博士論文

論文題目　**On Lagrangian caps and their applications**
（ラグランジュキャップとその応用について）

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全体の序文

Title On Lagrangian caps and their applications

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1. Background

There are several necessary conditions for closed submanifolds in symplectic manifolds to be Lagrangian. The first result is the non-existence of closed exact Lagrangian submanifolds in the standard symplectic space $\mathbb{R}^{2n}_{\mathrm{st}}$ proved by Gromov [4] using the technique of pseudo-holomorphic curves. A number of results are known until now. The results of Viterbo [9], Seidel [8], and Biran [1] are typical examples. On the other hand, Lagrangian immersions satisfy the h-principle [5] which states that the space of Lagrangian immersions is weakly homotopy equivalent to the space of Lagrangian monomorphisms. In particular, the h-principle for Lagrangian immersions gives a characterization of immersed Lagrangian submanifolds in terms of homotopy theory. Using the h-principle for Lagrangian immersions, we can construct the following examples:

- a Lagrangian regular homotopy class which does not contain any embedding.
- a homotopy class which contains a Lagrangian immersion and does not contain any Lagrangian embedding.

In symplectic topology, it is an important problem to study the difference between Lagrangian immersions and Lagrangian embeddings.

Eliashberg and Murphy [3] established the resolving theory of Lagrangian self-intersections by using the theory of loose Legendrian submanifolds [6]. As an application of the resolving theory, Ekholm, Eliashberg, Murphy, and Smith [2] gives an upper bound of the minimum of the number of self-intersection points in a Lagrangian regular homotopy class. These results ensure that the similarity between Lagrangian immersions and Lagrangian embeddings with a conical point, and that the minimum of the number of self-intersection points in a Lagrangian regular homotopy class is nearly independent of the symplectic structure.

2. Main result

The purpose of this thesis is to study the difference between Lagrangian immersions and Lagrangian embeddings in the symplectic manifolds $\mathbb{C}P^3$ and $\mathbb{C}P^1 \times \mathbb{C}P^2$. The main result is the following.

Theorem 1. Let X be either the complex projective 3-space $\mathbb{C}P^3$ or the product $\mathbb{C}P^1 \times \mathbb{C}P^2$ of the complex projective line and the complex projective plane, where the complex projective space $\mathbb{C}P^n$ is endowed with the

Fubini-Study form ω_n, $n = 1, 2, 3$. Then for a closed orientable connected 3-manifold L and a Lagrangian immersion $f\colon L\#(S^1 \times S^2) \to X$, there exists a Lagrangian embedding $L\#(S^1 \times S^2) \to X$ homotopic to f.

Theorem 1 is proved in the following two steps. First, a Lagrangian immersion $L\#(S^1 \times S^2) \to X$ induces a Lagrangian immersion $L \to X$. One can choose the Lagrangian immersion $L \to X$ to be of exactly one self-intersection point. Second, Polterovich's Lagrangian surgery [7] resolves the self-intersection point as an embedding of the connected sum of $S^1 \times S^2$. Choosing the homotopy class of the Lagrangian immersion in the first step, the resulting Lagrangian embedding realizes the given homotopy class. In the first step, we prove the resolving theory of Eliashberg and Murphy [3] also works on 6-dimensional compact symplectic manifolds. Namely, we prove and use the following theorems.

Theorem 2. Let (X, ω) be a 6-dimensional simply connected compact symplectic manifold, L a closed connected 3-manifold, and $f_0\colon L \to X$ a Lagrangian immersion. Then there exists a Hamiltonian regular homotopy $f_t\colon L \to X$, $0 \le t \le 1$, from the Lagrangian immersion f_0 to a self-transverse Lagrangian immersion f_1 such that

$$\mathrm{SI}(f_1) = \begin{cases} 1, & \text{if } I(f_0) = 1; \\ 2, & \text{if } I(f_0) = 0, \end{cases}$$

where $\mathrm{SI}(f_1)$ is the total number of self-intersection points of f_1 and $I(f_0)$ is the algebraic self-intersection number of f_0. Moreover, there exists a self-intersection point x of f_1, a point $p \in f_1(L)$, and a Darboux chart around p, symplectomorphic to the 6-ball $B^6_{\mathrm{st}}(\varepsilon)$ of radius ε, such that the self-intersection point x belongs to $B^6(\varepsilon/2)$ and near the 5-sphere $\partial B^6(\varepsilon/2)$ the Lagrangian immersion f_1 coincides with the Lagrangian cone over a loose Legendrian sphere $\phi = f_1(L) \cap \partial B^6(\varepsilon/2)$ in the 5-sphere $\partial B^6(\varepsilon/2)$ with the standard contact structure.

Theorem 3. Let (X, ω) be a 6-dimensional simply connected compact symplectic manifold, L a connected 3-manifold, and $f_0\colon L \to X$ a Lagrangian immersion with a conical point $p \in L$. Suppose that the Legendrian link of f_0 at p is loose. Then there exists a Hamiltonian regular homotopy $f_t\colon L \to X$, $0 \le t \le 1$, from the Lagrangian immersion f_0 to a self-transverse Lagrangian immersion f_1 with a conical point p such that f_t is the identity in a neighborhood of p and that $\mathrm{SI}(f_1) = |I(f_0)|$.

The key lemma for the proof of Theorems 2 and 3 is the following.

Lemma 4. Let $A = [0, 1] \times S^{n-1} \ni (x, z)$, $n \ge 3$, be the annulus with the coordinates (x, z). Take the dual coordinates (y, u) on the cotangent bundle T^*A so that the canonical Liouville form $\lambda = y\,dx + u\,dz$. Then for any integer $N \ge 10$ there exists a Lagrangian immersion $\Delta\colon A \to T^*A$ with the following properties:

- $\Delta(A) \subset \left\{ |y| \le \frac{12}{N},\ \|u\| \le \frac{12}{N} \right\}$;

- Δ coincides with the inclusion of the zero section $j_A\colon A \hookrightarrow T^*A$ near ∂A;
- there exists a Lagrangian regular homotopy which is the identity near ∂A and connects j_A to Δ in $\left\{|y| \leq \frac{12}{N},\ \|u\| \leq \frac{12}{N}\right\}$;
- for the Δ-image ζ of any path connecting $\{0\} \times S^{n-1}$ to $\{1\} \times S^{n-1}$ in A, $\int_\zeta \lambda = 1$;
- the action of any self-intersection point of Δ is $< \frac{2}{N}$;
- $\mathrm{SI}(\Delta) = 4N^2$.

The resolving theory of Lagrangian self-intersections by Eliashberg and Murphy [3] can be used for a Whitney pair such that the symplectic area of the Whitney disk is equal to zero. For a given Lagrangian immersion, they constructed such Whitney pairs by replacing the given self-intersection points by a large number of self-intersection points of small actions. In the step of constructing Whitney pairs, they took disjoint Darboux charts for each self-intersection points. Hence, the growth of self-intersection points plays an important role for the compactness of the symplectic manifold. Reducing the growth by using Lemma 4, we can prove that their resolving theory also works on 6-dimensional compact symplectic manifolds.

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ON LAGRANGIAN CAPS AND THEIR APPLICATIONS

TORU YOSHIYASU

Contents

1. Introduction

A symplectic structure is a closed non-degenerate 2-form ω on an even dimensional manifold X. We call the pair (X, ω) a symplectic manifold. If the symplectic structure ω is an exact 2-form $\omega = d\lambda$, then the pair (X, λ) is called an exact symplectic manifold. An embedding $f\colon L \to X$ of a half-dimensional manifold L to a symplectic manifold (X, ω) is called a Lagrangian embedding if $f^*\omega = 0$, and then $f(L)$ is called a Lagrangian submanifold. For a Lagrangian embedding $f\colon L \to X$ to an exact symplectic manifold (X, λ), the 1-form $f^*\lambda$ is closed. A Lagrangian embedding $f\colon L \to X$ to an exact symplectic manifold (X, λ) is called exact if the 1-form $f^*\lambda$ is exact, and then $f(L)$ is called an exact Lagrangian submanifold. A Lagrangian immersion, an exact Lagrangian immersion, and a Lagrangian regular homotopy are defined in a similar way.

Gromov developed the theory of h-principles about 50 years ago. Gromov [8] proved that Lagrangian immersions satisfy the h-principle. A precise

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statement is the following: Let (X, ω) be a $2n$-dimensional symplectic manifold and L an n-dimensional manifold. Then Lagrangian regular homotopy classes of L into (X, ω) are in one-to-one correspondence with the homotopy classes of Lagrangian monomorphisms $TL \to TX$. This result shows the flexibility of Lagrangian immersions. As a corollary, we obtain a characterization of immersed Lagrangian submanifolds in $\mathbb{R}^{2n}_{\mathrm{st}}$. For an n-dimensional manifold L, there exists a Lagrangian immersion $L \to \mathbb{R}^{2n}_{\mathrm{st}}$ if and only if the complexified tangent bundle $TL \otimes \mathbb{C}$ is trivial.

On the other hand, there are several necessary conditions for closed submanifolds in symplectic manifolds to be Lagrangian. The first result is the non-existence of closed exact Lagrangian submanifolds in the standard symplectic space $\mathbb{R}^{2n}_{\mathrm{st}}$ proved by Gromov [7] using the technique of pseudo-holomorphic curves. A number of results are known until now. Viterbo [14] proved that the non-existence of closed exact Lagrangian submanifolds in strong algebraic Weinstein manifolds. Seidel [13] proved that there are certain necessary conditions for the first cohomology group of closed Lagrangian submanifolds in the complex projective spaces. Biran [1] proved that the cohomology ring of a closed Lagrangian submanifold with the first cohomology group being 2-torsion in the complex projective spaces is isomorphic to that of the real projective space as graded vector spaces. Fukaya [6] proved that a closed orientable connected prime 3-manifold L can be a Lagrangian submanifold in $\mathbb{R}^6_{\mathrm{st}}$ if and only if L is diffeomorphic to the product $S^1 \times \Sigma_g$ of the circle and an oriented closed surface of genus $g \geq 0$. In particular, these necessary conditions show that there are no h-principles for Lagrangian embeddings in general. It was thought that to establish Whitney tricks for Lagrangian immersions is difficult.

However, recently Eliashberg and Murphy [5] established the Whitney trick for Lagrangian immersions by admitting Lagrangian immersions to have a conical singularity. Using this theory and the Lagrangian surgery introduced by Polterovich [11], Ekholm, Eliashberg, Murphy, and Smith [3] constructed a Lagrangian embedding $L\#(S^1 \times S^2) \to \mathbb{R}^6_{\mathrm{st}}$ for any closed orientable connected 3-manifold L.

The aim of this thesis is to study the homotopy classes of Lagrangian immersions containing a Lagrangian embedding. Specifically, we study the following problems:

(1) Which homotopy classes of a closed orientable connected 3-manifold of the form $L\#(S^1 \times S^2)$ into the complex projective 3-space $\mathbb{C}P^3$ contain a Lagrangian embedding?
(2) Which homotopy classes of a closed orientable connected 3-manifold of the form $L\#(S^1 \times S^2)$ into the product $\mathbb{C}P^1 \times \mathbb{C}P^2$ contain a Lagrangian embedding?

1.1. **Our results.**

In this thesis, we prove the following results. Theorem 1.1 is an answer to the problems (1) and (2).

Theorem 1.1. *Let X be either the complex projective 3-space $\mathbb{C}P^3$ or the product $\mathbb{C}P^1 \times \mathbb{C}P^2$ of the complex projective line and the complex projective plane, where the complex projective space $\mathbb{C}P^n$ is endowed with the*

Fubini-Study form ω_n, $n = 1, 2, 3$. Then for a closed orientable connected 3-manifold L and a Lagrangian immersion $f: L\#(S^1 \times S^2) \to X$, there exists a Lagrangian embedding $L\#(S^1 \times S^2) \to X$ homotopic to f.

To prove Theorem 1.1, we use the following theorems.

Theorem 1.2. *Let (X, ω) be a 6-dimensional simply connected compact symplectic manifold, L a closed connected 3-manifold, and $f_0: L \to X$ a Lagrangian immersion. Then there exists a Hamiltonian regular homotopy $f_t: L \to X$, $0 \le t \le 1$, from the Lagrangian immersion f_0 to a self-transverse Lagrangian immersion f_1 such that*

$$\mathrm{SI}(f_1) = \begin{cases} 1, & \text{if } I(f_0) = 1; \\ 2, & \text{if } I(f_0) = 0. \end{cases}$$

Moreover, there exists a self-intersection point x of f_1, a point $p \in f_1(L)$, and a Darboux chart around p, symplectomorphic to the 6-ball $B^6_{\mathrm{st}}(\varepsilon)$ of radius ε, such that the self-intersection point x belongs to $B^6(\varepsilon/2)$ and near the 5-sphere $\partial B^6(\varepsilon/2)$ the Lagrangian immersion f_1 coincides with the Lagrangian cone over a loose Legendrian sphere $\phi = f_1(L) \cap \partial B^6(\varepsilon/2)$ in the 5-sphere $\partial B^6(\varepsilon/2)$ with the standard contact structure.

Theorem 1.3. *Let (X, ω) be a 6-dimensional simply connected compact symplectic manifold, L a connected 3-manifold, and $f_0: L \to X$ a Lagrangian immersion with a conical point $p \in L$. Suppose that the Legendrian link of f_0 at p is loose. Then there exists a Hamiltonian regular homotopy $f_t: L \to X$, $0 \le t \le 1$, from the Lagrangian immersion f_0 to a self-transverse Lagrangian immersion f_1 with the conical point p such that f_t is the identity in a neighborhood of p and that $\mathrm{SI}(f_1) = |I(f_0)|$.*

Theorem 1.4. *Let (X, ω) be a 6-dimensional symplectic manifold with a negative Liouville end X_-, L a connected 3-manifold with a negative end, and $f: L \to X$ a proper embedding cylindrical at $-\infty$. Suppose that $X \setminus X_-$ is relatively compact, $[f^*\omega] = 0$ in $H^2(L; \mathbb{R})$, the asymptotic negative Legendrian boundary of f has a component which is loose in the complement of the other components, and there exists a homotopy of homomorphisms $\Psi_t: TL \to TX$ covering f such that $\Psi_0 = df$ and Ψ_1 is a Lagrangian monomorphism. Then for a cohomology class $A \in H^2(X, f(L); \mathbb{R})$ with $r_\infty(A) = A_\infty(f)$ and $j(A) = [\omega] \in H^2(X; \mathbb{R})$, there exists an isotopy $f_t: L \to X$ such that*

1. *$f_0 = f$;*
2. *f_1 is Lagrangian;*
3. *$A(f_1) = A$;*
4. *df_1 is homotopic to Ψ_1 through Lagrangian monomorphisms.*

Moreover, if (X, ω) is a Liouville manifold with a negative end, then for a cohomology class $a \in H^1(L; \mathbb{R})$ one can choose the isotopy f_t so that the absolute action class $a(f_1) = a$. In particular, one can make the Lagrangian embedding f_1 exact.

1.2. **Plan of the thesis.** In Section 2.1, we give definitions, examples, and basic theorems in symplectic and contact geometry. In Sections 2.2 and 2.3, following [3] we define loose Legendrian knots and Lagrangian immersions with a conical point, respectively. In Section 2.4, we construct a local deformation of Lagrangian immersions. With the help of this local deformation, the arguments in [5] and [3] ensure that Theorems 1.2, 1.3, and 1.4 hold. In Section 3.1, we review Polterovich's Lagrangian surgery [11]. In Sections 3.2 and 3.4, we characterize the homotopy classes of Lagrangian immersions of closed orientable connected 3-manifolds into $\mathbb{C}P^3$ and into $\mathbb{C}P^1 \times \mathbb{C}P^2$, respectively. In Sections 3.3 and 3.5, Theorem 1.1 is proved as an application of Theorems 1.2 and 1.3, Lemmas 3.4 and 3.8, and Polterovich's Lagrangian surgery [11].

1.3. **Notations and Conventions.** The following notations and conventions are used through the thesis.

Notations.

$\mathcal{L}_Z\omega$	the Lie derivative of a form ω by a vector field Z.
$\iota_Z\omega$	the interior product of a form ω by a vector field Z.
$\mathrm{SI}(f)$	the total number of self-intersection points of a self-transverse immersion f.
$I(f)$	the algebraic self-intersection number of an immersion f.
$J^1(\Lambda, \mathbb{R})$	the 1-jet bundle of C^1-smooth functions on a manifold Λ.
$[L, X]$	for topological spaces L and X, $[L, X]$ is a homotopy set of continuous maps of L to X.
$B^n(R)$	the ball of radius R centered at the origin of the n-dimensional Euclidean space $\mathbb{R}^n$.
$L_1 \# L_2$	for manifolds L_1 and L_2 of the same dimension, $L_1 \# L_2$ is the connected sum of them.
$\mathrm{Int}(U)$	for a subset U of a topological space V, $\mathrm{Int}(U)$ is the interior of the set U in V.
$\|A\|$	for a finite set A, $\|A\|$ is the total number of elements in A.

Conventions.

All manifolds and maps are supposed to be smooth.
A closed manifold is a compact manifold without boundary.
A simple point for a map is a point of multiplicity 1.
A double point for a map is a point of multiplicity 2.
A bundle monomorphism is a map between vector bundles which is fiberwise linear and fiberwise injective.

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2. Preliminaries

2.1. **Symplectic and contact basics.** In this section, we give definitions, examples, and basic theorems in symplectic and contact geometry.

Definition 2.1. A *symplectic structure* ω on a $2n$-dimensional manifold X is a closed non-degenerate 2-form on X. The pair (X, ω) is called a *symplectic* *manifold*. If the symplectic structure is an exact 2-form $\omega = d\lambda$, then ω, λ, and the pair (X, λ) are called an *exact* symplectic structure, a *Liouville form*, and an *exact* symplectic manifold, respectively.

Example 2.2. The $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$ has the standard symplectic structure

$$\omega_{\mathrm{st}} = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

where $(x_1, y_1, \dots, x_n, y_n)$ are the standard coordinates on $\mathbb{R}^{2n}$. The standard symplectic structure is exact. Actually,

$$\omega_{\mathrm{st}} = d\lambda_{\mathrm{st}}, \quad \lambda_{\mathrm{st}} = \frac{1}{2}\sum_{i=1}^{n}(x_i\, dy_i - y_i\, dx_i).$$

We denote by $\mathbb{R}^{2n}_{\mathrm{st}}$ the exact symplectic manifold $(\mathbb{R}^{2n}, \lambda_{\mathrm{st}})$. We call $\mathbb{R}^{2n}_{\mathrm{st}}$ and λ_{st} the *standard symplectic space* and the *standard Liouville form*, respectively.

Example 2.3. The complex projective space $\mathbb{C}P^n$ has the canonical symplectic structure ω_n called the *Fubini-Study form*. The complex projective space is defined as $\mathbb{C}P^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$ under the equivalence relation $(z_0, \dots, z_n) \sim (cz_0, \dots, cz_n)$, $c \in \mathbb{C} \setminus \{0\}$. We denote an equivalent class containing an element $(z_0, \dots, z_n)$ by $[z_0 : \cdots : z_n]$. Then subsets $U_j = \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n \mid z_j \neq 0\}$ are holomorphic charts in $\mathbb{C}P^n$ by maps

$$\varphi_j \colon U_j \to \mathbb{C}^n : [z_0 : \cdots : z_n] \mapsto \left(\frac{z_0}{z_j}, \dots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \dots, \frac{z_n}{z_j}\right), \quad j = 0, \dots, n.$$

The Fubini-Study form ω_n is defined on U_j by

$$\omega_n \mid_{U_j} = \varphi_j^*\left(\frac{i}{2\pi}\partial\bar{\partial}\log(\|w\|^2 + 1)\right), \quad w = (w_1, \dots, w_n) \in \mathbb{C}^n.$$

We can check that ω_n is well-defined on $\mathbb{C}P^n$. The cohomology class $[\omega_n]$ is the positive generator of the cohomology group $H^2(\mathbb{C}P^n; \mathbb{Z})$.

Indeed, the area of the submanifold $\mathbb{C}P^1 = \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n \mid z_2 = \cdots = z_n = 0\}$ is

$$\begin{aligned}\int_{\mathbb{C}P^1} \omega_n &= \int_{w_1 \in \mathbb{C}} \frac{i}{2\pi}\partial\bar{\partial}\log(|w_1|^2 + 1) \\ &= \frac{i}{2\pi}\int_{w_1 \in \mathbb{C}} \frac{1}{(|w_1|^2 + 1)^2} dw_1 \wedge d\bar{w}_1 \\ &= 1.\end{aligned}$$

We define an isomorphism between symplectic manifolds.

Definition 2.4. Let (X_1, ω_1) and (X_2, ω_2) be symplectic manifolds. A diffeomorphism $f \colon X_1 \to X_2$ is called a *symplectomorphism* if $f^*\omega_2 = \omega_1$. If a symplectomorphism exists, (X_1, ω_1) and (X_2, ω_2) are said to be *symplectomorphic*.

Any two symplectic manifolds of the same dimension are locally isomorphic. A precise statement is the following:

Theorem 2.5 (Darboux [2]). *Let (X, ω) be a $2n$-dimensional symplectic manifold. For a point $p \in X$, there exist neighborhoods $U \subset X$ of p and $V \subset \mathbb{R}^{2n}_{\mathrm{st}}$ of the origin such that symplectic manifolds (U, ω) and $(V, \omega_{\mathrm{st}})$ are symplectomorphic. We call U a Darboux chart around p.*

Next, we define a contact manifold.

Definition 2.6. A *contact structure* ξ on a $(2n-1)$-dimensional manifold Y is a maximally non-integrable hyperplane field on Y. The pair (Y, ξ) is called a *contact manifold*.

Example 2.7. The $(2n-1)$-dimensional Euclidean space $\mathbb{R}^{2n-1}$ has the standard contact structure

$$\xi_{\mathrm{st}} = \ker \alpha_{\mathrm{st}}, \quad \alpha_{\mathrm{st}} = dz - \sum_{i=1}^{n-1} y_i \, dx_i,$$

where $(x_1, y_1, \dots, x_{n-1}, y_{n-1}, z)$ are the standard coordinates of $\mathbb{R}^{2n-1}$. We denote by $\mathbb{R}^{2n-1}_{\mathrm{st}}$ the contact manifold $(\mathbb{R}^{2n-1}, \xi_{\mathrm{st}})$ and call it the *standard contact space*.

Example 2.8. For an $(n-1)$-dimensional manifold Λ, its 1-jet bundle $J^1(\Lambda, \mathbb{R}) = T^*\Lambda \times \mathbb{R}$ has the canonical contact structure ξ_{can}. The canonical contact structure ξ_{can} is of the form

$$\xi_{\mathrm{can}} = \ker \left(dz - \sum_{i=1}^{n-1} q_i \, dp_i \right),$$

where z is the coordinate on $\mathbb{R}$, and $(p_1, \dots, p_{n-1})$ are local coordinates on Λ and $(p_1, q_1, \dots, p_{n-1}, q_{n-1})$ are coordinates on the cotangent bundle $T^*\Lambda$.

Locally, any contact structure ξ is defined as the kernel of some 1-form α. A contact structure ξ is globally defined as the kernel of some 1-form α if and only if ξ is coorientable. In this case, we call the 1-form α a *contact form* of the contact structure ξ.

We define an isomorphism and an embedding between contact manifolds.

Definition 2.9. Let (Y_1, ξ_1) and (Y_2, ξ_2) be contact manifolds of the same dimension. A differentiable map $f \colon Y_1 \to Y_2$ is called *contact* if $df^{-1}(\xi_2) = \xi_1$. We call such a diffeomorphism a *contactomorphism* and such an embedding a *contact embedding*, respectively. If a contactomorphism exists, (Y_1, ξ_1) and (Y_2, ξ_2) are said to be *contactomorphic*.

Next, we define a Liouville manifold.

Definition 2.10. Let (X, ω) be a symplectic manifold. A vector field Z on X is called *Liouville* if $\mathcal{L}_Z \omega = \omega$. If a Liouville vector field exists, the pair $(X, \lambda = \iota_Z \omega)$ is said to be a *Liouville manifold*.

A Liouville structure induces a contact structure on the boundary if the Liouville vector field is transverse to the boundary.

Proposition 2.11. *Let $(X, \lambda = \iota_Z \omega)$ be a Liouville manifold with non-empty boundary ∂X. Suppose that the Liouville vector field Z is transverse to the boundary ∂X. Then a 1-form $\alpha = \lambda \,|_{\partial X}$ is a contact form on ∂X.*

Example 2.12. By the definition of the standard symplectic space $\mathbb{R}^{2n}_{\mathrm{st}}$ in Example 2.2, we can check that the vector field

$$Z_{\mathrm{st}} = \frac{1}{2}\sum_{i=1}^{n}\left(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}\right)$$

is a Liouville vector field for the standard symplectic space $\mathbb{R}^{2n}_{\mathrm{st}}$. The unit ball $B^{2n}(1)$ in the Liouville manifold $\mathbb{R}^{2n}_{\mathrm{st}}$ is also a Liouville manifold by restricting the structures. Since the Liouville vector field Z_{st} is outward transverse to the boundary of the unit ball $B^{2n}(1)$, the 1-form $\alpha_{\mathrm{st}} = \lambda_{\mathrm{st}} \mid_{S^{2n-1}}$ defines a contact structure $\xi_{\mathrm{st}} = \ker \alpha_{\mathrm{st}}$ on the unit sphere S^{2n-1}. We call ξ_{st} the *standard contact structure* and α_{st} the *standard contact form*, respectively, and denote by S^{2n-1}_{st} the contact manifold $(S^{2n-1}, \xi_{\mathrm{st}})$.

Next, we define specific submanifolds in symplectic and contact manifolds.

Definition 2.13. Let (X, ω) be a $2n$-dimensional symplectic manifold. An n-dimensional submanifold $L \subset X$ is called *Lagrangian* if the restriction of the symplectic structure vanishes as a 2-form; $\omega \mid_L = 0$. If the symplectic structure is exact; $\omega = d\lambda$, and if the restriction $\lambda \mid_L$ of the Liouville form λ is an exact 1-form on L, then we say the Lagrangian submanifold L is *exact*.

Definition 2.14. Let (X, ω) be a $2n$-dimensional symplectic manifold, L an n-dimensional manifold, and $f\colon L \to X$ an immersion. The immersion f is called *Lagrangian* if $f^*\omega = 0$. If the Lagrangian immersion f is an embedding, we say the embedding f is *Lagrangian*.

Definition 2.15. Let (X, λ) be a $2n$-dimensional exact symplectic manifold, L an n-dimensional manifold, and $f\colon L \to X$ a Lagrangian immersion. The Lagrangian immersion f is called *exact* if the 1-form $f^*\lambda$ is an exact 1-form on L. If the exact Lagrangian immersion f is an embedding, we say the Lagrangian embedding f is *exact*.

We define a formal version of a Lagrangian immersion. It is necessary to state Gromov's h-principle for Lagrangian immersions.

Definition 2.16. Let (X, ω) be a $2n$-dimensional symplectic manifold, L an n-dimensional manifold, and $F\colon TL \to TX$ a bundle monomorphism. The monomorphism F is *Lagrangian* if $\omega \mid_{F(TL)} = 0$.

Since a Lagrangian immersion f induces a Lagrangian monomorphism df, the space of Lagrangian immersions can be viewed as a subspace of the space of Lagrangian monomorphisms. Though the space of Lagrangian monomorphisms is much bigger than the space of Lagrangian immersions, the space of Lagrangian monomorphisms is weakly homotopy equivalent to the space of Lagrangian immersions by Gromov's h-principle for Lagrangian immersions.

Theorem 2.17 (Gromov [8])**.** *Let (X, ω) be a $2n$-dimensional symplectic manifold and L an n-dimensional manifold. If $h\colon L \to X$ is a continuous map with $[h^*\omega] = 0$ in $H^2(L;\mathbb{R})$ and $H\colon TL \to TX$ a Lagrangian monomorphism covering h, then there exists a Lagrangian immersion $f\colon L \to X$ homotopic to h. Moreover,*

(1) *one can choose f to be C^0-close to h;*
(2) *if h is an immersion, then one can choose f to be regularly homotopic to h;*
(3) *if h is a Lagrangian immersion on a neighborhood of a closed ball in L, then one can choose f to be equal to h on the closed ball.*

We define a Lagrangian property and an exact Lagrangian property for regular homotopies and isotopies.

Definition 2.18. Let (X, ω) be a $2n$-dimensional symplectic manifold, L an n-dimensional manifold, and $f_t \colon L \to X$ a regular homotopy, where $t \in [0, 1]$. The regular homotopy $\{f_t\}_{t\in[0,1]}$ is called *Lagrangian* if f_t is a Lagrangian immersion for $t \in [0, 1]$. If a Lagrangian regular homotopy $\{f_t\}_{t\in[0,1]}$ is an isotopy, we say the isotopy $\{f_t\}_{t\in[0,1]}$ is *Lagrangian*. If a symplectic manifold is exact and a Lagrangian regular homotopy $\{f_t\}_{t\in[0,1]}$ is an exact Lagrangian immersion for $t \in [0, 1]$, we say the Lagrangian regular homotopy $\{f_t\}_{t\in[0,1]}$ is *exact*.

We define a Legendrian submanifold in a contact manifold. It is a counterpart of a Lagrangian submanifold in a symplectic manifold.

Definition 2.19. Let (Y, ξ) be a $(2n-1)$-dimensional contact manifold. An $(n-1)$-dimensional submanifold $\Lambda \subset Y$ is called *Legendrian* if Λ is tangent to the contact structure ξ, $T\Lambda \subset \xi$.

Example 2.20. For an $(n-1)$-dimensional manifold Λ, the zero-section Λ of the contact manifold $(J^1(\Lambda, \mathbb{R}), \xi_{\mathrm{can}})$ is a Legendrian submanifold.

This example gives a normal form of a Legendrian submanifold. A precise statement is the following:

Theorem 2.21 (Weinstein [15])**.** *Let (Y, ξ) be a $(2n-1)$-dimensional contact manifold and $\Lambda \subset Y$ a Legendrian submanifold. There exist neighborhoods $U \subset Y$ of Λ and $V \subset J^1(\Lambda, \mathbb{R})$ of the zero-section Λ such that contact manifolds (U, ξ) and (V, ξ_{can}) are contactomorphic.*

2.2. **Loose Legendrian knots.** In this section, we define a loose Legendrian knot following [3, Section 2] with their notations.

First, we observe local models of a Legendrian submanifold. For the contact manifold $\mathbb{R}^{2n-1}_{\mathrm{st}}$, the subspace

$$\begin{aligned}\Lambda_0 =&\{(x_1, y_1, \dots, x_{n-1}, y_{n-1}, z) \in \mathbb{R}^{2n-1} \\ &\mid x_1 = y_2 = y_3 = \cdots = y_{n-1} = z = 0\}\end{aligned}$$

is a Legendrian submanifold. In view of Theorem 2.21, the pair $(\mathbb{R}^{2n-1}_{\mathrm{st}}, \Lambda_0)$ is a local model for any Legendrian submanifold. More precisely, for a $(2n-1)$-dimensional contact manifold (Y, ξ), a Legendrian submanifold $\Lambda \subset Y$, and a point $p \in \Lambda$, there exists a neighborhood $\Omega \subset Y$ of p which admits a contact embedding

$$\Phi \colon (\Omega, \Lambda \cap \Omega) \to (\mathbb{R}^{2n-1}_{\mathrm{st}}, \Lambda_0), \quad \Phi(p) = 0,$$

relative to Legendrian submanifolds.

A loose Legendrian knot will be defined as a Legendrian submanifold with a local model which is obtained by the *stabilization construction* [4] to

$(\mathbb{R}^{2n-1}_{\mathrm{st}}, \Lambda_0)$. Let us recall the stabilization construction. Let $F\colon \mathbb{R}^{2n-1}_{\mathrm{st}} \to \mathbb{R}^{2n-1}_{\mathrm{st}}$ be a contactomorphism given by the formula

$$F(x_1, y_1, \dots, x_{n-1}, y_{n-1}, z) = \left(x_1 + \frac{1}{2}y_1^2, y_1, x_2, y_2, \dots, x_{n-1}, y_{n-1}, z + \frac{1}{3}y_1^3\right).$$

The contactomorphism F maps Λ_0 to the Legendrian submanifold Λ_{cu}, where

$$\Lambda_{\mathrm{cu}} = \left\{ (x_1, y_1, \dots, x_{n-1}, y_{n-1}, z) \in \mathbb{R}^{2n-1} \;\middle|\; x_1 = \frac{1}{2}y_1^2,\ y_2 = \cdots = y_{n-1} = 0,\ z = \frac{1}{3}y_1^3 \right\}.$$

We denote by Γ_{cu} the *front* of Λ_{cu}. Namely, using a projection

$$\pi_F\colon \mathbb{R}^{2n-1}_{\mathrm{st}} \to \mathbb{R}^n : (x_1, y_1, \dots, x_{n-1}, y_{n-1}, z) \mapsto (x_1, \dots, x_{n-1}, z),$$

we define

$$\Gamma_{\mathrm{cu}} = \pi_F(\Lambda_{\mathrm{cu}}) = \{(x_1, \dots, x_{n-1}, z) \in \mathbb{R}^n \mid 9z_1^2 = 8x_1^3\}.$$

The two branches of the front Γ_{cu} are graphs of the functions $\pm h$, where

$$h(x) = h(x_1, \dots, x_{n-1}) = \frac{2\sqrt{2}}{3} x_1^{\frac{3}{2}}$$

defined on the half-space $\mathbb{R}^{n-1}_+ = \{(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} \mid x_1 \geq 0)\}$. The stabilization construction is to change the graph $-h$ as follows [3]. Let U be a closed domain with a smooth boundary contained in the interior $\mathrm{Int}(\mathbb{R}^{n-1}_+)$ in $\mathbb{R}^{n-1}$. We choose a non-negative function $\phi\colon \mathbb{R}^{n-1}_+ \to \mathbb{R}$ such that

- ϕ has compact support in $\mathrm{Int}(\mathbb{R}^{n-1}_+)$;
- $\tilde{\phi} = \phi - 2h$ is Morse;
- $U = \tilde{\phi}^{-1}([0, \infty))$;
- 0 is a regular value of $\tilde{\phi}$.

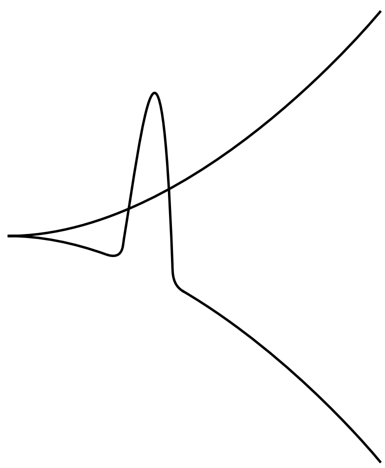

FIGURE 1. The graphs of the functions h and $\phi - h$ [3]

Let $\Gamma^U_{\mathrm{cu}} \subset \mathbb{R}^n$ be the front obtained from Γ_{cu} by replacing the graph $z = -h(x)$ by the graph $z = \phi(x) - h(x)$. Since ϕ has compact support, the front $\Gamma^U_{\rm cu}$ coincides with $\Gamma_{\rm cu}$ outside a compact set. The deformed front $\Gamma^U_{\rm cu}$ defines a Legendrian submanifold $\Lambda^U_{\rm cu} \subset \mathbb{R}^{2n-1}_{\rm st}$ as follows [3]. We recall that the standard contact structure $\xi_{\rm st}$ is defined as the kernel of the 1-form

$$\xi_{\rm st} = \ker \alpha_{\rm st}, \quad \alpha_{\rm st} = dz - \sum_{i=1}^{n-1} y_i \, dx_i,$$

and hence the tangency to the contact structure $\xi_{\rm st}$ is translated to the vanishing of the 1-form $\alpha_{\rm st}$. Thus the branches $\{z = h(x_1, \ldots, x_{n-1})\}$ and $\{z = \phi(x_1, \ldots, x_{n-1}) - h(x_1, \ldots, x_{n-1}))\}$ of the front $\Gamma^U_{\rm cu}$ is lifted to a Legendrian submanifold in $\mathbb{R}^{2n-1}_{\rm st}$ by the system of partial differential equations

$$y_1(x_1, \ldots, x_{n-1}) = \frac{\partial z}{\partial x_1}(x_1, \ldots, x_{n-1}),$$
$$\vdots$$
$$y_{n-1}(x_1, \ldots, x_{n-1}) = \frac{\partial z}{\partial x_{n-1}}(x_1, \ldots, x_{n-1}).$$

We call the lifted Legendrian submanifold $\Lambda^U_{\rm cu}$ defined by the above procedure the *Legendrian lift* of the front $\Gamma^U_{\rm cu}$ [3].

Pulling-back this construction to the standard local model $(\mathbb{R}^{2n-1}_{\rm st}, \Lambda_0)$, we define the stabilization construction for a general Legendrian submanifold. Let $\Lambda^U_0 = F^{-1}(\Lambda^U_{\rm cu})$. For a $(2n-1)$-dimensional contact manifold (Y, ξ), a Legendrian submanifold $\Lambda \subset Y$, and a point $p \in \Lambda$, we take a normal form of Λ around p and a contact embedding as above,

$$\Phi\colon (\Omega, \Lambda \cap \Omega) \to (\mathbb{R}^{2n-1}_{\rm st}, \Lambda_0).$$

Then we replace $\Lambda \cap \Omega$ by Λ^U_0 via the contact embedding Φ. Thus we obtain a Legendrian embedding Λ^U which coincides with Λ outside a compact set. We call Λ^U the *U-stabilization* of Λ in Ω [10].

Definition 2.22 ([10])**.** Let (Y, ξ) be a $(2n-1)$-dimensional contact manifold, $n \geq 3$. A connected Legendrian submanifold Λ of (Y, ξ) is called a *loose Legendrian knot* if it is Legendrian isotopic to the stabilization of another Legendrian submanifold. A Legendrian embedding $f\colon \Lambda \to Y$ is called *loose* if the image $f(\Lambda)$ is a loose Legendrian knot.

Remark 2.23. Murphy's h-principle for loose Legendrian knots [10] is used as one of the main tools in the proofs of Theorems 1.1 and 3.7 of [3] and of Theorem 2.2 of [5]. See [10], [5, Section 2], and [3, Section 2].

2.3. **Lagrangian immersions with a conical point.** In this section, we define a Lagrangian immersion with a conical point following [3, Section 3.2] with their notations.

For a positive integer $m \geq 2$, we denote by $(r, p) \in (0, \infty) \times S^{m-1}$ the polar coordinates in $\mathbb{R}^m \setminus \{0\}$, namely, p is the radial projection of a point to the unit sphere and r is its distance to the origin.

Definition 2.24 ([5])**.** A map $h\colon \mathbb{R}^n \to \mathbb{R}^{2n}_{\rm st}$ is called a *Lagrangian cone* if $h^{-1}(0) = 0$ and if it is given by the formula $h(r, p) = (cr^2, \phi(p))$ in the polar coordinates, where $\phi\colon S^{n-1} \to S^{2n-1}_{\rm st}$ is a Legendrian embedding and c is a positive constant.

Definition 2.25 ([3])**.** Let (X, ω) be a $2n$-dimensional symplectic manifold and L an n-dimensional manifold. A map $f\colon L \to X$ is called a *Lagrangian immersion with a conical point* at $p \in L$, if $f|_{L\setminus\{p\}}$ is an ordinary Lagrangian immersion, and if there exist a chart $U \subset L$ around p and a Darboux chart $V \subset X$ around $f(p)$ such that the map $f\colon U \to V$ is the restriction of a Lagrangian cone near the origin. A Legendrian embedding $\phi\colon S^{n-1} \to S^{2n-1}_{\mathrm{st}}$ corresponding to this cone is called the *link* of the conical point. If a Lagrangian immersion f with a conical point is a topological embedding, we say the map f is a *Lagrangian embedding with a conical point.*

We define a Lagrangian regular homotopy with a conical point. In general, a regular homotopy preserves the algebraic self-intersection number of its time-0 map. We therefore define it to preserve the algebraic self-intersection number.

Definition 2.26 ([3])**.** Let (X, ω) be a $2n$-dimensional symplectic manifold and L an n-dimensional manifold. A homotopy of continuous maps $f_t\colon L \to X$, $t \in [0, 1]$, is called a *Lagrangian regular homotopy with a conical point* at $p \in L$, if each f_t is a Lagrangian immersion with the conical point p, f_t is the identity in some neighborhood of the singular point p, and $\{f_t|_{L\setminus\{p\}}\}_{t\in[0,1]}$ is an ordinary Lagrangian regular homotopy.

Definition 2.27 ([3])**.** Let (X, ω) be a $2n$-dimensional symplectic manifold, L an n-dimensional manifold, and $f\colon L \to X$ a Lagrangian immersion with a conical point at $p \in L$. The map f is called *self-transverse* if $f|_{L\setminus\{p\}}$ is self-transverse and the point $f(p) \in X$ is a simple point. For a self-transverse Lagrangian immersion f with a conical point p, we define the self-intersection number $I(f)$ to be $I\big(f|_{L\setminus\{p\}}\big)$. Then $I(f)$ is invariant under regular homotopies which is the identity near p.

We define the action for a Lagrangian immersion with a conical point and the notion of a Hamiltonian regular homotopy with a conical point. They are defined by taking a Darboux chart around the conical point. The definitions in a Darboux chart are as follows. Let $g\colon \mathbb{R}^n \to \mathbb{R}^{2n}_{\mathrm{st}}$ be a Lagrangian immersion with a conical point $0 \in \mathbb{R}^n$. Suppose that the map g coincides with a Lagrangian cone over a Legendrian link $\phi\colon S^{n-1} \to S^{2n-1}_{\mathrm{st}}$ outside a compact set $K \subset \mathbb{R}^n$. For a path $\gamma\colon [0, 1] \to \mathbb{R}^n$ connecting a point $\gamma(0) \in \mathbb{R}^n \setminus K$ to the origin $\gamma(1) = 0 \in \mathbb{R}^n$, the integral $\int_{g\circ\gamma} \lambda_{\mathrm{st}}$ is independent of the choice of the path γ. The value $\int_{g\circ\gamma} \lambda_{\mathrm{st}}$ is called the *action of the singularity* 0 *with respect to infinity.* We denote it by $a(g, 0 \mid \infty)$. Let $g_t\colon \mathbb{R}^n \to \mathbb{R}^{2n}_{\mathrm{st}}$, $t \in [0, 1]$, be a Lagrangian regular homotopy with a conical point at the origin and which begins at $g_0 = g$ and is compactly supported away from 0. Then the homotopy g_t is *Hamiltonian* if $a(g_t, 0 \mid \infty)$ is independent of $t \in [0, 1]$.

2.4. **Local deformation of Lagrangian immersions.**

Proof of Theorem 1.4. Theorem 1.4 can be proved in a way similar to the proof of Theorem 2.2 of [5] for symplectic manifolds of dimension ≥ 8, by using the following lemma instead of Lemma 4.2 of [5]. □

Lemma 2.28. *Let* $A = [0, 1] \times S^{n-1} \ni (x, z)$*,* $n \geq 3$*, be the annulus with the coordinates* (x, z)*. Take the dual coordinates* (y, u) *on the cotangent bundle*

T^*A so that the canonical Liouville form $\lambda = y\,dx + u\,dz$. Then for any integer $N \geq 10$ there exists a Lagrangian immersion $\Delta\colon A \to T^*A$ with the following properties:

- $\Delta(A) \subset \left\{|y| \leq \frac{12}{N},\ \|u\| \leq \frac{12}{N}\right\}$;
- *Δ coincides with the inclusion of the zero section $j_A\colon A \hookrightarrow T^*A$ near ∂A;*
- *there exists a Lagrangian regular homotopy which is the identity near ∂A and connects j_A to Δ in $\left\{|y| \leq \frac{12}{N},\ \|u\| \leq \frac{12}{N}\right\}$;*
- *for the Δ-image ζ of any path connecting $\{0\} \times S^{n-1}$ to $\{1\} \times S^{n-1}$ in A, $\int_\zeta \lambda = 1$;*
- *the action of any self-intersection point of Δ is $< \frac{2}{N}$;*
- $\mathrm{SI}(\Delta) = 4N^2$.

Proof. We follow the proof of Lemma 4.2 of [5], where Δ was constructed by using the plane curves γ_1, γ_2, and γ_3. We change γ_1 so that $\mathrm{SI}(\Delta) = 4N^2$ as follows.

Consider in $\mathbb{R}^2$ with the coordinates (x, y) the curves $\zeta_k\colon [0,4] \to \mathbb{R}^2$, $k = 1, \dots, N$, defined by

$$\zeta_k(t) = \begin{cases} \left(\frac{1}{12} - \frac{k-1}{N^4}, \left(\frac{6}{N^2} + \frac{2(k-1)}{N^4}\right)t - \frac{k-1}{N^4}\right) & \text{if } 0 \leq t \leq 1, \\ \left(\left(\frac{1}{6} + \frac{2(k-1)}{N^4}\right)t - \frac{1}{12} - \frac{3(k-1)}{N^4}, \frac{6}{N^2} + \frac{k-1}{N^4}\right) & \text{if } 1 \leq t \leq 2, \\ \left(\frac{1}{4} + \frac{k-1}{N^4}, -\left(\frac{6}{N^2} + \frac{2k-1}{N^4}\right)t + \frac{18}{N^2} + \frac{5k-3}{N^4}\right) & \text{if } 2 \leq t \leq 3, \\ \left(-\left(\frac{1}{6} + \frac{2k-1}{N^4}\right)t + \frac{3}{4} + \frac{7k-4}{N^4}, -\frac{k}{N^4}\right) & \text{if } 3 \leq t \leq 4. \end{cases}$$

Then a product $\eta_N = \zeta_1 \cdot \zeta_2 \cdot \dots \cdot \zeta_N\colon [0,4] \to \mathbb{R}^2$ satisfies

$$\int_{\eta_N} y\,dx = \frac{1}{N} + \frac{1}{6N^2} + \frac{6}{N^4} - \frac{14}{3N^5} - \frac{1}{2N^6} + \frac{1}{6N^7}.$$

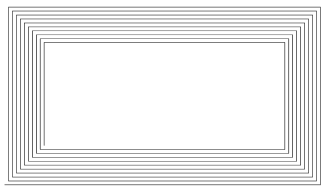

FIGURE 2. The curve η_N for $N = 10$

We denote by $T_\varepsilon\colon \mathbb{R}^2 \to \mathbb{R}^2$ the affine map $(x, y) \mapsto (x, y + \varepsilon)$ and let $l_N\colon [0,3] \to \mathbb{R}^2$ be a piecewise linear embedding connecting four points

$$\begin{aligned} l_N(0) &= \eta_N(4) = \left(\frac{1}{12} - \frac{1}{N^3}, -\frac{1}{N^3}\right), \\ l_N(1) &= \left(\frac{1}{12} - \frac{1}{N^3}, \frac{6}{N^2} + \frac{1}{N^3} - \frac{1}{2N^4}\right), \\ l_N(2) &= \left(\frac{1}{12}, \frac{6}{N^2} + \frac{1}{N^3} - \frac{1}{2N^4}\right), \text{ and} \\ l_N(3) &= T_{\delta_N}(\eta_N(0)) = \left(\frac{1}{12}, \frac{6}{N^2} + \frac{2}{N^3}\right), \end{aligned}$$

where $\delta_N = \frac{6}{N^2} + \frac{2}{N^3}$. We further let $k_N\colon [0,3] \to \mathbb{R}^2$ be a piecewise linear embedding connecting four points

$$\begin{aligned} k_N(0) &= T_{(N-1)\delta_N}(\eta_N(4)) = \left(\frac{1}{12} - \frac{1}{N^3}, \frac{6}{N} - \frac{4}{N^2} - \frac{3}{N^3}\right), \\ k_N(1) &= \left(\frac{1}{12} - \frac{1}{N^3}, \frac{6}{N} + \frac{2}{N^2} - \frac{2}{N^3} - \frac{1}{2N^4}\right), \\ k_N(2) &= \left(\frac{1}{4} + \frac{1}{N^3}, \frac{6}{N} + \frac{2}{N^2} - \frac{2}{N^3} - \frac{1}{2N^4}\right), \text{ and} \\ k_N(3) &= \left(\frac{1}{4} + \frac{1}{N^3}, 0\right). \end{aligned}$$

Then we define a curve $\gamma\colon [0,1] \to \mathbb{R}^2$ by connecting the straight line $\left[0, \frac{1}{12}\right] \times \{0\}$, N-copies $\eta_N, T_{\delta_N}(\eta_N), T_{2\delta_N}(\eta_N), \dots, T_{(N-1)\delta_N}(\eta_N)$ of η_N, $(N-1)$-copies $l_N, T_{\delta_N}(l_N), T_{2\delta_N}(l_N), \dots, T_{(N-2)\delta_N}(l_N)$ of l_N, the curve k_N, and the straight line $\left[\frac{1}{4} + \frac{1}{N^3}, \frac{1}{3}\right] \times \{0\}$. See Figure 3.

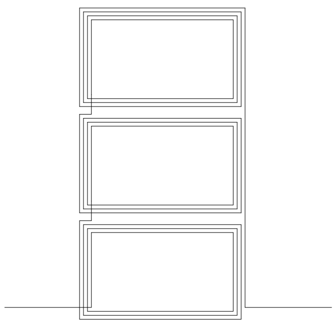

FIGURE 3. The curve γ for $N = 3$

By the construction, the curve γ satisfies the followings:

- $1 - \frac{2}{N} < \int_\gamma y\,dx < 1 + \frac{2}{N}$;
- the action of any self-intersection point of γ is $< \frac{2}{N}$;
- $\mathrm{SI}(\gamma) = N^2$.

Smoothing the corners of γ, we construct an immersed curve γ_1 with transverse self-intersections. We can arrange γ_1 to satisfy the followings:

- $\left| \int_{\gamma_1} y\,dx - 1 \right| < \frac{2}{N}$;
- the action of any self-intersection point of γ_1 is $< \frac{2}{N}$;
- $\mathrm{SI}(\gamma_1) = N^2$;
- the curve γ_1 is contained in the rectangle $\left\{0 \leq x \leq \frac{1}{3},\ |y| \leq \frac{7}{N}\right\}$.

We replace the plane curve γ_1 in the proof of Lemma 4.2 of [5] with the above γ_1. Then we define γ_2 and γ_3, and then Δ in a way similar to the proof of Lemma 4.2 of [5]. □

Remark 2.29. Lemma 4.2 of [5] only asserted the construction of such Δ with $\mathrm{SI}(\Delta) \sim N^3$, and hence Theorem 2.2 of [5] was shown for a symplectic manifold which is the negative completion of a compact symplectic manifold and of dimension $2n \geq 8$ in this way.

Proof of Theorem 1.3. Theorem 1.3 can be proved in a way similar to the proof of Theorem 3.7 of [3] for compact symplectic manifolds of dimension $2n \geq 8$. Considering the Lagrangian immersion $f_0 \mid_{L \setminus \{p\}}$, the case where $|I(f_0)| = 0$ is proved as a corollary of Theorem 1.4. The case where $|I(f_0)| = 1$ is deduced from the case where $|I(f_0)| = 0$ by applying Lemma 3.4 of [3] to f_0 near the point p and by replacing a neighborhood of the point p by the Lagrangian cone over a loose Legendrian knot. □

Proof of Theorem 1.2. Theorem 1.2 can be proved in a way similar to the proof of Theorem 1.1 of [3] for compact symplectic manifolds of dimension $2n \geq 8$. We take a simple point $q \in L$ of f_0. Applying Lemma 3.4 of [3] to f_0 near the point q and replacing a neighborhood of the point q by the Lagrangian cone over a loose Legendrian knot, we can use Theorem 1.3. Again replacing the Lagrangian cone around q by f_0, we obtain the desired Hamiltonian regular homotopy. □

Remark 2.30. Theorems 1.1 and 3.7 of [3] are proved by using Theorem 2.2 of [5]. Thus the theorems were shown for compact symplectic manifolds of dimension $2n \geq 8$.

3. Proof of Theorem 1.1

3.1. **Polterovich's Lagrangian surgery.** In this section, we review the Lagrangian surgery developed by Polterovich [11]. It is one of the main tools for the proof of Theorem 1.1.

Theorem 3.1 (Propositions 1 and 2 of [11])**.** *Let $n \geq 3$ be an odd integer, (X, ω) a $2n$-dimensional symplectic manifold, and L an n-dimensional manifold admitting a self-transverse Lagrangian immersion $L \to X$ with exactly one double point. Then, there exists a Lagrangian embedding $L \# (S^1 \times S^{n-1}) \to X$.*

Remark 3.2. Later, we use Theorem 3.1 with $n = 3$. We recall the outline of the proof. Polterovich constructed the model of a Lagrangian 1-handle $\Gamma = [0, 1] \times S^{n-1}$ which is embedded in $B^{2n}_{\mathrm{st}}(\varepsilon)$ and such that the boundary $\partial\Gamma$ joins $(\mathbb{R}^n \times \{0\}) \setminus B^{2n}(\varepsilon)$ and $(\{0\} \times \mathbb{R}^n) \setminus B^{2n}(\varepsilon)$ smoothly, where ε is a positive real number. Rescaling the handle Γ and taking a Darboux chart around the double point, we can resolve the double point of the Lagrangian immersion $L \to X$ by removing a small neighborhood of the double point and gluing the handle Γ.

3.2. **Lagrangian immersions into $\mathbb{C}P^3$.** In view of Gromov's h-principle, the classification of Lagrangian immersions is reduced to a pure algebro-topological problem. In this section, we characterize the homotopy classes of Lagrangian immersions of closed orientable connected 3-manifolds into $\mathbb{C}P^3$.

First the homotopy classes of continuous maps from a 3-manifold L to the complex projective 3-space $\mathbb{C}P^3$ are classified as follows. We denote by $\gamma_n \to \mathbb{C}P^n$ the tautological line bundle and by $c_1(\gamma_n)$ its first Chern class.

Proposition 3.3. *Let L be a 3-manifold. If $n \geq 2$ then the map*

$$[L, \mathbb{C}P^n] \to H^2(L;\mathbb{Z}) : [h] \mapsto -h^* c_1(\gamma_n)$$

is a bijection.

Proof. It follows from the fact that $\mathbb{C}P^\infty$ is the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$ and $\mathbb{C}P^n \subset \mathbb{C}P^\infty$ is the $2n$-skeleton. □

The two conditions in Theorem 2.17, $[h^*\omega] = 0 \in H^2(L;\mathbb{R})$ and the existence of a Lagrangian monomorphism covering h, are simplified in the case where (X, ω) is the complex projective 3-space $\mathbb{C}P^3$ and L is a closed orientable connected 3-manifold.

Lemma 3.4. *Let L be a closed orientable connected 3-manifold and $h\colon L \to \mathbb{C}P^3$ a continuous map. Then the followings are equivalent.*

1. *There exists a Lagrangian immersion $L \to \mathbb{C}P^3$ which is homotopic to h.*
2. *$h^*c_1(\gamma_3)$ is a 4-torsion element in $H^2(L;\mathbb{Z})$.*

Proof. By Example 2.3, the cohomology class $[\omega_3]$ is the positive generator of the cohomology group $H^2(\mathbb{C}P^3;\mathbb{Z})$, and hence $[\omega_3] = -c_1(\gamma_3) \in H^2(\mathbb{C}P^3;\mathbb{Z})$. For the computation of the first Chern class $c_1(\gamma_3)$, see [9]. The equality $[\omega_3] = -c_1(\gamma_3)$ and the naturality of coefficient homomorphisms imply that $[h^*\omega_3] = 0$ in $H^2(L;\mathbb{R})$ if and only if $h^*c_1(\gamma_3)$ is a torsion element in $H^2(L;\mathbb{Z})$.

Next, we fix a 3-frame of the tangent bundle TL. Let $P \to \mathbb{C}P^3$ be the principal U(3)-bundle associated to the tangent bundle $T\mathbb{C}P^3$. Then we can identify a Lagrangian homomorphism $H\colon TL \to T\mathbb{C}P^3$ covering h with a map $s\colon L \to P$ which is a lift of h. Thus there exists a Lagrangian monomorphism $H\colon TL \to T\mathbb{C}P^3$ covering h if and only if the principal U(3)-bundle $h^*P \to L$ admits a global section. Since $\dim L = 3$, the obstruction for the existence of a global section $L \to h^*P$ is only the first Chern class $c_1(h^*T\mathbb{C}P^3) = h^*c_1(T\mathbb{C}P^3) = -4h^*c_1(\gamma_3)$. □

Remark 3.5. Using part (2) of Theorem 2.17 and taking the connected sum of Whitney sphere, we can see that for the above pair (h, H) and a number $n \in \mathbb{Z}/2$, there exists a self-transverse Lagrangian immersion $f\colon L \to \mathbb{C}P^3$ which is homotopic to h and satisfies $I(f) = n$.

We state another lemma which is used in the proof of Theorem 1.1 for $\mathbb{C}P^3$. It directly follows from Theorem 1.2 and Lemma 3.4.

Lemma 3.6. *Let L be a closed orientable connected 3-manifold and $h\colon L \to \mathbb{C}P^3$ a continuous map with $4h^*c_1(\gamma_3) = 0$ in $H^2(L;\mathbb{Z})$. Then for an arbitrary Lagrangian immersion $f_0\colon L \to \mathbb{C}P^3$ which is homotopic to h, there exists a Lagrangian regular homotopy $f_t\colon L \to \mathbb{C}P^3$, $0 \le t \le 1$, such that f_1 is self-transverse and*

$$\mathrm{SI}(f_1) = \begin{cases} 1, & \text{if } I(f_0) = 1; \\ 2, & \text{if } I(f_0) = 0. \end{cases}$$

3.3. **Proof of Theorem 1.1 for $\mathbb{C}P^3$.** Let L be a closed orientable connected 3-manifold and $f\colon L\#(S^1 \times S^2) \to \mathbb{C}P^3$ a Lagrangian immersion. Lemma 3.4 provides the equality $4f^*c_1(\gamma_3) = 0$ in $H^2(L\#(S^1 \times S^2);\mathbb{Z})$. The Mayer-Vietoris exact sequence for $L\#(S^1 \times S^2) = (L \setminus \mathrm{Int}(D^3)) \cup (S^1 \times S^2 \setminus \mathrm{Int}(D^3))$, where D^3 is a closed 3-disk, gives the isomorphism $H^2(L\#(S^1\times S^2);\mathbb{Z}) \cong H^2(L\setminus \mathrm{Int}(D^3);\mathbb{Z}) \oplus H^2(S^1\times S^2\setminus \mathrm{Int}(D^3);\mathbb{Z})$. Since the isomorphism is induced by the inclusions and $H^2(S^1\times S^2\setminus \mathrm{Int}(D^3);\mathbb{Z}) \cong \mathbb{Z}$, the element $f^*c_1(\gamma_3)$ is of the form

$$f^*c_1(\gamma_3) = (h^*c_1(\gamma_3), 0) \in H^2(L \setminus \mathrm{Int}(D^3);\mathbb{Z}) \oplus H^2(S^1 \times S^2 \setminus \mathrm{Int}(D^3);\mathbb{Z}),$$

where $[h] = [f\,|_{L\setminus \mathrm{Int}(D^3)}] \in [L \setminus \mathrm{Int}(D^3), \mathbb{C}P^3]$.

In the following, we construct a self-transverse Lagrangian immersion of L into $\mathbb{C}P^3$ with exactly one double point and resolve the double point by Theorem 3.1 to obtain the desired Lagrangian embedding. Since $H^2(L \setminus \mathrm{Int}(D^3);\mathbb{Z}) \cong H^2(L;\mathbb{Z})$, we can identify $[L \setminus \mathrm{Int}(D^3), \mathbb{C}P^3]$ with $[L, \mathbb{C}P^3]$. Let $[\hat{h}]$ be the element of $[L, \mathbb{C}P^3]$ which is the extension of $[h]$. We note that $4\hat{h}^*c_1(\gamma_3) = 0$ in $H^2(L;\mathbb{Z})$. Applying Lemmas 3.4 and 3.6 to $\hat{h}$, we obtain a self-transverse Lagrangian immersion $f_1\colon L \to \mathbb{C}P^3$ which is homotopic to $\hat{h}$ and satisfies $\mathrm{SI}(f_1) = 1$. Using Theorem 3.1 to resolve the double point of f_1, we obtain a Lagrangian embedding $g\colon L\#(S^1 \times S^2) \to \mathbb{C}P^3$. We claim that g is homotopic to f. Indeed, it is enough to show that $g^*c_1(\gamma_3) = f^*c_1(\gamma_3)$, and by the definition of h,

$$\begin{aligned} g^*c_1(\gamma_3) &= ((g\,|_{L\setminus \mathrm{Int}(D^3)})^*c_1(\gamma_3), (g\,|_{S^1\times S^2\setminus \mathrm{Int}(D^3)})^*c_1(\gamma_3)) \\ &= ((f_1\,|_{L\setminus \mathrm{Int}(D^3)})^*c_1(\gamma_3), 0) \\ &= (h^*c_1(\gamma_3), 0) \\ &= f^*c_1(\gamma_3). \end{aligned}$$

The proof of Theorem 1.1 for $\mathbb{C}P^3$ is completed. □

3.4. **Lagrangian immersions into $\mathbb{C}P^1 \times \mathbb{C}P^2$.** In this section, we characterize the homotopy classes of Lagrangian immersions of closed orientable connected 3-manifolds into $\mathbb{C}P^1 \times \mathbb{C}P^2$.

We need a classification of homotopy classes of continuous maps from a 3-manifold L to the complex projective line $\mathbb{C}P^1$. In [12], Pontrjagin proved the following.

Theorem 3.7 (Pontrjagin [12])**.** *Let K be a 3-dimensional complex. Then there is a bijection*

$$[K, \mathbb{C}P^1] \approx \coprod_{z^2 \in H^2(K;\mathbb{Z})} H^3(K;\mathbb{Z})/(2z^2 \smile H^1(K;\mathbb{Z})),$$

where $\smile$ *denotes the cup product.*

We recall the correspondence of the elements in Theorem 3.7 for a closed orientable connected 3-manifold L. For an element $[h] \in [L, \mathbb{C}P^1]$, the cohomology class $z^2 \in H^2(L;\mathbb{Z})$ is equal to $-h^* c_1(\gamma_1)$. It represents the primary obstruction for continuous maps from a 3-manifold L to the complex projective line $\mathbb{C}P^1$ to be homotopic. The second obstruction is an element of $H^3(L;\pi_3(\mathbb{C}P^1)) \cong H^3(L;\mathbb{Z})$ modulo $2z^2 \smile H^1(L;\mathbb{Z})$. For continuous maps $f_1 \colon L \to \mathbb{C}P^1$ and $g_1 \colon L \to \mathbb{C}P^1$ with $f_1^* c_1(\gamma_1) = g_1^* c_1(\gamma_1)$, the difference between the homotopy classes $[f_1]$ and $[g_1]$ can be realized by the connected sum of an element of $\pi_3(\mathbb{C}P^1)$ since L is connected.

As in Section 3.2, we simplify the two conditions in Theorem 2.17.

Lemma 3.8. *Let L be a closed orientable connected 3-manifold and $h = (h_1, h_2) \colon L \to \mathbb{C}P^1 \times \mathbb{C}P^2$ a continuous map. Then the followings are equivalent.*

1. *There exists a Lagrangian immersion $L \to \mathbb{C}P^1 \times \mathbb{C}P^2$ which is homotopic to h.*
2. *$h_1^* c_1(\gamma_1)$ and $h_2^* c_1(\gamma_2)$ are torsion elements in $H^2(L;\mathbb{Z})$.*

Proof. By Example 2.3, the cohomology class $[\omega_n]$ is the positive generator of the cohomology group $H^2(\mathbb{C}P^n;\mathbb{Z})$, and hence $[\omega_n] = -c_1(\gamma_n) \in H^2(\mathbb{C}P^n;\mathbb{Z})$. For the computation of the first Chern class $c_1(\gamma_n)$, see [9]. Using the equalities $[\omega_1] = -c_1(\gamma_1)$, $[\omega_2] = -c_1(\gamma_2)$, $c_1(T\mathbb{C}P^1) = -2c_1(\gamma_1)$, and $c_1(T\mathbb{C}P^2) = -3c_1(\gamma_2)$, the proof can be done in a way similar to the proof of Lemma 3.4. □

Remark 3.9. As with Remark 3.5, the following statement holds. For the above pair (h, H) and a number $n \in \mathbb{Z}/2$, one can choose a self-transverse Lagrangian immersion $f \colon L \to \mathbb{C}P^1 \times \mathbb{C}P^2$ which is homotopic to h and satisfies $I(f) = n$.

We state another lemma which is used in the proof of Theorem 1.1 for the product $\mathbb{C}P^1 \times \mathbb{C}P^2$. It directly follows from Theorem 1.2 and Lemma 3.8.

Lemma 3.10. *Let $h \colon L \to \mathbb{C}P^1 \times \mathbb{C}P^2$ be a continuous map of a closed orientable connected 3-manifold L. Suppose that $h_1^* c_1(\gamma_1)$ and $h_2^* c_1(\gamma_2)$ are torsion elements in $H^2(L;\mathbb{Z})$. Then for an arbitrary Lagrangian immersion $f_0 \colon L \to \mathbb{C}P^1 \times \mathbb{C}P^2$ which is homotopic to h, there exists a Lagrangian regular homotopy $f_t \colon L \to \mathbb{C}P^1 \times \mathbb{C}P^2$, $0 \le t \le 1$, such that f_1 is self-transverse and*

$$\mathrm{SI}(f_1) = \begin{cases} 1, & \text{if } I(f_0) = 1; \\ 2, & \text{if } I(f_0) = 0. \end{cases}$$

3.5. **Proof of Theorem 1.1 for $\mathbb{C}P^1 \times \mathbb{C}P^2$.** Let L be a closed orientable connected 3-manifold and $f = (f_1, f_2) \colon L\#(S^1 \times S^2) \to \mathbb{C}P^1 \times \mathbb{C}P^2$ a Lagrangian immersion. By Lemma 3.8, the cohomology classes $f_1^* c_1(\gamma_1)$ and $f_2^* c_1(\gamma_2)$ are torsion elements in $H^2(L\#(S^1 \times S^2);\mathbb{Z})$. As in the proof of Theorem 1.1 for $\mathbb{C}P^3$, the cohomology classes $f_j^* c_1(\gamma_j)$ are of the forms

$$f_j^* c_1(\gamma_j) = (h_j^* c_1(\gamma_j), 0) \in H^2(L \setminus \mathrm{Int}(D^3);\mathbb{Z}) \oplus H^2(S^1 \times S^2 \setminus \mathrm{Int}(D^3);\mathbb{Z}),$$

where $h_j = f_j|_{L\setminus \mathrm{Int}(D^3)}\colon L\setminus \mathrm{Int}(D^3) \to \mathbb{C}P^j$ and $j \in \{1,2\}$. We take a continuous map $\tilde{h} = (\tilde{h}_1, \tilde{h}_2)\colon L \to \mathbb{C}P^1\times\mathbb{C}P^2$ such that $\tilde{h}_j^* c_1(\gamma_j) = h_j^* c_1(\gamma_j)$ via the isomorphism $H^2(L;\mathbb{Z}) \cong H^2(L\setminus\mathrm{Int}(D^3);\mathbb{Z})$, $j\in\{1,2\}$, as follows. In view of Proposition 3.3 and the isomorphism $H^2(L;\mathbb{Z}) \cong H^2(L\setminus\mathrm{Int}(D^3);\mathbb{Z})$, the cohomology class $h_2^* c_1(\gamma_2) \in H^2(L\setminus\mathrm{Int}(D^3);\mathbb{Z})$ determines the unique element in $[L,\mathbb{C}P^2]$. Choosing a representative $\tilde{h}_2$ of the homotopy class, we have $\tilde{h}_2^* c_1(\gamma_2) = h_2^* c_1(\gamma_2)$. Using Theorem 3.7 and the isomorphism $H^2(L;\mathbb{Z}) \cong H^2(L\setminus\mathrm{Int}(D^3);\mathbb{Z})$, we can take a continuous map $\tilde{h}_1\colon L \to \mathbb{C}P^1$ with $\tilde{h}_1^* c_1(\gamma_1) = h_1^* c_1(\gamma_1)$ in a similar way. We note that the equality is equivalent to that the maps h_1 and $\tilde{h}_1$ are homotopic on the 2-skeleton of L.

We construct a self-transverse Lagrangian immersion of L into $\mathbb{C}P^1\times\mathbb{C}P^2$ with exactly one double point. The continuous map $\tilde{h} = (\tilde{h}_1,\tilde{h}_2)\colon L \to \mathbb{C}P^1\times\mathbb{C}P^2$ satisfies the second condition of Lemma 3.8. Thus there exists a self-transverse Lagrangian immersion $\tilde{f}^0\colon L \to \mathbb{C}P^1\times\mathbb{C}P^2$ which is homotopic to $\tilde{h}$ and satisfies $I(\tilde{f}^0) = 1$. Applying Lemma 3.10 to $\tilde{f}^0$, we obtain a self-transverse Lagrangian immersion $\tilde{f}^1\colon L \to \mathbb{C}P^1\times\mathbb{C}P^2$ which is homotopic to $\tilde{h}$ and satisfies $\mathrm{SI}(\tilde{f}^1) = 1$. By Theorem 1.2, there exists a point $p \in \tilde{f}^1(L)$ and a Darboux chart around p, symplectomorphic to the 6-ball $B^6_{\mathrm{st}}(\varepsilon)$ of radius ε, such that the self-intersection point x belongs to $B^6(\varepsilon/2)$ and near the 5-sphere $\partial B^6(\varepsilon/2)$ the Lagrangian immersion f_1 coincides with the Lagrangian cone over a loose Legendrian sphere $\phi = f_1(L)\cap\partial B^6(\varepsilon/2)$ in the 5-sphere $\partial B^6(\varepsilon/2)$ with the standard contact structure.

We construct a Lagrangian embedding of $L\#(S^1\times S^2)$ into $\mathbb{C}P^1\times\mathbb{C}P^2$ which is homotopic to f. Using Theorem 3.1 to resolve the double point x of $\tilde{f}^1 = (\tilde{f}^1_1, \tilde{f}^1_2)$, we obtain a Lagrangian embedding $g = (g_1, g_2)\colon L\#(S^1\times S^2) \to \mathbb{C}P^1\times\mathbb{C}P^2$. Since $g_2^* c_1(\gamma_2) = f_2^* c_1(\gamma_2)$, g_2 is homotopic to f_2. We also have $g_1^* c_1(\gamma_1) = f_1^* c_1(\gamma_1)$. By Theorem 3.7, the difference between the homotopy classes $[g_1]$ and $[f_1]$ in $[L\#(S^1\times S^2), \mathbb{C}P^1]$ can be realized by the connected sum of an element of $\pi_3(\mathbb{C}P^1)$. Therefore, there exists a continuous map $a\colon S^3 \to \mathbb{C}P^1$ such that $g_1\#a$ is homotopic to f_1. We may assume that the disk in $L\#(S^1\times S^2)$ which is removed for the connected sum $g_1\#a$ does not intersect $g^{-1}(B^6(\varepsilon))$. We consider the continuous map $g\#a = (g_1\#a, g_2)\colon L\#(S^1\times S^2) \to \mathbb{C}P^1\times\mathbb{C}P^2$. Since $g\#a$ satisfies the assumption of Lemma 3.8, there exists a self-transverse Lagrangian immersion $g^a\colon L\#(S^1\times S^2) \to \mathbb{C}P^1\times\mathbb{C}P^2$ such that $I(g^a) = 0$ and g^a is homotopic to $g\#a$ relative to $(g\#a)^{-1}(B^6(\varepsilon)) = g^{-1}(B^6(\varepsilon))$. Since $\tilde{f}^1|_{g^{-1}(B^6(\varepsilon))}$ can be glued to $g^a|_{L\#(S^1\times S^2)\setminus g^{-1}(B^6(\varepsilon))}$, we obtain a self-transverse Lagrangian immersion $\tilde{g}^a\colon L \to \mathbb{C}P^1\times\mathbb{C}P^2$ of $I(\tilde{g}^a) = 1$. In the Darboux chart $B^6(\varepsilon)$, the Lagrangian immersion $\tilde{g}^a$ coincides with $\tilde{f}^1$. Hence, we can replace $\tilde{g}^a(L)\cap B^6(\varepsilon/2)$ by the Lagrangian cone over the loose Legendrian knot ϕ. Then we have a Lagrangian immersion $\tilde{g}^0\colon L \to \mathbb{C}P^1\times\mathbb{C}P^2$ with a conical point q such that the Legendrian link at q is loose and $I(\tilde{g}^0) = 0$. Applying Theorem 1.3 to $\tilde{g}^0$, we obtain a Lagrangian regular homotopy $\tilde{g}^t\colon L \to \mathbb{C}P^1\times\mathbb{C}P^2$, $t\in[0,1]$, that is the identity in a neighborhood of the conical point q and that connects $\tilde{g}^0$ to a self-transverse Lagrangian immersion $\tilde{g}^1$ with a conical point q and with $\mathrm{SI}(\tilde{g}^1) = I(\tilde{g}^0) = 0$.

Rescaling $\tilde{g}^a(L) \cap B^6(\varepsilon/2)$ and replacing the Lagrangian cone over the loose Legendrian knot ϕ by the rescaled $\tilde{g}^a(L) \cap B^6(\varepsilon/2)$, we obtain the self-transverse Lagrangian immersion $\tilde{g}^2 \colon L \to \mathbb{C}P^1 \times \mathbb{C}P^2$ with $\mathrm{SI}(\tilde{g}^2) = 1$. Finally, again resolving the double point x of $\tilde{g}^2$ by Theorem 3.1, we obtain a Lagrangian embedding $g^1 \colon L\#(S^1 \times S^2) \to \mathbb{C}P^1 \times \mathbb{C}P^2$ homotopic to g^a relative to a small neighborhood of the point q. In particular, g^1 is homotopic to f. The proof of Theorem 1.1 for $\mathbb{C}P^1 \times \mathbb{C}P^2$ is completed. □

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