time. We call the projective measurement at the initial and final times the pre- and post-selection, respectively.

which is derived by two Schrödinger equations for a system with a Hamiltonian H, one of which evolves forward in time and the other evolves backward in time:

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle &= H\left|\psi(t)\right\rangle, \\
-i\hbar\frac{\partial}{\partial t}\left\langle\phi(t)\right| &= \left\langle\phi(t)\right|H \quad \xrightarrow{-t\to t'} \quad i\hbar\frac{\partial}{\partial t'}\left\langle\phi(t)\right| = \left\langle\phi(t)\right|H.
\end{aligned}
$$

The Liouville equation is derived as

$$
\begin{aligned}
i\hbar\frac{\partial\varrho(t)}{\partial t} &= i\hbar\left(\frac{\partial}{\partial t}\left|\psi(t)\right\rangle\left\langle\phi(t)\right| + \left|\psi(t)\right\rangle\frac{\partial}{\partial t}\left\langle\phi(t)\right|\right) \\
&= H\left|\psi(t)\right\rangle\left\langle\phi(t)\right| - \left|\psi(t)\right\rangle\left\langle\phi(t)\right|H \\
&= [H, \varrho(t)]. \tag{3.7}
\end{aligned}
$$

If the Hamiltonian H depends on time, we need the extra term with dH/dt. Using the unitary operator of time evolution U, the two-state at time t is described as

$$
\varrho(t) := U(t, t_i)\left|\psi\right\rangle\left\langle\phi\right|U(t, t_f)^\dagger, \tag{3.8}
$$

for which the pre-selection is performed at t_i and the post-selection is performed at t_f (Fig. 3.1).

The ability of prediction by the two-state vector formalism is equivalent to that of the standard formalism of quantum mechanics. The standard formalism of quantum mechanics provides the way to predict the probability of obtaining a result of measurement. We can predict the probability of obtaining a certain result of an intermediate projective measurement by the two-state. If we measure the
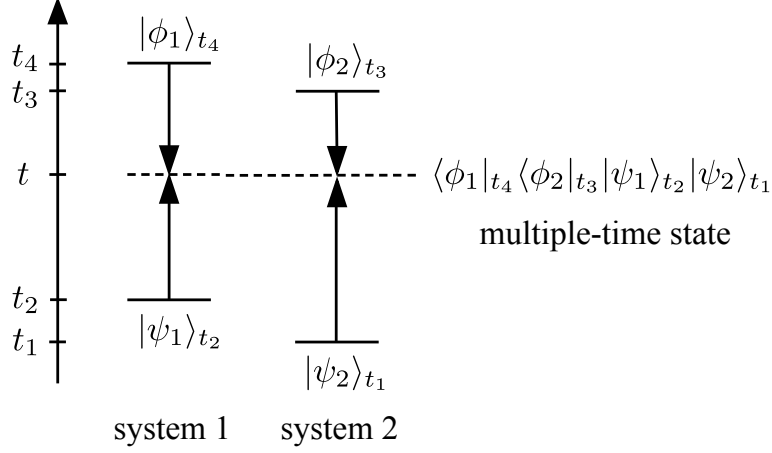
Figure 3.2: The multiple-time state in two systems. Each system has the initial and final boundaries. We here ignore time evolution of the systems. If there is non-zero Hamiltonian including the interaction with each other in the process, the initial and final states evolve according to the Hamiltonian.

observable A at time t in the quantum process determined by the two-state, the probability of obtaining the result a which is an eigenvalue of A is given by

$$\mathrm{Prob}(P_a) := \frac{1}{N} |\langle \phi| \, U(t_f, t) P_a U(t, t_i) \, |\psi\rangle \, |^2, \tag{3.9}$$

where P_a is a projective operator for the eigenstate of A corresponding to its eigenvalue a and $N := \sum_a |\langle \phi| \, U(t_f, t) P_a U(t, t_i) \, |\psi\rangle \, |^2$. This probability is called **the ABL probability**, which is suggested by Aharonov, Bergmann and Lebowitz [31]. This is just a conditional probability in the quantum processes, which is obtained by the standard calculation of quantum mechanics. The ABL probability (3.9) can be written by using the two-state,

$$\mathrm{Prob}(P_x) := \frac{1}{N'} |\mathrm{Tr}\left(\varrho(t) P_a\right)|^2 \tag{3.10}$$

where $N' := \sum_a |\mathrm{Tr}\left(\varrho(t) P_a\right)|^2$.

The two-state arises actually as the simplest case in a more general description of quantum processes. We can perform multiple pre- and post-selections to one quantum system. Multiple quantum systems can have these boundary conditions of each subspace. Furthermore, we can perform projective measurements on the same system repeatedly. To unify these two situations, Aharonov, Popescu, Tollaksen and Vaidman [36] discussed the multiple boundary conditions in this formalism. For simplicity, we hereafter ignore the time evolution of the system. They give the following definition of multiple-time states.

Definition 2 (Multiple-time states: generalized states in the TSVF) *The multiple-time state of quantum process is defined as*

$$\Pi_k \langle \phi_k|_{t_k} \Pi_l \, |\psi_l\rangle_{t_l} \tag{3.11}$$

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where $|\psi_l\rangle_{t_l}$ represents the pre-selected state at time t_l for $l = 1, 2, ...$ and $|\phi_k\rangle_{t_k}$ represents the post-selected state at time t_k for $k = 1, 2, ...$ (for example, see Fig. 3.2).

Here, each boundary condition corresponds to a projective measurement. We therefore need the results of projective measurements on the system to describe the system in terms of this multiple-time state. Since we can treat them similarly, we also call the formalism including the multiple-time state as the two-state vector formalism in this paper.

This multiple-time states can be defined even when the boundary conditions are partially specified. The unspecified boundaries of a system are described as the maximally entangled state with another system in the two-state vector formalism. If we specify only the initial boundary condition of the system, which is described as the one-time state, the two-state of the system is described by giving the final condition as a maximally entangled state between the system and the other system. For example, consider the N-level system which is pre-selected at time t_i by the projective measurement of $|\psi\rangle$ on the system to prepare the state $|\psi\rangle$. If we do not specify the post-selected state of the system, the state is effectively given by the maximally entangled state between the system and another copy of N-level system $\sum_k^N 1/\sqrt{N} |w_k\rangle \otimes |w_k\rangle$, where $\{|w_i\rangle\}$ is a complete orthonormal system on each system. Although the probability of obtaining the state $|a\rangle$ in this process is calculated by using the two-state (3.10), since the pre-selected state is only specified on the original system, we take the trace in the original system and consider the absolute value as the norm in the copied system.

$$
\begin{aligned}
\mathrm{Prob}(P_a) &= \frac{1}{N'} \left| \frac{1}{\sqrt{N}} \sum_k \langle w_k| \otimes \langle w_k| P_a |\psi\rangle \right|^2 \\
&= \frac{1}{N'} \frac{1}{N} \left| \sum_k \langle w_k| \langle w_k| P_a |\psi\rangle \right|^2 \\
&= \frac{1}{N'} \frac{1}{N} \sum_k |\langle w_k| P_a |\psi\rangle|^2 \\
&= \frac{1}{N'} \frac{1}{N} |\langle a|\psi\rangle|^2,
\end{aligned} \tag{3.12}
$$

where P_a is the projector of the quantum state $|a\rangle$ and $N' := \sum_a |\langle a|\psi\rangle|^2 /N$ is the normalization constant. This probability is equal to the probability of obtaining $|a\rangle$ after we prepare $|\psi\rangle$ on the system in standard quantum mechanics. Similarly, if we do not specify the initial boundary, we can describe the system by considering the pre-selected state as a maximally entangled state. This probability also represents the probability of realizing the quantum process from $|\psi\rangle$ to $|a\rangle$ as in the case of the one-state formalism where it is done by Born's rule.

3.3 Weak Value

Now, we consider the value of an observable in a quantum process. There is two kinds of values of an observable in a quantum process, In the two-sate vector formalism, we can calculate values of observables by using the ABL probability. Let us consider the spectral decomposition of an observable $A := \sum_a a P_a$, where $\{P_a\}$ are projectors of the eigenstates $|a\rangle$ of A. Using the ABL probability, we calculate the value of the observable A in the process from $|\psi\rangle$ to $|\phi\rangle$ as

$$\langle A \rangle_{\psi \to \phi} := \sum_a a \mathrm{Prob}(P_a) = \frac{1}{N} \sum_a a |\langle \phi| P_a |\psi\rangle|^2. \tag{3.13}$$

where $\mathrm{Prob}(P_a)$ is the ABL probability of obtaining the eigenvalue a and N is the normalization constant of the probability: $N := \sum_a |\langle \phi| P_a |\psi\rangle|^2$. This is just the conditional expectation value with the post-selection of $|\phi\rangle$. We can obtain the value by performing projective measurements in the quantum process.

Meanwhile, we can also consider another value of an observable, called "weak value", defined as follows.

Definition 3 (Weak value) *The weak value of the observable A is defined in the process from the quantum state $|\psi\rangle$ to $|\phi\rangle$ by*

$$A_w := \frac{\langle \phi| A |\psi\rangle}{\langle \phi|\psi\rangle}, \tag{3.14}$$

when we ignore the time evolution of the states.

Using the two-state, the weak value can be written as

$$A_w = \sum_a a \frac{\mathrm{Tr}\,(\varrho(t) P_a)}{\mathrm{Tr}\,(\varrho(t))}. \tag{3.15}$$

The weak value is the value of an observable which arises specifically in a quantum process in the two-state vector formalism [34, 14]. There is a connection between the expectation value in the standard formalism and the weak value in the two-state vector formalism [46],

$$\langle \psi| A |\psi\rangle = \sum_i |\langle \phi_i|\psi\rangle|^2 \frac{\langle \phi_i| A |\psi\rangle}{\langle \phi_i|\psi\rangle}, \tag{3.16}$$

where $\{|\phi_i\rangle\}$ is a complete orthonormal set of post-selected states. This means that the average of the weak value of A with the probability of finding the states $\{|\phi_i\rangle\}$ is equal to the expectation value of A. This allows us to interpret that the weak value is the value of an observable for a part of events which are prepared in the quantum state $|\psi\rangle$.

The weak value is measured by a new scheme called the weak measurement in a quantum process, which we describe next.
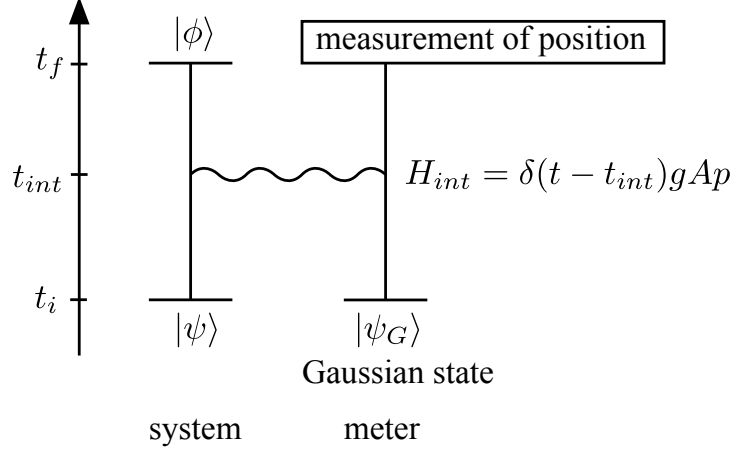
Figure 3.3: The simple boundary conditions for the weak measurement of a real part of the weak value of a system observable A. We obtain the real part of the weak value as the conditional expectation value of the position of the meter in the case where the strength of interaction is weak so that the approximation (3.18) holds.

3.4 Weak Measurement

Aharonov, Albert and Vaidman gave an operational meaning to the weak value by finding a new type of measurement called "weak measurement" [18], which is performed by using a weak interaction between a system and a meter.

First, we consider the weak measurement of the real part of the weak value (Fig. 3.3). Let the system be prepared in a state $|\psi\rangle$ and let the meter be prepared in a Gaussian distributed state around the origin

$$|\psi_G\rangle := (2\pi\sigma)^{(-1/4)} \int dx \exp\left(-x^2/4\sigma^2\right)|x\rangle$$

in $\mathcal{L}^2(\mathbb{R})$, where σ^2 is the variance of the position of the meter. The measurement is performed by an interaction between the system and the meter. We use the von-Neumann type interaction which includes the measuring observable A coupling with the momentum p of the meter. We impose the interaction with the strength g instantaneously at t_1, so that the interaction is described by $H_{int}(t) := \delta(t_1-t)gAp$. After the interaction, the quantum state becomes $\exp\left(-igAp\right)|\psi\rangle\otimes|\psi_G\rangle$. Then we perform post-selection for the system by the projective measurement $P_\phi := |\phi\rangle\langle\phi|$. The probability distribution of obtaining the position x of the meter after the post-selection is given by

$$\Pr(x) := \frac{1}{N}|\langle\phi|\otimes\langle x|\exp\left(-igAp\right)|\psi\rangle\otimes|\psi_G\rangle|^2, \qquad (3.17)$$

where the normalization constant $N := \int_{-\infty}^{\infty} |\langle\phi|\otimes\langle x|\exp\left(-igAp\right)|\psi\rangle\otimes|\psi_G\rangle|^2$.
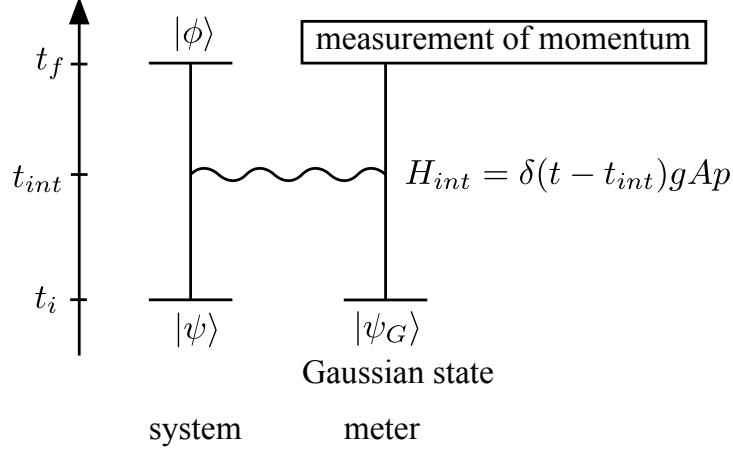
Figure 3.4: The simple boundary conditions for the weak measurement of the imaginary part of the weak value of a system observable A. The difference from the measurement of the real part of the weak value is in the readout observable of the meter.

If the strength of the interaction g is weak, we can approximate (3.17) as

$$
\begin{aligned}
\Pr(x) \; \simeq \; & \frac{1}{\sqrt{2\pi}\sigma N} \left| \langle\phi| \otimes \langle x| \left(1 - igAp\right) |\psi\rangle \otimes \int dx' \exp\left(-\frac{x'^2}{4\sigma^2}\right) |x'\rangle \right|^2 \\
= \; & \frac{1}{\sqrt{2\pi}\sigma N} \left| \langle x| \langle\phi|\psi\rangle \left(1 - ig\frac{\langle\phi| A |\psi\rangle}{\langle\phi|\psi\rangle}p\right) \int dx' \exp\left(-\frac{x'^2}{4\sigma^2}\right) |x'\rangle \right|^2 \\
\simeq \; & \frac{1}{\sqrt{2\pi}\sigma N} \left| \langle x| \langle\phi|\psi\rangle \exp\left(-ig\frac{\langle\phi| A |\psi\rangle}{\langle\phi|\psi\rangle}p\right) \int dx' \exp\left(-\frac{x^2}{4\sigma^2}\right) |x'\rangle \right|^2 \\
\simeq \; & \frac{1}{\sqrt{2\pi}\sigma N} \left| \langle\phi|\psi\rangle \exp\left(-\frac{\left(x - \mathrm{Re}\left(g\frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}\right)\right)^2}{4\sigma^2}\right) \right|^2 .
\end{aligned}
\tag{3.18}
$$

As a result, the Gaussian distribution of the meter shifts by the real part of the weak value $g\mathrm{Re}\left(A_w\right)$. We can thereby obtain the real part of the weak value as the shift in the expectation value of the position of the meter by choosing successful events in the post-selection.

The imaginary part of the weak value is also obtained by the post-selection technique (Fig. 3.4). If we measure the momentum of the meter at time t_f, the probability distribution of the momentum is calculated as

$$
\begin{aligned}
\Pr(p) \simeq \; & \frac{1}{\sqrt{2\pi}\sigma N} \left| \langle\phi| \otimes \langle p| \left(1 - igAp\right) |\psi\rangle \otimes \int dx' \exp\left(-\frac{x'^2}{4\sigma^2}\right) |x'\rangle \right|^2 \\
= \; & \frac{1}{\sqrt{2\pi}\sigma N} \left| \langle p| \langle\phi|\psi\rangle \left(1 - ig\frac{\langle\phi| A |\psi\rangle}{\langle\phi|\psi\rangle}p\right) \int dx' \exp\left(-\frac{x'^2}{4\sigma^2}\right) |x'\rangle \right|^2 \\
\simeq \; & \frac{1}{\sqrt{2\pi}\sigma N} \left| \langle\phi|\psi\rangle \exp\left(-ig\frac{\langle\phi| A |\psi\rangle}{\langle\phi|\psi\rangle}p\right) \int dx' \exp\left(-\frac{x^2}{4\sigma^2}\right) \langle p|x'\rangle \right|^2
\end{aligned}
$$

$$\simeq \frac{1}{\sqrt{2\pi}\sigma N} \left| \langle\phi|\psi\rangle \exp\left(-\sigma^2\left(p - \frac{g}{2\sigma^2}\mathrm{Im}\left(\frac{\langle\phi|\,A\,|\psi\rangle}{\langle\phi|\psi\rangle}\right)\right)\right) \right|^2, \qquad (3.19)$$

when g is small. The distribution of the momentum of the meter shifts by $g\mathrm{Im}\,(A_w)\,/2\sigma^2$, which is proportional to the imaginary part of the weak value.

The weak measurement is useful for precision measurements. Hosten and Kwiat use the weak measurement to detect the spin Hall effect of light [21]. The infinitesimal weak interaction of the effect was detected by the shift of the light, which is proportional to the weak value. In the standard measurement scheme, the shift caused by the interaction is restricted in the range of the spectrum of the observable appearing in the interaction Hamiltonian. However, if we use the post-selection technique, the shift is amplified beyond its spectrum. This amplification effect is called "weak value amplification", and the amplification effect is also used in detection of the beam deflection in optical experiments [22].

From a technical point of view, there are discussions whether the weak measurement scheme is useful for the detection or estimation of small effect of interactions in a system. Koike and Tanaka showed that the weak measurement is not useful in the estimation of the strength of interaction [23]. Since we dump unsuccessful events of post-selection in the weak measurement, it is reasonable that the weak measurement scheme is not useful in the statistical estimation. However, the weak measurement is helpful when the system has systematic errors. If we do not know what kind of errors exist in the system, we cannot treat the system by exact statistical error models. Lee and Tsutsui show that the weak measurement is helpful when we have such incomplete description of errors [24]. They call the indeterminacy of errors "uncertainty". If the system has the uncertainty, the weak measurement is useful to detect or estimate parameters in the system. For example, the experiment of detection of the spin Hall effect of light [21] was performed by a relatively simple equipment of an optical system which does not have an extraordinary noise proof. If the weak measurement can amplify the signal beyond the uncertainty of the system, it allows us to measure the small effect of interaction.

3.5   Danan's Interferometer

There is an interesting example where the interpretation of the forward and backward propagating states in the two-state vector formalism becomes relevant. Here we mention it to show how it works with the interpretation involving real (but negative) values of the generalized probability which is given by the weak value of the corresponding projection operator. Later, we discuss another, novel example with a modified setup in which the imaginary value of the generalized probability appears, and demonstrate how it can be detected experimentally.

The first example was proposed by Danan et al. [44], where they use beam splitters (BS) and a Mach-Zehnder interferometer (MZI) (Fig. 3.5). The interferometer is configured so that all incident photons go to a port below the interferometer in

Fig. 3.5 (a). In this setup, the detected photons on the terminal detector should come from the bottom path, which does not pass the interferometer.

Then we put a trick so that vertical oscillations are exerted on each mirror with different frequencies. The output signal vibrates by the frequencies of the mirrors which the detected photons passed. In the situation of Fig. 3.5 (a), the detection of the frequency of the mirror C was expected. However, the frequencies of the mirrors A and B were detected as well. Since the photons reflected by the mirrors A and B are expected to go to a port below the interferometer, which is not connected to the detector, this result is puzzling.

The detected photons seem to take the discontinued path, which does not contact with the mirrors E and F. To provide an intuitive interpretation of this result, the authors of [44] suggested to take account of both the forward and backward propagating photons simultaneously. The forward propagating photons which enter the interferometer go to a port below it and do not pass the mirror F. On the other hand, the backward propagating photons, which enter from the detector and go to the interferometer, go to the left port and do not pass the mirror E. From the outcome of the experiment, one realizes that the only paths where both the forward and backward propagating photons pass have affected the output signals.

This suggests that, in order to see where the photons pass, we should consider the forward and backward propagating photons simultaneously. The forward propagating state of the photon, when it is reflected by the mirrors A, B and C, is given by

$$|\Psi\rangle := \frac{1}{\sqrt{3}} \left(|A\rangle + i \, |B\rangle + |C\rangle \right), \tag{3.20}$$

where $|A\rangle, |B\rangle$ and $|C\rangle$ are the localized states around the mirrors A, B and C. The backward propagating state of the photon at the same time is given by

$$\langle\Phi| := \frac{1}{\sqrt{3}} \left(\langle A| + i \, \langle B| + \langle C| \right). \tag{3.21}$$

The weak value of the projection operators on each site is calculated by the two-states as

$$\begin{aligned} (P_A)_w &= 1, \\ (P_B)_w &= -1, \\ (P_C)_w &= 1, \\ (P_E)_w &= 0, \\ (P_F)_w &= 0. \end{aligned} \tag{3.22}$$

The absolute value of the weak value of projection operators is regarded to represent the strength of the output signal of the frequency characteristics to the mirror.
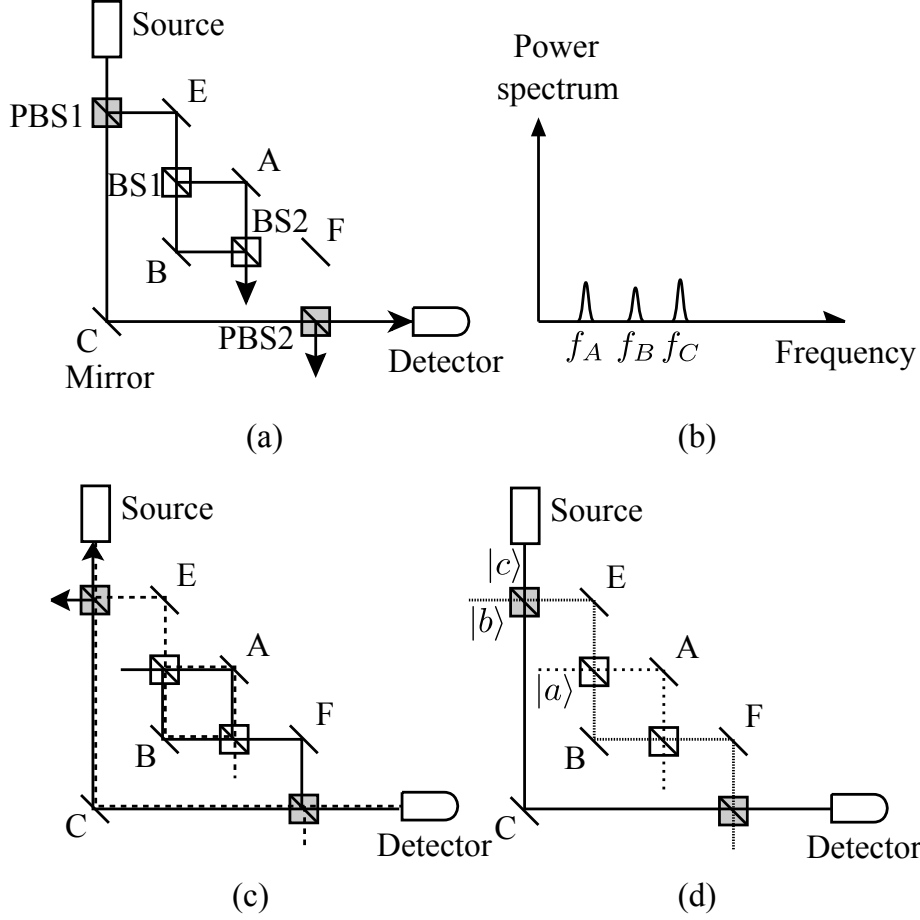
Figure 3.5: (a) The experimental setup for Danan's interferometer. Each labeled mirror from A to F is a piezo driven mirror which vibrates around its horizontal axis at each unique frequency. The Mach-Zehnder interferometer with the mirrors A and B is aligned to emit all photons to a port below the interferometer. It seems that all photons which are detected at the detector have passed the mirror C. (b) However, the detected signal shows the characteristic frequencies of the mirrors A, B and C. (c) Paths of the backward propagating photons. It suggests that the backward propagating state passing the mirrors A, B and C affect the output signal. (d) Paths in the vectorial representation.

Now we see how to reach the intriguing conclusion in detail for the setup Fig. 3.5 (a). It employs a polarized beam injected into the first polarization beam splitter (PBS), which is produced with polarization at angle $54.7°$ so that one third of the injected beam power goes to the lower arm and two thirds of the beam power goes to the right arm. After passing the PBS1, the state of the beam is then given by

$$\frac{1}{\sqrt{3}} \left(|H\rangle |C\rangle + i\sqrt{2} |V\rangle |E\rangle \right), \qquad (3.23)$$

where $|H\rangle$ and $|V\rangle$ are the horizontal and vertical polarized states and $|C\rangle$ and $|E\rangle$ are the localized states around the paths with the mirrors C and E. Now, BS1 splits the beam with equal power into two paths, $|A\rangle$ and $|B\rangle$, rendering the state

as

$$\frac{1}{\sqrt{3}}\left(\left|H\right\rangle\left|C\right\rangle+\left|V\right\rangle\left(i\left|B\right\rangle+\left|A\right\rangle\right)\right). \tag{3.24}$$

To describe the interference of the PBSs and BSs after passing mirrors A, B, and C, we consider a vectorial representation of the paths by the three vectors, $\left|a\right\rangle$, $\left|b\right\rangle$ and $\left|c\right\rangle$ (see Fig. 3.5 (d)) represented, for example, by $\left|a\right\rangle=\left|V\right\rangle\left|A\right\rangle$, $\left|b\right\rangle=\left|V\right\rangle\left|B\right\rangle$ and $\left|c\right\rangle=\left|H\right\rangle\left|C\right\rangle$ around the mirrors A, B and C.

Suppose that we configure the setup so that all injected photons go to the detector, and let the state of the injected photon be given by

$$\left|\Psi_0\right\rangle:=\begin{pmatrix}0\\0\\1\end{pmatrix}:=0\left|a\right\rangle+0\left|b\right\rangle+1\left|c\right\rangle. \tag{3.25}$$

PBS1 and PBS2 can then be written as

$$PBS_1=PBS_2=\begin{pmatrix}1&0&0\\0&1/\sqrt{3}&i\sqrt{2}/\sqrt{3}\\0&i\sqrt{2}/\sqrt{3}&1/\sqrt{3}\end{pmatrix}, \tag{3.26}$$

and BS1 and BS2 are

$$BS_1=BS_2=\begin{pmatrix}1/\sqrt{2}&-i/\sqrt{2}&0\\-i/\sqrt{2}&1/\sqrt{2}&0\\0&0&1\end{pmatrix}. \tag{3.27}$$

Thus, the pre-selected state around the mirrors A, B and C is found to be

$$\left|\Psi\right\rangle=BS_1PBS_1\left|\Psi_0\right\rangle=\begin{pmatrix}1/\sqrt{3}\\i/\sqrt{3}\\1\sqrt{3}\end{pmatrix}, \tag{3.28}$$

which is equal to (3.24). If the MZI is aligned in such a way that all photons go to the mirror E at BS2, it gives a phase π on the path $\left|b\right\rangle$ by $PS_B(\pi)$, which is defined by

$$PS_B(\theta)=\begin{pmatrix}1&0&0\\0&e^{i\theta}&0\\0&0&1\end{pmatrix}. \tag{3.29}$$

The state after passing BS2 is

$$BS_2PS_B(\pi)BS_1PBS_1\left|\Psi_0\right\rangle=\begin{pmatrix}0\\-i\sqrt{2}/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}, \tag{3.30}$$

from which one sees that indeed all photons end up in the detector,

$$PBS_2BS_2PS_B(\pi)BS_1PBS_1\left|\Psi_0\right\rangle=\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{3.31}$$

27

Let us consider that the internal MZI does not provide an extra phase on the path $|b\rangle$, which means that it passes all photons to the port below it

$$BS_2 PS_B(0) BS_1 PBS_1 |\Psi_0\rangle = \begin{pmatrix} \sqrt{2}/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \end{pmatrix}. \qquad (3.32)$$

In this situation, the state of the backward propagating photon is determined from the state at the detector,

$$\langle\Phi_0| := (0, 0, 1), \qquad (3.33)$$

as

$$\langle\Phi| = \langle\Phi_0| \, PBS_2 BS_2 PS_B(0) = (1/\sqrt{3}, i/\sqrt{3}, 1/\sqrt{3}). \qquad (3.34)$$

The pre- and post-selected states, $|\Psi\rangle$ and $|\Phi\rangle$, are equal to (3.20) and (3.21).

To find the power spectrum of the final outcome, one needs to look at the probability distribution of the position of the photons at the detector. For this purpose, we note that a small tilt θ_A of the mirror A along the vertical axis causes a vertical shift of the photon on the path of the mirror A, which is described by the interaction Hamiltonian $H_{intA} := \theta_A P_A p$, where P_A is a projector for the localized state around the mirror A and p is the vertical momentum of the photon.

To evaluate the effect explicitly, we prepare the injected photons in a Gaussian distribution of position $|\psi_G\rangle$ along the vertical axis. If all mirrors tilt independently by the angles θ_i for $i = A, B, C$, the total Hamiltonian reads $\theta_A P_A p + \theta_B P_B p + \theta_C P_C p$, where P_i for $i = A, B, C$ are projectors for the localized states around the mirror i. After the interaction, the distribution of the position $\mathrm{Pr}(x)$ of the detected photons, which passed the post-selection, turns out to be

$$\begin{aligned}
\mathrm{Pr}(x) &:= \left| \langle x| \, \langle\Phi| \, e^{-i(\theta_A P_A p + \theta_B P_B p + \theta_C P_C p)} \, |\Psi\rangle \, |\phi_G\rangle \right|^2 \\
&\simeq \left| \langle x| \, \langle\Phi| \, (\mathbb{1} - i\,(\theta_A P_A p + \theta_B P_B p + \theta_C P_C p)) \, |\Psi\rangle \, |\phi_G\rangle \right|^2 \\
&= \left| \langle x| \, \langle\Phi|\Psi\rangle \, (1 - i\,(\theta_A P_{Aw} p + \theta_B P_{Bw} p + \theta_C P_{Cw} p)) \, |\phi_G\rangle \right|^2 \\
&= \left| \langle\Phi|\Psi\rangle \right|^2 \left| \langle x| \, (1 - i\,(\theta_A P_{Aw} + \theta_B P_{Bw} + \theta_C P_{Cw})\, p) \, |\phi_G\rangle \right|^2, \qquad (3.35)
\end{aligned}$$

where $P_{iw} := \langle\Phi| P_i |\Psi\rangle / \langle\Phi|\Psi\rangle$ for $i = A, B, C$ are weak values of projectors. Using (3.22), we obtain

$$\begin{aligned}
\mathrm{Pr}(x) &\simeq \left| \langle\Phi|\Psi\rangle \right|^2 \left| \langle x| \, (1 - i\,(\theta_A - \theta_B + \theta_C)\, p) \, |\phi_G\rangle \right|^2 \\
&= \left| \langle\Phi|\Psi\rangle \right|^2 \left| \int dp \langle x|p\rangle \, (1 - i\,(\theta_A - \theta_B + \theta_C)\, p) \, \langle p|\phi_G\rangle \right|^2. \qquad (3.36)
\end{aligned}$$

Using the Gaussian distribution of the vertical momentum,

$$\langle p|\psi_G\rangle := \left(\frac{1}{2\pi\sigma_p^2} \right)^{\frac{1}{4}} e^{-\frac{p^2}{4\sigma_p^2}}, \qquad (3.37)$$

28

where σ_p^2 is the variance of p, we perform the integration over p in (3.36) to obtain

$$
\begin{aligned}
\Pr(x) &\simeq \sqrt{\frac{2\sigma_p^2}{\pi}} \left| 1 + 2\left(\theta_A - \theta_B + \theta_C\right) x\sigma_p^2 \right|^2 e^{-2\sigma_p^2 x^2} \\
&\simeq \sqrt{\frac{2\sigma_p^2}{\pi}} \left(1 + 4\left(\theta_A - \theta_B + \theta_C\right) x\sigma_p^2 \right) e^{-2\sigma_p^2 x^2},
\end{aligned}
\tag{3.38}
$$

where we have ignored the higher order of the tilts θ_i. The probability distribution of the position of the detected photons depends on the tilts of all mirrors, A, B and C. Since the mirrors vibrate with each frequency, the power spectrum of the position of detected photons has peaks around each frequency of the mirrors.

This illustrates an interesting relation between the interference and the post-selection, and the usefulness of the interpretation of considering both the forward and backward evolving states simultaneously as proposed in [44]. However, it should be noted that this effect may also be explained by the leakage of the photons from the interferometer caused by the oscillations of the mirrors, destroying the complete destruction of the interference due to the oscillation [45].

To argue the advantage of the proposed interpretation, consider that we perform a projective measurement on the paths around each mirror. In the setup, we cannot find photons around the mirrors E or F, but we can find photons around the mirrors A, B or C. Although each situation is counter-factual, and hence we cannot identify the path of each photon, it is counterintuitive to realize such outcomes with only the forward propagating picture. In this respect, one may think that the interpretation offers a simple, and possibly intuitive, way to understand the output of the measurement without being involved in the detailed consideration of the leakage or the imperfection of the setup.

3.6 Assignment of Values to Observables in the TSVF

We here discuss the assignment of the value of an observable in a quantum process to see a merit of introducing the two-state vector formalism.

Consider the quantum processes given by the pre-selection and the post-selection. The impossibility of assignment of values of observables argued in the previous section also exists in assignment of values of observables to the quantum processes. If we consider the assignments of all results of measurements to observables, we know that it is impossible since the Kochen-Specker theorem is based on the only simultaneous assignments of eigenvalues on non-commutative observables. The post-selected ensembles cannot help solve the difficulty of non-contextual assignments of the results of measurements.

However, we can consistently assign eigenvalues to more observables when we consider a quantum process. For example, consider a spin-1/2 system (Fig. 3.6). Let a process be pre-selected by $|x+\rangle$, which is a spin-up state in the x direction, and be post-selected by $|z+\rangle$ which is that in the z direction. P_{x+} and P_{z+} represent projective operators corresponding the states respectively. We can calculate
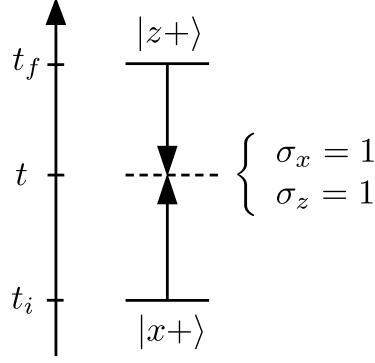
Figure 3.6: In a quantum process fixed from the spin-up state in the x direction to the spin-up state in the z direction, we can assign the values of observables, both σ_x and σ_z with probability unity.

the conditional probability of obtaining $|x+\rangle$ and that of obtaining $|z+\rangle$ at the intermediate time of the process. It is given by the ABL probability rule as

$$\text{Prob}(P_{x+}) = \frac{1}{N} |\langle z+| P_{x+} |x+\rangle|^2 = 1, \tag{3.39}$$

$$\text{Prob}(P_{z+}) = \frac{1}{N'} |\langle z+| P_{z+} |x+\rangle|^2 = 1, \tag{3.40}$$

where the normalization constants are defined by $N := |\langle z+| P_{x+} |x+\rangle|^2 + |\langle z+| P_{x-} |x+\rangle|^2$ and $N' := |\langle z+| P_{z+} |x+\rangle|^2 + |\langle z+| P_{z-} |x+\rangle|^2$. This means that the processes which pass these pre- and post-selections can be seen to have the values assigned to two non-commutative observables σ_x and σ_z. This assignment of eigenvalues to observables in a quantum process can be represented by the weak value. In fact, we can calculate the weak value of σ_x and σ_z as

$$\sigma_{xw} = \frac{\langle z+| P_{x+} |x+\rangle}{\langle z + |x+\rangle} = 1, \tag{3.41}$$

$$\sigma_{zw} = \frac{\langle z+| P_{z+} |x+\rangle}{\langle z + |x+\rangle} = 1, \tag{3.42}$$

which are consistent with the above results (3.39) and (3.40). This comes from the fact that the weak value is equal to the eigenvalue of the observable if either the initial or final state is an eigenstate of the observable.

Aharonov et al. suggested that the weak value is the value of an observable in a quantum process. There are reasons why the weak value is suitable for the value of an observable. One of the reasons is that the weak value represents the consistent assignment of eigenvalues to observables which can be obtained by intermediate projective measurements with probability unity. However, there is a crucial reason which comes from the concept of assignment of the value of an observable. If we suppose the weak value as the value of an observable in a quantum process, we find that the weak value has interesting features as the assignment of the value of an observable. Since we gives the projective measurement which will be performed

on the system as the boundary condition of the system in this formalism, we only assign eigenvalues of observables in a quantum process which are actually measured at the post-selection on the system. It means that this formalism naturally excludes the counter-factual assignment of eigenvalues to observables which is one of the assumptions of the Kochen-Specker theorem. To avoid the conflict with the Kochen-Specker theorem, we must assign either factual values to observables or assign eigenvalues not to all observables. The assignment of values to observables by the weak value avoids the conflict with the theorem by the factual assignment of eigenvalues to only a part of observables. This ensures that the weak value is capable of consistent for the assignment for the value of an observable.

3.7   Is Weak Value the Element of Reality?

If the weak value is succeeded for the assignment of the value of an observable in a quantum process, it may be possible to interpret the weak value as the value of an observable corresponding to the element of reality of the observable. However, the weak value does not satisfy the original definition of the value corresponding to the element of reality, and hence we need to discuss in what sense it can be regarded as the element of reality.

First, the value corresponding to the element of reality must be obtained without disturbing the physical system in its measurement according to the EPR argument [3]. We can practically measure the weak value in a quantum process by the weak measurement, which hardly disturb the system. The two-state vector formalism supposes that all boundary conditions of the system are given at first. These boundary conditions correspond to projective measurements at each time, which are pre-selections and post-selections of the system. If we perform a projective measurement A at an intermediate time of the process, the quantum process gains another boundary condition by the projective measurement of A. If we calculate the probability about another projective measurement B in the process, we need to consider the modified quantum process including the new boundary condition of the measurement of A. We call the modification of the quantum process the transition of the quantum process. It is due to the fact that the projective measurement disturb the system. It is difficult to consider the result obtained in the measurement which causes disturbance on the system as the value of an observable. On the other hand, if we perform a weak measurement of the observable A in a quantum process, we do not need to consider the transition of the quantum process by the weak measurement since the interaction of the weak measurement scarcely disturbs the quantum process. Ideally, we can perform the weak measurement with an infinitesimally weak interaction. This means that we can obtain the weak value as the value related to an observable by a practical measurement with an infinitesimal disturbance on the system.

Second, the non-contextuality is one of the properties required to be an element of reality. To examine the non-contextuality, we require that there is an appropriate measurement to measure a definite value of the observable which corresponds to the element of reality. The weak values of observables can be measured

simultaneously even if these observables are not mutually commutative. Since the measurement interactions are weak, we can ignore the order of the weak measurement in measurements of the weak value of observables which are not mutually commutative. This suggests that we can measure the weak values as the values of observables without depending on the context.

Although there are reasons to interpret the weak value as the value corresponding to the element of reality, we cannot conclude that the weak value corresponds to the element of reality in the original sense of the term given by EPR. Vaidman suggests a new definition of the element of reality which includes the weak value [14],

> "If, with scarcely disturbing a process, we can infer with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."

The word "predict" in the EPR version is replaced by "infer" in this definition. The weak measurement requires a number of trials to obtain the weak value except for the one corresponding to an eigenvalue of the observable which occurs when either the initial or final state is in its eigenstate. Thus, in general, the value of each event is randomized, and therefore we obtain the results from a distribution whose mean is the weak value. The distribution of the results allows us to "infer" the weak value of an observable. In the original definition of the element of reality [3], the value related to the element must be obtained with probability unity in an appropriate measurement. This is important for the traceability of the value of the element in the time evolution by the sequential measurements. However, the result of each event of the weak measurement does not represent the weak value of the measuring observable. Therefore, we cannot obtain a definite value of an observable in each event. This is one reason why the weak value is not the element of reality in its original definition.

Chapter 4

Complex Probability Measure in Quantum Process

4.1   Motivation

In Chapter 2, we have seen that the dependence of the value of an observable on the context, called the contextuality, prevents the assignment of value to the observable in a way required to obtain the classical ensemble picture. In Chapter 3, we have learned that the two-state vector formalism solves the contextuality problem, but it is not enough to realize the classical ensemble picture we want. When we consider the value of an observable associated with a quantum process, the weak value is shown to provide a non-contextual assignment of values. However, since the weak value cannot be determined by one shot of measurement, it cannot be regarded as the element of reality in the EPR sense. Accordingly, the weak value does not yield a definite value to an observable and, therefore, it does not realize the classical ensemble picture we want either.

Now, let us turn our attention to the probability distribution in the classical ensemble picture. In classical mechanics, the probability distribution of the value of an observable in a system is determined by objects having a definite value of the observable in the system. Conversely, we can obtain the distribution of the object which assigns a definite value of an observable in accordance with the probability distribution of the values of observables. The distribution of the objects and the probability distribution of the values of observables must be directly related, if we are to measure the value of an observable without disturbing the system, that is, without disarranging the distribution of the objects in the system. In quantum mechanics, on the other hand, due to the difficulty in the assignment of value for an observable, we cannot assume the object having definite values. It should be noted that the probability distribution used in quantum mechanics is simply to obtain the outcomes of measurement, without assuming the distribution of objects, like we do in the classical ensemble.

In this Chapter, in order to obtain the probability distribution of the value of an observable in a quantum system in accordance with the distribution of underlying objects, we shall introduce a generalized probability to a quantum process.

Namely, we allow the probability to take any complex value, rather than the real value in the range $[0, 1]$ as the ordinary probability does. It will be shown then that the possible form of the generalized probability can be pinned down from the consistency conditions required from the process under consideration. Interestingly, we shall then find that the weak value is just the expectation value of the observable under the generalized probability. The derivation of the generalized probability and the relation to the weak value are mentioned in [1], which we review below.

In fact, the generalized probability, which is equivalent to the weak value of a projector, is suitable for a probability distribution in the classical ensemble picture for a number of reasons. First of all, the non-contextuality is ensured for the weak value. Note that the ABL probability, which is also defined as a probability associated with a quantum process, is contextual and cannot be used for the realizing the classical ensemble. To see that the contextual dependence causes a problem for the distribution of objects, consider a measurement to examine whether the state is found in $|a\rangle$ or not in a given process. Since there are many ways to define the context by means of a complete orthonormal set of projectors to which $P_a := |a\rangle\langle a|$ belongs, the context dependent probability distribution cannot be determined uniquely by $|a\rangle$ only. It is not reasonable that the underlying ensemble depends on innumerable, virtual and counter-factual projectors.

Secondly, we can repeatedly measure the weak value of an observable without disturbing the system in any significant manner by the weak measurements. Since the measurement does not disturb the system, one may expect that the distribution in the quantum process is unchanged under sequential weak measurements. This allows us to interpret the generalized probability as a distribution of the physical objects like the probability associated with a classical system.

Third, we shall see that the generalized probability satisfies a set of conditions respected by the probability distribution of a classical system. In particular, it satisfies the law of total probability required for the conditional probability, unlike the ordinary probability in quantum mechanics which does not satisfy it. This implies that the generalized probability proposed here occupies a status which is closer to the probability distribution in the classical ensemble picture. We shall argue this point in Chapter 4.4.

4.2   Probability Measure for Single States

For our purposes, we first recall Gleason's theorem [39] which deduces Born's statistical rule based on a probability measure fulfilling certain logical conditions. Given a Hilbert space $\mathcal{H}$ of a finite dimension $d \geq 3$, consider a real-valued measure μ which is a map from the space of projection operators $\mathcal{P}(\mathcal{H})$ of $\mathcal{H}$ to non-negatives, *i.e.*, $\mu\colon \mathcal{P}(\mathcal{H}) \to [0, \infty) \subset \mathbb{R}$. The theorem states that, if the map is bounded $|\mu(P)| < \infty$ and satisfies the partial additivity condition,

$$\mu\left(\sum_i P_i\right) = \sum_i \mu\left(P_i\right), \tag{4.1}$$

for a set of projection operators $\{P_i\}$ which are mutually orthogonal $P_i P_j = \emptyset$ (null operator) for $i \neq j$, then it has the form,

$$\mu(P) = \mathrm{tr}(WP), \tag{4.2}$$

with a positive self-adjoint trace class operator W.

Since we have $\mu(\emptyset) = 0$ from (4.2), if the range of the map μ is restricted to $[0, 1]$, and if the condition $\mu(\mathbb{1}) = 1$ is further imposed, then obviously the map can be interpreted as a probability measure with the attached meaning that $\emptyset$ and $\mathbb{1}$ represent propositions which are identically false and true, respectively. Note that $\mu(\mathbb{1}) = 1$ implies $\mathrm{tr}(W) = 1$.

In quantum mechanics, the probability measure is indeed realized by such a measure μ, where the operator W corresponds to the density operator ρ that characterizes the state of the system, and this includes the case of a Hilbert space of dimension $d = 2$ as well. For instance, if the system is described by a pure state $\rho = P_\psi := |\psi\rangle\langle\psi|$ for some normalized $|\psi\rangle \in \mathcal{H}$, our probability measure is required to yield

$$\mu(P_\psi) = 1, \qquad \mu(P_{\psi^\perp}) = 0. \tag{4.3}$$

Here, the first condition states that the probability of being in the state $|\psi\rangle$ is unity, whereas the second states that there is no probability assigned for an arbitrary state $|\psi^\perp\rangle$ orthogonal to $|\psi\rangle$ for which the projection is given by $P_{\psi^\perp} := |\psi^\perp\rangle\langle\psi^\perp|$. Namely, the measure μ has no support for the subspace $\mathcal{P}(\mathcal{H}_\psi^\perp) \subset \mathcal{P}(\mathcal{H})$, where $\mathcal{H}_\psi^\perp$ is the orthogonal complement to the one-dimensional subspace $\mathcal{H}_\psi = \mathrm{span}\{|\psi\rangle\}$. We note that, because of (4.1) the second condition in (4.3) actually follows from the first for a non-negative map μ, but this will no longer be the case when the non-negativity is lifted.

From (4.3) one finds that W is uniquely determined as $W = \rho = P_\psi$, and this shows that the probability of the state $|\psi\rangle$ being in the subspace $\mathcal{H}_i \subset \mathcal{H}$ specified by the projection P_i reads $\mu(P_i) = \mathrm{tr}(\rho P_i)$, which is just Born's statistical rule. It follows that, if an observable A is measured in the pure state ρ, the expectation value is given by

$$\mathcal{E}(A) := \sum_i a_i\, \mu(P_i) = \langle\psi|A|\psi\rangle, \tag{4.4}$$

where a_i is an eigenvalue of A, and P_i is the corresponding projection appearing in the spectral decomposition $A = \sum_i a_i P_i$. One notable consequence of this is that the expectation value satisfies the sum rule, $\mathcal{E}(A + B) = \mathcal{E}(A) + \mathcal{E}(B)$ for any observables A, B which may not commute with each other. This implies that, although the sum of the individual measurement outcomes of A and B may not be an eigenvalue of $A + B$, on average they coincide.

4.3   Complex Measure for Double States

Now we extend the forgoing argument to a measure characterized by double states. Let $\{|\psi\rangle, |\phi\rangle\}$ be two states arbitrarily chosen from $\mathcal{H}$ except that they are neither identical (up to a phase) nor orthogonal to each other (*i.e.*, $\langle\phi|\psi\rangle \neq 0$).

Analogously to the single state case (4.3), given the two states $\{|\psi\rangle, |\phi\rangle\}$ we wish to require

$$
\begin{aligned}
\mu(P_\psi) &= 1, & \mu(P_{\psi^\perp}) &= 0, \\
\mu(P_\phi) &= 1, & \mu(P_{\phi^\perp}) &= 0,
\end{aligned}
\tag{4.5}
$$

where $P_\phi = |\phi\rangle\langle\phi|$ and $P_{\phi^\perp} = |\phi^\perp\rangle\langle\phi^\perp|$ with $|\phi^\perp\rangle \in \mathcal{H}_\phi^\perp$. Obviously, in view of the uniqueness of W, this is impossible unless the two states are identical. However, the condition (4.5) can be met if one promotes the measure to a complex one.

To see this, let us invoke the generalized Gleason's theorem [40] which extends the range of the map from $[0, \infty)$ to the entire reals $\mathbb{R}$. Demanding the condition (4.1), one finds that such a measure μ_R admits the same form,

$$
\mu_R(P) = \mathrm{tr}(W_R P),
\tag{4.6}
$$

but now W_R is a self-adjoint trace class operator, not necessarily positive. In order to extend the range of the map to complex numbers $\mathbb{C}$, we choose two such real maps $\mu_R, \mu_R' \colon \mathcal{P}(\mathcal{H}) \to \mathbb{R}$ and consider

$$
\mu_C(P) = \mu_R(P) + i\mu_R'(P),
\tag{4.7}
$$

with the imaginary unit i. Clearly, the map μ_C still fulfills (4.1) by linearity and is written as

$$
\mu_C(P) = \mathrm{tr}(W_C P), \qquad W_C = W_R + iW_R',
\tag{4.8}
$$

where W_R and W_R' are the self-adjoint trace class operators associated with μ_R and μ_R', respectively. We then have:

Theorem *If a map $\mu_C \colon \mathcal{P}(\mathcal{H}) \to \mathbb{C}$ for $\mathcal{H}$ with finite dimension $d \geq 3$ satisfies the partial additivity condition (4.1) and the consistency condition (4.5) for two non-identical states $|\psi\rangle, |\phi\rangle$ with $\langle\phi|\psi\rangle \neq 0$, then it has the form,*

$$
\begin{aligned}
\mu_C(P) &= \mathrm{tr}(W_C P), \\
W_C &= \alpha \frac{|\psi\rangle\langle\phi|}{\langle\phi|\psi\rangle} + (1 - \alpha)\frac{|\phi\rangle\langle\psi|}{\langle\psi|\phi\rangle},
\end{aligned}
\tag{4.9}
$$

for some $\alpha \in \mathbb{C}$.

Proof. The complex measure fulfilling (4.1) is given by (4.8) with a trace class operator W_C. Let $\{|e_i\rangle; i = 1, \ldots, d\}$ be a complete orthonormal basis in $\mathcal{H}$ with $|e_1\rangle = |\psi\rangle$. In terms of this we expand W_C and $|\psi^\perp\rangle \in \mathcal{H}_\psi^\perp$ as

$$
W_C = \sum_{i,j=1}^d \beta_{ij}|e_i\rangle\langle e_j|, \qquad |\psi^\perp\rangle = \sum_{i=2}^d \gamma_i|e_i\rangle,
\tag{4.10}
$$

with $\beta_{ij}, \gamma_i \in \mathbb{C}$. From (4.5) we have

$$
\begin{aligned}
0 = \mu_C(P_{\psi^\perp}) &= \sum_{i,j}\sum_{k,l \geq 2} \beta_{ij}\gamma_k\gamma_l^* \mathrm{tr}(|e_i\rangle\langle e_j|e_k\rangle\langle e_l|) \\
&= \sum_{i,j \geq 2} \beta_{ij}\gamma_j\gamma_i^*,
\end{aligned}
\tag{4.11}
$$

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which implies $\beta_{ij} = 0$ for $i, j \geq 2$ since γ_i can be chosen arbitrarily. The operator W_C is thus written, with some (unnormalized) states $|\xi_1\rangle, |\xi_2\rangle \in \mathcal{H}_\psi^\perp$, as

$$W_C = |\psi\rangle\langle\psi| + |\psi\rangle\langle\xi_1| + |\xi_2\rangle\langle\psi|. \tag{4.12}$$

Defining

$$|\eta\rangle = |\phi\rangle - \langle\psi|\phi\rangle |\psi\rangle \in \mathcal{H}_\psi^\perp, \tag{4.13}$$

we further decompose

$$|\xi_i\rangle = z_i |\eta\rangle + |\xi_i'\rangle, \qquad i = 1, 2, \tag{4.14}$$

with $z_i \in \mathbb{C}$ so that $\langle\xi_i'| \in \mathcal{H}_\eta^\perp$ in addition to $\langle\xi_i'| \in \mathcal{H}_\psi^\perp$. Similarly, if we define

$$|\zeta\rangle = |\psi\rangle - \langle\phi|\psi\rangle|\phi\rangle \in \mathcal{H}_\phi^\perp, \tag{4.15}$$

we find that both $|\phi\rangle$ and $|\zeta\rangle$ belong to the linear space spanned by $|\psi\rangle$ and $|\eta\rangle$. It follows that $|\xi_i'\rangle \in \mathcal{H}_\phi^\perp$ and $|\xi_i'\rangle \in \mathcal{H}_\zeta^\perp$ for $i = 1, 2$ as well. This observation motivates us to rewrite (4.12) in favor of $|\phi\rangle$, $|\zeta\rangle$ and $|\xi_i'\rangle$ to find

$$
\begin{aligned}
W_C &= \omega_{\zeta\zeta}|\zeta\rangle\langle\zeta| + \omega_{\phi\phi}|\phi\rangle\langle\phi| + \omega_{\phi\zeta}|\phi\rangle\langle\zeta| + \omega_{\zeta\phi}|\zeta\rangle\langle\phi| \\
&\quad + \omega_{\phi\xi_1}|\phi\rangle\langle\xi_1'| + \omega_{\xi_2\phi}|\xi_2'\rangle\langle\phi| + |\zeta\rangle\langle\xi_1'| + |\xi_2'\rangle\langle\zeta|,
\end{aligned} \tag{4.16}
$$

where

$$
\begin{aligned}
\omega_{\zeta\zeta} &= 1 - z_1^*\langle\phi|\psi\rangle - z_2\langle\psi|\phi\rangle, \\
\omega_{\phi\phi} &= 1 - \omega_{\zeta\zeta} + \omega_{\zeta\zeta}|\langle\psi|\phi\rangle|^2, \\
\omega_{\phi\zeta} &= z_2 + \omega_{\zeta\zeta}\langle\phi|\psi\rangle, \\
\omega_{\zeta\phi} &= z_1^* + \omega_{\zeta\zeta}\langle\psi|\phi\rangle, \\
\omega_{\phi\xi_1} &= \langle\phi|\psi\rangle, \qquad \omega_{\xi_2\phi} = \langle\psi|\phi\rangle.
\end{aligned} \tag{4.17}
$$

On the other hand, an analogous argument for the state $|\phi\rangle$ demanded by (4.5) shows that W_C must also be of the form,

$$W_C = |\phi\rangle\langle\phi| + |\phi\rangle\langle\chi_1| + |\chi_2\rangle\langle\phi|, \tag{4.18}$$

with some (unnormalized) states $|\chi_1\rangle, |\chi_2\rangle \in \mathcal{H}_\phi^\perp$. Since $|\psi\rangle$ and $|\phi\rangle$ are not identical, we have $|\zeta\rangle \neq 0$. Comparing (4.16) and (4.18), we obtain

$$\omega_{\zeta\zeta} = 0, \qquad |\xi_1'\rangle = |\xi_2'\rangle = 0, \tag{4.19}$$

which implies

$$
\begin{aligned}
W_C &= |\psi\rangle\langle\psi| + z_1^*|\psi\rangle\langle\eta| + z_2|\eta\rangle\langle\psi| \\
&= z_1^*|\psi\rangle\langle\phi| + z_2|\phi\rangle\langle\psi|.
\end{aligned} \tag{4.20}
$$

Since z_1, z_2 are free parameters but subject to $\omega_{\zeta\zeta} = 0$, we arrive at (4.9) after putting $\alpha = z_1^*\langle\phi|\psi\rangle$. ∎

Having found the complex measure μ_C for double states specified by (4.5), we may consider the expectation value of an observable A. Putting aside the question

of the meaning of complex probability for the moment, and assuming if necessary that (4.5) is valid also for the case of dimension $d = 2$, we just follow the standard construction of the expectation value as we did in (4.4) to find

$$
\begin{aligned}
\lambda(A) \quad &:= \quad \sum_i a_i \, \mu_C(P_i) \\
&= \quad \alpha \frac{\langle \phi | A | \psi \rangle}{\langle \phi | \psi \rangle} + (1 - \alpha) \frac{\langle \psi | A | \phi \rangle}{\langle \psi | \phi \rangle}.
\end{aligned}
\tag{4.21}
$$

We then notice that Aharonov's weak value $A_w = \langle \phi | A | \psi \rangle / \langle \phi | \psi \rangle$ arises at the choice $\alpha = 1$ of the expectation value $\lambda(A)$. Although $\lambda(A)$ is complex in general, it becomes real at $\alpha = 1/2$ where W_C becomes self-adjoint. This shows that one can find the measure μ which meets the condition (4.5) by extending the range of the map only to the entire $\mathbb{R}$, but we shall soon realize that the particular measure obtained by $\alpha = 1/2$ does not account for all possible cases when applied to quantum processes. Notice also that, as for $\mathcal{E}(A)$ the sum rule holds, $\lambda(A+B) = \lambda(A) + \lambda(B)$, for any α.

Interestingly, in the single state limit, that is, in the limit $|\phi\rangle \to |\psi\rangle$ the ambiguity of α disappears and our complex measure μ_C reduces to the real measure μ in (4.2) with the condition (4.3) enforced. Accordingly, the expectation value $\lambda(A)$ also reduces to the conventional one $\mathcal{E}(A)$ in (4.4).

Another observation worth mentioning is that, since under the single state $|\psi\rangle$ the probability $\mu(P_\phi) = |\langle \psi | \phi \rangle|^2$ represents the compatibility of the double states $|\phi\rangle$ and $|\psi\rangle$,

one obtains the overall expectation value of A by the weighted product, $|\langle \psi | \phi \rangle|^2 \lambda(A)$. The average value obtained after allowing the state $|\phi\rangle$ to vary freely may then be evaluated by

$$
\sum_{|\phi\rangle \in \mathcal{B}} |\langle \psi | \phi \rangle|^2 \lambda(A) = \langle \psi | A | \psi \rangle,
\tag{4.22}
$$

where the summation is over the states of a complete basis $\mathcal{B}$ of $\mathcal{H}$.

So far, we have considered complex measures with W_C of the type (4.9) which fulfills (4.5). As one can extend the scope of single states from pure states to mixed states by allowing W to be any positive self-adjoint operators with unit trace, one may similarly extend the scope of double states by allowing W_C to be any operators with unit trace. If we let $\mathcal{T}(\mathcal{H})$ be the space of operators with unit trace, we have

$$
\beta \, W_C + (1 - \beta) \, W_C' \in \mathcal{T}(\mathcal{H}),
\tag{4.23}
$$

for $W_C, W_C' \in \mathcal{T}(\mathcal{H})$ and $\beta \in \mathbb{C}$. This shows that, if we regard $\mathcal{T}(\mathcal{H})$ as the space of such generalized double states, the space is 'convex' in the complex sense. The measure $\mu_C(P)$ also provides the map $\mathrm{Pr}(P; W_C) := \mu_C(P) = \mathrm{tr}(W_C P)$, which enjoys the affine property,

$$
\begin{aligned}
\mathrm{Pr}&(P; \beta \, W_C + (1 - \beta) \, W_C') \\
&= \beta \, \mathrm{Pr}(P; W_C) + (1 - \beta) \, \mathrm{Pr}(P; W_C'),
\end{aligned}
\tag{4.24}
$$

analogous to the conventional probability map. Along with the property $\Pr(\emptyset; W_C) = 0$ and $\Pr(\mathbb{1}; W_C) = 1$, this may be considered as a formal support for $\mu_C(P)$ qualifying as a probability measure, albeit it is complex. To explore the possible use of the complex measure, and thereby examine the physical significance of the complex parameter α in (4.9) or (4.21), we now turn to the probability measure for a quantum process.

4.4   Probability Measure for a Quantum Process

The complex measure for double states can be used to furnish the probability measure for a quantum process $|\psi\rangle \to |\phi\rangle$ by taking the time dependence of the states properly into consideration. Let t_i and t_f be the initial time and the final time at which the states $|\psi\rangle$ and $|\phi\rangle$ are realized, respectively, and let t be the time of 'measuring' the observable A in the period, $t_i \leq t \leq t_f$. To evaluate the outcome of the measurement results, we would like to have the complex measure relevant at time t. Assuming that our system is closed during the process, we have a unitary operator U to describe the time development in the period. The forward time-developed state at t from the initial state is then given by $U(t - t_i)|\psi\rangle$, and the backward time-developed state at t from the final state is given by $U(t - t_f)|\phi\rangle$. This suggests that, instead of the two states $|\psi\rangle$, $|\phi\rangle$, we should use these forward and backward time-developed states to characterize the measure (4.9). We are thus led to

$$
\begin{aligned}
W_C(t) = {} &\alpha \frac{U(t - t_i)|\psi\rangle \langle\phi| U^\dagger(t - t_f)}{\langle\phi| U(t_f - t_i)|\psi\rangle} \\
&+ (1 - \alpha) \frac{U(t - t_f)|\phi\rangle \langle\psi| U^\dagger(t - t_i)}{\langle\psi| U^\dagger(t_f - t_i)|\phi\rangle},
\end{aligned}
\tag{4.25}
$$

from which we can obtain the time-dependent measure, $\mu_C(P; t) := \mathrm{tr}(W_C(t)P)$. The expectation value (4.21) then acquires the corresponding time-dependence by the use of $\mu_C(P; t)$, which is now characterized by the consistency condition at the initial and final times,

$$
\begin{array}{ll}
\mu_C(P_\psi; t_i) = 1, & \mu_C(P_{\psi^\perp}; t_i) = 0, \\
\mu_C(P_\phi; t_f) = 1, & \mu_C(P_{\phi^\perp}; t_f) = 0.
\end{array}
\tag{4.26}
$$

However, the identification of the operator W_C by (4.25) with the process $|\psi\rangle \to |\phi\rangle$ in the period $[t_i, t_f]$ is not quite correct, because our measure for double states is originally given at a single time t and does not involve the time direction in any intrinsic manner. In fact, one can also associate the same W_C in (4.25) with the 'dual' process $U(t_i - t_f)|\phi\rangle \to U(t_f - t_i)|\psi\rangle$ in the same period, since the two states that determine the double states at time t are equivalent in both cases (see Fig. 4.1). In fact, with $|\tilde\phi\rangle := U(t_i - t_f)|\phi\rangle$, $|\tilde\psi\rangle := U(t_f - t_i)|\psi\rangle$, one can equally characterize our measure by

$$
\begin{array}{ll}
\mu_C(P_{\tilde\phi}; t_i) = 1, & \mu_C(P_{\tilde\phi^\perp}; t_i) = 0, \\
\mu_C(P_{\tilde\psi}; t_f) = 1, & \mu_C(P_{\tilde\psi^\perp}; t_f) = 0,
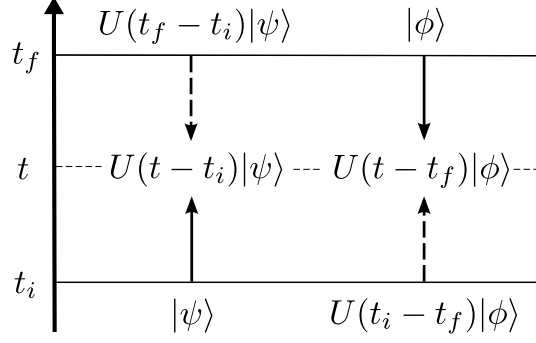\end{array}
\tag{4.27}
$$

39

Figure 4.1: The double state $W_C(t)$ in (4.25) can equally be associated with either of the two pure processes, $|\psi\rangle \to |\phi\rangle$ shown by the solid arrows or its dual $U(t_i - t_f)|\phi\rangle \to U(t_f - t_i)|\psi\rangle$ shown by the dashed arrows, representing the forward and backward time developments.

instead of (4.26). This indicates that the proper interpretation of W_C is that it is the measure corresponding to a linear superposition of the two processes, with the parameter of the superposition α. In particular, the choice $\alpha = 1$ yields the pure process $|\psi\rangle \to |\phi\rangle$ whereas the choice $\alpha = 0$ yields another pure process $|\tilde{\phi}\rangle \to |\tilde{\psi}\rangle$.

To establish a one-to-one correspondence between the measure and a quantum process, we remove, for the moment, the ambiguity of the complex probability measure by fixing the constant $\alpha = 1$. As a result, we obtain the generalized probability in a quantum process as

Definition 4 (Generalized probability measure in a quantum process) *The generalized probability in the quantum process from $|\psi\rangle$ to $|\phi\rangle$ is*

$$\text{Prob}_{(\psi \to \phi)}(P) := \frac{\langle\phi|P|\psi\rangle}{\langle\phi|\psi\rangle}, \tag{4.28}$$

where we ignore time evolution of the system.

After our suggestion was made, the same complex probability was also argued in [50] from a different viewpoint. Through this identification we can establish a one-to-one correspondence between the measure and the superposed process, and thereby remove the ambiguity in the expectation value $\lambda(A)$. As a result, $\lambda(A)$ agrees precisely with Aharonov's weak value A_w for $W_C(t)$ when the process is $|\psi\rangle \to |\phi\rangle$.

We note at this point that the generalized probability measure does not include the ABL-probability (3.9) which was introduced into the two-state vector formalism as the probability of obtaining a result in a process. The ABL-probability depends on the context of measuring projective operators and, hence, it cannot be calculated without fixing the complete orthogonal states including the measuring projector. Accordingly, the value of probability alters even if we exchange the complete orthogonal states which are orthogonal to the measuring projector.

4.4.1 Three-Box Paradox

We now argue that the generalized probability may deduce some paradoxical results, including the following three-box paradox [33, 35]. Consider three boxes which contain one electrically charged particle as superposition of localized states in each box. We describe the localized states of the particles in the three boxes as $|1\rangle$, $|2\rangle$ and $|3\rangle$, respectively. First, let us prepare the initial state in

$$|\psi\rangle := \frac{1}{\sqrt{3}} \left(|1\rangle + |2\rangle + |3\rangle \right). \tag{4.29}$$

Consider that we perform a post-selection by the projective measurement,

$$|\phi\rangle := \frac{1}{\sqrt{3}} \left(|1\rangle - |2\rangle + |3\rangle \right). \tag{4.30}$$

In this process, the generalized probability of the projective operator of each localized state, $P_1 := |1\rangle\langle 1|$, $P_2 := |2\rangle\langle 2|$ and $P_3 := |3\rangle\langle 3|$ is

$$
\begin{aligned}
\text{Prob}_{(\psi \to \phi)}(P_1) &= 1 \\
\text{Prob}_{(\psi \to \phi)}(P_2) &= -1 \\
\text{Prob}_{(\psi \to \phi)}(P_3) &= 1.
\end{aligned}
\tag{4.31}
$$

It suggests that the particle exists both in the box 1 and in the box 3. Using the weak measurement, we can shift meters in proportion to the weak value which is equal to the above generalized probability. This implies that, if we use the interaction which depends on the electric charge of the particle for the weak measurement, we can measure the weak value by a meter which interacts with each box. The amount of shift by the weak measurement is proportional to the shift, which will be observed when we prepare a particle in each box. Namely, the effect, which we see at the three boxes as the shifts of the meters, is equivalent and indicates that there exists precisely one particle in each box. This allows us to regard that the generalized probability represents the distribution of the physical objects in a quantum system. However, when examined carefully, we find that the effect seen at the box 2 is the opposite to the other boxes. On physical grounds, the effect observed at the box 2 is precisely the opposite effect observed at the other two boxes. A possible interpretation of this is offered by the scenario in which the particle in the box 2 has the opposite charge. This suggests that the physical meaning of the sign of the generalized probability is that it shows the positivity/negativity of the physical behavior it may yield.

4.4.2 Paths with Imaginary Generalized Probabilities

We here discuss the physical meaning of the imaginary value of the generalized probability through a novel experimental setup with interferometers. Consider the setup which consists of beam splitters (BS) and phase shifters (PS) as shown in Fig. 4.2. There are two Mach-Zehnder interferometers in this setup, which are configured so that, when we remove the phase shifters from the optical paths, all

incident photons go to each right port of the interferometers. After putting the phase shifters into the optical paths, which causes a phase shift that amounts to the multiplication by i to the state on each path, the incident photons go partly to the right port of each interferometer. This ensures that photons have gone through any optical paths in the interferometers before they are observed at the detector.

As before, let us tune the characteristic vertical oscillations of the mirrors with different frequencies from each other, so that the output signal of the detector is expected to arise at the frequency characteristic to the mirror on which the photon hit before it is detected. However, in reality, one finds in the experiment that the output signal arises only at the frequency of the mirror B (Fig. 4.2 (b)), not at the frequencies of the mirrors A and C.
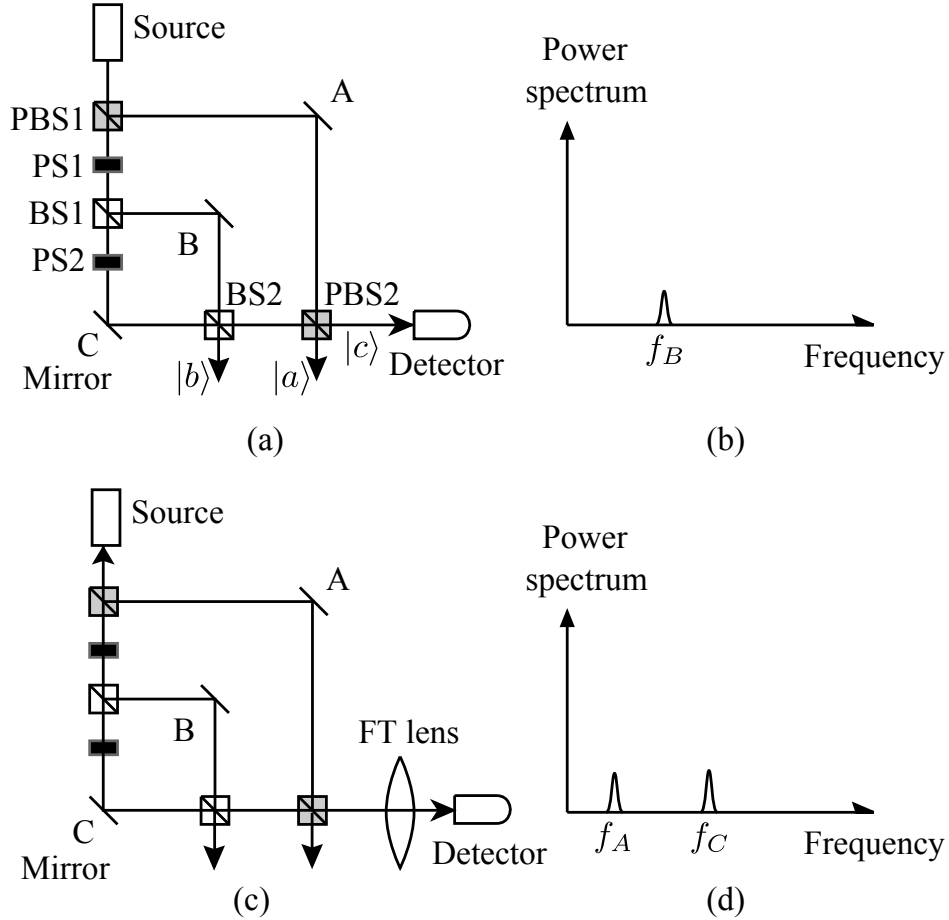

Figure 4.2: Experimental setup of the optical paths with imaginary generalized probabilities. The Mach-Zehnder interferometers are aligned to emit all photons to right ports of the interferometers when the phase shifters (PS) are removed. After inserting the PSs on each path, the photons pass any of the paths where the mirrors vibrate with different frequencies. (b) The detected signal vibrates only at the frequency of the mirror B. (c), (d) If we consider the detector of the vertical momentum of the photons, we find the signal from the mirrors A and C. The paths through the mirrors A and C have imaginary generalized probabilities.

We now show that the generalized probability with imaginary values provides a coherent interpretation of this result. To this end, we first note that, when the forward propagating state of the system is reflected by the mirrors A, B, and C, the state of the photon is given by

$$|\Psi\rangle := \frac{1}{\sqrt{3}} \left(i\,|A\rangle - |B\rangle - |C\rangle \right), \tag{4.32}$$

where $|A\rangle$, $|B\rangle$ and $|C\rangle$ are the localized states around the mirrors A, B and C. The backward propagating state of the photon at that time is similarly given by

$$|\Phi\rangle := \frac{1}{\sqrt{3}} \left(|A\rangle + |B\rangle - i\,|C\rangle \right). \tag{4.33}$$

The generalized probability on each site is thus found to be

$$\mathrm{Prob}_{(\Psi\to\Phi)}(P_A) := P_{Aw} = -i, \tag{4.34}$$

$$\mathrm{Prob}_{(\Psi\to\Phi)}(P_B) := P_{Bw} = 1, \tag{4.35}$$

$$\mathrm{Prob}_{(\Psi\to\Phi)}(P_C) := P_{Cw} = i. \tag{4.36}$$

Below, we shall find that the paths with the imaginary generalized probabilities are not detected in the setup mentioned above. However, this does not indicate that the photon does not exist in the paths with the imaginary generalized probability, because the signal of the power spectrum is sensitive only to the real part of the weak value in the present setup.

To observe the imaginary part, we consider the detection of the vertical momentum of the photons at the detector (see Fig. 4.2 (c)). This can be realized by inserting an achromatic lens (FT lens), which induces an optical Fourier transform, in front of the detector. We shall then obtain the output signal at the frequencies of the mirrors A and C (see Fig. 4.2 (d)) as promised. This shows that the imaginary part of the generalized probability also induces a physical effect on the system under consideration, which in the present case is detected in the the power spectrum of the vertical momentum. The measurement of the real part of the weak value, which corresponds to that of the generalized probability, has been considered in optical experiments [22], whereas we suggest that the imaginary part of the generalized probability is also measurable by the proposed procedure.

Now we argue how the interference occurs in this setup Fig. 4.2 in detail. As before, we arrange the setup in which the injected photons are polarized so that two thirds of the beam power go to the lower arm and one third of the injected beam power go to the mirror A. After passing PBS1, the state of the photon becomes

$$\frac{1}{\sqrt{3}}(\sqrt{2}\,|H\rangle\,|MZI\rangle + i\,|V\rangle\,|A\rangle), \tag{4.37}$$

where $|H\rangle$ and $|V\rangle$ are horizontal and vertical polarized states and $|A\rangle$ and $|MZI\rangle$ represent localized states going to the mirror A and the MZI, respectively. PS1 changes the phase on $|MZI\rangle$ by $\pi/2$, and BS1 splits the beam into

$$\frac{1}{\sqrt{3}}(|H\rangle\,(-\,|B\rangle + i\,|C\rangle) + i\,|V\rangle\,|A\rangle), \tag{4.38}$$

where $|B\rangle$ and $|C\rangle$ are localized states going to the mirror B and C. After PS2 changes the phase on the path with the mirror C by $\pi/2$, the state of the photon becomes

$$\frac{1}{\sqrt{3}}(|H\rangle\,(-|B\rangle - |C\rangle) + i\,|V\rangle\,|A\rangle), \tag{4.39}$$

which corresponds to the pre-selected state $|\Psi\rangle$ in (4.32). We use, as before, the vectorial representation of paths by three vectors, $|a\rangle$, $|b\rangle$ and $|c\rangle$ (see Fig. 4.2) given, for example, by $|a\rangle = |V\rangle\,|A\rangle$, $|b\rangle = |H\rangle\,|B\rangle$ and $|c\rangle = |H\rangle\,|C\rangle$ around the mirrors A, B and C.

We first check that all photons end up in the detector in the situation where two phase shifter are removed. First, the incident photon is given by the state,

$$|\Psi_0\rangle := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} := 0\,|a\rangle + 0\,|b\rangle + 1\,|c\rangle, \tag{4.40}$$

and PBS1 and PBS2 can be written as

$$PBS_1 = \begin{pmatrix} \sqrt{2}/\sqrt{3} & 0 & i/\sqrt{3} \\ 0 & 1 & 0 \\ i/\sqrt{3} & 0 & \sqrt{2}/\sqrt{3} \end{pmatrix}, PBS_2 = \begin{pmatrix} \sqrt{2}/\sqrt{3} & 0 & -i/\sqrt{3} \\ 0 & 1 & 0 \\ -i/\sqrt{3} & 0 & \sqrt{2}/\sqrt{3} \end{pmatrix}, \tag{4.41}$$

whereas BS1 and BS2 are

$$BS_1 = BS_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & i/\sqrt{2} \\ 0 & i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}. \tag{4.42}$$

The destructive interference on the port below BS2 requires a phase shift by π through PS in the optical length on $|b\rangle$,

$$PS = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.43}$$

In this situation, all photons go to the detector, which can be confirmed directly,

$$PBS_2 BS_2 PS BS_1 PBS_1\,|\Psi_0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.44}$$

We then consider installing PS1 and PS2, which are written as

$$PS_1 = PS_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}. \tag{4.45}$$

Accordingly, the pre-selected state is given by

$$|\Psi\rangle := PS_2 BS_1 PS_1 PBS_1\,|\Psi_0\rangle = \begin{pmatrix} i/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}. \tag{4.46}$$

Analogously, the post-selected state is obtained by building up necessary factors on the state at the detector,

$$\langle \Phi_0 | := (1, 0, 0), \tag{4.47}$$

as

$$\langle \Phi | := \langle \Phi_0 | \, PBS_2 BS_2 PS = \left(-i/\sqrt{3}, -i/\sqrt{3}, 1/\sqrt{3} \right). \tag{4.48}$$

The pre- and post-selected state are expected forms (4.32) and (4.33).

Then, the distribution of the position $\Pr(x)$ of the detected photons, which passed the post-selection, is evaluated as we have done before in (3.35). The result is

$$
\begin{aligned}
\Pr(x) &:= \left| \langle x | \langle \Phi | \, e^{-i(\theta_A P_A p + \theta_B P_B p + \theta_C P_C p)} \, |\Psi\rangle \, |\phi_G\rangle \right|^2 \\
&\simeq \left| \langle x | \langle \Phi | \, (\mathbb{1} - i \, (\theta_A P_A p + \theta_B P_B p + \theta_C P_C p)) \, |\Psi\rangle \, |\phi_G\rangle \right|^2 \\
&= \left| \langle x | \langle \Phi | \Psi \rangle \, (1 - i \, (\theta_A P_{Aw} p + \theta_B P_{Bw} p + \theta_C P_{Cw} p)) \, |\phi_G\rangle \right|^2 \\
&= \left| \langle \Phi | \Psi \rangle \right|^2 \left| \langle x | \, (1 - i \, (\theta_A P_{Aw} + \theta_B P_{Bw} + \theta_C P_{Cw}) \, p) \, |\phi_G\rangle \right|^2,
\end{aligned} \tag{4.49}
$$

where $P_{iw} := \langle \Phi | P_i | \Psi \rangle / \langle \Phi | \Psi \rangle$ for $i = A, B, C$ are the weak values of the projectors. Since these weak values are just the generalized probabilities (4.36), we have

$$
\begin{aligned}
\Pr(x) &\simeq \left| \langle \Phi | \Psi \rangle \right|^2 \left| \langle x | \, (1 - (\theta_A + i\theta_B - \theta_C) \, p) \, |\phi_G\rangle \right|^2 \\
&= \left| \langle \Phi | \Psi \rangle \right|^2 \left| \int dp \langle x | p \rangle \, (1 - (\theta_A + i\theta_B - \theta_C) \, p) \, \langle p | \phi_G \rangle \right|^2.
\end{aligned} \tag{4.50}
$$

Using the Gaussian distribution of the vertical momentum,

$$\langle p | \psi_G \rangle := \left(\frac{1}{2\pi\sigma_p^2} \right)^{\frac{1}{4}} e^{-\frac{p^2}{4\sigma_p^2}}, \tag{4.51}$$

where σ_p^2 is the variance of p, we perform the integration in (4.50), and ignoring the higher order of the tilts θ_i, we obtain

$$
\begin{aligned}
\Pr(x) &\simeq \sqrt{\frac{2\sigma_p^2}{\pi}} \left| 1 + 2 \, (\theta_B - i \, (\theta_A - \theta_C)) \, x \sigma_p^2 \right|^2 e^{-2\sigma_p^2 x^2} \\
&\simeq \sqrt{\frac{2\sigma_p^2}{\pi}} \left(1 + 4\theta_B x \sigma_p^2 \right) e^{-2\sigma_p^2 x^2}.
\end{aligned} \tag{4.52}
$$

This shows that the probability distribution of the position of the detected photons depends only on the tilt of the mirror B, which implies that the power spectrum of the position has a marked peak at the frequency of the mirror B which is in the path with a real generalized probability.

Now, to find the contribution of imaginary generalized probabilities, we consider the power spectrum of the vertical momentum $\Pr(p)$ given by

$$
\begin{aligned}
\Pr(p) &:= \left| \langle p | \langle \Phi | \, e^{-i(\theta_A P_A p + \theta_B P_B p + \theta_C P_C p)} \, |\Psi\rangle \, |\phi_G\rangle \right|^2 \\
&\simeq \left| \langle \Phi | \Psi \rangle \right|^2 \left| (1 - i \, (\theta_A P_{Aw} + \theta_B P_{Bw} + \theta_C P_{Cw}) \, p) \, \langle p | \phi_G \rangle \right|^2 \\
&= \left| \langle \Phi | \Psi \rangle \right|^2 \left| (1 - (\theta_A + i\theta_B - \theta_C) \, p) \right|^2 \left| \langle p | \phi_G \rangle \right|^2 \\
&\simeq \left| \langle \Phi | \Psi \rangle \right|^2 \left(1 - 2 \, (\theta_A - \theta_C) \, p \right) \left| \langle p | \phi_G \rangle \right|^2,
\end{aligned} \tag{4.53}
$$

where we have ignored the higher order of θ_i. In contrast to the previous case of the position, this time we do find that the probability distribution of the vertical momentum depends only on the tilts of the mirrors A and C which are in the paths with imaginary generalized probabilities.

Note that the setup of Danan's interferometer (Section 3.5), for which the real part of the generalized probability is significant, has symmetrical arrangement of paths of the photons, whereas the setup of the present interferometer, for which the imaginary part of the generalized probability is significant, has asymmetrical arrangement of paths due to the phase shifters. If we remove the phase shifter from our setup, these generalized probabilities become real-valued as one might expect.

Finally, we wish to remark on the implication of the complex nature of our generalized probability. We recall that the conjugate of a generalized probability of a process corresponds to that of the reversed process in time. This implies that, if the process is described by a complex-valued generalized probability, the reversed process has a different generalized probability distribution given by its conjugate. In this respect, the present experiment with the asymmetry on the paths of the photons provides an example of the system which has a counterpart system, that is, the system governed by the conjugate generalized probability with the forward and backward propagating states interchanged. The processes with real probabilities do not afford such conjugate processes.

4.5 Weak Value in a Quantum Process

The expectation value of the observable A with the spectral decomposition $A := \sum_i a_i P_i$, in a quantum process is given by

$$\langle A \rangle_{\psi \to \phi} := \sum_i a_i \mathrm{Prob}(P_i) \tag{4.54}$$

where $\mathrm{Prob}(P_i)$ is the ABL-probability (3.9). To obtain an expectation value, we perform a projective measurement of A in the intermediate time between the pre- and post-selections, then average the results with its frequency of the successful case of pre- and post-selections. This expectation value is based on the ABL probability which is easily understood from the operational point of view.

However, we have found another probability in the quantum process in the previous section. Let $\mathrm{Prob}_{(\psi \to \phi)}(P)$ be the generalized probability measure in a quantum process. The weak value of an observable A is given by using this probability measure (Fig. 4.3) as

$$A_w := \sum_i a_i \mathrm{Prob}_{(\psi \to \phi)}(P_i), \tag{4.55}$$

where A has a spectral decomposition $A := \sum_i a_i P_i$, $\{P_i\}$ are mutually orthogonal projective operators. The weak value is also a measurable quantity of an observable. Note that our generalized probability measure manifests itself in the weak

$$\sum_i a_i \frac{\langle\phi|P_{a_i}|\psi\rangle}{\langle\phi|\psi\rangle} = A_w$$
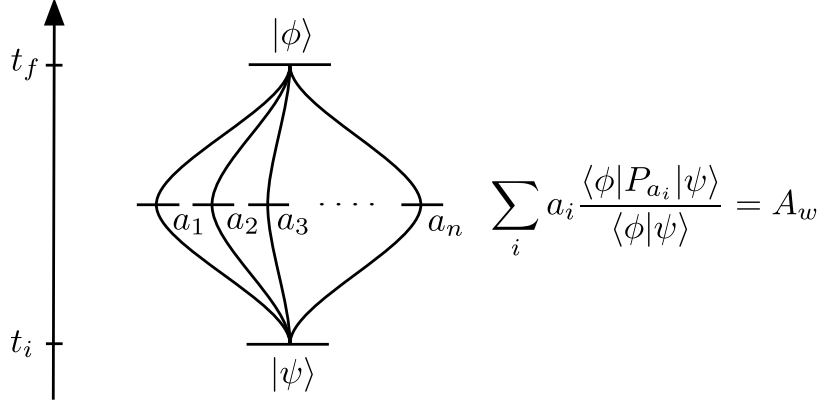
Figure 4.3: The expectation value with the generalized probability in a process becomes the weak value in the process.

value through the spectral decomposition of an observable. The observation of the weak value being an expectation value under the generalized probability measure has been pointed out earlier in our paper [1].

As we have argued, the weak value of a projective operator can be interpreted as the generalized probability of finding the state of the system. Even though the weak value is not always in the range [0,1], the probabilistic aspect of the weak value can 'explain away' the paradoxical situations. For example, Yokota et al. examine Hardy's paradox by the weak value based on the interpretation that it gives the probability of finding photons in optical paths [26].

4.6 Conditional Probability in a Quantum Process

In the multiple two-state formalism, we require all boundary conditions on a system which correspond to results of projective measurements in the system at each time. The transition law of the probability distributions for two-state can be explained by updating these boundaries. To see this, consider a quantum process from $|\psi\rangle$ to $|\phi\rangle$. If we perform a projective measurement of a observable A and obtain a result a as one of the eigenvalues of A, then the quantum process acquires a new boundary condition occurred by the measurement A at the measurement time. After the measurement of A, the boundary conditions of the quantum process are updated by the state $|a\rangle$, and the two-state is also updated to a new two-state corresponding the quantum process from $|a\rangle$ to $|\phi\rangle$.

We now take a look at this transition of probability distribution more closely based on the generalized probability in a quantum process. The probability of finding the state $|a\rangle$ in the quantum process from $|\psi\rangle$ to $|\phi\rangle$ is given by $\mathrm{Prob}_{(\psi\to\phi)}(P_a)$ and the probability of finding the state $|b\rangle$ in it is given by $\mathrm{Prob}_{(\psi\to\phi)}(P_b)$.

After a projective measurement of P_a, the probability of finding the state $|b\rangle$

is updated by **the conditional probability**, which is given by

$$\text{Prob}_{(\psi \to \phi)}(P_b|P_a) \quad := \quad \frac{\langle \phi | \, P_b \, |a\rangle}{\langle \phi | a \rangle}$$

$$= \quad \text{Prob}_{(a \to \phi)}(P_b). \tag{4.56}$$

This form of the conditional probability is derived from two assumptions, one of which is **the law of total probability**:

$$\text{Prob}_{(\psi \to \phi)}(P_b) = \sum_a \text{Prob}_{(\psi \to \phi)}(P_b|P_a)\text{Prob}_{(\psi \to \phi)}(P_a) \tag{4.57}$$

The from (4.56) satisfies this law,

$$\text{Prob}_{(\psi \to \phi)}(P_b) \quad = \quad \frac{\langle \phi | \, P_b \, |\psi\rangle}{\langle \phi | \psi \rangle}$$

$$= \quad \sum_a \frac{\langle \phi | \, P_b P_a \, |\psi\rangle}{\langle \phi | \psi \rangle}$$

$$= \quad \sum_a \frac{\langle \phi | \, P_b \, |a\rangle}{\langle \phi | a \rangle} \frac{\langle \phi | \, P_a \, |\psi\rangle}{\langle \phi | \psi \rangle}$$

$$= \quad \sum_a \text{Prob}_{(\psi \to \phi)}(P_b|P_a)\text{Prob}_{(\psi \to \phi)}(P_a) \tag{4.58}$$

The other assumption is **the independence of the updated probability** from the past boundary condition. For example, this indicates that the conditional probability $\text{Prob}_{(\psi \to \phi)}(P_b|P_a)$ does not depend on the initial boundary condition $|\psi\rangle$. If we measure P_a in a quantum process from $|\psi\rangle$ to $|\phi\rangle$, the transition of probability represents the acquisition of knowledge of the result of a measurement when the measurement causes no disturbance on a system.

The proof of the conditional probability form (4.56) is given as follows. Consider the conditional probability which is independent of the initial state $|\psi\rangle$. Let the initial state be described by $|\psi\rangle := \exp\left(iP_{b'}\theta\right)|\xi\rangle$, where $|\xi\rangle$ is a certain state and θ is a parameter. Since the conditional probability does not depend on θ, we find

$$\frac{\partial}{\partial \theta}\text{Prob}_{(\psi \to \phi)}(P_b|P_a) = 0. \tag{4.59}$$

If we take the derivative of the law of total probability with respect to θ, we find

$$\frac{\partial}{\partial \theta} \langle \phi | \, P_a \, |\psi\rangle \quad = \quad \sum_b \text{Prob}_{(\psi \to \phi)}(P_b|P_a)\frac{\partial}{\partial \theta} \langle \phi | \, P_b \, |\psi\rangle$$

$$\langle \phi | \, P_a P_{b'} e^{iP_{b'}\theta} \, |\xi\rangle \quad = \quad \sum_b \text{Prob}_{(\psi \to \phi)}(P_b|P_a) \langle \phi | \, P_b P_{b'} e^{iP_{b'}\theta} \, |\xi\rangle$$

$$\langle \phi | \, P_a P_{b'} \, |\psi\rangle \quad = \quad \text{Prob}_{(\psi \to \phi)}(P_{b'}|P_a) \langle \phi | \, P_{b'} \, |\psi\rangle. \tag{4.60}$$

As a result, we obtain

$$\text{Prob}_{(\psi \to \phi)}(P_{b'}|P_a) = \frac{\langle \phi | \, P_a P_{b'} \, |\psi\rangle}{\langle \phi | \, P_{b'} \, |\psi\rangle} = \frac{\langle \phi | \, P_a \, |b'\rangle}{\langle \phi | b'\rangle}, \tag{4.61}$$
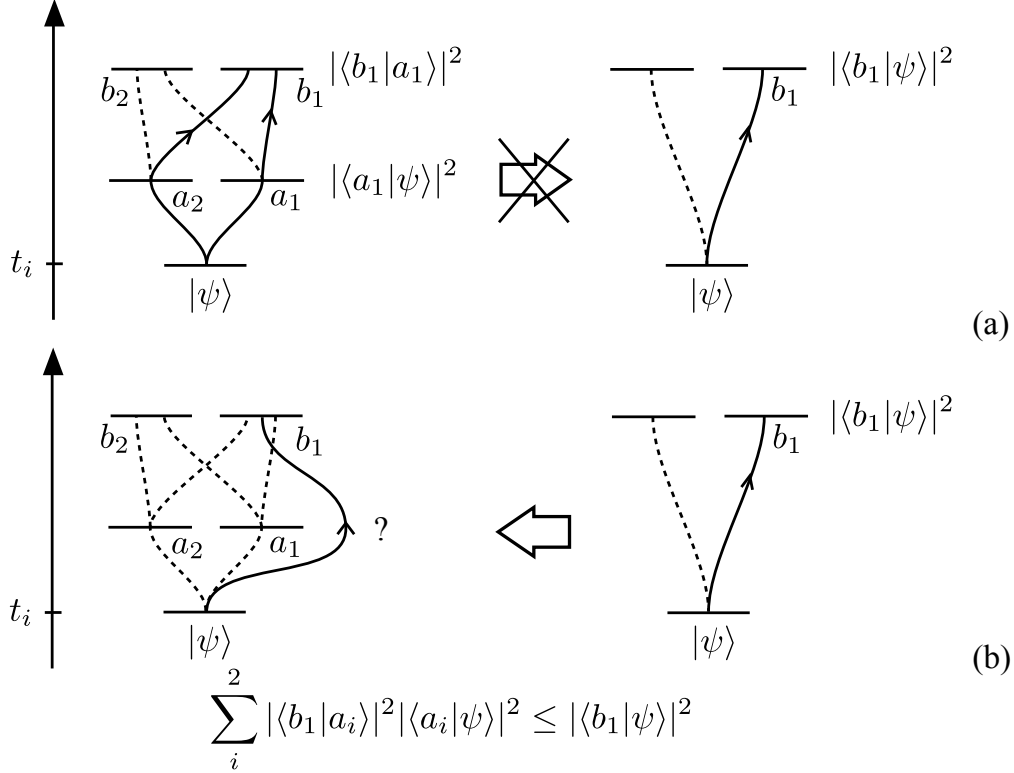
Figure 4.4: The probability in the one-state formalism does not satisfy the law of total probability. (a) If we ignore the first measurement of A, the number of events is less than that without the measurement of A. (b) Conversely, there are ensemble where we find $|b_1\rangle$ but do not find anywhere in the measurement A. This prevents us from recovering the underlying ensemble from the probability.

as expected.

The transition law of the probability in a quantum process, which is reasonable for the description of distribution, gives a clear advantage to the two-state vector formalism against the standard quantum mechanics. If all probabilities were $[0,1]$, the transition of the distribution could be interpreted just as a selection of the processes. Nonetheless, even though the generalized probability is complex valued, we may call the transition of the probability distribution by conditional probabilities as a selection of processes. Recall that the probability in the one-state formalism does not satisfy the transition law of the probability (Fig. 4.4), and the probability of obtaining a state $|a\rangle$ in a prepared state $|\psi\rangle$ is described by Born's rule as $p(a) := |\langle a|\psi\rangle|^2$. Similarly, the probability of obtaining a state $|b\rangle$ is given as $p(b) := |\langle b|\psi\rangle|^2$. The conditional probability of obtaining the state $|b\rangle$ after finding the state $|a\rangle$ is given by $p(b|a) := |\langle b|a\rangle|^2$. Obviously, these probabilities cannot satisfy the law of total probability:

$$p(b) = |\langle b|\psi\rangle|^2 \neq \sum_a |\langle b|a\rangle|^2 |\langle a|\psi\rangle|^2 = \sum_a p(b|a)p(a) \qquad (4.62)$$

We may ascribe this inconsistency of probability to the measurement disturbance incurred to a quantum state of the system. Otherwise, we may also regard it as

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introducing another transition law, which is described by using the instruments corresponding to a map between density operators of a system. In the one-state formalism, since we calculate the pullback of the probabilities represented by the the density operator, the probability itself does not follow the standard treatment of a probabilistic measure.

In this respect, it should also be mentioned that the complex amplitude probability satisfies the law of total probability:

$$\langle b|\psi\rangle = \sum_a \langle b|a\rangle\langle a|\psi\rangle \tag{4.63}$$

It should be noted, however, that it cannot be directly interpreted as probability, since it does not satisfy the normalization condition required for an ordinary probability of relative frequency. Note also that the square of the complex amplitude corresponds to the probability, but it does not satisfy the law of total probability. According to our derivation, the probability measure on the projective operators has two different forms. One of them is the density operator representation in the standard formulation, which is the square of the complex amplitude. The other is the generalized probability representation, which corresponds to the probability in a quantum process. We thus argue that, since the complex amplitude is not the probability measure on projective operators, it is not quite convenient for recovering the classical ensemble picture in a quantum system through probability.

4.7 Joint Probability in a Quantum Process

The joint probability in a quantum process is defined by the conditional probability as

$$
\begin{aligned}
\mathrm{Prob}_{(\psi\to\phi)}(P_a, P_b) &:= \mathrm{Prob}_{(\psi\to\phi)}(P_b|P_a)\mathrm{Prob}_{(\psi\to\phi)}(P_a) \\
&= \frac{\langle\phi|\,P_b\,|a\rangle}{\langle\phi|a\rangle}\frac{\langle\phi|\,P_a\,|\psi\rangle}{\langle\phi|\psi\rangle} \\
&= \frac{\langle\phi|\,P_bP_a\,|\psi\rangle}{\langle\phi|\psi\rangle}.
\end{aligned}
\tag{4.64}
$$

Although the probability in a quantum process is defined only for projective operators, the joint probability has the same form as the probability on a product of the projective operators P_aP_b (Fig. 4.5). Another form of the joint probability as the expectation value of the product form P_aP_b is defined at [49], which is the probability in the standard formulation.

The joint probability of an observables on distinct systems is also defined as above with the boundary conditions of the total system. For example, consider that the system $\mathcal{H}_1$ is a process from $|\psi\rangle$ to $|\phi\rangle$ and the system $\mathcal{H}_2$, which is $\mathcal{L}^2(\mathbb{R})$, is a process from a localized state around x_i, $|x_i\rangle$ to that around x_f, $|x_f\rangle$. After we let these two systems interact by the von-Neumann interaction gAp, the two-state of the total system at the time is given by the forward propagating state
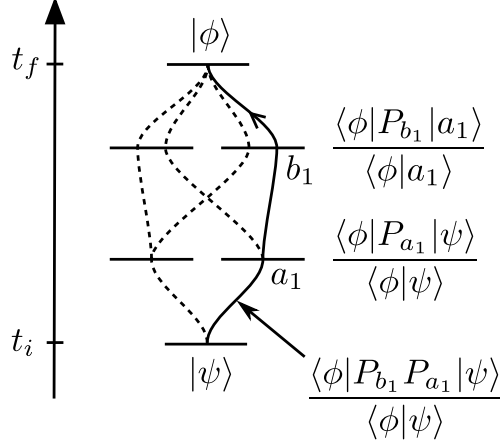
Figure 4.5: The generalized probability distributions of the forward propagating process. The joint probability for P_a and P_b is given by the product of the two probability. The joint probability depends on the order of measurements.

$|\Psi\rangle := e^{-igAp} |\psi\rangle |x_i\rangle$ and the backward propagating state $\langle\Phi| := \langle\phi| \langle x_f|$. Then the joint probability of $P_a \otimes \mathbb{1}$ and $\mathbb{1} \otimes |x\rangle\langle x|$ is given by

$$
\begin{aligned}
\mathrm{Prob}_{(\Psi\to\Phi)}(P_a, |x\rangle\langle x|) &:= \mathrm{Prob}_{(\Psi\to\Phi)}(|x\rangle\langle x||P_a)\mathrm{Prob}_{(\Psi\to\Phi)}(P_a) \\
&= \frac{\langle\Phi| \mathbb{1} \otimes |x\rangle\langle x| |\Psi'\rangle}{\langle\Phi|\Psi'\rangle} \frac{\langle\Phi| P_a \otimes \mathbb{1} |\Psi\rangle}{\langle\Phi|\Psi\rangle} \\
&= \frac{\langle\Phi| P_a \otimes |x\rangle\langle x| |\Psi\rangle}{\langle\Phi|\Psi\rangle}
\end{aligned}
\tag{4.65}
$$

where $|\Psi'\rangle := |a\rangle |x_i\rangle$.

The joint probability of the states separated in time can be also defined. Let the system evolve according to the Hamiltonian H. Then the joint probability of the state $|a\rangle$ at time t_1 and the state $|b\rangle$ at $t_2(> t_1)$ is described as

$$
\begin{aligned}
\mathrm{Prob}_{(\psi\to\phi)}(P_a, P_b) &:= \frac{\langle\phi| U(t_f - t_2)P_b U(t_2 - t_1)P_a U(t_1 - t_i) |\psi\rangle}{\langle\psi| U(t_f - t_i) |\phi\rangle} \\
&= \frac{\langle\phi| U(t_f - t_2)P_b U(t_2 - t_1) |a\rangle}{\langle\phi| U(t_f - t_1) |a\rangle} \frac{\langle\phi| U(t_f - t_1)P_a U(t_1 - t_i) |\psi\rangle}{\langle\phi| U(t_f - t_i) |\psi\rangle} \\
&=: \mathrm{Prob}_{(\psi\to\phi)}(P_b|P_a \ at \ t_1)\mathrm{Prob}_{(\psi\to\phi)}(P_a)
\end{aligned}
\tag{4.66}
$$

where $U(t) := \exp(-iHt)$.

We note that we cannot define the simultaneous joint probability for non-commutative projectors as we cannot do so in the standard formulation. Since the joint probability is defined by the product of two projectors, the value of the probability depends on the order of them. This is one of the reasons for preventing the construction of an underlying ensemble in a quantum system.

There is an experiment to measure the joint probability (4.64) for two non-commutative observables. The joint probability in a quantum process is measured by the sequential weak measurement. The sequential weak measurement is
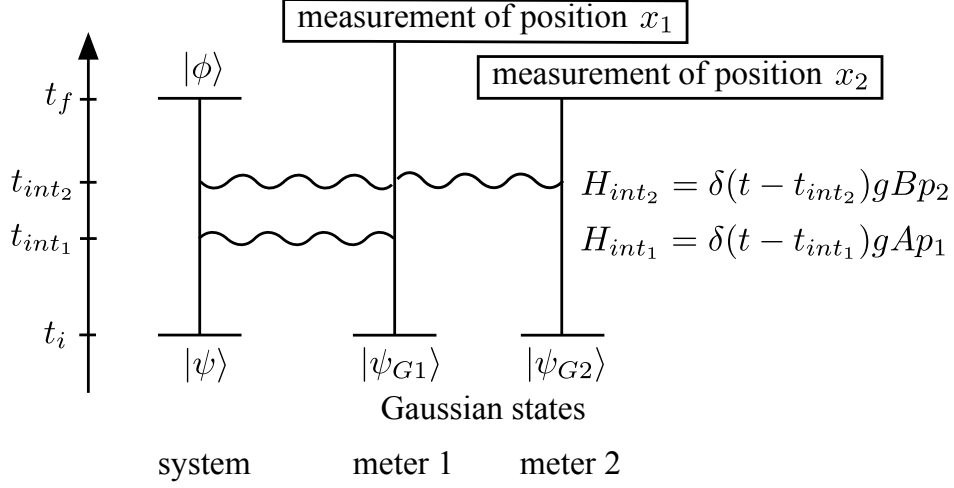
measurement of position x_1

measurement of position x_2

$|\phi\rangle$

t_f

t_{int_2} $H_{int_2} = \delta(t - t_{int_2})gBp_2$

t_{int_1} $H_{int_1} = \delta(t - t_{int_1})gAp_1$

t_i

$|\psi\rangle$ $|\psi_{G1}\rangle$ $|\psi_{G2}\rangle$

Gaussian states

system   meter 1   meter 2

Figure 4.6: The simple experimental setup for the sequential weak measurement of observables, A and B. We can obtain the joint probability from the correlation between the positions of meters, x_1 and x_2.

a method to measure the sequential weak value in a quantum process as a correlation of measured variables on a meter. This weak measurement scheme with multiple weak interactions, which is introduced by Mitchison et al [37], requires multiple meters corresponding to each interaction.

Let us consider the sequential weak measurement on a system with two meters (Fig. 4.6). The two meters couple with the system respectively by interactions, $H_{int_1} = gAp_1$, $H_{int_2} = gBp_2$, where A is the first measured observable and B is the second that. g is a common interaction strength and p_1, p_2 are momenta of each meters. We perform pointer measurements for each meter to obtain results x_1, x_2. After the post-selection, we calculate the average of the product of $x_1 x_2$, which is described by

$$E[x_1 x_2]_{selected} = \frac{g^2}{2}\mathrm{Re}\left((A, B)_w + (A)_w(B)_w^*\right) + \mathcal{O}(g^3), \qquad (4.67)$$

where $E[\cdot]_{selected}$ means the average over post-selected results. $(A, B)_w$ is called a sequential weak value, defined by

$$(A, B)_w = \frac{\langle\phi|AB|\psi\rangle}{\langle\phi|\psi\rangle}, \qquad (4.68)$$

when we ignore time evolution of the system. If we put $A = P_a$ and $B = P_b$, this is just the joint probability of $|a\rangle$ and $|b\rangle$ in the process.

Quantum Cheshire Cat

Even if we consider the joint probability of two commutative observables, it may still bring about nontrivial results, and the Quantum Cheshire cat is one of the examples [19] (Fig. 4.7). Consider the weak measurements of the spin of the spin

Figure 4.7: The experimental setup for Quantum Cheshire cat suggested in [19]. We weakly measure local spin components in the z direction, which corresponds to a polarization of photon in this setup, at each arm. If we pre-select the initial state by $|\Psi\rangle$ and post-select the final state by $|\Phi\rangle$, we find that the spin component exists in the right arm. However, the photon exists in the left arm of the interferometer according to the generalized probability.

1/2 particle system. We prepare the initial state as

$$|\Psi\rangle := (i\,|L\rangle + |R\rangle)\,|z+\rangle\,/\sqrt{2}, \tag{4.69}$$

where $|L\rangle$ and $|R\rangle$ are localized states around the two distant region, which are named Left and Right, and $|z+\rangle$ represents the spin-up state in the z direction. Let us perform the post-selection which consist of a phase shifter (PS), a half wave plate (HWP) and beam splitters (BS) by the state

$$|\Phi\rangle := (|L\rangle\,|z+\rangle + |R\rangle\,|z-\rangle)\,/\sqrt{2}. \tag{4.70}$$

The probabilities for the states $|L\rangle$ and $|R\rangle$ are

$$\mathrm{Prob}_{(\Psi\to\Phi)}(P_L) = 1$$
$$\mathrm{Prob}_{(\Psi\to\Phi)}(P_R) = 0.$$

Also we can calculate the joint probability of the position and the spin. The expectation value for the spin components in the z direction is defined by the joint probability as

$$\langle\sigma_z^L\rangle := \mathrm{Prob}_{(\Psi\to\Phi)}(P_{(z+)}P_L) - \mathrm{Prob}_{(\Psi\to\Phi)}(P_{(z-)}P_L) = 0,$$
$$\langle\sigma_z^R\rangle := \mathrm{Prob}_{(\Psi\to\Phi)}(P_{(z+)}P_R) - \mathrm{Prob}_{(\Psi\to\Phi)}(P_{(z-)}P_R) = 1.$$

This shows that the particle is in the Left region while the spin component exists in the Right region, which is impossible in classical mechanics. This result is demonstrated by the experiment using a neutron system [20], where the weak measurement is used for the local region to measure the joint probability.

Before proceeding, however, we mention that, in terms of the transition of probability in measurements, the two formulations contain clearly different treatments. In quantum mechanics, there is the projection postulate to describe the transition of probability distribution. The projection postulate determined the quantum state after a projective measurement. For example, after a projective measurement of a projector, the quantum state becomes an eigenstate of the projector. It derives from the uniqueness of the probability distribution which indicates unity for the measuring projector. In the two-state vector formalism, we do not consider the transition of the two-state on a system since the two-state is given by the factual boundary conditions which correspond to pre- and post-selections on the system.

The generalized probability introduced in this paper provides another formulation which differs form the two-state vector formalism in the transition of probability. We derived a generalized probability measure from mathematical requirements of the probability in a quantum process and derived a conditional probability in a quantum process. Since the conditional probability represents the transition of the probability by the acquisition of knowledge, we expect that the transition of the generalized probability given by the conditional probability provides an explanation of the transition of the probability in measurements. That this is in fact the case will be seen when we discuss the description of measurement procedures by means of the generalized probability next in Chapter 5.

Chapter 5

Measurement in Quantum Process

In this chapter, we see the measurement scheme in the time-symmetric formulation in terms of the generalized probability.

5.1 Direct Measurement Scheme of Standard Quantum Mechanics

For comparison, we begin by reviewing the measurement scheme in the standard quantum mechanics with a simple measurement setup. Consider a system prepared in the state $|\psi\rangle$ to measure the observable A which has the spectral decomposition $A = \sum_a a P_a$. The probability of obtaining a in the measurement is given by the probability postulate [2] as

$$p(a) := \mathrm{Tr}\left(P_a |\psi\rangle\langle\psi|\right). \tag{5.1}$$

This is Born's rule, which is the fundamental postulate of quantum mechanics. The expectation value of the measurement outcome of A is thus given by

$$\langle A \rangle := \sum_a a \mathrm{Tr}\left(P_a |\psi\rangle\langle\psi|\right) = \mathrm{Tr}\left(A |\psi\rangle\langle\psi|\right). \tag{5.2}$$

This definition is quite natural so that the ensemble of the system is divided by the results of measurement outcomes according to their probability. However, this probability has no underlying ensemble as mentioned previously.

5.2 Indirect Measurement Scheme of Standard Quantum Mechanics

The scheme mentioned above is of a direct measurement without considering the measurement apparatus or meter explicitly. The measurement with a meter is described in the indirect measurement scheme. A standard description of it goes
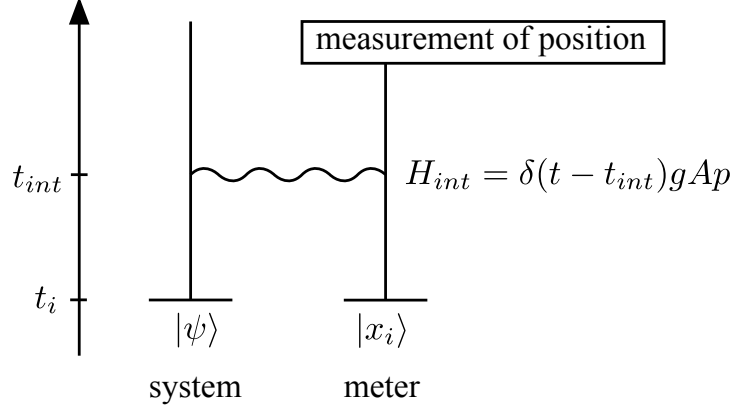
Figure 5.1: The indirect measurement scheme in the standard quantum mechanics. We prepare a localized state around x_i as the initial state of the meter.

as follows. Consider a system prepared in the state $|\psi\rangle$, and also consider a meter prepared in the state $|x_i\rangle$ which is a localized state around $x = x_i$ in $\mathcal{L}^2(\mathbb{R})$. The projective measurement of the observable A on the system uses the instantaneous von-Neumann interaction with the Hamiltonian $H_{int} := \delta(t - t_{int})gAp$, where p is the momentum of the meter and g is the strength of the interaction.

After the interaction between the system and the meter, we measure the position of the meter. The probability of obtaining the meter in x_f is calculated as

$$
\begin{aligned}
\Pr(x_f) & := \mathrm{Tr}_{sys}\left(|x_f\rangle\langle x_f| e^{-igAp}|\psi\rangle|x_i\rangle\langle\psi|\langle x_i| e^{igAp}\right) \\
& = \left(\sum_a \int dp dp' \langle x_f|p\rangle\langle p|x_i\rangle\langle x_i|p'\rangle\langle p'|x_f\rangle e^{iga(p'-p)}|\langle\psi|a\rangle|^2\right) \\
& = \sum_a \delta(x_f - (x_i + ga))|\langle\psi|a\rangle|^2. & (5.3)
\end{aligned}
$$

After tracing out the system state, we find that the shifts of the position of the mater agree with the results in the direct measurement scheme. Indeed, the expectation value of the results leads

$$
\begin{aligned}
\langle x_f\rangle & = \int dx_f x_f p(x_f) \\
& = \sum_a (x_i + ga)|\langle\psi|a\rangle|^2 \\
& = x_i + g\langle A\rangle, & (5.4)
\end{aligned}
$$

from which we obtain the expectation value of the measuring observable A by the amount of the shift in the position of the meter.

5.3 Projective Measurement and the Weak Measurement in Quantum Processes

In the two-state vector formalism, since a system is described by the forward and backward propagating states, the conditional expectation value determined from these states is introduced as the value to be measured for a observable. If we take the ABL-probability rule in place of Born's rule, we obtain

$$\langle A \rangle_{\psi \to \phi} := \sum_i a_i \mathrm{Prob}(P_i), \tag{5.5}$$

where $\mathrm{Prob}(P_i)$ is the ABL probability. This is the conditional expectation value of the observable in the two-state vector formalism.

If, instead, we want to measure the expectation value of the initial state in the formalism, we need to use the one-state described in [36], which corresponds to the situation where only the pre-selection is made for the system. Following discussion given before, we prepare the maximally entangled state between the system and the meter as the final boundary condition, and use the ABL-probability formula to derive the expectation value.

If, on the other had, we choose to take the generalized probability in a quantum process in place of the probability given by Born's rule, we find the weak value as the value of an observable,

$$A_w := \sum_i a_i \mathrm{Prob}_{(\psi \to \phi)}(P_i) \tag{5.6}$$

Now, let us consider the relation between the projective measurement and the weak measurement in a process. For this, we first note that, since we interpret the generalized probability as the distribution of the objects on a quantum system, the generalized probability does not have to directly correspond to the result of measurement. However, the transition of the generalized probability given by the conditional probability may correspond to the transition caused by the projective measurement. If so, the description of such measurements in terms of the generalized probability should be consistent with the transition of the generalized probability.

Moreover, one may expect that there is an unified treatment for these measurements in terms of the generalized probability in a process. In fact, one knows that the conditional expectation value $\langle A \rangle_{\psi \to \phi}$ can be obtained operationally by the projective measurement, whereas the weak value A_w can also be obtained operationally by the weak measurement. This suggests that one can perform the above two different measurements by changing the strength of measuring interaction, that is, if one weakens the interaction strength of the interaction of a projective measurement, one observes that the measurement tends gradually to the weak measurement. In other words, these two measurements are loosely connected by changing the strength of the measuring interaction. We shall see that this is indeed the case below.
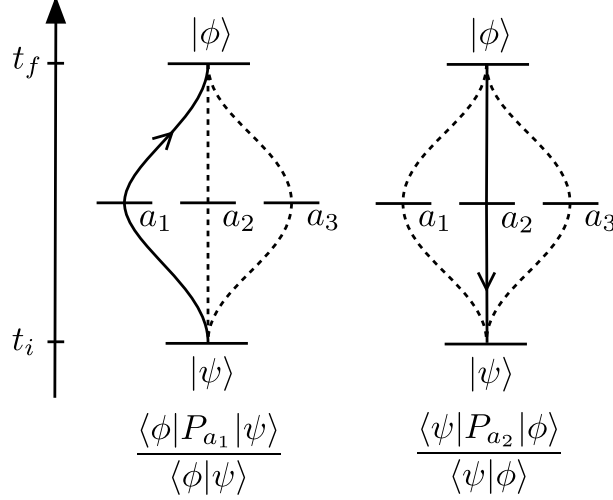
Figure 5.2: Measurement in a process is described by the underlying distribution given by the generalized probability. Both the forward and backward propagating processes appear in the description of the measurement.

5.4 Direct Measurement Scheme in Quantum Processes

To confirm our conjecture stated above, let us express the above two expectation values in terms of the generalized probability measure we introduced. First, the conditional expectation value is rewritten as

$$\langle A \rangle_{\psi \to \phi} = \frac{1}{N} \sum_i a_i \text{Prob}_{(\psi \to \phi)}(P_i) \text{Prob}_{(\phi \to \psi)}(P_i), \tag{5.7}$$

where N is a normalization constant given by $N = \sum_i \text{Prob}_{(\psi \to \phi)}(P_i) \text{Prob}_{(\phi \to \psi)}(P_i)$. Second, from the weak value which is already expressed in terms of the generalized probability (5.6), we notice that the real part of the weak value is written as

$$\text{Re}\,(A_w) = \frac{1}{N'} \sum_{i,j} \frac{a_i + a_j}{2} \text{Prob}_{(\psi \to \phi)}(P_i) \text{Prob}_{(\phi \to \psi)}(P_j), \tag{5.8}$$

where N' is a normalization constant given by $N' = \sum_{i,j} \text{Prob}_{(\psi \to \phi)}(P_i) \text{Prob}_{(\phi \to \psi)}(P_j) = 1$.

One then observes from the expression (5.13) that the real part of weak value is independently affected by the two quantum processes, $|\psi\rangle$ to $|\phi\rangle$ and $|\phi\rangle$ to $|\psi\rangle$. On the other hand, the conditional expectation value is also affected by the two processes, but these are completely correlated with each other.

The above observation allows us to interpret that the conditional expectation value (5.7) and the real part of the weak value (5.13) are given by the contributions coming from the forward and backward propagating processes of the system which are considered as two independent processes. Namely, we here interpret the complex conjugate of the generalized probability as the backward propagating process.

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Put differently, we regard the complex conjugate of a generalized probability as the same generalized probability in a reversed process. This is in fact consistent with the conjugation rule,

$$\left(\frac{\langle\phi|\,P\,|\psi\rangle}{\langle\phi|\psi\rangle}\right)^* = \frac{\langle\psi|\,P\,|\phi\rangle}{\langle\psi|\phi\rangle}. \tag{5.9}$$

More precisely, in the terms of the wave function, we have an analogy that the backward propagating process is represented by the complex conjugate of the generalized probability. To see this, consider a process on $\mathcal{L}^2(\mathbb{R})$ which is pre-selected by $|\psi\rangle$ and post-selected by a zero-momentum state $|p=0\rangle$. The generalized probability of finding the position of the system at x is described by

$$\mathrm{Prob}_{(\psi\to p=0)}(P_x) = \frac{\langle p=0|x\rangle\langle x|\psi\rangle}{\langle p=0|\psi\rangle} = \frac{\psi(x)}{\int dx'\psi(x')}, \tag{5.10}$$

where $\psi(x)$ is a wave function of the state $|\psi\rangle$. The complex conjugate of this generalized probability is

$$\mathrm{Prob}_{(\psi\to p=0)}(P_x)^* = \frac{\langle\psi|x\rangle\langle x|p=0\rangle}{\langle\psi|p=0\rangle} = \frac{\psi(x)^*}{\int dx'\psi(x')^*}. \tag{5.11}$$

This all stems from the simple fact that the complex conjugate of the wave function represents a backward propagating wave function according to the Schrödinger equation,

$$i\hbar(-1)\frac{\partial}{\partial t}\psi(x)^* = H\psi(x)^*. \tag{5.12}$$

The conjugate of the generalized probability which corresponds to the backward propagating state bring another measurement, which is for the imaginary part of the weak value. The imaginary part of weak value is rewritten as

$$\mathrm{Im}\,(A_w) = -i\sum_{i,j}\frac{a_i-a_j}{2}\mathrm{Prob}_{(\psi\to\phi)}(P_i)\mathrm{Prob}_{(\phi\to\psi)}(P_j), \tag{5.13}$$

This implies that if the meter which couples with a system by a measuring interaction behaves to the forward and backward propagating states in an opposite way, we can measure the imaginary part of the weak value on the system. The measurement of the imaginary part of weak value by the measurement of the momentum of a meter is one of the realizations of it.

To sum up, here we interpret that we can obtain the result of the projective measurement when the forward propagating state meets the backward propagating state at the same time, and we obtain the result of the weak measurement wherever these propagating states are present. We shall show next that this interpretation provides a reasonable explanation for the measurement procedure in the indirect measurement in a process, in which we explain the reason why the normalization constant of the conditional expectation value is different from that of the real part of weak value.

5.5 Indirect Measurement Scheme in Quantum Processes

There is an ideal measurement procedure to measure the conditional expectation value and the weak value in a system through the indirect measurement of a meter. To discuss it, we first consider a setup of the ideal measurement in a process.

Recall that in the standard formulation, a projective measurement is achieved by the von Neumann interaction between a system and a meter, where we obtain the expectation value in the system from the measurement of the position in the meter state. Similarly, to obtain the conditional expectation value and the weak value of a system from a meter, we consider the quantum process of the meter, which is pre-selected by a localized state $|x_i\rangle$ and post-selected by a zero momentum state $|p = 0\rangle$. As before, we suppose that the system is pre-selected in the state $|\psi\rangle$ and post-selected in the state $|\phi\rangle$.

After the system interacts instantaneously with the meter by the von-Neumann type interaction gAp, in which A is the observable measured in the system and p is the momentum of the meter, we perform a projective measurement of the position x on the meter before the momentum post-selection on the meter. The weak value of the position of the meter $\mathbb{1} \otimes |x\rangle\langle x|$ at the time is given by

$$
\begin{aligned}
(\mathbb{1} \otimes |x\rangle\langle x|)_w &= \frac{\langle\phi|\,\langle p = 0|\,(\mathbb{1} \otimes |x\rangle\langle x|)e^{-igAp}\,|\psi\rangle\,|x_i\rangle}{\langle\phi|\,\langle p = 0|\,e^{-igAp}\,|\psi\rangle\,|x_i\rangle} \\
&= \frac{\sum_a \delta(x_i + ga - x)\langle\phi|a\rangle\langle a|\psi\rangle}{\langle\phi|\psi\rangle} \\
&= \sum_a \delta(x_i + ga - x)P_{aw'},
\end{aligned}
\tag{5.14}
$$

where $P_{aw'}$ is the weak value of the projective operator $P_a := |a\rangle\langle a|$ on the original quantum process of the system which is pre-selected by $|\psi\rangle$ and post-selected by $|\phi\rangle$. Since the joint generalized probability in the process is equal to the weak value in the process in the total system, we define the generalized probability for the position of the meter by the weak values

$$
\begin{aligned}
\mathrm{Prob}_{(\Psi\to\Phi)}(|x\rangle\langle x|) &:= (\mathbb{1} \otimes |x\rangle\langle x|)_w & (5.15) \\
\mathrm{Prob}_{(\Phi\to\Psi)}(|x\rangle\langle x|) &:= (\mathbb{1} \otimes |x\rangle\langle x|)_w^*, & (5.16)
\end{aligned}
$$

where we write the initial state of the total system as $|\Psi\rangle := |\psi\rangle\,|x_i\rangle$ and the final state of that as $|\Phi\rangle := |\phi\rangle\,|p = 0\rangle$. The conditional expectation value of the position x is calculated by (5.7) as

$$
\begin{aligned}
\langle x \rangle_{x_i \to p=0} &= \frac{1}{N} \int dx\, x\, \mathrm{Prob}_{(\Psi\to\Phi)}(|x\rangle\langle x|)\mathrm{Prob}_{(\Phi\to\Psi)}(|x\rangle\langle x|) \\
&= \frac{1}{N} \int dx\, x \sum_a \delta(x_i + ga - x)P_{aw'} \sum_{a'} \delta(x_i + ga' - x)P_{a'w'}^\dagger \\
&= \frac{1}{N} \sum_a (x_i + ga)P_{aw'}P_{aw'}^\dagger
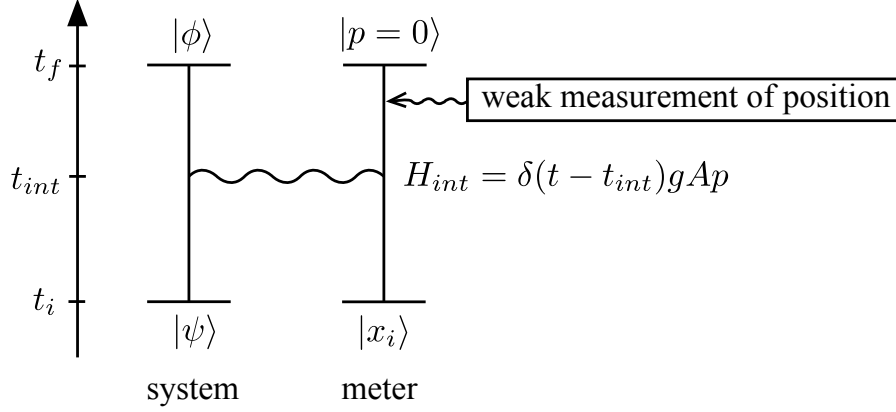\end{aligned}
$$

Figure 5.3: The setup for the ideal weak measurement. We can extract the weak value of the system by the weak measurement on the meter only. If we use the projective measurement instead of the weak measurement, we can extract the conditional expectation value of the system.

$$
\begin{aligned}
&= x_i + \frac{g}{N} \sum_a a P_{aw'} P_{aw'}^{\dagger} \\
&= x_i + \frac{g}{N} \sum_a a \mathrm{Prob}_{(\psi \to \phi)}(P_a) \mathrm{Prob}_{(\phi \to \psi)}(P_a) \\
&= x_i + g \langle A \rangle_{\psi \to \phi}.
\end{aligned}
\tag{5.17}
$$

We thus find that the conditional expectation value of the shift of the position of the meter is equal to the conditional expectation value of the observable A on the system $\langle A \rangle_{\psi \to \phi}$, which shows that this is indeed the ideal projective measurement in a process.

Now, the weak value of the position of the meter is calculated from (5.13) as

$$
\begin{aligned}
\mathrm{Re}\left(|x\rangle\langle x|_w\right) &= \int dx dx' \frac{x + x'}{2} \mathrm{Prob}_{(\Psi \to \Phi)}(|x\rangle\langle x|) \mathrm{Prob}_{(\Phi \to \Psi)}(|x'\rangle\langle x'|) \\
&= \int dx dx' \frac{(x + x')}{2} \sum_a \delta(x_i + ga - x) P_{aw'} \sum_{a'} \delta(x_i + ga' - x') P_{a'w'}^{\dagger} \\
&= \sum_{a,a'} \left(x_i + g \frac{a + a'}{2} \right) P_{aw'} P_{a'w'}^{\dagger} \\
&= x_i + g \sum_{a,a'} \frac{(a + a')}{2} P_{aw'} P_{a'w'}^{\dagger} \\
&= x_i + g \sum_{a,a'} \frac{(a + a')}{2} \mathrm{Prob}_{(\psi \to \phi)}(P_a) \mathrm{Prob}_{(\phi \to \psi)}(P_{a'}) \\
&= x_i + g \mathrm{Re}\left(A_{w'}\right).
\end{aligned}
\tag{5.18}
$$

This shows that the real part of weak value of the system is obtained from the weak measurement of the position of the meter. Since the weak value of the system is transported to the meter, this procedure is again considered to be the

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ideal weak measurement in a process (Fig. 5.3). We note that the real part of weak value of the observable on system is extracted by the weak measurement of the meter observable. It means that we need another weak measurement to measure the weak value of the meter observable.

We next consider the weak measurement in the indirect measurement scheme, in which the weak value of a system is obtained form the probability distribution of results of a projective measurement on a meter. To this end, we prepare a Gaussian distributed meter state $|\psi_G\rangle$ and a system state $|\psi\rangle$. After the instantaneous von-Neumann type measurement interaction gAp, we post-select the meter by the zero momentum state $|p = 0\rangle$ and the system by $|\phi\rangle$. The generalized probability of finding the meter in the position x after the interaction is given by

$$
\begin{aligned}
\mathrm{Prob}_{(\Psi \to \Phi)}(|x\rangle\langle x|) &= \frac{\langle \Phi| (\mathbb{1} \otimes |x\rangle\langle x|) |\Psi\rangle}{\langle \Phi|\Psi\rangle} \\
&= \frac{\langle \phi| \langle p = 0| (\mathbb{1} \otimes |x\rangle\langle x|) e^{-igAp} |\psi\rangle |\psi_G\rangle}{\langle \phi| \langle p = 0| |\psi\rangle |\psi_G\rangle} \\
&= \frac{\sum_a \int dx' \delta(x - ga - x') \psi_G(x') \langle \phi|a\rangle\langle a|\psi\rangle}{\langle \phi|\psi\rangle\langle p = 0|\psi_G\rangle} \\
&= \frac{\sum_a \psi_G(x - ga)\langle \phi|a\rangle\langle a|\psi\rangle}{\langle \phi|\psi\rangle\langle p = 0|\psi_G\rangle} \\
&= \frac{\sum_a \psi_G(x - ga)\mathrm{Prob}_{(\psi \to \phi)}(P_a)}{\langle p = 0|\psi_G\rangle},
\end{aligned}
\tag{5.19}
$$

where $\mathrm{Prob}_{(\psi \to \phi)}(P_a)$ is the generalized probability of finding the state $|a\rangle$ in the process and $\psi_G(x) := \langle x|\psi_G\rangle$. If we measure directly the position of the meter at the time, the conditional expectation value of the position x is calculated as

$$
\begin{aligned}
\langle x \rangle_{\Psi \to \Phi} &= \frac{1}{N} \int dx\, x \mathrm{Prob}_{(\Psi \to \Phi)}(|x\rangle\langle x|)\mathrm{Prob}_{(\Phi \to \Psi)}(|x\rangle\langle x|) \\
&= \frac{1}{N} \sum_{a,a'} \int dx\, x \psi(x - ga)\psi(x - ga')^* \mathrm{Prob}_{(\psi \to \phi)}(P_a)\mathrm{Prob}_{(\phi \to \psi)}(P_{a'})
\end{aligned}
$$

where N is the normalization constant given by $N := \int dx \mathrm{Prob}_{(\Psi \to \Phi)}(|x\rangle\langle x|)$ $\mathrm{Prob}_{(\Phi \to \Psi)}(|x\rangle\langle x|)$. This conditional expectation value of the position of the meter is exactly the same as the output of the weak measurement for the real part of the weak value since the pre- and post-selected state of the system and the pre-selected state of the meter are exactly the same as the original weak measurement setup (Fig. 3.3). This gives the procedure to obtain the real part of weak value of an observable on the system as the conditional expectation value of the position of the meter, which is the indirect measurement scheme of the weak measurement of the real part of the weak value.

Let us compare the weak measurement for the real part of weak value and the projective measurement, which is shown as the ideal projective measurement in a process (5.17), in the indirect measurement scheme. The difference of the two procedures is only in the pre-select state of the meter. In both of the procedures,
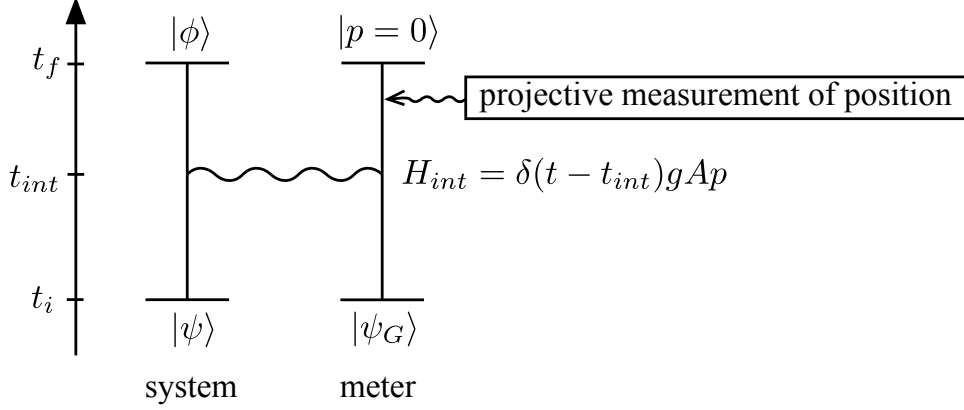
Figure 5.4: To perform the weak measurement for the real part of weak value, we prepare the initial state of meter by a Gaussian distributed state $|\psi_G\rangle$.

we perform a projective measurement of the position to obtain the result of the measurement after the measurement interaction, and then post-select the meter by $|p = 0\rangle$. These two indirect scheme use the virtual final boundary condition $|p = 0\rangle$ to extract the output probability distribution of the position of meter. Since the process cannot give the frequency of the finding the process itself, the zero momentum state $|p = 0\rangle$ which causes no bias for the position is the most suitable boundary for calculation of frequency of processes.

5.6   Consistency of the Probability in the Forward and Backward Processes

The indirect measurement scheme in a process explains the difference of the normalization constants of the projective measurement and weak measurement in the direct measurement scheme in a process. We consider the forward propagating process and the backward propagating process in a system. If the two processes are independent on each other, any pair of processes should appear in the calculation. The probability of finding the pair of processes where forward propagating process is found in $|a_i\rangle$ and the backward propagating process is found in $|a_j\rangle$ is given by $\mathrm{Prob}_{(\psi \to \phi)}(P_i)\mathrm{Prob}_{(\phi \to \psi)}(P_j)$. This probability of finding the pair of processes satisfies the normalization,

$$\sum_{i,j} \mathrm{Prob}_{(\psi \to \phi)}(P_i)\mathrm{Prob}_{(\phi \to \psi)}(P_j) = 1. \qquad (5.20)$$

In a measurement of the real part of the weak value, all processes in the system contributes to the meter through the measurement interaction. On the other hand, the coincident pairs of processes, which are the pairs of the forward propagating process and the backward propagating process found in the same state, contribute to the meter when we perform the projective measurement. If we take account only of these coincident pairs in the calculation, we obtain the normalization constant
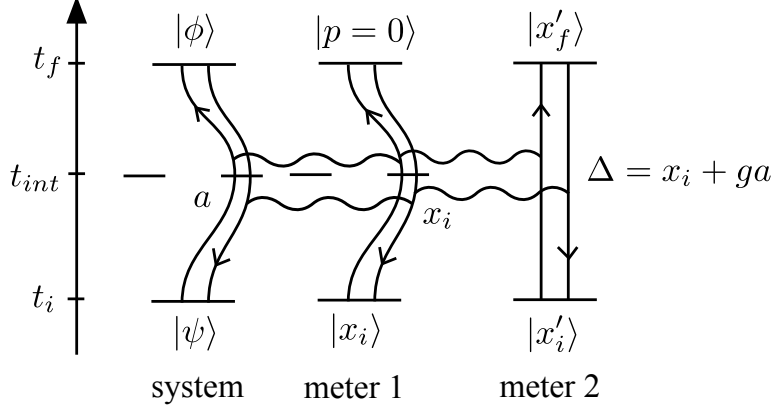
Figure 5.5: The selection on the meter 2 restricts the processes in the system in the measurement procedure.

as

$$N = \sum_i \mathrm{Prob}_{(\psi \to \phi)}(P_i)\mathrm{Prob}_{(\phi \to \psi)}(P_i) \neq 1. \tag{5.21}$$

This does not mean that we cannot find a intermediate state, or we find a intermediate state without finding events at the pre- or post-selections. The selection of the coincident pairs, which occur in a projective measurement, comes from the selection of the meter which interacts with the system. This implies that we need to consider another set of boundary conditions on another meter system required to obtain the result of the projective measurement on the meter.

To meet such an extra requirement, consider a process of a system from $|\psi\rangle$ to $|\phi\rangle$, and a process of a meter from $|x_i\rangle$ to $|p = 0\rangle$ (see Fig. 5.5). To measure the position of the meter in an intermediate time, we use another meter named meter 2, which interacts with the original meter, named meter 1. We thus prepare the setup in which the meter 2 is prepared in $|x_i'\rangle$ and the result of the measurement of the position of the meter 2 is x_f'. The measuring interaction between the system and the meter 1 is given by gAp, where g is the measurement strength, A is a measured observable on the system and p is the momentum of the meter 1. In addition, another measuring interaction between the meter 1 and the meter 2 is given by xp', where x is the position of the meter 1 and p' is the momentum of the meter 2.

Now, suppose that the forward propagating processes affect the forward propagating processes of other systems through the measuring interactions, and similarly the backward propagating processes affect the backward propagating processes of the other systems, respectively. We then discuss how the forward propagating states interacts each other in this situation. First, if the forward propagating state of the system is found in the state $|a\rangle$, the meter 1 shifts by ga. Since the meter is pre-selected in $|x_i\rangle$, the position of the meter 1 changes to $x_i + ga$. Then if we interact the meter 1 and the meter 2, the meter 2 shifts by $\Delta := x_i + ga$. In this situation, there is only the process of the meter 2 whose shift is equal to $x_f' - x_i'$ because of the pre- and post-selections on the meter 2. Thus the forward propagating state of the meter 1 is selected as $|x = \Delta\rangle$, where $\Delta = x_i + ga$. Similarly, the
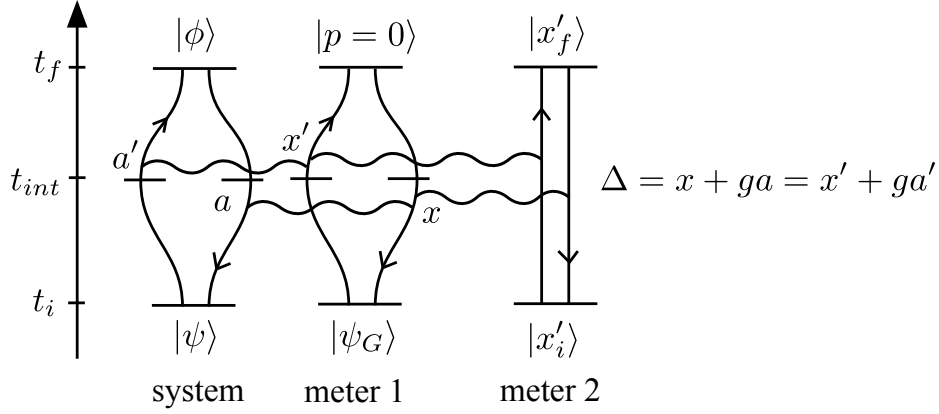
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Figure 5.6: Weak measurement is preformed by arranging the pre-selected state of the meter 1 by a Gaussian distributed state $|\psi_G\rangle$. Even if we pre- and post-select the meter 2, the process of the system is not restricted to a coincident pair.

backward propagating state of the meter 1 is selected as $|x = \Delta\rangle$. This indicates the forward and backward propagating states on the meter 1 shift by ga, that is, that the forward and backward propagating states in the system are found in $|a\rangle$.

It follows that the probability of finding the forward propagating and backward propagating process of meter 1 at $|x = \Delta\rangle$ is given by

$$\mathrm{Prob}_{(\Psi\to\Phi)}(|a\rangle\langle a|)\mathrm{Prob}_{(\Phi\to\Psi)}(|a\rangle\langle a|), \tag{5.22}$$

where $a = (\Delta - x_i)/g$. Since we perform many projective measurements on the meter 1 to obtain the probability distribution, the shift of the position of the meter 2 Δ is dispersed randomly. Therefore, the total number of the selected processes is calculated as

$$\sum_a \mathrm{Prob}_{(\Psi\to\Phi)}(|a\rangle\langle a|)\mathrm{Prob}_{(\Phi\to\Psi)}(|a\rangle\langle a|), \tag{5.23}$$

which is the expected result form (5.21). We obtain only the subset of the forward and backward processes which form the coincident pairs, because the results of measurement of the position of the meter are obtained by using another meter which has the boundary conditions determined by the measurement itself.

This explanation of the coincident pairs of the forward and backward processes is compatible with the weak measurement scheme (Fig. 5.6). Indeed, if we prepare the initial state of meter 1 as a Gaussian distributed state $|\psi_G\rangle$, the shift of the position of the meter 2 is not limited to one value. Since the meter 1 evolves from the Gaussian distributed state and is post-selected at $p = 0$, the generalized probability of the position of meter 1 distributes according to the Gaussian distribution and hence any amount of the shift on meter 1 is allowed.

Here, it is important to recognize that the shift of the backward propagating process in the system can take different values from that of the forward propagating process. As a result, the total number of the selected processes is calculated as

$$\sum_{a,a'} \mathrm{Prob}_{(\Phi\to\Psi)}(|a\rangle\langle a|)\mathrm{Prob}_{(\Phi\to\Psi)}(|a'\rangle\langle a'|) = 1. \tag{5.24}$$

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However, we remark that the initial Gaussian distribution causes a biased selection in the processes, and hence the selection with the Gaussian distributed meter is not ideal for obtaining the real part of the weak value as the conditional expectation value. This is an inevitable consequence of the fact that the ideal weak measurement is performed by the infinitely broad Gaussian distributed state of the meter.

Chapter 6

Conclusion and Discussion

We wish to have a proper description of the microscopic system, ideally by means of the classical ensemble picture. However, the contextuality of the value of observables poses a problem for realizing the classical ensemble picture. Quantum mechanics, which describes the microscopic world, avoids the problem of contextuality by introducing the probability distribution which does not presuppose underlying objects. The probability distribution in quantum mechanics is described by states which are just a mathematical entity to represent probabilities, not something existing in physical reality. Without underlying objects, we cannot interpret the probability distribution attached to a quantum system as that of the distribution of underlying objects. Thus, to understand the dynamics of probability distributions in the microscopic world, we need the solution for the contextuality problem in one way or the other.

The time-symmetric formulation of quantum mechanics advocated by Aharonov et al. [31, 34], which is also called the two-state vector formalism, does indeed solve the contextuality problem by introducing the forward and backward propagating states. The two-state defined by the two propagating states gives the probability of obtaining a result on the system, which agrees with that of the standard formulation of quantum mechanics. In the time-symmetric formalism, the non-contextual assignment of the value is achieved by the weak value which is specified by the boundary conditions of a quantum process. The boundary conditions restrict the context of the system, and give a consistent assignment of values to observables in a given process. However, we are not sure whether the weak value gives the true value assigned to a system which fulfills the requirement of the element of reality in the sense of EPR. After all, we cannot measure the weak values in a single trial in general, except for the particular case where the weak value happens to be equal to the eigenvalue corresponding either to the initial or to the final state of the process. The weak value is therefore an intrinsically statistical quantity and cannot escape from the ambiguous status when it comes to the reality question.

In the present paper, we turn our attention to the question of probability itself and have tried to provide a probability distribution which can reasonably be close to the one associated with the classical ensemble as much as possible. To this end, in Chapter 4 we have introduced a generalized probability defined on a quantum

process, where we allowed the probability to take complex values, not just the ordinary values within the range $[0, 1]$. We also assumed that the probability is objective, rather than subjective as in the recent interpretation of QBism [17]. From a set of consistency requirements, including the boundary conditions of the given process, we have succeeded to derive the possible form of the generalized probability as shown in (4.28). An interesting byproduct of this is that the weak value turns out to be precisely the expectation value (4.55) under the generalized probability we have just found.

More importantly, we have found that the generalized probability satisfies the law of total probability (4.57), which is a basic requirement for conditional probability. Obviously, the law of total probability is respected by the classical probability, but it is not so by the ordinary probability in quantum mechanics. This implies that, unlike the ordinary probability, the generalized probability allows for the updating law of distribution of the processes which occurs in the acquisition of information concerning a result of measurement. Thus, although it does not admit the ordinary relative frequency interpretation, the generalized probability shares an important common feature with the probability associated with the classical ensemble picture.

Another advantage of our generalized probability is found in describing quantum measurement in a process. Indeed, as shown in (5.7), (5.13) in Chapter 5, with the use of the generalized probability, the two distinct measurements, the projective measurement and the weak measurement which are treated differently in the conventional description of measurement, can be treated in a unified manner. This has been made possible thanks to the fact that the generalized probability describes a process in which the initial pre-selected state evolves forward in time and the final post-selected state evolves backward in time in a coherent framework. We emphasize here that nevertheless the generalized probability yields outcomes of joint measurements which are in complete agreement with the standard probability of quantum mechanics and cause no discrepancy in prediction. It has the property being closer to the classical ensemble picture in that it is non-contextual and respects the requirement for conditional probability, both enjoyed by the classical probability.

Finally, we give some remarks which are related to the interpretation of the weak value. It is known that the weak value of an observable can exceed the range of the spectrum of the observable, in which case the weak value is called 'anomalous'. The anomalous weak value of the projectors occurs when the generalized probability exceeds the ordinary range $[0, 1]$. Since the anomalous weak value appears in the system where quantum nature becomes significant [46], the generalized probability which exceeds $[0, 1]$ may be regarded as something signifying the quantum nature. Another point to be noted concerns with the physical significance of the imaginary part of the weak value. We have seen that its imaginary part appears in the experimental setup with the asymmetric on the process, where the time evolution of the forward propagating state and that of the backward propagating state differ. These connections seem to suggest that, if we can obtain a deeper understanding of the generalized probability, we can learn more

about the weak value and possibly its physical significance. These are certainly part of the important issues awaiting for future investigation in the foundation of quantum mechanics.

Acknowledgments

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