Calculation of the reflection function of an optically thick scattering layer for a Henyey–Greenstein phase function

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Simple analytical methods are proposed for calculating the reflection function of a semi-infinite and conservative scattered layer, the value of which is needed to solve many atmospheric optics problems. The methods are based on approximations of the exact values obtained with a strict numerical method. For a Henyey–Greenstein phase function, knowledge of the zeroth and sixth higher harmonics appears to be sufficient for a quite accurate approximation of the angle range, which is acceptable for solution of direct and inverse problems in atmospheric optics when a plane atmosphere is assumed. An error estimation and a comparison with the exact solution are presented. © 2000 Optical Society of America

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1. Introduction

Knowledge of the value of the reflection function is necessary for many problems whose solutions involve using satellite radiance measurements. There are several complicated strict numerical methodologies for calculating the reflection function[1–4] that are quite suitable for determining the reflected radiance for solution of direct problems in atmospheric optics. For remote sensing of cloud layers, analysis of an inverse problem[5–9] requires knowledge of the value of the reflection function for a semi-infinite conservatively scattering medium, preferably presented analytically.

Here we illustrate an analytical method for solution of inverse problems. It is based on an analysis of the numerical results of calculating the zeroth and higher harmonics of the reflection function for a set of Henyey–Greenstein phase function $\chi(\gamma)$ parameters g.[3,10,11] A brief description of this method is given in Appendix A. Two approximations are analyzed here. The first consists of fitting regressions by linear and exponential functions for the harmonics that make the largest contributions to the value of the reflection function. The second consists in accurately taking into account the first order of scattering for all harmonics and the higher scattering orders for only the zeroth harmonic of the reflection function.

We point out that real cloud is characterized by a complicated phase function that might be different from the Henyey–Greenstein model. In this case the reflection function might differ from the model function by as much as 10%.[5] Here we do not treat the problem of calculating the real cloudy phase function, but it is necessary to have in mind when one is dealing with real clouds that errors of application of these methods might be larger than those shown here for the Henyey–Greenstein phase function.

2. Approximation of Numerical Calculation Results

As is usually done,[1–4] let us describe the reflection function by its expansion on the azimuth angle cosine:

$$\rho(\varphi, \mu, \mu_0) = \rho^0(\mu, \mu_0) + 2 \sum_{m=1}^{\infty} \rho^m(\mu, \mu_0)\cos m\varphi, \quad (1)$$

where μ and μ_0 are zenith-viewing and solar-angle cosines, φ is the azimuth angle, and $\rho^m(\mu, \mu_0)$ are the harmonics of the reflection function of order m. As was mentioned above, here we use the phase function described by the Henyey–Greenstein formula:

$$\chi(\gamma) = \frac{1 - g^2}{(1 + g^2 - 2g \cos \gamma)^{3/2}}, \quad (2)$$

where γ is the scattering angle.

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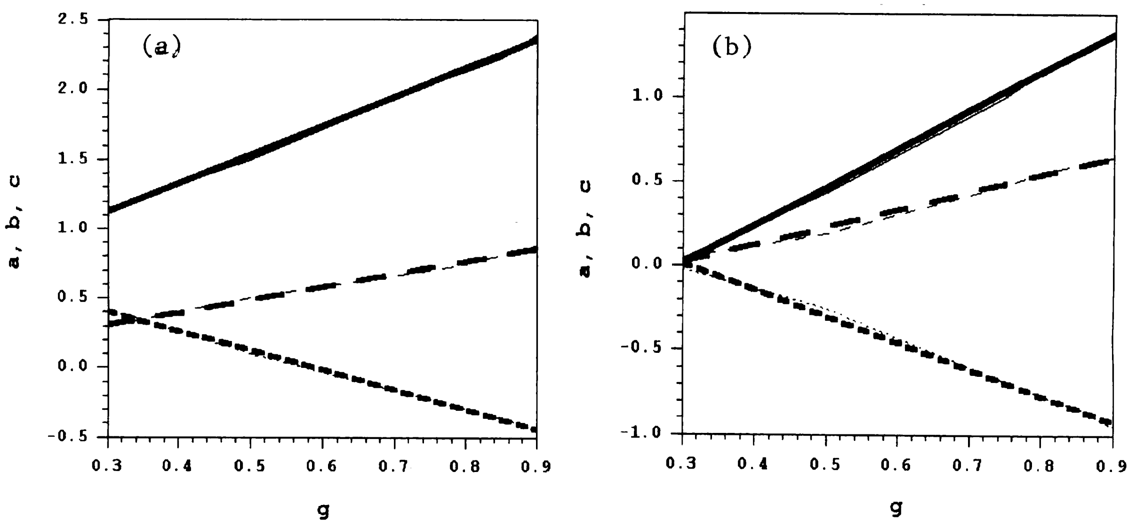

Fig. 1. Coefficients a^m, b^m, and c^m (thinner solid, short-dashed, and long-dashed curves, respectively) in Eq. (3) versus phase function parameter g: (a) for the zeroth ($m = 0$) harmonic of the reflection function, (b) for the second ($m = 2$) harmonics of the reflected function and their linear approximation according to Table 1 (thicker curves). The thinner and thicker curves are almost coincident.

Numerical calculations[10,11] show that for an accurate description of the function $\rho(\varphi, \mu, \mu_0)$ it is enough to know the first six harmonics, even for $g = 0.9$ if μ and μ_0 are greater than 0.15. This limitation does not restrict our calculations because for the small zenith solar and viewing angles it is also necessary to use a complicated model of a spherical atmosphere and to take into account the refraction of solar rays. Those cases are not studied here.

The values of $\rho^m(\mu, \mu_0)$ for $m = 0, \ldots, 6$ are analyzed in this study. The following formula, which is similar to the formula for the zeroth harmonic,[1] is used here for a description of high harmonics also:

$$\rho^m(\mu, \mu_0) = [a^m \mu\mu_0 + b^m(\mu + \mu_0) + c^m]/(\mu + \mu_0). \quad (3)$$

This presentation shows the reciprocity of the reflection function for the zenith-viewing and solar angles.

In Fig. 1 we present coefficients a^m, b^m, and c^m for the zeroth and second harmonics as functions of phase function parameter g. One can see that the approximation with the linear functions yields a satisfactory fit in the $0.3 \leq g \leq 0.9$ range of the phase function parameter with $m = 0, 1, 2$. This approximation by linear functions gives errors of less than 5%. It is necessary, however, to point out that for g in the $0 \leq g \leq 0.9$ range there is no linearity, even for these harmonics. The approximation of the coefficients a^m, b^m, and c^m in the g $0.3 \leq g \leq 0.9$ range is called the linear approximation method and is presented in Table 1. It can be seen that the values of the first and second harmonics for $g \sim 0.3$ are close to zero.

The well-known relation of the strict theory[1,2,4] is assumed for isotropic and conservative scattering ($g = 0$, $\omega_0 = 1$), namely,

$$\rho^0(\mu, \mu_0) = \frac{\varphi(\mu)\varphi(\mu_0)}{4(\mu + \mu_0)}, \quad (4)$$

where $\varphi(\mu)$ is Ambartsumian's function.[1] In this case the approximation $\varphi(\mu) = 1.874\mu + 1.058$ is correct, and $a^0 = 0.88$, $b^0 = 0.47$, and $c^0 = 0.28$.[12] It is known that the reflection function for isotropic scattering does not differ much from the anisotropic values of $\rho^0(\mu, \mu_0)$ if $\mu, \mu_0 > 0.25$,[2,13] so it is possible to improve this approach and make it applicable for an enlarged angle. The simple formula for isotropic scattering can be approximately corrected by a linear dependence on the phase function, as follows:

$$\rho^0(\mu, \mu_0) = \frac{\varphi(\mu)\varphi(\mu_0) + g[4.8\mu_0\mu - 3.0(\mu_0 + \mu) + 1.9]}{4(\mu_0 + \mu)}, \quad (5)$$

which is further named the corrected isotropic (CI) method. It is shown below in Table 6 that this approach yields results that are closer to the exact values of the zeroth harmonic than does the linear fit, which is why the final approximation is derived according to Eq. (5).

It is useful to test the above zeroth-harmonic approximation by a formula that connects $\rho^0(\mu, \mu_0)$ and escape function $K_0(\mu)$ for the conservative case[1]:

$$K_0(\mu_0) = \frac{3}{2}\int_0^1 \rho^0(\mu, \mu_0)(\mu + \mu_0)\mu d\mu. \quad (6)$$

Table 1. LA's for Coefficients a^m, b^m, and c^m of the Zeroth, First, and Second Harmonics of the Analytical Presentation of the Reflection Function

m	a_m	b_m	c_m	μ_{limit}
0	$2.051g + 0.508$	$-1.420g + 0.831$	$0.930g + 0.023$	—
1	$1.821g - 0.558$	$-1.413g + 0.387$	$1.150g - 0.239$	0.80
2	$2.227g - 0.669$	$-1.564g + 0.481$	$1.042g - 0.293$	0.55

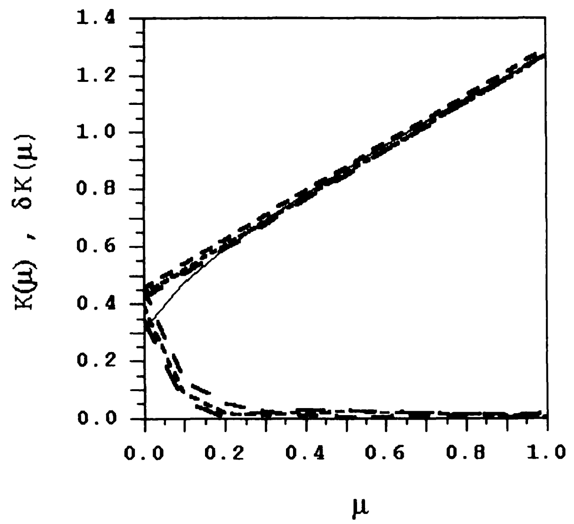

Fig. 2. Exact (solid, thin curve) and approximate (thick curves) values of the escape function $K_0(\mu)$ for $g = 0.85$ [long-dashed curve for Eq. (7)] and for $g = 0.85, 0.9$ [short- and long-dashed curves for Eq. (8)] and relative errors $\Delta K_0(\mu)/K_0(\mu)$ (lower curves) of the approximations.

After calculations it is easy to express the escape function from the approximation of the numerical calculation, Eq. (3), and Table 1 as

$$K_0(\mu) = (0.780 + 0.090g)\mu + 0.437 - 0.017g \quad (7)$$

or from Eq. (5) (corrected isotropic formula) as

$$K_0(\mu) = (0.793 + 0.048g)\mu + 0.445 - 0.003g. \quad (8)$$

Both of the approximations are shown in Fig. 2, and it can be seen that their agreement with the angular dependence of the escape function[4] is rather good. Thus it is reasonable to consider that Eqs. (3) and (5) are suitable for these calculations if $\mu \geq 0.15$.

Figure 3 shows the dependence of coefficients a^m, b^m, and c^m on parameter g for higher harmonics ($m > 2$), which indicates that a linear fitting for a range of the phase function parameter of $0.3 \leq g \leq 0.9$ is impossible. Although it appears impossible to find a linear presentation for the third through the sixth harmonics for any values of parameter g, it is possible to approximate the dependence of coefficients a^m, b^m, and c^m for $m \geq 3$ by exponents with powers 2 and 3. These fitting regressions are presented in Table 2. We refer to this method below as to the power-fitting (PF) method.

In the case of the Henyey–Greenstein phase function the high harmonics are close to zero [$\rho^m(\mu, \mu_0) \approx 0$, $m > 0$] if either of the zenith angle cosines μ or μ_0 is greater than μ_{limit}. The values of μ_{limit} differ for different harmonics, and they are shown in Tables 1 and 2.

The approximation by Eq. (1) with coefficients a^m, b^m, and c^m in Tables 1 and 2 gives an acceptable presentation for all harmonics of the reflection function considered here. Errors in this approximation depend on values of zenith-viewing and solar-angle cosines, number of harmonics m, and phase function parameter g. Results for the zeroth harmonic are shown in Fig. 4. The figure indicates that the errors are less than 2% for the values of zenith angle cosines μ, $\mu_0 > 0.02$ when $g \leq 0.75$. For g in the 0.8–0.9 range the error of our approximation is less than 3% if either of the values of μ and μ_0 is greater than 0.3 and the error is less than 10% for arbitrary values of μ and μ_0. The first and the second harmonics are approximated with an error of less than 1–2% if g is less than 0.8 and either μ or μ_0 is greater than 0.12.

The approximations for the high harmonics are proposed in the range of the zenith angle cosines, where $\rho^m(\mu, \mu_0)$ differs from zero ($\mu < \mu_{\text{limit}}$). The relative error is less 10% even for $g = 0.9$ and small values of μ, as Fig. 5 indicates for the sixth harmonic and $g = 05, 0.9$.

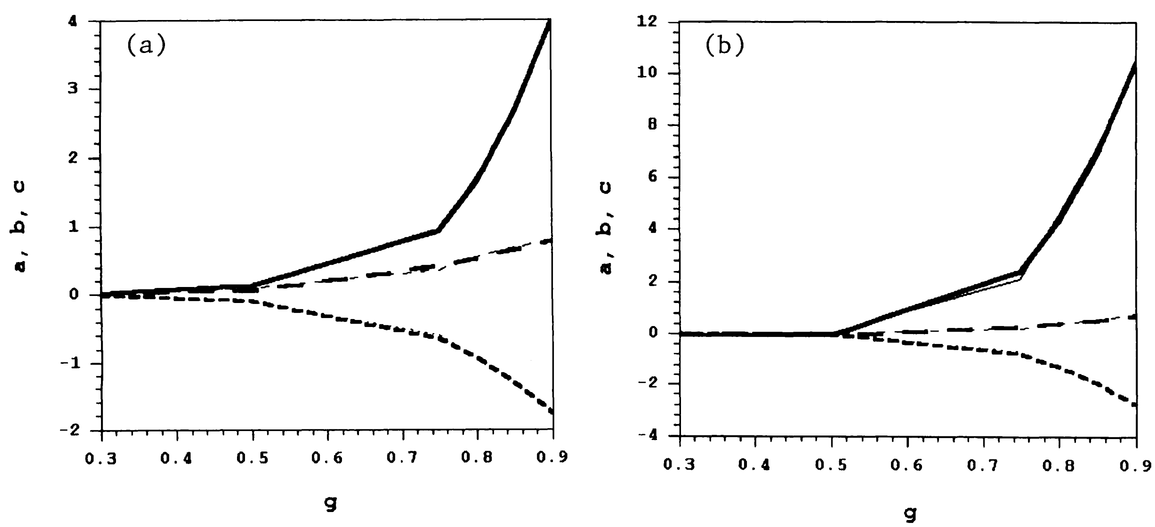

Fig. 3. Dependence of coefficients a^m, b^m, and c^m of harmonics with numbers (a) $m = 3$ and (b) $m = 6$ on phase function parameter g (thinner curves) and power fit approximations according to Table 2 (thicker curves).

Table 2. Exponential Approximations of Coefficients a^m, b^m, and c^m of the Third through Sixth Harmonics of the Reflection Function (PF Method)

	$0.3 \le g \le 0.9$			
m	a^m	b^m	c^m	μ_{limit}
3	$62.00g^3 - 90.28g^2 + 42.42g - 6.26$	$-15.24g^3 + 19.70g^2 - 8.73g + 1.25$	$2.75g^2 - 2.03g + 0.39$	0.50
4	$105.26g^3 - 155.06g^2 + 72.93g - 10.76$	$-30.30g^3 + 43.04g^2 - 19.83g + 2.89$	$3.70g^2 - 3.20g + 0.65$	0.45
5	$120.63g^3 - 177.60g^2 + 83.48g - 12.32$	$-25.84g^3 + 35.15g^2 - 15.61g + 2.22$	$3.23g^2 - 2.75g + 0.55$	0.35
6	$144.92g^3 - 202.16g^2 + 90.48g - 12.85$	$-32.60g^3 + 43.88g^2 - 19.15g + 2.67$	$3.90g^2 - 3.41g + 0.70$	0.35

It is necessary to point out that for the zenith-angle cosine $\mu = 0.67$ (which corresponds to 48°), for which the zeroth harmonic of the reflection function is very close to 1, especially in the case of the Henyey–Greenstein phase function, the other harmonics are close to zero and the escape function $K_0(\mu)$ is equal to 1. Thus the reflected radiance measured in viewing angles close to 48° is equal to the reflected irradiance (the same is true for the transmitted radiation). The radiance and irradiance measured at a solar angle of 48° also approximately coincide with the spherical albedo of the cloud layer. The values of $|1 - \rho^0(\mu, \mu_0)|$ for recent approximations are listed in Table 3 for $g = 0.3$–0.9 and $\mu = \mu_0 = 0.67$. The small deviations *from* 1 show rather small approximation errors. An analysis of the reflection function's zeroth harmonic $\rho^0(\mu, 0.67)$ for the various values of g shows that the deviation from 1 is ~8% for $g \le 0.5$ and 10% for $g = 0.85$–0.9.

The reflection function has been calculated for the Mie phase function that corresponds to the model of fair-weather cumulus clouds.[5] These results indicate that, at the zenith angles in the 47–50° range, the reflection function differs from 1 by 2–5%. Hence it is possible to conclude that the reflection function is close to 1 at these zenith angles, even for a complicated phase function.

It was shown[15] that the effects of the phase function on radiative forcing are almost the same for several phase functions if the solar incident angle is approximately 45–50° ($\mu_0 = 0.643 - 0.707$). The influence of particle-size distribution on the cloud phase function for scattering angles of ~90° (Ref. 16) (which approximately correspond to the zenith-angle cosines $\mu_0 = \mu = 0.67$) is slight.

These facts can be explained because in this angular range the reflection function (and the escape function) depends on a phase function that is weaker than for other angles. Thus it is useful to measure the reflected radiation for either the zenith-viewing or solar angle in the 45–50° range for retrieval of the optical thickness and the single-scattering albedo. Otherwise it is better to use other zenith angles, at which the radiance is susceptible to the phase function, to estimate the phase function parameter, as was pointed out earlier.[15]

3. First and Higher Orders of Scattering

High harmonics of the radiation reflected by a thick cloud form mainly in the upper part of the cloud. Calculations of the radiation field in the optically thick layer[16] showed that almost all harmonics with numbers $m > 0$ are approximately zero at optical depth $\tau \approx 2$. The ratio of the high harmonic to the zeroth harmonic is in the 10–20% range for $\tau \sim 1$ and is applicable only for a zenith angle μ of less than

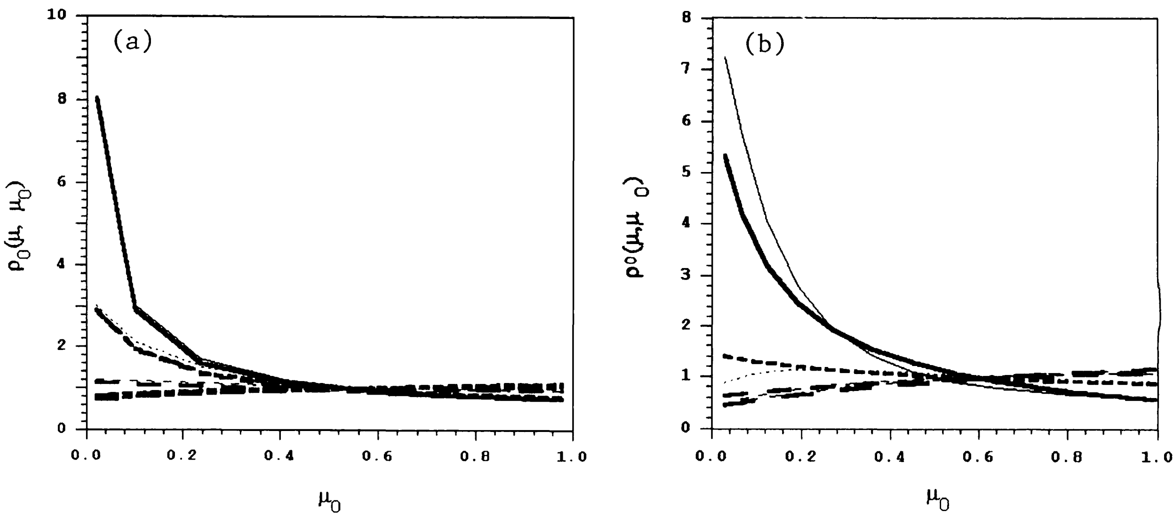

Fig. 4. Exact (thinner curves) and approximate according to Table 3 (thicker curves) angular dependence of the zeroth harmonic of the reflected function on (a) phase function parameter $g = 0.3$ and $\mu_0 = 0.01986, 0.1017, 0.4083, 0.7628, 0.9801$, (b) phase function parameter $g = 0.9$ and $\mu_0 = 0.1223, 0.4525, 0.8089, 0.9947$.

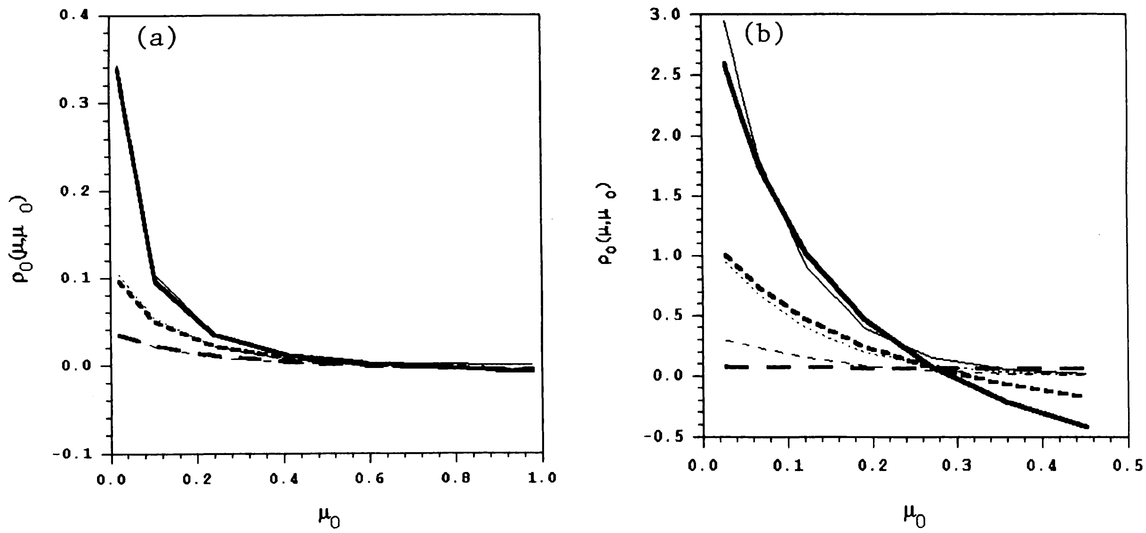

Fig. 5. Exact (thinner curves) and approximate according to Table 3 (thicker curves) angular dependence of the sixth harmonic of the reflected function on (a) phase function parameter $g = 0.5$ and $\mu_0 = 0.01986, 0.4083, 0.7628, 0.9801$, (b) phase function parameter $g = 0.9$ and $\mu_0 = 0.1223, 0.1911, 0.271$.

0.25. For an optical thickness of $\tau \sim 2^{-14}$ the first, second, and third harmonics are 50%, 30%, and 10%, respectively, of the zeroth harmonic. Thus it is clear that first-order scattering includes most of the high harmonics and that multiple scattering corresponds mostly to the zeroth harmonic.[17] It is possible to express the reflection function as

$$\rho(\mu, \mu_0, \varphi) = \rho_1(\mu, \mu_0, \varphi) + [\rho^0(\mu, \mu_0) - \rho_1^0(\mu, \mu_0)] + 2 \sum_{m=1}^{N} [\rho^m(\mu, \mu_0) - \rho_1^m(\mu, \mu_0)]\cos m\varphi, \tag{9}$$

where $\rho_1(\mu, \mu_0, \varphi)$ is the first-order scattering term of the reflection function for a semi-infinite and conservatively scattering medium:

$$\rho_1(\mu, \mu_0, \varphi) = \frac{\chi(\gamma)}{4} \frac{1}{\mu + \mu_0}, \tag{10}$$

and the first-order scattering part of the zeroth harmonic is as follows:

$$\rho_1^0(\mu, \mu_0) = \frac{1}{8\pi(\mu_0 + \mu)} \int_0^{2\pi} \chi(\gamma)d\varphi = \frac{\rho^0(-\mu_0, \mu)}{4(\mu_0 + \mu)}. \tag{11}$$

Table 3. **Deviation of the Approximation of Zeroth Harmonic $\rho^0(0.67, 0.67)$ from 1**

	Value of g					
Variable	0.3	0.5	0.75	0.8	0.85	0.9
Deviation of $\lvert 1 - \rho^0(0.67, 0.67)\rvert$	0.0037	0.024	0.021	0.0059	0.013	0.0046

In the case of first-order scattering the scattering-angle cosine is expressed in terms of the solar-angle and zenith-viewing cosines by $\cos \gamma = \mu\mu_0 + [(1 - \mu^2)(1 - \mu_0{}^2)]^{1/2} \cos \varphi$. If the phase function is assumed to have a Henyey–Greenstein formula, the function $p^0(-\mu, \mu_0)$ can be presented as[4]

$$p^0(-\mu_0, \mu) = \frac{2(1-g^2)}{\pi} \int_0^{\pi} \frac{d\varphi}{(e \pm f \cos \varphi)^{3/2}} = \frac{2(1-g^2)\sqrt{e+f}}{\pi(e^2 - f^2)} E\left(\sqrt{\frac{2f}{e+f}}\right), \tag{12}$$

where $E(x)$ is the complete elliptical integral of the second kind and the following notation is used:

$$e = 1 + g^2 + 2g\mu\mu_0, \quad f = 2g[(1 - \mu^2)(1 - \mu_0^2)]^{1/2}. \tag{13}$$

The results of calculations of exact values of the four reflection function harmonics for phase function parameter $g \doteq 0.5$ and ten harmonics for $g = 0.85$ have been presented, as well as values of the first and second orders of scattering for the same harmonics.[18] The ratio of each azimuth harmonic ($m > 0$) to the zeroth harmonic is taken as follows:

$$\delta(\mu, \mu_0, m) = \frac{\rho^m(\mu, \mu_0)}{\rho^0(\mu, \mu_0)}, \quad \Delta(\mu, \mu_0, m, i) = \frac{\rho^m(\mu, \mu_0) - \rho_i^m(\mu, \mu_0)}{\rho^0(\mu, \mu_0)}, \tag{14}$$

where i and m indicate the scattering order and the harmonic order, respectively. Here δ is the relative contribution of each harmonic ($m = 0, 1, \ldots$) to the value of the reflection function and Δ is the contribution of ith-order scattering in azimuth-angle-dependent terms relative to the zeroth harmonic.

Table 4. Contributions of the First through Fourth Harmonics Relative to the Zeroth Harmonic to the First and Second Scattering Orders for Phase Function Parameter $g = 0.5$

	Parameter δ			Parameter Δ ($i = 1$)			Parameter Δ ($i = 2$)		
Variable	Case I	Case II	Case III	Case I	Case II	Case III	Case I	Case II	Case III
μ; μ_0	0; 0.1	0; 0.9	0.9; 0.9	0; 0.1	0; 0.9	0.9; 0.9	0; 0.1	0; 0.9	0.9; 0.9
$m = 1$	0.600	0.089	0.0078	0.048	0.008	0.0055	0.009	0.007	0.003
$m = 2$	0.345	0.018	2.9×10^{-4}	0.014	0.001	1.5×10^{-4}	0.001	3.2×10^{-4}	5.7×10^{-5}
$m = 3$	0.192	0.0035	1.2×10^{-5}	0.004	1.1×10^{-4}	5.1×10^{-6}	2.3×10^{-4}	1.6×10^{-5}	1.8×10^{-6}
$m = 4$	0.105	6.9×10^{-4}	6.0×10^{-6}	0.001	1.3×10^{-5}	1.6×10^{-6}	0	0	2.4×10^{-7}

The values of parameters δ and Δ for the cosines of zenith angles $\mu = 0$ and $\mu_0 = 0.1$ (case I), $\mu = 0$ and $\mu_0 = 0.9$ (case II), and $\mu = 0.9$ and $\mu_0 = 0.9$ (case III) are listed in Tables 4 ($g = 0.5$) and 5 ($g = 0.85$) for the first and second orders of scattering. These tables indicate that the third term in Eq. (9) is close to zero if either angle cosine μ or μ_0 is close to or greater than 0.9. We refer to this methodology as the LS method.

A comparison of the zeroth harmonic calculated by the CI approximation, the LA method, the discrete ordinate method, and the method described in Refs. 10 and 11 is presented in Table 6. This table indicates that the simplest way to calculate the zeroth harmonic of the reflection function is the CI method (Eqs. 4) and (5). Errors of the LA method are in the 2–3% range for zenith-angle cosines μ_0 and μ and are greater than 0.2 and ~5% for one of the cosines equal to 0.2. The CI approximation yields errors of less than 1% for $\mu > 0.2$ and approximately 3–5% for $\mu = 0.2$. We compare the discrete ordinate method, the method of Refs. 10 and 11, and the two methodologies proposed above in Table 7 and calculate the reflection function. Two solar-angle cosines, five viewing-angle cosines, and an azimuth angle of 0 are considered. We found that the exactness of the PF approximation together with the CI method for the zeroth harmonic is quite good for all the angles considered. The LS method can be suitable for calculation of the reflection function, preferably in cases of nadir observation or when the Sun is close to zenith, because it shows good agreement with the result of numerical calculation for $\mu = 1$ and a much worse result for $\mu = 0.5$.

We propose the following algorithm for calculation of the reflection function $\rho(\varphi, \mu, \mu_0)$:

(1) If cosine μ or μ_0 is greater than 0.8, the zeroth harmonic is calculated with Eqs. (4) and (5) and is approximately equal to the reflection function.

(2) In the opposite case the approximations in Tables 1 and 2 for high harmonics must be added to the value of the zeroth harmonic.

4. Parameterization of Cloud Horizontal Inhomogeneity

A simple approximate parameterization of the cloud top boundary heterogeneity was proposed earlier.[19] A rough cloud top causes an increase of the part of diffuse radiation in the incident flux. Hence knowledge of this increase is essential for calculation of the radiative characteristics that depend on solar incident angle. The escape function together with the reflection function describes this dependence for the reflected radiance, and the escape function together with the plane albedo of the semi-infinite medium describes the reflected irradiance.[1,2] Thus we propose to replace all functions that depend on the incident-angle cosine μ_0 by their modifications according to the following expressions:

$$\begin{aligned} \rho^0(\mu, \mu_0) &= \rho^0(\mu, \mu_0)(1 - r) + ra(\mu), \\ K(\mu_0) &= K(\mu_0)(1 - r) + rn, \\ a(\mu_0) &= a(\mu_0)(1 - r) + ra^{\infty}, \end{aligned} \tag{15}$$

Table 5. Contributions of the First through the Seventh Harmonics Relative to the Zeroth Harmonic to the First and Second Scattering Orders for Phase Function Parameter $g = 0.85$

	Parameter δ			Parameter Δ ($i = 1$)			Parameter Δ ($i = 2$)		
Variable	Case I	Case II	Case III	Case I	Case II	Case III	Case I	Case II	Case III
μ; μ_0	0; 0.1	0; 0.9	0.9; 0.9	0; 0.1	0; 0.9	0.9; 0.9	0; 0.1	0; 0.9	0.9; 0.9
$m = 1$	0.905	0.086	0.016	0.158	0.050	0.016	0.062	0.040	0.015
$m = 2$	0.805	0.017	5.8×10^{-4}	0.121	0.007	5.9×10^{-4}	0.041	0.005	5.5×10^{-4}
$m = 3$	0.705	0.0038	2.5×10^{-5}	0.088	0.001	2.3×10^{-5}	0.026	8.8×10^{-4}	2.1×10^{-5}
$m = 4$	0.611	9.1×10^{-4}	1.1×10^{-6}	0.064	4.0×10^{-4}	9.2×10^{-6}	0.016	1.2×10^{-4}	7.9×10^{-6}
$m = 5$	0.528	2.3×10^{-4}	5.1×10^{-7}	0.047	3.8×10^{-5}	4.5×10^{-7}	0.010	1.8×10^{-5}	3.5×10^{-7}
$m = 6$	0.454	5.1×10^{-5}	2.8×10^{8}	0.034	6.9×10^{-6}	2.6×10^{-8}	0.006	3.1×10^{-6}	2.1×10^{-8}
$m = 7$	0.389	1.1×10^{-5}	1.0×10^{-9}	0.024	1.0×10^{-6}	7.8×10^{-9}	0.003	4.3×10^{-7}	5.9×10^{-9}

Table 6. Results of Approximate and Numerical Method Calculations of the Reflection Function Zeroth Harmonic[a]

μ	Isotropic Scattering	Error (%)	CI Method	Error (%)	LA Method	Error (%)	Ref. 4	Discrete Ordinate
$\mu_0 = 1$								
1	1.074	5.0	1.145	1.3	1.161	2.2	1.128	1.130
0.8	1.041	3.7	1.075	0.5	1.081	0.01	1.073	1.072
0.6	1.000	0.8	0.995	0.3	0.980	1.6	0.995	0.995
0.4	0.946	6.7	0.886	0.2	0.850	3.7	0.882	0.882
0.2	0.875	24.8	0.720	5.5	0.676	4.0	0.708	0.707
$\mu_0 = 0.5$								
1	0.975	2.5	0.944	0.1	0.919	2.6	0.943	0.942
0.8	0.981	0.9	0.965	0.7	0.944	2.9	0.974	0.973
0.6	0.990	2.6	0.993	0.8	0.978	3.7	1.010	1.012
0.4	1.002	4.6	1.035	1.4	1.028	2.2	1.062	1.063
0.2	1.021	4.3	1.099	3.0	1.106	3.6	1.064	1.063

[a]The CI method calculates the isotropic zeroth harmonic and adds the item that is linearly dependent on phase function parameter g for correction; the LA method calculates the zeroth harmonic with the linear approximation on the parameter g according to Table 1.

where spherical albedo a^{∞}, plane albedo $a(\mu_0)$, and the value of n are defined as

$$a^{\infty} = 2\int_0^1 a(\mu_0)\mu_0 d\mu_0 = 4\int_0^1 \mu_0 d\mu_0 \int_0^1 \rho^0(\mu, \mu_0)\mu d\mu,$$

$$n = 2\int_0^1 K(\mu_0)\mu_0 d\mu_0, \tag{16}$$

and the parameter r describes the completely diffuse part of the light in the incident flux.

The influence of the overlying atmospheric layers (including high thin clouds), the difference between the reflection functions of the real cloud (described by Mie phase function) and the model (described by the Henyey–Greenstein phase function) and the other factors that affect the angular dependence of the radiation are also partly corrected by the same parameter.

Let us consider the numerical and analytical results that concern cloud heterogeneity. There have been many studies of this subject in the past several years.[20–23] It was shown that the influence of geometric variations of the cloud parameters is greater than the internal influence for one order.[22] Analytical solutions[20,23] show that cloud heterogeneity greatly affects radiance and irradiance, and we can actually describe it by modifying the escape function (or analogous functions) with an expression similar to Eqs. (15).

There are different estimations of the power of such effects. In our case it is expressed by a value of the parameter r, and an analysis based on the studies mentioned above[20,23] allows us to let $r \sim 0.01$–0.1. Most results also show that the minimal disturbance in the radiation field caused by cloud heterogeneity is at a solar angle of 48–49°. As was mentioned above, all the functions that depend on the incident angle are approximately equal to the integrals of these angles, which is why the value of the parameter r is small if the measurement is made at these incident angles.

Parameter r can be estimated from the ground ra-

Table 7. Results of Approximations and Exact Methods of the Reflection Function Calculations[a]

μ	LS Method	Error (%)	PF Method	Error (%)	Ref. 4	Discrete Ordinate	Difference between Two Strict Methods (%)
$\mu_0 = 1$							
1	1.128	0	1.145	1.8	1.128	1.131	0.3
0.8	1.073	0.1	1.075	0.7	1.074	1.072	0.1
0.6	0.995	0.1	0.995	0.1	0.996	0.995	0.1
0.4	0.881	0.1	0.886	0.7	0.882	0.882	0
0.2	0.711	0.1	0.720	1.4	0.708	0.706	0.2
$\mu_0 = 0.5$							
1	0.950	0.4	0.944	1.1	0.943	0.942	0.1
0.8	1.006	10	1.104	1.7	1.124	1.123	0.1
0.6	1.044	22	1.328	1.1	1.347	1.350	0.2
0.4	1.146	33	1.684	1.4	1.710	1.713	0.2
0.2	1.309	42	2.245	1.6	2.178	2.183	0.2

[a]The LS method takes into account all the scattering orders for the zeroth harmonic and only the first scattering order for high harmonics; the PF method fits the power regression on parameter g for harmonics with numbers higher than 2.

diance or irradiance measurements in stable overcast conditions in the following way: Ground-based and satellite observations indicate that the dependence of radiance or irradiance on solar incident angle is less than was calculated or than the dependence on viewing angle[24] and termed a "violation of the directional reciprocity"[24] for reflected radiation. It is known that the dependence of both the incident-angle and the viewing-angle cosines on the radiation that escapes from an optically thick layer is described by the escape function $K(\mu_0)$, which has been presented in tabular and analytical form.[1,2,4] Thus time data taken for a period of several hours may give us the dependence of the escape function on the solar incident angle. If it differs from the dependence of radiance on the viewing angle it is possible to obtain a value of r as follows:

$$r = \frac{I(\mu_1, \mu_2) - I(\mu_2, \mu_1)}{1 - I(\mu_1, 0.67)} \frac{K_0(\mu_1)}{K_0(\mu_1) - K_0(\mu_2)}, \qquad (17)$$

where $I(\mu_0, \mu)$ is the observed (reflected or transmitted) radiance. In addition, we assume here that $\rho^0(\mu, 0.67) = K_0(0.67) = 1$ and that there is a small amount of radiation absorption. Certainly we need here a high cloud stability that is not often but sometimes possible, especially at northern latitudes. This method seems preferable for ground-based observation.

There is another way to estimate parameter r from multidirectional radiance measurements [e.g., with the Polarization and Directionality of the Earth's Reflectance (POLDER) instrument]. Approximate values of the optical thickness of the cloud layer are obtained for each viewing direction available for each pixel; conservative scattering is assumed at the first stage of data processing. Then the average value of the optical thickness is calculated for each pixel. The relative deviations from average of the optical thickness obtained for each direction can be taken as a measure of the deviation of the cloud top from the plane. It is necessary to have in mind that the deviation also includes the influence of the other factors mentioned above. Then we propose to evaluate parameter r as follows:

$$r = \frac{1}{N\bar{\tau}} \sum_{i=1}^{N} |\bar{\tau} - \tau_i|, \qquad (18)$$

where N is the number of viewing directions for each pixel and $\bar{\tau}$ is the average optical thickness over the viewing directions. This methodology was applied to POLDER level-2 data containing the reflected radiance in 14 directions. Pixels with greater than 0.5% cloud were chosen. Only averaged data for a low-resolution grid were available, and the size of a pixel was ~59 km. The date of the observations was 24 June 1997. Figure 6 shows the dependence of parameter r on the pixel number for two cloud fields. In fact, the value of r was found to range from 0.01 to 0.06.

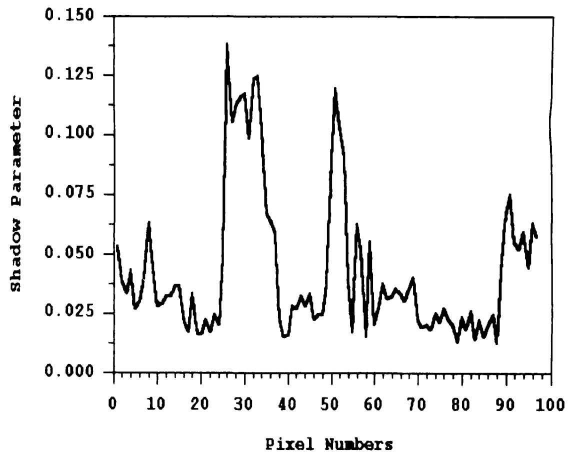

Fig. 6. Shadow parameter r for a cloud field at latitude 59.75° N on 24 June 1997 versus pixel number. Pixel size, approximately 60 km.

5. Conclusion

Two ways to calculate the reflection function in semi-infinite and conservative media (the PF and LS methods) have been proposed for which a Henyey–Greenstein phase function was assumed but the real phase function of the cloud was unknown. On the one hand the Mie phase function may describe natural clouds better than the Henyey–Greenstein phase function does. On the other hand, the influence of molecular scattering may be increased in clouds by multiple scattering and could smooth out the real phase function and the real reflection function. Thus the proposed method may be applicable to real clouds with better precision than we suspect.

The PF method together with the CI method takes into account the zeroth and the sixth azimuth harmonics. They are suitable for use over a wide range of phase function parameter g and zenith-viewing and solar angles if at least one cosine angle is greater than 0.15 (angle is less than 80°). This assumption does not restrict solution of the problem because the same requirements for the plane model of the atmosphere are usually made. It seems enough to calculate only the zeroth harmonic to obtain the reflection function in the case when at least one cosine angle is greater than 0.8 (angle is less than 37°). A similar condition is necessary for use of the LS method. Thus the PF method is more convenient than the LS method for calculating the reflected radiance at a larger angular range.

It is certain that analytical methodology is especially important for solution of the inverse problem, namely, for retrieval of the optical cloud parameter from radiance satellite observations. In such a case it is better to choose for measurements a viewing angle in the 45–50° range because this will result in the least effect of the phase function and the highest harmonics on the reflection function. Errors in the proposed approximations usually do not exceed 5%.

The way to take into account the heterogeneity of the cloud top border, the influence of the overlying atmosphere and high cirrus clouds, the differences between the real phase function and the Henyey–Greenstein phase function, and other factors is developed by use of a shadow parameter. It is certain that this method is approximate and partly takes into account all the above factors. Two ways to retrieve the shadow parameter from ground or satellite measurements were proposed. Results of estimating this parameter for two latitudes from POLDER observations were presented relative to pixel numbers. This parameter was found to range from 0.01 to 0.1; variations are in the 40–50% range for one site. No spectral dependence of shadow parameter r was found.

Appendix A

Here we briefly summarize the method[10] that we used for strict computation of the Fourier components of reflection function $\rho^m(\mu, \mu_0)$. Specifically, we numerically solved Ambartsumian's nonlinear integral equation[1]:

$$\begin{aligned}(\mu + \mu_0)\rho^m(\mu, \mu_0) = {} & \frac{\omega}{4} p^m(-\mu, \mu_0) \\ & + \frac{\omega}{2}\mu_0 \int_0^1 p^m(\mu, \mu')\rho^m(\mu', \mu_0)\mathrm{d}\mu' \\ & + \frac{\omega}{2}\mu \int_0^1 p^m(\mu', \mu_0)\rho^m(\mu, \mu')\mathrm{d}\mu' \\ & + \omega\mu\mu_0 \int_0^1 \int_0^1 \rho^m(\mu, \mu') \\ & \times p^m(-\mu', \mu'')\rho^m(\mu'', \mu_0)\mathrm{d}\mu'\mathrm{d}\mu'',\end{aligned} \qquad \text{(A1)}$$

where ω_0 is the single-scattering albedo and $p^m(\mu, \mu_0)$ are the Fourier components of the phase function. For $m \geq 1$, the solution of Eq. (A1) can easily be found by simple iterations. For $m = 0$, the standard scheme of successive iterations works well only if ω_0 is sufficiently far from unity, and it becomes inapplicable when $1 - \omega_0 \ll 1$. To ameliorate this convergence problem, it was suggested[10] that the zeroth harmonic of the reflection function $\rho^0(\mu, \mu_0)$ be modified after each iteration by use of the so-called Sobolev–van de Hulst relation[1]:

$$i(-\mu) = 2\int_0^1 \rho^0(\mu, \mu_0)\mu_0\mathrm{d}\mu_0. \qquad \text{(A2)}$$

The function $i(\mu)$ is the solution of the equation

$$i(\mu)(1 - k\mu) = \frac{\omega_0}{2}\int_{-1}^1 i(\mu')p^0(\mu, \mu')\mathrm{d}\mu', \qquad \text{(A3)}$$

in which we found the diffusion exponent k by satisfying the normalization condition

$$\frac{\omega_0}{2}\int_{-1}^1 i(\mu)\mathrm{d}\mu = 1. \qquad \text{(A4)}$$

The algorithm[10] of the solution is constructed in such a way that each successive iteration satisfies Eq. (A2) also. As a result, for each n (n is the number of iterations), the modified value of $\rho_n{}^0(\mu, \mu_0)$ is substituted into the right-hand side of Eq. (A1). For $m \geq 1$, we used the standard iterative procedure. This technique has proved to be highly efficient and gave accurate results even in the case of strong anisotropic scattering.[4,11]

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