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A magnetospheric energy principle extended to include neutral atmosphere

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The problem of ideal magnetohydrodynamic stability of plasmas in a magnetosphere-atmosphere system, in which the unperturbed magnetic field is assumed to be perpendicular to the plasma-atmosphere interface (ionospheric surface), is investigated by means of an extended magnetospheric energy principle. The derivation of the principle and conditions under which it applies to a real terrestrial magnetosphere is given. In the principle, the atmosphere is considered to be a very heavy and compressible gas with finite pressure. A thin ionospheric layer is taken into account as boundary conditions, but energetics within it are neglected. The solid-earth surface is assumed to be a perfectly conducting wall for perturbations. For a perturbation that satisfies either rigid or horizontally free boundary conditions at the plasma-atmosphere interface, the self-adjointness of the force operator is satisfied and an extended magnetospheric energy principle can be developed on the basis of the extended energy principle for fusion plasmas. These two boundary conditions are shown to be realized in the magnetosphere when the ionospheric conductivity is either very large or very small. Whereas in fusion plasmas the perturbed magnetic energy in the vacuum makes a stabilizing contribution to the potential energy, in the magnetosphere the perturbed magnetic energy in the atmosphere makes no such stabilizing contribution. This is due to the difference of the assumed field configurations of the magnetospheric and fusion plasmas. The ionospheric surface makes a destabilizing negative contribution to the potential energy owing to a horizontal plasma displacement on the spherical ionospheric surface. The method is applied to magnetospheric ballooning and interchange instabilities. The existence of a new type of magnetospheric interchange instability is shown and its structure in the magnetosphere-atmosphere system is clarified. Possible consequences of the instabilities and their relevance to magnetospheric physics are discussed. © *2011 American Institute of Physics*. [doi:10.1063/1.3570646]

I. INTRODUCTION

The investigation of hydromagnetic stability of plasmas is of interest in such varied fields as the study of fusion plasmas, magnetospheric plasmas, and space and astrophysical plasmas. An extended energy principle, which is a powerful minimizing principle based on the self-adjointness of the force operator, has been developed[1] and used extensively for the study of the stability of fusion plasmas.[1,2] For a fusion plasma, which is separated from an outer conducting wall by a vacuum region, the self-adjointness of the force operator is proven[1–4] by using three boundary conditions satisfied at the unperturbed plasma-vacuum interface and at the surface of the outer perfectly conducting wall. With a few judicious integrations by parts, it was shown[1] that the potential energy functional δW consists of three parts, i.e., the potential energy of the plasma-fluid δW_F, the perturbed magnetic energy in the vacuum region δW_V, and a contribution δW_S from the free plasma-vacuum interface.

Such an energy principle has been used for investigating ideal magnetohydrodynamic (MHD) stability[5] of plasma in a levitated dipole,[6,7] which has a magnetosphere-like configuration. However, unlike real magnetospheres in nature it has no bounding surfaces across field lines such as the earth's ionosphere and the solid-earth surface. Therefore, there is no boundary contribution to the potential energy for such a closed field line configuration, and the potential energy only has δW_F.[5] Thus, such a model cannot clarify magnetospheric stability peculiar to the magnetospheres with bounding surfaces across field lines.

Recently, on the basis of the self-adjointness of the force operator, a magnetospheric energy principle has been formulated[8] and used for the study of ideal MHD stability of plasma on field lines threading the ionosphere in the polar latitudes.[8,9] The magnetospheric energy principle is based on a single assumption that the unperturbed magnetic field is incident vertically on the unperturbed ionospheric surface.[8] It neglects the existence of the neutral atmosphere below the ionosphere. According to this principle, the ionosphere makes a negative contribution to the potential energy owing to a horizontal plasma displacement on a spherical ionospheric surface.[8] Furthermore, it was shown that owing to this negative contribution to the potential energy, an ionosphere-driven interchange instability occurs.[9] This is a new ideal MHD instability peculiar to the magnetosphere-spherical ionosphere system.

Nevertheless, since the existence of the neutral atmosphere below the ionosphere is neglected entirely in the magnetospheric energy principle,[8] it has not been clear whether the atmosphere contributes to the stability of the ionosphere-driven magnetospheric interchange instability just as there is a stabilizing vacuum contribution δW_V in δW for fusion plasmas. Therefore, the purpose of this study is to formulate an extended magnetospheric energy principle which includes

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the neutral atmosphere based on the extended energy principle in fusion plasmas.[1–4]

Since the magnetospheric plasma is separated from a solid-earth surface by the neutral atmosphere, the magnetospheric plasma is in a sense similar to a fusion plasma, which is separated from a conducting wall by a vacuum region. However, the magnetospheric case is more difficult than magnetic confinement devices, because the solid earth is not a perfect conductor and the neutral atmosphere is not a vacuum region. In order to avoid these difficulties, we assume in this study that the neutral atmosphere is a very heavy and compressible neutral gas with finite pressure and that the solid-earth surface is a perfectly conducting wall for perturbations. Three boundary conditions, which are similar to the ones used in the extended energy principle,[1–4] i.e., the pressure balance and the continuity of the tangential component of the electric field at the unperturbed ionospheric surface, and a condition for the vector potential at the solid-earth surface, are used in the extended magnetospheric energy principle. Like the magnetospheric energy principle,[8] the extended principle is also based on an assumption that the unperturbed magnetic field is incident vertically on the unperturbed spherical ionospheric surface.

In this study, we show that the difference of the extended energy principle in fusion and magnetospheric plasmas is caused by the difference of the configuration of the unperturbed magnetic field with respect to the plasma-vacuum or plasma-atmosphere interface. While in fusion plasmas, the vacuum region is known to make a stabilizing positive contribution to the potential energy,[1,2] there is no contribution from the perturbed magnetic energy in the atmosphere in the extended magnetospheric energy principle.

According to the fusion plasma terminology,[2,4] in this paper we will call the energy principle, in which the neutral atmosphere is neglected, "the magnetospheric energy principle," and call the energy principle, in which the neutral atmosphere is taken into account, "the extended magnetospheric energy principle."

The organization of this paper is as follows. The magnetospheric configuration is described in Sec. II. Linearized equations are introduced in Sec. III. The relationship between the self-adjointness of the force operator and the energy conservation is described in Sec. IV. A magnetospheric energy principle is briefly reviewed in Sec. V. The physical meaning of the ionospheric contribution to the potential energy is discussed in Sec. VI. A model of the neutral atmosphere and thin ionospheric layer is given in Sec. VII. Boundary conditions for an ideal MHD plasma-atmosphere system are obtained in Sec. VIII. An extended magnetospheric energy principle is formulated in Sec. IX. The condition for minimization of the change in the potential energy with respect to $\xi_\parallel$ is obtained in Sec. X. Discussion is given in Sec. XI. Summary is given in Sec. XII.

II. MAGNETOSPHERIC CONFIGURATION

Figure 1 shows a bird's-eye view and a cross-section of a three-dimensional (3D) magnetospheric configuration. The solid circle is the solid-earth surface. The dotted circle is the ionospheric surface. The region between the solid-earth surface and the ionospheric surface is a neutral atmosphere. In the unperturbed state, the magnetospheric plasma satisfies the static ideal MHD equations. Figure 1 shows a realistic unaxisymmetric magnetospheric configuration of the earth, which is compressed in the dayside (left-hand side of the panel) and extended in the nightside (right-hand side of the panel). However, any axisymmetric configuration can also be used in the energy principle as long as the plasma satisfies the static ideal MHD equations.

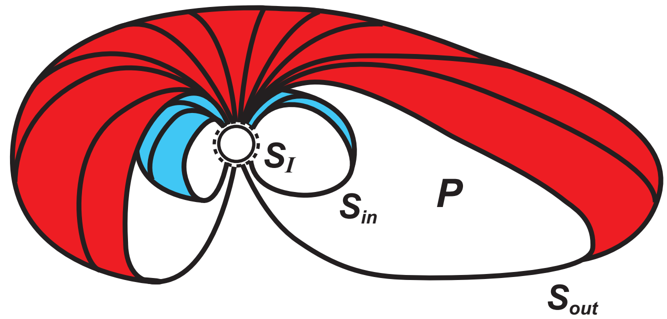

FIG. 1. (Color) A bird's-eye view and a cross-section of a 3D magnetosphere-atmosphere system are shown. The solid circle is the earth's surface. The dotted circle is the ionospheric surface. Magnetic field lines on two different flux surfaces $S_{\rm out}$ and $S_{\rm in}$, which are shown by red and blue surfaces, respectively, are plotted by solid lines. The plasma volume P is surrounded by $S_{\rm out}$, $S_{\rm in}$ and the ionospheric surface.

As shown in Fig. 1, the magnetospheric region of interest P is surrounded by two lateral boundaries, $S_{\rm out}$ and $S_{\rm in}$, which are shown by red and blue surfaces, respectively, and by the ionospheric surface S_I, which is shown by the dotted circle. These boundary surfaces surround a part of the magnetospheric plasma, which is in a static equilibrium. The virtual boundaries $S_{\rm out}$ and $S_{\rm in}$ are taken to be flux surfaces of the static magnetospheric equilibrium and are located far enough from disturbed field lines in region P, so that physics in region P are not affected. Hence, these boundaries are not perturbed and there is no magnetic field component threading them. Therefore, these virtual boundaries, $S_{\rm out}$ and $S_{\rm in}$, are in effect considered to be thin perfectly conducting walls.

The earthward ends of the magnetospheric region of interest P are bounded by an ionospheric surface S_I, which is a plasma-atmosphere interface and is shown by the dotted circle in Fig. 1. Unlike $S_{\rm out}$ and $S_{\rm in}$, magnetic field lines thread the ionospheric surfaces. As was assumed in the magnetospheric energy principle, which was formulated earlier,[8] a basic assumption of this extended magnetospheric energy principle is that the unperturbed magnetic field is incident vertically on the unperturbed ionospheric surface. This assumption enables one to formulate an extended magnetospheric energy principle analytically. Since we are interested in instabilities occurring in the high-latitude of the earth's magnetosphere, this assumption is considered to be a good approximation. For the sake of simplicity, the solid earth is considered to be a perfect conductor for perturbations and the neutral atmosphere is assumed to be a very heavy and compressible gas with finite pressure (see Secs. VII and VIII for details).

In Fig. 1, P also represents the unperturbed plasma vol-

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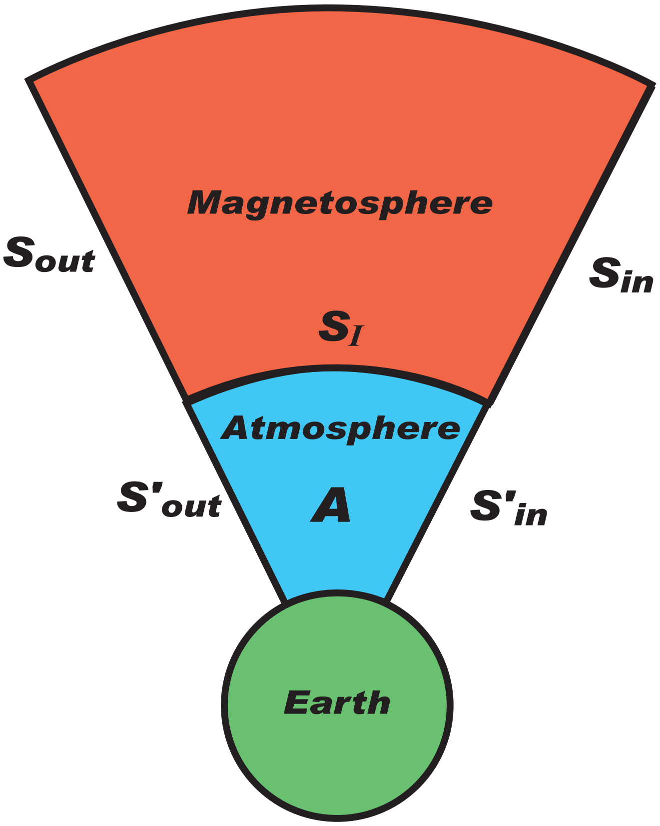

FIG. 2. (Color) A magnetosphere-atmosphere system is shown. Magnetospheric plasma is shown by the red area. The atmosphere is shown by the blue area. The solid earth is shown by the green circle. S_I is the plasma-atmosphere interface. $S_{\rm out}$ and $S_{\rm in}$ are the outer and inner flux surfaces. $S'_{\rm out}$ and $S'_{\rm in}$ are extensions of $S_{\rm out}$ and $S_{\rm in}$ to the atmosphere.

ume for integration, which is surrounded by the unperturbed ionospheric surface S_I and by the outer flux surface $S_{\rm out}$ and the inner flux surface $S_{\rm in}$. S is the unperturbed plasma surface surrounding the plasma volume P. This means that S is the sum of S_I, $S_{\rm out}$, and $S_{\rm in}$.

Figure 2 shows a cross-section of a magnetosphere-atmosphere system considered in this investigation. A sketch of only one hemisphere is shown in this figure. Magnetospheric plasma, which is shown by the red area, is bounded by ionospheric surfaces S_I at earthward ends. Below the ionospheric surface S_I, there is a neutral atmosphere, which is shown by the blue area. Therefore, the ionospheric surface, which is shown by S_I, is a plasma-atmosphere interface. The neutral atmosphere is also bounded at the bottom by a solid-earth surface, as shown in this figure. The solid earth is shown by the green circle. In this figure, A is the unperturbed atmospheric volume surrounded by the solid-earth surface, the plasma-atmosphere interface S_I, and outer and inner virtual surfaces $S'_{\rm out}$ and $S'_{\rm in}$, which are extensions of the virtual surfaces $S_{\rm out}$ and $S_{\rm in}$ extending into the atmospheric volume. Notice that A represents the atmospheric volumes in northern and southern hemispheres. For the sake of simplicity, we assume in this study that there are no ions and electrons in the neutral atmosphere and there are no neutral components in the magnetospheric plasma. Thus, there is no electric current in the atmosphere.

This configuration is somewhat similar to fusion plasmas in magnetic confinement devices, in which the plasma is separated from a perfectly conducting wall by a vacuum region. However, the magnetospheric case is more complicated, since the neutral atmosphere is not a vacuum region and the solid earth is not a perfect conductor. In spite of these differences, the similarity between the magnetospheric plasma and the fusion plasma in magnetic confinement devices enables one to formulate an extended magnetospheric energy principle in analogy with the formulation of the extended energy principle for magnetically confined plasmas.

III. LINEARIZED EQUATIONS

We assume that a static ideal MHD magnetospheric equilibrium is given. All quantities are linearized about this background state: $\mathbf{Q}(\mathbf{r},t)=\mathbf{Q}_0(\mathbf{r})+\tilde{\mathbf{Q}}_1(\mathbf{r},t)$ with $\tilde{\mathbf{Q}}_1/\mathbf{Q}_0 \ll 1$. Following the formulation and notation of Freidberg,[2] all perturbed quantities denoted by subscript 1 are expressed in terms of a displacement vector $\tilde{\boldsymbol{\xi}}$. Using $\tilde{\boldsymbol{\xi}}$ the general linearized equations of motion are cast into an initial value problem.

A more efficient way to investigate linear stability is to reformulate the initial value problem as a normal mode problem. To do this, all perturbed quantities are assumed to vary as follows:

$$\tilde{\mathbf{Q}}_1(\mathbf{r},t) = \mathbf{Q}_1(\mathbf{r})\exp(-i\omega t), \tag{1}$$

where $\mathbf{Q}_1(\mathbf{r})$ is a complex quantity in general, whereas $\tilde{\mathbf{Q}}_1(\mathbf{r},t)$ is defined in the real time domain. From now on, we shall drop the subscript zeros from the equilibrium quantities. The momentum equation can be written as

$$-\omega^2\rho\boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi}), \tag{2}$$

where the force operator $\mathbf{F}(\boldsymbol{\xi})$ is given by

$$\mathbf{F}(\boldsymbol{\xi}) = \mu_0^{-1}(\nabla\times\mathbf{Q})\times\mathbf{B} + \mu_0^{-1}(\nabla\times\mathbf{B})\times\mathbf{Q} + \nabla(\boldsymbol{\xi}\cdot\nabla p + \gamma p\nabla\cdot\boldsymbol{\xi}). \tag{3}$$

Here,

$$\mathbf{Q} \equiv \mathbf{B}_1 = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}). \tag{4}$$

In this approach, the only requirements are appropriate boundary conditions on $\boldsymbol{\xi}$.

IV. SELF-ADJOINTNESS OF THE FORCE OPERATOR AND ENERGY CONSERVATION

The self-adjointness of the force operator is defined as

$$\int_P \boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi})d\mathbf{r} = \int_P \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\eta})d\mathbf{r}, \tag{5}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are two arbitrary vectors and the integral is calculated for the unperturbed plasma volume P. The potential energy $\delta W(\boldsymbol{\xi}^*,\boldsymbol{\xi})$ is defined as

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$$\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = -\frac{1}{2}\int_P \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi}) d\mathbf{r}, \qquad (6)$$

where $\boldsymbol{\xi}(\mathbf{r})$ is treated as complex in the following equations in anticipation of cases where, because of symmetry, several spatial coordinates can be Fourier analyzed.

The physical basis for the energy principle is the fact that energy is exactly conserved in the ideal MHD model. The conservation of energy is proven by the following calculation,[1,2] which is carried out in the real time domain.

Consider the energy $H(t)$, which is the sum of the kinetic energy K and the potential energy δW, given by

$$H = K\left(\frac{\partial \tilde{\boldsymbol{\xi}}}{\partial t}, \frac{\partial \tilde{\boldsymbol{\xi}}}{\partial t}\right) + \delta W(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}) = \frac{1}{2}\int_P d\mathbf{r}\left[\rho\left(\frac{\partial \tilde{\boldsymbol{\xi}}}{\partial t}\right)^2 - \tilde{\boldsymbol{\xi}} \cdot \mathbf{F}(\tilde{\boldsymbol{\xi}})\right], \qquad (7)$$

where $\tilde{\boldsymbol{\xi}} = \tilde{\boldsymbol{\xi}}(\mathbf{r}, t)$. A simple calculation that makes use of the self-adjoint property of $\mathbf{F}$ yields

$$\frac{dH}{dt} = \int_P d\mathbf{r} \frac{\partial \tilde{\boldsymbol{\xi}}}{\partial t} \cdot \left[\rho \frac{\partial^2 \tilde{\boldsymbol{\xi}}}{\partial t^2} - \mathbf{F}(\tilde{\boldsymbol{\xi}})\right] = 0. \qquad (8)$$

This equation corresponds to energy conservation. The validity of this energy conservation is further discussed on the basis of a local energy conservation equation in Sec. VI B.

V. MAGNETOSPHERIC ENERGY PRINCIPLE AND IDEAL IONOSPHERIC BOUNDARY CONDITIONS

In order to investigate the ideal MHD stability of magnetospheric plasmas, a magnetospheric energy principle has been formulated[8] on the basis of the formulation of the energy principle[1,2] used in fusion plasmas. The magnetospheric energy principle is a specific energy principle, which completely neglects the existence of a neutral atmosphere below the ionosphere. In this section, the formulation of the magnetospheric energy principle is briefly reviewed, since this is the basis for the formulation of an extended magnetospheric energy principle including a neutral atmosphere. First, we seek ideal ionospheric boundary conditions, which are necessary to validate the self-adjointness of the force operator, and then calculate a specific form of δW.

A. Ionospheric boundary conditions for self-adjointness of the force operator

Since the existence of the neutral atmosphere is not taken into account in the magnetospheric energy principle,[8] ideal ionospheric boundary conditions on two arbitrary vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ at the unperturbed ionospheric surfaces are obtained, so that the force operator $\mathbf{F}$ becomes self-adjoint. Here, we define the ionosphere as the bottom plane of the magnetospheric plasma. Since we do not consider any dissipation in the ionosphere, in the magnetospheric energy principle we call the present ionosphere an ideal ionosphere. For such an ideal MHD magnetosphere-ionosphere system, we can formulate an energy principle.

Let us assume that the unperturbed ionospheric surface is located at $\mathbf{r} = \mathbf{r}_0$. We assume that the unperturbed magnetic field $\mathbf{B}(\mathbf{r}_0)$ is everywhere perpendicular to the local unperturbed ionospheric surface. Therefore, the normal vector $\mathbf{n}$ on the unperturbed plasma volume P at the ionospheric surface satisfies $\mathbf{n} = \mathbf{b}$, where $\mathbf{b} = \mathbf{B}/|\mathbf{B}|$, in the northern hemisphere and $\mathbf{n} = -\mathbf{b}$ at the ionospheric surface in the southern hemisphere. This normal incidence of the unperturbed magnetic field on the ionosphere is the only assumption in the magnetospheric energy principle. On the lateral boundaries $S_{\rm out}$ and $S_{\rm in}$, which are thin perfectly conducting walls, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfy $\mathbf{n} \cdot \boldsymbol{\xi} = \mathbf{n} \cdot \boldsymbol{\eta} = 0$, since $\mathbf{n} \cdot \mathbf{v} = 0$ on these boundaries.

When the plasma is assumed to be everywhere surrounded by a perfectly conducting wall, all the boundary terms (surface integrals) arising from the volume integral of Eq. (5) vanish and the force operator $\mathbf{F}$ becomes evidently self-adjoint. This is shown in detail in Bernstein *et al.*[1] and Appendix A of Freidberg.[2] However, since the ionosphere is not a perfectly conducting wall, the self-adjointness of the force operator $\mathbf{F}$ is not satisfied for an arbitrary perturbation and the energy conservation in the magnetosphere is not guaranteed in general. The self-adjointness of the force operator $\mathbf{F}$ is satisfied only for some specific boundary conditions on ionospheric boundaries.

One obtains[2,8]

$$\begin{aligned}\int_P \boldsymbol{\eta} \cdot \mathbf{F}(\boldsymbol{\xi}) d\mathbf{r} = -\int_P d\mathbf{r}\{&\mu_0^{-1}[(\mathbf{B} \cdot \nabla)\boldsymbol{\xi}_\perp] \\ &\cdot [(\mathbf{B} \cdot \nabla)\boldsymbol{\eta}_\perp] + \gamma p(\nabla \cdot \boldsymbol{\xi})(\nabla \cdot \boldsymbol{\eta}) \\ &+ \mu_0^{-1}B^2(\nabla \cdot \boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})(\nabla \cdot \boldsymbol{\eta}_\perp + 2\boldsymbol{\eta}_\perp \cdot \boldsymbol{\kappa}) \\ &- 4\mu_0^{-1}B^2(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})(\boldsymbol{\eta}_\perp \cdot \boldsymbol{\kappa}) \\ &+ (\boldsymbol{\eta}_\perp \boldsymbol{\xi}_\perp : \nabla\nabla)(p + \mu_0^{-1}B^2/2)\} + \mathrm{BT},\end{aligned} \qquad (9)$$

where BT is the boundary term arising from the surface integral over the unperturbed surface S. Here, $\boldsymbol{\xi} = \boldsymbol{\xi}_\perp + \xi_\parallel \mathbf{b}$, $\boldsymbol{\eta} = \boldsymbol{\eta}_\perp + \eta_\parallel \mathbf{b}$, and $\boldsymbol{\kappa} = (\mathbf{b} \cdot \nabla)\mathbf{b}$. The BT is given by[8]

$$\begin{aligned}\mathrm{BT} = &\int_{\rm north}\{\gamma p \eta_\parallel(\nabla \cdot \boldsymbol{\xi}) + \mu_0^{-1}B\boldsymbol{\eta}_\perp \cdot [(\mathbf{B} \cdot \nabla)\boldsymbol{\xi}_\perp]\}dS \\ &- \int_{\rm south}\{\gamma p \eta_\parallel(\nabla \cdot \boldsymbol{\xi}) + \mu_0^{-1}B\boldsymbol{\eta}_\perp \cdot [(\mathbf{B} \cdot \nabla)\boldsymbol{\xi}_\perp]\}dS,\end{aligned} \qquad (10)$$

where "north" and "south" denote unperturbed ionospheric surfaces in the northern hemisphere and the southern hemisphere, respectively.

On the right-hand side of Eq. (9), the volume integral over the plasma volume P is clearly a self-adjoint form. Since BT cannot be further reduced to a self-adjoint form, the force operator $\mathbf{F}$ becomes self-adjoint only when BT is zero. Since $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are independent and arbitrary except for satisfying boundary conditions, BT vanishes only when the condition $\eta_\parallel = 0$ or $\nabla \cdot \boldsymbol{\xi} = 0$ and the condition $\boldsymbol{\eta}_\perp = 0$ or $(\mathbf{b} \cdot \nabla)\boldsymbol{\xi}_\perp = 0$ are satisfied on unperturbed ionospheric surfaces in both the hemispheres. These are not only sufficient conditions but also necessary conditions for BT to vanish. This

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means that these are the necessary and sufficient conditions for the force operator to become self-adjoint.

Therefore, when the vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfy these boundary conditions on the unperturbed ionospheric surfaces, the force operator $\mathbf{F}$ becomes self-adjoint and energy conservation is valid. It follows that in order for the force operator to become self-adjoint, the plasma displacement vector $\boldsymbol{\xi}$ at the ionospheric surface must satisfy one of the following boundary conditions, i.e.,

$$\xi_{\parallel}=0 \quad \text{and} \quad (\mathbf{b}\cdot\nabla)\boldsymbol{\xi}_{\perp}=0, \tag{11}$$

$$\nabla\cdot\boldsymbol{\xi}=0 \quad \text{and} \quad (\mathbf{b}\cdot\nabla)\boldsymbol{\xi}_{\perp}=0, \tag{12}$$

$$\nabla\cdot\boldsymbol{\xi}=0 \quad \text{and} \quad \boldsymbol{\xi}_{\perp}=0, \tag{13}$$

$$\xi_{\parallel}=0 \quad \text{and} \quad \boldsymbol{\xi}_{\perp}=0. \tag{14}$$

It is obvious that among the above four boundary conditions, the magnetospheric interchange mode requires nonzero $\boldsymbol{\xi}_{\perp}$ on the ideal ionospheric surfaces. Therefore, only boundary conditions (11) and (12) satisfy the requirement for the interchange mode. The boundary conditions (11) and (12) are called the horizontally free and free boundary conditions, respectively. The boundary condition (14) is called the rigid boundary condition. It is important to point out here that the horizontally free boundary condition (11) becomes an insulating boundary condition[10] for a flat ionosphere, since there is no finite $\mathbf{B}_{1\perp}$ at the flat ionosphere and hence no ionospheric surface current for a flat ionospheric surface.[8]

B. Potential energy δW

The potential energy δW defined by Eq. (6) has been calculated[8] for possible boundary conditions (11)–(14). For both horizontally free and free boundary conditions, δW is given by $\delta W(\boldsymbol{\xi}^*,\boldsymbol{\xi})=\delta W_F+\delta W_I$. For boundary conditions (13) and (14), δW is simply equal to δW_F. Here, δW_F is the variational change of the potential energy for the magnetospheric plasma, which is calculated for the unperturbed plasma volume P. The specific form of δW_F for the magnetospheric energy principle is the same as δW_F given by Bernstein *et al.*[1] and Freidberg[2] and can be written as[8]

$$\delta W_F=\frac{1}{2}\int_P d\mathbf{r}\big[\mu_0^{-1}|\mathbf{Q}_{\perp}|^2+\mu_0^{-1}B^2|\nabla\cdot\boldsymbol{\xi}_{\perp}+2\boldsymbol{\xi}_{\perp}\cdot\boldsymbol{\kappa}|^2 + \gamma p|\nabla\cdot\boldsymbol{\xi}|^2-2(\boldsymbol{\xi}_{\perp}\cdot\nabla p)(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_{\perp}^*)-J_{\parallel}(\boldsymbol{\xi}_{\perp}^*\times\mathbf{b})\cdot\mathbf{Q}_{\perp}\big], \tag{15}$$

where γ is the ratio of specific heats. This is the intuitive form of δW_F.[11,12]

δW_I is the ionospheric surface contribution to the variational change of the potential energy and is calculated for the unperturbed spherical ionospheric surface. The specific form of δW_I for the three-dimensional configuration assuming a spherical ionospheric surface is[8]

$$\delta W_I=-\frac{1}{2\mu_0}\left(\int_{\text{north}}\frac{B^2|\boldsymbol{\xi}_{\perp}|^2}{R_I}dS+\int_{\text{south}}\frac{B^2|\boldsymbol{\xi}_{\perp}|^2}{R_I}dS\right), \tag{16}$$

where R_I is the sum of the earth's radius R_E and the ionospheric height h ($R_I=R_E+h\sim R_E$).

Notice that the surface contribution δW_I is clearly negative for both the horizontally free and free boundary conditions. Therefore, this term is destabilizing for these boundary conditions. Since R_I appears in the denominator of Eq. (16), this destabilizing effect by δW_I occurs only for a spherical ionospheric surface for a three-dimensional configuration. The magnetospheric energy principle states that a plasma equilibrium is stable if and only if $\delta W(\boldsymbol{\xi}^*,\boldsymbol{\xi})=\delta W_F+\delta W_I \geq 0$ for all allowable displacements $\boldsymbol{\xi}$.

The existence of negative δW_I suggests that a magnetospheric plasma can be ideal MHD unstable under proper conditions even without potential sources of pressure-driven modes or current-driven modes, which give rise to negative δW_F.

VI. PHYSICAL MEANING OF THE IONOSPHERIC CONTRIBUTION δW_I TO THE POTENTIAL ENERGY

A. Intuitive explanation

For a horizontally free ionospheric boundary condition (11), the relationship between the Poynting vector across the ionospheric surface and the temporal change of $\tilde{\boldsymbol{\xi}}_{\perp}$ is important to understand the energy balance in the magnetospheric system, since there is no other means to carry energy flux across the ionospheric surface under the assumption of $\xi_{\parallel}=0$.

Let us consider a physical picture of this relationship. We first note that the second order linear perturbation of Poynting vector $\tilde{\mathbf{s}}_2$ is expressed by

$$\tilde{\mathbf{s}}_2=\mu_0^{-1}(\tilde{\mathbf{E}}_1\times\tilde{\mathbf{B}}_1+\tilde{\mathbf{E}}_2\times\mathbf{B}_0), \tag{17}$$

where the subscripts 1 and 2 denote the first order linear perturbation and the second order linear perturbation, respectively, and the subscript 0 denotes an unperturbed quantity. Since one is interested in the component of the Poynting vector parallel to the unperturbed magnetic field, only the first term in Eq. (17) is relevant in the following discussion.

Let us consider a magnetospheric interchange mode, which occurs for the horizontally free ionospheric boundary condition and is characterized by $\tilde{\mathbf{B}}_{1\perp}\simeq 0$ in the magnetosphere.[9] Figure 3 shows a physical picture to explain the direction of Poynting vector across the ionospheric surface when the amplitude of $\tilde{\boldsymbol{\xi}}_{\perp}$ is increasing. The circular arc S_I in this figure represents an unperturbed spherical ionospheric surface in the northern hemisphere. Let A be a point on the unperturbed ionospheric surface S_I. At $t=0$ the unperturbed field line passing through point A is shown by the line AB, which is incident vertically on the unperturbed ionospheric surface. At time $t=\Delta t$, this field line is moved to the position of line $A'B'$ by a horizontal lateral plasma motion, since the horizontal free boundary condition $(\mathbf{b}\cdot\nabla)\tilde{\boldsymbol{\xi}}_{\perp}=0$ must be satisfied. Therefore,

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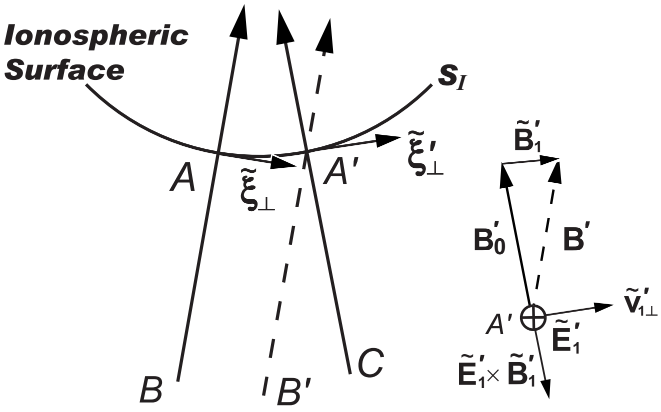

FIG. 3. A physical picture showing the direction of Poynting vector across the ionospheric surface when the amplitude of $\tilde{\xi}_\perp$ is increasing. The circular arc S_I shows an unperturbed spherical ionospheric surface in the northern hemisphere. An enlarged plot of the neighborhood of point A' is shown in the lower-right corner.

$$\mathbf{r}_{A'}(\Delta t) = \mathbf{r}_A(0) + \tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_A,\Delta t) \tag{18}$$

is satisfied. Although in Fig. 3 the position of A' is shown on the ionospheric spherical surface S_I, actually A' is located infinitesimally above the ionospheric surface S_I, because $\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_A,\Delta t)$ is tangent to the spherical surface S_I at point A. If $A'C$ is the z-axis and z'_A is the z component of $\mathbf{r}_{A'}$, $z_{A'}=0_+$, where $z=0$ corresponds to the ionospheric surface and z increases in the direction of $A'C$. Here, 0_+ means $0+\delta$ and $\delta > 0$ is an infinitesimally small quantity. The straight line $A'C$ also represents an unperturbed field line passing through point A', which is also incident vertically on the ionospheric surface at point A'. An enlarged plot of the neighborhood of point A' is shown in the lower-right corner of this figure. Let us assume that the amplitude of $\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_A,\Delta t)$ is increasing. Since the amplitude of $\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_A,\Delta t)$ is increasing, $\tilde{\mathbf{v}}_{1\perp}(\mathbf{r}_A,\Delta t)$ is parallel to $\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_A,\Delta t)$.

Then, $\tilde{\mathbf{v}}'_{1\perp} \equiv \tilde{\mathbf{v}}_{1\perp}(\mathbf{r}_{A'},\Delta t)$ is also directed in the same direction as $\tilde{\mathbf{v}}_\perp(\mathbf{r}_A,\Delta t)$, since A' is very close to A. Notice that $\tilde{\mathbf{v}}_\perp=\tilde{\mathbf{v}}_{1\perp}$, since there is no unperturbed velocity. Therefore, $\tilde{\mathbf{E}}'_1 \equiv \tilde{\mathbf{E}}_1(\mathbf{r}_{A'},\Delta t) = -\tilde{\mathbf{v}}'_{1\perp}\times\mathbf{B}_0(\mathbf{r}_{A'})$ is directed into the page, as shown in the lower-right corner of this figure, where $\mathbf{B}'_0 \equiv \mathbf{B}_0(\mathbf{r}_{A'})$ is the unperturbed magnetic field at point A'.

Let us assume that the magnetic field line at point A' at $t=\Delta t$ is shown by the dotted line $A'B'$, which is convected from the position of line AB at $t=0$. The magnetic field at point A' at time $t=\Delta t$ is shown by the dashed vector denoted by $\mathbf{B}' \equiv \mathbf{B}(\mathbf{r}'_A,\Delta t)$ in the lower-right corner of this figure. Therefore, this assumption means that $\mathbf{B}(\mathbf{r}'_A,\Delta t)=\mathbf{B}(\mathbf{r}_A)$. Since the magnetic field perturbation $\tilde{\mathbf{B}}_1(\mathbf{r}_{A'},\Delta t)$ at point A' is given by

$$\tilde{\mathbf{B}}_1(\mathbf{r}_{A'},\Delta t) = \mathbf{B}(\mathbf{r}_{A'},\Delta t) - \mathbf{B}_0(\mathbf{r}_{A'}), \tag{19}$$

$\tilde{\mathbf{B}}_1(\mathbf{r}_{A'},\Delta t)$ is shown by $\tilde{\mathbf{B}}'_1$. Thus, it is obvious that $\tilde{\mathbf{E}}'_1 \times \tilde{\mathbf{B}}'_1$, which has a component parallel to the unperturbed magnetic field, is directed into the magnetosphere at point A', as shown in the lower-right corner. Therefore, the second order linear perturbation of Poynting vector at point A' has a component into the magnetosphere when the amplitude of $\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_A,t)$ is increasing. Thus, the magnetospheric interchange mode grows by absorbing energy carried by this Poynting vector, which is directed into the magnetosphere.

The important assumption in the above physical picture is that $\mathbf{B}(\mathbf{r}_{A'},\Delta t)=\mathbf{B}(\mathbf{r}_A)$. Therefore, we must show that this relationship is valid. From Eq. (18), we obtain by expansion

$$\mathbf{B}(\mathbf{r}_{A'},\Delta t) = \mathbf{B}(\mathbf{r}_A+\tilde{\boldsymbol{\xi}}_\perp,\Delta t) \simeq \mathbf{B}(\mathbf{r}_A,0) + \Delta t\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r}_A,0) + (\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla)\mathbf{B}(\mathbf{r}_A,\Delta t), \tag{20}$$

where $\tilde{\boldsymbol{\xi}}_\perp=\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_A,\Delta t)$ and $\partial\mathbf{B}(\mathbf{r}_A,0)/\partial t \equiv \partial\mathbf{B}(\mathbf{r}_A,t)/\partial t|_{t=0}$. From the induction equation, we have

$$\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r}_A,t) = -\mathbf{B}(\mathbf{r}_A,t)[\nabla\cdot\mathbf{v}(\mathbf{r}_A,t)] + [\mathbf{B}(\mathbf{r}_A,t)\cdot\nabla]\mathbf{v}(\mathbf{r}_A,t) - [\mathbf{v}(\mathbf{r}_A,t)\cdot\nabla]\mathbf{B}(\mathbf{r}_A,t). \tag{21}$$

Substitution of Eq. (21) into Eq. (20) yields

$$\begin{aligned}\mathbf{B}(\mathbf{r}_{A'},\Delta t) = {}& \mathbf{B}(\mathbf{r}_A,0) - \Delta t\cdot\mathbf{B}(\mathbf{r}_A,0)[\nabla\cdot\mathbf{v}(\mathbf{r}_A,0)] \\ &+ \Delta t\cdot[\mathbf{B}(\mathbf{r}_A,0)\cdot\nabla]\mathbf{v}(\mathbf{r}_A,0) \\ &- \Delta t\cdot[\mathbf{v}(\mathbf{r}_A,0)\cdot\nabla]\mathbf{B}(\mathbf{r}_A,0) \\ &+ (\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla)\mathbf{B}(\mathbf{r}_A,\Delta t).\end{aligned} \tag{22}$$

Since

$$\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r},t) = \int_0^t dt'\mathbf{v}_\perp(\mathbf{r},t'), \tag{23}$$

we obtain

$$\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_A,\Delta t) = \int_0^{\Delta t} dt'\mathbf{v}_\perp(\mathbf{r}_A,t') \simeq \mathbf{v}_\perp(\mathbf{r}_A,0)\Delta t. \tag{24}$$

Therefore, in the lowest order, Eq. (22) becomes $\mathbf{B}(\mathbf{r}'_A,\Delta t) = \mathbf{B}(\mathbf{r}_A,0)=\mathbf{B}(\mathbf{r}_A)$. This completes the proof of $\mathbf{B}(\mathbf{r}'_A,\Delta t) \simeq \mathbf{B}(\mathbf{r}_A)$. This means that when there is a plasma displacement $\boldsymbol{\xi}$, the magnetic field at $\mathbf{r}'_A(\Delta t)$ is considered to be a magnetic field which is convected from $\mathbf{r}=\mathbf{r}_A$.

Figure 4 shows a physical picture to explain the direction of Poynting vector across the ionospheric surface when the amplitude of $\tilde{\boldsymbol{\xi}}_\perp$ is decreasing in the same format as Fig. 3. Since the amplitude of $\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_A,\Delta t)$ is decreasing, $\tilde{\mathbf{v}}_{1\perp}(\mathbf{r}_{A'},\Delta t)$ is directed opposite of $\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_A,t)$. Therefore, $\tilde{\mathbf{E}}'_1 \equiv \tilde{\mathbf{E}}_1(\mathbf{r}_{A'},\Delta t)$ is directed out of page, as shown in the lower-right corner of this figure. Hence, the direction of $\tilde{\mathbf{E}}_1(\mathbf{r}_{A'},\Delta t)$ is opposite to that shown in Fig. 3. Since $\tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_A,\Delta t)$ is still directed rightward, as shown in Fig. 4, $\tilde{\boldsymbol{\xi}}'_\perp \equiv \tilde{\boldsymbol{\xi}}_\perp(\mathbf{r}_{A'},\Delta t)$ is also directed rightward. Therefore, at time Δt, the dashed line, which is parallel to field line AB, represents a perturbed field line passing through point A'. Hence, the direction of $\tilde{\mathbf{B}}'_1 \equiv \tilde{\mathbf{B}}_1(\mathbf{r}_{A'},\Delta t)$ is the same as that shown in Fig. 3. Thus, the Poynting vector $\mu_0^{-1}\tilde{\mathbf{E}}'_1\times\tilde{\mathbf{B}}'_1$ is directed into the ionosphere at point A'. Thus, one notes physically that Poynting

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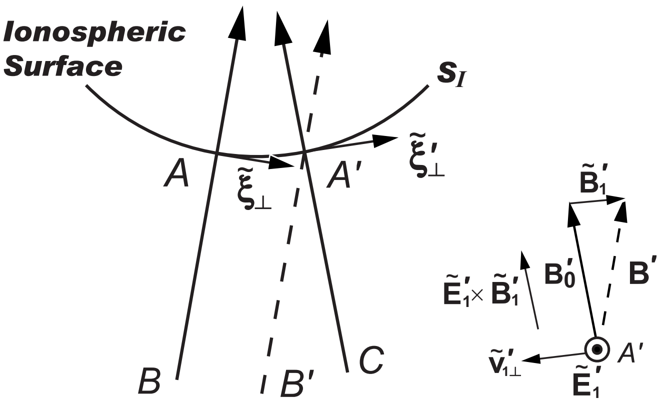

FIG. 4. A physical picture showing the direction of Poynting vector across the ionospheric surface when the amplitude of $\tilde{\xi}_\perp$ is decreasing. The circular arc S_I shows an unperturbed spherical ionospheric surface in the northern hemisphere. An enlarged plot of the neighborhood of point A' is shown in the lower-right corner.

vector $\mu_0^{-1}\tilde{\mathbf{E}}_1' \times \tilde{\mathbf{B}}_1'$ is directed into the ionosphere from the magnetosphere when the amplitude of $\tilde{\xi}_\perp$ is decreasing at point A. Thus, the magnetospheric interchange mode loses energy and its amplitude decays.

Let us assume that $\delta W_F = 0$ and $\delta W = \delta W_I < 0$. Then, in the ideal MHD there are purely growing and purely decaying modes. Figures 3 and 4 show that when $\xi_\perp \neq 0$, there are two possible modes, one growing and the other decaying. These two modes correspond to the purely growing and decaying modes predicted by the energy principle.

B. Explanation by using the local energy conservation equation

The energy conservation equation (8) is derived from the self-adjointness of the force operator and it means that $H(t) = K(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_1) + \delta W(\tilde{\xi}, \tilde{\xi}) = K + \delta W_F + \delta W_I$, where $\tilde{\mathbf{v}}_1 = \partial \tilde{\xi}/\partial t$ and $\delta W(\tilde{\xi}, \tilde{\xi}) = -\frac{1}{2}\int_P \tilde{\xi} \cdot \mathbf{F}(\tilde{\xi}) d\mathbf{r}$, is conserved. This is the basis of the present magnetospheric energy principle.[8] In this subsection, we show that $\delta W(\tilde{\xi}, \tilde{\xi}) = \delta W_F + \delta W_I$ is valid based on a rigorous local energy conservation equation. Thus, we give an alternative indication that ionosphere-driven interchange instability[9] may exist. Although the details of the calculation are shown elsewhere,[8] the calculation is briefly reviewed here, since the understanding of the origin of δW_I is essential for understanding the magnetospheric energy principle. In this subsection, K, δW, δW_F, and δW_I denote $K(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_1)$, $\delta W(\tilde{\xi}, \tilde{\xi})$, $\delta W_F(\tilde{\xi}, \tilde{\xi})$, and $\delta W_I(\tilde{\xi}_\perp, \tilde{\xi}_\perp)$, respectively, and the calculation is carried out in the real time domain.

The time evolution of total energy at any point in the magnetospheric plasma is described by the local energy conservation equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho \mathbf{v}^2 + \frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0}\right) = -\nabla \cdot \left[\left(\frac{1}{2}\rho \mathbf{v}^2 + p + \frac{p}{\gamma - 1}\right)\mathbf{v} + \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}\right], \tag{25}$$

where, contrary to the notation used thus far, ρ, p, B, $\mathbf{v}$, $\mathbf{E}$, and $\mathbf{B}$ are all total quantities, which are functions of the position $\mathbf{r}$ and time t and not unperturbed quantities.

Taking the second order linear perturbation of this equation and then integrating over the unperturbed plasma volume P, we obtain

$$\frac{\partial}{\partial t}\int_P \left(\frac{1}{2}\rho_0 \tilde{\mathbf{v}}_1^2 + \tilde{w}_2\right) d\mathbf{r} = -\int_S \tilde{\mathbf{u}}_2 \cdot \mathbf{n} dS, \tag{26}$$

where the energy flux density $\tilde{\mathbf{u}}_2$ is given by

$$\begin{aligned}\tilde{\mathbf{u}}_2 &= \frac{\gamma}{\gamma - 1}(\tilde{p}_1 \tilde{\mathbf{v}}_1) - \frac{1}{\mu_0}[(\tilde{\mathbf{v}}_1 \times \tilde{\mathbf{B}}_1) \times \mathbf{B}_0 + (\tilde{\mathbf{v}}_1 \times \mathbf{B}_0) \times \tilde{\mathbf{B}}_1] \\ &= \frac{\gamma}{\gamma - 1}(\tilde{p}_1 \tilde{\mathbf{v}}_1) + \frac{1}{\mu_0}[2\tilde{\mathbf{v}}_1(\mathbf{B}_0 \cdot \tilde{\mathbf{B}}_1) - \mathbf{B}_0(\tilde{\mathbf{v}}_1 \cdot \tilde{\mathbf{B}}_1) \\ &\quad - \tilde{\mathbf{B}}_1(\tilde{\mathbf{v}}_1 \cdot \mathbf{B}_0)]\end{aligned} \tag{27}$$

and

$$\tilde{w}_2 = \frac{\tilde{p}_2}{\gamma - 1} + \frac{1}{2\mu_0}(2\mathbf{B}_0 \cdot \tilde{\mathbf{B}}_2 + \tilde{\mathbf{B}}_1^2). \tag{28}$$

Contrary to the notation used thus far in Secs. III–V, the subscript 0 is added explicitly to the unperturbed quantity in this subsection in order to avoid confusion, and subscripts 1 and 2 denote the first order linear perturbation and the second order linear perturbation, respectively, as was used in the previous subsection VI A. The tilde on the perturbation means that the calculation is done in the real time domain and the perturbation is a function of position $\mathbf{r}$ and time t.

Using vector identities, it is straightforward to show that $\tilde{\mathbf{u}}_2 \cdot \mathbf{n} = 0$ on S_{out} and S_{in}, since $\tilde{\xi}_\perp = 0$ on S_{out} and S_{in}. Therefore, only the integral over the ionospheric surface contributes to the right-hand side of Eq. (26). Since $\mathbf{n} = \mathbf{b}$ is assumed on the ionosphere of the northern hemisphere in the present magnetospheric energy principle, we obtain

$$\begin{aligned}&[2\tilde{\mathbf{v}}_1(\mathbf{B}_0 \cdot \tilde{\mathbf{B}}_1) - \mathbf{B}_0(\tilde{\mathbf{v}}_1 \cdot \tilde{\mathbf{B}}_1) - \tilde{\mathbf{B}}_1(\tilde{\mathbf{v}}_1 \cdot \mathbf{B}_0)] \cdot \mathbf{n} \\ &\quad = -B_0^2\{\tilde{\mathbf{v}}_{1\perp} \cdot [(\mathbf{b} \cdot \nabla)\tilde{\xi}_\perp] - \tilde{\mathbf{v}}_{1\perp} \cdot [(\tilde{\xi}_\perp \cdot \nabla)\mathbf{b}]\}\end{aligned} \tag{29}$$

in the northern hemisphere by using vector identities. Notice that this is a contribution to the energy flux density normal to the ionospheric surface by Poynting vector. Therefore, on the ionosphere of the northern hemisphere, we obtain

$$\begin{aligned}\tilde{\mathbf{u}}_2 \cdot \mathbf{n} = &-\frac{\gamma}{\gamma - 1}(\tilde{\xi}_\perp \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \tilde{\xi})\tilde{v}_{1\parallel} \\ &- \frac{B_0^2}{\mu_0}\{\tilde{\mathbf{v}}_{1\perp} \cdot [(\mathbf{b} \cdot \nabla)\tilde{\xi}_\perp] - \tilde{\mathbf{v}}_{1\perp} \cdot [(\tilde{\xi}_\perp \cdot \nabla)\mathbf{b}]\}.\end{aligned} \tag{30}$$

On the ionosphere of the southern hemisphere, the sign of the right-hand side of Eq. (30) reverses.

For the horizontally free ionospheric boundary condition, which is valid for interchange instability, Eq. (30) shows that

$$\tilde{\mathbf{u}}_2 \cdot \mathbf{n} = \mu_0^{-1} B_0^2 \tilde{\mathbf{v}}_{1\perp} \cdot [(\tilde{\xi}_\perp \cdot \nabla)\mathbf{b}] \tag{31}$$

is valid on the ionosphere of the northern hemisphere. On the ionosphere of the southern hemisphere, the sign of the right-

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hand side reverses. Owing to the assumption of the normal incidence of the unperturbed magnetic field on the ionospheric surface, $(\tilde{\boldsymbol{\xi}}_\perp \cdot \nabla)\mathbf{b}$ is equal to $-\tilde{\boldsymbol{\xi}}_\perp / R_I$ and $\tilde{\boldsymbol{\xi}}_\perp / R_I$ on the ionospheres of the northern hemisphere and southern hemisphere, respectively.[8] Therefore, substitution of Eq. (31) into Eq. (26) yields

$$\frac{\partial}{\partial t}\int_P \left(\frac{1}{2}\rho_0 \tilde{\mathbf{v}}_1^2 + \tilde{w}_2\right) d\mathbf{r} = -\frac{\partial}{\partial t}\delta W_I(\tilde{\boldsymbol{\xi}}_\perp, \tilde{\boldsymbol{\xi}}_\perp), \tag{32}$$

where

$$\delta W_I(\tilde{\boldsymbol{\xi}}_\perp, \tilde{\boldsymbol{\xi}}_\perp) = -\frac{1}{2\mu_0}\left(\int_{\text{north}} \frac{B_0^2 \tilde{\boldsymbol{\xi}}_\perp^2}{R_I} dS + \int_{\text{south}} \frac{B_0^2 \tilde{\boldsymbol{\xi}}_\perp^2}{R_I} dS\right). \tag{33}$$

This means that for the horizontally free boundary condition, a second order contribution to the normal component of Poynting vector, which is integrated over the ionospheric surface, is expressed by the temporal change of δW_I. Therefore, in this case, Eq. (32) means that $K + W + \delta W_I$ is conserved, where $W \equiv \int_P \tilde{w}_2 d\mathbf{r}$. Thus, δW_I, which is included in δW, arises from the integration over the ionospheric surface of the normal component of Poynting vector. This appearance of δW_I in δW is due to the assumption of the normal incidence of the unperturbed magnetic field on the spherical ionospheric surface and a frozen-in plasma displacement on the spherical ionospheric surface.

The comparison of Eq. (32) with the energy conservation equation (8) resulting from the self-adjointness of the force operator yields $\delta W_F(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}) \equiv \delta W - \delta W_I = W$ for the horizontally free ionospheric boundary condition, since $H = K + \delta W = K + \delta W_F + \delta W_I$ is conserved from Eq. (8).

VII. MODEL OF THE NEUTRAL ATMOSPHERE AND THIN IONOSPHERIC LAYER

In the extended energy principle for fusion plasmas, a plasma is assumed to be separated from a conducting wall by a vacuum region.[1] Since the plasma pressure vanishes in the vacuum region, there is a thin layer between the vacuum region and the plasma, in which the plasma pressure must decrease to zero toward the vacuum region.[4,13] Since the plasma in this thin layer is also described by ideal MHD equations, this thin layer is considered to be within the plasma-vacuum interface used in the extended energy principle.[1,2] Therefore, a stabilizing positive potential energy contribution by magnetic tension force in this layer for fusion plasma is automatically included in the calculation of δW. However, in the extended magnetospheric energy principle, the magnetospheric plasma is separated from a conducting solid-earth surface by a neutral atmosphere. Between the atmosphere and the magnetospheric plasma, there is an ionosphere, which is a thin layer consisting of a partially ionized gas. We call this layer a thin ionospheric layer.

Figure 5 shows a sketch of the thin ionospheric layer between the atmosphere and the magnetosphere in the northern hemisphere. **B** denotes the direction of the unperturbed magnetic field. Let us assume that ℓ is the distance along the field line and it increases with the distance from the magnetosphere to the atmosphere. We assume that $\ell = \epsilon (\epsilon > 0)$ is located at the atmospheric edge between the neutral atmosphere and the thin ionospheric layer, as shown in Fig. 5, and $\ell = 0$ is located at the plasma edge between the ideal MHD magnetospheric plasma and the thin ionospheric layer. We take the limit of $\epsilon \to 0$ in the extended magnetospheric energy principle. Thus, in this limit, the thin ionospheric layer located in the region $0 < \ell < \epsilon$ becomes the ionospheric surface S_I shown in Figs. 1 and 2.

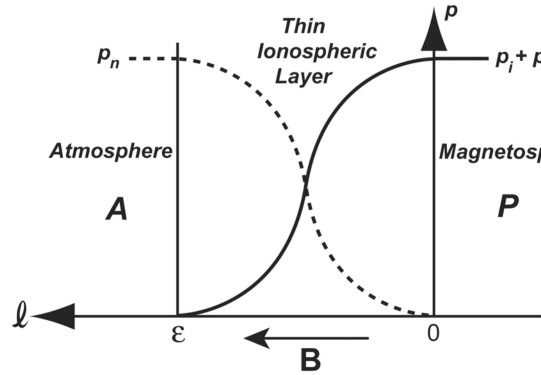

FIG. 5. A physical picture showing the thin ionospheric layer in the northern hemisphere. The thin layer is located at $0 < \ell < \epsilon$, where ℓ is the distance along a field line. **B** shows the direction of the unperturbed magnetic field. The ionospheric surface S_I is located at $\ell = 0$ and this is the edge of the magnetosphere. The atmospheric edge is located at $\ell = \epsilon$. The solid and dashed lines show the plasma pressure $(p_i + p_e)$ and the neutral gas pressure (p_n), respectively.

In the thin ionospheric layer, the plasma pressure $(p_i + p_e)$ shown by the solid line must decrease to zero toward the neutral atmosphere, as shown in Fig. 5. The neutral gas pressure must also decrease to zero toward the magnetosphere, as shown in Fig. 5. Since the plasma pressure must change along the field line, this thin ionospheric layer is clearly not an ideal MHD layer. Since there is a partially ionized gas in this thin ionospheric boundary layer, we need multifluid equations to describe the partially ionized gas.

For fusion plasmas, the pressure balance condition across the thin layer at the plasma edge can be obtained by assuming that the ideal MHD equations are valid inside the thin layer and by taking the limit of $\epsilon \to 0$.[4,13] Since the ideal MHD equations are valid in the thin layer in fusion plasmas, this thin layer is included in the plasma volume P used in the volume integration for calculation of δW in the extended energy principle for fusion plasmas.

In the extended magnetospheric energy principle, the thin ionospheric layer consisting of a partially ionized gas ought not be included in the plasma volume P, since the ideal MHD equations are not valid in the region $0 < \ell < \epsilon$. Therefore, we need to consider that $\ell = 0$ is the edge of the plasma volume P and S_I is located at $\ell = 0$ in the extended magnetospheric energy principle. Since the atmosphere consists of a neutral gas and there is no electric current in the atmosphere just as there is no electric current in the vacuum region surrounding the fusion plasma, we ought to think that the thin ionospheric boundary layer is also not included in the atmospheric volume A defined in Sec. II.

The equation of motion for s species in the multispecies

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gas in the thin ionospheric layer is given by[14]

$$m_sN_s\frac{\partial \mathbf{u}_s}{\partial t}+m_sN_s(\mathbf{u}_s\cdot\nabla)\mathbf{u}_s+\nabla p_s-q_sN_s(\mathbf{E}+\mathbf{u}_s\times\mathbf{B})$$
$$=\sum_t m_sN_s\bar{\nu}_{st}(\mathbf{u}_t-\mathbf{u}_s), \tag{34}$$

where

$$\bar{\nu}_{st}=\frac{m_t}{m_s+m_t}\nu_{st}. \tag{35}$$

Here, the subscript s denotes i(ion), e(electron), and n(neutral), q_s is the charge of s species, m_s is the mass of s species particles, N_s is the density of s species, and ν_{st} is the collision frequency of elastic collisions between species s and t. We neglected the effects of ionization, recombination, and chemical reactions in the thin ionospheric layer. We further assume that $N_i=N_e=N$.

The scalar pressure of the gas mixture is given by[14]

$$p=\sum_s p_s+\frac{1}{3}\left(\sum_s m_sn_su_s^2-\rho_mu^2\right), \tag{36}$$

where

$$\rho_m=\sum_s m_sn_s,$$

$$\rho_m\mathbf{u}=\sum_s m_sn_s\mathbf{u}_s.$$

Therefore, at the atmospheric edge ($\ell=\epsilon$) $p=p_n$ and at the magnetospheric edge ($\ell=0$) $p=p_i+p_e$.

In the following, we first explain how the plasma pressure $p=p_i+p_e$ changes from a finite value at $\ell=0$ to zero at $\ell=\epsilon$ in the equilibrium state ($\partial/\partial t=0$) in the thin ionospheric layer, as shown in Fig. 5. In order to show this, we add the i and e components of the equation of motion (34) and obtain

$$\nabla\left(p_i+p_e+\frac{B^2}{2\mu_0}\right)=\mu_0^{-1}(\mathbf{B}\cdot\nabla)\mathbf{B}+\mathbf{f}_p, \tag{37}$$

where

$$\mathbf{f}_p=-N[m_i(\mathbf{u}_i\cdot\nabla)\mathbf{u}_i+m_e(\mathbf{u}_e\cdot\nabla)\mathbf{u}_e]$$
$$-Nm_i\bar{\nu}_{\text{in}}(\mathbf{u}_i-\mathbf{u}_n)-Nm_e\bar{\nu}_{en}(\mathbf{u}_e-\mathbf{u}_n)$$
$$-Nm_i\bar{\nu}_{ie}(\mathbf{u}_i-\mathbf{u}_e)-Nm_e\bar{\nu}_{ei}(\mathbf{u}_e-\mathbf{u}_i), \tag{38}$$

and we used that $\mathbf{J}=\mu_0^{-1}\nabla\times\mathbf{B}$. By taking the dot product of this equation with $\mathbf{n}$ and integrating from $\ell=0$ to $\ell=\epsilon$, we obtain in the northern hemisphere

$$\left[\left[p_i+p_e+\frac{B^2}{2\mu_0}\right]\right]=\lim_{\epsilon\to0}\int_0^\epsilon \mu_0^{-1}\mathbf{n}\cdot[(\mathbf{B}\cdot\nabla)\mathbf{B}]d\ell$$
$$+\lim_{\epsilon\to0}\int_0^\epsilon \mathbf{n}\cdot\mathbf{f}_pd\ell, \tag{39}$$

where $[[G]]=G_A-G_P$, G_A and G_P being the values of G in the atmosphere and in the plasma, respectively, in the neighborhood of the plasma-atmosphere interface S_I. Notice that this equation is valid regardless of the geometrical relationship between $\mathbf{n}$ and $\mathbf{B}$. Since the normal component of the magnetic field is continuous,

$$\lim_{\epsilon\to0}\int_0^\epsilon \mathbf{n}\cdot[(\mathbf{B}\cdot\nabla)\mathbf{B}]d\ell\to0. \tag{40}$$

Thus, if we assume, as assumed in the magnetospheric energy principle, that $\mathbf{n}$ and $\mathbf{B}$ are parallel, we obtain from Eq. (39)

$$p_i(0)+p_e(0)=-\lim_{\epsilon\to0}\int_0^\epsilon \mathbf{n}\cdot\mathbf{f}_pd\ell, \tag{41}$$

since the plasma pressure is zero at the atmospheric side of the interface S_I and $[[B^2]]=0$. Therefore, for the plasma pressure to decrease to zero at the atmospheric side of the interface S_I, $|\mathbf{n}\cdot\mathbf{f}_p|$ must become infinite in $0<\ell<\epsilon$ in the limit of $\epsilon\to0$.

Next, in the equilibrium state, the n component of Eq. (34) becomes

$$\nabla p_n=\mathbf{f}_n, \tag{42}$$

where

$$\mathbf{f}_n=-N_n[m_n(\mathbf{u}_n\cdot\nabla)\mathbf{u}_n]-N_nm_n\bar{\nu}_{ni}(\mathbf{u}_n-\mathbf{u}_i)$$
$$-N_nm_n\bar{\nu}_{ne}(\mathbf{u}_n-\mathbf{u}_e). \tag{43}$$

Therefore, we obtain in the northern hemisphere

$$\lim_{\epsilon\to0}p_n(\epsilon)=\lim_{\epsilon\to0}\int_0^\epsilon \mathbf{n}\cdot\mathbf{f}_nd\ell, \tag{44}$$

where we used that the pressure of the neutral component vanishes at the plasma side of the interface S_I. Notice that this equation is valid regardless of the relationship between $\mathbf{n}$ and $\mathbf{B}$. Therefore, for the neutral pressure to become zero at the plasma side of the interface S_I, $|\mathbf{n}\cdot\mathbf{f}_n|$ must become infinite in $0<\ell<\epsilon$ in the limit of $\epsilon\to0$. Thus, we find that if the multispecies gas equation of motion (34) is taken into account to describe the partially ionized gas in the thin ionospheric layer, the transition from the plasma to the neutral gas across the thin ionospheric layer, as shown in Fig. 5, can reasonably be described.

Let us now consider how the total pressure is balanced across the thin ionospheric layer. By adding Eqs. (37) and (42), we obtain

$$\nabla\left(p_i+p_e+p_n+\frac{B^2}{2\mu_0}\right)=\mu_0^{-1}(\mathbf{B}\cdot\nabla)\mathbf{B}+\mathbf{f}_p+\mathbf{f}_n. \tag{45}$$

Therefore, in the limit of $\epsilon\to0$, one obtains in the northern hemisphere

$$\left[\left[p_i+p_e+p_n+\frac{B^2}{2\mu_0}\right]\right]=\lim_{\epsilon\to0}\int_0^\epsilon \mathbf{n}\cdot(\mathbf{f}_p+\mathbf{f}_n)d\ell. \tag{46}$$

Notice that this equation is valid regardless of the geometrical relationship between $\mathbf{n}$ and $\mathbf{B}$.

In the unperturbed state used in the extended magnetospheric energy principle, $\mathbf{n}$ and $\mathbf{B}$ are assumed to be parallel in the northern hemisphere and $[[B^2]]=0$. Therefore, from Eq. (46) we obtain

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$$[[p_i + p_e + p_n]] = \lim_{\epsilon \to 0} \int_0^{\epsilon} \mathbf{n} \cdot (\mathbf{f}_p + \mathbf{f}_n) d\ell. \quad (47)$$

We further assume in the extended magnetospheric energy principle that the plasma pressure and the neutral gas pressure are balanced at the interface S_I, i.e.,

$$p_i(0) + p_e(0) = \lim_{\epsilon \to 0} p_n(\epsilon). \quad (48)$$

Then, from Eq. (47) the right-hand side of Eq. (47) must vanish. In such a case, the pressure balance condition across the ionosphere can be written simply as

$$\left[\left[p_i + p_e + p_n + \frac{B^2}{2\mu_0} \right]\right] = 0. \quad (49)$$

This is similar to the pressure balance equation used in the extended energy principle for fusion plasmas, although $p_n = 0$ in fusion plasmas. As noted above, in the equilibrium configuration of the extended magnetospheric energy principle, Eq. (48) is assumed and thus the pressure balance condition (49) is assumed to be valid.

The magnetic field and the normal vector at the perturbed plasma-atmosphere interface are not necessarily parallel. Since Eq. (46) is valid regardless of the relation between the magnetic field and the normal vector, we assume that the right-hand side of Eq. (46) vanishes, and therefore the same pressure balance condition (49) holds in the perturbed state. Thus, like the energy principle for fusion plasmas, we impose a pressure balance condition at the perturbed plasma-atmosphere interface.

Notice that the pressure balance condition is satisfied at the plasma-vacuum interface in fusion plasmas,[4,13] but in the extended magnetospheric energy principle, the pressure balance is an assumption, since we cannot justify in general that the right-hand side of Eq. (46) can be so small that it can be neglected compared with the left-hand side. We note, however, that if $N_n = N$, the collisional parts of $\mathbf{f}_p$ and $\mathbf{f}_n$ exactly cancel out. Therefore, the assumption of pressure balance for the case of $\mathbf{B}$ parallel to $\mathbf{n}$ may be a reasonable assumption, even though we cannot rigorously prove its validity in general.

Let $\mathbf{r}_0$ be the equilibrium position of the plasma-atmosphere interface S_I. Then, from Eq. (49) the equilibrium variables must satisfy

$$p(\mathbf{r}_0) + \frac{B^2(\mathbf{r}_0)}{2\mu_0} = \hat{p}(\mathbf{r}_0) + \frac{\hat{B}^2(\mathbf{r}_0)}{2\mu_0}, \quad (50)$$

where the caret denotes a quantity in the atmosphere. Therefore, in the expected pressure balance at the interface, $\hat{p}$ of the neutral gas must be explicitly included in the total pressure on the atmospheric side.

As an extreme simplification, the atmosphere may also be represented by a vacuum region. However, from the pressure balance condition (49) and the continuity of the normal component of the magnetic field, such a model must assume that the magnetospheric plasma is cold. Therefore, such a model cannot describe pressure-driven modes in the magnetosphere.

Here, we should mention that in the extended magnetospheric energy principle, there is an important assumption about m_n. We have seen from Eq. (41) that in order for the plasma pressure to decrease to zero at the atmospheric edge between the thin ionospheric layer and the neutral gas, m_n ought not be zero. Since in the extended magnetospheric energy principle we neglect the kinetic energy of the neutral gas, the neutral gas must be considered to be heavy enough, so that it is almost immobile and its kinetic energy is negligible. Thus, we assume that m_n is very large. In order to sustain the plasma pressure in the equilibrium state, we also have to assume that the neutral gas pressure in the atmosphere must be finite. If we assume that the neutral gas in the atmosphere obeys the adiabatic gas law, we have

$$\frac{d}{dt}\left(\frac{\hat{p}}{\hat{\rho}^{\hat{\gamma}}}\right) = 0. \quad (51)$$

Since $\hat{\rho}$ is assumed to be very large, we have to assume that $\hat{\gamma}$ is very small. This means that the neutral gas is a very compressible gas. In the case of the terrestrial atmosphere, neutral components consist of heavy molecules such as nitrogen molecules (N_2) and oxygen molecules (O_2). Therefore, the assumption of a very large m_n might well be justified.

VIII. BOUNDARY CONDITIONS FOR PLASMA-ATMOSPHERE SYSTEM

The unperturbed pressure balance at the plasma-atmosphere interface is shown by Eq. (50). The unperturbed magnetic field must satisfy the following two boundary conditions at the plasma-atmosphere interface:

$$\mathbf{n} \cdot [[\mathbf{B}]] = 0 \quad (52)$$

and

$$\mathbf{n} \times [[\mathbf{B}]] = \mu_0 \mathbf{K}, \quad (53)$$

where $\mathbf{K}$ is the surface current density at the interface. We assume that the unperturbed surface current density is zero at the interface and $\mathbf{B}$ is incident vertically on the unperturbed ionospheric surface. Then, we obtain from Eq. (52)

$$\mathbf{B}(\mathbf{r}_0) = \hat{\mathbf{B}}(\mathbf{r}_0). \quad (54)$$

Therefore, from Eq. (50) we have $p(\mathbf{r}_0) = \hat{p}(\mathbf{r}_0)$ at the interface.

Boundary conditions on perturbations in the plasma-atmosphere system can be obtained in analogy with boundary conditions in the plasma-vacuum system in the extended energy principle for fusion plasmas.[1–4] First, the tangential component of the electric field seen in any frame of reference must be continuous at the interface. In the present problem, there are two frames of interest. One is the frame moving with the plasma and the other is the stationary frame of reference. We obtain boundary conditions for electric field perturbations in those two frames.

In the frame of reference moving with the plasma, $\mathbf{E}_1^* = \mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}$ is the perturbed electric field on the magnetospheric plasma side. Obviously, $\mathbf{E}_1^* = 0$ in the plasma because $\mathbf{E}_1 = -\mathbf{v}_1 \times \mathbf{B}$ owing to the frozen-in law of the plasma.

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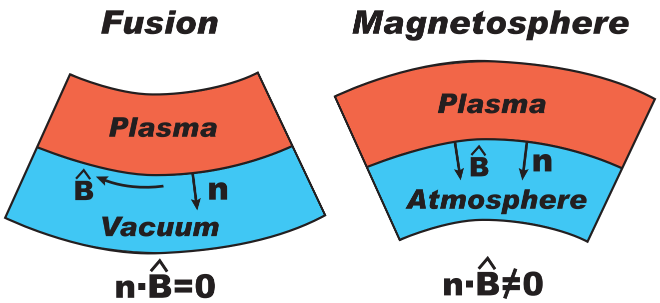

FIG. 6. (Color) The left panel shows an unperturbed magnetic field configuration in a fusion plasma such as a tokamak. The red and blue areas show plasma and vacuum regions, respectively. The right panel shows an unperturbed magnetic field configuration in the magnetosphere-atmosphere system. The red and blue areas show plasma and atmosphere regions, respectively.

Therefore, from the continuity of the tangential electric field, i.e., $\mathbf{n}\times\mathbf{E}_1^*=\mathbf{n}\times\hat{\mathbf{E}}_1^*$, we obtain $\mathbf{n}\times\hat{\mathbf{E}}_1^*=0$, where $\hat{\mathbf{E}}_1^*=\hat{\mathbf{E}}_1+\mathbf{v}_1\times\hat{\mathbf{B}}$ is the electric field seen in the frame moving with the plasma on the atmospheric side. Therefore, we obtain

$$\mathbf{n}\times\hat{\mathbf{E}}_1=\hat{\mathbf{B}}(\mathbf{n}\cdot\mathbf{v}_1)-\mathbf{v}_1(\mathbf{n}\cdot\hat{\mathbf{B}}). \tag{55}$$

This is a general boundary condition at the plasma-atmosphere interface, which includes as a special case the plasma-vacuum interface in fusion plasmas. However, in the plasma-vacuum interface in fusion plasmas, the second term on the right-hand side vanishes because the unperturbed magnetic field is assumed to be parallel to the interface,[1] i.e., $\mathbf{n}\cdot\hat{\mathbf{B}}=0$, as shown in the left panel of Fig. 6. We obtain from Eq. (55) by setting $\hat{\mathbf{E}}_1=-\partial\hat{\mathbf{A}}_1/\partial t$ by the assumption of Coulomb gauge, and $\mathbf{v}_1=\partial\xi/\partial t$,

$$\mathbf{n}\times\hat{\mathbf{A}}_1=-\hat{\mathbf{B}}(\mathbf{n}\cdot\boldsymbol{\xi})+\boldsymbol{\xi}(\mathbf{n}\cdot\hat{\mathbf{B}}). \tag{56}$$

This is a boundary condition for $\hat{\mathbf{A}}_1$, which must be satisfied at the plasma-atmosphere interface. We notice that in the extended magnetospheric energy principle, the first term on the right-hand side vanishes because $\xi_\parallel=0$ at the plasma-atmosphere interface. Therefore, the two terms on the right-hand side of this equation represent two extreme cases for the extended energy principle in fusion plasmas and for the extended magnetospheric energy principle.

Since the above boundary condition (56) is the key to understanding the difference of the extended energy principle for fusion plasmas and the extended magnetospheric energy principle, we next derive Eq. (56) by looking at the electric field in the stationary frame of reference. In the stationary frame of reference, the perturbed electric field must satisfy $\mathbf{E}_1+\mathbf{v}_1\times\mathbf{B}=0$ owing to the frozen-in law. Therefore,

$$\mathbf{E}_1=-\mathbf{v}_1\times\mathbf{B}. \tag{57}$$

On the other hand, on the atmospheric side the perturbed electric field $\hat{\mathbf{E}}_1$ seen in the stationary frame of reference is $\hat{\mathbf{E}}_1=-\partial\hat{\mathbf{A}}_1/\partial t$. Owing to the continuity of the tangential component of the electric field we have $\mathbf{n}\times\mathbf{E}_1=\mathbf{n}\times\hat{\mathbf{E}}_1$. Therefore, we obtain

$$\mathbf{n}\times\hat{\mathbf{A}}_1=-\mathbf{B}(\mathbf{n}\cdot\boldsymbol{\xi})+\boldsymbol{\xi}(\mathbf{n}\cdot\mathbf{B}). \tag{58}$$

In the extended magnetospheric energy principle, $\mathbf{B}=\hat{\mathbf{B}}$ at the plasma-atmosphere interface S_I. Therefore, this boundary condition agrees with Eq. (56).

Figure 6 shows the difference between the unperturbed magnetic field configuration in fusion and magnetospheric plasmas. The left panel shows an unperturbed magnetic field configuration in a fusion plasma such as a tokamak. The red area shows a plasma and the blue area shows a vacuum. The unperturbed magnetic field $\hat{\mathbf{B}}$ on the plasma-vacuum interface is assumed to be parallel to the unperturbed plasma-vacuum interface.[1] Therefore, $\mathbf{n}\cdot\hat{\mathbf{B}}=\mathbf{n}\cdot\mathbf{B}=0$ at the plasma-vacuum interface. The right panel shows an unperturbed magnetic field configuration in the magnetosphere-atmosphere system. The red area shows a plasma and the blue area shows the atmosphere. The unperturbed magnetic field $\hat{\mathbf{B}}$ on the plasma- atmosphere interface is assumed to be perpendicular to the unperturbed plasma-atmosphere interface. Therefore, $\mathbf{n}\cdot\hat{\mathbf{B}}\neq 0$. From these panels, it is obvious that the last term on the right-hand side of Eq. (56) vanishes in fusion plasmas. However, at the plasma-atmosphere interface the last term on the right-hand side of Eq. (56) does not vanish, since $\mathbf{n}$ is parallel to $\hat{\mathbf{B}}$ at the interface in the northern hemisphere.

The boundary condition at the perfectly conducting wall, which represents the solid-earth surface, is

$$\hat{\mathbf{n}}\times\hat{\mathbf{A}}_1=0. \tag{59}$$

This boundary condition is also valid at side boundaries S'_{in} and S'_{out}, because $\hat{\mathbf{A}}_1\to 0$ at those boundaries. If the solid earth is a perfect conductor, the unperturbed magnetic field must also circumvent the solid-earth surface and must be parallel to it, since the unperturbed magnetic field cannot penetrate into the perfect conductor. However, actually, the solid earth is not a perfect conductor and, therefore, the unperturbed magnetic field can penetrate into the solid earth. Therefore, we consider that the solid earth is an insulator for a slow variation representing the unperturbed magnetic field, but it is a perfect conductor for a rapid variation representing the perturbed field.

During the displacement of the plasma, the plasma-atmosphere interface will also be displaced to a new location $\mathbf{r}_s=\mathbf{r}_0+\boldsymbol{\xi}$. As we discussed in Sec. VII, we assume that a pressure balance condition holds in the perturbed state. Therefore, at this perturbed boundary, the pressure balance condition is

$$p(\mathbf{r}_0+\boldsymbol{\xi})+p_1(\mathbf{r}_0+\boldsymbol{\xi})+\frac{1}{2\mu_0}[\mathbf{B}(\mathbf{r}_0+\boldsymbol{\xi})+\mathbf{B}_1(\mathbf{r}_0+\boldsymbol{\xi})]^2$$
$$=\hat{p}(\mathbf{r}_0+\boldsymbol{\xi})+\hat{p}_1(\mathbf{r}_0+\boldsymbol{\xi})+\frac{1}{2\mu_0}[\hat{\mathbf{B}}(\mathbf{r}_0+\boldsymbol{\xi})+\hat{\mathbf{B}}_1(\mathbf{r}_0+\boldsymbol{\xi})]^2. \tag{60}$$

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By expanding in powers of a small quantity $\boldsymbol{\xi}$ and taking the lowest order in small quantities, we have

$$(\boldsymbol{\xi}\cdot\nabla)p + p_1 + \frac{1}{\mu_0}\mathbf{B}\cdot[(\boldsymbol{\xi}\cdot\nabla)\mathbf{B} + \mathbf{B}_1] = (\boldsymbol{\xi}\cdot\nabla)\hat{p} + \hat{p}_1 + \frac{1}{\mu_0}\hat{\mathbf{B}}\cdot[(\boldsymbol{\xi}\cdot\nabla)\hat{\mathbf{B}} + \hat{\mathbf{B}}_1], \tag{61}$$

where $\hat{p}(\mathbf{r}_0+\boldsymbol{\xi})=\hat{p}(\mathbf{r}_0)+(\boldsymbol{\xi}\cdot\nabla)\hat{p}(\mathbf{r}_0)$ is used and p, p_1, $\hat{p}$, $\hat{p}_1$, $\mathbf{B}$, $\mathbf{B}_1$, $\hat{\mathbf{B}}$, and $\hat{\mathbf{B}}_1$ represent those quantities at $\mathbf{r}=\mathbf{r}_0$. This boundary condition must be satisfied at the equilibrium ionospheric surface S_I. Notice that if the pressures in the neutral atmosphere $\hat{p}$ and $\hat{p}_1$ are neglected, this boundary condition becomes the same as that used for the plasma-vacuum interface in fusion plasmas,[1–4] which are surrounded by a vacuum region.

IX. EXTENDED MAGNETOSPHERIC ENERGY PRINCIPLE

In this section, an extended magnetospheric energy principle, which takes into account the existence of the neutral atmosphere below the ionospheric surface, is formulated.

A. Conditions for self-adjointness of the force operator at the plasma-atmosphere interface

For fusion plasmas, in which the unperturbed magnetic field at the plasma-vacuum interface is assumed to be parallel to the interface,[1] the self-adjointness of the force operator can be rigorously proven by taking into account the three boundary conditions given in Sec. VIII, i.e., the perturbed pressure balance condition at the plasma-vacuum interface, the continuity of the tangential electric field at the plasma-vacuum interface, and the boundary condition for vector potential at the conducting boundary surface.[1–4] However, for the present magnetosphere-atmosphere system, in which the unperturbed magnetic field is assumed to be incident vertically on the ionospheric surface, the self-adjointness of the force operator cannot be proven rigorously for an arbitrary perturbation, even if the three boundary conditions given in Sec. VIII are satisfied. Thus, the self-adjointness of the force operator is satisfied only for certain perturbations that satisfy certain boundary conditions at the plasma-atmosphere interface. We clarify in this subsection those specific conditions at the plasma-atmosphere interface, for which the self-adjointness of the force operator is satisfied.

Although a procedure used in Bernstein *et al.*,[1] which is also explained in detail in Freidberg,[2] is used in order to show the self-adjointness of the force operator in the magnetospheric energy principle,[8] here we follow a different procedure, which is similar to Kadomtsev,[3] in order to obtain the necessary and sufficient conditions for the self-adjointness of the force operator in the extended magnetospheric energy principle.

By using vector formulas, the integrand of Eq. (5) can be written as

$$\begin{aligned}\boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi}) = &-\gamma p(\nabla\cdot\boldsymbol{\eta})(\nabla\cdot\boldsymbol{\xi}) - \mu_0^{-1}[\nabla\times(\boldsymbol{\eta}\times\mathbf{B})]\\ &\cdot[\nabla\times(\boldsymbol{\xi}\times\mathbf{B})] - (\boldsymbol{\xi}\cdot\nabla p)\,\nabla\cdot\boldsymbol{\eta} + \mu_0^{-1}\boldsymbol{\eta}\\ &\cdot\{(\nabla\times\mathbf{B})\times[\nabla\times(\boldsymbol{\xi}\times\mathbf{B})]\}\\ &+\nabla\cdot[\mu_0^{-1}(\mathbf{B}_1\cdot\boldsymbol{\eta})\mathbf{B} - (p_1+\mu_0^{-1}\mathbf{B}\cdot\mathbf{B}_1)\boldsymbol{\eta}].\end{aligned} \tag{62}$$

This is a general expression, which is valid for any static plasma equilibrium. Nevertheless, for fusion plasmas, a contribution from the term $\mu_0^{-1}\nabla\cdot[(\mathbf{B}_1\cdot\boldsymbol{\eta})\mathbf{B}]$ on the right-hand side is absent in the calculation of Eq. (5), since upon integration over the plasma volume this term vanishes because of $\mathbf{B}\cdot d\mathbf{S}=\mathbf{B}\cdot\mathbf{n}dS=0$, which is satisfied for fusion plasmas (see the left panel in Fig. 6).

Let us now assume that ∇p does not vanish identically. Then, the direction of current $\mathbf{J}$ does not coincide with that of the magnetic field and $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ can be expanded in terms of vectors $\mathbf{B}$, $\nabla\times\mathbf{B}$, and $\mathbf{e}=\nabla p/|\nabla p|$. Therefore, we write

$$\boldsymbol{\xi}_\perp = \xi_1\nabla\times\mathbf{B} + \xi_2\mathbf{e}, \tag{63}$$

$$\boldsymbol{\eta}_\perp = \eta_1\nabla\times\mathbf{B} + \eta_2\mathbf{e}. \tag{64}$$

After some vector calculation, we obtain from Eqs. (62)–(64)

$$\begin{aligned}\boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi}) = \mathrm{SP} &- \nabla\cdot[\eta_\parallel\mathbf{b}(\boldsymbol{\xi}\cdot\nabla p)]\\ &+\nabla\cdot[\mu_0^{-1}(\mathbf{B}_1\cdot\boldsymbol{\eta})\mathbf{B} - (p_1+\mu_0^{-1}\mathbf{B}\cdot\mathbf{B}_1)\boldsymbol{\eta}],\end{aligned} \tag{65}$$

where SP is the symmetric part given by

$$\begin{aligned}\mathrm{SP} = &-\gamma p(\nabla\cdot\boldsymbol{\eta})(\nabla\cdot\boldsymbol{\xi}) - \mu_0^{-1}[\nabla\times(\boldsymbol{\eta}\times\mathbf{B})]\\ &\cdot[\nabla\times(\boldsymbol{\xi}\times\mathbf{B})] - (\boldsymbol{\xi}_\perp\cdot\nabla p)\,\nabla\cdot\boldsymbol{\eta}_\perp\\ &-(\boldsymbol{\eta}_\perp\cdot\nabla p)\,\nabla\cdot\boldsymbol{\xi}_\perp + \mu_0^{-1}\eta_2\xi_2\mathbf{e}\\ &\cdot\{(\nabla\times\mathbf{B})\times[(\mathbf{B}\cdot\nabla)\mathbf{e} - (\mathbf{e}\cdot\nabla)\mathbf{B}]\}.\end{aligned} \tag{66}$$

All terms in the symmetric part SP are symmetric with respect to the change of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. Therefore, the integration of SP over the plasma volume P gives a self-adjoint contribution.

The integration of Eq. (65) over the plasma volume P gives

$$\begin{aligned}\int_P \boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi})d\mathbf{r} = \int_P \mathrm{SP}d\mathbf{r} &- \int_{S_I}\eta_\parallel\mathbf{b}(\boldsymbol{\xi}\cdot\nabla p)\cdot\mathbf{n}dS\\ &+\int_{S_I}\mu_0^{-1}(\mathbf{B}_1\cdot\boldsymbol{\eta})\mathbf{B}\cdot\mathbf{n}dS\\ &-\int_{S_I}(p_1+\mu_0^{-1}\mathbf{B}\cdot\mathbf{B}_1)\boldsymbol{\eta}\cdot\mathbf{n}dS.\end{aligned} \tag{67}$$

Since the pressure balance condition (50) holds everywhere on the plasma-atmosphere interface, the following equation is valid:

$$(\boldsymbol{\xi}_\perp\cdot\nabla)\left(\hat{p} + \frac{\hat{\mathbf{B}}^2}{2\mu_0}\right) = (\boldsymbol{\xi}_\perp\cdot\nabla)\left(p + \frac{\mathbf{B}^2}{2\mu_0}\right). \tag{68}$$

Therefore, from Eq. (61) we obtain

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$$p_1+\mu_0^{-1}\mathbf{B}\cdot\mathbf{B}_1=\xi_n\frac{\partial}{\partial n}\left(\hat{p}+\frac{\hat{\mathbf{B}}^2}{2\mu_0}\right)-\xi_n\frac{\partial}{\partial n}\left(p+\frac{\mathbf{B}^2}{2\mu_0}\right)+\hat{p}_1+\mu_0^{-1}\hat{\mathbf{B}}\cdot\hat{\mathbf{B}}_1. \quad (69)$$

Substitution of Eq. (69) into Eq. (67) yields

$$\begin{aligned}\int_P \boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi})d\mathbf{r}
&=\int_P \mathrm{SP}d\mathbf{r}-\int_{S_I}\xi_n\eta_n\frac{\partial}{\partial n}\left[\left(\hat{p}+\frac{\hat{\mathbf{B}}^2}{2\mu_0}\right)-\left(p+\frac{\mathbf{B}^2}{2\mu_0}\right)\right]dS\\
&\quad-\int_{S_I}\eta_\parallel(\boldsymbol{\xi}\cdot\nabla p)\mathbf{b}\cdot\mathbf{n}dS\\
&\quad-\int_{S_I}(\hat{p}_1+\mu_0^{-1}\hat{\mathbf{B}}\cdot\hat{\mathbf{B}}_1)\boldsymbol{\eta}\cdot\mathbf{n}dS\\
&\quad+\mu_0^{-1}\int_{S_I}(\mathbf{B}_{1\perp}\cdot\boldsymbol{\eta}_\perp)(\mathbf{n}\cdot\mathbf{B})dS\\
&\quad+\mu_0^{-1}\int_{S_I}B_{1\parallel}\eta_\parallel\mathbf{B}\cdot\mathbf{n}dS. \quad (70)\end{aligned}$$

In order to further make the surface integral of $\mu_0^{-1}(\hat{\mathbf{B}}\cdot\hat{\mathbf{B}}_1)(\boldsymbol{\eta}\cdot\mathbf{n})$ in the fourth integral on the right-hand side self-adjoint, let $\hat{\mathbf{B}}_1(\boldsymbol{\xi})$ be a perturbed magnetic field on the atmospheric side of the interface caused by the plasma displacement $\boldsymbol{\xi}$ and $\hat{\mathbf{B}}_1(\boldsymbol{\xi})=\nabla\times\hat{\mathbf{A}}_1(\boldsymbol{\xi})$. Then, from Eq. (56) we obtain by replacing $\boldsymbol{\xi}$ with $\boldsymbol{\eta}$

$$\mathbf{n}\times\hat{\mathbf{A}}_1(\boldsymbol{\eta})=-\hat{\mathbf{B}}(\mathbf{n}\cdot\boldsymbol{\eta})+\boldsymbol{\eta}(\mathbf{n}\cdot\hat{\mathbf{B}}). \quad (71)$$

By taking the dot product of $\hat{\mathbf{B}}_1(\boldsymbol{\xi})$ with Eq. (71) and then integrating, we obtain

$$\int_{S_I}\hat{\mathbf{B}}_1(\boldsymbol{\xi})\cdot\hat{\mathbf{B}}(\mathbf{n}\cdot\boldsymbol{\eta})dS=-\int_{S_I}[\hat{\mathbf{A}}_1(\boldsymbol{\eta})\times\hat{\mathbf{B}}_1(\boldsymbol{\xi})]\cdot\mathbf{n}dS+\int_{S_I}\hat{\mathbf{B}}_1(\boldsymbol{\xi})\cdot\boldsymbol{\eta}(\mathbf{n}\cdot\hat{\mathbf{B}})dS. \quad (72)$$

Since $\nabla\times\hat{\mathbf{B}}_1(\boldsymbol{\xi})=0$ in the atmosphere, we obtain

$$\nabla\cdot[\hat{\mathbf{A}}_1(\boldsymbol{\eta})\times\hat{\mathbf{B}}_1(\boldsymbol{\xi})]=\hat{\mathbf{B}}_1(\boldsymbol{\xi})\cdot\hat{\mathbf{B}}_1(\boldsymbol{\eta}). \quad (73)$$

Therefore, by using $\mathbf{n}=-\hat{\mathbf{n}}$ on the plasma-atmosphere interface and using the boundary condition (59) on the ground and on the side surfaces, we can rewrite the first term on the right-hand side of Eq. (72) by the volume integral of $\hat{\mathbf{B}}_1(\boldsymbol{\xi})\cdot\hat{\mathbf{B}}_1(\boldsymbol{\eta})$ in the atmospheric volume A. Notice that $\hat{\mathbf{B}}_1(\boldsymbol{\xi})\cdot\hat{\mathbf{B}}_1(\boldsymbol{\eta})$ is defined only on the atmospheric side of the interface S_I and a contribution to the atmospheric volume integral from the thin ionospheric layer, $0<\ell<\epsilon$, vanishes in the limit of $\epsilon\to 0$. Thus, from Eq. (72) we obtain

$$\int_{S_I}\hat{\mathbf{B}}_1(\boldsymbol{\xi})\cdot\hat{\mathbf{B}}(\mathbf{n}\cdot\boldsymbol{\eta})dS=\int_A\hat{\mathbf{B}}_1(\boldsymbol{\xi})\cdot\hat{\mathbf{B}}_1(\boldsymbol{\eta})d\mathbf{r}+\int_{S_I}\hat{\mathbf{B}}_1(\boldsymbol{\xi})\cdot\boldsymbol{\eta}(\mathbf{n}\cdot\hat{\mathbf{B}})dS. \quad (74)$$

It follows that Eq. (70) can be rewritten as

$$\begin{aligned}\int_P \boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi})d\mathbf{r}
&=\int_P \mathrm{SP}d\mathbf{r}-\int_{S_I}\xi_n\eta_n\frac{\partial}{\partial n}\left[\left(\hat{p}+\frac{\hat{\mathbf{B}}^2}{2\mu_0}\right)-\left(p+\frac{\mathbf{B}^2}{2\mu_0}\right)\right]dS\\
&\quad-\mu_0^{-1}\int_A\hat{\mathbf{B}}_1(\boldsymbol{\xi})\cdot\hat{\mathbf{B}}_1(\boldsymbol{\eta})d\mathbf{r}\\
&\quad-\int_{S_I}\eta_\parallel(\boldsymbol{\xi}\cdot\nabla p)\mathbf{b}\cdot\mathbf{n}dS-\int_{S_I}\hat{p}_1\boldsymbol{\eta}\cdot\mathbf{n}dS\\
&\quad-\mu_0^{-1}\int_{S_I}\hat{\mathbf{B}}_1(\boldsymbol{\xi})\cdot\boldsymbol{\eta}(\mathbf{n}\cdot\hat{\mathbf{B}})dS\\
&\quad+\mu_0^{-1}\int_{S_I}(\mathbf{B}_{1\perp}\cdot\boldsymbol{\eta}_\perp)(\mathbf{n}\cdot\mathbf{B})dS+\mu_0^{-1}\int_{S_I}B_{1\parallel}\eta_\parallel\mathbf{B}\cdot\mathbf{n}dS. \quad (75)\end{aligned}$$

This equation is obtained by using the pressure balance condition (61), the boundary condition (56) on $\hat{\mathbf{A}}_1$ at the plasma-atmosphere interface, and the boundary condition (59) on the conducting surface. This equation is valid regardless of the relationship between $\mathbf{n}$ and $\mathbf{B}$. Thus, if we consider S_I as the plasma-vacuum interface and let A be the vacuum volume, this equation is valid for fusion plasmas. Since the first three integrals on the right-hand side of Eq. (75) are self-adjoint forms and $\hat{p}_1=0$ and $\mathbf{n}\cdot\mathbf{B}=\mathbf{n}\cdot\hat{\mathbf{B}}=0$ in the energy principle for fusion plasmas (see the left panel in Fig. 6), we find at once that the force operator for the fusion plasma surrounded by a vacuum region is rigorously self-adjoint for perturbations satisfying fusion counterparts of the three boundary conditions (56), (59), and (61).

Since

$$\begin{aligned}\mathbf{B}_1&=-B\mathbf{b}(\nabla\cdot\boldsymbol{\xi}_\perp)+B(\mathbf{b}\cdot\nabla)\boldsymbol{\xi}_\perp-B(\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{b}\\
&\quad-\mathbf{b}(\boldsymbol{\xi}_\perp\cdot\nabla)B, \quad (76)\end{aligned}$$

we obtain

$$\mathbf{B}_{1\perp}\cdot\boldsymbol{\eta}_\perp=[B(\mathbf{b}\cdot\nabla)\boldsymbol{\xi}_\perp]\cdot\boldsymbol{\eta}_\perp-[B(\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{b}]\cdot\boldsymbol{\eta}_\perp. \quad (77)$$

Owing to the assumption of the normal incidence of the unperturbed magnetic field on the ionospheric surface, $(\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{b}=\boldsymbol{\xi}_\perp/R_I$ holds in the ionosphere of the southern hemisphere and $(\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{b}=-\boldsymbol{\xi}_\perp/R_I$ holds in the ionosphere of the northern hemisphere.[8] Substitution of these relations into Eqs. (70) and (77) yields

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$$\int_P \boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi})d\mathbf{r} = \int_P \mathrm{SP}d\mathbf{r} - \int_{S_I} \xi_n\eta_n \frac{\partial}{\partial n}\left[\left(\hat{p}+\frac{\hat{\mathbf{B}}^2}{2\mu_0}\right)-\left(p+\frac{\mathbf{B}^2}{2\mu_0}\right)\right]dS + \mu_0^{-1}\int_{\mathrm{north}} \frac{B^2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\eta}_\perp}{R_I}(\mathbf{b}\cdot\mathbf{n})dS - \mu_0^{-1}\int_{\mathrm{south}} \frac{B^2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\eta}_\perp}{R_I}(\mathbf{b}\cdot\mathbf{n})dS + \mathrm{NSA}, \tag{78}$$

where NSA is the nonself-adjoint part given by

$$\mathrm{NSA} = \int_{S_I} \{\mu_0^{-1}B^2\boldsymbol{\eta}_\perp\cdot[(\mathbf{b}\cdot\nabla)\boldsymbol{\xi}_\perp] + [\mu_0^{-1}BB_{1\parallel} - (\boldsymbol{\xi}\cdot\nabla p) - (\hat{p}_1+\mu_0^{-1}\hat{\mathbf{B}}\cdot\hat{\mathbf{B}}_1)]\eta_\parallel\} \times(\mathbf{b}\cdot\mathbf{n})dS. \tag{79}$$

It is obvious that the second, third, and fourth integrals on the right-hand side of Eq. (78) are also symmetric with respect to the change of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. Therefore, those integrals give self-adjoint contributions. Notice that $\hat{\mathbf{B}}\cdot\hat{\mathbf{B}}_1 = B\hat{B}_{1\parallel}$ does not cancel out $BB_{1\parallel}$, since the perturbed interface is not necessarily perpendicular to the unperturbed field $\mathbf{B}=\hat{\mathbf{B}}$. Here, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are arbitrary vectors satisfying boundary conditions on S_I. From Eq. (61), $\mu_0^{-1}BB_{1\parallel}-(\boldsymbol{\xi}\cdot\nabla p)-(\hat{p}_1+\mu_0^{-1}\hat{\mathbf{B}}\cdot\hat{\mathbf{B}}_1)$ cannot identically vanish on S_I. Therefore, NSA vanishes only when either the condition $\eta_\parallel=0$ and $\boldsymbol{\eta}_\perp=0$ or the condition $\eta_\parallel=0$ and $(\mathbf{b}\cdot\nabla)\boldsymbol{\xi}_\perp=0$ is satisfied on S_I. Consequently, the necessary and sufficient condition for the force operator to become self-adjoint is that the arbitrary displacement vector $\boldsymbol{\xi}$ must satisfy on S_I,

$$\xi_\parallel = 0 \quad \text{and} \quad \boldsymbol{\xi}_\perp = 0, \tag{80}$$

$$\xi_\parallel = 0 \quad \text{and} \quad (\mathbf{b}\cdot\nabla)\boldsymbol{\xi}_\perp = 0. \tag{81}$$

These are boundary conditions on $\boldsymbol{\xi}$ at the plasma-atmosphere interface in the extended magnetospheric energy principle. Notice that in the magnetospheric energy principle, in which the existence of the atmosphere below the ionosphere is entirely neglected, there are four necessary and sufficient conditions (11)–(14) for the force operator to become self-adjoint. However, in the extended magnetospheric energy principle taking account of the pressure balance condition at the plasma-atmosphere interface, the possible boundary conditions are reduced to only two, i.e., Eqs. (11) and (14). Thus, we find an important difference between the magnetospheric energy principle[8] and the extended magnetospheric energy principle.

This difference can be easily understood if we note the difference of the treatment of the atmosphere in the two energy principles. In the magnetospheric energy principle,[8] there is no consideration given to the physical nature of the boundary between the plasma and the atmosphere. Thus, all the possible boundary conditions, either compressible, Eqs. (11) and (14), or incompressible, Eqs. (12) and (13), are allowed as long as they guarantee the self-adjointness. However, in the extended magnetospheric energy principle, the magnetospheric plasma is in contact with a very compressible atmospheric gas, as was discussed in Sec. VII. Thus, the incompressible boundary conditions (12) and (13) are removed and the possible boundary conditions are reduced to only two.

In order to confirm this important difference, we note that we obtain directly from Eq. (67)

$$\int_P \boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi})d\mathbf{r} = \int_P \mathrm{SP}d\mathbf{r} + \mu_0^{-1}\int_{\mathrm{north}} \frac{B^2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\eta}_\perp}{R_I}(\mathbf{b}\cdot\mathbf{n})dS - \mu_0^{-1}\int_{\mathrm{south}} \frac{B^2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\eta}_\perp}{R_I}(\mathbf{b}\cdot\mathbf{n})dS + \mathrm{NSA}, \tag{82}$$

where NSA is the nonself-adjoint part given by

$$\mathrm{NSA} = \int_{S_I} \{\mu_0^{-1}B^2\boldsymbol{\eta}_\perp\cdot[(\mathbf{b}\cdot\nabla)\boldsymbol{\xi}_\perp] + \gamma p\,\eta_\parallel\nabla\cdot\boldsymbol{\xi}\} \times(\mathbf{b}\cdot\mathbf{n})dS. \tag{83}$$

In deriving this equation, we used that

$$p_1 = -\boldsymbol{\xi}\cdot\nabla p - \gamma p\nabla\cdot\boldsymbol{\xi}. \tag{84}$$

Notice that in deriving Eq. (82) we did not use any physical boundary conditions given in Sec. VIII. Thus, this equation could also be obtained in the magnetospheric energy principle neglecting the existence of the neutral atmosphere.[8]

Equation (83) is equal to Eq. (10) obtained in the magnetospheric energy principle[8] using the procedure of Bernstein *et al.*,[1] which is described in detail in Freidberg.[2] However, the derivation of the boundary term BT given by Eq. (10), which is equal to NSA given by Eq. (83), is based on the procedure used by Kadomtsev.[3] Thus, this derivation of Eq. (10) gives an independent check of the magnetospheric energy principle.[8]

Although the self-adjointness of the force operator is proven rigorously for fusion plasmas by using three boundary conditions, which are the fusion plasma counterparts of Eqs. (56), (59), and (61), the self-adjointness cannot be proven for an arbitrary perturbation in magnetospheric plasmas unless we assume Eq. (80) or Eq. (81) at the plasma-atmosphere interface in the extended magnetospheric energy principle. This difference is due to our assumption in the extended magnetospheric energy principle that the unperturbed magnetic field is incident vertically on the ionospheric surface (see the right panel in Fig. 6). For fusion plasmas, the unperturbed magnetic field at the plasma-vacuum interface is assumed to be parallel to the interface (see the left panel in Fig. 6).

B. Potential energy δW

By using the force operator (3) and vector formulas, $\delta W(\boldsymbol{\xi}^*,\boldsymbol{\xi})$ given by Eq. (6) can be written as $\delta W(\boldsymbol{\xi}^*,\boldsymbol{\xi}) = \delta W_F + \mathrm{BT}$, where δW_F and the boundary term BT are given by

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$$\delta W_F = \frac{1}{2}\int_P [\mu_0^{-1}|\mathbf{Q}|^2 + \mu_0^{-1}(\nabla\times\mathbf{B})\cdot(\boldsymbol{\xi}^*\times\mathbf{Q}) + (\nabla\cdot\boldsymbol{\xi}^*)(\boldsymbol{\xi}\cdot\nabla p + \gamma p\nabla\cdot\boldsymbol{\xi})]d\mathbf{r}, \quad (85)$$

$$\mathrm{BT} = \frac{1}{2}\int_S \{\mu_0^{-1}[(\mathbf{Q}\cdot\mathbf{B})\boldsymbol{\xi}^* - (\mathbf{Q}\cdot\boldsymbol{\xi}^*)\mathbf{B}] - \boldsymbol{\xi}^*(\boldsymbol{\xi}\cdot\nabla p + \gamma p\nabla\cdot\boldsymbol{\xi})\}\cdot d\mathbf{S}. \quad (86)$$

These are general expressions, which do not depend on the configuration of the interface between the plasma and the surrounding volume such as vacuum and atmosphere. Therefore, these expressions are also valid for fusion plasmas.[1] However, in the extended energy principle for fusion plasmas, the term $\mu_0^{-1}(\mathbf{Q}\cdot\boldsymbol{\xi}^*)\mathbf{B}$ in the integrand of BT vanishes because $\mathbf{B}\cdot d\mathbf{S}=\mathbf{B}\cdot\mathbf{n}dS=0$ at the plasma-vacuum interface (see Fig. 6). In the present extended magnetospheric energy principle, the surface S in the integral of Eq. (86) can be replaced by S_I, since the contributions from the side boundaries S_{out} and S_{in} vanish because $\mathbf{B}\cdot d\mathbf{S}=0$ and $\boldsymbol{\xi}_\perp=0$ on those boundaries.

The integrand of BT in Eq. (86) can be written as

$$\mu_0^{-1}[(\mathbf{Q}\cdot\mathbf{B})\boldsymbol{\xi}^* - (\mathbf{Q}\cdot\boldsymbol{\xi}^*)\mathbf{B}] - \boldsymbol{\xi}^*(\boldsymbol{\xi}\cdot\nabla p + \gamma p\nabla\cdot\boldsymbol{\xi}) = \boldsymbol{\xi}^*\left\{(\boldsymbol{\xi}\cdot\nabla)\left(\hat{p}+\frac{\hat{\mathbf{B}}^2}{2\mu_0}\right) - (\boldsymbol{\xi}\cdot\nabla)\left(p+\frac{\mathbf{B}^2}{2\mu_0}\right) + \hat{p}_1 + \mu_0^{-1}\hat{\mathbf{B}}\cdot\hat{\mathbf{B}}_1\right\} - \mu_0^{-1}(\mathbf{Q}\cdot\boldsymbol{\xi}^*)\mathbf{B}. \quad (87)$$

Since the unperturbed pressure balance condition (50) holds everywhere on the plasma-atmosphere interface, we obtain on the plasma-atmosphere interface

$$\boldsymbol{\xi}^*\cdot d\mathbf{S}(\boldsymbol{\xi}\cdot\nabla)\left[\left(\hat{p}+\frac{\hat{\mathbf{B}}^2}{2\mu_0}\right) - \left(p+\frac{\mathbf{B}^2}{2\mu_0}\right)\right] = |\boldsymbol{\xi}\cdot\mathbf{n}|^2\left[\left[\nabla\left(p+\frac{\mathbf{B}^2}{2\mu_0}\right)\right]\right]\cdot d\mathbf{S}. \quad (88)$$

Since $\hat{\mathbf{B}}(\mathbf{r}_0)=\mathbf{B}(\mathbf{r}_0)$ on the ionospheric surface, we obtain

$$[(\hat{\mathbf{B}}\cdot\hat{\mathbf{B}}_1)\boldsymbol{\xi}^* - (\mathbf{B}_1\cdot\boldsymbol{\xi}^*)\mathbf{B}]\cdot d\mathbf{S} = [(\hat{\mathbf{B}}_{1\parallel} - \mathbf{B}_1)\cdot\boldsymbol{\xi}^*](\mathbf{n}\cdot\mathbf{B})dS. \quad (89)$$

Furthermore, since

$$\mathbf{B}_1\cdot\boldsymbol{\xi}_\perp^* = \boldsymbol{\xi}_\perp^*\cdot[(\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp] - B\boldsymbol{\xi}_\perp^*\cdot[(\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{b}], \quad (90)$$

it follows from Eqs. (88)–(90) that δW can be written as

$$\delta W = \delta W_F + \delta W_S + \delta W_I + \frac{1}{2}\int_{S_I}\hat{p}_1\boldsymbol{\xi}^*\cdot d\mathbf{S} - \frac{1}{2\mu_0}\int_{S_I}\{\boldsymbol{\xi}_\perp^*\cdot[(\mathbf{B}\cdot\nabla)\boldsymbol{\xi}_\perp]\}(\mathbf{n}\cdot\mathbf{B})dS + \frac{1}{2\mu_0}\int_{S_I}(\hat{B}_{1\parallel} - B_{1\parallel})\xi_\parallel^*(\mathbf{n}\cdot\mathbf{B})dS, \quad (91)$$

where

$$\delta W_S = \frac{1}{2}\int_{S_I}|\boldsymbol{\xi}\cdot\mathbf{n}|^2\left[\left[\nabla\left(p+\frac{\mathbf{B}^2}{2\mu_0}\right)\right]\right]\cdot d\mathbf{S}, \quad (92)$$

$$\delta W_I = \frac{1}{2\mu_0}\int_{S_I} B\{\boldsymbol{\xi}_\perp^*\cdot[(\boldsymbol{\xi}_\perp\cdot\nabla)\mathbf{b}]\}(\mathbf{B}\cdot\mathbf{n})dS. \quad (93)$$

We note that both δW_S and δW_I have contributions from both the southern and northern hemispheres and δW_I is reduced to Eq. (16) for a spherical ionospheric surface by assuming that the unperturbed magnetic field is incident vertically on the spherical ionospheric surface.

Thus, δW becomes the sum of δW_F, δW_S, δW_I and a few additional integrations over the ionospheric surface. Particularly noted here is the appearance of δW_I in δW, which did not occur in the energy principle for fusion plasmas, wherein $\mathbf{B}$ is perpendicular to $\mathbf{n}$ at the plasma-vacuum interface (see the left panel in Fig. 6). Therefore, this appearance of δW_I in δW is entirely due to the geometrical assumption that the unperturbed magnetic field is incident vertically on the ionospheric surface in the present magnetospheric plasmas (see the right panel in Fig. 6).

Another important difference of Eq. (91) from its counterpart for fusion plasmas is that there is no perturbed magnetic energy contribution δW_A in the atmosphere, which corresponds to the perturbed magnetic energy δW_V in the vacuum region for fusion plasmas. Since this difference is very important, this point is discussed on the basis of a different derivation of Eq. (91) in the next subsection IX C.

In the present formulation of an extended magnetospheric energy principle, the ionospheric surface is treated as a free surface in the magnetosphere-atmosphere system. Nevertheless, owing to the single assumption of the normal incidence of the unperturbed magnetic field on the ionospheric surface, the self-adjointness of the force operator cannot be proven rigorously for an arbitrary perturbation and we must impose either Eq. (80) or Eq. (81) at the unperturbed ionospheric surface in order to guarantee the self-adjointness. Under such a constraint, we obtain from Eq. (91)

$$\delta W = \delta W_F + \delta W_I. \quad (94)$$

Thus, δW obtained in the extended magnetospheric energy principle is simply equal to δW obtained in the normal magnetospheric energy principle,[8] which was reviewed in Sec. V B.

C. Alternative derivation of δW and its physical meaning

In order to make the physical meaning of Eq. (91) more transparent and to clarify its relevance to the extended energy principle for fusion plasmas,[1,2] we derive Eq. (91) by explicitly taking into account three boundary conditions satisfied in the magnetosphere-atmosphere system, i.e., Eqs. (56), (59), and (61). In particular, we clarify why δW_A, which is the perturbed magnetic energy in the atmosphere, does not appear in the final expression of δW contrary to the appearance of δW_V in δW of the extended energy principle for fusion plasmas.[1,2]

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First, using Eq. (87), BT in Eq. (86) can be written as

$$\mathrm{BT} = \delta W_S + \frac{1}{2}\int_{S_I} \hat{p}_1 \boldsymbol{\xi}^* \cdot d\mathbf{S} + \frac{1}{2\mu_0}\int_{S_I} (\hat{\mathbf{B}} \cdot \hat{\mathbf{B}}_1)\boldsymbol{\xi}^* \cdot d\mathbf{S} - \frac{1}{2\mu_0}\int_{S_I} (\mathbf{B}_1 \cdot \boldsymbol{\xi}^*)\mathbf{B} \cdot d\mathbf{S}. \quad (95)$$

Using Eq. (56), we can reduce the third term in this equation to

$$\frac{1}{2\mu_0}\int_{S_I} (\hat{\mathbf{B}} \cdot \hat{\mathbf{B}}_1)\boldsymbol{\xi}^* \cdot \mathbf{n} dS = \frac{1}{2\mu_0}\int_{S_I} (\mathbf{n} \cdot \hat{\mathbf{B}})(\hat{\mathbf{B}}_1 \cdot \boldsymbol{\xi}^*) dS - \frac{1}{2\mu_0}\int_{S_I} \mathbf{n} \cdot [\hat{\mathbf{A}}_1^* \times (\nabla \times \hat{\mathbf{A}}_1)] dS. \quad (96)$$

In the last integral in this equation, we can replace $\mathbf{n}$ by $-\hat{\mathbf{n}}$ and add the integral contributions from the side boundaries S'_{out}, S'_{in} and a solid-earth surface, since those contributions are zero owing to Eq. (59). Thus,

$$\frac{1}{2\mu_0}\int_{S_I} \mathbf{n} \cdot [\hat{\mathbf{A}}_1^* \times (\nabla \times \hat{\mathbf{A}}_1)] dS = -\frac{1}{2\mu_0}\int_A \nabla \cdot [\hat{\mathbf{A}}_1^* \times (\nabla \times \hat{\mathbf{A}}_1)] d\mathbf{r}, \quad (97)$$

where A is the atmospheric volume. Notice that the integral in Eq. (97) has two contributions from both the southern and northern hemispheres.

Since one has a vector identity

$$\nabla \cdot [\hat{\mathbf{A}}_1^* \times (\nabla \times \hat{\mathbf{A}}_1)] = (\nabla \times \hat{\mathbf{A}}_1) \cdot (\nabla \times \hat{\mathbf{A}}_1^*) - \hat{\mathbf{A}}_1^* \cdot [\nabla \times (\nabla \times \hat{\mathbf{A}}_1)], \quad (98)$$

we can rewrite Eq. (96) by using Eq. (98) as

$$\frac{1}{2\mu_0}\int_{S_I} (\hat{\mathbf{B}} \cdot \hat{\mathbf{B}}_1)\boldsymbol{\xi}^* \cdot \mathbf{n} dS = \frac{1}{2\mu_0}\int_{S_I} (\mathbf{n} \cdot \hat{\mathbf{B}})(\hat{\mathbf{B}}_1 \cdot \boldsymbol{\xi}^*) dS + \delta W_A, \quad (99)$$

where

$$\delta W_A = \frac{1}{2\mu_0}\int_A |\hat{\mathbf{B}}_1|^2 d\mathbf{r} \quad (100)$$

is the perturbed magnetic energy in the atmosphere and we used $\hat{\mathbf{j}}_1 = 0$ in the atmosphere. Substitution of Eq. (99) into Eq. (95) yields

$$\mathrm{BT} = \delta W_S + \delta W_A + \frac{1}{2}\int_{S_I} \hat{p}_1 \boldsymbol{\xi}^* \cdot d\mathbf{S} + \frac{1}{2\mu_0}\int_S (\mathbf{n} \cdot \mathbf{B})[(\hat{\mathbf{B}}_{1\perp} - \mathbf{B}_{1\perp}) \cdot \boldsymbol{\xi}_\perp^*] dS + \frac{1}{2\mu_0}\int_{S_I} (\hat{B}_{1\parallel} - B_{1\parallel})\xi_\parallel^* (\mathbf{n} \cdot \mathbf{B}) dS. \quad (101)$$

Thus, δW_A appears as a positive stabilizing term just as δW_V appeared in δW for fusion plasmas.[1,2] Notice that this equation is valid regardless of the relationship between $\mathbf{n}$ and $\mathbf{B}$. Therefore, we find that for fusion plasmas, $\delta W = \delta W_F + \delta W_S + \delta W_V$, since $\hat{p}_1 = 0$, $\delta W_A = \delta W_V$, and $\mathbf{n}$ and $\mathbf{B}$ are perpendicular at the plasma-vacuum interface in fusion plasmas. Thus, we recover the extended energy principle for fusion plasmas.

However, in the case of the extended magnetospheric energy principle, the appearance of δW_A in BT is only apparent as shown below. δW_A given by Eq. (100) has a physical meaning of perturbed magnetic energy in the atmosphere. We note in Eq. (99) that

$$[\boldsymbol{\xi}^* (\hat{\mathbf{B}} \cdot \hat{\mathbf{B}}_1) - \hat{\mathbf{B}}(\hat{\mathbf{B}}_1 \cdot \boldsymbol{\xi}^*)] \cdot \mathbf{n} dS = -(\hat{\mathbf{B}}_{1\perp} \cdot \boldsymbol{\xi}_\perp^*)\hat{\mathbf{B}} \cdot \mathbf{n} dS. \quad (102)$$

Thus, we obtain from Eq. (99)

$$\delta W_A = -\frac{1}{2\mu_0}\int_{S_I} (\mathbf{n} \cdot \hat{\mathbf{B}})(\hat{\mathbf{B}}_{1\perp} \cdot \boldsymbol{\xi}_\perp^*) dS. \quad (103)$$

Therefore, a part of the fourth term in Eq. (101) can be written by using δW_A. Thus, using Eq. (90), we obtain

$$\delta W = \delta W_F + \delta W_S + \delta W_I - \delta W_A + \delta W_A + \frac{1}{2}\int_{S_I} \hat{p}_1 \boldsymbol{\xi}^* \cdot d\mathbf{S} - \frac{1}{2\mu_0}\int_{S_I} \{\boldsymbol{\xi}_\perp^* \cdot [(\mathbf{B} \cdot \nabla)\boldsymbol{\xi}_\perp]\}(\mathbf{n} \cdot \mathbf{B}) dS + \frac{1}{2\mu_0}\int_{S_I} (\hat{B}_{1\parallel} - B_{1\parallel})\xi_\parallel^* (\mathbf{n} \cdot \mathbf{B}) dS. \quad (104)$$

Since δW_A terms cancel each other, this expression for δW becomes the same as Eq. (91). This cancellation of δW_A terms arises, since in Eq. (56) the last term does not vanish because of the assumption that the unperturbed magnetic field is incident vertically on the ionospheric surface (see the right panel in Fig. 6). The existence of the last term in Eq. (56), which is not present in fusion plasmas, is the origin of the above cancellation of δW_A terms.

Equation (103) shows that the perturbed magnetic energy in the atmosphere is given by $\hat{\mathbf{B}}_{1\perp}(\mathbf{r}_0) \cdot \boldsymbol{\xi}_\perp^*(\mathbf{r}_0)$ at the plasma-atmosphere interface. Since $\xi_\parallel = 0$ is valid at the plasma-atmosphere interface, we obtain from Eq. (56)

$$\mathbf{n} \times \hat{\mathbf{A}}_1 = \boldsymbol{\xi}(\mathbf{n} \cdot \hat{\mathbf{B}}). \quad (105)$$

This means that the component of $\hat{\mathbf{A}}_1$ parallel to the interface S_I is perpendicular to $\boldsymbol{\xi}$. Since $\hat{\mathbf{A}}_1$ must decrease toward the earth's surface, if we assume that the radial change of $\hat{\mathbf{A}}_1$ contributes most to $\hat{\mathbf{B}}_1 = \nabla \times \hat{\mathbf{A}}_1$, we find that $\hat{\mathbf{B}}_{1\perp}(\mathbf{r}_0)$ is antiparallel to $\boldsymbol{\xi}_\perp$ and thus to $\mathbf{B}_{1\perp}(\mathbf{r}_0)$ (see Fig. 3). Thus, in the ionosphere of the northern hemisphere, the right-hand side of Eq. (103) gives a positive contribution. In the southern hemisphere, it also gives a positive contribution. Therefore, δW_A defined by Eq. (103) is positive. Since the Poynting vector at the ionospheric surface at $\ell = 0$ becomes upward for growing perturbations, as shown in Fig. 3, the perturbed magnetic energy in the atmosphere δW_A is not directly provided from the magnetosphere. Since the direction of $\hat{\mathbf{B}}_{1\perp}$ at $\ell = \epsilon$ is

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opposite to $\mathbf{B}_{1\perp}$ at $\ell=0$, the direction of the Poynting vector below the ionospheric surface becomes downward for growing perturbations. Thus, in the present formulation, the perturbed magnetic energy in the atmosphere is considered to be provided from the thin ionospheric layer.

Thus, in order to clarify the origin of $\delta W_A > 0$, we should also take into account the energetics in the thin ionospheric layer. Nevertheless, so long as we are discussing the ideal MHD stability of the magnetospheric plasma, such a consideration of the energetics in the thin ionospheric layer is not necessary, since the origin of δW_I (<0) exists within the magnetospheric plasma volume P and not in the thin ionospheric layer outside the surface S_I. Therefore, the magnetospheric energy principle, which totally neglects the spatial regions below S_I, can also give the same result concerning magnetospheric stability. We conjecture that once an ideal MHD instability is set in the magnetosphere, physics below the ionospheric surface are automatically adjusted so that they become consistent with the consequences of $\delta W = \delta W_F + \delta W_I$.

X. MINIMIZATION OF THE CHANGE IN THE POTENTIAL ENERGY δW WITH RESPECT TO $\xi_\parallel$

In order to obtain a condition which is satisfied by the most unstable mode, the change in the potential energy $\delta W = \delta W_F + \delta W_I$ is minimized with respect to $\xi_\parallel$. Like the procedure in obtaining the minimizing condition in the magnetospheric energy principle,[8] the minimizing condition for the extended principle follows from letting $\xi_\parallel \to \xi_\parallel + \delta\xi_\parallel$ in $\delta W = \delta W_F + \delta W_I$ and then setting the corresponding variation $\delta(\delta W)_{\xi_\parallel}$ to zero. Since $\xi_\parallel = 0$ at the plasma-atmosphere interface in the extended magnetospheric energy principle, it is necessary to choose $\delta\xi_\parallel$ so as to satisfy $\delta\xi_\parallel = 0$ at the plasma-atmosphere interface. Except for that constraint on $\delta\xi_\parallel$ at the ionospheric surface S_I, $\delta\xi_\parallel$ is arbitrary everywhere else.

Keeping this constraint on $\delta\xi_\parallel$ in mind, let us calculate the variation of δW with respect to $\xi_\parallel$. Following Freidberg[2] and noting that $\xi_\parallel$ appears only in the $\gamma p |\nabla \cdot \boldsymbol{\xi}|^2$ term, one finds

$$\begin{aligned}\delta(\delta W)_{\xi_\parallel} &= \delta W_F(\xi_\parallel + \delta\xi_\parallel) - \delta W_F(\xi_\parallel) \\ &= \frac{1}{2}\int_P d\mathbf{r}\,\gamma p\{|\nabla\cdot\boldsymbol{\xi}_\perp + \nabla\cdot[(\xi_\parallel + \delta\xi_\parallel)\mathbf{b}]|^2 \\ &\quad - |\nabla\cdot\boldsymbol{\xi}_\perp + \nabla\cdot(\xi_\parallel\mathbf{b})|^2\} \\ &= \frac{1}{2}\int_P d\mathbf{r}\,\gamma p[|\nabla\cdot\boldsymbol{\xi} + \nabla\cdot(\delta\xi_\parallel\mathbf{b})|^2 - |\nabla\cdot\boldsymbol{\xi}|^2] \\ &\cong \frac{1}{2}\int_P d\mathbf{r}\,\gamma p[(\nabla\cdot\boldsymbol{\xi})\,\nabla\cdot(\delta\xi_\parallel^*\mathbf{b}) \\ &\quad + (\nabla\cdot\boldsymbol{\xi}^*)\,\nabla\cdot(\delta\xi_\parallel\mathbf{b})]. \end{aligned} \tag{106}$$

In Eq. (106), one obtains

$$\begin{aligned}(\nabla\cdot\boldsymbol{\xi}^*)\left[\nabla\cdot\left(\frac{\delta\xi_\parallel}{B}\mathbf{B}\right)\right] &= \nabla\cdot\left[(\nabla\cdot\boldsymbol{\xi}^*)\frac{\delta\xi_\parallel}{B}\mathbf{B}\right] \\ &\quad - [\nabla(\nabla\cdot\boldsymbol{\xi}^*)]\cdot\frac{\delta\xi_\parallel}{B}\mathbf{B}. \end{aligned} \tag{107}$$

It follows from Eqs. (106) and (107) that

$$\begin{aligned}\delta(\delta W)_{\xi_\parallel} &= -\frac{1}{2}\int_P d\mathbf{r}\,\gamma p\left[\frac{\delta\xi_\parallel}{B}(\mathbf{B}\cdot\nabla)(\nabla\cdot\boldsymbol{\xi}^*)\right. \\ &\quad \left. + \frac{\delta\xi_\parallel^*}{B}(\mathbf{B}\cdot\nabla)(\nabla\cdot\boldsymbol{\xi})\right] \\ &\quad + \frac{1}{2}\int_S \gamma p\left[\frac{\delta\xi_\parallel}{B}(\nabla\cdot\boldsymbol{\xi}^*) + \frac{\delta\xi_\parallel^*}{B}(\nabla\cdot\boldsymbol{\xi})\right]\mathbf{B}\cdot\mathbf{n}dS. \end{aligned} \tag{108}$$

In Eq. (108), the second integral represents the boundary term. For the horizontally free and rigid ionospheric boundary conditions allowed in the extended magnetospheric energy principle, this boundary term becomes zero because $\delta\xi_\parallel = 0$ at the ionospheric surface S_I. Therefore, we have

$$\begin{aligned}\delta(\delta W)_{\xi_\parallel} &= -\frac{1}{2}\int_P d\mathbf{r}\,\gamma p\left[\frac{\delta\xi_\parallel}{B}(\mathbf{B}\cdot\nabla)(\nabla\cdot\boldsymbol{\xi}^*)\right. \\ &\quad \left. + \frac{\delta\xi_\parallel^*}{B}(\mathbf{B}\cdot\nabla)(\nabla\cdot\boldsymbol{\xi})\right]. \end{aligned} \tag{109}$$

Since $\delta\xi_\parallel$ and $\delta\xi_\parallel^*$ are arbitrary in the integrand of Eq. (109), Eq. (109) becomes zero only when

$$(\mathbf{B}\cdot\nabla)\,\nabla\cdot\boldsymbol{\xi} = 0 \tag{110}$$

within the unperturbed plasma volume P.

The condition (110) represents the minimizing condition of the change in the potential energy δW with respect to $\xi_\parallel$. Equation (110) means that for the most unstable mode, $\nabla\cdot\boldsymbol{\xi}$=const along the field line.

XI. DISCUSSION

In this section, we first discuss the framework of the extended magnetospheric energy principle and the validity of several of its assumptions. Next, we discuss the validity of specific boundary conditions obtained for the principle in a real magnetosphere-atmosphere system. Then, in order to show the usefulness of the magnetospheric energy principle and its extended form in studying the ideal MHD stability of static magnetospheric configurations, we apply the principles to magnetospheric ballooning and interchange instabilities and discuss the relevance of the consequences of those instabilities in magnetospheric physics.

The magnetospheric energy principle[8] needs the single assumption of the vertical incidence of the unperturbed magnetic field on the spherical ionospheric surface and obtains the ideal ionospheric boundary conditions purely mathematically from the requirement of the self-adjointness of the force operator.[8] Thus, this principle gives no physical consideration to the atmosphere existing under the ionospheric surface. On the other hand, the extended magnetospheric en-

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ergy principle takes account of the existence of the atmosphere and the thin ionospheric layer under the ionospheric surface S_I, but it needs several other assumptions besides the assumption of the vertical incidence of the unperturbed magnetic field on the ionospheric surface. It assumes a pressure balance at the ionospheric surface and also makes assumptions that the atmosphere is so heavy and immobile that the atmospheric kinetic energy is negligible, and that for a rapid variation the solid-earth surface is a perfect conductor, but for a very slow variation representing the unperturbed magnetic field, it is an insulator.

Even with these simplified assumptions, the extended magnetospheric energy principle is not a fully consistent description of the MHD stability of the magnetosphere-atmosphere system, since the plasma volume P does not include the thin ionospheric layer, within which the ideal MHD is not valid. Energetics in the thin ionospheric layer are not self-consistently included in the principle. This is a difficult problem which does not exist in fusion plasmas. The extended energy principle for fusion plasmas[1,2] is a fully consistent description of the ideal MHD stability of a plasma-vacuum system, since a thin layer existing between the plasma and the vacuum region is an ideal MHD layer and thus included in the plasma volume P. If the dissipation in the thin ionospheric layer were included, the energy conservation $K+\delta W=\text{const}$ in the magnetosphere would be violated and the self-adjointness of the force operator could not be laid as a solid foundation of a stability analysis.

It follows that an underlying basic assumption of the extended magnetospheric energy principle is that the thin ionospheric layer automatically adjusts its dynamics so that its energetics become consistent with the consequences of the extended magnetospheric energy principle. In other words, the extended magnetospheric energy principle takes into account the thin ionospheric layer as boundary conditions at the edge of the plasma volume P and at the edge of the atmospheric volume A, but does not take into account energetics occurring inside the thin ionospheric layer. According to the magnetospheric energy principle[8] and its extended form, there is a global energy invariant $H=K+\delta W=K+\delta W_F+\delta W_I$ defined in the plasma volume P, which is conserved regardless of the existence of the thin ionospheric layer and the neutral atmosphere outside of P. Since the calculation of δW is made by integration in the plasma volume P, the origin of δW_I is definitely ascribable to the ideal MHD dynamics in the magnetospheric plasma, i.e., the frozen-in motion of field lines over the spherical ionospheric surface. As shown in Fig. 3, such a horizontal motion of field lines over the spherical ionospheric surface yields an upward Poynting vector when $\boldsymbol{\xi}$ is increasing. Once the horizontally free field line motion is set in the magnetosphere over the spherical ionospheric surface, $\delta W_I<0$ arises regardless of detailed energetics in the thin ionospheric layer.

While $\delta W=\delta W_F+\delta W_S+\delta W_V$ for fusion plasmas,[1,2] $\delta W=\delta W_F+\delta W_I$ for magnetospheric plasmas. Although there is δW_S in Eq. (91) for magnetospheric plasmas just as in the energy principle for fusion plasmas,[1,2] $\delta W_S=0$ in the extended magnetospheric energy principle, since $\xi_\parallel=0$ must be satisfied at the plasma-atmosphere interface. The absence of δW_A in δW in the extended magnetospheric energy principle originates from the difference in the unperturbed magnetic field configuration in magnetospheric plasmas and fusion plasmas (see Fig. 6). As far as the expression of δW is concerned, there is no difference between the extended magnetospheric energy principle and the normal magnetospheric energy principle,[8] which neglects the existence of the neutral atmosphere below the ionosphere.

Since the extended magnetospheric energy principle is based on several simplified assumptions for the magnetosphere-atmosphere system, a test of the validity of the extended magnetospheric energy principle only from the point of self-consistency by taking into account energetics in the thin ionospheric layer would be very difficult. Instead, just as experimental tests of ideal MHD instabilities predicted by the energy principle for fusion plasmas have been important, an observational test of the validity of the extended magnetospheric energy principle would be of vital importance. Nevertheless, in the following, the validity of those assumptions is checked from the point of consistency.

We first test the validity of the assumption that the unperturbed magnetic field is incident vertically on the unperturbed spherical ionospheric surface. This assumption is justified if the incident angle of the field line at the earth's surface is close to 90°. For the dipole field, the angle Θ between the unperturbed magnetic field vector and the horizontal ionospheric plane is given by $\tan(\pi/2-\Theta)=(2\tan\Phi_0)^{-1}$, where Φ_0 is the latitude of the intersection of the field line and the earth's surface. For $\Phi_0=45°$, 60°, and 70°, Θ are 63°, 74°, and 80°, respectively. Thus, the assumption of the normal incidence of the unperturbed magnetic field on the ionospheric surface is justified with high accuracy at least for $\Phi_0\geq 60°$, which is the region of our main physical interest. However, this does not mean that the magnetospheric energy principles are not valid for $\Phi_0\leq 45°$, since as is obvious from Fig. 3, as long as the unperturbed magnetic field at the ionospheric surface has a component perpendicular to the spherical ionospheric surface, there is always a destabilizing contribution to δW. Nevertheless, in such a configuration there is also a magnetic field component parallel to the ionospheric surface. Therefore, a real field line configuration of magnetospheres is an intermediate between the two extreme configurations shown in Fig. 6.

Next, we test the validity of the complete neglect of energetics in the thin ionospheric layer for the horizontally free $[(\mathbf{b}\cdot\nabla)\boldsymbol{\xi}_\perp=0]$ and for the rigid $(\boldsymbol{\xi}_\perp=0)$ boundary conditions. Since the thin ionospheric layer consists of a weakly ionized gas, there are ion-neutral collisions in the thin ionospheric layer (see Sec. VII). Thus, there is a finite energy dissipation described by Pedersen conductivity Σ_P. If the Joule dissipation due to Pedersen conductivity can be neglected compared with the potential energy δW_I, the neglect of energetics in the thin ionospheric layer is justified. In the limit of $\epsilon\to 0$, the Joule dissipation δW_{Joule} in the thin ionospheric layer is proportional to $\Sigma_P|\mathbf{E}_1|^2$.

Thus, one of the conditions under which the ionospheric Joule dissipation can be neglected is the limit of $\mathbf{E}_{1\perp}\to 0$. This corresponds to the case of $\Sigma_P\to\infty$, so field lines are almost rigidly tied to the ionosphere and the rigid boundary

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condition ($\boldsymbol{\xi}_\perp=0$) holds. Since $|\tilde{\boldsymbol{\xi}}|=|\tilde{\mathbf{E}}_1|/(\Gamma B_0)$, where Γ is the growth rate of magnetospheric interchange instability, which is either pressure-driven or ionosphere-driven, we obtain from Eq. (33)

$$\frac{\delta W_{\mathrm{Joule}}}{|\delta W_I|} \cong 2\mu_0 V_A \Sigma_P \Gamma \frac{R_I}{V_A}, \tag{111}$$

where V_A is the average Alfvén velocity along magnetospheric field lines. The dimensionless parameter $\mu_0 V_A \Sigma_P$ is typically an order of 1 for the earth's magnetosphere and $\simeq 0.1$ for a low conductivity in the nightside ionosphere. For $R_I=6400$ km, $V_A \simeq 1000$ km/s, and $\mu_0 V_A \Sigma_P=0.1$, Eq. (111) is equal to 1.28Γ. Since $\Gamma^{-1}>10$ s is quite reasonable for magnetospheric interchange instability, we find that for low ionospheric conductivity in the nightside ionosphere, the ionospheric dissipation $\delta W_{\mathrm{Joule}}$ can be neglected in comparison with δW_I. Thus, the limit of $\Sigma_P \to 0$ is the other condition under which the ionospheric Joule dissipation can be neglected compared with δW_I. In this limit, field lines can move freely in the ionospheric surface. Thus, this limit of $\mu_0 V_A \Sigma_P \ll 1$ corresponds to the horizontally free boundary condition $(\mathbf{b}\cdot\nabla)\boldsymbol{\xi}_\perp=0$ at the plasma-atmosphere interface. This small ionospheric conductivity $\mu_0 V_A \Sigma_P \ll 1$ is necessary for magnetospheric interchange instability, because for a large Σ_P, field lines are almost rigidly tied to the ionosphere and no displacement of field lines, which is necessary for interchange modes, is possible.

Thus, we find that there are two limits of Σ_P, i.e., a very large $\mu_0 V_A \Sigma_P$ and a very small $\mu_0 V_A \Sigma_P$, in which the ionospheric Joule dissipation $\delta W_{\mathrm{Joule}}$ can be neglected in comparison with δW_I. The large $\mu_0 V_A \Sigma_P$ corresponds to the boundary condition $\boldsymbol{\xi}_\perp=0$ and a very small $\mu_0 V_A \Sigma_P$ corresponds to $(\mathbf{b}\cdot\nabla)\boldsymbol{\xi}_\perp=0$. Thus, the complete neglect of energetics in the thin ionospheric layer can be justified for the rigid [Eq. (80)] and the horizontally free [Eq. (81)] boundary conditions.

Consequently, the validity of the neglect of energetics in the thin ionospheric layer is related to the boundary condition on $\boldsymbol{\xi}_\perp$ at the plasma-atmosphere interface. On the other hand, the boundary condition on $\xi_\parallel$ is related to the compressibility at the magnetosphere-atmosphere interface. As we found in Sec. IX A, in the extended magnetospheric energy principle, $\xi_\parallel=0$ must be satisfied at the magnetosphere-atmosphere interface, since the magnetospheric plasma is assumed to be in contact with a very compressible atmospheric gas.

The magnetospheric energy principle[8] and its extended form are useful for studying the ideal MHD stability of magnetospheric plasmas. Magnetospheric ideal MHD instabilities include pressure-driven instabilities such as ballooning and interchange instabilities, and a current-driven instability such as kink instability, which is driven by parallel-currents. The magnetospheric energy principle[8] clearly showed that in the magnetosphere there are two different ballooning modes, i.e., incompressible and compressible ballooning modes. The incompressible ballooning mode satisfies the ionospheric boundary condition (13) and the compressible ballooning mode satisfies the ionospheric boundary condition (14).

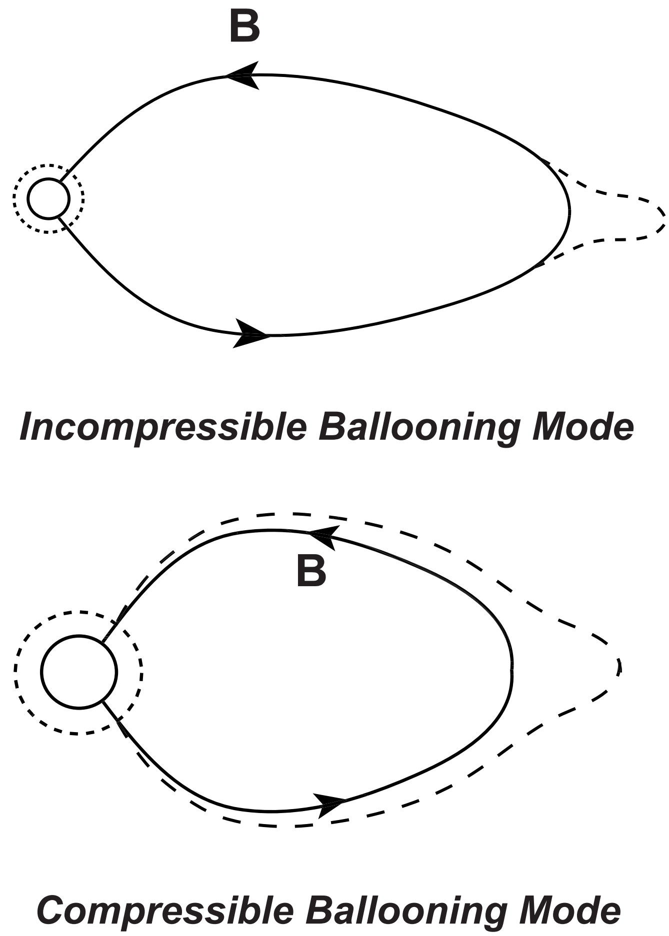

FIG. 7. The upper and lower panels show conceptual sketches of incompressible and compressible ballooning modes in the meridian plane, respectively. The dotted circles in both panels show ionospheric boundaries. The solid circles show the earth. The solid and dashed lines show unperturbed and perturbed magnetic field lines, respectively.

Since the compressibility factor $\nabla\cdot\boldsymbol{\xi}$ is constant along the field line in both the magnetospheric energy principle[8] and its extended one, the compressibility of magnetospheric ballooning modes is determined uniquely by boundary conditions at the ionospheric surface.

Figure 7 shows the conceptual sketches of incompressible ballooning mode (upper panel) and compressible ballooning mode (lower panel), which are drawn[8] based on previous numerical eigenmode analyses of magnetospheric ballooning instabilities. The solid and dashed lines in Fig. 7 show unperturbed and perturbed magnetic field lines, respectively. The solid and dotted circles are the earth and the ionospheric surfaces, respectively. This figure shows that while the incompressible ballooning mode is highly bounded near the equatorial plane, the compressible ballooning mode has a broad structure along the magnetic field line.[8] This is because in the compressible ballooning mode the wave energy is carried far from the equatorial plane by compressional waves.

According to the extended magnetospheric energy principle, there is only one ionospheric boundary condition which is suitable for ballooning modes, i.e., the rigid boundary condition given by Eq. (80) ($\boldsymbol{\xi}=0$). This is a compress-

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ible boundary condition and the compressibility factor given by $\nabla \cdot \boldsymbol{\xi}$ is constant along the field line in the extended magnetospheric energy principle (see Sec. X). Although this seems to inhibit the occurrence of an incompressible ballooning mode in the magnetosphere, the upper panel of Fig. 7 shows that the displacement of the incompressible ballooning mode decreases so rapidly with distance toward the ionosphere that practically all the three components of the displacement vector are zero at the ionosphere. Thus, even if a rigid boundary condition is imposed at the plasma-atmosphere interface, an incompressible ballooning mode can still exist in the magnetosphere[8] and may play an important role in magnetospheric dynamics. Notice that if such an incompressible ballooning mode, as shown in the upper panel of Fig. 7, really exists, the boundary condition (12) might also be satisfied by such a mode,[8] since $\boldsymbol{\xi}_{\perp}$ is nearly equal to zero near the ionospheric surface, so $(\mathbf{b}\cdot\nabla)\boldsymbol{\xi}_{\perp}$ is also nearly equal to zero. Thus, for such an incompressible ballooning mode, which is strongly bounded near the equatorial plane, the ionospheric boundary condition may not be important.

The existence of such an incompressible ballooning mode has been suggested with regard to problems of substorm onset in the terrestrial magnetosphere.[15,16] In particular, it has been suggested that when the magnetospheric configuration becomes more tail-like, the incompressible ballooning mode becomes more easily destabilized and thus possibly responsible for a very fast growth of ideal MHD perturbations in the near-earth tail.[15,16] A magnetospheric substorm is a relaxation process,[15] which reconfigures a magnetospheric configuration with highly stretched tail fields to a potential field configuration (dipole field). An onset of substorm, which is characterized by a sudden increase in auroral brightness at high latitudes, is considered to be a start of a sudden release of energy stored in a highly stressed magnetospheric magnetic field configuration with a geomagnetic tail. Although the problem of what physical processes are responsible for the rapid relaxation has not been solved completely, it has recently been recognized that the onset of ballooning instability in the near-earth tail may be a key to understanding this rapid relaxation process. The extended magnetospheric energy principle shows that there are two stabilizing mechanisms for ballooning modes. One is the field line tension, which is expressed by $\mu_0^{-1}|\mathbf{Q}_{\perp}|^2$ in Eq. (15), and the other is the stabilization by compressibility, which is expressed by the $\gamma p|\nabla\cdot\boldsymbol{\xi}|^2$ term in Eq. (15). Thus, a very likely scenario of substorm related ballooning mode is that as the magnetosphere becomes more tail-like before the onset of substorms, the near-earth tail becomes more easily susceptible to incompressible ballooning mode because the compressibility stabilization becomes smaller.[15,16]

Although $\delta W_I=0$ for ballooning modes and thus the existence of δW_I term is irrelevant to magnetospheric ballooning modes, the existence of δW_I suggests that the magnetospheric plasma is ideal MHD unstable even without pressure-driven or current-driven modes. Such a possibility has been studied in detail[9] and it has been shown that besides a conventional pressure-driven magnetospheric interchange instability,[17] there is an ionosphere-driven magnetospheric interchange instability.[9] This is a new type of magnetospheric interchange instability, which involves a motion of a whole flux tube from the equator down to ionospheres.

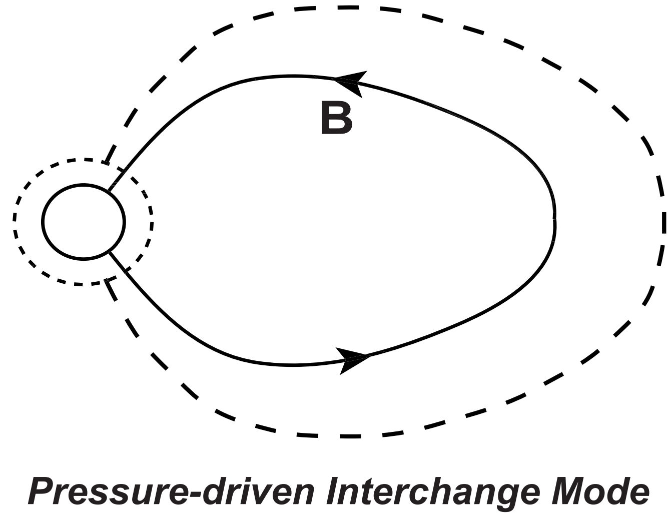

FIG. 8. A conceptual sketch of pressure-driven interchange mode in the meridian plane. The solid and dotted circles show the earth and the ionospheric surface, respectively. The solid and dashed lines show unperturbed and perturbed magnetic field lines, respectively.

Figure 8 shows a conceptual sketch of a pressure-driven interchange mode[9,17] in the meridian plane. The solid and dotted circles are the earth and the ionospheric surface, respectively. The solid line is an unperturbed field line and the dashed line is a perturbed field line. Notice that unlike the ballooning modes shown in Fig. 7, the interchange mode has no field line deformation or $\mathbf{B}_{1\perp}\simeq 0$ in the magnetosphere.[2,8,9] This pressure-driven interchange mode is destabilized by the unperturbed pressure gradient existing in the meridian plane of the terrestrial magnetosphere.[9,17] So, as shown in Fig. 8, the pressure-driven interchange mode involves latitudinal interchange of flux tubes. On the other hand, the ionosphere-driven interchange mode[9] involves longitudinal interchange of flux tubes. Although there is no pressure gradient in the azimuthal direction for an axisymmetric magnetosphere, a longitudinal motion of flux tubes yields an upward directed Poynting vector, as shown in Fig. 3. According to a stability analysis[9] based on the magnetospheric energy principle[8] and an eikonal approximation, an interchange mode with $m\leq 2$, where m is the azimuthal mode number, can easily be destabilized in the inner magnetosphere when the equatorial plasma beta value is smaller than a critical beta value of order 1.

Since magnetospheric interchange modes, both pressure-driven[9,17] and ionosphere-driven,[9] are compressible modes, they accompany the change of magnetic field strength. They are considered to be important sources of MHD fluctuations in the earth's radiation belts. Although the interaction of trapped particles in the earth's radiation belts with MHD fluctuations has been a long standing problem in magnetospheric physics since the discovery of Van Allen radiation belts, there has been little investigation of the origin of ideal MHD fluctuations in these belts. While in the inner magnetosphere, where radiation belt particles are trapped,

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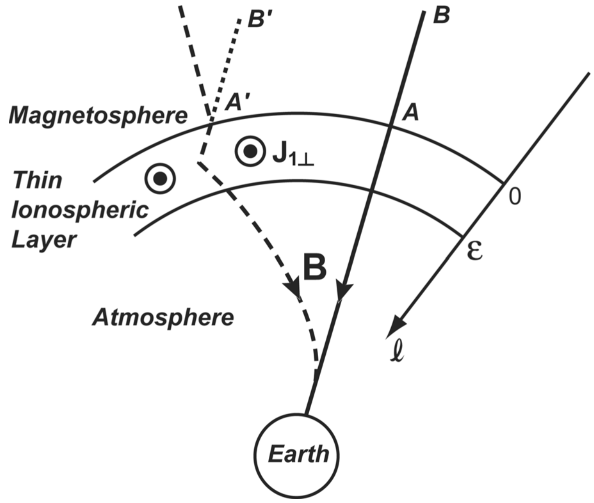

FIG. 9. A conceptual sketch of ionosphere-driven interchange mode in the neighborhood of the thin ionospheric layer in the northern hemisphere. The solid circle is the earth. The thin ionospheric layer is located between $0<\ell<\epsilon$. In this layer, the perturbed current $\mathbf{J}_{1\perp}$ is directed out of the page. The thick solid and dashed lines show unperturbed and perturbed magnetic field lines, respectively. The motion from the solid line to the dashed line in the magnetosphere occurs in the longitudinal direction. $\mathbf{B}$ shows the direction of the magnetic field. The dotted line $A'B'$ is parallel to the unperturbed field line AB, as shown in Fig. 3.

the pressure gradient is so small that a pressure-driven interchange instability cannot easily be destabilized,[9] an ionosphere-driven interchange instability is easily destabilized in the inner magnetosphere.[9] Thus, ionosphere-driven magnetospheric interchange instablity[9] is considered to be important in discussing radiation belt dynamics.

With the extended magnetospheric energy principle, it is now possible to study the structure of magnetic field perturbations in the atmosphere associated with the ionosphere-driven interchange mode.[9] Figure 9 shows a conceptual sketch of ionosphere-driven interchange mode in the neighborhood of the thin ionospheric layer in the northern hemisphere. The solid circle is the earth. In this figure, $\ell=0$ is the plasma edge, $\ell=\epsilon$ is the atmospheric edge, and the region $0<\ell<\epsilon$ is the thin ionospheric layer. The thick solid line is an unperturbed field line and the dashed line is a perturbed field line. $\mathbf{B}$ shows the direction of the magnetic field. Notice that the dotted line $A'B'$ is parallel to the unperturbed field line AB, as shown in Fig. 3. Unlike the pressure-driven interchange mode,[9,17] the flux interchange in the longitudinal direction in the magnetosphere is most susceptible to ionosphere-driven interchange instability.[9] Thus, we note that the field line motion from the thick solid line to the dashed line in the magnetosphere in Fig. 9 occurs in the longitudinal direction.

According to the extended magnetospheric energy principle, a horizontal magnetic perturbation $\mathbf{B}_{1\perp}$ is induced at the ionospheric surface due to a horizontally free motion of field lines by an electric field perturbation when the unperturbed field line is incident vertically on the spherical ionospheric surface. In the magnetosphere, the magnetospheric interchange mode has no horizontal magnetic field perturbation, i.e., $\mathbf{B}_{1\perp}\simeq 0$.[2,9] Thus, as shown in Fig. 9, field lines are kinked at the ionospheric surface located at $\ell=0$ in an ionosphere-driven interchange instability. Owing to the continuity of a tangential electric field across the ionospheric surface, which is shown by Eq. (105), there also arises a horizontal magnetic field perturbation $\hat{\mathbf{B}}_{1\perp}$ just below the thin ionospheric layer at $\ell=\epsilon$ and this magnetic field perturbation directs opposite to $\mathbf{B}_{1\perp}$. Thus, there is a reversal of the horizontal component of magnetic field perturbations across this layer, i.e., from $\mathbf{B}_{1\perp}$ to $\hat{\mathbf{B}}_{1\perp}$. This means that there is an electric current in $0<\ell<\epsilon$ and the current is flowing out of this page, as shown by $\mathbf{J}_{1\perp}$ in Fig. 9. Since the magnetic field in the atmosphere is a potential field, the horizontal component of the magnetic field $\hat{\mathbf{B}}_{1\perp}$ must decrease toward the earth's surface. Thus, the perturbed field line shown by the dashed line asymptotically approaches the unperturbed field line shown by the thick solid line toward the earth's surface. If such a field line structure, as shown in Fig. 9, is truly observed, the existence of the ionosphere-driven interchange instability[9] predicted by the magnetospheric energy principle[8] is confirmed and the validity and usefulness of the extended magnetospheric energy principle in studying magnetospheric stability are verified.

XII. SUMMARY

An extended magnetospheric energy principle has been developed on the basis of the extended energy principle for fusion plasmas.[1,2] The extended energy principle for fusion plasmas and the extended magnetospheric energy principle investigate the ideal MHD stability of two extreme limits of static plasma configurations in magnetic fields. In the extended energy principle for fusion plasmas, the unperturbed magnetic field is assumed to be parallel to the plasma-vacuum interface,[1] and in the extended magnetospheric energy principle, it is assumed to be perpendicular to the plasma-atmosphere interface (see Fig. 6). Unlike the magnetospheric energy principle,[8] which was previously developed by neglecting the existence of a neutral atmosphere below the ionospheric surface, the extended magnetospheric energy principle uses three physical boundary conditions at the plasma-atmosphere interface and at the earth's surface. In order to realize the transition from the plasma to the atmosphere, the existence of a thin ionospheric layer consisting of a partially ionized gas is necessary. Although this thin ionospheric layer is taken into account as boundary conditions in the extended magnetospheric energy principle, energetics occurring in this thin layer are neglected based on the assumption that the ionospheric conductivity is either very small or very large.

Whereas in the magnetospheric energy principle there are four possible boundary conditions at the ionospheric surface,[8] which are necessary and sufficient for the self-adjointness of the force operator, in the extended magnetospheric energy principle there are only two possible boundary conditions, i.e., the rigid ($\boldsymbol{\xi}=0$) and horizontally free $[\xi_{\parallel}=0$ and $(\mathbf{b}\cdot\nabla)\boldsymbol{\xi}_{\perp}=0]$ boundary conditions at the plasma-atmosphere interface. The two incompressible boundary conditions in the magnetospheric energy principle are removed in the extended magnetospheric energy principle because the

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plasma is considered to be in contact with a very compressible atmospheric gas. The two boundary conditions allowed in the extended principle are realized in a real magnetosphere when the ionospheric conductivity is very large or very small.

Unlike the extended energy principle for fusion plasmas,[1,2] in the extended magnetospheric energy principle there is no stabilizing positive contribution δW_A representing the perturbed atmospheric magnetic energy. This difference arises because the geometrical relationship between **n** and **B** is different in the magnetosphere and in fusion plasmas (see Fig. 6). Thus, like the magnetospheric energy principle,[8] the extended magnetospheric energy principle shows that $\delta W = \delta W_F + \delta W_I$, where the negative destabilizing contribution δW_I arises as a consequence of a horizontal plasma displacement on the spherical ionospheric surface.[8]

It follows that we can investigate the ideal MHD stability of magnetospheric plasmas regardless of the existence of the atmosphere and the thin ionospheric layer existing between the plasma and the solid-earth surface as long as we assume the vertical incidence of the unperturbed magnetic field on the spherical ionospheric surface. Consequently, the previous investigation of magnetospheric interchange instabilities[9] using the magnetospheric energy principle neglecting the existence of the atmosphere[8] is justified, and thus the finding of the existence of the ionosphere-driven interchange instability[9] using the magnetospheric energy principle[8] is justified.

The extended magnetospheric energy principle is applied to magnetospheric ballooning[8,15,16] and interchange[9] instabilities in order to show its usefulness for studying the ideal MHD stability of magnetospheric plasmas. The ballooning and interchange modes satisfy the rigid and horizontally free boundary conditions, respectively, at the plasma-atmosphere interface. The extended magnetospheric energy principle can clarify the structure of perturbed magnetic fields of the ionosphere-driven interchange mode in the atmosphere (see Fig. 9), which was unclear in the normal magnetospheric energy principle neglecting the existence of the atmosphere.[8]

In order to show the validity and applicability of the extended magnetospheric energy principle to a real terrestrial magnetosphere, possible consequences of the extended magnetospheric energy principle need to be tested against observations.

From the point of magnetospheric plasma theory, the extended magnetospheric energy principle merits further generalization for possible application to a wide variety of magnetospheres existing in nature.

ACKNOWLEDGMENTS

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