博士論文

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FUKAYA CATEGORIES AND BLOW-UPS

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Abstract. In this paper we study the Lagrangian Floer theory of Kähler manifolds and their blow-ups. We present a method to describe the quantum cohomology from the Lagrangian Floer theory. As an example we determine the quantum cohomology of a blow-up from its corresponding potential function.

1. Introduction

1.1. **Backgrounds.** The motivation of this paper comes from Homological Mirror Symmetry conjecture(HMS) proposed by Kontsevich [16]. Although this conjecture was initially stated as a duality between Calabi-Yau manifolds, it was extended to Fano manifolds and manifolds of general type [15]. For these manifolds it is known that the mirror counterparts are given by so-called Landau-Ginzburg models, that is, complex manifolds with holomorphic functions(potential functions). For a Landau Ginzburg model corresponding to X, we have two different types of categories; one is the category of D-branes of Landau Ginzburg A-models and the other is that of Landau Ginzburg B-models. The HMS relates these two categories, respectively, to the bounded derived category of coherent sheaves $D^bCoh(X)$ and the derived Fukaya category $D^\pi Fuk(X)$. The categories of D-branes of Landau-Ginzburg A-models are called Fukaya-Seidel categories in mathematics, whose objects consist of Lefschetz thimbles [21]. It is expected that Fukaya-Seidel categories admit semi-orthogonal decompositions whose summands correspond to the connected components of the critical loci of the potential functions. On the other hand, the categories of D-branes of Landau-Ginzburg B-models are categories of singularities due to Orlov[19]. These categories admit orthogonal decompositions whose summands correspond to the connected components of the critical loci of the potential functions. Here observing that both two categories admit (semi-)orthogonal decompositions described by the critical loci of the same potential functions , it is natural to expect that there exists some correspondence between semi-orthogonal decompositions of Fukaya-Seidel categories and orthogonal decompositions of categories of singularities. By considering mirror symmetry, we can also expect that there exists some correspondence between orthogonal decompositions of Fukaya categories and semi-orthogonal decompositions of categories of coherent sheaves [15]. To state our observation above more concretely, let us consider the geometry of blow-ups. Let X be a smooth projective variety and C be a smooth subvariety of X. Assume, for simplicity, codimension of C is two. We denote by X_C the blow-up of X along C. Then by the theorem of Orlov [18] we have $D^bCoh(X_C) = \langle D^bCoh(C), D^bCoh(X)\rangle$. This theorem suggests that the derived Fukaya category $D^\pi Fuk(X)$ admits some corresponding orthogonal decomposition. Moreover, by taking Hochschild cohomology of $D^\pi Fuk(X)$, we expect that Quantum cohomology $QH^*(X)$ decomposes into corresponding factors.

1.2. **Results.** The relationship between counting of holomorphic disks of Lagrangian torus fibrations and the corresponding Landau-Ginzburg models was first studied by Cho-Oh [4]. They proved that for toric Fano manifolds, the generating functions of counting of holomorphic disks give the potential functions of the Landau-Ginzburg mirrors. Recently, Abouzaid, Auroux, and Katzarkov constructed Lagrangian torus fibrations on blow-ups of toric manifolds and computed the potential functions of the Landau-Ginzburg mirrors[1].

In this paper, we slightly generalize the construction of Abouzaid-Auroux-Katzarkov and obtain some evidences of the above observation in the following two statements.

Let V be a compact toric Kähler manifold and H be a submanifold of V with some positivity condition. We put

$$X = V \times \mathbb{P}^{r-1},$$

and

$$C = H \times \overbrace{[0 : \cdots : 0 : 1]}^{r} \subset X.$$

Our first statement is that the potential function of X_C is a bulk deformed potential of X (Theorem 4.2, Theorem 4.8). We refer to Subsection 4.2 for a more precise statement. This statement implies that the bulk deformed derived Fukaya category of X is contained in the derived Fukaya category of X_C.

As an example of the above geometry, we consider the case where $V = \mathbb{P}^2, r = 2$. Our second statement is the following.

Theorem 1.1. (Theorem 5.15) When H is a smooth curve of degree three, then the quantum cohomology ring of X_C is described as follows:

$$QH^*(X_C;\Lambda) = \overbrace{\Lambda \times \cdots \times \Lambda}^{6} \times H^*(C;\Lambda).$$

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2. Notations and Preliminaries

2.1. **Notations.** We denote by Λ the universal Novikov field over $\mathbb{C}$. Namely,

$$\Lambda = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \middle| a_i \in \mathbb{C} \ \lambda_i \in \mathbb{R} \ \lim_{i\to\infty} \lambda_i = \infty \right\}.$$

This is an algebraically closed valuation field. We denote by Λ_0 the valuation ring of Λ and by Λ_+ the maximal ideal of Λ_0. Let $N \cong \mathbb{Z}^n$ be a free lattice of rank n, and let M be the dual lattice of N. We write the scalar extensions by

$$N_{\mathbb{R}} = N \otimes \mathbb{R}, \ M_{\mathbb{R}} = M \otimes \mathbb{R}.$$

Let P be a smooth polytope in $M_{\mathbb{R}}$, i.e., P is a polytope which defines a smooth toric Kähler manifold. We denote by Σ the normal fan of P. P may be given from Σ by

$$P = \{u \in M_{\mathbb{R}} | \langle u, v_i \rangle + c_i \geq 0 \ (1 \leq i \leq m)\},$$

where $\{v_1, \ldots, v_m\}$ is the set of all primitive generators of one dimensional cones in Σ and $c_1, \ldots, c_m$ are some constants. We denote by

$$\Lambda\langle\langle y, y^{-1}\rangle\rangle_0^{\overset{\circ}{P}}$$

the ring of formal Laurent series convergent on $\overset{\circ}{P}$ (see [7, Definition 1.2.1]). We have the surjective ring homomorphism

$$\psi : \Lambda_0[[Z_1, \ldots Z_m]] \to \Lambda\langle\langle y, y^{-1}\rangle\rangle_0^{\overset{\circ}{P}}$$

which sends Z_i to $T^{c_i} y^{v_i}$, where we identify the lattice of monomials with N, i.e., $\mathbb{C}[y, y^{-1}] = \mathbb{C}[N]$. We put

$$\Lambda\langle\langle y, y^{-1}\rangle\rangle_+^{\overset{\circ}{P}} = \psi(\Lambda_+[[Z_1, \ldots Z_m]]).$$

A formal Laurent power series $f \in \Lambda\langle\langle y, y^{-1}\rangle\rangle_0^{\overset{\circ}{P}}$ is said to be gapped if f is G-gapped for a discrete submonoid G of $\mathbb{R}_{\geq 0}$(see [7, Definition 1.2.5] for the definition of G-gapped elements). We denote by

$$\mathrm{Aut}(\Lambda\langle\langle y, y^{-1}\rangle\rangle_0^{\overset{\circ}{P}})$$

the set of gapped coordinate changes on $\overset{\circ}{P}$ (see [7, Definition2.5.1]).

2.2. **Potential functions of Kähler manifolds.** In this subsection, we recall some statements about the so-called potential functions and Floer cohomology of intersections of Lagrangian submanifolds (see [6] for the definition of Floer cohomology and [10] for the construction of cyclic filtered A_∞ algebras and their canonical models.)

Let (X, ω) be a compact Kähler manifold equipped with a Kähler form ω, and $L \subset X$ be a compact oriented spin Lagrangian submanifold. We assume the following conditions.

Conditon 2.1. Every rational curve has non-negative Chern number.

Conditon 2.2. Every holomorphic disk bounded by L has positive Maslov number.

For $\beta \in H_2(X, L; \mathbb{Z})$, we denote by $\mathcal{M}^X_{k;\ell}(L, \beta)$ the moduli space of the genus zero stable bordered holomorphic maps with the image in class β with k boundary marked points and also ℓ interior marked points. By Conditons 2.1 and 2.2, if the Maslov number of $\beta \in H_2(X, L; \mathbb{Z})$ is two, then $\mathcal{M}^X_{1;0}(L, \beta)$ has an oriented Kuranishi structure without boundary.

Definition 2.3. We define n_β by the following formula.

$$n_\beta = \deg[\mathrm{ev}_1 : \mathcal{M}^{\mathrm{X}}_{1;0}(\mathrm{L}, \beta) \to \mathrm{L}],$$

where ev_1 is the evaluation map of the boundary marked point.

Remark 2.4. To define the degree of evaluation map, we need perturb $\mathcal{M}^X_{1;0}(L, \beta)$ by choosing a continuous family of multisections. However, by Conditions 2.1 and 2.2, if perturbation is sufficiently small, n_β is independent of the choice of perturbation.

Definition 2.5. We call

$$W^X_L(b) = \sum_{\mu_{(X,L)}(\beta)=2} n_\beta T^{\frac{\omega(\beta)}{2\pi}} e^{\langle \partial\beta, b\rangle}$$

the potential function of (X, L), where b is an element of $H^1(L; \Lambda_0)$.

Remark 2.6. By Conditions 2.1 and 2.2, $W^X_L(b)$ is equal to the obstruction $\mathfrak{m}^{\mathrm{can},b}_0$ of the canonical model of the Floer complex.

We denote by $\mathrm{Crit}(W^X_L)$ the set of critical points of W^X_L. Let $e_1, \ldots e_n$ be an integral basis of $H^1(L; \mathbb{Q})$, where $n = \dim H^1(L, \mathbb{Q})$. We put $y_i =$

$e^{\langle e_i, b\rangle}$. Then we can consider W_L^X as an element of $\Lambda_0[[y_1, y_1^{-1}, \ldots, y_n, y_n^{-1}]]$. $b \in \mathrm{Crit}(W_L^X)$ is said to be non-degenerate if

$$\det \left[\frac{\partial^2 W_L^X}{\partial y_i \partial y_j}\right]_{i,j=1}^{i,j=n} (b) \neq 0.$$

We use a forgetful-map-compatible Kuranishi structure to define a gapped filtered A_∞ structure on $H^*(L;\Lambda)$. Hence $b \in H^1(L;\Lambda_0)$ is a weak Maurer-Cartan element of L. Therefore we can define Floer cohomology $HF^*((L,b);\Lambda)$ for $b \in H^1(L;\Lambda_0)$.

Lemma 2.7. Assume $L \cong T^n$, n-dimensional torus with $n = \dim_{\mathbb{C}} X$. Then,

(1) $HF^*((L,b);\Lambda) \neq 0$ if and only if $b \in \mathrm{Crit}(W_L^X)$.

(2) If $HF^*((L,b);\Lambda) \neq 0$, then

$$HF^*((L,b);\Lambda) \cong H^*(L;\Lambda)$$

as a $\mathbb{Z}/2\mathbb{Z}$ graded vector space.

(3) If $b \in \mathrm{Crit}(W_L^X)$ is non-degenerate, then

$$HF^*((L,b);\Lambda) \cong C\ell_n$$

as a $\mathbb{Z}/2\mathbb{Z}$ graded algebra, where $C\ell_n$ is the Clifford algebra of $\dim 2^n$ over Λ.

Proof. The proof is the same as [7, Section 3.6]. □

Remark 2.8. Since $C\ell_n$ is intrinsically formal([22, Corollary 6.4]),

$$HF^*((L,b);\Lambda) \cong C\ell_n$$

as a $\mathbb{Z}/2\mathbb{Z}$ graded A_∞ algebra.

Lemma 2.9. Suppose that L is diffeomorphic to n-dimensional torus T^n. Let b_1 and b_2 be non-degenerate critical points of W_L^X with $b_1 \neq b_2, W_L^X(b_1) = W_L^X(b_2)$. Then $HF^*((L,b_1),(L,b_2);\Lambda) = 0$.

Proof. The Floer differential on $H^*(L;\Lambda)$ is not equal to 0 since $b_1 \neq b_2$. Thus we obtain $\dim HF^*((L,b_1),(L,b_2);\Lambda) < 2^n$. Since b_1 and b_2 are non-degenerate critical points, $HF^*((L,b_1),(L,b_2);\Lambda)$ has a $\mathbb{Z}/2\mathbb{Z}$ graded Clifford bimodule structure. By an argument similar to Theorem [11, 2.11], we see that the dimensions of finite dimensional $\mathbb{Z}/2\mathbb{Z}$ graded irreducible Clifford representations are 2^n. Hence the claim follows. □

3. Geometry of blow-ups

3.1. **Constructions.** Let $F_1, \dots, F_k$ be holomorphic vector bundles on X with hermitian metrics and s_i be a holomorphic section of F_i. We denote by $C_i \subset X$ the zero set of s_i. We assume $C_1 \dots C_k$ intersect transversely. We put $C = \bigcup_{i=1}^k C_i$. We consider the following set.

$$X_C = \left\{ (x, [v_1], \dots, [v_k]) \in \mathbb{P}(F_1) \underset{X}{\times} \dots \underset{X}{\times} \mathbb{P}(F_k) \middle| s_i(x) \in \mathbb{C}v_i \ (i = 1, \dots k) \right\}$$

Since $C_1, \dots, C_k$ intersect transversally, we can describe X_C as an iterated blow-up along smooth submanifolds. Hence X_C is a smooth complex manifold. We denote by π the blow-down map from X_C to X. We put

$$E_i = \pi^{-1}(C_i), E = \bigcup_{i=1}^k E_i.$$

These are exceptional divisors of X_C. We choose a tubular neighborhood U of C and a smooth cut-off function χ supported in U with $\chi = 1$ near C.

Lemma 3.1. Let $\epsilon_1, \dots \epsilon_k \in \mathbb{R}$ be sufficiently small constants. We define a two form ω_ϵ on X_C by

$$\pi^*\omega + \sqrt{-1}\partial\bar{\partial}(\chi \sum_{i=1}^k \epsilon_i \log |s_i|),$$

where $|s_i|$ is the norm of s_i with respect to the hermitian metric. Then ω_ϵ is a Kähler form on X_C.

Proof. Since $\epsilon_1, \dots \epsilon_k$ are sufficiently small, ω_ϵ is non-degenerate on $X \setminus E$. On E,

$$\sqrt{-1}\partial\bar{\partial}(\chi \sum_{i=1}^k \epsilon_i \log |s_i|)$$

is non-degenerate and also $\pi^*\omega$ is positive. Hence, ω_ϵ is non-degenerate on E, too. □

Recall that $L \subset X$ is a compact oriented spin Lagrangian submanifold. From now on, we assume the following condition.

Conditon 3.2. *L is contained in $X \setminus U$.*

Then we can consider L as a compact oriented spin Lagrangian submanifold in X_C. The divisor $E_i \subset X_C \setminus L$ defines the relative cohomology class

$$[E_i] \in H^2(X_C, L; \mathbb{Z}).$$

We denote by $\mu_{(X,L)}$ the Maslov class of (X, L). The next lemma compute the relative cohomology group and the Maslov class of (X_C, L).

Lemma 3.3. (1) $H^2(X_C, L; \mathbb{Z}) = \pi^* H^2(X, L; \mathbb{Z}) \oplus \bigoplus_{i=1}^{k} \mathbb{Z}[E_i]$.

(2) $\mu_{(X_C,L)} = \pi^*\mu_{(X,L)} - 2\sum_{i=1}^{k}(\mathrm{rk}F_i - 1)[E_i]$, if X has a meromorphic volume form with pole along some normal crossing divisors which intersect transversely with C, and L is a special Lagrangian submanifoldsd with respect to this meromorphic volume form.

Proof. (1) By the Mayer-Vietoris sequence and the Leray-Hirsch theorem, we obtain

$$H^1(X_C; \mathbb{Z}) = \pi^* H^1(X; \mathbb{Z}),$$

$$H^2(X_C; \mathbb{Z}) = \pi^* H^2(X; \mathbb{Z}) \oplus \bigoplus_{i=1}^{k} \mathbb{Z}[E_i].$$

We consider the following diagram

$$\begin{tikzcd}
H^1(X;\mathbb{Z}) \arrow[r] \arrow[d, "\pi^*"'] & H^1(L;\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{id}"'] & H^2(X,L;\mathbb{Z}) \arrow[r] \arrow[d, "\pi^*"'] & H^2(X;\mathbb{Z}) \arrow[r] \arrow[d, "\pi^*"'] & H^2(L;\mathbb{Z}) \arrow[d, "\mathrm{id}"'] \\
H^1(X_C;\mathbb{Z}) \arrow[r] & H^1(L;\mathbb{Z}) \arrow[r] & H^2(X_C,L;\mathbb{Z}) \arrow[r] & H^2(X_C;\mathbb{Z}) \arrow[r] & H^2(L;\mathbb{Z}).
\end{tikzcd}$$

Then we have

$$H^2(X_C, L; \mathbb{Z}) = \pi^* H^2(X, L; \mathbb{Z}) \oplus \bigoplus_{i=1}^{k} \mathbb{Z}[E_i]$$

by the snake lemma([14]).

(2) By assumption, the pole of the meromorphic volume form defines the Maslov class $\mu_{(X,L)}$. The pole of the lift of this meromorphic volume form defines the Maslov class $\mu_{(X_C,L)}$. Since

$$-K_{X_C} \cong (-\pi^* K_X) \otimes \bigotimes_{i=1}^{k} \mathcal{O}(-E_i)^{\otimes(\mathrm{rk}F_i - 1)},$$

we have the desired formula. □

Proposition 3.4. Assume $F_i = \bigoplus_{j=1}^{\mathrm{rk}F_i} \mathcal{L}_{i,j}$ and $s_i = \oplus_{j=1}^{\mathrm{rk}F_i} s_{i,j}$, where $\mathcal{L}_{i,j}$ is a nef line bundle on X and $s_{i,j}$ is a holomorphic section of $\mathcal{L}_{i,j}$.

Suppose that

$$c_1(TX)(C) - \sum_{i=1}^{k}(\mathrm{rk}F_i - 1)\max_{1\le j\le \mathrm{rk}F_i} c_1(\mathcal{L}_{i,j})(C) \ge 0$$

for every rational curve $C \subset X$, then X_C satisfies Condition 2.1.

Proof. For $I \subset \{1,\dots,k\}$, we put $C_I = \bigcap_{i\in I} C_i$. We define X_I by

$$X_I = \left\{ (x, \prod_{i\in I}[v_i]) \in \prod_{i\in I}\mathbb{P}(F_i) \middle| s_i(x) \in \mathbb{C}v_i \ \text{ for } \ i \in I \right\}.$$

We denote by π_I the blow-down map from X_C to X_I. We prove

$$c_1(TX_C)(f) \ge 0$$

for an arbitrary holomorphic map f from $\mathbb{P}^1$ to X_C. Put $f_I = \pi_I \circ f$. We can assume

$$\mathrm{Im} f_\emptyset \subset C_I, \ \mathrm{Im} f_\emptyset \not\subset \bigcup_{i\notin I} C_i$$

for some I.

We first consider the case of $k = 1$ and $\mathrm{Im} f_\emptyset \not\subseteq C_1$. Put $D_{i,j} = s_{i,j}^{-1}(0)$. There exists $J \subsetneq \{1,\dots,\mathrm{rk}F_1\}$ such that

$$\mathrm{Im} f_\emptyset \subset \bigcap_{j\in J} D_{1,j}, \ \mathrm{Im} f_\emptyset \not\subset \bigcup_{j\notin J} D_{1,j}.$$

Then we see that

$$\begin{aligned} c_1(TX_C)(f) &= c_1(TX)(f_\emptyset) - (\mathrm{rk}F_1 - 1)[E_1](f) \\ &\ge c_1(TX)(f_\emptyset) - (\mathrm{rk}F_1 - 1)\min_{j\notin J} c_1(\mathcal{L}_{1,j})(f_\emptyset) \\ &\ge c_1(TX)(f_\emptyset) - (\mathrm{rk}F_1 - 1)\max_{1\le j\le \mathrm{rk}F_1} c_i(\mathcal{L}_{1,j})(f_\emptyset). \end{aligned}$$

By an inductive argument, we have

$$c_1(TX_C)(f) \ge c_1(TX_I)(f_I) - \sum_{i\notin I}(\mathrm{rk}F_i - 1)\max_{1\le j\le (\mathrm{rk}F_i)} c_1(\mathcal{L}_{i,j})(f_\emptyset).$$

We next prove

$$c_1(TX_I)(f_I) \ge c_1(TX|_{C_I})(f_\emptyset) - \sum_{i\in I}(\mathrm{rk}F_i - 1)\max_{1\le j\le (\mathrm{rk}F_i)} c_1(\mathcal{L}_{i,j})(f_\emptyset).$$

By abuse of notation, we continue to write E_i $(i \in I)$ for the exceptional divisors of X_I. We put

$$E_I = \bigcap_{i\in I} E_i.$$

This is a subset of X_I. We can describe E_I as the fiber product

$$\prod_{i\in I}\mathbb{P}_{C_I}(F_i).$$

We observe $\mathrm{Im}f_I \subset E_I$. We denote by H_i the hyperplane class of $\mathbb{P}_{C_I}(F_i)$.

Then, we have

$$\begin{aligned}
c_1(TX_I|_{E_I})(f_I) &= c_1(TE_I)(f_I) - \sum_{i\in I} H_i(f_I)\\
&= c_1(TC_I)(f_\emptyset) + \sum_{i\in I} c_1(F_i)(f_\emptyset) + \sum_{i\in I}(\mathrm{rk}F_i - 1)H_i(f_I)\\
&= c_1(TX|_{C_I})(f_\emptyset) + \sum_{i\in I}(\mathrm{rk}F_i - 1)H_i(f_I)\\
&= c_1(TX|_{C_I})(f_\emptyset) - \sum_{i\in I}(\mathrm{rk}F_i - 1)\max_{1\le j\le \mathrm{rk}F_i} c_1(\mathcal{L}_{i,j})(f_\emptyset)\\
&\quad + \sum_{i\in I}(\mathrm{rk}F_i - 1)\max_{1\le j\le \mathrm{rk}F_i}(c_1(\mathcal{L}_{i,j})(f_\emptyset) + H_i(f_I)),
\end{aligned}$$

where we use

$$c_1(TE_I)(f_\emptyset) = c_1(TC_I)(f_\emptyset) + \sum_{i\in I} c_1(F_i)(f_\emptyset) + \sum_{i\in I}(\mathrm{rk}F_i)H_i(f_I).$$

Since

$$\max_{1\le j\le \mathrm{rk}F_i}(c_1(\mathcal{L}_{i,j})(f_\emptyset) + H_i(f_I)) \ge 0,$$

we have

$$c_1(TX_I)(f_I) \ge c_1(TX|_{C_I})(f_\emptyset) - \sum_{i\in I}(\mathrm{rk}F_i - 1)\max_{1\le j\le (\mathrm{rk}F_i)} c_1(\mathcal{L}_{i,j})(f_\emptyset).$$

From these statements, we have

$$\begin{aligned}
c_1(TX_C)(f) &\ge c_1(TX)(f_\emptyset) - \sum_{i=1}^{k}(\mathrm{rk}F_i - 1)\max_{1\le j\le (\mathrm{rk}F_i)} c_1(\mathcal{L}_{i,j})(f_\emptyset)\\
&\ge 0.
\end{aligned}$$

This is the desired inequality. □

3.2. **The potential function of** (X_C, L)**.** In this subsection, we study the potential function of (X_C, L). We assume (X_C, L) satisfies Conditions 2.1 and 2.2.

Definition 3.5. $\beta \in H_2(X, L;\mathbb{Z})$ is said to be simple if $\mu_{(X,L)}(\beta) = 2$ and the domain of each element of $\mathcal{M}^X_{1;0}(L,\beta)$ is a disk (no bubble components). We denote by $H_2(X, L;\mathbb{Z})^{\mathrm{simp}}$ the set of simple homology classes.

For $\beta \in H_2(X,L;\mathbb{Z})$, we denote by $\hat{\beta}$ the element of $H_2(X_C,L;\mathbb{Z})$ with $\pi_*(\hat{\beta}) = \beta$ and $[E_i](\hat{\beta}) = 0$.

Lemma 3.6. If $\mu_{(X,L)}(\beta) = 2$, then

$$\mathcal{M}_{1;0}^{X_C}(L,\hat{\beta}) \cong \mathcal{M}_{1;0}^{X}(L,\beta)$$

as an oriented Kuranishi manifold.

Proof. For $f \in \mathcal{M}_{1;0}^{X_C}(L,\hat{\beta})$, $\pi \circ f$ is a stable disk in X with Maslov number two. Since X_C and L satisfy Conditions 2.1 and 2.2, both holomorphic disks in X bounded by L with Maslov number two and holomorphic spheres in X with Chern number zero are contained in $X \setminus C$ or C. Since $L \cap C = \emptyset$, we must have $\mathrm{Im}(\pi \circ f) \subset X \setminus C$. The claim follows from this. $\square$

We define by $W^{\mathrm{Ex}}(b)$ the part of the potential function $W_L^{X_C}(b)$ given by

$$\sum_{\substack{\beta \in H_2(X_C,L;\mathbb{Z}) \\ ([E_1](\beta),\dots,[E_k](\beta)) \neq (0,\dots,0)}} n_\beta T^{\frac{\omega_{\epsilon(\beta)}}{2\pi}} e^{\langle \partial\beta, b\rangle}.$$

Proposition 3.7.

$$W_L^{X_C}(b) = W_L^X(b) + W^{\mathrm{Ex}}(b)$$

Proof. This follows immediately from Lemma 3.6. $\square$

4. Main theorem

4.1. **Statements.** To state the main theorem, we need some definitions. We recall that (X,ω) is a compact Kähler manifold equipped with a Kähler form ω and L is a compact oriented spin Lagrangian submanifold. For $\delta \in \mathbb{R}_{\le 0}$, we put

$$P_L^\delta = \left\{ u \in H^1(L;\mathbb{R}) \,\middle|\, \langle u, \partial\beta\rangle + \frac{\omega(\beta)}{2\pi} \ge \delta \text{ for } \beta \in H_2(X,L;\mathbb{Z})^{\mathrm{simp}} \right\}.$$

Definition 4.1. (1) $\beta \in H_2(X,L;\mathbb{Z})^{\mathrm{simp}}$ is said to be fake if

$$\langle u, \partial\beta\rangle + \frac{\omega(\beta)}{2\pi} > \delta + \eta$$

for all $\eta > 0$ and $u \in P_L^{\delta+\eta}$.

(2) $\beta \in H_2(X,L;\mathbb{Z})^{\mathrm{simp}}$ is said to be essential if β is not fake.

We denote by $H_2(X,L;\mathbb{Z})^{\mathrm{ess}}$ the set of essential classes. For

$$v \in \partial H_2(X,L;\mathbb{Z})^{\mathrm{ess}} \subset H_1(L;\mathbb{Z}),$$

we put

$$n_v = \sum_{\substack{\beta \in H_2(X,L;\mathbb{Z})^{\mathrm{ess}} \\ \partial\beta = v}} n_\beta,$$

and

$$\omega_v = \omega(\beta),$$

where β is an element of $H_2(X,L;\mathbb{Z})^{\mathrm{ess}}$ with $\partial\beta = v$. Note that it is easy to check that if

$$\beta_1, \beta_2 \in H_2(X,L;\mathbb{Z})^{\mathrm{ess}} \ \text{ and } \ \partial\beta_1 = \partial\beta_2,$$

then

$$\omega(\beta_1) = \omega(\beta_2).$$

Hence ω_v is well defined.

The next statement is the main theorem of this paper.

Theorem 4.2. Assume the following conditions for X, C, L, and δ.

(1) P_L^δ is a smooth polytope in the sense of toric geometry.
(2) $\{\beta \in H_2(X,L;\mathbb{Z})^{\mathrm{ess}}\}$ is a finite set.
(3) For $\beta \in H_2(X,L;\mathbb{Z})^{\mathrm{ess}}$, $\partial\beta$ is a primitive element in $H_1(L,\mathbb{Z})$.
(4) For $v \in \partial H_2(X,L;\mathbb{Z})^{\mathrm{ess}}, n_v \neq 0$.
(5) $T^{-\delta}W_L^X$ is a gapped element of $\Lambda\langle\langle y, y^{-1}\rangle\rangle_0^{\overset{\circ}{P}{}_L^\delta}$.
(6) $T^{-\delta}W^{\mathrm{Ex}}$ is a gapped element of $\Lambda\langle\langle y, y^{-1}\rangle\rangle_+^{\overset{\circ}{P}{}_L^\delta}$.

Then, for a small constant $\eta \in \mathbb{R}_{>0}$ such that $P_L^{\delta+\eta}$ is combinatorially equivalent to P_L^δ, there exists

$$\mathfrak{b} \in H^*(X_{P_L^{\delta+\eta}}; \Lambda_0)$$

and

$$\varphi \in \mathrm{Aut}(\Lambda\langle\langle y, y^{-1}\rangle\rangle_0^{\overset{\circ}{P}{}_L^{\delta+\eta}})$$

which satisfy

$$T^{-\delta-\eta}W_L^{X_C} = W_{\mathfrak{b}}^{X_{P_L^{\delta+\eta}}} \circ \varphi,$$

where $X_{P_L^{\delta+\eta}}$ is the symplectic toric manifold defined by the smooth polytope $P_L^{\delta+\eta}$, and $W_{\mathfrak{b}}^{X_{P_L^{\delta+\eta}}}$ is the bulk deformed potential function of $X_{P_L^{\delta+\eta}}$ with a bulk parameter $\mathfrak{b}$(cf.[8]).

Proof. By the lemma below, if $\beta \in H_2(X,L;\mathbb{Z})^{\mathrm{simp}}$ is fake, then

$$T^{\frac{\omega(\beta)}{2\pi}-\delta-\eta}y^{\partial\beta}$$

is contained in $\Lambda\langle\langle y, y^{-1}\rangle\rangle_{+}^{\overset{\circ}{P}{}_L^{\delta+\eta}}$. Assume that $\beta \notin H_2(X, L;\mathbb{Z})^{\mathrm{simp}}$, $\mu_{(X,L)}(\beta) = 2$, and $\mathcal{M}_{1;0}^{X}(L, \beta) \neq \emptyset$. Then β decomposes into a simple class β' and an effective class α with the Chern number zero. Hence we have

$$T^{\frac{\omega(\beta)}{2\pi}-\delta-\eta}y^{\partial\beta} \in \Lambda\langle\langle y, y^{-1}\rangle\rangle_{+}^{\overset{\circ}{P}{}_L^{\delta+\eta}}.$$

By Assumptions (1) and (6), we have

$$T^{-\delta-\eta}W^{\mathrm{Ex}} \in \Lambda\langle\langle y, y^{-1}\rangle\rangle_{+}^{\overset{\circ}{P}{}_L^{\delta+\eta}}.$$

Then it follows that

$$T^{-\delta-\eta}W_L^{X_C} \equiv \sum_{v\in\partial H_2(X,L;\mathbb{Z})^{\mathrm{ess}}} n_v T^{\frac{\omega_v}{2\pi}-\delta-\eta}y^v \pmod{\Lambda\langle\langle y, y^{-1}\rangle\rangle_{+}^{\overset{\circ}{P}{}_L^{\delta+\eta}}}.$$

Now the existence of $\mathfrak{b}$ and the automorphism φ follow by the versality theorem([7, Theorem2.8.1]). □

Lemma 4.3. Assume Conditions (1)–(6) in Theorem 4.2 hold. Suppose that $\omega \in \mathbb{R}$ and $\alpha \in H_1(L;\mathbb{Z})$ satisfies the inequality

$$\langle \alpha, u\rangle + \omega > \delta + \eta$$

for all $u \in P_L^{\delta+\eta}$. Then

$$T^{\omega-\delta-\eta}y^{\alpha} \in \Lambda\langle\langle y, y^{-1}\rangle\rangle_{+}^{\overset{\circ}{P}{}_L^{\delta+\eta}}.$$

Proof. We denote by $\Sigma_L^{\delta+\eta}$ the normal fan of $P_L^{\delta+\eta}$. Since $P_L^{\delta+\eta}$ is a smooth polytope, there exists $\sigma \in \Sigma_L^{\delta+\eta}$ such that $\alpha \in \sigma$. Let $\{v_1, \ldots, v_l\}$ be the set of all primitive generators of σ. We write $\alpha = \sum_{i=1}^{l} \alpha_i v_i$, where $\alpha_i \in \mathbb{Z}_{\geq 0}$. Choose $u_\sigma \in P_L^{\delta+\eta}$ such that

$$\langle u_\sigma, v_i\rangle + \frac{\omega_{v_i}}{2\pi} = \delta + \eta$$

for $i = 1, \ldots, l$. By the morphism

$$\psi : \Lambda_0[[Z_1, \ldots, Z_m]] \to \Lambda\langle\langle y, y^{-1}\rangle\rangle_{0}^{\overset{\circ}{P}{}_L^{\delta+\eta}},$$

$T^{\omega-\sum_{i=1}^{l}(\frac{\omega_{v_i}}{2\pi}-\delta-\eta)\alpha_i-\delta-\eta}Z_1^{\alpha_1}\ldots Z_l^{\alpha_l}$ is send to $T^{\omega-\delta-\eta}y^{\alpha}$. By assumption, we have

$$\langle \alpha, u_\sigma\rangle + \omega > \delta + \eta.$$

Thus we have

$$\omega - \sum_{i=1}^{l}(\frac{\omega_{v_i}}{2\pi} - \delta - \eta)\alpha_i - \delta - \eta > 0,$$

which proves the lemma. □

Remark 4.4. If X is a toric manifold and L is a torus orbit, then Conditions (2), (3), (4) of Theorem4.2 are satisfied(cf.[4]). Moreover if δ is sufficiently small, then Condition (1) is also satisfied.

The next lemma is useful to check Condition (6) of Theorem 4.2.

Lemma 4.5. Assume that $W^{\mathrm{Ex}} \in \Lambda\langle\langle y, y^{-1}\rangle\rangle_0^{\overset{\circ}{P}{}_L^\delta}$. If $\beta \in H_2(X, L; \mathbb{Z})$ with $\mathcal{M}_{1;0}^X(L, \beta) \neq \emptyset$ is described as a sum of simple homology classes, and if $\epsilon_1, \dots, \epsilon_k$ and δ satisfy

$$\delta > \max_{1 \le i \le k}\{\epsilon_i\},$$

then

$$W^{\mathrm{Ex}} \in \Lambda\langle\langle y, y^{-1}\rangle\rangle_+^{\overset{\circ}{P}{}_L^\delta}.$$

Proof. Let $\tilde{\beta}$ be an element of $H_2(X_C, L; \mathbb{Z})$ with $\mathcal{M}_{1;0}^{X_C}(L, \tilde{\beta}) \neq \emptyset$ and $\mu_{(X_C,L)}(\tilde{\beta}) = 2$. Since $\tilde{\beta}$ decomposes into a relative homology class of some holomorphic disk and an effective class with Chern number zero, we can assume $\tilde{\beta}$ is a relative homology class of some holomorphic disk. We put $\beta = \pi_*\tilde{\beta}$ and $a_i = [E_i](\tilde{\beta})$. Then we have

$$\frac{\mu_{(X,L)}(\beta)}{2} \ge 1 + a_1 + \cdots + a_k,$$

and

$$\omega_\epsilon(\tilde{\beta}) = \omega(\beta) - 2\pi a_1\epsilon_1 - \cdots - 2\pi a_k\epsilon_k.$$

By assumption, we can write

$$\beta = \sum_{i=1}^{\frac{\mu_{(X,L)}(\beta)}{2}} \beta_i,$$

where β_i is an element of $H_2(X, L; \mathbb{Z})^{\mathrm{simp}}$. For $u \in P_L^\delta$, we have

$$\begin{aligned}
\langle\partial\beta, u\rangle + \frac{\omega_\epsilon(\tilde{\beta})}{2\pi} - \delta &= \sum_{i=1}^{\frac{\mu_{(X,L)}(\beta)}{2}} \left(\langle\partial\beta_i, u\rangle + \frac{\omega(\beta_i)}{2\pi} - \delta\right) + \frac{\mu_{(X,L)}(\beta)}{2}\delta - \delta - \sum_{i=1}^{k} a_i\epsilon_i \\
&> \sum_{i=1}^{\frac{\mu_{(X,L)}(\beta)}{2}} \left(\langle\partial\beta_i, u\rangle + \frac{\omega(\beta_i)}{2\pi} - \delta\right) + \sum_{i=1}^{k} a_i(\delta - \epsilon_i) \\
&> 0.
\end{aligned}$$

Thus

$$T^{\frac{\omega_\epsilon(\tilde{\beta})}{2\pi}} y^{\partial\beta} \in \Lambda\langle\langle y, y^{-1}\rangle\rangle_+^{\overset{\circ}{P}{}_L^\delta}.$$

$\square$

Remark 4.6. If X is a toric manifold and L is a torus orbit, then the assumptions of above lemma are satisfied.

4.2. **Examples.** The construction of this subsection slightly generalize that of [1]. Let (V, ω_V) be a compact toric Kähler manifold of complex dimension n defined by a fan

$$\Sigma_V \subset N_{\mathbb{R}}$$

and a polytope

$$P = \{u \in M_{\mathbb{R}} \mid \langle v_i, u \rangle + \lambda(v_i) \geq 0 \ \forall 1 \leq i \leq m\},$$

where $\{v_1, \ldots, v_m\}$ is the set of all primitive generators of one dimensional cones of Σ_V and λ is a strictly convex function on Σ_V. We denote by $L(u)$ the torus orbit of V corresponding to $u \in \overset{\circ}{P}$. We equip $L(u)$ with the standard spin structure(cf.[4, Section9]). We choose a small $\delta > 0$ such that P^δ is combinatorially equivalent to P, where

$$P^\delta = \{u \in M_{\mathbb{R}} \mid \langle v_i, u \rangle + \lambda(v_i) \geq \delta \ \forall 1 \leq i \leq m\}.$$

Let $\mathcal{L}_1, \ldots \mathcal{L}_k$ be nef line bundles on V defined, respectively, by integral convex functions $\rho_1, \ldots, \rho_k$ on Σ_V. Denote by $A_i \subset M$ the set of all integral points of the section polytope of $\mathcal{L}_i$. Consider integral convex functions σ_i on $\operatorname{Conv}(A_i)$, which determine regular triangulations of $\operatorname{Conv}(A_i)$. Then we define the sections s_{i,τ_i} of $\mathcal{L}_i$ by

$$s_{i,\tau_i} = \sum_{\alpha \in A_i} c_\alpha \tau_i^{\sigma_i(\alpha)} s_\alpha,$$

where $c_\alpha \in \mathbb{C}^*$ and $\tau_i \in \mathbb{R}_{>0}$ are arbitrary constants, and s_α are the sections of $\mathcal{L}_i$ corresponding to α. We put

$$H_{i,\tau_i} = s_{i,\tau_i}^{-1}(0) \subset V$$

and choose a tubular neighborhood U_V of H_{i,τ_i}. Assume σ_i attains unique minimum value at some point of A_i. By [17, Corollary6.4], we can choose τ_i and U_V such that the moment map image of U_V is contained in $P \setminus P^\delta$. We consider the following product

$$X = V \times \mathbb{P}^{r_1 - 1} \times \cdots \times \mathbb{P}^{r_k - 1}$$

with the natural projections p_V and p_i $(i = 1 \leq i \leq k)$ to each factor. We equip $\mathbb{P}^{r_i - 1}$ with the standard Fubini-Study Kähler form with moment polytope

$$\Delta_i = \left\{ (u_1, \ldots, u_{r_i - 1}) \in \mathbb{R}^{r_i - 1} \,\middle|\, \begin{matrix} 0 \leq u_1, \ldots, 0 \leq u_{r_i - 1} \\ u_1 + \cdots + u_{r_i - 1} \leq a_i \end{matrix} \right\},$$

where $a_i \in \mathbb{R}$ $(1 \leq i \leq k)$ are arbitrary constants with $a_i > r_i\delta$. Then X is equipped with the product Kähler form. For $i = 1, \dots, k$, put

$$F_i = p_i^*\mathcal{O}_{\mathbb{P}^{r_i-1}}(1)^{\oplus(r_i-1)} \oplus p_V^*\mathcal{L}_i,$$

and

$$s_{i,j} = \begin{cases} p_i^* z_{i,j} & (1 \leq j \leq r_i - 1) \\ p_V^* s_{i,\tau_i} & (j = r_i) \end{cases}$$

where $z_{i,1}, \dots, z_{i,r_i}$ are the homogeneous coordinates on $\mathbb{P}^{r_i-1}$. Then we have

$$C_i = s_i^{-1}(0) \cong H_{i,\tau_i},$$

where

$$s_i = \oplus_{j=1}^{r_i} s_{i,j}.$$

We assume that $C_1, \dots, C_k$ intersect transversally in X. Then $C_1, \dots C_k$ define our geometry X_C of the blow-up $\pi : X_C \to X$ defined in Section 3.

Let us choose a tubular neighborhood U_i of $[0 : \cdots : 0 : 1] \in \mathbb{P}^{r_i-1}$ such that the moment map image of U_i is contained in Δ_i^δ. Put

$$U = U_V \times U_1 \times \cdots \times U_k.$$

We assume that $\omega_\epsilon = \omega$ outside of U and $\epsilon_i < \delta$.

Proposition 4.7. Assume that

$$c_1(TV) - \sum_{i=1}^{k}(r_i - 1)c_1(\mathcal{L}_i) \geq 0.$$

Then

(1) X_C satisfies Condition 2.1.
(2) Suppose that

$$(u, u_1, \dots, u_k) \in P^\delta \times \Delta_1^\delta \times \cdots \times \Delta_k^\delta.$$

Then $(X_C, L(u, u_1, \dots, u_k))$ satisfies Condition 2.2, where $L(u, u_1, \dots, u_k)$ is the torus orbit of X corresponding to $(u, u_1, \dots, u_k)$.

Proof. (1) This follows immediately from Proposition 3.4.

(2) Let

$$f : (D, \partial D) \to (X_C, L(u, u_1, \dots, u_k))$$

be a non-constant holomorphic disk. By assumption, we obtain

$$L(u, u_1, \dots u_k) \subset X_C \setminus E.$$

Thus $f^{-1}(E_i) \subset D$ is isolated. Let p be a point of $f^{-1}(E_i) \subset D$ with multiplicity $m(p)$. Then $p_i \circ f$ passes through $[0 : \cdots : 0 : 1]$ with multiplicity $m(p)$. The contribution of this point to the Maslov number of $p_i \circ f$ is at least $(r_i - 1)m(p)$. Hence we have

$$\mu_{(X_C, L(u,u_1,\dots u_k))}(f) \geq \mu_{(V, L(u))}(p_V \circ f).$$

Therefore we see that $\mu_{(X_C, L(u,u_1,\dots u_k))}(f) \geq 2$ if $p_V \circ f$ is a non-constant map. On the other hand, if $p_V \circ f$ is a constant map, then we have

$$\mathrm{Im}(f) \subset X_C \setminus E,$$

and hence

$$\mu_{(X_C, L(u,u_1,\dots u_k))}(f) = \mu_{(X, L(u,u_1,\dots u_k))}(f) \geq 2.$$

This completes the proof. □

Theorem 4.8. Assume the positivity as in Proposition 4.7. Then $(X_C, L(u, u_1, \dots, u_k))$ satisfies the conditions (1)–(6) of Theorem 4.2.

Proof. $(X_C, L(u, u_1, \dots, u_k))$ satisfies Conditions (1)–(4) of Theorem 4.2, since X is toric. By Proposition 4.7, $(X_C, L(u, u_1, \dots, u_k))$ satisfies Condition 2.2 for all

$$(u, u_1, \dots u_k) \in P^\delta \times \Delta_1^\delta \times \cdots \times \Delta_k^\delta.$$

Hence we have

$$W^{X_C}_{L(u,u_1,\dots,u_k)} \in \Lambda\langle\langle y, y^{-1}\rangle\rangle_0^{\overset{\circ}{P}{}^\delta_L}.$$

Similarly, we have

$$W^{X}_{L(u,u_1,\dots u_k)} \in \Lambda\langle\langle y, y^{-1}\rangle\rangle_0^{\overset{\circ}{P}{}^\delta_L}.$$

From this we obtain

$$W^{\mathrm{Ex}}_{L(u,u_1,\dots u_k)} \in \Lambda\langle\langle y, y^{-1}\rangle\rangle_0^{\overset{\circ}{P}{}^\delta_L}.$$

By Lemma 4.5, we conclude that $(X_C, L(u, u_1, \dots, u_k))$ satisfies Conditions (5) and (6) of Theorem 4.2. □

Remark 4.9. This theorem implies that the derived Fukaya category of (X, ω) deformed by some bulk parameter is embedded in the derived Fukaya category of (X_C, ω_ϵ).

In the next section, we consider the special case of this example.

5. Quantum cohomology from Fukaya categories

5.1. **Some properties of quantum Gysin maps.** In this subsection, we consider a general compact symplectic manifold (X, ω) of real dimension $2n$ and compact oriented spin Lagrangian submanifolds with weak Maurer-Cartan elements (L_i, b_i) $(i = 1, 2)$. We assume that Floer cohomology $HF^*((L_1, b_1), (L_2, b_2); \Lambda)$ is well-defined. We take a basis e_I of

$$HF^*((L_1, b_1), (L_2, b_2); \Lambda).$$

and denote by e^I the dual basis of e_I. By using the natural duality, we consider e^I as an element of

$$HF^*((L_2, b_2), (L_1, b_1); \Lambda).$$

.

Definition 5.1. (see[7]) Let x_i be an element of $HF^*((L_i, b_i); \Lambda)$. We define a map

$$Z : HF^*((L_1, b_1); \Lambda) \to HF^*((L_2, b_2); \Lambda)$$

by the following property

$$\langle Z(x_1), x_2 \rangle_{PD_{L_2}} = \sum_I \pm \langle \mathfrak{m}_2(e^I, x_1), \mathfrak{m}_2(e_I, x_2) \rangle,$$

where $\mathfrak{m}_2$ is the product of Floer cohomology and $\langle , \rangle_{\mathrm{PD}_{L_2}}$ is the pairing on $HF^*((L_2, b_2); \Lambda)$ induced by the natural pairing on the set of differential forms on L_2. See Proposition 3.10.17 of [7] for the sign$\pm$ in the case where (L_1, b_1) is equal to (L_2, b_2).

Remark 5.2. Since we do not need precise definition of the sign, we do not fix it.

We denote by $QH^*((X, \omega); \Lambda)$ the small quantum cohomology ring of (X, ω) defined over Λ. Let

$$i_{(L_i, b_i)*} : HF^*((L_i, b_i); \Lambda) \to QH^{*+n}((X, \omega); \Lambda)$$

be the quantum Gysin map(see [6], [7]) and $\star$ be the small quantum product of (X, ω).

Assumption 5.3. We assume the following equality.

$$i_{(L_1, b_1)*}(x_1) \star i_{(L_2, b_2)*}(x_2) = i_{(L_2, b_2)*}(\mathfrak{m}_2(Z(x_1), x_2)),$$

where $\mathfrak{m}_2$ is the product of $HF^*((L_2, b_2); \Lambda)$.

Remark 5.4. It is expected that there is a ring morphism

$$i^*_{(L,b)} : QH^*((X,\omega);\Lambda) \to HF^*((L,b);\Lambda)$$

(See [9, Section 17] for the construction of this map). With this $i^*_{(L,b)}, HF^*((L,b);\Lambda)$ is a $QH^*((X,\omega);\Lambda)$ module. The quantum Gysin map $i_{(L,b)*}$ is expected to preserves this module structure. Assumption 5.3 should follow from this property of the quantum Gysin map $i_{(L,b)*}$ and the standard annulus argument(see [7] for toric cases, [12] for exact cases, and [22] for monotone cases).

Remark 5.5. The definition of $i_{(L,b)*}$ contains an issue about the cyclic symmetry, which should be settled in the line of [10].

Corollary 5.6. If

$$HF^*((L_1,b_1),(L_2,b_2);\Lambda) = 0,$$

then

$$i_{(L_1,b_1)*}(x_1) \star i_{(L_2,b_2)*}(x_2) = 0.$$

Proof. This follows immediately from Assumption 5.3. □

We use the next lemma to construct an idempotent of $QH^*((X,\omega);\Lambda)$.

Lemma 5.7. Suppose that (X, L) satisfies Conditions 2.1 and 2.2. Let $b \in H^1(L;\Lambda)$ be a critical point of $W^X_L(b)$. If $L \cong T^n$, then we have

$$Z([\mathrm{pt}]) = \det\left[y_i y_j \frac{\partial^2 W^X_L}{\partial y_i \partial y_j}\right]^{i,j=n}_{i,j=1}(b).$$

Proof. The proof is the same as [7] □

We assume the following formula.

Assumption 5.8. Assume that (X, L) satisfy Conditions 2.1 and 2.2. Then for $x \in HF^*((L,b);\Lambda)$,

$$c_1(TX) \star i_{(L,b)*}(x) = W^X_L(b) i_{(L,b)*}(x).$$

Remark 5.9. It is expected that $i_{(L,b)*}$ and $i^*_{(L,b)}$ satisfy the following equation(see [7, Theorem 3.3.8] for toric cases).

$$\langle i_{(L,b)*}(x), y\rangle_{\mathrm{PD}_X} = \langle x, i^*_{(L,b)}(y)\rangle_{\mathrm{PD}_L}$$

Assumption 5.8 follows from this equality and [9, Theorem 23.13].

Corollary 5.10. Assume that (X, L_i) (i=1,2) satisfy Conditions 2.1 and 2.2. If

$$W^X_{L_1}(b_1) \neq W^X_{L_2}(b_2)$$

then

$$i_{(L_1,b_1)}(x_1) \star i_{(L_2,b_2)}(x_2) = 0$$

5.2. **An example of computation of quantum cohomology of blow ups.** In this subsection, we consider an example of the geometry $\pi : X_C \to X$ given in Subsection 4.2, i.e., when $V = \mathbb{P}^2$, $k = 1$, $r_1 = 2$, and $F_1 = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(3,0)\oplus\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(0,1)$. In this case, the set of the integral points of the section polytope of F_1 is

$$\left\{(a,b,c) \in \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \middle| -1 \le a,\ -1 \le b,\ a+b \le 1,\ -1 \le c \le 1\right\}.$$

We assume that the integral convex function σ_1 attains the unique minimal value at $\{0\}$. Assume also that the moment polytope of (X,ω) is

$$\Delta_a \times [0,b],$$

where

$$\Delta_a = \left\{(u_1,u_2) \in \mathbb{R}^2 \middle| u_1 \ge 0, u_2 \ge 0, 0 \le u_1+u_2 \le a\right\}.$$

The complex torus $\mathbb{C}^*$ acts on the $\mathbb{P}^1$ factor of $X = \mathbb{P}^2\times\mathbb{P}^1$ and extend to the blow-up X_C. We can assume that ω_ϵ is invariant under this action. The next proposition describes the potential function of the torus orbit $L(u_1,u_2,u_3)$.

Proposition 5.11. Suppose that If $(u_1,u_2,u_3) \in \Delta_a^\delta \times [\delta, b-\delta]$, then

$$W^{X_C}_{L(u_1,u_2,u_3)} = (1+T^{-\epsilon+u_3}z)(T^{u_1}x+T^{u_2}y+\frac{T^{a-u_1-u_2}}{xy})+(1+c)T^{u_3}z+\frac{T^{b-u_3}}{z},$$

where $c \in \Lambda_+$ is the contribution of holomorphic spheres with $c_1 = 0$.

Proof. Holomorphic disks with Maslov number two contained in the blow-up of $\mathbb{P}^2\times(\mathbb{P}^1\setminus\{\infty\})$ are classified by [1]. The contribution of these disks to $W^{X_C}_{L(u_1,u_2,u_3)}$ is

$$(1+T^{-\epsilon+u_3}z)(T^{u_1}x+T^{u_2}y+\frac{T^{a-u_1-u_2}}{xy})+T^{u_3}z.$$

Let f be a holomorphic disk with Maslov number two which intersects with $\mathbb{P}^2\times\{\infty\}$. By the same argument of Proposition 4.7, we obtain

$$\mathrm{Im}(f) \subset \mathbb{P}^2\times(\mathbb{P}^1\setminus\{0\}).$$

Hence we see that $p_V\circ f$ is a constant map and $p_1\circ f$ is the northern half of $\mathbb{P}^1$. The contribution of these disks to $W^{X_C}_L$ is

$$\frac{T^{b-u_3}}{z}.$$

Let g be a holomorphic map from $\mathbb{P}^1$ to X_C with Chern number zero. By an argument similar to Proposition 4.7, $p_1\circ g$ is a constant map. If the image of this map is not equal to $\{0\}\subset\mathbb{P}^1$, then this image is contained in $X_C\setminus E$ and the Chern number of this map is positive.

Hence $\mathrm{Im}(g)$ is contained in the proper transform of $\mathbb{P}^2 \times \{0\}$. The constant $c \in \Lambda_+$ in the coefficient of $T^{u_3}z$ is the contribution from these holomorphic spheres. $\square$

We do not use explicit formula of the constant c. However, we have

Conjecture 5.12. Let $K_{\mathbb{P}^2}^{\mathbb{C}^*}$ be a canonical bundle of $\mathbb{P}^2$ with $\mathbb{C}^*$ action by the scalar multiplication. We denote by

$$\langle\ ,\dots,\ \rangle_{0,\ell,d}^{K_{\mathbb{P}^2}^{\mathbb{C}^*}}$$

the genus-zero equivariant Gromov-Witten invariants twisted by the equivariant Euler class of $K_{\mathbb{P}^2}^{\mathbb{C}^*}$, where ℓ is a number of marked points and d is an element of the set of effective classes $H_2^{\mathrm{eff}}(\mathbb{P}^2,\mathbb{Z}) \cong \mathbb{Z}_{\geq 0}$. Then,

$$c = \sum_{d=1}^{\infty} - \left\langle \frac{[\mathrm{pt}]}{\lambda+\psi} \right\rangle_{0,1,d}^{K_{\mathbb{P}^2}^{\mathbb{C}^*}} \Bigg|_{\lambda=0} T^{(a-3\epsilon)d},$$

where λ is an equivariant parameter.

Remark 5.13. (1) By using the Quantum-Lefschetz theorem for concave bundles ([5]), we can explicitly compute the right hand side of the above equation.
(2) We expect that we can prove the above formula by using Chan's capping argument ([3]) and the virtual localization technique([13]).

We put $\tilde{x} = T^{u_1}x, \tilde{y} = T^{u_2}y, \tilde{z} = T^{u_3}z$. We also put

$$W^{X_C} = (1 + T^{-\epsilon}\tilde{z})(\tilde{x} + \tilde{y} + \frac{T^a}{\tilde{x}\tilde{y}}) + (1+c)\tilde{z} + \frac{T^b}{\tilde{z}}.$$

The following statements are corollary of Proposition 5.11.

Corollary 5.14. W^{X_C} has six non-degenerate isolated critical points such that the tropicalization of these points are $(\frac{a}{3}, \frac{a}{3}, \frac{b}{2})$.

Proof. We can check directly. $\square$

Theorem 5.15. $QH^*((X_C,\omega_\epsilon);\Lambda) \cong \overbrace{\Lambda \times \cdots \times \Lambda}^{6} \times H^*(C,\Lambda)$ as a ring.

Proof. By using Assumption 5.3, Lemma 5.7, and Corollary 5.14, we can construct idempotents $e_1, \dots e_6$. Put

$$e_7 = 1 - e_1 - \cdots - e_6$$

and

$$QH_i^* = QH^*((X_C,\omega_\epsilon);\Lambda)e_i.$$

We first prove that $QH_i^{\mathrm{odd}} = 0$ for $i = 1,\dots,6$. Let α be an element of QH_i^{odd} $(1 \leq i \leq 6)$. Then we have $c_1(TX_C) \star \alpha = \lambda_i\alpha$, where λ_i is the critical value of W^{X_C} corresponding to e_i. By direct computation, we see that the valuation of λ_i is $\min\{\frac{a}{3}, \frac{b}{2}\}$. If $\alpha \neq 0$, we can choose $\beta \in QH^{\mathrm{odd}}((X_C, \omega_\epsilon); \Lambda)$ with $\langle \alpha, \beta \rangle_{\mathrm{PD}_{X_C}} = 1$. Then we see that

$$\lambda_i = \sum_{d \in H_2^{\mathrm{eff}}(X_C)} \langle c_1(TX_C), \alpha, \beta \rangle_{0,3,d} T^{\frac{\omega_\epsilon(d)}{2\pi}},$$

where $H_2^{\mathrm{eff}}(X_C)$ is the set of effective class of X_C. By degree counting, if $\langle c_1(TX_C), \alpha, \beta \rangle_{0,3,d} \neq 0$, then $c_1(TX_C)(d) = 1$. Hence the valuation of λ_i is contained in

$$\left\{ \frac{\omega_\epsilon(d)}{2\pi} \mid d \in H_2^{\mathrm{eff}}(X_C),\ c_1(d) = 1 \right\}.$$

By classification of holomorphic spheres with Chern number one (see proof of Proposition 3.4), we have

$$\left\{ \frac{\omega_\epsilon(d)}{2\pi} \mid d \in H_2^{\mathrm{eff}}(X_C),\ c_1(d) = 1 \right\} = \{b - \epsilon\} \cap \{\epsilon + k(a - 3\epsilon) \mid k \geq 0\}.$$

This contradicts $\mathrm{val}(\lambda_i) = \min\{\frac{a}{3}, \frac{b}{2}\}$. So we have $\alpha = 0$ and $QH_i^{\mathrm{odd}} = 0$ for $i = 1, \ldots, 6$.

Since $\dim QH^{\mathrm{even}}((X_C, \omega_\epsilon); \Lambda) = 8$ and $\dim QH^{\mathrm{odd}}((X_C, \omega_\epsilon); \Lambda) = 2$, it follows that $QH_7^* \cong H^*(C, \Lambda)$ and $QH_i^* \cong \Lambda$ for $i = 1, \ldots, 6$ as rings. $\square$

Remark 5.16. We expect that for an idempotent of $QH^*((X, \omega); \Lambda)$, we can construct a direct summand of the derived Fukaya category of (X, ω)(see[2],[20]). Moreover we expect that if this idempotent is constructed by a non-degenerate critical point, then the object of the derived Fukaya category corresponding to this point split generates the summand of the derived Fukaya category(see[20]) .

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