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Index Theorem and Overlap Formalism with Naive and Minimally Doubled Fermions

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We present a theoretical foundation for the Index theorem in naive and minimally doubled lattice fermions by studying the spectral flow of a Hermitean version of Dirac operators. We utilize the point splitting method to implement flavored mass terms, which play an important role in constructing proper Hermitean operators. We show the spectral flow correctly detects the index of the would-be zero modes which is determined by gauge field topology. Using the flavored mass terms, we present new types of overlap fermions from the naive fermion kernels, with a number of flavors that depends on the choice of the mass terms. We succeed to obtain a single-flavor naive overlap fermion which maintains hypercubic symmetry.

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I. INTRODUCTION

Studies of topological aspects have been attracting a great deal of attention in lattice QCD. Issues here include whether the index theorem relating gauge field topology and fermionic zero modes is well defined. Lattice fermion actions without doublers do illustrate the index theorem nicely. For example, one old-fashioned way to study topology in QCD is to use Wilson fermion [1, 2], although it requires fine-tuning of a mass parameter because of explicit breaking of chiral symmetry. The overlap fermion [3, 4], which is built on the Wilson fermion kernel, resolves this problem since it possesses both exact chiral symmetry and exact zero modes related to gauge field topology, agreeing with the index theorem [5, 6]. The domain wall formalism [7, 8], closely related to the overlap fermion, is also a useful tool to study topological effects in QCD. On the other hand, the index is hidden in lattice fermions with species doublers since the index effect cancels between doubling pairs. Although a theoretical approach to the index of the staggered fermions [9–11] were developed in [2, 12], it does not give an integer value from the beginning and requires a renormalization depending on the full ensemble of the gauge fields. Thus it is not easy to define the index theorem in naive, staggered and minimally-doubled fermions and thereby study topological effects.

Ref. [13] recently presented how to identify the would-be zero modes and their chiralities with staggered fermions away from the continuum limit using the spectral flow of a certain Hermitean version of the Dirac operator. The integer index obtained here correctly illustrates the gauge topological charge up to a factor coming from tastes. In a sequent paper [14], the author presents a new version of the overlap fermion, the "staggered overlap fermion" constructed from the staggered kernel. It is a two-flavor overlap fermion in four dimensions and the associated index of exact zero modes illustrates the gauge field topology correctly again. In Ref. [15] the possibility to construct a single-flavor version of the staggered overlap fermion is also discussed. The key for theoretical foundation of the index theorem and the sequent overlap formalism here is to introduce a taste sensitive mass which is essential to reveal the hidden index with the spectral flow of the Hermitean operator. The new mass term, which assigns positive and negative masses to tastes depending on their flavor-chiralities, removes the cancellation of the index between tastes. When the staggered kernel with this kind of mass is substituted into the overlap formalism, some tastes with positive mass decouple while other tastes with negative mass yield massless modes, which correspond to the two-flavor staggered overlap fermion. It is a universal feature for fermions with species doublers that you can obtain the associated overlap fermion with a proper flavored mass term illustrating the correct index.

One natural extension of this approach is to apply it to the naive lattice fermion. As is well-known, the naive fermion includes 16 species while staggered has 4 species in four dimension thus it is expected we can take a parallel process to establish the integer index theorem without renormalization and construct overlap fermion with the naive kernel. However one obstacle in this case is the individual species appear most easily only in momentum space, which leads to difficulty of identifying them as flavors and introducing flavored mass terms. We have a similar problem with minimally doubled fermions [11, 16–19, 22, 23], where there are two species in momentum space. So far there has been no theoretical foundation of the index theorem for naive and minimally doubled fermions [24].

In this paper, we successfully identify species in naive and minimally doubled fermions as flavors by using the point splitting method, which is proposed in Ref. [20] to define up and down quark fields in minimally doubled fermions. By this identification, we define proper flavored mass terms to extract the index in the spectral flow of the associated Hermitean version of the Dirac operator. Here we follow a parallel approach for identifying the would-be zero modes and their chiralities to that proposed in the staggered case [13]. Then we find the spectral flow correctly illustrates the index determined by the gauge field topology in naive and minimally doubled fermions. In addition we present new versions of overlap fermions built from the naive fermion kernel with added flavored mass terms. Especially we construct a single-flavor naive overlap fermion by choosing a certain flavored mass which assigns negative mass to one and positive mass to all other species. One good property of this fermion over the minimally doubled case is it maintains hypercubic symmetry. Furthermore it appears to have more stability of the index against disorder than the Wilson overlap fermion. Thus this overlap fermion might be more suitable for practical simulations.

II. POINT SPLITTING AND FLAVORED MASS TERMS

With the d-dimensional naive lattice fermion there are 2^d species and the associated massless Dirac operator is anti-Hermitean. These properties are common with the staggered fermion except that there are only $2^{d/2}$ tastes in the latter case. Thus it seems easy to apply the approach for the index of staggered fermion [13] to the naive fermion. In this spectral-flow approach the flavored mass term is essential, which gives + and − masses to flavors(species) depending on their chiral charges. However the species in the naive fermion appear naturally only in momentum space. Therefore in this case we cannot implement flavored mass terms straightforwardly since we need to identify species in momentum space as independent flavors. To resolve this problem we apply a method called "point splitting," originally proposed to define up and down quark fields in minimally doubled fermion [20]. By using this method we obtain flavored mass terms to give proper Hermitean operators for the spectral flow both in minimally doubled and naive fermions. In this section we first introduce the point splitting for minimally doubled fermions and implement flavored mass terms. Secondly we perform the parallel approach for naive fermions.

A. Minimally doubled action

Here we introduce the point splitting method to obtain flavored mass terms in minimally doubled fermions. We focus only on the Karsten-Wilczek type [16, 17] all through this paper. Now we begin with the introduction of minimally doubled fermions. The $d = 4$ Karsten-Wilczek action is obtained by introducing a Wilson-like term proportional to $i\gamma_4$. Its position-space expression is

$$\begin{aligned} S_{\mathrm{md}} \;=\; \sum_x \Bigg[& \frac{1}{2}\sum_{\mu=1}^{3} \bar{\psi}_x\gamma_\mu \left(U_{x,x+e_\mu}\psi_{x+e_\mu} - U_{x,x-e_\mu}\psi_{x-e_\mu}\right) \\ & + \frac{i}{\sin\alpha}\left((\cos\alpha+3)\bar{\psi}_x\gamma_4\psi_x - \frac{1}{2}\sum_{\mu=1}^{4}\bar{\psi}_x\gamma_4\left(U_{x,x+e_\mu}\psi_{x+e_\mu} + U_{x,x-e_\mu}\psi_{x-e_\mu}\right)\right)\Bigg] \end{aligned} \tag{1}$$

where the link variables satisfy $U_{xy} = U_{yx}^\dagger$. For the free theory, the associated Dirac operator in momentum space is given by

$$D_{\mathrm{md}}(p) = i\sum_{\mu=1}^{3}\gamma_\mu \sin p_\mu + \frac{i\gamma_4}{\sin\alpha}\left(\cos\alpha + 3 - \sum_{\mu=1}^{4}\cos p_\mu\right), \tag{2}$$

where the parameter α adjusts the relative positions of zeros. It has only two zeros located at $p = (0, 0, 0, \pm\alpha)$. These two species are not equivalent since the gamma matrices are differently defined between them as $\gamma'_\mu = \Gamma^{-1}\gamma_\mu\Gamma$. In the above case the transformation matrix is given by $\Gamma = i\gamma_4\gamma_5$. This means the chiral symmetry possessed by this action is identified as a flavored one given by $\gamma_5 \otimes \tau_3$.

The point splitting identifies these inequivalent species as independent flavors. In this method each flavor field is defined so that the associated propagator includes only a single pole. Thus we point-split the original fermion field by multiplying factors removing another undesired pole in momentum space,

$$u(p - \alpha e_4) = \frac{1}{2}\left(1 + \frac{\sin p_4}{\sin\alpha}\right)\psi(p), \tag{3}$$

$$d(p + \alpha e_4) = \frac{1}{2}\Gamma\left(1 - \frac{\sin p_4}{\sin\alpha}\right)\psi(p). \tag{4}$$

Here $u(p - \alpha e_4)$ and $d(p + \alpha e_4)$ fields correspond to the poles at $p = (0, 0, 0, \alpha)$ and $(0, 0, 0, -\alpha)$, respectively. We remark both of them yield single fermionic modes and the associated point-split fields in position space are composed of the original field and the two kinds of nearest neighbors [20]. With these flavor fields we obtain a flavor-multiplet field as following,

$$\Psi(p) = \begin{pmatrix} u(p - \alpha e_4) \\ d(p + \alpha e_4) \end{pmatrix}. \tag{5}$$

Here γ_5 multiplication on the original Dirac field is identified as

$$\gamma_5\psi(p) \quad \longrightarrow \quad \begin{pmatrix} +\gamma_5 & \\ & -\gamma_5 \end{pmatrix}\Psi(p) = (\gamma_5 \otimes \tau_3)\,\Psi(p). \tag{6}$$

Here we introduce a multiplet representation as a direct product of the Pauli matrix to express the 2-flavor structure. It means the flavored chiral symmetry generated by $\gamma_5 \otimes \tau_3$ is exactly preserved in terms of the flavor multiplet while the flavor singlet chiral symmetry given by $\gamma_5 \otimes \mathbf{1}$ is broken in higher order terms. It is also the case of the staggered fermion [23].

Now we can introduce a flavor breaking mass into minimally doubled fermions by a parallel approach to the case of staggered fermions in Refs. [13, 14]. We have to choose a flavored mass term so that it reveals the hidden index when looking into the spectral flow of a certain Hermitean version of the Dirac operator $H(m) = \gamma_5(D - M)$, which we will explain in detail in Sec. III. (This γ_5 is regarded as $\gamma_5 \otimes \tau_3$ in terms of the flavor multiplet as shown in (6).) We need to impose that $\gamma_5 M$ should be flavor-singlet as $\gamma_5 M = \gamma_5 \otimes \mathbf{1}$. Thus the corresponding flavored mass term here is $M = m\,(\mathbf{1} \otimes \tau_3)$ with a mass parameter

m. To implement this flavored mass term into the action, we rewrite the mass term with the original Dirac field as

$$\bar{\Psi}(p)\,(\mathbf{1}\otimes\tau_3)\,\Psi(p) = \bar{u}u(p-\alpha e_4) - \bar{d}d(p+\alpha e_4) = \frac{\sin p_4}{\sin\alpha}\bar{\psi}(p)\psi(p). \tag{7}$$

It is straightforward to obtain the mass term in the position space with the link variables present,

$$M_{\rm md} = \frac{m}{2i\sin\alpha}\bar{\psi}_x\left(U_{x,x+e_4}\psi_{x+e_4} - U_{x,x-e_4}\psi_{x-e_4}\right). \tag{8}$$

Here we also write a flavor-singlet mass term

$$\bar{\Psi}(p)\,(\mathbf{1}\otimes\mathbf{1})\,\Psi(p) = \bar{u}u(p-\alpha e_4) + \bar{d}d(p+\alpha e_4) = \frac{1}{2}\left(1+\frac{\sin^2 p_4}{\sin^2\alpha}\right)\bar{\psi}(p)\psi(p), \tag{9}$$

which is not required for the study of the index. The associated massive Dirac operator $D_{\rm md} - M_{\rm md}$ is non-Hermitean, and when gauge fields are present the mass term does not commute with the kinetic term $[D_{\rm md}, M_{\rm md}] \neq 0$. Thus the Dirac operator eigenvalues are complex. Indeed it is essential for the purpose to detect the index from the spectral flow of the Hermitean operator since it relies on real eigenvalues of the Dirac operator.

In Fig. 1 we show a numerical result of complex eigenvalues of the Dirac operator for the $d=2$ free case with a parameter $\alpha=\pi/2$. Here the low-lying spectrum is split into two branches crossing the real axis at the magnitude of the mass parameter $|m|$. It means that the flavored mass $-M_{\rm md} = {\rm diag}(-m,+m)$ assigns $-m(+m)$ to modes depending on $+1(-1)$ chiral charges, or equivalently $+1(-1)$ eigenvalues for $\mathbf{1}\otimes\tau_3$. In other words the flavored mass term splits the minimally doubled fermion into two single Dirac fermions with $-m$ and $+m$ as in the staggered case. In Sec. III we will see this flavored mass gives the spectral flow of the Hermitean operator illustrating the correct index related to the gauge topology.

B. Naive action

We can apply the same approach to the naive lattice fermion [1] to obtain a proper flavored mass term. The action of the naive fermions in general dimensions is simply given by

$$S_{\rm n} = \frac{1}{2}\sum_x\sum_{\mu=1}^{d}\bar{\psi}_x\gamma_\mu\left(U_{x,x+e_\mu}\psi_{x+e_\mu} - U_{x,x-e_\mu}\psi_{x-e_\mu}\right). \tag{10}$$

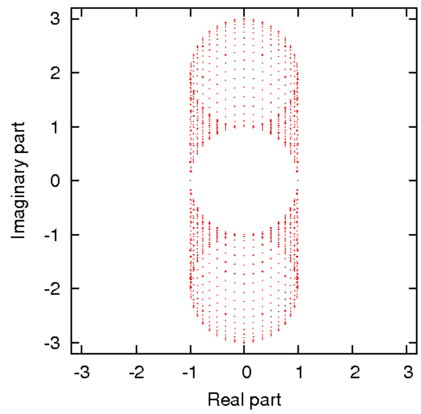

FIG. 1: Complex spectrum of the free non-Hermitean Dirac operator $D_{\rm md} - M_{\rm md}$ for the $d = 2$ free field case on a 36×36 lattice with a mass parameter $m = 1$. The spectrum is split into two branches crossing the real axis at $|m|$.

For simplicity here we consider the $d = 2$ naive fermions. The Dirac operator has four zeros $(0, 0)$, $(\pi, 0)$, $(0, \pi)$ and (π, π) in momentum space, thus we introduce four associated point-split fields

$$\begin{aligned}
\psi_{(1)}(p - p_{(1)}) &= \frac{1}{4}(1 + \cos p_1)(1 + \cos p_2)\Gamma_{(1)}\psi(p), \\
\psi_{(2)}(p - p_{(2)}) &= \frac{1}{4}(1 - \cos p_1)(1 + \cos p_2)\Gamma_{(2)}\psi(p), \\
\psi_{(3)}(p - p_{(3)}) &= \frac{1}{4}(1 + \cos p_1)(1 - \cos p_2)\Gamma_{(3)}\psi(p), \\
\psi_{(4)}(p - p_{(4)}) &= \frac{1}{4}(1 - \cos p_1)(1 - \cos p_2)\Gamma_{(4)}\psi(p),
\end{aligned} \tag{11}$$

whose locations of zeros, chiral charges and transformation matrices Γ giving the corresponding set of gamma matrices, $\gamma_\mu^{(i)} = \Gamma_{(i)}^\dagger \gamma_\mu \Gamma_{(i)}$, are listed in Table I. The flavor-multiplet field is given by

$$\Psi(p) = \begin{pmatrix} \psi_{(1)}(p - p_{(1)}) \\ \psi_{(2)}(p - p_{(2)}) \\ \psi_{(3)}(p - p_{(3)}) \\ \psi_{(4)}(p - p_{(4)}) \end{pmatrix}. \tag{12}$$

Here the operation of γ_5 on the original fermion field again means the flavored chiral

label	position	χ charge	Γ
1	$(0,0)$	$+$	$\mathbf{1}$
2	$(\pi,0)$	$-$	$i\gamma_1\gamma_5$
3	$(0,\pi)$	$-$	$i\gamma_2\gamma_5$
4	(π,π)	$+$	γ_5

TABLE I: Chiral charges and transformation matrices for each of zeros in the $d = 2$ naive fermions with $\gamma_1 = \sigma_1$, $\gamma_2 = \sigma_2$ and $\gamma_5 = \sigma_3$.

transformation in the sense of the flavor multiplet as

$$\gamma_5\psi(p) \quad \longrightarrow \quad \begin{pmatrix} +\gamma_5 & & & \\ & -\gamma_5 & & \\ & & -\gamma_5 & \\ & & & +\gamma_5 \end{pmatrix} \Psi(p) = (\gamma_5 \otimes (\tau_3 \otimes \tau_3))\,\Psi(p). \tag{13}$$

Here we introduce a multiplet representation as two direct products of the Pauli matrix to express the 4-flavor structure. Now we adopt the criterion for the proper flavored mass term that $\gamma_5 M$ should be flavor-singlet as $\gamma_5 M = \gamma_5 \otimes \mathbf{1}$. Therefore the desirable flavored mass for the naive fermion is given by

$$\bar{\Psi}(p)\,(\mathbf{1} \otimes (\tau_3 \otimes \tau_3))\,\Psi(p) = \cos p_1 \cos p_2 \bar{\psi}(p)\psi(p). \tag{14}$$

Please note there are also other kinds of flavored mass terms in this case:

$$\bar{\Psi}(p)\,(\mathbf{1} \otimes (\tau_3 \otimes \mathbf{1}))\,\Psi(p) = \frac{1}{2}\cos p_1(1 + \cos^2 p_2)\bar{\psi}(p)\psi(p), \tag{15}$$

$$\bar{\Psi}(p)\,(\mathbf{1} \otimes (\mathbf{1} \otimes \tau_3))\,\Psi(p) = \frac{1}{2}(1 + \cos^2 p_1)\cos p_2 \bar{\psi}(p)\psi(p), \tag{16}$$

$$\bar{\Psi}(p)\,(\mathbf{1} \otimes (\mathbf{1} \otimes \mathbf{1}))\,\Psi(p) = \frac{1}{4}(1 + \cos^2 p_1)(1 + \cos^2 p_2)\bar{\psi}(p)\psi(p). \tag{17}$$

These varieties are not useful to detect the index via the spectral flow of the Hermitean operator. However we will show they are essential when we search for a single-flavor naive overlap fermion in Sec. IV. All of these mass terms are spread over several nearby sites, and therefore their position-space expressions include hopping terms with associated gauge field matrices. It is convenient to introduce the usual translation operators $T_{\pm\mu}\psi_x = U_{x,x\pm e_\mu}\psi_{x\pm e_\mu}$

and $C_\mu = (T_{+\mu} + T_{-\mu})/2$. Then the flavored mass for the index (14) is written in the position space as

$$M_{\tau_3\otimes\tau_3} = m\sum_{sym.} C_1 C_2 \equiv M_{\rm n}, \tag{18}$$

where $\sum_{sym.}$ stands for symmetric summation over the order of the factors C_1 and C_2, and m stands for a mass parameter. We can also write the extra mass terms in the position space as

$$M_{\tau_3\otimes\mathbf{1}} = \frac{m_{\tau_3\otimes\mathbf{1}}}{2}\sum_{sym.}\left(1+C_1^2\right)C_2, \tag{19}$$

$$M_{\mathbf{1}\otimes\tau_3} = \frac{m_{\mathbf{1}\otimes\tau_3}}{2}\sum_{sym.}C_1\left(1+C_2^2\right), \tag{20}$$

$$M_{\mathbf{1}\otimes\mathbf{1}} = \frac{m_{\mathbf{1}\otimes\mathbf{1}}}{4}\sum_{sym.}\left(1+C_1^2\right)\left(1+C_2^2\right). \tag{21}$$

Here we comment on the possibility that the two-step hoppings in the same directions would be reduced to zero-step terms with reduction in the effects of gauge field fluctuations.

Now let us look into the eigenvalues of the $d=2$ naive Dirac operator $D_{\rm n} - M_{\rm n}$ with the flavored mass $M_{\rm n}$ in Eq. (18). In Fig. 2 we show a numerical result of the complex eigenvalues. Here the low-lying spectrum is again split into two branches crossing the real axis at the magnitude of the mass parameter $|m|$. However in this case both of the two branches are doubled. (We will be convinced of this doubling in Fig. 6 where it is lifted by the other mass terms.) It means that the flavored mass term $-M_{\rm n} = \mathrm{diag}(-m,+m,+m,-m)$ assigns $-m(+m)$ to modes depending on $+1(-1)$ chiral charges, or $+1(-1)$ eigenvalues for $\mathbf{1}\otimes(\tau_3\otimes\tau_3)$. Thus the $d=2$ naive fermion with 4 species is split into two pairs of Dirac fermions with $-m$ and $+m$. In the next section we will show the spectral flow of the associated Hermitean operator gives the correct index related to the gauge topology.

In the $d=4$ case we can apply the same approach to obtain a proper flavored mass term. In this case there are more possibilities for flavored mass. So we will not discuss details here, and just show the desirable flavored mass term to reveal the index. It is given in the momentum space by

$$\bar{\Psi}\left(\mathbf{1}\otimes(\tau_3\otimes\tau_3\otimes\tau_3\otimes\tau_3)\right)\Psi = \left(\prod_{\mu=1}^{4}\cos p_\mu\right)\bar{\psi}\psi, \tag{22}$$

and in the position space by

$$M_{\rm P} = m_{\rm P}\sum_{sym.}\prod_{\mu=1}^{4} C_\mu. \tag{23}$$

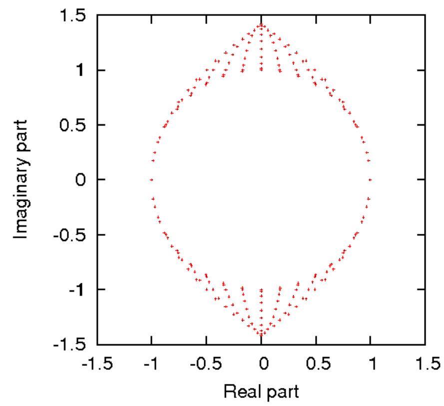

FIG. 2: Complex spectra of non-Hermitean Dirac operator $D_{\rm n} - M_{\tau_3\otimes\tau_3}$ for the $d = 2$ free field case on a 36×36 lattice with mass parameter $m = 1$.The spectrum is split into two doubled branches crossing the real axis at $|m|$.

Appendix A is devoted to details of the flavored mass terms in $d = 4$ naive fermions. In Fig. 8(a) for the eigenvalues of the Dirac operator we can verify the modes are split into two 8-degenerate branches, which means the flavored mass decomposes the 4d naive fermions into two pairs with negative and positive masses.

III. SPECTRAL FLOW AND THE INDEX THEOREM

In this section we obtain the integer index related with gauge field topology for the naive and minimally doubled fermions. As in the cases of Wilson [5] and staggered fermions [13, 14] we utilize the spectral flow of the Hermitean operators with the flavored mass terms introduced in Sec. II. Here the would-be zero modes of the Dirac operators are identified as low-lying crossings of the eigenvalue flow of the Hermitean operators.

Let us begin with explaining what the spectral flow is. In the continuum field theory the index is defined as the difference between the numbers of zero modes of the massless Dirac operator with positive and negative chirality, n_+ and n_-. The statement of the index theorem is that the index is just equal to a topological charge Q of a background gauge configuration up to a sign factor depending on its dimensionality,

$$n_+ - n_- = (-1)^{d/2} Q. \tag{24}$$

Here the question is how to obtain the index of the Dirac operator. We can of course calculate the zero-mode chiralities straightforwardly, but there is a useful way called *spectral flow*. To introduce it we first introduce a certain Hermitean version of the Dirac operator

$$H(m) = \gamma_5(D - m). \tag{25}$$

where any zero modes of the Dirac operator with $\pm$ chirality correspond to some eigenmodes of this Hermitean operator with eigenvalues $\lambda(m) = \mp m$. If we now consider the flow of the eigenvalues $\lambda(m)$ as the mass varies, those corresponding to zero modes will cross the origin with slopes ∓ 1 depending on their $\pm$ chirality. The non-zero eigenmodes of D, in contrast, occur in pairs which are mixed by H and cannot cross zero. Therefore the index of the Dirac operator is given by minus the spectral flow of the Hermitean operator, which stands for the net number of eigenvalues crossing the origin, counted with sign $\pm$ depending on the slope.

The index with lattice Wilson fermions [5] can similarly be obtained from the spectral flow, which in this case means the net number of eigenvalues crossing zero at low-lying values of m, counted with signs of the slopes. In the continuum limit, we are only interested in the crossings at small mass; the massive doublers also eventually cross zero, but only for large values of m. However for lattice fermions with massless species doublers the index cancels between pairs, so that an eigenvalue flow with a simple mass term does not properly capture gauge field topology. This difficulty has been recently resolved by introducing a proper flavored ("tasted") mass term in staggered fermions [13, 14]. The associated Hermitean operator is given by

$$H_{\mathrm{st}}(m) = \Gamma_{55}\left(D_{\mathrm{st}} - m\Gamma_{55}\Gamma_5\right) = \Gamma_{55}D_{\mathrm{st}} - m\Gamma_5, \tag{26}$$

where Γ_{55} and Γ_5 stand for $\Gamma_{55} = \gamma_5 \otimes \gamma_5$ and $\gamma_5 \otimes \mathbf{1}$ with discretization error $\mathcal{O}(a)$ in terms of a taste multiplet. Indeed it was shown that the index of the staggered fermion is given by minus the spectral flow of this Hermitean operator. And it was also pointed out that it illustrates correctly the gauge topology up to a integer factor from the number of tastes as $\mathrm{Index}(D_{\mathrm{st}}) = 2^{d/2}(-1)^{d/2}Q$. Here the spectral flow again means the net number of eigenvalues crossing zero at low-lying values of m, counted with signs of the slopes. Thus the theoretical foundation of the index theorem with staggered fermions is established without a renormalization depending on the gauge ensemble. Here the mass part in the Hermitean operator Eq. (26) is approximately proportional to flavor-singlet gamma-5. Note we constructed flavored mass terms for our cases in the previous section following this.

Now we can symbolically write a formula for the index as

$$\mathrm{Index}(D) = -\mathrm{Spectral\ \ flow}(H). \tag{27}$$

It is quite natural to consider whether this formula is also available to detect the index of minimally doubled and naive fermions with the flavored mass terms we proposed. We will from now show this spectral flow method can be also applied to these cases. The associated Hermitean operators for minimally doubled and naive fermions are given by

$$H_{\mathrm{md}}(m) = \gamma_5(D_{\mathrm{md}} - M_{\mathrm{md}}), \tag{28}$$

$$H_{\mathrm{n}}(m) = \gamma_5(D_{\mathrm{n}} - M_{\mathrm{n}}), \tag{29}$$

where the matrix γ_5 is regarded as a flavored one, $\gamma_5 \otimes \tau_3$ for minimally doubled fermions and $\gamma_5 \otimes (\tau_3 \otimes \tau_3)$ for two dimensional naive fermions in terms of the flavor multiplet as shown in Sec. II. The desirable flavored mass terms M_{md} and M_{n} for $d = 2$ have been already given in Eq. (8) and (18). For now we focus on the two dimensional case.

We then numerically calculate the eigenvalue flows of two dimensional minimally doubled and naive fermions. We consider background configurations proposed in [2] for the staggered case [13]: we start with a smooth $U(1)$ gauge field with topological charge Q,

$$U_{x,x+e_1} = e^{i\omega x_2}, \qquad U_{x,x+e_2} = \begin{cases} 1 & (x_2 = 1, 2, \cdots, L-1) \\ e^{i\omega L x_1} & (x_2 = L) \end{cases}, \tag{30}$$

where L is the lattice size and ω is the curvature given by $\omega = 2\pi Q$. Then, to emulate a typical gauge configuration of a practical simulation, we introduce disorder effects to link variables by random phase factors, $U_{x,y} \to e^{ir_{x,y}} U_{x,y}$, where $r_{x,y}$ is a random number uniformly distributed in $[-\delta\pi, \delta\pi]$. The parameter δ determines the magnitude of disorder.

Fig. 3(a) shows the eigenvalue flow of the minimally doubled Hermitean operator (28). It is calculated with a $Q = 1$ and $\delta = 0.25$ background configuration on a 16×16 lattice. There are two low-lying crossings around $m = 0$ with positive slopes, which correspond to would-be zero modes. With the formula (27), it means the index of the Dirac operator of the minimally doubled fermion in this case is -2. This result is consistent with the index theorem for the minimally doubled fermions given by

$$\mathrm{Index}(D_{\mathrm{md}}) = 2(-1)^{d/2} Q, \tag{31}$$

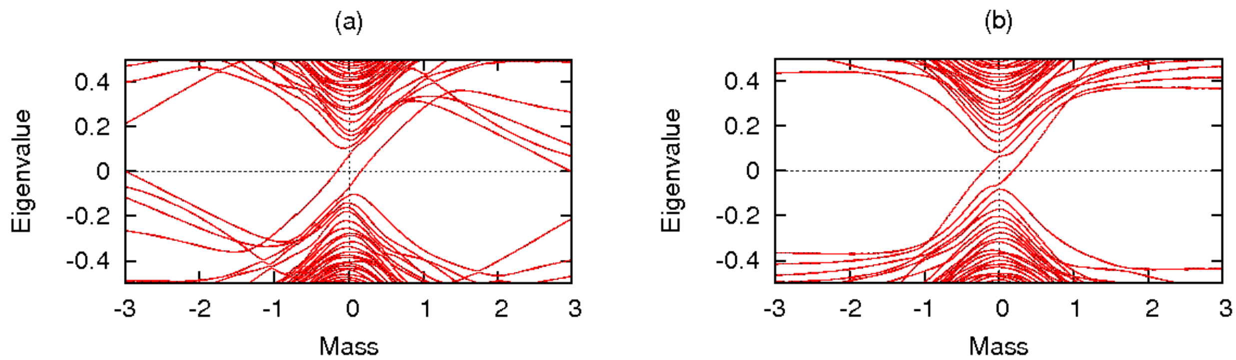

FIG. 3: Spectral flows of (a) Minimally doubled and (b) naive Hermitean operators with a $Q = 1$, $\delta = 0.25$ background configuration on a 16×16 lattice. Two single crossings with positive slopes are seen in (a), which means the index is -2. Two doubled crossings with positive slopes are seen in (b), which means the index is -4.

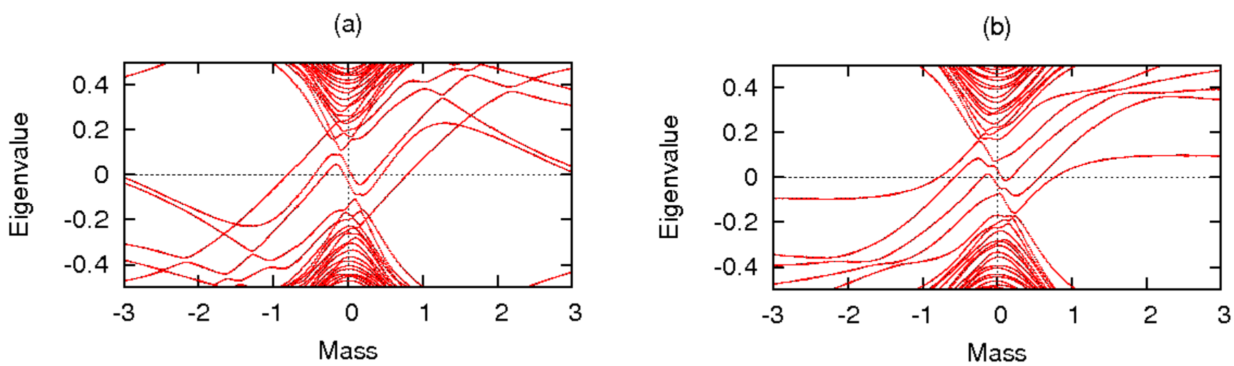

FIG. 4: Spectral flows of (a) Minimally doubled and (b) naive Hermitean operators with a $Q = 2$, $\delta = 0.2$ background configuration on a 16×16 lattice. Six single crossings with positive slopes and two single crossings with negative slopes are seen in (a), which means the index is -4. Six doubled crossings with positive slopes and two doubled crossings with negative slopes are seen in (b), which means the index is -8.

which contains a factor 2 reflecting two species. This relation is also satisfied by cases with other topological charges, as shown in Fig. 4(a) for the case for $Q = 2$. Here the net number of crossings counted with $\pm$ depending on the slopes is 4. It means the corresponding index is -4, which is consistent with (31). We also emphasize that there is a clear separation between low- and high-lying crossings in Fig. 3(a) where low-lying ones are localized about

$m = 0$ and high-lying ones are located at large $|m|$. Although further numerical study with realistic gauge configurations is required, it indicates the zero modes and the index tend to be robust against randomness of the gauge configuration. Now we have established the index theorem with minimally doubled fermions.

Next results for the naive fermion case are shown in Fig. 3(b). The calculation is done with the same background configuration as the minimally doubled case. Fig. 3(b) shows the eigenvalue flow of the naive Hermitean operator (29). There are two doubled crossings around the origin: Here we can verify they are doubled and there totally exist four crossings as shown in Fig. 7(a) by introducing other kinds of mass terms given in Sec. II. Again with the formula (27) we obtain the index of the Dirac operator of the naive fermion in this case is -4. This result satisfies the index theorem for the naive fermion given by

$$\mathrm{Index}(D_{\mathrm{n}}) = 2^d(-1)^{d/2}Q, \tag{32}$$

where it contains a factor 2^d reflecting 2^d species. This theorem is also satisfied by the cases with other topological charges, as shown in Fig. 4(b) for $Q = 2$. Here the spectral flow is 8 since all the crossings are doubled in this case. It means the corresponding index is -8, which is consistent with (32). There is also large separation between low-lying and high-lying crossings. It indicates the zero modes and the index tend to be robust as in the case of minimally doubled fermions. Now we have established the index theorem with the naive fermion.

We remark both of the Hermitean operators, (28) and (29), satisfy a relation of $\gamma_5 H(m)\gamma_5 = -H(-m)$, reflecting the fact that the original actions have an exact chiral symmetry while the Wilson fermion does not. We can see low-lying and high-lying crossings get closer as the randomness effect becomes larger in the both cases, but this original chiral symmetry actually improves the lower bound of $H^2(m)$ satisfying $H^2(m) = D^\dagger D + m^2$, which enhances the stability of the index against disorder. Thus overlap formalisms with originally chirally symmetric fermions would be more applicable to practical simulations as we will discuss later. In the end of this section, let us comment on $d = 4$ cases. We can perform the same argument for the 4-dimensional minimally doubled fermions and naive fermions. Especially in the case of the 4d naive fermion, we can show the integer index is obtained from the spectral flow and it correctly illustrates the topological charge in accordance with

Eq. (32) by introducing the flavored mass term

$$M_{\mathrm{P}} = m_{\mathrm{P}} \sum_{sym.} \prod_{\mu=1}^{4} C_{\mu}, \tag{33}$$

In Appendix A we will explain this mass term in detail and show the eigenvalues of the associated Dirac operator in Fig. 8(a).

IV. OVERLAP FORMALISM

In this section we discuss new versions of overlap fermions constructed from the naive and minimally doubled Dirac kernels with the flavored mass terms. The main result here is we obtain a single-flavor overlap fermion with hypercubic symmetry from the naive fermion kernel, which may be somewhat simpler than the Wilson overlap fermion.

Firstly we show the index of exact zero modes of the naive and minimally doubled overlap fermions also illustrate the topological charge correctly. We now introduce minimally doubled and naive versions of overlap Dirac operators,

$$D_{\mathrm{mo}} = 1 + \gamma_5 \frac{H_{\mathrm{md}}}{\sqrt{H_{\mathrm{md}}^2}}, \qquad D_{\mathrm{no}} = 1 + \gamma_5 \frac{H_{\mathrm{n}}}{\sqrt{H_{\mathrm{n}}^2}}, \tag{34}$$

where these Hermitean operators have been introduced in (28) and (29). These depend on the mass parameter, which should be chosen in an appropriate region. Then we can obtain the corresponding Ginsparg-Wilson relations

$$\{\gamma_5, D_{\mathrm{mo}}\} = D_{\mathrm{mo}}\gamma_5 D_{\mathrm{mo}}, \qquad \{\gamma_5, D_{\mathrm{no}}\} = D_{\mathrm{no}}\gamma_5 D_{\mathrm{no}}, \tag{35}$$

where, as we have discussed, the γ_5 here can be identified as a flavored one in terms of the flavor multiplet, for example $\gamma_5 \rightarrow \gamma_5 \otimes \tau_3$ for the minimally doubled fermion and $\gamma_5 \rightarrow \gamma_5 \otimes (\tau_3 \otimes \tau_3)$ for the $d = 2$ naive fermion. Note that only half of the original flavors(species) with negative mass are converted into physical massless modes in these overlap fermions while the others with positive mass become massive and decouple in the continuum limit. This is because the flavored mass terms we introduced assign negative mass to half of species and positive mass to the others. Thus there is only a single physical mode for the minimally doubled overlap fermion in any dimensions while there are $2^d/2$ physical modes in d-dimensional naive overlap fermions. Here the $2^d/2$ massless modes in the naive overlap fermion have the same chiral charge, or equivalently the same eigenvalue for the "flavor-chirality" matrix given by $\mathbf{1}\otimes(\tau_3\otimes\tau_3)$ in the $d=2$ case. As a consequence, the flavored Ginsparg-Wilson relations in (35) reduce to unflavored relations for massless modes as in the staggered case [14].

This reduction of flavored degrees also affects the index of the Dirac operators. We can obtain the indices of the minimally doubled and naive overlap Dirac operators from the Ginsparg-Wilson relations (35),

$$\mathrm{Index}(D_{\mathrm{mo}}) = -\frac{1}{2}\mathrm{Tr}\left(\frac{H_{\mathrm{md}}}{\sqrt{H_{\mathrm{md}}^2}}\right), \qquad \mathrm{Index}(D_{\mathrm{no}}) = -\frac{1}{2}\mathrm{Tr}\left(\frac{H_{\mathrm{n}}}{\sqrt{H_{\mathrm{n}}^2}}\right). \tag{36}$$

Now we can easily calculate these quantities analytically: The main part of the above equations is a sign function as $H/\sqrt{H^2}=\mathrm{sgn}(H)$. Thus the trace of this operator $\mathrm{Tr}\,(\mathrm{sgn}(H))$ gives the difference between the number of positive and negative eigenvalues at some value of the mass parameter. It is essential to fix the mass parameter between low- and high-lying crossings in the eigenvalue flows of H_{md} and H_{n}. Then the index of the above overlap Dirac operator becomes just a half of that of the original Dirac operator,

$$\mathrm{Index}\,(D_{\mathrm{mo}}) = \frac{1}{2}\mathrm{Index}\,(D_{\mathrm{md}})\,, \qquad \mathrm{Index}\,(D_{\mathrm{no}}) = \frac{1}{2}\mathrm{Index}\,(D_{\mathrm{n}})\,. \tag{37}$$

This relation relies on the property of the Hermitean operator $\gamma_5 H(m)\gamma_5 = -H(-m)$, as discussed in the end of Sec. III. This is also the case with the staggered overlap case.

As shown here, we can construct a single-flavor overlap fermion from minimally doubled fermions, which however lacks enough discrete symmetry for a simple continuum limit [18, 19]. On the other hand, the naive overlap fermion includes lots of flavors (8 for $d=4$) although it possess hypercubic symmetry. Now we will proceed to obtain a single-flavor overlap fermion from the naive fermion kernel by choosing a proper mass term. The key is to utilize the extra flavored mass terms given in Eqs. (19)(20).

For the purpose of constructing a single flavor naive overlap fermion with the hypercubic symmetry, we make use of the fact that only species(flavors) with negative mass lead to massless modes in the overlap formalism. It means if we introduce a flavored mass term to assign negative mass to one flavor and positive or zero mass to other flavors we can derive a single-flavor naive overlap fermion. For example, in the case of the 2d naive fermion with the original flavored mass (18), there are two degenerate species with negative mass. Now we need to lift this degeneracy by adding other kinds of flavored mass terms. From now we focus on the 2d case for a while. Now we have two possibilities of flavored mass terms to lift

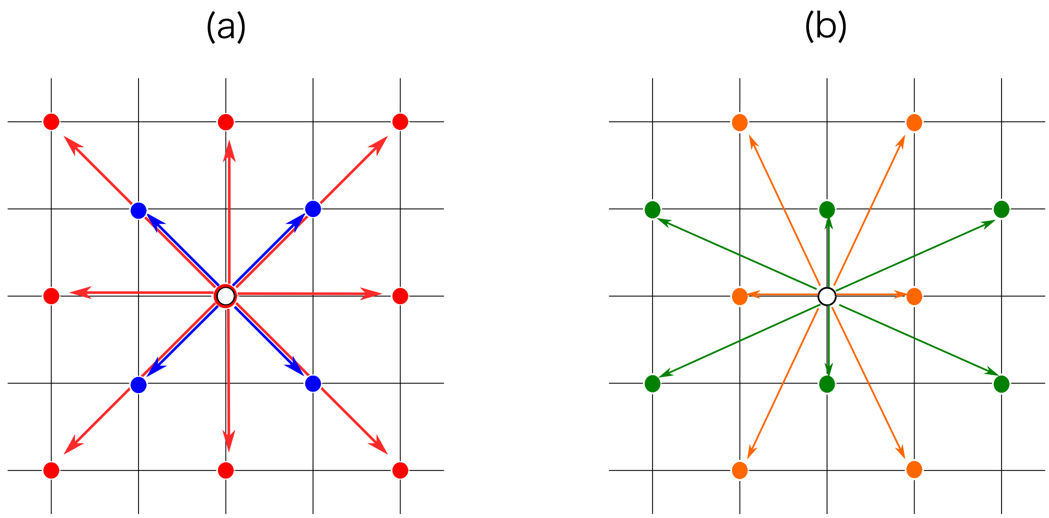

FIG. 5: (Color online) Hopping terms included in naive flavored mass terms: (a) isotropic ones, $M_{\tau_3\otimes\tau_3}$ (blue) and $M_{\mathbf{1}\otimes\mathbf{1}}$ (red), and (b) anisotropic ones, $M_{\tau_3\otimes\mathbf{1}}$ (orange) and $M_{\mathbf{1}\otimes\tau_3}$ (green).

the degeneracy

$$M_{\tau_3\otimes\mathbf{1}} = \frac{m_{\tau_3\otimes\mathbf{1}}}{2}\sum_{sym.}\left(1+C_1^2\right)C_2, \tag{38}$$

$$M_{\mathbf{1}\otimes\tau_3} = \frac{m_{\mathbf{1}\otimes\tau_3}}{2}\sum_{sym.}C_1\left(1+C_2^2\right), \tag{39}$$

which have been already shown in Eqs. (19)(20). Here we do not consider the mass term of $M_{\mathbf{1}\otimes\mathbf{1}}$ because it just gives an overall shift of the spectrum.

There is a criterion about how to add these mass terms to the original one $M_{\tau_3\otimes\tau_3}$ (18) with preserving the hypercubic symmetry. As shown in Fig. 5(b), the extra mass terms $M_{\mathbf{1}\otimes\tau_3}$and $M_{\tau_3\otimes\mathbf{1}}$ stand for anisotropic hoppings in position space. And the only way to hold hypercubic symmetry is add them with the same coefficient to the original one $M_{\tau_3\otimes\tau_3}$, which has isotropic hoppings as shown in Fig. 5(a). Thus the possible form of the flavored mass to give single negative mass to the naive fermion is given by

$$-M_{\mathrm{n}}(c) = -M_{\tau_3\otimes\tau_3} - c\left(M_{\tau_3\otimes\mathbf{1}} + M_{\mathbf{1}\otimes\tau_3}\right). \tag{40}$$

where c is an overall coefficient of the two extra mass terms and we take account of the convention $D-M$. Then we find the simplest case $c=1$ leads to single negative mass as following,

$$-M_{\mathrm{n}}(c=1) = \mathrm{diag}(-3,1,1,1) \equiv -\tilde{M}_{\mathrm{n}}. \tag{41}$$

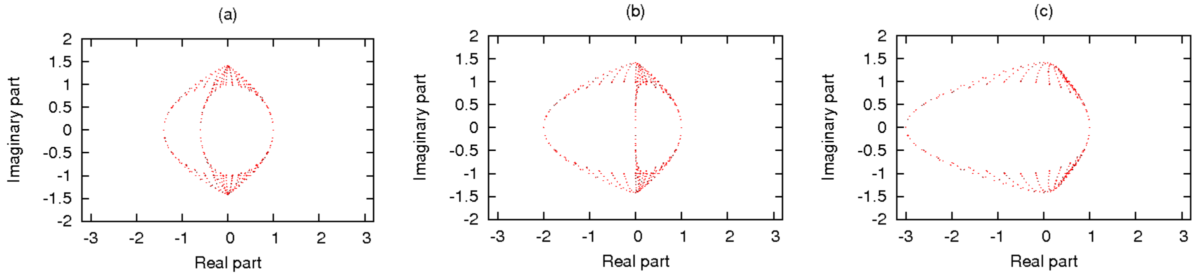

FIG. 6: Complex spectra of the naive Dirac operators for the $d = 2$ free field case on a 36×36 lattice: (a) $c = 0.2$, (b) $c = 0.5$ and (c) $c = 1$. In (a) the doubled negative branch is lifted and one of them goes to the positive direction. In (b) this branch enters a positive range. In (c) it coincides with the positive branch.

This assigns only one negative mass to the eigenvalues of the Dirac operator $D_{\rm n} - \tilde{M}_{\rm n}$. Figure 6(c) shows there is a single negative branch and a tripled positive branch. This means there is only one flavor with negative mass. We also depict figures for $c = 0.2$ in Fig. 6(a) and $c = 0.5$ in Fig. 6(b) to convince you that one of the branch is singled out and the others are tripled. It describes the situation that the doubled negative branch is split and one branch goes towards the the positive branch. Then for the case of $c \sim 0.5$ this branch enters a positive range. Thus it is clear that the negative branch with $c = 1$ is singled out and the positive one should be triply degenerate.

Now we find one example of the single-flavor naive overlap operator as following,

$$\tilde{D}_{\rm no} = 1 + \frac{D_{\rm n} - r\tilde{M}_{\rm n}}{\sqrt{(D_{\rm n} - r\tilde{M}_{\rm n})^\dagger (D_{\rm n} - r\tilde{M}_{\rm n})}}. \tag{42}$$

where we introduce an overall mass parameter r. Here we also study the spectral flow of the Hermitean operator with this mass term. Indeed it gives us a consistency check between the number of flavors and the index: Now we have a single flavor overlap fermion (42), thus the associated index should be exactly equal to minus the topological charge without a flavor factor in this dimension. The eigenvalue flow of the naive Hermitean operator with $c = 0.2$, $c = 0.5$ and $c = 1$ for $Q = 1$ are depicted in Fig. 7. It is obvious that the doubled flows are separated in the case of $c = 0.2$, and we find two of the four flows no longer cross zeros for $c = 0.5$. Thus only two crossings remain in the case of $c = 1$. As we have discussed in Eq. (36), the index of the overlap version is given by minus half difference of positive and

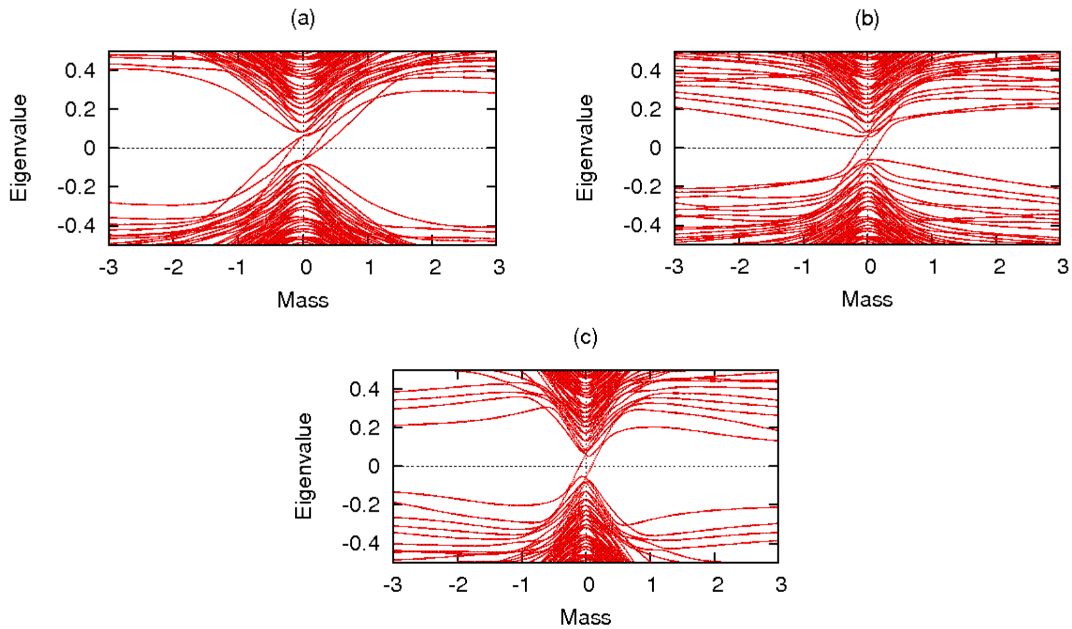

FIG. 7: Spectral flows of the $d = 2$ naive Hermitean operators with a $Q = 1$, $\delta = 0.25$ background configuration on a 16×16 lattice: (a) $c = 0.2$, (b) $c = 0.5$ and (c) $c = 1$. The doubled flows are lifted in (a). Two of the four flows no longer cross zeros in (b). The two single crossings are shown in (c).

negative eigenvalues of the Hermitean version of the original Dirac operator at some mass parameter. It means if you take an overall mass parameter r in Eq. (42) between low and high-lying crossings, the index of exact zero modes of the $d = 2$ naive overlap fermion (42) with $c = 1$ is given by -1 for $Q = 1$. It generally means

$$\mathrm{Index}\left(\tilde{D}_{\mathrm{no}}\right) = -Q, \tag{43}$$

which is a quarter of the original index of the 4-species naive fermion and there is no flavor factor. We can also check this formula for other topological charges. Thus we checked the consistency with the index. In the extension to general dimensions the theorem is given by

$$\mathrm{Index}\left(\tilde{D}_{\mathrm{no}}\right) = \frac{1}{2^d}\mathrm{Index}\left(D_{\mathrm{n}}\right) = (-1)^{d/2}Q. \tag{44}$$

Now we can also follow same approach in 4 dimensions, which is slightly more complicated. Here we do not show details but just present the flavored mass to give only one negative mass with preserving hypercubic symmetry,

$$M_{\mathrm{P}} + M_{\mathrm{T}} + M_{\mathrm{V}} + M_{\mathrm{A}}, \tag{45}$$

where the meaning of each mass term is shown in Appendix. A. In Fig. 8(c) the eigenvalues of the Dirac operator with this mass term are depicted. Here the negative branch is non-degenerate and the positive one is 15-fold degenerate. The associated overlap fermion produces only one massless mode. Thus the associated overlap form is a 4d single-flavored naive overlap fermion with hypercubic symmetry. You can also verify that this 4d version also satisfies the theorem in Eq. (44).

In the end of this section, let us comment on the symmetries and practical issues of this fermion. As long as the hopping symmetry of the mass terms is concerned in Fig. 5, hypercubic symmetry is maintained. Thus, unlike the minimally doubled case, we do not need to fine-tune additional parameters for the continuum limit. Regarding the numerical speed, it should be computationally comparable to the Wilson overlap. Furthermore the naive overlap fermion we proposed may have stability of the index against disorder due to the original chiral symmetry, although we should also take account of large fluctuations and renormalization effects in the realistic gauge configurations due to non-nearest hopping terms [26]. Now we have three examples for overlap varieties: Wilson, Staggered and Naive. Which is the best among them is an open question.

V. SUMMARY AND DISCUSSION

In this paper we have shown how the index theorem is realized in naive and minimally doubled fermions by considering the spectral flow of the Hermitean version of Dirac operators. The key is to make use of a point splitting for flavored mass terms. We also presented a new version of overlap fermions composed from the naive fermion kernel, which is single-flavored and maintains the hypercubic symmetry essential for a good continuum limit.

In Sec. II we introduce the point-splitting method to identify species in momentum space as flavors. By using this method, we succeed to define the proper flavored mass terms in minimally doubled and naive fermions. In Sec. III we study the spectral flow of the Hermitean operators in these fermions. Then it is shown that the spectral flow correctly illustrates the integer index determined by gauge field topology both in naive and minimally doubled fermions up to overall integer factors reflecting the number of species. In Sec. IV we study new versions of overlap fermions composed from these fermion kernels with flavored mass terms. These fermions satisfy Ginsparg-Wilson relations instead of usual chiral symmetry. We show the topological charge can be also obtained as the index of exact zero modes of the naive and minimally doubled overlap fermions. Then we show by choosing the flavored mass term so that only one of species has negative mass, the associated overlap fermion produces one massless fermionic mode. This single-flavor naive overlap fermion should possess the hypercubic symmetry and other properties possessed by the original naive fermion. Furthermore it may possess more stability for the index against disorder than Wilson overlap fermion since the naive fermion originally has chiral symmetry. However we also need to consider potential large fluctuations and renormalization effects in the realistic gauge configurations. Thus further investigation is required to show whether this type of overlap fermions has advantages for practical simulations over the Wilson overlap fermion.

In the end of this paper, let us discuss realistic possibility to use this "Naive Overlap fermion". Compared to Wilson overlap fermion, this one begins with only a simple kinetic term, or just a "naive action". Once you fix a flavored mass term such as the one we have shown, to simulate QCD with this fermion may be easier than with the Wilson overlap. On the other hand, the staggered overlap fermion is based on one-component fermionic filed per site while our formulation uses four-component fields. This means staggered overlap fermion is likely to be somewhat better than the naive overlap in terms of numerical expense. However, to fully answer to this question we need more detailed numerical research.

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Appendix A: Naive flavored mass in 4 dimension

In this appendix we show details of the point splitting and flavored mass terms for the $d = 4$ naive fermion. We introduce 16 point-split fields, corresponding to 16 species doublers of the $d = 4$ naive fermions,

$$\begin{aligned}
\psi_{(1)}(p - p_{(1)}) &= \frac{1}{2^4}(1 + \cos p_1)(1 + \cos p_3)(1 + \cos p_3)(1 + \cos p_4)\Gamma_{(1)}\psi(p), \\
\psi_{(2)}(p - p_{(2)}) &= \frac{1}{2^4}(1 - \cos p_1)(1 + \cos p_2)(1 + \cos p_3)(1 + \cos p_4)\Gamma_{(2)}\psi(p), \\
\psi_{(3)}(p - p_{(3)}) &= \frac{1}{2^4}(1 + \cos p_1)(1 - \cos p_2)(1 + \cos p_3)(1 + \cos p_4)\Gamma_{(3)}\psi(p), \\
&\vdots \\
\psi_{(16)}(p - p_{(16)}) &= \frac{1}{2^4}(1 - \cos p_1)(1 - \cos p_2)(1 - \cos p_3)(1 - \cos p_4)\Gamma_{(16)}\psi(p),
\end{aligned} \tag{A1}$$

where the positions of zeros in the momentum space, chiral charges and definitions of transformation matrices $\Gamma_{(i)}$ are listed in Table II. Here a set of gamma matrices $\gamma_\mu^{(i)}$ defined for each zero is given by this $\Gamma_{(i)}$ as $\Gamma_{(i)}^{-1}\gamma_\mu\Gamma_{(i)} = \gamma_\mu^{(i)}$. We classify these zeros depending on this $\Gamma_{(i)}$. For example we denote A: axial for the case of $\Gamma_{(2)} = i\gamma_1\gamma_5$ while we assign T: Tensor for $\Gamma_{(4)} = i\gamma_1\gamma_2$. We introduce a flavor multiplet field with 16 components as

$$\Psi(p) = \begin{pmatrix} \psi_{(1)}(p - p_{(1)}) \\ \psi_{(2)}(p - p_{(2)}) \\ \vdots \\ \psi_{(16)}(p - p_{(16)}) \end{pmatrix}. \tag{A2}$$

label	position	χ charge	Γ	type
1	$(0,0,0,0)$	$+$	$\mathbf{1}$	S
2	$(\pi,0,0,0)$	$-$	$i\gamma_1\gamma_5$	A
3	$(0,\pi,0,0)$	$-$	$i\gamma_2\gamma_5$	A
4	$(\pi,\pi,0,0)$	$+$	$i\gamma_1\gamma_2$	T
5	$(0,0,\pi,0)$	$-$	$i\gamma_3\gamma_5$	A
6	$(\pi,0,\pi,0)$	$+$	$i\gamma_1\gamma_3$	T
7	$(0,\pi,\pi,0)$	$+$	$i\gamma_2\gamma_3$	T
8	$(\pi,\pi,\pi,0)$	$-$	γ_4	V
9	$(0,0,0,\pi)$	$-$	$i\gamma_4\gamma_5$	A
10	$(\pi,0,0,\pi)$	$+$	$i\gamma_1\gamma_4$	T
11	$(0,\pi,0,\pi)$	$+$	$i\gamma_2\gamma_4$	T
12	$(\pi,\pi,0,\pi)$	$-$	γ_3	V
13	$(0,0,\pi,\pi)$	$+$	$i\gamma_3\gamma_4$	T
14	$(\pi,0,\pi,\pi)$	$-$	γ_2	V
15	$(0,\pi,\pi,\pi)$	$-$	γ_1	V
16	(π,π,π,π)	$+$	γ_5	P

TABLE II: Positions of zeros, chiral charges and definitions of transformation matrices for the $d = 4$ naive fermions. Letters of S, V, T, V and P stand for Scalar, Vector, Tensor, Axial-vector and Pseudo-scalar, respectively.

Then 16 flavored mass terms are given by

$$
\begin{aligned}
\mathrm{S}\ :\quad \bar{\Psi}\left(\mathbf{1}\otimes(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1})\right)\Psi &= \frac{1}{2^4}\left(\prod_{\mu=1}^{4}\left(1+\cos^2 p_\mu\right)\right)\bar{\psi}\psi \\
\mathrm{V}\ :\quad \bar{\Psi}\left(\mathbf{1}\otimes(\tau_3\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1})\right)\Psi &= \frac{1}{2^3}\cos p_1\left(\prod_{\mu=2}^{4}\left(1+\cos^2 p_\mu\right)\right)\bar{\psi}\psi \\
\mathrm{T}\ :\quad \bar{\Psi}\left(\mathbf{1}\otimes(\tau_3\otimes\tau_3\otimes\mathbf{1}\otimes\mathbf{1})\right)\Psi &= \frac{1}{2^2}\cos p_1\cos p_2\left(\prod_{\mu=3}^{4}\left(1+\cos^2 p_\mu\right)\right)\bar{\psi}\psi \qquad \text{etc,} \\
\mathrm{A}\ :\quad \bar{\Psi}\left(\mathbf{1}\otimes(\mathbf{1}\otimes\tau_3\otimes\tau_3\otimes\tau_3)\right)\Psi &= \frac{1}{2}(1+\cos^2 p_1)\left(\prod_{\mu=2}^{4}\cos p_\mu\right)\bar{\psi}\psi \\
\mathrm{P}\ :\quad \bar{\Psi}\left(\mathbf{1}\otimes(\tau_3\otimes\tau_3\otimes\tau_3\otimes\tau_3)\right)\Psi &= \left(\prod_{\mu=1}^{4}\cos p_\mu\right)\bar{\psi}\psi
\end{aligned}
$$

where we introduce a multiplet representation as four direct products of the Pauli matrix to express the 16 flavor structure. In this representation the chiral transformation matrix γ_5 is converted to $\gamma_5\otimes(\tau_3\otimes\tau_3\otimes\tau_3\otimes\tau_3)$, under which the naive action is invariant. Here the V-type includes 4 varieties while the T-type has 6 varieties and the A-type 4 varieties, giving then the total number as 16. Each of these varieties breaks the hypercubic symmetry of the lattice since their hoppings are anisotropic. Thus we need to take a proper combination of them in order to construct a flavored mass term with hypercubic symmetry. For example we consider the sum of the 4 varieties with the same ratios for the V-type as in the case of $d=2$. It is also the case with the T- and A-types. Thus the 5 types of flavored masses with the symmetry in terms of the original fermion field are given by

$$M_{\mathrm{S}} = \frac{m_{\mathrm{S}}}{2^4}\sum_{sym.}\prod_{\mu=1}^{4}\left(1+C_\mu^2\right), \tag{A3}$$

$$M_{\mathrm{V}} = \frac{m_{\mathrm{V}}}{2^3}\sum_{perm.}\sum_{sym.}C_\mu\prod_{\nu(\neq\mu)}\left(1+C_\nu^2\right), \tag{A4}$$

$$M_{\mathrm{T}} = \frac{m_{\mathrm{T}}}{2^2}\sum_{perm.}\sum_{sym.}C_\mu C_\nu\left(1+C_\rho^2\right)\left(1+C_\sigma^2\right), \tag{A5}$$

$$M_{\mathrm{A}} = \frac{m_{\mathrm{A}}}{2}\sum_{perm.}\sum_{sym.}\left(1+C_\mu^2\right)\prod_{\nu(\neq\mu)}C_\nu, \tag{A6}$$

$$M_{\mathrm{P}} = m_{\mathrm{P}}\sum_{sym.}\prod_{\mu=1}^{4}C_\mu, \tag{A7}$$

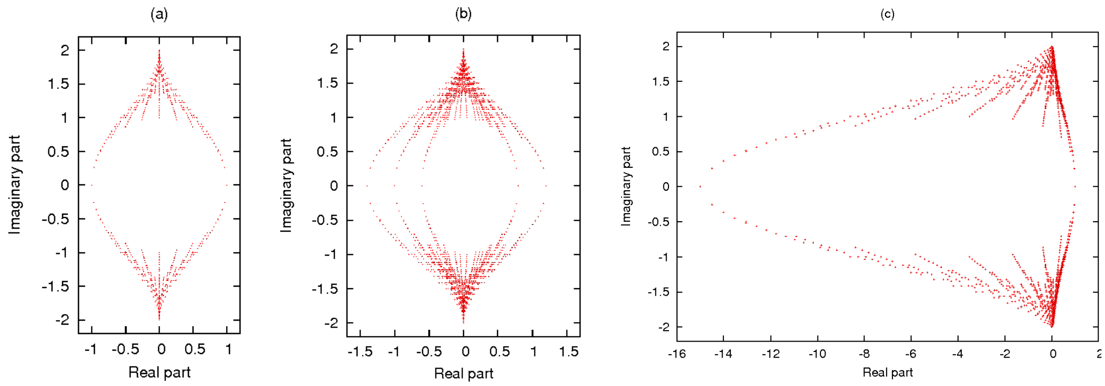

FIG. 8: Complex spectra of non-Hermitean Dirac operators for the $d = 4$ free field case in momentum space with 16^4 grids of the brillouin zone. (a) $D_{\mathrm{n}} - M_{\mathrm{P}}$. (b) $D_{\mathrm{n}} - (M_{\mathrm{P}} + 0.1 M_{\mathrm{A}})$. (c) $D_{\mathrm{n}} - (M_{\mathrm{P}} + M_{\mathrm{V}} + M_{\mathrm{T}} + M_{\mathrm{A}})$.

where $\sum_{perm.}$ means summation over permutations of the space-time indices.

Now we derive the flavored mass terms required to detect the index from the spectral flow of the Hermitean operator. As in the $d = 2$ case, it should be constructed so that the associated Hermitean operator has a flavor-singlet mass part as $\gamma_5 M \sim \gamma_5 \otimes (\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1})$. Such a mass term is just the P-type mass (A7). Thus the flavored mass term for the Hermitean operator is given by

$$M_{\mathrm{P}} = m_{\mathrm{P}} \sum_{sym.} \prod_{\mu=1}^{4} C_\mu. \tag{A8}$$

With the Hermitean operator $H_{\mathrm{n}} = \gamma_5 (D_{\mathrm{n}} - M_{\mathrm{P}})$, we reveal the index theorem with the naive fermion as in the $d = 2$ case. Here we only show the figure for eigenvalues of the free Dirac operator $D_{\mathrm{n}} - M_{\mathrm{P}}$ in Fig. 8(a). The mass term splits the modes into two branches, which are 8 fold degenerate. If we introduce other types of mass terms, the degeneracy is lifted as seen in Fig. 8(b).

Next we show the flavored mass term to yield a single-flavor naive overlap fermion in 4d. As in the case of 2d there are some possibilities to realize it. The simplest example of the mass term to yield a single-flavor naive overlap fermion with hypercubic symmetry is given by

$$M_{\mathrm{P}} + M_{\mathrm{V}} + M_{\mathrm{T}} + M_{\mathrm{A}}. \tag{A9}$$

The eigenvalues of the Dirac operator with this mass term is depicted in Fig. 8(c). Here the negative branch is single and the positive one is 15 fold degenerate. It means the associated overlap fermion produces only one massless mode. Furthermore, as seen from the construction, it should possess hypercubic symmetry. Thus it could be a useful fermion action for lattice QCD simulation.

We now comment on some connections with other works. The P-type mass term actually corresponds to the modified gamma matrix $\Gamma_5 = (\prod_{\mu=1}^{4} \cos p_\mu)\gamma_5$, which is introduced in Ref. [25] to study the effects of topological charge with the 4d naive fermion. In some sense we have given its theoretical foundation by the point-splitting method. It is also similar to $\Gamma_5 \sim \gamma_5 \otimes \mathbf{1}$ of the staggered fermions [23]. Regarding the T-type masses, they are similar to the flavored mass terms for the single-flavor staggered overlap fermion proposed in [15]. These similarities are reasonable since the staggered fermions can be obtained from the naive fermions [11].

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