博士論文

論文題目:
Autonomous limit of 4-dimensional Painlevé-type equations
and the singular fibers of their spectral curve fibrations
(4 次元 Painlevé 型方程式の自励極限と
スペクトラル曲線ファイブレーションの特異ファイバー)

氏名: 中村 あかね

AUTONOMOUS LIMIT OF 4-DIMENSIONAL PAINLEVÉ-TYPE EQUATIONS AND DEGENERATION OF CURVES OF GENUS TWO

AKANE NAKAMURA

ABSTRACT. Higher dimensional analogs of the Painlevé equations have been proposed from various aspects. In recent studies, 4-dimensional analogs of the Painlevé equations were classified into 40 types. The aim of the present paper is to geometrically characterize these 40 types of equations. For this purpose, we study the autonomous limit of these equations and degeneration of their spectral curves. We obtain two functionally independent conserved quantities H_1 and H_2 for each system. We construct fibrations whose fiber at a general point h_i is the spectral curve of the system with $H_i = h_i$ for $i = 1, 2$. The singular fibers at $H_i = \infty$ are one of the degenerate curves of genus 2 classified by Namikawa and Ueno. Liu's algorithm enables us to give degeneration type of spectral curves for our 40 types of integrable systems. This result is analogous to the following observation; spectral curve fibrations of the autonomous 2-dimensional Painlevé equations $P_{\rm I}$, $P_{\rm II}$, $P_{\rm IV}$, $P_{\rm III}^{D_8}$, $P_{\rm III}^{D_7}$, $P_{\rm III}^{D_6}$, P_V and $P_{\rm VI}$ are elliptic surfaces with the singular fiber at $H = \infty$ of Dynkin type $E_8^{(1)}$, $E_7^{(1)}$, $E_6^{(1)}$, $D_8^{(1)}$, $D_7^{(1)}$, $D_6^{(1)}$, $D_5^{(1)}$ and $D_4^{(1)}$, respectively.

1. Introduction

The Painlevé equations are 8 types of nonlinear second-order ordinary differential equations with the Painlevé property[1] which are not solvable by elementary functions. The Painlevé equations have various interesting features; they can be derived from isomonodromic deformation of certain linear equations, they are linked by degeneration process, they have affine Weyl group symmetries, they can be derived from reductions of soliton equations, and they are equivalent to nonautonomous Hamiltonian systems. Furthermore, their autonomous limits are integrable systems solvable by elliptic functions. We call such systems the autonomous Painlevé equations.

Various generalization of the Painlevé equations have been proposed by focusing on one of the features of the Painlevé equations. The main two directions of generalizations are higher-dimensional analogs and difference analogs. We focus on higher-dimensional analogs in this paper. The eight types of the Painlevé equations are called the 2-dimensional Painlevé equations in this context, and higher-dimensional analogs are $2n$-dimensional Painlevé-type equations ($n = 1, 2, \dots$).

Recently, classification theory of linear equations up to some transformations have been developed. Since these transformations of linear equations leave isomonodromic deformation equations invariant [23], we can make use of classification theory of linear equations to the classification of Painlevé-type equations[2]. As for the 4-dimensional case, classification and derivation of the Painlevé-type equations from isomonodromic deformation have been completed [62, 32, 30]. According to their result, there are 40 types of 4-dimensional Painlevé-type equations. Among these 40 types of equations, some of them coincides with previously-known equations from different derivations. Such well-known equations along with new equations are all organized in a unified way: isomonodromic deformation and degeneration.

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[1]A differential equation is said to have the Painlevé property if it has no movable singularities other than poles.

[2]In this paper we use the term Painlevé-type equations synonymously with isomonodromic deformation equations.

There is a problem to their classification. They say that there are “at most” 40 types of 4-dimensional Painlevé-type equations. There is no guarantee that these 40 equations are actually different. Some Hamiltonians of these equations look very similar to each other. Since the appearance of Hamiltonians or equations may change significantly by changes of variables, we can not classify equations by their appearances. Intrinsic or geometrical studies of these equations may be necessary to distinguish these Painlevé-type equations. The aim of this paper is to start such intrinsic studies of 4-dimensional Painlevé-type equations.

Let us first review how the 2-dimensional Painlevé equations are geometrically classified. Okamoto initiated the studies of the space of initial conditions of the Painlevé equations [58]. He constructed rational surfaces, whose points correspond to the germs of meromorphic solutions of the Painlevé equations, by resolving singularities of the differential equations. Sakai extracted the key features of Okamoto’s spaces of Painlevé equations and classified what he calls “generalized Halphen surfaces” with such features [61]. The classification of the 2-dimensional difference Painlevé equations correspond to the classification of generalized Halphen surfaces, and 8 types of such surfaces produce Painlevé differential equations. Such surfaces are distinguished by their anticanonical divisors. For the autonomous 2-dimensional Painlevé equations, the spaces of initial conditions are elliptic surfaces and their anticanonical divisors are one of the Kodaira types [63]. We can say that the 2-dimensional autonomous Painlevé equations are characterized or distinguished by the corresponding Kodaira types.

It is expected to carry out a similar study for the 4-dimensional Painlevé-type equations. However, the straightforward generalization of 2-dimensional case seems to contain many difficulties. One of the reason is that the classification of 4-folds is much more difficult than that of surfaces. In this paper, we try to propose a way to overcome such difficulties. We consider the autonomous limit of these Painlevé-type equations. While the geometry is made simple considerably in the autonomous limit, the autonomous systems still retain the important characteristics of the original non-autonomous equations.

Integrable systems are Hamiltonian systems on symplectic manifolds (M^{2n}, ω, H) with n functions $f_1 = H, \dots, f_n$ in involution $\{f_i, f_j\} = 0$. The regular level sets of the momentum map $F = (f_1, \dots, f_n)\colon M \to \mathbb{C}^n$ are Liouville tori. The image under F of critical points are called the bifurcation diagram, and it is studied to characterize integrable systems [10]. However, studying the bifurcation diagrams may become complicated for higher dimensional cases. When an integrable system is expressed in a Lax form, it is not difficult to determine the discriminant locus of spectral curves. We can often find correspondence between the bifurcation diagrams and the discriminant locus of spectral curves [7]. Therefore, we mainly study degeneration of spectral curves in this paper.

Let $\{H_i\}_{i=1,\dots,g}$ be a set of functionally independent conserved quantities of an integrable system (M, ω, H). We construct relatively minimal smooth projective surfaces over $\mathbb{P}^1$, whose fiber at a general point $h_i \in \mathbb{P}^1$ is the spectral curve with $H_i = h_i$. We examine the singular fibers at $H_i = \infty$ of such “spectral curve fibrations”. One of the advantage in studying the spectral curve fibrations is that the following theorem holds for the autonomous 2-dimensional Painlevé equations.

Theorem (cf. Theorem 5)**.** *Each autonomous 2-dimensional Painlevé equation defines an elliptic surface, whose general fibers are spectral curves of the system. The Kodaira types of the singular fibers at* $H_{\mathrm{J}} = \infty$ *are listed as follows.*

Hamiltonian	H_{VI}	H_{V}	$H_{\mathrm{III(D_6)}}$	$H_{\mathrm{III(D_7)}}$	$H_{\mathrm{III(D_8)}}$	H_{IV}	H_{II}	H_{I}
Kodaira type	I_0^*	I_1^*	I_2^*	I_3^*	I_4^*	IV^*	III^*	II^*
Dynkin type	$D_4^{(1)}$	$D_5^{(1)}$	$D_6^{(1)}$	$D_7^{(1)}$	$D_8^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$

Table 1: The singular fiber at $H_{\mathrm{J}} = \infty$ of spectral curve fibrations of the autonomous 2-dimensional Painlevé equations.

The Dynkin's types in the above diagram are well-known to appear as the configurations of vertical leaves of the Okamoto's spaces of initial conditions [58].

For the autonomous 4-dimensional Painlevé-type equations, general invariant sets $\cap_{i=1,2} H_i^{-1}(h_i)$ are 2-dimensional Liouville tori. Such Liouville tori are related to the Jacobian varieties of the corresponding spectral curves. Instead of studying the degenerations of 2-dimensional Liouville tori, we study the degenerations of the spectral curves of genus 2. As an analogy of the above theorem for the 2-dimensional Painlevé equations, we find the following:

Main theorem (cf. Theorem 9). *Each autonomous 4-dimensional Painlevé-type equation defines two smooth surfaces with relatively minimal fibrations $\phi_i \colon X_i \to \mathbb{P}^1$ $(i = 1, 2)$, whose general fibers are the spectral curves of genus 2. The Namikawa-Ueno type of their singular fibers at $H_1 = \infty$ and $H_2 = \infty$ are as in Table 7 and Table 8 in Section 4 respectively*[3].

The table shows, for example, that the spectral curve fibration of the matrix Painlevé equations $H_{\mathrm{VI}}^{\mathrm{Mat}}$, $H_{\mathrm{V}}^{\mathrm{Mat}}$, $H_{\mathrm{III(D_6)}}^{\mathrm{Mat}}$, $H_{\mathrm{III(D_7)}}^{\mathrm{Mat}}$ and $H_{\mathrm{III(D_8)}}^{\mathrm{Mat}}$ have the singular fibers of the Namikawa-Ueno type $\mathrm{I}_0 - \mathrm{I}_0^* - 1$, $\mathrm{I}_0 - \mathrm{I}_1^* - 1$, $\mathrm{I}_0 - \mathrm{I}_2^* - 1$, $\mathrm{I}_0 - \mathrm{I}_3^* - 1$ and $\mathrm{I}_0 - \mathrm{I}_4^* - 1$, respectively. Those of $H_{\mathrm{IV}}^{\mathrm{Mat}}$, $H_{\mathrm{II}}^{\mathrm{Mat}}$ and $H_{\mathrm{I}}^{\mathrm{Mat}}$ are $\mathrm{I}_0 - \mathrm{IV}^* - 1$, $\mathrm{I}_0 - \mathrm{III}^* - 1$ and $\mathrm{I}_0 - \mathrm{II}^* - 1$. Therefore, the singular fibers at $H_{\mathrm{J}}^{\mathrm{Mat}} = \infty$ of the spectral curve fibrations of the autonomous 4-dimensional matrix Painlevé equations are characterized by adding one additional component of elliptic curve to those counterpart of the 2-dimensional Painlevé systems.

There are various ways to have integrable systems in general. To identify the isomorphic equations from different origins is often not easy; equations may change their appearance by transformations. It is hoped that such intrinsic geometrical studies will be helpful for such identification problems.

Contents. The organization of this paper is as follows. In Section 2, after summarizing preliminaries, we review the classification of the 4-dimensional Painlevé-type equations. In Section 3, we consider the autonomous limit of these 40 equations. In Section 4, we introduce the main tool of our study: "spectral curve fibration". We study the fibers at $H_i = \infty$ $(i = 1, 2)$ of these fibrations to characterize these integrable systems. In Appendix A.1, we have conserved quantities of autonomous 4-dimensional Painlevé-type equations. In Appendix A.2, we review the local data of linear equations and explain the symbols used in this paper to express spectral types. In Appendin A.3, we list the dual graph of the singular fibers appeared in our table.

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[3]When we consider a fibration $\phi_i \colon X_i \to \mathbb{P}^1$, we assume that the other conserved quantity H_{i+1} to take a general value.

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2. Classification of 4-dimensional Painlevé-type equations

In this section, we review some of the recent progresses in classification of the 4-dimensional Painlevé-type equations, and introduce notation we use in this paper. The contents of this section is a summary of the other papers [62, 32, 30] and references therein.

The Painlevé equations were found by Painlevé through his classification of the second order algebraic differential equations with the "Painlevé property": equations possess at most poles as movable singularities. However, a straightforward application of Painlevé's classification method to higher-dimensional cases seems to face difficulties.[4] Therefore other properties which characterize the Painlevé equations become important for further generalization. The Painlevé equations P_{J} ($\mathrm{J} = \mathrm{I}, \dots, \mathrm{VI}$) can be expressed as Hamilton systems [51, 58].

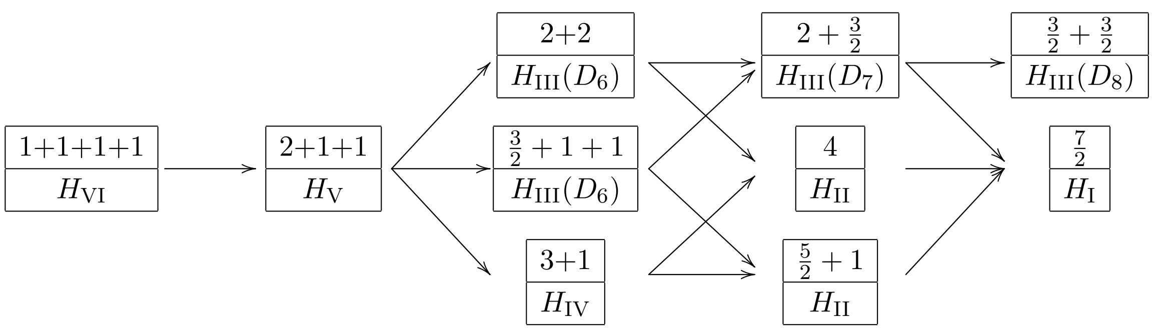
$$\frac{dq}{dt} = \frac{\partial H_{\mathrm{J}}}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H_{\mathrm{J}}}{\partial q}.$$

We list the Hamiltonian functions for all the 2-dimensional Painlevé equations for later use.

$$\begin{aligned}
t(t-1)H_{\mathrm{VI}}\begin{pmatrix}\alpha,\beta \\ \gamma,\epsilon\end{pmatrix}; t; q, p\Big) &= q(q-1)(q-t)p^2 \\
&\quad + \{\epsilon q(q-1) - (2\alpha+\beta+\gamma+\epsilon)q(q-t) + \gamma(q-1)(q-t)\}p \\
&\quad + \alpha(\alpha+\beta)(q-t), \\
tH_{\mathrm{V}}\left(\begin{matrix}\alpha,\beta \\ \gamma\end{matrix}; t; q, p\right) &= p(p+t)q(q-1) + \beta pq + \gamma p - (\alpha+\gamma)tq, \\
H_{\mathrm{IV}}(\alpha,\beta; t; q, p) &= pq(p-q-t) + \beta p + \alpha q, \\
tH_{\mathrm{III}}(D_6)(\alpha,\beta; t; q, p) &= p^2q^2 - (q^2 - \beta q - t)p - \alpha q, \\
tH_{\mathrm{III}}(D_7)(\alpha; t; q, p) &= p^2q^2 + \alpha qp + tp + q, \quad tH_{\mathrm{III}}(D_8)(t; q, p) = p^2q^2 + qp - q - \frac{t}{q}, \\
H_{\mathrm{II}}(\alpha; t; q, p) &= p^2 - (q^2+t)p - \alpha q, \quad H_{\mathrm{I}}(t; q, p) = p^2 - q^3 - tq.
\end{aligned}$$

The Painlevé equations have another important aspect initiated by R. Fuchs [19]. Namely, they can be derived from (generalized) isomonodromic deformation of linear equations [28]. This aspect is crucial for the classification of the 4-dimensional Painlevé-type equations.

Furthermore, these eight types of Painlevé equations are linked by processes called degenerations. In fact, $P_{\mathrm{I}}, \cdots, P_{\mathrm{V}}$ can be all derived from P_{VI} through degenerations. Along with the degenerations of the Hamiltonian systems, the corresponding linear equations degenerate too. The notations used in the degeneration scheme are also explained in Appendix A.2.



[4] Works of Chazy [11] and Cosgrove [14, 15] are famous in this direction.

2.1. **Isomonodromic deformation and classification of linear equations.** We review the classification of the 4-dimensional Painlevé-type equations [62, 32, 30] based on the classification of (Fuchsian) linear equations.

2.1.1. *The classification of Fuchsian equations and isomonodromic deformation.* If we fix the number of accessory parameters and identify the linear equations that transforms into one another by Katz's operations (addition and middle convolutions) [29, 16], we have only finite types of linear equations. The symbols for spectral types are explained in Appendix A.2.

Theorem 1 (Kostov[43]). *Irreducible Fuchsian equations with two accessory parameters result in one of the four types by successive additions and middle convolutions:*

- $11, 11, 11, 11$
- $111, 111, 111 \quad 22, 1111, 1111 \quad 33, 222, 111111.$

Remark 1. Note that only the equation of the type $11, 11, 11, 11$ has four singular points and the other three types have three singular points. The three of the singular points can be fixed at $0, 1, \infty$ by a Möbius transformation. Thus the three equations with only three singularities do not admit the continuous deformation of position of singularities. Only $11, 11, 11, 11$ admit the isomonodromic deformation. It corresponds to the fact that the only 2-dimensional Painlevé equation that can be derived from isomonodromic deformation of Fuchsian equation is P_{VI}.

The Katz's operations are important for studying Painlevé-type equations, because the following theorem holds.

Theorem 2 (Haraoka-Filipuk [23]). *Isomonodromic deformation equations are invariant under Katz's operations.*

Remark 2. Katz's operations that do not change the type of the linear equation induce the corresponding Bäcklund transformations on the isomonodromic deformation equation. In fact, all of the $D_4^{(1)}$-type affine Weyl group symmetry that P_{VI} possesses can be derived from Katz's operations and the Schlesinger transformations on the linear equations [52].

2.2. **The classification of 4-dimensional Painlevé-type equations.** In this subsection, we review the classification of 4-dimensional Painlevé-type equations. From the classification of Fuchsian type equations, we have four equations from which 4-dimensional Painlevé-type equations follow. Degenerations of these four equations yield 40 types of 4-dimensional Painlevé-type equations. We also refer to some different origins of the systems in our list.

2.2.1. *The classification of Fuchsian linear equations and isomonodromic deformation.* Analogously, Fuchsian equations with four accessory parameters are also classified.

Theorem 3 (Oshima [59]). *Irreducible Fuchsian equations with four accessory parameters result in one of the following 13 types by successive additions and middle convolutions:*

- $11, 11, 11, 11, 11$
- $21, 21, 111, 111 \quad 31, 22, 22, 1111 \quad 22, 22, 22, 211$
- $211, 1111, 1111 \quad 221, 221, 11111 \quad 32, 11111, 11111$
  $222, 222, 2211 \quad 33, 2211, 111111 \quad 44, 2222, 22211$
  $44, 332, 11111111 \quad 55, 3331, 22222 \quad 66, 444, 2222211.$

Remark 3. Note that the equation of type $11, 11, 11, 11, 11$ has five singular points, and the next three types have four singular points, and the rest nine types have three singular points. The equation of type $11, 11, 11, 11, 11$ has two singularities t_1 and t_2 to deform after fixing three of singularities to $0, 1, \infty$. The next three types of equation with four singularities have a singularity t to deform after fixing three of singularities to $0, 1, \infty$. The nine equations with only three singularities do not admit isomonodromic deformation.

Sakai derived explicit Hamiltonians of the above mentioned four equations with four accessory parameters.

Theorem 4 (Sakai [62])**.** *There are four 4-dimensional Painlevé-type equations governed by Fuchsian equations.*

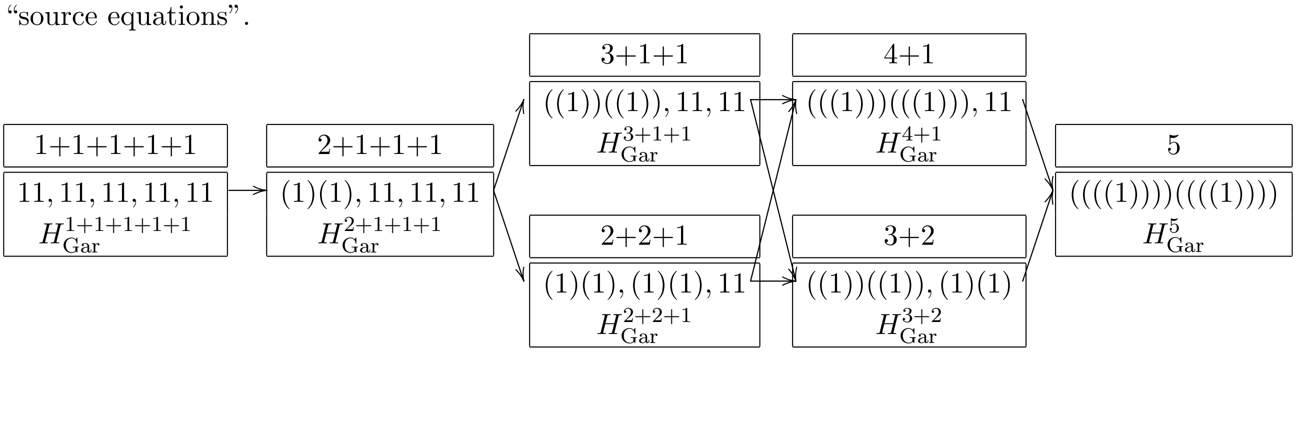
- *The Garnier system in two variables (11,11,11,11,11).*
- *The Fuji-Suzuki system (21,21,111, 111).*
- *The Sasano system (31,22,22,1111).*
- *The Sixth Matrix Painlevé equation of size 2 (22,22,22,211).*

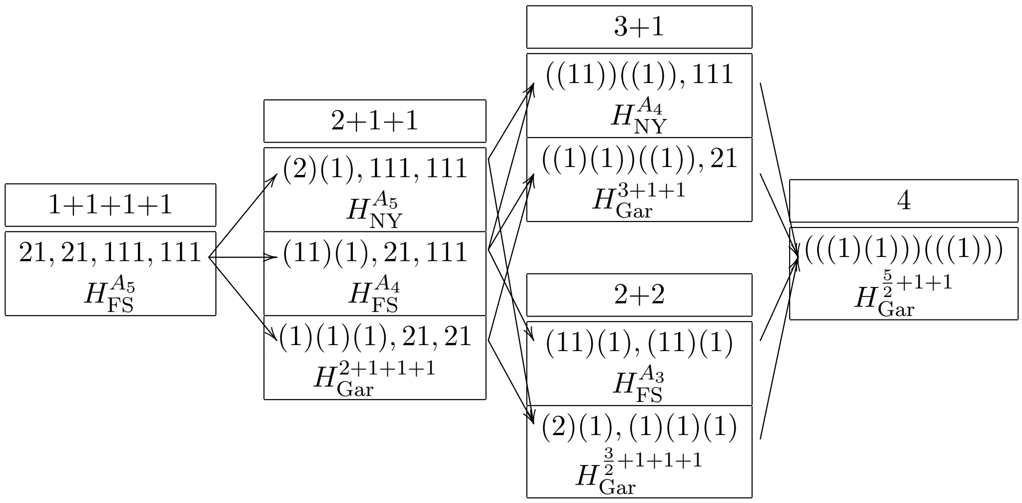
In this paper, we call these 4 equations derived from Fuchsian equation the "source equations".

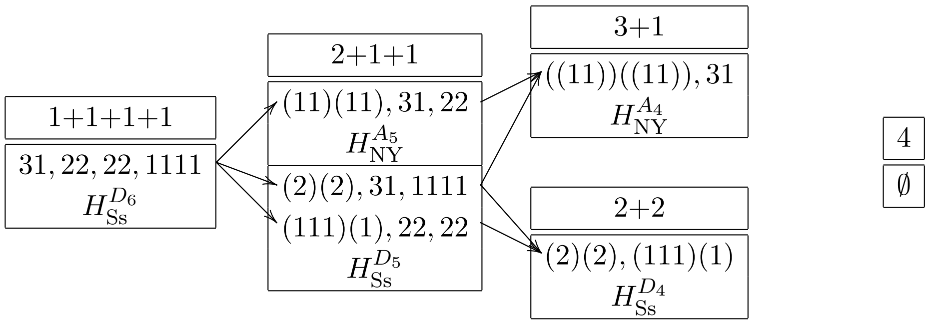
2.2.2. *Degeneration scheme of 4-dimensional Painlevé-type equations.* In order to obtain all the 4-dimensional Painlevé-type equations, we now present the degeneration scheme of these source equations. The degenerations corresponding to unramified linear equations are treated in Kawakami-Nakamura-Sakai [32]. The degenerations corresponding to ramified linear equations are written in Kawakami [30].

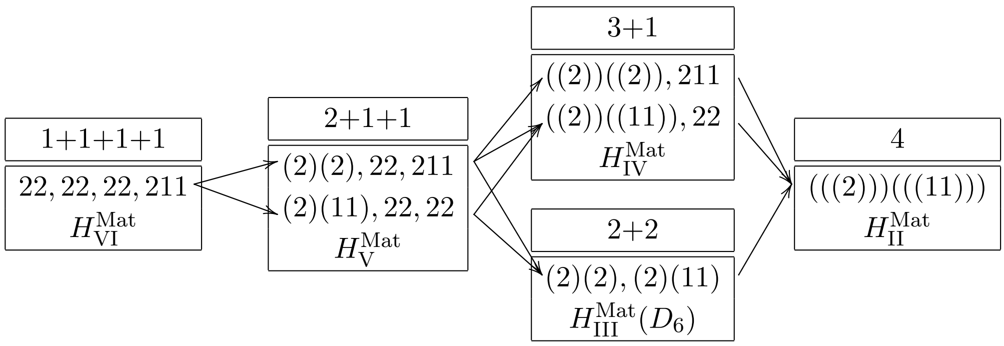
	Fuchsian	Non-Fuchsian (unramified)	Non-Fuchsian (ramified)
PDE	$H_{\mathrm{Gar}}^{1+1+1+1+1}$	$H_{\mathrm{Gar}}^{2+1+1+1}$ H_{Gar}^{3+1+1} H_{Gar}^{2+2+1} H_{Gar}^{4+1} H_{Gar}^{3+2} H_{Gar}^{5} $H_{\mathrm{Gar}}^{\frac{3}{2}+1+1+1}$ $H_{\mathrm{Gar}}^{\frac{5}{2}+1+1}$	$H_{\mathrm{Gar}}^{2+\frac{3}{2}+1}$ $H_{\mathrm{Gar}}^{\frac{3}{2}+\frac{3}{2}+1}$ $H_{\mathrm{Gar}}^{\frac{5}{2}+2}$ $H_{\mathrm{Gar}}^{\frac{7}{2}+1}$ $H_{\mathrm{Gar}}^{\frac{3}{2}+3}$ $H_{\mathrm{Gar}}^{\frac{5}{2}+\frac{3}{2}}$ $H_{\mathrm{Gar}}^{\frac{9}{2}}$
ODE	$H_{\mathrm{FS}}^{A_5}$ $H_{\mathrm{Ss}}^{D_6}$ $H_{\mathrm{VI}}^{\mathrm{Mat}}$	$H_{\mathrm{FS}}^{A_4}$ $H_{\mathrm{FS}}^{A_3}$ $H_{\mathrm{NY}}^{A_5}$ $H_{\mathrm{NY}}^{A_4}$ $H_{\mathrm{Ss}}^{D_5}$ $H_{\mathrm{Ss}}^{D_4}$ $H_{\mathrm{V}}^{\mathrm{Mat}}$ $H_{\mathrm{IV}}^{\mathrm{Mat}}$ $H_{\mathrm{III}(D_6)}^{\mathrm{Mat}}$ $H_{\mathrm{II}}^{\mathrm{Mat}}$	$H_{\mathrm{Suz}}^{2+\frac{3}{2}}$ $H_{\mathrm{KFS}}^{2+\frac{4}{3}}$ $H_{\mathrm{KFS}}^{\frac{3}{2}+\frac{4}{3}}$ $H_{\mathrm{KFS}}^{\frac{4}{3}+\frac{4}{3}}$ $H_{\mathrm{KSs}}^{2+\frac{3}{2}}$ $H_{\mathrm{KSs}}^{2+\frac{4}{3}}$ $H_{\mathrm{KSs}}^{2+\frac{5}{4}}$ $H_{\mathrm{KSs}}^{\frac{3}{2}+\frac{5}{4}}$ $H_{\mathrm{III}(D_7)}^{\mathrm{Mat}}$ $H_{\mathrm{III}(D_8)}^{\mathrm{Mat}}$ $H_{\mathrm{I}}^{\mathrm{Mat}}$

As shown in the following diagrams, there are 4 series of degeneration diagram corresponding to 4 "source equations".









Here the symbol in the each upper box indicates singularity pattern of the linear equation. There are one or more than one spectral types corresponding to each singularity pattern. In the each lower box, we write the spectral types and the corresponding Hamiltonian. Explicit forms of the Hamiltonians and the Lax pairs can be found in other papers [32, 30]. Some pairs of linear equations transform one into another by the Laplace transformations. Since isomonodromic equations are invariant under the Laplace transformations, we identify isomonodromic equations corresponding to linear equations that transform one into another by a Laplace transformation. This is why the some Hamiltonians such as $H_{\mathrm{NY}}^{A_5}$, $H_{\mathrm{Gar}}^{2+1+1+1}$ appear twice in the diagram. This also explains the fact that there are boxes in the diagram with two types of linear equations and only one Hamiltonian.

Remark 4. The naming of these Hamiltonians is temporary. As Sakai called the 2-dimensional systems by the types of the spaces of initial conditions, it may be natural to call the 4-dimensional systems from geometrical characterization.

Remark 5. While Kawakami-Nakamura-Sakai studied the degeneration from Fuchsian types, the theory of unramified non-Fuchsian linear equations has developed. Hiroe and Oshima have classified all the unramified linear equations with 4 accessory parameters up to some transformations [24, Thm3.29].

Yamakawa proved that analogous theorem of Haraoka-Filpuk (Theorem 2) holds for unramified non-Fuchsian equations [76]. From their studies, we can assure that the all unramified equations come from 4 Fuchsian source equations by degenerations, and our list of the unramified case is complete.[5]

Remark 6. Some of these 40 equations look similar to each other. For instance, H. Chiba pointed out that H_{Gar}^{4+1} and $\tilde{H}_{\text{II}}^{\text{Mat}}$ look almost the same after a symplectic transformations of the variables.[6]

$$\begin{aligned} H_{\text{Gar},t_1}^{4+1} =& p_1^2 - \left(q_1^2 + t_1\right) p_1 + \kappa_1 q_1 + p_1 p_2 + p_2 q_2 \left(q_1 - q_2 + t_2\right) + \theta_0 q_2, \\ \tilde{H}_{\text{II}}^{\text{Mat}} =& p_1^2 - \left(\frac{q_1^2}{4} + t\right) p_1 - \left(\theta_0 + \frac{\kappa_2}{2}\right) q_1 + p_1 p_2 + p_2 q_2 \left(q_1 - q_2\right) + \theta_0 q_2. \end{aligned}$$

One of the key motivations of the present paper is to geometrically distinguish such cases. We will show in a later chapter that the types of singular fibers in their spectral curve fibrations are different.[7]

Let us review some of the historical origins of these equations.

The classification of the Garnier systems are studied beforehand by Kimura [36]. The cases corresponding to ramified linear equations are completed by Kawamuko [35]. Koike clearfied some of the relations between the Painlevé hierarchies and the degenerate Garnier systems. He showed that P_{I}, P_{34}, P_{II} and P_{IV}-hierarchies can also be obtained by the restrictions of certain Garnier systems [41, 42]. These relations are listed in Table 2

H_{Gar}	$(P_{\text{J}})_2$	other derivations	references
$H_{\text{Gar}}^{9/2}$	$(P_{\text{I}})_2$	self-similarity reduction of the KdV (first Painlevé hierarchy)	Kudryashov [44], Kimura [36]
$H_{\text{Gar}}^{1+7/2}$	$(P_{34})_2$	self-similarity reduction of the mKdV (second Painlevé hierarchy)	Airault [6], Flaschka-Newell [18], Ablowitz-Segur [1] Clarkson-Joshi-Pickering [12], Kawamuko [35]
H_{Gar}^{5}	$(P_{\text{II}})_2$	A_m-system	Kimura [36], Gordoa-Joshi-Pickering [22], Liu-Okamoto [47]
H_{Gar}^{1+4}	$(P_{\text{IV}})_2$	Kawamuko's system	Kimura [36], Gordoa-Joshi-Pickering [22], Kawamuko [34]

Table 2: Other derivations of some of the degenerate Garnier systems

What we call A_5 and A_4-type Noumi-Yamada equations also appear in the degenerations of the A_5-type Fuji-Suzuki system. A_4-type Noumi-Yamada system is well known in the following expression:

$$\text{(2.1)} \qquad \text{NY}^{A_4} \; : \frac{df_i}{dt} = f_i(f_{i+1} - f_{i+2} + f_{i+3} - f_{i+4}) + \alpha_i, \quad (i \in \mathbb{Z}/5\mathbb{Z}),$$

as systems of equations for the unknown functions $f_0, \ldots, f_l (l = 4)$ [55]. This systems coincide with the Hamiltonian system $H_{\text{NY}}^{A_4}$ by putting $p_1 = f_2$, $q_1 = -f_1$, $p_2 = f_4$, and $q_2 = -f_1 - f_3$. Here the parameters are $\alpha = -\alpha_1$, $\beta = -\alpha_2$, $\gamma = -\alpha_3$, $\delta = -\alpha_4$, $\epsilon = -\alpha_5$ [65]. Adler [5] and Veselov-Shabat [74] also studied these equations independently prior to Noumi-Yamada [55]. In their terminology, equations (2.1) is the Darboux chain with period 5. Note that the A_5 and A_4-type Noumi-Yamada systems also appear in the degeneration diagram of D_6-Sasano equation. Kawakami [30] further obtained $H_{\text{KSs}}^{\frac{3}{3}+2}$, $H_{\text{KSs}}^{\frac{4}{3}+2}$, $H_{\text{KSs}}^{\frac{5}{4}+2}$, $H_{\text{KSs}}^{\frac{3}{2}+\frac{5}{4}}$ through degenerations.

[5]To the author's knowledge, the classification of ramified equations is still an open problem.

[6]The following transformation to the $H_{\text{II}}^{\text{Mat}}$ in [32] yields $\tilde{H}_{\text{II}}^{\text{Mat}}$: $p_1 \to 2p_1, q_1 \to \frac{q_1}{2}, p_2 \to -q_2, q_2 \to p_2, \kappa_1 \to -\theta_0 - \kappa_2$.

[7]It is hoped to prove a theorem stating "If the types of singular fibers in the spectral curve fibrations are different, the equations are not equivalent."

$H_{\mathrm{FS}}^{A_5}$	Drinfeld-Sokolov hierarchy UC hierarchy	Fuji-Suzuki [21] Tsuda [71, 72]
$H_{\mathrm{NY}}^{A_n}$	dressing chain representation theory UC hierarchy	Adler [5], Veselov-Shabat [74] Noumi-Yamada [55] Tsuda [71, 72]
$H_{\mathrm{Ss}}^{D_n}$	holomorphy conditions Drinfeld-Sokolov hierarchy	Sasano [64] Fuji-Suzuki [20]
$H_{\mathrm{VI}}^{\mathrm{Mat}}$, $H_{\mathrm{V}}^{\mathrm{Mat}}$, $H_{\mathrm{IV}}^{\mathrm{Mat}}$	quiver variety	Boalch [8, 9]

Table 3: Other derivations of some of the Painevé-type systems.

The last series is the degeneration of the sixth matrix Painlevé equation. They are called the "matrix Painlevé equations", since their Hamiltonians have beautiful expressions using matrix analogs of canonical valuables. For instance, the $H_{\mathrm{VI}}^{\mathrm{Mat}}$ can be expressed as follows:

$$t(t-1)H_{\mathrm{VI}}^{\mathrm{Mat}}\begin{pmatrix}\alpha,\beta,\gamma & ;t; & q_1,p_1\\ \delta,\omega & & q_2,p_2\end{pmatrix} = \mathrm{tr}\Big[Q(Q-1)(Q-t)P^2+\{(\delta-(\alpha-\omega)K)Q(Q-1)$$
$$+\gamma(Q-1)(Q-t)-(2\alpha+\beta+\gamma+\delta)Q(Q-t)\}P+\alpha(\alpha+\beta)Q\Big].$$

Here the matrices P and Q satisfy the relation $[P,Q]=(\alpha-\omega)K \quad (K=\mathrm{diag}(1,-1))$, and canonical variables can be written as

$$p_1=\mathrm{tr}P,\quad q_1=\frac{1}{2}\mathrm{tr}Q,\quad p_2=\frac{P_{12}}{Q_{12}},\quad q_2=-Q_{12}Q_{21}.$$

We have denoted the $(1,2)$-component of matrix P as P_{12}, and so on. This explicit Hamiltonian $H_{\mathrm{VI}}^{\mathrm{Mat}}$ is derived by Sakai [62] as the isomonodromic deformation equation of $22,22,22,211$-type Fuchsian equation. Kawakami [31] further generalized the matrix Painlevé systems to the higher dimensions through isomonodromic deformation of $mm,mm,m;m-1;1$-type equation and its degenerations. Kawakami [30] also considered the degenerations to ramified cases and obtained $H_{\mathrm{III}}^{\mathrm{Mat}}(D_7)$, $H_{\mathrm{III}}^{\mathrm{Mat}}(D_8)$ and $H_{\mathrm{I}}^{\mathrm{Mat}}$. The matrix Painlevé equations are also found by Boalch from a different approach [8, 9].

3. Autonomous limit of Painlevé type equations

In the previous section, we saw that there are 40 types of 4-dimensional Painlevé-type equations. In this section, we consider the autonomous limit of these 40 types equations. We consider isospectral limit of isomonodromic deformation equations. Using the Lax pair, we obtain two functionally independent conserved quantities for each system. Therefore, the autonomous limit of 4-dimensional Painlevé-type equations are integrable in Liouville's sense.

3.1. **Intgrable system and Lax pair representation.** Let us recall the definition of integrability in Liouville's sense. A Hamiltonian system is a triple (M,ω,H), where (M,ω) is a symplectic manifold and H is a Hamiltonian function on M; $\iota_{X_H}\omega=dH$. It is easily verified that $\{f,H\}=0$ if and only if f is constant along integrable curve of the Hamiltonian vector field X_H. Such a function f is called a conserved quantity or first integral of motion. A Hamiltonian system is (completely) integrable in Liouville's sense if it possesses $n:=\frac{1}{2}\dim M$ independent integrals of motion, $f_1=H, f_2,\ldots,f_n$, which are pairwise in involution with respect to the Poisson bracket; $\{f_i,f_j\}=0$ for all i,j. This definition of integrability is motivated by Liouville's theorem. Let (M,ω,H) be a real integrable system of dimension $2n$ with integral of motion $f_1,\ldots,f_n$, and let $c\in\mathbb{R}^n$ be a regular value of $f=(f_1,\ldots,f_n)$. Liouville's theorem states that any compact component of the level set $f^{-1}(c)$ is a torus. The complex Liouville theorem is also known [4].

Many integrable systems are known to have Lax pair expressions:

$$\frac{dA(x)}{dt} + [A(x), B(x)] = 0, \tag{3.1}$$

where $A(x)$ and $B(x)$ are m by m matrices and x is a spectral parameter. From this differential equation, $\mathrm{tr}\left(A(x)^k\right)$ are conserved quantities of the system:

$$\frac{d}{dt}\,\mathrm{tr}\left(A(x)^k\right) = \mathrm{tr}\left(k\,[A(x), B(x)]\,A(x)^{k-1}\right) = 0.$$

Therefore, the eigenvalues of $A(x)$ are all conserved quantities since the coefficients of the characteristic polynomial are expressible in terms of these traces. In fact, the Lax pair is equivalent to the following isospectral problem:

$$\begin{cases} A(x) = Y A_0(x) Y^{-1}, \\ \dfrac{dY}{dt} = B(x)Y, \end{cases}$$

where $A_0(x)$ is a matrix satisfying $\frac{dA_0(x)}{dt} = 0$. The curve defined by the characteristic polynomial is called the spectral curve: $\det(yI_m - A(x)) = 0$. Spectral curves are the main tools in this paper.

3.2. **Isomonodromic deformation to isospectral deformation.** Recall from the previous section that the isomonodromic problems have the following forms:

$$\begin{cases} \dfrac{\partial Y}{\partial x} = A(x,t)Y, \\ \dfrac{\partial Y}{\partial t} = B(x,t)Y, \end{cases}$$

and the deformation equation is expressed as

$$\frac{\partial A(x,t)}{\partial t} - \frac{\partial B(x,t)}{\partial x} + [A(x,t), B(x,t)] = 0. \tag{3.2}$$

We find the similarities in isospectral and isomonodromic problems; the only difference is the existence of the term $\frac{\partial B}{\partial x}$ in isomonodromic deformation equation. In fact, we can consider isospectral problems as the special limit of isomonodromic problem with a parameter δ. We restate the isomonodromic problem as follows[8]:

$$\begin{cases} \delta\dfrac{\partial Y}{dx} = A(x,\tilde{t})Y, \\ \dfrac{\partial Y}{\partial t} = B(x,\tilde{t})Y, \end{cases}$$

where $\tilde{t}$ is a variable which satisfies $\frac{d\tilde{t}}{dt} = \delta$. The integrability condition $\frac{\partial^2 Y}{\partial x \partial t} = \frac{\partial^2 Y}{\partial t \partial x}$ is equivalent to the following:

$$\frac{\partial A(x,\tilde{t})}{\partial t} - \delta\frac{\partial B(x,\tilde{t})}{\partial x} + [A(x,\tilde{t}), B(x,\tilde{t})] = 0. \tag{3.3}$$

The case when $\delta = 1$ is the usual one.[9] When $\delta = 0$, the term $\delta\frac{\partial B}{\partial x}$ drops off from the deformation equation and we have a Lax pair in a narrow sense.[10] The deformation equation 3.2 with δ is solved by a Hamiltonian $H(\delta)$. When $\delta = 1$, the Hamiltonian $H(1)$ coincides with the original Hamiltonian

[8] In some literature such as Levin-Olshanetsky [46], it is customary to use κ instead of δ.

[9] The Montreal group studied finite dimensional integrable systems by embedding them into rational coadjoint orbits of loop algebras [3, 2]. Harnad further generalized their theory as applicable to the isomonodromic systems. Such nonautonomous isomonodromic systems are obtained by identifying the time flows of the integrable system with parameters determining the moment map.

[10] In another word, we mean a Lax pair in the sense of integrable systems. The isomonodromic problems are often called as Lax pairs, but they do not give conserved quantities.

of the isomonodromic problem. Therefore, $H(\delta)$ is a slight modification of the Hamiltonian. When $\delta = 0$, $H(0)$ is a conserved quantity of the system.

Taking the isospectral limit of 8 types of 2-dimensional Painlevé equations, we can state the following classically-known result.

Proposition 1. *As the autonomous limits of 2-dimensional Painlevé equations, we obtain 8 types of integrable systems with a conserved quantity for each system*[11].

Proof. We take the second Painlevé equation as an example to demonstrate a proof. Proofs of the other equations are similar.

$$\begin{cases} \delta \dfrac{\partial Y}{\partial x} = A(x,\tilde{t})Y, \quad A(x,\tilde{t}) = \left(A_\infty^{(-3)}(\tilde{t})x^2 + A_\infty^{(-2)}(\tilde{t})x + A_\infty^{(-1)}(\tilde{t})\right), \\ \dfrac{\partial Y}{\partial t} = B(x,\tilde{t})Y \quad B(x,\tilde{t}) = \left(A_\infty^{(-3)}(\tilde{t})x + B_1(\tilde{t})\right), \end{cases}$$

where

$$\hat{A}_\infty^{(-3)}(\tilde{t}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{A}_\infty^{(-2)}(\tilde{t}) = \begin{pmatrix} 0 & 1 \\ -p+q^2+\tilde{t} & 0 \end{pmatrix},$$

$$\hat{A}_\infty^{(-1)}(\tilde{t}) = \begin{pmatrix} -p+q^2+\tilde{t} & q \\ (p-q^2-\tilde{t})q - \kappa_2 & p-q^2 \end{pmatrix}, \quad \hat{B}_1(\tilde{t}) = \begin{pmatrix} q & 1 \\ p-q^2-\tilde{t} & 0 \end{pmatrix},$$

$$A_\infty^{(-i)} = U^{-1}\hat{A}_\infty^{(-i)}U \quad \text{for } i = 1,2,3, \quad B_1 = U^{-1}\hat{B}_1 U, \quad U = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

The deformation equation (3.3) is equivalent to the following differential equations.

$$\frac{dq}{dt} = 2p - q^2 - \tilde{t}, \qquad \frac{dp}{dt} = 2pq + \delta - \kappa_1, \qquad \frac{du}{dt} = 0.$$

The first two equations are equivalent to the Hamiltonian system

$$\frac{dq}{dt} = \frac{\partial H_{\mathrm{II}}(\delta)}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H_{\mathrm{II}}(\delta)}{\partial q},$$

with the Hamiltonian $H_{\mathrm{II}}(\delta) := p^2 - (q^2+\tilde{t})p + (\kappa_1 - \delta)q$. When $\delta = 1$, it is the usual Hamiltonian of H_{II}. Taking the limit $\delta \to 0$, we obtain an autonomous system with a Hamiltonian $H_{\mathrm{II}}(0) = p^2 - (q^2+\tilde{t})p + \kappa_1 q$. Since it is an autonomous system, the Hamiltonian is a conserved quantity. The dimension of the phase space is two, and we have the half the number of conserved quantity. Therefore, the autonomous second Painlevé equation is integrable in Liouville's sense. The Lax pair[12] and the spectral curve of the autonomous second Painlevé equation are

$$\frac{dA(x)}{dt} + [A(x), B(x)] = 0, \quad \det(yI - A(x)) = y^2 - (x^2+\tilde{t})y - \kappa_1 x - H_{\mathrm{II}}(0) = 0. \tag{3.4}$$

□

Remark 7. For the 2-dimensional cases, parameters of the Painlevé equations can be thought as roots of affine root systems, and δ corresponds to the nulroot [61, 63].

[11]These conserved quantities are the autonomous Hamiltonians. They are rational in the phase variables q, p.

[12]We rewrite $A(x) = A(x,\tilde{t})$ and $B(x) = B(x,\tilde{t})$.

3.3. **The autonomous limit of the 4-dimensional Painlevé-type equations.** We can also consider such autonomous limit for higher dimensional Painlevé-type equations. From the coefficients of the spectral curves, we obtain conserved quantities.

Theorem 5. *As the autonomous limits of 4-dimensional Painlevé-type equations, we obtain 40 types of integrable systems with two conserved quantities for each system*[13].

Proof. One of the simplest 4-dimensional Painlevé-type equation is the first matrix Painlevé equation [30]. The linear equation is given by

$$\frac{dA(x)}{dt}+[A(x),B(x)]=0,\quad A(x)=\left(A_0x^2+A_1x+A_2\right),\quad B(x)=A_0x+B_1.$$

where

$$A_0=\begin{pmatrix}O_2 & I_2\\ O_2 & O_2\end{pmatrix},\quad A_1=\begin{pmatrix}O_2 & Q\\ I_2 & O_2\end{pmatrix},\quad A_2=\begin{pmatrix}-P & Q^2+\tilde{t}I_2\\ -Q & P\end{pmatrix},\quad B_1=\begin{pmatrix}O_2 & 2Q\\ I_2 & O_2\end{pmatrix},$$
$$Q=\begin{pmatrix}q_1 & u\\ -q_2/u & q_1\end{pmatrix},\quad P=\begin{pmatrix}p_1/2 & -p_2u\\ (p_2q_2-\kappa_2)/u & p_1/2\end{pmatrix},\quad \Theta=\begin{pmatrix}\kappa_2 & 0\\ 0 & \kappa_3\end{pmatrix}.$$

The spectral curve is defined by the characteristic polynomial of the matrix $A(x)$;

$$\det\left(yI_4-A(x)\right)=y^4-\left(2\,x^3+2\tilde{t}x+h\right)y^2+x^6+2\tilde{t}x^4+hx^3+\tilde{t}^2x^2+\left(\tilde{t}h-\kappa_2^2\right)x+g.$$

The explicit forms of h and g are

$$h:=H_{\mathrm{I}}^{\mathrm{Mat}}=\mathrm{tr}\left(P^2-Q^3-\tilde{t}Q\right)=-2p_2\left(p_2q_2-\kappa_2\right)+\frac{p_1^2}{2}-2q_1\tilde{t}-2q_1\left(q_1^2-q_2\right)+4q_1q_2,$$
$$g:=G_{\mathrm{I}}^{\mathrm{Mat}}=q_2\left(p_1p_2+3q_1^2-q_2+\tilde{t}\right)^2-\kappa_2p_1\left(p_1p_2+3q_1^2-q_2+\tilde{t}\right)-2\kappa_2^2q_1.$$

Since h and g are coefficient of the spectral curve, they are conserved quantities of the autonomous system. We can also check that h and g are conserved quantities by direct computation:

$$\dot{h}=X_hh=\{h,h\}=0,\quad \dot{g}=X_hg=\{g,h\}=0,$$

where X_h is the Hamiltonian vector field associated to the Hamiltonian h. The Poisson bracket in the above equations is defined by

$$\{F,G\}:=\sum_{i=1}^{2}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i}\right).$$

Since

$$\mathrm{rank}\begin{pmatrix}\frac{\partial h}{\partial q_1} & \frac{\partial h}{\partial p_1} & \frac{\partial h}{\partial q_2} & \frac{\partial h}{\partial p_2}\\ \frac{\partial g}{\partial q_1} & \frac{\partial g}{\partial p_1} & \frac{\partial g}{\partial q_2} & \frac{\partial g}{\partial p_2}\end{pmatrix}=2$$

for the general value of (q_1,p_1,q_2,p_2), we have two functionally independent conserved quantities of the system. Hence the number of independent conserved quantities is the half the dimension of the phase space. Thus the autonomous Hamiltonian system of Hamiltonian $H_{\mathrm{I}}^{\mathrm{Mat}}$ is integrable in Liouville's sense.

From similar direct computations, we obtain the desired results for all the rest of 4-dimensional Painlevé-type equations. We have listed Hamiltonians and the other conserved quantities for the ramified types in Appendix A.1. These spectral curves and conserved quantities can be calculated from the data in the papers [62, 32, 30]. The only troublesome part is to find appropriate modifications of the Hamiltonians in the presence of δ. Otherwise, the computation is straightforward. □

[13]These conserved quantities are rational in the phase variables q_1,p_1,q_2,p_2.

Remark 8. It is an interesting problem to study the invariant surfaces defined by $H^{-1}(c_1) \cap G^{-1}(c_2) \subset \mathbb{C}^4$ for $c_1, c_2 \in \mathbb{C}$, where H and G are functionally independent conserved quantities of the system. As in other integrable systems [4], these Liouville tori would be completed into Abelian surfaces by adjoining suitable divisors. However, the actual computation seems tough in our cases. Therefore, the spectral curves are the main tool in this paper.

4. Spectral curve fibrations of integrable systems and their singular fibers

This section is the main part of this paper. We construct "spectral curve fibrations" associated with integrable Lax equations and study their singular fibers. Applying this construction to our 40 types of autonomous 4-dimensional Painlevé-type equations, we obtain the list of Namikawa-Ueno types of fibers at $H_i = \infty$ for $i = 1, 2$.

4.1. **Genus 1 fibration and Tate's algorithm.** Before discussing the genus 2 cases, corresponding to autonomous limit of 4-dimensional Painlevé type equations, we discuss the genus 1 cases, corresponding to autonomous 2-dimensional Painlevé equations.

Let us recall some of the basics we need: the Kodaira-Néron model and Tate's algorithm. We can construct the Kodaira-Néron model of elliptic curve E over the function field $\mathbb{C}(C)$ of a curve C by compactification and minimal desingularization of the affine surface defined by the defining equation of E in the Weierstrass form. The possible types of singular fibers of elliptic surfaces were classified by Kodaira [40]. Tate's algorithm provides a way to determine the Kodaira type of singular fibers without actually resolving the singularities [70].

In our case, we take $\mathbb{P}^1(\mathbb{C})$ as the curve C. Let E be an elliptic curve over $\mathbb{C}(C)$ in the Weierstrass form:

$$y^2 = x^3 + a(h)x + b(h), \qquad (a(h), b(h) \in \mathbb{C}[h]). \tag{4.1}$$

We may assume that for $a(h)$ and $b(h)$, the polynomials $l(h)$ such that $l(h)^4 | a(h)$, $l(h)^6 | b(h)$ are only constants. Otherwise, we may divide both sides of the equations by $l(h)^6$ and replace x, y by $x/l(h)^2, y/l(h)^3$ if necessary. Let X_1 be the affine surface defined by the equation (4.1):

$$X_1 = \left\{ (x, y, h) \in \mathbb{A}^2_{(x,y)} \times \mathbb{A}^1_h \;\middle|\; y^2 = x^3 + a(h)x + b(h) \right\}.$$

In general, a fiber of a morphism defined by the projection $\varphi_1 \colon X_1 \to \mathbb{A}^1_h$ is an affine part of an elliptic curve in general. Let n be the minimal positive integer satisfying $\deg a(h) \le 4n$ and $\deg b(h) \le 6n$. Dividing equation (4.1) by h^{6n} and replacing $\bar{x} = x/h^{2n}$, $\bar{y} = y/h^{3n}$, $\bar{h} = 1/h$, we obtain the "∞-model":

$$\bar{y}^2 = \bar{x}^3 + \bar{a}(\bar{h})\bar{x} + \bar{b}(\bar{h}), \tag{4.2}$$

where $\bar{a}(\bar{h}) = a(h)/h^{4n}$, $\bar{b}(\bar{h}) = b(h)/h^{6n}$ are polynomials in h. Let X_2 be the affine surface defined by equation (4.2). Let $\overline{X_1}$ and $\overline{X_2}$ be the projectivized surfaces in $\mathbb{P}^2 \times \mathbb{A}^1$; $\overline{X_2} \subset \mathbb{P}^2 \times \mathbb{A}^1_{\bar{h}}$, $\overline{\varphi_2} \colon \overline{X_2} \to \mathbb{A}^1_{\bar{h}}$. We glue $\overline{X_1}$ and $\overline{X_2}$ by identifying (x, y, h) and $(\bar{x}, \bar{y}, \bar{h})$ by the equations above. Let us denote the surface obtained this way by W. We call W the Weierstrass model. The surface W has a morphism to $\phi \colon W \to \mathbb{P}^1 = \mathbb{A}^1_h \cup \mathbb{A}^1_{\bar{h}}$. After the minimal resolution of the singular points of W, we obtain a nonsingular surface X. This nonsingular projective surface X together with the fibration $\phi \colon X \to \mathbb{P}^1$ is called the Kodaira-Néron model of the elliptic curve E over $\mathbb{C}(h)$.

The types of the singular fibers of the elliptic surface X can be computed from the equation (4.1) using Tate's algorithm. From the Weierstrass form equation (4.1), we can associate two quantities: $\Delta := 4a^3 + 27b^2$, $j := 4a^3/\Delta$. Here, Δ is the discriminant of the cubic $x^3 + a(h)x + b(h)$ and j is the j-invariant. The Kodaira types of the singular fibers are determined as in Table 4 by the order of Δ and j which we denote $\mathrm{ord}_v(\Delta)$, $\mathrm{ord}_v(j)$.

fiber type	Dynkin type	$\mathrm{ord}_v(\Delta)$	$\mathrm{ord}_v(j)$	fiber type	Dynkin type	$\mathrm{ord}_v(\Delta)$	$\mathrm{ord}_v(j)$
I_0	-	0	≥ 0	I_0^*	$D_4^{(1)}$	6	≥ 0
I_m	$A_{m-1}^{(1)}$	m	$-m$	I_m^*	$D_{4+m}^{(1)}$	$6+m$	$-m$
II	-	2	≥ 0	IV^*	$E_6^{(1)}$	8	≥ 0
III	$A_1^{(1)}$	3	≥ 0	III^*	$E_7^{(1)}$	9	≥ 0
IV	$A_2^{(1)}$	4	≥ 0	II^*	$E_8^{(1)}$	10	≥ 0

Table 4: Tate's algorithm and Kodaira types

4.1.1. *Spectral curve fibration.* We introduce the main subject of this article, spectral curve fibrations associated with integrable Lax equations. Let us consider a $2n$-dimensional integrable system with n functionally independent conserved quantities $H_1, \ldots, H_n$ and a Lax pair. Spectral curve fibrations of this integrable system are surfaces $\phi_i \colon X_i \to \mathbb{P}^1$ whose general fiber at $h_i \in \mathbb{P}^1$ is a spectral curve of the system with $H_i = h_i$ $(i = 1, \ldots, n)$.

Theorem 6. *Each autonomous 2-dimensional Painlevé equation defines elliptic surface as the spectral curve fibration. The types of singular fiber at $H = \infty$ are listed as follows.*

Hamiltonian	H_{VI}	H_{V}	$H_{\mathrm{III(D_6)}}$	$H_{\mathrm{III(D_7)}}$	$H_{\mathrm{III(D_8)}}$	H_{IV}	H_{II}	H_{I}
Kodaira type	I_0^*	I_1^*	I_2^*	I_3^*	I_4^*	IV^*	III^*	II^*
Dynkin type	$D_4^{(1)}$	$D_5^{(1)}$	$D_6^{(1)}$	$D_7^{(1)}$	$D_8^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$

Table 5: The singular fiber at $H = \infty$ of spectral curve fibrations of autonomous 2-dimensional Painlevé equations.

Proof. First, let us consider the first Painlevé equation

$$\frac{d^2q}{dt^2} = 6q^2 + t.$$

The first Painlevé equation has a Lax form

$$\frac{\partial A}{\partial t} - \delta\frac{\partial B}{\partial x} + [A, B] = 0, \quad A(x) = \begin{pmatrix} -p & x^2 + qx + q^2 + \tilde{t} \\ x - q & p \end{pmatrix}, \quad B(x) = \begin{pmatrix} 0 & x + 2q \\ 1 & 0 \end{pmatrix}.$$

The spectral curve associated with its autonomous equation is defined by $\det(yI_2 - A(x)) = 0$. This is equivalent to

$$y^2 = x^3 + \tilde{t}x + H_{\mathrm{I}}, \qquad H_{\mathrm{I}} = p^2 - q^3 + \tilde{t}q.$$

Let us write H_{I} as h for short. We view it as the defining equation of an elliptic curve over $\mathbb{A}^1_h$. We compactify this affine surface to obtain the Weierstrass model. After the minimal desingularization, we obtain the desired elliptic surface.

The actual computation we need to carry out are as follows. Upon replacing $\bar{h} = h^{-1}$, $\bar{x} = h^{-2}x$, $\bar{y} = h^{-3}y$ we obtain the "∞-model":

$$\bar{y}^2 = \bar{x}^3 + \tilde{t}\bar{h}^4\bar{x} + \bar{h}^5. \tag{4.3}$$

Thus we get fibration $\varphi \colon W \to \mathbb{P}^1 = \mathbb{A}^1_h \cup \mathbb{A}^1_{\bar{h}}$. The Kodaira-Néron model $\phi \colon X \to \mathbb{P}_1 = \mathbb{A}^1_h \cup \mathbb{A}^1_{\bar{h}}$ is obtained from W by the minimal desingularization. The Kodaira-type of singular fiber at $h = \infty$ can be computed using the equation (4.3). The discriminant of the cubic $\bar{x}^3+\tilde{t}\bar{h}^4\bar{x}+\bar{h}^5$ and the j-invariant are

$$\Delta = 4\left(\tilde{t}\bar{h}^4\right)^3+27\left(\bar{h}^5\right)^2=\bar{h}^{10}\left(27+4\tilde{t}^3\bar{h}^2\right),\quad j=\frac{4}{\Delta}\left(\tilde{t}\bar{h}^4\right)^3=\frac{4\tilde{t}^3\bar{h}^{12}}{\bar{h}^{10}(27+2\tilde{t}^3\bar{h}^2)}=\bar{h}^2\frac{4\tilde{t}^3}{27+4\tilde{t}^3\bar{h}^2}.$$

Thus order of zero of Δ and j at $\bar{h}=0$ is $\mathrm{ord}_\infty(\Delta)=10, \mathrm{ord}_\infty(j)=2$. Using Tate's algorithm, we find that the surface $X\to\mathbb{P}^1$ has the singular fiber of type II^*. In Dynkin's notation, this fiber is of type $\mathrm{E}_8^{(1)}$. Let us express the other two zeros of the discriminant Δ by h_+ and h_-. Since h_+ and h_- are both simple zeros of Δ, the Kodaira type of the fibers at h_+ and h_- are I_0, from Tate's algorithm.

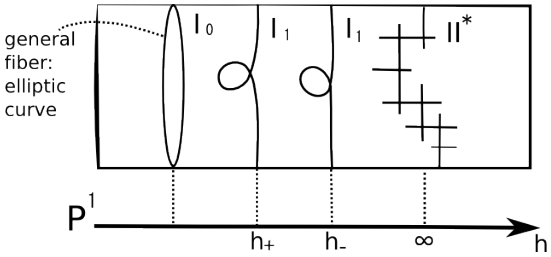
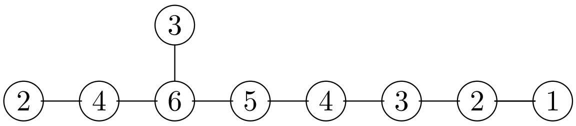


FIGURE 1. The elliptic surface constructed as the spectral curve fibration of the autonomous P_{I}.



The dual graph of the singular fiber of the Kodaira type II^* (Dynkin type $E_8^{(1)}$). The numbers in circles denote the multiplicities of components in the reducible fibers.

In the same manner, we construct the spectral curve fibrations from integrable Lax equations. As in the case of the first Painlevé equation, spectral curves of the other autonomous 2-dimensional Painlevé equations define elliptic surfaces. Other spectral curves too are curves of genus one for the general values of the Hamiltonians. It is well known that curves with genus one can always be transformed into the Weierstrass normal form. With the aid of computer programs, we can transform the spectral curves of autonomous 2-dimensional Painlevé equations into the Weierstrass normal form[14]. Once the spectral curves are in the Weierstrass form, we are able to construct the Weierstrass model. After the minimal desingularizations, we obtain the elliptic surfaces. We apply Tate's algorithm to find the Kodaira types of the singular fibers of these elliptic surfaces. □

4.1.2. *Liouville torus fibration.* We introduce another type of fibrations, the Liouville torus fibrations. It is to view the defining equations of the autonomous Hamiltonians as elliptic fibrations. We think of the time variable $\tilde{t}$ as a constant.

Theorem 7. *Each Hamiltonian of autonomous 2-dimensional Painlevé equation defines elliptic surface. Each function field k of $\mathbb{P}^1$ and singular fiber at $h=\infty$ is as same as in the case of spectral curve fibration.*

Proof. Let us first consider the easiest case: the first Painlevé equation. We write $h=H_{\mathrm{I}}$ for short. The Hamiltonian of the first Painlevé equation is $h=p^2-(q^3+\tilde{t}q)$. We view it as an elliptic curve over $\mathbb{C}(h)$:

$$\left\{(q,p,h)\in\mathbb{A}^2_{(q,p)}\times\mathbb{A}^1_h \;\middle|\; p^2=q^3+\tilde{t}q+h\right\}\to\mathbb{A}^1_h.$$

As in the case of the spectral curve fibration, we can construct the Kodaira-Néron model of the elliptic surface from the equation. Replacing $\bar{q}=q/h^2,\ \bar{p}=p/h^3,\ \bar{h}=1/h$, we obtain the ∞-model:

$$\bar{p}^2=\bar{q}^3+\tilde{t}\bar{h}^4\bar{q}+\bar{h}^5.$$

[14] Magma, Sage and Maple serve this purpose. Magma even calculates Kodaira types from the equations.

After a compactification and the minimal desingularization, we obtain a regular elliptic surface whose general fiber at h is the elliptic curve defined by $p^2 = q^3 + \tilde{t}q + h$. The discriminant and the j-invariant are:

$$\Delta = 4\left(\tilde{t}\bar{h}^4\right)^3 + 27\left(\bar{h}^5\right)^2 = \bar{h}^{10}(27 + 4\tilde{t}^3\bar{h}^2), \quad j = \frac{4(\tilde{t}\bar{h}^4)^3}{4\bar{h}^{10}(27 + 4\tilde{t}^3\bar{h}^2)} = \bar{h}^2\frac{\tilde{t}^3}{27 + 4\tilde{t}^3\bar{h}^2}.$$

Thus the order of zero of Δ and j at $\bar{h} = 0$ are $\mathrm{ord}_\infty(\Delta) = 10, \mathrm{ord}_\infty(j) = 2$. It follows from Tate's algorithm that the singular fiber at $h = \infty$ of the autonomous P_{I}-Hamiltonian fibration is of type II^*, or $E_8^{(1)}$ in the Dynkin's notation.

We use computer programs to transform the other Hamiltonians into the Weierstrass normal form. The rest of the proof is similar to the proof of H_{I}. □

The agreement of singular fibers at $h = \infty$ of spectral curve fibrations and Hamiltonian fibrations is not a coincidence. Autonomous Liouville fibrations are genus 1 cases of the Liouville tori fibrations. Liouville tori are related to the Jacobian varieties of spectral curves, and taking Jacobians are isomorphism in genus 1 cases by Abel's theorem. It might be natural to study the Liouville tori fibration, but we have to deal with families of 2-dimensional Abelian varieties to study the autonomous 4-dimensional Painlevé-type equations. So the actual computation might be harder compared to cases of 2-dimensional Painlevé equations. On the other hand, we only need to deal with genus 2 curves to study spectral curve fibrations of 4-dimensional autonomous Painlevé type equations. Thus, spectral curve fibration is the main object of the rest of this paper.

Using blowing-up process, Okamoto resolved the singularities of 2-dimensional Painlevé differential equations and constructed the "spaces of initial conditions" [58]. While he deals with singularities of the systems of differential equations, we deal with spectral curves or Hamiltonians themselves for autonomous cases.

A space of initial conditions can be characterized by a pair (X, D) of a rational surface X and the anti-canonical divisor D of X. Each irreducible component of D is a rational curve and, in the case of the Painlevé equations, is called as a vertical leaf [61]. The intersection diagram of D is given by that of the certain root lattice listed above.

Remark 9. The spaces of initial conditions are also considered for several cases of 4-dimensional Garnier systems and Noumi-Yamada systems [37, 38, 68, 69].

Restricting our attention to autonomous cases as in this paper, the geometrical studies are much simpler. The autonomous 2-dimensional Painlevé equations constructed from the spaces of initial conditions were studied by Sakai [63].

4.2. **Genus 2 fibration and Liu's algorithm.** We apply a similar method as in the previous subsection to our 40 kinds of autonomous 4-dimensional Painlevé-type equations. While the genus of spectral curves of autonomous 2-dimensional Painlevé equations are one, those of 4-dimensional Painlevé-type equations are two. We construct spectral curve fibrations from explicit forms of spectral curves in the Weierstrass form. Gluing two affine models by $\bar{x} = x/h,\ \bar{y} = y/h^3,\ \bar{h} = 1/h$, we obtain the Weierstrass models. As we have used Tate's algorithm to determine the fibers at $h = \infty$ of minimal models for genus 1 cases, we use Liu's algorithm for the genus 2 cases.

Let E be an hyperelliptic curve of genus 2 over the functional field $\mathbb{C}(h)$ with one variable h in the Weierstrass form:

$$y^2 = a_0(h)x^6 + a_1(h)x^5 + a_2(h)x^4 + a_3(h)x^3 + a_4(h)x^2 + a_5(h)x + a_6(h), \quad (a_i(h) \in \mathbb{C}[h]). \tag{4.4}$$

We may assume that for $a_i(h)$, the polynomials $l \in k[h]$ such that $l^i | a_i(h)$ are only constants. (We can divide both sides of equations by l^6 and replace x, y by $x/l, y/l^3$ if necessary.) Let X_1 be the affine surface defined by equation (4.4):

$$X_1 = \left\{ (x, y, h) \in \mathbb{A}^2_{(x,y)} \times \mathbb{A}^1_h \;\middle|\; y^2 = a_0x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6 \right\}$$

A general fiber of the morphism defined by the projection $\varphi_1 \colon X_1 \to \mathbb{A}^1_h$ is an affine part of genus 2 hyperelliptic curve. From projectivized equation

$$y^2z^4 = a_0x^6 + a_1x^5z + a_2x^4z^2 + a_3x^3z^3 + a_4x^2z^4 + a_5xz^5 + a_6z^6$$

of (4.4), we also obtain

$$\overline{X_1} \subset \mathbb{P}^2 \times \mathbb{A}^1_h, \qquad \overline{\varphi_1} \colon \overline{X_1} \to \mathbb{A}^1_h.$$

Let n be the minimal positive integer satisfying $\deg \bar{a}_i(h) \leq i$. Dividing the both sides of equation (4.4) by h^{6n} and replacing

$$\bar{x} = \frac{x}{h^n},\ \bar{y} = \frac{y}{h^{3n}},\ \bar{h} = \frac{1}{h}, \tag{4.5}$$

we obtain the "∞-model":

$$\bar{y}^2 = \bar{a}_0(\bar{h})\bar{x}^6 + \bar{a}_1(\bar{h})\bar{x}^5 + \bar{a}_2(\bar{h})\bar{x}^4 + \bar{a}_3(\bar{h})\bar{x}^3 + \bar{a}_4(\bar{h})\bar{x}^2 + \bar{a}_5(\bar{h})\bar{x} + \bar{a}_6(\bar{h}), \tag{4.6}$$

where $\bar{a}(\bar{h}) = a(h)/h^{4n}$, $\bar{b}(\bar{h}) = b(h)/h^{6n}$ are polynomials in the variable h. Let X_2 be the affine surface defined by the equation (4.6):

$$X_2 \subset \mathbb{A}^2_{(\bar{x},\bar{y})} \times \mathbb{A}^1_{\bar{h}} = \mathbb{A}^3_{(\bar{x},\bar{y},\bar{h})}, \qquad \varphi_2 \colon X_2 \to \mathbb{A}^1_{\bar{h}}.$$

Similarly, we have the closure $\overline{X_2} \subset \mathbb{P}^2 \times \mathbb{A}^1_{\bar{h}}$, $\overline{\varphi_2} \colon \overline{X_2} \to \mathbb{A}^1_{\bar{h}}$. We glue $\overline{X_1}$ and $\overline{X_2}$ by identifying (x, y, h) and $(\bar{x}, \bar{y}, \bar{h})$ by equations (4.5). Let us denote the surface obtained this way by W. The surface W has morphism to $\mathbb{A}^1_h \cup \mathbb{A}^1_{\bar{h}}$ $(h \cdot \bar{h} = 1)$, $\phi \colon W \to \mathbb{P}^1$. After the minimal resolution of singular points of W, we obtain a nonsingular surface X.

4.2.1. *Liu's algorithm.* We briefly review related works. The numerical classification of the fibers in pencils of genus 2 curves are given by Ogg [56] and Iitaka [27]. Namikawa and Ueno [53] have completed the geometrical classification of such fibers (and a few missing types in [56] and [27] are also added). There are 120 types in Namikawa-Ueno's classification, while there are only 10 types in Kodaira's classification of the fibers in pencils of genus 1 curves.[15] Liu gave an algorithm similar to Tate's algorithm for genus 2 case [48, 49]. Using his result, we can determine the Namikawa-Ueno type of singular fibers from explicit equations of pencils of genus 2 hyperelliptic curves in the Weierstrass form[16].

	genus of spectral curve	types of singular fibers in pencils	algorithm to determine types of fibers
2-dim. Painlevé	1	Kodaira	Tate's algorithm
4-dim. Painlevé	2	Namikawa-Ueno	Liu's algorithm

[15] Kodaira type I_n $(n \geq 1)$ and I^*_n $(n \geq 1)$ are counted as 1 type, respectively.

[16] This algorithm is implemented by Liu and Cohen using PARI/GP library for the arithmetic situation: underlying curve is $\operatorname{Spec}\mathbb{Z}$ and consider the reduction at a prime p. This command can also be used from Sage. Although our situation is different from the above arithmetic situation and such useful implementation is not applicable, Liu's paper is written in very general setting, so we can apply the algorithm written in his paper.

We actually use the case when $R = \mathbb{C}[\![\bar{h}]\!],\ K = \mathbb{C}(\!(\bar{h})\!),\ \mathfrak{m} = \bar{h}\,\mathbb{C}[\![\bar{h}]\!],\ k = \mathbb{C}$ in this paper. Let C be a smooth projective curve, geometrically connected and of genus 2 over K. The curve C is defined by the equation

$$y^2 = a_0x^6 + a_1x^5 + \cdots + a_6 \in K[x]$$

with $a_0 \neq 0$ or $a_1 \neq 0$. The curve C admits a minimal model $\mathscr{X}$ over R. That is proper flat and regular R-scheme, where generic fiber $\mathscr{X}_\eta$ is isomorphic to C, such that for all R-scheme $\mathscr{X}'$ satisfying these properties, the isomorphism $\mathscr{X}'_\eta \to \mathscr{X}_\eta$ extends to a morphism $\mathscr{X}' \to \mathscr{X}$, where $\mathscr{X}_\eta$ and $\mathscr{X}'_\eta$ are the generic fibers.

Since the notion of stable model is effectively used in Liu's paper [49] for the identification of the minimal model, we review basic notions.

Definition 1. Let C be an algebraic curve over an algebraically closed field k. We say that C is *semi-stable* if it is reduced, and if its singular points are ordinary double points. We say that C is *stable* if, moreover, the following conditions are verified:

(1) C is connected and projective, of arithmetic genus $p_a(C) \geq 2$.
(2) Let Γ be an irreducible component of C that is isomorphic to $\mathbb{P}^1(k)$. Then it intersects the other irreducible components at at least three points.

It is easily shown that there exist seven possible stable curves of genus 2 over k [50].

Definition 2. Let C be a curve over K, R' a dominant discrete valuation ring for R. A stable model of C over R' is a stable curve $\mathscr{C}$ over R' with generic fiber isomorphic to $C \times_K \operatorname{Fr} R'$, where $\operatorname{Fr} R'$ is the field of fractions of R'.

Let F be a finite extension of K and R_F the integral closure of R in F. We call C *has stable reduction over* F if it admit a stable model $\mathscr{C}$ over the localization R' of R_F at the maximal ideal. We denote $\mathscr{C}_{\bar{s}}$ the special geometric fiber of $\mathscr{C}$ over R'.

According to Viehweg [75], the special fiber $\mathscr{X}_s$ is completely determined by the three data:

(1) $\mathscr{C}_s$ (Liu [48, Thm 1]),
(2) "degree" (which is the thickness of singular points of $\mathscr{C}$) (Liu [48, Prop 2]),
(3) $\operatorname{Gal}(L/K)$ over $\mathscr{C}_s$ where L is the minimal extension of K defined above (Liu [49, Thm 1,2,3]).

The point is that all three information can be computed from an explicit equation in the Weierstrass form thanks to Liu's theorems indicated in parentheses. [17] Let us cite one of them. In the following theorem, J_{2i} $(i = 1, \ldots, 5)$ are Igusa invariants [26] and I_{2i} $(i = 1, \ldots, 6)$ are degree $2i$ homogeneous elements of $\mathbb{Z}[J_2, J_4, J_6, J_8, J_{10}]$ as in Liu [48].

Theorem 8 (Liu [48])**.** *Let C be a smooth projective curve over K, geometrically connected and of genus 2, and let J_{2i}, $1 \leq i \leq 5$, be the invariants of C associated to the equation $y^2 = f(x)$. Then we have*

(I) (Igusa) *$\mathscr{C}_{\bar{s}}$ is smooth if and only if: $J_{2i}^5 J_{10}^{-i} \in R$ for all $i \leq 5$,*
(II) *$\mathscr{C}_{\bar{s}}$ is an irreducible elliptic curve with an unique double point if and only if: $J_{2i}^6 I_{12}^{-i} \in R$ for all i and $J_{10}^6 I_{12}^{-5} \in \mathfrak{m}$,*
(III) *$\mathscr{C}_{\bar{s}}$ is an irreducible projective line with two double points if and only if: $J_{2i}^6 I_{12}^{-i} \in R$ for all $i \leq 5$, $J_{10}^2 I_4^{-5} \in \mathfrak{m}$, $I_{12} I_4^{-3} \in \mathfrak{m}$, and $J_4 I_4^{-1}$ or $J_6^2 I_4^{-3}$ is invertible in R,*
(IV) *$\mathscr{C}_{\bar{s}}$ constitutes of two projective crossing transversally at three points if and only if: $J_{2i}^2 I_4^{-i} \in \mathfrak{m}$ for all $2 \leq i \leq 5$,*

[17] We here note a typo in Liu [49, Prop 4.3.1(d)](page 144). The term $a_0^{-20} A_2^6 J_2^{-5}$ should be $a_0^{-20} A_5^6 J_2^{-5}$ as in the code"genus2reduction" written by Liu and Cohen.

Type	Order of monodromy	stable type (Liu's notation)
elliptic[1]	finite	I
elliptic[2]	finite	V
parabolic[3]	infinite	II, VI
parabolic[4]	infinite	III,VII
parabolic[5]	infinite	IV

Table 6: Namikawa-Ueno's elliptic and parabolic types and Liu's stable model types

(V_*) *$\mathscr{C}_{\bar{s}}$ is the union of two irreducible components intersecting transversally at one point if and only if:*

$$I_4 I_2^{-2},\ J_{10} I_2^{-5},\ I_{12} I_2^{-6} \in \mathfrak{m}. \tag{4.7}$$

In addition,

(V) *$\mathscr{C}_{\bar{s}}$ is the union of two elliptic curves intersecting transversally at one point if and only if: in addition to (4.7), $I_4^3 J_{10}^{-1} I_2^{-1} \in R$, $I_{12} J_{10}^{-1} I_2^{-1} \in R$.*

(VI) *$\mathscr{C}_{\bar{s}}$ is the union of an elliptic curve and a projective line which has an ordinary double point if and only if: in addition to (4.7), $I_4^3 I_{12}^{-1} \in R$, $J_{10} I_2 I_{12}^{-1} \in \mathfrak{m}$,*

(VII) *$\mathscr{C}_{\bar{s}}$ is the union of two singular curves if and only if: in addition to (4.7), $I_{12} I_4^{-3} \in \mathfrak{m}$, and $J_{10} I_2 I_4^{-3} \in \mathfrak{m}$.*

4.3. **List of singular fibers of the spectral curve fibrations.** Let us state the main theorem of this paper. We write $H_1 = h$, $H_2 = g$ for simplicity.[18]

Theorem 9. *Each autonomous 4-dimensional Painlevé-type equation defines two smooth surfaces with relatively minimal fibrations $\phi_i \colon X_i \to \mathbb{P}^1$ $(i = 1, 2)$, whose general fibers are the spectral curves of genus 2. The Namikawa-Ueno type of their singular fibers at $H_1 = \infty$ and $H_2 = \infty$ are as in Table 7 and Table 8 respectively.*

Remark 10. Let us explain the notations used in Table 7. The Hamiltonians are the Hamiltonians of 4-dimensional Painlevé-type equations. The explicit forms of non-autonomous counterparts can be found in [62, 33, 30, 36, 35]. The spectral types indicate the type of the corresponding linear equations. Such notations are explained in Subsection 2.1. "N-U type" means the Namikawa-Ueno type in the paper [53]. When the fiber contains components expressible by the Kodaira-type, we also write its Dynkin's name in the column noted "Dynkin". The column named "stable" tells us the type of the stable model. This stable model is determined by the Liu's theorem cited above (Theorem 8). The "Φ" indicates the group of connected components of the Néron model of the Jacobian $J(C)$. The symbol (n) means the cyclic group with n elements.[19] H_n is isomorphic to $(2) \times (2)$ if n is even and to (4) otherwise. We also write Ogg's type written in "On pencils of curves of genus two" [56]. Ogg uses the notation "Kod" to express Kodaira-type and do not distinguish them, while Namikawa and Ueno does. Ogg's type might be helpful to see the rough classification. For example, all 8 types of matrix Painlevé equations have the same Ogg's type 14. The column "NU" means 5 types in Namikawa-Ueno. Elliptic types are those with finite degrees of monodromy, while parabolic types have infinite degrees. Elliptic[1] are those with stable model "I" in Liu's notation (Theorem 8). We abbreviate as "ell[1]". We summarize such correspondences in the following table. The column named "page" indicate the page number of Namikawa-Ueno's paper where some data of the corresponding type can be found.

18 $H_1 = h = H_{\mathrm{Gar}, \tilde{t_1}}$, $H_2 = g = H_{\mathrm{Gar}, \tilde{t_2}}$ for Garnier equations and $H_1 = h = H$, $H_2 = g = G$ for the other ordinary differential systems.

19 When $n = 0$, (n) is the trivial group.

Hamiltonian	the Hamiltonian of the Painlevé-type equation
spectral type	a spectral type of corresponding linear equation
N-U type	the type of fiber in the minimal model following the notation in Namikawa-Ueno
Dynkin	indicate Dynkin type when the fiber contains Kodaira-type component
stable	the type of stable model of the fiber
Φ	the group of connected components of the Néron model of the Jacobian $J(C)$.
Ogg	the type of fiber in the minimal model following the notation in Ogg
NU	5 types of monodromy (elliptic[1],[2],Parabolic[3],[4],[5]) as in Namikawa-Ueno
page	the page number of the fiber in Namikawa-Ueno's paper

Proof of Theorem 8. Let us take $\mathrm{Gar}\frac{9}{2}$, the most degenerated Garnier system, to demonstrate our proof. The characteristic polynomial of a Lax equation of $\mathrm{Gar}\frac{9}{2}$ is expressed as

$$y^2 = 9x^5 + 9\tilde{t_1}x^3 + 3\tilde{t_2}x^2 - hx + g,$$

where $h = H^{9/2}_{\mathrm{Gar},\tilde{t_1}}$, $g = H^{9/2}_{\mathrm{Gar},\tilde{t_2}}$. Note that it is already in the Weierstrass form. Upon a replacement $\bar{x} = x/h,\ \bar{y} = y/h^3,\ \bar{h} = 1/h$, we have an equation of the "∞"-model:

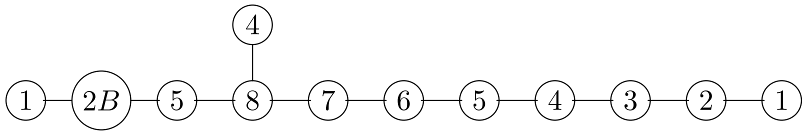
$$\bar{y}^2 = 9\bar{h}^6x^5 + 9\tilde{t_1}\bar{h}^6x^3 + 3\bar{h}^6\tilde{t_2}x^2 - \bar{h}^5x + g\bar{h}^6.$$

We obtain the Weierstrass model by gluing these affine surfaces. After the minimal desingularization, we obtain the desired surface. In order to see the type of the singular fiber at $h = \infty$ of the surface, we apply Liu's algorithm. The Igusa invariants of the quintic can be calculated as follows.

$$J_2 = \frac{3^2}{2^2}\bar{h}^{11}\left(-20 + 27\tilde{t_1}^2\bar{h}\right), \quad J_4 = \frac{3^5}{2^7}\bar{h}^{22}\left(80 + 24\tilde{t_1}^2\bar{h} + \left(-400g\tilde{t_2} + 48\tilde{t_1}\tilde{t_2}^2 + 81\tilde{t_1}^4\right)\bar{h}^2\right)$$
$$J_6 = \frac{3^6}{2^{10}}\bar{h}^{33}\left(320 + O(\bar{h})\right), \quad J_8 = -\frac{3^8}{2^{16}}\bar{h}^{44}(83200 + O(\bar{h})), \quad J_{10} = \frac{3^{10}}{2^{12}}\bar{h}^{55}\left(-2^8 + O(\bar{h})\right).$$

Since $5 \cdot \mathrm{ord}_\infty J_{2i} - i \cdot \mathrm{ord}_\infty J_{10} = 0$ for $i \leq 5$, the stable model has smooth fiber (type(I)) at $h = \infty$ from Theorem 8. With further computation, we find that the Namikawa-Ueno type of the singular fiber at $h = \infty$ is VII* from Liu's algorithm. This type is type 22 in Ogg's notation [56].

VII*: $H^{\frac{9}{2}}_{\mathrm{Gar},\tilde{t_1}}$

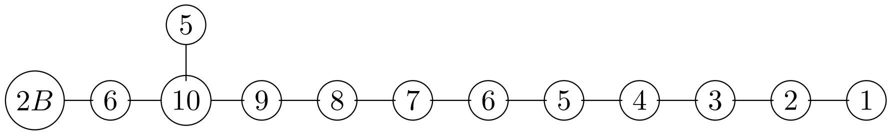


Similarly, we can associate another surface to this system: the spectral curve fibration with respect to another conserved quantity "g". After replacing $\bar{x} = x/g,\ \bar{y} = y/g^3,\ \bar{g} = 1/g$ in the above equation, we obtain an affine equation around $g = \infty$;

$$\bar{y}^2 = 9\bar{g}x^5 + 9\tilde{t_1}\bar{g}^3x^3 + 3\bar{g}^4\tilde{t_2}x^2 - \bar{g}^5hx + \bar{g}^6.$$

From Liu's algorithm, the fiber at $g = \infty$ is type VIII $-$ 4 in Namikawa-Ueno's notation.

VIII $-$ 4: $H^{\frac{9}{2}}_{\mathrm{Gar},\tilde{t_2}}$

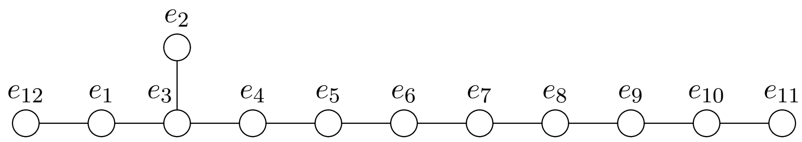


The numbers in circles denote the multiplicities of components in the reducible fibers. All curves are (-2)-curves except the one expressed as "B", which is a (-3)-curve.

With the help of computer, we can transform the spectral curves of the other autonomous 4-dimensional Painlevé equations into the Weierstrass normal form.[20]

□

Remark 11. The dual graph of the singular fiber at $H^{9/2}_{\mathrm{Gar},\tilde{t_2}}$ of the spectral curve fibration contains, as a subgraph, the extended Dynkin diagram of the unimodular integral lattice[21] D^+_{12}. The Dynkin diagram of D^+_{12} is as follows.[22]



It follows that the Mordell-Weil group of $f\colon X \to \mathbb{P}^1$ is trivial ([39, Thm 3.1]). It can be thought as a generalization of the fact (Thm 5) that the spectral curve fibration defined by the autonomous $\mathrm{P_I}$ (the most degenerated 2-dimensional Painlevé equation) has the singular fiber of type $E_8 = D^+_8$.

As E_8 lattice is important as the "frame lattice" for the Mordell-Weil lattice[23] of rational elliptic surfaces [57], D^+_{4g+4} lattice is important for fibrations of genus g on rational surfaces [60]. The classification of rational elliptic surface by Oguiso-Shioda [57] owes to the classification of root lattices contained in E_8, which is equivalent to the classification of regular semisimple subalgebras of the exceptional Lie algebra of type E_8 by Dynkin [17]. Sakai [63] studied autonomous 2-dimensional Painlevé equations corresponding to rational elliptic surfaces in Oguiso-Shioda's list. We can speculate that there are (autonomous) 4-dimensional Painlevé-type equations corresponding to sublattices of D^+_{12} lattice. Thus the classification of sublattices of D^+_{12} might be meaningful for the classification of (autonomous) 4-dimensional Painlevé-type equations.[24]

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[20]We used Maple's command "Weierstrassform" implemented by van Hoeij.

[21]The notation is as in Conway-Sloane [13]. In some literatures, this lattice is expressed as Γ_{12}. Recall that the lattice D_n is defined by $D_n = \{(x_1, \ldots, x_n) \in \left(\mathbb{Z} + \frac{1}{2}\right)^n : x_1 + \cdots + x_n \equiv 0 \bmod 2\}$. Then, D^+_n is defined to be $D^+_n = D_n \cup \left(\left(\frac{1}{2}, \ldots, \frac{1}{2}\right) + D_n\right)$, which is lattice when n is even.

[22]$e_i^2 = 2$ for $i \neq 12$ and $e_{12}^2 = 3$.

[23]The Mordell-Weil group is the group of $\mathbb{C}(t)$-rational points of the Jacobian of the generic fiber of $\phi\colon X \to \mathbb{P}^1$. Under suitable conditions, the Mordell-Weil group is a finitely generated abelian group. Shioda [66, 67] viewed Mordell-Weil groups as Euclidean lattices and studied them using their connection between intersection theory.

[24]Saitō-Sakakibara [60] gave the bound of Mordell-Weil rank for fibration on curves of genus g as $4g + 4$, and studied the case when the Mordell-Weil rank is maximal. Nguen [54] studied Mordell-Weil lattices of hyperelliptic type with higher ranks rank $J(K) \geq 4g + 1$.

Hamiltonian	spectral type	N-U type	Dynkin	stable	Φ	Ogg	NU	page
$H_{\mathrm{Gar},\tilde{t_1}}^{1+1+1+1+1}$	11,11,11,11,11	I^*_{1-0-0}	-	II	$(2)^2 \times H_1$	33	par[3]	p.171
$H_{\mathrm{Gar},\tilde{t_1}}^{2+1+1+1}$	(1)(1),11,11,11	I^*_{1-1-0}	-	III	$H_1 \times H_1$	33	par[4]	p.180
$H_{\mathrm{Gar},\tilde{t_1}}^{3/2+1+1+1}$	(2)(1),(1)(1)(1)	I^*_{1-2-0}	-	III	$H_1 \times H_2$	33	par[4]	p.180
$H_{\mathrm{Gar},\tilde{t_1}}^{3/2+2+1}$	$(1)_2,(1)(1),11$	I^*_{1-2-0}	-	III	$H_1 \times H_2$	33	par[4]	p.180
$H_{\mathrm{Gar},\tilde{t_1}}^{3/2+3/2+1}$	$(1)_2,(1)_2,11$	I^*_{2-2-0}	-	III	$H_2 \times H_2$	33	par[4]	p.180
$H_{\mathrm{Gar},\tilde{t_1}}^{3+1+1}$	((1))((1)),11,11	$\mathrm{IV}^* - \mathrm{I}_1^* - (-1)$	$E_6 - D_5 - (-1)$	VI	$(3) \times H_1$	29a	par[3]	p.175
$H_{\mathrm{Gar},\tilde{t_1}}^{2+2+1}$	(1)(1),(1)(1),11	$\mathrm{IV}^* - \mathrm{I}_1^* - (-1)$	$E_6 - D_5 - (-1)$	VI	$(3) \times H_1$	29a	par[3]	p.175
$H_{\mathrm{Gar},\tilde{t_1}}^{5/2+1+1}$	(((1)(1)))(((1)))	$\mathrm{III}^* - \mathrm{I}_1^* - (-1)$	$E_7 - D_5 - (-1)$	VI	$(2) \times H_1$	29a	par[3]	p.177
$H_{\mathrm{Gar},\tilde{t_1}}^{4+1}$	(((1)))(((1))),11	$\mathrm{III}^* - \mathrm{II}_1^*$	$E_7 - \mathrm{II}_1^*$	II	(8)	23	par[3]	p.178
$H_{\mathrm{Gar},\tilde{t_1}}^{3+2}$	((1))((1)),(1)(1)	$\mathrm{III}^* - \mathrm{II}_1^*$	$E_7 - \mathrm{II}_1^*$	II	(8)	23	par[3]	p.178
$H_{\mathrm{Gar},\tilde{t_1}}^{5/2+2}$	$(((1)))_2,(1)(1)$	$\mathrm{II}^* - \mathrm{II}_1^*$	$E_8 - \mathrm{II}_1^*$	II	H_2	25	par[3]	p.176
$H_{\mathrm{Gar},\tilde{t_1}}^{7/2+1}$	$(((((1)))))_2,11$	$\mathrm{II}^* - \mathrm{II}_1^*$	$E_8 - \mathrm{II}_1^*$	II	H_2	25	par[3]	p.176
$H_{\mathrm{Gar},\tilde{t_1}}^{3/2+3}$	$(1)_2,((1))((1))$	$\mathrm{IV}^* - \mathrm{III}^* - (-1)$	$E_6 - E_7 - (-1)$	V	(6)	29	ell[2]	p.168
$H_{\mathrm{Gar},\tilde{t_1}}^{5/2+3/2}$	$(((1)))_2,(1)_2$	$\mathrm{III}^* - \mathrm{III}^* - (-1)$	$E_7 - E_7 - (-1)$	V	$(2)^2$	29	ell[2]	p.169
$H_{\mathrm{Gar},\tilde{t_1}}^{5}$	((((1))))((((1))))	IX-3	-	I	(5)	21	ell[1]	p.157
$H_{\mathrm{Gar},\tilde{t_1}}^{9/2}$	$(((((((1)))))))_2$	VII^*	-	I	(2)	22	ell[1]	p.156
$H_{\mathrm{FS}}^{A_5}$	21,21,111,111	II_{4-0}	-	II	(16)	41	par[3]	p.171
$H_{\mathrm{FS}}^{A_4}$	(11)(1),21,111	II_{4-1}	-	IV	(17)	41	par[5]	p.183
$H_{FS}^{A_3}$	$(1)_2,21,111$	II_{4-2}	-	IV	(18)	41	par[5]	p.183
$H_{\mathrm{Suz}}^{\frac{3}{2}+2}$	$(11)(1),(1)_2 1$	II_{4-3}	-	IV	(19)	41	par[5]	p.183
$H_{\mathrm{KFS}}^{\frac{3}{2}+\frac{3}{2}}$	$(1)_3,(11)(1)$	II_{4-4}	-	IV	(20)	41	par[5]	p.183
$H_{\mathrm{KFS}}^{\frac{4}{3}+\frac{3}{2}}$	$(1)_3,(1)_2 1$	II_{4-5}	-	IV	(21)	41	par[5]	p.183
$H_{\mathrm{KFS}}^{\frac{4}{3}+\frac{4}{3}}$	$(1)_3,(1)_3$	II_{3-6}	-	IV	(18)	41	par[5]	p.183
$H_{\mathrm{NY}}^{A_5}$	(2)(1),111,111	II_{5-1}	-	III	$(2) \times (2)$	41a	par[4]	p.182
$H_{\mathrm{NY}}^{A_4}$	((11))((1)),111	$\mathrm{IV}^* - \mathrm{II}_4$	$E_6 - \mathrm{II}_4$	II	(13)	41b	par[3]	p.175
$H_{\mathrm{Ss}}^{D_6}$	31,22,22,1111	$\mathrm{I}_3 - \mathrm{I}_0^* - 0$	$A_2 - D_4 - 0$	VI	$(3) \times H_0$	2	par[3]	p.171
$H_{\mathrm{Ss}}^{D_5}$	(111)(1),22,22	$\mathrm{I}_3 - \mathrm{I}_1^* - 0$	$A_2 - D_5 - 0$	VII	$(3) \times H_1$	2	par[4]	p.180
$H_{\mathrm{Ss}}^{D_4}$	(2)(2),(111)(1)	$\mathrm{I}_3 - \mathrm{I}_2^* - 0$	$A_2 - D_6 - 0$	VII	$(3) \times H_2$	2	par[4]	p.180
$H_{\mathrm{KSs}}^{\frac{3}{2}+2}$	$(1)_2 11,(2)(2)$	$\mathrm{I}_3 - \mathrm{I}_3^* - 0$	$A_2 - D_7 - 0$	VII	$(3) \times H_3$	2	par[4]	p.180
$H_{\mathrm{KSs}}^{\frac{4}{3}+2}$	$(1)_3 1,(2)(2)$	$\mathrm{I}_3 - \mathrm{I}_4^* - 0$	$A_2 - D_8 - 0$	VII	$(3) \times H_4$	2	par[4]	p.180
$H_{\mathrm{KSs}}^{\frac{5}{4}+2}$	$(1)_4,(2)(2)$	$\mathrm{I}_3 - \mathrm{I}_5^* - 0$	$A_2 - D_9 - 0$	VII	$(3) \times H_5$	2	par[4]	p.180
$H_{\mathrm{KSs}}^{\frac{3}{2}+\frac{5}{4}}$	$(1)_4,(2)_2$	$\mathrm{I}_2 - \mathrm{I}_6^* - 0$	$A_1 - D_{10} - 0$	VII	$(2) \times H_6$	2	par[4]	p.180
$H_{\mathrm{VI}}^{\mathrm{Mat}}$	22,22,22,211	$\mathrm{I}_0 - \mathrm{I}_0^* - 1$	$\mathrm{I}_0 - D_4 - 1$	V	$(2)^2$	14	ell[2]	p.159
$H_{\mathrm{V}}^{\mathrm{Mat}}$	(2)(11),22,22	$\mathrm{I}_0 - \mathrm{I}_1^* - 1$	$\mathrm{I}_0 - D_5 - 1$	VI	H_1	14	par[3]	p.170
$H_{\mathrm{III(D_6)}}^{\mathrm{Mat}}$	(2)(2),(2)(11)	$\mathrm{I}_0 - \mathrm{I}_2^* - 1$	$\mathrm{I}_0 - D_6 - 1$	VI	H_2	14	par[3]	p.170
$H_{\mathrm{III(D_7)}}^{\mathrm{Mat}}$	$(2)(2),(11)_2$	$\mathrm{I}_0 - \mathrm{I}_3^* - 1$	$\mathrm{I}_0 - D_7 - 1$	VI	H_3	14	par[3]	p.170
$H_{\mathrm{III(D_8)}}^{\mathrm{Mat}}$	$(2)_2,(11)_2$	$\mathrm{I}_0 - \mathrm{I}_4^* - 1$	$\mathrm{I}_0 - D_8 - 1$	VI	H_4	14	par[3]	p.170
$H_{\mathrm{IV}}^{\mathrm{Mat}}$	((2))((11)),22	$\mathrm{I}_0 - \mathrm{IV}^* - 1$	$\mathrm{I}_0 - E_6 - 1$	V	(3)	14	ell[2]	p.160
$H_{\mathrm{II}}^{\mathrm{Mat}}$	(((2)))(((11)))	$\mathrm{I}_0 - \mathrm{III}^* - 1$	$\mathrm{I}_0 - E_7 - 1$	V	(2)	14	ell[2]	p.162
$H_{\mathrm{I}}^{\mathrm{Mat}}$	$(((((11)))))_2$	$\mathrm{I}_0 - \mathrm{II}^* - 1$	$\mathrm{I}_0 - E_8 - 1$	V	0	14	ell[2]	p.160

Table 7: The singular fibers at $H_1 = \infty$ of spectral curve fibrations of autonomous 4-dimensional Painlevé-type equations

Hamiltonian	spectral type	N-U type	Dynkin	stable	Φ	Ogg	NU	page
$H^{1+1+1+1+1}_{\mathrm{Gar},\tilde{t_2}}$	11,11,11,11,11	I^*_{1-0-0}	-	II	$(2)^2 \times H_1$	33	par[3]	p.171
$H^{2+1+1+1}_{\mathrm{Gar},\tilde{t_2}}$	(1)(1),11,11,11	I^*_{1-1-0}	-	III	$(1) \times (1)$	33	par[3]	p.180
$H^{3/2+1+1+1}_{\mathrm{Gar},\tilde{t_2}}$	(2)(1),(1)(1)(1)	I^*_{1-2-0}	-	III	$(1) \times (2)$	33	par[4]	p.180
$H^{2+2+1}_{\mathrm{Gar},\tilde{t_2}}$	(1)(1),(1)(1),11	I^*_{1-1-1}	-	IV	$H_3 \times H_1$	33	par[5]	p.183
$H^{3/2+2+1}_{\mathrm{Gar},\tilde{t_2}}$	$(1)_2,(1)(1),11$	$\mathrm{IV}^* - \mathrm{I}^*_2 - (-1)$	$E_6 - D_6 - (-1)$	VI	$(3) \times H_2$	29a	par[3]	p.175
$H^{3/2+3/2+1}_{\mathrm{Gar},\tilde{t_2}}$	$(1)_2,(1)_2,11$	$\mathrm{III}^* - \mathrm{I}^*_2 - (-1)$	$E_7 - D_6 - (-1)$	VI	$(2) \times H_2$	29a	par[3]	p.177
$H^{3+1+1}_{\mathrm{Gar},\tilde{t_2}}$	((1))((1)),11,11	$\mathrm{IV}^* - \mathrm{I}^*_1 - (-1)$	$E_6 - D_5 - (-1)$	VI	$(3) \times H_1$	29a	par[3]	p.175
$H^{5/2+1+1}_{\mathrm{Gar},\tilde{t_2}}$	(((1)(1)))(((1)))	$\mathrm{III}^* - \mathrm{I}^*_1 - (-1)$	$E_7 - D_5 - (-1)$	VI	$(2) \times H_1$	29a	par[3]	p.177
$H^{3+2}_{\mathrm{Gar},\tilde{t_2}}$	((1))((1)),(1)(1)	$\mathrm{IV}^* - \mathrm{IV}^* - (-1)$	$E_6 - E_6 - (-1)$	V	$(3)^2$	29	ell[2]	p.166
$H^{5/2+2}_{\mathrm{Gar},\tilde{t_2}}$	$(((1)))_2,(1)(1)$	$\mathrm{IV}^* - \mathrm{III}^* - (-1)$	$E_6 - E_7 - (-1)$	V	(6)	29	ell[2]	p.168
$H^{3/2+3}_{\mathrm{Gar},\tilde{t_2}}$	$(1)_2,((1))((1))$	$\mathrm{III}^* - \mathrm{II}^*_2$	$E_7 - \mathrm{II}^*_2$	II	(8)	23	par[3]	p.178
$H^{5/2+3/2}_{\mathrm{Gar},\tilde{t_2}}$	$(((1)))_2,(1)_2$	$\mathrm{II}^* - \mathrm{II}^*_2$	$E_8 - \mathrm{II}^*_2$	II	H_3	25	par[3]	p.176
$H^{4+1}_{\mathrm{Gar},\tilde{t_2}}$	(((1)))(((1))),11	IX − 3	-	I	(5)	21	ell[1]	p.157
$H^{7/2+1}_{\mathrm{Gar},\tilde{t_2}}$	$(((((1)))))_2,11$	VII^*	-	I	(2)	22	ell[1]	p.156
$H^{5}_{\mathrm{Gar},\tilde{t_2}}$	((((1))))((((1))))	V^*	-	I	(3)	19	ell[1]	p.156
$H^{9/2}_{\mathrm{Gar},\tilde{t_2}}$	$(((((((1)))))))_2$	VIII − 4	-	I	(0)	20	ell[1]	p.157
$G^{A_5}_{\mathrm{FS}}$	21,21,111,111	III	-	I	$(3)^2$	42	ell[1]	p.155
$G^{A_4}_{\mathrm{FS}}$	(11)(1),21,111	III_1 (par[5])	-	IV	(9)	43	par[5]	p.184
$G^{A_3}_{\mathrm{FS}}$	$(1)_21,21,111$	III_2 (par[5])	-	IV	(9)	43	par[5]	p.184
$G^{\frac{3}{2}+2}_{\mathrm{Suz}}$	$(11)(1),(1)_21$	III_3 (par[5])	-	IV	$(3)^2$	43	par[5]	p.184
$G^{\frac{3}{2}+\frac{3}{2}}_{\mathrm{KFS}}$	$(1)_3,(11)(1)$	III_4 (par[5])	-	IV	(9)	43	par[5]	p.184
$G^{\frac{4}{3}+\frac{3}{2}}_{\mathrm{KFS}}$	$(1)_3,(1)_21$	III_5 (par[5])	-	IV	(9)	43	par[5]	p.184
$G^{\frac{4}{3}+\frac{4}{3}}_{\mathrm{KFS}}$	$(1)_3,(1)_3$	III_6 (par[5])	-	IV	$(3)^2$	43	par[5]	p.184
$G^{A_5}_{\mathrm{NY}}$	(2)(1),111,111	$\mathrm{IV} - \mathrm{III}^* - (-1)$	$A_2 - E_7 - (-1)$	V	(6)	42	ell[2]	p.167
$G^{A_4}_{\mathrm{NY}}$	((11))((1)),111	IX − 4	-	I	(5)	44	ell[1]	p.158
$G^{D_6}_{\mathrm{Ss}}$	31,22,22,1111	VI	-	I	$(2)^2$	4	ell[1]	p.156
$G^{D_5}_{\mathrm{Ss}}$	(111)(1),22,22	III_1 (par[4])	-	III	H_1	5	par[4]	p.182
$G^{D_4}_{\mathrm{Ss}}$	(2)(2),(111)(1)	III_2 (par[4])	-	III	H_2	5	par[4]	p.182
$G^{\frac{3}{2}+2}_{\mathrm{KSs}}$	$(1)_211,(2)(2)$	III_3 (par[4])	-	III	H_3	5	par[4]	p.182
$G^{\frac{4}{3}+2}_{\mathrm{KSs}}$	$(1)_31,(2)(2)$	III_4 (par[4])	-	III	H_4	5	par[4]	p.182
$G^{\frac{5}{4}+2}_{\mathrm{KSs}}$	$(1)_4,(2)(2)$	III_5 (par[4])	-	III	H_5	5	par[4]	p.182
$G^{\frac{3}{2}+\frac{5}{4}}_{\mathrm{KSs}}$	$(1)_4,(2)_2$	III_6 (par[4])	-	III	H_6	5	par[4]	p.182
$G^{\mathrm{Mat}}_{\mathrm{VI}}$	22,22,22,211	$2\mathrm{I}^*_0 - 0$	$2D_4 - 0$	V	$(2)^2$	24a	ell[2]	p.159
$G^{\mathrm{Mat}}_{\mathrm{V}}$	(2)(11),22,22	$2\mathrm{I}^*_1 - 0$	$2D_5 - 0$	VII	H_1	24	par[4]	p.181
$G^{\mathrm{Mat}}_{\mathrm{III(D_6)}}$	(2)(2),(2)(11)	$2\mathrm{I}^*_2 - 0$	$2D_6 - 0$	VII	H_2	24	par[4]	p.181
$G^{\mathrm{Mat}}_{\mathrm{III(D_7)}}$	$(2)(2),(11)_2$	$2\mathrm{I}^*_3 - 0$	$2D_7 - 0$	VII	H_3	24	par[4]	p.181
$G^{\mathrm{Mat}}_{\mathrm{III(D_8)}}$	$(2)_2,(11)_2$	$2\mathrm{I}^*_4 - 0$	$2D_8 - 0$	VII	H_4	24	par[4]	p.181
$G^{\mathrm{Mat}}_{\mathrm{IV}}$	((2))((11)),22,	$2\mathrm{IV}^* - 0$	$2E_6 - 0$	V	(3)	26	ell[2]	p.165
$G^{\mathrm{Mat}}_{\mathrm{II}}$	(((2)))(((11)))	$2\mathrm{III}^* - 0$	$2E_7 - 0$	V	(2)	27	ell[2]	p.168
$G^{\mathrm{Mat}}_{\mathrm{I}}$	$(((((11)))))_2$	$2\mathrm{II}^* - 0$	$2E_8 - 0$	V	0	28	ell[2]	p.163

Table 8: The singular fibers at $H_2 = \infty$ of spectral curve fibrations of autonomous 4-dimensional Painlevé-type equations

Appendix

A.1. **Conserved quantities.** The autonomous 4-dimensional Painlevé-type equations have two functionally independent conserved quantities. In this subsection, we list these conserved quantities for the ramified equations.[25] One of the reason is that the other conserved quantities than the Hamiltonians have long expressions. Writing them for "less-degenerated" systems take huge spaces. But they are easily computable from data in the previous paper [32]. We only give conserved quantities for autonomous version of equations in Kawakami [30]. The Lax pairs for these ramified equations will be written in his forthcoming paper [30]. We list Hamiltonians H's with δ,[26] and the other conserved quantities G's.

There are 5 ramified cases from the degeneration of A_5 Fuji-Suzuki system.[27]

$$
\begin{aligned}
H^{A_3}_{\mathrm{FS}} &= H_{\mathrm{III}}(D_6)(-\theta^{\infty}_2, \delta+\theta^0_1+\theta_1; \tilde{t}; q_1, p_1) + H_{\mathrm{III}}(D_6)(\theta^0_2-\theta^0_1, \theta^0_2-\theta^0_1-\theta^1; \tilde{t}; q_2, p_2) - \frac{1}{\tilde{t}} p_1 p_2 (q_1 q_2 + \tilde{t}),\\
G^{A_3}_{\mathrm{FS}} &= \theta^0_2 \tilde{t} \left(H_{\mathrm{III}}(D_6)(-\theta^{\infty}_2, \theta_1+\theta^0_1; \tilde{t}; q_1, p_1) - p_1 p_2 \right)\\
&\quad + \left(q_1 q_2 - \tilde{t}\right)\left(\theta^{\infty}_2 p_2 - \left(\theta^0_1 - \theta^0_2\right) p_1 + p_1 p_2 \left(\theta_1 + \theta^0_1 - \theta^0_2 + (p_1-1) q_1 - (p_2-1) q_2\right)\right),\\
H^{\frac{3}{2}+2}_{\mathrm{KFS}} &= H_{\mathrm{III}}(D_7)(-\theta^0_1; \tilde{t}; q_1, p_1) + H_{\mathrm{III}}(D_7)(\theta^0_2-\theta^0_1; \tilde{t}; q_2, p_2) + \frac{1}{\tilde{t}}\left(p_2 q_1 (p_1(q_1+q_2) + \theta^{\infty}_2) - q_1\right),\\
G^{\frac{3}{2}+2}_{\mathrm{KFS}} &= (q_2 - q_1)\left(\left(\theta^0_2 - \theta^0_1\right) p_1 p_2 q_1 - \theta^{\infty}_2 p_2^2 q_1 + p_1 p_2^2 q_1 q_2 + p_1^2 p_2 q_1^2 + p_1 q_1 + p_1 p_2 \tilde{t}\right)\\
&\quad + \theta^0_2 \left(\tilde{t} H_{\mathrm{III}}(D_7)(-\theta^0_1; \tilde{t}; q_1, p_1) - q_1 + p_1 p_2 q_1^2\right) - \theta^{\infty}_2 \tilde{t} H_{\mathrm{III}}(D_7)(-\theta^0_1; \tilde{t}; q_1, p_2) + p_1 p_2 q_1^2 (\theta^0_2 - \theta^{\infty}_2),\\
H^{\frac{4}{3}+\frac{3}{2}}_{\mathrm{KFS}} &= H_{\mathrm{III}}(D_7)(\theta^{\infty}_1; \tilde{t}; q_1, p_1) + H_{\mathrm{III}}(D_7)(\delta - \theta^{\infty}_1; \tilde{t}; q_2, p_2) - \frac{1}{\tilde{t}} p_1 q_1 p_2 q_2 - \left(\frac{p_2}{q_1} + p_1 + p_2\right),\\
G^{\frac{4}{3}+\frac{3}{2}}_{\mathrm{KFS}} &= \frac{1}{q_1}\left(p_2\left(\theta^{\infty}_1 + p_1 q_1 - p_2 q_2\right)\left(\tilde{t} - p_1 q_1^2 q_2\right) + (p_1 - p_2) q_2 q_1^2 - \tilde{t}\right),\\
H^{\frac{3}{2}+\frac{3}{2}}_{\mathrm{KFS}} &= H_{\mathrm{III}}(D_7)(\theta^{\infty}_1 - \theta^{\infty}_2; \tilde{t}; q_1, p_1) + H_{\mathrm{III}}(D_7)(\delta - \theta^{\infty}_1; \tilde{t}; q_2, p_2) - \frac{1}{\tilde{t}} p_1 q_1 p_2 q_2 - (p_1 p_2 + p_1 + p_2),\\
G^{\frac{3}{2}+\frac{3}{2}}_{\mathrm{KFS}} &= \left(p_1 p_2 q_1 - p_2^2 q_2 + \theta^{\infty}_1 p_2 - 1\right)\left(-p_1\left(q_1 q_2 - \tilde{t}\right) + \theta^{\infty}_2 q_2\right) - p_2\left(q_1 q_2 - \tilde{t}\right),\\
H^{\frac{4}{3}+\frac{4}{3}}_{\mathrm{KFS}} &= \frac{1}{\tilde{t}}\left(p_1^2 q_1^2 + \delta q_1 p_1 - q_1 - \frac{\tilde{t}}{q_1}\right) + H_{\mathrm{III}}(D_8)(\tilde{t}; q_2, p_2) + \frac{1}{\tilde{t}}\left(-p_1 q_1 p_2 q_2 + \frac{q_1 q_2}{\tilde{t}} + q_1 + q_2\right),\\
G^{\frac{4}{3}+\frac{4}{3}}_{\mathrm{KFS}} &= \frac{1}{\tilde{t} q_1 q_2}\left((p_1 q_1 - p_2 q_2)\left(p_1 p_2 q_2^2 q_1^2 \tilde{t} + p_1 q_2 q_1^2 \tilde{t} + q_2^2 q_1^2\right) + \tilde{t}^2\left(p_1 q_1^2 - p_2 q_2^2 - q_2\right)\right).
\end{aligned}
$$

There are also 4 systems that are ramified derived from D_6-Sasano system.

$$
\begin{aligned}
H^{\frac{3}{2}+2}_{\mathrm{KSs}} &= H_{\mathrm{III}}(D_7)(\theta_0 + 2\theta^{\infty}_2; \tilde{t}; q_1, p_1) + H_{\mathrm{III}}(D_7)(-\theta_0; \tilde{t}; q_2, p_2) + \frac{1}{\tilde{t}}(2 p_2 q_1 (p_1 q_1 - \theta_0 - \theta^{\infty}_1) - q_1),\\
G^{\frac{3}{2}+2}_{\mathrm{KSs}} &= p_1^2 p_2^2 q_1^4 - p_1^2 q_1^3 - 2 p_1^2 p_2^2 q_2 q_1^3 - 2 p_1 p_2^2 \theta_0 q_1^3 + p_1^2 p_2 \theta_0 q_1^3 - 2 p_1 p_2^2 \theta^{\infty}_1 q_1^3 + \tilde{t} p_1 p_2^2 q_1^2 + p_1^2 p_2^2 q_2^2 q_1^2\\
&\quad + p_2^2 \theta_0^2 q_1^2 - p_1 p_2 \theta_0^2 q_1^2 + p_2^2 (\theta^{\infty}_1)^2 q_1^2 + p_1^2 q_2 q_1^2 + p_1 \theta_0 q_1^2 + p_1 p_2^2 q_2 \theta_0 q_1^2 - p_1^2 p_2 q_2 \theta_0 q_1^2 + p_1 \theta^{\infty}_1 q_1^2\\
&\quad + 2 p_1 p_2^2 q_2 \theta^{\infty}_1 q_1^2 + 2 p_2^2 \theta_0 \theta^{\infty}_1 q_1^2 - p_1 p_2 \theta_0 \theta^{\infty}_1 q_1^2 - p_1 \theta^{\infty}_2 q_1^2 - 2 p_1 p_2^2 q_2 \theta^{\infty}_2 q_1^2 + p_1 p_2 \theta_0 \theta^{\infty}_2 q_1^2\\
&\quad + p_2^2 q_2 \theta_0^2 q_1 - p_1 p_2 q_2 \theta_0^2 q_1 - \tilde{t} p_1 q_1 - 2 \tilde{t} p_1 p_2^2 q_2 q_1 - \tilde{t} p_2^2 \theta_0 q_1 + p_1 p_2^2 q_2^2 \theta_0 q_1 + \tilde{t} p_1 p_2 \theta_0 q_1\\
&\quad + p_1 q_2 \theta_0 q_1 - \tilde{t} p_2^2 \theta^{\infty}_1 q_1 + p_2^2 q_2 \theta_0 \theta^{\infty}_1 q_1 + 2 p_1 p_2^2 q_2^2 \theta^{\infty}_2 q_1 - p_2 \theta_0^2 \theta^{\infty}_2 q_1 + 2 p_1 q_2 \theta^{\infty}_2 q_1\\
&\quad + 2 p_2^2 q_2 \theta_0 \theta^{\infty}_2 q_1 - 2 p_1 p_2 q_2 \theta_0 \theta^{\infty}_2 q_1 + \theta_0 \theta^{\infty}_2 q_1 + 2 p_2^2 q_2 \theta^{\infty}_1 \theta^{\infty}_2 q_1 - p_2 \theta_0 \theta^{\infty}_1 \theta^{\infty}_2 q_1 + \theta^{\infty}_1 \theta^{\infty}_2 q_1
\end{aligned}
$$

[25] We do not write conserved quantities of the Garnier equations here, since the conserved quantities are just autonomous limit of two Hamiltonians. Such Hamiltonians are listed by Kimura [36] and Kawamuko [35].

[26] Hamiltonians for the case $\delta = 0$ is the conserved quantity.

[27] Although we obtain Garnier equations of ramified types from degenerations of A_5-type Fuji-Suzuki system, we are excluding Garnier systems.

$$
\begin{aligned}
&+\tilde{t}p_1p_2^2q_2^2+p_2^2q_2^2(\theta_2^\infty)^2+\tilde{t}p_2(\theta_2^\infty)^2+q_2(\theta_2^\infty)^2-p_2q_2\theta_0(\theta_2^\infty)^2+\tilde{t}p_1q_2+\tilde{t}p_2^2q_2\theta_0-\tilde{t}p_1p_2q_2\theta_0\\
&+\tilde{t}p_2^2q_2\theta_1^\infty-p_2q_2\theta_0^2\theta_2^\infty+p_2^2q_2^2\theta_0\theta_2^\infty+2\tilde{t}p_2\theta_0\theta_2^\infty+q_2\theta_0\theta_2^\infty+\tilde{t}p_2\theta_1^\infty\theta_2^\infty,
\end{aligned}
$$

$$
\begin{aligned}
H_{\mathrm{KSs}}^{\frac{4}{3}+2} &= H_{\mathrm{III}}(D_7)(\theta_0+2\theta_2^\infty;\tilde{t};q_1,p_1)+H_{\mathrm{III}}(D_7)(-\theta_0;\tilde{t};q_2,p_2)-\frac{1}{\tilde{t}}\left(2p_2q_1+q_1+\tilde{t}p_2\right),\\
G_{\mathrm{KSs}}^{\frac{4}{3}+2} &= -3\theta_0^2\theta_1^\infty p_2q_2-\theta_0(\theta_1^\infty)^2p_2q_2+3\theta_0\theta_1^\infty p_2^2q_2^2+\theta_0\theta_1^\infty p_2q_1+2\theta_0\theta_1^\infty p_1p_2q_1q_2-2\theta_0^3p_2q_2\\
&+2\theta_0^2p_2^2q_2^2+2\theta_0^2p_2q_1+3\theta_0^2p_1p_2q_1q_2-\theta_0p_1p_2q_1^2-3\theta_0p_1p_2^2q_1q_2^2-\theta_0p_1^2p_2q_1^2q_2\\
&-3\theta_0p_2^2q_1q_2-3\theta_0p_1q_1q_2+(\theta_1^\infty)^2p_2^2q_2^2-2\theta_1^\infty p_1p_2^2q_1q_2^2-2\theta_1^\infty p_2^2q_1q_2-2\theta_1^\infty p_1q_1q_2\\
&-\theta_0p_1p_2q_2\tilde{t}+p_1p_2^2q_2^2\tilde{t}+p_2^2q_2\tilde{t}+p_1q_2\tilde{t}+p_2^2q_1^2+p_1q_1^2+p_1^2p_2^2q_1^2q_2^2+p_1^2q_1^2q_2\\
&+2p_1p_2^2q_1^2q_2-2\theta_0p_2\tilde{t}-\theta_1^\infty p_2\tilde{t}+3\theta_0\theta_1^\infty q_2+2\theta_0^2q_2-2\theta_0q_1+(\theta_1^\infty)^2q_2-\theta_1^\infty q_1,\\
H_{\mathrm{KSs}}^{\frac{5}{4}+2} &= H_{\mathrm{III}}(D_8)(\tilde{t};q_1,p_1)+H_{\mathrm{III}}(D_7)(-\theta_0,\tilde{t};q_2,p_2)+2\frac{p_2}{q_1}-\frac{(\theta_0+1-\delta)p_1q_1}{\tilde{t}}+\frac{1}{q_1}-p_2,\\
G_{\mathrm{KSs}}^{\frac{5}{4}+2} &= \frac{1}{q_1^2}\Bigg(p_2^2\left(p_1q_2q_1^2\left(2\tilde{t}-\theta_0q_1q_2\right)+p_1^2q_2^2q_1^4-\theta_0q_2q_1\tilde{t}-q_2^2q_1^3+\tilde{t}^2\right)+q_1^3q_2(p_1^2q_1-1)\\
&-p_2q_1^2\left(\theta_0p_1^2q_2q_1^2+\theta_0p_1\left(\tilde{t}-\theta_0q_1q_2\right)-\theta_0q_2q_1+\tilde{t}\right)+q_1^2p_1\left(\tilde{t}-\theta_0q_1q_2\right)\Bigg),\\
H_{\mathrm{KSs}}^{\frac{3}{2}+\frac{5}{4}} &= \frac{1}{\tilde{t}}\left(p_1^2q_1^2+\delta q_1p_1-q_1-\tilde{t}/q_1\right)+H_{\mathrm{III}}(D_8)(\tilde{t};q_2,p_2)-2\frac{q_1q_2}{\tilde{t}^2}+\frac{q_1+q_2}{\tilde{t}},\\
G_{\mathrm{KSs}}^{\frac{3}{2}+\frac{5}{4}} &= \frac{q_1^2q_2^2}{\tilde{t}^2}-\frac{p_1^2q_1^2\tilde{t}}{q_2}+q_1+q_2+\frac{\tilde{t}^2}{q_1q_2}-\frac{1}{4q_1\tilde{t}}\left(2p_2q_2+1\right)\left(\tilde{t}\left(2p_2q_2+1\right)\left(\tilde{t}-p_1^2q_1^3\right)+4p_1q_2q_1^3\right).
\end{aligned}
$$

We have three ramified systems of matrix Painlevé equations. $G_{\mathrm{III}(D_7)}^{\mathrm{Mat}}$ and $G_{\mathrm{III}(D_8)}^{\mathrm{Mat}}$ are too long and $H_{\mathrm{I}}^{\mathrm{Mat}}$ is already written in the main part of this paper. So we skip writing conserved quantities of autonomous matrix Painlevé equations.

A.2. **Local data of linear equations.** In this subsection, we explain the notion of spectral type used in this paper. We follow the notion used in Oshima [59] for Fuchsian linear equations, Kawakami-Nakamura-Sakai [32] for unramified linear equations and Kawakami [30] for ramified equations.

For the classification of linear equations, we need to discern the types of linear equations. We review the local normal forms of linear equations, and introduce symbols to express such data. We study linear systems of first-order equations

$$\frac{dY}{dx}=A(x)Y. \tag{A.8}$$

We first consider the Fuchsian case where

$$A(x)=\sum_{i=1}^{n}\frac{A_i}{x-t_i}.$$

We assume that each matrix A_i is diagonalizable. The equation can be transformed into

$$\frac{d\hat{Y}(x)}{dx}=\frac{T^{(i)}}{x-t_i}\hat{Y}(x),\quad T^{(i)}\colon \text{diangonal}$$

by the transformation $Y=P(x)\hat{Y}$. We express the multiplicity of the eigenvalues by a non-increasing sequence of numbers.

Example 1. When $T^{(i)}=\mathrm{diag}(a,a,a,b,b,c)$, we write the multiplicity as 321.

Collecting such multiplicity data for all the singular points, the spectral type of the linear equation is defined as the $n+1$-tuples of partitions of m,

$$\underbrace{m^1_1 m^1_2 \ldots m^1_{l_1}}, \underbrace{m^2_1 \ldots m^2_{l_2}}, \ldots, \underbrace{m^n_1 \ldots m^n_{l_n}}, \underbrace{m^\infty_1 \ldots m^\infty_{l_\infty}}, \qquad \left(\sum_{j=1}^{l_i} m^i_j = m \text{ for } 1 \leq \forall i \leq n \text{ or } i = \infty \right),$$

where m is the size of matrices.

Example 2. When $T^{(1)}(x) = \mathrm{diag}(a,a,a,b)$, $T^{(2)}(x) = \mathrm{diag}(c,c,d,d)$, $T^{(3)}(x) = \mathrm{diag}(f,f,g,g)$, $T^{(4)}(x) = \mathrm{diag}(h,i,j,k)$ we write 31,22,22,1111.

Now we explain the way to obtain the formal canonical form around each non-Fuchsian singular point. We also explain that these canonical forms can be expressed by the refining sequences of partition. Let us assume that the coefficient matrix $A(x)$ of the equation has a singularity at the origin, and that $A(x)$ is expanded in Laurent series as follows:

$$\frac{dY}{dx} = \left(\frac{A^0}{x^{r+1}} + \frac{A^1}{x^r} + \cdots \right) Y. \tag{A.9}$$

Here, A^j $(j = 0, 1, \ldots)$ are $m \times m$ matrices. We assume that A^0 is diagonalizable. With an appropriate choice of the gauge matrix, we can assume that A^0 is diagonal and that its eigenvalues are $t^0_1, \ldots, t^0_m$. When $r = 0$, then the origin is a regular singular point. Let us assume that $r > 0$. If $t^0_i \neq t^0_j$ $(1 \leq i \leq l,\ l+1 \leq j \leq m)$, then a gauge transformation by a formal power series $Y = P(x)Z$ $(P(x) = I + P_1 x + P_2 x^2 + \cdots)$ leads to the following form:

$$\frac{dZ}{dx} = \left(\frac{B^0}{x^{r+1}} + \frac{B^1}{x^r} + \cdots \right) Z.$$

Here, we can transform B^i into the following form:

$$B^i = \begin{pmatrix} B^i_{11} & O \\ O & B^i_{22} \end{pmatrix}, \ B^i_{11} \in M_l(\mathbb{C}),\ B^i_{22} \in M_{m-l}(\mathbb{C}).$$

With successive application of this process, the equation (A.9) is formally decomposed to direct sum of equations whose leading terms have only one eigenvalues respectively. When the leading term of the block is diagonalizable, that is when it is scalar matrix, then this part can be canceled by a gauge transformation by a scalar function, so that the equation is reduced to a equation with smaller r.

Remark 12. When A^0 is not diagonalizable, in order to decompose the system into equations of smaller sizes, we need to take an appropriate covering $x = \xi^k$. In that case, the transformation matrix $P(x)$ is a Puiseux series in x. The equations with this property are called ramified. When we do not need to take coverings, that is when $k = 1$, we say that the equations are unramified. □

Unramified non-Fuchsian case. When the equation (A.9) is unramified, it can be transformed into the following form:

$$\frac{dY}{dx} = \left(\frac{T_0}{x^{r+1}} + \frac{T_1}{x^r} + \cdots + \frac{T_r}{x} + \cdots \right) Y.$$

We can assume that T_js are diagonal matrices and that $T_0 = A_0$. Furthermore, we can eliminate the regular terms by an appropriate diagonal matrix with formal power series components. Thus, the equation (A.9) can be transformed into the following form by gauge transformation of a formal power series:

$$\frac{dY}{dx} = \left(\frac{T_0}{x^{r+1}} + \frac{T_1}{x^r} + \cdots + \frac{T_r}{x} \right) Y. \tag{A.10}$$

If we write the diagonal components of T_i as t^i_j $(j = 1, \dots, m)$, then a canonical form around the origin can be described by the following data:

$$\overbrace{\begin{matrix} t^0_1 & t^1_1 & \dots & t^r_1 \\ \vdots & \vdots & & \vdots \\ t^0_m & t^1_m & \dots & t^r_m \end{matrix}}^{x=0}.$$

We write down such formal canonical forms for each singular point, and put them together. This kind of table is called the Riemann scheme of the linear equation. As we can see from the procedure to obtain the canonical form, the leftmost column splits into several groups as equivalence class of values. In the second column from the left, these groups splits further, and so on, we get a nested columns.

We describe such nesting structure by refining sequences of partitions of m and call it the spectral type of the singular point. We line up such spectral types of each singular point, and separate them by commas. We call it the spectral type of the equation. In such a case,

$$\exp\left(-\frac{T_0}{rx^r} + \dots - \frac{T_{r-1}}{x}\right) x^{T_r}$$

is the fundamental solution matrix for the formal canonical form (A.10). The degree r of the polynomial is called the Poincaré rank of the singular point. When the singular point is of regular type, the Poincaré rank is 0. When the equation is ramified, this part is a polynomial in $x^{-1/k}$, and the Poincaré rank is non-integer rational number. If we want to express only Poincaré ranks at each singularity, we attach Poincaré rank plus 1 to each singularity, and line them up, and separate them by + signs. When the equation is unramified, Poincaré rank plus 1 is as same as numbers of columns appeared in refining sequences of partitions at each singularities.

Example 3. For instance, let us consider the following normal form:

$$\frac{d\hat{Y}(x)}{dx} = \{\frac{1}{x^3}\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} + \frac{1}{x^2}\begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & e \end{pmatrix} + \frac{1}{x}\begin{pmatrix} f & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & i \end{pmatrix}\}\hat{Y}.$$

We align the diagonal entries as $\begin{cases} a & a & b & b \\ c & c & d & e \\ f & g & h & i \end{cases}$. We express the degeneracy of the eigenvalues by $\begin{Bmatrix} 22 \\ 211 \\ 1111 \end{Bmatrix}$. Each row expresses a partition of the matrix size m. The partitions in the lower rows are a refinement of a partition in the upper rows.

In order to express the degeneracy of the eigenvalues shortly, we use parentheses. Firstly, write the finest partition of m in the lowest row, which expresses the degeneracy of the eigenvalue of $T^{(i)}$. Secondly, put the numbers that are grouped together in the second lowest partition in parentheses. We continue this process until the highest row.

Example 4. The local data of the example above can be expressed concisely using the parentheses.

$$\overbrace{\begin{matrix} a & c & f \\ a & c & g \\ b & d & h \\ b & e & i \end{matrix}}^{x=0} = \begin{cases} a & a & b & b \\ c & c & d & e \\ f & g & h & i \end{cases} \longrightarrow \begin{Bmatrix} 22 \\ 211 \\ 1111 \end{Bmatrix} \longrightarrow ((11))((1)(1)),$$

level 2	((11))((1)(1))
level 1	(11)(1)(1)
level 0	1111

The way to restore the degeneracy of eigenvalues from the symbol ((11))((1)(1)) is as follows.

- Add the numbers in the outermost parenthesis → 22
- Add the numbers in the inner parenthesis →211
- Write the numbers in the innermost parentheses →1111

We express the types of linear equations by aligning such data for each singular point.
Ramified non-Fuchsian case. We have the following formal normal form at each singularities (Hukuhara [25], Levelt [45] and Turrittin [73]). Let us assume that $x = 0$ is an irregular singular point. Then, there exist a positive integer q, rational numbers with the common denominator q such that $r_0 < r_1 < \cdots < r_{n-1} < r_n = -1$, diagonal matrices $T_0, \dots, T_n$, a transformation $z = F(x^{1/q})$ in the class of formal series in $x^{1/q}$, such that the transformed system have the following form;

$$\frac{dz}{dx} = (T_0 x^{r_0} + \cdots + T_{n-1} x^{r_{n-1}} + T_n x^{-1})z. \tag{A.11}$$

Let us assume that the diagonal matrix T_k has t_i^k for $i = 1, \dots, m$ as diagonal components. We express the local data by the following table:

$$\overbrace{\begin{matrix} t_1^0 & t_1^1 & \dots & t_1^n \\ \vdots & \vdots & & \vdots \\ t_m^0 & t_m^1 & \dots & t_m^n \end{matrix}}^{x=0 \ \left(\frac{n}{q}\right)}.$$

Let us introduce a way to express such local data compactly with examples.[28]

Example 5. Let us assume the following normal form;

$$\frac{dz}{dx} = \left\{ c_0 \Omega x^{-\frac{8}{3}} + c_1 \Omega^2 x^{-\frac{7}{3}} + c_2 I_3 x^{-2} + c_3 \Omega x^{-\frac{5}{3}} + c_4 \Omega^2 x^{-\frac{4}{3}} + c_5 I_3 x^{-1} \right\} z,$$

where $\Omega = \mathrm{diag}(1, \omega, \omega^2)$. This normal form can be expressed as

$$\overbrace{\begin{matrix} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \\ c_0\omega & c_1\omega & c_2 & c_3\omega & c_4\omega^2 & c_5 \\ c_0\omega^2 & c_1\omega & c_2 & c_3\omega^2 & c_4\omega & c_5 \end{matrix}}^{x=0 \ \left(\frac{5}{3}\right)}.$$

Two systems in the lower rows $\frac{dz_2}{dx} = \left(c_0 \omega x^{-\frac{8}{3}} + \dots\right) z_2$ and $\frac{dz_3}{dx} = \left(c_0 \omega x^{-\frac{8}{3}} + \dots\right) z_3$ can be obtained by the first row $\frac{dz_1}{dx} = \left(c_0 x^{-\frac{8}{3}} + \dots\right) z_1$ upon replacement $x^{\frac{1}{3}} \mapsto \omega x^{\frac{1}{3}} \mapsto \omega^2 x^{\frac{1}{3}}$. Since we have 3 copies of the first equation, we express the local data as $(((((1)))))_3$.

[28] Kawakami [30] devised such notation.

Example 6. We show some other examples.

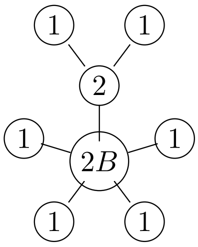
$$\underbrace{x=0\ \left(\tfrac{1}{2}\right)}_{\begin{matrix}\alpha & \beta\\ -\alpha & \beta\\ 0 & \gamma\\ 0 & \epsilon\end{matrix}} \to (1)_2 11, \quad \underbrace{x=0\ \left(\tfrac{1}{2}\right)}_{\begin{matrix}\alpha & \beta\\ \alpha\omega & \beta\\ \alpha\omega^2 & \beta\\ 0 & \gamma\end{matrix}} \to (1)_3 1, \quad \underbrace{x=0\ \left(\tfrac{1}{2}\right)}_{\begin{matrix}\alpha & \beta\\ \alpha & \beta\\ -\alpha & \beta\\ -\alpha & \beta\end{matrix}} \to (2)_2, \quad \underbrace{x=0\ \left(\tfrac{1}{2}\right)}_{\begin{matrix}\alpha & \beta\\ \alpha & \gamma\\ -\alpha & \beta\\ -\alpha & \gamma\end{matrix}} \to (11)_2.$$

A.3. **The dual graphs of the singular fibers.** We list the dual graphs of singular fibers appeared in the table. The numbers in circles indicate multiplicities of the components. We adopt, as in Ogg [56], the following symbol for component Γ of singular fibers (Table 9). K_X is the canonical divisor of surface X. The matrices next to or below the dual graphs are the monodromy [53].

Symbol	Genus	Γ^2	$\Gamma \cdot K_X$
A	1	-1	1
B	0	-3	1
C	1	-2	2
D	0	-4	2
none	0	-2	0

Table 9: Components of singular fibers

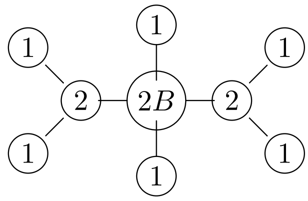
I^*_{1-0-0}: $H^{1+1+1+1+1+1}_{\mathrm{Gar},\tilde{t_1}}$, $H^{1+1+1+1+1+1}_{\mathrm{Gar},\tilde{t_2}}$



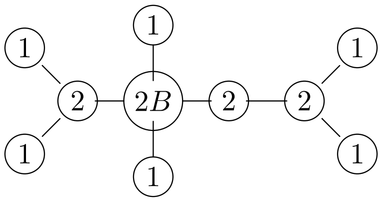
$$\begin{pmatrix} -1 & 0 & -p-q & q \\ 0 & -1 & q & -n-q \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

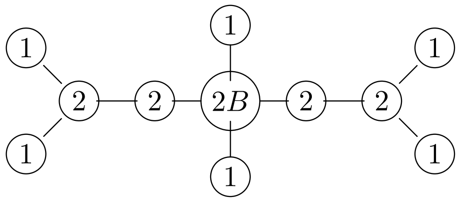
for $\mathrm{I}^*_{\mathrm{n-p-q}}$

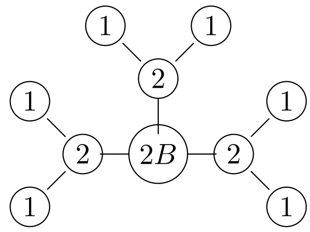
I^*_{1-1-0}: $H^{2+1+1+1+1}_{\mathrm{Gar},\tilde{t_1}}$, $H^{2+1+1+1+1}_{\mathrm{Gar},\tilde{t_2}}$



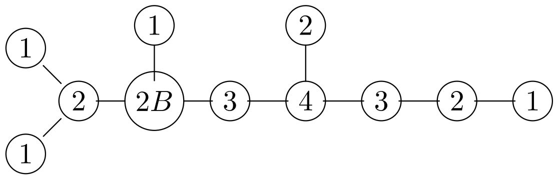
I^*_{1-2-0}: $H^{\frac{3}{2}+1+1+1+1}_{\mathrm{Gar},\tilde{t_1}}$, $H^{\frac{3}{2}+1+1+1+1}_{\mathrm{Gar},\tilde{t_2}}$, I^*_{2-2-0}: $H^{\frac{3}{2}+\frac{3}{2}+1}_{\mathrm{Gar},\tilde{t_1}}$ I^*_{1-1-1}: $H^{2+2+1}_{\mathrm{Gar},\tilde{t_2}}$
$H^{\frac{3}{2}+2+1}_{\mathrm{Gar},\tilde{t_1}}$







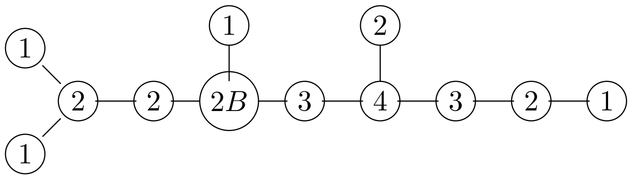
$\mathrm{III}^* - \mathrm{I}^*_1 - (-1)$: $H^{\frac{5}{2}+1+1}_{\mathrm{Gar},\tilde{t_1}}, H^{\frac{5}{2}+1+1}_{\mathrm{Gar},\tilde{t_2}}$



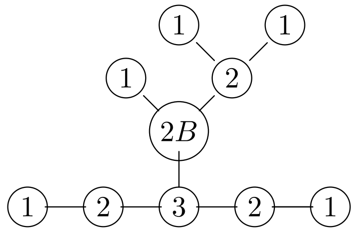
$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -n \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

for $\mathrm{III} - \mathrm{I_n} - \mathrm{m}$

$\mathrm{III}^* - \mathrm{I}^*_2 - (-1)$: $H^{\frac{3}{2}+\frac{3}{2}+1}_{\mathrm{Gar},\tilde{t_2}}$



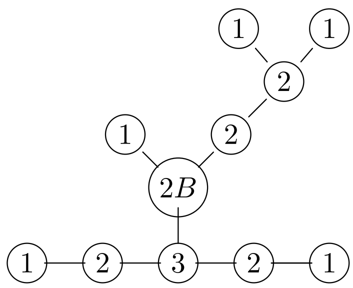
$\mathrm{IV}^* - \mathrm{I}^*_1 - (-1)$: $H^{3+1+1}_{\mathrm{Gar},\tilde{t_1}}$, $H^{3+1+1}_{\mathrm{Gar},\tilde{t_2}}, H^{2+2+1}_{\mathrm{Gar},\tilde{t_1}}$



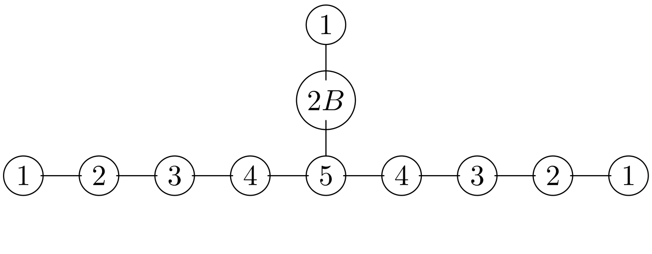
$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -n \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

for $\mathrm{IV}^* - \mathrm{I}^*_\mathrm{n} - \mathrm{m}$

$\mathrm{IV}^* - \mathrm{I}^*_2 - (-1)$: $H^{\frac{3}{2}+2+1}_{\mathrm{Gar},\tilde{t_2}}$

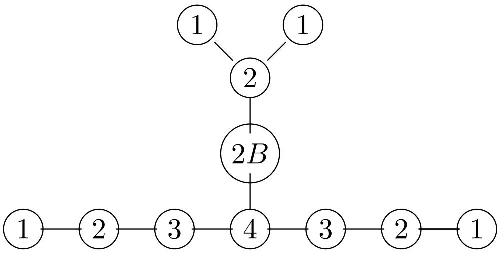


$\mathrm{IX} - 3$: $H^{4+1}_{\mathrm{Gar},\tilde{t_2}}, H^{5}_{\mathrm{Gar},\tilde{t_1}}$



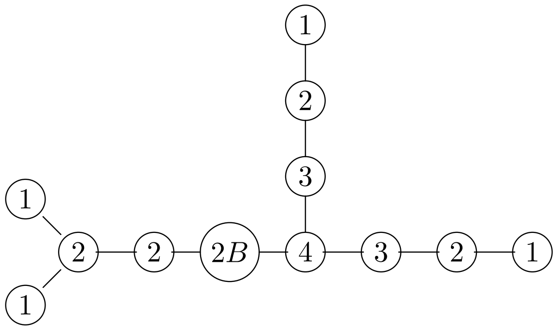
$$\begin{pmatrix} 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\mathrm{III}^* - \mathrm{II}_1^*$: $H^{4+1}_{\mathrm{Gar},\tilde{t_1}}, H^{3+2}_{\mathrm{Gar},\tilde{t_1}}$

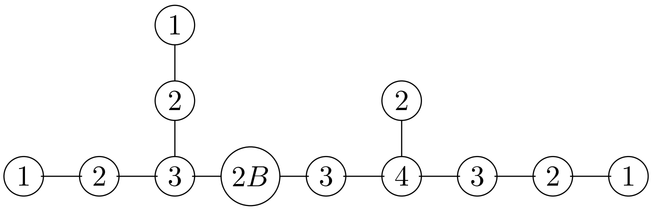


$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & -n-1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\mathrm{III}^* - \mathrm{II}_2^*$: $H^{\frac{3}{2}+3}_{\mathrm{Gar},\tilde{t_2}}$

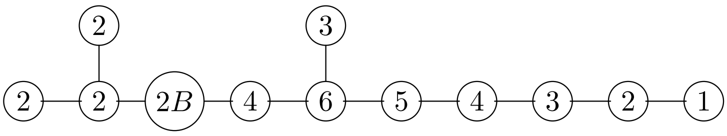


$\mathrm{IV}^* - \mathrm{III}^* - (-1)$: $H^{\frac{5}{2}+2}_{\mathrm{Gar},\tilde{t_2}}, H^{\frac{3}{2}+3}_{\mathrm{Gar},\tilde{t_1}}$



$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

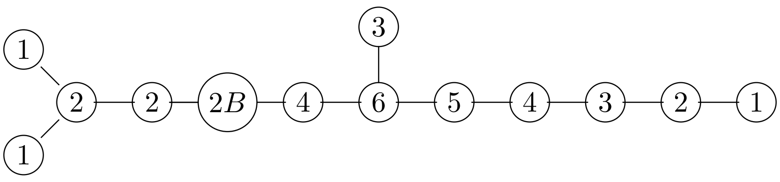
$\mathrm{II}^* - \mathrm{II}_1^*$: $H^{\frac{5}{2}+2}_{\mathrm{Gar},\tilde{t_1}}, H^{\frac{7}{2}+1}_{\mathrm{Gar},\tilde{t_1}}$



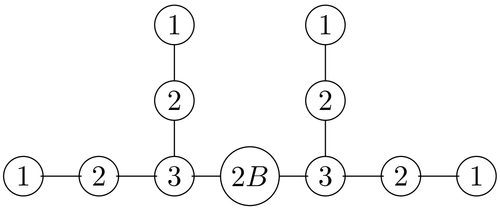
$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & -n \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

for $\mathrm{II}^* - \mathrm{II}_\mathrm{n}^*$

$\mathrm{II}^* - \mathrm{II}_2^*$: $H^{\frac{5}{2}+\frac{3}{2}}_{\mathrm{Gar},\tilde{t_2}}$

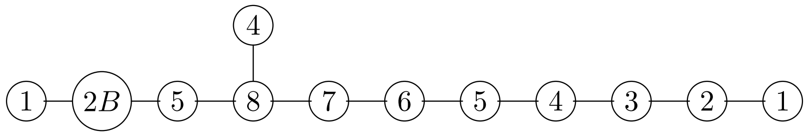


$\mathrm{IV}^* - \mathrm{IV}^* - (-1)$: $H^{3+2}_{\mathrm{Gar},\tilde{t_2}}$



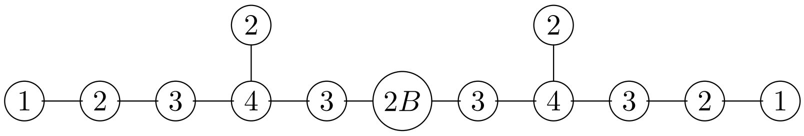
$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

VII*: $H^{\frac{7}{2}+1}_{\mathrm{Gar},\tilde{t_2}}, H^{\frac{9}{2}}_{\mathrm{Gar},\tilde{t_1}}$



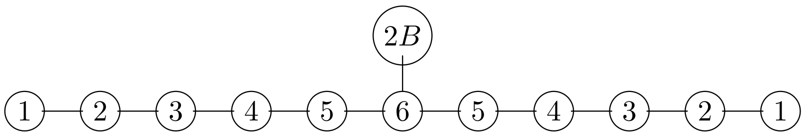
$$\begin{pmatrix} 0 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

III* − III* − (−1): $H^{\frac{5}{2}+\frac{3}{2}}_{\mathrm{Gar},\tilde{t_1}}$



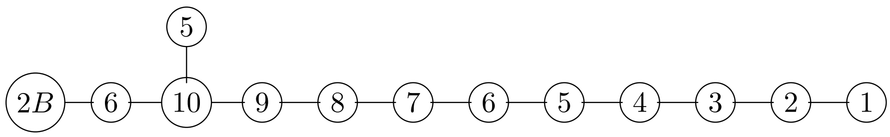
$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

V*: $H^{5}_{\mathrm{Gar},\tilde{t_2}}$



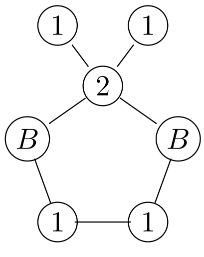
$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

VIII − 4: $H^{\frac{9}{2}}_{\mathrm{Gar},\tilde{t_2}}$



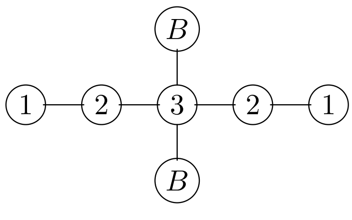
$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix}$$

II_{4-0}: $H^{A_5}_{\mathrm{FS}}$



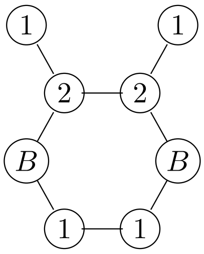
$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 4 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

III: $G^{A_5}_{\mathrm{FS}}$



$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

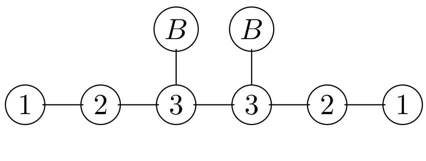
II_{4-1}: $H^{A_4}_{\mathrm{FS}}$



$$\begin{pmatrix} -1 & 0 & -p & 0 \\ 1 & 1 & p & n \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for $\mathrm{II_{n-p}}(\mathrm{p} \geq 1)$

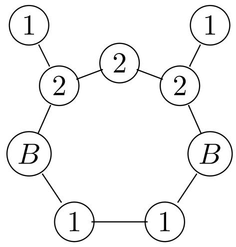
5 − III_1: $G^{A_4}_{\mathrm{FS}}$



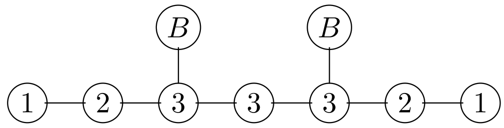
$$\begin{pmatrix} 0 & 1 & -n & n \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

for $\mathrm{III_n}$

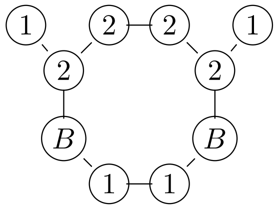
$\mathrm{II}_{4-2}\colon H^{A_3}_{\mathrm{FS}}$



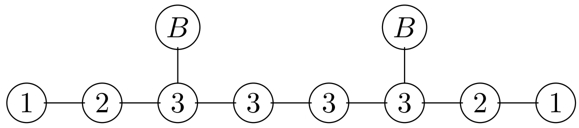
$5-\mathrm{III}_2\colon G^{A_3}_{\mathrm{FS}}$



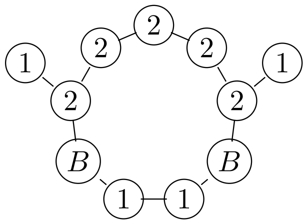
$\mathrm{II}_{4-3}\colon H^{\frac{3}{2}+2}_{\mathrm{KFS}}$



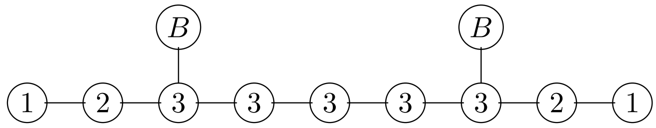
$5-\mathrm{III}_3\colon G^{\frac{3}{2}+2}_{\mathrm{KFS}}$



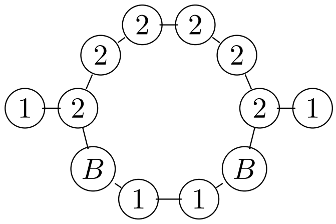
$\mathrm{II}_{4-4}\colon H^{\frac{3}{2}+\frac{3}{2}}_{\mathrm{KFS}}$



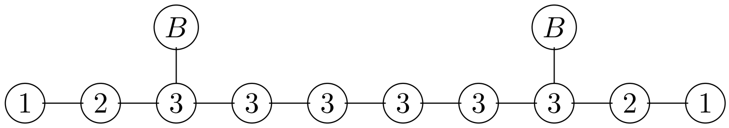
$5-\mathrm{III}_4\colon G^{\frac{3}{2}+\frac{3}{2}}_{\mathrm{KFS}}$



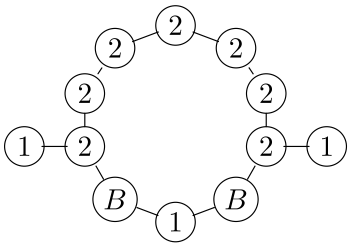
$\mathrm{II}_{4-5}\colon H^{\frac{4}{3}+\frac{3}{2}}_{\mathrm{KFS}}$



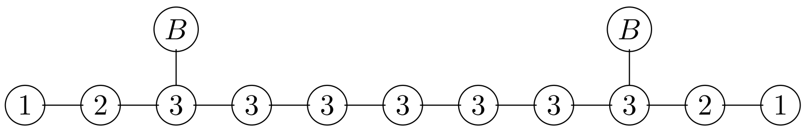
$5-\mathrm{III}_5\colon G^{\frac{4}{3}+\frac{3}{2}}_{\mathrm{KFS}}$



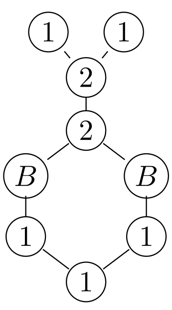
$\mathrm{II}_{3-6}\colon H^{\frac{4}{3}+\frac{4}{3}}_{\mathrm{KFS}}$



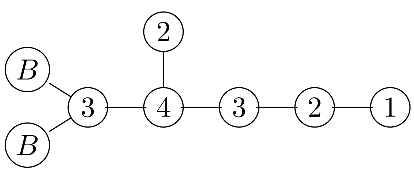
$5-\mathrm{III}_6\colon G^{\frac{4}{3}+\frac{4}{3}}_{\mathrm{KFS}}$



$\mathrm{II}_{5-1}\colon H^{A_5}_{\mathrm{NY}}$

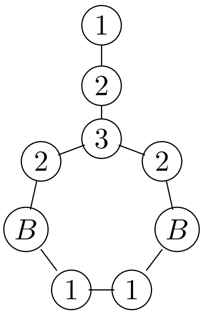


$\mathrm{IV}-\mathrm{III}^*-(-1)\colon G^{A_5}_{\mathrm{NY}}$



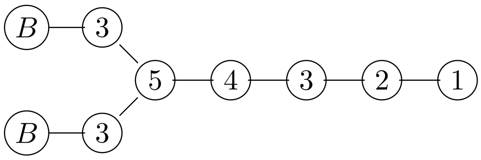
$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\mathrm{IV}^* - \mathrm{II}_4$: $H_{\mathrm{NY}}^{A_4}$



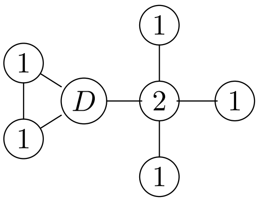
$$\begin{pmatrix} -1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{IX} - 4$: $G_{\mathrm{NY}}^{A_4}$



$$\begin{pmatrix} 0 & 0 & -1 & -1 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

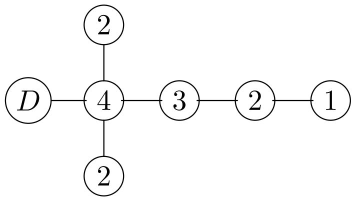
$\mathrm{I}_3 - \mathrm{I}_0^* - 0$: $H_{\mathrm{Ss}}^{D_6}$



$$\begin{pmatrix} -1 & 0 & -p & 0 \\ 0 & 1 & 0 & n \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

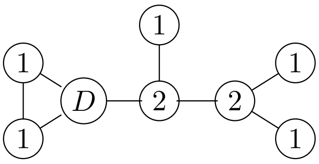
for $\mathrm{I_n} - \mathrm{I_p^*} - \mathrm{m}$

VI: $G_{\mathrm{Ss}}^{D_6}$

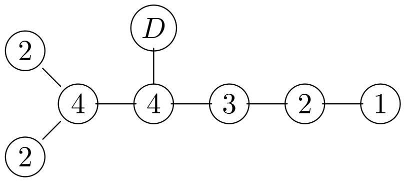


$$\begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\mathrm{I}_3 - \mathrm{I}_1^* - 0$: $H_{\mathrm{Ss}}^{D_5}$



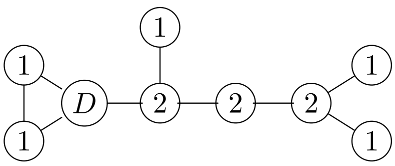
$4 - \mathrm{III}_1$: $G_{\mathrm{Ss}}^{D_5}$



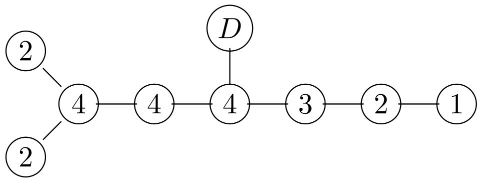
$$\begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & n & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

for $\mathrm{III_n}$

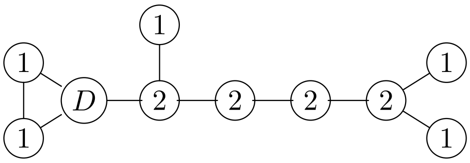
$\mathrm{I}_3 - \mathrm{I}_2^* - 0$: $H_{\mathrm{Ss}}^{D_4}$



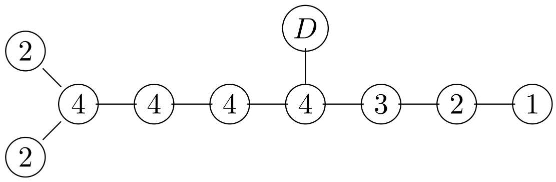
$4 - \mathrm{III}_2$: $G_{\mathrm{Ss}}^{D_4}$



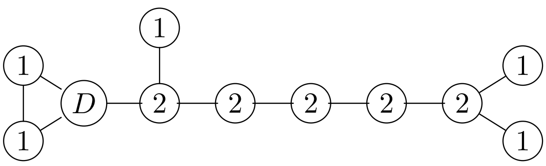
$\mathrm{I}_3 - \mathrm{I}_3^* - 0$: $H_{\mathrm{KSs}}^{\frac{3}{2}+2}$



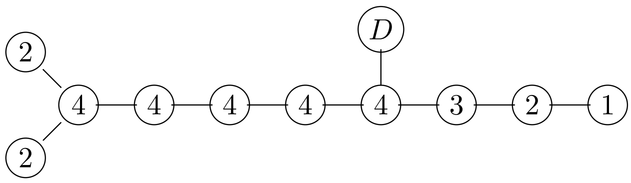
$4 - \mathrm{III}_3$: $G_{\mathrm{KSs}}^{\frac{3}{2}+2}$



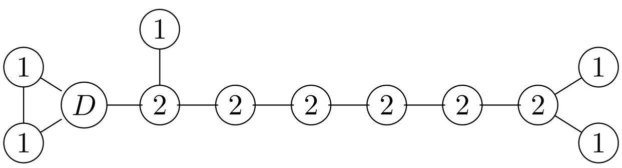
$\mathrm{I}_3 - \mathrm{I}_4^* - 0$: $H_{\mathrm{KSs}}^{\frac{4}{3}+2}$



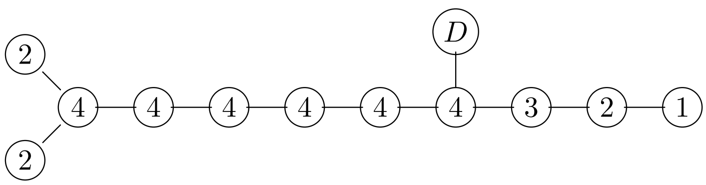
$4 - \mathrm{III}_4$: $G_{\mathrm{KSs}}^{\frac{4}{3}+2}$



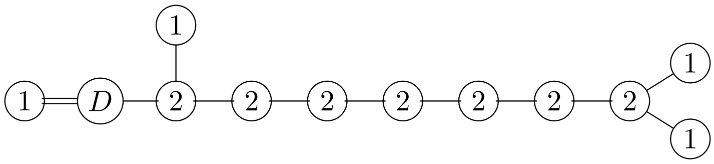
$\mathrm{I}_3 - \mathrm{I}_5^* - 0$: $H_{\mathrm{KSs}}^{\frac{5}{4}+2}$



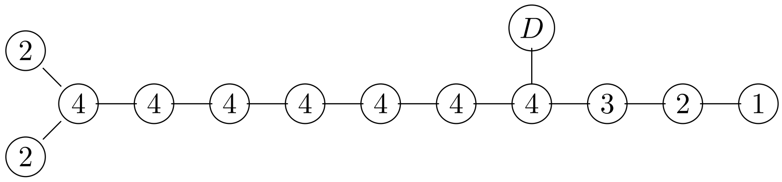
$4 - \mathrm{III}_5$: $G_{\mathrm{KSs}}^{\frac{5}{4}+2}$



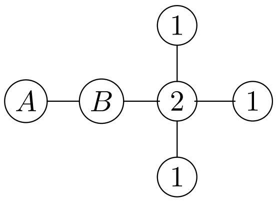
$\mathrm{I}_2 - \mathrm{I}_6^* - 0\colon H_{\mathrm{KSs}}^{\frac{3}{2}+\frac{5}{4}}$



$4 - \mathrm{III}_6\colon G_{\mathrm{KSs}}^{\frac{3}{2}+\frac{5}{4}}$



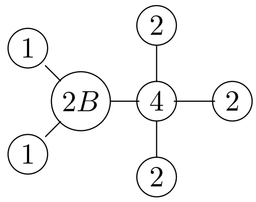
$\mathrm{I}_0 - \mathrm{I}_0^* - 1\colon H_{\mathrm{VI}}^{\mathrm{Mat}}$



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -n \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

for $\mathrm{I}_0 - \mathrm{I}_\mathrm{n}^* - \mathrm{m}$

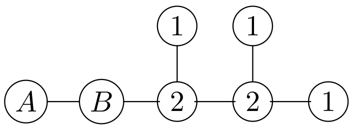
$2\mathrm{I}_0^* - 0\colon G_{\mathrm{VI}}^{\mathrm{Mat}}$



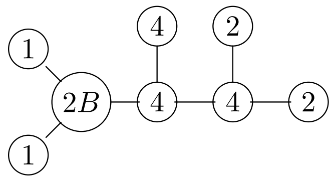
$$\begin{pmatrix} 0 & -1 & 0 & -n \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

for $2\mathrm{I}_\mathrm{n}^* - \mathrm{m}$

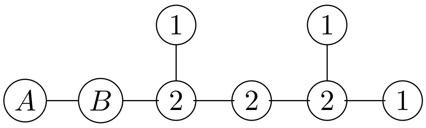
$\mathrm{I}_0 - \mathrm{I}_1^* - 1\colon H_{\mathrm{V}}^{\mathrm{Mat}}$



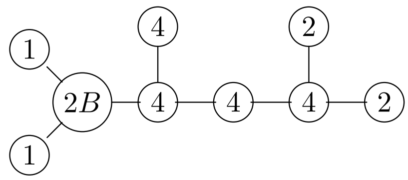
$2\mathrm{I}_1^* - 0\colon G_{\mathrm{V}}^{\mathrm{Mat}}$



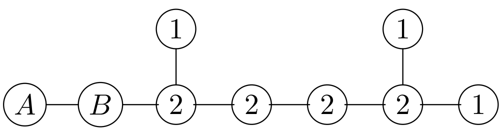
$\mathrm{I}_0 - \mathrm{I}_2^* - 1\colon H_{\mathrm{III}(D_6)}^{\mathrm{Mat}}$



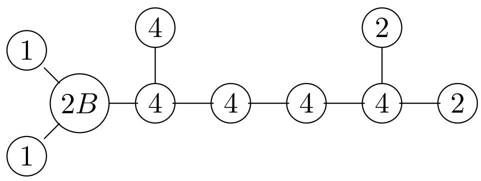
$2\mathrm{I}_2^* - 0\colon G_{\mathrm{III}(D_6)}^{\mathrm{Mat}}$



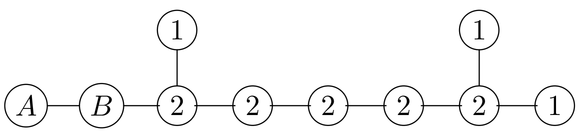
$\mathrm{I}_0 - \mathrm{I}_3^* - 1\colon H_{\mathrm{III}(D_7)}^{\mathrm{Mat}}$



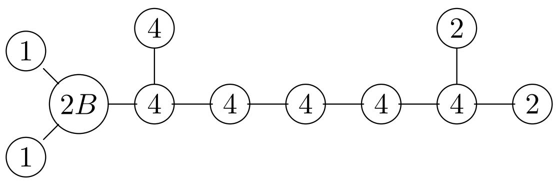
$2\mathrm{I}_3^* - 0\colon G_{\mathrm{III}(D_7)}^{\mathrm{Mat}}$



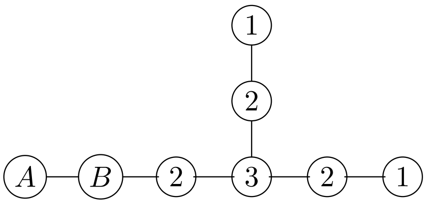
$\mathrm{I}_0 - \mathrm{I}_4^* - 1\colon H_{\mathrm{III}(D_8)}^{\mathrm{Mat}}$



$2\mathrm{I}_4^* - 0\colon G_{\mathrm{III}(D_8)}^{\mathrm{Mat}}$

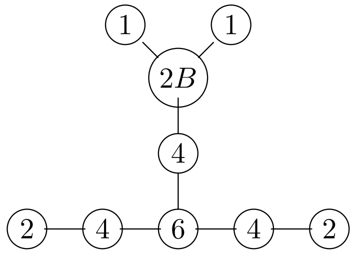


$\mathrm{I}_0 - \mathrm{IV}^* - 1$: $H_{\mathrm{IV}}^{\mathrm{Mat}}$



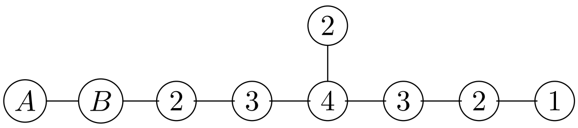
$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$2\mathrm{IV}^* - 0$: $G_{\mathrm{IV}}^{\mathrm{Mat}}$



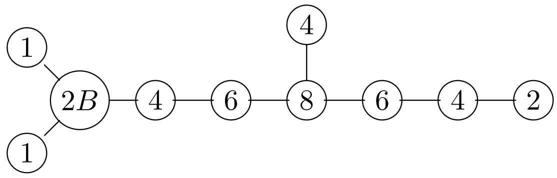
$$\begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\mathrm{I}_0 - \mathrm{III}^* - 1$: $H_{\mathrm{II}}^{\mathrm{Mat}}$



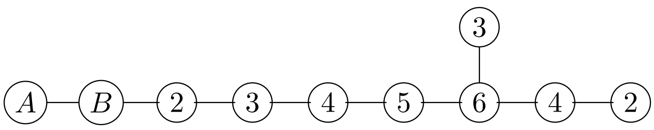
$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$2\mathrm{III}^* - 0$: $G_{\mathrm{II}}^{\mathrm{Mat}}$



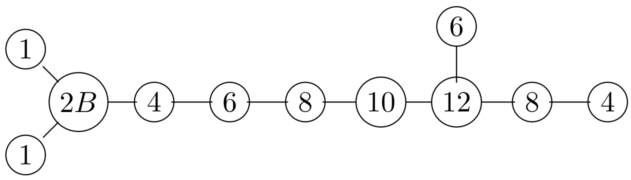
$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\mathrm{I}_0 - \mathrm{II}^* - 1$: $H_{\mathrm{I}}^{\mathrm{Mat}}$



$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$2\mathrm{II}^* - 0$: $G_{\mathrm{I}}^{\mathrm{Mat}}$



$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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Graduate School of Mathematical Sciences, University of Tokyo, Komaba, Tokyo 153-8914, Japan

E-mail address: akanna@ms.u-tokyo.ac.jp