博士論文

題目：Some results concerning the range of random walk of several types

（複数の種類のランダムウォークの訪問点に関連する結果）

氏名　　　岡村　和樹

Chapter 1

Introduction and Main results

This thesis consists of four papers and one preprint by the author [81], [82], [83], [84] and [85]. Chapter 1 is devoted to state introduction and main results of these papers. Sections 1.1 to 1.5 consist of [81] to [85], respectively. Sections 1.2, 1.3, and 1.4 relate to each other. Some results and arguments in Sections 1.3 and 1.4 depend on some results and arguments in Section 1.2. Chapters 2 to 6 are devoted to proofs of main results and related results of [81] to [85], respectively.

First, we briefly summarize contents of these papers.

In Section 1.1, we consider the range of random walks up to time n, R_n, on graphs satisfying a certain uniformity condition, characterized by potential theory. Not only all vertex transitive graphs but also many non-regular graphs satisfy the condition. We will state certain weak laws of R_n from above and below. We will also state that there is a graph satisfying the condition such that the sequence of means of R_n/n fluctuates. By noting the construction of the graph, we see that under the condition, the weak laws are best in a sense. We will give proofs of these statements and related results in Chapter 2.

In Section 1.2, we consider functional equations driven by linear fractional transformations, which are special cases of de Rham's functional equations. We consider Hausdorff dimension of the measure whose distribution function is the solution. We will state a necessary and sufficient condition for singularity. We will also state that the functional equations have a relationship with stationary measures. These arise from a research for the range of self-interacting walks on an interval in the author [83], which will be stated in Section 1.3. We will give proofs of these statements and related results in Chapter 3.

In Section 1.3, we consider the range of a one-parameter family of self-interacting walks on the integers up to the time of exit from an interval. We will state a weak convergence of an appropriately scaled range. We will state that the distribution functions of the limits of the scaled range satisfy a certain class of de Rham's functional equations. We examine the regularity of the limits. We will give proofs of these statements and related results in Chapter 4.

In Section 1.4, we define a probability measure on the Cantor space by using two linear fractional transformations consisting of computable real numbers. Specifically, this measure is defined in the same manner as in Section 1.2. We will consider the constructive dimensions for the points which are random with respect to the measure. We will examine limit frequencies of the outcome of 0 for such random points. These results corresponds to the results in Section 1.2. We will give proofs of these statements in Chapter 5.

In Section 1.5, we state quenched large deviations for simple random walk on a certain class of percolations with long-range correlations. This class contains supercritical Bernoulli percolation, the model considered by Drewitz, Ráth, and Sapozhnikov [28], and, the random-cluster model up to the slab critical point. Our result is an extension of Kubota's result [69] for supercritical Bernoulli percolation. We will also state a shape theorem for the chemical distance, which is an extension of Garet and Marchand's result [38] for supercritical Bernoulli percolation. We will give proofs of these statements and related results in Chapter 6.

1.1 On the range of random walk on graphs satisfying a uniform condition

This section will be based on the author's paper [81].

The range of random walk R_n is simply the number of sites which the random walk visits up to time n. We review known results for R_n. If we do not refer state spaces of random walks, they are always the integer lattices $\mathbb{Z}^d$, and, if we do not refer random walks $(S_n)_n$, they are always given by the sum of i.i.d. $\mathbb{Z}^d$-valued random variables $(X_i)_i$ (That is, $S_n = \sum_{i=1}^n X_i$.) such that $E[X_1] = 0$ and it satisfies some integrable conditions. (Integrable conditions for X_i vary depending on settings, but in the following review we do not state them precisely.)

Dvoretzky and Erdös [29] derived the strong law of large numbers (SLLN) for the simple random walks by considering the variances $\mathrm{Var}(R_n)$. An unpublished work by Kesten, Spitzer and Whitman showed it for general random walks by using Birkoff's ergodic theorem. See Spitzer's book [101] p35-40 for details. A discussion Kingman with Spitzer [102], and, Dekking [21] showed the SLLN for the case that $(X_i)_i$ is a stationary ergodic sequence. If $d = 1, 2$, then, $R_n/n \to 0$, almost surely, and therefore the asymptotic for $R_n/E[R_n]$ is considered alternatively. Jain and Pruitt [57], [60] considered the asymptotic for the variances and showedthe SLLN for the general random walks for $d = 1, 2$. Pitt [92] considered the multiple point range on any discrete Abelian group. Derriennic [25] characterized recurrence for random walk on groups by the range of it.

Jain and Orey [56] showed the central limit theorem (CLT) for R_n for strongly transient random walks (including the simple random walk on $d \geq 5$). Jain and Pruitt [58], [61] showed the CLT for $d = 3, 4$ by improving estimates of the variances by [29]. Le Gall [71] showed the CLT for $d = 2$.

An almost sure invariance principle was shown by Hamana [44] for $d \geq 4$, Bass and Kumagai [9] for $d = 3$, and Bass and Rosen [11] for $d = 2$. It is a further extension of the Donsker-type invariance principle by [61]. Jain and Pruitt [59] showed the law of the iterated logarithm (LIL) for $d \geq 4$ and Bass and Kumagai [9] showed the LIL for $d = 2, 3$. Bass, Chen, and Rosen [10] solved some parts which are not investigated in [9] for $d = 2$. A very rough outline of their proofs for $d \geq 3$ is described as follows : First, decompose R_n into a main process and an error process, second, apply the standard LIL (for the sum of i.i.d. random variables or Brownian motion[1]) to the main process, and, finally, give a nice estimate for the error process and then neglect it.

Donsker and Varadhan [27] considered the asymptotic for $\log E[\exp(-\theta R_n)]$ as $n \to \infty$ for a fixed $\theta > 0$. Hamana and Kesten [47], [48] showed the large deviations. [47] deals with $d \geq 2$, and, [48] deals with $d = 1$. Hamana [46] considered the asymptotic for $\Lambda(\theta) := \inf_n \log E[\exp(\theta R_n)]/n$ as $\theta \to 0, \theta > 0$ for $d \geq 2$. Chen [18] obtained moderate and small deviations for $d = 1$. [10][2] shows moderate deviations for $d = 2$.

[27] has an application to knowing about the return probability of random walk on a certain class of discrete groups, specifically, the wreath product of a finitely generated group G with $\mathbb{Z}^d$. This research yields the asymptotic for $\log E[\exp(-\theta R_n)]$ as $n \to \infty$ on more general graphs than $\mathbb{Z}^d$. Erschler [31] showed that for the simple random walk on the Cayley graph of finitely generated groups whose volume growth is polynomial degree d,

[1] The Skorokhod embedding is used in [14]

[2] Chapter 2 in [10] is devoted to review for the random walk range.

$-\log E[\exp(-\theta R_n)] \simeq n^{d/(d+2)}$. (Here $f \simeq g$ means that $c_1 g \le f \le c_2 g$ for some two constants $c_1, c_2 > 0$.) Gibson [39] extended the result and obtained an estimate applicable to graphs satisfying a certain (upper and lower) sub-Gaussian heat kernel estimates. The class of graphs contains not only the Cayley graphs of finitely generated groups but also some fractal graphs (e.g. the graphical Sierpinski gaskets and carpets, Vicsek trees).

The asymptotic of R_n conditioned that the random walk returns to a fixed point at time n has been considered. Hamana [45] showed the weak law of large numbers and the large deviations of R_{2n} conditioned that the simple random walk returns to the origin at time $2n$. Uchiyama [107] derived a detailed asymptotic expansion of the mean of R_n conditioned that the random walk returns to a point which is not too far from the origin at time n. Benjamini, Izkovsky and Kesten [13] showed the conditioned weak law of large numbers on vertex-transitive graphs (in particular Cayley graphs of finitely generated groups) under some assumptions for the return probabilities.

We consider the range of nearest-neighbor random walk on graphs satisfying a uniform condition (U). See Definition 1.1.1. This condition is characterized by potential theory, specifically, effective resistances. Not only all vertex transitive graphs but also some *non-regular* graphs satisfy (U). See Section 2.3 for details. We state certain weak laws of R_n from above and below in Theorem 1.1.2. Under a stronger assumption, a certain strong law holds for R_n. In Theorem 1.1.3, we state the existence of a graph such that it satisfies (U) and the sequence of the mean of R_n/n fluctuates. This construction shows that under (U), the two convergences are best in a sense. The initial motivation for this work is an attempt to extend the results by [13] to non-regular graphs. In Corollary 2.1.4, we extend Theorem 1 in [13] to graphs satisfying (U).

1.1.1 Settings and Main results

Now we describe the settings. Let (X, μ) be a weighted graph. That is, X is an infinite graph and X is endowed with weights μ_{xy} which form a symmetric nonnegative function on $X \times X$ such that $\mu_{xy} > 0$ if and only if x and y are connected. We write $x \sim y$ if x and y are connected by an edge. Let $\mu_x = \sum_{y \in X} \mu_{xy}$, $x \in X$. Let $\mu(A) = \sum_{x \in A} \mu_x$ for $A \subset X$.

In this section we assume that $\sup_{x \in X} \deg(x) < +\infty$ and $0 < \inf_{x,y \in X, x \sim y} \mu_{xy} \le \sup_{x,y \in X, x \sim y} \mu_{xy} < +\infty$. *Whenever we do not refer to weights, we assume that* $\mu_{xy} = 1$ *for any* $x \sim y$.

Let $\{S_n\}_{n \ge 0}$ be a Markov chain on X whose transition probabilities are given by $P(S_{n+1} = y | S_n = x) = \mu_{xy}/\mu_x$, $n \ge 0$, $x, y \in X$. We write

$P = P_x$ if $P(S_0 = x) = 1$. We say that (X, μ) is recurrent (resp. transient) if $(\{S_n\}_{n \ge 0}, \{P_x\}_{x \in X})$ is recurrent (transient). Let the size of random walk range $R_n = |\{S_0, \dots, S_{n-1}\}|$.

Let $T_A = \inf\{n \ge 0 : S_n \in A\}$ and $T_A^+ = \inf\{n \ge 1 : S_n \in A\}$ for $A \subset X$. For $x, y \in X$, $n \ge 0$ and $B \subset X$, let $p_n^B(x, y) = P_x(S_n = y, T_{B^c} > n)/\mu_y$ and $g^B(x, y) = \sum_{n \ge 0} p_n^B(x, y)$. Let $p_n(x, y) = p_n^X(x, y)$ and $g(x, y) = g^X(x, y)$. g is the Green function.

Let $F_1 = \inf_{x \in X} P_x(T_x^+ < +\infty)$ and $F_2 = \sup_{x \in X} P_x(T_x^+ < +\infty)$.

Let d be the graph metric on X. Let $B(x, n) = \{y \in X : d(x, y) < n\}$, $x \in X$, $n \in \mathbb{N}_{\ge 1}$. Let $V(x, n) = \mu(B(x, n))$. Let $\mathcal{E}(f, f) = \frac{1}{2} \sum_{x, y \in X, x \sim y} (f(x) - f(y))^2 \mu_{xy}$ for $f : X \to \mathbb{R}$. Let us define the effective resistance by $R_{\text{eff}}(A, B)^{-1} = \inf\{\mathcal{E}(f, f) : f|_A = 1, f|_B = 0\}$ for $A, B \subset X$ with $A \cap B = \emptyset$.

Let $\rho(x, n) = R_{\text{eff}}(\{x\}, B(x, n)^c)$, $x \in X, n \in \mathbb{N}$. Let $\rho(x) = \lim_{n \to \infty} \rho(x, n)$. If (X, μ) is recurrent (resp. transient), then, $\rho(x) = +\infty$ (resp. $\rho(x) < +\infty$) for any $x \in X$. It is known that

$$g^{B(x,n)}(x, x) = \rho(x, n) = \frac{1}{\mu_x P_x(T_x^+ > T_{B(x,n)^c})}, \; x \in X, n \ge 1.$$

We refer the readers to Kumagai [70] or Peres [88] for the theory of electrical networks. Now we define a uniform condition for weighted graphs.

Definition 1.1.1 (uniform condition)**.** We say that a weighted graph (X, μ) satisfies (U) if $\rho(x, n)$ converges *uniformly* to $\rho(x)$, $n \to \infty$. More precisely, if (X, μ) is recurrent, then, the above uniform convergence means that for any $M > 0$, there exists N such that for any $n \ge N$ and any $x \in X$, $\rho(x, n) \ge M$; if (X, μ) is transient, then, it means that for any $\epsilon > 0$, there exists N such that for any $n \ge N$ and any $x \in X$, $|\rho(x, n) - \rho(x)| \le \epsilon$.

Not only vertex transitive graphs (e.g. $\mathbb{Z}^d$, the M-regular tree T_M, Cayley graphs of finitely generated groups) but also some non-regular graphs (e.g. graphs which are roughly isometric with $\mathbb{Z}^d$, Sierpiński gasket or carpet) satisfy (U) if all weights are equal to 1. See Section 2.3 for detail.

Now we describe the main results.

Theorem 1.1.2. *Let (X, μ) be a weighted graph satisfying (U). Then, for any $x \in X$ and any $\epsilon > 0$, we have that*

$$\lim_{n \to \infty} P_x(R_n \ge n(1 - F_1 + \epsilon)) = 0, \tag{1.1.1}$$

and,

$$\lim_{n \to \infty} P_x(R_n \le n(1 - F_2 - \epsilon)) = 0. \tag{1.1.2}$$

For a fixed $\epsilon > 0$, these convergences are uniform with respect to x. The convergence in (1.1.1) is exponentially fast.

We do not know whether the convergence in (1.1.2) is exponentially fast.

If (X, μ) satisfies an assumption which is stronger than (U), then, certain strong laws hold for R_n, that is,

$$1 - F_2 \le \liminf_{n \to \infty} \frac{R_n}{n} \le \limsup_{n \to \infty} \frac{R_n}{n} \le 1 - F_1, \ P_x\text{-a.s.}$$

See Corollary 2.1.3 for details.

Theorem 1.1.3. *There exists an infinite weighted graph (X, μ) with a reference point o which satisfies $F_1 < F_2$, (U),*

$$\liminf_{n \to \infty} \frac{E_o[R_n]}{n} = 1 - F_2, \ \textit{and,} \ \limsup_{n \to \infty} \frac{E_o[R_n]}{n} = 1 - F_1. \tag{1.1.3}$$

Remark 1.1.4. (i) If X is vertex transitive, then, $F_1 = F_2$ and hence $R_n/n \to 1 - F_1 \in [0, 1]$ in probability. On the other hand, by noting Theorem 1.1.3, there exists an infinite weighted graph (X, μ) with a reference point o which satisfies (U) and R_n/n does *not* converge to any $a \in [0, 1]$ in probability under P_o.
(ii) If we replace F_1 (resp. F_2) with a real number larger than F_1 (resp. smaller than F_2), (1.1.1) (resp. (1.1.2)) fails for a weighted graph in Theorem 1.1.3. In this sense, the convergences (1.1.1) and (1.1.2) are best.

The main difficulty of the proof of Theorem 1.1.2 is that $P_x \neq P_y$ can happen for $x \neq y$. On the other hand, we use the fact in order to show Theorem 1.1.3. We will give proofs of Theorems 1.1.2 and 1.1.3 in Chapter 2.

1.2 Singularity results for functional equations driven by linear fractional transformations

This section will be based on the author's paper [82].

De Rham [96][3] considered the following functional equation.

$$f(x) = \begin{cases} F_0(f(2x)) & 0 \le x \le 1/2 \\ F_1(f(2x-1)) & 1/2 \le x \le 1. \end{cases} \tag{1.2.1}$$

He showed that there exists a unique, continuous and strictly increasing solution f of (1.2.1), if F_0 and F_1 are strictly increasing contractions on $[0, 1]$ such that $0 = F_0(0) < F_0(1) = F_1(0) < F_1(1) = 1$.

[3] An English translation of [96] is included in Edger [30].

Let μ_p, $p \in (0,1)$, be the probability measure on $\{0,1\}$ with $\mu_p(\{0\}) = p$ and $\mu_p(\{1\}) = 1-p$. Let $\mu_p^{\otimes \mathbb{N}}$ be the infinite product measure of μ_p on $\{0,1\}^{\mathbb{N}}$. Let $\varphi : \{0,1\}^{\mathbb{N}} \to [0,1]$ be a function defined by $\varphi((x_n)_n) = \sum_{n=1}^{\infty} x_n/2^n$. Let f_p be the distribution function of the image measure of $\mu_p^{\otimes \mathbb{N}}$ by φ. We see that f_p is a singular function on $[0,1]$ if $p \neq 1/2$ and $f_{1/2}(x) = x$.

De Rham [96] studied f_p as a solution of the functional equation (1.2.1) for $F_0(x) = px$ and $F_1(x) = (1-p)x + p$. This is a typical example of (1.2.1). We will consider this case in Example 3.4.1. In the case, both F_0 and F_1 are affine maps on $\mathbb{R}$. It is natural to consider singularities for the solution of (1.2.1) for more general F_0, F_1. However, it is difficult to see singularities for general cases, because we do not see that what properties of F_0 and F_1 definitely affect singularities.

It seems that the first study of $\{f_p\}_{p \in (0,1)}$ was done by Césaro in 1906 and Heillinger [53] in 1907. $\{f_p\}_{p \in (0,1)\setminus\{1/2\}}$ are examples for singular functions (that is, continuous strictly[4] increasing functions whose derivatives take zero almost everywhere). They are called the *Lebesgue's singular function*. We briefly review studies for singular functions.

Minkowski [78] introduced a function called Minkowski's question-mark function, which is denoted by $?(x)$. We define $? : [0,1] \to [0,1]$ in the following way : $?(0) = 0, ?(1) = 1$, $?((a+c)/(b+d)) = (?(a/b) + ?(c/d))/2$ for two consecutive fractions a/b and c/d in the Farey sequence. We can extend this to a unique continuous function on $[0,1]$. This function is considered by an arithmetic motivation. There is a relationship between the function ? and quadratic irrationals and continued fractions. x is a quadratic irrational if and only if $?(x)$ is a non-dyadic rational, and, x is a rational number if and only if $?(x)$ is a dyadic rational. Singularity for $?(x)$ was shown by Denjoy [22], [23]. Salem [98] used a geometric approach and gave a short proof of the singularity for $?(x)$. This approach yields many results for singular functions, for example, the first edition of Riesz and Sz. Nagy's book [97] Hewitt and Stromberg's book [54], Takacs [105], recently, Paradis, Viader, and Bibiloni [86], [87], Okamoto and Wunsch [80], Fernández-Sánchez, Viader, Paradis, and Díaz Carrillo [33], [34].

De Rham [96] regarded $?(x)$ as a solution of a functional equation. Girgensohn [40] constructed a class of singular functions by using a certain class of functional equations, which includes $?(x)$. In this section, we focus on this approach. A survey by Kairies [62] and the section 14.4 in a monograph by Kannappan [65] study a class of functional equations containing the Lebesgue singular functions and the Minkowski question-mark functions. This approach to singular functions is not taken many times. Some recent

[4]Here we do not deal with non-monotone functions such as the Cantor-type functions.

results concerning this approach are Berg and Krüppel [14], Kawamura [66], Krüppel [68], Protasov [94], and, De Amo, Díaz-Carrillo and Fernández-Sánchez [1]. Conley [19] surveyed Minkowski's question-mark function.

We return to the description of settings. In this section, we consider the equation (1.2.1) under the assumption that both F_0 and F_1 are linear fractional transformations. Let $\Phi(A; z) = \dfrac{az+b}{cz+d}$ for a 2×2 real matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z \in \mathbb{R}$. Let $F_i(x) = \Phi(A_i; x)$, $x \in [0,1]$, $i = 0, 1$, such that 2×2 real matrices $A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, $i = 0, 1$, satisfy the following conditions (A1) - (A3).

(A1) $0 = b_0 < \dfrac{a_0 + b_0}{c_0 + d_0} = \dfrac{b_1}{d_1} < \dfrac{a_1 + b_1}{c_1 + d_1} = 1$.

(A2) $a_i d_i - b_i c_i > 0$, $i = 0, 1$.

(A3) $(a_i d_i - b_i c_i)^{1/2} < \min\{d_i, c_i + d_i\}$, $i = 0, 1$.

The conditions (A1) - (A3) guarantee that $F_i := \Phi(A_i; \cdot)$, $i = 0, 1$, satisfy de Rham's conditions. Let us denote μ_f be the probability measure such that the solution f of (1.2.1) is the distribution function of μ_f.

Let $\alpha = \min\{0, c_0/(d_0 - a_0), c_1/b_1\}$, $\beta = \max\{0, c_0/(d_0 - a_0), c_1/b_1\}$ and $\gamma = 1/\Phi(A_0; 1) > 1$. Let $p_0(x) = (x+1)/(x+\gamma)$ and $p_1(x) = 1 - p_0(x)$ for $x > -\gamma$. Let $s(p) = -p \log p - (1-p)\log(1-p)$ for $p \in [0,1]$. We denote the s-dimensional Hausdorff measure, $s \in (0,1]$, of $E \subset \mathbb{R}$ by $H_s(E)$ and the Hausdorff dimension of E by $\dim_H(E)$.

The following theorems are main results in this section.

Theorem 1.2.1. (1) *There exists a Borel set K_0 such that $\mu_f(K_0) = 1$ and $\dim_H(K_0) \le \max\{s(p_0(y)); y \in [\alpha, \beta]\}/\log 2$.*
(2) *We have that $\mu_f(K) = 0$ for any Borel set K with $\dim_H(K) < \min\{s(p_0(y)); y \in [\alpha, \beta]\}/\log 2$.*

Denote the set of Borel subsets of $[0,1]$ by $\mathcal{B}([0,1])$. Let us define the Hausdorff dimension of μ_f by $\dim_H \mu_f := \inf\{\dim_H(E) : E \in \mathcal{B}([0,1]), \mu(E) > 0\}$. Then, by the above theorem, we have that

$$\frac{\min\{s(p_0(y)); y \in [\alpha, \beta]\}}{\log 2} \le \dim_H \mu_f \le \frac{\max\{s(p_0(y)); y \in [\alpha, \beta]\}}{\log 2}.$$

Theorem 1.2.2. (1) *If both (i) $(c_0 + d_0 - 2a_0)(d_0 - a_0) = a_0 c_0$, and (ii) $(a_1 - 2c_1)(d_1 - 2b_1) = b_1 c_1$ are satisfied, then $\mu_f(dx) = (1 + 2c_0)/(-2c_0 x + 1 + 2c_0)^2 dx$. In particular, μ_f is absolutely continuous.*
(2) *If either (i) or (ii) fails, then there exists a Borel set K_1 such that $\mu_f(K_1) = 1$ and $\dim_H(K_1) < 1$. In particular, μ_f is singular.*

We remark that singularity is robust as a function of a_i, b_i, c_i, d_i, $i = 0, 1$, on the other hand, absolute continuity is not robust.

We will give proofs of Theorems 1.2.1 and 1.2.2 in Chapter 3. The key points of Theorems 1.2.1 and 1.2.2 are Lemmas 3.2.1 and 3.2.3, respectively. Our proofs are quite probabilistic and different from the proofs in [98][5], [40]. The study in this section is motivated by the study of a range of self-interacting random walks on an integer interval. We will state it in the following section.

1.3 On the range of self-interacting random walks on an integer interval

This section will be based on the author's paper [83].

The range of random walk has been studied for a long time. Examining the range at the time the random walk leaves an interval is a simple and natural concern. Recently, Athreya, Sethuraman and Tóth [3] considered questions of this kind. They studied the range, local times and periodicity or parity statistics of some nearest-neighbor Markov random walks up to the time of exit from an interval of N sites. They derived several associated scaling limits as $N \to \infty$ and related the limits to various notions such as the entropy of an exit distribution, generalized Ray-Knight constructions, and Bessel and Ornstein-Uhlenbeck square processes.

Inspired by [3], we consider the ranges of a certain class of self-interacting random walks up to the time of exit from an interval. The study of self-interacting walks originated from the modeling of polymer chains in chemical physics. There are various models in this study. We consider the model defined by Denker and Hattori [24], Hambly, Hattori and Hattori [49], Hattori and Hattori [51], [52]. They constructed a natural one-parameter family of self-repelling and self-attracting walks on $\mathbb{Z}$ and the infinite pre-Sierpiński gasket. It interpolates continuously between self-avoiding walk and simple random walk in the sense of exponents.

In general, most of the studies of self-interacting walks are difficult due to the lack of Markov property, even if they are one-dimensional. In the studies of Markov walks, we can use techniques in analysis, especially, potential theory. However, in the case of non-Markov walks, we cannot use most of the techniques used in the studies of Markov walks. Most of the arguments in [3] depend heavily on the Markov property. Therefore, we have to use

[5]The fact that almost all numbers in $(0, 1)$ is normal is used.

alternative methods for our study. We apply the result in Section 1.2 which considers a certain class of de Rham's functional equations.

Now we state our settings and results briefly. Let W_∞ be the path space of the nearest-neighbor walk starting at 0 on $\mathbb{Z}$. Let $\{P^u\}_{u \geq 0}$ be a one-parameter family of probability measures on W_∞ defined by [26] and [52]. We will give the precise definitions of them in Section 4.1. P^0 defines the self-avoiding walk on $\mathbb{Z}$ and P^1 defines the standard simple random walk. If $u \neq 1$, P^u defines a non-Markov random walk on $\mathbb{Z}$.

Definition 1.3.1. Let $n \in \mathbb{N} = \{1, 2, \dots\}$ and $\omega \in W_\infty$. Let $R_n(\omega)$ be the range of ω up to the time of exit from $\{-2^n, \dots, 2^n\}$, that is,

$$R_n(\omega) = (\text{the number of points which } \omega \text{ visits before it hits the points} \{\pm 2^n\}) .$$

Note that $2^n \leq R_n \leq 2^{n+1} - 1$.

Then, we have the following results which are analogous to [3], Proposition 2.1.

Theorem 1.3.2. (1) *Let $u \geq 0$. Then, the random variables $\{(R_n/2^n) - 1\}_n$ converges weakly to a distribution function f_u on $[0, 1]$, $n \to \infty$.*
(2) *Let $u > 0$. Then f_u satisfies a certain class of de Rham's functional equations [96] :*

$$f(x) = \begin{cases} \Phi(A_{u,0}; f(2x)) & 0 \leq x \leq 1/2 \\ \Phi(A_{u,1}; f(2x-1)) & 1/2 \leq x \leq 1, \end{cases} \tag{1.3.1}$$

where we let

$$\Phi(A; z) = \frac{az + b}{cz + d} \text{ for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ and,}$$

$$A_{u,0} = \begin{pmatrix} x_u & 0 \\ -u^2 x_u^2 & 1 \end{pmatrix}, \; A_{u,1} = \begin{pmatrix} 0 & x_u \\ -u^2 x_u^2 & 1 - u^2 x_u^2 \end{pmatrix}, \; x_u = \frac{2}{1 + \sqrt{1 + 8u^2}}.$$

(3) *Let $\widetilde{P}^u$ be the probability measure on $[0, 1]$ such that its distribution function is f_u. If $u = 1$, $\widetilde{P}^u$ is absolutely continuous with respect to the Lebesgue measure on $[0, 1]$. If $u \neq 1$, $\widetilde{P}^u$ is singular.*

We remark that $\widetilde{P}^0 = \widetilde{P}^0_n = \delta_{\{0\}}$, where δ denotes a point mass.

Let us denote the Hausdorff dimension of $K \subset [0, 1]$ by $\dim_H(K)$. Let $s(p) = -p \log p - (1 - p) \log(1 - p)$ for $p \in [0, 1]$.

If $0 < u < \sqrt{3}$, $(A_{u,0}, A_{u,1})$ satisfies the conditions (A1) - (A3) in Section 1.2 (See also Example 3.4.2), so we can apply the results in Section 1.2 to this case and obtain the following results.

Theorem 1.3.3. (1) *If $u \neq 1$ and $0 < u < \sqrt{3}$, then, then, there exists a Borel set K_0 such that $\widetilde{P}^u(K_0) = 1$ and $\dim_H(K) < 1$.*
(2) *If $0 < u < 1$, then, there exists a Borel set K_0 such that $\widetilde{P}^u(K_0) = 1$ and $\dim_H(K) \leq s(x_u)/\log 2 < 1$. Moreover, $\widetilde{P}^u(K) = 0$ for any Borel set K with $\dim_H(K) < s(2x_u/(1 + x_u))/\log 2$.*

Hence, if $0 < u < 1$,

$$\frac{s(2x_u/(1 + x_u))}{\log 2} \leq \dim_H \widetilde{P}^u \leq \frac{s(x_u)}{\log 2}.$$

We also examine whether $\widetilde{P}^u$ has atoms.

Theorem 1.3.4. (1) *Let $u \leq \sqrt{3}$. Then, $\widetilde{P}^u$ has no atoms.*
(2) *Let $u > \sqrt{3}$. Then, $\widetilde{P}^u(\{x\}) > 0$ for any $x \in D \cap (0, 1]$. Here D is the set of dyadic rationals on $[0, 1]$.*

We will give proofs of Theorems 1.3.2-1.3.4 in Chapter 4.

1.4 Random sequences with respect to a measure defined by two linear fractional transformations

This section will be based on the author's paper [84].

In terms of mathematics and the theory of computing, it is interesting to define random sequences in the space of infinite binary sequences $\{0,1\}^{\mathbb{N}}$ rigorously. Martin-Löf [77] gave a mathematically rigorous definition of randomness of individual infinite binary sequences. By using the theory of computing, he introduced the notion of constructive null sets for a computable measure on $\{0,1\}^{\mathbb{N}}$, and defined the randomness by being not contained in any constructive null set. Levin [72] and Chaitin [17] introduced the notion of the prefix-free Kolmogorov complexity K, which is a modification of the plain complexity considered by Kolmogolov [67]. They defined random sequences by an incompressibility of $K(x \upharpoonright n)$, where $x \upharpoonright n$ denotes the first n bits of $x \in \{0,1\}^{\mathbb{N}}$. This definition is equivalent to Martin-Löf's one if we consider constructive null sets for the Lebesgue measure on $\{0,1\}^{\mathbb{N}}$.

It is natural to consider limit behaviors of $K(x \upharpoonright n)$ for a random sequence x with respect to a more general computable measure. Lutz [75] introduced the notion of constructive dimensions by using supergales, which are generalizations of supermartingales. The constructive dimension is a constructive version of Hausdorff dimension and has properties similar to it. However, the constructive dimension of one point set $\{x\}$ can be positive, whereas the Hausdorff dimension of $\{x\}$ is always 0. There are relationships between constructive dimension and Kolmogorov complexity. In particular, the constructive dimension of $\{x\}$ is equal to the limit infimum of $K(x \upharpoonright n)/n$ as $n \to \infty$. This is a consequence of Levin's result. See Zvonkin and Levin [112], Staiger [104], and Section 13.3 in Downey and Hirschfeldt [26] for details. Constructive dimensions for strongly positive computable measures have been considered in [75], Lutz and Mayordomo [76], and, Nandakumar [79], for example. [75] Theorem 7.7 states that there is a relationship between the constructive dimension of $\{x\}$ for μ-random x and the Shannon entropy if μ belongs to a class of product measures on $\{0,1\}^{\mathbb{N}}$. In [75], this issue is considered for product measures on $\{0,1\}^{\mathbb{N}}$. Under any product measure, the projection mappings from the Cantor space to the coordinates are independent. However, *non-product* measures do not have such nice property.

In this section, we consider the constructive dimensions for μ_{A_0,A_1}-random points, where μ_{A_0,A_1} is a measure on $\{0,1\}^{\mathbb{N}}$ defined by a pair (A_0, A_1) of two linear fractional transformations satisfying some conditions. We define this by following Section 1.2, which considers regularity for a functional equation driven by (A_0, A_1). Such functional equations form a class of de Rham's functional equations [96]. The initial motivation is the range of a class of self-interacting random walks on an integer interval considered in Section 1.3. In Section 1.3, it is shown that an appropriately scaled range of one-dimensional self-interacting (that is, non-Markov) random walks before exiting intervals converges weakly to μ_{A_0,A_1} for some (A_0, A_1). We emphasize that μ_{A_0,A_1} can be a *non-product* measure. In Theorems 1.4.1 and 1.4.2, we will show that there are relationships between the constructive dimensions for μ_{A_0,A_1}-random points and the Shannon entropy. These statements are similar to [75] Theorem 7.7. On the other hand, μ_{A_0,A_1} can also be the Bernoulli measure, which is a very typical product measure on the Cantor space. We will consider this case in Corollary 1.4.4, which is also a corollary of [75], Theorem 7.7. Specifically, Corollary 1.4.4 is the case that all of the biases in the assumption of [75] Theorem 7.7 are the same.

The main ingredients of proofs are to show that if all elements in A_0 and A_1 are computable numbers, then, some μ_{A_0,A_1}-null sets in the arguments in Section 1.2 are constructive. The proofs are different from the proof of [75], Theorem 7.7. Some ingredients of the proofs rely on the results and arguments in [82]. In the proof of Theorem 1.4.2, we will show a lemma which describes the limit frequency of the outcome of 0 for μ_{A_0,A_1}-random points. See Proposition 5.1.3 for details.

Let 1_B be the characteristic function of a set B. Let $\{0,1\}^*$ be the set of finite binary sequences and $\{0,1\}^{\mathbb{N}}$ be the set of infinite binary sequences. Let λ be the empty sequence. We remark that $\lambda \in \{0,1\}^*$. Let $|\sigma|$ be its length for $\sigma \in \{0,1\}^*$. Let $x \restriction n = (x(0), \ldots, x(n-1)) \in \{0,1\}^*$, $n \geq 1$, and, $x \restriction 0 = \lambda$, for $x = (x(k))_{k \in \mathbb{N}} \in \{0,1\}^{\mathbb{N}}$. We define $\sigma \restriction n$ for $\sigma \in \{0,1\}^*$ and $n \leq |\sigma|$ in the same manner. Let $[\sigma] = \{x \in \{0,1\}^{\mathbb{N}} : x \restriction |\sigma| = \sigma\}$ for $\sigma \in \{0,1\}^*$. Let $[V] = \cup_{\sigma \in V} [\sigma]$ for $V \subset \{0,1\}^*$.

Let $D = \mathbb{N}$, $\{0,1\}^*$, or, $\mathbb{N} \times \{0,1\}^*$. We say that a function $f : D \to \mathbb{R}$ is *lower semicomputable* if there exists a computable function $g : D \times \mathbb{N} \to \mathbb{Q}$ such that for any $x \in D$ and $n \in \mathbb{N}$, $g(x,n) \leq g(x,n+1) < f(x)$, and, $f(x) = \lim_{n \to \infty} g(x,n)$. We say that a function $f : D \to \mathbb{R}$ is *computable* if both f and $-f$ are lower semicomputable. We say that a real number $r \in \mathbb{R}$ is *computable* if the constant function $f \equiv r$ is computable. We say that a Borel probability measure μ on $\{0,1\}^{\mathbb{N}}$ is *computable* if $\sigma \mapsto \mu([\sigma])$ is a computable function on $\{0,1\}^*$. We say that a sequence $\{V_m\}_{m \in \mathbb{N}} \subset \{0,1\}^*$ is *uniformly c.e.* if $\{(m, \sigma) \in \mathbb{N} \times \{0,1\}^* : \sigma \in V_m\}$ is a c.e. subset of $\mathbb{N} \times \{0,1\}^*$.

We say that $N \subset \{0,1\}^{\mathbb{N}}$ is an *constructive μ-null* set if there exists a uniformly c.e. sequence $\{U_n\}_n \subset \{0,1\}^*$ such that $N \subset [U_n]$ and $\mu([U_n]) \leq 2^{-n}$, $n \in \mathbb{N}$. Let μ be a Borel probability measure on $\{0,1\}^{\mathbb{N}}$. We say that $x \in \{0,1\}^{\mathbb{N}}$ is *μ-random* if $x \notin N$ for any constructive μ-null set N.

Let μ be a Borel probability measure on $\{0,1\}^{\mathbb{N}}$. We say that a function $d : \{0,1\}^* \to [0, +\infty)$ is a *μ-martingale* if $d(\sigma)\mu([\sigma]) = d(\sigma 0)\mu([\sigma 0]) + d(\sigma 1)\mu([\sigma 1])$ for any $\sigma \in \{0,1\}^*$. We simply call the function $d(\cdot)$ *martingale* if μ is the Lebesgue measure.

Let $s \geq 0$. We say that a function $d : \{0,1\}^* \to [0, +\infty)$ is *s-gale* if $d(\sigma) = 2^{-s}(d(\sigma 0) + d(\sigma 1))$, $\sigma \in \{0,1\}^*$. We note that 1-gales are martingales. Let $\mathcal{G}_c(A)$, $A \subset \{0,1\}^{\mathbb{N}}$, be the set of $s \geq 0$ such that there exists lower semicomputable s-gale d such that $\limsup_{n \to \infty} d(x \restriction n) = +\infty$ for any $x \in A$. Let $\mathrm{cdim}_H(A) = \inf \mathcal{G}_c(A)$. We call this the *constructive dimension* of A.[6] We call the value $\mathrm{cdim}_H(\{x\})$ the *constructive dimension* of a sequence x. We denote this by $\mathrm{cdim}(x)$ for simplicity.

Now we define μ_{A_0, A_1}. Let $\Phi(A; z) = \dfrac{az+b}{cz+d}$ for a real matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z \in \mathbb{R}$ with $cz + d \neq 0$. Let $A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, $i = 0, 1$, be two real matrices satisfying the following conditions.

(A1) $0 = b_0 < (a_0 + b_0)/(c_0 + d_0) = b_1/d_1 < (a_1 + b_1)/(c_1 + d_1) = 1$.

(A2) $\det A_i = a_i d_i - b_i c_i > 0$, $i = 0, 1$.

[6] In [75], Lutz defined constructive dimension by using supergales instead of gales. Fenner [32] and Hitchcock [55] showed that gales can be used to define the dimension.

(A3) $(a_i d_i - b_i c_i)^{1/2} < \min\{d_i, c_i + d_i\}$, $i = 0, 1$.
(A4) a_i, b_i, c_i, d_i, $i = 0, 1$, are computable real numbers.

Let $\alpha = \min\{0, c_0/(d_0 - a_0), c_1/b_1\}$, $\beta = \max\{0, c_0/(d_0 - a_0), c_1/b_1\}$, $\gamma = 1/\Phi(A_0; 1)$, $p_0(t) = (t+1)/(t+\gamma)$, $p_1(t) = 1 - p_0(t)$, $t > -\gamma$. These definitions and assumptions other than (A4) are the same as in Section 1.2.

Let $F(i, \sigma)$ be a partial function on $\mathbb{N} \times \{0,1\}^*$ to $\mathbb{R}$ defined by $F(i, \sigma) = \Phi({}^t A_{\sigma(i-1)}; F(i-1, \sigma))$, $1 \le i \le |\sigma|$, and, $F(0, \sigma) = 0$. Here ${}^t A$ denotes the transpose matrix of A. F is not defined for $i > |\sigma|$. This is a computable function by (A4). We will give a proof of computability of F in Section 5.2. We remark that $F(i, \sigma) = F(i, \sigma \upharpoonright i)$. By using (A1)-(A3), we have that $\alpha \le F(i, \sigma) \le \beta$ for any i, σ, and, $0 < p_0(\alpha) \le p_0(\beta) < 1$. See also Chapter 3, in particular, Lemma 3.2.1. These will be used later.

Let μ_{A_0, A_1} be the probability measure on $\{0,1\}^{\mathbb{N}}$ defined by

$$\mu_{A_0,A_1}([\sigma]) = \prod_{i=0}^{|\sigma|-1} p_{\sigma(i)}(F(i, \sigma)) \text{ for } \sigma \in \{0,1\}^* \text{ with } |\sigma| \ge 1.$$

This is well-defined due to (A1)-(A3), and, a computable measure due to (A4) and the computability of F. This corresponds to μ_f in Section 1.2. We have that

$$0 < \min\{p_0(\alpha), p_1(\beta)\} \le \frac{\mu_{A_0,A_1}([\sigma i])}{\mu_{A_0,A_1}([\sigma])} \le \max\{p_0(\beta), p_1(\alpha)\} < 1$$

for any $\sigma \in \{0,1\}^*$ and $i \in \{0,1\}$. Therefore, any μ_{A_0,A_1} is a well-balanced measure in [32], an additive geometric premeasure in Reimann and Stephan [95], and, a balanced measure in Staiger [103].

Let $\mathcal{H}(p) = -p \log_2 p - (1-p) \log_2 (1-p)$, $p \in [0,1]$. Let M be a total function on $\{0,1\}^*$ defined by $M(\lambda) = 0$, and,

$$M(\sigma) = \sum_{i=0}^{|\sigma|-1} \left(-\log_2 p_{\sigma(i)}(F(i,\sigma)) - \mathcal{H}(p_{\sigma(i)}(F(i,\sigma))) \right) \text{ for } \sigma \text{ with } |\sigma| \ge 1.$$

This is a computable function and a μ_{A_0,A_1}-martingale. This corresponds to $(M_n)_n$ in Chapter 3.

Now we describe our main results.

Theorem 1.4.1. *Let $x \in \{0,1\}^{\mathbb{N}}$ be μ_{A_0,A_1}-random. Then we have the following assertions.*
(1) *If $p_0(\alpha) \ge 1/2$, then,* $\mathrm{cdim}(x) \in [\mathcal{H}(p_0(\beta)), \mathcal{H}(p_0(\alpha))]$.
(2) *If $p_0(\beta) \le 1/2$, then,* $\mathrm{cdim}(x) \in [\mathcal{H}(p_0(\alpha)), \mathcal{H}(p_0(\beta))]$.

Theorem 1.4.2. *Let* $(i) : (c_0 + d_0 - 2a_0)(d_0 - a_0) = a_0c_0$ *and* $(ii) : (a_1 - 2c_1)(d_1 - 2b_1) = b_1c_1$. *Let* $x \in \{0,1\}^{\mathbb{N}}$ *be* μ_{A_0,A_1}*-random. Then we have the following assertions.*
(1) *If both* (i) *and* (ii) *hold, then,* $\mathrm{cdim}(x) = 1$.
(2) *If* (i) *fails or* (ii) *fails, then,* $\mathrm{cdim}(x) < 1$.

Theorems 1.4.1 and 1.4.2 correspond to Theorems 1.2.1 and 1.2.2 respectively.

Now we give two examples of μ_{A_0,A_1}.

Example 1.4.3. (i) Let $p \in (0,1)$ be a computable number. Let $A_0 = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 1-p & p \\ 0 & 1 \end{pmatrix}$. Then, (A_0, A_1) satisfies (A1)-(A4). We have that $\alpha = \beta = 0$, $F(i,\sigma) = 0$, and $\gamma = 1/p$. In this case, μ_{A_0,A_1} is the Bernoulli measure with parameter p.
(ii) Let $A_0 = \begin{pmatrix} 3 & 0 \\ -1 & 6 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & 3 \\ -2 & 5 \end{pmatrix}$. Then, (A_0, A_1) satisfies (A1)-(A4). We also have that $\alpha = -2/3$, $\beta = 0$, $\gamma = 5/3$, $F(1,00) = F(1,01) = -1/6$ and $F(1,10) = F(1,11) = -2/5$. Therefore,

$$\mu_{A_0,A_1}([0]) = \frac{3}{5}, \quad \mu_{A_0,A_1}([1]) = \frac{2}{5},$$

$$\frac{\mu_{A_0,A_1}([00])}{\mu_{A_0,A_1}([0])} = \frac{5}{9}, \quad \frac{\mu_{A_0,A_1}([01])}{\mu_{A_0,A_1}([0])} = \frac{4}{9}, \quad \frac{\mu_{A_0,A_1}([10])}{\mu_{A_0,A_1}([1])} = \frac{9}{19}, \quad \frac{\mu_{A_0,A_1}([11])}{\mu_{A_0,A_1}([1])} = \frac{10}{19}.$$

Therefore, μ_{A_0,A_1} is *not* a product measure.

Applying Theorem 1.4.1 to the case in Example 1.4.3(i), we have the following.

Corollary 1.4.4. *Let* $p \in (0,1)$ *be a computable number. Let* μ_p *be the probability measure on* $\{0,1\}^{\mathbb{N}}$ *such that* $\mu_p([\sigma 0]) = p\mu_p([\sigma])$ *for any* $\sigma \in \{0,1\}^*$. *Let* $x \in \{0,1\}^{\mathbb{N}}$ *be* μ_p*-random. Then,* $\mathrm{cdim}(x) = \mathcal{H}(p)$.

The case in Example 1.4.3(i) is special, because $\alpha = \beta$. If $\alpha = \beta$, then, either $p_0(\alpha) \geq 1/2$ or $p_0(\beta) = p_0(\alpha) \leq 1/2$ happens, and hence, Theorem 1.4.1 is always applicable. However, in general, $\alpha < \beta$ and $p_0(\alpha) < 1/2 < p_0(\beta)$ can happen. Theorem 1.4.1 is not applicable to such cases, but we can apply Theorem 1.4.2 instead.

We can see such example in Example 1.4.3(ii). In that case, $p_0(\alpha) = 1/3$, $p_0(\beta) = 3/5$, and, the condition (ii) in Theorem 1.4.2 fails.

We will give proofs of Theorems 1.4.1 and 1.4.2 in Chapter 5.

1.5 Large deviations for simple random walk on percolations with long-range correlations

This section will be based on the author's preprint [85].

In the research of percolation, it is important to understand geometric properties of clusters and behaviors of random walks on the clusters. In the case of supercritical Bernoulli percolation, Antal and Pisztora [2] gave large deviation estimates for the graph distance of two sites lying in the same cluster. Kubota [69] showed quenched large deviations for the simple random walks on the supercritical Bernoulli percolations on $\mathbb{Z}^d$. The strategy of proof in [69] is similar to the one in Zerner [111], which showed large deviations for random walks in random environment. However, the configurations of percolations fluctuate and the random walk has non-elliptic transition probability. These obstructions were overcome by using [2] Theorem 1.1.

In this section, we state quenched large deviation principles for simple random walk on a certain class of percolations on $\mathbb{Z}^d$ with long-range correlations. Our result is an extension of Kubota's result for supercritical Bernoulli percolations. We can apply this result to the model considered by Drewitz, Ráth, and Sapozhnikov [28]. The model contains supercritical Bernoulli site percolations, random interlacements, the vacant set of random interlacements and the level set of the Gaussian free field. We can also apply this result to the random cluster model up to the slab critical point. See Section 6.1 for details.

Our strategy of proof follows the one in [111] and [69]. In [69], the fact that the Bernoulli measure P_p is a product measure on the configuration space is essentially used in order to show that the Lyapunov exponent $\alpha_\lambda(\cdot)$ is subadditive. However, in the case under consideration, a probability measure $\mathbb{P}$ on the configuration space is *not* necessarily a product measure. In [85], in order to get over this obstruction, we use some ergodic theoretical results for commutative transformations, specifically, Furstenberg and Katznelson's theorem [37] and Tao [106] Theorem 1.1. However, an anonymous referee in a journal to which the author submitted [85] pointed us a smarter proof than the author's proof in [85]. We write the referee's proof here. See Sections 6.3 and 6.4 for the proof of the large deviation. The author's original proof in [85] will be discussed in Section 6.7.

By using the technique, we can also show a shape theorem for the chemical distance, which is an extension of Garet and Marchand [38] Corollary 5.4. We discuss this in Section 6.5.

In Section 6.6, we briefly discuss the asymptotics for the rate function $I(x)$ as $x \to 0$ in the settings of [28].

Now we describe the setting. We consider both bond and site percolations on $\mathbb{Z}^d$, $d \geq 2$. Let $E(\mathbb{Z}^d)$ be the set of edges of the graph $\mathbb{Z}^d$. We write $|x|_\infty = \max_{1 \leq i \leq d} |x_i|$, and, $|x|_1 = \sum_{1 \leq i \leq d} |x_i|$ for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Let $B(z, r) := \{y \in \mathbb{Z}^d : |y - z|_1 \leq r\}$ and $B_\infty(z, r) = \{y \in \mathbb{Z}^d : |y - z|_\infty \leq r\}$. Denote the configuration space by Ω. Ω is $\{0,1\}^{E(\mathbb{Z}^d)}$ or $\{0,1\}^{\mathbb{Z}^d}$, according to bond or site percolation respectively. Denote by ω a configuration on Ω. We write $x \leftrightarrow y$ if x and y are in the same open cluster. Let $D(x, y)$ be the graph distance on the vertices of open clusters between x and y. If x and y are in different open clusters, we let $D(x, y) = +\infty$. We often call D the chemical distance. Let θ_x, $x \in \mathbb{Z}^d$, be the shifts on Ω, that is, $\theta_x(\omega)(\cdot) = \omega(x + \cdot)$.

Assumption 1.5.1. Let $\mathbb{P}$ be a probability measure on Ω. We assume the following conditions :
(i) $\mathbb{P}$ is invariant and ergodic with respect to θ_x for any $x \in \mathbb{Z}^d \setminus \{0\}$.
(ii) $\mathbb{P}$-a.s. ω, there exists a unique infinite open cluster $\mathcal{C}_\infty = \mathcal{C}_\infty(\omega)$.
(iii) There exist constants $c_1, c_2, c_3 > 0$ such that for any $x \in \mathbb{Z}^d$,

$$\mathbb{P}(0 \leftrightarrow x, D(0, x) \geq c_1 |x|_1) \leq c_1 \exp(-c_2 (\log |x|_1)^{1+c_3}).$$

Denote the set of vertices of the infinite connected graph $\mathcal{C}_\infty$ by the same symbol $\mathcal{C}_\infty$. Let the event $\Omega_0 := \{0 \in \mathcal{C}_\infty\}$. Thanks to (i) and (ii), $\mathbb{P}(\Omega_0) > 0$. Let $\overline{\mathbb{P}} := \mathbb{P}(\cdot | \Omega_0)$.

Let $((X_n)_{n \geq 0}, (P^x_\omega)_{x \in \mathcal{C}_\infty(\omega)})$ be the Markov chain on $\mathcal{C}_\infty(\omega)$ whose transition probabilities are given by $P^x_\omega(X_0 = x) = 1$,

$$P^z_\omega(X_{n+1} = x + e | X_n = x) = \frac{1}{2d} \text{ if } |e|_1 = 1 \text{ and } x + e \in \mathcal{C}_\infty(\omega),$$

and,

$$P^z_\omega(X_{n+1} = x | X_n = x) = \frac{1}{2d} |\{e' : |e'|_1 = 1, x + e' \notin \mathcal{C}_\infty(\omega)\}|,$$

for any $x, z \in \mathcal{C}_\infty(\omega)$.

Let H_y be the first hitting time to $y \in \mathcal{C}_\infty$ for $(X_n)_n$. For $x, y, z \in \mathcal{C}_\infty$, we define the Laplace transform of the hitting time by

$$a_\lambda(x, y) = a^\omega_\lambda(x, y) := -\log E^x_\omega[\exp(-\lambda H_y) 1_{\{H_y < +\infty\}}], \ \lambda \geq 0.$$

Let $x \in \mathbb{Z}^d \setminus \{0\}$. Let $T_x : \Omega \to \mathbb{N} \cup \{+\infty\}$ be the map defined by $T_x(\omega) = \inf\{n \geq 1 : nx \in \mathcal{C}_\infty(\omega)\}$, where we let $\inf \emptyset = +\infty$. We define the maps $\Theta_x : \Omega_0 \to \Omega_0$ by $\Theta_x \omega = \theta_x^{T_x(\omega)} \omega$. Due to the Poincaré recurrence theorem (See Theorem 9.2 in Pollicott and Yuri [93]), Θ_x is well-defined up to sets of measure 0 under $\overline{\mathbb{P}}$. By using Lemma 3.3 in Berger and Biskup [15], Θ_x is invertible measure-preserving and ergodic with respect to $\overline{\mathbb{P}}$. Let $T_x^{(n)} := \sum_{k=0}^{n-1} T_x \circ \Theta_x^k$.

Theorem 1.5.2 (Existence of the Lyapunov exponents)**.** *Assume that $\mathbb{P}$ satisfies Assumption 1.5.1. Let $\lambda \geq 0$. Then, there exists a function $\alpha_\lambda(\cdot)$ on $\mathbb{Z}^d$such that $\alpha_\lambda(0) = 0$ and for any $x \in \mathbb{Z}^d \setminus \{0\}$,*

$$\lim_{n\to\infty} \frac{a_\lambda(0, T_x^{(n)} x)}{T_x^{(n)}} = \alpha_\lambda(x),\ \overline{\mathbb{P}}\text{-a.s.}$$

We can extend α_λ uniquely to a continuous function on $\mathbb{R}^d$. $\alpha_\lambda(\cdot)$ satisfies the following properties : for any $x, y \in \mathbb{R}^d$ and for any $q \in (0, +\infty)$, $\alpha_\lambda(qx) = q\alpha_\lambda(x)$, $\alpha_\lambda(x + y) \leq \alpha_\lambda(x) + \alpha_\lambda(y)$, and, $\lambda|x|_1 \leq \alpha_\lambda(x) \leq (\lambda + \log(2d))C\mathbb{P}(\Omega_0)|x|_1$, where C is a constant which does not depend on (λ, x).

$\alpha_\lambda(\cdot)$ is called the *Lyapunov exponent.* This is an extension of Theorem 1.1 in [69] and this is the key ingredient of the proof of the following result.

Theorem 1.5.3 (Quenched large deviation principles)**.** *Assume that $\mathbb{P}$ satisfies Assumption 1.5.1. Then, the law of X_n/n obeys the following large deviation principles with rate function $I(x) = \sup_{\lambda\geq 0}(\alpha_\lambda(x) - \lambda)$, $x \in \mathbb{R}^d$, where $\alpha_\lambda(\cdot)$ is the function on $\mathbb{R}^d$ in Theorem 1.5.2.*
(1) *Upper bound : For any closed set A in $\mathbb{R}^d$, we have $\overline{\mathbb{P}}$-a.s. ω,*

$$\limsup_{n\to\infty} \frac{\log P_\omega^0(X_n/n \in A)}{n} \leq -\inf_{x\in A} I(x). \tag{1.5.1}$$

(2) *Lower bound : For any open set B in $\mathbb{R}^d$, we have $\overline{\mathbb{P}}$-a.s. ω,*

$$\liminf_{n\to\infty} \frac{\log P_\omega^0(X_n/n \in B)}{n} \geq -\inf_{x\in B} I(x). \tag{1.5.2}$$

The rate function I is $+\infty$ outside the ball of radius 1 in the l^1-norm.

In Section 6.1, we give examples of models satisfying Assumption 1.5.1. In Section 6.2, we give some preliminaries. In Sections 6.3 and 6.4, we show Theorems 1.5.2 and 1.5.3 respectively. In Section 6.5, we discuss a shape theorem for the chemical distance. In Section 6.6, we briefly discuss some properties for the rate function, which is based on [99]. In Section 6.7, we discuss the author's original proof of Theorem 1.5.2, which is described in

[85].

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Chapter 2

On the range of random walk on graphs satisfying a uniform condition

This chapter will be based on [81]. In Sections 2.1 and 2.2, we give the proof of Theorems 1.1.2 and 1.1.3, respectively. However, the proof of Theorem 1.1.3 is simpler than [81]. In Section 2.3, we give some examples for graphs satisfying (U).

2.1 Proof of Theorem 1.1.2

First, we show the following lemma.

Lemma 2.1.1. *Let (X, μ) be a weighted graph satisfying (U). Then,*

$$\lim_{n \to \infty} \sup_{x \in X} P_x(n < T_x^+ < +\infty) = 0.$$

Proof. By [70] Theorem 2.2.5, $\rho(x,n)^{-1} = \mu_x P_x(T_x^+ > T_{B(x,n)^c})$, $x \in X$, $n \geq 1$. Letting $n \to \infty$, we have $\rho(x)^{-1} = \mu_x P_x(T_x^+ = +\infty)$.

Since $\rho(x,1)^{-1} = \mu_x$,

$$\begin{aligned} P_x(T_{B(x,n)^c} < T_x^+ < +\infty) &= \mu_x^{-1}(\rho(x,n)^{-1} - \rho(x)^{-1}) \\ &\leq \mu_x(\rho(x) - \rho(x,n)). \end{aligned}$$

Since $\mu_x \leq \sup_{y \in X} \deg(y) \sup_{y,z \in X, y \sim z} \mu_{yz} < +\infty$ and (X, μ) satisfies (U), we see that

$$\lim_{n \to \infty} \sup_{x \in X} P_x(T_{B(x,n)^c} < T_x^+ < +\infty) = 0. \tag{2.1.1}$$

Since $\sup_x \deg(x) < +\infty$ and $\sup_{y,z \in X, y \sim z} \mu_{yz} < +\infty$, we have that $\sup_{x \in X} V(x,n) < +\infty$, $n \geq 1$. Since $\rho(x,n)^{-1} \geq \inf_{y,z \in X, y\sim z} \mu_{yz}/n > 0$, we have that $\sup_{x \in X} \rho(x,n) < +\infty$, $n \geq 1$.

Thus we can let $f(n) = \sup_{x\in X} \rho(x,n) \sup_{x \in X} V(x,n)$, $n \geq 1$.

By [70] Lemma 4.1.1(v),

$$P_x(T_{B(x,n)^c} \geq nf(n)) \leq \frac{E_x[T_{B(x,n)^c}]}{nf(n)} \leq \frac{\rho(x,n)V(x,n)}{nf(n)} \leq \frac{1}{n}.$$

Hence,

$$\lim_{n\to\infty} \sup_{x \in X} P_x(T_{B(x,n)^c} \geq nf(n)) = 0. \tag{2.1.2}$$

We have that

$$P_x(nf(n) < T_x^+ < +\infty) \leq P_x(T_{B(x,n)^c} < T_x^+ < +\infty) + P_x(T_{B(x,n)^c} \geq nf(n)).$$

By noting (2.1.1) and (2.1.2), we have that

$$\lim_{n\to\infty} \sup_{x \in X} P_x(nf(n) < T_x^+ < +\infty) = 0.$$

This completes the proof of Lemma 2.1.1. □

Let $Y_{i,j}$ be the indicator function of $\{S_i \neq S_{i+k}$ for any $1 \leq k \leq j\}$. Let $Y_{i,\infty}$ be the indicator function of $\{S_i \neq S_{i+k}$ for any $k \geq 1\}$.

Proof of Theorem 1.1.2. We show this assertion in a manner which is partially similar to the proof of Theorem 1 in Benjamini, Izkovsky and Kesten [13]. However $P_x \neq P_y$ can happen for $x \neq y$ and hence the random variables $\{Y_{k+aM,M}\}_{a\in\mathbb{N}}$ are *not* necessarily independent. The details are different from the proof of Theorem 1 in [13].

First, we will show (1.1.1). Let $\epsilon > 0$. Let M be a positive integer such that $\sup_{x\in X} P_x(M < T_x^+ < +\infty) < \epsilon/4$. We can take such M by Lemma 2.1.1.

By counting the random walk range up to time n from the terminal time n (it is called a last exit decomposition in [13]),

$$R_n = 1 + \sum_{i=0}^{n-2} Y_{i,n-1-i} \leq M + \sum_{i=0}^{n-1-M} Y_{i,n-1-i} \leq M + \sum_{i=0}^{n-1-M} Y_{i,M}.$$

Hence for $n > 2M/\epsilon$,

$$\begin{aligned}
P_x(R_n \geq n(1 - F_1 + \epsilon)) &\leq P_x\left(\sum_{i=0}^{n-1-M} Y_{i,M} > n\left(1 - F_1 + \frac{\epsilon}{2}\right)\right) \\
&= P_x\left(\sum_{a=0}^{M} \sum_{i \equiv a \bmod (M+1)} Y_{i,M} > n\left(1 - F_1 + \frac{\epsilon}{2}\right)\right) \\
&\leq \sum_{a=0}^{M} P_x\left(\sum_{i \equiv a \bmod (M+1)} Y_{i,M} > \frac{n}{M+1}\left(1 - F_1 + \frac{\epsilon}{2}\right)\right).
\end{aligned}$$

Therefore it is sufficient to show that for each $a \in \{0, 1, \ldots, M\}$,

$$P_x\left(\sum_{i \equiv a \bmod (M+1)} Y_{i,M} > \frac{n}{M+1}\left(1 - F_1 + \frac{\epsilon}{2}\right)\right) \to 0,\ n \to \infty, \text{ exponentially fast.} \tag{2.1.3}$$

For any $t > 0$, we have that

$$\begin{aligned}
&P_x\left(\sum_{i \equiv a \bmod (M+1)} Y_{i,M} > \frac{n}{M+1}\left(1 - F_1 + \frac{\epsilon}{2}\right)\right) \\
&\leq \exp\left(-t\frac{n}{M+1}\left(1 - F_1 + \frac{\epsilon}{2}\right)\right) E_x\left[\exp\left(t \sum_{i \equiv a \bmod (M+1)} Y_{i,M}\right)\right].
\end{aligned} \tag{2.1.4}$$

By using the Markov property of $\{S_n\}_n$,

$$\begin{aligned}
E_x\left[\exp\left(t \sum_{i \equiv a \bmod (M+1)} Y_{i,M}\right)\right] &= E_x\left[\prod_{i \equiv a \bmod (M+1)} \exp(tY_{i,M})\right] \\
&\leq \left(\sup_{y \in X} E_y[\exp(tY_{0,M})]\right)^{n/(M+1)} \\
&= \left(1 + (\exp(t) - 1) \sup_{y \in X} P_y(T_y^+ > M)\right)^{n/(M+1)}.
\end{aligned}$$

By noting the definition of M and F_1,

$$\begin{aligned}
\sup_{y \in X} P_y(T_y^+ > M) &\leq \sup_{y \in X} P_y(M < T_y^+ < +\infty) + \sup_{y \in X} P_y(T_y^+ = +\infty) \\
&\leq \frac{\epsilon}{4} + 1 - F_1.
\end{aligned}$$

Hence, for any $t \geq 0$ and $x \in X$,

$$E_x \left[\exp \left(t \sum_{i \equiv a \bmod (M+1)} Y_{i,M} \right) \right] \leq \left(1 + (\exp(t) - 1) \left(\frac{\epsilon}{4} + 1 - F_1 \right) \right)^{n/(M+1)}.$$

Hence, the right hand side of the inequality (2.1.4) is less than or equal to

$$\left[\exp \left(-t \left(1 - F_1 + \frac{\epsilon}{2} \right) \right) \left\{ 1 + (\exp(t) - 1) \left(\frac{\epsilon}{4} + 1 - F_1 \right) \right\} \right]^{n/(M+1)}.$$

It is easy to see that for sufficiently small $t_1 = t_1(F_1, \epsilon) > 0$,

$$\left\{ 1 + (\exp(t_1) - 1) \left(\frac{\epsilon}{4} + 1 - F_1 \right) \right\} < \exp \left(t_1 \left(1 - F_1 + \frac{\epsilon}{2} \right) \right).$$

Thus we have (2.1.3) and this convergence is uniform with respect to x. This completes the proof of (1.1.1).

Second, we will show (1.1.2). Let $\epsilon > 0$. Let M be a positive integer. By considering a last exit decomposition,

$$\begin{aligned} P_x(R_n \leq n(1 - F_2 - \epsilon)) &= P_x(n - R_n \geq n(F_2 + \epsilon)) \\ &= P_x \left(\sum_{i=0}^{n-2} (1 - Y_{i,n-1-i}) \geq n(F_2 + \epsilon) \right) \\ &\leq P_x \left(\sum_{i=0}^{n-2} (1 - Y_{i,\infty}) \geq n(F_2 + \epsilon) \right). \end{aligned}$$

Now we have $1 - Y_{i,\infty} = 1 - Y_{i,M} + Y_{i,M} - Y_{i,\infty}$ and

$$\begin{aligned} P_x \left(\sum_{i=0}^{n-2} (1 - Y_{i,\infty}) \geq n(F_2 + \epsilon) \right) &\leq P_x \left(\sum_{i=0}^{n-2} (1 - Y_{i,M}) \geq n \left(F_2 + \frac{\epsilon}{2} \right) \right) \\ &\quad + P_x \left(\sum_{i=0}^{n-2} (Y_{i,M} - Y_{i,\infty}) \geq \frac{n\epsilon}{2} \right). \end{aligned} \tag{2.1.5}$$

We have that $Y_{i,M} - Y_{i,\infty}$ is the indicator function of

$$\{S_i \neq S_{i+k} \text{ for any } 1 \leq k \leq M, \ S_i = S_{i+k} \text{ for some } k > M\},$$

and hence, $E_x[Y_{i,M} - Y_{i,\infty}] \leq \sup_{y \in X} P_y(M < T_y^+ < +\infty)$.

Then for any n,

$$P_x\left(\sum_{i=0}^{n-2}(Y_{i,M}-Y_{i,\infty})\geq\frac{n\epsilon}{2}\right)\leq\frac{2}{n\epsilon}\sum_{i=0}^{n-2}E_x[Y_{i,M}-Y_{i,\infty}]$$
$$\leq\frac{2}{\epsilon}\sup_{y\in X}P_y(M<T_y^+<+\infty).\qquad(2.1.6)$$

On the other hand,

$$P_x\left(\sum_{i=0}^{n-2}(1-Y_{i,M})\geq n\left(F_2+\frac{\epsilon}{2}\right)\right)$$
$$\leq\sum_{a=0}^{M}P_x\left(\sum_{i\equiv a \bmod M+1}(1-Y_{i,M})\geq\frac{n}{M+1}\left(F_2+\frac{\epsilon}{2}\right)\right).$$

By using the Markov property of $\{S_n\}_n$, we have that for any $t>0$ and any $a\in\{0,1,\dots,M\}$,

$$P_x\left(\sum_{i\equiv a \bmod M+1}(1-Y_{i,M})\geq\frac{n}{M+1}\left(F_2+\frac{\epsilon}{2}\right)\right)$$
$$\leq\exp\left(-t\frac{n}{M+1}\left(F_2+\frac{\epsilon}{2}\right)\right)E_x\left[\prod_{i\equiv a \bmod M+1}\exp(t(1-Y_{i,M}))\right]$$
$$\leq\exp\left(-t\frac{n}{M+1}\left(F_2+\frac{\epsilon}{2}\right)\right)\left(\sup_{y\in X}E_y\left[\exp(t(1-Y_{0,M}))\right]\right)^{n/(M+1)}$$
$$=\left[\exp\left(-t\left(F_2+\frac{\epsilon}{2}\right)\right)\left\{1+(\exp(t)-1)\sup_{y\in X}P_y(T_y^+\leq M)\right\}\right]^{n/(M+1)}.$$

Since $\sup_{y\in X}P_y(T_y^+\leq M)\leq F_2$, we have that for sufficiently small $t_2=t_2(F_2,\epsilon)>0$,

$$\exp\left(-t_2\left(F_2+\frac{\epsilon}{2}\right)\right)\left\{1+(\exp(t_2)-1)\sup_{y\in X}P_y(T_y^+\leq M)\right\}<1.$$

Therefore for any $a\in\{0,1,\dots M\}$,

$$P_x\left(\sum_{i\equiv a \bmod M+1}(1-Y_{i,M})\geq\frac{n}{M+1}\left(F_2+\frac{\epsilon}{2}\right)\right)\to 0, n\to\infty.$$

Thus we see that

$$P_x\left(\sum_{i=0}^{n-2}(1-Y_{i,M}) \geq n\left(F_2+\frac{\epsilon}{2}\right)\right) \to 0, n \to \infty. \qquad (2.1.7)$$

This convergence is uniform with respect to x.

By using (2.1.5), (2.1.6) and (2.1.7), we have

$$\limsup_{n\to\infty} P_x(R_n \leq n(1-F_2-\epsilon)) \leq \frac{2}{\epsilon}\sup_{y\in X} P_y(M < T_y^+ < +\infty).$$

By letting $M \to \infty$, it follows from Lemma 2.1.1 that

$$\limsup_{n\to\infty} P_x(R_n \leq n(1-F_2-\epsilon)) = 0.$$

This convergence is uniform with respect to x. This completes the proof of (1.1.2). □

Remark 2.1.2. If $F_1 = F_2$, then (1.1.2) is easy to see by noting (1.1.1) and $E_x[R_n] \geq n(1-F_2)$, $n \geq 1$, $x \in X$.

Corollary 2.1.3. *If* $\sup_x P_x(M < T_x^+ < +\infty) = O(M^{-1-\delta})$ *for some* $\delta > 0$*, then, certain strong laws hold. More precisely, for any* $x \in X$*,*

$$1-F_2 \leq \liminf_{n\to\infty}\frac{R_n}{n} \leq \limsup_{n\to\infty}\frac{R_n}{n} \leq 1-F_1,\ P_x\text{-}a.s.$$

Proof. By noting the Borel-Cantelli lemma, we see that it suffices to show that for any $x \in X$ and $\epsilon > 0$,

$$\sum_{n\geq 1} P_x(R_n \geq n(1-F_1+\epsilon)) < +\infty, \qquad (2.1.8)$$

and,

$$\sum_{n\geq 1} P_x(R_n \leq n(1-F_2-\epsilon)) < +\infty. \qquad (2.1.9)$$

(2.1.8) follows from that the convergence (1.1.1) is exponentially fast.

By noting (2.1.5), (2.1.6) and (2.1.7), we have that there exists $a = a(F_2,\epsilon) \in (0,1)$ such that for any n and $M < n$,

$$P_x(R_n \leq n(1-F_2-\epsilon)) \leq \frac{2}{\epsilon}O(M^{-1-\delta}) + a^{n/(M+1)}.$$

If we let $M = n^{1-\delta/2} - 1$ for each n, then, we see (2.1.9). □

The assumption in Corollary 2.1.3 is used only in the proof of the lower bound. The upper bound holds whenever the graph satisfies (U).

Since the convergence in (1.1.1) is exponentially fast, we can extend Theorem 1 in [13], which considers the range of the random walk bridge on vertex transitive graphs.

Corollary 2.1.4. *Let (X, μ) be a weighted graph satisfying (U). Let $x \in X$. We assume that $\limsup_{n\to\infty} P_x(S_{2n} = x)^{1/n} = 1$. Let $\epsilon > 0$. Then,*

$$\lim_{n\to\infty} P_x\left(R_n \geq n(1 - F_1 + \epsilon) | S_n = x\right) = 0.$$

The limit is taken on n such that $P_x(S_n = x) > 0$. This convergence is exponentially fast.

2.2 Proof of Theorem 1.1.3

To begin with, we state a very rough sketch of the proof.

Let N_1, N_2 be integers such that $3 \leq N_1 < N_2$. First, we prepare a finite tree with degree N_1 and denote it $X^{(1)}$. Second, we surround $X^{(1)}$ with finite trees with degree N_2. We denote the graph we obtain by $X^{(2)}$. Third, we surround $X^{(2)}$ with finite trees with degree N_1. We denote the graph we obtain by $X^{(3)}$. Repeating this construction, we obtain an increasing sequence of finite trees $(X^{(n)})_n$. $X^{(2n+1)} \setminus X^{(2n)}$ (resp. $X^{(2n+2)} \setminus X^{(2n+1)}$) is a ring-like object consisting of the N_1 (resp. N_2) -trees. Let r_{2n+1} (resp. r_{2n+2}) be the width of the ring. Assume $r_i \ll r_{i+1}$ for any i. Let X be the infinite graph of the limit of $(X^{(n)})_n$. This satisfies (U). Lemma 2.2.2 in below states this formally. X also satisfies $F_1 < F_2$ and (1.1.3), because $r_i \ll r_{i+1}$ for any i.

In this section, we assume that any weight is equal to 1, that is, $\mu_{xy} = 1$ for any $x \sim y$. In the following proof, we use the theory of flow. See [88] for terminologies.

Let X be an infinite tree. For a connected subgraph Y of X, we denote the restriction of $\mathcal{E}$, deg, and ρ to Y by $\mathcal{E}_Y$, $\deg_Y$, and ρ_Y respectively.

Let $x \in X$. Let

$$D_x(y) = \{z \in X : \text{the path between } x \text{ and } z \text{ contains } y\}, \; y \in X.$$

We remark that $y \in D_x(y)$ and $D_x(x) = X$. Let $I_x(y, n) = \rho_{D_x(y)}(y, n)^{-1}$, $y \in X$. We remark that $I_x(x, n) = \rho_X(x, n)^{-1} = \rho(x, n)^{-1}$. By using the series and parallel laws for conductances, we see the following easily.

Lemma 2.2.1. *Let X be an infinite tree. Let $x, y \in X$. Let $n \geq 1$. Let y_i, $1 \leq i \leq \deg_{D_x(y)}(y)$, be the neighborhoods of y in $D_x(y)$. Then,*

$$I_x(y, n+1) = \sum_{i=1}^{\deg_{D_x(y)}(y)} \frac{I_x(y_i, n)}{1 + I_x(y_i, n)}.$$

Lemma 2.2.2. *Let X be an infinite tree with minimal degree at least 3. Then, X satisfies (U).*

Proof. Let $x \in X$ and $n \geq 1$. Let x_i, $1 \leq i \leq \deg_X(x)$, be the neighborhoods of x. Let X_i be the subtree of X which consists of a single edge from x to x_i and $D_x(x_i) \cap B(x, n+1)$. Then, by using the series and parallel laws, we see that the conductance of X_i is $I_x(x_i, n-1)/(1 + I_x(x_i, n-1))$. By using Lemma 2.2.1 and that the minimal degree of X is at least 3, we see that $I_x(x_i, n-1) \geq 1$. Therefore we have that the conductance of X_i is between $1/2$ and 1. Let θ be the unit current flow from x to $B(x,n)^c$. Then, $0 \leq \theta(\overrightarrow{xx_i}) \leq 2/3$ for each i.

Let $\{x_{k,i}\}_i$ be the vertices of the outer boundary of $B(x, k)$, $1 \leq k \leq n$, and let $a_{k,i}$ be the amount of θ going into $x_{k,i}$ from a point in the outer boundary of $B(x, k-1)$. Since θ is the unit current flow, $a_{k,i} \geq 0$. By using the same argument as in the case $k = 1$, we see that $a_{k,i} \leq (2/3)^k$ by induction on k. Hence $\sum_i a_{n,i}^2 \leq (2/3)^n$.

Now we extend the flow θ to a unit flow on X from x to ∞. Since the minimal degree of X is at least 3, we can construct a flow θ on $D_x(x_{n,i})$ such that it starts at $x_{n,i}$, the strength of the flow is $a_{n,i}$, and, $\sum_{e:\text{ edge in } D_x(x_{n,i})} \theta(e)^2 \leq a_{n,i}^2$. By using Thomson's principle, we have that

$$\rho(x) \leq \sum_{e:\text{ edge in } X} \theta(e)^2 = \rho(x,n) + \sum_i \sum_{e:\text{ edge in } D_x(x_i)} \theta(e)^2 \leq \rho(x,n) + \left(\frac{2}{3}\right)^n.$$

$\square$

Let $N \geq 3$. Let T_N be the infinite N-regular tree. Let $\tilde{T}_N(o)$ be the infinite tree T such that $\deg(o) = N-1$ for $o \in T$ and $\deg(x) = N$ for any $x \in T \setminus \{o\}$. For the simple random walk on T_N, we let $g_N = P_x(T_x^+ = +\infty) = (N-2)/(N-1)$. and $g_N(n) = P_x(T_x^+ > n)$ for some (or any) $x \in T_N$.

Definition 2.2.3. Let Y be a finite tree. Let $L(Y) = \{y \in Y : \deg(y) = 1\}$. Let $N \geq 3$. We define an infinite tree Y_N as follows : We prepare Y and $|L(Y)|$ copies of $\tilde{T}_N(o)$. Let Y_N be the infinite tree obtained by attaching $o \in \tilde{T}_N(o)$ to each $y \in L(Y)$.

Lemma 2.2.4. *Let $N \geq 3$. Let Y be a finite tree with a reference point o such that $\deg(y) \geq 3$ for any $y \in Y \setminus L(Y)$. Let Y_N be the infinite tree in Definition 2.2.3. Let R_n be the range of the simple random walk up to time $n-1$ on Y_N. Then,*

$$\lim_{n\to\infty} \frac{E_o[R_n]}{n} = g_N.$$

Proof. First, we remark that Y_N satisfies (U) thanks to Lemma 2.2.2. For $m \in \mathbb{N}$, let $F_{m,1} := \inf_{y \in Y_N \setminus B(o,m)} P_y(T_y^+ < +\infty)$ and $F_{m,2} := \sup_{y \in Y_N \setminus B(o,m)} P_y(T_y^+ < +\infty)$. We will show that for any m,

$$1 - F_{m,2} \leq \liminf_{n\to\infty} \frac{E_o[R_n]}{n} \leq \limsup_{n\to\infty} \frac{E_o[R_n]}{n} \leq 1 - F_{m,1}. \tag{2.2.1}$$

By considering a last exit decomposition, we have that

$$\begin{aligned} E_o[R_n] &= 1 + \sum_{i=0}^{n-2} \sum_{y \in Y_N} P_o(S_i = y) P_y(T_y^+ > n-1-i) \\ &= 1 + \sum_{i=0}^{n-2} \sum_{y \in B(o,m)} P_o(S_i = y) P_y(T_y^+ > n-1-i) \\ &\quad + \sum_{i=0}^{n-2} \sum_{y \in Y_N \setminus B(o,m)} P_o(S_i = y) P_y(T_y^+ > n-1-i) \end{aligned}$$

Therefore, we have that

$$E_o[R_n] \leq 1 + \sum_{i=0}^{n-2} \left(P_o(S_i \in B(o,m)) + \sup_{y \in Y_N \setminus B(o,m)} P_y(T_y^+ > n-1-i) \right), \tag{2.2.2}$$

and,

$$E_o[R_n] \geq \sum_{i=0}^{n-2} P_o(S_i \in Y_N \setminus B(o,m)) \inf_{y \in Y_N \setminus B(o,m)} P_y(T_y^+ > n-1-i). \tag{2.2.3}$$

We have that for any $x, y \in X$,

$$P_x(S_i = y) \leq \left(\frac{\deg(y)}{\deg(x)} P_x(S_{2i} = x) \right)^{1/2} \to 0, \;\; i \to \infty. \tag{2.2.4}$$

By using (2.2.2), (2.2.3), (2.2.4), and, that Y_N satisfies (U), we have (2.2.1).

We have that if m is sufficiently large, then, for any $y \in Y_N \setminus B(o,m)$,

$$\begin{aligned} P_y(T_y^+ = +\infty) &= P_y(T_y^+ > m/2) - P_y(m/2 < T_y^+ < +\infty) \\ &= g_N(m/2) - P_y(m/2 < T_y^+ < +\infty). \end{aligned}$$

Since Y_N satisfies (U) and $\lim_{k\to\infty} g_N(k) = g_N$,

$$\lim_{m\to\infty} F_{m,1} = \lim_{m\to\infty} F_{m,2} = 1 - g_N. \tag{2.2.5}$$

(2.2.1) and (2.2.5) complete the proof. $\square$

Proof of Theorem 1.1.3. First, we will construct an increasing sequence of finite trees $(X^{(n)})_n$ by induction on n. Second, we will show that the limit infinite graph X of $(X^{(n)})_n$ satisfies (U), $F_1 < F_2$ and (1.1.3).

Let $3 \le N_1 < N_2$. Let $X^{(1)}$ be a finite tree such that $\deg(x) = N_1$ for any $x \in X^{(1)} \setminus E(X^{(1)})$ and $X^{(1)} = B(o, k_1)$ for a point $o \in X^{(1)}$ and a positive integer k_1.

We assume that $X^{(2n-1)}$ is constructed and $X^{(2n-1)} = B_{X^{(2n-1)}}(o, k_{2n-1})$ for a positive integer k_{2n-1}. Thanks to Lemma 2.2.4, there exists $k_{2n} > 2k_{2n-1}$ such that for the simple random walk on $(X^{(2n-1)})_{N_2}$ starting at o,

$$\frac{E_o[R_{k_{2n}}]}{k_{2n}} \ge g_{N_2} - \frac{1}{n}. \tag{2.2.6}$$

Then we let $X^{(2n)} = (X^{(2n-1)})_{N_2} \cap B_{(X^{(2n-1)})_{N_2}}(o, k_{2n})$.

We assume that $X^{(2n)}$ is constructed and $X^{(2n)} = B_{X^{(2n)}}(o, k_{2n})$ for a positive integer k_{2n}. Thanks to Lemma 2.2.4, there exists $k_{2n+1} > 2k_{2n}$ such that for the simple random walk on $(X^{(2n)})_{N_1}$ starting at o,

$$\frac{E_o[R_{k_{2n+1}}]}{k_{2n+1}} \le g_{N_1} + \frac{1}{n}. \tag{2.2.7}$$

Then we let $X^{(2n+1)} = (X^{(2n)})_{N_1} \cap B_{(X^{(2n)})_{N_1}}(o, k_{2n+1})$.

Let X be the infinite graph obtained by the limit of a sequence of $(X^{(n)})$. Then, X is a tree with minimal degree at least 3. By Lemma 2.2.2, X satisfies (U).

Now we show (1.1.3). We remark that the distribution of the simple random walk up to time $k-1$ on X starting at o is determined by $B_X(o,k)$, $k \ge 1$. By the definition of X, (2.2.6) and (2.2.7) hold also for the simple random walk on X. Hence,

$$\liminf_{n\to\infty} \frac{E_o[R_n]}{n} \le g_{N_1}, \text{and}, \limsup_{n\to\infty} \frac{E_o[R_n]}{n} \ge g_{N_2}. \tag{2.2.8}$$

By considering a last exit decomposition as in the proof of Theorem 1.1.2, and, noting that X satisfies (U), we have

$$1 - F_2 = \inf_{x \in X} P_x(T_x^+ = +\infty) \leq \liminf_{n \to \infty} \frac{E_o[R_n]}{n}, \tag{2.2.9}$$

and,

$$\limsup_{n \to \infty} \frac{E_o[R_n]}{n} \leq \sup_{x \in X} P_x(T_x^+ = +\infty) = 1 - F_1. \tag{2.2.10}$$

In order to see (1.1.3), it is sufficient to show that for any $x \in X$,

$$g_{N_1} \leq P_x(T_x^+ = +\infty) \leq g_{N_2}. \tag{2.2.11}$$

We can regard T_{N_1} as a subtree of X, and, X as a subtree of T_{N_2}. By using [70] Theorem 2.2.7 and the monotonicity of the effective resistance, we see that if $\deg_X(x) = N_1$, then,

$$g_{N_1} = N_1^{-1} \rho_{T_{N_1}}(x)^{-1} \leq N_1^{-1} \rho_X(x)^{-1} = P_x(T_x^+ = +\infty) \leq N_2^{-1} \rho_{T_{N_2}}(x)^{-1} = g_{N_2},$$

and, if $\deg_X(x) = N_2$, then,

$$g_{N_1} = N_1^{-1} \rho_{T_{N_1}}(x)^{-1} \leq N_2^{-1} \rho_X(x)^{-1} = P_x(T_x^+ = +\infty) \leq N_2^{-1} \rho_{T_{N_2}}(x)^{-1} = g_{N_2}.$$

Thus the proof of (2.2.11) completes and we obtain (1.1.3).

By using [110] Lemma 1.24 and $N_1 < N_2$, we see that $g_{N_1} = (N_1 - 2)/(N_1 - 1) < g_{N_2} = (N_2 - 2)/(N_2 - 1)$. By using (2.2.8), (2.2.9), (2.2.10) and (2.2.11), we see $g_{N_1} = 1 - F_2$ and $g_{N_2} = 1 - F_1$. Hence $F_1 < F_2$.

Thus we see that X satisfies (U), $F_1 < F_2$, and, (1.1.3). □

2.3 Examples of graphs satisfying the uniform condition

In this section, we give some examples of graphs satisfying (U). We assume that all weights are equal to 1.

Here we follow [70] Definition 2.1.8 for the definition of rough isometry introduced by Kanai [63].

Definition 2.3.1. Let X_i be weighted graphs and d_i be the graph metric of X_i, $i = 1, 2$. We say that a map $T : X_1 \to X_2$ is a $((A, B, M)$-$)$*rough isometry* if there exist constants $A > 1$, $B > 0$, and, $M > 0$ satisfying the following inequalities.

$$A^{-1} d_1(x, y) - B \leq d_2(T(x), T(y)) \leq A d_1(x, y) + B, \; x, y \in X_1.$$

$$d_2(T(X_1), z) \leq M, \; z \in X_2.$$

We say that X_1 is *roughly isometric* to X_2 if there exists a rough isometry between them. We say that a property is *stable under rough isometry* if whenever X_1 satisfies the property and is roughly isometric to X_2, then X_2 also satisfies the property.

2.3.1 Recurrent graphs

Proposition 2.3.2. *The condition (U) is stable under rough isometry between recurrent graphs.*

We do not know whether (U) is stable under rough isometry between transient graphs.

Proof. Assume that X_1 is a recurrent graph satisfying (U) and X_2 is a (recurrent) graph which is roughly isometric to X_1. We would like to show that X_2 satisfies (U).

Since rough isometry is an equivalence relation, there exists a (A, B, M)-rough isometry $T : X_2 \to X_1$. Fix $n \in \mathbb{N}$ and $x \in X_2$. Let f be a function on X_1 such that $f(T(x)) = 1$ and $f = 0$ on $X_1 \setminus B(T(x), A^{-1}n - B)$. Since T is a (A, B, M)-rough isometry, we have that for any $y \in X_2 \setminus B(x, n)$, $T(y) \in X_1 \setminus B(T(x), A^{-1}n - B)$, and hence, $f \circ T = 0$ on $X_2 \setminus B(x, n)$.

By using Theorem 3.10 in [110], we see that there exists a constant $c > 0$ such that $\mathcal{E}_{X_1}(f, f) \geq c\mathcal{E}_{X_2}(f \circ T, f \circ T)$. This constant does not depend on (x, n, f). Therefore,

$$\inf\left\{\mathcal{E}_{X_1}(f, f) : f(T(x)) = 1, f = 0 \text{ on } X_1 \setminus B(T(x), A^{-1}n - B)\right\}$$
$$\geq c \inf\left\{\mathcal{E}_{X_2}(g, g) : g(x) = 1, g = 0 \text{ on } X_2 \setminus B(x, n)\right\}.$$

Hence, $\rho_{X_2}(x, n) \geq c\rho_{X_1}(T(x), A^{-1}n - B)$. By recalling that X_1 satisfies (U), we see that X_2 satisfies (U). $\square$

Proposition 2.3.3. *Let X be a graph such that there exists $C > 0$ such that $V(x, n) \leq Cn^2$ for any $x \in X$ and $n \geq 1$. Then, X satisfies (U).*

We can show the above assertion in the same manner as in the proof of [110], Lemma 3.12, so we omit the proof.

Proposition 2.3.4. *Let X be a graph such that*

$$\lim_{n\to\infty} \inf_{x \in X} \sum_{k=0}^{n} p_k(x, x) = +\infty. \tag{2.3.1}$$

Then, X satisfies (U).

Proof. By noting [70] Lemma 4.1.1(iv), we see that $\rho(x,n) = g^{B(x,n)}(x,x)$, $x \in X$, $n \geq 1$. Since $p_k^{B(x,n)}(x,x) = p_k(x,x)$ for $k < n$,

$$\rho(x,n) = g^{B(x,n)}(x,x) \geq \sum_{0 \leq k < n} p_k(x,x).$$

By noting (2.3.1), we see X satisfies (U). $\square$

By using Section 5 in Barlow, Coulhon and Kumagai [8], we see that the d-dimensional standard graphical Sierpiński gaskets, $d \geq 2$, and Vicsek trees (See Barlow [5] for definition) satisfies (2.3.1). Thus we have

Example 2.3.5. The graphs which are roughly isometric with the following graphs satisfy (U).
(i) Infinite connected subgraphs in $\mathbb{Z}^2$.
(ii) Infinite connected subgraphs in the planer triangular lattice.
(iii) The d-dimensional standard graphical Sierpiński gaskets, $d \geq 2$.
(iv) Vicsek trees.

2.3.2 Transient graphs

We say that X satisfies (UC_α), $\alpha > 2$, if there exist $C > 0$ such that $\sup_{x \in X} p_n(x,x) \leq Cn^{-\alpha/2}$ for any $n \geq 1$. The stability of the property (UC_α), $\alpha > 2$, under rough isometry follows from Varopoulous [108] Theorem 1 and 2, and, Kanai [64] Proposition 2.1.

Proposition 2.3.6. *If X satisfies (UC_α) for some $\alpha > 2$, then, X satisfies (U).*

Proof. Let $m > n$. Then, by using [70] Lemma 4.1.1(iv) and $p_k^{B(x,m)}(x,x) = p_k^{B(x,n)}(x,x) = p_k(x,x)$ for $k < n$,

$$\begin{aligned}\rho(x,m) - \rho(x,n) &= g^{B(x,m)}(x,x) - g^{B(x,n)}(x,x) \\ &= \sum_{k \geq n} (p_k^{B(x,m)}(x,x) - p_k^{B(x,n)}(x,x)) \\ &\leq \sum_{k \geq n} p_k(x,x).\end{aligned}$$

Letting $m \to \infty$,

$$\rho(x) - \rho(x,n) \leq \sum_{k \geq n} p_k(x,x), x \in X, n \geq 1.$$

Thus we see that if X satisfies (UC_α) for some $\alpha > 2$, then X satisfies (U). $\square$

$\mathbb{Z}^d$ satisfies (UC_d). By using Barlow and Bass [6], [7], we see that if $d \geq 3$, then d-dimensional standard graphical Sierpiński carpet satisfies (UC_α) for some $\alpha > 2$. Therefore we have

Example 2.3.7. The graphs which are roughly isometric with the following graphs satisfy (U).
(i) $\mathbb{Z}^d$, $d \geq 3$.
(ii) d-dimensional standard graphical Sierpiński carpet, $d \geq 3$.

2.3.3 A graph which does not satisfy (U)

Finally, we give examples of recurrent and transient graphs which do not satisfy (U).

Remark 2.3.8. The recurrent tree T considered in [110], Example 6.16 does not satisfy (U). T is constructed as follows : Let the vertex set $V := \{\{x_{n,i} : 1 \leq i \leq 2^n, n \geq 0\}\}$ and the edge set $E := \{\{x_{0,1}, x_{1,1}\}, \{x_{0,1}, x_{1,2}\}\} \cup \{\{x_{n,i}, x_{n+1,j}\} : 3i-2 \leq j \leq 3i, 1 \leq i \leq 2^{n-1}, n \geq 1\}$
$\cup \{\{x_{n,i}, x_{n+1,2^n+i}\} : 2^{n-1}+1 \leq i \leq 2^n, n \geq 1\}$. Then, $T = (V, E)$. For any $n \geq 1$, there exists $x_n \in T$ such that $\rho(x_n, n) = \rho_{T_4}(x_n, n)$, where T_4 is the 4-regular tree. Since T_4 is vertex transitive and transient, we have that $\rho_{T_4}(x_n, n) \leq \rho_{T_4}(x_n) = \rho_{T_4}(o) < +\infty$, $n \geq 1$, for a reference point $o \in T_4$. However, T is recurrent and hence $\rho(x, n) \to \infty$, $n \to \infty$, $x \in T$. Thus we see that T does not satisfy (U).

Now we give an example of a transient graph which does not satisfy (U). Let $T_3 = (V(T_3), E(T_3))$ be the 3-regular tree and o be a point of T_3. Let $C = \{c_i : i \geq 1\}$ be an infinite countable set. Let $V := V(T_3) \cup C$ and $E := E(T_3) \cup \{o, c_1\} \cup \{\{c_i, c_{i+1}\} : i \geq 1\}$ and define a new tree $T' := (V, E)$. Then, T' is transient. $R_{\text{eff}}(c_n, V \setminus B(c_n, n)) = n/2$ and $R_{\text{eff}}(c_n, V \setminus B(c_n, n-1)) = (n-1)/2$ and hence T' does not satisfy (U).

Chapter 3

Singularity results for functional equations driven by linear fractional transformations

This chapter will be based on [82]. This chapter is organized as follows. In Section 3.1, we state some lemmas. In Section 3.2, we show the main results. In Section 3.3, we state a relationship between these functional equations and stationary measures. In Section 3.4, we give examples and remarks.

3.1 Lemmas

First, we introduce some notation.

Let $X_n : [0,1) \to \{0,1\}$, $n \geq 1$ be given by $X_n(x) = [2^n x] - 2[2^{n-1}x]$, $x \in [0,1)$. Let $\rho_n(i_1,\dots,i_n) = \mu_f(\{X_j = i_j, 1 \leq j \leq n\})$ for $n \geq 1$, $i_1,\dots,i_n \in \{0,1\}$ and $R_n(x) = \rho_n(X_1(x),\dots,X_n(x))$ for $n \geq 1$ and $x \in [0,1)$. Let
$I_n(x) = [\sum_{i=1}^n 2^{-j} X_j(x), \sum_{i=1}^n 2^{-j} X_j(x) + 2^{-n}) = [2^{-n}[2^n x], 2^{-n}([2^n x]+1))$. Then, $x \in I_n(x)$, $x \in [0,1)$, and, $X_n(y) = X_n(x)$ and $I_n(y) = I_n(x)$ for $y \in I_n(x)$. We have that $R_n(x) = \mu_f(\{X_j = X_j(x), 1 \leq j \leq n\}) = \mu_f(I_n(x))$. Let

$$\begin{pmatrix} p_n(x) & q_n(x) \\ r_n(x) & s_n(x) \end{pmatrix} = A_{X_1(x)} \cdots A_{X_n(x)}, x \in [0,1).$$

Lemma 3.1.1. *Let $n \geq 1$ and $i_1,\dots,i_n \in \{0,1\}$. Then we have the following.*

(1) $f\left(\sum_{i=1}^{n} 2^{-j} i_j\right) = \Phi(A_{i_1} \cdots A_{i_n}; 0)$ *and* $f\left(\sum_{i=1}^{n} 2^{-j} i_j + 2^{-n}\right) = \Phi(A_{i_1} \cdots A_{i_n}; 1)$.
(2) $R_{n+1}(x)/R_n(x) = p_{X_{n+1}(x)}(r_n(x)/s_n(x))$.

Proof. (1) By recalling (1.2.1), we easily show the assertion by induction in n.

(2) By the assertion (1), we have that

$$\begin{aligned} R_k(x) &= \Phi(A_{X_1(x)} \cdots A_{X_k(x)}; 1) - \Phi(A_{X_1(x)} \cdots A_{X_k(x)}; 0) \\ &= \frac{p_k(x)s_k(x) - q_k(x)r_k(x)}{s_k(x)(r_k(x) + s_k(x))}. \end{aligned}$$

By computation, we have that

$$\begin{aligned} \frac{R_{n+1}(x)}{R_n(x)} = & \frac{(\det A_{X_{n+1}(x)})s_n(x)}{b_{X_{n+1}(x)}r_n(x) + d_{X_{n+1}(x)}s_n(x)} \\ & \times \frac{r_n(x) + s_n(x)}{\left(a_{X_{n+1}(x)} + b_{X_{n+1}(x)}\right) r_n(x) + \left(c_{X_{n+1}(x)} + d_{X_{n+1}(x)}\right) s_n(x)}. \end{aligned}$$

By noting (A2), we have that

$$\frac{R_{n+1}(x)}{R_n(x)} = p_{X_{n+1}(x)}\left(\frac{r_n(x)}{s_n(x)}\right).$$

Thus we obtain the assertion (2). □

Now we state some properties of $\Phi({}^tA_i; \cdot)$, $i = 0, 1$.

We remark that $\Phi({}^tA_0; \cdot)$ (resp. $\Phi({}^tA_1; \cdot)$) is well-defined and continuous on $\mathbb{R}$ (resp. $(-\gamma, \infty)$).

Lemma 3.1.2. (1) $d_0 > a_0 > 0$, $b_1 + c_1 > 0$ *and* $\alpha > -1$.
(2) $\Phi({}^tA_0; z) = z$ *if and only if* $z = c_0/(d_0 - a_0)$.
(3) $\Phi({}^tA_1; z) = z$ *if and only if* $z = -1$ *or* $z = c_1/b_1$.

Proof. (1) By (A2) and (A3), we have that $d_0 > 0$, and then $a_0 > 0$. By (A3) and (A1), we have that $0 < (a_0 d_0)^{1/2} = (a_0 d_0 - b_0 c_0)^{1/2} < d_0$ and then $0 < a_0 < d_0$.

By (A1), we have that $a_1 + b_1 = c_1 + d_1$ and then $a_1 d_1 - b_1 c_1 = (c_1 + d_1)(d_1 - b_1)$. By (A2) and (A3), we have that $c_1 + d_1 > 0$, and then $d_1 - b_1 > 0$. By (A3), we have that $0 < (c_1 + d_1)^{1/2}(d_1 - b_1)^{1/2} < c_1 + d_1$. Hence we have that $d_1 - b_1 < c_1 + d_1$, and then $b_1 + c_1 > 0$.

By (A2) and (A3), we have that $d_1 > 0$. By (A1), we have that $b_1 > 0$. Since $b_1 + c_1 > 0$, we see that $c_1/b_1 > -1$. Then, we have that $c_0/(d_0 - a_0) >$

-1 by noting (A1) and $a_0 < d_0$. Now we have that $\alpha = \min\{0, c_0/(d_0 - a_0), c_1/b_1\} > -1$.

(2) Since $b_0 = 0$, we have that $\Phi({}^tA_0; z) - z = -(d_0 - a_0)z/d_0 + c_0/d_0$. Since $d_0 > a_0$, we see that $\Phi({}^tA_0; z) = z$ if and only if $z = c_0/(d_0 - a_0)$.

(3) Since

$$\begin{aligned}\Phi({}^tA_1; z) - z &= \frac{-b_1z^2 - (d_1 - a_1)z + c_1}{b_1z + d_1} \\ &= \frac{(-b_1z + c_1)(z+1)}{b_1z + d_1} = -\frac{(z+1)(z - c_1/b_1)}{z + \gamma},\end{aligned}$$

we see that $\Phi({}^tA_1; z) = z$ if and only if $z = -1$ or $z = c_1/b_1$. □

Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, $n \geq 1$. Let $L_n = \sum_{i=1}^n E^{\mu_f}[-\log(R_i/R_{i-1})|\mathcal{F}_{i-1}]$ and $M_n = -\log R_n - L_n$, $n \geq 1$. Then we have the following.

Lemma 3.1.3. *We have that*
(1) $L_{n+1}(x) - L_n(x) = s(p_0(r_n(x)/s_n(x)))$ for μ_f-a.s.$x \in [0,1)$.
(2) $M_n/n \to 0$, $(n \to \infty)$ for μ_f-a.s.

Proof. (1) It is sufficient to show that for any $x \in [0,1)$,

$$\int_{I_n(x)} s\left(p_0\left(\frac{r_n(y)}{s_n(y)}\right)\right)\mu_f(dy) = \int_{I_n(x)} -\log\left(\frac{R_{n+1}(y)}{R_n(y)}\right)\mu_f(dy).$$

Since $r_n(y)/s_n(y) = r_n(x)/s_n(x)$ for $y \in I_n(x)$, we see that

$$\int_{I_n(x)} s\left(p_0\left(\frac{r_n(y)}{s_n(y)}\right)\right)\mu_f(dy) = \mu_f(I_n(x))s\left(p_0\left(\frac{r_n(x)}{s_n(x)}\right)\right).$$

By Lemma 3.1.1(2), we see that

$$-\log\frac{\mu_f(I_{n+1}(y))}{\mu_f(I_n(y))} = -\log\frac{R_{n+1}(y)}{R_n(y)} = -\log p_{X_{n+1}(y)}\left(\frac{r_n(x)}{s_n(x)}\right),$$

and,

$$\int_{I_n(x)} -\log\left(\frac{R_{n+1}(y)}{R_n(y)}\right)\mu_f(dy) = \int_{I_n(x)} -\log\left(p_{X_{n+1}(y)}\left(\frac{r_n(y)}{s_n(y)}\right)\right)\mu_f(dy)$$

$$= -\mu_f\left(I_n(x) \cap \{X_{n+1} = 0\}\right)\log p_0\left(\frac{r_n(x)}{s_n(x)}\right) - \mu_f\left(I_n(x) \cap \{X_{n+1} = 1\}\right)\log p_1\left(\frac{r_n(x)}{s_n(x)}\right)$$

$$= \mu_f\left(I_n(x)\right)s\left(p_0(r_n(x)/s_n(x))\right),$$

which implies the assertion (1).

(2) By noting Jensen's inequality, we have that

$$
\begin{aligned}
E^{\mu_f}\left[(M_k - M_{k-1})^2\right] &\le 2\left(E^{\mu_f}\left[(-\log R_k + \log R_{k-1})^2\right] + E^{\mu_f}\left[(L_k - L_{k-1})^2\right]\right) \\
&\le 4E^{\mu_f}\left[(-\log R_k + \log R_{k-1})^2\right].
\end{aligned}
$$

Let $C_0 = \sup\{x(\log x)^2 + (1-x)(\log(1-x))^2 : x \in [0,1]\} < +\infty$. We will show that $E^{\mu_f}[(\log(R_{n+1}/R_n))^2] \le C_0$ for any $n \ge 1$.

Let $\tau(p) = p(\log p)^2 + (1-p)(\log(1-p))^2$ for $p \in [0,1]$. We remark that $\tau(p) = \tau(1-p)$. Then we have that

$$
E^{\mu_f}\left[(-\log R_n + \log R_{n-1})^2\right] = \sum_{k=0}^{2^n-1} \mu_f\left(I_n\left(\frac{k}{2^n}\right)\right)\left(\log\frac{R_n(k/2^n)}{R_{n-1}(k/2^n)}\right)^2
$$

$$
\begin{aligned}
= \sum_{k=0}^{2^{n-1}-1} \mu_f\left(I_n\left(\frac{2k}{2^n}\right)\right)&\left(\log\frac{R_n(2k/2^n)}{R_{n-1}(2k/2^n)}\right)^2 \\
&+ \mu_f\left(I_n\left(\frac{2k+1}{2^n}\right)\right)\left(\log\frac{R_n(2k+1/2^n)}{R_{n-1}(2k+1/2^n)}\right)^2.
\end{aligned}
$$

By noting that $R_{n-1}(2k/2^n) = R_{n-1}(2k+1/2^n) = R_{n-1}(k/2^{n-1})$, $\mu_f\left(I_n(2k/2^n)\right) = R_n(2k/2^n)$ and $\mu_f\left(I_n(2k+1/2^n)\right) = R_n(2k+1/2^n)$, we have that

$$
E^{\mu_f}\left[\left(\log\frac{R_n}{R_{n-1}}\right)^2\right] = \sum_{k=0}^{2^{n-1}-1} R_{n-1}\left(\frac{k}{2^{n-1}}\right)\tau\left(\frac{R_n(k/2^{n-1})}{R_{n-1}(k/2^{n-1})}\right) \le C_0.
$$

Thus we have that $\sup_{k\ge1} E^{\mu_f}[(M_k - M_{k-1})^2] \le 4C_0 < +\infty$. Since $\{M_n\}$ is an $\{\mathcal{F}_n\}$-martingale, $\{M_n^2\}$ is an $\{\mathcal{F}_n\}$-submartingale. Noting that $M_0 = 0$, we have that $E^{\mu_f}[M_n^2] = \sum_{k=1}^n E^{\mu_f}[(M_k - M_{k-1})^2]$.

By Doob's submartingale inequality, we have that

$$
\mu_f\left(\max_{1\le k\le 2^l} M_k^2 \ge \epsilon 4^l\right) \le \frac{E^{\mu_f}[M_{2^l}^2]}{\epsilon 4^l} \le \frac{4C_0}{\epsilon 2^l},\ l \ge 1,\ \epsilon > 0.
$$

Now we have that for μ_f-a.s.x, there exists $m = m(x) \in \mathbb{N}$ such that $\max_{1\le k\le 2^l}(M_k(x)/2^l)^2 \le \epsilon$, $l \ge m$, and then, $(M_n(x)/n)^2 \le 4\epsilon$, $n \ge 2^m$. Then we see that $\limsup_{n\to\infty}(M_n/n)^2 \le \epsilon$, μ_f-a.s., which implies our assertion. □

Lemma 3.1.4. (1) *Suppose that* $\limsup_{n\to+\infty}(-\log R_n)/n \le \theta_1$ *for a constant* θ_1*, then there exists a Borel set* K_0 *such that* $\mu_f(K_0) = 1$ *and* $\dim_H(K_0) \le \theta_1/\log 2$.
(2) *Suppose that* $\liminf_{n\to+\infty}(-\log R_n)/n \ge \theta_2$ *for a constant* θ_2*, then we have that* $\mu_f(K) = 0$ *for any Borel set* K *with* $\dim_H(K) < \theta_2/\log 2$.

Proof. We denote the diameter of a set $G \subset \mathbb{R}$ by $\mathrm{diam}(G)$.

(1) Let $Y_{\epsilon,n} = \bigcap_{k\geq n} \{(-\log R_k)/k \leq \theta_1 + \epsilon\}$. Then we have that $\mu_f\left(\bigcup_{n\geq 1} Y_{\epsilon,n}\right) = 1$. Let $\mathcal{A}_{\epsilon,k}$ be the set of $I_k(x)$, $x \in [0,1)$, such that $R_k(x) \geq \exp(-k(\theta_1+\epsilon))$. Then, for any $k \geq n$, $\{I_k(x) \in \mathcal{A}_{\epsilon,k} : x \in Y_{\epsilon,n}\}$ is a 2^{-k}-covering of $Y_{\epsilon,n}$.

Since $\mu_f([0,1)) = 1$, we see that $\sharp(\mathcal{A}_{\epsilon,k})\exp(-k(\theta_1+\epsilon)) \leq 1$. Then

$$\sum_{I\in\mathcal{A}_{\epsilon,k}} \mathrm{diam}(I)^{(\theta_1+2\epsilon)/\log 2} = \sharp(\mathcal{A}_{\epsilon,k})\exp\left(-k(\theta_1+2\epsilon)\right) \leq \exp(-k\epsilon).$$

By letting $k \to +\infty$, we see $H_{(\theta_1+2\epsilon)/\log 2}(Y_{\epsilon,n}) = 0$.

Let $K_0 = \bigcap_{k\geq 1}\bigcup_{n\geq 1} Y_{1/k,n}$. Then, we have that $\mu_f(K_0) = 1$ and $H_{(\theta_1+2\epsilon)/\log 2}(K_0) = 0$ for any $\epsilon > 0$. Hence $\dim_H(K_0) \leq \theta_1/\log 2$.

(2) Let K be a Borel set such that $\dim_H(K) < \theta_2/\log 2$. Then, there exists $\epsilon > 0$ such that $H_{(\theta_2-\epsilon)/\log 2}(K) = 0$. Then, for any $n \geq 1$ and $\delta > 0$, there exist intervals $\{U(n,l)\}_{l=1}^{\infty}$ on $[0,1)$ such that $K \subset \bigcup_{l\geq 1} U(n,l)$ and $\mathrm{diam}(U(n,l)) < 2^{-n}$ for $l \geq 1$ and $\sum_{l\geq 1}\mathrm{diam}(U(n,l))^{(\theta_2-\epsilon)/\log 2} \leq \delta$. For each $l \geq 1$, let $k(n,l) > n$ be the integer such that $2^{-k(n,l)} \leq \mathrm{diam}(U(n,l)) < 2^{-(k(n,l)-1)}$.

Let $Z_{\epsilon,n} = \bigcap_{k\geq n}\{(-\log R_k)/k \geq \theta_2 - \epsilon\}$. Then we have that $\lim_{n\to\infty}\mu_f(Z_{\epsilon,n}) = \mu_f\left(\bigcup_{n\geq 1} Z_{\epsilon,n}\right) = 1$, and,

$$\mu_f\left(I_{k(n,l)}(y)\right) = R_{k(n,l)}(y) \leq \exp\left(-k(n,l)(\theta_2-\epsilon)\right) \leq \mathrm{diam}(U(n,l))^{(\theta_2-\epsilon)/\log 2},$$

for $y \in Z_{\epsilon,n}$ and $l \geq 1$.

Since $\mathrm{diam}(I_{k(n,l)}(x)) = 2^{-k(n,l)}$ and $\mathrm{diam}(U(n,l)) < 2^{-(k(n,l)-1)}$, we see that $\sharp\left\{I_{k(n,l)}(x); I_{k(n,l)}(x)\cap U(n,l) \neq \emptyset\right\} \leq 3$ and that $\mu_f\left(K \cap Z_{\epsilon,n}\cap U(n,l)\right) \leq 3\mathrm{diam}(U(n,l))^{(\theta_2-\epsilon)/\log 2}$.

Noting that $K \subset \bigcup_{l\geq 1} U(n,l)$, we see that

$$\mu_f(K\cap Z_{\epsilon,n}) \leq \sum_{l\geq 1}\mu_f(K\cap Z_{\epsilon,n}\cap U(n,l)) \leq 3\sum_{l\geq 1}\mathrm{diam}(U(n,l))^{(\theta_2-\epsilon)/\log 2} \leq 3\delta.$$

Since δ is taken arbitrarily, we see that $\mu_f(K \cap Z_{\epsilon,n}) = 0$. Recalling $\mu_f\left(\bigcup_{n\geq 1} Z_{\epsilon,n}\right) = 1$, we see that $\mu_f(K) = 0$. □

3.2 Proofs of Main Theorems

Lemma 3.2.1. *Let $n \geq 1$ and $i_1,\ldots,i_n \in \{0,1\}$. Then,*
$\alpha \leq \Phi({}^tA_{i_n}\cdots{}^tA_{i_1};\alpha) \leq \Phi({}^tA_{i_n}\cdots{}^tA_{i_1};\beta) \leq \beta$.
In particular, $r_n(x)/s_n(x) \in [\alpha,\beta]$ for $n \geq 1$ and $x \in [0,1)$.

Proof. By noting Lemma 3.1.2, we have that $\Phi({}^tA_0; z) - z = -(d_0 - a_0)z/d_0 + c_0/d_0$ and $\Phi({}^tA_1; z) - z = -(z+1)(z - c_1/b_1)/(z+\gamma)$. We remark that $\alpha > -1 > -\gamma$. Since $\alpha \le c_0/(d_0 - a_0), c_1/b_1 \le \beta$, we see that $\alpha \le \Phi({}^tA_i; \alpha) \le \Phi({}^tA_i; \beta) \le \beta$ for $i = 0, 1$.

Since $\Phi({}^tA_0; \cdot)$ and $\Phi({}^tA_1; \cdot)$ are increasing, we obtain the assertion by induction in n.

We have that $\alpha \le 0 \le \beta$ by the definition of α and β. Since $r_n(x)/s_n(x) = \Phi({}^tA_{X_n(x)} \cdots {}^tA_{X_1(x)}; 0)$, we see that $r_n(x)/s_n(x) \in [\alpha, \beta]$. □

Now we show Theorem 1.2.1.

By noting Lemma 3.1.3 and Lemma 3.2.1, we see that for μ_f-a.s.,

$$\limsup_{n\to+\infty} \frac{-\log R_n}{n} = \limsup_{n\to\infty} \frac{L_n}{n} = \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} s\left(p_0\left(\frac{r_n(x)}{s_n(x)}\right)\right) \le \max\left\{s(p_0(y)); y \in [\alpha, \beta]\right\},$$

and,

$$\liminf_{n\to+\infty} \frac{-\log R_n}{n} = \liminf_{n\to\infty} \frac{L_n}{n} = \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} s\left(p_0\left(\frac{r_n(x)}{s_n(x)}\right)\right) \ge \min\left\{s(p_0(y)); y \in [\alpha, \beta]\right\}.$$

Let $\theta_1 = \max\{s(p_0(y)); y \in [\alpha, \beta]\}$ and $\theta_2 = \min\{s(p_0(y)); y \in [\alpha, \beta]\}$. Then, by Lemma 3.1.4(1) (resp. (2)), we obtain the assertion (1) (resp. (2)).

These complete the proof of Theorem 1.2.1.

Lemma 3.2.2. *Let $\mathbb{N}_i(x) = \{n \in \mathbb{N} : X_n(x) = i\}$ for $x \in [0,1)$, $i = 0, 1$. Then,*

$$\liminf_{N\to\infty} \frac{|\mathbb{N}_0(x) \cap \{1, \dots, N\}|}{N} \ge p_0(\alpha) > 0, \ \mu_f\text{-a.s.}x.$$

Proof. Let $\zeta_N(x) = |\mathbb{N}_0(x) \cap \{1, \dots, N\}|$. Then, $\zeta_N(x) = \sum_{n=1}^{N} 1_{\{0\}}(X_n(x))$. Let $M_n = \sum_{i=1}^{n} \left(1_{\{0\}}(X_n) - p_0(\alpha)\right)$. Then, $\{M_n\}$ is an $\{\mathcal{F}_n\}$-submartingale because

$$E^{\mu_f}[M_{n+1} - M_n | \mathcal{F}_n](x) = E^{\mu_f}[1_{\{0\}}(X_{n+1}) - p_0(\alpha) | \mathcal{F}_n](x) = p_0\left(\frac{r_n(x)}{s_n(x)}\right) - p_0(\alpha) \ge 0.$$

We remark that $|M_{n+1} - M_n| = |1_{\{0\}}(X_{n+1}) - p_0(\alpha)| \le 1 + p_0(\alpha)$ for μ_f-a.s.. By Azuma's inequality[1] [4], we see that for $N \in \mathbb{N}$ and $0 < c < 1$,

$$\mu_f(\zeta_N < Ncp_0(\alpha)) = \mu_f(M_N < -N(1-c)p_0(\alpha)) \le \exp\left(-\frac{N(1-c)^2 p_0(\alpha)^2}{2(1+p_0(\alpha))^2}\right).$$

[1] It is also called Azuma-Hoeffding inequality. See Exercise 14.2 in Williams' book [109]

Hence, for any $0 < c < 1$, $\liminf_{N\to\infty} \zeta_N/N \geq cp_0(\alpha)$ for μ_f-a.s.. Thus we obtain the assertion. $\square$

Lemma 3.2.3. *We assume that the condition* (i) *in Theorem 1.2.2 fails. Then,*
(1) *There exists* $\epsilon_0 \in (0, 2(\gamma-1))$ *such that for any* $z \in \mathbb{R}$ *with* $|z-(\gamma-2)| \leq \epsilon_0$, $|\Phi({}^tA_0; z) - (\gamma-2)| > \epsilon_0$.

Let $A(x) = \{n \in \mathbb{N} : |r_n(x)/s_n(x) - (\gamma-2)| \leq \epsilon_0\}$, $B(x) = \mathbb{N} \setminus A(x)$, $C(x) = \{n \in A(x) : n-1 \in B(x)\}$ *and* $D(x) = B(x) \cup C(x)$. *Then we have the following.*
(2) $\mathbb{N}_0(x) \subset D(x)$ *for* $x \in [0,1)$.
(3) $\liminf_{N\to\infty} |B(x) \cap \{1,\dots,N\}|/N \geq p_0(\alpha)/2$, μ_f*-a.s.x.*
(4) *Let* $e_0 = s(p_0(\gamma - 2 + \epsilon_0)) < \log 2$. *Then,*

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} s\left(p_0\left(\frac{r_n(x)}{s_n(x)}\right)\right) \leq \log 2 - \frac{(\log 2 - e_0)p_0(\alpha)}{2}, \ \mu_f\text{-a.s.x.}$$

Proof. (1) This is a direct consequence of the assumption that the condition (i) in Theorem 1.2.2 fails, that is, $\Phi({}^tA_0; \gamma-2) \neq \gamma - 2$.

(2) It is sufficient to show that $\mathbb{N} \setminus D(x) \subset \mathbb{N}_1(x)$. We see that $\mathbb{N} \setminus D(x) = A(x) \cap (\mathbb{N} \setminus C(x)) = \{n \in A(x) : n-1 \in A(x)\}$. We assume that there exists $n \in \mathbb{N} \setminus D(x)$ such that $n \in \mathbb{N}_0(x)$. Since $n-1 \in A(x)$, we have that $|r_{n-1}(x)/s_{n-1}(x) - (\gamma-2)| \leq \epsilon_0$. Since $n \in \mathbb{N}_0(x)$, $r_n(x)/s_n(x) = \Phi({}^tA_0; r_{n-1}(x)/s_{n-1}(x))$. By the assertion (1), we see that $|r_n(x)/s_n(x) - (\gamma-2)| > \epsilon_0$. But this is contradict to $n \in A(x)$.

(3) By the assertion (2), we see that $|\mathbb{N}_0(x) \cap \{1,\dots,N\}| \leq |D(x) \cap \{1,\dots,N\}|$. We have that $|C(x) \cap \{1,\dots,N\}| \leq |B(x) \cap \{1,\dots,N\}|$ for any $N \geq 1$, by the injectivity of the map $h : C(x) \to B(x)$ given by $h(n) = n-1$. Then we see that $|D(x) \cap \{1,\dots,N\}| \leq 2|B(x) \cap \{1,\dots,N\}|$, and then, $|\mathbb{N}_0(x) \cap \{1,\dots,N\}| \leq 2|B(x) \cap \{1,\dots,N\}|$, for any $N \geq 1$.

By Lemma 3.2.2,

$$\liminf_{N\to\infty} \frac{|B(x) \cap \{1,\dots,N\}|}{N} \geq \frac{p_0(\alpha)}{2}, \mu_f\text{-a.s.x.}$$

Thus we obtain the assertion (3).

(4) By noting the definition of $B(x)$, we see that
$s(p_0(r_n(x)/s_n(x))) < \max\{s(p_0(\gamma-2-\epsilon_0)), s(p_0(\gamma-2+\epsilon_0))\} = e_0$ for any $x \in [0,1)$ and $n \in B(x)$.

Now we have that

$$\frac{1}{N}\sum_{n=1}^{N} s\left(p_0\left(\frac{r_n(x)}{s_n(x)}\right)\right) = \frac{1}{N}\left(\sum_{n\in A(x), n\leq N} + \sum_{n\in B(x), n\leq N}\right) s\left(p_0\left(\frac{r_n(x)}{s_n(x)}\right)\right).$$

Let $\xi_N(x) = |B(x) \cap \{1, \ldots, N\}|/N$. Then, by noting that $s(p_0(r_n(x)/s_n(x))) \leq \log 2$, we see that

$$\frac{1}{N} \sum_{n \in A(x), n \leq N} s\left(p_0\left(\frac{r_n(x)}{s_n(x)}\right)\right) \leq \frac{|A(x) \cap \{1, \ldots, N\}|}{N} \log 2 = (1 - \xi_N(x)) \log 2.$$

Now we have that

$$\frac{1}{N} \sum_{n \in B(x), n \leq N} s\left(p_0\left(\frac{r_n(x)}{s_n(x)}\right)\right) \leq \xi_N(x) e_0.$$

By noting that $e_0 < \log 2$, we see that

$$\limsup_{N \to \infty} ((1 - \xi_N(x)) \log 2 + \xi_N(x) e_0) \leq \log 2 - (\log 2 - e_0) \liminf_{N \to \infty} \xi_N(x).$$

By the assertion (3), we see that $\liminf_{N \to \infty} \xi_N(x) \geq p_0(\alpha)/2 > 0$ for μ_f-a.s.x. Thus we obtain the assertion (4). $\square$

Now we show Theorem 1.2.2 (1). We remark that $\Phi(cA; z) = \Phi(A; z)$ for any constant $c > 0$ and the conditions (A1) - (A3) remain valid for (cA_0, cA_1). Then, we can assume that $d_0 = 1$ and $b_1 = 1$.

By computation, we see that

$$A_0 = \begin{pmatrix} 1/2 & 0 \\ c_0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 4c_0 + 1 & 1 \\ 2c_0 & 2(1 + c_0) \end{pmatrix},$$

and $f(x) = \dfrac{x}{-2c_0 x + 1 + 2c_0}$ satisfies the equation (1.2.1). This completes the proof of Theorem 1.2.2 (1).

Now we show Theorem 1.2.2 (2). We assume that the condition (i) fails. Then, by Lemma 3.1.3, we have that for μ_f-a.s.x,

$$\limsup_{N \to +\infty} \frac{-\log R_N(x)}{N} = \limsup_{N \to \infty} \frac{L_N(x)}{N} = \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} s\left(p_0\left(\frac{r_n(x)}{s_n(x)}\right)\right).$$

Then, by noting Lemma 3.2.3(4) and Lemma 3.1.4(1), we obtain the desired result.

We can show the assertion in the same manner if the condition (ii) fails.

These complete the proof of Theorem 1.2.2(2).

3.3 A relationship with stationary measures

In this section, we state a relationship between a certain class of de Rham's functional equations and stationary measures.

We state a general setting. Let G be a semigroup and μ be a probability measure on G. Let M be a topological space. We assume that G acts on M measurably, that is, there is a map from $(g, x) \in G \times M$ to $g \cdot x \in M$ satisfying the following conditions :
(1) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for any $g_1, g_2 \in G$ and $x \in M$.
(2) $x \mapsto g \cdot x$ is measurable map on M for any $g \in G$.

We say that a probability measure ν on M is a *μ-stationary measure* if

$$\nu(B) = \int_G \nu(h^{-1}B)\mu(dh), \tag{3.3.1}$$

for any $B \in \mathcal{B}(M)$. Furstenberg [36] Lemma 1.2 showed that if M is a compact metric space, then there exists a μ-stationary measure.

Let

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2; \mathbb{R}) : ad > bc, b \geq 0, d > 0, 0 < a + b \leq c + d \right\},$$

and, $M = [0, 1]$. Then G is a semigroup. We define a continuous action of G to M by $A \cdot z = \Phi(A; z)$. For (A_0, A_1) satisfying (A1)-(A3), we see that $A_0, A_1 \in G$. Let μ be a probability measure on G such that $\mu(\{A_0\}) = \mu(\{A_1\}) = 1/2$. Then we have the following.

Lemma 3.3.1. (1) *For $k \geq 1$,*

$$\begin{cases} A_0^{-1}(f(I_k(x))) = f(I_{k-1}(2x)), \ A_1^{-1}(f(I_k(x))) = \emptyset & x \in [0, 1/2) \\ A_0^{-1}(f(I_k(x))) = \emptyset, \ A_1^{-1}(f(I_k(x))) = f(I_{k-1}(2x - 1)) & x \in [1/2, 1). \end{cases}$$

(2) *For any μ-stationary measure ν and $k \geq 1$,*

$$\nu(f(I_k(x))) = \begin{cases} \nu(f(I_{k-1}(2x)))/2 & x \in [0, 1/2) \\ \nu(f(I_{k-1}(2x - 1)))/2 & x \in [1/2, 1). \end{cases}$$

(3) *There exists exactly one μ-stationary measure ν.*

Proof. (1) By Lemma 3.1.1(1), we see that
$f(I_k(x)) = \Phi(A_{X_1(x)} \cdots A_{X_k(x)}; [0, 1)) = \Phi(A_{X_1(x)}; \Phi(A_{X_2(x)} \cdots A_{X_k(x)}; [0, 1)))$.
We see that $f(I_{k-1}(2x)) = \Phi(A_{X_2(x)} \cdots A_{X_k(x)}; [0, 1)) = A_0^{-1}(f(I_k(x)))$, $x \in [0, 1/2)$, and, $f(I_{k-1}(2x - 1)) = \Phi(A_{X_2(x)} \cdots A_{X_k(x)}; [0, 1)) = A_1^{-1}(f(I_k(x)))$,

$x \in [1/2,1)$. Since $\Phi(A_0;[0,1)) \cap \Phi(A_1;[0,1)) = \emptyset$, $A_1^{-1}(f(I_k(x))) = \emptyset$, $x \in [0,1/2)$, and, $A_0^{-1}(f(I_k(x))) = \emptyset$, $x \in [1/2,1)$. Thus we have the assertion (1).

(2) By noting the assertion (1) and (3.3.1), we obtain the desired result.

(3) Let ν_i, $i = 0,1$, be two μ-stationary measures. By the assertion (2), we see that $\nu_0(f(I_k(x))) = \nu_1(f(I_k(x)))$ for $k \geq 1$, $x \in [0,1)$. Let $\mathcal{C} = \left\{ f(\sum_{i=1}^k 2^{-j}X_j(x)) : k \geq 1, x \in [0,1) \right\} = \left\{ f(l/2^k) : 0 \leq l \leq 2^{k-1}, k \geq 1 \right\}$. Then, we have that $\nu_0([a,b)) = \nu_1([a,b))$ for $a,b \in \mathcal{C}$. Since f is continuous on $[0,1]$, $\mathcal{C}$ is dense in $[0,1]$. Thus we see that $\nu_0 = \nu_1$. $\square$

Lemma 3.3.2. *Let $g : [0,1] \to [0,1]$ be the inverse function of the solution f of (1.2.1). Then,*
(1) *g is continuous and strictly increasing. Hence, μ_g is well-defined.*
(2) *μ_f is singular if and only if μ_g is so.*

Proof. (1) Noting that f is continuous and strictly increasing on $[0,1]$, $f(0) = 0$ and $f(1) = 1$, we obtain the desired result.

(2) Since $l([a,b)) = \mu_f\left(f^{-1}([a,b))\right) = \mu_g\left(g^{-1}([a,b))\right)$ for $0 \leq a \leq b \leq 1$, we see that $l(B) = \mu_f\left(f^{-1}(B)\right) = \mu_g\left(g^{-1}(B)\right)$ for any Borel set B.

We assume that μ_f is singular. Then, there exists a Borel set B_0 such that $\mu_f(B_0) = 0$ and $l(B_0) = 1$. Then, $\mu_g\left(g^{-1}(B_0)\right) = 1$ and $l\left(g^{-1}(B_0)\right) = \mu_f\left(f^{-1}(g^{-1}(B_0))\right) = \mu_f(B_0) = 0$. Thus we see that μ_g is singular.

We assume that μ_g is singular. Then, we see that μ_f is singular in the same manner as in the above argument. $\square$

The following theorem gives a necessary and sufficient condition for the regularity of the stationary measure in this setting.

Theorem 3.3.3. *Let the conditions* (*i*) *and* (*ii*) *as in Theorem 1.2.2 and ν be a unique μ-stationary measure. Then, we have*
(1) *ν is absolutely continuous if and only if both* (*i*) *and* (*ii*) *hold.*
(2) *ν is singular if and only if either* (*i*) *or* (*ii*) *fails.*

Proof. It is sufficient to show "if" parts.

(1) By noting Theorem 1.2.2(1), we have that $f(x) = x/(-2c_0x + 2c_0 + 1)$ and then $g(y) = (2c_0+1)y/(2c_0y+1)$. By Lemma 3.3.2(2), we have that μ_g is absolutely continuous and obtain the assertion (1).

(2) We see that $\mu_g(f(I_k(x))) = \mu_g(g^{-1}(I_k(x))) = 2^{-k}$, $x \in [0,1)$, $k \geq 1$. By Lemma 3.3.1(1),
$\mu_g\left(f(I_k(x))\right) = \dfrac{1}{2}\left(\mu_g\left(A_0^{-1}(f(I_k(x)))\right) + \mu_g\left(A_1^{-1}(f(I_k(x)))\right)\right)$, $x \in [0,1)$, $k \geq 1$. Then we see that (3.3.1) holds for $[a,b)$, $a,b \in \mathcal{C}$ and that μ_g is a μ-stationary measure. By noting Theorem 1.2.2(2), we have that μ_f is singular.

By Lemma 3.3.2(2), we have that μ_g is singular and obtain the assertion (2). $\square$

3.4 Examples and Remarks

The following example concerns Lebesgue's singular functions.

Example 3.4.1. Let us define 2×2 real matrices $A_{p,0}, A_{p,1}$, $p \in (0,1)$, by

$$A_{p,0} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \ A_{p,1} = \begin{pmatrix} 1-p & p \\ 0 & 1 \end{pmatrix}.$$

Then, $(A_0, A_1) = (A_{p,0}, A_{p,1})$ satisfies the conditions (A1)-(A3).

Let f_p be the solution of (1.2.1) for $(A_0, A_1) = (A_{p,0}, A_{p,1})$. By the main theorems, we immediately have the following.
(1) μ_{f_p} is absolutely continuous if $p = 1/2$, and μ_{f_p} is singular if $p \neq 1/2$.
(2) There exists a Borel set K_p such that $\mu_{f_p}(K_p) = 1$ and $\dim_H(K_p) \leq s(p)/\log 2$.
(3) $\mu_{f_p}(K) = 0$ for any Borel set K with $\dim_H(K) < s(p)/\log 2$.

The following example concerns the range of self-interacting walks on an interval in [83]. This is used in Section 1.3 and Chapter 4.

Example 3.4.2. Let $x_u = 2/(1 + \sqrt{1+8u^2})$, $u \geq 0$. Let $\tilde{A}_{u,i}$, $i = 0, 1$, be two 2×2 real matrices given by

$$\tilde{A}_{u,0} = \begin{pmatrix} x_u & 0 \\ -u^2 x_u^2 & 1 \end{pmatrix}, \ \tilde{A}_{u,1} = \begin{pmatrix} 0 & x_u \\ -u^2 x_u^2 & 1 - u^2 x_u^2 \end{pmatrix}, \ u \geq 0.$$

Let $0 < u < \sqrt{3}$. Then $(A_0, A_1) = (\tilde{A}_{u,0}, \tilde{A}_{u,1})$ satisfies the conditions (A1)-(A3). Let g_u be the solution of (1.2.1) for $(A_0, A_1) = (\tilde{A}_{u,0}, \tilde{A}_{u,1})$. We remark that $\gamma = (1 - u^2 x_u^2)/x_u = (1 + x_u)/2x_u$. By the definition of x_u, we see that each of the conditions in Theorem 1.2.2 is equivalent to $x_u \neq 1/2$, that is, $u \neq 1$. Then, by Theorem 1.2.2, we have that μ_{g_u} is singular for $0 < u < \sqrt{3}$ and $u \neq 1$, and absolutely continuous for $u = 1$.

Let $0 < u < 1$. Then we have that $x_u > 1/2$, $\alpha = \min\{0, -1/2, -u^2 x_u\} = -1/2$, $\beta = 0$ and $\gamma < 3/2$. Hence we see that $\gamma - 2 < \alpha$, in particular, $\gamma - 2 \notin [\alpha, \beta]$. By Theorem 1.2.1, we see that there exists a Borel set $\tilde{K}_u$ such that $\dim_H(\tilde{K}_u) \leq s(p_0(\alpha))/\log 2 = s(x_u)/\log 2$ and $\mu_{g_u}(\tilde{K}_u) = 1$ and that $\mu_{g_u}(K) = 0$ for any Borel set K with $\dim_H(K) < s(p_0(\beta))/\log 2 = s(2x_u/(1 + x_u))/\log 2$.

Remark 3.4.3. (1) Pincus [89], [90] obtained results similar to Theorem 3.3.3. Hata [50] Corollary 7.4 showed the singularity of the solution of (1.2.1) under the assumptions similar to the ones in [90] Theorem 2.1.
(2) Let $T : [0,1) \to [0,1)$ be given by $T(x) = 2x \mod 1$. Then, by computation,

$$\mu_f\left(T^{-1}(A)\right) = \int_A \left(\frac{d\Phi(A_0; \cdot)}{dz}\left(f(y)\right) + \frac{d\Phi(A_1; \cdot)}{dz}\left(f(y)\right) \right) \mu_f(dy), \; A \in \mathcal{B}([0,1)).$$

We see that T is a non-singular transformation on $[0,1)$ with respect to μ_f, that is, $\mu_f \circ T^{-1} \ll \mu_f$ and $\mu_f \ll \mu_f \circ T^{-1}$. We remark that μ_f is *not* invariant with respect to T in some cases.

Chapter 4

On the range of self-interacting random walks on an integer interval

This chapter will be based on [83]. In Section 4.1, we give some preliminaries. In Section 4.2, we show the main results.

4.1 Preliminaries

We briefly state our settings by following [24] and [51]. See the references for details.

For each $n \in \mathbb{N} \cup \{0\}$, let

$$W(n) = \{(\omega(0), \omega(1) \dots \omega(n)) \in \mathbb{Z}^{n+1} : \omega(0) = 0, |\omega(i) - \omega(i+1)| = 1,\ 0 \le i \le n-1\}.$$

Let $W^* = \cup_{n=0}^{\infty} W(n)$. Let $L(\omega) = n$ for $\omega \in W(n)$. For $\omega \in W^*$, we define $T_i^M(\omega)$, $i, M \in \mathbb{N} \cup \{0\}$, by $T_0^M(\omega) = 0$,

$$T_i^M(\omega) = \min\left\{ j > T_{i-1}^M(\omega) : \omega(j) \in 2^M \mathbb{Z} \setminus \{\omega(T_{i-1}^M(\omega))\} \right\}, i \ge 1.$$

Let $T_i^M(\omega) = +\infty$ if the above minimum does not exist.

We define a decimation map $Q_M : W^* \to W^*$, $M \in \mathbb{N}$, by $(Q_M \omega)(i) = \omega(T_i^M(\omega))$ for i such that $T_i^M(\omega) < +\infty$. Let Q_0 be the identity map on W^*. Let $(2^{-M} Q_M \omega)(i) = 2^{-M} \omega(T_i^M(\omega))$. Then, $2^{-M} Q_M \omega \in W^*$ and $L(2^{-M} Q_M \omega) = k$, where $k = \max\{i : T_i^M(\omega) < \infty\}$. Let $W_{N,+(resp.-)} = \{\omega \in W^* : L(\omega) = T_1^N(\omega), \omega(T_1^N(\omega)) = +(\text{resp.}-)2^N\}$ and $W_N = W_{N,+} \cup W_{N,-}$.

For $\omega \in W_{N+n,+}$, let $\omega' = 2^{-N} Q_N \omega$. For $1 \le j \le L(\omega')$, we let $\omega_j = \left(0,\, \omega(T_{j-1}^N(\omega) + 1) - \omega(T_{j-1}^N(\omega)), \dots, \omega(T_j^N(\omega)) - \omega(T_{j-1}^N(\omega))\right) \in W_N$, and, $\widetilde{\omega}_j = \text{sign}\left(\omega(T_j^N(\omega)) - \omega(T_{j-1}^N(\omega))\right) \omega_j \in W_{N,+}$.

Now we will define a probability measure $P^u_{N,\pm}$, $u \geq 0$, on $W_{N,\pm}$ by induction on N in the following manner. We recall that $x_u = 2/(1+\sqrt{1+8u^2})$. Let $P^u_{1,+}(\{\omega\}) = u^{L(\omega)-2} x_u^{L(\omega)-1}$, $\omega \in W_{1,+}$, where we adopt the conventions $0^0 = 1$ and $0^n = 0$, $n \geq 1$. For $\omega \in W_{N+1,+}$, let

$$P^u_{N+1,+}(\{\omega\}) = P^u_{1,+}(\{\omega'\}) \prod_{i=1}^{L(\omega')} P^u_{N,+}(\{\widetilde{\omega}_i\}). \tag{4.1.1}$$

We define $P^u_{N,-}(\{\omega\}) = P^u_{N,+}(\{-\omega\})$ for $\omega \in W_{N,-}$, $N \in \mathbb{N}$. Let P^u_N be a probability measure on W_N given by $P^u_N = (P^u_{N,+} + P^u_{N,-})/2$.

We denote the set of the paths of infinite length by

$$W_\infty = \left\{ (\omega(0), \omega(1), \dots) \in \mathbb{Z}^{\mathbb{N}\cup\{0\}} : \omega(0) = 0,\ |\omega(i) - \omega(i+1)| = 1,\ i \geq 0 \right\}.$$

Let the σ-algebra on this set be the family of subsets which is generated by cylinder sets. By [26], Proposition 2.5, there exists a probability measure P^u on W_∞ such that

$$P^u\left(\{\omega \in W_\infty : \omega(j) = \widetilde{\omega}(j),\ 0 \leq j \leq L(\widetilde{\omega})\}\right) = \frac{1}{2} P^u_{N,+(\text{resp.}-)}(\{\widetilde{\omega}\}),$$

for any $\widetilde{\omega} \in W_{N,+(\text{resp.}-)}$, $N \geq 1$.

4.2 Range of random walk on the interval $[-2^n, 2^n]$ and its scaling limit

Here and henceforth, we assume that $u > 0$.

First we will show Theorem 1.3.2. The main ingredient of the proof is to show that $g_u(k/2^n) := P^u_{n,+}(R_n \leq 2^n + k - 1)$ satisfies (1.3.1) on the dyadic rationals. This depends heavily on the definition of $P^u_{n,+}$ in Section 2. Then, we will see that the right continuous modification of g_u satisfies (1.3.1) on $[0, 1]$. Next, we will show that the distribution of $R_n/2^n - 1$ converges to g_u weakly as $n \to \infty$ and examine the regularity of g_u.

We remark that $P^u(R_n = 2^n + k) = P^u_{n,+}(R_n = 2^n + k)$, $0 \leq k \leq 2^n$, $n \geq 1$.

Lemma 4.2.1.

$$P^u_{N,+}\left(\frac{R_N}{2^N} - 1 \geq \frac{k}{2^n}\right) = P^u_{n,+}\left(\frac{R_n}{2^n} - 1 \geq \frac{k}{2^n}\right),$$

for any $N \geq n$, $0 \leq k \leq 2^n$ and $n \geq 1$.

Proof. Let $N > n$. Then,

$$\begin{aligned}
P^u_{N,+}\left(\frac{R_N}{2^N}-1\geq\frac{k}{2^n}\right) &= P^u_{N,+}\left(\left\{\omega\in W_{N,+}:\omega\,\text{hits the point}\,\{-2^{N-n}k\}\right\}\right)\\
&= P^u_{N,+}\left(\left\{\omega:Q_{N-n}\omega\,\text{hits the point}\,\{-2^{N-n}k\}\right\}\right)\\
&= P^u_{N,+}\left(\left\{\omega:2^{-(N-n)}Q_{N-n}\omega\,\text{hits the point}\,\{-k\}\right\}\right)\\
&= P^u_{n,+}\left(\{\zeta\in W_{n,+}:\zeta\,\text{hits the point}\,\{-k\}\}\right)\\
&= P^u_{n,+}\left(\frac{R_n}{2^n}-1\geq\frac{k}{2^n}\right),
\end{aligned}$$

where in the fourth equality we have used [26] Proposition 2.2. □

Definition 4.2.2. (1) Let g_u be a function on D given by $g_u((k+1)/2^n) = P^u_{n,+}(R_n \leq 2^n + k)$, $-1 \leq k \leq 2^n - 1$. By Lemma 4.2.1, this is well-defined. We immediately see that $g_u(x)$ is increasing and $g_u(0) = 0, g_u(1) = 1$.
(2) Let $\tilde{g}_u$ be a function on $[0,1]$ given by $\tilde{g}_u(x) = \lim_{y\in D, y>x, y\to x} g_u(y)$, $0 \leq x < 1$ and $\tilde{g}_u(1) = 1$. This is right continuous.

The following is a key proposition.

Proposition 4.2.3. *The function g_u satisfies* (1.3.1) *on D, that is,*

$$P^u_{n+1,+}(R_{n+1}\leq 2^{n+1}+k) = \begin{cases}\Phi\left(A_{u,0};P^u_{n,+}(R_n\leq 2^n+k)\right) & -1\leq k\leq 2^n-1\\ \Phi\left(A_{u,1};P^u_{n,+}(R_n\leq k)\right) & 2^n-1\leq k\leq 2^{n+1}-1.\end{cases}$$

Proof. If $k = -1$, we have that $\Phi\left(A_{u,0}; P^u_{n,+}(R_n \leq 2^n + k)\right) = \Phi(A_{u,0}; 0) = 0 = P^u_{n+1,+}(R_{n+1} \leq 2^{n+1} + k)$. If $k = 2^n - 1$, we have that
$\Phi\left(A_{u,0}; P^u_{n,+}(R_n \leq 2^n + k)\right) = \Phi(A_{u,0}; 1) = \Phi(A_{u,1}; 0) = \Phi\left(A_{u,1}; P^u_{n,+}(R_n \leq k)\right)$.
Then, it is sufficient to show this assertion in the following two cases. For any $\omega \in W_{n+1,+}$, define $(\omega', \widetilde{\omega}_1, \ldots, \widetilde{\omega}_{L(\omega')})$ as in Section 4.1.

Case 1. $0 \leq k \leq 2^n - 1$. We have

$$P^u_{n+1,+}(R_{n+1}\leq 2^{n+1}+k) = \sum_{m=1}^{\infty} P^u_{n+1,+}\left(\left\{\omega:L(\omega')=2m,\,R_{n+1}(\omega)\leq 2^{n+1}+k\right\}\right).$$

Since $0 \leq k \leq 2^n - 1$, we see that $\omega' \in W_{1,+}$ does not hit -1 for any $\omega \in W_{n+1,+}$ with $R_{n+1}(\omega) \leq 2^{n+1} + k$. Then we see that

$$\left\{\omega:L(\omega')=2m,\,R_{n+1}(\omega)\leq 2^{n+1}+k\right\}$$

$$= \{\omega:\omega'=(0,1,0,1,\ldots,0,1,2),\,L(\omega')=2m,\,R_n(\widetilde{\omega}_{2i-1})\leq 2^n+k,\,1\leq i\leq m\}.$$

By (4.1.1), we see that

$$P^u_{n+1,+}\left(\left\{\omega : L(\omega') = 2m,\ R_{n+1}(\omega) \le 2^{n+1} + k\right\}\right)$$
$$= P^u_{1,+}\left(\{\zeta : \zeta = (0,1,0,1,\dots,0,1,2),\ L(\zeta) = 2m\}\right) \cdot P^u_{n,+}(R_n \le 2^n + k)^m.$$
$$= u^{2m-2} x_u^{2m-1} P^u_{n,+}(R_n \le 2^n + k)^m.$$

Then,

$$\begin{aligned} P^u_{n+1,+}(R_{n+1} \le 2^{n+1} + k) &= \sum_{m=1}^{\infty} u^{2m-2} x_u^{2m-1} P^u_{n,+}(R_n \le 2^n + k)^m \\ &= \Phi\left(A_{u,0}; P^u_{n,+}(R_n \le 2^n + k)\right), \end{aligned}$$

which is the desired result.

Case 2. $2^n \le k \le 2^{n+1} - 1$.

Since $L(\omega') = 2m$, we can write $\omega' = (0, \epsilon_1, 0, \epsilon_2, \dots, 0, \epsilon_{m-1}, 0, 1, 2)$, $\epsilon_i \in \{\pm 1\}$, $1 \le i \le m-1$. Then we see that

$$\left\{\omega : L(\omega') = 2m,\ R_{n+1}(\omega) \le 2^{n+1} + k\right\}$$
$$= \bigcup_{i=0}^{m-1} \left\{\omega : \sharp(j : \epsilon_j = -1) = i,\ L(\omega') = 2m,\ R_{n+1}(\omega) \le 2^{n+1} + k\right\}.$$

We remark that the union in the above is disjoint.

For $1 \le i \le m-1$,

$$\left\{\omega : \sharp(j : \epsilon_j = -1) = i, L(\omega') = 2m, R_{n+1}(\omega) \le 2^{n+1} + k\right\}$$

$$\begin{aligned} = \bigcup_{1 \le n_1 < n_2 < \cdots < n_i \le m-1} &\{\omega : \{j : \epsilon_j = -1\} = \{n_1 < n_2 < \cdots < n_i\}, \\ &L(\omega') = 2m, R_{n+1}(\omega) \le 2^{n+1} + k\}. \end{aligned}$$

We remark that the union in the above is disjoint.

By (4.1.1),

$$\begin{aligned} P^u_{n+1,+}(&\{\omega : \{j : \epsilon_j = -1\} = \{n_1 < n_2 < \cdots < n_i\}, \\ &L(\omega') = 2m, R_{n+1}(\omega) \le 2^{n+1} + k\}) \end{aligned}$$

$$\begin{aligned} = P^u_{n+1,+}(&\{\omega : \{j : \epsilon_j = -1\} = \{n_1 < n_2 < \cdots < n_i\}, \\ &L(\omega') = 2m, R_n(\widetilde{\omega}_{2n_j}) \le k, 1 \le j \le i\}) \end{aligned}$$

$$= P^u_{1,+}\left(\{\omega' : \{j : \epsilon_j = -1\} = \{n_1 < n_2 < \cdots < n_i\}, L(\omega') = 2m\}\right) P^u_{n,+}(R_n \le k)^i$$
$$= u^{2m-2} x_u^{2m-1} \left(P^u_{n,+}(R_n \le k)\right)^i.$$

Since the number of choices $\{n_1 < n_2 < \cdots < n_i\} \subset \{1, \ldots, m-1\}$ is equal to $\binom{m-1}{i}$, we see that

$$P^u_{n+1,+}(\{\omega : \sharp(j : \epsilon_j = -1) = i, L(\omega') = 2m, R_{n+1}(\omega) \le 2^{n+1} + k\})$$

$$= \sum_{1 \le n_1 < n_2 < \cdots < n_i \le m-1} P^u_{n+1,+}(\{\omega : \{j : \epsilon_j = -1\} = \{n_1 < n_2 < \cdots < n_i\},$$
$$L(\omega') = 2m, R_{n+1}(\omega) \le 2^{n+1} + k\})$$
$$= \binom{m-1}{i} u^{2m-2} x_u^{2m-1} \left(P^u_{n,+}(R_n \le k)\right)^i, \ 1 \le i \le m-1.$$

This is also true for $i = 0$.

Therefore, by summing up over i, we see that

$$P^u_{n+1,+}\left(\left\{\omega : L(\omega') = 2m, R_{n+1}(\omega) \le 2^{n+1} + k\right\}\right) = u^{2m-2} x_u^{2m-1} \left(1 + P^u_{n,+}(R_n \le k)\right)^{m-1}.$$

By summing up over m, we see that

$$P^u_{n+1,+}(R_{n+1} \le 2^{n+1} + k) = \sum_{m=1}^{\infty} u^{2m-2} x_u^{2m-1} \left(1 + P^u_{n,+}(R_n \le k)\right)^{m-1}$$
$$= \Phi\left(A_{u,1}; P^u_{n,+}(R_n \le k)\right).$$

This completes the proof. □

Next, we will show that $\tilde{g}_u$, which is the right continuous modification of g_u, satisfies (1.3.1) on $[0, 1]$, not only on D. We define some notation. Let $X_n(x) = \lfloor 2^n x \rfloor - 2\lfloor 2^{n-1} x \rfloor$ and $\zeta_n(x) = \sum_{k=1}^n 2^{-k} X_k(x)$, $x \in [0, 1)$, $n \ge 1$. Then, $\zeta_n(x) \le x < \zeta_n(x) + 2^{-n}$, $x \in [0, 1)$, $n \ge 1$. Let $\gamma_u = 1/\Phi(A_{u,0}; 1)$. Let $p_{u,0}(z) = (z + 1)/(z + \gamma_u)$ and $p_{u,1}(z) = 1 - p_{u,0}(z)$ for $z > -\gamma_u$. Let

$$\begin{pmatrix} p_{u,n}(x) & q_{u,n}(x) \\ r_{u,n}(x) & s_{u,n}(x) \end{pmatrix} = A_{u,X_1(x)} \cdots A_{u,X_n(x)}, \ x \in [0, 1), n \ge 1.$$

Proposition 4.2.4. (1) *$g_u(\zeta_m(x)) = \Phi(A_{u,X_1(x)} \cdots A_{u,X_m(x)}; 0)$ and $g_u(\zeta_m(x) + 2^{-m}) = \Phi(A_{u,X_1(x)} \cdots A_{u,X_m(x)}; 1)$, $x \in [0, 1)$, $m \ge 1$.*
(2) *$\tilde{g}_u = g_u$ on D.*
(3) *$\tilde{g}_u$ satisfies the equation (1.3.1) on $[0, 1]$.*

Proof. (1) Using (1.3.1), we can show the assertion by induction in n.

(2) By noting the definition of g_u and $\tilde{g}_u$, we have that $\tilde{g}_u(1) = 1 = g_u(1)$. Let $x \in D \cap [0,1)$. Then, there exists N such that $X_n(x) = 0$, $n > N$.

Then, by using the assertion (1),

$$\lim_{l\to\infty} g_u(x + 2^{-l}) = \lim_{l\to\infty} g_u(\zeta_l(x) + 2^{-l})$$
$$= \lim_{m\to\infty} \Phi\left(A_{u,X_1(x)} \cdots A_{u,X_N(x)}; \Phi(A_{u,0}^m; 1)\right).$$

Since $\Phi(A_{u,0}; \cdot)$ is a contraction map on $[0,1]$, $\lim_{m\to\infty} \Phi(A_{u,0}^m; 1) = 0$. Then, by using the assertion (1),

$$\lim_{m\to\infty} \Phi\left(A_{u,X_1(x)} \cdots A_{u,X_N(x)}; \Phi(A_{u,0}^m; 1)\right) = \Phi\left(A_{u,X_1(x)} \cdots A_{u,X_N(x)}; 0\right) = g_u(x).$$

Thus we obtain the assertion (2).

(3) Since $\tilde{g}_u(1) = 1$ and $\Phi(A_{u,1}; 1) = 1$, (1.3.1) holds for $x = 1$.

Let $x \in [0, 1/2)$. Then there exists a sequence $\{x_n\}_n \subset D \cap [0, 1/2)$ such that $x_n \downarrow x$. By using Proposition 4.2.3 and the assertion (2), $\tilde{g}_u(x_n) = \Phi\left(A_{u,0}; \tilde{g}_u(2x_n)\right)$, $n \ge 1$. Since $\Phi(A_{u,0}; \cdot)$ is continuous and $\tilde{g}_u$ is right continuous, we have that $\tilde{g}_u(x) = \Phi\left(A_{u,0}; \tilde{g}_u(2x)\right)$.

In the same manner, we see that $\tilde{g}_u(x) = \Phi\left(A_{u,1}; \tilde{g}_u(2x-1)\right)$ for $x \in [1/2, 1)$. Thus we obtain the assertion (3). $\square$

Proof of Theorem 1.3.2. First, we show the assertion (1). Let $\widetilde{P}_n^u = P^u \circ ((R_n/2^n) - 1)^{-1}$. Let $\widetilde{P}^u$ be the probability measure on $[0,1]$ whose distribution function is $\tilde{g}_u$ and satisfying $\widetilde{P}^u(\{0\}) = 0$. In other words, we will show that the function f_u in the statement in Theorem 1.3.2 is equal to $\tilde{g}_u$. It suffices to show that $\widetilde{P}_n^u$ converges weakly to $\widetilde{P}^u$, that is, for any continuous function f on $[0,1]$,

$$\lim_{n\to\infty} \int_{[0,1]} f(x)\widetilde{P}_n^u(dx) = \int_{[0,1]} f(x)\widetilde{P}^u(dx). \tag{4.2.1}$$

Let $\epsilon > 0$. Then, $\max_{1\le k\le 2^m} |f(k/2^m) - f((k-1)/2^m)| < \epsilon$ for some m. We have that

$$\left|\int_{[0,1]} f(x)\widetilde{P}_n^u(dx) - \sum_{k=1}^{2^m} f\left(\frac{k}{2^m}\right) \widetilde{P}_n^u\left(\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right)\right)\right| < \epsilon, \tag{4.2.2}$$

and,

$$\left|\int_{[0,1]} f(x)\widetilde{P}^u(dx) - \sum_{k=1}^{2^m} f\left(\frac{k}{2^m}\right) \widetilde{P}^u\left(\left(\frac{k-1}{2^m}, \frac{k}{2^m}\right]\right)\right| < \epsilon, \tag{4.2.3}$$

where we have used $\widetilde{P}^u_n(\{1\}) = P^u(R_n = 2^{n+1}) = P^u_{n,+}(R_n = 2^{n+1}) = 0$ for the first inequality, and, $\widetilde{P}^u(\{0\}) = 0$ for the second.

Let $n > m$. Then, by using Lemma 4.2.1, we see that for $1 \le k \le 2^m$,

$$\widetilde{P}^u_n\left(\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right)\right) = \widetilde{P}^u_m\left(\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right)\right) = g_u\left(\frac{k}{2^m}\right) - g_u\left(\frac{k-1}{2^m}\right).$$

By using Proposition 4.2.4(2), we see that for $1 \le k \le 2^m$,

$$\widetilde{P}^u\left(\left(\frac{k-1}{2^m}, \frac{k}{2^m}\right]\right) = \tilde{g}_u\left(\frac{k}{2^m}\right) - \tilde{g}_u\left(\frac{k-1}{2^m}\right) = g_u\left(\frac{k}{2^m}\right) - g_u\left(\frac{k-1}{2^m}\right).$$

Therefore, we see that

$$\sum_{k=1}^{2^m} f\left(\frac{k}{2^m}\right)\widetilde{P}^u_n\left(\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right)\right) = \sum_{k=1}^{2^m} f\left(\frac{k}{2^m}\right)\widetilde{P}^u\left(\left(\frac{k-1}{2^m}, \frac{k}{2^m}\right]\right).$$

Recalling (4.2.2) and (4.2.3), we see that for any $n > m$,

$$\left|\int_{[0,1]} f(x)\widetilde{P}^u_n(dx) - \int_{[0,1]} f(x)\widetilde{P}^u(dx)\right| < 2\epsilon.$$

Thus we see (4.2.1) and the proof of (1) completes.

The assertion (2) immediately follows from the definition of $\widetilde{P}^u$ and Proposition 4.2.4(3).

Finally, we show the assertion (3). Let $u = 1$. Then, the absolute continuity of $\widetilde{P}^1$ follows from Theorem 1.2.2(1) in Section 1.2.

Lemma 4.2.5. *Let $u \neq 1$. Let $x \in [0,1] \setminus D$. If $\tilde{g}_u$ is differentiable at x and $\tilde{g}'_u(x) \in [0,+\infty)$, then, $\tilde{g}'_u(x) = 0$.*

Proof. We assume that there exists a point $x \in [0,1] \setminus D$ such that $\tilde{g}_u$ is differentiable at x and $\tilde{g}'_u(x) \in (0,+\infty)$.

Since $\tilde{g}_u$ is strictly increasing and $x \notin D$, we have that

$$\tilde{g}'_u(x) = \lim_{n\to\infty} 2^n(\tilde{g}_u(\zeta_n(x)+2^{-n}) - \tilde{g}_u(\zeta_n(x))) = \lim_{n\to\infty} 2^n(g_u(\zeta_n(x)+2^{-n}) - g_u(\zeta_n(x))).$$

Since $\tilde{g}'_u(x) \in (0,+\infty)$,

$$\lim_{n\to\infty} \frac{g_u(\zeta_{n+1}(x) + 2^{-(n+1)}) - g_u(\zeta_{n+1}(x))}{g_u(\zeta_n(x) + 2^{-n}) - g_u(\zeta_n(x))} = \frac{1}{2}.$$

Then, by using Proposition 4.2.4(1),

$$p_{u,X_{n+1}(x)}\left(\frac{r_{u,n}(x)}{s_{u,n}(x)}\right) = \frac{g_u(\zeta_{n+1}(x) + 2^{-(n+1)}) - g_u(\zeta_{n+1}(x))}{g_u(\zeta_n(x) + 2^{-n}) - g_u(\zeta_n(x))},$$

and, $\lim_{n\to\infty} p_{u,X_{n+1}(x)}(r_{u,n}(x)/s_{u,n}(x)) = 1/2$. Since $p_{u,1} = 1 - p_{u,0}$, $\lim_{n\to\infty} p_{u,i}(r_{u,n}(x)/s_{u,n}(x)) = 1/2$ for $i = 0, 1$. Now we see that $\lim_{n\to\infty} r_{u,n}(x)/s_{u,n}(x) = \gamma_u - 2$. Since $x \notin D$, there exists infinitely many natural numbers n such that $X_n(x) = i$ for each $i = 0, 1$. Since $r_{u,n+1}(x)/s_{u,n+1}(x) = \Phi\left({}^tA_{u,X_{n+1}(x)}; r_{u,n}(x)/s_{u,n}(x)\right)$, we see that $\Phi\left({}^tA_{u,i}; \gamma_u - 2\right) = \gamma_u - 2$ for each $i = 0, 1$. This is true if and only if $u = 1$. But this contradicts the assumption. □

Let $u \neq 1$. Then, by noting Lemma 4.2.5 and the Lebesgue differentiation theorem, we see that $\tilde{g}'_u = 0$ a.e. and $\widetilde{P}^u$ is singular. These complete the proof of (3). □

Proof of Theorem 1.3.4. In this proof, we write $\Phi_{u,i}(z) = \Phi(A_{u,i}; z)$, $i = 0, 1$. We first explain the meaning of the value $u = \sqrt{3}$. By explicit calculation, we see that if $u < \sqrt{3}$, then, $0 < \Phi'_{u,1}(z) < 1$, $z \in [0, 1]$, namely, $\Phi_{u,1}(\cdot)$ is a contraction map on $[0, 1]$, and de Rham's theory [96] is applicable to $(A_{u,0}, A_{u,1})$ in the form of Section 1.2. In contrast, this property fails if $u \geq \sqrt{3}$. In fact, $\Phi'_{\sqrt{3},1}(z) \leq 1$, with $\Phi'_{\sqrt{3},1}(z) = 1$ implying $z = 1$. If $u > \sqrt{3}$, there exists $z_0 = z_0(u) \in (0, 1)$ such that $\Phi'_{u,1}(z) < 1$ for $z < z_0$, and $\Phi'_{u,1}(z) > 1$ for $z > z_0$.

We now turn to the proof of the theorem. We denote $f^{m+1} = f \circ f^m$, $m \geq 1$, for $f : [0, 1] \to [0, 1]$.

(1) If $0 < u < \sqrt{3}$, then, $(A_{u,0}, A_{u,1})$ satisfies the conditions (A1) - (A3) in Section 1.2 and hence $\widetilde{P}^u$ has no atoms.

Let $u = \sqrt{3}$. Let $h_i = \Phi_{\sqrt{3},i}$, $i = 0, 1$. Then we have the following results by computations.

Lemma 4.2.6. (1) *$h_0(z) < h_1(z)$ for $z \in [0, 1]$.*
(2) *h'_i, $i = 0, 1$, are strictly increasing on $(0, 1)$.*
(3) *$h'_0(z) \leq 3h'_1(z)$ for $z \in (0, 1)$.*
(4) *$h'_0(z) \leq h'_1(z)$ for $z \geq h_1^2(0)$.*

Now it is sufficient to show the following.

$$\lim_{m\to\infty} \max_{1\leq k\leq 2^m} \left\{ g_{\sqrt{3}}\left(\frac{k}{2^m}\right) - g_{\sqrt{3}}\left(\frac{k-1}{2^m}\right) \right\} = 0. \tag{4.2.4}$$

Let $m \geq 3$ and $1 \leq k \leq 2^m$. Let $x_i = X_i((k-1)/2^m)$, $1 \leq i \leq m$. Then,

$$\begin{aligned}
g_{\sqrt{3}}\left(\frac{k}{2^m}\right) - g_{\sqrt{3}}\left(\frac{k-1}{2^m}\right) &= h_{x_1} \circ \cdots \circ h_{x_m}(1) - h_{x_1} \circ \cdots \circ h_{x_m}(0) \\
&= \int_0^1 (h_{x_1} \circ \cdots \circ h_{x_m})'(x)dx \\
&= \int_0^1 h'_{x_1}(h_{x_2} \circ \cdots \circ h_{x_m}(x)) \cdots h'_{x_{m-1}}(h_{x_m}(x))h'_{x_m}(x)dx \\
&\leq \int_0^1 h'_{x_1}(h_1^{m-1}(x)) \cdots h'_{x_{m-1}}(h_1(x))h'_{x_m}(x)dx \\
&\leq \int_0^1 h'_1(h_1^{m-1}(x)) \cdots 3h'_1(h_1(x))3h'_1(x)dx \\
&= 9\int_0^1 (h_1^m)'(x)dx = 9(1 - h_1^m(0)),
\end{aligned}$$

where we have used Proposition 4.2.4 (1) for the first equality, Lemma 4.2.6 (1) and (2) for the fourth inequality, and, Lemma 4.2.6 (3) and (4) for the fifth. Since $h_1^n(0) = n/(n+1)$, $n \geq 1$, we see that $\lim_{n\to\infty} h_1^n(0) = 1$. Thus we see (4.2.4) and the proof of the assertion (1) completes.

(2) Let $x \in D \cap (0,1)$. Let $x_i = X_i(x)$, $i \geq 1$. Then, there exists a unique $m \geq 1$ such that $x_m = 1$ and $x_i = 0$, $i \geq m+1$. Let $\phi = \Phi_{u,x_1} \circ \cdots \circ \Phi_{u,x_{m-1}} \circ \Phi_{u,0}$. Let $n > m$ and $y_i = X_i(x - (1/2^n))$. Then, we have that $y_i = x_i$, $1 \leq i \leq m-1$, $y_m = 0$, $y_i = 1$, $m+1 \leq i \leq n$, and, $y_i = 0$, $i > n$. By noting Proposition 4.2.4 (1) and $\Phi_{u,0}(1) = \Phi_{u,1}(0)$, we have that

$$g_u(x) = \phi(1), \ g_u\left(x - \frac{1}{2^n}\right) = \phi(\Phi_{u,1}^{n-m}(0)). \tag{4.2.5}$$

Note that $\Phi_{u,1}$ is increasing and strictly convex, $\Phi_{u,1}(0) > 0$, $\Phi_{u,1}(1) = 1$, and, $\Phi'_{u,1}(1) > 1$. Therefore, there exists $z_1 \in (0,1)$ such that

$$\Phi_{u,1}(z_1) = z_1, \ \Phi_{u,1}(z) > z, \ z \in (0, z_1), \ \Phi_{u,1}(z) < z, \ z \in (z_1, 1).$$

Then, $z_1 = \lim_{n\to\infty} \Phi_{u,1}^n(0)$ and $\Phi_{u,1}^n(0) \leq z_1 < 1$, $n \geq 1$.

We have that for $n > m$,

$$\begin{aligned}
\widetilde{P}^u\left(\left(x - \frac{1}{2^n}, x\right]\right) &= g_u(x) - g_u\left(x - \frac{1}{2^n}\right) \\
&= \phi(1) - \phi(\Phi_{u,1}^{n-m}(0)) \\
&\geq \phi(1) - \phi(z_1),
\end{aligned}$$

where we have used Proposition 4.2.4 (2) for the first equality, and, (4.2.5) for the second. Letting $n \to \infty$, we have that $\widetilde{P}^u(\{x\}) \geq \phi(1) - \phi(z_1) > 0$.

We can show that $\widetilde{P}^u(\{1\}) > 0$ in the same manner. These complete the proof of the assertion (2). $\square$

Chapter 5

Random sequences with respect to a measure defined by two linear fractional transformations

This chapter will be based on [84]. In Section 5.1, we give proofs of Theorems 1.4.1 and 1.4.2. In Section 5.2, we give a proof of computability of F appearing in Section 1.4.

5.1 Proofs of Theorems 1.4.1 and 1.4.2

We will use the following well-known result.

Proposition 5.1.1 (Schnorr [100])**.** *Let μ be a computable Borel probability measure on $\{0,1\}^{\mathbb{N}}$. Then, $x \in \{0,1\}^{\mathbb{N}}$ is μ-random if and only if $\limsup_{n\to\infty} d(x \upharpoonright n) < +\infty$ for any lower semicomputable μ-martingale d.*

The main ingredients of proofs are to show that N_1 and N_2 defined in the arguments below are constructive μ_{A_0,A_1}-null sets. Hereafter, we fix (A_0, A_1) and let $\mu = \mu_{A_0,A_1}$ for simplicity. The following assertion corresponds to Lemma 3.1.3 (2) in Chapter 3.

Proposition 5.1.2. *Let $x \in \{0,1\}^{\mathbb{N}}$ be μ-random. Then,*

$$\lim_{n\to\infty} \frac{M(x \upharpoonright n)}{n} = 0.$$

Proof. Let

$$N_1 = \bigcup_{k\geq 1}\bigcap_{N\geq 1}\bigcup_{n\geq N}\left\{x\in\{0,1\}^{\mathbb{N}} : \frac{|M(x\restriction n)|}{n} > \frac{1}{k}\right\}.$$

It is sufficient to show that N_1 is a constructive μ-null set. Let

$$V_m^1 = \bigcup_{n\geq 2^m}\left\{\sigma\in\{0,1\}^* : |M(\sigma)|^4 > n^3,\ |\sigma| = n\right\}.$$

Since M is computable, $\{V_m^1\}_m$ is uniformly c.e. It is sufficient to show that the following assertions.
(i) $N_1 \subset [V_m^1]$, $m\geq 1$.
(ii) There exists a computable function $g_1 : \mathbb{N}\to\mathbb{N}$ such that $\mu([V_{g_1(m)}^1]) \leq 2^{-m}$, $m\geq 1$.

Let $x\in N_1$. Then, there exists $k\geq 1$ such that $|M(x\restriction n))| > n/k$ for infinitely many n. Since $(n/k)^4 > n^3$ for large n, $|M(x\restriction n)|^4 > n^3$ for infinitely many n. Hence $x\in[V_m^1]$, $m\geq 1$. This completes the proof of (i).

Now we will show (ii). Here $E^\mu[\cdot]$ denotes the expectation with respect to μ. Let $\mathcal{F}_n = \sigma([\tau] : \tau\in\{0,1\}^n)$. Let $M_n(x) = M(x\restriction n)$, $x\in\{0,1\}^{\mathbb{N}}$.

We now give an upper bound of $\mu([V_m^1])$, $m\geq 1$. Let $E_n = \{x\in\{0,1\}^{\mathbb{N}} : M_n(x)^4 > n^3\}$, $n\geq 1$, and, $F_{k,n} = E_n\cap\left(\bigcap_{2^k\leq i<n} E_i^c\right)$, $2^k\leq n<2^{k+1}$, $k\geq 0$. By using that $\{F_{k,n}\}_n$ are pairwise disjoint and $F_{k,n}\subset E_n$, we see that

$$\begin{aligned}\mu([V_m^1]) &\leq \sum_{k\geq m}\mu\left(\bigcup_{2^k\leq n<2^{k+1}} E_n\right) = \sum_{k\geq m}\sum_{2^k\leq n<2^{k+1}}\mu(F_{k,n})\\ &\leq \sum_{k\geq m}\sum_{2^k\leq n<2^{k+1}} n^{-3/2}E^\mu[M_n^2, F_{k,n}].\end{aligned}$$

Since M is a μ-martingale, we have that

$$E^\mu[M_n^2, F_{k,n}] \leq E^\mu[M_{2^{k+1}}^2, F_{k,n}],\ 2^k\leq n<2^{k+1}.$$

Since $\{F_{k,n}\}_n$ are pairwise disjoint,

$$\sum_{2^k\leq n<2^{k+1}} n^{-3/2}E^\mu[M_n^2, F_{k,n}] \leq 2^{-3k/2}E^\mu\left[M_{2^{k+1}}^2, \bigcup_{2^k\leq n<2^{k+1}} E_n\right].$$

Let $C = 2(-\log_2(p_0(\alpha)) - \log_2 p_1(\beta) + 1)^2$. Then, $|M_{i+1}(x) - M_i(x)|^2 \leq C/2$, and,

$$2^{-3k/2}E^\mu\left[M_{2^{k+1}}^2, \bigcup_{2^k\leq n<2^{k+1}} E_n\right] \leq 2^{-3k/2}E^\mu\left[M_{2^{k+1}}^2\right] \leq 2^{-k/2}C.$$

Thus we have that

$$\mu([V_m^1]) \leq C \sum_{k \geq m} 2^{-k/2} \leq 4C2^{-m/2}.$$

Let $g_1(m) = 2(m + [4C])$. Then, this is computable and $\mu([V_{g_1(m)}^1]) \leq 2^{-m}$. This completes the proof of (ii).

Thus the proof of Proposition 5.1.2 is completed. □

By noting Proposition 5.1.2, the definition of M, and, $\mathcal{H}(p_0(t)) = \mathcal{H}(1 - p_0(t)) = \mathcal{H}(p_1(t))$, we have that for μ-random x,

$$\limsup_{n \to +\infty} \frac{-\log_2 \mu([x \upharpoonright n])}{n} = \limsup_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathcal{H}\left(p_0(F(i, x \upharpoonright n))\right). \tag{5.1.1}$$

$$\liminf_{n \to +\infty} \frac{-\log_2 \mu([x \upharpoonright n])}{n} = \liminf_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathcal{H}\left(p_0(F(i, x \upharpoonright n))\right). \tag{5.1.2}$$

Proof of Theorem 1.4.1. We show the assertion (1) only. The assertion (2) can be shown in the same manner; so we omit the proof of it.

By using the arguments the proof of Lemma 3.2.1 in Chapter 3, $\alpha \leq F(i, x \upharpoonright n) \leq \beta$. Then we have that $p_0(\beta) \geq p_0(F(i, x \upharpoonright n)) \geq p_0(\alpha) \geq 1/2$, and,

$$\mathcal{H}\left(p_{x(i)}(F(i, x \upharpoonright n))\right) = \mathcal{H}\left(p_0(F(i, x \upharpoonright n))\right) \in \left[\mathcal{H}(p_0(\beta)), \mathcal{H}(p_0(\alpha))\right].$$

By noting (5.1.1) and (5.1.2),

$$\mathcal{H}(p_0(\beta)) \leq \liminf_{n \to +\infty} \frac{-\log_2 \mu([x \upharpoonright n])}{n} \leq \limsup_{n \to +\infty} \frac{-\log_2 \mu([x \upharpoonright n])}{n} \leq \mathcal{H}(p_0(\alpha)) \tag{5.1.3}$$

First, we will show that $\mathrm{cdim}(x) \leq \mathcal{H}(p_0(\alpha))$. Let $t > \mathcal{H}(p_0(\alpha))$. By using (5.1.3),

$$\limsup_{n \to +\infty} \frac{-\log_2 \mu([x \upharpoonright n])}{n} < t. \tag{5.1.4}$$

Let $d(\sigma) = 2^{|\sigma|}\mu([\sigma])$, $\sigma \in \{0,1\}^*$. Then, this is a martingale. By noting (5.1.4),

$$\limsup_{n \to \infty} \frac{d(x \upharpoonright n)}{2^{(1-t)n}} = \limsup_{n \to \infty} 2^{tn}\mu([x \upharpoonright n]) = +\infty.$$

Since d is a martingale, the map $\sigma \mapsto d(\sigma)/2^{(1-t)|\sigma|}$ is a t-gale. By noting the definition of the constructive dimension, we see that $\mathrm{cdim}(x) \leq t$. Since $t > \mathcal{H}(p_0(\alpha))$ is taken arbitrarily, $\mathrm{cdim}(x) \leq \mathcal{H}(p_0(\alpha))$.

Second, we will show that $\mathrm{cdim}(x) \geq \mathcal{H}(p_0(\beta))$. Let $s < \mathcal{H}(p_0(\beta))$. Then, by noting (5.1.3),

$$\liminf_{n\to+\infty} \frac{-\log_2 \mu([x \restriction n])}{n} > s. \tag{5.1.5}$$

Let d be a lower semicomputable martingale. Let $\tilde{d}(\sigma) = \dfrac{d(\sigma)}{2^{|\sigma|}\mu([\sigma])}$, $\sigma \in \{0,1\}^*$. Then, $\tilde{d}$ is a lower semicomputable μ-martingale. By using (5.1.5),

$$\begin{aligned} \log_2 \tilde{d}(x \restriction n) &= \log_2 d(x \restriction n) - n - \log_2 \mu([x \restriction n]) \\ &> \log_2 d(x \restriction n) - (1-s)n, \end{aligned} \tag{5.1.6}$$

for sufficiently large n. By noting that x is μ-random and Proposition 5.1.1, $\limsup_{n\to\infty} \tilde{d}(x \restriction n) < +\infty$. By using (5.1.6), $\limsup_{n\to\infty} \dfrac{d(x \restriction n)}{2^{(1-s)n}} < +\infty$. By the definition of constructive dimensions, we see that $\mathrm{cdim}(x) \geq s$. Since $s < \mathcal{H}(p_0(\beta))$ is taken arbitrarily, $\mathrm{cdim}(x) \geq \mathcal{H}(p_0(\beta))$. □

Proof of Theorem 1.4.2 (1). Let $\phi : \{0,1\}^{\mathbb{N}} \to [0,1]$ be the map defined by $\phi(x) = \sum_{i=0}^{\infty} 2^{-i-1} x(i)$. Let f be the distribution function of the image measure $\mu \circ \phi^{-1}$. Then, by using the assumption and arguments in the proof of Theorem 1.2.2 (1) in Section 1.2, we see that f is differentiable, and, its derivative is in $(0, +\infty)$ at any points in $[0,1]$. Hence, we see that for any $x \in \{0,1\}^{\mathbb{N}}$,

$$\lim_{i\to\infty} 2^i \mu([x \restriction i]) \text{ exists and is in } (0, +\infty).$$

Hence,

$$\lim_{n\to\infty} \frac{-\log_2 \mu([x \restriction n])}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} -\log_2 \frac{\mu([x \restriction i])}{\mu([x \restriction i-1])} = 1.$$

By using the same argument in the proof of Theorem 1.4.1, we have that $\mathrm{cdim}(x) = 1$ for μ-random point x. □

Proof of Theorem 1.4.2 (2). We assume that the condition (i) fails. We can show the assertion in the same manner if the condition (ii) fails, so, we omit the proof in the case. The main ingredient of this proof is to show the following proposition which states the frequency of the outcome of 0 for μ-random points. The following assertions corresponds to Lemma 3.2.2 in Chapter 3.

Proposition 5.1.3. *Let $x \in \{0,1\}^{\mathbb{N}}$ be μ-random. Then,*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_{\{0\}}(x(i)) \geq p_0(\alpha).$$

Proof. We will show this assertion in a manner similar to the proof of Proposition 5.1.2. It is sufficient to show that

$$N_2 = \left\{ x \in \{0,1\}^{\mathbb{N}} : \liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} 1_{\{0\}}(x(i)) < p_0(\alpha) \right\}$$

is a constructive μ_{A_0,A_1}-null set. Let

$$V_n^2 = \bigcup_{k\geq n} \left\{ \sigma \in \{0,1\}^k : \frac{1}{k}\sum_{i=0}^{k-1} 1_{\{0\}}(\sigma(i)) < \left(1 - \frac{1}{k^{1/4}}\right) p_0(\alpha) \right\},\ n \in \mathbb{N}.$$

Since $p_0(\alpha)$ is a computable real number, $\{V_m^2\}_m$ is uniformly c.e. Now it is sufficient to show the following assertions :
(i) $N_2 \subset [V_m^2]$, $m \geq 1$.
(ii) There exists a computable function $g_2 : \mathbb{N} \to \mathbb{N}$ such that $\mu([V^2_{g_2(m)}]) \leq 2^{-m}$, $m \geq 1$.

Let $x \in N_2$. Since $p_0(\alpha) > 0$, there exists $\epsilon \in (0,1)$ such that $\liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} 1_{\{0\}}(x(i)) < (1-\epsilon)p_0(\alpha)$. Hence,

$$\frac{1}{n}\sum_{i=0}^{n-1} 1_{\{0\}}(x(i)) < (1-\epsilon)p_0(\alpha) < \left(1 - \frac{1}{n^{1/4}}\right) p_0(\alpha),$$

for infinitely many n. This completes the proof of (i).

We will show the assertion (ii).

Let $(\Omega, \mathcal{F}, P) = (\{0,1\}^{\mathbb{N}}, \mathcal{B}(\{0,1\}^{\mathbb{N}}), \mu)$, $X_k(x) = \sum_{i=0}^{k-1}(1_{\{0\}}(x(i)) - p_0(\alpha))$, and, $t = k^{3/4}p_0(\alpha)$. Here $\mathcal{B}(\{0,1\}^{\mathbb{N}})$ denotes the Borel σ-algebra. By applying Azuma's inequality [4] to this case, we have that

$$\begin{aligned}
\mu([V_m^2]) &\leq \sum_{k\geq m} \mu\left(\left\{ x : \frac{1}{k}\sum_{i=0}^{k-1} 1_{\{0\}}(x(i)) < \left(1 - \frac{1}{k^{1/4}}\right) p_0(\alpha) \right\}\right) \\
&\leq \sum_{k\geq m} \exp\left(-\frac{p_0(\alpha)^2}{2}k^{1/2}\right) \\
&\leq \sum_{k\geq m} \frac{120}{C^5}k^{-5/2} \leq \frac{120}{C^5} \times \frac{1}{m-1},
\end{aligned}$$

where we let $C = p_0(\alpha)^2/2$. Therefore, for sufficiently large $N \in \mathbb{N}$, $g_2(m) = N^m$ satisfies $\mu([V^2_{g_2(m)}]) \leq 2^{-m}$. This completes the proof of (ii). Thus the proof of Proposition 5.1.3 is completed. □

Let x be a μ-random point. Then, by noting this proposition and the argument in the proof of Theorem 1.2.2(2) in Section 1.2, in particular, Lemma 3.2.3, we have that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathcal{H}(p_0(F(i, x \restriction n))) < 1.$$

By using (5.1.1), we see that

$$\limsup_{n\to\infty} \frac{-\log_2 \mu([x \restriction n])}{n} < 1.$$

Now we see $\mathrm{cdim}(x) < 1$ by using the same argument as in the proof of Theorem 1.4.1. □

5.2 Computability of $F(i, \sigma)$

We will give a proof of that $F(i, \sigma)$ defined in Section 1.4 is computable. Since $-\gamma < -1 < \alpha$, there exist $\alpha' \in (-1, \alpha)$ and $\beta' > \beta$ such that $b_i x + d_i \neq 0$ for any $x \in [\alpha', \beta']$, $i = 0, 1$. Then,

Lemma 5.2.1. $\alpha' < \Phi({}^t A_i; \alpha') \leq \Phi({}^t A_i; \beta') < \beta'$, $i = 0, 1$.

Proof. By using the proof of Lemma 3.1.2 in Chapter 3, we have

$$\Phi({}^t A_0; z) - z = \frac{-(d_0 - a_0)z + c_0}{d_0}, \text{ and, } \Phi({}^t A_1; z) - z = -\frac{(z+1)(z - c_1/b_1)}{z + \gamma}.$$

Since $d_0 > a_0 > 0$, $\Phi({}^t A_0; z) - z$ is strictly decreasing. By using $\alpha' < \alpha$, $\beta' > \beta$, and, $\alpha \leq \Phi({}^t A_0; \alpha) \leq \Phi({}^t A_0; \beta) \leq \beta$, we have that $\alpha' < \Phi({}^t A_0; \alpha') \leq \Phi({}^t A_0; \beta') < \beta'$. By using $-\gamma < -1 < \alpha' < c_1/b_1 < \beta'$, we have that $\alpha' < \Phi({}^t A_1; \alpha') \leq \Phi({}^t A_1; \beta') < \beta'$. □

There exists $L \in \mathbb{N}$ such that for any $x, y \in [\alpha', \beta']$ and $i \in \{0, 1\}$, $|\Phi({}^t A_i; x) - \Phi({}^t A_i; y)| \leq L|x - y|$. Since $a_0, \ldots, d_1$ are computable numbers, there exist computable functions $F_x : \mathbb{N} \to \mathbb{Q}$ such that $|F_x(n) - x| \leq (L+1)^{-n}$, $x = a_0, \ldots, d_0, a_1, \ldots, d_1$.

Let $\tilde{A}_{i,n} = \begin{pmatrix} F_{a_i}(n) & F_{b_i}(n) \\ F_{c_i}(n) & F_{d_i}(n) \end{pmatrix}$, $i = 0, 1$, $n \in \mathbb{N}$. Then,

Lemma 5.2.2. *There exists $N \in \mathbb{N}$ such that for any $n \geq N$ and $i = 0, 1$,*
(i) $\Phi({}^t \tilde{A}_{i,n}; z)$ is well-defined on $[\alpha', \beta']$.
(ii) $\Phi({}^t \tilde{A}_{i,n}; z)$ is increasing on $[\alpha', \beta']$.
(iii) $\alpha' < \Phi({}^t \tilde{A}_{i,n}; \alpha') \leq \Phi({}^t \tilde{A}_{i,n}; \beta') < \beta'$.
(iv) $\Phi({}^t \tilde{A}_{i,n}; z) \in [\alpha', \beta'], \forall z \in [\alpha', \beta']$.

Proof. (i) By noting $\lim_{n\to\infty} F_{b_i}(n) = b_i$, $\lim_{n\to\infty} F_{d_i}(n) = d_i$, and, $\inf_{x\in[\alpha',\beta'],i=0,1} |b_i x + d_i| > 0$, we have that $\inf_{x\in[\alpha',\beta'],i=0,1} |F_{b_i}(n)x + F_{d_i}(n)| > 0$ for any sufficiently large n.

(ii) By using $\det A_i > 0$ and $\lim_{n\to\infty} F_x(n) = x$, $x = a_0, \dots, d_1$, we have that $\det \tilde{A}_{i,n} > 0$ for any sufficiently large n.

(iii) This follows from $\lim_{n\to\infty} \Phi({}^t\tilde{A}_{i,n}; \alpha') = \Phi({}^tA_i; \alpha')$, $\lim_{n\to\infty} \Phi({}^t\tilde{A}_{i,n}; \beta') = \Phi({}^tA_i; \beta')$, $i = 0, 1$, and Lemma 5.2.1.

(iv) This follows from (ii) and (iii). $\square$

Let $D := \{(i, \sigma) \in \mathbb{N}\times\{0, 1\}^* : i \le |\sigma|\}$. We define a function $\tilde{F} : D\times\mathbb{N} \to \mathbb{Q}$ by $\tilde{F}(0, \sigma, n) := 0$, and, $\tilde{F}(i, \sigma, n) := \Phi({}^t\tilde{A}_{\sigma(i-1),n+N}; \tilde{F}(i-1, \sigma, n))$, $1 \le i \le |\sigma|$. Due to Lemma 5.2.2, this is well-defined and $\tilde{F}(i, \sigma, n) \in [\alpha', \beta']$. This is a computable function.

We let $G(m) := \max_{x\in[\alpha',\beta'],j=0,1} |\Phi({}^tA_j; x) - \Phi({}^t\tilde{A}_{j,m+N}; x)|$, $H(0, n) := 0$, and, $H(i, m) := \max_{\sigma:|\sigma|\ge i} |F(i, \sigma) - \tilde{F}(i, \sigma, m)|$, $i \ge 1$, $m \in \mathbb{N}$. Then,

$$\begin{aligned} H(i, m) &\le \max_{\sigma:|\sigma|\ge i} |\Phi({}^tA_{\sigma(i-1)}; F(i-1, \sigma)) - \Phi({}^tA_{\sigma(i-1)}; \tilde{F}(i-1, \sigma, m))| \\ &\quad + \max_{\sigma:|\sigma|\ge i} |\Phi({}^tA_{\sigma(i-1)}; \tilde{F}(i-1, \sigma, m)) - \Phi({}^t\tilde{A}_{\sigma(i-1),m+N}; \tilde{F}(i-1, \sigma, m))| \\ &\le LH(i-1, m) + G(m). \end{aligned}$$

Hence, $H(i, m) \le (L+1)^i G(m)$. By noting that $|F_x(n)-x| \le (L+1)^{-m}$, $x = a_0, \dots, d_1$, we see that there exists a constant $C > 0$ and $M \in \mathbb{N}$ such that $G(m) \le C(L + 1)^{-m}$ for any $m \ge M$. Therefore, there exists a computable function $g : \mathbb{N} \to \mathbb{N}$ such that $g(0) \ge N$, and, $G(g(m)) \le m^{-1}$, $m \ge 1$.

Let $f(i, n) := g((L + 2)^i 2^n)$ and define $u : D \times \mathbb{N} \to \mathbb{Q}$ by $u(0, \sigma, n) := 0$, $u(i, \sigma, n) := \tilde{F}(i, \sigma, f(i, n))$, $n \in \mathbb{N}, 1 \le i \le |\sigma|$. Then, u is a computable function and $|F(i, \sigma) - u(i, \sigma, n)| \le H(i, f(i, n)) \le (L + 1)^i G(f(i, n)) \le 2^{-n}$.

Thus we see that $F : D \to \mathbb{R}$ is a computable function.

Chapter 6

Large deviations for simple random walk on percolations with long-range correlations

This chapter will be based on [85], however, as we announce in Section 1.5, the proof of the crucial part, which concerns the subadditivity of the Lyapunov exponent, is simplified essentially. Moreover, we give details for some parts of the proof which we omit in [85].

In Section 6.1, we state two examples of models. In Section 6.2, we give preliminary results. In Section 6.3, we give a proof of Theorem 1.5.2. The subadditivity of the Lyapunov exponent is shown in this section. In Section 6.4, we give a proof of Theorem 1.5.3, by following the strategy in [69]. In Section 6.5, we state a shape theorem for the chemical distance. In Section 6.6, we state the asymptotics for the rate function near the origin. In Section 6.7, we state the proof of the subadditivity of the Lyapunov exponent given in [85].

6.1 Examples of models

In this section, we state two examples of models satisfying Assumption 1.5.1.

6.1.1 The model considered by Drewitz, Ráth, and Sapozhnikov

Drewitz, Ráth, and Sapozhnikov [28] considered a certain class of percolation models on $\mathbb{Z}^d$ with long range correlations. They obtained large deviation estimates for the chemical distance, which is similar to [2] Theorem 1.1.

By using the result, they also obtained a shape theorem for the chemical distance.

We now state the conditions **(P1)-(P3)** and **(S1)-(S2)** introduced by [28]. Let $0 < a < b$. We define conditions **(P1)-(P3)** and **(S1)-(S2)** for a family of probability measures $\{P_u\}_{a<u<b}$ on $\{0,1\}^{\mathbb{Z}^d}$:

(P1) P_u is invariant and ergodic with respect to the lattice shifts θ_x, $x \in \mathbb{Z}^d \setminus \{0\}$, $u \in (a,b)$.

(P2) For any $u_1 < u_2$ and any increasing event G, $P_{u_1}(G) \leq P_{u_2}(G)$.

(P3) There exist constants $R_P, L_P < +\infty$, $\epsilon_P, \chi_P > 0$, and a real valued function f_P with $f_P(t) \geq \exp((\log t)^{\epsilon_P})$, $t \geq L_P$, such that
$P_{u_2}(A_1 \cap A_2) \leq P_{u_1}(A_1)P_{u_1}(A_2) + \exp(-f_P(L))$, and,
$P_{u_1}(B_1 \cap B_2) \leq P_{u_2}(B_1)P_{u_2}(B_2) + \exp(-f_P(L))$
for any pair $(R, L, u_1, u_2, x_1, x_2, A_1, A_2, B_1, B_2)$ satisfying the following five conditions :
(i) $R \geq R_P$ is an integer.
(ii) $L \geq 1$ is an integer.
(iii) u_1, u_2 are real numbers such that $a < u_1 < u_2 < b$ and $u_2 \geq (1+R^{-\chi_P})u_1$.
(iv) $x_1, x_2 \in \mathbb{Z}^d$ such that $|x_1 - x_2|_\infty \geq RL$.
(v) A_i (resp. B_i), $i = 1, 2$, are decreasing (resp. increasing) events such that A_i (resp. B_i) $\in \sigma(\Phi_y : y \in B(x_i, 10L))$.

(S1) (connectivity) There exists $f_S : (a,b) \times \mathbb{Z}_+ \to \mathbb{R}$ such that for any $u \in (a,b)$, there exist $\Delta_S(u)$ and $R_S(u)$ such that $f_S(u,R) \geq (\log R)^{1+\Delta_S(u)}$ for $R \geq R_S(u)$. Moreover, for any $R \geq 1$,

$$P_u(\mathcal{C}_R \cap B(0,R) \neq \emptyset) \geq 1 - \exp(-f_S(u,R)), \text{ and,}$$

$$P_u\left(\bigcap_{x,y \in \mathcal{C}_{R/10} \cap B(0,R)} \{x \text{ and } y \text{ are connected in } \mathcal{C} \cap B(0,2R)\}\right) \geq 1 - \exp(-f_S(u,R)).$$

(S2) (density) $u \mapsto P_u(0 \in \mathcal{C}_\infty)$ is positive and continuous.

Proposition 6.1.1. *If a family of probability measures $\{P_u\}_{a<u<b}$ on $\{0,1\}^{\mathbb{Z}^d}$ satisfies the conditions **(P1)-(P3)** and **(S1)-(S2)** in the above. then, for each $u \in (a,b)$, P_u satisfies Assumption 1.5.1.*

Proof. Assumption 1.5.1(i) follows from **(P1)**, Assumption 1.5.1(ii) follows from **(S1)** and **(S2)**, and, Assumption 1.5.1(iii) follows from Theorem 1.3 in [28]. □

6.1.2 The random-cluster model

Now we state our setting. See Grimmett's book [42] for basic definitions and properties of the random-cluster model. Let $d \geq 2$, $p \in [0,1]$ and $q \geq 1$. Let $\mathbb{P}^{\xi}_{\Lambda,p,q}$ be the random-cluster measure on a box Λ in $\mathbb{Z}^d$ with boundary condition $\xi \in \{0,1\}^{E(\mathbb{Z}^d)}$. Let $\mathbb{P}^{b}_{p,q}$, $b = 0, 1$, be the extremal infinite-volume limit random-cluster measures.

Let $p_c^b(q) = \inf\{p \in [0,1] : \mathbb{P}^b_{p,q}(0 \leftrightarrow \infty) > 0\}$, $b = 0, 1$. Then, $p_c^0(q) = p_c^1(q)$ and we write this as $p_c(q)$. We have $p_c(q) \in (0,1)$. For any $p > p_c(q)$, there exists a unique infinite cluster $\mathcal{C}_\infty$, $\mathbb{P}^b_{p,q}$-a.s. See Chapter 5 in [42] for the results in this paragraph.

We define the *slab critical point* $\hat{p}_c(q)$ as follows : If $d \geq 3$, we let

$$S(L,n) := [0, L-1] \times [-n,n]^{d-1},$$

$$\hat{p}_c(q,L) := \inf\left\{p : \liminf_{n\to\infty} \inf_{x \in S(L,n)} \mathbb{P}^0_{S(L,n),p,q}(0 \leftrightarrow x) > 0\right\},$$

and,

$$\hat{p}_c(q) := \lim_{L\to\infty} \hat{p}_c(q,L).$$

If $d = 2$, we let

$$p_g(q) := \sup\left\{p : \lim_{n\to\infty} \frac{-\log \mathbb{P}^0_{p,q}(0 \leftrightarrow e_n)}{n} > 0\right\}, \quad e_n = (n, 0, \dots, 0) \in \mathbb{R}^d,$$

and,

$$\hat{p}_c(q) := \frac{q(1-p_g(q))}{p_g(q) + q(1-p_g(q))}.$$

It is known that $\hat{p}_c(q) \geq p_c(q)$.

Proposition 6.1.2. *If $p > \hat{p}_c(q)$, then, $\mathbb{P}^b_{p,q}$, $b = 0,1$, satisfies Assumption 1.5.1.*

Proof. $\mathbb{P}^b_{p,q}$, $b = 0, 1$, satisfies Assumption 1.5.1 (i) for all $p \in [0,1]$ due to [42] (4.19) and (4.23), and, Assumption 1.5.1 (ii) for $p > p_c(q)$ due to [42] (5.99).

Now we show Assumption 1.5.1(iii). We follow the strategy taken in the proof of [2] Theorem 1.1. The key point is to show the random-cluster version of (2.14) in [2]. Let $B_0(r) := [-r, r]^d$, $r \geq 0$. Let $B_i(N) := \tau_{(2N+1)i} B_0(N)$ and $B'_i(N) := \tau_{(2N+1)i} B_0(5N/4)$, $i \in \mathbb{Z}^d$, where τ_i is the transformation on $\mathbb{Z}^d$ defined by $\tau_i(x) := i + x$. Let $Y_z : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}$ be the projection mapping to the coordinate $z \in \mathbb{Z}^d$.

Let $R_i^{(N)}$ be the event in $\{0,1\}^{E(\mathbb{Z}^d)}$ satisfying the following conditions (i) - (iii) :

(i) There exists a unique crossing open cluster for $B'_i(N)$. That is, there is a connected subset $\mathcal{C}$ of an open cluster such that it is contained in $B'_i(N)$, and, for all d directions there is a path in $\mathcal{C}$ connecting the left face and the right face of $B'_i(N)$.

(ii) The cluster in (i) intersects all boxes with diameter larger than $N/10$.

(iii) All open clusters with diameter larger than $N/10$ are connected in $B'_i(N)$.

Let $\phi_N : \{0,1\}^{E(\mathbb{Z}^d)} \to \{0,1\}^{\mathbb{Z}^d}$ be the map defined by $(\phi_N \omega)_i := 1_{R_i^{(N)}}(\omega)$, $i \in \mathbb{Z}^d$. Let $\mathbb{P}^b_{p,q,N}$ be the image measure of $\mathbb{P}^b_{p,q}$ by ϕ_N. Let $\mathbb{P}^*_p$ be the Bernoulli measure on $\{0,1\}^{\mathbb{Z}^d}$ with parameter p. By using Pisztora [91] Theorem 3.1 for $d \geq 3$ and Couronné and Messikh [20] Theorem 9 for $d = 2$, we see that there exist constants $c'_1, c'_2 > 0$ depending only on (d, p, q) such that for any $N \geq 1$ and $i \in \mathbb{Z}^d$,

$$\sup_{\xi \in \Omega} \mathbb{P}^{\xi}_{B'_i(N),p,q} \left((R_i^{(N)})^c \right) \leq c'_1 \exp(-c'_2 N).$$

This inequality corresponds to (2.24) in [2]. By using the DLR property for the random-cluster model (See Section 4.4 in [42] for details),

$$\lim_{N \to \infty} \sup_{z \in \mathbb{Z}^d} \text{ess.sup}\, \mathbb{P}^b_{p,q,N} \left(Y_z = 0 | \sigma(Y_x : |x - z|_\infty \geq 2) \right) = 0.$$

By using Liggett, Schonmann and Stacey [74] Theorem 1.3, we see that there exists a function $\overline{p}(\cdot)$ such that $\overline{p}(N) \to 1$ as $N \to \infty$ and $\mathbb{P}^*_{\overline{p}(N)}$ is dominated by $\mathbb{P}_{p,q,N}$ for each N. This claim corresponds to (2.14) in [2]. The rest of the proof goes in the same way as in the proof of [2] Theorem 1.1. $\square$

Remark 6.1.3. If $q = 1$, then, $\hat{p}_c(1) = p_c(1)$ by Grimmett and Marstrand [43]. If $d \geq 3$, then, by Bodineau [16], $\hat{p}_c(2) = p_c(2)$. If $d = 2$, then, by Beffara and Duminil-Copin [12], $\hat{p}_c(q) = p_c(q)$ for any $q \geq 1$. Therefore, Theorem 1.2 and Theorem 1.3 hold on the whole supercritical regime if $q = 1$ (the Bernoulli percolation case), $q = 2$ (the FK-Ising case), or, $d = 2$.

6.2 Preliminaries

By noting the strong Markov property of $(X_n)_n$,

$$a_\lambda(x, z) \leq a_\lambda(x, y) + a_\lambda(y, z),\ x, y, z \in \mathcal{C}_\infty. \tag{6.2.1}$$

By considering a path from x to y of length $D(x, y)$ in $\mathcal{C}_\infty$,

$$a_\lambda(x, y) \leq (\lambda + \log(2d))D(x, y), \; x, y \in \mathcal{C}_\infty. \tag{6.2.2}$$

By using Birkhoff's ergodic theorem and Kac's theorem (See Chapter 9 in [93] for the statements of these results), we see that for any $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\lim_{n\to\infty} \frac{T_x^{(n)}}{n} = E_{\overline{\mathbb{P}}}[T_x] = \mathbb{P}(\Omega_0)^{-1}, \; \overline{\mathbb{P}} \text{ -a.s. and in } L^1(\overline{\mathbb{P}}). \tag{6.2.3}$$

Here we denote the expectation with respect to $\overline{\mathbb{P}}$ by $E_{\overline{\mathbb{P}}}$.

We now describe some assertions derived from Assumption 1.5.1(iii). The following assertions correspond to Garet and Marchand [38] Lemmas 2.2 and Lemma 2.4 respectively. By using Assumption 1.5.1(iii), we can show them in the same manner as in the proof of [38] Lemmas 2.2 and Lemma 2.4. See [38] for details.

Lemma 6.2.1. *Let $\mathbb{P}$ satisfy Assumption 1.5.1. Then, there exist $C_1, C_2 > 0$ such that for any $r \geq 1$ and for any y with $|y|_1 \leq r$,*

$$\mathbb{P}\left(D(0, y) \geq (3r)^d, 0 \leftrightarrow y\right) \leq C_1 \exp(-C_2(\log r)^{1+c_3}).$$

Lemma 6.2.2. *Let $\mathbb{P}$ satisfy Assumption 1.5.1. Then, there exists $C_3 > 0$ such that $E_{\overline{\mathbb{P}}}[D(0, T_x x)] \leq C_3|x|_1$ for any $x \in \mathbb{Z}^d$.*

Noting Assumption 1.5.1(iii) and Lemma 6.2.1, we can show the following by using the arguments in the proof of [69], Lemma 3.1, or, in the proof of [111] Lemma 6.

Let $d_\lambda(x, y) := \max\{a_\lambda(x, y), a_\lambda(y, x)\}$.

Lemma 6.2.3. *Let $\lambda \geq 0$. Then the following holds $\mathbb{P}$-a.s. : For any $\epsilon \in \mathbb{Q} \cap (0, +\infty)$, there exists a positive number N such that for any $x \in \mathcal{C}_\infty$ with $|x|_1 \geq N$,*

$$\sup\{d_\lambda(x, y) : y \in \mathcal{C}_\infty, |x - y|_1 \leq \epsilon|x|_1\} \leq (\lambda + \log(2d))C_4\epsilon|x|_1.$$

Here C_4 is a positive constant.

6.3 Proof of Theorem 1.5.2

Let

$$\alpha_\lambda(x) := \mathbb{P}(\Omega_0) \inf_{n\geq 1} \frac{E_{\overline{\mathbb{P}}}[a_\lambda(0, T_x^{(n)} x)]}{n},$$

for $\lambda \geq 0$ and $x \in \mathbb{Z}^d$. They are also obtained by using Liggett's subadditive ergodic theorem [73] as the following.

Proposition 6.3.1. *Let $\lambda \geq 0$ and $x \in \mathbb{Z}^d \setminus \{0\}$. Then,*

$$\lim_{n\to\infty} \frac{a_\lambda(0, T_x^{(n)}x)}{T_x^{(n)}} = \alpha_\lambda(x),\ \overline{\mathbb{P}}\text{-}a.s.$$

Proof. Fix $\lambda \geq 0$ and $x \in \mathbb{Z}^d \setminus \{0\}$. Let $W_{m,n} = a_\lambda(T_x^{(m)}x, T_x^{(n)}x)$, $0 \leq m < n$. Then, by using (6.2.1), (6.2.2) and Lemma 6.2.2, we see that $W_{m+1,n+1} = W_{m,n} \circ \Theta_x$, $W_{0,n} \leq W_{0,m} + W_{m,n}$, and, $W_{m,n} \in L^1(\overline{\mathbb{P}})$, $0 \leq m < n$. Therefore we can apply Liggett's subadditive ergodic theorem [73] to $\{W_{m,n}\}_{0 \leq m < n}$ and obtain

$$\lim_{n\to\infty} \frac{a_\lambda(0, T_x^{(n)}x)}{n} = \inf_{n\geq 1} \frac{E_{\overline{\mathbb{P}}}[a_\lambda(0, T_x^{(n)}x)]}{n},\ \overline{\mathbb{P}}\text{-a.s.}$$

By using (6.2.3), we have that

$$\lim_{n\to\infty} \frac{a_\lambda(0, T_x^{(n)}x)}{T_x^{(n)}} = \alpha_\lambda(x),\ \overline{\mathbb{P}}\text{-a.s.}$$

□

Thus we have the first part of Theorem 1.5.2. We now proceed to a proof of the second part.

Proposition 6.3.2. *Let $x, y \in \mathbb{Z}^d$ and $q \in \mathbb{N}$. Then, we have that*
(i) $\alpha_\lambda(x+y) \leq \alpha_\lambda(x) + \alpha_\lambda(y)$.
(ii) $\alpha_\lambda(qx) = q\alpha_\lambda(x)$.
(iii) $\lambda|x|_1 \leq \alpha_\lambda(x) \leq (\lambda + \log(2d))C_3\mathbb{P}(\Omega_0)|x|_1$, where C_3 is the constant in Lemma 6.2.2.

Proof. We can see the assertion (ii) by using the methods taken in the proof of [69], Corollary 2.4.

By using (6.2.2) and Lemma 6.2.2, we have that $E_{\overline{\mathbb{P}}}[a(0, T_x x)] \leq (\lambda + \log(2d))C_3|x|_1$ and hence $\alpha_\lambda(x) \leq (\lambda + \log(2d))C_3\mathbb{P}(\Omega_0)|x|_1$. We see that $\lambda|x|_1 \leq \alpha_\lambda(x)$ by using the methods taken in the proof of [69], Lemma 2.2. Thus we have the assertion (iii).

Now we show the assertion (i). *As we announce in Section 1.5, the following proof is suggested by an anonymous referee.*

Using (6.2.1),

$$a_\lambda(0, T_x^{(n)}(\omega)x) + a_\lambda\left(T_x^{(n)}(\omega)x, T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y\right)$$

$$+a_\lambda\left(T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y, T_{x+y}^{(n)}(\omega)(x+y)\right)$$

$$\geq a_\lambda\left(0, T_{x+y}^{(n)}(\omega)(x+y)\right).$$

Taking expectations with respect to $\overline{\mathbb{P}}$ and dividing by n,

$$\frac{E_{\overline{\mathbb{P}}}[a_\lambda(0, T_x^{(n)}(\omega)x)]}{n} + \frac{E_{\overline{\mathbb{P}}}\left[a_\lambda\left(T_x^{(n)}(\omega)x, T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y\right)\right]}{n}$$

$$+\frac{E_{\overline{\mathbb{P}}}\left[a_\lambda\left(T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y, T_{x+y}^{(n)}(\omega)(x+y)\right)\right]}{n}$$

$$\geq \frac{E_{\overline{\mathbb{P}}}[a_\lambda\left(0, T_{x+y}^{(n)}(\omega)(x+y)\right)]}{n}.$$

Hence it suffices to show that

$$\lim_{n\to\infty}\frac{E_{\overline{\mathbb{P}}}\left[a_\lambda\left(T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y, T_{x+y}^{(n)}(\omega)(x+y)\right)\right]}{n} = 0.$$

Using (6.2.2), it suffices to show that

$$\lim_{n\to\infty}\frac{E_{\overline{\mathbb{P}}}\left[D\left(T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y, T_{x+y}^{(n)}(\omega)(x+y)\right)\right]}{n} = 0. \qquad (6.3.1)$$

We have

$$E_{\overline{\mathbb{P}}}\left[D\left(T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y, T_{x+y}^{(n)}(\omega)(x+y)\right)\right]$$

$$= \int_0^\infty \overline{\mathbb{P}}\left(D\left(T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y, T_{x+y}^{(n)}(\omega)(x+y)\right) > r\right) dr.$$

$$= \int_0^\infty \overline{\mathbb{P}}(D(T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y, T_{x+y}^{(n)}(\omega)(x+y)) > r,$$
$$|T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y - T_{x+y}^{(n)}(\omega)(x+y)| \leq r/c_1)dr.$$

$$+\int_0^\infty \overline{\mathbb{P}}\left(|T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y - T_{x+y}^{(n)}(\omega)(x+y)| > r/c_1\right) dr.$$

$$= \int_0^\infty \overline{\mathbb{P}}(D(T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y, T_{x+y}^{(n)}(\omega)(x+y)) > r,$$
$$|T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y - T_{x+y}^{(n)}(\omega)(x+y)| \leq r/c_1)dr.$$

$$+c_1 E_{\overline{\mathbb{P}}}\left[\left|T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y - T_{x+y}^{(n)}(\omega)(x+y)\right|\right]. \qquad (6.3.2)$$

We will show that

$$\lim_{n\to\infty}\frac{1}{n}E_{\overline{\mathbb{P}}}\left[\left|T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y - T_{x+y}^{(n)}(\omega)(x+y)\right|\right] = 0. \qquad (6.3.3)$$

$$\begin{aligned}
&\frac{1}{n}E_{\overline{\mathbb{P}}}\left[\left|T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y - T_{x+y}^{(n)}(\omega)(x+y)\right|\right] \\
&\qquad \le \frac{|x|}{n}E_{\overline{\mathbb{P}}}\left[\left|T_x^{(n)}(\omega) - T_{x+y}^{(n)}(\omega)\right|\right] \\
&\qquad + \frac{|y|}{n}E_{\overline{\mathbb{P}}}\left[\left|T_y^{(n)}(\Theta_x^n\omega) - T_{x+y}^{(n)}(\omega)\right|\right].
\end{aligned}$$

By using Kac's theorem, the following three convergences hold in $L^1(\overline{\mathbb{P}})$:

$$\frac{T_x^{(n)}(\omega)}{n} \to \mathbb{P}(\Omega_0)^{-1}, \frac{T_y^{(n)}(\Theta_x^n\omega)}{n} \to \mathbb{P}(\Omega_0)^{-1}, \text{ and, } \frac{T_{x+y}^{(n)}(\omega)}{n} \to \mathbb{P}(\Omega_0)^{-1}.$$

Hence we see (6.3.3).
Recall (6.3.2). In order to obtain (6.3.1), it suffices to show that

$$\begin{aligned}
&\int_0^\infty \overline{\mathbb{P}}\big(D(T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y, T_{x+y}^{(n)}(\omega)(x+y)) > r/c_1, \\
&\qquad |T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y - T_{x+y}^{(n)}(\omega)(x+y)| \le r/c_1\big)dr. < +\infty. \qquad (6.3.4)
\end{aligned}$$

We have that

$$\begin{aligned}
&\int_0^\infty \overline{\mathbb{P}}\big(D(T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y, T_{x+y}^{(n)}(\omega)(x+y)) > r/c_1, \\
&\qquad\qquad |T_x^{(n)}(\omega)x + T_y^{(n)}(\Theta_x^n\omega)y - T_{x+y}^{(n)}(\omega)(x+y)| \le r/c_1\big)dr.
\end{aligned}$$

$$\begin{aligned}
&\le \int_0^\infty \overline{\mathbb{P}}\big(\exists z \in B_\infty(T_{x+y}^{(n)}(\omega)(x+y), r/c_1), \\
&\qquad\qquad D(z, T_{x+y}^{(n)}(\omega)(x+y)) > r, z \leftrightarrow T_{x+y}^{(n)}(\omega)(x+y)\big)dr. \qquad (6.3.5)
\end{aligned}$$

Using that Θ_{x+y} is invariant under $\overline{\mathbb{P}}$,

$$\begin{aligned}
(6.3.5) &= \int_0^\infty \overline{\mathbb{P}}\left(\exists z \in B_\infty(0, r/c_1), D(z,0) > r, z \leftrightarrow 0\right) dr \\
&= c_1 \int_0^\infty \overline{\mathbb{P}}\left(\exists z \in B_\infty(0, r), D(z,0) > c_1 r, z \leftrightarrow 0\right) dr
\end{aligned}$$

$$\leq c_1 \int_0^\infty \sum_{z \in B_\infty(0,r)} \overline{\mathbb{P}}\left(D(z,0) > c_1 r, z \leftrightarrow 0\right) dr$$

$$\leq c_1' \sum_{r \geq 0} \sum_{z \in B_\infty(0,r)} \overline{\mathbb{P}}\left(D(z,0) > c_1 r, z \leftrightarrow 0\right).$$

In order to obtain (6.3.4), it suffices to show that

$$\sum_{z \in \mathbb{Z}^d} \sum_{r=|z|}^{\infty} \overline{\mathbb{P}}\left(D(z,0) > c_1 r, z \leftrightarrow 0\right) < +\infty. \tag{6.3.6}$$

By using Lemma 6.2.1,

$$\sum_{r=3|z|^d}^{\infty} \overline{\mathbb{P}}\left(D(z,0) > c_1 r, z \leftrightarrow 0\right) \leq c \exp(-c(\log|z|)^{1+\delta}),\ z \neq 0.$$

We also have that

$$\sum_{r=|z|}^{3|z|^d} \overline{\mathbb{P}}\left(D(z,0) > c_1 r, z \leftrightarrow 0\right) \leq 3|z|^d c_1 \exp(-c_1(\log|z|)^{1+\delta}),\ z \neq 0.$$

Therefore we have (6.3.6). The proof of (i) completes. □

Now we can easily extend the Lyapunov exponent $\alpha_\lambda(\cdot)$ to a unique continuous function on $\mathbb{R}^d$. Thus we have Theorem 1.5.2.

6.4 Proof of Theorem 1.5.3

In the author's preprint [85], the proof of Theorem 1.5.3 is omitted, because the proof is essentially the same as in the proof of large deviation in [69]. Here we write down details. First, we state a shape theorem for the Lyapunov exponent α_λ, which is essentially shown by Zerner [111]. Second, we show the upper bound for Theorem 1.5.3. Third, we show the lower bound for Theorem 1.5.3.

6.4.1 A shape theorem

First, we state the following lemma, which is essentially the same as Garet and Marchand [38] Lemma 5.5.

Lemma 6.4.1. *Let $z \in \mathbb{Z}^d \setminus \{0\}$. Let $\eta > 0$. Then, we have $\mathbb{P}$-a.s. that there exists a positive integer N such that for any $r \geq N$ there exists $k \in [(1-\eta)r, (1+\eta)r]$ such that $kz \in \mathcal{C}_\infty$.*

Let $\Omega_{1,\lambda,z}$ be a set such that $\mathbb{P}(\Omega_{1,\lambda,z}) = 1$ and the conclusion in Proposition 6.3.1 holds on $\Omega_{1,\lambda,z}$ for a fixed (λ, z). Let $\Omega_{2,z,\eta}$ be a set such that $\mathbb{P}(\Omega_{2,z,\eta}) = 1$ and the conclusion in Lemma 6.4.1 holds on $\Omega_{2,z,\eta}$ for a fixed (z, η). Let $\Omega_{3,\lambda}$ be a set such that $\mathbb{P}(\Omega_{3,\lambda}) = 1$ and the conclusion in Lemma 6.2.3 holds on the set for a fixed λ. For $\lambda \geq 0$, we let

$$\Omega(\lambda) := \left(\bigcap_{z \in \mathbb{Z}^d} \Omega_{1,\lambda,z} \right) \cap \left(\bigcap_{z \in \mathbb{Z}^d \setminus \{0\}, \eta \in \mathbb{Q} \cap (0,\infty)} \Omega_{2,z,\eta} \right) \cap \Omega_{3,\lambda}.$$

We remark that $\mathbb{P}(\Omega(\lambda)) = 1$ for any $\lambda \geq 0$.

Proposition 6.4.2 (Shape theorem)**.** *We have $\mathbb{P}$-a.s. that for any $\lambda \geq 0$,*

$$\lim_{|x|_1 \to \infty,\, x \in \mathcal{C}_\infty} \frac{a_\lambda(0,x) - \alpha_\lambda(x)}{|x|_1} = 0.$$

Proof. The following proof is the same as the proof of [69] Theorem 1.2. By using the continuity of $\alpha_\lambda(x)$ with respect to (λ, x) and the argument in the final part of [111] Theorem A, we see that it is sufficient to show that for any *fixed* $\lambda \geq 0$ and $\epsilon \in \mathbb{Q} \cap (0,1)$, the following holds $\mathbb{P}$-a.s., there exists a positive integer N such that for any $x \in \mathcal{C}_\infty$ with $|x|_1 \geq N$, $|a_\lambda(0,x) - \alpha_\lambda(x)| \leq \epsilon |x|_1$.

Assume this statement fails. Then, there exist $\lambda_0 \geq 0$ and $\epsilon_0 > 0$ and an event A with positive probability such that on A, there exists a sequence $(x_n)_n \subset \mathcal{C}_\infty$ satisfying $|x_n|_1 \to \infty$, and $|a_{\lambda_0}(0,x_n) - \alpha_{\lambda_0}(x_n)| \geq \epsilon_0 |x_n|_1$, $n \geq 1$.

Take a configuration $\omega \in A \cap \Omega(\lambda_0)$ and a sequence $(x_n)_n$ in $\mathcal{C}_\infty(\omega)$ described as above. By taking a subsequence if necessary, we can assume that $x_n/|x_n|_1$ converges to a point $v \in \{z \in \mathbb{R}^d : |z|_1 \leq 1\}$.

Take $\eta \in \mathbb{Q} \cap (0,\infty)$, which is chosen small enough later. Let $v' \in S^{d-1} \cap \mathbb{Q}^d$ such that $|v - v'| < \eta$. Let $M \in \mathbb{N}_{\geq 1}$ such that $Mv' \in \mathbb{Z}^d$. Let $x'_n = \lfloor |x_n|_1/M \rfloor Mv'$, $n \geq 1$. By using Lemma 6.4.1 and $\omega \in \Omega(\lambda_0)$, we have that for any n, there exists $k_n = k_n(\eta, \omega)$ such that $(1-\eta)\lfloor |x_n|_1/M \rfloor \leq k_n \leq \lfloor |x_n|_1/M \rfloor$, and, $k_n Mv' \in \mathcal{C}_\infty(\omega)$. Let $x''_n = k_n Mv'$. Then,

$$\begin{aligned} |x_n - x''_n|_1 &\leq |x_n - x'_n|_1 + |x'_n - x''_n|_1 \\ &\leq |x_n - |x_n|_1 v'|_1 + ||x_n|_1 v' - x'_n|_1 + M(\lfloor |x_n|_1/M \rfloor - k_n) \\ &\leq |x_n|_1 \left| \frac{x_n}{|x_n|_1} - v' \right| + M + \eta |x_n|_1. \end{aligned}$$

Hence $|x_n - x''_n| \leq 3\eta |x_n|_1$ for sufficiently large n.

Recalling $x''_n = k_n Mv' \in \mathcal{C}_\infty(\omega)$, Proposition 6.3.1 and $\omega \in \Omega(\lambda_0)$, we have that

$$\lim_{n \to \infty} \frac{a_{\lambda_0}(0, x''_n)}{k_n} = \lim_{n \to \infty} \frac{a_{\lambda_0}(0, k_n Mv')}{k_n} = \alpha_{\lambda_0}(Mv').$$

Since $k_n \le |x_n|_1$ and $\alpha_{\lambda_0}(Mv') = \alpha_{\lambda_0}(x''_n)/k_n$,

$$\lim_{n\to\infty} \frac{a_{\lambda_0}(0, x''_n) - \alpha_{\lambda_0}(x''_n)}{|x_n|_1} = 0.$$

Hence,

$$\limsup_{n\to\infty} \frac{|a_{\lambda_0}(0, x_n) - \alpha_{\lambda_0}(x_n)|}{|x_n|_1}$$
$$\le \limsup_{n\to\infty} \frac{|a_{\lambda_0}(0, x_n) - a_{\lambda_0}(0, x''_n)|}{|x_n|_1} + \limsup_{n\to\infty} \frac{|\alpha_{\lambda_0}(x_n) - \alpha_{\lambda_0}(x''_n)|}{|x_n|_1}.$$

By recalling $|x_n - x''_n| \le 3\eta |x_n|_1$ for sufficiently large n, it follows from (6.2.1), Lemma 6.2.3 and $\omega \in \Omega(\lambda_0)$ that

$$\limsup_{n\to\infty} \frac{|a_{\lambda_0}(0, x_n) - a_{\lambda_0}(0, x''_n)|}{|x_n|_1} \le \limsup_{n\to\infty} \frac{d_{\lambda_0}(x_n, x''_n)}{|x_n|_1} \le 3\eta(\lambda_0 + \log(2d))C_4.$$

By using Proposition 6.3.1, and $|a_{\lambda_0}(0, x_n) - a_{\lambda_0}(0, x''_n)| \le \alpha_{\lambda_0}(x_n - x''_n) \vee \alpha_{\lambda_0}(x''_n - x_n)$,

$$\limsup_{n\to\infty} \frac{\alpha_{\lambda_0}(x_n - x''_n) \vee \alpha_{\lambda_0}(x''_n - x_n)}{|x_n|_1} \le 3\eta(\lambda_0 + \log(2d))C_3.$$

Thus we have

$$\limsup_{n\to\infty} \frac{|a_{\lambda_0}(0, x_n) - \alpha_{\lambda_0}(x_n)|}{|x_n|_1} \le 3(\lambda_0 + \log(2d))(C_3 + C_4)\eta.$$

By recalling the definition of $(x_n)_n$, we have that $\epsilon_0 \le 3(\lambda_0 + \log(2d))(C_3 + C_4)\eta$. However we can take $\eta < \epsilon_0/(3(\lambda_0 + \log(2d))(C_3 + C_4))$. This is a contradiction. □

As a corollary, we have that

Corollary 6.4.3 (Directionally shape theorem)**.** *Let $x \in \mathbb{Q}^d \setminus \{0\}$ and $\beta \in (0, 1)$. Let $v = x/|x|$ and $M \in \mathbb{N}$ such that $Mv \in \mathbb{Z}^d$. Then,*

$$\lim_{n\to\infty} \frac{1}{n} a_\lambda \left(T_{Mv}^{\lfloor \mathbb{P}(\Omega_0)\beta n|x|/M \rfloor} Mv, T_{Mv}^{\lfloor \mathbb{P}(\Omega_0) n|x|/M \rfloor} Mv \right) = (1-\beta)\alpha_\lambda(x),\ \overline{\mathbb{P}}\text{-a.s.}$$

This statement is the same as Corollary 3.3 in [69]. We can show this in the same manner as in the proof of Corollary 3.3 in [69].

Before we proceed to the proof of Theorem 1.5.3, we define the rate function I and describe some properties. Let

$$I(z) := \sup_{\lambda \geq 0}(\alpha_\lambda(x) - \lambda),\ z \in \mathbb{R}^d.$$

Let $\mathcal{D}_I = (I < +\infty)$. By Proposition 6.3.2(iii), we see that there exists $C_5 > 0$ such that $0 \leq I(x) \leq C_5|x|_1$ for $x \in B(0, C_5^{-1})$. Hence $\mathcal{D}_I$ contains an open neighborhood of 0.

Lemma 6.4.4. *(i) I is convex on $\mathbb{R}^d$.*
(ii) I is lower semicontinuous (on $\mathcal{D}_I$).
(iii) I is upper semicontinuous on $\mathrm{int}\mathcal{D}_I$.
(iv) $I(x) = +\infty$ if $|x|_1 > 1$.

Proof. (i) By using Theorem 1.5.2, $\alpha_\lambda(\cdot)$ is convex and then (i) follows.

(ii) Let $l \in \mathbb{R}$. Let $x_n \in \{I \leq l\}$ and $x_n \to x \in \mathbb{R}^d$. Let $\lambda \geq 0$. Since $\alpha_\lambda(\cdot)$ is convex, it is continuous (on $\mathbb{R}^d$). Therefore,

$$\alpha_\lambda(x) - \lambda = \lim_{n\to\infty} \alpha_\lambda(x_n) - \lambda \leq \limsup_{n\to\infty} I(x_n) \leq l.$$

This completes the proof of the assertion (ii).

(iii) Let $x \in \mathrm{int}\mathcal{D}_I$. Then, I is bounded on a neighborhood of x. Then, I is continuous on a neighborhood of x.

(iv) It is sufficient to show that for any $x \in \mathbb{R}^d$ with $|x|_1 > 1$, $\alpha_\lambda(x) \geq \lambda|x|_1$. This follows from Theorem 1.5.2. □

6.4.2 Proof of the upper bound

Let A be a closed set in $\mathbb{R}^d$. By Lemma 6.4.4, we can assume without loss of generality that A is contained in the closed l_1-ball centered at 0 with radius 1 in $\mathbb{R}^d$. If $0 \in A$, then $\inf_{z\in A} I(z) = 0$ and hence the assertion holds. Hereafter we assume that $0 \notin A$.

Let $I^\delta(z) := (I(z) - \delta) \wedge (1/\delta)$ and $A_\lambda(\delta) := \{z \in A : \alpha_\lambda(z) - \lambda > \inf_{x\in A} I^\delta(x) - \delta\}$, $\lambda \geq 0$, $\delta > 0$. Since A is compact, there exist $\lambda_1, \dots, \lambda_m$ such that $A = \cup_{i=1}^m A_{\lambda_i}(\delta)$. Hence we have

$$\limsup_{n\to\infty} \frac{\log P_\omega^0(X_n \in nA)}{n} \leq \max_{1\leq i\leq m} \limsup_{n\to\infty} \frac{\log P_\omega^0(X_n \in nA_{\lambda_i}(\delta))}{n}. \tag{6.4.1}$$

We will show that for $\lambda \geq 0$ and $\delta > 0$, the following holds $\mathbb{P}$-a.s.ω :

$$\limsup_{n\to\infty} \frac{\log P_\omega^0(X_n \in nA_\lambda(\delta))}{n} \leq \delta - \inf_{z\in A} I^\delta(z). \tag{6.4.2}$$

We can assume without loss of generality that $nA_\lambda(\delta)\cap\mathcal{C}_\infty(\omega) \neq \emptyset$. Then,

$$\begin{aligned}
P^0_\omega(X_n \in nA_\lambda(\delta)) &= \sum_{y\in nA_\lambda(\delta)\cap\mathcal{C}_\infty(\omega)} P^0_\omega(X_n = y) \\
&\leq \sum_{y\in nA_\lambda(\delta)\cap\mathcal{C}_\infty(\omega)} P^0_\omega(H_y \leq n) \\
&\leq \sum_{y\in nA_\lambda(\delta)\cap\mathcal{C}_\infty(\omega)} \exp(\lambda n - a^\omega_\lambda(0,y)) \\
&\leq \left|nA_\lambda(\delta)\cap\mathbb{Z}^d\right| \exp(\lambda n - a^\omega_\lambda(0,y_{n,\lambda})),
\end{aligned}$$

for some $y_{n,\lambda} \in nA_\lambda(\delta)\cap\mathcal{C}_\infty(\omega)$.

Since $A_\lambda(\delta)$ is bounded, we have

$$\begin{aligned}
\frac{\log P^0_\omega(X_n \in nA_\lambda(\delta))}{n} &\leq o(1) + \lambda - \frac{a_\lambda(0,y_{n,\lambda})}{n} \\
&= o(1) + \lambda - \alpha_\lambda\left(\frac{y_{n,\lambda}}{n}\right) - \frac{a_\lambda(0,y_{n,\lambda}) - \alpha_\lambda(y_{n,\lambda})}{|y_{n,\lambda}|_1}\frac{|y_{n,\lambda}|_1}{n}.
\end{aligned} \tag{6.4.3}$$

Since $A_\lambda(\delta) \subset A$, $0 \notin A$ and A is compact, $\mathrm{dist}(0, A_\lambda(\delta)) > 0$. Hence $|y_{n,\lambda}|_1 \to \infty$, $n \to \infty$. Then, by Proposition 6.4.2 and boundedness of $A_\lambda(\delta)$, we have $\mathbb{P}$-a.s. that

$$\frac{a_\lambda(0,y_{n,\lambda}) - \alpha_\lambda(y_{n,\lambda})}{|y_{n,\lambda}|_1}\frac{|y_{n,\lambda}|_1}{n} \to 0, n \to \infty.$$

Recalling (6.4.3) and $y_{n,\lambda}/n \in A_\lambda(\delta)$, we have $\mathbb{P}$-a.s. that

$$\limsup_{n\to\infty} \frac{\log P^0_\omega(X_n \in nA_\lambda(\delta))}{n} \leq \lambda - \inf_{z\in A_\lambda(\delta)} \alpha_\lambda(z) \leq \delta - \inf_{z\in A} I^\delta(z).$$

Thus we see that (6.4.2) holds $\mathbb{P}$-a.s. for fixed $\lambda \geq 0$ and $\delta > 0$. By (6.4.1), we see that for fixed $\delta > 0$ the following holds $\mathbb{P}$-a.s. :

$$\limsup_{n\to\infty} \frac{\log P^0_\omega(X_n \in nA)}{n} \leq \delta - \inf_{z\in A} I^\delta(z).$$

By letting $\delta \to 0$, we see that (1.5.1) holds $\overline{\mathbb{P}}$-a.s.

6.4.3 Proof of the lower bound

For $\lambda \geq 0$, $\omega \in \Omega_0$, $x, y \in \mathcal{C}_\infty(\omega)$, let

$$Q^{x,y}_{\lambda,\omega}(dX.) = \frac{\exp(-\lambda H_y(X.))1_{\{H_y(X.)<+\infty\}}}{E^x_\omega[\exp(-\lambda H_y)1_{\{H_y<+\infty\}}]} P^x_\omega(dX.).$$

Then we have the following lemma, which is essentially the same as [69] Lemma 4.1 and Fukushima and Kubota [35] Lemma 4.1. See the references for proof.

Lemma 6.4.5. *Let $x \in \mathbb{Q}^d \setminus \{0\}$. Let $\beta \in [0,1)$. Denote $v = x/|x|_1$. Denote $M \in \mathbb{N}_{\geq 1}$ such that $Mv \in \mathbb{Z}^d$. Denote $y_n^{(1)} = T_{Mv}^{(\lfloor \mathbb{P}(\Omega_0)\beta n|x|/M \rfloor)} Mv$ and $y_n^{(2)} = T_{Mv}^{(\lfloor \mathbb{P}(\Omega_0)n|x|/M \rfloor)} Mv$. Then, the following holds $\overline{\mathbb{P}}$-a.s. : for any $\lambda \geq 0$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ with $0 \leq \gamma_1 < \alpha'_{\lambda+}(x) \leq \alpha'_{\lambda-}(x) < \gamma_2$,*

$$\lim_{n\to\infty} Q_{\lambda,\omega}^{y_n^{(1)}, y_n^{(2)}} \left(\frac{H_{y_n^{(2)}}}{(1-\beta)n} \in (\gamma_1, \gamma_2) \right) = 1.$$

Now we proceed to the proof of the lower bound.

Step 1. We will show that it suffices to show that for any fixed $z \in \mathbb{Q}^d \setminus \{0\} \cap \mathcal{D}_I$ and $r \in (0,\infty) \cap \mathbb{Q}$, the following holds $\overline{\mathbb{P}}$-a.s. :

$$\liminf_{n\to\infty} \frac{\log P_\omega^0(X_n \in nB(z,r))}{n} \geq -I(z). \tag{6.4.4}$$

Let $B \subset \mathbb{R}^d$ be open. If $B \cap \mathcal{D}_I = \emptyset$, $-\inf_{z\in B} I(z) = -\infty$ and hence the assertion holds. Assume $B \cap \mathcal{D}_I \neq \emptyset$. Since $\mathcal{D}_I$ is convex and B is open, we see $B \cap \mathrm{int}\mathcal{D}_I \neq \emptyset$ and for any $z \in B \cap \mathcal{D}_I$, there exists $u < 1$ such that $uz \in B \cap \mathrm{int}\mathcal{D}_I$. Therefore, $\inf_{z\in B\cap\mathcal{D}_I} I(z) = \inf_{z\in B\cap \mathrm{int}\mathcal{D}_I} I(z)$. By the continuity of I on $\mathrm{int}\mathcal{D}_I$, $\inf_{z\in B} I(z) = \inf_{z\in B\cap\mathrm{int}\mathcal{D}_I\cap\mathbb{Q}^d} I(z)$. Take a point $z \in B \cap \mathrm{int}\mathcal{D}_I \cap \mathbb{Q}^d$ and $r > 0$ with $B(z,r) \subset B$ arbitrarily. By applying (6.4.4) to $B(z,r)$, we see that (1.5.2) holds $\mathbb{P}$-a.s. for B.

Step 2. We will show (6.4.4). Hereafter we fix z and r.
Let

$$\lambda_*(z) := \sup\left\{\lambda \geq 0 : \alpha'_\lambda(z) \text{ exists and } \alpha'_\lambda(z) \geq 1\right\},$$

where $\alpha'_\lambda(z)$ denotes the derivative of $\alpha_\lambda(z)$ with respect to λ if it exists. Let $v = z/|z|_1$ and M be the least integer such that $Mv \in \mathbb{Z}^d$.

Let $\Omega_{4,x,\beta}$ be a set with probability 1 such that the following (i) and (ii) hold on $\Omega_{4,x,\beta}$:
(i) The conclusions in Corollary 6.4.3 and Lemma 6.4.5 hold.
(ii) $y_n^{(2)}/n \to x$, $n \to \infty$.

Case 1. $\lambda_*(z) = 0$. In this case, we use the methods described in the proof of [69] Theorem 1.3. Let $y_n = y_n^{(2)}$, where $y_n^{(2)}$ is defined in Lemma

6.4.5 for $(x, \beta) = (z, 0)$. Then, $y_n/n \to z$ on $\Omega_{4,z,0}$. Let $R > 0$ be an even integer. Then, for all sufficiently large n, $B(y_n, R) \subset nB(z, r)$.

$$\begin{aligned} P^0(X_n \in nB(z,r)) &\geq P^0(H_{y_n} \leq n,\, X_{m+H_{y_n}} \in B(y_n, R), \forall m \in [0, n]) \\ &\geq P^0(H_{y_n} \leq n) P^{y_n}(X_m \in B(y_n, R), \forall m \in [0, n]) \\ &\geq E^0[\exp(-\lambda H_{y_n}), H_{y_n} \leq n] P^{y_n}(X_R = y_n)^{n/R}. \quad (6.4.5) \end{aligned}$$

Applying Lemma 6.4.5 to the case $(x, \beta, \gamma_1, \gamma_2) = (z, 0, 0, 1)$, we have that on $\Omega_{4,z,0}$, for any $\lambda > 0$ such that $\alpha'_\lambda(z)$ exists,

$$E^0\left[\exp(-\lambda H_{y_n}), H_{y_n} \leq n\right] \sim E^0[\exp(-\lambda H_{y_n})] = \exp(-a_\lambda(0, y_n)).$$

Here $f(n) \sim g(n)$ means that $f(n)/g(n) \to 1, n \to +\infty$.

By using Proposition 6.3.1 and (6.4.5), we see that for any $\lambda > 0$ such that $\alpha'_\lambda(z)$ exists, we have that on $\Omega_{4,z,0} \cap \Omega_{1,\lambda,Mv}$,

$$\begin{aligned} \liminf_{n\to\infty} \frac{\log P^0(X_n \in nB(z,r))}{n} &\geq \liminf_{n\to\infty} \frac{-a_\lambda(0, y_n)}{n} + \liminf_{n\to\infty} \frac{\log P^{y_n}(X_R = y_n)}{R} \\ &= -\alpha_\lambda(z) + \liminf_{n\to\infty} \frac{\log P^{y_n}(X_R = y_n)}{R}. \end{aligned}$$

Since $\mathcal{C}_\infty$ is a subgraph of $\mathbb{Z}^d$, $P^{y_n}(X_R = y_n) \geq c_d R^{-d}$ for *any* $n \geq 1$, where c_d is a positive constant depending only on d. See [70] Proposition 4.3.4 for proof. Therefore, by letting $R \to \infty$, we see that for any $\lambda > 0$ such that $\alpha'_\lambda(z)$ exists, on $\Omega_{4,z,0} \cap \Omega_{1,\lambda,Mv}$,

$$\liminf_{n\to\infty} \frac{\log P^0(X_n \in nB(z,r))}{n} \geq -\alpha_\lambda(z).$$

Since $\lambda_*(z) = 0$, we have that $I(z) = \lim_{\lambda \downarrow 0} \alpha_\lambda(z)$ and the following holds $\overline{\mathbb{P}}$-a.s. :

$$\liminf_{n\to\infty} \frac{\log P^0(X_n \in nB(z,r))}{n} \geq -I(z).$$

This completes the proof of Case 1.

Case 2. $\lambda_*(z) \in (0, +\infty)$. In this case, we follow the strategy of proof of [111] Theorem B. Let $\epsilon \in (0, \lambda_*(z) \wedge 1)$. By noting the assumption $\lambda_*(z) \in (0, \infty)$, and, $(\lambda, x) \mapsto \alpha_\lambda(x)$ is continuous, there are $\rho \in (0, 1)$, $\eta > 0$, and, λ_0, λ_2 such that

(1) $\alpha'_{\lambda_0}(z)$ and $\alpha'_{\lambda_2}(z)$ exist.
(2) $\lambda_*(z) - \epsilon < \lambda_0 \leq \lambda_*(z) \leq \lambda_2$.
(3) $\alpha_{\lambda_2}(z) < \alpha_{\lambda_*(z)}(z) + \epsilon$.
(4) $\rho\alpha'_{\lambda_0}(z) + (1 - \rho)\alpha'_{\lambda_2}(z) + [-\eta, +\eta] \subset (1 - \epsilon r/2, 1 + \epsilon r/2)$.

Let $y_n^{(1)}$, $y_n^{(2)}$ as defined in Lemma 6.4.5 for $(x, \beta) = (z, \rho)$. Since $y_n^{(2)}/n \to z$ on $\Omega_{4,z,\rho}$, $B(y_n^{(2)}, nr/2) \subset nB(z, r)$ for sufficiently large n. Let

$$A_n = \left\{ H_{y_n^{(1)}} \in n\rho(\alpha'_{\lambda_0}(z) + [-\eta, +\eta]) \right\} \\ \bigcap \left\{ \inf\{m : X_{H_{y_n^{(1)}}+\cdot}\} \in n(1-\rho)(\alpha'_{\lambda_2}(z) + [-\eta, +\eta]) \right\}.$$

Then,

$$\frac{1}{n} \log P^0(X_n \in nB(z, r)) \geq \frac{1}{n} \log P^0(A_n).$$

Thanks to the strong Markov property of $(X_n)_n$,

$$P^0(A_n) = P^0(H_{y_n^{(1)}} \in n\rho(\alpha'_{\lambda_0}(z)+[-\eta, \eta]))P^{y_n^{(1)}}(H_{y_n^{(2)}} \in n(1-\rho)(\alpha'_{\lambda_2}(z)+[-\eta, \eta])).$$

$$\geq E^0 \left[\exp\left(-\lambda_*(z) H_{y_n^{(1)}} \right), H_{y_n^{(1)}} \in n\rho(\alpha'_{\lambda_0}(z) + [-\eta, \eta]) \right].$$

$$\times \exp\left(\lambda_*(z) n\rho(\alpha'_{\lambda_0}(z) - \eta) \right).$$

$$\times E^{y_n^{(1)}} \left[\exp\left(-\lambda_*(z) H_{y_n^{(2)}} \right), H_{y_n^{(2)}} \in n(1-\rho)(\alpha'_{\lambda_2}(z) + [-\eta, \eta]) \right].$$

$$\times \exp\left(\lambda_*(z) n(1-\rho)(\alpha'_{\lambda_2}(z) - \eta) \right).$$

Since $-\lambda_*(z) H_{y_n^{(1)}} \geq -\lambda_0 H_{y_n^{(1)}} + (\lambda_0 - \lambda_*(z)) n\rho(\alpha'_{\lambda_2}(z) + \eta)$ on the set $\{H_{y_n^{(1)}} \in n\rho(\alpha'_{\lambda_0}(z) + [-\eta, +\eta])\}$, and, $\lambda_*(z) \leq \lambda_2$, we have that on $\Omega_{4,z,\rho}$,

$$\liminf_{n\to\infty} \frac{1}{n} \log E^0 \left[\exp(-\lambda_*(z) H_{y_n^{(1)}}), A_n \right] \\ \geq \lambda_*(z)(1 - \epsilon r/2) + (\lambda_0 - \lambda_*(z))\rho(\alpha'_{\lambda_2}(z) + \eta) + a_1 + a_2, \quad (6.4.6)$$

where we let

$$a_1 = \liminf_{n\to\infty} \frac{1}{n} \log E^0 \left[\exp(-\lambda_0 H_{y_n^{(1)}}), H_{y_n^{(1)}} \in n\rho(\alpha'_{\lambda_0}(z) + [-\eta, +\eta]) \right], \text{ and,}$$

$$a_2 = \liminf_{n\to\infty} \frac{1}{n} \log E^{y_n^{(1)}} \left[\exp(-\lambda_2 H_{y_n^{(2)}}), H_{y_n^{(2)}} \in n(1-\rho)(\alpha'_{\lambda_2}(z) + [-\eta, +\eta]) \right].$$

By using Lemma 6.4.5 for $(x, \beta) = (z, 0)$, and then by using Proposition 6.3.1,

$$a_1 = \lim_{n\to\infty} \frac{\log E^0[\exp(-\lambda_0 H_{y_n^{(1)}})]}{n} = -\rho\alpha_{\lambda_0}(z), \text{ on } \Omega_{4,z,0} \cap \Omega_{1,\lambda_0}.$$

By using Lemma 6.4.5 for $(x, \beta) = (z, \rho)$, and then, by using Corollary 6.4.3,

$$a_2 = \lim_{n\to\infty} \frac{\log E^{y_n^{(1)}}[\exp(-\lambda_2 H_{y_n^{(2)}})]}{n} = -(1-\rho)\alpha_{\lambda_2}(z), \text{ on } \Omega_{4,z,\rho} \cap \Omega_{1,\lambda_2}.$$

Therefore we have that on $\Omega_{4,z,\rho} \cap \Omega_{4,z,0} \cap \Omega_{1,\lambda_0} \cap \Omega_{1,\lambda_2}$, the right hand side of (6.4.6) is larger than or equal to

$$\lambda_*(z)\left(1 - \frac{\epsilon r}{2}\right) + (\lambda_0 - \lambda_*(z))\rho(\alpha'_{\lambda_2}(z) + \eta) - \rho\alpha_{\lambda_0}(z) - (1-\rho)\alpha_{\lambda_2}(z).$$

By using the assumption $\lambda_*(z) \in (0, +\infty)$, we have that $I(z) = \alpha_{\lambda_*(z)}(z) - \lambda_*(z)$. Recalling the properties (1) - (4) which ρ, λ_0 and λ_2 satisfy, we see that on $\Omega_{4,z,\rho} \cap \Omega_{4,z,0} \cap \Omega_{1,\lambda_0} \cap \Omega_{1,\lambda_2}$,

$$\liminf_{n\to\infty} \frac{\log P^0(X_n \in nB(z,r))}{n} \geq -I(z) - \lambda_*(z)\epsilon r - \epsilon(2 + \epsilon r).$$

By letting $\epsilon \to 0$, we see that (6.4.4) holds $\overline{\mathbb{P}}$-a.s.

Case 3. $\lambda_*(z) = +\infty$. In this case, we use the methods taken in the proof of [35] Theorem 1.4. Since $\lambda_*(z) = +\infty$, we have $\lim_{\lambda\to\infty} \alpha'_\lambda(z) \leq 1$. Then, for any $u \in \mathbb{Q} \cap (0,1)$, there exists $\lambda(u) < \infty$ such that for any $\lambda \geq \lambda(u)$, $\alpha'_\lambda(uz) < 1$, and hence, $\lambda_*(uz) \in [0, \infty)$. If $u \in (0 \vee (1 - r/|z|), 1)$, we can take $r(u) \in \mathbb{Q}$ with $B(uz, r(u)) \subset B(z,r)$. By using Case 1 or 2 according to the value of $\lambda_*(uz)$, we have $\mathbb{P}$-a.s. that

$$\liminf_{n\to\infty} \frac{\log P^0(X_n \in nB(z,r))}{n} \geq \liminf_{n\to\infty} \frac{\log P^0(X_n \in nB(uz, r(u)))}{n} \geq -I(uz).$$

Since $I(uz) \leq uI(z) \leq I(z)$, (5.4) holds $\overline{\mathbb{P}}$-a.s.

Thus the proof of the lower bound (1.5.2) completes.

Remark 6.4.6. The proof of Theorem B in Zerner's paper [111] and the first version of the author's preprint contain a minor error in the case $\lambda_*(z) \in (0, +\infty)$. Specifically, it concerns the definition of A_n. The above proof is one way to fix it.

6.5 A shape theorem for the chemical distance

In this section, we briefly discuss a shape theorem *for the chemical distance.*

Theorem 6.5.1 (Existence of directional constants)**.** *Assume that $\mathbb{P}$ satisfies Assumption 1.5.1. Then, there exists a non-negative function $\mu(\cdot)$ on $\mathbb{Z}^d$ such that $\mu(0) = 0$, and, for any $x \in \mathbb{Z}^d \setminus \{0\}$,*

$$\lim_{n\to\infty} \frac{D(0, T_x^{(n)}x)}{T_x^{(n)}} = \mu(x),\ \overline{\mathbb{P}}\text{-}a.s.$$

We can extend this to a continuous function on $\mathbb{R}^d$ uniquely. $\mu(\cdot)$ satisfies the following properties : for any $x, y \in \mathbb{R}^d$ and for any $q \in (0, +\infty)$, $\mu(qx) = q\mu(x)$, $\mu(x + y) \leq \mu(x) + \mu(y)$, and, $|x|_1 \leq \mu(x) \leq C_3|x|_1$, where C_3 is the constant in Lemma 6.2.2.

This is an extension of [38], Corollary 3.3. By replacing the Lyapunov exponent $a_\lambda(\cdot, \cdot)$ with the chemical distance $D(\cdot, \cdot)$, and modifying the definition of A_{z_1,z_2} slightly, the proof goes in the same way as in the proof of Theorem 1.5.2.

Let $\mathcal{D}$ be the Hausdorff distance on $\mathbb{R}^d$. For $t > 0$, we let a random subset $B_t := \{x \in \mathcal{C}_\infty : D(0, x) \leq t\}$ of $\mathcal{C}_\infty$ on Ω_0.

Theorem 6.5.2 (Shape theorem)**.** *Assume that $\mathbb{P}$ satisfies Assumption 1.5.1. Then,*

$$\lim_{t\to+\infty} \mathcal{D}(B_t/t, B_\mu) = 0,\ \overline{\mathbb{P}}\text{-}a.s.,$$

where we let $B_\mu := \{y \in \mathbb{R}^d : \mu(y) \leq 1\}$ for the function μ in Theorem 6.5.1.

This assertion is an extension of Corollary 6.4.4 in [38]. Thanks to Assumption 1.5.1, Lemma 6.2.1 and $|x|_1 \leq \mu(x)$, we can show this in a manner similar to the proof of Theorem 5.3 in [38]. In our case, $\mu(x) \neq \mu(-x)$ may happen, but this is a minor difference and does not affect the argument.

Theorem 6.5.2 holds for the Drewitz, Ráth and Sapozhnikov model and the random-cluster model up to the slab critical point. For the Drewitz, Ráth and Sapozhnikov model, Theorem 1.5 in [28] also states a shape theorem. However, our approach is different from the one in [28]. [28] introduces a pseudo-metric, which is equal to the chemical distance on $\mathcal{C}_\infty$. On the other hand, we do not use the notion.

6.6 The asymptotics for the rate function

Proposition 6.6.1. *Assume that a family of probability measures $\{P_u\}_{a<u<b}$ on $\{0,1\}^{\mathbb{Z}^d}$ satisfies the conditions **(P1)-(P3)** and **(S1)-(S2)** in Subsection 6.1.1 and $\mathbb{P} = P_u$ for some $u \in (a, b)$. Let I be the rate function in Theorem 1.5.3. Then, there exist a constant $c > 0$ such that*

$$I(x) > c|x|^2,\ x \in \mathcal{D}_I.$$

We can show this in the same strategy as in the proof of Proposition 4.2 in [69]. This proof depends heavily on Theorem 1.15 in [99].

Proof. Let $x \in \mathcal{D}_I$ and $\delta \in (0, |x|)$. Then, by using (1.5.2) and Theorem 1.15 in [99], the following inequalities hold $\overline{\mathbb{P}}$-a.s.,

$$-I(x) \leq -\inf_{y \in B(x,\delta)} I(y) \leq \liminf_{n \to +\infty} \frac{\log P_\omega^0(X_n \in nB(x,\delta))}{n}$$

$$\leq \limsup_{n \to +\infty} \frac{\log Cn^d \exp(-cn(|x|-\delta)^2)}{n} = -c(|x|-\delta)^2.$$

In the above $C, c > 0$ are constants depending on the model only. $\square$

Remark 6.6.2. We are not sure $\limsup_{x \to 0} \frac{I(x)}{|x|^2} < +\infty$. The Gaussian heat kernel lower bounds in [99] is not sufficient to apply the strategy in [69] to this case. See Remark 1.21 (3) in [99].

6.7 Appendix : The author's proof of the subadditivity of the Lyapunov exponent

We give the proof of Proposition 6.3.2(i) by following [85].

Lemma 6.7.1. *Let $z_1, z_2 \in \mathbb{Z}^d$. Then,*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \mathbb{P}(\Omega_0 \cap \theta_{z_1}^{-i}\Omega_0 \cap \theta_{z_2}^{-i}\Omega_0) \textit{ exists and is positive.}$$

We denote this limit by b_{z_1,z_2}.

Proof. By using Tao [106] Theorem 1.1, there exists a function $g \in L^2(\mathbb{P})$ such that

$$\frac{1}{n}\sum_{i=1}^n (1_{\Omega_0} \circ \theta_0^i) \cdot (1_{\Omega_0} \circ \theta_{z_1}^i) \cdot (1_{\Omega_0} \circ \theta_{z_2}^i) \to g,\ n \to \infty, \text{ in } L^2(\mathbb{P}).$$

Hence,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \mathbb{P}(\Omega_0 \cap \theta_{z_1}^{-i}\Omega_0 \cap \theta_{z_2}^{-i}\Omega_0) = \int g d\mathbb{P}.$$

Since $\mathbb{P}(\Omega_0) > 0$, it follows from Furstenberg and Katznelson's theorem [37] that

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \mathbb{P}(\Omega_0 \cap \theta_{z_1}^{-i}\Omega_0 \cap \theta_{z_2}^{-i}\Omega_0) > 0.$$

These complete the proof. $\square$

We can assume without loss of generality that $x, y, x+y \in \mathbb{Z}^d \setminus \{0\}$. For $z_1, z_2 \in \mathbb{Z}^d$, let

$$A_{z_1,z_2} := \{z_1, z_2 \in \mathcal{C}_\infty,\ a_\lambda(z_1, z_2) \le c_1(\lambda + \log(2d))|z_1 - z_2|_1\},$$

where c_1 is the constant in Assumption 1.5.1(iii).

Let

$$A_i := A_{0,ix} \cap A_{0,i(x+y)} \cap A_{ix,i(x+y)}.$$

By using (6.2.1),

$$\frac{1}{n}\sum_{i=1}^{n} E_{\overline{\mathbb{P}}}\left[\frac{a_\lambda(0, i(x+y))}{i}, A_i\right]$$

$$\le \frac{1}{n}\sum_{i=1}^{n} E_{\overline{\mathbb{P}}}\left[\frac{a_\lambda(0, ix)}{i}, A_i\right] + \frac{1}{n}\sum_{i=1}^{n} E_{\overline{\mathbb{P}}}\left[\frac{a_\lambda(ix, i(x+y))}{i}, A_i\right].$$

Now it is sufficient to show the following convergences.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} E_{\overline{\mathbb{P}}}\left[\frac{a_\lambda(0, i(x+y))}{i}, A_i\right] = \alpha_\lambda(x+y)\frac{b_{x,x+y}}{\mathbb{P}(\Omega_0)}. \tag{6.7.1}$$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} E_{\overline{\mathbb{P}}}\left[\frac{a_\lambda(0, ix)}{i}, A_i\right] = \alpha_\lambda(x)\frac{b_{x,x+y}}{\mathbb{P}(\Omega_0)}. \tag{6.7.2}$$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} E_{\overline{\mathbb{P}}}\left[\frac{a_\lambda(ix, i(x+y))}{i}, A_i\right] = \alpha_\lambda(y)\frac{b_{x,x+y}}{\mathbb{P}(\Omega_0)}. \tag{6.7.3}$$

Here b denotes the constant in Lemma 6.7.1.

Now we prepare the following lemma.

Lemma 6.7.2.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \overline{\mathbb{P}}(A_i) = \frac{b_{x,x+y}}{\mathbb{P}(\Omega_0)}.$$

Proof. By using Lemma 6.7.1, it is sufficient to show that

$$\lim_{i\to\infty} \mathbb{P}(A_i^c \cap \Omega_0 \cap \theta_x^{-i}\Omega_0 \cap \theta_{x+y}^{-i}\Omega_0) = 0.$$

By noting (6.7.2) and Assumption 1.5.1(iii),

$$\begin{aligned}\mathbb{P}\left(A_i^c \cap \Omega_0 \cap \theta_x^{-i}\Omega_0 \cap \theta_{x+y}^{-i}\Omega_0\right) &\le \mathbb{P}\left(\Omega_0 \cap \theta_x^{-i}\Omega_0 \cap A_{0,ix}^c\right) + \mathbb{P}\left(\Omega_0 \cap \theta_{x+y}^{-i}\Omega_0 \cap A_{0,i(x+y)}^c\right)\\ &\quad + \mathbb{P}\left(\theta_x^{-i}\Omega_0 \cap \theta_{x+y}^{-i}\Omega_0 \cap A_{ix,i(x+y)}^c\right)\\ &\le \mathbb{P}\left(D(0,ix) > c_1 i|x|_1, 0 \leftrightarrow ix\right)\\ &\quad + \mathbb{P}\left(D(0,i(x+y)) > c_1 i|x+y|_1, 0 \leftrightarrow i(x+y)\right)\\ &\quad + \mathbb{P}\left(D(ix,i(x+y)) > c_1 i|y|_1, ix \leftrightarrow i(x+y)\right)\\ &\le 3c_1 \exp\left(-c_2(\log(i\min\{|x|_1,|x+y|_1,|y|_1\}))^{1+c_3}\right).\end{aligned}$$

Since $x, y, x+y \neq 0$, $\exp\left(-c_2(\log(i\min\{|x|_1,|x+y|_1,|y|_1\}))^{1+c_3}\right) \to 0$, $i \to \infty$. This completes the proof of Lemma 6.7.2. □

We show (6.7.2). First, we have that

$$E_{\overline{\mathbb{P}}}\left[\frac{a_\lambda(0,ix)}{i}, A_i\right] = E_{\overline{\mathbb{P}}}\left[\frac{a_\lambda(0,ix)}{i} - \alpha_\lambda(x), A_i\right] + \alpha_\lambda(x)\overline{\mathbb{P}}(A_i).$$

By noting Lemma 6.7.1, it is sufficient to show that

$$E_{\overline{\mathbb{P}}}\left[\left|\frac{a_\lambda(0,ix)}{i} - \alpha_\lambda(x)\right| 1_{A_i}\right] \to 0,\ i \to \infty. \tag{6.7.4}$$

By using Proposition 6.3.1, we have that

$$\left|\frac{a_\lambda(0,ix)}{i} - \alpha_\lambda(x)\right| 1_{A_i} \le \left|\frac{a_\lambda(0,ix)}{i} - \alpha_\lambda(x)\right| 1_{\{0,ix \in \mathcal{C}_\infty\}} \to 0,\ i \to \infty,\ \overline{\mathbb{P}}\text{-a.s.}$$

By recalling the definition of A_i,

$$\left|\frac{a_\lambda(0,ix)}{i} - \alpha_\lambda(x)\right| 1_{A_i} \le c_1(\lambda + \log(2d)) + \alpha_\lambda(x),\ i \ge 1.$$

By using the Lebesgue convergence theorem, we obtain (6.7.4). Thus (6.7.2) is shown.

We can show (6.7.1) in the same manner.

Finally we show (6.7.3). By noting Lemma 6.7.2, it is sufficient to show that

$$E_{\mathbb{P}}\left[\left|\frac{a_\lambda(ix,i(x+y))}{i} - \alpha_\lambda(y)\right| 1_{A_i}\right] \to 0,\ i \to \infty. \tag{6.7.5}$$

Here we denote the expectation with respect to $\mathbb{P}$ by $E_{\mathbb{P}}$.

By using the shift invariance of $\mathbb{P}$, we have

$$E_{\mathbb{P}}\left[\left|\frac{a_\lambda(ix,i(x+y))}{i} - \alpha_\lambda(y)\right| 1_{A_i}\right] = E_{\mathbb{P}}\left[\left|\frac{a_\lambda(0,iy)}{i} - \alpha_\lambda(y)\right| 1_{\theta_x^i A_i}\right].$$

Now we have that $a_\lambda(0, iy) \le c_1(\lambda + \log(2d))i|y|_1$ on $\theta_x^i A_i$. Hence,

$$\left|\frac{a_\lambda(0, iy)}{i} - \alpha_\lambda(y)\right| 1_{\theta_x^i A_i} \le c_1(\lambda + \log(2d))|y|_1 + \alpha_\lambda(y).$$

By noting Proposition 6.3.1,

$$\left|\frac{a_\lambda(0, iy)}{i} - \alpha_\lambda(y)\right| 1_{\theta_x^i A_i} \le \left|\frac{a_\lambda(0, iy)}{i} - \alpha_\lambda(y)\right| 1_{\{0, iy \in \mathcal{C}_\infty\}} \to 0,\ i \to \infty, \mathbb{P}\text{-a.s.}$$

Thus we obtain (6.7.5) by using the Lebesgue convergence theorem and hence (6.7.3) is shown.

Thus we see that α_λ is subadditive.

Bibliography

[1] E. de Amo, M. Díaz Carrillo, and J. Fernández-Sánchez, Singular functions with applications to fractal dimensions and generalized Takagi functions, Acta Appl. Math. 119 (2012), 129-148.

[2] P. Antal and A. Pisztora, On the chemical distance for supercritical Bernoulli percolation, Ann. Probab. 24 (1996) 1036-1048.

[3] S. Athreya, S. Sethuraman, and B. Tóth, On the range, local times and periodicity of random walk on an interval, ALEA, Lat. Am. J. Probab. Math. Stat. 8 (2011), 269-284.

[4] K. Azuma, Weighted sums of certain random variables, Tôhoku Math. J. (2) 19 (1967) 357-367.

[5] M. T. Barlow, Which values of the volume growth and escape time exponent are possible for a graph?, Rev. Mat. Iberoamericana, 20 (2004) 1-31.

[6] M. T. Barlow and R. F. Bass, Brownian motion and harmonic analysis on Sierpinski carpet, Canad. J. Math. 51(1999), 673-744.

[7] M. T. Barlow and R. F. Bass, "Random walks on graphical Sierpinski carpets" in Random Walks and Discrete Potential Theory (Cortona, Italy, 1997), Sympos. Math. 39, Cambridge University Press, Cambridge, (1999), 26-55.

[8] M. T. Barlow, T. Coulhon, and, T. Kumagai, Characterization of sub-Gaussian heat kernel estimates on strongly recurrent graphs, Comm. Pure Appl. Math. 58 (2005) 1642-1677.

[9] R. F. Bass and T. Kumagai, Laws of the iterated logarithm for the range of random walks in two and three dimensions, Ann. Probab. 30 (2002) 1369-1396.

[10] R. F. Bass, X. Chen, and, J. Rosen, Moderate deviations for the range of planer random walks, Mem. Amer. Math. Soc. 198 (2009) no. 929.

[11] R. F. Bass and J. Rosen, An almost sure invariance principle for the range of planer random walks, Ann. Probab. 33 (2005) 1856-1885.

[12] V. Beffara and H. Duminil-Copin, The self-dual point of the two-dimensional random-cluster model is critical for $q \geq 1$, Probab. Theory Related Fields 153 (2012) 511-542.

[13] I. Benjamini, R. Izkovsky and H. Kesten, On the range of the simple random walk bridge on groups, Elec. J. Prob., 12 (2007), 591-612.

[14] L. Berg and M. Krüppel, De Rham's singular function and related functions, Z. Anal. Anwendungen 19 (2000), 227-237.

[15] N. Berger and M. Biskup, Quenched invariance principles for simple random walk on percolation clusters, Probab. Theory Related Fields 137 (2007), 83-120.

[16] T. Bodineau, Slab percolation for the Ising model, Probab. Theory and Relat. Fields 132 (2005) 83-118.

[17] G. J. Chaitin, A theory of program size formally identical to information theory, J. Assoc. Comput. Mach. 22 (1975) 329-340.

[18] X. Chen, Moderate and small deviations for the ranges of one-dimensional random walks, J. Theoret. Probab. 19 (2006) 721-739.

[19] R. M. Conley, A survey of the Minkowski ?(x) Function, Master thesis, West Virginia University, 2003.

[20] O. Couronné and R. J. Messikh, Surface order large deviations for 2D FK-percolation and Potts models, Stoc. Proc. Appl. 113 (2004) 81-99.

[21] F. M. Dekking, On transience and recurrence of generalized random walks, Z. Wahr. verw. Gebiate 61 (1982) 459-465.

[22] A. Denjoy, Sur une fonction de Minkowski, C. R. Acad. Sci. Paris 194 (1932) 44-46.

[23] A. Denjoy, Sur une fonction réelle de Minkowski, J. Math. Pures Appl. 17 (1938) 105-151.

[24] M. Denker, and K. Hattori, Recurrence of self-repelling and self-attracting walks on the pre-Sierpiński gasket and $\mathbb{Z}$, Stoc. Dyn. 8 (2008), 155-172.

[25] Y. Derriennic, Quelques applications du théorème ergodique sous-additif, Astérisque 74 (1980) 183-201.

[26] R. G. Downey and D. R. Hirschfeldt, Algorithmic randomness and complexity, Springer, New York, 2010.

[27] M. D. Donsker and S. R. S. Varadhan, On the number of distinct sites visited by a random walk, Comm. Pure Appl. Math. 32 (1979) 721-747.

[28] A. Drewitz, B. Ráth and A. Sapozhnikov, On chemical distances and shape theorems in percolation models with long-range correlations, J. Math. Phys. 55, 083307, DOI : 10. 1063/1.4886515.

[29] A. Dvoretzky and P. Erdös, Some problems on random walk in space, Proc. Second Berkeley Symp. on Math. Stat. and Prob., 353-368, Univ. of California Press, 1951.

[30] G. Edger, Classics on Fractals, Addison-Wesley, Reading, MA, 1993.

[31] A. Erschler, Isoperimetry for wreath products of markov chains and multiplicity of self-interactions of random walks, Probab. Theory Related Fields 136 (2006) 560-586.

[32] S. A. Fenner, Gales and supergales are equivalent for defining constructive Hausdorff dimension, available at arXiv:cs/0208044.

[33] J. Fernández-Sánchez, P. Viader, J. Paradís and M. Díaz-Carrillo, A singular function with a non-zero finite derivative, Nonlinear Anal. 75 (2012) 5010-5014.

[34] J. Fernández-Sánchez, P. Viader, J. Paradís and M. Díaz-Carrillo, A singular function with a non-zero finite derivative on a dense set, Nonlinear Anal. 95 (2014) 703-713.

[35] R. Fukushima and N. Kubota, Quenched large deviations for multidimensional random walk in random environment with holding times, J. Theoret. Probab. 27 (2014) 1140-1166.

[36] H. Furstenberg, Noncommuting random products, Trans. Amer. Math. Soc. 108 (1963), 377-428.

[37] H. Furstenberg and Y. Katznelson, An ergodic Szeméredi teorem for commuting transformations, J. Analyse. Math. 34 (1978), 275-291.

[38] O. Garet and R. Marchand, Asymptotic shape for the chemical distance and first-passage percolation on the infinite Bernoulli cluster, ESAIM Probab. Stat. 8 (2004), 169-199.

[39] L. R. Gibson, The mass of sites visited by a random walk on an infinite graph, Elec. J. Probab. 13 (2008) no. 44 1257-1282.

[40] R. Girgensohn, Constructing singular functions via Farey fractions, J. Math. Anal. Appl. 203 (1996) 127-141.

[41] G. R. Grimmett, Percolation, Springer-Verlag, 2nd ed., 1999.

[42] G. R. Grimmett, The random-cluster model, Springer-Verlag, 2006.

[43] G. Grimmett and J. Marstrand, The supercritical phase of percolation is well behaved, Proc. R. Soc. Lond. Ser. A 430 (1990) 439-457.

[44] Y. Hamana, An almost sure invariance principle for the range of random walks, Stochastic Process Appl. 78 (1998) 131-143.

[45] Y. Hamana, On the range of pinned random walks, Tôhoku Math. J. 58 (2006) 329-357.

[46] Y. Hamana, Asymptotics of the moment generating function for the range of random walks, J. Theoret. Probab. 14 (2001) 189-197.

[47] Y. Hamana and H. Kesten, A large-deviation result for the range of random walk and for the Wiener sausage, Probab. Theory Related Fields 120 (2001) 183-208.

[48] Y. Hamana and H. Kesten, Large deviations for the range of an integer-valued random walk, Ann. Inst. H. Poincaré Probab. Statist. 38 (2002) 17-58.

[49] B. M. Hambly, K. Hattori, and T. Hattori, Self-repelling walk on the Sierpiński gasket, Probab. Theory Relat. Fields 124 (2002), 1-25.

[50] M. Hata, On the structure of self-similar sets, Japan J. Appl. Math. 2 (1985), 381-414.

[51] K. Hattori, and T. Hattori, Displacement exponents of self-repelling walks and self-attracting walks on the pre-Sierpiński Gasket, J. Math. Sci. Univ. Tokyo 12 (2005), 417-443.

[52] T. Hattori, and K. Hattori, Renormalization group approach to a generalization of the law of iterated logarithms for one-dimensional (non-Markovian) stochastic chains, Kokyuroku (Kyoto Univ.) 1386 (2004), 35-49.

[53] E. Heillinger, Die othogonalinvarianten quadratischer formen von unendlichvielen variablen, PhD thesis, Göttingen, 1907.

[54] E. Hewitt, K. Stromberg, Real and Abstract Analysis. A Modern Treatment of the Theory of Functions of a Real Variable, Springer, New York, 1965.

[55] J. M. Hitchcock, Gales suffice for constructive dimension, Inform. Process. Lett. 86 (2003) 9-12.

[56] N. C. Jain and S. Orey, On the range of random walk, Israel J. Math. 6 (1978) 373-380.

[57] N. C. Jain and W. E. Pruitt, The range of recurrent random walk in the plane, Z. Wahrsch. Verw. Gebiate 16 (1970) 279-292.

[58] N. C. Jain and W. E. Pruitt, The range of transient random walk, J. Anal. Math. 24 (1971) 369-393.

[59] N. C. Jain and W. E. Pruitt, The law of the iterated logarithm for the range of random walk, Ann. Math. Statist. 43 (1972) 1692-1697.

[60] N. C. Jain and W. E. Pruitt, The range of random walk, Proc. Sixth Berkeley Symp. Math. Statis. Probab. 3 (1972) 31-50. Univ. California Press, Berkeley.

[61] N. C. Jain and W. E. Pruitt, Further limit theorems for the range of random walk, J. Anal. Math. 27 (1974) 94-117.

[62] H.-H. Kairies, Functional equations for peculiar functions, Aequ. Math. 53 (1997) 207-241.

[63] M. Kanai, Rough isometries, and combinatorial approximations of geometries of non-compact riemannian manifolds, J. Math. Soc. Japan, 37 (1985), 391-413.

[64] M. Kanai, Analytic inequalities, and rough isometries between non-compact riemannian manifolds. Lect. Notes Math. 1201, 122-137, Springer, 1986.

[65] Pl. Kannappan, Functional Equations and Inequalities with Applications, Springer Monographs in Mathematics, Springer 2009.

[66] K. Kawamura, On the set of points where Lebesgue's singular function has the derivative zero, Proc. Japan Acad. Ser. A Math Sci. 87 (2011), 162-166.

[67] A. N. Kolmogorov, Three approaches to the quantitative definition of 'information', Probl. Inf. Transm. 1 (1965) 1-7.

[68] M. Krüppel, De Rham's singular function, its partial derivatives with respect to the parameter and binary digit sums, Rostock. Math. Kolloq. 64 (2009), 57-74.

[69] N. Kubota, Large deviations for simple random walk on supercritical percolation clusters, Kodai Math. J. 35 (2012), 560-575.

[70] T. Kumagai, Random walks on disordered media and their scaling limits, École d'Été de Probabilités de Saint-Flour XL 2010, Springer, 2014.

[71] J.-F. Le Gall, Propriétés d'intersection des marches alḿatoirs, I. Convergence vers le temps local d'intersection, Comm. Math. Phys. 104 (1986) 471-507.

[72] L. A. Levin, On the notion of a random sequence, Sov. Math. Dokl. 14 (1973) 1413-1416.

[73] T. Liggett, An improved subadditive ergodic theorem, Ann. Probab. 13 (1985) 1279-1285.

[74] T. Liggett, R. Schonmann and A. Stacey, Domination by product measures, Ann. Probab. 25 (1997) 71-95.

[75] J. H. Lutz, The dimensions of individual strings and sequences, Inform. and Comput. 187 (2003) 49-79.

[76] J. H. Lutz and E. Mayordomo, Dimension of points in self-similar fractals, SIAM J. Comput. 38 (2008) 1080-1112.

[77] P. Martin-Löf, The definition of random sequences, Information and Control 9 (1966) 602-619.

[78] H. Minkowski, Zur Geometrie der Zahlen, in "Verhandlungen des III. Internationalen Mathematiker-Kongress in Heidenberg," 1904, pp. 164-173.

[79] S. Nandakumar, A characterization of constructive dimension, Math. Logic Quart. 55 (2009) 271-286.

[80] H. Okamoto and M. Wunsch, A geometric construction of continuous, strictly increasing singular functions, Proc. Japan Acad. Ser. A Math. Sci. 83 (2007) 114-118.

[81] K. Okamura, On the range of random walk on graphs satisfying a uniform condition, ALEA, Lat. Am. J. Probab. Math. Stat. 11 (2014) 341-357.

[82] K. Okamura, Singularity results for functional equations driven by linear fractional transformations, J. Theoret. Probab., 27 (2014) 1316-1328.

[83] K. Okamura, On the range of self-interacting random walks on an integer interval, Tsukuba J. Math. 38 (2014) 123-135.

[84] K. Okamura, Random sequences with respect to a measure defined by two linear fractional transformations, to appear in Theory of Computing Systems, DOI 10.1007/s00224-014-9585-1.

[85] K. Okamura, Large deviations for simple random walk on percolations with long-range correlations, preprint, available at arXiv 1308.2282v2.

[86] J. Paradís, P. Viader, and, L. Bibiloni, Riesz-Nágy singular fnuctions revisited, J. Math. Anal. Appl. 329 (2007) 592-602.

[87] J. Paradís, P. Viader, and, L. Bibiloni, A new singular function, Amer. Math. Monthly 118 (2011) 344-354.

[88] Y. Peres, Probability on trees : an introductory climb, Lectures on probability theory and statistics (Saint-Flour 1997) 193-280. Lect. Notes Math. 1717, Springer, Berlin, 1999.

[89] S. Pincus, A class of Bernoulli random matrices with continuous singular stationary measures, Ann. Probab. 11 (1983) 931-938.

[90] S. Pincus, Singular stationary measures are not always fractal, J. Theoret. Probab. 7 (1994) 199-208.

[91] A. Pisztora, Surface order large deviations for Ising, Potts and percolation models, Probab. Theory Relat. Fields 104 (1996) 427-466.

[92] J. H. Pitt, Multiple points of transient random walks, Proc. Amer. Math. Soc. 43 (1974) 195-199.

[93] M. Pollicott and M. Yuri, Dynamical systems and ergodic theory, Cambridge university press, 1998.

[94] V. Y. Protasov, On the regularity of de Rham curves, Izv. Math. 68 (2004) 567-606.

[95] J. Reiman and F. Stephan, Hierarchies of randomness tests, Mathematical logic in Asia, 215-232, World Sci. Publ., Hackensack, NJ. 2006.

[96] G. de Rham, Sur quelques courbes définies par des équations fonctionalles, Rend. Sem. Mat. Torino 16 (1957), 101-113.

[97] F. Riesz, B. Sz.-Nágy, Functional Analysis, Dover Books on Advanced Mathematics, Dover, New York, 1990. Translated from the second French edition by Leo F. Boron, reprint of the 1955 original. First published in French in 1952 by the Hungarian Academy of Science.

[98] R. Salem, On some singular monotonic functions which are strictly increasing, Trans. Amer. Math. Soc. 53 (1943) 427-439.

[99] A. Sapozhnikov, Random walks on infinite percolation clusters in models with long-range correlations, preprint, available at arXiv 1410. 0605v2.

[100] C. P. Schnorr, A unified approach to the definition of random sequences, Mathematical Systems Theory, 5 (1971) 246-258.

[101] F. Spitzer, Principles of Random Walk, 2nd ed., Springer-Verlag, 1976.

[102] F. Spitzer, in Discussion of subadditive ergodic theory by J. F. C. Kingman, Ann. Probab. 1 (1973) 904-905.

[103] L. Staiger, Rich ω-words and monadic second-order arithmetic, Computer science logic (Aarhus 1997), Lecture Notes in Comput. Sci. 1414, 478-490, Springer, Berlin, 1998.

[104] L. Staiger, Constructive dimension equals Kolmogorov complexity, Inform. Process. Lett. 93 (2005) 149-153.

[105] L. Takács, An increasing continuous singular function, Amer. Math. Monthly 85 (1978) 35-37.

[106] T. Tao, Norm convergence of multiple ergodic averages for commuting transformations, Ergodic Theory Dynam. Systems, 28 (2008), 657-688.

[107] K. Uchiyama, The mean number of sites visited by a pinned random walk, Math. Z. 261 (2009) 277-295.

[108] N. Varopoulos, Isoperimetric inequalities and Markov chains, J. Func. Anal., 63 (1985), 215-239.

[109] D. Williams, Probability with martingales, Cambridge University Press, 1991.

[110] W. Woess, Random walks on infinite graphs and groups, Cambridge University Press, Cambridge, 2000.

[111] M. P. W. Zerner, Lyapunov exponents and quenched large deviations for multidimensional random walk in random environment, Ann. Probab. 26 (1998), 1446-1476.

[112] A. K. Zvonkin and L. A. Levin, The complexity of finite objects and the basing of the concepts of information and randomness on the theory of algorhithms , Uspehi Mat. Nauk., 25 (1970) 85-127.