Vortices on Orbifolds

TARO KIMURA[*1,2] and MUNETO NITTA[†3]

[1]*Department of Basic Science, University of Tokyo, Tokyo 153-8902, Japan*

[2]*Mathematical Physics Lab., RIKEN Nishina Center, Saitama 351-0198, Japan*

[3]*Department of Physics, and Research and Education Center for Natural Sciences,*
Keio University, Hiyoshi 4-4-1, Kanagawa 223-8521, Japan

Abstract

The Abelian and non-Abelian vortices on orbifolds are investigated based on the moduli matrix approach, which is a powerful method to deal with the BPS equation. The moduli space and the vortex collision are discussed through the moduli matrix as well as the regular space. It is also shown that a quiver structure is found in the Kähler quotient, and *a half of ADHM* is obtained for the vortex theory on the orbifolds as the case before orbifolding.

[*]E-mail address: `kimura@dice.c.u-tokyo.ac.jp`

[†]E-mail address: `nitta@phys-h.keio.ac.jp`

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Contents

1 Introduction

Vortices in Abelian gauge theory are essential degrees of freedom in superconductors under the magnetic fields [1] while they also provide a model of strings in relativistic field theories [2]. They are refered as Abrikosov-Nielsen-Olesen (ANO) vortices. In the case of the critical coupling between type I and II superconductors they saturate Bogomol'nyi bound and become Bogomol'nyi-Prasad-Sommerfield (BPS) states [3]. While in this case they can be embedded into supersymmetric theories and preserve a half of supersymmetry [4] on one hand, the whole solutions admit integration constants (moduli parameters, or collective coordinates) which constitute the moduli space on the other hand. A merit of the latter is that low-energy dynamics of vortices can be described by geodesics on the moduli space, see [5] as a review.

Since the discovery of non-Abelian vortices [6, 7], various aspects of vortices have been investigated from both of string theory and gauge field theory perspectives. One peculiar property is that a non-Abelian vortex in $U(N)$ gauge theory admits a $\mathbb{C}\mathbf{P}^{N-1}$ moduli space which corresponds to Nambu-Goldstone modes in the internal space as for Yang-Mills instantons. Non-Abelian vortices provide a connection between BPS spectra of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories and two dimensional $\mathcal{N} = (2, 2)$ $\mathbb{C}\mathbf{P}^{N-1}$ model [8, 9]. While 't Hooft-Polyakov monopoles become kinks inside a non-Abelian vortex [10], Yang-

Mills instantons become sigma model instantons there [9, 11]. As for the moduli space perspective, by applying a useful treatment of BPS equations to the non-Abelian vortex [12], several properties on the moduli space and low-energy dynamics have been studied in detail [13, 14, 15, 16, 17, 18]. Recently further developments, for example, vortex counting [19, 20, 21, 22], volume of the moduli space [23] and so on, have been performed by applying the localization formula as well as the four dimensional gauge theory. They play an important role in determining the low energy behavior of the supersymmetric gauge theory.

Other than the flat space $\mathbb{R}^2 \simeq \mathbb{C}$, studies of non-Abelian vortices have been so far restricted to those on regular spaces, such as a cylinder [24], a torus [25, 26], Riemann surfaces [27, 28, 29, 30], and hyperbolic surfaces [31]. (For Abelian vortices on various geometry, see [5] as a review, after which there have been developments in those on Riemann surfaces [32] and hyperbolic surfaces [33, 34].)

In this paper we consider Abelian and non-Abelian vortices on two dimensional singular spaces, namely the orbifolds $\mathbb{C}/\mathbb{Z}_n$. Although the four dimensional gauge theory on the singular space, or the smooth space given by resolving its singularity, which is called the asymptotically locally Euclidean (ALE) space, has been investigated in detail, various studies on the vortex theory are mainly based on the regular space as denoted above. Indeed, in the case of the four dimensional theory, the ALE space has a hyper-Kähler metric [35], so that almost the same procedure to construct instantons on such a space as the usual Euclidean space $\mathbb{R}^4$, or the four sphere S^4, can be performed [36, 37]. Furthermore the instanton counting and its matrix model description have been discussed for the ALE space [38, 39]. The recent development on the four dimensional theory, which is called the AGT relation [40], is also studied for this case [41, 42, 43, 44, 45]. Therefore it is natural to expect that study of vortices on orbifolds would give a novel perspective to the vortex theory.

To deal with the orbifold theory we first apply the moduli matrix approach [46, 12, 13] rather than Hanany-Tong's method. It is because, for the former one, we can easily see the space-time structure of the vortex configuration and the dynamics of vortices [16, 18] while it is slightly difficult to see it for the latter one, which is based on the dual configuration. We first consider the fields on the usual regular space $\mathbb{R}^2 \simeq \mathbb{C}$, namely the universal cover of the orbifold, and then take the orbifold projection to obtain the $\mathbb{Z}_n$ invariant sector of the configuration. Because of the singular property of the space, we have to assign the boundary condition, which breaks the original gauge symmetry. This symmetry breaking affects the symmetry of the moduli space, and leads to its decomposition.[1] We find that k-vortices on $\mathbb{C}/\mathbb{Z}_n$ with $k < n$ are fixed at the orbifold singularity, as Yang-Mills instantons [36, 37] and fractional D-branes [47] on the orbifolds or their resolution to ALE spaces. We call them

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[1] We remark that a similar but different situation is found when the twisted mass term is introduced [10]. The gauge symmetry is dynamically broken due to the twisted mass, and then interesting topological excitations such as confined monopoles [10] and vortices stretched between domain walls [46] can be observed.

fractional vortices borrowing the terminology of D-branes.[2] If we combine n vortices properly they can be free from the singularity. We show that mirror images have the same internal moduli $\mathbb{C}\mathbf{P}^{N-1}$ for $U(N)$ gauge theory. We also find that ANO vortices at the singularity do not give any moduli, while fractional non-Abelian vortices at the singularity give internal moduli.

We then discuss the vortex collision for the orbifold theory. Since elements of the moduli matrix are directly related to smooth coordinates of the moduli space, we can follow vortex dynamics along geodesics on the moduli space. Especially when we concentrate on a short time behavior at collision moment, the geodesics can be approximated by straight lines in smooth coordinates and therefore we do not need the actual metric [16]. In this paper we concentrate on the collision at the origin of the complex plane, which corresponds to the singular point of the orbifold, because other points are essentially the same as the usual regular space. Due to the singularity of the orbifold, we can see unusual scattering behavior in this case.

Finally we also comment on Hanany-Tong's approach, or the Kähler quotient for the orbifold theory. Starting with the Kähler quotient for vortices on the flat space $\mathbb{C}$, we implement the orbifold projection as well as the moduli matrix approach. We can perform a quite similar procedure as the instantons on the ALE space, and then obtain *a half of ADHM on the ALE spaces of the A_{n-1} type* as discussed in the regular case.

This paper is organized as follows. In section 2 we first discuss how to deal with the orbifold theory through the boundary condition. Section 3 is the main part of this paper, which is devoted to the moduli matrix approach. We investigate both of Abelian and non-Abelian vortices, and extract the moduli space for such a vortex configuration. We show how to implement the orbifold projection to the moduli matrix. We consider the vortex collision in section 4. The vortex dynamics is obtained by studying the geodesics on the moduli space. In section 5 we also discuss the Kähler quotient for the orbifold theory, according to Hanany-Tong's approach. Stressing the similarity between vortices and instantons on the orbifold, we propose a generic form of the moduli space of the vortices on the orbifold. We finally summarize the results and comment on some applications in section 6.

2   Orbifolding and boundary conditions

An orbifold $\mathbb{C}/\mathbb{Z}_n$ is constructed by identifying $z \sim \omega z$, where z is a coordinate of the covering space $\mathbb{C}$ and $\omega = \exp(2\pi i/n)$ is the primitive n-th root of unity, as shown in Fig. 1. For the Abelian case the following boundary condition on a Higgs scalar field $H(z, \bar{z})$ is allowed

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[2] Rather different types of fractional vortices appear in different settings such as gauge theories which flow in infrared to nonlinear sigma models with deformed [48, 49] or singular [49, 50] target spaces.
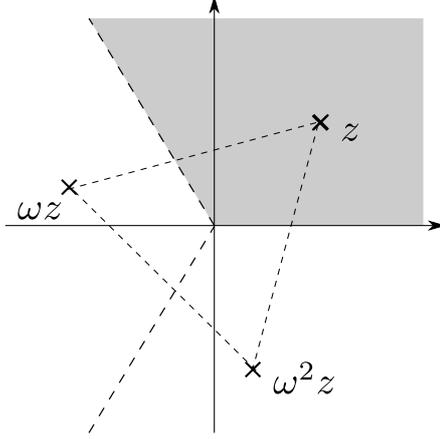
Figure 1: Fundamental region of orbifold $\mathbb{C}/\mathbb{Z}_n$ for $n = 3$. The orbifolding theory is obtained by identifying $z \sim \omega z$ in the universal cover $\mathbb{C}$. $\omega = \exp(2\pi i/3)$ is the primitive third root of unity. The origin is the singular point.

under this identification,

$$H(z, \bar{z}) \to H(\omega z, \overline{\omega z}) = e^{i\alpha} H(z, \bar{z}). \tag{2.1}$$

Due to the consistency condition for single valuedness of the Higgs field,

$$H(z, \bar{z}) = H(\omega^n z, \overline{\omega^n z}) = e^{in\alpha} H(z, \bar{z}), \tag{2.2}$$

the phase factor α has to be quantized as

$$\alpha = \frac{2\pi}{n} m, \qquad m = 0, \cdots, n - 1. \tag{2.3}$$

Therefore the boundary condition (2.1) is characterized by an integer $0 \leq m \leq n - 1$, and can be rewritten as

$$H(\omega z, \overline{\omega z}) = \omega^m H(z, \bar{z}). \tag{2.4}$$

This phase factor suggests that the singularity of the spacetime assigns a flux on the singular point via the boundary condition.

We then generalize this result to the non-Abelian theory. More precisely we introduce $U(N)$ gauge field and N Higgs scalar fields in the fundamental representation, summarized as an $N \times N$ matrix $H(z, \bar{z})$ on which gauge symmetry acting from the left. In this case the Higgs and gauge field should be transformed as

$$H(\omega z, \overline{\omega z}) = \Omega H(z, \bar{z}), \qquad A_z(\omega z, \overline{\omega z}) = \Omega A_z(z, \bar{z}) \Omega^{-1}, \tag{2.5}$$

where the orbifold transformation matrix Ω, satisfying $\Omega^n = \mathbb{1}$, can be diagonalized without loss of generality due to the gauge symmetry,

$$\Omega = \mathrm{diag}(\omega^{m_1}, \cdots, \omega^{m_N}). \tag{2.6}$$

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A set of integers $(m_1, \cdots, m_N) \in \{0, \cdots, n-1\}^N$ characterizes the boundary condition of the Higgs field. Changing the order of the diagonal component, it can be written as

$$
\Omega = \begin{pmatrix} \omega^0 \mathbb{1}_{N^{(0)}} & & & 0 \\ & \omega^1 \mathbb{1}_{N^{(1)}} & & \\ & & \ddots & \\ 0 & & & \omega^{n-1} \mathbb{1}_{N^{(n-1)}} \end{pmatrix}. \tag{2.7}
$$

This implies that (the global part of) the original gauge group $U(N)$ is broken by the boundary condition as

$$
U(N) \longrightarrow U(N^{(0)}) \times \cdots \times U(N^{(n-1)}) \tag{2.8}
$$

where the rank of each gauge group is given by $N^{(m)} = \# \{ m_i = m, i = 1, \cdots, N \}$, satisfying

$$
N^{(0)} + \cdots + N^{(n-1)} = N. \tag{2.9}
$$

3   Moduli matrix approach

The model we study in this paper is the following,

$$
\mathcal{L} = \mathrm{Tr} \left[-\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} - \mathcal{D}_\mu H \mathcal{D}^\mu H^\dagger - \lambda (c\mathbb{1}_N - HH^\dagger)^2 \right]. \tag{3.1}
$$

Especially we focus on the critical coupling case $\lambda = g^2/4$. In this case, the model can be endowed with supersymmetry by adding suitable bosonic fields and fermionic superpartners at least if we consider on the flat space. In supersymmetric context the parameter c is called the Fayet-Illiopoulos parameter. However supersymmetry is broken in orbifold theory [52] and so we do not discuss supersymmetry in this paper. Note that BPS properties remain at least semi-clasically even without supersymmetry.

Then we can obtain the BPS equation from this Lagrangian,

$$
(D_1 + iD_2)H = 0, \qquad F_{12} + \frac{g^2}{2}(c\mathbb{1}_N - HH^\dagger) = 0. \tag{3.2}
$$

We then introduce the moduli matrix approach to solve this BPS equation. The first equations can be solved as [46, 12, 13]

$$
H(z, \bar{z}) = S^{-1}(z, \bar{z}) H_0(z), \qquad A_z = A_1 + iA_2 = S^{-1}(z, \bar{z}) \partial_z S(z, \bar{z}), \tag{3.3}
$$

where the holomorphic matrix $H_0(z)$ is called the moduli matrix. The rank of this matrix is N for $U(N)$ gauge theory, thus it becomes just a holomorphic function for the Abelian case. Then the second equation can be recast into a gauge invariant equation, which has a unique solution [53]. Therefore all moduli parameters are contained in the moduli matrix $H_0(z)$ up to the following transformation; This construction is invariant under

$$
(H_0(z), \ S(z, \bar{z})) \longrightarrow (V(z)H_0(z), \ V(z)S(z, \bar{z})) \tag{3.4}
$$

with $V(z) \in GL(N, \mathbb{C})$ being holomorphic with respect to z. This is called V-transformation and plays an important role on classifying the moduli space.

Applying the boundary condition (2.5) the moduli matrix behaves as

$$H_0(\omega z) = \Omega H_0(z), \tag{3.5}$$

since we have $\Omega H = \Omega S^{-1}(\Omega^{-1}\Omega)H_0$. Especially the boundary condition for the moduli matrix plays an important role on studying the moduli space of vortices on the orbifold.

Since the energy of this configuration is given by

$$T = \frac{2\pi ck}{n} = -i\frac{c}{2} \oint dz \ \partial_z \log \det H_0(z), \tag{3.6}$$

where the integral is performed only on the fundamental region, the moduli matrix for k-vortex solution is written as

$$\det H_0(z) = \prod_{i=1}^{k}(z - z_i). \tag{3.7}$$

Here k is the number of vortices on the whole complex plane $\mathbb{C}$ and z_i parametrizes the position of the vortex. We remark the winding number k/n is different from the vortex number k. The former one can be fractional while the latter one is always integral.

3.1 Abelian case

We start with the simplest case $k = 1$. The moduli matrix, which is just a holomorphic function in this case, can be generally written as

$$H_0(z) = \prod_{i=1}^{k}(z - z_i). \tag{3.8}$$

The parameter z_i is regarded as a position of the i-th vortex. For $k = 1$, applying the boundary condition (3.3), this function has to satisfy

$$\omega z - z_1 = \omega^m(z - z_1). \tag{3.9}$$

This condition is satisfied only if $m = 1$ and $z_1 = 0$. This means that the vortex is fixed at the singular point of $\mathbb{C}/\mathbb{Z}_n$, and the boundary condition is automatically determined as $H_0(\omega z) = \omega H_0(z)$.

We can easily generalize this result for $k < n$. The boundary condition (2.4) reduces the moduli matrix (3.8) to a trivial form

$$H_{0;n}(z) = z^k, \tag{3.10}$$

satisfying $H_0(\omega z) = \omega^k H_0(z)$. As the case of $k = 1$, positions of vortices are fixed at the origin, and thus the position moduli does not exist for $k < n$. This means the fractional vortices are fixed at the origin.

We then study the case of $k = n$. In this case the moduli matrix (3.8) is reduced to

$$H_{0;n}(z) = z^n - z_1^n = \prod_{p=0}^{n-1}(z - \omega^p z_1), \tag{3.11}$$

which satisfies $H_0(\omega z) = H_0(z)$. This configuration has only one position modulus z_1 (see Fig. 1).

Finally we provide the moduli matrix for generic vortex number $k = ln + m \equiv m \pmod{n}$,

$$H_{0;n}(z) = z^m \prod_{i=1}^{l}(z^n - z_i^n) = z^m \prod_{i=1}^{l}\prod_{p=0}^{n-1}(z - \omega^p z_i). \tag{3.12}$$

We can see that this solution satisfies the boundary condition $H_0(\omega z) = \omega^m H_0(z)$. Since there is only position, but internal moduli, the moduli space for Abelian vortices on the orbifold is given by

$$\mathcal{M}_{N=1,k;n} = (\mathbb{C}/\mathbb{Z}_n)^{[k/n]} / \mathfrak{S}_{[k/n]}. \tag{3.13}$$

Here $[x]$ denotes the largest integer not greater than x.

We have found that k fractional vortices with $k < n$ are fixed at the orbifold singularity of $\mathbb{C}/\mathbb{Z}_n$, but once n vortices are combined together a set of them can be free from the singularity. The same property can be found in Yang-Mills instantons [36, 37] and fractional D-branes [47] on orbifolds .

3.2 $U(2)$ gauge theory

3.2.1 $k = 1$ vortices

We now consider the simplest non-Abelian gauge group $U(2)$ and the orbifold $\mathbb{C}/\mathbb{Z}_2$ for convenience. The moduli matrix for $k = 1$ vortex in $U(2)$ gauge theory on $\mathbb{C}$ is given by [11, 15]

$$H_0^{(1,0)}(z) = \begin{pmatrix} z - z_1 & 0 \\ -b' & 1 \end{pmatrix}, \qquad H_0^{(0,1)}(z) = \begin{pmatrix} 1 & -b \\ 0 & z - z_1 \end{pmatrix}. \tag{3.14}$$

The corresponding orbifold transformation matrix can be simply determined by reading the diagonal components of the moduli matrix as

$$\Omega^{(1,0)} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}, \qquad \Omega^{(0,1)} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.15}$$

On the other hand, the moduli matrices (3.14) themselves do not satisfy the boundary condition in a consistent way because we have different expressions as follows,

$$H_0^{(1,0)}(-z) = \begin{pmatrix} -z - z_1 & 0 \\ -b' & 1 \end{pmatrix}, \qquad \Omega^{(1,0)} H_0^{(1,0)}(z) = \begin{pmatrix} -z + z_1 & 0 \\ -b' & 1 \end{pmatrix}. \tag{3.16}$$

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By equating them, $H_0^{(1,0)}(-z) = \Omega^{(1,0)} H_0^{(1,0)}(z)$, we have to assign $z_1 = 0$ to satisfy the boundary condition consistently. We obtain the same condition from the other one $H_0^{(0,1)}(z)$. Thus under this orbifold projection the moduli matrix for $\mathbb{C}/\mathbb{Z}_2$ theory becomes

$$H_{0;n=2}^{(1,0)}(z) = \begin{pmatrix} z & 0 \\ -b' & 1 \end{pmatrix}, \qquad H_{0,n=2}^{(0,1)}(z) = \begin{pmatrix} 1 & -b \\ 0 & z \end{pmatrix}. \tag{3.17}$$

We remark since b and b' are related as $b = 1/b'$ via the V-transformation, which is regarded as the complexified gauge transformation [15], there is only one parameter b interpreted as the internal moduli. Therefore $k = 1$ fractional vortex is fixed at the singular point, and still has the internal degree of freedom, which turns out to be a coordinate of $\mathbb{C}\mathbf{P}^1$, as the usual non-Abelian vortex. Indeed we can extract the orientation of the vortex via the following equations [13],

$$H_{0;n=2}^{(1,0)}(z=0)\vec{\phi}^{(1,0)} = 0, \tag{3.18}$$

$$H_{0;n=2}^{(0,1)}(z=0)\vec{\phi}^{(0,1)} = 0. \tag{3.19}$$

Thus we have the orientational vectors

$$\vec{\phi}^{(1,0)} = \begin{pmatrix} 1 \\ b' \end{pmatrix}, \qquad \vec{\phi}^{(0,1)} = \begin{pmatrix} b \\ 1 \end{pmatrix}. \tag{3.20}$$

These are nothing but two patches of $\mathbb{C}\mathbf{P}^1$. Thus the moduli space for $k = 1$ vortex solution in $U(2)$ gauge theory on the orbifold $\mathbb{C}/\mathbb{Z}_2$ is given by

$$\mathcal{M}_{N=2,k=1;n=2}^{\square} \simeq \{0\} \times \mathbb{C}\mathbf{P}^1. \tag{3.21}$$

Here we introduce a useful notation to characterize the moduli matrix by using the Young diagram as discussed in [51]: N entries of the partition correspond to the powers of the diagonal components of the moduli matrix for $U(N)$ gauge theory. We can always obtain a descending ordered matrix by the V-transformation. We again remark that there is no position moduli $\mathbb{C}/\mathbb{Z}_2$. Unlike the Abelian case it still has internal moduli $\mathbb{C}\mathbf{P}^1$, as in Yang-Mills instantons on orbifolds or their resolutions to ALE spaces [36, 37].

3.2.2 $k = 2$ vortices

Let us study a next example, configurations of $k = 2$ vortices. In this case generic forms of moduli matrices on $\mathbb{C}$ are given by

$$H_0^{(2,0)}(z) = \begin{pmatrix} z^2 - \alpha' z - \beta' & 0 \\ -a'z - b' & 1 \end{pmatrix}, \tag{3.22}$$

$$H_0^{(1,1)}(z) = \begin{pmatrix} z - \phi & -\eta \\ -\tilde{\eta} & z - \tilde{\phi} \end{pmatrix}, \tag{3.23}$$

$$H_0^{(0,2)}(z) = \begin{pmatrix} 1 & -az - b \\ 0 & z^2 - \alpha z - \beta \end{pmatrix}. \tag{3.24}$$

The corresponding orbifold transformation matrices are given by

$$\Omega^{(2,0)} = \Omega^{(0,2)} = \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix}, \qquad \Omega^{(1,1)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.25}$$

Performing the orbifold projection in a similar manner, the moduli matrices are reduced to

$$H^{(2,0)}_{0;n=2}(z) = \begin{pmatrix} z^2 - \beta' & 0 \\ -b' & 1 \end{pmatrix} = H^{(1,0)}_0(z^2), \tag{3.26}$$

$$H^{(1,1)}_{0;n=2}(z) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} = z\mathbb{1}_2, \tag{3.27}$$

$$H^{(0,2)}_{0;n=2}(z) = \begin{pmatrix} 1 & -b \\ 0 & z^2 - \beta \end{pmatrix} = H^{(0,1)}_0(z^2). \tag{3.28}$$

We now comment on two remarkable facts. First by imposing

$$\det H^{(2,0)}_{0;n=2} = \det H^{(0,2)}_{0;n=2} \Rightarrow \beta = \beta' \tag{3.29}$$

in order for these two of the moduli matrices, $H^{(2,0)}_{0;n=2}$ and $H^{(0,2)}_{0;n=2}$, to describe the vortices in the same positions as can be seen in Eq. (3.7), they are almost the same as those for $k = 1$ on $\mathbb{C}$ shown in Eq. (3.14). The difference is its argument, $z \to z^2$. This means that two vortices located at $z = z_1 = \sqrt{\beta}$ and $z = -z_1 = -\sqrt{\beta}$ have the same internal moduli $\mathbb{C}\mathbf{P}^1$, and thus they turn out to be identical. The vortex orientation can be obtained in a similar way as the case $k = 1$; from the equations

$$H^{(2,0)}_{0;n=2}(z^2 = \beta)\vec{\phi}^{(2,0)} = 0, \tag{3.30}$$

$$H^{(0,2)}_{0;n=2}(z^2 = \beta')\vec{\phi}^{(0,2)} = 0. \tag{3.31}$$

we obtain the orientational vectors

$$\vec{\phi}^{(2,0)} = \begin{pmatrix} 1 \\ b' \end{pmatrix}, \qquad \vec{\phi}^{(0,2)} = \begin{pmatrix} b \\ 1 \end{pmatrix}. \tag{3.32}$$

They are nothing but coordinates of $\mathbb{C}\mathbf{P}^1$.

Second is that we cannot perform a V-transformation which connects $H^{(1,1)}_{0;n=2}$ with $H^{(2,0)}_{0;n=2}$ or $H^{(0,2)}_{0;n=2}$, while $H^{(2,0)}_{0;n=2}$ and $H^{(0,2)}_{0;n=2}$ are connected via V-transformation. This is obvious from the fact that we cannot impose $\det H^{(1,1)}_{0;n=2} = \det H^{(2,0)}_{0;n=2}(= \det H^{(0,2)}_{0;n=2})$ except for $\beta = 0$. Therefore the part, corresponding to $H^{(1,1)}_{0;n=2}$, is disconnected to the continuous one connecting $H^{(2,0)}_{0;n=2}$ and $H^{(0,2)}_{0;n=2}$ in the moduli space $\mathcal{M}_{N=2,k=2;n=2}$. In general, we can see if the orbifold transformation matrices are different, the corresponding parts are disconnected. Actually in this case we have $\Omega^{(2,0)} = \Omega^{(0,2)} \neq \Omega^{(1,1)}$.

Furthermore the moduli matrix $H^{(1,1)}_{0;n=2}$ has neither of position nor internal moduli, so that it corresponds to just an isolated point in the moduli space. The absence of internal

moduli also means that it can be regarded as an Abelian (ANO) vortex. Actually we cannot extract the vortex orientation for this case because $H^{(1,1)}_{0;n=2}(z=0)=0$. Thus implying $\mathbb{Z}_2$ transformation of a complex coordinate $z \leftrightarrow -z$, the moduli space for $k=2$ turns out to be

$$\mathcal{M}_{N=2,k=2;n=2} \simeq \mathcal{M}^{\square\square}_{N=2,k=2;n=2} \cup \mathcal{M}^{\square}_{N=2,k=2;n=2}, \qquad (3.33)$$

where each sector is given by

$$\mathcal{M}^{\square\square}_{N=2,k=2;n=2} \simeq (\mathbb{C}/\mathbb{Z}_2) \times \mathbb{C}\mathbf{P}^1, \qquad \mathcal{M}^{\square}_{N=2,k=2;n=2} \simeq \{0\} \times \{0\}. \qquad (3.34)$$

3.2.3 $k=3$ vortices

We then consider $k=3$ vortices configurations. The generic forms of the moduli matrices on $\mathbb{C}$ are written as

$$H^{(3,0)}_0(z) = \begin{pmatrix} z^3 - \alpha' z^2 - \beta' z - \gamma' & 0 \\ -a' z^2 - b' z - c' & 1 \end{pmatrix}, \qquad (3.35)$$

$$H^{(2,1)}_0(z) = \begin{pmatrix} z^2 - \eta' z - \kappa' & 0 \\ -\lambda' z - \xi' & z - \zeta' \end{pmatrix}, \qquad (3.36)$$

$$H^{(1,2)}_0(z) = \begin{pmatrix} z - \zeta & -\lambda z - \xi \\ 0 & z^2 - \eta z - \kappa \end{pmatrix}, \qquad (3.37)$$

$$H^{(0,3)}_0(z) = \begin{pmatrix} 1 & -a z^2 - b z - c \\ 0 & z^3 - \alpha z^2 - \beta z - \gamma \end{pmatrix}. \qquad (3.38)$$

The orbifold transformation matrices, characterizing the boundary conditions, can be simply obtained from powers of the diagonal components of the moduli matrices,

$$\Omega^{(3,0)} = \Omega^{(1,2)} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}, \qquad \Omega^{(2,1)} = \Omega^{(0,3)} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (3.39)$$

We then apply the orbifold projection to the moduli matrices by removing components whose powers are different by modulo n from that of the diagonal component in the same row. The moduli matrices for the orbifolding theory are obtained as

$$H^{(3,0)}_{0;n=2}(z) = \begin{pmatrix} z(z^2 - \beta') & 0 \\ -a' z^2 - c' & 1 \end{pmatrix}, \qquad (3.40)$$

$$H^{(2,1)}_{0;n=2}(z) = \begin{pmatrix} z^2 - \kappa' & 0 \\ -\lambda' z & z \end{pmatrix}, \qquad (3.41)$$

$$H^{(1,2)}_{0;n=2}(z) = \begin{pmatrix} z & -\lambda z \\ 0 & z^2 - \kappa \end{pmatrix}, \qquad (3.42)$$

$$H^{(0,3)}_{0;n=2}(z) = \begin{pmatrix} 1 & -a z^2 - c \\ 0 & z(z^2 - \beta) \end{pmatrix}. \qquad (3.43)$$

As in the previous example we have to choose $\beta = \beta'$ and $\kappa = \kappa'$ to obtain the same determinant $\det H_{0:n=2}^{(3,0)} = \det H_{0:n=2}^{(0,3)}$ and $\det H_{0:n=2}^{(2,1)} = \det H_{0:n=2}^{(1,2)}$, respectively. We can see that $H_{0;n=2}^{(3,0)}$ and $H_{0;n=2}^{(0,3)}$ or $H_{0;n=2}^{(2,1)}$ and $H_{0;n=2}^{(1,2)}$ are connected via V-transformations. However we cannot connect a different combination, for example, $H_{0;n=2}^{(3,0)}$ and $H_{0;n=2}^{(2,1)}$, and they are disconnected as in the $k = 2$ case. Indeed the numbers of internal moduli are different: two for the former and one for the latter. Furthermore, the orbifold transformation matrices for them are different by comparing in the *descending* order. Although we have the same orbifold transformation matrices for $H_{0;n=2}^{(3,0)}$ and $H_{0;n=2}^{(1,2)}$, we have to compare $\Omega^{(3,0)}$ with $\Omega^{(2,1)}$ because $\Omega^{(1,2)}$ is not in the *descending* order.

The vortex orientations are given by

$$\vec{\phi}^{(3,0)} = \begin{pmatrix} 1 \\ a'\beta' + c' \end{pmatrix} \quad \text{for} \quad z^2 = \beta, \qquad \vec{\phi}^{(3,0)} = \begin{pmatrix} 1 \\ c' \end{pmatrix} \quad \text{for} \quad z = 0, \tag{3.44}$$

$$\vec{\phi}^{(0,3)} = \begin{pmatrix} a\beta + c \\ 1 \end{pmatrix} \quad \text{for} \quad z^2 = \beta, \qquad \vec{\phi}^{(0,3)} = \begin{pmatrix} c \\ 1 \end{pmatrix} \quad \text{for} \quad z = 0. \tag{3.45}$$

The two internal moduli for $H_{0;n=2}^{(3,0)}$ and $H_{0;n=2}^{(0,3)}$ are interpreted as coordinates of $(\mathbb{CP}^1)^2$ for the separated $k = 2$ vortices as well as the usual $k = 2$ configuration on $\mathbb{C}$ studied in[15]. When these $k = 2$ vortices coincide, we have $\beta = 0$. In this case the remaining orientation is just a $\mathbb{CP}^1$. This situation is essentially different from the the coincident $k = 2$ vortices on $\mathbb{C}$ where the moduli space is given by $W\mathbb{CP}^2_{(2,1,1)} \simeq \mathbb{CP}^2/\mathbb{Z}_2$ [54, 15].

On the other hand, since there is the only one internal moduli for $H_{0;n=2}^{(2,1)}$ and $H_{0;n=2}^{(1,2)}$, we have to discuss which vortex possesses it. Then taking an asymptotic limit $z \sim \pm\sqrt{\kappa} \to \infty$, we have

$$H_{0;n=2}^{(2,1)}(z) \longrightarrow \pm\sqrt{\kappa} \begin{pmatrix} 2(z \mp \sqrt{\kappa}) & 0 \\ -\lambda' & 1 \end{pmatrix}, \tag{3.46}$$

$$H_{0;n=2}^{(1,2)}(z) \longrightarrow \pm\sqrt{\kappa} \begin{pmatrix} 1 & -\lambda \\ 0 & 2(z \mp \sqrt{\kappa}) \end{pmatrix}. \tag{3.47}$$

This means that an internal moduli parameter $\lambda = 1/\lambda'$, regarded as a coordinate of $\mathbb{CP}^1$, belongs to a vortex at $\pm\sqrt{\kappa}$. Indeed the vortex orientations at $z^2 = \kappa$ turn out to be

$$\vec{\phi}^{(2,1)} = \begin{pmatrix} 1 \\ \lambda' \end{pmatrix}, \qquad \vec{\phi}^{(1,2)} = \begin{pmatrix} \lambda \\ 1 \end{pmatrix}. \tag{3.48}$$

In summary, in this case the moduli space is given by

$$\mathcal{M}_{N=2,k=3,n=2} \simeq \mathcal{M}_{N=2,k=3,n=2}^{\boxed{}} \cup \mathcal{M}_{N=2,k=3,n=2}^{\boxed{}}. \tag{3.49}$$

Here, for the first sector we have to write it as follows,

$$\mathcal{M}_{N=2,k=3,n=2}^{\boxed{}} \simeq \mathcal{M}_{\text{separate}}^{\boxed{}} \cup \mathcal{M}_{\text{coincident}}^{\boxed{}}, \tag{3.50}$$

where each part

$$\mathcal{M}_{\text{separate}}^{\square\square\square} \simeq (\mathbb{C}^*/\mathbb{Z}_2) \times (\mathbb{C}\mathbf{P}^1)^2, \qquad \mathcal{M}_{\text{coincident}}^{\square\square\square} \simeq \{0\} \times \mathbb{C}\mathbf{P}^1 \qquad (3.51)$$

is glued to each other. Since the first sector is defined only when $\beta \neq 0$, we denote the position moduli as $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. The second sector is simply given by

$$\mathcal{M}_{N=2,k=3,n=2}^{\square} \simeq (\mathbb{C}/\mathbb{Z}_2) \times \mathbb{C}\mathbf{P}^1. \qquad (3.52)$$

3.3 $U(N)$ gauge theory

Let us consider the moduli matrix for the generic $U(N)$ gauge theory. Decomposing the total vortex number, $k \to (k_1, \cdots, k_N)$ with $k_1 + \cdots + k_N = k$ and $k_1 \geq k_2 \geq \cdots \geq k_N$, it is written as a lower triangle matrix,

$$H_0(z) = \begin{pmatrix} z^{k_1} + \cdots & & & 0 \\ * & z^{k_2} + \cdots & & \\ \vdots & & \ddots & \ddots & \\ * & & \cdots & * & z^{k_N} + \cdots \end{pmatrix}. \qquad (3.53)$$

We can perform the orbifold projection as discussed in the previous section. Writing each decomposed vortex number as $k_i = l_i n + m_i \equiv m_i \pmod{n}$, the orbifold transformation matrix is given by

$$\Omega = \begin{pmatrix} \omega^{m_1} & & & 0 \\ & \omega^{m_2} & & \\ & & \ddots & \\ 0 & & & \omega^{m_N} \end{pmatrix}. \qquad (3.54)$$

In order for the moduli matrix to satisfy the boundary condition consistently, we have to remove factors whose powers are different by modulo n from that of the diagonal component in the same row. Thus the moduli matrix (3.53) becomes

$$H_{0;n}(z) = \begin{pmatrix} z^{m_1}\left(Z^{l_1} + \cdots\right) & & & 0 \\ z^{m_2}f_{1,2}(Z) & z^{m_2}\left(Z^{l_2} + \cdots\right) & & \\ \vdots & & \ddots & \ddots & \\ z^{m_N}f_{1,N}(Z) & \cdots & z^{m_N}f_{N-1,N}(Z) & z^{m_N}\left(Z^{l_N} + \cdots\right) \end{pmatrix} \qquad (3.55)$$

where $f_{i,j}(Z)$ is a polynomial of $Z = z^n$ whose degree is lower than l_i. Its determinant is given by

$$\det H_{0;n}(z) = z^{m_1+\cdots+m_N} \prod_{i=1}^{l_1+\cdots+l_N} (Z - Z_i) = z^{m_1+\cdots+m_N} \prod_{i=1}^{l_1+\cdots+l_N} (z^n - z_i^n). \qquad (3.56)$$

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This shows that the number of the position moduli is $l_1 + \cdots + l_N$, and there are $m_1 + \cdots + m_N$ vortices fixed at the origin. In terms of broken gauge group the number of fixed vortices is given by

$$\sum_{i=1}^{N} m_i = \sum_{m=0}^{n-1} m N^{(m)} \leq (n-1)N. \tag{3.57}$$

This inequality is saturated when $N^{(0)} = \cdots = N^{(n-2)} = 0$, $N^{(n-1)} = N$.

4 Vortex collision

Vortex collision is an important aspect of vortex dynamics. The merit for the study with the moduli matrix is that it is directly related to coordinates of the moduli space. It has been investigated by studying geodesics in the moduli space for vortices on $\mathbb{C}$. Using a general formula for the moduli space metric [14] head-on collision during short time has been studied [16], and asymptotic dynamics have been studied [18] by using the asymptotic metric for well-separated vortices [17]. Here we concentrate on collision dynamics on the orbifold singularity by applying the method in [16].

Let us start with $k = n$ Abelian vortices on the orbifold $\mathbb{C}/\mathbb{Z}_n$. We now rewrite the moduli matrix, just a holomorphic function, given by (3.11) as

$$H_{0;n}(z, t) = Z - \Xi t = \prod_{m=0}^{n-1} (z - \omega^m \xi t^{1/n}) \tag{4.1}$$

where $Z = z^n$, $\Xi = \xi^n$ and $t \in \mathbb{R}$. After changing $t \to -t$ we have

$$H_{0;n}(z, -t) = Z + \Xi t = \prod_{m=0}^{n-1} (z - \omega^m e^{i\pi/n} \xi t^{1/n}). \tag{4.2}$$

Here we have an extra factor $e^{i\pi/n}$. This means that vortices are colliding at $t = 0$ at the origin with a scattering angle $\theta = \pi/n$.[3] Fig. 2 shows collision of Abelian vortices on the orbifolds $\mathbb{C}/\mathbb{Z}_2$ and $\mathbb{C}/\mathbb{Z}_3$. The moduli matrix approach correctly reproduces the results in [55, 56, 5] in which $\mathbb{Z}_n$ symmetric collisions are studied in $\mathbb{C}$.

Let us then discuss the non-Abelian vortex collision. This scattering property is also found for the non-Abelian cases because it is only related to the position moduli. On the other hand, we can see an interesting behavior of the internal moduli.

For the case of $k = 2$, $n = 2$, the vortex orientation is given by (3.32). Although we expand the moduli matrices, (3.26) and (3.28), as $\beta = \mathfrak{b}t$ and $\beta' = \mathfrak{b}'t$, the vortex orientation is not affected at all. This means that even though each collides and scatters with its mirror image with an angle $\theta = \pi/n = \pi/2$ in space, but its internal moduli do not change.

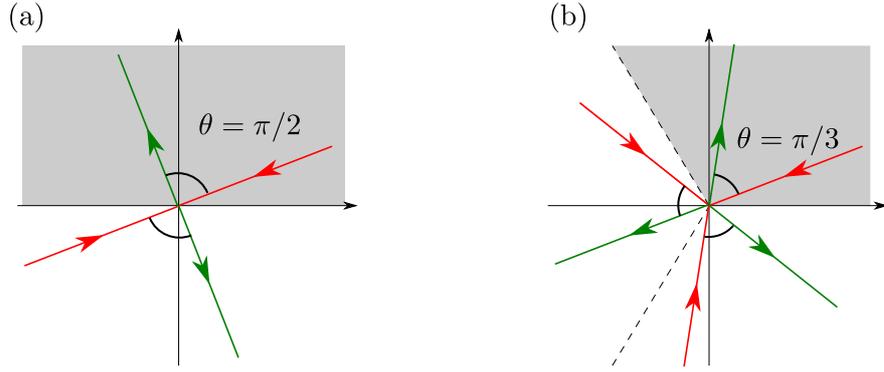[3]Such $\mathbb{Z}_n$ symmetric collisions are also studied on hyperbolic surfaces [33].

Figure 2: Collision of vortices on the orbifold (a) $\mathbb{C}/\mathbb{Z}_2$ with $\theta = \pi/3$ and (b) $\mathbb{C}/\mathbb{Z}_3$ with $\theta = \pi/3$. For $\mathbb{Z}_n$ with an odd n, vortices just look passing through the origin of the universal covering space $\mathbb{C}$.

For the $k = 3$ case we substitute $\beta = \mathfrak{b}t$ and $\beta' = \mathfrak{b}'t$ as before. The orientations, (3.44) and (3.45), are expanded as

$$\vec{\phi}^{(3,0)} = \begin{pmatrix} 1 \\ a'\mathfrak{b}'t + c' \end{pmatrix} \quad \text{for} \quad z^2 = \mathfrak{b}'t, \qquad \vec{\phi}^{(3,0)} = \begin{pmatrix} 1 \\ c' \end{pmatrix} \quad \text{for} \quad z = 0, \qquad (4.3)$$

$$\vec{\phi}^{(0,3)} = \begin{pmatrix} a\mathfrak{b}t + c \\ 1 \end{pmatrix} \quad \text{for} \quad z^2 = \mathfrak{b}t, \qquad \vec{\phi}^{(0,3)} = \begin{pmatrix} c \\ 1 \end{pmatrix} \quad \text{for} \quad z = 0. \qquad (4.4)$$

We can see that they coincide at $t = 0$ as

$$\vec{\phi}^{(3,0)} = \begin{pmatrix} 1 \\ c' \end{pmatrix}, \qquad \vec{\phi}^{(0,3)} = \begin{pmatrix} c \\ 1 \end{pmatrix}, \qquad (4.5)$$

and go through the original direction even after the collision. This behavior is different from the usual vortex collision on $\mathbb{C}$ [16]. In that case, after orientations of two colliding vortices coincide, they change their directions in the internal space. On the other hand, in this case, because they have only one parameter, $\mathfrak{b}$ or $\mathfrak{b}'$, the direction of the time-evolution of the internal moduli does not change. We remark that when we consider the $k = 4$ configuration on $\mathbb{C}/\mathbb{Z}_2$, we will see the same situation as the usual $k = 2$ vortex collision on $\mathbb{C}$.

5 Kähler quotient

We then discuss the Kähler quotient description of the moduli space, which has been originally studied in terms of string theory [6] and later proven from field theory [12, 13]. It is obtained from the D-term condition for the effective theory on D-branes,

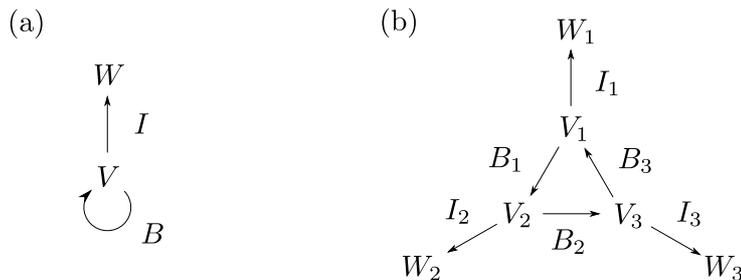$$[B, B^\dagger] + II^\dagger = c\mathbb{1}_k. \qquad (5.1)$$

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Figure 3: Quiver diagrams for the moduli spaces of vortices on the orbifolds: (a) $\mathbb{C}$ and (b) $\mathbb{C}/\mathbb{Z}_3$.

Here we have $B \in \mathrm{Hom}(V,V)$, $I \in \mathrm{Hom}(V,W)$ for two vector spaces V and W. The winding number and the rank of the gauge group are given by their dimensions, dim $V = k$ and dim $W = N$. Since we have $U(k)$ gauge symmetry for these data, $(B,I) \to (gBg^{-1}, gI)$, $g \in U(k)$, the moduli space is given by

$$\mathcal{M}_{N,k} \simeq \{(B,I)|[B,B^\dagger] + II^\dagger = c\mathbb{1}_k\}/U(k). \tag{5.2}$$

Let us consider the moduli space of vortices on orbifolds with respect to this Kähler quotient description. We introduce the decomposed vector spaces in order to characterize the representations under $\mathbb{Z}_n$ action,

$$V = \bigoplus_{m=0}^{n-1} V_m, \qquad W = \bigoplus_{m=0}^{n-1} W_m. \tag{5.3}$$

Their dimensions are dim $V_m = k^{(m)}$, dim $W_m = N^{(m)}$ with $\sum_{m=0}^{n-1} k^{(m)} = k$ and $\sum_{m=0}^{n-1} N^{(m)} = N$. This decomposition corresponds to the gauge symmetry breaking in Eq. (2.8) due to the boundary condition as discussed in the previous section.

The isometry $z \to e^\epsilon z$ of $\mathbb{C}$ acts on the data as $(B,I) \to (e^\epsilon B, I)$. Therefore we have to consider components of $B_m \in \mathrm{Hom}(V_m, V_{m+1})$ and $I_m \in \mathrm{Hom}(V_m, W_m)$ where we identify $V_n = V_0$ and $W_n = W_0$. Fig. 3 shows quiver diagrams for vector spaces. Comparing with the ADHM method for the ALE spaces, as shown in Appendix A, we can see the quiver for the orbifold vortex theory is *a half of ADHM* as well as the usual case on $\mathbb{C}$. We remark that, if the two dimensional orbifold $\mathbb{C}/\Gamma$ is considered, we have to set Γ as a finite subgroup of $U(1)$. This means it must be the cyclic group $\Gamma = \mathbb{Z}_n$, corresponding to the A_{n-1} type root system.

The algebraic condition (5.1) yields

$$B_{m-1}B_{m-1}^\dagger - B_m^\dagger B_m + I_m I_m^\dagger = c\mathbb{1}_{k^{(m)}}, \qquad m = 0, \cdots, n-1. \tag{5.4}$$

We have $U(k^{(m)})$ symmetry for m-th equation. Note that as well as the ALE spaces, if we could resolve the singularity of the orbifold $\mathbb{C}/\mathbb{Z}_n$, a different FI parameter c_m would be

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applied to each block. However such a resolution does not exist for one dimensional orbifolds $\mathbb{C}/\mathbb{Z}_n$. Thus the total moduli space is given by

$$\mathcal{M}_{\vec{N},\vec{k};n} \simeq \frac{\left\{(B_0,\cdots,B_{n-1},I_0,\cdots,I_{n-1})|B_{m-1}B_{m-1}^\dagger - B_m^\dagger B_m + I_m I_m^\dagger = c\mathbb{1}_{k^{(m)}}, m = 0,\cdots,n-1\right\}}{U(k^{(0)}) \times \cdots \times U(k^{(n-1)})}.$$

(5.5)

This moduli space is not determined by just choosing the rank of the original gauge group and the total winding number of vortices. We have to specify a way of decomposition, thus it is labeled by $\vec{N} = (N^{(0)},\cdots,N^{(n-1)})$ and $\vec{k} = (k^{(0)},\cdots,k^{(n-1)})$, satisfying $N^{(0)}+\cdots+N^{(n-1)} = N$ and $k^{(0)}+\cdots+k^{(n-1)} = k$. Since this decomposition corresponds to the boundary condition as discussed above, the moduli spaces under different decompositions are disconnected. The moduli space of k-vortex on the orbifold is given by

$$\mathcal{M}_{N,k;n} = \bigcup_{|\vec{N}|=N,|\vec{k}|=k} \mathcal{M}_{\vec{N},\vec{k};n}.$$

(5.6)

6　Summary and Discussion

We have investigated several properties of vortices on orbifolds $\mathbb{C}/\mathbb{Z}_n$. First we have considered consistent boundary conditions for fields on the orbifolds. Then we have performed the moduli matrix approach to characterize the moduli space of vortices. Under the orbifold boundary condition the moduli matrices should be determined consistently. We have clarified some restrictions to obtain the moduli matrices for the orbifolds from the usual ones for $\mathbb{C}$, which we call the orbifold projection.

We have investigated the moduli spaces based on the orbifolded moduli matrices with some examples. The most remarkable point which we can find in the both of Abelian and non-Abelian cases is that the fractional vortices are just fixed at the singular point of the orbifold, namely the origin of the complex plane. They lose position moduli but fractional non-Abelian vortices still possess internal moduli $\mathbb{C}\mathbf{P}^{N-1}$ for $U(N)$ gauge theory. Indeed a similar situation for a fractional object is observed in other models; Yang-Mills instantons [36, 37] and fractional D-branes [47] on orbifolds or their resolution to ALE spaces.

In the non-Abelian theory we have the internal moduli as well as the usual vortices on $\mathbb{C}$. However, while the whole moduli space is connected via the V-transformation before orbifolding, the moduli space for the orbifold theory can be decomposed to disconnected parts. This decomposition is due to the gauge symmetry breaking via the boundary condition. We have found that the decomposed sectors one to one correspond to the orbifold transfomation matrices Ω in Eq. (2.7), which transform fields at z and ωz, in the *descending* order. Although it is similar to the theory in presence of the twisted mass term [10], an essential difference exists between them; each sector is completely decoupled from each other in the case of orbifolds while it is just separated by a potential in the case of twisted mass.

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Accordingly, two vortex states can be connected along a string making a confined monopole in the case of the twisted masses while it is impossible for the orbifolds.

We have also discussed vortex collision for the orbifold theory with the moduli matrix perspective. We have shown the scattering angle is directly reflecting the orbifold structure. Furthermore we can observe an unusual behavior of the internal moduli for the non-Abelian vortex. The internal orientations of two vortices coincide when they are colliding, but their behavior after the collision is different from the usual one on $\mathbb{C}$.

Finally the relation to Hanany-Tong's approach is discussed. As well as the instantons on the orbifold, or the ALE space given by resolving its singularity, we have a similar quiver theory for the Kähler quotient. We have seen that it is just a half of ADHM for instantons on the type-A orbifold $\mathbb{C}^2/\mathbb{Z}_n$.

Here before closing the paper, we give several discussions.

In this paper we have studied short time dynamics around the collision moment for which we have not needed explicit metric on the moduli space, but we need it for studying a long time behavior. The moduli space metric has been explicitly obtained for non-Abelian vortices on $\mathbb{C}$ [17] and on Riemann surfaces [29] when vortices are well separated, and dynamics and scattering have been studied on $\mathbb{C}$ [18]. Beyond our work it will be an interesting and important future problem to construct an explicit metric of the moduli space in our case of the orbifold $\mathbb{C}/\mathbb{Z}_n$. One nontrivial effect will be the existence of the fractional vortices fixed at the orbifold singularity. It does not appear in the moduli space in some cases, but even in that case the existence of such a flux at the singularity will change the moduli space metric. For instance, ANO vortices fixed at the singularity in both Abelian and non-Abelian gauge theories have no moduli but the fluxes do exist there.

In this paper we have concentrated on local vortices, namely vortices in $U(N)$ gauge theory with $N_{\mathrm{F}} = N$ fundamental scalar fields. When the number of flavor is greater than the number of color, $N_{\mathrm{F}} > N$, vortices are called semi-local [57]. However we have to be careful on (non-)normalizability of moduli parameters in this case [58]. Other extensions would be changing gauge symmetry from $U(N)$ to $G \times U(1)$ such as $G = SO, USp$ [59].

Let us make a comment on a relation to Yang-Mills instantons. In five dimensional gauge theory in the Higgs phase, instantons on $\mathbb{C}^2$ can live stably in a vortex world-volume in which Yang-Mills instantons are regarded as lumps or sigma model instantons [11]. It becomes Amoeba in more general on $(\mathbb{C}^*)^2$ [60]. In our case of the orbifold theory, instantons live on a vortex in $\mathbb{C} \times \mathbb{C}/\mathbb{Z}_n$ where the vortex world-volume extends to $\mathbb{C}$. In the case of $\mathbb{C}^2$ we obtain usual instantons in the limit of vanishing Fayet-Iliopoulos parameter c in which the model goes to unbroken phase. It will be interesting to study what happens in the same limit for instantons on $\mathbb{C} \times \mathbb{C}/\mathbb{Z}_n$. Also this may be related to surface operators in the AGT relation [61].

Finally we comment on physical applications of the vortices on orbifolds. The vortex polygon and crystal, discussed in a few body vortex system [62, 63], have a similar property

to the orbifold theory studied in this paper. We have to discuss relation between them in a future work. Another proposal is application to the two dimensional carbon system. The conical structure, mimicking the orbifolds $\mathbb{C}/\mathbb{Z}_n$, could be obtained by manipulating the graphene sheet or the carbon nanotube.

Acknowledgments

A   Instanton construction on the ALE spaces

In this appendix we comment on the ADHM construction for the ALE spaces [36, 37, 35]. First let us start with the ADHM method without orbifolding. We now introduce the ADHM data (B_1, B_2, I, J) satisfying the following ADHM equations given by

$$
\begin{aligned}
\mu_{\mathbb{C}} &:= [B_1, B_2] + IJ = 0, & \text{(A.1)} \\
\mu_{\mathbb{R}} &:= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger + J^\dagger J = 0, & \text{(A.2)}
\end{aligned}
$$

where $B_1, B_2 \in \mathrm{Hom}(V, V)$, $I \in \mathrm{Hom}(W, V)$ and $J \in \mathrm{Hom}(V, W)$. The dimensions of these vector spaces are $\dim V = k$, $\dim W = N$ for k-instanton configuration of $SU(N)$ gauge theory on $\mathbb{R}^4$. The instanton moduli space is given by

$$
\mathcal{M}_{N,k} = \{(B_1, B_2, I, J) | \mu_{\mathbb{C}} = \mu_{\mathbb{R}} = 0\} / U(k). \tag{A.3}
$$

We have $U(k)$ symmetry for the ADHM data such that

$$
(B_1, B_2, I, J) \longrightarrow (gB_1 g^{-1}, gB_2 g^{-1}, gI, Jg^{-1}), \qquad g \in U(k). \tag{A.4}
$$

The ALE space is given by resolving the singularity of the orbifold $\mathbb{C}/\Gamma$ where Γ is a finite subgroup of $SU(2)$. We now discuss only the case of $\Gamma = A_{n-1}$ for simplicity. Let (z_1, z_2) be a coordinate of $\mathbb{C}^2$, thus the orbifold $\mathbb{C}^2/\mathbb{Z}_n$ is obtained by identification $(z_1, z_2) \sim (\omega z_1, \bar{\omega} z_2)$ with $\omega = \exp(2\pi i/n)$. To study how this identification affects the ADHM equation, we then consider action of the isometry $(z_1, z_2) \to (e^{\epsilon_1} z_1, e^{\epsilon_2} z_2)$ on the ADHM data, $(B_1, B_2, I, J) \to (e^{\epsilon_1} B_1, e^{\epsilon_2} B_2, I, e^{\epsilon_1 + \epsilon_2} J)$. Decomposing the vector spaces as well as the vortex theory (5.3) due to the irreducible representation of $\mathbb{Z}_n$, again these dimensions are $\dim V_m = k^{(m)}$, $\dim W_m = N^{(m)}$. Then we have $B_{1,m} \in \mathrm{Hom}(V_m, V_{m+1})$,
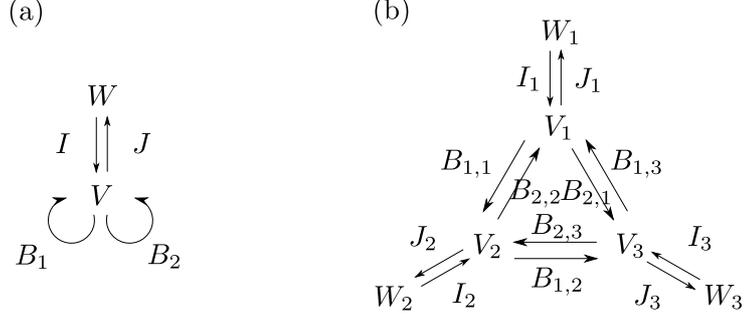
Figure 4: Quiver diagrams for the moduli spaces of instantons on the orbifolds: (a) $\mathbb{C}^2$ and (b) $\mathbb{C}^2/\mathbb{Z}_3$.

$B_{2,m} \in \mathrm{Hom}(V_m, V_{m-1})$, $I_m \in \mathrm{Hom}(W_m, V_m)$ and $J_m \in \mathrm{Hom}(V_m, W_m)$. We can write down the ADHM equation for the ALE space in terms of these components,

$$B_{1,m}B_{2,m+1} - B_{2,m}B_{1,m-1} + I_m J_m = -\zeta_{\mathbb{C}}^{(m)},$$
(A.5)

$$B_{1,m-1}B_{1,m-1}^\dagger - B_{1,m}^\dagger B_{1,m} + B_{2,m}B_{2,m}^\dagger - B_{2,m+1}^\dagger B_{2,m+1} + I_m I_m^\dagger - J_m^\dagger J_m = -\zeta_{\mathbb{R}}^{(m)},$$
(A.6)

where $\zeta_{\mathbb{C}}^{(m)} \in \mathbb{C}$ and $\zeta_{\mathbb{R}}^{(m)} \in \mathbb{R}$ are related to blow-up parameters of orbifold singularities, satisfying

$$\sum_{m=0}^{n-1} \zeta_{\mathbb{C}}^{(m)} = \sum_{m=0}^{n-1} \zeta_{\mathbb{R}}^{(m)} = 0.$$
(A.7)

When all $\zeta_{\mathbb{C}}^{(m)} = \zeta_{\mathbb{R}}^{(m)} = 0$, the ALE space goes back to the singular orbifold.

Fig. 4 shows the quiver diagram for the ADHM data. Note that it is directly related to Dynkin diagram for the A_{n-1} root system. If we consider other types of orbifolds, type D or E, we have the corresponding quivers.

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