# 博士論文

論文題目　The Stokes phenomena of additive linear difference equations
(加法的線形差分方程式のストークス現象)

氏　　名　勝島　義史

# The Stokes phenomena of additive linear difference equations

Yoshifumi Katsushima[a]

[a] *Graduate School of Mathematical Sciences, The University of Tokyo*
*3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan*
*e-mail: katsushi@ms.u-tokyo.ac.jp*

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**Abstract.** We show that the Stokes phenomena of difference equations are calculated by Borel-Laplace analysis, if the inverse Mellin transformation of the equations are Fuchsian differential equations. The Stokes multipliers of the fundamental solutions are expressed by the connection coefficients of the Fuchsian differential equations and local monodromies of the differential equations.

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## 1 Introduction

In this article, we deal with linear additive difference equations whose coefficients are rational functions, as subjects of complex analysis similar to differential equations. Algebraic multiplicative difference equations, so called *q-difference equations* are well studied recently. An additive difference equation with rational coefficients, in this text, means the equation such as

$$F(f(x),\, f(x+1),\, \dots,\, f(x+N),\, x) = 0. \tag{1}$$

Here, $F(y_0, y_1, \dots, y_N, x)$ is a polynomial in $\mathbb{C}[y_0, y_1, \dots, N, x]$. Indeed, additive difference equations are traditional, common difference equations. Although we get a multiplicative difference equation

$$F(g(t),\, g(qt),\, \dots,\, g(q^N t), \log_q(t)) = 0, \tag{2}$$

by changing the variable $x$ to $t$, $t = q^x$, we regard them as different subjects. The equation (2) is not a difference equation with rational coefficients, so it is difficult to treat the equation (2), though Poincaré or Picard looked them as the same objects. Therefore, it is meaningful to treat the equation (1) as an additive difference equation. The purpose of this paper is to investigate analytic continuations of solutions of linear difference equations whose coefficients are rational functions. Linear difference equations whose coefficients are rational, in this paper, means equations of which form are

$$Lf(x) = \sum_{n=0}^{N} a_n(x) f(x+n) = 0. \tag{3}$$

Here, $\{a_n(x)\}$ are polynomials of $x$, $a_n(x) \in \mathbb{C}[x]$. In order to compare difference equations with differential equations, we look over what is well known in the studies of algebraic linear ordinary differential equations.

Studies about the linear ordinary differential equations of Fuchsian types on the surface $\mathbb{P}^1$ with respect to the *global monodromy*, have been continued one hundred or more years. The global monodromy is a concept which describes transition of fundamental solutions when we continue them analytically along the loops in $\mathbb{P}^1$. However, most of differential equations' monodromies are very difficult to calculate. We call the differential equations *rigid* if their global monodromies are determined by the data of locations of the singularity points and characteristic exponents (or, we may say that the global monodromies of rigid equations are determined by local monodromies). For example, as is well known, the hypergeometric equation is a rigid equation. In this manner, the solutions of the linear ordinary differential equations are investigated from the view point of the complex analysis.

In a history of the study of difference equations, G.D.Birkhoff tried to solve the connection problem about the singular points $-\infty$ and $\infty$ or, *the generalized Riemann problem*[1]. The connection problem is the following.

**Problem 1.** *Let $A(x)$ be a $n \times n$ matrix of which elements are polynomials of $x$. Assume that $S(x)$ is a formal solution of a equation*

$$Y(x+1) = A(x)Y(x). \tag{4}$$

*Assume that there exists 2 analytic solutions $Y^-(x)$ and $Y^+(x)$ which have the asymptotic expansions $Y^-(x) \sim S(x)$ in the left half plane ($x \to -\infty$),* *and $Y^+(x) \sim S(x)$ in the right half plane $(x \to \infty)$. Calculate the connection matrix $P(x)$ which satisfies the relation*

$$Y^-(x) = Y^+(x)P(x). \tag{5}$$

The generalized Riemann problem, which is introduced by Birkhoff, is as follows.

**Problem 2.** *Are the difference equations determined uniquely by giving the characteristic constants?*

Where *the characteristic constants* in Birkhoff's sense, are specific parameters included in the matrices $S(x)$ and $P(x)$. For reference, to mention the notation Birkhoff made, the formal solution matrix $S(x) = (s_{i,j}(x))_{i,j}$ has a form

$$s_{i,j}(x) = x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\{s_{i,j}^{(0)} + s_{i,j}^{(1)} x^{-1} + (lower\, order)\}, \tag{6}$$

where $\mu$ is a degree of $A(x)$ with respect to $x$, and $P(x) = (p_{i,j}(x))_{i,j}$ has a form

$$\begin{aligned} p_{i,i} &= 1 + c_{i,i}^{(1)} e^{2\pi\sqrt{-1}x} + \cdots + c_{i,i}^{(\mu-1)} e^{2\pi\sqrt{-1}(\mu-1)x} + e^{2\pi\sqrt{-1}r_i} e^{2\pi\sqrt{-1}\mu x}, \\ p_{i,j} &= e^{2\pi\sqrt{-1}\lambda_{i,j}x}[c_{i,j}^{(0)} + \cdots + c_{i,j}^{(\mu-1)} e^{2\pi\sqrt{-1}(\mu-1)x}]. \end{aligned}$$

The characteristic constants are a set $\{\rho_j, r_j, c_{i,j}^{(k)}\}$ which appear in $S(x)$ and $P(x)$. Birkhoff announced that he solved the generalized Riemann problem of difference equations. Definitely, by diagonalizing $A_\mu$, the highest term of $A(x)$, we find the number of parameters included in $A(x)$ is $n^2\mu + n$, and the number of characteristic constants is $n(\mu-1) + (n^2-n)\mu + 2n = n^2\mu + n$, so the correspondence must be there. However, there remains a question. *Can we make the space of characteristic constants smaller*? There are too many parameters in the set of characteristic constants to explain them from the view point of analysis of difference equations. We must determine what kind of parameters are essential.

In this article, we do not touch the generalized Riemann problem, but we consider the connection problem of single difference equation (3). This correspond to the Problem 1 of the case that the elements of the coefficient matrix are rational functions.

Loosely speaking, Birkhoff "solved" the connection problem by using the following method.

$$
\begin{aligned}
Y(x) &= A(x)^{-1}Y(x+1) = \dots \\
&= A(x)^{-1}A(x+1)^{-1}\cdots A(x+k-1)^{-1}Y(x+k), \\
Y(x) &= A(x-1)Y(x-1) = \dots \\
&= A(x-1)A(x-2)\cdots A(x-k)Y(x-k).
\end{aligned}
$$

Therefore, if we can control the infinite products $\prod_{k=0}^{\infty} A(x+k)^{-1}$ and $\prod_{k=1}^{\infty} A(x-k)$, we get the connection matrix

$$P(x) = (\prod_{k=1}^{\infty} A(x-k))^{-1} \prod_{k=0}^{\infty} A(x+k)^{-1}.$$

That trial is natural, because the difference equation includes the difference operator $\sigma : f(x) \mapsto f(x+1)$. Indeed, we can bring the information of the domain $U^+$ to $U$, or $U^-$ to $U$.

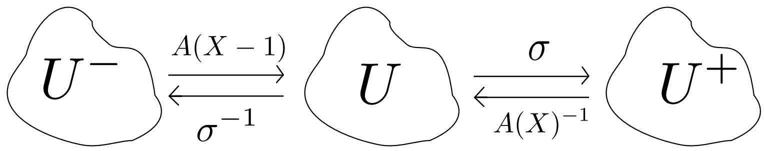


Figure 1: the acts of $\sigma$

Then, we should have questions; can we connect the functions analytically to the *different directions*, or, from the very first, is there any analytic solutions of difference equations? The question about the existence of analytic solutions is quite difficult. However, there exist formal power series solutions of linear difference equations whose coefficients are polynomials, with their growth estimates of Gevrey order. In this paper, we study difference equations in the aspect of *Stokes phenomena* of the formal fundamental solutions, because we are concerned with the analytic continuation of the solutions in wide domains or sectors. Birkhoff tried to connect solutions only along the line $\operatorname{Im} x = (constant)$ or a strip region including this line, but in this text, we try to connect solutions to the direction of rotation; $x \mapsto x \cdot e^{i\theta}$. An advantage of this new method is giving us asymptotical information of difference equations thoroughly. For example, *the hypergeometric difference equation* has infinitely many Stokes directions. We can calculate the Stokes multipliers on each directions by that method. Here, we used a word "the hypergeometric difference equation". In the study of $q$-difference equations, *the basic hypergeometric difference equation* is well known. That is a second order $q$-difference equation, whose coefficients are polynomials of degree 1, and have some generic conditions. We can find the similar equation, in the case of additive difference equations. We call it the hypergeometric difference equation. In fact, the hypergeometric difference equation is well known as a contiguity relation of the hypergeometric series (which is a solution of the hypergeometric differential equation).

This text consists of 3 sections. In the first section, we compare formal solutions of difference equations with formal solutions of differential equations of Fuchsian type. Differential equations of Fuchsian type have local solutions around their regular singular points, which are composed by Frobenius method. We usually calculate characteristic exponents at regular singular points, and create power series solutions recursively. In short, singular points and characteristic exponents are important to make a basis of local solutions. We suggest formal solutions of linear difference equations similar to differential equations of Fuchsian type, and we reveal that what should we calculate for consisting formal solutions, in the first section.

In the second section, we review the Borel-Laplace analysis and Stokes phenomena quickly. The Stokes phenomena is a term of asymptotic analysis; in short, the Stokes phenomena is a phenomena: "*the change of asymptotic expansion of the analytic functions, when we change a sector for asymptotic expansion*". Or we may say the phenomena "*exponentially small terms appears when we connect the analytic functions*". The Borel transformation is a formal inverse Laplace transformation of the formal power series. We can create analytic functions by combining the Borel transform and Laplace transform for formal power series, on some conditions. The Borel-Laplace analysis is a studies for these newly created analytic functions in the aspects of Stokes phenomena or analytic continuations, often applied to differential equations in a complex domain. We see these facts briefly, and show that the Stokes phenomena also appear in the analysis of difference equations in this section. At the last of this section, although the discussion leaves from the main subject, we apply the Borel-Laplace analysis to the hypergeometric difference equation and we get the monodromies of the solutions of the hypergeometric "*differential*" equation. In other words, the difference equation possess the analytic property of the differential equation. The author wonders why the analysis of difference equations gives us informations of differential equations. We leave this issue for the future.

In the third section, we will see how we can calculate Stokes multipliers of solutions. Unfortunately, we cannot understand the calculus of the Stokes multipliers by seeing only Borel transform of the solutions. Therefore we introduce *the inverse Mellin transformation* of difference equations, and watch them in details. The Borel transformation and the inverse Mellin transformation are equivalent each other when we change variables, but their merits are different. Roughly speaking, the Borel-Laplace analysis is an useful tool for seeing when the Stokes phenomena occurs, and by taking an inverse Mellin transform, we can specify what kind of characters we should calculate for revealing the Stokes multipliers of the solutions of difference equations. There is a technical difficulty to calculate the connection formulae of Borel transformed equations' solutions, and also there is a difficulty to make sure that which direction a Stokes phenomenon occurs, only by seeing the inverse Mellin transformation. By combining these two transformations with making up for each other's weak points, we can verify the analytic continuation of solutions in details. We also see benefits of these calculus in examples. In the last of this section, we confirm that we can calculate the connection formulae of the beta function's difference equation and the hypergeometric function's difference equation, for the results of Stokes phenomena.

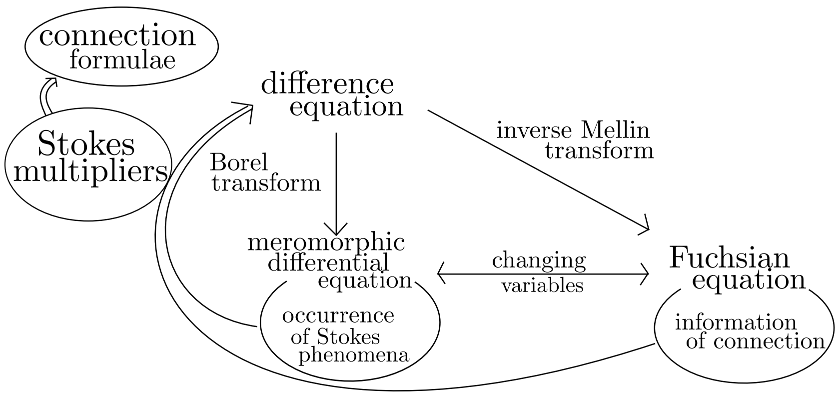


Figure 2: conceptual diagram of theory

The main theorem of this paper is as follows.

**Theorem 3.** *Let $Lf(x) = 0$ be a difference equation. Assume that the inverse Mellin transformation of $L$, $\mathcal{M}^{-1}L$ be a Fuchsian type differential operator. If we have connection formulae between the singular points of the equation $\mathcal{M}^{-1}L(\mathcal{M}^{-1}f)(t) = 0$, then the following holds.*

1. *The Stokes multipliers of the difference equation can be calculated by using the connection formulae of the Fuchsian differential equation $\mathcal{M}^{-1}L(\mathcal{M}^{-1}f)(t) = 0$.*

2. *In particular, the connection of the solutions of the difference equation can be calculated by using the Stokes multipliers which are obtained by 1. Especially, the connection is represented by Fourier series.*

# 2 The basic parameters and formal solutions.

In this paper, we denote $\vartheta$ as a Euler operator,

$$\vartheta = x\frac{d}{dx} \tag{7}$$

for simplifying. The Euler operator is useful for the analysis of regular singular points of differential equations. It is important calculating the characteristic exponents for the construction of the convergent power series solutions around the regular singular point of the differential equations of fuchsian types, because they determine the major parts of the local solutions. At the beginning, we remember the proposition about the singular points of the linear differential equations.

**Proposition 4.** *Let $L$ be a differential operator of which coefficients are polynomials:*

$$L = \sum_{j=0}^{N} a_j(z)\frac{d^j}{dz^j}, \quad a_j(z) \in \mathbb{C}[z]. \tag{8}$$

*Assume that $\{a_j(z)\}$ has no common factors. Then the singular points of the solutions of the linear differential equation $Lf(z) = 0$ are limited at most to the points $z = z_1, z_2, \ldots, z_n, \infty$, where $z_1, \ldots, z_n, z_j \neq z_k (j \neq k)$ are the zero points of $a_N(z) = A(z - z_1)^{l_1} \ldots (z - z_n)^{l_n}$, $A \in \mathbb{C}$.*

The proof is clear because we can construct analytic solutions around $z = z_0, a(z_0) \neq 0$ by substituting the power series $f(z) = \sum_{k=0}^{\infty} f_k(z - z_0)^k$, here we can choose $N$-parameters $\{f_k\}_{k=0}^{k=N-1}$ freely. Therefore the singular points of differential equations' solutions are limited to the zero points of $a_N(z)$.

**Proposition 5.** *Let $L$ be a differential operator which satisfies the conditions of Propisition 4. We may assume that $z_1 = 0$. Then $z_1 = 0$ is at most a regular singular point if there exists a natural number $l_1$, such that $z^{N-l_1}L$ becomes the following form.*

$$z^{N-l_1}L = \sum_{s \geq 0} z^s P_s(\vartheta). \tag{9}$$

*Here, $P_s(x) \in \mathbb{C}[x]$, and $\deg P_0(x) = N$. We call the roots $\nu_{1,1}, \ldots, \nu_{1,N}$ of the indicial polynomial $P_0(\nu)$, the characteristic exponents of the regular singular point $z = 0$.*

If the polynomial $P_0(\vartheta)$ has a degree $\deg P_0 < N$, the point $z = 0$ is called a *irregular singular* point. We do not touch the case that differential equations have irregular singular points. We call the differential equations *Fuchsian types* when all of their singular points are regular singular points. The table of the set of the singular points and characteristic exponents is called the *Riemann scheme*:

$$\left\{ \begin{array}{cccc} z = z_1 & \cdots & z_n & \infty \\ \nu_{1,1} & \cdots & \nu_{n,1} & \nu_{\infty,1} \\ \vdots & & \vdots & \vdots \\ \nu_{1,N} & \cdots & \nu_{1,N} & \nu_{\infty,N} \end{array} \right\}. \tag{10}$$

The Riemann scheme seems to include almost all important data of the local solutions, because it determine the form of the local solutions in generic cases. For example, the following proposition holds.

**Proposition 6.** *Let $\nu_{l,1}, \ldots, \nu_{l,N}$ be characteristic exponents of the regular singular point $z = z_l$ which satisfy the generic condition*

$$\forall j \neq k, \quad \nu_{l,j} - \nu_{l,k} \notin \mathbb{Z}. \tag{11}$$

*Then there exist $N$ solutions of the form*

$$f_j^l(z) = (z - z_l)^{\nu_{l,j}} \sum_{n=0}^{\infty} a_{n,j}^l (z - z_l)^n \qquad (j = 1, 2, \ldots, N). \tag{12}$$

*Here $a_{0,j}^l$ is an arbitrary number in $\mathbb{C} \setminus \{0\}$, and $\{a_{n,j}^l\}$ are determined by the Frobenius method.*

Thus, the Riemann scheme holds almost all information around the regular singular points. The monodromies of rigid equations are determined by these data. More precise discussion will be done in section 4, where we extend the definition of such data to the *generalized Riemann scheme* which is recently mentioned by T. Oshima[2].

Now, let us consider the similar situation about additive difference equations, namely, we clarify the important parameters of difference equations to determine formal solutions. In the case of differential equations the characteristic exponents immediately correspond to the solutions, hence we examine the same parameters which correspond to the solutions. We prepare some definitions and a proposition about formal solutions of linear difference equations.

**Definition 7.** Let L be an additive difference operator

$$L = \sum_{j=0}^{N} a_j(x)\sigma^j \tag{13}$$

where $\sigma$ is the difference operator $\sigma : f(x) \mapsto f(x+1)$, and coefficients $\{a_j(x)\}$ are polynomials of $x$, of which coefficients are denoted as $\{a_{j,l}\}_{l=0,\ldots,l_j}$, $a_j(x) = \sum_{l=0}^{l_j} a_{j,l}x^l \in \mathbb{C}[x]$, $a_{j,l_j} \neq 0$. We assume about the degrees of the coefficients that $M := l_0 = l_N$ and $l_0 \geq l_j$ for all $j$, $1 \leq j \leq N-1$. Then we call the following polynomial *the characteristic polynomial.*

$$D(\lambda) := \sum_{j=0}^{N} a_{j,M}\lambda^j. \tag{14}$$

Here, some coefficients $a_{j,M}$ may not be defined, then we interpret them to be 0. We call the roots of $D(\lambda) = 0$ *the deterministic roots.*

The deterministic roots are used to solve difference equations with constant coefficients.

**Example 8.** Assume that $A_j \in \mathbb{C}$, $A_N$ and $A_0$ are not 0, and

$$A_N\lambda^N + A_{N-1}\lambda^{N-1}\cdots + A_0 = 0$$

has roots $\lambda_1, \ldots, \lambda_N$ which differ from each other. Then the equation

$$A_N f(x+N) + A_{N-1}f(x+N-1) + \cdots + A_0 f(x) = 0$$

has N solutions

$$f_k(x) = \lambda_k^x \qquad (k = 1, \ldots, N).$$

The proof is trivial; we can verify that they are solutions by substituting them into the equation.

The proof of this example is easy, however, it is useful for solving formally the difference equations, or analyzing the difference equations asymptotically like the theorem which is called Poincaré-Perron. Now, we define the *characteristic exponents* of the difference equations. Roughly speaking, the deterministic roots shows how the solution of the difference equation growth exponentially, and the characteristic exponents indicates the order of the polynomial growth (although the term turned out to be not polynomials). Let $L$ be a difference operator satisfying the condition denoted in Definition 7., and we assume that the roots of $D(\lambda) = 0$ differ from each other. We denote $L_k$ as the difference operator

$$L_k : = \lambda_k^{-x} L \lambda_k^x = \sum_{j=0}^{N} \lambda_k^j a_j(x)\sigma^j \tag{15}$$

$$= \sum_{j=0}^{N}\sum_{l=0}^{M} \lambda_k^j a_{j,l} x^l \sigma^j. \tag{16}$$

We also denote $\hat{L}_k$ as the differential operator

$$\hat{L}_k = \sum_{j=0}^{N}\sum_{l=0}^{M} \lambda_k^j a_{j,l} x^l (1 + j\frac{d}{dx}) \tag{17}$$

Then, *the highest weight operator* of $L$ is written as

$$(\hat{L_k})_{\max} = x^{M-1}H_k := x^{M-1}\left(\sum_{j=0}^{N} j\lambda_k^j a_{j,M} x\frac{d}{dx} + \sum_{j=0}^{N}\lambda_k^j a_{j,M-1}\right). \tag{18}$$

The highest weight operator in this context, is defined as follows. Let $D$ be a differential operator (including infinite order differential operator) which has polynomial coefficients

$$D = \sum_{p=0}^{finite} \sum_{q \geq 0} C_{p,q} x^p \frac{d^q}{dx^q}.$$

Let $\Delta_D$ be the subset of $\mathbb{N}^2$, $\Delta_D = \{(p,q) | C_{p,q} \neq 0\}$. We take the weight $(x, \frac{d}{dx}) \mapsto (1, -1)$. Let $H$ be a integer $H = \max_{(p,q) \in \Delta_D}(p-q)$. The highest weight operator $D_{\max}$ is defined as

$$D_{\max} = \sum_{(p,q) \in \Delta_D} C_{p,q} x^p \frac{d^q}{dx^q}.$$

In these conditions ,

**Definition 9.** Let $\nu_k$ be a complex number

$$\nu_k := \frac{\sum_{j=0}^{N} \lambda_k^j a_{j,M-1}}{\sum_{j=0}^{N} j \lambda_k^j a_{j,M}}. \tag{19}$$

We call $\nu_k$ the characteristic exponent of $L$ related to a deterministic root $\lambda_k$.

This definition is not essential. This characteristic exponent is obtained from the highest weight operator; $\nu_k$ is a characteristic exponent of $H_k$ at the singular point $x = \infty$. Such sets of constants $\{\lambda_k\}, \{\nu_k\}$ play an important roles to construct the formal power series solutions of the difference equations. Indeed, the next proposition holds.

**Proposition 10.** *Let $L$ be a difference operator satisfying the condition denoted in* Definition 7.*, and the deterministic roots of $L$ are differ from each other. Then, the equation $Lf(x) = 0$ has formal solutions written in the form*

$$f^k(x) = \lambda_k^x x^{-\nu_k} g_k(x), \tag{20}$$

*where $g_k(x)$ is a formal power series of $x^{-1}$, $g_k(x) = \sum_{n=0}^{\infty} g_{k,n} x^{-n}$.*

*Proof.* Let us take a gauge transformation of $L$, $L_{gauge} = \lambda_k^{-x} x^{\nu_k} L \lambda_k x^{-\nu_k}$. Then, have a look at the acts of $L_{gauge}$ on the space of formal power series $\mathbb{C}[[x^{-1}]] = \{g(x) | g(x) = \sum_{j=0}^{\infty} g_j x^{-j}\}$. By taking a simple calculus, we find

$$\begin{aligned} L_{gauge} &= \sum_{n=0}^{N} \lambda_k^n (1+\frac{n}{x})^{-\nu_k} a_n(x) \sigma^n \\ &= \sum_{n=0}^{N} \lambda_k^n \sum_{m=0}^{\infty} \frac{(-1)^m (\nu_k)_m}{m!} \left(\frac{n}{x}\right)^m \sum_{l=0}^{M} a_{n,l} x^l \sigma^n \\ &= x^M \sum_{n=0}^{N} \lambda_k^n \left[a_{n,M} + (-n\nu_k a_{n,M} + a_{n,M-1}) x^{-1} + x^{-2} r_n(x)\right] \sigma^n. \end{aligned}$$

Where $r_n(x)$ means a power series of $x^{-1}$. $\sigma^n$ acts to $x^{-j}$ in this way;

$$\sigma^n x^{-j} = x^{-j} \sum_{p=0}^{\infty} \frac{(-1)^p (j)_p}{p!} n^p x^{-p} = x^{-j} - jnx^{-j-1} + \dots.$$

We get the formal Laurent series $h(x) = L_{gauge} g(x)$ has a degree at most $M-1$, because the coeficient of the term $x^M$ is $(\sum_{n=0}^{N} a_{n,M} \lambda_k^n) g_0 = 0$, from the definition of deterministic roots $\{\lambda_k\}$. The coefficient of $x^{M-1}$ is

$$g_0 \sum_{n=0}^{N} \lambda_k^n (-n\nu_k a_{n,M} + a_{n,M-1}) + g_1 (\sum_{n=0}^{N} a_{n,M} \lambda_k^n) = 0.$$

Here, we used the definition of characteristic exponents $\{\nu_k\}$. By calculating the coefficients of $h(x) = \sum_{p=-M+2}^{\infty} h_p x^{-p}$, we find

$$\begin{aligned} h_p &= g_{p+M-1} \sum_{n=0}^{N} \lambda_k^n (-n(\nu_k + p + M - 1) + a_{n,M-1}) + R_{k,p} \\ &= -g_{p+M-1} (\sum_{n=0}^{N} n \lambda_k^n)(p + M - 1) + R_{k,p} \end{aligned}$$

where $R_{k,p}$ means the terms of $\{g_0, \dots, g_{p+M-2}\}$. Hence, if we hold the formal Laurent series $h(x) = 0$, then we get

$$g_{p+M-1} = \frac{R_{k,p}}{(\sum_{n=0}^{N} n \lambda_k^n)(p + M - 1)}, \qquad p + M - 1 = 1, 2, \dots.$$

Let us see the denominator of the right-hand side of this equation. We can write down it as

$$\sum_{n=0}^{N} n\lambda_k^n(p+M-1) = \lambda_k D'(\lambda_k)(p+M-1).$$

Here, $D$ is the characteristic polynomial of the difference equation $Lf(x) = 0$. By the assumption that $L$ satisfies the condition denoted in Definition 7, we find $\lambda_k \neq 0$, and also we assumed that the characteristic polynomial $D$ does not have any multiple root, we get $D'(\lambda_k) \neq 0$. Consequently, we get the formal power series $g(x)$ satisfying $L_{gauge}g(x) = 0$, where we can choose $g_0 \in \mathbb{C}$ arbitrarily. □

In brief, the set $\{f_k(x)\}$ is a basis of the space of the solutions formally. In this sense, $\{\lambda_k\}, \{\nu_k\}$ are the basic parameters of the linear difference equations like the characteristic exponents of differential equations. However, there must be more parameters to identify the difference equations. In other words, the difference equations are not specified only by their formal solutions around $x = \infty$. We can construct two difference equations $Lf = 0$ and $\tilde{L}f = 0$, having the same deterministic roots and the same characteristic exponents, but they are different from each other; for example, the coefficients polynomials' degree about the variable $x$ can be different, $\deg_x(L) \neq \deg_x(\tilde{L})$. It is doubtful that they have the same analytic aspects, so we should make an effort to specify difference equations. To find another parameters of the difference equations, we recall the case of the differential equations. A special affair of rigid differential equations is that we can calculate their quantities which shows us analytic properties for a global condition. The global monodromy is such a property. Therefore we introduce the global analytic properties on the difference equations, by following a precedent of the case of differential equations. An analytic property we already know about the formal power series (or we can say asymptotic expansions of analytic functions) is *the Stokes phenomenon*. We will see *the Borel Laplace analysis* in the next section to verify how the Stokes phenomena occur, and we will see the Stokes structures of the difference equations in the section 3.

**Remark 11.** *We assumed that the deterministic polynomial has $N$ roots which are different from each other, however, we can construct formal solutions when the deterministic roots are not different from each other. In such cases, we must define the characteristic exponents of difference equations in* *another way. We replaced the difference operator* $\sigma^j$ *with* $1+j\frac{d}{dx}$ *in the operator* (19)*, but we should replace them with* $\sigma^j \mapsto \sum_{p=0}^{n_k} \frac{1}{p!}(j\frac{d}{dx})^p$*, where* $n_k$ *is a multiplicity of the deterministic root* $\lambda_k$*. When the equation has* $n_k$ *solutions of the form* (20)*, the highest weight operator becomes the fuchsian operator of order* $n_k$*. If there does not exist the solution of the form* (20)*, the highest weight operator does not include the differential term.*

# 3 The Borel-Laplace analysis and the Stokes phenomena

In this section, we introduce the Borel resummation of the formal power series, and its analytic continuation . The Borel resummation is a method which is used for asymptotic analysis. We also describe an example of difference equations of which Stokes phenomena can be calculated easily. In fact, that difference equation is the equation which corresponds to the Gauss' hypergeometric functions, therefore, connecting Stokes multipliers matches to the local monodromy of the Gauss' hypergeometric equation.

## 3.1 Borel resummation

First of all, we define the Borel transform of the formal power series. The Borel transform is defined as a map from the formal power series to the formal power series. However, it acts to their coefficients to decrease the growth of them. Therefore, if a formal power series satisfies a good condition for coefficients (the condition is called Gevrey 1) , then it can be transformed to an analytic function. The definitions and Proposition 15 written in this subsection are detailed in [4].

**Definition 12.** Let $x$ be a positive constant and $f$ be a formal power series which has the form

$$f(x) = e^{\lambda x} x^{-\nu} \sum_{n=0}^{\infty} f_n x^{-n} \tag{21}$$

where $\lambda, \nu, f_n \in \mathbb{C}$, and $\nu$ is not the negative integer. We define the Borel transform of $f$ as

$$\mathcal{B}(f)(\xi) := f_B(\xi) := \sum_{n=0}^{\infty} \frac{f_n}{\Gamma(n+\nu)} (\xi+\lambda)^{n+\nu-1}. \tag{22}$$

The Borel transform is a formal inverse Laplace transform. To make it clear, calculating the Laplace transform of $\mathcal{B}(f)(\xi)$ term by term formally, we get formal power series $f(x)$ again. $f_B(\xi)$ is an analytic function if and only if the coefficients $\{f_n\}$ satisfies the condition

$$\exists A, C > 0, \quad \forall n, \quad |f_n| \leq AC^n n!. \tag{23}$$

**Definition 13.** Assume that the function $f_B(\xi)$ is continued analytically along the line $(-\lambda, \infty)$. The Borel resummation of $f(x)$ is defined by

$$F(x) = \mathcal{L} \circ \mathcal{B}(f)(x) = \int_{-\lambda}^{\infty} e^{-x\xi} f_B(\xi) d\xi \tag{24}$$

when the integral converges.

Under these definitions the variable $x$ is limited to a real variable, but we can remove the restriction as follows. Let us denote the complex variable $x$ in the polar form $x = re^{i\theta}$. Then we find the formal power series $f(re^{i\theta})$ is

$$f(re^{i\theta}) = e^{(\lambda e^{i\theta})r} \sum_{n=0}^{\infty} (f_n e^{-(n+\nu)i\theta}) r^{-n-\nu}. \tag{25}$$

The Borel transform of (25) with respect to the variable $r$ is

$$\mathcal{B}(f)(\rho) = \sum_{n=0}^{\infty} \frac{f_n e^{-(n+\nu)i\theta}}{\Gamma(n+\nu)} (\rho + \lambda e^{i\theta})^{n+\nu-1} \tag{26}$$

$$= \sum_{n=0}^{\infty} \frac{f_n}{\Gamma(n+\nu)} (e^{-i\theta}\rho + \lambda)^{n+\nu-1} \cdot e^{-i\theta} = f_B(e^{-i\theta}\rho) \cdot e^{-i\theta}. \tag{27}$$

Therefore the Laplace transform of $\mathcal{B}f(\rho)$ is

$$\int_{-\lambda e^{i\theta}}^{\infty} e^{-r\rho} f_B(e^{-i\theta}\rho) \cdot e^{-i\theta} d\rho = \int_{-\lambda}^{e^{-i\theta}\infty} e^{-x\xi} f_B(\xi) d\xi \tag{28}$$

here, we chose the variable $\xi = e^{-i\theta}\rho$. Thus, we saw that the Borel resummation can be extended to the complex variable naturally. The advantage of the Borel resummation is a possibility of analyzing the function in the Borel plane. Although the original function $f(x)$ is a formal power series, its Borel transform $f_B(\xi)$ can be an analytic function, so we can obtain the analytic information from it. Before we see them, we prepare a proposition for the relation of the difference operator and the Borel transform.

**Proposition 14.** *Let $f(x)$ be a formal power series of the form* (21) *satisfying the condition* (23)*. Then $\sigma f(x)$ satisfies the condition* (23) *again, and $\mathcal{B}(\sigma(f))(\xi) = e^{-\xi} f_B(\xi)$ holds.*

*Proof.* First of all, we confirm the action of $\sigma$ to a formal power series, and later, we prove that $\sigma$ holds the condition (23). The difference operator $\sigma$ acts naturally to the monomial $x^{-n-\nu}$ as

$$\sigma : x^{-n-\nu} \mapsto x^{-n-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k (n+\nu)_k}{k!} x^{-k}. \tag{29}$$

Then we find the relation

$$\sigma \left( e^{\lambda x} x^{-nu} \sum_{n=0}^{\infty} f_n x^{-n} \right) = e^{\lambda x} x^{-\nu} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k e^{\lambda} (n+\nu)_k}{k!} x^{-n-k}. \tag{30}$$

Hence we get by confirming the Borel transform of this power series

$$\mathcal{B}(\sigma f)(\xi) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k e^{\lambda} (n+\nu)_k}{k! \Gamma(n+k+\nu)} (\xi+\lambda)^{n+k+\nu-1} \tag{31}$$

$$= e^{\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (\xi+\lambda)^k \cdot \sum_{n=0}^{\infty} \frac{f_n}{\Gamma(n+\nu)} (\xi+\lambda)^{n+\nu-1} \tag{32}$$

$$= e^{\lambda} e^{-\xi-\lambda} \mathcal{B} f(\xi). \tag{33}$$

Thus we get $\mathcal{B}(\sigma f)(\xi) = e^{-\xi} f_B(\xi)$. Now it is clear that $\sigma$ holds the condition (23) because $(\xi+\lambda)^{1-\nu} e^{-\xi} f_B(\xi)$ is a convergent power series at $\xi = -\lambda$, when $f(x)$ satisfies the condition (23). □

In the same way, we can prove the relation $\mathcal{B}(x \cdot f)(\xi) = \partial_\xi f_B(\xi)$, where $\partial_\xi = \frac{d}{d\xi}$.

**Proposition 15.** *Let $f(x)$ be a formal power series of the form* (21)*. Assume that $f$ satisfies the condition* (23)*. Then the following holds.*

$$\mathcal{B}(x \cdot f)(\xi) = \partial_\xi f_B(\xi). \tag{34}$$

*Proof.* Let us see the left-hand side of the equation (34). We find

$$\begin{aligned}\mathcal{B}(x\cdot f)(\xi) &= \mathcal{B}(e^{\lambda x}x^{-\nu+1}\sum_{n=0}^{\infty} f_n x^{-n}) \\ &= \sum_{n=0}^{\infty}\frac{f_n}{\Gamma(n+\nu-1)}(\xi+\lambda)^{n+\nu-2} \\ &= \sum_{n=0}^{\infty}\frac{f_n}{\Gamma(n+\nu)}(n+\nu-1)(\xi+\lambda)^{n+\nu-2} \\ &= \partial_\xi \mathcal{B}f(\xi).\end{aligned}$$

□

We immediately get the following proposition about the solution of the linear difference equation.

**Proposition 16.** *Let the formal power series* (21) *satisfies formally the linear difference equation*

$$Lf(x) = \sum_{j=0}^{N} a_j(x)\sigma^j f(x) = 0. \tag{35}$$

*Assume that* $L$ *satisfies the condition denoted in Definition 7, and the characteristic polynomial* $D$ *does not have any multiple root. Then* $f_B(\xi)$*, the Borel transform of the formal power series, satisfies the differential equation*

$$\sum_{j=0}^{N} a_j(\partial_\xi)e^{-j\xi} f_B(\xi) = \sum_{j=0}^{N} e^{-j\xi}a_j(\partial_\xi - j)f_B(x) = 0. \tag{36}$$

*Furthermore,* $f(x)$ *satisfies the condition* (23)*.*

*Proof.* It is obvious that the Borel transform of the formal solution of (35) satisfies the differential equation (36) from the Propositions 14 and 15. Now, the formal power series $f_B(\xi)$ has a singular point $\xi = -\lambda$. From the general theory of meromorphic differential equations, $-\lambda$ is limited to the singular points of the differential equation, and if the singular point is a regular singular point, the formal power series solutions of the differential equation converge. The singular points of the differential equation is the zero point of the coefficient of the highest order differential term. By following the notation of the Definition 7 in the previous section, the coefficient of the highest order differential term is

$$\sum_{j=0}^{N} e^{-j\xi} a_{j,M} = D(e^{-\xi}).$$

Therefore, there exists a deterministic root $\lambda_k$ such that $e^{\lambda} = \lambda_k$. Because $D$ does not have any multiple root, the point $\xi = \lambda = -\log \lambda_k$ is a regular singular point. At that regular singular point, we can calculate the characteristic exponents of the differential equation. They are

$$0, 1, \ldots, M-2,\ \nu_k - 1.$$

Hence, the Borel transform of the formal solution of the difference equation correspond to the convergent power series solution of (36). Consequently, we get the estimate

$$\exists A,\ C > 0\ , \forall n \in \mathbb{N},\ \left| \frac{f_n}{\Gamma(n+\nu)} \right| \leq AC^n.$$

After that, by changing $A, C$ slightly greater, we get (23). □

We call the equation (36) the Borel transform of the difference equation (35). In the next subsection, we apply this proposition to the simple difference equation which is obtained from the Gauss' hypergeometric function ${}_2F_1(\alpha, \beta, \gamma; x)$.

## 3.2 A simple example of a difference equation and the Stokes phenomenon

The hypergeometric differential equation has shift operators which adds $\pm 1$ to the parameter $\alpha$, $\beta$ or $\gamma$, for example, $x\frac{d}{dx} + \alpha = \vartheta + \alpha$ acts to the Riemann scheme as follows:

$$\left\{ \begin{matrix} x=0 & 1 & \infty \\ 0 & 0 & \alpha \\ 1-\gamma & \gamma-\alpha-\beta & \beta \end{matrix} \right\}$$
$$\mapsto \left\{ \begin{matrix} 0 & 1 & \infty \\ 0 & 0 & \alpha+1 \\ 1-\gamma & \gamma-(\alpha+1)-\beta & \beta \end{matrix} \right\}.$$

We can interpret this fact in an aspect of operators. Let $L$ be the differential operator

$$L = x(\vartheta+\alpha)(\vartheta+\beta) - \vartheta(\vartheta+\gamma-1).$$

We call this operator the hypergeometric differential operator in this text. Then, the following conjugate transform holds;

$$\begin{aligned}(\vartheta+\alpha)L(\vartheta+\alpha)^{-1} &= [x(\vartheta+\alpha+1)(\vartheta+\alpha)(\vartheta+\beta) \\ &\quad -\vartheta(\vartheta+\gamma-1)(\vartheta+\alpha)](\vartheta+\alpha)^{-1} \\ &= x(\vartheta+\alpha+1)(\vartheta+\beta) - \vartheta(\vartheta+\gamma-1).\end{aligned}$$

Thus, we see the action of operator $(\vartheta+\alpha)$ to the space of solutions is denoted as the changing of the Riemann scheme above. In this time, we examine this operator to the hypergeometric series. The hypergeometric series ${}_2F_1(\alpha,\beta,\gamma;x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}x^n$ satisfies the next relation

$$\frac{\vartheta+\alpha}{\alpha}\cdot\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}x^n = \sum_{n=0}^{\infty}\frac{(\alpha+1)_n(\beta)_n}{(\gamma)_n n!}x^n. \tag{37}$$

Therefore, by setting the difference operator $\sigma = \frac{\vartheta+\alpha}{\alpha}$ and $\sigma^2 = \frac{(\vartheta+\alpha+1)(\vartheta+\alpha)}{(\alpha+1)\alpha}$, and dividing the differential operator $L$ by $\sigma^2$ first, and then by $\sigma$, we get the next difference equation about the hypergeometric series.

$$[(1-x)(\alpha+1)\sigma^2 + \{(x-2)(\alpha+1)+\gamma-\beta x\}\sigma \tag{38}$$

$$+\alpha+1-\gamma]f(\alpha) = 0. \tag{39}$$

**Remark 17.** *We usually introduce a raising operator and lowering operator for a solution of the differential equation and make a recurrence equation in the elementally quantum mechanics, however, the method for making a difference equation denoted in the above context doesn't need to introduce a lowering operator. In fact, these two methods are different essentially. The former method acts only to the subspace of the solutions' space, but the latter acts for all over the space of the solutions. See the Hermitian differential equation for example.*

We analyze this difference equation about the independent variable $\alpha$. We give formal power series solutions of the form (21) and see that the Stokes phenomenon occurs. The characteristic polynomial is calculated as

$$D(e^\lambda) = (1-x)e^{2\lambda} + (x-2)e^\lambda + 1 \tag{40}$$

$$= (1-x)(e^\lambda - 1)(e^\lambda - \frac{1}{1-x}) \tag{41}$$

therefore the deterministic roots are determined as $e^\lambda = 1, \frac{1}{1-x}$. The characteristic exponent of $L$ related to the deterministic roots 1 is

$$\begin{aligned}\nu_1 &= \frac{(1-x)+(x-2)+\gamma-\beta x+1-\gamma}{2(1-x)+(x-2)} \\ &= \frac{-\beta x}{-x} = \beta.\end{aligned}$$

In the same way, the characteristic exponent of $L$ related to the deterministic roots $\frac{1}{1-x}$ is calculated as follows:

$$\begin{aligned}\nu_2 &= \frac{\left(\frac{1}{1-x}\right)^2(1-x)+\frac{1}{1-x}(x-2+\gamma-\beta x)+1-\gamma}{2(\frac{1}{1-x})^2(1-x)+\frac{1}{1-x}(x-2)} \\ &= \frac{1+(x-2+\gamma-\beta x)+(1-\gamma)(1-x)}{2+(x-2)} = \frac{(\gamma-\beta)x}{x} = \gamma-\beta.\end{aligned}$$

Thus we conclude that there are two solutions $f^1$ and $f^2$ denoted as follows.

$$f^1(\alpha) = \alpha^{-\beta}(1+\cdots), \tag{42}$$

$$f^2(\alpha) = (\frac{1}{1-x})^\alpha \alpha^{\beta-\gamma}(1+\cdots) \tag{43}$$

where $\cdots$ means formal power series of $x^{-1}$, starting with $(constant) \times x^{-1}$, which is determined uniquely. The Borel transform of these power series are

$$f_B^1(\xi) = \xi^{\beta-1}(1+\cdots), \tag{44}$$

$$f_B^2(\xi) = (\xi-\log(1-x))^{\gamma-\beta-1}(1+\cdots). \tag{45}$$

We verify that the function $f_B^1(\xi)$ and $f_B^2(\xi)$ are local solutions of the differential equation (36) around the regular singular points. The Borel transform of the difference equation (38) is following.

$$\begin{aligned}&[(1-x)(\partial_\xi+1)e^{-2\xi}+\{(x-2)(\partial_\xi+1)+\gamma-\beta x\}e^{-\xi}+\partial_\xi+1-\gamma]f_B(\xi) \\ &= [D(e^{-\xi})\partial_\xi+\{-(1-x)e^{-2\xi}+(\gamma-\beta x)e^{-\xi}+1-\gamma\}]f_B(\xi) = 0.\end{aligned}$$

In this equation, we used a notation $D(e^{-\xi}) = ((1-x)e^{-\xi}-1)(e^{-\xi}-1)$. This differential equation is a first order linear differential equation because the difference equation's coefficients are all first-degree polynomials. We can solve this equation by separating variable method. We get

$$f_B(\xi) = \exp\left(\int \frac{(1-x)e^{-2\xi}-(\gamma-\beta x)e^{-\xi}+\gamma-1}{((1-x)e^{-\xi}-1)(e^{-\xi}-1)}d\xi\right). \tag{46}$$

Here, the integral included in $\exp\cdot$, is calculated by substitution $e^{-\xi} = Y$, $d\xi = -dY/Y$, and we obtain

$$\begin{aligned} f_B(\xi) &= \exp\left(-\int \frac{(1-x)Y^2 - (\gamma - \beta x)Y + \gamma - 1}{Y\{(1-x)Y - 1\}(Y-1)} dY\right) \\ &= (Const) \cdot Y^{1-\gamma}(Y-1)^{\beta-1}(Y - \frac{1}{1-x})^{\gamma-\beta-1} \\ &= (Const) \cdot e^{(\gamma-1)\xi}(e^{-\xi}-1)^{\beta-1}(e^{-\xi} - \frac{1}{1-x})^{\gamma-\beta-1}. \end{aligned}$$

It is clear that the singular points of this function are located at $\xi = 2\pi i\mathbb{Z}$ and $\xi = \log(1-x) + 2\pi i\mathbb{Z}$. Taking a series expansion around there, we find

$$(47)\quad e^{(\gamma-1)\xi}(e^{-\xi}-1)^{\beta-1}(e^{-\xi} - \frac{1}{1-x})^{\gamma-\beta-1} = (-1)^{\beta-1}\left(\frac{-x}{1-x}\right)^{\gamma-\beta-1} f_B^1(\xi)$$

around $\xi = 0$, and

$$(48)\quad e^{(\gamma-1)\xi}(e^{-\xi}-1)^{\beta-1}(e^{-\xi} - \frac{1}{1-x})^{\gamma-\beta-1} = (-1)^{\gamma-\beta-1}(1-x)^{-1}x^{\beta-1} f_B^2(\xi)$$

around $\xi = \log(1-x)$. Hence we got the relation of the difference equation's formal power series solutions and the solutions of the differential equation which is obtained by the Borel transform. We consider the Borel resummation of $f_1$ and $f_2$.

$$\begin{aligned} F_1(\alpha) &= \mathcal{L} \circ \mathcal{B}(f_1)(\alpha) = \int_0^\infty e^{-\alpha\xi} f_B^1(\xi) d\xi \\ F_2(\alpha) &= \mathcal{L} \circ \mathcal{B}(f_2)(\alpha) = \int_{\log(1-x)}^\infty e^{-\alpha\xi} f_B^2(\xi) d\xi. \end{aligned}$$

We analyze these functions as an example of the Stokes phenomena. Because the monodromy of the Gauss' hypergeometric equation is known, we analyze $F_1$ and $F_2$ with respect to the parameter $x$, for instance, we consider the case that the parameter $x$ turn around the point $x = 0$ or $x = 1$. The move of the parameter $x$ corresponds the move of the initial point of the Laplace integral in (44). We assume $\alpha$ is a positive real number for simplicity.

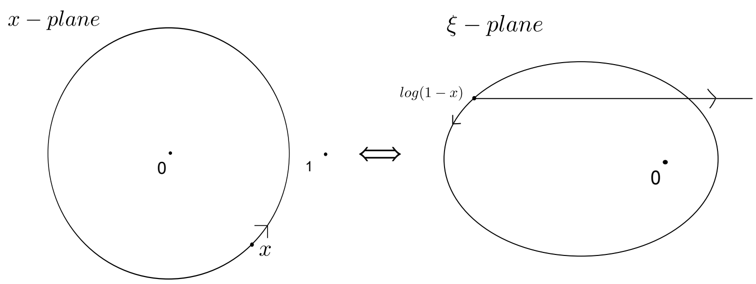


Figure 3: Movements of $x$.

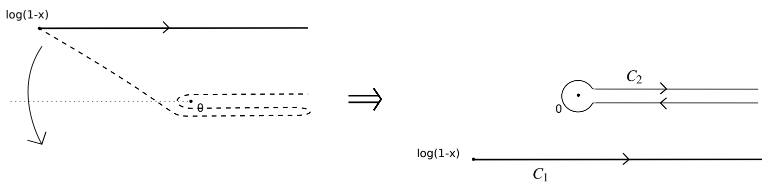


Figure 4: The path bifurcation.

In the loop (Figure 3), we assume that $x$ is in the 4th quadrant at the beginning, and it goes around counterclockwisely the point $\xi = 0$. Then, the integral interval of $F_2(\alpha)$, $[\log(1-x), \infty]$, moves parallel. When the initial point $\log(1-x)$ approaches to the line $\operatorname{Im}\xi = 0$, the integral path is devided into two curves, because $f_B^2(\xi)$ has branch point at $\xi = 0$. One curve $C_1$ is the line with the interval $[\log(1-x)\,, \infty]$, and another curve $C_2$ is a curve which comes from $\infty$, rotate the point $\xi = 0$ and goes toward $\infty$ again (Figure 2).

Thus, continuing $F_2(\alpha)$ analytically across the line $\operatorname{Im}\xi = 0$, we find that *discontinuous* analytic continuation occurs.

$$\int_{\log(1-x)}^{\infty} e^{-\alpha\xi} f_B^2(\xi) d\xi \mapsto \int_{C_1} e^{-\alpha\xi} f_B^2(\xi) d\xi + \int_{C_2} e^{-\alpha\xi} f_B^2(\xi) d\xi \tag{49}$$

From the relation (47) and (48), we get

$$f_B^2(\xi) = (-1)^{\beta-1}(1-x)^{\beta-\gamma+2} x^{\gamma-2\beta} f_B^1(\xi) \tag{50}$$

and the relation (49) turns out to be following

$$F_2(\alpha) \mapsto F_2(\alpha) + 2i\sin(\pi(\beta-1))(1-x)^{\beta-\gamma+2}x^{\gamma-2\beta}F_1(\alpha). \tag{51}$$

We call a phenomenon occurring *discontinuous* analytic continuation, the Stokes phenomenon. The non-trivial coefficients like $2i\sin(\pi(\beta-1))(1-x)^{\beta-\gamma+2}x^{\gamma-2\beta}$ in (51), is called *Stokes multipliers*. In the same way, we can verify that the Stokes phenomenon occurs when the point $\log(1-x)$ across the interval $(0\,,\,+\infty)$. In this case, the non-trivial term appears to the function $F_1(\alpha)$ as follows.

$$F_1(\alpha) \mapsto F_1(\alpha)+2i(-1)^{\gamma-2\beta}\sin(\pi(\gamma-\beta-1))(1-x)^{-\beta+\gamma-2}x^{-\gamma+2\beta}F_2(\alpha). \tag{52}$$

As a result, we find the (local) monodromy of the difference equation's solutions, with respect to the behavior of the parameter $x$. Namely, by the move of $x$, $x$ rotates near the point 0, the functions $\langle F_1, F_2\rangle$ changes linearly

$$\langle F_1(\alpha),\, F_2(\alpha)\rangle \mapsto \langle F_1(\alpha),\, F_2(\alpha)\rangle\, A$$

where the elements of the matrix $A \in M_2$ doesn't depend on the independent variable $\alpha$,

$$A = \begin{pmatrix} 1 & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}.$$

Here, we denoted $A_{i,j}$

$$\begin{aligned} A_{1,2} &= 2i\sin(\pi(\beta-1))(1-x)^{\beta-\gamma+2}x^{\gamma-2\beta}, \\ A_{2,1} &= 2i(-1)^{\gamma-2\beta}\sin(\pi(\gamma-\beta-1))(1-x)^{\gamma-\beta-2}x^{2\beta-\gamma}, \\ A_{2,2} &= -4sin(\pi(\beta-1))\sin(\pi(\gamma-\beta-1))+1. \end{aligned}$$

**Remark 18.** *In this discussion, we held the formal solutions $f^1(\alpha)$ and $f^2(\alpha)$ to the form which the coefficients of the leading term is* 1 *so the Stokes coefficients are not simple. We may have an exchange of the basis of the solutions' space, $f^2(\alpha) = (1-x)^{\beta-\gamma+2}x^{\gamma-2\beta}\tilde{f}^2(\alpha)$, then we find that the Stokes multiplier does not include the parameter $x$. Then the monodromy is calculated by using only the* sin *and* exp *function of the parameter $\beta$ and $\gamma$.*

**Remark 19.** *Although we calculated the monodromy of the function when the parameter $x$ goes around the point $x = 0$, we can get the monodromy of the case $x$ rotates the point $x = 1$ in the same way (see Figure 3). It is rather easy to discuss, because the circle rounding only the singular point $x = 1$ correspond to the changes of integral path in the Borel plane parallel for upper(see the figure 5). Therefore we should calculate one Stokes multiplier to clarify the monodromy. By incorporating these affairs, we can construct the global monodromy of the hypergeometric functions from the view point of the Stokes phenomena of the difference equation. Indeed, the hypergeometric difference equation includes an important information of the hypergeometric differential equation. The author loves this harmony of functional equations.*

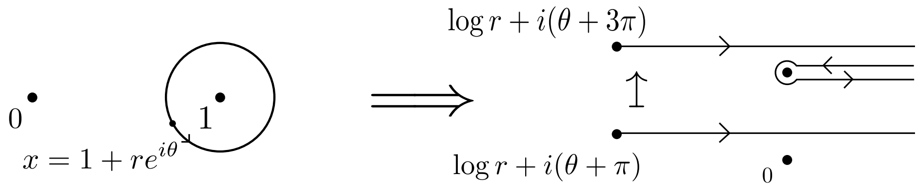


Figure 5: $x$ goes around 1.

In summary, we obtained the Stokes multipliers of the difference equation of the hypergeometric functions about the variable $\alpha$, by investigating the Borel transform of the formal power series solutions. What we used for calculating the Stokes multipliers are, the deterministic roots of the difference equation, the characteristic exponents of it, and finally, *the connection formula* (50) of the solutions of the differential equation. Although the connection formula does not seem important because the differential equation is first order so the formula is trivial, but in general, the connection formula is quite important; almost all differential equations' connection formulae is not trivial, or can't be calculated easily. Also, we should consider that the differential equation which is transformed from the difference equation has infinite number of regular singular points. We manage to transform the differential equations to the case of finite number of the regular singular points in the next section, and show how we get the Stokes multipliers of the solutions of the difference equations.

# 4 Mellin transform and Stokes phenomena.

In the previous section, the Borel resummation (or we may say that the Borel Laplace analysis) turned out to be useful to calculate the Stokes multipliers which denote the global characters of the equations. However, in the method of the Borel transform, there is a problem that the number of singular points is infinite. To dispose this problem, we introduce a change of the independent variable. In the Borel plane, the description of the Stokes phenomena is quite simple, but in the another variable, the connection formulae becomes simple. We mix these analytic aspects and get the Stokes multipliers in some cases corresponds to the rigid cases. On the process of these analysis, we reveal the important parameters of the difference equations which are not clarified in the section 2, namely, we get the parameters which are not obtained from the analysis of the formal power series solutions.

## 4.1 The inverse Mellin transform

In this subsection, we change the variable of the differential equations introduced by the Borel transformation of the difference equation. Let $L$ be a difference operator of which deterministic roots differ from each other, and we assume that the deterministic polynomial of $L$ have $N$ roots which are not 0. We consider the difference equation

$$Lf(x) = \sum_{j=0}^{N} a_j(x)\sigma^j f(x) = 0. \tag{53}$$

The Borel transform of this difference equation is the following.

$$\sum_{j=0}^{N} a_j(\partial_\xi)e^{-j\xi} f_B(\xi) = 0.$$

We mentioned that this equation has infinitely many singular points. However, by exchanging the variable $\xi$ to the variable $Y = e^{-\xi}$, we get the differential equation

$$\sum_{j=0}^{N} a_j(-\vartheta_Y)Y^j \hat{f}_B(Y) = \sum_{j=0}^{N} Y^j a_j(-\vartheta_Y - j)\hat{f}_B(Y) = 0 \tag{54}$$

where $\vartheta_Y = Y\frac{d}{dY}$. This equation has *finitely many* singular points, because its coefficients are polynomials. We call this differential equation the *inverse Mellin transform* of the difference equation. Remember that $a_j(x)$ was written as $a_j(x) = \sum_{l=0}^{l_j} a_{j,l}x^l$ in section 1. We defined $M$ as the maximal degree of $\{a_j(x)\}$: $M = \max_j \deg a_j(x) = \deg a_0(x) = \deg a_N(x)$. Then, the differential equation (54) is written as follows:

$$\sum_{j=0}^{N}\sum_{l=0}^{l_j} Y^j a_{j,l}(-1)^l(\vartheta + j)^l \hat{f}_B(Y) = 0. \tag{55}$$

Let us confirm the coefficients of the highest order term of the differential operator. The following relation is clear.

$$\vartheta_Y^M = Y^M \partial_Y^M + (\text{lower order operators}). \tag{56}$$

Therefore, the highest order term is

$$Y^M \sum_{j=0}^{N} Y^j a_{j,M}(-1)^M \partial_Y^M = (-1)^M Y^M D(Y)\partial_Y^M. \tag{57}$$

Here, $D$ indicates the characteristic polynomial of the difference equation $Lf(x) = 0$. Namely, setting the roots of $D(\lambda) = 0$ to be $\lambda = \lambda_1, \cdots, \lambda_N$, we find that the singular points of the differential equaltion (54) is limited to the points $0, \{\lambda_j\}_{j=0}^{N}$, and $\infty$ at most. In fact, these singular points are turned out to be regular singular points at most.

**Proposition 20.** *Let $L$ be a difference operator satisfying the condition of Definition 7, and assume that its deterministic roots $\lambda_1, \lambda_2, \cdots, \lambda_N$ are differ from each other. Then, differential equation (54) constructed as above has at most $N + 2$ regular singular points.*

*Proof.* It is obvious that $\lambda_j$ becomes the regular singular point, because the coefficient of the highest order is factorized as $Y^M \Pi_{j=0}^{N}(\lambda - \lambda_j) \times Const$. We prove that $Y = 0$ and $Y = \infty$ are the regular singular points. Let us pay attention to the equation (54) and confirm the assumption that the degree of the coefficient $a_0$ is $M$. Then, by Proposition 5, we find that the point $Y = 0$ is at most a regular singular point. In the same manner, we find that $Y = \infty$ is at most a regular singular point. □

In short, we saw that the difference equation was transformed to the differential equation of a Fuchsian type. To see that equation more precisely, we calculate its characteristic exponents, and give the *generalized Riemann scheme* of it. Before we define a generalized Riemann scheme, we define the *generalized characteristic exponents.*

**Definition 21.** Let $P$ be a differential operator $P = b_n(x)\partial_x^n + b_{n-1}(x)\partial_x^{n-1} + \cdots + b_0(x)$, where $\{b_j(x)\}$ are polynomials. Assume that the operator $P$ has a regular singularity at the point $x = 0$, and $b_n(x)$ satisfies the condition $b_n(0) = b_n'(0) = \cdots = b_n^{(n-1)}(0) = 0$, $b_n^{(n)}(0) \neq 0$. Furthermore, we assume that there exists a positive integer $k$ and differential operator $R$, of which coefficients are all holomorphic at $x = 0$, and $P$ is written as $P = x^k R$ (of course $k$ is smaller than $n$, because $P$ has a regular singularity at $x = 0$). We also assume that $k$ is the maximal integer satisfying such a condition. Then, we call $P$ has a generalized characteristic exponents $[0]_k$ at the point $x = 0$.

If the operator $P$ has a generalized characteristic exponents at $x = 0$, then $P$ has characteristic exponents $0, 1, \cdots, k-1$. In the same way, if the operator $P$ satisfies the condition $x^{-\nu} P x^{\nu} = x^k R$, $R$ is holomorphic operator at $x = 0$, then we call the operator $P$ has generalized exponents $[\nu]_k$ at $x = 0$. We limited to the regular singular point $x = 0$ in this arguments, but we can generalize to the regular singular point $x = x_0 \in \mathbb{C}$, because we can exchange the independent variable $x$ to $\tilde{x} = x - x_0$. Moreover, we may argue about the regular singular point $\infty$ because we can exchange the variable $w = 1/x$. Thus, we finished the definition of the generalized exponents in $\mathbb{P}^1$. The generalized characteristic exponents correspond to the characteristic exponents if $k = 1$, so we regard the characteristic exponent as the generalized one in such occasion, and denote $[\lambda]_1 = \lambda$ for simplifying. Now, we define the *generalized Riemann scheme.*

**Definition 22.** Let the Fuchsian differential operator $P$ has regular singular points $x_1, x_2, \cdots, x_m \in \mathbb{P}^1$. We assume that $P$ has the generalized characteristic exponents $[\nu_1^{(j)}]_{n_1^{(j)}}, \cdots, [\nu_l^{(j)}]_{n_{l_j}{}^{(j)}}$ at $x = x_j$, and also assume $n_1^{(j)} + \cdots + n_{l_j}^{(j)} = n$ for any $j = 1, \cdots, m$. We define the generalized Riemann scheme as the table of the singular points and generalized exponents corresponding to them, denoted as follows.

$$
\left\{
\begin{array}{ccc}
x = x_1 & \cdots & x = x_m \\
\left[\nu_1^{(1)}\right]_{n_1^{(1)}} & \cdots & \left[\nu_1^{(m)}\right]_{n_1^{(m)}} \\
\vdots & & \vdots \\
\left[\nu_{l_1}^{(1)}\right]_{n_{l_1}^{(1)}} & \cdots & \left[\nu_{l_m}^{(m)}\right]_{n_{l_m}^{(m)}}
\end{array}
\right\}. \tag{58}
$$

**Remark 23.** *In Proposition* 6, *we assumed that the characteristic exponents are satisfying the generic condition* $\nu_{l,j} - \nu_{l,k} \notin \mathbb{Z}$ *for simplify the problem. In general, if the characteristic exponents don't satisfy the generic condition, then, the local solution of the equation includes a logarithmic term. However, if the operator* $P$ *has the generalized characteristic exponents* $[\nu_1^{(j)}]_{n_1^{(j)}}, \cdots, [\nu_l^{(j)}]_{n_{l_j}{}^{(j)}}$ *at* $x = x_j$, *and* $\nu_i^{(j)} - \nu_k^{(j)} \notin \mathbb{Z}$, *then the local solutions are written in the form of the power series only. This means that the local monodromy matrix of the singular point* $x_j$ *is diagonalizable* (*semi-simple*).

**Remark 24.** *It is important to reveal the generalized Riemann scheme of the Fuchsian differential equation, because we can specify the equation rigid or not rigid from* $m$ *set of partition of* $n$, $n_1^{(1)}, \dots, n_{l_1}^{(1)}; \dots; n_1^{(m)}, \dots, n_{l_m}^{(m)}$.

Let us see the generalized Riemann scheme of the equation (54). Seeing the relation (57), we conclude that the regular singular point $Y = \lambda_k$ has the generalized exponents $[0]_{M-1}$. This is because the deterministic roots are differ from each other. Another characteristic exponent is calculated by following manner. Pay attention only to the highest order term and the second highest order terms, because lower order terms do not affect to the nontrivial characteristic exponents. The operator (56) is refined as $\vartheta_Y^M = Y^M \partial_Y^M + \frac{M(M-1)}{2} Y^{M-1} \partial_Y^{M-1} + \cdots$. Therefore we find (55) is

$$
\begin{aligned}
\Bigg[ & (-1)^M D(Y) \left\{ Y^M \partial_Y^M + \frac{M(M-1)}{2} Y^{M-1} \partial_Y^{M-1} + \cdots \right\} \\
& + (-1)^M \Big(\sum_{j=0}^{M} j M a_{j,M} Y^j\Big) Y^{M-1} \partial_Y^{M-1} \\
& + (-1)^{M-1} \Big(\sum_{j=0}^{M} a_{j,M-1} Y^j\Big) Y^{M-1} \partial_Y^{M-1} + \cdots \Bigg] \hat{f}_B(Y) = 0
\end{aligned}
$$

The characteristic exponent $s$ of this equation at the regular singular point $Y = \lambda_k$ satisfies

$$\lambda_k D'(\lambda_k)s(s-1)\cdots(s-M+1) + M\lambda_k D'(\lambda_k)s(s-1)\cdots(s-M+2) - \sum_{j=0}^{M} a_{j,\,M-1}\lambda_k^j s(s-1)\cdots(s-M+2) = 0.$$

Indeed, we get $s = 0, 1, \cdots, M-2, \frac{\sum_{j=0}^{M} a_{j,\,M-1}\lambda_k^j}{\lambda_k D'(\lambda_k)} - 1$. Remember the definition of the characteristic exponents of the difference equations mentioned in Definition 9, then, the nontrivial exponent is written as $s = \nu_k - 1$. Thus, we conclude that the generalized Riemann scheme of the equation (54) is

$$\left\{ \begin{array}{ccccc} Y=0 & Y=\lambda_1 & \cdots & Y=\lambda_N & Y=\infty \\ -\nu_{0,\,1} & [0]_{M-1} & \cdots & [0]_{M-1} & \nu_{\infty,\,1}+N \\ \vdots & & & & \vdots \\ -\nu_{0,\,M} & \nu_1 - 1 & \cdots & \nu_N - 1 & \nu_{\infty,\,M}+N \end{array} \right\}. \tag{59}$$

Here, $\{\nu_{0,l}\}$ are roots of $a_0(x) = 0$, and $\{\nu_{\infty,\,l}\}$ are roots of $a_N(x) = 0$.

The characteristic exponents at $Y = 0$ and $Y = \infty$ are not become known to the generalized characteristic exponents by themselves. To reveal that they are generalized one or not, we prepare a lemma.

**Lemma 25.** *The differential operator of $M$-th order $P = \sum_{l=0}^{\infty} x^l p_l(\vartheta)$ has generalized characteristic exponents $[\nu]_k$ at the point $x = 0$ if and only if the polynomial $p_l(s)$ has a form*

$$p_l(s) = q_l(s) \prod_{m=0}^{k-l} (\vartheta - \nu - m) \tag{60}$$

*and $k$ is a maximal integer that $P$ is written as above.*

We can import this fact to the difference equation simply. We set the difference equation $L = \sum_{j=0}^{N} a_j(x)\sigma^j$ satisfying the condition mentioned in Definition 4. Then the proof of the next proposition is clear.

**Proposition 26.** *Suppose that the coefficients $\{a_j(x)\}$ are factorized to the form $a_j(x) = q_j(x)\prod_{l=0}^{n-j}(x+\nu+l)$, and $q_0(-\nu) \neq 0$. Then the Mellin transform of the equation $Lf(x) = 0$ has a generalized characteristic exponent $[\nu]_n$ at $Y = 0$.*

**Example 27.** We see the difference equation obtained from the hypergeometric series ${}_4F_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3; x)$, about the main variable $a_1$. The function ${}_4F_3$ has the shift operator $\sigma = \frac{1}{\alpha}(\vartheta + \alpha) : {}_4F_3(a_1) \mapsto {}_4F_3(a_1 + 1)$. Therefore we get $\sigma^k = \prod_{j=0}^{k-1} \frac{1}{\alpha+j}(\vartheta + \alpha + j)$. To get the difference equation, we divide the differential operator $P_4$,

$$P_4 = x(\vartheta + a_1)(\vartheta + a_2)(\vartheta + a_3)(\vartheta + a_4) - (\vartheta + b_1 - 1)(\vartheta + b_2 - 1)(\vartheta + b_3 - 1)\vartheta,$$

by difference operators. Let $L[K]$, $K = 0, \ldots, 4$ be differential operators obtained by following algorithm.

(61) $$L[0] = P_4.$$

for $K = 1, \ldots, 4$, there exist a unique polynomial $Q[K-1]$ and a unique differential operator $L[K]$ of which order is $4 - K$, such that

(62) $$L[K-1] = Q[K-1]\sigma^{5-K} + L[K],$$

(63) $$R = L[4].$$

This algorithm gives us the representation of $P_4$ by using difference operators

(64) $$L = P_4 = \sum_{K=0}^{3} Q[K]\sigma^{4-K} + R.$$

The difference operator obtained by this algorithm, has the following form, 4-*th* order difference operator with its coefficients are polynomials of degree 4.

$$
\begin{aligned}
&L = \\
&a_1[(x-1)(a_1+1)(a_1+2)(a_1+3)\sigma^4 \\
&-(a_1+1)(a_1+2)\{(3a_1-a_2-a_3-a_4+6)x+b_1+b_2+b_3-4a_1-9\}\sigma^3 \\
&+(a_1+1)\{(3a_1^2+(-2a_2-2a_3-2a_4+9)a_1 \\
&\quad +(a_3+a_4-3)a_2+(a_4-3)a_3-3a_4+7)x \\
&\quad +(-b_2-b_3+3a_1+5)b_1+(-b_3+3a_1+5)b_2 \\
&\quad +(3a_1+5)b_3-6a_1^2-21a_1-19\}\sigma^2 \\
&-(a_1^3+(-a_2-a_3-a_4+3)a_1^2 \\
&\quad +((a_3+a_4-2)a_2+(a_4-2)a_3-2a_4+3)a_1 \\
&\quad +((-a_4+1)a_3+a_4-1)a_2+(a_4-1)a_3-a_4+1)x+((b_3-2a_1-2)b_2 \\
&\quad +(-2a_1-2)b_3+3a_1^2+7a_1+4)b1+((-2a_1-2)b_3+3a_1^2+7a_1+4)b_2 \\
&\quad +(3a_1^2+7a_1+4)b_3-4a_1^3-15a_1^2-19a_1-8,1]\sigma \\
&+(b_3-a_1-1)(b_2-a_1-1)(b_1-a_1-1)].
\end{aligned}
$$

Infact, this operator can be divided by $a_1$, so the essential operator becomes $L/a_1$, 4-$th$ order, degree 3 operator. This shows that the inverse Mellin transform of $L/a_1$ has generalized characteristic exponents $[1]_3$ at $Y=\infty$ and characteristic exponents $\{1-b_1, 1-b_2, 1-b_3\}$ at $Y=0$. The other singular points' characteristic exponents are difficult to calculate, but we get $\{a_2, a_3, a_4\}$ at $Y=1$, and $\{[0]_2, a_2+a_3+a_4-b_1-b_2-b_3-1\}$ at $Y=\frac{1}{1-x}$. For the better understanding of this facts, we suggest the program code written in the calculating language Risa/Asir in the Appendix. In summary, the inverse Mellin transform of the hypergeometric difference equation obtained from ${}_4F_3(a_1,a_2,a_3,a_4;b_1,b_2,b_3;x)$ has a Riemann scheme

$$
\left\{\begin{array}{cccc}
Y=0 & 1 & \frac{1}{1-x} & Y=\infty \\
1-b_1 & a_2-1 & [0]_2 & [1]_3 \\
1-b_2 & a_3-1 & & \\
1-b_3 & a_4-1 & \nu_{\frac{1}{1-x}} &
\end{array}\right\} \tag{65}
$$

where $\nu_{\frac{1}{1-x}} = b_1+b_2+b_3-a_2-a_3-a_4-1$. Of course, this Riemann scheme satisfies the Fuchs relation. In generic case, this type of differential equation is reduced to the hypergeometric equation $P_3 f = 0$ by using Möbius transformation.

Indeed, for clarifying the generalized Riemann scheme of the Mellin transform of a difference equation, it is important seeing the information of roots of coefficients. In this subsection, we saw that the Mellin transform of the difference equation becomes a Fuchsian differential equation, and we clarified the relation between the information of the difference equation and the differential equation's generalized Riemann scheme. In the next subsection, we show how we can calculate the Stokes multipliers of difference equations by using these information obtained in this subsection.

## 4.2 The Stokes phenomena

We saw the structure of differential equations which is transformed from difference equations by the inverse Mellin transformation. In this subsection, we look over how we can calculate the Stokes phenomena of the difference equation. In the case there is 1 or 2 deterministic roots, we investigate a connection of the solutions of the difference equation from $\infty$ and $-\infty$. The case that there is only one deterministic root, includes the difference equation of the beta function. The case the number of deterministic roots is 2, includes the difference equation of the (general) hypergeometric functions. The specific calculus of each equation's connection will be appear in the next subsection. We confirm the relation between the connection of the solutions of two differential equations. One is the equation which is obtained by the inverse Mellin transformation. The other one is the equation which is obtained by the Borel transformation. The connection matrix is defined as follows. Assume we have a basis of solutions $\langle f_1, f_2, \cdots, f_M \rangle$ at $\xi = \xi_0$, and $\langle \tilde{f_1}, \tilde{f_2}, \cdots, \tilde{f_M} \rangle$ at $\xi = \xi_1$, they can be connected each other along the path $\gamma$. We are considering the solutions of a linear differential equation, so that the relations of them are represented by a matrix $C \in GL_M(\mathbb{C})$. We call this matrix a *connection matrix* along the path $\gamma$. In this subsection, we assume that

$$\forall j, \forall k, \quad j \neq k \Longrightarrow \arg(\lambda_j) \neq \arg(\lambda_k),\ |\lambda_j| \neq |\lambda_k|. \tag{66}$$

And we arrange the index $j$ of the deterministic roots by following rules.

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_N|, \tag{67}$$

for simplicity.

The inverse Mellin transformation of difference equation was defined as the differential equation, which is obtained from the Borel transformed equation by changing variable $Y = e^{-\xi}$. We associate the connecting path in the inverse Mellin plane(the $Y$-plane) with the connecting path in the Borel plane(the $\xi$-plane). In the inverse Mellin plane, the equation has $N+2$ regular singular points $0, \lambda_1, \cdots, \lambda_N, \infty$. We set the path $\gamma_j$ in the inverse Mellin plane, which is a loop starting from near the point $Y = \lambda_j$, goes around counterclockwise along the circle $|Y| = \lambda_j$ until the path getting near the point $Y = \lambda_j$ again, and rotate clockwisely avoiding $\lambda_j$, to the starting point. We denote the connection matrix of the fundamental solutions along the path $\gamma_j$ as $C_{j^{(0)},j^{(1)}}$. This connection corresponds to a connection in the Borel plane, of the path from $-\log\lambda_j$ to $-\log\lambda_j - 2\pi i$, as in the following figure.

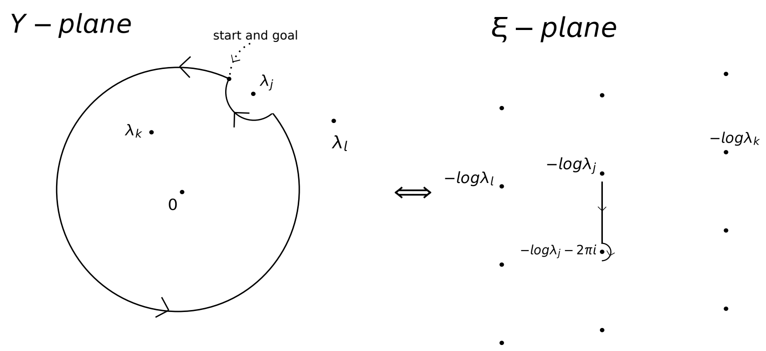


Figure 6: Connection from $-log\lambda_j$ to $-log\lambda_j - 2\pi i$

The connection matrix $C_{j^{(0)},j^{(1)}}$ is written by using the connection matrix $C_{l,0}$ which is the connection matrix from the singular point $\lambda_l$ to 0 in the Mellin plane, and the local monodromy matrix $M_0$, $M_l$, $l \geq j+1$. For the index $l > j$, we prepare the permutation as follows:

(68) $s_j(l) = \#\{k \in \{j+1, \ldots, N\} | \arg(\lambda_j) \leq A_j(\lambda_k) \leq A_j(\lambda_l)\} + j.$

Here, we denoted $A_j$

$$A_j(\lambda_k) = \begin{cases} \arg(\lambda_k) & (\arg(\lambda_k) > \arg(\lambda_j)), \\ \arg(\lambda_k) + 2\pi & (\arg(\lambda_k) \leq \arg(\lambda_j)). \end{cases} \tag{69}$$

The map $s$ is a permutation of the set $\{j+1, \ldots, N\}$. For example, when $\{\lambda_1, \ldots, \lambda_5\}$ locates as following figure 7, we get $s_1$ is

$$(s_1(2), s_1(3), s_1(4), s_1(5)) = (3, 5, 4, 2). \tag{70}$$

We denote $t$ as the inverse permutation of $s$, that is, $t(s(k)) = k$.

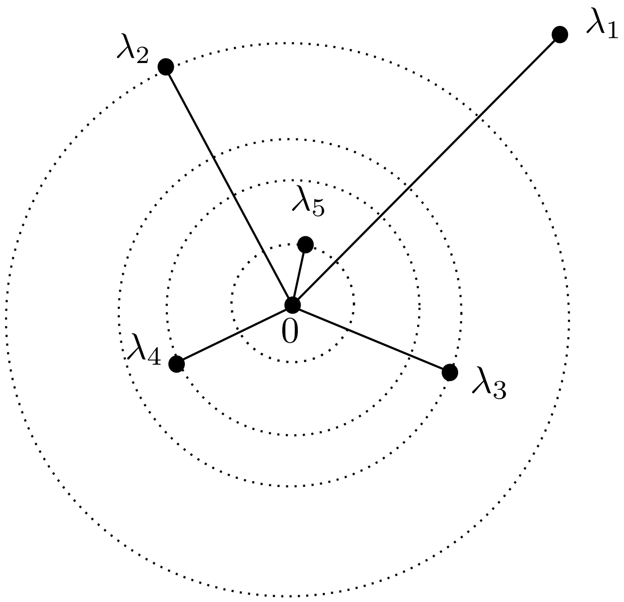


Figure 7: 5-singular points

In such an assumption, the connection matrix $C_{j^{(0)},j^{(1)}}$ is written as

$$C_{j^{(0)},j^{(1)}} = C_{j,0}^{-1} M_0 (T_{t(N)}) \cdot (T_{t(N-1)}) \cdots (T_{t(j+1)}) C_{j,0}. \tag{71}$$

Where we set $T_l = C_{l,0} M_l C_{l,0}^{-1}$ for simplicity, i.e. $T_l$ is the connection matrix which corresponds to the loop from 0, goes around the point $\lambda_l$, and return to the point 0. Indeed, the connection $C_{j^{(0)},j^{(1)}}$ is described by the connection between the regular singular points of the inverse Mellin plane. We define the connection from $-\log \lambda_j - 2\pi ik$ to $-\log \lambda_j - 2\pi il$ as $C_{j^{(k)},j^{(l)}} = (C_{j^{(0)},j^{(1)}})^{l-k}$. The connection from $-\log \lambda_j$ to $-\log \lambda_{j+1}$ in the Borel plane is equal to the connection $\lambda_j$ to $\lambda_{j+1}$ in the inverse Mellin plane (if $0 < \mathrm{Im}(-\log \lambda_j + \log \lambda_{j+1}) \leq \pi$ then rotate counterclockwisely, and else if $\pi < \mathrm{Im}(-\log \lambda_j + \log \lambda_{j+1}) < 2\pi$ then rotate clockwisely). This connection is also calculated by using the connections in the inverse Mellin plane, in terms of $C_{l,0}$ and $T_l$.

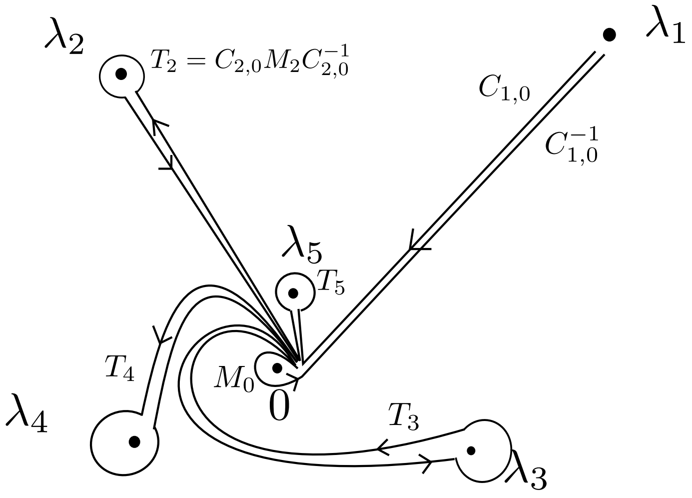


Figure 8: The connection path and matrices.

More generally, we define $C_{j^{(m)},\,k^{(n)}}$ as the connection matrix which represent the analytic continuation of the basis of the solutions from $-\log\lambda_j - 2\pi i m$ to $-\log\lambda_k - 2\pi i n$, as below. Take a line segment from $-\log\lambda_j - 2\pi i m$ to $-\log\lambda_k - 2\pi i n$, when the singular point $\lambda_j$ and $\lambda_k$ satisfies an inequality $j < k$. Calculate the intersection point $p_l$ of this line segment and the line $\{\xi\,|\,\mathrm{Re}\,\xi = -\log\lambda_l\}$ for $l = j+1,\ldots,k-1$. Find an integer $h_l$ which is the minimal integer satisfying the condition $-\log\lambda_l - 2\pi i h_l < p_l$ for each $l$. Then, we define $C_{j^{(m)},\,k^{(n)}}$ as

$$\begin{aligned} C_{j^{(m)},\,k^{(n)}} &= C_{k-1^{(n)},k^{(n)}} C_{k-1^{(h_{k-1})},k-1^{(n)}} \cdots M_{j+2}^{-1} C_{j+1^{(h_{j+2})},j+2^{(h_{j+2})}} \\ &\quad \times C_{j+1^{(h_{j+1})},j+1^{(h_{j+2})}} M_{j+1}^{-1} C_{j^{(h_{j+1})},j+1^{(h_{j+1})}} C_{j^{(m)},j^{(h_{j+1})}}. \end{aligned}$$

In the case $j > k$, we need to define the connection the same way basically, but we should be aware of the changes of sequenses.

$$\begin{aligned} C_{j^{(m)},\,k^{(n)}} &= (M_{k+1} C_{k+1^{(0)},k+1^{(1)}})^{n-h_{k+1}} \cdots C_{j-1^{(h_{j-2})},j-2^{(h_{j-2})}} \\ &\quad \times (M_{j-2} C_{j-1^{(0)},j-1(1)})^{h_{j-2},h_{j-1}} M_{j-1} C_{j^{(h_{j-1})},j-1^{(h_{j-1})}} (M_j C_{j(0),j^{(1)}})^{h_{j-1}-m}. \end{aligned}$$

In short, we defined these connection matrix by using connections of singular points and local monodromies of the inverse Mellin plane. In another words, connections are completely written in the words of Fuchsian differential equations. Now, we show how the Stokes phenomena occur, and how we calculate the Stokes multiplier by using connection matrices. For simplicity, we consider the situation that there are only two sequenses of singular points in the Borel plane. In more general cases, we obtain the Stokes multipliers in similar calculus. Assume that the singular points are denoted as $\xi = a = -\log\lambda_1$, $\xi = b = -\log\lambda_2$. Then points $\xi = a + 2\pi i\mathbb{Z}$ and $\xi = b + 2\pi i\mathbb{Z}$ are also singular points. By continuing the solutions analytically about the variable $x$ from $\arg x = 0$ to $\arg x = \theta < \arg(b - a)$, the integral path of the Borel ressumation of the formal solutions changes as in the following figure.

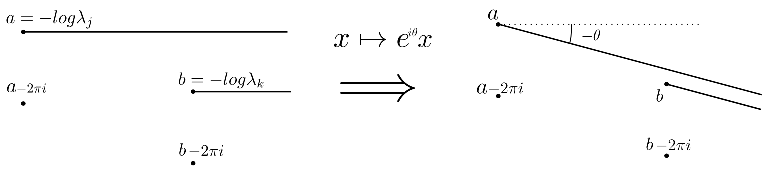


Figure 9: changes of paths when we rotate $x$

Then, we consider the case connecting them from $\arg x < \arg(b - a)$ to $\arg x > \arg(b-a)$. We find that the path of integral passes across the singular point $\xi = b$. Therefore we change the path from $a$ to $e^{-i\theta}\infty$ as follows. Let the path be deformed avoiding the point $\xi = b$, go along the left hand side of it, and enclose the singular point. Extend the superfluous path to $e^{-i\theta}\infty$. Then the path which starts from the point $a$, becomes devided into two paths (see Figure 10).

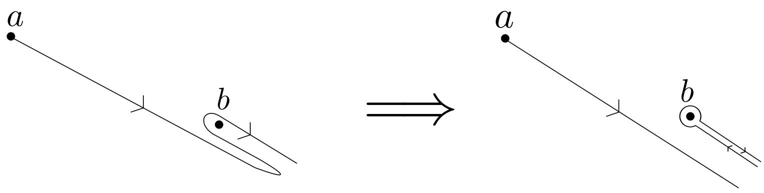


Figure 10: the path devided into two

The problem is how we calculate the connection. We now know that the Borel transform of the formal solutions are connected from $a$ to $b$. The analytic continuation of $\mathcal{B}f_a$ is written as $\mathcal{B}f_bC_{1^{(0)},2^{(0)}}$. The difference of the integral between the upper path and the lower path, is denoted as $M_2^{-1} - I$, so that the connection formula is

$$F_a(x) \mapsto F_a(x) + F_b(x)(M_k^{-1} - I)C_{j^{(0)},\,k^{(0)}}. \tag{72}$$

Therefore, we get the following proposition by considering in the same way to the above calculus.

**Proposition 28.** *Let $L$ be a difference operator of which inverse Mellin transformation is a Fuchsian differential operator. We consider a connection of solutions from $\arg x = \theta^-$ to $\arg x = \theta^+$. Assume that there exists only one triple of integers $(j, k, n)$ such that the path of Laplace integral $\{\xi = -\log\lambda_j + re^{-i\theta} | r \in [0, \infty)\}$ pass across the point $\xi = -\log\lambda_k - 2\pi i n$, when we connect $\theta$ from $\theta^-$ to $\theta^+$. We note the Borel ressumation of $e^{2\pi i p x} f^j$ as $F_j^{(p)}$. Then, the Stokes multipliers are denoted as follows.*

$$F_l^{(p)}(x) \to F_l^{(p)}(x) \;\; (l \neq j,\, p \in \mathbb{Z}), \tag{73}$$

$$F_j^{(p)}(x) \to F_j^{(p)}(x) + F_k^{(n+p)}\, {}^t\mathrm{e}_k\, \hat{M}_k C_{j^{(m)},\,k^{(n)}}\, \mathrm{e}_j. \tag{74}$$

*Here, we denoted $\mathrm{e}_j$ as $N \times 1$ matrix of which elements are*

$$(\mathrm{e}_j)_{r,1} = \begin{cases} 0 & (r \neq j), \\ 1 & (r = j), \end{cases}$$

*and $\hat{M}_k$ is*

$$\hat{M}_k = \begin{cases} M_k^{-1} - I & (j < k), \\ I - M_k & (j > k). \end{cases} \tag{75}$$

By this proposition, we comfirmed that the Stokes phenomena of difference equation is calculated by the information of the inverse Mellin plane, except for the connection from $\arg(x) = \frac{\pi}{2} - 0 \to \frac{\pi}{2} + 0$. Now, let us calculate the connection.

**Proposition 29.** *Let $L$ be a difference operator which satisfies the condition of the previous Proposition 28. Then the connection from $\arg(x) = \frac{\pi}{2} - 0 \to \frac{\pi}{2} + 0$ is*

$$F_j^{(p)}(x) \to F_j^{(p)}(x) + \sum_{n=1}^{\infty} F_j^{(p+n)}(x) {}^t\mathrm{e}_j (I - M_j) C_{j^{(p)},j^{(p+n)}} \mathrm{e}_j. \tag{76}$$

This fact can be obtained from the results written in [5 section 2]. They showed the proposition in some cases, different from this paper (they treated the problem of Abelian equation in their sense, and it includes some difference equations). Although, they did not consider the connection between the singular points generally. The outline of the proof is included in Figure 11.

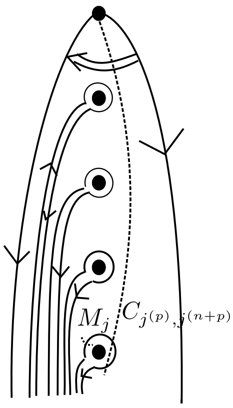


Figure 11: the connection from $\frac{\pi}{2} - 0$ to $\frac{\pi}{2} + 0$

From these propositions, we can calculate the Stokes multipliers of difference equations in principle. Therefore, we obtain the main theorem.

**Theorem** 3. *Let $Lf(x) = 0$ be a difference equation. Assume that the inverse Mellin transformation of $L$, $\mathcal{M}^{-1}L$ be a Fuchsian type differential operator. If we have connection formulae between the singular points of the differential equation $\mathcal{M}^{-1}L(\mathcal{M}^{-1}f)(Y) = 0$, then the following holds.*

1. *The Stokes multipliers of the difference equation can be calculated by using the connection formulae of the equation $\mathcal{M}^{-1}L(\mathcal{M}^{-1}f)(Y) = 0$.*
2. *In particular, the connection of the solutions of the difference equation can be calculated by using the Stokes multipliers which are obtained by 1. Especially, the connection is represented by Fourier series.*

The connection is represented by Fourier series, because $F_j^{(p)} = e^{2\pi ipx}F_j^{(0)}$, so the equation (76) can be written as

$$(77)\quad e^{2\pi ipx}F_j^{(0)}(x) \to e^{2\pi ipx}F_j^{(0)}(x)^t\mathrm{e}_j(I - M_j)(1 + \sum_{n=1}^{\infty}(e^{2\pi inx})(C_{j^{(0)},j^{(1)}})^n)\mathrm{e}_j.$$

In particular, we calculate the connection formulae of the difference equations which have 1 or 2 deterministic roots. The case that there is only one deterministic root is exactly the proposition 29, i.e., the equation (77) is the formula of connection from $\infty$ to $-\infty$ (the angle which the Stokes phenomena occur is only $\frac{\pi}{2} + \pi\mathbb{Z}$).

Then, think about the connection problem of the case the number of deterministic roots is 2. We take complex numbers $a = -\log \lambda_1$, and $b = -\log \lambda_2$. Assume that $\operatorname{Re} a < \operatorname{Re} b$ and $\operatorname{Im} a > \operatorname{Im} b$. We connect $\theta = \arg(x)$ 0 to $\frac{\pi}{2} - 0$, then we find

$$F_1(x) \mapsto F_1(x) + F_2(x)(1 + \sum_{s=1}^{\infty} e^{2\pi i x s}\,({}^t\mathrm{e}_2)(M_k^{-1} - I)C_{1^{(0)},\,2^{(s)}})\,\mathrm{e}_1. \tag{78}$$

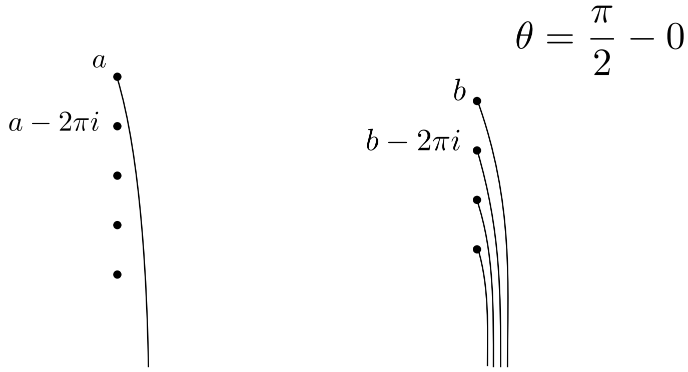


Figure 12: the connection from 0 to $\frac{\pi}{2} - 0$

And then, take $\theta$ to be $\frac{\pi}{2} + 0$. In this time, we can use Proposition 29.

$$F_j^{(0)}(x) \mapsto F_j^{(0)}(x)(1 + \sum_{r=1}^{\infty} e^{2\pi i x r}\,({}^t\mathrm{e}_j)(I - M_j)(C_{j^{(0)},j^{(1)}})^r)\,\mathrm{e}_j, \tag{79}$$

where $j = 1, 2$.

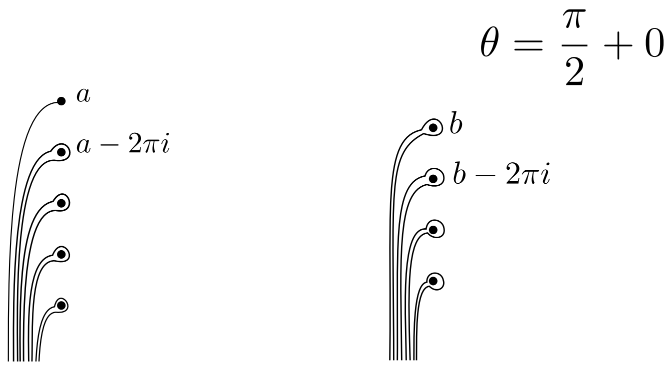


Figure 13: the connection from $\frac{\pi}{2} - 0$ to$\frac{\pi}{2} + 0$

Finally, we take a connection of the argument of $x$ from $\frac{\pi}{2} + 0$ to $\pi$, we find the following relation by using Proposition 28:

$$F_2^{(0)}(x) \to F_2^{(0)}(x) + F_1^{(0)} \sum_{n=1}^{\infty} e^{2\pi inx} \left({}^t\mathrm{e}_1\right) (I - M_1) C_{2^{(0)},\,1^{(n)}}\, \mathrm{e}_2 \tag{80}$$

From the connection relations $(78), (79)$ and $(80)$, we get a connection from $\infty$ to $-\infty$. Looking over these three connection relations, we have the connection formulae of the difference equation with 1 or 2 deterministic roots, in the form of rational function of $e^{2\pi ix}$. This result was suggested by Birkhoff[1], but we get this from the view point of the Stokes phenomena of difference equations, and obtained the coefficients of connection formulae by using the connection coefficients of Fuchsian differential equations.

We apply these propositions to the cases of the beta-function and the hypergeometric function, and see how we can calculate the connection formula in details from $\infty$ to $-\infty$ in next subsection.

## 4.3 Examples

We calculated the Stokes multipliers of the hypergeometric difference equation in subsection 3.2, though they are only one Stokes multipliers. To calculate a connection formula of the difference equation from $\infty$ to $-\infty$, we should calculate infinitely many Stokes multipliers. The connection formula between $\infty$ to $-\infty$ of the hypergeometric difference equation is a bit difficult to calculate. Therefore, we search for the connection formula of the beta function first as the simplest example, and then the hypergeometric equation.

The beta-function $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ satisfies the following difference equation.

$$(x+y)B(x+1, y) = xB(x, y). \tag{81}$$

We regard the variable $y$ as a parameter. This equation's Mellin transform is

$$(-\vartheta_t + y)t\hat{f}(t) = -\vartheta_t\hat{f}(t), \tag{82}$$

and the Borel transform is

$$(\partial_\xi + y)e^{-\xi}f_B(\xi) = \partial_\xi f_B(\xi). \tag{83}$$

We take a base of the space of solutions $f_B(\xi) = (1 - e^{-\xi})^{y-1}$. The singular points of the equation (82) are $t = 1, \infty$. The point $t = 0$ is a removable singular point, because the roots of $a_0(x) = x$ has a root $x = 0$ only, and this correspond to the characteristic exponents of $t = 0$ is 0. The equation is rewritten as follows.

$$(1-t)\partial_t\hat{f}(t) = (1-y)\hat{f}(t) \tag{84}$$

Therefore we find the characteristic exponent $y-1$ at $t = 1$. The Riemann scheme of this equation is

$$\left\{ \begin{array}{ccc} x=0 & 1 & \infty \\ 0 & y-1 & 1-y \end{array} \right\}. \tag{85}$$

This indicates the connection matrices (in this situation however, they are $1 \times 1$ matrices) $C_{1^{(0)},1^{(j)}} = 1$ because there is any singular point inside of a circle $|t| = 1 - 0$, so that the connection along this circle is trivial. Therefore, accepting the connection formula (79), we find

$$F(x) \mapsto F(x)\left(1 + \sum_{r=1}^{\infty} e^{2\pi i r x}(1 - e^{2\pi i(y-1)})\right) \tag{86}$$

when we connect analytically the Borel resummation of the formal solution from sector $(0, \frac{\pi}{2})$ to $(\frac{\pi}{2}, \pi)$.

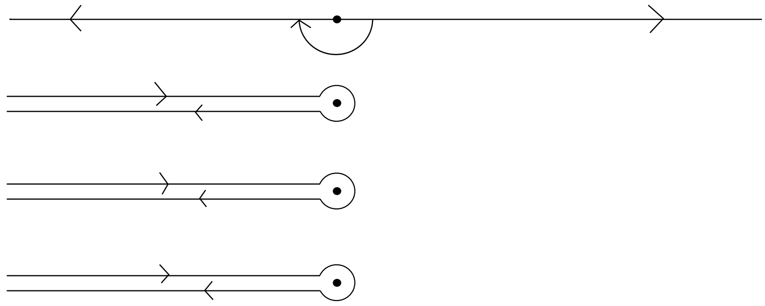

Figure 14: connection from $\theta = 0$ to $\pi$ in the case of beta function

Let us calculate the summation appeared in the right hand side of the relation (86). If the variable $x$ satisfies $\mathrm{Im}\, x > 0$, then the series converges uniformly on compact sets.

$$1 + \sum_{r=1}^{\infty} e^{2\pi irx}(1 - e^{2\pi i(y-1)}) = 1 + (1 - e^{2\pi i(y-1)})\frac{e^{2\pi ix}}{1 - e^{2\pi ix}}$$
$$= \frac{1 - e^{2\pi i(x+y)}}{1 - e^{2\pi ix}} = e^{\pi iy}\frac{\sin \pi(\mathrm{x} + \mathrm{y})}{\sin \pi x} = e^{\pi iy}\frac{\Gamma(x)\Gamma(1-x)}{\Gamma(x+y)\Gamma(1-x-y)}.$$

This sum is analytic at $\mathbb{C}\backslash\mathbb{Z}$. On the other hand, the integral is represented as

$$\int_0^{e^{-\pi i}\infty} e^{-\xi x}(1 - e^{-\xi})^{y-1}d\xi = -\int_0^{\infty} e^{\xi' x}(1 - e^{\xi'})^{y-1}d\xi'$$
$$= -e^{-\pi i(y-1)}\int_0^{\infty} e^{-\xi'(1-x-y)}(1 - e^{-\xi'})^{y-1}d\xi'$$
$$= e^{-\pi iy}B(1-x-y,\, y)$$
$$= e^{-\pi iy}\frac{\Gamma(1-x-y)\Gamma(y)}{\Gamma(1-x)}.$$

Therefore, we obtain the connection formula

$$F(x) = \int_0^\infty e^{-\xi x}(1-e^{-\xi})^{y-1}d\xi \;\mapsto\; \frac{\Gamma(x)\Gamma(1-x)\Gamma(1-x-y)\Gamma(y)}{\Gamma(x+y)\Gamma(1-x-y)\Gamma(1-x)}$$
$$= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(x, y).$$

Namely, the beta function is connected as the identity. This identity is a result of the Stokes phenomena, and of course it consistents with the identity theorem.

Although the connection formula of the difference equation of the Beta function was quite simple, the connection formulae of the hypergeometric difference equation becomes more difficult to calculate, because the Mellin transform of the difference equation has at most 4 singular points, and at least 2 singular points. We now show the connection formulae of the hypergeometric difference equation by calculating the Stokes multipliers.

First of all, we see the Riemann scheme of the Mellin transformed hypergeometric difference equation. The hypergeometric difference equation was the equation (38). Therefore, we get the Riemann scheme

$$\left\{ \begin{array}{cccc} 0 & 1 & \frac{1}{1-x} & \infty \\ 1-\gamma & \beta-1 & \gamma-\beta-1 & 1 \end{array} \right\}. \tag{87}$$

The Mellin transform of the hypergeometric difference equation is

$$[(Y-1)((1-x)Y-1)\vartheta_Y + (1-x)Y^2 + (\beta x-\gamma)Y + \gamma - 1]F(\alpha) = 0. \tag{88}$$

In the previous subsection, we discussed about the connection formulae with using the information of the Riemann scheme and connection formulae of the differential equation. We remember the formal solutions of the difference equation:

$$f^1(\alpha) = \alpha^{-\beta}(1+\cdots),$$
$$\tilde{f}^2(\alpha) = (1-x)^{\gamma-\beta+2}x^{2\beta-\gamma}(\frac{1}{1-x})^{\alpha}\alpha^{\beta-\gamma}(1+\cdots).$$

We denote $f^2$ as $\tilde{f}^2$ for simplicity. As in Remark 18, by setting the formal solutions as above, the connection of the Borel transform of $f^2$ and $f^1$ from $\xi = \log(1-x)$ to $\xi = 0$ in the Borel plane, becomes trivial :

$$\mathcal{B}f^2 \mapsto \mathcal{B}f^1.$$

The local monodromies of the equation (88) is

$$M_0 = e^{2\pi i(1-\gamma)}, \quad M_1 = e^{2\pi i(\beta-1)},$$
$$M_2 = e^{2\pi i(\gamma-\beta-1)}, \quad M_\infty = e^{2\pi i} = 1.$$

We see the connection of the solutions in the Borel plane. In the Borel plane, there exist singular points $2\pi i\mathbb{Z}$ and $\log(1-x) + 2\pi i\mathbb{Z}$.

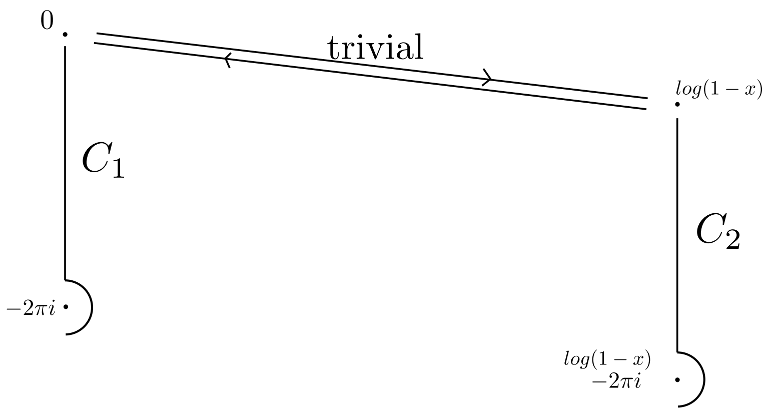


Figure 15: $C_1$ and $C_2$

Trivial connections are between the points $-2\pi im$ and $\log(1-x) - 2\pi im$ ($m \in \mathbb{Z}$), so what we have to calculate is the connections between the points 0 and $-2\pi i$, and the points $\log(1-x)$ and $\log(1-x) - 2\pi i$. Seeing a figure 16, we conclude that the connection $C_2 = C_{2^{(0)},2^{(1)}}$ is

$$C_2 = A^{-1}M_0A = M_0 = e^{2\pi i(1-\gamma)}.$$

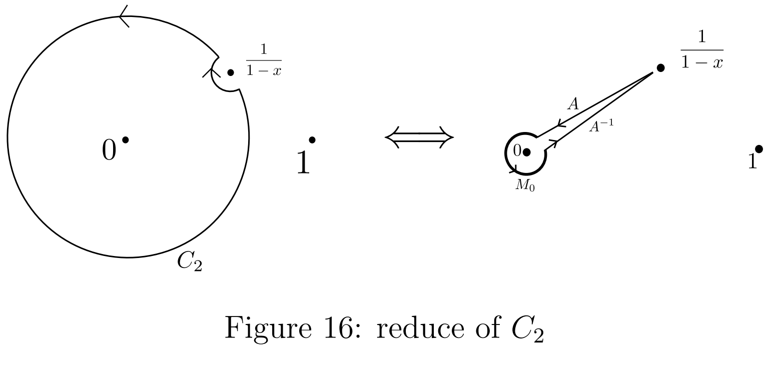


Figure 16: reduce of $C_2$

Here, we denoted $A$ as the connection of the solution between the singular points $\frac{1}{1-x}$ and 0 in the Mellin plane. Infact, $A$ is not important in this argument, because the Mellin transform of the difference equation is a first order differential equation : $A$ commutes with $M_0$. In the same way, we find that $C_1 = C_{1^{(0)},1^{(1)}}$ is following (see figure 17) :

$$C_1 = B^{-1}M_0AM_2A^{-1}B = M_0M_2 = e^{-2\pi i\beta}.$$

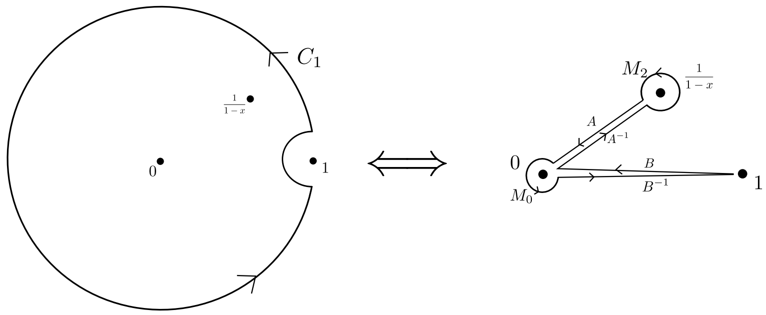


Figure 17: reduce of $C_1$

In this way, we revealed the fundamental quantities for calculating the connections.

| $M_0$ | $M_1$ | $M_2$ | $C_{1^{(0)},2^{(0)}}$ | $C_1$ | $C_2$ |
|---|---|---|---|---|---|
| $e^{2\pi i(1-\gamma)}$ | $e^{2\pi i(\beta-1)}$ | $e^{2\pi i(\gamma-\beta-1)}$ | 1 | $e^{-2\pi i\beta}$ | $e^{2\pi i(1-\gamma)}$ |

Table 1: connections between singular points and local monodromies

Now we calculate the connection formulae from $\arg(\alpha) = 0$ to $\arg(\alpha) = \pi$. To calculate them, we devide discussion in 3 steps ; ① : $\arg(\alpha) = 0 \to \frac{\pi}{2} - 0$, ② : $\arg(\alpha) = \frac{\pi}{2} - 0 \to \frac{\pi}{2} + 0$, and ③ : $\arg(\alpha) = \frac{\pi}{2} + 0 \to \pi$.

① : $\arg(\alpha) = 0 \to \frac{\pi}{2} - 0$.

Seeing the equation (78), we find the connection from $\theta = 0$ to $\frac{\pi}{2} - 0$ is

$$
\begin{aligned}
F^1(\alpha) &\mapsto F^1(\alpha) + F^2(\alpha)\sum_{k=0}^{\infty} e^{2\pi i k\alpha}(M_2^{-1} - 1)C_{1^{(0)},2^{(k)}} \\
&= F^1(\alpha) + F^2(\alpha)\sum_{k=0}^{\infty} e^{2\pi i k\alpha}(e^{-2\pi i(1-\gamma+\beta)} - 1)e^{-2\pi i\beta k} \\
&= F^1(\alpha) + F^2(\alpha)\frac{e^{-2\pi i(1-\gamma+\beta)} - 1}{1 - e^{2\pi i(\alpha-\beta)}},
\end{aligned}
$$

and

$$F^2(\alpha) \mapsto F^2(\alpha).$$

②: $\arg(\alpha) = \frac{\pi}{2} - 0 \to \frac{\pi}{2} + 0$.
The connection from $\frac{\pi}{2} - 0$ to $\frac{\pi}{2} + 0$ is, by using (79),

$$
\begin{aligned}
F^1(\alpha) &\mapsto F^1(\alpha)(1 + \sum_{j=1}^{\infty} e^{2\pi i j\alpha}(1 - M_1)C_1^j) \\
&= F^1(\alpha)(1 + e^{2\pi i(\alpha-\beta)}\frac{1 - e^{2\pi i(\beta-1)}}{1 - e^{2\pi i(\alpha-\beta)}}) \\
&= F^1(\alpha)\frac{1 - e^{2\pi i(\alpha-1)}}{1 - e^{2\pi i(\alpha-\beta)}},
\end{aligned}
$$

$$
\begin{aligned}
F^2(\alpha) &\mapsto F^2(\alpha)(1 + \sum_{j=1}^{\infty} e^{2\pi i j\alpha}(1 - M_2)C_2^j) \\
&= F^2(\alpha)(1 + e^{2\pi i(1-\gamma+\alpha)}\frac{1 - e^{2\pi i(\gamma-\beta-1)}}{1 - e^{2\pi i(1-\gamma+\alpha)}}) \\
&= F^2(\alpha)\frac{1 - e^{2\pi i(\alpha-\beta)}}{1 - e^{2\pi i(1-\gamma+\alpha)}}.
\end{aligned}
$$

③: $\arg(\alpha) = \frac{\pi}{2} + 0 \to \pi$.
Finally, the connection from $\frac{\pi}{2} + 0$ to $\pi$ is

$$F^1(\alpha) \mapsto F^1(\alpha),$$

$$
\begin{aligned}
F^2(\alpha) &\mapsto F^2(\alpha) + F^1(\alpha)(\sum_{l=1}^{\infty} e^{2\pi i l\alpha}(1 - M_1)C_{2^{(0)},1^{(l)}}) \\
&= F^2(\alpha) + F^1(\alpha)e^{2\pi i(\alpha-\beta)}\frac{1 - e^{2\pi i(\beta-1)}}{1 - e^{2\pi i(\alpha-\beta)}}.
\end{aligned}
$$

Combining these results, we get the connection formulae from $\theta = 0$ to $\theta = \pi$ as follows.

$$
\begin{aligned}
(F^1, F^2) &\mapsto (F^1, F^2) \begin{pmatrix} 1 & \frac{e^{2\pi i(\alpha-\beta)}(1-e^{2\pi i(\beta-1)})}{1-e^{2\pi i(\alpha-\beta)}} \\ 0 & 1 \end{pmatrix} \\
&\times \begin{pmatrix} \frac{1-e^{2\pi i(\alpha-1)}}{1-e^{2\pi i(\alpha-\beta)}} & 0 \\ 0 & \frac{1-e^{2\pi i(\alpha-\beta)}}{1-e^{2\pi i(1-\gamma+\alpha)}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{e^{-2\pi i(1-\gamma+\beta)}-1}{1-e^{2\pi i(\alpha-\beta)}} & 1 \end{pmatrix} \\
&= (F^1, F^2) \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \qquad (89)
\end{aligned}
$$

where $A_{i,j}$ are

$$
\begin{aligned}
A_{1,1} &= \frac{1-e^{2\pi i(\alpha-1)}}{1-e^{2\pi i(\alpha-\beta)}} + \frac{e^{2\pi i(\alpha-\beta)}(1-e^{2\pi i(\beta-1)})(e^{2\pi i(\gamma-\beta-1)}-1)}{(1-e^{2\pi i(\alpha-\beta)})(1-e^{2\pi i(\alpha-\gamma+1)})} \\
A_{2,1} &= \frac{e^{2\pi i(\gamma-\beta-1)}-1}{1-e^{2\pi i(\alpha-\gamma+1)}} \\
A_{1,2} &= \frac{e^{2\pi i(\alpha-\beta)}(1-e^{2\pi i(\beta-1)})}{1-e^{2\pi i(\alpha-\gamma+1)}} \\
A_{2,2} &= \frac{1-e^{2\pi i(\alpha-\beta)}}{1-e^{2\pi i(\alpha-\gamma+1)}}.
\end{aligned}
$$

We translate this result to the terms of hypergeometric functions. Let us write down the functions $F^1(\alpha)$ and $F^2(\alpha)$ when the argument of $\alpha$ is 0 or $\pi$, by using the hypergeometric function. Remember the integral representation of the hypergeometric function.

$$
{}_2F_1\left(\begin{matrix}\alpha, \beta \\ \gamma\end{matrix}; x\right) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-\alpha} dx. \qquad (90)
$$

The function $F^1$ is written as follows, when the argument of the variable $\alpha$ is 0.

$$
\begin{aligned}
F^1(\alpha) &= \int_0^\infty (-1)^{1-\beta}(\frac{-x}{1-x})^{1-\gamma+\beta} e^{(-\alpha+\gamma-1)\xi} \\
&\qquad \times (e^{-\xi}-1)^{\beta-1}(e^{-\xi}-\frac{1}{1-x})^{\gamma-\beta-1} d\xi \\
&= \int_0^1 x^{1-\gamma+\beta} Y^{\alpha-\gamma}(1-Y)^{\beta-1}(1-(1-x)Y)^{\gamma-\beta-1} dY \\
&= x^{1-\gamma+\beta} B(\alpha-\gamma+1, \beta) {}_2F_1\left(\begin{matrix}\beta-\gamma+1,\ \alpha-\gamma+1 \\ \alpha+\beta-\gamma+1\end{matrix}; 1-x\right).
\end{aligned}
$$

Here, we changed the variable $e^{-\xi} = Y$. The function $F^2(\alpha)$ is written as

$$\begin{aligned} F^2(\alpha) &= \int_{\log(1-x)}^{\infty} (-1)^{1+\beta-\gamma}(1-x)^{\gamma-\beta-1}x^{\beta-\gamma+1} \\ &\qquad \times e^{(-\alpha+\gamma-1)\xi}(e^{-\xi}-1)^{\beta-1}(e^{-\xi}-\frac{1}{1-x})^{\gamma-\beta-1}d\xi \\ &= (-1)^{\beta-1}(1-x)^{\gamma-\alpha+1}x^{\beta-\gamma+1}B(\alpha-\gamma+1,\gamma-\beta) \\ &\qquad \times {}_2F_1\left(\begin{matrix}1-\beta,\ \alpha-\gamma+1\\ \alpha-\beta+1\end{matrix};\frac{1}{1-x}\right), \end{aligned}$$

by changing variable $(1-x)e^{-\xi} = Y$. In the same way, when the argument of the variable $\alpha$ is $\pi$, $F^1$ and $F^2$ are written as follows.

$$\begin{aligned} F^1(\alpha) &= \int_0^{e^{-i\pi}\infty} (-1)^{1-\beta}(\frac{-x}{1-x})^{1-\gamma+\beta}e^{(-\alpha+\gamma-1)\xi}(e^{-\xi}-1)^{\beta-1} \\ &\qquad \times(e^{-\xi}-\frac{1}{1-x})^{\gamma-\beta-1}d\xi \\ &= (-1)^{2-\beta}(\frac{-x}{1-x})^{1-\gamma+\beta}\int_0^1 Y^{-\alpha}(1-Y)^{\beta-1}(1-\frac{1}{1-x}Y)^{\gamma-\beta-1}dY \\ &= (-1)^{2-\beta}(\frac{-x}{1-x})^{1-\gamma+\beta}B(1-\alpha,\beta){}_2F_1\left(\begin{matrix}\beta-\gamma+1,\ 1-\alpha\\ \beta-\alpha+1\end{matrix};\frac{1}{1-x}\right), \end{aligned}$$

by replacing the variable $e^{\xi} = Y$. To take an integration by substitution $(1-x)e^{\xi} = Y$, we have

$$\begin{aligned} F^2(\alpha) &= \int_{\log(1-x)}^{e^{-i\pi}\infty} (-1)^{1+\beta-\gamma}(1-x)^{\gamma-\beta-1}x^{\beta-\gamma+1}e^{(-\alpha+\gamma-1)\xi} \\ &\qquad \times(e^{-\xi}-1)^{\beta-1}(e^{-\xi}-\frac{1}{1-x})^{\gamma-\beta-1}d\xi \\ &= (-1)^{2+\beta-\gamma}(1-x)^{\gamma-\beta-\alpha}x^{\beta-\gamma+1}B(1-\alpha,\gamma-\beta) \\ &\qquad \times{}_2F_1\left(\begin{matrix}1-\beta\,,\ 1-\alpha\\ 1-\alpha-\beta+\gamma\end{matrix};1-x\right). \end{aligned}$$

Combining these fact, we get formulae of the hypergeometric functions. For example, connect $F^2(\alpha)$ analytically from the argument $\arg(\alpha) = 0$ to $\pi$, we get

$$
\begin{aligned}
(91) \qquad & \frac{\Gamma(1-\beta)\Gamma(1-\alpha)}{\Gamma(1-\alpha-\beta+\gamma)} {}_2F_1\left(\begin{matrix} 1-\beta\,,\, 1-\alpha \\ 1-\alpha-\beta+\gamma \end{matrix}\,;\, 1-x\right) \\
= \quad & \frac{\Gamma(1-\beta)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)} (x-1)^{\beta-1} {}_2F_1\left(\begin{matrix} 1-\beta,\ \alpha-\gamma+1 \\ \alpha-\beta+1 \end{matrix}\,;\, \frac{1}{1-x}\right) \\
& + \frac{\Gamma(1-\alpha)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\beta)} (x-1)^{\alpha-1} {}_2F_1\left(\begin{matrix} \beta-\gamma+1,\, 1-\alpha \\ \beta-\alpha+1 \end{matrix}\,;\, \frac{1}{1-x}\right).
\end{aligned}
$$

This result is the Barnes' connection formula[3, p152]. The important fact is that we obtained this equation (91) by the analysis of the Stokes phenomena of the difference equation.

# 5 Appendix

In the Example 27., we didn't give the detail of the calculus. In this Appendix, we give the calculating program of determining the hypergeometric difference equation, written in Risa/Asir.

```
/* Program main.rr                                            */
/* Hypergeometric differential operator P_4=L[0], where t is*/
/* a euler operator t=x\frac{d}{dx}.                          */
/* Following code, we may assume x and t is commute, though */
/* it's not commute. After calculating, we back x to the     */
/* front of the operator.                                     */

L=newvect(4)$
L[0]=x*(t+a1)*(t+a2)*(t+a3)*(t+a4)
-(t+b1-1)*(t+b2-1)*(t+b3-1)*t$

/* D[K]=\prod_{j=0}^{3-K} (a_1+j) \sigma^{4-k+1}            */
/* Q[K] is the coefficient of \prod_{j=0}^{3-K}(a_1+j)      */
/* \sigma^{4-K}                                               */

D=newvect(4)$

D[0]=(t+a1+3)*(t+a1+2)*(t+a1+1)*(t+a1)$
for (K=1; K<=3; K++) {
D[K]=sdiv(D[K-1],t+a1+4-K,t)$
}

Q=newvect(4)$
for (K=0; K<=2; K++){
Q[K]=sdiv(L[K],D[K],t)$
L[K+1]=srem(L[K],D[K],t)$
}
Q[3]=sdiv(L[3],D[3],t)$
R=srem(L[3],D[3],t)$

/* R is the constant term (does not depend on \sigma). We    */
/* get P_4 = \sum_{K=0}^{3} \prod_{j=0}^{3-K} (a_1+j) Q[K]   */
/* \sigma^{4-K} +R.                                           */
/* We denote P_4 with the difference operator s=\sigma.      */
```

```
S = newvect(4)$
P = newvect(4)$

P[3]=a1$
P[2]=P[3]*(a1+1)$
P[1]=P[2]*(a1+2)$
P[0]=P[1]*(a1+3)$

S[0]=P[0]*s4$
S[1]=P[1]*s3$
S[2]=P[2]*s2$
S[3]=P[3]*s1$

/* sj is a difference operator \sigma^j                       */
/* We will substitute sj the differential operator later.     */
/* O is a difference operator which is gotten from HG         */
/* differential op.                                           */

O=0$

for (K=0; K<=3; K++) {
O=O+Q[K]*S[K]$
}
O=O+R$
/* O has a factor a1, i.e. O/a1 is a difference operator      */
/* with its coefficients polynomial. So we devide O by a1     */
/* and name it U.                                             */

U=sdiv(O,a1)$
/* U is the hypergeometric difference operator.               */
T = newvect(5)$
T[4]=coef(U,1,s4)$
T[3]=coef(U,1,s3)$
T[2]=coef(U,1,s2)$
T[1]=coef(U,1,s1)$
T[0]=coef(coef(coef(coef(U,0,s1),0,s2),0,s3),0,s4)$

print(" ")$
print("the hypergeometric difference operator is")$
print(" ")$
```

```
for (K=0; K<=3; K++){
print("(",0)$ print(fctr(T[4-K]),0)$ print(") sigma^",0)$
print(4-K,0)$ print(" +")$
}
print(fctr(T[0]))$


/* T[K] is the coefficients of \sigma^K.                        */
/* The deterministic polynomial of the hypergeometric           */
/* difference equation is obtained as follows.                  */

F=0$
for (K=0; K<=4; K++) {
if ( deg(T[K],a1) > F ) {
F=deg(T[K],a1);
}
}

/* F is the degree of the difference equation.                  */

G=0$
for (K=0; K<=4; K++){
G=G+coef(T[K],F,a1)*l^K;
}
print(" ")$
print("the deterministic polynomial is")$
print(" ")$
fctr(G);
print(" ")$
/* G is the deterministic polynomial, and the roots l is the*/
/*  deterministic roots. If the calculus is correct, then,  */
/* the deterministic roots must be triplet roots l=1 and    */
/* simple root l=1/1-x.                                     */
/* For the purpose of calculating the characteristic        */
/* exponents related to the triplet roots l=1, we substitute*/
/* the differential operator to the difference operator.    */

W=newvect(5)$
E=1+q+(1/2)*q^2+(1/6)*q^3$
for (K=0; K<=4; K++){
```

```
W[K]=subst(E,q,K*da1)$
}
Z=0$
for (K=0; K<=4; K++){
Z=Z+T[K]*W[K]$
}

C=newvect(4)$
for (K=0; K<=3; K++) {
C[K]= coef(coef(Z ,K, da1),K,a1)$
}
B=newvect(4)$
B[0]=1$
for (K=1; K<=3; K++) {
B[K]=B[K-1]*(n-K+1)$
}

A=0$
for (K=0; K<=3; K++) {
A=A+B[K]*C[K]$
}
fctr(A);

/* A is the characteristic polynomial of the hypergeometric */
/* difference equation and the root of A is the             */
/* characteristic index of the equation related to the      */
/* deterministic root \lambda=1.                            */

/* We calculate a characteristic exponents related to the   */
/* deterministic root \lambda=1/(1-x). Let N be a difference*/
/* operator which is obtained by gauge transform of the     */
/* hypergeometric operator P_4 with (1/(1-x))^{a1}. We      */
/* denote M[K] as the coefficient of the difference operator*/
/* \sigma^K, in the operator (1-x)^4*M.                     */

M=newvect(5)$
for (K=0; K<=4; K++) {
M[K]=(1-x)^(4-K)*T[K];
}
```

```
/* W is a vector of differential operator, which should be  */
/* substituted to the difference operator [1,s,s^2,s^3,s^4].*/

W=newvect(5)$
E=1+q$

for (K=0; K<=4; K++) {
W[K]=subst(E,q,K*da1);
}
Z=0$
for (K=0; K<=4; K++) {
Z=Z+M[K]*W[K];
}
deg(coef(Z,1,da1),a1);
deg(coef(Z,0,da1),a1);
/* if the calculus is correct, the result of deg(**) must be*/
/* 3, and 2.                                                */

fctr(coef(coef(Z,1,da1),3,a1)*n+coef(coef(Z,0,da1),2,a1));
/* This fctr(**) is the characteristic polynomial related to*/
/* the deterministic root 1/(1-x).                          */
end$
```

The result is

```
the hypergeometric difference operator is

([[1,1],[a1+1,1],[a1+2,1],[a1+3,1],[x-1,1]]) sigma^4 +
([[-1,1],[a1+1,1],[a1+2,1],[(3*a1-a2-a3-a4+6)*x+b1+b2+b3
-4*a1-9,1]]) sigma^3 +
([[1,1],[a1+1,1],[(3*a1^2+(-2*a2-2*a3-2*a4+9)*a1+(a3+a4-
3)*a2+(a4-3)*a3-3*a4+7)*x+(-b2-b3+3*a1+5)*b1+(-b3+
3*a1+5)*b2+(3*a1+5)*b3-6*a1^2-21*a1-19,1]]) sigma^2 +
([[-1,1],[(a1^3+(-a2-a3-a4+3)*a1^2+((a3+a4-2)*a2+(a4-2)*a3
-2*a4+3)*a1+((-a4+1)*a3+a4-1)*a2+(a4-1)*a3-a4+1)*x+((b3-
2*a1-2)*b2+(-2*a1-2)*b3+3*a1^2+7*a1+4)*b1+((-2*a1-2)*b3
+3*a1^2+7*a1+4)*b2+(3*a1^2+7*a1+4)*b3-4*a1^3-15*a1^2
-19*a1-8,1]]) sigma^1 +
```

```
[[1,1],[b3-a1-1,1],[b2-a1-1,1],[b1-a1-1,1]]

the deterministic polynomial is

[[1,1],[l-1,3],[l*x-l+1,1]]

The characteristic polynomial of the hypergeometric
 difference equation related to the triplet deterministic
 root lambda=1 is

[[1,1],[n+a4,1],[n+a3,1],[n+a2,1],[x,1]]

The characteristic polynomial of the hypergeometric
 difference equation related to the simple deterministic
 root lambda=frac{1}{1-x} is

[[1,1],[n+b1+b2+b3-a2-a3-a4,1],[x,3],[x-1,1]]
```

These results show that the Riemann scheme of the inverse Mellin transformation of the hypergeometric difference equation is (65).

# References


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