博　士　論　文

論文題目

Numerical analysis of various domain-penalty and boundary-penalty methods

（様々な領域処罰法および境界処罰法の数値解析）

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Contents

1 The fictitious domain method with L^2-penalty **3**
1.1 Introduction 3
1.2 The L^2-penalty problem 6
1.2.1 The fictitious domain method with L^2-penalty 6
1.2.2 The regularity and error estimates of the penalty problem 8
1.3 The finite element approximation to the L^2-penalty method . 14

2 The penalty method to the Stokes and Navier-Stokes equations with slip boundary condition **22**
2.1 Introduction 22
2.2 The penalty method to the Stokes problem 27
2.2.1 The error estimates of H^1 norm 29
2.2.2 The error estimates of H^m norm 31
2.2.3 Finite element approximation with penalty 33
2.2.4 Error estimates: nonreduced-integration scheme 40
2.2.5 Error estimates: reduced-integration scheme 41
2.2.6 Numerical examples 42
2.3 The penalty method to the non-stationary Navier-Stokes problem 45
2.3.1 The well-posedness of penalty problem 46
2.3.2 The error estimates of penalty 51
2.4 The penalty method to the stationary Navier-Stokes problem 54
2.4.1 The penalty problems $(\mathbf{NS}_\epsilon)$ and $(\mathbf{NS}'_\epsilon)$ 54
2.4.2 The well-posedness of $(\mathbf{NS}_\epsilon)$ and $(\mathbf{NS}'_\epsilon)$ 56
2.4.3 The iteration methods for $(\mathbf{NS}'_\epsilon)$ and $(\mathbf{NS}_\epsilon)$ 63
2.4.4 Error estimates of $(\mathbf{NS}'_\epsilon)$ 68
2.4.5 The finite element method to $(\mathbf{NS}'_\epsilon)$ 70

3 The Stokes/Navier-Stokes equations with a unilateral boundary condition of Signorini's type and its penalty method 74

3.1 Introduction 74
3.2 The energy inequality and the variational inequality 77
3.2.1 The re-definition of traction vectors 80
3.2.2 Variational form of (**NS**). 81
3.3 The well-posedness of (**NSI**) 84
3.4 Penalty method 88
3.5 The completion the proof of Theorem 3.3.1 and 3.4.1 92
3.6 The Stokes problem with a unilateral boundary condition of Signorini's type 108
3.6.1 Penalty method for the Stokes problem 110
3.6.2 Error estimate of penalty method 116

Acknowledgement 119

Bibliography 119

Chapter 1

The fictitious domain method with L^2-penalty

1.1 Introduction

The fictitious domain method is a powerful technique for solving partial differential equations. It is based on a reformulation of the original problem in a larger spatial domain, called the *fictitious domain*, with a simple shape. One of the advantages of this approach is that we can avoid the time-consuming construction of a boundary-fitted mesh. Thus, the fictitious domain is discretized by a simple-shaped mesh, independent of the original boundary. Consequently, we can directly apply a large class of numerical methods, for example, the finite element, finite volume, finite difference methods as well. Furthermore, this approach will be useful to solve time-dependent moving-boundary problems.

Actually, the fictitious domain reformulation combined with the finite volume and finite difference methods are successfully applied in numerical simulations for real-world problems, for example, a blood flow and fluid-structure interactions in thoracic aorta ([40]) and a simulation of spilled oil on coastal ecosystems ([39]). The aim of our work is to establish a mathematical study of the penalty fictitious domain method which can be applied to these time-dependent moving-boundary problems. As a primary step towards this final end, herein we examine the error analysis for elliptic problems.

In a previous work, Zhou and Saito [53], we studied a class of the fictitious domain methods with a penalty for elliptic problems with various boundary conditions. Therein, we introduce a fictitious domain reformulation by considering a discontinuous diffusion coefficient, which we call the H^1*-penalty fictitious domain method* or, simply, the H^1*-penalty method.* As is reported in [53], this reformulation and its finite element discretization enjoy finite mathematical properties. However, it is rather difficult to apply the finite volume and finite difference methods to the H^1-penalty method since the treatment of a discontinuous diffusion coefficient is not straightforward. Moreover, solutions of the H^1-penalty problem are not smooth across the original boundary that may cause some difficulties in actual computations.

In this chapter, we study another type of the fictitious domain method by introducing a *discontinuous reaction term*, which we call the L^2*-penalty fictitious domain method* or, simply, the L^2*-penalty method.* This method can be directly discretized not only by the finite element but also finite volume and finite difference methods. Moreover, the penalty solution has the H^2 regularity in the whole fictitious domain.

In Section 1.2, we study the L^2-penalty method by examining the H^2 regularity and some estimates for solutions of the L^2-penalty problem. Then, we derive error estimates of H^1 and L^2 norms. In summary, we have (cf. Theorem 1.2.1) the error estimates

$$\|u - u_\epsilon\|_{H^1(\Omega)} \le C\epsilon^{\frac{1}{4}}\|f\|_{L^2(\Omega)}, \qquad \|u - u_\epsilon\|_{L^2(\Omega)} \le C\epsilon^{\frac{1}{2}}\|f\|_{L^2(\Omega)},$$

where u and u_ϵ denote the solutions of the original elliptic problem (1.2.1) defined in a bounded domain $\Omega \subset \mathbb{R}^2$ and its L^2-penalty problem (1.2.19) for a given $f \in L^2(\Omega)$, ϵ is the penalty parameter with $\epsilon \to 0$. Moreover, the Dirichlet boundary condition posed on the original boundary $\Gamma = \partial\Omega$ is approximated in the sense that

$$\|u_\epsilon\|_{H^{\frac{1}{2}}(\Gamma)} + \frac{1}{\sqrt{\epsilon}}\|u_\epsilon\|_{L^2(\Omega_1)} \le C\epsilon^{\frac{1}{4}}\|f\|_{L^2(\Omega)},$$

where D denotes the fictitious domain such that $\overline{\Omega} \subset D$ and $\Omega_1 = D\backslash\overline{\Omega}$.

Thanks to our regularity results and error estimates, the finite element analysis becomes easy to treat. In Section 1.3, we derive the error estimates of the finite element approximation of the L^2-penalty problem. We have (cf. Theorem 1.3.1)

$$\|\nabla(u_\epsilon - u_{\epsilon h})\|_{L^2(D)} + \frac{1}{\sqrt{\epsilon}}\|u_\epsilon - u_{\epsilon h}\|_{L^2(\Omega_1)} \le C\|f\|_{L^2(\Omega)}(h^{\frac{1}{2}} + \epsilon^{\frac{1}{4}}),$$
$$\|u_\epsilon - u_{\epsilon h}\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}(h^{\frac{1}{2}} + \epsilon^{\frac{1}{4}})^2,$$

where $u_{\epsilon h}$ denotes the solution of the finite element approximation (1.3.1) for the L^2-penalty problem (1.2.19) with the mesh parameter h.

Consequently, we obtain (cf. Theorem 1.3.2)

$$\|u - u_{\epsilon h}\|_{H^1(\Omega)} \le C(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})\|f\|_{L^2(\Omega)}, \ \|u - u_{\epsilon h}\|_{L^2(\Omega)} \le C(\epsilon^{\frac{1}{2}} + h)\|f\|_{L^2(\Omega)},$$
$$\|u_{\epsilon h}\|_{H^{\frac{1}{2}}(\Gamma)} + \frac{1}{\sqrt{\epsilon}}\|u_{\epsilon h}\|_{L^2(\Omega_1)} \le C(h^{\frac{1}{2}} + \epsilon^{\frac{1}{4}})\|f\|_{L^2(\Omega)}.$$

From these results, we see that the optimal choice of ϵ is $\epsilon = h^2$, when h fixed.

According to the fictitious domain method, we solve the discrete L^2-penalty problem (1.3.1) instead of the original problem of (1.2.1). Since the domain Ω has smooth boundary, we provide an approximation scheme for the computation of the inner-product $(u_{\epsilon h}, v_h)_{\Omega_1}$. We find a polygon $\hat{\Omega}$ approximating to Ω, with $\max_{x\in\partial\Omega} \text{dist } (x, \partial\hat{\Omega}) = O(h^2)$. For example, the $\hat{\Omega}$ is constructed by connecting the intersection points between $\partial\Omega$ and the mesh for every triangle of the mesh. Then, instead of (1.3.1), we solve its approximation problem (1.3.6), and we have the error estimate (cf. Theorem 1.3.3)

$$\|u - \hat{u}_{\epsilon,h}\|_{H^1(\Omega)} \le C(h^{\frac{1}{2}} + \epsilon^{\frac{1}{4}} + \epsilon^{-\frac{1}{2}}h^{\frac{3}{2}})\|f\|_{L^2(\Omega)},$$
$$\|u - \hat{u}_{\epsilon,h}\|_{L^2(\Omega)} \le C(h + \epsilon^{\frac{1}{2}} + \epsilon^{-\frac{1}{2}}h^2 + \epsilon^{-\frac{1}{4}}h^{\frac{3}{2}})\|f\|_{L^2(\Omega)},$$

which show the approximation scheme shares the same error order with the error of finite element method for $\epsilon = h^2$; however, $\epsilon \ll h^2$ would enlarge errors.

The convergence of L^2-penalty for elliptic and parabolic problems has been proved in [31]; however, no error estimate has been found, neither the finite element analysis. A similar penalty problem for the Navier-Stokes equations is considered without any numerical results in [2]. Our error estimates in the H^1 norm maintain the sharpness of those for Navier-Stokes problems in [2]. It should be kept in mind that our method of analysis presented here can also be applied to Stokes and Navier-Stokes problems with little difficulty. Furthermore, the results presented in this paper are applied to analysis of L^2 and H^1-penalty fictitious domain methods for parabolic problems in cylindrical and non-cylindrical domains in [49].

Notation

Throughout this chapter, we follow the notation of [29]. Namely we use standard Lebesgue and Sobolev spaces $L^2(\omega)$, $H^m(\omega)$ $(m > 0)$ and $H_0^1(\omega)$, where ω denotes a domain in $\mathbb{R}^2$. We write as

$$\begin{aligned}
(u,v)_\omega &= (u,v)_{L^2(\omega)} = \int_\omega u(x)v(x)\ dx; \\
\|u\|_{0,\omega} &= \|u\|_{L^2(\omega)} = \left(\int_\omega |u(x)|^2\ dx\right)^{1/2}; \\
|u|_{m,\omega} &= \left(\sum_{|\alpha|=m} \|\partial^\alpha u\|_{0,\omega}^2\right)^{1/2}; \\
\|u\|_{m,\omega} &= \left(\|u\|_{m-1,\omega}^2 + |u|_{m,\omega}^2\right)^{1/2},
\end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2)$ denotes a multi-index with $|\alpha| = \alpha_1 + \alpha_2$ and set $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1}(\partial/\partial x_2)^{\alpha_2}$.

We also use standard Lebesgue and Sobolev spaces $L^2(\gamma)$ and $H^s(\gamma)$ $(s > 0)$ defined on a part γ of the boundary $\partial\omega$. The unit outer normal vector to the boundary under consideration is always denoted by n. Finally, we use the same letter C to express a generic constant independent of the penalty parameter ϵ and the discretization parameter h.

1.2 The L^2-penalty problem

Throughout this chapter, we assume that Ω is a bounded domain in $\mathbb{R}^2$ with the C^2 boundary $\Gamma = \partial\Omega$. As a model problem, we consider the Poisson equation with the homogeneous Dirichlet boundary condition,

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma, \tag{1.2.1}$$

where f is a given function of $L^2(\Omega)$. The weak form reads as:

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ (\nabla u, \nabla v)_\Omega = (f, v)_\Omega \qquad \forall v \in H_0^1(\Omega). \end{cases} \tag{1.2.2}$$

1.2.1 The fictitious domain method with L^2-penalty

We take a convex polygonal domain $D \subset \mathbb{R}^2$, which we call the fictitious domain, such that $\overline{\Omega} \subset D$ and set $\Omega_1 = D \backslash \overline{\Omega}$ (see Figure 1.2.1). Then, the fictitious domain formulation with the L^2 penalization for (1.2.2) is given as

$$\begin{cases} \text{Find } u_\epsilon \in H_0^1(D) \text{ such that} \\ (\nabla u_\epsilon, \nabla v)_D + \dfrac{1}{\epsilon}(u_\epsilon, v)_{\Omega_1} = (\tilde{f}, v)_D \qquad \forall v \in H_0^1(D), \end{cases} \tag{1.2.3}$$

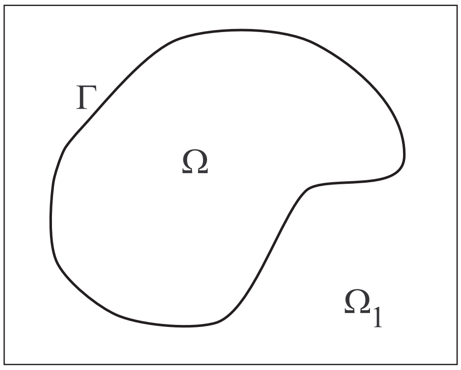

Figure 1.2.1: The original domain Ω and the fictitious domain D.

where

$$0 < \epsilon \leq 1 \tag{1.2.4}$$

is the penalty parameter and $\tilde{f} \in L^2(D)$ is any extension of f into D such that

$$\tilde{f} = f \text{ a.e. in } \Omega, \quad \|\tilde{f}\|_{0,D} \leq C\|f\|_{0,\Omega}$$

with a positive constant C depending only on D and Ω.

According to the Lax and Milgram's theory, there exists a unique solution u_ϵ of (1.2.3) for any $\epsilon \in (0,1]$. Substituting $v = u_\epsilon$ in (1.2.3) and then using Schwarz, Poincaré and Young's inequalities, we have

$$\begin{aligned} &\|\nabla u_\epsilon\|_{0,\Omega}^2 + \|\nabla u_\epsilon\|_{0,\Omega_1}^2 + \frac{1}{\epsilon}\|u_\epsilon\|_{0,\Omega_1}^2 \\ &\qquad \leq \frac{C^2}{2}\|f\|_{0,\Omega}^2 + \frac{1}{2}\|\nabla u_\epsilon\|_{0,\Omega}^2 + \frac{1}{2}\epsilon\|\tilde{f}\|_{0,\Omega_1}^2 + \frac{1}{2\epsilon}\|u_\epsilon\|_{0,\Omega_1}^2. \end{aligned}$$

This gives

$$\|u_\epsilon\|_{1,D} + \frac{1}{\sqrt{\epsilon}}\|u_\epsilon\|_{0,\Omega_1} \leq C\|f\|_{0,\Omega}. \tag{1.2.5}$$

In particular, we have $\|u_\epsilon\|_{0,\Omega_1} \leq C\sqrt{\epsilon}$.

Furthermore, the function u_ϵ solves the variational problem

$$(\nabla u_\epsilon, \nabla v)_D = \left(\tilde{f} - \frac{1}{\epsilon}1_{\Omega_1}u_\epsilon, v\right)_D \qquad \forall v \in H_0^1(D),$$

where $1_{\Omega_1} \in L^\infty(D)$ denotes the characteristic function of Ω_1 defined as

$$1_{\Omega_1}(x) = \begin{cases} 0 & (x \in \Omega) \\ 1 & (x \in \Omega_1). \end{cases} \tag{1.2.6}$$

Hence, we can apply regularity results of elliptic problems in convex domains (cf. [20, Theorem 3.2.1.2] for example) to obtain

$$u_\epsilon \in H^2(D) \tag{1.2.7}$$

and

$$\|u_\epsilon\|_{2,D} \le C \left\| \tilde{f} - \frac{1}{\epsilon}\chi u_\epsilon \right\|_{0,D} \le C\left(1 + \frac{1}{\sqrt{\epsilon}}\right) \|f\|_{0,\Omega}. \tag{1.2.8}$$

This estimate is meaningless for a sufficiently small ϵ; However, we can deduce better a priori bounds for $\|u_\epsilon\|_{2,\Omega}$ and, by using this, we can derive some error estimate for u_ϵ.

1.2.2 The regularity and error estimates of the penalty problem

We present the main result of this section

Theorem 1.2.1. *Let $u_\epsilon \in H_0^1(D)$ be the solution of* (1.2.3). *Then, we have $u_\epsilon \in H^2(D)$ and*

$$\begin{aligned} \|u_\epsilon\|_{2,\Omega} &\le C\|f\|_{0,\Omega}, && (1.2.9)\\ \|u_\epsilon\|_{2,\Omega_1} &\le C\epsilon^{-\frac{1}{4}}\|f\|_{0,\Omega}, && (1.2.10)\\ \|u_\epsilon\|_{1,\Omega_1} &\le C\epsilon^{\frac{1}{4}}\|f\|_{0,\Omega}, && (1.2.11)\\ \|u_\epsilon\|_{0,\Omega_1} &\le C\epsilon^{\frac{3}{4}}\|f\|_{0,\Omega}. && (1.2.12) \end{aligned}$$

Furthermore,

$$\begin{aligned} \|u - u_\epsilon\|_{1,\Omega} &\le \epsilon^{\frac{1}{4}}\|f\|_{0,\Omega}, && (1.2.13)\\ \|u - u_\epsilon\|_{0,\Omega} &\le \epsilon^{\frac{1}{2}}\|f\|_{0,\Omega}, && (1.2.14)\\ \|u_\epsilon\|_{\frac{1}{2},\Gamma} &\le C\epsilon^{\frac{1}{4}}\|f\|_{0,\Omega}, && (1.2.15) \end{aligned}$$

where $u \in H_0^1(\Omega)$ denotes the solution of (1.2.2).

Remark 1.2.1. In [31, Theorem I-4], it has already proved

$$\|u_\epsilon - u\|_{1,\Omega} \to 0, \quad \frac{1}{\sqrt{\epsilon}}\|u_\epsilon\|_{0,\Omega_1} \to 0 \ \text{ as } \ \epsilon \to 0 \tag{1.2.16}$$

for $\tilde{f}$ being the zero extension of f.

In the proof of Theorem 1.2.1, we use the following regularity result for a linear elliptic equation. Although it seems not to be new, we give its proof for readers' convenience.

Lemma 1.2.1. *For $\phi \in L^2(\Omega_1)$ and $g \in H^{1/2}(\Gamma)$, let $w \in H^2(\Omega_1)$ be a solution of*

$$-\Delta w + \frac{1}{\epsilon}w = \phi \text{ in } \Omega_1, \quad \frac{\partial w}{\partial n} = g \text{ on } \Gamma, \quad w = 0 \text{ on } \partial D.$$

Then, we have

$$\begin{aligned} \|w\|_{0,\Omega_1} &\leq C(\epsilon\|\phi\|_{0,\Omega_1} + \epsilon^{\frac{3}{4}}\|g\|_{\frac{1}{2},\Gamma}), \\ \|w\|_{2,\Omega_1} &\leq C(\|\phi\|_{0,\Omega_1} + \epsilon^{-\frac{1}{4}}\|g\|_{\frac{1}{2},\Gamma}). \end{aligned}$$

In order to prove this, we need the following auxiliary lemma. .

Lemma 1.2.2. *For $g \in H^{\frac{1}{2}}(\Gamma)$ and $\eta > 0$, there exists $v = v_\eta \in H^2(\Omega_1)$ such that,*

$$\frac{\partial v}{\partial n} = g \ \text{on } \Gamma, \qquad v = 0 \ \text{on } \partial D$$

with estimates

$$\|v\|_{0,\Omega} \leq C\eta^3\|g\|_{\frac{1}{2},\Gamma}, \quad |v|_{1,\Omega} \leq C\eta\|g\|_{\frac{1}{2},\Gamma}, \quad |v|_{2,\Omega} \leq C\eta^{-1}\|g\|_{\frac{1}{2},\Gamma}.$$

Proof of Lemma 1.2.2. It suffices to consider the case $\Omega = \mathbb{R}^N_+$, since then the general case is proved by the standard argument of using partition of the unity and localization technique (see, for example, [47, §20]).

We suppose that $\hat{h}(\xi')$ is the Fourier transform of a function $h(x_1, \ldots, x_{N-1})$, where $\xi' = (\xi_1, \ldots, \xi_{N-1})$. Similarly, let $\hat{w}(\xi)$ be the Fourier transform of a function $w(x)$ in variables $(x_1, \ldots, x_{N-1})$, where $\xi = (\xi', x_N)$. We apply the extension formula in [32, Theorem 5.2, Chapter 2] with a slightly modification. Thus, we propose

$$\hat{v}(\xi', x_N) = x_N \exp\left(-(1 + |\xi'|)\eta^{-2}, x_N\right) \hat{g}(\xi'). \tag{1.2.17}$$

Indeed, let $|\alpha| \leq 2$, let us consider $w_\alpha = \partial^\alpha v$ in $\mathbb{R}^N_+$ and set $w_\alpha = 0$ for $x_N < 0$. Let us denote $\alpha = (\alpha_1, \ldots, \alpha_N,)$, and $\alpha = (\alpha', \alpha_N)$. Hence $\hat{w}_\alpha(\xi)$ is a finite sum of expressions like

$$aI(\xi) = a \int_0^\infty e^{(-ix_N\xi_N)}(\xi')^{\alpha'}((1+|\xi'|)\eta^{-2})^{\alpha_N - j} x_N^{1-j}. \\ \exp\left(-(1+|\xi'|)\eta^{-2}, x_N\right)\hat{g}(\xi')dx_N,$$

where a is a constant, $j = 0, 1$. We have:

$$I(\xi) = \frac{(\xi')^{\alpha'}((1+|\xi'|)\eta^{-2})^{\alpha_N - j}\hat{g}(\xi')}{((1+|\xi'|)\eta^{-2} + i\xi_N)^{2-j}},$$

and so

$$\|I(\xi)\|^2_{0,\mathbb{R}^N} = C \int_{\mathbb{R}^{N-1}} (\xi')^{2\alpha'}((1+|\xi'|)\eta^{-2})^{2\alpha_N - 3}|\hat{g}(\xi')|^2 d\xi' \\ \leq \begin{cases} C\eta^{-2}\|g\|^2_{\frac{1}{2},\Gamma}, & \alpha_N = 2, \\ C\eta^{2}\|g\|^2_{\frac{1}{2},\Gamma}, & \alpha_N = 1, \\ C\eta^{6}\|g\|^2_{\frac{1}{2},\Gamma}, & \alpha_N = 0. \end{cases}$$

This completes the proof. □

Proof of Lemma 1.2.1. By Lemma 1.2.2 with $\eta = \epsilon^{\frac{1}{4}}$, there exists $\psi \in H^2(\Omega)$ such that $\partial\psi/\partial n = g$ on Γ, $\psi = 0$ on ∂D, $\|\psi\|_{0,\Omega_1} \leq C\epsilon^{\frac{3}{4}}\|g\|_{\frac{1}{2},\Gamma}$ and $\|\psi\|_{2,\Omega_1} \leq C\epsilon^{-\frac{1}{4}}\|g\|_{\frac{1}{2},\Gamma}$. Setting $u = w - \psi$, we have

$$-\Delta u + \frac{1}{\epsilon}u = \phi + \Delta\psi + \frac{1}{\epsilon}\psi \text{ in } \Omega_1, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma, \quad u = 0 \text{ on } \partial D.$$

Multiplying the both sides by u and integrating over Ω_1, we have

$$\|\nabla u\|^2_{0,\Omega_1} + \frac{1}{\epsilon}\|u\|^2_{0,\Omega_1} \leq \|\phi\|_{0,\Omega_1}\|u\|_{0,\Omega_1} + \left(\|\psi\|_{2,\Omega_1} + \frac{1}{\epsilon}\|\psi\|_{0,\Omega_1}\right)\|u\|_{0,\Omega_1}.$$

Hence,

$$\begin{aligned} \|u\|_{0,\Omega_1} &\leq \epsilon\|\phi\|_{0,\Omega_1} + \epsilon\|\psi\|_{2,\Omega_1} + \|\psi\|_{0,\Omega_1} \\ &\leq \epsilon\|\phi\|_{0,\Omega_1} + \epsilon \cdot C\epsilon^{-\frac{1}{4}}\|g\|_{\frac{1}{2},\Gamma} + C\epsilon^{\frac{3}{4}}\|g\|_{\frac{1}{2},\Gamma}. \end{aligned}$$

This implies

$$\|w\|_{0,\Omega_1} \leq \|\psi\|_{0,\Omega_1} + \epsilon\|\phi\|_{0,\Omega_1} + C\epsilon^{\frac{3}{4}}\|g\|_{\frac{1}{2},\Gamma} \leq \epsilon\|\phi\|_{0,\Omega_1} + C\epsilon^{\frac{3}{4}}\|g\|_{\frac{1}{2},\Gamma}.$$

On the other hand,

$$
\begin{aligned}
\|w\|_{2,\Omega_1} &\le C\left\|\phi+\Delta\psi+\frac{1}{\epsilon}\psi\right\|_{0,\Omega_1}+C\|g\|_{\frac{1}{2},\Gamma}\\
&\le C\|\phi\|_{0,\Omega_1}+C\|\psi\|_{2,\Omega_1}+C\frac{1}{\epsilon}\|\psi\|_{0,\Omega_1}+C\|g\|_{\frac{1}{2},\Gamma}\\
&\le C\|\phi\|_{0,\Omega_1}+C\epsilon^{-\frac{1}{4}}\|g\|_{\frac{1}{2},\Gamma}+C\|g\|_{\frac{1}{2},\Gamma},
\end{aligned}
$$

which implies the desired estimate. □

Now we can state the following proof.

Proof of Theorem 1.2.1. First, we prove inequalities (1.2.10)–(1.2.15) by using (1.2.9).

Applying Green's formula, we observe that (1.2.3) is equivalent to the following problem:

$$-\Delta u_\epsilon = f \text{ in } \Omega, \quad u_\epsilon|_\Omega = u_\epsilon|_{\Omega_1} \text{ on } \Gamma, \quad u_\epsilon = 0 \text{ on } \partial D; \tag{1.2.18}$$

$$-\Delta u_\epsilon + \frac{1}{\epsilon}u_\epsilon = \tilde{f} \text{ in } \Omega_1, \quad \left.\frac{\partial u_\epsilon}{\partial n}\right|_\Omega = \left.\frac{\partial u_\epsilon}{\partial n}\right|_{\Omega_1} \text{ on } \Gamma. \tag{1.2.19}$$

In view of the trace theorem, we have

$$\left\|\frac{\partial u_\epsilon}{\partial n}\right\|_{\frac{1}{2},\Gamma} \le C\|u_\epsilon\|_{2,\Omega} \le C\|f\|_{0,\Omega}.$$

Hence, we apply Lemma 1.2.1 to the problem (1.2.19) in order to obtain

$$\|u_\epsilon\|_{0,\Omega_1} \le C(\epsilon^{\frac{3}{4}}\|f\|_{0,\Omega}+\epsilon\|\tilde{f}\|_{0,\Omega_1}), \tag{1.2.20}$$

$$\|u_\epsilon\|_{2,\Omega_1} \le C(\epsilon^{-\frac{1}{4}}\|f\|_{0,\Omega}+\|\tilde{f}\|_{0,\Omega_1}) \tag{1.2.21}$$

which imply (1.2.10) and (1.2.12), respectively.

We recall that in general we have (cf. [18, Theorem 7.27])

$$|v|_{1,\Omega_1} \le C(\eta|v|_{2,\Omega_1}+\eta^{-1}\|v\|_{0,\Omega})$$

for any $\eta > 0$ and $v \in H^2(\Omega)$. Setting $\eta = \epsilon^{\frac{1}{2}}$, we deduce (1.2.11).

Estimates (1.2.13) and (1.2.15) are readily obtainable consequences of (1.2.11) and trace theorems. Thus,

$$
\begin{aligned}
\|u_\epsilon - u\|_{1,\Omega} &\le C\|u_\epsilon - u\|_{\frac{1}{2},\Gamma} = C\|u_\epsilon\|_{\frac{1}{2},\Gamma}\\
&\le C\|u_\epsilon\|_{1,\Omega_1} \le C\epsilon^{\frac{1}{4}}\|f\|_{0,\Omega}.
\end{aligned}
$$

We proceed to derive (1.2.14). To this end, we introduce the adjoint problems for (1.2.2) and (1.2.3) which are given as

$$\begin{cases} \text{Find } u_F \in H_0^1(\Omega) \text{ such that} \\ (\nabla u_F, \nabla v)_\Omega = (F, v)_\Omega \quad \forall v \in H_0^1(\Omega) \end{cases} \tag{1.2.22}$$

and

$$\begin{cases} \text{Find } u_{F\epsilon} \in H_0^1(D) \text{ such that} \\ (\nabla u_{F\epsilon}, \nabla v)_D + \dfrac{1}{\epsilon}(u_{F\epsilon}, v)_{\Omega_1} = (\tilde{F}, v)_D \quad \forall v \in H_0^1(D), \end{cases} \tag{1.2.23}$$

for any $F \in L^2(\Omega)$, and the extension of F, $\tilde{F} \in L^2(D)$, satisfying $\|\tilde{F}\|_{0,\Omega_1} \leq C\|F\|_{0,\Omega}$.

Apparently, we can obtain the a priori estimates and H^1 norm penalization error estimate, like (1.2.21), (1.2.21) and (1.2.13), for the adjoint problems (1.2.22) and (1.2.23). Thus we have

$$\|u_{F\epsilon}\|_{2,\Omega} \leq C(\epsilon^{-\frac{1}{4}}\|F\|_{0,\Omega} + \|\tilde{F}\|_{0,\Omega_1}), \tag{1.2.24}$$

$$\|u_{F\epsilon}\|_{0,\Omega} \leq C(\epsilon^{\frac{3}{4}}\|F\|_{0,\Omega} + \epsilon\|\tilde{F}\|_{0,\Omega_1}), \tag{1.2.25}$$

$$\|u_{F\epsilon}|_\Omega - u_F\|_{1,\Omega} \leq C\epsilon^{\frac{1}{4}}\|F\|_{0,\Omega}. \tag{1.2.26}$$

Denoting by $\tilde{u}$ and $\tilde{u}_F$ the zero extension of u and u_F, respectively, one can show that

$$\begin{aligned}(\nabla u_\epsilon, \nabla \tilde{u}_F)_D = (\tilde{u}_F, \tilde{f})_D = (u_F, f)_\Omega = (\nabla u_F, \nabla u)_\Omega \\ = (F, u)_\Omega = (\tilde{F}, \tilde{u})_D = (\nabla u_{F\epsilon}, \nabla \tilde{u})_D,\end{aligned}$$

and hence

$$(\nabla(u_{F\epsilon} - \tilde{u}_F), \nabla(u_\epsilon - \tilde{u}))_D = (\tilde{F}, u_\epsilon - \tilde{u})_D - \frac{1}{\epsilon}(u_{F\epsilon}, u_\epsilon)_{\Omega_1}.$$

At this stage, we let $\tilde{F} = u_\epsilon - \tilde{u}$. Then,

$$\|u_\epsilon - \tilde{u}\|_{0,\Omega}^2 + \|u_\epsilon\|_{0,\Omega_1}^2 = (\nabla(u_{F\epsilon} - \tilde{u}_F), \nabla(u_\epsilon - \tilde{u}))_D + \frac{1}{\epsilon}(u_{F\epsilon}, u_\epsilon)_{\Omega_1}.$$

Combining those estimates, we get

$$\|u_\epsilon|_\Omega - u\|_{0,\Omega} \leq C\epsilon^{\frac{1}{2}}\|f\|_{0,\Omega}. \tag{1.2.27}$$

Thus, we have proved (1.2.14).

Now, we go back to the beginning of the proof; It remains to show (1.2.9). To this end, let us consider the interface problem composed of (1.2.18) and (1.2.19) and apply the standard method of tangential difference quotients; See, for example, [20, Theorem 2.2.2.3], [33, Appendix] or [53, Theorem 3.1].

We take a set $\{U_j\}_{j=1}^N$ of open subsets in $\mathbb{R}^2$ enjoying the following properties. With U_j and $1 \le j \le N$, we associate a C^2 diffeomorphism $\Phi_j : U_j \to \mathbb{R}^2$ that satisfies

$$\overline{\Omega} \subset \bigcup_{j=1}^N \Phi_j(U_j) \subset D,$$

$$U_{j0} = \Psi_j(\Phi_j(U_j) \cap \Omega) = \mathbb{R}^2_+ \cap U_j, \quad U_{j1} = \Psi_j(\Phi_j(U) \cap \Omega_1) = \mathbb{R}^2_- \cap U_j,$$

where $\mathbb{R}^2_\pm = \mathbb{R}^2 \cap \{\pm x_2 > 0\}$ and $\Psi_j = \Phi_j^{-1}$. Further, we take $\{\theta_j\}_{j=1}^N \subset C_0^\infty(\overline{\Omega})$ such that supp $\theta_j \subset \Phi_j(U_j)$ and

$$\sum_{j=1}^N \theta_j = 1 \text{ on } \overline{\Omega} \quad \text{and} \quad \delta = \min_{1\le j\le N} \text{dist } (\text{supp } \theta_j, \partial\Phi_j(U_j)) > 0.$$

We note that $(\theta_j u_\epsilon)\circ\Phi_j \in H_0^1(U_j)$ for $j = 1, 2, \ldots, N$. We drop the subscript j and write $U = U_j$, $U_1 = U_{j1}$, $U_0 = U_{j0}$, $\Phi = \Phi_j$, $\Psi = \Psi_j$, and $\theta = \theta_j$ for short.

Set $u_1 = \theta u_\epsilon$ and $u_2 = (\theta u_\epsilon) \circ \Phi$.

First, if $U_1 = \emptyset$, then $u_1 \in H^2(\Omega)$ and $\|u_1\|_{2,\Omega} \le C\|\tilde{f}\|_{0,D}$. In what follows, we consider the case $U_0 \ne \emptyset$ and $U_1 \ne \emptyset$. Set $D_i = \partial/\partial x_i$, $(i = 1, 2)$. We observe that $u_2 \in H_0^1(U)$ satisfies, for all $v \in H_0^1(U)$,

$$\sum_{i,k=1}^2 \int_U a_{ik} D_i u_2 D_k v dx + \frac{1}{\epsilon} \sum_{i,k=1}^2 \int_{U_1} u_2 v |D\Phi| dx = (f_2, v), \qquad (1.2.28)$$

where $f_2 = (\theta\tilde{f} + \nabla u_\epsilon \nabla\theta + \nabla\cdot(u_\epsilon\nabla\theta)) \circ \Phi|D\Phi|$ and

$$a_{ik} = (\sum_{l=1}^2 D_l\psi_i D_l\psi_k) \circ \Phi|D\Phi| \quad (i, k = 1, 2), \quad \Psi = (\psi_1, \psi_2).$$

Let $\tilde{u}_2$ be the zero extension of u_2 onto $\mathbb{R}^2$ and let $|h| \le \delta/4$. Substituting $v = \frac{\tau_h - 1}{h}\frac{\tau_{-h}-1}{h}\tilde{u}_2 \in H_0^1(U)$ into (1.2.28), where τ_h is the translation operator with $\tau_h\phi(x) = \phi(x_1 + h, x_2),\ \phi(x) \in L^2(\mathbb{R}^2)$, we have after some calculation

$$\sum_{i=1}^{2} \left\| D_i \left(\frac{\tau_h - 1}{h} \tilde{u}_2 \right) \right\|_{0,U}^2 + \frac{1}{\epsilon} \sum_{i=1}^{2} \left\| \frac{\tau_h - 1}{h} \tilde{u}_2 \right\|_{0,U_1}^2$$
$$\leq C \sum_{i=1}^{2} \left\| D_i \left(\frac{\tau_h - 1}{h} \tilde{u}_2 \right) \right\|_{0,U} + C \frac{1}{\epsilon} \|\tilde{u}_2\|_{0,U_1}^2 + C\|f_2\|_{0,U}^2,$$

applying (1.2.16) or (1.2.5), we have $\sum_{i=1}^{2} \left\| D_i \left(\frac{\tau_h - 1}{h} \tilde{u}_2 \right) \right\|_{U_0} \leq C\|f\|_{0,\Omega}$. On letting $h \downarrow 0$, we conclude $D_i D_1 u_2 \in L^2(U_0)$ and $\|D_i D_1 u_2\|_{0,U_0} \leq C\|\tilde{f}\|_{0,\Omega}$ for $i = 1, 2$.

Finally, we see that

$$D_2^2 u_2 = \frac{1}{a_{22}}(f_2 - \sum_{k+l\leq 3} D_l(a_{kl} D_k u_2) - D_2 a_{22} D_2 u_2) \quad \text{ in } U_0.$$

This implies that $D_2^2 u_2 \in L^2(U_0)$ and $\|u_2\|_{2,U_0} \leq C\|\tilde{f}\|_{0,\Omega}$.

Summing up, we conclude that $u_\epsilon|_\Omega \in H^2(\Omega)$ and $\|u\|_{2,\Omega} \leq C\|f\|_{0,\Omega}$. This completes the proof of Theorem 1.2.1. □

1.3 The finite element approximation to the L^2-penalty method

We introduce a shape-regular family of triangulations $\{\mathcal{T}_h\}_{h>0}$ to the convex polygonal domain D, where h is the maximum diameter of the triangles of $\mathcal{T}_h$. That is, there exists a positive constant ν_1 such that

$$\frac{h_T}{\rho_T} \leq \nu_1 \qquad (\forall T \in \forall \mathcal{T}_h \in \{\mathcal{T}_h\}_h),$$

where h_T and ρ_T, respectively, denote the diameters of circumscribe and inscribe circles of T. Let $V_h(D) \subset H_0^1(D)$ be the set of all continuous piecewise-affine functions subordinate to $\mathcal{T}_h$. A finite element approximation for (1.2.3) reads as

$$\begin{cases} \text{Find } u_{\epsilon h} \in V_h(D) \text{ such that} \\ (\nabla u_{\epsilon h}, \nabla v_h)_D + \dfrac{1}{\epsilon}(u_{\epsilon h}, v_h)_{\Omega_1} = (\tilde{f}, v_h)_D \quad \forall v_h \in V_h(D), \end{cases} \tag{1.3.1}$$

Thus, applying the fictitious domain method, we compute (1.3.1) instead of (1.2.2). According to Theorem 1.2.1, the error satisfies

$$\|u - u_{\epsilon h}\|_{1,\Omega} \leq \|u - u_\epsilon\|_{1,\Omega} + \|u_\epsilon - u_{\epsilon h}\|_{1,D} \leq C\epsilon^{\frac{1}{4}} + C\|\nabla(u_\epsilon - u_{\epsilon h})\|_{0,D},$$
$$\|u - u_{\epsilon h}\|_{0,\Omega} \leq \|u - u_\epsilon\|_{0,\Omega_1} + \|u_\epsilon - u_{\epsilon h}\|_{0,\Omega} \leq C\epsilon^{\frac{1}{2}} + \|u_\epsilon - u_{\epsilon h}\|_{0,\Omega}.$$

Hence, it suffices to examine $u_\epsilon - u_{\epsilon h}$. First, we give the following lemma.

Lemma 1.3.1. *Let u_ϵ and $u_{\epsilon h}$ be the solutions of (1.2.3) and (1.3.1), respectively. Then, we have*

$$\|\nabla(u_\epsilon - u_{\epsilon h})\|_{0,D} + \frac{1}{\sqrt{\epsilon}}\|u_\epsilon - u_{\epsilon h}\|_{0,\Omega_1}$$
$$\leq C \inf_{v_h \in V_h(D)} \left(\|\nabla(u_\epsilon - v_h)\|_{0,D} + \frac{1}{\sqrt{\epsilon}}\|u_\epsilon - v_h\|_{0,\Omega_1} \right). \quad (1.3.2)$$

Proof. It is a consequence of the Galerkin orthogonality

$$(\nabla(u_\epsilon - u_{\epsilon h}), \nabla v_h)_D + \frac{1}{\epsilon}(u_\epsilon - u_{\epsilon h}, v_h) = 0 \qquad \forall v_h \in V_h(D).$$

□

Theorem 1.3.1. *Suppose that u_ϵ and $u_{\epsilon h}$ are the solutions of (1.2.3) and (1.3.1), respectively. Then, we have*

$$\|\nabla(u_\epsilon - u_{\epsilon h})\|_{0,D} + \frac{1}{\sqrt{\epsilon}}\|u_\epsilon - u_{\epsilon h}\|_{0,\Omega_1} \leq C(h^{\frac{1}{2}} + \epsilon^{\frac{1}{4}})\|f\|_{0,\Omega}, \quad (1.3.3)$$
$$\|u_\epsilon - u_{\epsilon h}\|_{0,\Omega} \leq C(h^{\frac{1}{2}} + \epsilon^{\frac{1}{4}})^2\|f\|_{0,\Omega}. \quad (1.3.4)$$

Proof. We introduce some notations first. A generic (closed) triangle of $\mathcal{T}_h$ is denoted by K, and the set of all vertices of K is denoted by $\Lambda(K) = (\nu_1^K, \nu_2^K, \nu_3^K)$. Set $T_\Gamma = \{K \mid K \cap \Gamma \neq \emptyset\}$ and $T' = \{K \subset \Omega | K \cap T_\Gamma = \emptyset\}$. The standard P1 Lagrange interpolation of $v \in H^2(D)$ is denoted by $I_h v$. We define $v_h \in V_h(D)$ by setting,

$$v_h(\nu) = \begin{cases} 0 & \text{for } \nu \in \Lambda(K), K \subset T_\Gamma \cup \overline{\Omega_1}, \\ u_\epsilon(\nu) & \text{for all other vertices } \nu. \end{cases}$$

substitute v_h into (1.3.2) and using the a priori estimates in Theorem 1.2.1, we have

$$\|u_\epsilon - v_h\|_{0,\Omega_1} = \|u_\epsilon\|_{0,\Omega_1} \leq C\epsilon^{\frac{3}{4}}\|f\|_{0,\Omega}$$

and

$$
\begin{aligned}
&\|\nabla(u_\epsilon - v_h)\|^2_{0,\Omega} \\
\le\ & C(\|\nabla(u_\epsilon - I_h u)\|^2_{0,T'} + \|\nabla u_\epsilon\|^2_{0,\Omega\backslash T'} + \|\nabla v_h\|^2_{0,\Omega\backslash T'}) \\
\le\ & C(\|\nabla(u_\epsilon - u)\|^2_{0,T'} + \|\nabla(u - I_h u)\|^2_{0,T'} + \|\nabla u_\epsilon\|^2_{0,\Omega\backslash T'} + \|\nabla v_h\|^2_{0,\Omega\backslash T'}) \\
\le\ & C\left(h^2\|u\|^2_{2,\Omega} + h\|u_\epsilon\|^2_{2,\Omega} + h\|u\|^2_{2,\Omega}\right) \\
\le\ & Ch\|f\|^2_{0,\Omega},
\end{aligned}
$$

where $u \in H^2(\Omega)$ is the solution of (1.2.2). Therefore,

$$
\begin{aligned}
\|\nabla(u_\epsilon - v_h)\|^2_{0,D} &= \|\nabla(u_\epsilon - v_h)\|^2_{0,\Omega} + \|\nabla(u_\epsilon - v_h)\|^2_{0,\Omega_1} \\
&= \|\nabla(u_\epsilon - v_h)\|^2_{0,\Omega} + \|\nabla u_\epsilon\|^2_{0,\Omega_1} \\
&\le Ch\|f\|^2_{0,\Omega} + C\epsilon^{\frac{1}{2}}\|f\|^2_{0,\Omega},
\end{aligned}
$$

which implies (1.3.3). See the proof of [53, Theorem 4.4] for the detailed proof of this estimate; Especially, the estimate $\|\nabla u_\epsilon\|_{0,\Omega\backslash T'} \le Ch^{\frac{1}{2}}\|u_\epsilon\|_{2,\Omega}$ follows from [53, Lemma 4.2] or [48, Lemma 2.1], and for the proof of $\|\nabla v_h\|_{0,\Omega\backslash T'} \le Ch^{\frac{1}{2}}\|u\|_{2,\Omega}$, one can refer to the proof of [53, Theorem 4.4], with aware of $u = 0$ on Γ, which gives (1.3.3).

Then, setting $\tilde{F} = 1_\Omega(u_\epsilon - u_{\epsilon h})$ and $v = u_\epsilon - u_{\epsilon h}$ in the adjoint problem (1.2.23), where $1_\Omega = 1$ in Ω, and $1_\Omega = 0$ in otherwise, applying (1.3.3) and the prior estimates in Theorem 1.2.1, we have for any $v_h \in V_h(D)$

$$
\begin{aligned}
\|F\|^2_{0,\Omega} &= \|u_\epsilon - u_{\epsilon h}\|^2_{0,\Omega} = (\nabla u_{F\epsilon}, \nabla(u_\epsilon - u_{\epsilon h}))_D + \frac{1}{\epsilon}(u_{F\epsilon}, u_\epsilon - u_{\epsilon h})_{\Omega_1} \\
&= (\nabla u_{F\epsilon} - v_h, \nabla(u_\epsilon - u_{\epsilon h}))_D + \frac{1}{\epsilon}(u_{F\epsilon} - v_h, u_\epsilon - u_{\epsilon h})_{\Omega_1} \\
&\le C(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})\|F\|_{0,\Omega}(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})\|f\|_{0,\Omega} \\
&\qquad + C\frac{1}{\epsilon}\epsilon^{\frac{1}{2}}(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})\|F\|_{0,\Omega}\epsilon^{\frac{1}{2}}(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})\|f\|_{0,\Omega},
\end{aligned}
$$

which implies (1.3.4), and the proof is completed. □

Combining Theorems 1.2.1 and 1.3.1, we obtain the following estimates.

Theorem 1.3.2. *Let that u and $u_{\epsilon h}$ be the solutions of (1.2.2) and (1.3.1), respectively. Then, we have*

$$
\|\nabla(u - u_{\epsilon h})\|_{0,\Omega} \le C(h^{\frac{1}{2}} + \epsilon^{\frac{1}{4}})\|f\|_{0,\Omega}, \quad \|u - u_{\epsilon h}\|_{0,\Omega} \le C(h + \epsilon^{\frac{1}{2}})\|f\|_{0,\Omega},
$$
$$
\|u_{\epsilon h}\|_{\frac{1}{2},\Gamma} + \frac{1}{\sqrt{\epsilon}}\|u_{\epsilon h}\|_{0,\Omega_1} \le C(h^{\frac{1}{2}} + \epsilon^{\frac{1}{4}})\|f\|_{\Omega}.
$$

Due to the smooth boundary of Ω, the inner-product $(u_{\epsilon,h}, v_h)_{\Omega_1}$ cannot be computed exactly. Therefore we need an approximation scheme for computation of the problem (1.3.1).

As we mentioned in Introduction, we find a polygonal domain $\hat{\Omega}$ for Ω such that the vertices of $\partial\hat{\Omega}$ are situated on $\partial\Omega$ and assume that there are $h_1 > 0$ and $c_0 > 0$ such that

$$\text{dist}\,(\Omega, \hat{\Omega}) \leq c_0 h^2 \qquad (h \in (0, h_1)). \tag{1.3.5}$$

We set $\hat{\Omega}_1 = D \backslash \overline{\hat{\Omega}}$.

Then, we consider

$$\begin{cases} \text{Find } \hat{u}_{\epsilon h} \in V_h(D) \text{ such that} \\ (\nabla \hat{u}_{\epsilon h}, \nabla v_h)_D + \dfrac{1}{\epsilon}(\hat{u}_{\epsilon h}, v_h)_{\hat{\Omega}_1} = (\tilde{f}, v_h)_D \quad \forall v_h \in V_h(D). \end{cases} \tag{1.3.6}$$

We have the error estimate of the approximation

Theorem 1.3.3. *Let u and $\hat{u}_{\epsilon,h}$ be the solutions of* (1.2.2) *and* (1.3.6), *respectively. Then, we have*

$$\|u - \hat{u}_{\epsilon,h}\|_{1,\Omega} \leq C\|\hat{u}_{\epsilon,h}\|_{\frac{1}{2},\Gamma} \leq C(h^{\frac{1}{2}} + \epsilon^{\frac{1}{4}} + \epsilon^{-\frac{1}{2}} h^{\frac{3}{2}})\|f\|_{0,\Omega},$$
$$\|u - \hat{u}_{\epsilon,h}\|_{0,\Omega} \leq C(h + \epsilon^{\frac{1}{2}} + \epsilon^{-\frac{1}{2}} h^2 + \epsilon^{-\frac{1}{4}} h^{\frac{3}{2}})\|f\|_{0,\Omega}.$$

Remark 1.3.1. For $\epsilon = h^2$, we have $\|u - \hat{u}_{\epsilon,h}\|_{1,\Omega} \leq Ch^{\frac{1}{2}} = C\epsilon^{\frac{1}{4}}$ and $\|u - \hat{u}_{\epsilon,h}\|_{0,\Omega} \leq Ch = C\epsilon^{\frac{1}{2}}$.

Proof of Theorem 1.3.3. In view of Theorem 1.3.2, it suffices to prove

$$\|\hat{u}_{\epsilon,h} - u_{\epsilon,h}\|_{1,\Omega} \leq C\epsilon^{-\frac{1}{2}} h^{\frac{3}{2}} \|f\|_{0,\Omega}, \tag{1.3.7}$$
$$\|\hat{u}_{\epsilon,h} - u_{\epsilon,h}\|_{0,\Omega} \leq C(\epsilon^{-\frac{1}{2}} h^2 + \epsilon^{-\frac{1}{4}} h^{\frac{3}{2}})\|f\|_{0,\Omega}. \tag{1.3.8}$$

Subtracting (1.3.1) from (1.3.6), we have

$$\begin{aligned} (\nabla(u_{\epsilon,h} - \hat{u}_{\epsilon,h}), v_h)_D &+ \frac{1}{\epsilon}(u_{\epsilon,h} - \hat{u}_{\epsilon,h}, v_h)_{\Omega_1 \cap \hat{\Omega}_1} \\ &+ \frac{1}{\epsilon}(u_{\epsilon,h}, v_h)_{\Omega_1 \backslash \hat{\Omega}_1} - \frac{1}{\epsilon}(\hat{u}_{\epsilon,h}, v_h)_{\hat{\Omega}_1 \backslash \Omega_1} = 0. \end{aligned} \tag{1.3.9}$$

for any $v_h \in V_h(D)$. We also have

$$\|\hat{u}_{\epsilon,h}\|_{0,\hat{\Omega}_1} \leq C\sqrt{\epsilon}\|f\|_{0,\Omega}, \quad \|u_{\epsilon,h}\|_{0,\Omega_1} \leq C\sqrt{\epsilon}\|f\|_{0,\Omega}$$

which be obtained by substituting $v = \hat{u}_{\epsilon,h}$ and $v = u_{\epsilon,h}$, respectively, into (1.3.6) into (1.3.1).

Since we assume that (1.3.5) hold true, we have

$$\|\hat{u}_{\epsilon,h}\|_{0,\hat{\Omega}_1\setminus\Omega_1} \le Ch^{\frac{1}{2}}\|\hat{u}_{\epsilon,h}\|_{0,\hat{\Omega}_1\cap T_\Gamma},$$
$$\|v_h\|_{0,\hat{\Omega}_1\setminus\Omega_1} \le Ch^{\frac{1}{2}}\|v_h\|_{0,\hat{\Omega}_1\cap T_\Gamma} \le Ch\|v_h\|_{1,D},$$
$$\|u_{\epsilon,h}\|_{0,\Omega_1\setminus\hat{\Omega}_1} \le Ch^{\frac{1}{2}}\|\hat{u}_{\epsilon,h}\|_{0,\Omega_1\cap T_\Gamma},$$
$$\|v_h\|_{0,\Omega_1\setminus\hat{\Omega}_1} \le Ch^{\frac{1}{2}}\|v_h\|_{0,\Omega_1\cap T_\Gamma} \le Ch\|v_h\|_{1,D},$$

where $T_\Gamma = \{K \in \mathcal{T} \mid K \cap \Gamma \ne \emptyset\}$, and these estimates can be found in [44]. Substituting $v_h = u_{\epsilon,h} - \hat{u}_{\epsilon,h}$ into (1.3.9), and applying these estimates and Poincaré's inequality, we obtain that

$$\begin{aligned}
& \|u_{\epsilon,h} - \hat{u}_{\epsilon,h}\|^2_{1,D} + \frac{1}{\epsilon}\|u_{\epsilon,h} - \hat{u}_{\epsilon,h}\|^2_{0,\Omega_1\cap\hat{\Omega}_1} \\
\le\ & (\nabla(u_{\epsilon,h} - \hat{u}_{\epsilon,h}), \nabla(u_{\epsilon,h} - \hat{u}_{\epsilon,h}))_D + \frac{1}{\epsilon}(u_{\epsilon,h} - \hat{u}_{\epsilon,h}, u_{\epsilon,h} - \hat{u}_{\epsilon,h})_{0,\Omega_1\cap\hat{\Omega}_1} \\
\le\ & \frac{1}{\epsilon}\|\hat{u}_{\epsilon,h}\|_{0,\hat{\Omega}_1\setminus\Omega_1}\|u_{\epsilon,h} - \hat{u}_{\epsilon,h}\|_{0,\hat{\Omega}_1\setminus\Omega_1} + \frac{1}{\epsilon}\|u_{\epsilon,h}\|_{0,\Omega_1\setminus\hat{\Omega}_1}\|u_{\epsilon,h} - \hat{u}_{\epsilon,h}\|_{0,\Omega_1\setminus\hat{\Omega}_1} \\
\le\ & C\frac{1}{\epsilon}h^{\frac{1}{2}}\epsilon^{\frac{1}{2}}h\|u_{\epsilon,h} - \hat{u}_{\epsilon,h}\|_{1,D},
\end{aligned}$$

which gives (1.3.7). Setting $\tilde{f} = u_{\epsilon,h} - \hat{u}_{\epsilon,h}$ in (1.3.1) and (1.3.6), applying (1.3.7) we finally get (1.3.8). □

At this stage, we give numerical experiments to show that the L^2-error is bounded by $(\sqrt{\epsilon} + h)$ and the H^1-norm error is bounded by $(\epsilon^{\frac{1}{4}} + h^{\frac{1}{2}})$, which is according to our analysis on L^2-penalization and finite element error estimates. We consider the problem

$$-\Delta u = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma,$$

where $\Omega = \{(x, y) \mid x^2+y^2 < 1\}$ and the exact solution is $u = -\frac{1}{4}(x^2+y^2-1)$. To implement the fictitious domain method, we set the domain $D = \{-1.2 < x, y < 1.2\}$. We show a example of mesh (see Figure 1.3.1) and the numerical solution (see Figure 1.3.2). We solve the problem (1.3.6). First, fixing $h = 0.01$, we show the errors for different ϵ, see Figure 1.3.3; then, setting $\epsilon = 10^{-6}$, we observe the errors dependents on different h, see Figure 1.3.4. The logarithm is of base 10 for all the figures.

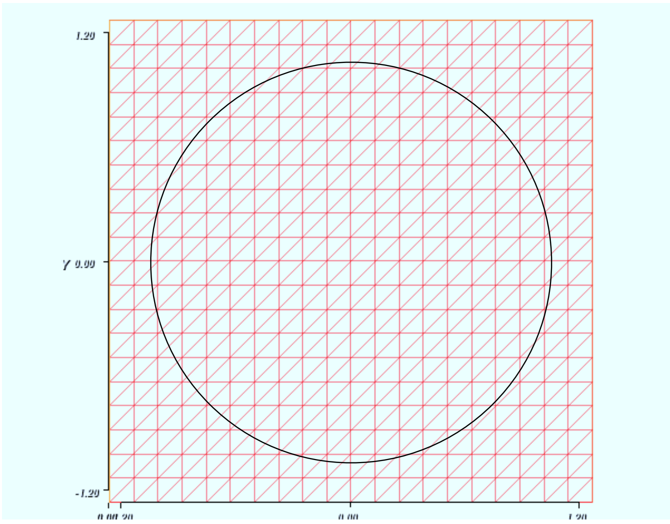

Figure 1.3.1: Ω, D and mesh

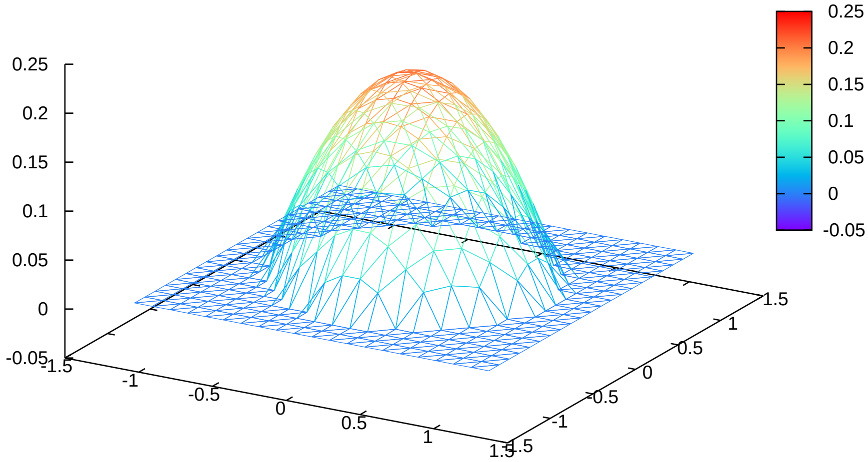

Figure 1.3.2: $\hat{u}_{\epsilon,h}$

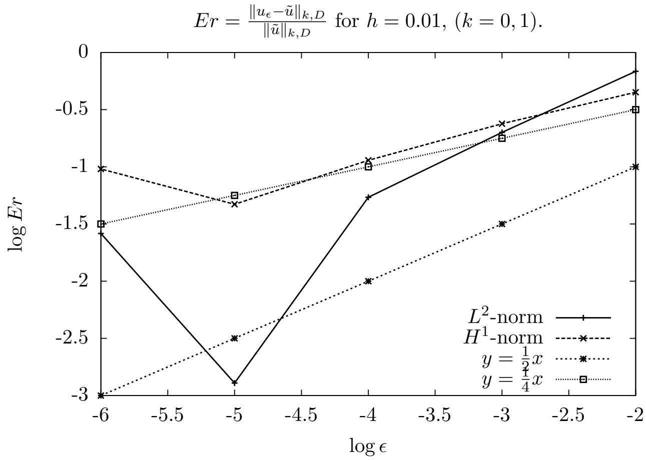

Figure 1.3.3: $\frac{\|\hat{u}_{\epsilon,h} - \tilde{u}\|_{k,D}}{\|\tilde{u}\|_{k,D}}$ for $h = 0.01$, $k = 0, 1$

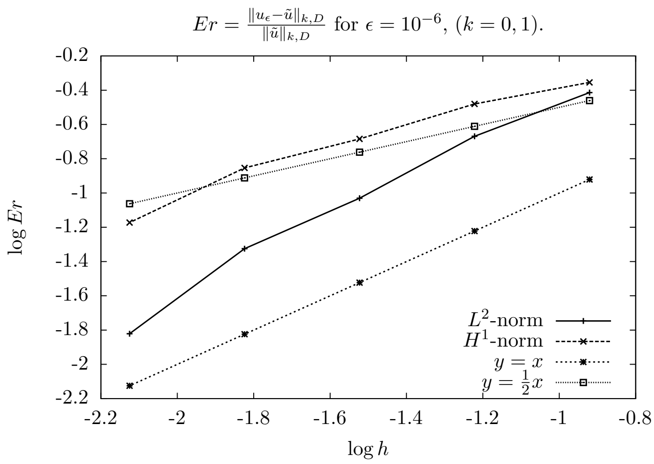

Figure 1.3.4: $\frac{\|\hat{u}_{\epsilon,h} - \tilde{u}\|_{k,D}}{\|\tilde{u}\|_{k,D}}$ for $\epsilon = 1e - 6$, $k = 0, 1$.

Remark

This chapter is based on [35].

Chapter 2

The penalty method to the Stokes and Navier-Stokes equations with slip boundary condition

2.1 Introduction

Let us consider the Navier-Stokes equations with slip boundary condition. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded smooth domain, with $\partial\Omega = D \cup \Gamma$, $D \cap \Gamma = \emptyset$ (see Figure 2.1.1). Given arbitrary $T > 0$, the Navier-Stokes problem read as:

$$
\begin{aligned}
&u' - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f, && \text{in } \Omega \times (0, T), && (2.1.1a)\\
&\nabla \cdot u = 0, && \text{in } \Omega \times (0, T), && (2.1.1b)\\
&u = 0, && \text{on } D \times (0, T), && (2.1.1c)\\
&u_n = 0, \quad \tau_T(u) = 0, && \text{on } \Gamma \times (0, T), && (2.1.1d)\\
&u(0, x) = u_0, && \text{on } \Omega, && (2.1.1e)
\end{aligned}
$$

where $\nu > 0$, $u_n = u \cdot n$, n is the unit outer normal vector to Γ, and $\tau_T(u)$ is the tangential component of traction vector on Γ defined below. Here, we set $\tau_T(u) = 0$ for simplicity. f and u_0 are given functions.

For velocity u and pressure p, we set the stress tensor,

$$\sigma(u,p) = (\sigma_{i,j}(u,p)) = -pI + 2\nu\mathcal{E}(u), \tag{2.1.2a}$$

$$\mathcal{E}(u) = \frac{1}{2}(\nabla u + \nabla u^T), \tag{2.1.2b}$$

where I denotes the identity. We set the traction vector together with its normal and tangential components:

$$\tau(u,p) = \sigma(u,p)n, \tag{2.1.3a}$$

$$\tau_n(u,p) = \tau(u,p)\cdot n, \quad \tau_T(u) = \tau(u,p) - \tau_n(u,p)n. \tag{2.1.3b}$$

Also, we set the normal and tangential component of velocity u:

$$u_n = u \cdot n, \quad u_T = u - u_n n.$$

The slip boundary condition $u_n = 0$ plays important roles in physical fluid models (cf. [5, 41]). To solve the Stokes/Navier-Stokes equations with the slip boundary condition by the finite element method is not as easy as the case of non-slip boundary problems(e.g. Dirichlet boundary condition). It is known that the variational crimes (cf. [3, 26]) may occur if the finite element spaces or the implementation method are not chosen properly to approximate the slip boundary condition.

To make a brief explanation about the variational crimes, we introduce a polygon or polyhedral domain Ω_h (see Figure 2.1.2) to approximate the smooth boundary domain Ω, with a triangulation $\mathcal{T}_h$ to Ω_h. $\partial\Omega_h = D_h \cup \Gamma_h$, $D_h \cap \Gamma_h = \emptyset$. We denote n_h as the unit outer normal vector to Γ_h. Let us consider the $P1$-element in finite element method to velocity u, which is to find a piecewise linear continuous function u_h defined on $\mathcal{T}_h$ to approximate u. We see that

$$u \in V_h = \{v_h \in C(\Omega_h)^d \mid v_h|_T \in P_1(T), \forall T \in \mathcal{T}_h, v_h = 0 \text{ on } D_h\},$$

where $P_i(T)$ is the set of polynomials of degree i on T. If we set

$$V_{hn} = \{v_h \in V_h \mid v_h \cdot n_h = 0 \text{ on } \Gamma_h\},$$

as the finite element space with slip boundary information. Since n_h is discontinuous on Γ_h, V_{hn} coincides with V_{h0}, where

$$V_{h0} = \{v_h \in V_h \mid v_h = 0 \text{ on } \Gamma_h\}.$$

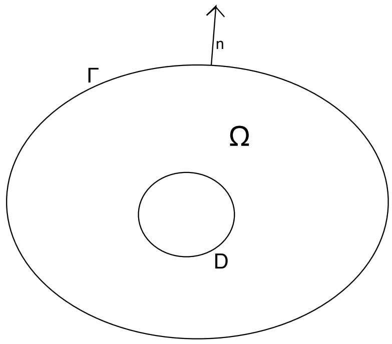

Figure 2.1.1: Ω, Γ and D.

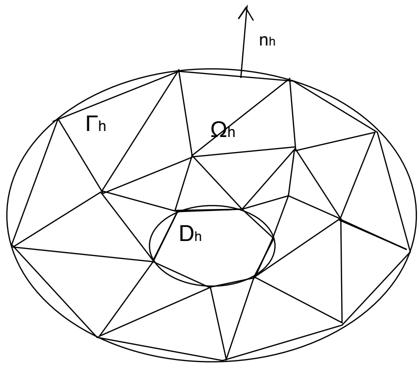

Figure 2.1.2: Ω_h, $\partial\Omega_h = \Gamma_h \cup D_h$ and triangulation $\mathcal{T}_h$.

Therefore, we cannot approximate $u_n|_\Gamma = 0$ by $u_h \cdot n_h|_{\Gamma_h} = 0$ naively. Several methods have been proposed to tackle this problem. For example, Verfürth (cf. [45, 46]) enforces the slip boundary condition in a weak sense:

$$\int_\Gamma u_n \mu \, ds = 0, \quad \forall \mu \in H^{-1/2}(\Gamma),$$

where a discrete coupled *inf-sup condition* is required for the finite element method. We have to mention that the discrete coupled *inf-sup condition* is nontrivial to verify or even may not be satisfies for general finite element spaces, for example, the $P1/P1$ element.

Let Ω_h be the polygon/polyhedral domain approximating to the smooth domain Ω, with $\partial\Omega_h = \Gamma_h \cup D_h$, $\Gamma_h \cap D_h = \emptyset$ (see Figure 2.1.2). The approach proposed in [41, 42, Tabata and Suzuki] is to use $P1/P1$ element with stabilization, and implement the slip boundary condition as $u_h(p) \cdot n(p) = 0$, where u_h is the finite element solution, and p are the vertices on Γ_h. A similar method presented in [16] using $P2/P1$-element is to introduce a homeomorphism $G_h : \Omega_h \to \Omega$, and implement the slip boundary condition as $u_h(G(p)) \cdot n(G(p)) = 0$, where p are the vertices or the midpoints of edges on Γ_h, These two implementation methods avoid the variational crimes; however, G_h and n are not easy to obtain in numerical computation for complex domain Ω. In FEM, it is more convenience to use n_h (the unit outer normal vector to Γ_h) than n. Also, we have to mention that it is technical to implement $u_h(p) \cdot n(p) = 0$ in finite element code.

Instead of enforcing $u_n|_\Gamma = 0$ into weak sense, or searching for the suitable implement method to avoid the variational crimes, an alternative way is to introduce a penalty term to approximate $u_n|_\Gamma = 0$. Here we present the penalty problem to (2.1.3),

$$
\begin{aligned}
&u_\epsilon' - \nu\Delta u_\epsilon + (u_\epsilon \cdot \nabla)u_\epsilon + \nabla p_\epsilon = f, && \text{in } \Omega, && \text{(2.1.4a)}\\
&\nabla \cdot u_\epsilon = 0, && \text{in } \Omega, && \text{(2.1.4b)}\\
&u_\epsilon|_D = 0, \quad \tau(u_\epsilon, p_\epsilon) + \epsilon^{-1} u_{\epsilon n} n = 0, && \text{on } \Gamma, && \text{(2.1.4c)}\\
&u_\epsilon(0, x) = u_{\epsilon 0}, && \text{on } \Omega. && \text{(2.1.4d)}
\end{aligned}
$$

where $0 < \epsilon \ll 1$ is the penalty parameter, and $u_{\epsilon 0}$ is some approximation to u_0. In view of (2.1.4c), the idea of penalty method is to approximate $u_n|_\Gamma = 0$ by a Robin boundary condition. In the variational form of (2.1.4), the penalty term becomes $\frac{1}{\epsilon}\int_\Gamma u_{\epsilon n} v_n ds$ (see (2.3.8)), where

$$v \in V \equiv \{v \in H^1(\Omega)^d \mid v|_D = 0\}$$

is the test function. For u_ϵ the solution of (2.1.4), it is apparently that $u_{\epsilon n} \to 0$ in $L^2(\Gamma)$ as $\epsilon \to 0$, which approximate to $u_n|_\Gamma = 0$.

The penalty method has several advantages. The technical implementation of $u_n|_\Gamma = 0$ (cf. [42, 16]) to avoid the variational crimes is unnecessary. In cost we need to compute the integration $\int_{\Gamma_h} (u_h \cdot n_h)(v_h \cdot n_h) ds$, where u_h, v_h are the solution and test function for finite element approximation. The integration on Γ_h can be easily implemented by popular FEM softwares (Freefem++, FeniCS, cf. [21, 30]), and here only n_h (instead of n) is involved. The penalty method is well applicable to various types of finite element spaces, such as $P1/P1$ and $P1b/P1$ (cf. [24]), $P2/P1$ (cf. [12, 14]) and so on.

In this chapter, we first consider the penalty method for the Stokes equations with slip boundary condition (see Section 2.2). We prove the error estimates (see Theorem 2.2.3)

$$\|u - u_\epsilon\|_{H^1(\Omega)} + \|p - p_\epsilon\|_{H^1(\Omega)/\mathbb{R}} \le C\epsilon,$$

which has already been obtained in [14]; however, we give a different proof based on the separation of $p_\epsilon \in L^2(\Omega)$:

$$p_\epsilon = \mathring{p}_\epsilon + l_\epsilon, \quad \mathring{p}_\epsilon \in L_0^2(\Omega), \quad k_\epsilon = \int_\Omega p_\epsilon dx/|\Omega|, \tag{2.1.5}$$

and we show

$$\|\tau_n(u, p) - \epsilon^{-1} u_{\epsilon n} + l_\epsilon\|_{H^{-\frac{1}{2}}(\Gamma)} \le C\epsilon.$$

Moreover, we show the regularity of the penalty problem (see Theorem 2.2.4)

$$\|u_\epsilon\|_{H^m(\Omega)} + \|p_\epsilon\|_{H^{m-1}(\Omega)} \leq C\|f\|_{H^{m-2}(\Omega)},$$

under the C^{m-} smoothness assumption of Ω, for any integer $m \geq 2$. Furthermore, we obtain a new result of the error estimates (see Theorem 2.2.5)

$$\|u - u_\epsilon\|_{H^m(\Omega)} \leq C\epsilon, \quad \forall m \in \mathbb{N}.$$

We then apply the finite element approximation to the penalty problem (2.2.9) with $P1b/P1$ element, and we proved the error estimates (see Theorem 2.2.7 and 2.2.8). We show the best error estimates we obtain:

$$\|\tilde{u} - u_h\|_{1,\Omega_h} + \|\tilde{p} - p_h\|_{\Omega_h} \leq C(h + \sqrt{\epsilon} + h^2/\sqrt{\epsilon}), \qquad \text{for } d = 2,$$

$$\|\tilde{u} - u_h\|_{1,\Omega_h} + \|\tilde{p} - p_h\|_{\Omega_h} \leq C(\sqrt{h} + \sqrt{\epsilon} + h/\sqrt{\epsilon}), \qquad \text{for } d = 3,$$

where h is the mesh size of triangulation.

In Section 2.3, we consider the penalty method to the Navier-Stokes problem (2.1.1). For the slip boundary condition $u_n|_\Gamma = 0$, we have

$$\int_\Omega (u \cdot \nabla)u \cdot u \; dx = \frac{1}{2}\int_\Gamma u_n|u^2| \; ds = 0,$$

which implies the energy inequality of u:

$$\|u(T)\|^2_{L^2(\Omega)^d} + \int_0^T \|u(t)\|_{H^1(\Omega)^d} \; dt \leq C.$$

Since $u_{\epsilon n}|_\Gamma \neq 0$, we have

$$\int_\Omega (u_\epsilon \cdot \nabla)u_\epsilon \cdot u_\epsilon \; dx = \frac{1}{2}\int_\Gamma u_{\epsilon n}|u_\epsilon^2| \; ds \neq 0,$$

and the energy inequality (or the well-posedness) of u_ϵ is not apparent. Our first job is to prove the well-posedness of the penalty problem (2.1.4) (see Theorem 2.3.1). We show the estimates of u_ϵ, p_ϵ are bounded independent on the penalty coefficient ϵ^{-1}.

Besides of the well-posedness, we derive the error estimates of the penalty method (see Theorem 2.3.3):

$$\|u' - u'_\epsilon\|_{L^2(0,T;L^2(\Omega)^d)} + \|u - u_\epsilon\|_{L^\infty(0,T;H^1(\Omega)^d)} \leq C\epsilon.$$

Section 2.4 is devoted to the penalty method for stationary Navier-Stokes equations. We investigate the well-posedness of penalty problem, the error estimates of penalty, and the finite element method for penalty problem.

Notations

Throughout this chapter, we write $\|\cdot\|_{H^k}$ as the norm of Sobolev spaces $H^k(\Omega)$ or $H^k(\Omega)^d$, and $\|\cdot\|_{W^{k,p}}$ for $W^{k,p}(\Omega)$ or $W^{k,p}(\Omega)^d$. Let ω be some open set of $\mathbb{R}^d$, we denote $(\cdot,\cdot)_\omega$ as the inner-product of $L^2(\omega)$, and we write $(\cdot,\cdot)$ for the case $\omega = \Omega$. Sometimes, we use $L^m(0,T;H^k)$ instead of $L^m(0,T;H^k(\Omega)^d)$ for short.

2.2 The penalty method to the Stokes problem

Let $f \in L^2(\Omega)$. We consider the Stokes equations with slip boundary condition:

$$-\nu\Delta u + \nabla p = f \qquad \text{in } \Omega, \tag{2.2.1a}$$
$$\nabla \cdot u = 0 \qquad \text{in } \Omega, \tag{2.2.1b}$$
$$u_n = 0, \quad \tau_T(u) = 0 \qquad \text{on } \Gamma, \tag{2.2.1c}$$
$$u = 0 \qquad \text{on } D. \tag{2.2.1d}$$

Remark 2.2.1 (cf. [37])**.** Assume $f \in L^2(\Omega)$ and Ω is C^3-smooth, then there exists a unique solution $(u,p) \in H^2(\Omega)^d \times (H^1(\Omega)/\mathbb{R})$ to (2.2.1).

Function spaces.

$$V = \{v \in H^1(\Omega)^d \mid v|_D = 0\}, \quad V_n = \{v \in V \mid v_n|_\Gamma = 0\}, \tag{2.2.2a}$$
$$V^\sigma = \{v \in V \mid \nabla\cdot v = 0\}, \quad V_n^\sigma = V_n \cap V^\sigma, \tag{2.2.2b}$$
$$Q = L^2(\Omega), \quad \mathring{Q} = L_0^2(\Omega), \tag{2.2.2c}$$
$$M = H^{1/2}(\Gamma). \tag{2.2.2d}$$

We denote X' as the dual of Banach space X, for example $M' = H^{-\frac{1}{2}}(\Gamma)$.

For any $u, v, w \in H^1(\Omega)^d$, $p \in Q$, $\eta \in M$ and $\mu \in M'$, we set

$$a(u,v) = 2\nu(\mathcal{E}(u), \mathcal{E}(u)), \tag{2.2.3a}$$
$$a_1(u,v,w) = \int_\Omega (u\cdot\nabla)\cdot w \; dx, \tag{2.2.3b}$$
$$b(v,p) = -(\nabla\cdot v, p), \tag{2.2.3c}$$
$$c(\mu,\eta) = \int_\Gamma \mu\eta \; ds. \tag{2.2.3d}$$

Some properties of bilinear and trilinear forms.(cf. [8, 19, 45])

- *Coercivity of* a: there exists $\alpha > 0$ such that

$$a(u,u) \geq \alpha \|u\|_{H^1}^2, \quad \forall u \in V. \tag{2.2.4}$$

- The *inf-sup condition* of b: there exists $\beta > 0$ such that

$$\inf_{p \in L_0^2(\Omega)\backslash\{0\}} \sup_{v \in H_0^1(\Omega)^d \backslash\{0\}} \frac{b(v,p)}{\|v\|_{H^1}\|p\|_{L^2}} \geq \beta. \tag{2.2.5}$$

- The *inf-sup condition* of c: there exists $\gamma_0 > 0$ such that

$$\inf_{\mu \in M'\backslash\{0\}} \sup_{v \in V\backslash\{0\}} \frac{c(\mu, v_n)}{\|v\|_{H^1}\|\mu\|_{M'}} \geq \gamma_0. \tag{2.2.6}$$

The variational form of (2.2.1) reads as: find $(u,p) \in V_n \times \mathring{Q}$ such that,

$$a(u,v) + b(v,p) = (f,v), \quad \forall v \in V_n, \tag{2.2.7a}$$

$$b(u,q) = 0, \quad \forall q \in \mathring{Q}. \tag{2.2.7b}$$

Let $0 < \epsilon \ll 1$, the penalty method for (2.2.1) reads as:

$$-\Delta u_\epsilon + \nabla p_\epsilon = f \qquad \text{in } \Omega, \tag{2.2.8a}$$

$$\nabla \cdot u_\epsilon = 0 \qquad \text{in } \Omega, \tag{2.2.8b}$$

$$\tau_n(u_\epsilon, p_\epsilon) + \frac{1}{\epsilon} u_{\epsilon n} = 0, \quad \tau_T(u_\epsilon) = 0 \qquad \text{on } \Gamma, \tag{2.2.8c}$$

$$u_\epsilon = 0 \qquad \text{on } D. \tag{2.2.8d}$$

The variational form of (2.2.8) reads as: find $(u_\epsilon, p_\epsilon) \in V \times Q$ such that

$$a(u_\epsilon, v) + b(v, p_\epsilon) + \frac{1}{\epsilon} c(u_{\epsilon n}, v_n) = (f,v), \quad \forall v \in V, \tag{2.2.9a}$$

$$b(u_\epsilon, q) = 0, \quad \forall q \in Q. \tag{2.2.9b}$$

Remark 2.2.2. $p_\epsilon \notin \mathring{Q}$. For non-homogeneous slip boundary condition $u_n = g$ on Γ, we set the penalty term $\frac{1}{\epsilon} c(u_{\epsilon n} - g, v_n)$ in (2.2.9a), or equivalently, $\tau_n(u_\epsilon, p_\epsilon) + \frac{1}{\epsilon}(u_{\epsilon n} - g) = 0$ in (2.2.8c).

The following theorem gives the well-posedness of penalty problem (2.2.9), also it shows the estimates of u_ϵ, p_ϵ are independent on ϵ^{-1}.

Theorem 2.2.1. *Given $f \in V'$, there exists a unique solution $(u_\epsilon, p_\epsilon) \in V \times Q$ to (2.2.9), with*

$$\|u_\epsilon\|_{H^1} + \|p_\epsilon\|_{L^2} \le C\|f\|_{V'}.$$

Proof. From the coercivity of a (2.2.4), we conclude the existence of u_ϵ and $\|u_\epsilon\|_V \le C\|f\|_{V^*}$. Set $p_\epsilon = \mathring{p}_\epsilon + l_\epsilon$, where $\mathring{p}_\epsilon \in \mathring{Q}$ and $l_\epsilon = \int_\Omega p_\epsilon \, dx/|\Omega|$. From the *inf-sup condition* of b (2.2.5), we have $\|\mathring{p}_\epsilon\|_\Omega \le C\|f\|_{V'}$. To estimate l_ϵ, we choose a trace lifting $v \in V$ satisfying $v = l_\epsilon n$ on Γ, and $\|v\|_{1,\Omega} \le C|l_\epsilon|$. Substituting this v into (2.2.9), in view of the fact $\int_\Gamma u_{\epsilon n} \, ds = 0$, we have

$$|\Gamma| l_\epsilon^2 = k_\epsilon \int_\Gamma v_n dx = -b(v, k_\epsilon) = a(u_\epsilon, v) + b(v, \mathring{p}_\epsilon) - (f, v),$$

which implies

$$|l_\epsilon| \le C(\|u_\epsilon\|_{H^1} + \|\mathring{p}_\epsilon\|_{L^2} + \|f\|_{V'}) \le C\|f\|_{V'}.$$

□

2.2.1 The error estimates of H^1 norm

To show the error estimates of penalty method, we introduce the Largrange multipliers $\lambda = -\tau_n(u, p)$ and $\lambda_\epsilon = \frac{1}{\epsilon} u_{\epsilon n}$, then (2.2.7) and (2.2.9) are rewritten into the following two equations, respectively.

(1) Find $(u, p, \lambda) \in V \times Q \times M'$ such that,

$$a(u, v) + b(v, p) + c(\lambda, v_n) = (f, v), \quad \forall v \in V, \tag{2.2.10a}$$
$$b(u, q) = 0, \quad \forall q \in Q, \tag{2.2.10b}$$
$$c(u_n, \eta) = 0, \quad \forall \eta \in M; \tag{2.2.10c}$$

(2) Find $(u_\epsilon, p_\epsilon, \lambda_\epsilon) \in V \times Q \times M'$ such that,

$$a(u_\epsilon, v) + b(v, p_\epsilon) + c(\lambda_\epsilon, v_n) = (f, v), \quad \forall v \in V, \tag{2.2.11a}$$
$$b(u_\epsilon, q) = 0, \quad \forall q \in Q, \tag{2.2.11b}$$
$$c(u_{\epsilon n}, \eta) = \epsilon c(\lambda_\epsilon, \eta), \quad \forall \eta \in M. \tag{2.2.11c}$$

We state the error estimates of penalty method.

Theorem 2.2.2. *Let (u, p) and (u_ϵ, p_ϵ) be the solutions of (2.2.1) and (2.2.8), respectively, then we have*

$$\|u - u_\epsilon\|_{H^1} + \|p - \mathring{p}_\epsilon\|_{L^2} + \sqrt{\epsilon}\|\lambda - \lambda_\epsilon\|_{L^2(\Gamma)} \le c\sqrt{\epsilon}\|\lambda\|_{L^2(\Gamma)}. \tag{2.2.12}$$

Proof. Substituting $v = u - u_\epsilon$ into (2.2.10a)−(2.2.11a), we have

$$a(u - u_\epsilon, u - u_\epsilon) + c(\lambda - \lambda_\epsilon, u_n - u_{\epsilon n}) = 0. \tag{2.2.13}$$

Since $u_n = 0$ and $u_{\epsilon n} = \epsilon\lambda_\epsilon$, we have

$$c(\lambda - \lambda_\epsilon, u_n - u_{\epsilon n}) = \epsilon c(\lambda - \lambda_\epsilon, \lambda - \lambda_\epsilon) - \epsilon c(\lambda, \lambda - \lambda_\epsilon). \tag{2.2.14}$$

From the *coercivity* of a (2.2.4), (2.2.13) and (2.2.14) we obtain

$$\begin{aligned}&\alpha\|u - u_\epsilon\|^2_{1,\Omega} + \epsilon\|\lambda - \lambda_\epsilon\|^2_{L^2(\Gamma)}\\ \le&\epsilon c(\lambda, \lambda - \lambda_\epsilon) \le \frac{\epsilon}{2}\|\lambda - \lambda_\epsilon\|^2_{L^2(\Gamma)} + \frac{\epsilon}{2}\|\lambda\|^2_{L^2(\Gamma)},\end{aligned}$$

which implies

$$\|u - u_\epsilon\|_{H^1} + \sqrt{\epsilon}\|\lambda - \lambda_\epsilon\|_{L^2(\Gamma)} \le c\sqrt{\epsilon}\|\lambda\|_{L^2(\Gamma)}. \tag{2.2.15}$$

From the *inf-sup condition* of b (2.2.5) and

$$b(p - \mathring{p}_\epsilon, v) = -a(u - u_\epsilon, v), \quad \forall v \in (H^1_0(\Omega))^d, \tag{2.2.16}$$

we have

$$\|p - \mathring{p}_\epsilon\|_{L^2} \le C\|u - u_\epsilon\|_{H^1}, \tag{2.2.17}$$

which gives (2.2.12). □

Theorem 2.2.3. *Let* (u, p) *and* (u_ϵ, p_ϵ) *be the solutions of (2.2.1) and (2.2.8), respectively, then we have*

$$\|u - u_\epsilon\|_{H^1} + \|p - \mathring{p}_\epsilon\|_{L^2} + \sqrt{\epsilon}\|\lambda - \lambda_\epsilon + l_\epsilon\|_{L^2(\Gamma)} \le C\epsilon(\|\lambda\|_{H^{\frac{1}{2}}(\Gamma)} + 1). \tag{2.2.18}$$

Proof. Subtracting (2.2.10a) from (2.2.11a), we have, for any $v \in V$,

$$c(\lambda - \lambda_\epsilon + l_\epsilon, v_n) = -a(u - u_\epsilon, v) - b(v, p - \mathring{p}_\epsilon).$$

In view of the *inf-sup condition* of c (2.2.6) and (2.2.17), it yields

$$\|\lambda - \lambda_\epsilon + l_\epsilon\|_{M'} \le C\|u - u_\epsilon\|_{H^1} \tag{2.2.19}$$

Noticing that $\int_\Gamma u_{\epsilon n}\, ds = 0$, instead of (2.2.14), we derive

$$c(\lambda - \lambda_\epsilon, u_n - u_{\epsilon n}) = \epsilon c(\lambda - \lambda_\epsilon + k_\epsilon, \lambda - \lambda_\epsilon + k_\epsilon) - \epsilon c(\lambda + k_\epsilon, \lambda - \lambda_\epsilon + k_\epsilon). \tag{2.2.20}$$

From the *coercivity* of a (2.2.4), (2.2.13) and (2.2.20), we obtain

$$\begin{aligned}&\alpha\|u-u_\epsilon\|_{H^1}^2+\epsilon\|\lambda-\lambda_\epsilon+l_\epsilon\|_{L^2(\Gamma)}^2\\ \le&\epsilon c(\lambda+l_\epsilon,\lambda-\lambda_\epsilon+l_\epsilon)\le\epsilon\|\lambda+l_\epsilon\|_M\|\lambda-\lambda_\epsilon+l_\epsilon\|_{M'}.\end{aligned} \tag{2.2.21}$$

From (2.2.21) and (2.2.19), we obtain

$$\|u-u_\epsilon\|_{H^1}\le C\epsilon\|\lambda+l_\epsilon\|_M,$$

which implies (2.2.18) because l_ϵ is bounded independent of ϵ (see Theorem 2.2.1). □

Remark 2.2.3. From (2.2.19), we have $\|\lambda-\lambda_\epsilon+l_\epsilon\|_{H^{-1/2}(\Gamma)}\le C\epsilon$.

2.2.2 The error estimates of H^m norm

In view of

$$\|u_{\epsilon n}\|_{H^{\frac12}(\Gamma)}=\|u_{\epsilon n}-u_n\|_{H^{\frac12}(\Gamma)}\le C\|u_\epsilon-u\|_{H^1}\le C\epsilon,$$

we have

$$\|\tau_n(u_\epsilon,p_\epsilon)\|_{H^{\frac12}(\Gamma)}=\|\epsilon^{-1}u_{\epsilon n}\|_{H^{\frac12}(\Gamma)}\le C,$$

which implies

$$\|u_\epsilon\|_{H^2}+\|p_\epsilon\|_{H^1}\le C.$$

In fact, we have the following regularity result for penalty problem (2.2.8).

Theorem 2.2.4. *For arbitrary integer $m\ge 0$, let $\Omega\in C^{m+3}$, $f\in H^m(\Omega)^d$, then there exists a unique solution $(u_\epsilon,p_\epsilon)\in H^{m+2}(\Omega)^d\times H^{m+1}(\Omega)$ to (2.2.8), with*

$$\|u_\epsilon\|_{H^{m+2}}+\|p_\epsilon\|_{H^{m+1}}\le C\|f\|_{H^m}. \tag{2.2.22}$$

Proof. For general domain $\Omega\in C^{m+2}$, the regularity in interior or near C is well known(cf. [13, 27]); that is $\|u_\epsilon\|_{H^{m+2}(\omega)}+\|p_\epsilon\|_{H^{m+1}(\omega)}\le C(\omega)\|f\|_{H^m(\omega)}$, where $\omega\subset\Omega$ and $\mathrm{dist}(\overline{\omega},\Gamma)\ge\delta>0$.

For the regularity near Γ, there exists a set of smooth sub-domain in $\mathbb{R}^d$, denoted as $\{U_i\}_{i=1}^N$, satisfying $\Gamma\subset\cup_{i=1}^N U_i$.

We introduce a cut-off function $\theta_i\in C^\infty(\mathbb{R}^d)$ with $\mathrm{supp}\theta_i\subset U_i$, and consider the equations of $(\theta_i^2u_\epsilon,\theta_i^2p_\epsilon)$ in $U_i\cap\Omega$.

There exists a C^{k+3}-diffeomorphism(cf. [47]) $\mathbf{\Phi}_i: U_i\to Q_R:=\mathbb{R}^d_{d,+}\cap\{\tilde{x}\in\mathbb{R}^d,\mid|\tilde{x}|<R\}$, where $\mathbb{R}^d_{d,+}:=\{\tilde{x}=(\tilde{x}',\tilde{x}_d)\in\mathbb{R}^d\mid\tilde{x}'\in\mathbb{R}^{d-1},\tilde{x}_d>0\}$ is the half-plane, and we also have $\mathbf{\Phi}_i:\Gamma\cap U_i\to\tilde{\Gamma}_i:=\{\tilde{x}\mid|\tilde{x}|<R,\tilde{x}_d=0\}$.

Then we consider the equation of $(\tilde{u}_\epsilon, \tilde{p}_\epsilon) := ((\theta_i^2 u_\epsilon)\circ\mathbf{\Phi}_i, (\theta_i^2 p_\epsilon)\circ\mathbf{\Phi}_i)$ in domain Q_R, to which we apply the famous Agmon-Douglis-Nirenberg' method(cf. [1]) and obtain $\|D_i D_j \tilde{u}_\epsilon\|_{L^2} \le C(\|f\|_{L^2} + \|u_\epsilon\|_{H^1})$, $i = 1, \ldots, d-1$; $j = 1, \ldots, d$, where $D_i v = \nabla_{x_i} v$. Hence, we can conclude $\|\tilde{u}_\epsilon\|_{H^{\frac{3}{2}}(\tilde{\Gamma}_i)} \le C\|f\|_{H^k}$, which implies $\|u_{\epsilon n}\|_{\frac{3}{2},\Gamma} \le C\|f\|_\Omega$. Following from well-known regularity result for Stokes equation by Cattabriga [13], it yields $\|u_\epsilon\|_{H^2} + \|p_\epsilon\|_{H^1} \le C\|f\|_{L^2}$. For $m \ge 1$, (2.2.22) can be proved by induction method.

In above, we briefly sketch the strategy of proof. The key point is to consider the equation in the half-plane via some transformations. We refer the readers to [34, Saito, proof of Lemma 4.1] for detailed arguments on those techniques. Here, to make the argument brief, we only prove the case of $k = 0$ and the half-plane domain $\Omega = \mathbb{R}^d_{d,+} := \{x = (x', x_d) \in \mathbb{R}^d \mid x' \in \mathbb{R}^{d-1}, x_d > 0\}$.

Set $D_h^i v = (v(x_1, \cdots, x_i + h, \cdots, x_d) - v(x))/h$, $h > 0$. Substituting $v = D_{-h}^i D_h^i u_\epsilon$ into (2.2.8), $i = 1, \ldots, d-1$, we have, with $\Gamma = \{x \mid x_d = 0\}$,

$$a(u_\epsilon, D_{-h}^i D_h^i u_\epsilon) + b(D_{-h}^i D_h^i u_\epsilon, p_\epsilon) + \frac{1}{\epsilon}\int_\Gamma u_{\epsilon n} D_{-h}^i D_h^i u_\epsilon \cdot n ds = (f, D_{-h}^i D_h^i u_\epsilon).$$

Using the fact $(w, D_{-h}^i v) = (D_h^i w, v)$, $\forall w, v \in H^1(\mathbb{R}^d_{d,+})$, we get

$$a(D_h^i u_\epsilon, D_h^i u_\epsilon) + \frac{1}{\epsilon}\int_\Gamma |D_h^i u_{\epsilon n}|^2 ds = (f, D_{-h}^i D_h^i u_\epsilon) \le C\|f\|_{L^2}\|D_{-h}^i D_h^i u_\epsilon\|_{L^2}.$$

Since $\|D_h^i v\|_{L^2} \le C\|\nabla_{x_i} v\|_{L^2}$, from the *coercivity* of a (2.2.4), we have,

$$\|D_h^i u_\epsilon\|_{H^1} + \epsilon^{-1/2}\|D_h^i u_{\epsilon n}\|_{L^2(\Gamma)} \le C\|f\|_{L^2}, \quad i = 1, \ldots, d-1.$$

Let $h \to 0$, and we have

$$\|D_i D_j u_\epsilon\|_{L^2} + \epsilon^{-1/2}\|D_i u_{\epsilon n}\| \le C\|f\|_{L^2}, \quad i = 1, \ldots, d-1;\ j = 1, \ldots, d.$$

By trace theorem and $n = (0, \ldots, 0, 1)$, we have

$$\|u_{\epsilon n}\|_{H^{\frac{3}{2}}(\Gamma)} \le C\|f\|_{L^2}.$$

And we can conclude $(u_\epsilon, p_\epsilon) \in H^2(\Omega)^d \times H^1(\Omega)$ and (2.2.22) for $m = 0$(cf. [13]). □

Theorem 2.2.5. *For any integer $m \ge 0$, assume $f \in H^m(\Omega)^d$ and Ω has C^{m+3} smoothness. Let (u, p) and (u_ϵ, p_ϵ) of $H^{m+2}(\Omega)^d \times H^{m+1}(\Omega)$ be the solutions of (2.2.1) and (2.2.8), respectively, then we have,*

$$\|u - u_\epsilon\|_{H^{m+2}} + \|p - \mathring{p}_\epsilon\|_{H^{m+1}} \le C\epsilon\|\lambda\|_{H^{m+\frac{3}{2}}}. \tag{2.2.23}$$

Proof. To make the argument brief, we only prove the case of $m = 0$($m \geq 1$ follows form induction method) and the half-plane domain $\Omega = \mathbb{R}^d_{d,+}$. For general domain, we can applied the transformation introduced in Theorem 2.2.4. Substituting $v = D^i_{-h}D_h(u - u_\epsilon)$, $i = 1, \dots, d-1$, into (2.2.10a)–(2.2.11a), we have

$$a(u - u_\epsilon, D^i_{-h}D^i_h(u - u_\epsilon)) + c(\lambda - \lambda_\epsilon + l_\epsilon, D^i_{-h}D^i_h(u - u_\epsilon) \cdot n) = 0,$$

which yields,

$$\begin{aligned} &a(D^i_h(u - u_\epsilon), D^i_h(u - u_\epsilon)) + \epsilon c(D^i_h(\lambda - \lambda_\epsilon + l_\epsilon), D^i_h(\lambda - \lambda_\epsilon + l_\epsilon)) \\ &\quad = \epsilon c(D^i_h(\lambda - \lambda_\epsilon + l_\epsilon), D^i_h(\lambda + l_\epsilon)). \end{aligned}$$

Since l_ϵ is a constant, $D^i_h l_\epsilon = 0$. Therefore, we have

$$\begin{aligned} &\alpha \|D^i_h(u - u_\epsilon)\|^2_{H^1} + \epsilon \|D^i_h(\lambda - \lambda_\epsilon)\|^2_{L^2(\Gamma)} \\ \leq &C\epsilon \|D^i_h(\lambda - \lambda_\epsilon + l_\epsilon)\|_{H^{-\frac{1}{2}}(\Gamma)} \|D^i_h \lambda\|_{H^{\frac{1}{2}}(\Gamma)}. \end{aligned} \tag{2.2.24}$$

Via *inf-sup condition* of b, and the equation

$$b(D^i_h(p - \mathring{p}_\epsilon), v) = -a(D^i_h(u - u_\epsilon), v), \quad \forall v \in H^1_0(\mathbb{R}^d_{d,+}),$$

we have $\|D^i_h(p - \mathring{p}_\epsilon)\|_{L^2} \leq C\|D^i_h(u - u_\epsilon)\|_{H^1}$.

Via *inf-sup condition* of c, and the equation

$$c(D^i_h(\lambda - \lambda_\epsilon + l_\epsilon), v) = -a(D^i_h(u - u_\epsilon), v) - b(D_i(p - \mathring{p}_\epsilon), v),$$

we have

$$\|D^i_h(\lambda - \lambda_\epsilon + l_\epsilon)\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C\|D^i_h(u - u_\epsilon)\|_{H^1}.$$

In views of (2.2.24), we obtain

$$\|D^i_h(u - u_\epsilon)\|_{H^1} \leq C\epsilon \|D^i_h \lambda\|_{H^{\frac{1}{2}}(\Gamma)},$$

then letting $h \to 0$, we proved (2.2.23). □

2.2.3 Finite element approximation with penalty

A regular triangulation $\mathcal{T}_h$ is introduced to the smooth domain Ω, where $h = \max_{K \in \mathcal{T}_h} \operatorname{diam}(K)$. $\Omega_h = \cup_{K \in \mathcal{T}_h} \overline{K}$, $\partial\Omega_h = \Gamma_h \cup D_h$, $\Gamma_h \cap D_h = \emptyset$ (see Figure 2.1.2). The boundary mesh $\mathcal{S}_h$ inherited from $\mathcal{T}_h$ is also a regular triangulation of Γ_h in $d-1$ dimension. n_h is the outer unit normal assigned to Γ_h. We assume $D = D_h$ for simplicity. Suppose Γ is C^3 smooth, then we have

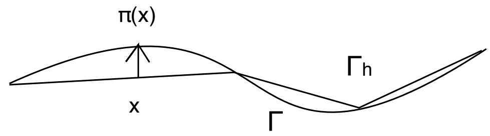

Figure 2.2.1: $\pi : \Gamma_h \to \Gamma$.

(1) $\max_{x\in\Gamma} \mathrm{dist}(x, \Gamma_h) \le Ch^2$.

(2) There exists a continuous bijective mapping

$$\pi : \Gamma_h \to \Gamma; \quad x \mapsto \pi(x).$$

Moreover, for any element S of $\mathcal{S}_h$, we have $\pi, \pi^{-1} \in C^2(S)$ and

$$||D\pi| - 1|, ||D\pi^{-1}| - 1| \le Ch^2, \tag{2.2.25}$$

where $|D\pi|$ satisfies $\int_\Gamma v ds = \int_{\Gamma_h} v \circ \pi |D\pi^{-1}| ds$. And we also have

$$|n_h - n \circ \pi| \le Ch. \tag{2.2.26}$$

Finite element spaces:

We consider the $P1/P1$ and $P1b/P1$ finite element spaces.

$$V_h = \{v_h \in C(\overline{\Omega_h})^d \mid v_h|_K \in P_1(K),\ K \in \mathcal{T}_h,\ v_h|_{D_h} = 0\},\ \text{for } P1$$

$$V_h = \{v_h \in C(\overline{\Omega_h})^d \mid v_h|_K \in P_1(K) \oplus B(K),\ K \in \mathcal{T}_h,\ v_h|_{D_h} = 0\},\ \text{for } P1b,$$

$$Q_h = \{v_h \in C(\overline{\Omega_h})^d \mid v_h|_K \in P_1(K),\ K \in \mathcal{T}_h\},$$

$$V_{h0} = \{v_h \in V_h \mid v_h = 0 \text{ on } \Gamma_h\}, \quad \mathring{Q}_h = Q_h \cap L_0^2(\Omega_h),$$

$$\Lambda_h = \{v_h \cdot n_h \mid v_h \in V_h\},$$

where $P_l(K)$ is the set of polynomial of order l in K, and $B(K)$ stands for the space spanned by the bubble function on K. We define the following bilinear and trilinear forms:

$$a_h(u_h, v_h) = \int_{\Omega_h} 2\nu \mathcal{E}(u_h)\mathcal{E}(v_h), \quad \forall u_h, v_h \in V_h; \tag{2.2.27}$$

$$b_h(v_h, p_h) = -\int_{\Omega_h} \nabla \cdot v_h p_h dx, \quad \forall v_h \in V_h, p_h \in Q_h, \tag{2.2.28}$$

$$d_h(p_h, q_h) = \gamma h^2 (\nabla p_h, \nabla q_h)_{\Omega_h}, \quad \begin{cases} \gamma = 1 \text{ for } P1/P1, \\ \gamma = 0 \text{ for } P1b/P1. \end{cases} \tag{2.2.29}$$

Choice of c_h.

(1) *Nonreduced-integration*: For any $\lambda_h, \mu_h \in \Lambda_h$.

$$c_h(\lambda_h, \mu_h) := \int_{\Gamma_h} \lambda_h \mu_h ds. \tag{2.2.30}$$

$\|\mu_h\|_{c_h} := c_h(\mu_h, \mu_h)^{\frac{1}{2}}$ is equivalent to $\|\mu_h\|_{L^2(\Gamma_h)}$, for any $\mu_h \in \Lambda_h$.

(2) *Reduced-integration*: For any $\lambda_h, \mu_h \in \Lambda_h$,

$$c_h(\mu_h, \eta_h) = \sum_{s \in \mathcal{S}_h} |s| \mu_h(m_s) \eta_h(m_s), \; m_s = \begin{cases} \text{midpoint of } s \text{ if } d = 2, \\ \text{barycenter of } s \text{ if } d = 3. \end{cases} \tag{2.2.31}$$

$\|\mu_h\|_{c_h} = c_h(\mu_h, \mu_h)^{\frac{1}{2}}$ is a semi-norm of Λ_h(there exists $\mu_h \neq 0$ but $c_h(\mu_h, \mu_h) = 0$).

Coercivity and inf-sup conditions.

- *Coercivity* of a_h:

$$a_h(v_h, v_h) \geq \alpha_1 \|v_h\|^2_{H^1(\Omega_h)}, \quad \alpha_1 > 0, \quad \forall v_h \in V_h. \tag{2.2.32}$$

- *inf-sup condition* of b_h, $\beta_1, \tilde{\beta}_1 > 0$:

$$\inf_{p_h \in \mathring{Q}_h \backslash \{0\}} \sup_{v_h \in V_{h0} \backslash \{0\}} \frac{b_h(v_h, p_h)}{\|v_h\|_{H^1(\Omega)} \|p_h\|_{L^2(\Omega_h)}} \geq \beta_1, \text{ for } P1b/P1. \tag{2.2.33}$$

$$\sup_{v_h \in V_{h0} \backslash \{0\}} \frac{b_h(v_h, p_h)}{\|v_h\|_{H^1(\Omega)}} \geq \tilde{\beta}_1 \|p_h\|_{L^2(\Omega_h)} - \gamma C h \|\nabla p_h\|_{L^2(\Omega_h)}, \quad \forall p_h \in \mathring{Q}_h, \text{ for } P1/P1. \tag{2.2.34}$$

- *inf-sup condition* of c_h defined by (2.2.30):

$$\inf_{\mu_h \in \Lambda_h \backslash \{0\}} \sup_{v_h \in V_h \backslash \{0\}} \frac{\int_{\Gamma_h} v_h \cdot n_h \mu_h}{\|v_h\|_{H^1(\Omega_h)} \|\mu_h\|_{M'}} \geq \gamma_1 > 0. \tag{2.2.35}$$

Finite element penalty scheme.

The finite element approximation to penalty problem (2.2.9) reads as: find $(u_h, p_h) \in V_h \times Q_h$ such that,

$$a_h(u_h, v_h) + b_h(v_h, p_h) + \frac{1}{\epsilon} c_h(u_h \cdot n_h, v_h \cdot n_h) = (\tilde{f}, v_h)_{\Omega_h}, \quad \forall v_h \in X_h, \tag{2.2.36a}$$

$$b_h(u_h, q_h) = d_h(p_h, q_h), \quad \forall q_h \in M_h, \tag{2.2.36b}$$

where $\tilde{f}$ is some extension of f onto $\tilde{\Omega} = \Omega \cup \Omega_h$ with $\|\tilde{f}\|_{L^2(\tilde{\Omega})} \leq C\|f\|_{L^2}$.

In the following we only discuss the $P1b/P1$ element approximation ($\gamma = 0$, $b_h(u_h, q_h) = 0$), since the analysis method and results of $P1/P1$ with stabilization ($b_h(u_h, q_h) = h^2(\nabla p_h, \nabla q_h)$) are very similar to the case of $P1b/P1$.

Well-posedness and a priori estimate

Theorem 2.2.6. *There exists a unique solution* $(u_h, p_h) \in V_h \times Q_h$ *to (2.2.36) with* c_h *defined by both (2.2.30) and (2.2.31), and the solution satisfies*

$$\|u_h\|_{H^1(\Omega_h)} + \|\mathring{p}_h\|_{L^2(\Omega_h)} + \epsilon^{-1/2}\|u_h \cdot n_h\|_{c_h} \leq C\|\tilde{f}\|_{L^2(\Omega_h)}, \tag{2.2.37}$$

where $p_h = \mathring{p}_h + l_h$, $\mathring{p}_h \in \mathring{Q}_h$, $l_h = \int_{\Omega_h} p_h dx / |\Omega_h|$, *and*

$$|l_h| \leq C\left(\|\tilde{f}\|_{L^2(\Omega_h)} + \|u_h\|_{H^1(\Omega_h)} + \|u_h\|^2_{H^1(\Omega_h)} + \frac{h}{\epsilon}\right). \tag{2.2.38}$$

Proof. The existence and uniqueness of solution $(u_h, \mathring{p}_h)$ and (2.2.37) follow from the coercivity of a_h, the *inf-sup conditions* of b_h. Here, we only check the estimate (2.2.38) of l_h. In views of (2.2.36b) of $\gamma = 0$, we obtain, for c_h defined by both (2.2.30) and (2.2.31),

$$c_h(u_h \cdot n_h, 1) = \int_{\Gamma_h} u_h \cdot n_h ds = \sum_{s \in \mathcal{S}_h} |s|(u_h \cdot n_h)(m_s) = -b_h(u_h, 1) = 0. \tag{2.2.39}$$

Since n_h is discontinuous on Γ_h, we cannot choose the trace lifting $v_h \in V_h$ with $v_h = l_h n_h$ on Γ. Let $\{P_i\}_{i=1}^N$ be the set of the vertices of polygon or polyhedral domain Ω_h(nodes of Γ_h), $\Gamma_i = \{s \in \mathcal{S}_h \mid P_i \in \overline{s}\}$(faces/edges contain the vertex P_i), we then define a $v_h \in X_h$ satisfying

$$v_h(P_i) = l_h \frac{1}{\Gamma_i^{\#}} \sum_{s \in \Gamma_i} n_h(s), \quad \|v_h\|_{H^1(\Omega_h)} \leq C l_h,$$

where $\Gamma_i^{\#}$ equals to the number of faces s in Γ_i, and $n_h(s)$ is the value of n_h on s. Since Γ has C^3 smoothness, we have $|v_h - l_h n_h| \leq Ch$ on Γ_h. Then, substituting this v_h into (2.2.36a), it yields,

$$l_h \int_{\Gamma_h} v_h \cdot n_h = -b_h(v_h, l_h) = a_h(u_h, v_h) + b_h(v_h, \mathring{p}_h) + \frac{1}{\epsilon} c_h(u_h \cdot n_h, v_h \cdot n_h).$$

In view of (2.2.39), we have

$$\frac{1}{\epsilon} c_h(u_h \cdot n_h, v_h \cdot n_h) = \frac{l_h}{\epsilon} \underbrace{c_h(u_h \cdot n_h, 1)}_{=0} + \frac{1}{\epsilon} c_h(u_h \cdot n_h, (v_h - l_h n_h) \cdot n_h).$$

Therefore, we have

$$\begin{aligned} l_h^2 |\Gamma_h| &= l_h \int_{\Gamma_h} l_h n_h \cdot n_h = l_h \int_{\Gamma_h} (l_h n_h - v_h + v_h) \cdot n_h \\ &= l_h \int_{\Gamma_h} (l_h n_h - v_h) \cdot n_h + a_h(u_h, v_h) + b_h(v_h, \mathring{p}_h) \\ &\quad + \frac{1}{\epsilon} c_h(u_h \cdot n_h, (v_h - l_h n_h) \cdot n_h), \end{aligned}$$

which implies (2.2.38) since $|v_h - l_h n_h| \leq Ch$ on Γ_h. □

Extension operators and skin domain estimates

We denote the skin domain $\Omega \triangle \Omega_h = (\Omega \backslash \Omega_h) \cup (\Omega_h \backslash \Omega)$, $\tilde{\Omega} := \Omega \cup \Omega_h$.

Lemma 2.2.1 (cf. [29]). *There exists an extension operator*

$$P \in \mathcal{L}(H^m(\Omega)^d, H^m(\mathbb{R}^d)^d), \quad (0 \leq m \in \mathbb{N}0), \quad v \mapsto Pv =: \tilde{v}$$

such that,

$$\|\tilde{v}\|_{H^k(\mathbb{R}^d)} \leq C_m \|v\|_{H^k(\Omega)}, \quad 0 \leq k \leq m, \quad \forall v \in H^m(\Omega)^d.$$

Moreover, if $\nabla \cdot v = 0$, then we can take the extension $\tilde{v}$ satisfying $\nabla \cdot v = 0$ in $\mathbb{R}^d$.

Lemma 2.2.2 (cf. [44, 48, 53]). *Under the assumption* $\max_{x \in \Gamma}$ *dist*$(x, \Gamma_h) \leq Ch^2$, *we have*

$$\|\tilde{v}\|_{H^k(\Omega \triangle \Omega_h)} \leq Ch \|v\|_{H^{k+1}(\Omega)}, \quad 0 \leq k \leq m-1, \quad \forall v \in H^m(\Omega)^d.$$

Lemma 2.2.3 (cf. [44]). *There exists an extension operator* $P_h \in \mathcal{L}(V_h, H^1(\tilde{\Omega}))$, *such that,* $\forall v_h \in V_h$,

$$\|P_h v_h\|_{H^1(\tilde{\Omega})} \leq C\|v_h\|_{H^1(\Omega_h)},$$

$$\|P_h v_h\|_{H^k(\Omega\Delta\Omega_h)} \leq Ch^{\frac{1}{2}}\|v_h\|_{H^k(K_{\Gamma_h})}, \quad k = 0, 1,$$
$$\|P_h v_h\|_{L^2(\tilde{\Omega})} \leq Ch\|v_h\|_{H^1(\Omega_h)},$$

where $K_{\Gamma_h} := \{K \in \mathcal{T}_h \mid \overline{K} \cap \Gamma_h \neq \emptyset\}$.

Lagrange interpolation and projection operators

We employ the Lagrange interpolation operator I_h and projection operator P_{L^2}(cf. [19, 46]).

$$I_h : C(\overline{\Omega_h}) \to V_h, \quad v \mapsto I_h v,$$
$$\|v - I_h v\|_{L^p(\Omega_h)} + h\|v - I_h v\|_{W^{1,p}(\Omega_h)} \leq Ch^2\|v\|_{W^{2,p}(\tilde{\Omega})}, \quad \forall v \in W^{2,p}(\Omega_h).$$
$$P_{L^2} : H^1(\Omega_h) \to V_h, \quad v \mapsto P_{L^2} v,$$
$$(v - P_{L^2} v, v_h)_{L^2(\Omega_h)} = 0, \quad \forall v_h \in V_h,$$
$$\|v - P_{L^2} v\|_{L^2(\Omega_h)} \leq Ch\|v\|_{H^1(\Omega_h)}.$$

Consistency error estimates

Lemma 2.2.4 (cf. [24]). *Let* $\pi \in C^2(\Gamma_h)$, *then we have, for any* $v \in H^1(\tilde{\Omega})$,

(i) $\|v \circ \pi\|_{L^2(\Gamma_h)} \leq C\|v\|_{L^2(\Gamma)}$.

(ii) $|\int_\Gamma v ds - \int_{\Gamma_h} v \circ \pi ds| \leq Ch^2\|v\|^2_{L^2(\Gamma_h)}$.

(iii) $\|v - v \circ \pi\|_{L^2(\Gamma_h)} \leq Ch\|v\|_{H^1(\tilde{\Omega})}$.

Proof. The proof has been derived in [24]. Here, we present a brief proof for the convenience of readers. (i) is obvious. (ii) follows from the properties of π (2.2.25),

$$\int_\Gamma v ds - \int_{\Gamma_h} v \circ \pi ds = \int_{\Gamma_h} v \circ \pi(|D\pi^{-1}| - 1) ds \leq Ch^2\|v\|_{L^2(\Gamma_h)}.$$

(iii) is from [45]((5.1), Verfürth). □

Lemma 2.2.5 (cf. [24]). *Assume* $\lambda \in L^2(\Gamma)$*(resp.* $W^{1,\infty}(\Gamma)$*) for* c_h *defined by (2.2.30) (resp. (2.2.31)), and let* $\tilde{\lambda} = \lambda \circ \pi$*, then we have*

$$|c(v_n, \lambda) - c_h(v \cdot n_h, \tilde{\lambda})| \le Ch\|v\|_{H^1(\tilde{\Omega})}, \quad \forall v \in H^1(\tilde{\Omega})^d. \tag{2.2.40}$$

Proof. For c_h defined by (2.2.30), we have, from (2.2.26) and (iii) of Lemma 2.2.4,

$$\begin{aligned}
&|c(v_n, \lambda) - c_h(v \cdot n_h, \tilde{\lambda})| = |c(v_n, \lambda) - \int_{\Gamma_h} v \cdot n_h \tilde{\lambda} ds| \\
\le &\left| \int_\Gamma v_n \lambda - \int_{\Gamma_h} (v_n \lambda) \circ \pi \right| \\
&+ \left| \int_{\Gamma_h} (v_n \lambda) \circ \pi - v \cdot (n\lambda) \circ \pi + v \cdot (n\lambda) \circ \pi - v \cdot n_h \tilde{\lambda} \right| \\
\le &Ch\|v\|_{H^1(\tilde{\Omega})} \|\lambda\|_{L^2(\Gamma_h)}.
\end{aligned}$$

For c_h defined by (2.2.31), we have

$$\begin{aligned}
&\left| \int_{\Gamma_h} v \cdot n_h \tilde{\lambda} ds - c_h(v \cdot n_h, \tilde{\lambda}) \right| \\
\le &\sum_{s \in \mathcal{S}_h} \int_s v \cdot n_h |\tilde{\lambda} - \tilde{\lambda}(m_s)| ds \le Ch\|v\|_{H^1(\tilde{\Omega})} \|\lambda\|_{W^{1,\infty}(\Gamma)}.
\end{aligned}$$

□

Proposition 2.2.1. *Let* (u, p) *and* (u_h, p_h) *be solutions of (2.2.1) and (2.2.36), respectively. Set* $\lambda = -\tau_n(u, p)$*,* $\lambda_h = \frac{1}{\epsilon} u_h \cdot n_h$*. We assume* $f \in L^2(\Omega)$*, and* $(u, p) \in H^2(\Omega)^d \times H^1(\Omega)$*, and the same assumption of Lemma 2.2.5. For any* $v_h \in V_h$*, we set the consistency error*

$$E(v_h) := a_h(\tilde{u} - u_h, v_h) + b_h(v_h, \tilde{p} - p_h) + c_h(v_h \cdot n_h, \tilde{\lambda} - \lambda_h),$$

where $(\tilde{u}, \tilde{p})$ *is the extension(Lemma 2.2.1) of* (u, p) *onto* $\tilde{\Omega} = \Omega \cup \Omega_h$*. Then, we have*

$$|E(v_h)| \le Ch\|v_h\|_{H^1(\Omega_h)}. \tag{2.2.41}$$

Proof. We denote

$$a_\omega(u, v) := 2\nu(\mathcal{E}(u), \mathcal{E}(v))_\omega,$$

$$b_\omega(v, q) = -(\nabla \cdot v, q)_\omega,$$

for some subset ω of $\tilde{\Omega}$.

From (2.2.7) and (2.2.36), we have

$$\begin{aligned}E(v_h) = &- a_{\Omega\backslash\Omega_h}(u, P_h v_h) + a_{\Omega_h\backslash\Omega}(\tilde{u}, v_h)\\ &- b_{\Omega\backslash\Omega_h}(P_h v_h, u) + b_{\Omega_h\backslash\Omega}(v_h, \tilde{u}) + (f, P_h v_h)_{\Omega\backslash\Omega_h} - (\tilde{f}, v_h)_{\Omega_h\backslash\Omega}\\ &- c(P_h v_h \cdot n, \lambda) + c_h(v_h \cdot n_h, \tilde{\lambda}).\end{aligned}$$

(2.2.41) follows from Lemma 2.2.2, 2.2.3 and 2.2.5. □

2.2.4 Error estimates: nonreduced-integration scheme

Theorem 2.2.7. *c_h is defined by (2.2.30). Let (u, p) and (u_h, p_h) be solutions of (2.2.1) and (2.2.36), respectively. Assuming $f \in L^2(\Omega)$, $(u, p) \in H^2(\Omega)^d \times H^1(\Omega)$, we have*

$$\|\tilde{u} - u_h\|_{H^1(\Omega_h)} + \|\tilde{p} - p_h\|_{L^2(\Omega_h)} \le C(\sqrt{h} + \sqrt{\epsilon} + h/\sqrt{\epsilon}). \tag{2.2.42}$$

Proof. Set $v_h = I_h\tilde{u}$. Since $\|\tilde{u}-u_h\|_{H^1(\Omega_h)} \le \|\tilde{u}-v_h\|_{H^1(\Omega_h)}+\|u_h-v_h\|_{H^1(\Omega_h)}$ and $\|\tilde{u} - v_h\|_{H^1(\Omega_h)} \le Ch\|\tilde{u}\|_{H^2(\tilde{\Omega})}$, we only need to show the estimate of $\|u_h - v_h\|_{H^1(\Omega_h)}$.

$$\begin{aligned}\alpha_1\|u_h - v_h\|^2_{H^1(\Omega_h)} &\le a_h(u_h - v_h, u_h - v_h)\\ &= a_h(v_h - \tilde{u}, v_h - u_h) + a_h(\tilde{u} - u_h, v_h - u_h).\end{aligned} \tag{2.2.43}$$

$$\begin{aligned}&a_h(\tilde{u} - u_h, v_h - u_h)\\ =&E(v_h - u_h) - b_h(v_h - u_h, \tilde{p} - p_h) - c_h((v_h - u_h) \cdot n_h, \tilde{\lambda} - \lambda_h).\end{aligned}$$

In the following, we are aim to prove

$$\begin{aligned}a_h(\tilde{u} - u_h, v_h - u_h) \le Ch\|v_h - u_h\|_{H^1(\Omega_h)}&\\ - \frac{\epsilon}{4}\|\tilde{\lambda} - \lambda_h\|^2_{L^2(\Gamma_h)} + C\frac{h^2}{\epsilon} + \epsilon\|\tilde{\lambda}\|^2_{L^2(\Gamma_h)},&\end{aligned} \tag{2.2.44}$$

which implies (2.2.42).

From Proposition 2.2.1, we have $|E(v_h-u_h)| \le Ch\|v_h-u_h\|_{H^1(\Omega_h)}$. Since we can replace p by $p+l$ for any constant l, we set $\tilde{p}$ satisfies $\tilde{p}-p_h \in L^2_0(\Omega_h)$ and $q_h = P_{L^2}\tilde{p}$, $q_h - p_h \in \mathring{Q}_h$. With $b_h(u_h, q_h) = 0$ and $\nabla \cdot \tilde{u} = 0$, we have

$$\begin{aligned}&- b_h(v_h - u_h, \tilde{p} - p_h)\\ =&b_h(\tilde{u} - v_h, \tilde{p} - q_h) + b_h(\tilde{u} - v_h, q_h - p_h) + b_h(u_h, \tilde{p} - q_h)\\ =&b_h(\tilde{u} - v_h, q_h - p_h) - b_h(v_h - u_h, \tilde{p} - q_h)\\ \le&Ch\|\tilde{u}\|_{H^2(\tilde{\Omega})}\|q_h - p_h\|_{L^2(\Omega_h)} + Ch\|\tilde{p}\|_{H^1(\tilde{\Omega})}\|v_h - u_h\|_{H^1(\Omega_h)}.\end{aligned}$$

Since $q_h - p_h \in \mathring{Q}_h$, by *inf-sup condition* of b_h, we obtain

$$\|q_h - p_h\|_{L^2(\Omega_h)} \le Ch(\|\tilde{u}\|_{H^2(\tilde{\Omega})} + \|\tilde{p}\|_{H^1(\tilde{\Omega})}) + C\|v_h - u_h\|_{H^1(\Omega_h)}.$$

Therefore, we have $|b_h(v_h - u_h, \tilde{p} - p_h)| \le Ch^2 + Ch\|v_h - u_h\|_{H^1(\Omega_h)}$. We are left to estimate $-c_h((v_h - u_h)\cdot n_h, \tilde{\lambda} - \lambda_h)$. In views of $\lambda_h = \frac{1}{\epsilon} u_h \cdot n_h$,

$$\begin{aligned}
&- c_h((v_h - u_h)\cdot n_h, \tilde{\lambda} - \lambda_h) = -\epsilon c_h(\tilde{\lambda} - \lambda_h, \tilde{\lambda} - \lambda_h) + \epsilon c_h(\tilde{\lambda}, \tilde{\lambda} - \lambda_h) \\
&\quad + c_h((\tilde{u} - v_h)\cdot n_h, \tilde{\lambda} - \lambda_h) - c_h(\tilde{u}\cdot n_h, \tilde{\lambda} - \lambda_h) \\
&\le -\epsilon\|\tilde{\lambda} - \lambda_h\|^2_{L^2(\Gamma_h)} + \epsilon\|\tilde{\lambda}\|^2_{L^2(\Gamma_h)} + \frac{\epsilon}{4}\|\tilde{\lambda} - \lambda_h\|^2_{L^2(\Gamma_h)} \\
&\quad + \frac{1}{\epsilon}\|(\tilde{u} - v_h)\cdot n_h\|^2_{L^2(\Gamma_h)} + \frac{1}{\epsilon}\|\tilde{u}\cdot n_h\|^2_{L^2(\Gamma_h)} + \frac{\epsilon}{2}\|\tilde{\lambda} - \lambda_h\|^2_{L^2(\Gamma_h)}.
\end{aligned} \tag{2.2.45}$$

Since $\|(\tilde{u} - v_h)\cdot n_h\|_{L^2(\Gamma_h)} \le C\|\tilde{u} - v_h\|_{H^1(\tilde{\Omega})} \le Ch\|\tilde{u}\|_{H^2(\tilde{\Omega})}$ and

$\|\tilde{u}\cdot n_h\|_{L^2(\Gamma_h)} \le \|\tilde{u}\cdot(n_h - n\circ\pi) + (\tilde{u} - u\circ\pi)n\circ\pi\|_{L^2(\Gamma_h)} \le Ch, \quad (\because u_n|_\Gamma = 0)$

it yields

$$-c_h((v_h - u_h)\cdot n_h, \tilde{\lambda} - \lambda_h) \le -\frac{\epsilon}{4}\|\tilde{\lambda} - \lambda_h\|^2_{L^2(\Gamma_h)} + C\frac{h^2}{\epsilon} + \epsilon\|\tilde{\lambda}\|^2_{L^2(\Gamma_h)},$$

Combining those inequalities, we proved (2.2.44). From (2.2.43), (2.2.44), we conclude (2.2.42). □

2.2.5 Error estimates: reduced-integration scheme

Lemma 2.2.6 (cf. [24]). *Let $u \in W^{2,\infty}(\Omega)$ with $u_n|_\Gamma = 0$. For any $s \in \mathcal{S}_h$, $\tilde{u}$ is the extension of u according to Lemma 2.2.1, then we have*

(i) For $d = 2$, there exists π such that $|n\circ\pi(m_s) - n_h(m_s)| \le Ch^2$; moreover

$$|(I_h\tilde{u}\cdot n_h)(m_s)| \le Ch^2\|\tilde{u}\|_{W^{2,\infty}(\tilde{\Omega})}.$$

(ii) For $d = 3$, if $\tilde{u} \in W^{2,\infty}(\tilde{\Omega})$ satisfies $\nabla\cdot\tilde{u} = 0$, and $\tilde{u}_n = 0$ on Γ, then we have $|(I_h\tilde{u}\cdot n_h)(m_s)| \le Ch\|\tilde{u}\|_{W^{2,\infty}(\tilde{\Omega})}$.

Proof. (i) For $d = 2$, since Γ has C^3 smoothness, there exists $\pi : \Gamma_h \to \Gamma$ satisfying $|n\circ\pi(m_s) - n_h(m_s)| \le Ch^2$ is obvious. In view of $\tilde{u}_n = 0$ on Γ, we have

$$\begin{aligned}
&|(I_h\tilde{u}\cdot n_h)(m_s)| \\
\le&|(I_h\tilde{u}\cdot n_h)(m_s) - I_h\tilde{u}(m_s)\cdot n\circ\pi(m_s)| \\
&\quad + |I_h\tilde{u}(m_s)\cdot n\circ\pi(m_s) - (\tilde{u}_n)\circ\pi(m_s)| \\
\le&Ch^2\|\tilde{u}\|_{W^{1,\infty}(\tilde{\Omega})} + Ch^2\|\tilde{u}\|_{W^{2,\infty}(\tilde{\Omega})}.
\end{aligned}$$

(ii) It follows from (2.2.26) and the fact $\tilde{u}_n = 0$ on Γ. $\square$

Theorem 2.2.8. *Let (u, p) and (u_h, p_h) be the unique solutions of (2.2.1) and (2.2.36), respectively. We assume $f \in L^2(\Omega)$, $(u, p) \in W^{2,\infty}(\Omega)^d \times W^{1,\infty}(\Omega)$. We also assume $(\tilde{u}, \tilde{p})$, the extension of (u, p), satisfies (i)(ii) of Lemma 2.2.6, then we have*

$$\|\tilde{u}-u_h\|_{H^1(\Omega_h)}+\|\tilde{p}-p_h\|_{L^2(\Omega_h)} \leq C(h+\sqrt{\epsilon}+h^2/\sqrt{\epsilon}), \quad \textit{for } d=2, \tag{2.2.46}$$

$$\|\tilde{u}-u_h\|_{H^1(\Omega_h)}+\|\tilde{p}-p_h\|_{L^2(\Omega_h)} \leq C(\sqrt{h}+\sqrt{\epsilon}+h/\sqrt{\epsilon}), \quad \textit{for } d=3. \tag{2.2.47}$$

Proof. In views of the proof of Theorem 2.2.7, the only difference here is the estimate of $-c_h((v_h - u_h) \cdot n_h, \tilde{\lambda} - \lambda_h)$ in (2.2.45). We have, noticing that $v_h = I_h \tilde{u}$,

$$\begin{aligned}
& - c_h((v_h - u_h) \cdot n_h, \tilde{\lambda} - \lambda_h) + \epsilon c_h(\tilde{\lambda} - \lambda_h, \tilde{\lambda} - \lambda_h) \\
=& \epsilon c_h(\tilde{\lambda}, \tilde{\lambda} - \lambda_h) - c_h(v_h \cdot n_h, \tilde{\lambda} - \lambda_h) \\
\leq& -\frac{\epsilon}{2}\|\tilde{\lambda} - \lambda_h\|_{c_h}^2 + C\epsilon\|\tilde{\lambda}\|_{c_h}^2 + C\frac{1}{\epsilon}\|I_h\tilde{u} \cdot n_h\|_{L^\infty(\Gamma_h)}^2.
\end{aligned} \tag{2.2.48}$$

The error estimates (2.2.46) and (2.2.47) follow from Lemma 2.2.6. $\square$

Remark 2.2.4. For $d = 2$, from the error estimates (2.2.42) and (2.2.46), we conclude the optimal choices of ϵ and h:

(1) Nonreduced-integration scheme: $\epsilon \simeq h$, and the error estimate is $O(\sqrt{h})$;

(2) Reduced-integration scheme: $\epsilon \simeq h^2$, and the error estimate is $O(h)$.

And we notice that for nonreduced-integration, if $\epsilon \ll h$, then the scheme is not convergence. For $d = 3$, we choose $\epsilon \simeq h$, and the error estimate is $O(\sqrt{h})$.

2.2.6 Numerical examples

Let $\Omega = \{(x, y) \mid 1 < x^2 + y^2 < 4\}$, with

$$D = \{(x, y) \mid x^2 + y^2 = 1\}, \quad \Gamma = \{(x, y) \mid x^2 + y^2 = 1\}.$$

We consider the Stokes problem in Ω with solution:

$$u = (x^2 + y^2 - 1)(y, -x)^T, \quad p = xy.$$

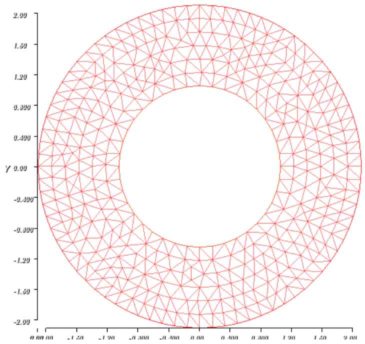

Figure 2.2.2: Ω and mesh

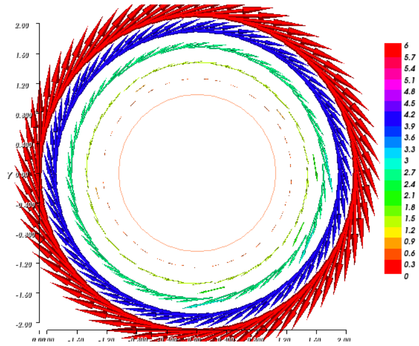

Figure 2.2.3: u

We see that $u|_D = 0$ and $u_n|_\Gamma = 0$, for $n = (x, y)^T$ on Γ. Here, $\tau_T(u) = H \neq 0$, therefore, we have to add $\int_\Gamma H v_T \, ds$ to the RHS of the variational form (2.2.7), and make some corresponding changes to the penalty problem (2.2.9), and the finite element schemes.

We show some figures of mesh (see Figure 2.2.2) and solutions. Figure 2.2.3 is the exact solution u.

Figure 2.2.4 is the numerical solution of reduced-integration scheme, with $\epsilon = 0.1h^2$.

Figure 2.2.5 is the numerical solution of non-reduced-integration scheme, with $\epsilon = 0.1h$.

Figure 2.2.6 is the numerical solution of non-reduced-integration scheme, with $\epsilon = 0.01h^2$, which fails to approximate the exact solution.

We show the error estimates results for both reduced and non-reduced-integration scheme.

Figure 2.2.7 shows the errors of $\|u_h - u\|_{L^2}$, $\|u_h - u\|_{H^1}$ and $\|p_h - p\|_{L^2/\mathbb{R}}$, when $\epsilon = 0.1h$. We observe the $O(h)$ convergence of u in H^1-norm.

Figure 2.2.8 shows the errors of $\|u_h - u\|_{L^2}$, $\|u_h - u\|_{H^1}$ and $\|p_h - p\|_{L^2/\mathbb{R}}$, when $\epsilon = 0.1h^2$. And it fails to converge.

Figure 2.2.9 shows the errors of $\|u_h - u\|_{L^2}$, $\|u_h - u\|_{H^1}$ and $\|p_h - p\|_{L^2/\mathbb{R}}$, when $\epsilon = 0.1h$. We see the error of $u_h - u$ in H^1-norm is bounded by $O(h)$.

Figure 2.2.10 shows the errors of $\|u_h - u\|_{L^2}$, $\|u_h - u\|_{H^1}$ and $\|p_h - p\|_{L^2/\mathbb{R}}$, when $\epsilon = 0.1h^2$. We observe the error estimates $\|u - u_h\|_{L^2} \leq Ch^2$ and $\|u - u_h\|_{H^1} \leq Ch$.

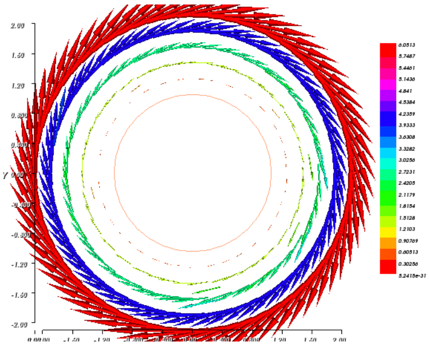

Figure 2.2.4: u_h: reduced

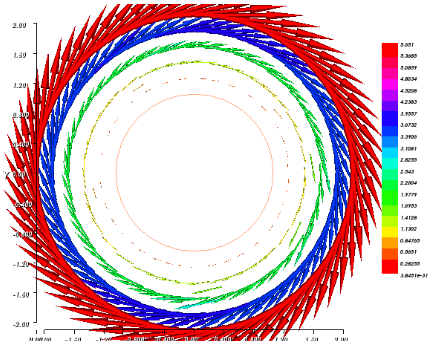

Figure 2.2.5: u_h: nonreduced

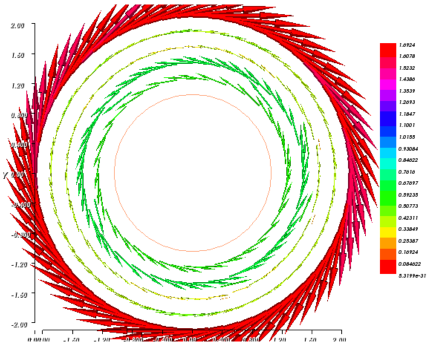

Figure 2.2.6: u_h: nonreduced, $\epsilon = 0.01h^2$

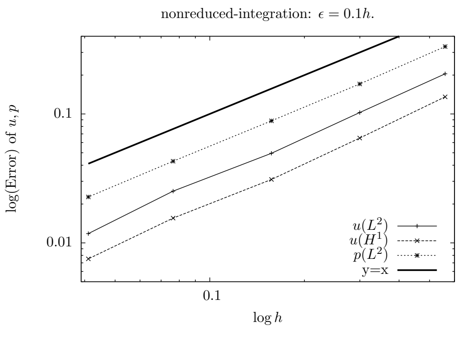

Figure 2.2.7: *nonreduced*, $\epsilon = 0.1h$

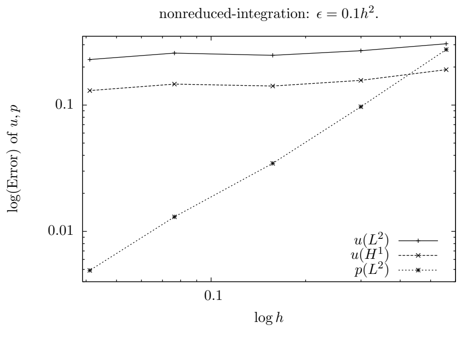

Figure 2.2.8: *nonreduced*, $\epsilon = 0.1h^2$

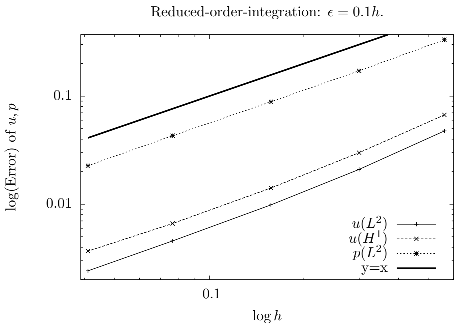

Figure 2.2.9: *reduced-order*, $\epsilon = 0.1h$

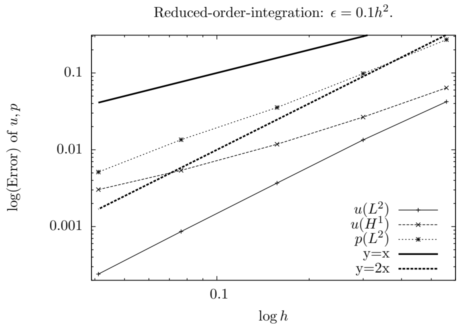

Figure 2.2.10: *reduced-order*, $\epsilon = 0.1h^2$

2.3 The penalty method to the non-stationary Navier-Stokes problem

Variational form of (2.1.1).

Find $(u(t), p(t)) \in V_n \times \mathring{Q}$, with $u'(t) \in L^2(\Omega)^d$, for any $t \in (0, T)$, such that,

$$(u', v) + a(u, v) + a_1(u, u, v) + b(v, p) = (f, v), \quad \forall v \in V_n, \tag{2.3.1a}$$
$$b(u, q) = 0, \quad \forall q \in \mathring{Q}, \tag{2.3.1b}$$
$$u(0, x) = u_0. \tag{2.3.1c}$$

Assumptions.

(A) The initial value u_0 and f satisfies,

(i) $f \in H^1(0, T; L^2(\Omega)^d)$;

(ii) $u_0 \in H^2(\Omega)^d \cap V_n^\sigma$, such that we have the *compatibility condition*

$$a(u_0, v) = -\nu(\Delta u_0, v), \quad \forall v \in V_n^\sigma. \tag{2.3.2}$$

Lemma 2.3.1 (The well-posedness of (2.3.1))**.** *Under the assumptions* **(A)** *and $\partial\Omega$ is of C^3-class, when $d = 2$, for any $T \in (0, \infty)$, there exists a unique solution (u, p) to (2.3.1) satisfying*

$$\|u\|_{L^\infty(0,T;H^2)} + \|u'\|_{L^\infty(0,T;L^2(\Omega)^d)} + \|u'\|_{L^2(0,T;V_n^\sigma)} \leq C, \tag{2.3.3}$$

$$\|p\|_{L^\infty(0,T;L_0^2(\Omega))} \le C, \tag{2.3.4}$$

where C depends on Ω, f and u_0. When $d = 3$, the conclusion holds for a small time interval $(0, T')$.

Lemma 2.3.2 (The regularity of (2.3.1))**.** *Let (u, p) be the solution of (2.3.1) satisfies Lemma 2.3.1. Assume $\partial\Omega$ is of C^{m+2} class, m, s are integers, with $2s \le m$, and u_0, $f^{(s)} = \partial^s f/\partial t^s$, satisfy*

$$u_0 \in H^m(\Omega)^d \cap V_n^\sigma, \quad f^{(s)} \in L^2(0,T; H^{m-2s-1}(\Omega)^d).$$

We also assume the compatibility condition

$$u^{(k)}|_D = 0, \quad u_n^{(k)}|_\Gamma = 0, \quad \tau_T(u^{(k)})|_\Gamma = 0, \quad k = 0, \dots, s. \tag{2.3.5}$$

Then we have

$$\|u^{(s)}\|_{L^2(0,T;H^{m-2s+1}(\Omega)^d)} + \|u^{(s)}\|_{L^\infty(0,T;H^{m-2s}(\Omega)^d)} \le C, \tag{2.3.6}$$

$$\|p^{(s)}\|_{L^2(0,T;H^{m-2s})} \le C. \tag{2.3.7}$$

The well-posedness and regularity of Navier-Stokes problem with Dirichlet boundary condition are well known (cf. [7, 22, 43]). With a similar argument to the case of the Dirichlet boundary condition, one can prove Lemma 2.3.1 and Lemma 2.3.2. We write the weak form of penalty problem (2.1.4). Find $(u_\epsilon(t), p_\epsilon(t)) \in V \times Q$, with $u'_\epsilon(t) \in L^2(\Omega)^d$, for all $t \in (0, T)$ such that

$$\begin{aligned}&(u'_\epsilon, v) + a(u_\epsilon, v) + a_1(u_\epsilon, u_\epsilon, v) + b(v, p_\epsilon) + \frac{1}{\epsilon}c(u_{\epsilon n}, v_n)\\ &\qquad = (f, v), \quad \forall v \in V,\end{aligned} \tag{2.3.8a}$$

$$b(u_\epsilon, q) = 0, \quad \forall q \in Q, \tag{2.3.8b}$$

$$u_\epsilon(0, x) = u_{\epsilon 0}, \tag{2.3.8c}$$

2.3.1 The well-posedness of penalty problem

Assumption.

(A′ii) The initial value $u_{\epsilon 0}$ satisfies $u_{\epsilon 0} \in V^\sigma \cap H^2(\Omega)^d$, and the *compatibility condition*

$$a(u_{\epsilon 0}, v) + \frac{1}{\epsilon}c(u_{\epsilon 0} \cdot n, v_n) = -\nu(\Delta u_{\epsilon 0}, v), \quad \forall v \in V^\sigma, \tag{2.3.9}$$

which also implies $\|u_{\epsilon 0} \cdot n\|_{L^2(\Gamma)} \le C\sqrt{\epsilon}$.

Theorem 2.3.1 (The well-posedness and regularity of (2.3.8)). *We assume* **(Ai)(A′ii)**, *and* $\partial\Omega$ *is of* C^2 *class, then we have, when* $d = 2$, *for any* $T \in (0,\infty)$, *there exists a unique solution* (u_ϵ, p_ϵ) *to (2.2.9) for sufficiently small* ϵ, *which satisfies*

$$\|u_\epsilon\|_{L^\infty(0,T;V^\sigma\cap H^2)} + \|u'_\epsilon\|_{L^\infty(0,T;L^2)} + \|u'_\epsilon\|_{L^2(0,T;V^\sigma)} \le C, \tag{2.3.10}$$

$$\|p_\epsilon\|_{L^\infty(0,T;L^2)} \le C, \tag{2.3.11}$$

where C *depends on* Ω, f *and* $u_{\epsilon 0}$.

When $d = 3$, *the same conclusion holds for a small time interval* $(0, T')$.

We introduce the variational equation without p_ϵ.

Find $u_\epsilon(t) \in V^\sigma$, with $u'_\epsilon(t) \in L^2(\Omega)^d$, for all $t \in (0,T)$ such that

$$\begin{aligned} &(u'_\epsilon, v) + a(u_\epsilon, v) + a_1(u_\epsilon, u_\epsilon, v) + \frac{1}{\epsilon}c(u_{\epsilon n}, v_n) \\ &\quad = (f, v), \quad \forall v \in V^\sigma, \end{aligned} \tag{2.3.12a}$$

$$u_\epsilon(0,x) = u_{\epsilon 0}, \tag{2.3.12b}$$

We see that u_ϵ of (2.3.8) satisfies (2.3.12).

Proposition 2.3.1 (The existence of p_ϵ). *Let* u_ϵ *be the solution of (2.3.12) with (2.3.3), then there exists a unique* p_ϵ, *such that* (u_ϵ, p_ϵ) *is the solution of (2.3.8) and* p_ϵ *satisfies (2.3.4).*

Proof. From the *inf-sup condition* of b (2.2.5), there exists a unique $\mathring{p}_\epsilon \in \mathring{Q}$ such that

$$\begin{aligned} -b(v, \mathring{p}_\epsilon) = &(u'_\epsilon, v) + a(u_\epsilon, v) + a_1(u_\epsilon, u_\epsilon, v) + b(v, \mathring{p}_\epsilon) \\ &- (f, v), \quad \forall v \in H_0^1(\Omega)^d, \end{aligned} \tag{2.3.13}$$

and $\mathring{p}_\epsilon$ satisfies, for any $t \in (0,T)$ (for $d = 3$, T is replaced by T'),

$$\|\mathring{p}_\epsilon(t)\|_{L^2} \le C(\|u'_\epsilon(t) + (u_\epsilon \cdot \nabla u_\epsilon)(t) - f(t)\|_{H^{-1}} + \|u_\epsilon(t)\|_{H^1}), \tag{2.3.14}$$

where $H^{-1}(\Omega)^d = (H_0^1(\Omega)^d)^*$.

Next, we find some function $l_\epsilon(t) \in \mathbb{R}$, such that $p_\epsilon = \mathring{p}_\epsilon + l_\epsilon$ is the solution to (2.3.8). To do so, we choose any $\phi \in V$ with $\phi_n|_\Gamma = 1$, and define l_ϵ by

$$\begin{aligned} l_\epsilon|\Gamma| = l_\epsilon \int_\Gamma \phi_n ds = -b(\phi, l_\epsilon) \\ = - b(\phi, \mathring{p}_\epsilon) + (u'_\epsilon, \phi) + a(u_\epsilon, \phi) + a_1(u_\epsilon, u_\epsilon, \phi) - (f, \phi), \end{aligned} \tag{2.3.15}$$

then $(u_\epsilon, \mathring{p}_\epsilon + l_\epsilon)$ satisfies (2.3.8). From (2.3.13), we see that the l_ϵ determined by (2.3.15) is unique (independent on the choice of ϕ).

To show the boundedness of l_ϵ, we substitute $v = w \in V$ into (2.3.8) with $w_n|_\Gamma = l_\epsilon n$ and $\|w\|_{H^1} \le C|l_\epsilon|$, and we have

$$\begin{aligned} |l_\epsilon|^2|\Gamma| = l_\epsilon \int_\Gamma w_n ds = -b(w, l_\epsilon) \\ = -\, b(w, \mathring{p}_\epsilon) + (u'_\epsilon, w) + a(u_\epsilon, w) + a_1(u_\epsilon, u_\epsilon, w) - (f, w), \end{aligned} \tag{2.3.16}$$

which implies, for all $t \in (0, T)$,

$$|l_\epsilon(t)| \le C(\|\mathring{p}_\epsilon(t)\|_{L^2} + \|u'_\epsilon(t) + (u_\epsilon \cdot \nabla u_\epsilon)(t) - f(t)\|_{H^{-1}} + \|u_\epsilon(t)\|_{H^1}). \tag{2.3.17}$$

We complete the proof. □

Proposition 2.3.2 (The uniqueness of u_ϵ)**.** *If there exist two solutions u_ϵ^1 and u_ϵ^2 to (2.3.12) with (2.3.3), then $u_\epsilon^1 = u_\epsilon^2$.*

Proof. It follows from the standard argument (cf. [23, Proposition 3.1],[43]). □

Proof of Theorem 2.3.1. We only need to show the existence of solution u_ϵ to (2.3.12) with (2.3.3). The existence of p_ϵ and the uniqueness of solution follow from Proposition 2.3.1 and 2.3.2.

We apply the Galerkin's approximation method. There exists a linear base $\{w_k\}_{k=1}^\infty$ to V^σ with $w_1 = u_{\epsilon 0}$, such that $\overline{\cup_{m=1}^\infty \text{span}\{w_k\}_{k=1}^m}$ is dense in V^σ. For $m \in \mathbb{N}_+$, we consider the Galerkin's approximation problem to (2.3.12): find $u_{\epsilon m} = \sum_{k=1}^m c_k(t) w_k$, with $c_k(t) \in C^2([0,T])$, such that $u_{\epsilon m}(0) = u_{\epsilon 0}$, and

$$\begin{aligned} (u'_{\epsilon m}, w_k) + a(u_{\epsilon m}, w_k) + a_1(u_{\epsilon m}, u_{\epsilon m}, w_k) + \frac{1}{\epsilon} c(u_{\epsilon mn}, w_{kn}) \\ = (f, w_k), \quad \forall k = 1, \ldots, m, \end{aligned} \tag{2.3.18}$$

where $u_{\epsilon mn} = u_{\epsilon m} \cdot n$ and $w_{kn} = w_k \cdot n$. We see that

$$a_1(u_{\epsilon m}, u_{\epsilon m}, u_{\epsilon m}) = \frac{1}{2} \int_\Gamma u_{\epsilon mn} |u_{\epsilon m}|^2 ds \le c_1 \|u_{\epsilon mn}\|_{L^2(\Gamma)} \|u_{\epsilon m}\|_{H^1}^2.$$

Multiplying (2.3.18) with $c_k(t)$ and taking the summation of k, it yields,

$$\frac{1}{2}\frac{d}{dt}\|u_{\epsilon m}\|_{L^2}^2 + (\alpha - c_1\|u_{\epsilon mn}\|_{L^2(\Gamma)})\|u_{\epsilon m}\|_{H^2}^2 + \frac{1}{\epsilon}\|u_{\epsilon mn}\|_{L^2(\Gamma)}^2 \le (f, u_{\epsilon m}). \tag{2.3.19}$$

Since $\|u_{\epsilon mn}(0)\|_{L^2(\Gamma)} = \|u_{\epsilon 0} \cdot n\|_{L^2(\Gamma)} \leq C\sqrt{\epsilon}$, for sufficiently small ϵ, there exists a maximum time $T_1 > 0$, such that

$$\alpha - c_1\|u_{\epsilon mn}\|_{L^2(\Gamma)} \geq \alpha/2, \quad \forall t \in [0, T_1]. \tag{2.3.20}$$

From (2.3.19) and (2.3.20), we have

$$\|u_{\epsilon m}\|^2_{L^\infty(0,T_1;L^2)} + \|u_{\epsilon m}\|^2_{L^2(0,T_1;V^\sigma)} + \epsilon^{-1}\|u_{\epsilon mn}\|^2_{L^2(0,T_1;L^2(\Gamma))} \leq C. \tag{2.3.21}$$

Differentiating (2.3.18) with respect to t, multiplying it with $c'_k(t)$ and taking the summation of k, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u'_{\epsilon m}\|^2_{L^2} + (\alpha - c_1\|u_{\epsilon mn}\|_{L^2(\Gamma)})\|u'_{\epsilon m}\|^2_{H^1} + \frac{1}{\epsilon}\|u'_{\epsilon mn}\|^2_{L^2(\Gamma)} \\ &\quad \leq (f', u'_{\epsilon m}) - a_1(u'_{\epsilon m}, u_{\epsilon m}, u'_{\epsilon m}). \end{aligned} \tag{2.3.22}$$

From the *compatibility condition* (2.3.9), we see that

$$\begin{aligned} (u'_{\epsilon m}(0), u'_{\epsilon m}(0)) = &(\nu\Delta u_{\epsilon 0}, u'_{\epsilon m}(0)) \\ &- a_1(u_{\epsilon 0}, u_{\epsilon 0}, u'_{\epsilon m}(0)) - (f(0), u'_{\epsilon m}(0)), \end{aligned} \tag{2.3.23}$$

which shows

$$\|u'_{\epsilon m}(0)\|_{L^2} \leq C(\|u_{\epsilon 0}\|_{H^2} + \|f(0)\|_{L^2} + \|u_{\epsilon 0} \cdot \nabla u_{\epsilon 0}\|_{L^2}). \tag{2.3.24}$$

(1) Let us consider the case of $d = 2$. From (2.3.22) and Sobolev's inequality, we have, for arbitrary $\eta_0 > 0$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u'_{\epsilon m}\|^2_{L^2} + (\alpha - c_1\|u_{\epsilon mn}\|_{L^2(\Gamma)} - \eta_0)\|u'_{\epsilon m}\|^2_{H^1} + \frac{1}{\epsilon}\|u'_{\epsilon mn}\|^2_{L^2(\Gamma)} \\ \leq &\|f'\|_{L^2}\|u'_{\epsilon m}\|_{L^2} + C\eta_0^{-1}\|u_{\epsilon m}\|^2_{H^1}\|u'_{\epsilon m}\|^2_{L^2}, \end{aligned} \tag{2.3.25}$$

which implies

$$\|u'_{\epsilon m}\|^2_{L^\infty(0,T_1;L^2)} + \|u'_{\epsilon m}\|^2_{L^2(0,T_1;V^\sigma)} + \epsilon^{-1}\|u'_{\epsilon mn}\|^2_{L^2(0,T_1;L^2(\Gamma))} \leq C. \tag{2.3.26}$$

Multiplying (2.3.18) with $c'_k(t)$ and taking summation of k, it yields

$$\begin{aligned} &\|u'_{\epsilon m}\|^2_{L^2} + \frac{1}{2}\frac{d}{dt}a(u_{\epsilon m}, u_{\epsilon m}) + \frac{1}{\epsilon}\frac{1}{2}\frac{d}{dt}c(u_{\epsilon mn}, u_{\epsilon mn}) \\ \leq &\|f'\|_{L^2}\|u'_{\epsilon m}\|_{L^2} + C\|u'_{\epsilon m}\|_{H^1}\|u_{\epsilon m}\|^2_{H^1}. \end{aligned} \tag{2.3.27}$$

From (2.3.26) and (2.3.27), we conclude

$$\|u'_{\epsilon m}\|^2_{L^2(0,T_1;L^2)} + \|u_{\epsilon m}\|^2_{L^\infty(0,T_1;V^\sigma)} + \epsilon^{-1}\|u_{\epsilon mn}\|^2_{L^\infty(0,T_1;L^2(\Gamma))} \leq C. \tag{2.3.28}$$

Therefore, $\|u_{\epsilon mn}(T_1)\|_\Gamma \le C\sqrt{\epsilon}$, and for sufficiently small ϵ, there exists a time $T_2 > T_1$, such that $\alpha - c_1\|u_{\epsilon mn}\|_\Gamma \ge \alpha/2$ for all $t \in [0, T_2]$. With the same argument from (2.3.20) with T_1 replaced by T_2, we show the solution $u_{\epsilon m}$ exists in time interval $(0, T_2]$ satisfying (2.3.21), (2.3.26) and (2.3.28) with T_1 replaced by T_2.

By induction method, we continue this process with a sufficiently small ϵ to reach a time $T_k \ge T$, such that u_ϵ exists in $[0, T_k]$, and satisfies (2.3.21), (2.3.26) and (2.3.28) with T_1 replaced by T_k.

Hence, there exists a subsequence $\{u_{\epsilon m}\}_{m=1}^\infty$ such that, as $m \to \infty$,

$$u_{\epsilon m} \to u_\epsilon, \ \text{weakly* in } L^\infty(0,T;V^\sigma),$$

$$u'_{\epsilon m} \to u'_\epsilon, \ \text{weakly* in } L^\infty(0,T;L^2(\Omega)^d), \ \text{weakly in } L^2(0,T;V^\sigma),$$

and u_ϵ is the solution of (2.3.8) with

$$\|u_\epsilon\|_{L^\infty(0,T;V^\sigma)} + \|u'_\epsilon\|_{L^\infty(0,T;L^2)\cap L^2(0,T;V^\sigma)} \le C,$$

Follows form the same argument of [43, Theorem 3.6], we can obtain

$$\|u_\epsilon\|_{L^\infty(0,T;H^2)} \le C,$$

which complete the proof of case $d = 2$.

(2) When $d = 3$, the argument before (2.3.25) is the same. From (2.3.22) and Sobolev's inequality, we have, for arbitrary $\eta_0 > 0$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u'_{\epsilon m}\|_{L^2}^2 + (\alpha - c_1\|u_{\epsilon mn}\|_{L^2(\Gamma)} - \eta_0\|u_{\epsilon m}\|_{H^1})\|u'_{\epsilon m}\|_{H^1}^2 \\ &\quad + \frac{1}{\epsilon}\|u'_{\epsilon mn}\|_{L^2(\Gamma)}^2 \le \|f'\|_{L^2}\|u'_{\epsilon m}\|_{L^2} + C\eta_0^{-3}\|u_{\epsilon m}\|_{H^1}\|u'_{\epsilon m}\|_{L^2}^2. \end{aligned} \tag{2.3.29}$$

For sufficiently small η_0 and ϵ, there exists $T'_1 > 0$ such that

$$\alpha - c_1\|u_{\epsilon mn}\|_{L^2(\Gamma)} - \eta_0\|u_{\epsilon m}\|_{H^1} \ge \alpha/2, \quad \forall t \in [0, T'_1]. \tag{2.3.30}$$

From (2.3.29) and (2.3.30), we obtain (2.3.26), and furthermore (2.3.28), with T_1 replaced by T'_1. With a similar argument to the case of $d = 2$ from (2.3.28), we conclude the existence of u_ϵ in $(0, T']$, where T' is the maximum time such that $\sup_{t\in(0,T')}\|u_\epsilon(t)\|_{H^1} < \infty$. □

Remark 2.3.1. When $d = 3$, the solution u_ϵ exists locally in time. For sufficiently small initial value $u_{\epsilon 0}$ and f, one can prove the existence of solution u_ϵ in $(0, \infty)$.

2.3.2 The error estimates of penalty

We show the error estimates of $u_\epsilon - u$.

Recalling that $l_\epsilon(t) = \frac{1}{|\Omega|}\int_\Omega p_\epsilon(t)dx$, and $\mathring{p}_\epsilon(t) = p_\epsilon(t) - l_\epsilon(t) \in \mathring{Q}$, we set

$$\lambda = -\tau_n(u,p)|_\Gamma, \quad \lambda_\epsilon = \epsilon^{-1}u_{\epsilon n}|_\Gamma - l_\epsilon(t).$$

We shall study the estimates of

$$e_u(t) = u(t) - u_\epsilon(t), \quad e_p(t) = p(t) - \mathring{p}_\epsilon(t),$$
$$e_\lambda(t) = \lambda(t) - \lambda_\epsilon(t).$$

We assume the error of initial value

$$\|e_u(0)\|_{H^2} = \|u_0 - u_{\epsilon 0}\|_{H^2} \leq C\epsilon. \tag{2.3.31}$$

Error estimates at $t = 0$

Subtracting (2.3.8) from (2.3.1) at $t = 0$ yields,

$$\mathcal{P}(u'(0) - u'_\epsilon(0)) = \nu\mathcal{P}\Delta(u_0 - u_{\epsilon 0}) - \mathcal{P}(u_0 \cdot \nabla u_0 - u_{\epsilon 0} \cdot \nabla u_{\epsilon 0}),$$

which implies, from the assumption (2.3.31),

$$\|e'_u(0)\|_{L^2} \leq C\|u_0 - u_{\epsilon 0}\|_{H^2} \leq C\epsilon. \tag{2.3.32}$$

Then, from the *inf-sup conditions* (2.2.5), (2.2.6), and

$$\begin{aligned}(e'_u(0), v) + a(e_u(0), v) + b(v, e_p(0)) + c(e_\lambda(0), v_n)\\ + a_1(e_u(0), u_0, v) + a_1(u_{\epsilon 0}, e_u(0), v) = 0, \quad v \in V,\end{aligned} \tag{2.3.33}$$

we have

$$\|e_p(0)\|_{L^2} \leq C(\|e'_u(0)\|_{L^2} + \|e_u(0)\|_{H^1}) \leq C\epsilon, \tag{2.3.34}$$

$$\|e_\lambda(0)\|_{H^{-1/2}} \leq C(\|e'_u(0)\|_{L^2} + \|e_u(0)\|_{H^1} + \|e_p(0)\|_{L^2}) \leq C\epsilon. \tag{2.3.35}$$

Substituting $v = e_u(0)$ into (2.3.33), it yields,

$$\begin{aligned}\epsilon\|e_\lambda(0)\|^2_{L^2(\Gamma)} =& \epsilon c(e_\lambda(0), \lambda + l_\epsilon) - (e'_u(0), e_u(0))\\ &- a(e_u(0), e_u(0)) - a_1(e_u(0), u_0, e_u(0)) + a_1(u_{\epsilon 0}, e_u(0), e_u(0))\\ \leq& C\epsilon\|e_\lambda(0)\|_{H^{-1/2}} + \|e'_u(0)\|_{L^2}\|e_u(0)\|_{L^2} + C\|e_u(0)\|^2_{H^1} \leq C\epsilon^2,\end{aligned}$$

which shows

$$\|e_\lambda(0)\|^2_{L^2(\Gamma)} \leq C\epsilon. \tag{2.3.36}$$

Theorem 2.3.2. *Let (u,p) and (u_ϵ, p_ϵ) be the unique solutions to (2.3.1) and (2.3.8), respectively. Under the assumption that*

$$\tau_n(u,p) \in L^2(0,T;L^2(\Gamma)), \quad u_\epsilon \in L^4(0,T;V), \quad l_\epsilon \in L^2((0,T)),$$

we have

$$\|e_u\|^2_{L^\infty(0,t;L^2)} + \|e_u\|^2_{L^2(0,t;H^1)} \leq C\epsilon. \tag{2.3.37}$$

Under the assumption that

$$\tau_n(u',p') \in L^2(0,T;L^2(\Gamma)), \quad u', u'_\epsilon \in L^2(0,T;V), \quad l'_\epsilon \in L^2((0,T)),$$

we have

$$\|e'_u\|^2_{L^\infty(0,t;L^2)} + \|e'_u\|^2_{L^2(0,t;H^1)} \leq C\epsilon. \tag{2.3.38}$$

To state the proof, we rewrite (2.3.1) and (2.3.8) into the following. forms

Find $(u(t), p(t), \lambda(t)) \in V \times \mathring{Q} \times M'$, with $u'(t) \in L^2(\Omega)^d$, for any $t \in (0,T)$, such that,

$$(u',v) + a(u,v) + a_1(u,u,v) + b(v,p) + c(\lambda, v_n) = (f,v), \quad \forall v \in V, \tag{2.3.39a}$$

$$b(u,q) = 0, \quad \forall q \in \mathring{Q}, \tag{2.3.39b}$$

$$c(u_n,\mu) = 0, \quad \forall \mu \in M, \tag{2.3.39c}$$

$$u(0,x) = u_0. \tag{2.3.39d}$$

Find $(u_\epsilon(t), p_\epsilon(t), \lambda_\epsilon(t)) \in V \times Q \times M'$, with $u'_\epsilon(t) \in L^2(\Omega)^d$, for all $t \in (0,T)$ such that

$$\begin{aligned}(u'_\epsilon, v) + a(u_\epsilon, v) + a_1(u_\epsilon, u_\epsilon, v) + b(v, \mathring{p}_\epsilon) + c(\lambda_\epsilon, v_n) \\ = (f,v), \quad \forall v \in V,\end{aligned} \tag{2.3.40a}$$

$$b(u_\epsilon, q) = 0, \quad \forall q \in Q, \tag{2.3.40b}$$

$$c(u_{\epsilon n}, \mu) = \epsilon c(\lambda_\epsilon + l_\epsilon(t), \mu) \quad \forall \mu \in M, \tag{2.3.40c}$$

$$u_\epsilon(0,x) = u_{\epsilon 0}. \tag{2.3.40d}$$

Proof of Theorem 2.3.2. Subtracting (2.3.39) from (2.3.40) yields, for all $v \in V$,

$$(e'_u, v) + a(e_u, v) + b(v, e_p) + a_1(u, e_u, v) + a_1(e_u, u_\epsilon, v) + c(e_\lambda, v_n) = 0. \tag{2.3.41}$$

In view of $u_n|_\Gamma = 0$ and $\int_\Gamma u_{\epsilon n} ds = 0$, we have $e_n \cdot n|_\Gamma = -u_{\epsilon n}$ and

$$c(e_\lambda, e_u \cdot n) = c(\lambda - \epsilon^{-1} u_{\epsilon n}, -u_{\epsilon n}) = \epsilon \|\lambda - \epsilon^{-1} u_{\epsilon n}\|^2_{L^2(\Gamma)} - \epsilon c(\lambda - \epsilon^{-1} u_{\epsilon n}, \lambda).$$

Substituting $v = e_u$ to (2.3.41), we obtain, for any $\eta_0 > 0$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|e_u\|_{L^2}^2 + \alpha\|e_u\|_{H^1}^2 + \epsilon\|\lambda - \epsilon^{-1}u_{\epsilon n}\|_{L^2(\Gamma)}^2 \\
\le&\epsilon c(\lambda - \epsilon^{-1}u_{\epsilon n}, \lambda) - a_1(e_u, u_\epsilon, e_u) \\
\le&\eta_0\epsilon\|\lambda - \epsilon^{-1}u_{\epsilon n}\|_{L^2(\Gamma)}^2 + C\eta_0^{-1}\epsilon\|\lambda\|_{L^2(\Gamma)}^2 + \eta_0\|e_u\|_{H^1}^2 + C\eta_0^{-3}\|e_u\|_{L^2}^2\|u_\epsilon\|_{H^1}^4,
\end{aligned} \tag{2.3.42}$$

which gives (2.3.37).

Differentiating (2.3.41) with respect to t and substituting $v = e'_\lambda(t)$, we have

$$\begin{aligned}
&\frac{d}{dt}\|e'_u\|_{L^2}^2 + \alpha\|e'_u\|_{H^1}^2 + \epsilon\|\lambda' - \epsilon^{-1}u'_{\epsilon n}\|_{L^2(\Gamma)}^2 \\
\le&C(\|u'\|_{H^1}^2 + \|u'_\epsilon\|_{H^1}^2)\|e_u\|_{H^1}^2 + C\epsilon\|\lambda'\|_{L^2(\Gamma)}^2 + C\|u_\epsilon\|_{H^1}^4\|e'_u\|_{L^2}^2.
\end{aligned} \tag{2.3.43}$$

From (2.3.32), (2.3.37) and (2.3.43), we conclude (2.3.38). □

Theorem 2.3.3. *Let (u, p) and (u_ϵ, p_ϵ) be the unique solutions to (2.3.1) and (2.3.8), repectivelty. Assume*

$$(u, p), (u_\epsilon, p_\epsilon) \in H^1(0, T; H^2(\Omega)^d) \times H^1(0, T; H^1(\Omega)),$$

we have,

$$\|e'_u\|_{L^2(0,T;L^2)} + \|e_u\|_{L^\infty(0,t;V)} \le C\epsilon, \tag{2.3.44}$$

$$\|e_p\|_{L^2(0,T;L^2)} + \|e_\lambda\|_{L^2(0,T;M^*)} \le C\epsilon. \tag{2.3.45}$$

Proof of Theorem 2.3.3. From the assumption, we see that

$$\lambda \in H^1(0, T; H^{1/2}(\Gamma)), \quad l_\epsilon \in H^1((0, T)).$$

From (2.3.41), we have, for all $t \in (0, T)$, and for any $v \in H_0^1(\Omega)^d$,

$$b(v, e_p(t)) = -(e'_u(t), v) - a(e_u(t), v) - a_1(u(t), e_u(t), v) - a_1(e_u(t), u_\epsilon(t), v). \tag{2.3.46}$$

Applying the *inf-sup condition* (2.2.5) to (2.3.46), it gives

$$\|e_p(t)\|_{L^2} \le C(\|e'_u(t)\|_{L^2} + \|e_u(t)\|_{H^1}). \tag{2.3.47}$$

Applying the *inf-sup condition* (2.2.6) to (2.3.41), we have

$$\|e_\lambda(t)\|_{M^*} \le C(\|e'_u(t)\|_{L^2} + \|e_u(t)\|_{H^1} + \|e_p(t)\|_{L^2}). \tag{2.3.48}$$

We see that

$$c(e_\lambda, e_u') = \epsilon \frac{1}{2}\frac{d}{dt}\|e_\lambda\|^2_{L^2(\Gamma)} - \epsilon c(e_\lambda, \lambda' + l_\epsilon'). \tag{2.3.49}$$

Substituting $v = e_u'(t)$ into (2.3.41), it yields

$$\begin{aligned} &\|e_u'\|^2_{L^2} + \frac{1}{2}\frac{d}{dt}a(e_u, e_u) + \epsilon\frac{1}{2}\frac{d}{dt}\|e_\lambda\|^2_\Gamma \\ \le& \epsilon c(e_\lambda, \lambda' + l_\epsilon') - a_1(u, e_u, e_u') - a_1(e_u, u_\epsilon, e_u') \\ \le& C\epsilon\|e_\lambda\|_{M'}(\|\lambda'\|_{H^{1/2}(\Gamma)} + |l_\epsilon'|) + C\|e_u\|_{H^1}\|e_u'\|_{L^2}. \end{aligned} \tag{2.3.50}$$

From (2.3.47), (2.3.48), and $\alpha\|e_u(t)\|^2_{H^1} \le a(e_u(t), e_u(t))$, we get

$$\|e_u'\|^2_{L^2} + \frac{d}{dt}a(e_u, e_u) + \epsilon\frac{d}{dt}\|e_\lambda\|^2_{L^2(\Gamma)} \le Ca(e_u(t), e_u(t)) + C\epsilon^2. \tag{2.3.51}$$

From (2.3.31) and (2.3.36), we see that (2.3.51) implies (2.3.44). (2.3.45) follows directly from (2.3.47) and (2.3.48). □

2.4 The penalty method to the stationary Navier-Stokes problem

We consider the stationary Navier-Stokes problem (**NS**) with slip boundary condition.

$$-\nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \qquad \text{in } \Omega, \tag{2.4.1a}$$
$$\nabla\cdot u = 0, \qquad \text{in } \Omega, \tag{2.4.1b}$$
$$u_n = 0, \quad \tau_T(u) = 0, \qquad \text{on } \Gamma, \tag{2.4.1c}$$
$$u = 0 \qquad \text{on } D. \tag{2.4.1d}$$

In this section, we consider two penalty problem to (**NS**) (also (2.4.1)). The well-posedness, regularity and error estimates of the penalty problems are investigated.

2.4.1 The penalty problems ($\mathbf{NS}_\epsilon$) and ($\mathbf{NS}'_\epsilon$)

First, we give the variational forms of (**NS**) (also (2.4.1)) and the penalty problem ($\mathbf{NS}_\epsilon$) (also (2.4.2)).

The variational forms of (NS) and ($\mathbf{NS}_\epsilon$)

We write the the penalty problem ($\mathbf{NS}_\epsilon$):

$$
\begin{aligned}
&-\nu\Delta u_\epsilon + (u_\epsilon\cdot\nabla)u_\epsilon + \nabla p_\epsilon = f, && \text{in } \Omega, && (2.4.2a)\\
&\nabla\cdot u_\epsilon = 0, && \text{in } \Omega, && (2.4.2b)\\
&\tau_n(u_\epsilon,p_\epsilon) + \frac{1}{\epsilon}u_{\epsilon n} = 0, \quad \tau_T(u_\epsilon) = 0, && \text{on } \Gamma, && (2.4.2c)\\
&u_\epsilon = 0 && \text{on } D. && (2.4.2d)
\end{aligned}
$$

The variational form of (2.4.1) reads as: find $(u,p)\in V_n\times \mathring{Q}$ such that

$$
\begin{aligned}
&a(u,v) + a_1(u,u,v) + b(v,p) = (f,v), \quad \forall v\in V_n, && (2.4.3a)\\
&b(u,q) = 0, \quad \forall q\in\mathring{Q}. && (2.4.3b)
\end{aligned}
$$

Remark 2.4.1 (cf. [19])**.** For $f=0$, (2.4.3) admits a unique solution $u=0$. For any $f\in V'$ and $f\neq 0$, there exists a solution $(u,p)\in V_n\times\mathring{Q}$ for (2.4.3), with

$$\|u\|_{H^1} \le \|f\|_{V'}/\alpha, \quad \|p\|_{L^2} \le C\|f\|_{V'}. \qquad (2.4.4)$$

If $\alpha^2 > \|f\|_{V'}$, then the solution is unique.

The variational form of (2.4.2) reads as: find $(u_\epsilon,p_\epsilon)\in V\times Q$ such that

$$a(u_\epsilon,v) + a_1(u_\epsilon,u_\epsilon,v) + b(v,p_\epsilon) + \frac{1}{\epsilon}\int_\Gamma u_{\epsilon n}v_n ds = (f,v), \quad \forall v\in V, \qquad (2.4.5a)$$

$$b(u_\epsilon,q) = 0, \quad \forall q\in Q. \qquad (2.4.5b)$$

The penalty problem ($\mathbf{NS}'_\epsilon$)

We also consider the penalty problem with skew symmetric term, denoted as ($\mathbf{NS}'_\epsilon$): find $(u_\epsilon,p_\epsilon)\in V\times Q$ such that,

$$
\begin{aligned}
&a(u_\epsilon,v) + \frac{1}{2}[a_1(u_\epsilon,u_\epsilon,v) - a_1(u_\epsilon,v,u_\epsilon)] + \frac{1}{\epsilon}\int_\Gamma u_{\epsilon n}v_n ds\\
&\qquad + b(v,p_\epsilon) = (f,v), \quad \forall v\in V, && (2.4.6a)\\
&b(u_\epsilon,q) = 0, \quad \forall q\in Q. && (2.4.6b)
\end{aligned}
$$

The strong form of (2.4.6) reads as:

$$
\begin{aligned}
&-\nu\Delta u_\epsilon + (u_\epsilon \cdot \nabla)u_\epsilon + \nabla p_\epsilon = f, && \text{in } \Omega, && (2.4.7a)\\
&\nabla \cdot u_\epsilon = 0, && \text{in } \Omega, && (2.4.7b)\\
&\tau(u_\epsilon, p_\epsilon) + \frac{1}{\epsilon}u_{\epsilon n}n - \frac{1}{2}u_{\epsilon n}u_\epsilon = 0, && \text{on } \Gamma, && (2.4.7c)\\
&u_\epsilon = 0 && \text{on } D. && (2.4.7d)
\end{aligned}
$$

Remark 2.4.2. If we replace $u_n|_\Gamma = 0$ in (**NS**) with the non-homogeneous boundary condition $u_n|_\Gamma = g \neq 0$, we have to replace the penalty term $\tau_n(u_\epsilon + p_\epsilon) + \epsilon^{-1}u_{\epsilon n} = 0$ of ($\mathbf{NS}_\epsilon$) with $\tau_n(u_\epsilon + p_\epsilon) + \epsilon^{-1}(u_{\epsilon n} - g) = 0$. Correspondently, we have to replace the penalty term $\frac{1}{\epsilon}\int_\Gamma u_{\epsilon n}v_n\, ds$ in (2.4.5) with $\frac{1}{\epsilon}\int_\Gamma (u_{\epsilon n} - g)v_n\, ds$. In this case, the skew-symmetric term

$$
\frac{1}{2}[a_1(u,u,v) - a_1(u,v,u)] = a_1(u,u,v) - \frac{1}{2}\int_\Gamma g(u\cdot v)\, ds,
$$

Therefore, instead of (2.4.6), we have to consider the penalty problem

$$
\begin{aligned}
a(u_\epsilon, v) &+ \frac{1}{2}[a_1(u_\epsilon,u_\epsilon,v) - a_1(u_\epsilon,v,u_\epsilon)] + \frac{1}{2}\int_\Gamma g(u_\epsilon\cdot v)\\
&+ b(v,p_\epsilon) + \epsilon^{-1}c(u_{\epsilon n} - g, v_n) = (f,v), \quad \forall v \in V.
\end{aligned}
$$

Correspondently, we replace (2.4.7c) with $\tau(u_\epsilon,p_\epsilon) + \frac{1}{\epsilon}(u_{\epsilon n} - g)n - \frac{1}{2}(u_{\epsilon n} - g)u_\epsilon = 0$.

2.4.2 The well-posedness of ($\mathbf{NS}_\epsilon$) and ($\mathbf{NS}'_\epsilon$)

For ($\mathbf{NS}_\epsilon$) (also (2.4.5)), we consider the equation without p_ϵ, denoted as ($\mathbf{NS}^\sigma_\epsilon$): find $u_\epsilon \in V^\sigma$ such that,

$$
a(u_\epsilon, v) + a_1(u_\epsilon,u_\epsilon,v) + \frac{1}{\epsilon}\int_\Gamma u_{\epsilon n}v_n ds = (f,v), \quad \forall v \in V^\sigma. \tag{2.4.8}
$$

For ($\mathbf{NS}'_\epsilon$) (also (2.4.6)), we consider the equation without p_ϵ, denoted as ($\mathbf{NS}'^\sigma_\epsilon$): find $u_\epsilon \in V^\sigma$ such that,

$$
\begin{aligned}
a(u_\epsilon, v) &+ \frac{1}{2}[a_1(u_\epsilon,u_\epsilon,v) - a_1(u_\epsilon,v,u_\epsilon)] + \frac{1}{\epsilon}\int_\Gamma u_{\epsilon n}v_n ds\\
&= (f,v), \quad \forall v \in V^\sigma.
\end{aligned} \tag{2.4.9}
$$

Remark 2.4.3. Let (u_ϵ, p_ϵ) be the solution of (2.4.5) (*resp.* (2.4.6)), then u_ϵ satisfies (2.4.8) (*resp.* (2.4.9)).

Proposition 2.4.1. *Let* u_ϵ *be the solution of* (2.4.8) *(*resp. (2.4.9)*), then there exists a unique* p_ϵ *associated to* u_ϵ*, such that* (u_ϵ, p_ϵ) *satisfies* (2.4.5) *(*resp. (2.4.6)*), with*

$$\|p_\epsilon\|_{L^2} \leq C(\|u_\epsilon\|_{H^1} + \|u_\epsilon\|^2_{H^1} + \|f\|_{V'}).$$

Proof. (1) First, let us prove the case of (2.4.8). In view of the *inf-sup condition* of b (3.2.7), for any $u_\epsilon \in V$, there exists a unique $\mathring{p}_\epsilon \in \mathring{Q}$ such that

$$a(u_\epsilon, v) + a_1(u_\epsilon, u_\epsilon, v) + b(v, \mathring{p}_\epsilon) = (f, v), \quad v \in H^1_0(\Omega)^d, \tag{2.4.10}$$

and we have

$$\beta\|\mathring{p}_\epsilon\|_{L^2} \leq \sup_{v\in H^1_0(\Omega)^d\setminus\{0\}} \frac{b(v, \mathring{p}_\epsilon)}{\|v\|_{H^1}} \leq C(\|u_\epsilon\|_{H^1} + \|(u_\epsilon \cdot \nabla)u_\epsilon\|^2_{V'} + \|f\|_{V'}).$$

For arbitrary $\phi \in C^\infty(\Gamma)$ with $\int_\Gamma \phi_n \, ds = 1$, we set

$$k_\epsilon = \frac{1}{|\Gamma|}\left(a(u_\epsilon, \phi) + a_1(u_\epsilon, u_\epsilon, v) + b(\phi, \mathring{p}_\epsilon) - \epsilon^{-1}c(u_{\epsilon n}, \phi_n) - (f, \phi)\right). \tag{2.4.11}$$

One can verify that k_ϵ is independent of ϕ, and (u_ϵ, p_ϵ) with $p_\epsilon = \mathring{p}_\epsilon + k_\epsilon$ satisfies (2.4.5).

Substituting $v = \varphi$ into (2.4.5), where $\varphi \in V$ with $\varphi|_\Gamma = k_\epsilon n$ and $\|v\|_{H^1} \leq C|k_\epsilon|$, we have

$$\begin{aligned}|k_\epsilon|^2|\Gamma| &= k_\epsilon \int_\Gamma \varphi_n \, ds = -b(\varphi, k_\epsilon) \\ &= a(u_\epsilon, \varphi) + a_1(u_\epsilon, u_\epsilon, \varphi) + b(\varphi, \mathring{p}_\epsilon) + \epsilon^{-1}c(u_{\epsilon n}, \varphi_n) - (f, v),\end{aligned}$$

which implies

$$|k_\epsilon| \leq C(\|u_\epsilon\|_{H^1} + \|(u_\epsilon \cdot \nabla)u_\epsilon\|_{V'} + \|f\|_{V'}).$$

(2) For the case of (2.4.9), we have there exists a unique $\mathring{p}_\epsilon \in \mathring{Q}$ such that

$$a(u_\epsilon, v) + \frac{1}{2}[a_1(u_\epsilon, u_\epsilon, v) - a_1(u_\epsilon, v, u_\epsilon)] + b(v, \mathring{p}_\epsilon) = (f, v), \quad v \in H^1_0(\Omega)^d, \tag{2.4.12}$$

and we have

$$\beta\|\mathring{p}_\epsilon\|_{L^2} \leq C(\|u_\epsilon\|_{H^1} + \|(u_\epsilon \cdot \nabla)u_\epsilon\|_{V'} + \|u_\epsilon\|_{L^3}\|u_\epsilon\|_{L^6} + \|f\|_{V'}).$$

For arbitrary $\phi \in C^\infty(\Gamma)$ with $\int_\Gamma \phi_n \, ds = 1$, setting

$$\begin{aligned}|\Gamma| k_\epsilon =& a(u_\epsilon, \phi) + \frac{1}{2}[a_1(u_\epsilon, u_\epsilon, v) - a_1(u_\epsilon, v, u_\epsilon)] \\ &+ b(\phi, \mathring{p}_\epsilon) - \epsilon^{-1} c(u_{\epsilon n}, \phi_n) - (f, \phi),\end{aligned} \tag{2.4.13}$$

one can verify that k_ϵ is the constant independent of ϕ, with

$$|k_\epsilon| \leq C(\|u_\epsilon\|_{H^1} + \|(u_\epsilon \cdot \nabla) u_\epsilon\|_{V'} + \|u_\epsilon\|_{L^3} \|u_\epsilon\|_{L^6} + \|f\|_{V'})$$

and (u_ϵ, p_ϵ) with $p_\epsilon = \mathring{p}_\epsilon + k_\epsilon$ satisfies (2.4.6). □

From Solbolev's embedding theorem and trace theorem:

$$\|v\|_{L^4(\Gamma)} \leq C_1 \|v\|_{H^{\frac{1}{2}}(\Gamma)}, \quad \|v\|_{H^{\frac{1}{2}}(\Gamma)} \leq C_2 \|v\|_{H^1}, \quad \forall v \in V, \ d = 2, 3,$$

we set the constant $c_1 > 0$ such that

$$a_1(w, v, v) = \frac{1}{2} \int_\Gamma w_n |v|^2 \, ds \leq c_1 \|w_n\|_{L^2(\Gamma)} \|v\|_{H^1}^2, \quad \forall w \in V^\sigma, v \in V. \tag{2.4.14}$$

Proposition 2.4.2. *(1) For arbitrary* η $(0 < \eta \ll 1)$*, when* ϵ *is sufficiently small, there exists a solution* $u_\epsilon \in V^\sigma$ *of* $(\mathbf{NS}_\epsilon{}^\sigma)$ *(also* (2.4.8)*), with*

$$\|u_\epsilon\|_{H^1} \leq \|f\|_{V'}(1+\eta)/\alpha, \quad \|u_{\epsilon n}\|_{L^2(\Gamma)} \leq \sqrt{2\epsilon(1+\eta)/\alpha} \|f\|_{V'}. \tag{2.4.15}$$

Moreover, if $\|f\|_{V'}$ *is sufficiently small (equivalently,* α *or* ν *is large enough) such that*

$$\alpha - \|a_1\| \frac{1+\eta}{\alpha} \|f\|_{V'} - c_1 \sqrt{\frac{2\epsilon(1+\eta)}{\alpha}} \|f\|_{V'} > 0,$$

then u_ϵ *is unique in* $\{v \in V \mid \|v\|_{H^1} \leq \|f\|_{V'}(1+\eta)/\alpha\}$*.*

(2) There exists a solution $u_\epsilon \in V^\sigma$ *of* $(\mathbf{NS}'^\sigma_\epsilon)$ *(also* (2.4.9)*), with*

$$\|u_\epsilon\|_{H^1} \leq \|f\|_{V'}/\alpha, \quad \|u_{\epsilon n}\|_{L^2(\Gamma)} \leq \sqrt{\epsilon/\alpha} \|f\|_{V'}. \tag{2.4.16}$$

Moreover, if $\|f\|_{V'}$ *is sufficiently small such that* $\alpha - \|a_1\| \|f\|_{V'}/\alpha > 0$*, then the solution* u_ϵ *is unique.*

Proof. The proof is similar to the standard argument (cf. [19, Chapter IV, Theorem 1.2]). We construct the approximate solutions by Galerkin's method. Since V^σ is separable, there exists a sequence $\{w_i\}_{i=1}^\infty \subset V^\sigma$ such that, for any $m \geq 1$, $w_1, \ldots, w_m$ are linearly independent, and $\overline{\cup_{m=1}^\infty V_m}$ is dense in V^σ, where $V_m = \text{span}\{w_i\}_{i=1}^m$.

Let us first prove (2). For any $m \geq 1$, we consider the Galerkin's approximate problem, denoted as $(\mathbf{NS}^{\sigma}_{\epsilon\mathbf{m}}{}')$: find $u_{\epsilon m} \in V_m$ such that

$$
\begin{aligned}
a(u_{\epsilon m}, w_i) + \frac{1}{2}[a_1(u_{\epsilon m}, u_{\epsilon m}, w_i) - a_1(u_{\epsilon m}, w_i, u_{\epsilon m})] + \frac{1}{\epsilon}c(u_{\epsilon mn}, w_{in}) \\
= (f, w_i), \quad \forall i = 1, \ldots, m,
\end{aligned}
\tag{2.4.17}
$$

where $u_{\epsilon mn} = u_{\epsilon m} \cdot n$, $w_{in} = w_i \cdot n$.

We define the mapping $\Phi_m : V_m \to V_m$:

$$
\begin{aligned}
(\Phi_m(v), w_i) =& a(v, w_i) + \frac{1}{2}[a_1(v, v, w_i) - a_1(v, w_i, v)] \\
&+ \frac{1}{\epsilon}c(v_n, w_{in}) - (f, w_i).
\end{aligned}
$$

We have

$$
\begin{aligned}
(\Phi_m(v), v) &= a(v, v) + \epsilon^{-1}\|v_n\|^2_{L^2(\Gamma)} - (f, v) \\
&\geq (\alpha\|v\|_{H^1} - \|f\|_{V'})\|v\|_{H^1} + \epsilon^{-1}\|v_n\|^2_{L^2(\Gamma)}.
\end{aligned}
$$

Hence, $(\Phi_m(v), v) \geq 0$ for all $v \in V_m$ with $\|v\|_{H^1} = \|f\|_{V'}/\alpha$. Applying the Browser's fixed point theorem (cf. [19, Chapter IV, Theorem 1.1]), there exists a solution $u_{\epsilon m}$ of $(\mathbf{NS}^{\sigma}_{\epsilon\mathbf{m}}{}')$, with $\|u_{\epsilon m}\|_{H^1} \leq \|f\|_{V'}/\alpha$. Then there exists a subsequence of $\{u_{\epsilon m}\}_{m=1}^{\infty}$, which we also denoted as $\{u_{\epsilon m}\}_{m=1}^{\infty}$, satisfies

$$
u_{\epsilon m} \to \bar{u}_\epsilon, \text{ weakly in } V^\sigma, \quad u_{\epsilon m} \to \bar{u}_\epsilon \text{ in } L^2(\Omega),
$$

as $m \to \infty$. Passing the limit $m \to \infty$ of (2.4.17), we see that $u_\epsilon = \bar{u}_\epsilon$ is the solution of $(\mathbf{NS}'^{\sigma}_{\epsilon})$.

For any solution u_ϵ of $(\mathbf{NS}^{\sigma\prime}_{\epsilon})$, substituting $v = u_\epsilon$ into (2.4.6), we have

$$
\begin{aligned}
\alpha\|u_\epsilon\|^2_{H^1} + \epsilon^{-1}\|u_{\epsilon n}\|^2_{L^2(\Gamma)} \leq & a(u_\epsilon, u_\epsilon) + \epsilon^{-1}c(u_{\epsilon n}, u_{\epsilon n}) \\
=& (f, u_\epsilon) \leq \|f\|_{V'}\|u_\epsilon\|_{H^1},
\end{aligned}
$$

which implies (2.4.16).

We then consider the uniqueness of solution. Assume there exist two solutions u_ϵ and U_ϵ of $(\mathbf{NS}^{\sigma\prime}_{\epsilon})$. Setting $w = u_\epsilon - U_\epsilon$, we see that

$$
\begin{aligned}
a(w, v) &+ \frac{1}{2}[a_1(U_\epsilon, w, v) - a_1(U_\epsilon, v, w)] \\
&+ \frac{1}{2}[a_1(w, u_\epsilon, v) - a_1(w, v, u_\epsilon)] + \frac{1}{\epsilon}c(w_n, v_n) = 0, \quad \forall v \in V^\sigma.
\end{aligned}
\tag{2.4.18}
$$

Substituting $v = w$ into (2.4.18), we have

$$\begin{aligned}0 &= a(w,w) + \epsilon^{-1}\|w_n\|^2_{L^2(\Gamma)} + \frac{1}{2}[a_1(w,u_\epsilon,w) - a_1(w,w,u_\epsilon)] \\ &\geq \alpha\|w\|^2_{H^1} + \epsilon^{-1}\|w_n\|^2_{L^2(\Gamma)} - \|a_1\|\|w\|^2_{H^1}\|u_\epsilon\|_{H^1}.\end{aligned}$$

If $\alpha > \|a_1\|\|f\|_{V'}/\alpha \geq \|a_1\|\|u_\epsilon\|_{H^1}$, then $w = 0$. We finish the proof of (2).

Next, we prove (1). Similar to the argument above, we have the Galerkin's approximate problem, denoted as ($\mathbf{NS}^\sigma_{\epsilon\mathbf{m}}$): find $u_{\epsilon m} \in V_m$ such that

$$\begin{aligned}&a(u_{\epsilon m}, w_i) + a_1(u_{\epsilon m}, u_{\epsilon m}, w_i) + \epsilon^{-1}c(u_{\epsilon mn}, w_{in}) \\ &\qquad = (f, w_i), \quad \forall i = 1,\dots,m,\end{aligned} \tag{2.4.19}$$

and the associate mapping $\Phi_m : V_m \to V_m$:

$$(\Phi_m(v), w_i) = a(v, w_i) + a_1(v, v, w_i) + \epsilon^{-1}c(v_n, w_{in}) - (f, w_i).$$

In view of (2.4.14), we have

$$a_1(v,v,v) \leq c_1\|v_n\|_{L^2(\Gamma)}\|v\|^2_{H^1} \leq \frac{1}{2\epsilon}\|v_n\|^2_{L^2(\Gamma)} + \frac{c_1^2\epsilon}{2}\|v\|^4_{H^1},$$

applying which we can obtain

$$(\Phi_m(v), v) \geq (\alpha\|v\|_{H^1} - \frac{c_1^2\epsilon}{2}\|v\|^3_{H^1} - \|f\|_{V'})\|v\|_{H^1} + \frac{1}{2\epsilon}\|v_n\|^2_{L^2(\Gamma)}. \tag{2.4.20}$$

For any $\eta > 0$ ($\eta \ll 1$), and for any $v \in V_m$ with $\|v\|_{H^1} = \frac{(1+\eta)\|f\|_{V'}}{\alpha}$, if

$$\epsilon \leq \frac{2\eta\alpha^3}{c_1^2(1+\eta)^3\|f\|^2_{V'}}, \tag{2.4.21}$$

we have

$$(\alpha\|v\|_{H^1} - \frac{c_1^2\epsilon}{2}\|v\|^3_{H^1} - \|f\|_{V'}) \geq 0.$$

Hence, there exists a solution $u_{\epsilon m}$ of ($\mathbf{NS}^\sigma_{\epsilon\mathbf{m}}$), with $\|u_{\epsilon m}\|_{H^1} \leq \frac{(1+\eta)\|f\|_{V'}}{\alpha}$.

Substituting $w_i = u_{\epsilon m}$ in (2.4.19), it yields

$$(\alpha - \frac{c_1^2\epsilon}{2}\|u_{\epsilon m}\|^2_{H^1})\|u_{\epsilon m}\|^2_{H^1} + \frac{1}{2\epsilon}\|u_{\epsilon mn}\|^2_{L^2(\Gamma)} \leq \|f\|_{V'}u_{\epsilon m}\|_{H^1}.$$

In view of $\epsilon \leq \frac{2\eta\alpha^3}{c_1^2(1+\eta)^3\|f\|^2_{V'}}$ and $\|u_{\epsilon m}\|_{H^1} \leq \frac{(1+\eta)\|f\|_{V'}}{\alpha}$, we have

$$\alpha - \frac{c_1^2\epsilon}{2}\|u_{\epsilon m} \geq \alpha - \frac{\alpha\eta}{1+\eta} = \frac{\alpha}{1+\eta} > 0,$$

which implies

$$\|u_{\epsilon mn}\|_{L^2(\Gamma)} \le \sqrt{2\epsilon(1+\eta)/\alpha}\|f\|_{V'}.$$

After passing the limit $m \to \infty$, we have $u_{\epsilon m} \to u_\epsilon$ weakly in V^σ, with $\|u_\epsilon\|_{H^1} \le \frac{(1+\eta)\|f\|_{V'}}{\alpha}$, $\|u_{\epsilon n}\|_{L^2(\Gamma)} \le \sqrt{2\epsilon(1+\eta)/\alpha}\|f\|_{V'}$, and u_ϵ is a solution of ($\mathbf{NS}_\epsilon^\sigma$). We proved (2.4.16). Now, for u_ϵ the solution of

We then consider the uniqueness of u_ϵ. Assume u_ϵ and U_ϵ are two solutions of ($\mathbf{NS}_\epsilon^\sigma$) satisfying (2.4.15). Setting $w = u_\epsilon - U_\epsilon$, we see that

$$a(w,v) + a_1(U_\epsilon, w, v) + a_1(w, u_\epsilon, v) + \frac{1}{\epsilon}c(w_n, v_n) = 0, \quad \forall v \in V^\sigma. \quad (2.4.22)$$

Substituting $v = w$ into (2.4.22), we have

$$\begin{aligned} 0 &= a(w,w) + \epsilon^{-1}\|w_n\|^2_{L^2(\Gamma)} + a_1(U_\epsilon, w, v) + a_1(w, u_\epsilon, v) \\ &\ge (\alpha - c_1\|U_{\epsilon n}\|_{L^2(\Gamma)})\|w\|^2_{H^1} + \epsilon^{-1}\|w_n\|^2_{L^2(\Gamma)} - \|a_1\|\|w\|^2_{H^1}\|u_\epsilon\|_{H^1}. \end{aligned}$$

Since u_ϵ and U_ϵ satisfy (2.4.15), if $\alpha > \frac{\|a_1\|(1+\eta)\|f\|_{V'}}{\alpha} + c_1\sqrt{\frac{2\epsilon(1+\eta)}{\alpha}}\|f\|_{V'}$, then $w = 0$. We finish the proof of (1). □

From Proposition 2.4.2 and 2.4.1, we conclude the theorem of the well-posedness of ($\mathbf{NS}_\epsilon$) and ($\mathbf{NS}'_\epsilon$).

Theorem 2.4.1. *(1) For arbitrary small positive number η, there exists a solution $(u_\epsilon, p_\epsilon) \in V \times Q$ of ($\mathbf{NS}_\epsilon$) (also (2.4.5)) for sufficiently small ϵ (see (2.4.21)), satsifying*

$$\|u_\epsilon\|_{H^1} \le \frac{\|f\|_{V'}(1+\eta)}{\alpha}, \quad \epsilon^{-1/2}\|u_{\epsilon n}\|_{L^2(\Gamma)} + \|p_\epsilon\|_{L^2} \le C. \quad (2.4.23)$$

where C is dependent on $\eta, \|f\|_{V'}$ and α. Moreover, if

$$\alpha - \|a_1\|\frac{1+\eta}{\alpha}\|f\|_{V'} - c_1\sqrt{\frac{2\epsilon(1+\eta)}{\alpha}}\|f\|_{V'} > 0,$$

then (u_ϵ, p_ϵ) is unique in $\{v \in V \mid \|v\|_{H^1} \le \|f\|_{V'}(1+\eta)/\alpha\} \times Q$.

(2) There exists a solution $(u_\epsilon, p_\epsilon) \in V \times Q$ of ($\mathbf{NS}'_\epsilon$) (also (2.4.6)), with

$$\|u_\epsilon\|_{H^1} \le \|f\|_{V'}/\alpha, \quad \epsilon^{-1/2}\|u_{\epsilon n}\|_{L^2(\Gamma)} + \|p_\epsilon\|_{L^2} \le C. \quad (2.4.24)$$

Moreover, if $\alpha - \|a_1\|\|f\|_{V'}/\alpha > 0$, then the solution u_ϵ is unique.

Remark 2.4.4. In Theorem 2.4.1, we show that all solutions of $(\mathbf{NS}'_\epsilon)$ satisfies the estimate $\|u_\epsilon\|_{H^1} \leq \|f\|_{V'}/\alpha$; however, we cannot conclude all solutions of $(\mathbf{NS}_\epsilon)$ satisfies $\|u_\epsilon\|_{H^1} \leq \frac{(1+\eta)\|f\|_{V'}}{\alpha}$. Even when the solution u_ϵ is unique in $\{v \in V \mid \|v\|_{H^1} \leq \frac{(1+\eta)\|f\|_{V'}}{\alpha}\}$, there may still exists other solutions in with $\|u_\epsilon\|_{H^1} > (1+\eta)\|f\|_{V'}/\alpha$.

The following proposition is to discuss the solutions of $(\mathbf{NS}_\epsilon)$.

Proposition 2.4.3. *We consider the problem* $(\mathbf{NS}_\epsilon)$. *For arbitrary positive small* η, *let* ϵ *satisfy* (2.4.21), *and*

$$\epsilon < \frac{8\alpha^3}{27c_1^2\|f\|_{V'}}.$$

Then there exist two positive roots $a < b$ *of the cubic equation*

$$\Psi(x) = 0, \qquad \text{with } \Psi(x) := -\frac{c_1^2\epsilon}{2\alpha}x^3 + x - \frac{\|f\|_{V'}}{\alpha}. \tag{2.4.25}$$

Moreover, we have

(i) there exists a solution u_ϵ *with* $\|u_\epsilon\|_{H^1} \leq a$;

(ii) there is no solution u_ϵ *with* $a < \|u_\epsilon\|_{H^1} < b$;

(iii) there may exists a solution u_ϵ *with* $\|u_\epsilon\|_{H^1} \geq b$,

where

$$\frac{\|f\|_{V'}}{\alpha} \leq a \leq \frac{(1+\eta)\|f\|_{V'}}{\alpha}, \qquad \sqrt{\frac{2\alpha}{3c_1^2\epsilon}} \leq b \leq \sqrt{\frac{2\alpha}{c_1^2\epsilon}}.$$

Proof. (i) is proved in Theorem 2.4.1. Let u_ϵ be any solution of $(\mathbf{NS}_\epsilon)$. Substituting $v = u_\epsilon$ into $(\mathbf{NS}_\epsilon)$ (also 2.4.5), it yields, similar to the derivation of (2.4.20),

$$\begin{aligned}
&(\alpha\|u_\epsilon\|_{H^1} - \frac{\epsilon c_1}{2}\|u_\epsilon\|_{H^1}^3 - \|f\|_{V'})\|u_\epsilon\|_{H^1} + \frac{1}{2\epsilon}\|u_{\epsilon n}\|_{L^2(\Gamma)}^2 \\
\leq& a(u_\epsilon, u_\epsilon) + a_1(u_\epsilon, u_\epsilon, u_\epsilon) + \epsilon^{-1}c(u_{\epsilon n}, u_{\epsilon n}) - (f, u_\epsilon) \\
=&0,
\end{aligned}$$

which implies $\alpha\|u_\epsilon\|_{H^1} - \frac{\epsilon c_1}{2}\|u_\epsilon\|_{H^1}^3 - \|f\|_{V'} \leq 0$. Taking $\|u_\epsilon\|_{H^1} = x$, it is equivalent to consider the inequality

$$\Psi(x) \leq 0, \qquad \text{for } x \geq 0.$$

Since $\Psi'(x) = 1 - \frac{3c_1^2\epsilon}{2\alpha}$, there are two critical points $x_1 = -\sqrt{\frac{2\alpha}{3c_1^2\epsilon}}$, $x_2 = \sqrt{\frac{2\alpha}{3c_1^2\epsilon}}$ of $\Psi(x)$. Under the assumption $\epsilon < \frac{8\alpha^3}{27c_1^2\|f\|_{V'}}$, we have

$$\Psi(x_2) = \sqrt{\frac{8\alpha}{27c_1^2\epsilon}} - \frac{\|f\|_{V'}}{\alpha} > 0,$$

which implies there exist two positive roots a, b $(a < b)$ of (2.4.25). And see that

$$\Phi(x) \le 0 \text{ for } x \in [0,a] \cup [b,\infty], \quad \Phi(x) \le 0 \text{ for } x \in (a,b),$$

which proves (i)(ii)(iii). As $\Psi(a) = 0$, $\Psi(0) = -\frac{\|f\|_{V'}}{\alpha} \le 0$, we have

$$a - \frac{\|f\|_{V'}}{\alpha} = \frac{c_1^2\epsilon}{2\alpha}a^3 \ge 0.$$

Under the assumption (2.4.21), we have $\Psi(\frac{(1+\eta)\|f\|_{V'}}{\alpha}) \ge 0$, which implies $a \le \frac{(1+\eta)\|f\|_{V'}}{\alpha}$.

$\Psi(b) = 0$ gives $b(1 - b^2\frac{c_1^2\epsilon}{2\alpha}) = \frac{\|f\|_{V'}}{\alpha} > 0$, from which we obtain $b \le \sqrt{\frac{2\alpha}{c_1^2\epsilon}}$. Since $\Psi(x_2) > 0$, we have $b \ge x_2$. The proof is completed. □

2.4.3 The iteration methods for $(\mathbf{NS}'_\epsilon)$ and $(\mathbf{NS}_\epsilon)$

According to (iii) of Proposition 2.4.3, even when $(\mathbf{NS}_\epsilon)$ has a unique solution in $\{v \in V \mid \|v\|_{H^1} \le \frac{(1+\eta)\|f\|_{V'}}{\alpha}\}$, there may still exists other solution in $\{v \in V \mid \|v\|_{H^1} > C\epsilon^{-1/2}\}$. It seems $(\mathbf{NS}'_\epsilon)$ is more reliable to approximate $(\mathbf{NS})$ than $(\mathbf{NS}_\epsilon)$. However, when we apply the iteration methods to solve $(\mathbf{NS}'_\epsilon)$ and $(\mathbf{NS}_\epsilon)$ in numerical computation, the convergence behavior of them are not so much different.

We consider two iteration methods to both $(\mathbf{NS}_\epsilon)$ and $(\mathbf{NS}'_\epsilon)$.

Let $(u_\epsilon^0, p_\epsilon^0)$ be the solution of the penalty Stokes problem $(\mathbf{S}_\epsilon)$, with

$$\|u_\epsilon^0\|_{1,\Omega} \le \frac{\|f\|_{V'}}{\alpha}, \quad \|u_{\epsilon n}^0\|_{L^2(\Gamma)} \le \sqrt{\epsilon}\|f\|_{V'}. \tag{2.4.26}$$

We set $(u_\epsilon^0, p_\epsilon^0) \in V \times Q$ as the initial value of iteration.

Iteration method (i) for $(\mathbf{NS}_\epsilon)$

For $k = 1, 2, \ldots, M_{max}$, find $(u_\epsilon^k, p_\epsilon^k) \in V \times Q$ such that,

$$a(u_\epsilon^k, v) + a_1(u_\epsilon^{k-1}, u_\epsilon^k, v) + b(v, p_\epsilon^k) + \frac{1}{\epsilon\alpha'} \int_\Gamma u_{\epsilon n}^k v_n \, ds = (f, v), \quad \forall v \in V, \tag{2.4.27a}$$

$$b(u_\epsilon^k, q) = 0, \quad \forall q \in Q, \tag{2.4.27b}$$

$$\text{if } \|u_\epsilon^k - u_\epsilon^{k-1}\|_{1,\Omega} \le \eta_0, \text{ then stop the iteration,} \tag{2.4.27c}$$

where M_{max} is the maximum iteration number, η_0 is the error of iteration, and $\alpha' := \alpha - c_1\sqrt{\epsilon}\|f\|_{V'} > 0$ (with sufficiently small ϵ).

Lemma 2.4.1. *For sufficiently small ϵ such that $\alpha' := \alpha - c_1\sqrt{\epsilon}\|f\|_{V'} > 0$, we have*

$$\|u_\epsilon^k\|_{1,\Omega} \le \frac{\|f\|_{V'}}{\alpha'}, \quad \|u_{\epsilon n}^k\|_{L^2(\Gamma)} \le \sqrt{\epsilon}\|f\|_{V'}, \quad \forall k \ge 1. \tag{2.4.28}$$

Furthermore, if $(\alpha')^2 > \|a_1\|\|f\|_{V'}$, then $u_\epsilon^k \to u_\epsilon$ in V.

Proof. Substituting $v = u_\epsilon^1$ into (2.4.1) for $k = 1$, with (2.4.26), and $\alpha' := \alpha - c_1\sqrt{\epsilon}\|f\|_{V'} > 0$, it yields

$$\|u_\epsilon^1\|_{1,\Omega} \le \frac{\|f\|_{V'}}{\alpha'}, \quad \|u_{\epsilon n}^1\|_{L^2(\Gamma)} \le \sqrt{\epsilon}\|f\|_{V'}.$$

(2.4.28) follows from the induction method. (2.4.28) implies the existence of a subsequence $\{u_\epsilon^m\}_{m\ge 0}$ such that $u_\epsilon^m \to u_\epsilon$ weakly in V as $m \to \infty$.

Next, we show the convergence $u_\epsilon^k \to u_\epsilon$ in V.

Setting $w^k = u_\epsilon^k - u_\epsilon^{k-1}$, we have

$$a(w^{k+1}, v) + a_1(u_\epsilon^k, w^{k+1}, v) + \frac{1}{\alpha'\epsilon} \int_\Gamma w_n^{k+1} v_n ds = -a_1(w^k, u_\epsilon^k, v), \quad \forall v \in V^\sigma.$$

Substituting $v = w^{k+1}$, we obtain

$$\begin{aligned} &\alpha\|w^{k+1}\|_{H^1}^2 - c_1\|u_{\epsilon n}^k\|_{L^2(\Gamma)}\|w^{k+1}\|_{H^1} + (\alpha'\epsilon)^{-1}\|w_n^{k+1}\|_{L^2(\Gamma)}^2 \\ \le& - a_1(w^k, u_\epsilon^k, w^{k+1}) \le \|a_1\|\|u_\epsilon^k\|_{H^1}\|w^k\|_{H^1}\|w^{k+1}\|_{H^1}, \end{aligned}$$

which gives

$$\alpha'\|w^{k+1}\|_{H^1} \le \frac{\|a_1\|\|f\|_{V'}}{\alpha'}\|w^k\|_{H^1}.$$

If $\alpha'^2 > \|a_1\|\|f\|_{V'}$, then $\|w^k\|_{H^1} \to 0$ as $k \to \infty$, which implies $u_\epsilon^k \to u_\epsilon$ in V. □

Iteration method (i) for ($\mathbf{NS}'_\epsilon$)

For $k = 1, 2, \ldots, M_{max}$, find $(u_\epsilon^k, p_\epsilon^k) \in V \times Q$ such that,

$$\begin{aligned} a(u_\epsilon^k, v) + \frac{1}{2}[a_1(u_\epsilon^{k-1}, u_\epsilon^k, v) - a_1(u_\epsilon^{k-1}, v, u_\epsilon^k)] + \frac{1}{\epsilon}\int_\Gamma u_{\epsilon n}^k v_n \, ds \\ + b(v, p_\epsilon^k) = (f, v), \quad \forall v \in V, \end{aligned} \tag{2.4.29a}$$

$$b(u_\epsilon^k, q) = 0, \quad \forall q \in Q, \tag{2.4.29b}$$

$$\text{if } \|u_\epsilon^k - u_\epsilon^{k-1}\|_{1,\Omega} \le \eta_0, \text{ then stop the iteration.} \tag{2.4.29c}$$

Lemma 2.4.2. *Let* $\{u_\epsilon^k\}_{k\ge1}$ *be the solution of* (2.4.29)*, we have*

$$\|u_\epsilon^k\|_{1,\Omega} \le \|f\|_{V'}/\alpha, \quad \|u_{\epsilon n}^k\|_{L^2(\Gamma)} \le \sqrt{\epsilon}\|f\|_{V'}, \quad \forall k \ge 1. \tag{2.4.30}$$

Furthermore, if $\alpha^2 > \|a_1\|\|f\|_{V'}$*, then* $u_\epsilon^k \to u_\epsilon$ *in* V*.*

Proof. Substituting $v = u_\epsilon^k$ into (2.4.29), it yields (2.4.30), which implies the existence of a subsequence $\{u_\epsilon^m\}_{m\ge0}$ such that $u_\epsilon^m \to u_\epsilon$ weakly in V as $m \to \infty$.

Setting $w^k = u_\epsilon^k - u_\epsilon^{k-1}$, we have

$$\begin{aligned} a(w^{k+1}, v) + \frac{1}{2}[a_1(u_\epsilon^k, w^{k+1}, v) - a_1(u_\epsilon^k, v, w^{k+1})] + \frac{1}{\epsilon}\int_\Gamma w_n^{k+1} v_n ds \\ = -\frac{1}{2}[a_1(w^k, u_\epsilon^k, v) - a_1(w^k, v, u_\epsilon^k)], \quad \forall v \in V^\sigma. \end{aligned}$$

Substituting $v = w^{k+1}$, we obtain

$$\begin{aligned} &\alpha\|w^{k+1}\|_{H^1}^2 + \epsilon^{-1}\|w_n^{k+1}\|_{L^2(\Gamma)}^2 \\ =& - a_1(w^k, u_\epsilon^k, w^{k+1}) \le \|a_1\|\|u_\epsilon^k\|_{H^1}\|w^k\|_{H^1}\|w^{k+1}\|_{H^1}, \end{aligned}$$

which implies $\|w^{k+1}\|_{H^1} \le \frac{\|a_1\|\|f\|_{V'}}{\alpha^2}\|w^k\|_{H^1}$. And we conclude if $\alpha^2 > \|a_1\|\|f\|_{V'}$, then $u_\epsilon^k \to u_\epsilon$ in V as $k \to \infty$. □

Remark 2.4.5. In view of Lemma 2.4.1, the convergence condition $\alpha'^2 > \|a_1\|\|f\|_{V'}$ is similar to the assumption of unique solution in (1) of Theorem 2.4.1. According to Lemma 2.4.2, the convergence condition $\alpha^2 > \|a_1\|\|f\|_{V'}$ is the same condition to prove the unique solution in (2) of Theorem 2.4.1.

Iteration method (ii) for $(\mathbf{NS}_\epsilon)$

We consider the Newton's method. For $k = 1, 2, \ldots, M_{max}$, find $(\delta u^k, \delta p^k) \in V \times Q$ such that,

$$
\begin{aligned}
&a(\delta u^k, v) + a_1(\delta u^k, u_\epsilon^{k-1}, v) + a_1(u_\epsilon^{k-1}, \delta u^k, v) + b(v, \delta p^k) \\
&\qquad + \epsilon^{-1} c(\delta u^k \cdot n, v_n) = (f, v) - a(u_\epsilon^{k-1}, v) - a_1(u_\epsilon^{k-1}, u_\epsilon^{k-1}, v) \\
&\qquad - b(v, p_\epsilon^{k-1}) - \epsilon^{-1} c(u_\epsilon^{k-1} \cdot n, v_n), \quad \forall v \in V_\sigma,
\end{aligned} \tag{2.4.31a}
$$

$$
b(\delta u_\epsilon^k, q) = 0, \quad \forall q \in M, \tag{2.4.31b}
$$

$$
u_\epsilon^k = u_\epsilon^{k-1} + \delta u^k, \quad p_\epsilon^k = p_\epsilon^{k-1} + \delta p^k, \tag{2.4.31c}
$$

$$
\text{if } \|\delta u^k\| \le \eta_0, \text{ then stop the iteration.} \tag{2.4.31d}
$$

Via calculation, we have, for each k,

$$
\begin{aligned}
&a(\delta u_\epsilon^k, v) + a_1(\delta u_\epsilon^k, u_\epsilon^{k-1}, v) + a_1(u_\epsilon^{k-1}, \delta u^k, v) + \epsilon^{-1} c(\delta u_{\epsilon n}^k, v_n) \\
=& - a_1(\delta u^{k-1}, \delta u^{k-1}, v), \quad \forall v \in V^\sigma,
\end{aligned} \tag{2.4.32}
$$

where $a_1(\delta u^0, \delta u^0, v) := a_1(u_\epsilon^0, u_\epsilon^0, v)$. Substituting $v = \delta u_\epsilon^k$ into (2.4.32), it yields

$$
\begin{aligned}
&\underbrace{\Big(\alpha - \|a_1\| \|u_\epsilon^{k-1}\|_{H^1} - c_1 \|u_{\epsilon n}^{k-1}\|_{L^2(\Gamma)}\Big)}_{=:\alpha_k} \|\delta u_\epsilon^k\|_{H^1}^2 + \frac{1}{\epsilon} \|\delta u_{\epsilon n}^k\|_{L^2(\Gamma)}^2 \\
\le& \|a_1\| \|\delta u_\epsilon^{k-1}\|_{H^1}^2 \|\delta u_\epsilon^k\|_{H^1}.
\end{aligned}
$$

If $\alpha_k \ge \tilde{\alpha} > 0$, for all $k \ge 1$, then we obtain

$$
\|\delta u_\epsilon^k\|_{H^1} \le \frac{\|a_1\|}{\tilde{\alpha}} \|\delta u_\epsilon^{k-1}\|_{H^1}^2,
$$

which shows the second order convergence of the Newton's method. However, we have to admit that there is no explicit choice of u_ϵ^0 and ϵ, such that the convergence condition $\alpha_k \ge \tilde{\alpha} > 0$ is satisfied. All we know is that if ϵ is sufficiently small the initial value u_ϵ^0 is sufficiently close to u_ϵ, then the Newton's method converges.

Iteration method (ii) for $(\mathbf{NS}'_\epsilon)$

For $k = 1, 2, \ldots, M_{max}$, find $(\delta u^k, \delta p^k) \in V \times Q$ such that,

$$\begin{aligned}
&a(\delta u^k, v) + \frac{1}{2}[a_1(\delta u^k, u_\epsilon^{k-1}, v) - a_1(\delta u^k, v, u_\epsilon^{k-1})] + b(v, \delta p^k)\\
&\qquad + \frac{1}{2}[a_1(u_\epsilon^{k-1}, \delta u^k, v) - a_1(u_\epsilon^{k-1}, v, \delta u^k)] + \epsilon^{-1} c(\delta u^k \cdot n, v_n)\\
=&(f, v) - a(u_\epsilon^{k-1}, v) - \frac{1}{2}[a_1(u_\epsilon^{k-1}, u_\epsilon^{k-1}, v) - a_1(u_\epsilon^{k-1}, v, u_\epsilon^{k-1})]\\
&\qquad - b(v, p_\epsilon^{k-1}) - \epsilon^{-1} c(u_\epsilon^{k-1} \cdot n, v_n), \quad \forall v \in V_\sigma,
\end{aligned} \tag{2.4.33a}$$

$$b(\delta u_\epsilon^k, q) = 0, \quad \forall q \in M, \tag{2.4.33b}$$

$$u_\epsilon^k = u_\epsilon^{k-1} + \delta u^k, \quad p_\epsilon^k = p_\epsilon^{k-1} + \delta p^k, \tag{2.4.33c}$$

$$\text{if } \|\delta u^k\| \le \eta_0, \text{ then stop the iteration.} \tag{2.4.33d}$$

Via calculation, we have, for each k,

$$\begin{aligned}
&a(\delta u_\epsilon^k, v) + \frac{1}{2}[a_1(\delta u_\epsilon^k, u_\epsilon^{k-1}, v) - a_1(\delta u_\epsilon^k, v, u_\epsilon^{k-1})]\\
&\qquad + \frac{1}{2}[a_1(u_\epsilon^{k-1}, \delta u^k, v) - a_1(u_\epsilon^{k-1}, v, \delta u^k)] + \epsilon^{-1} c(\delta u_{\epsilon n}^k, v_n)\\
=& - \frac{1}{2}[a_1(\delta u^{k-1}, \delta u^{k-1}, v) - a_1(\delta u^{k-1}, v, \delta u^{k-1})], \quad \forall v \in V^\sigma,
\end{aligned} \tag{2.4.34}$$

where $a_1(\delta u^{k-1}, \delta u^{k-1}, v) - a_1(\delta u^{k-1}, v, \delta u^{k-1}) := a_1(u_\epsilon^0, u_\epsilon^0, v) - a_1(u_\epsilon^0, v, u_\epsilon^0)$. Substituting $v = \delta u_\epsilon^k$ into (2.4.34), it yields

$$\begin{aligned}
&\underbrace{\left(\alpha - \|a_1\| \|u_\epsilon^{k-1}\|_{H^1}\right)}_{=:\alpha_k} \|\delta u_\epsilon^k\|_{H^1}^2 + \frac{1}{\epsilon} \|\delta u_{\epsilon n}^k\|_{L^2(\Gamma)}^2\\
\le& \|a_1\| \|\delta u_\epsilon^{k-1}\|_{H^1}^2 \|\delta u_\epsilon^k\|_{H^1}.
\end{aligned}$$

If ϵ is sufficiently small the initial value u_ϵ^0 is sufficiently close to u_ϵ such that $\alpha_k \ge \tilde{\alpha} > 0$, for all $k \ge 1$, then we obtain

$$\|\delta u_\epsilon^k\|_{H^1} \le \frac{\|a_1\|}{\tilde{\alpha}} \|\delta u_\epsilon^{k-1}\|_{H^1}^2.$$

The method convergence at second order.

2.4.4 Error estimates of $(\mathbf{NS}'_\epsilon)$

Let $f \in L^2(\Omega)$, we assume there exists a unique solution $(u,p) \in H^2(\Omega) \times H^1(\Omega)$ of (2.4.1).

Theorem 2.4.2. *Let u and u_ϵ be the solutions of (2.4.1) and (2.4.6), respectively. Assume $\tau_n(u,p) \in L^2(\Gamma)$, and α is sufficiently large(or $\|f\|_{V'}$ is small enough) such that $\alpha^2 > \|a_1\|\|f\|_{V'}$, then we have*

$$\|u-u_\epsilon\|_{H^1} + \|p-\mathring{p}_\epsilon\|_{L^2} + \sqrt{\epsilon}\|\lambda-\lambda_\epsilon\|_{L^2(\Gamma)} \leq C\sqrt{\epsilon}\|\tau_n(u,p)\|_{L^2(\Gamma)}, \quad (2.4.35)$$

where $p_\epsilon = \mathring{p}_\epsilon + k_\epsilon$, $\mathring{p}_\epsilon \in \mathring{Q}$, and $k_\epsilon = \frac{1}{|\Omega|}\int_\Omega p_\epsilon dx$.

Proof. Introducing the Lagrange multiplier $\lambda = -\tau_n(u,p)$ and $\lambda_\epsilon = \frac{1}{\epsilon}u_{\epsilon n}$, we rewrite the variational equations (2.4.3) and (2.4.6) into

(1) find $(u,p,\lambda) \in V \times Q \times M'$ such that,

$$a(u,v) + a_1(u,u,v) + b(v,p) + c(\lambda, v_n) = (f,v), \quad \forall v \in V, \qquad (2.4.36a)$$
$$b(u,q) = 0, \quad \forall q \in Q, \qquad (2.4.36b)$$
$$c(u_n,\mu) = 0, \quad \forall \mu \in M; \qquad (2.4.36c)$$

(2) find $(u_\epsilon, p_\epsilon, \lambda_\epsilon) \in V \times Q \times M'$ such that,

$$\begin{aligned} a(u_\epsilon,v) + \frac{1}{2}a_1(u_\epsilon,u_\epsilon,v) - \frac{1}{2}a_1(u_\epsilon,v,u_\epsilon) \\ + b(v,p_\epsilon) + c(\lambda_\epsilon,v_n) = (f,v), \quad \forall v \in V, \end{aligned} \qquad (2.4.37a)$$
$$b(u_\epsilon,q) = 0, \quad \forall q \in Q, \qquad (2.4.37b)$$
$$c(u_{\epsilon n},\mu) = \epsilon c(\lambda_\epsilon,\mu), \quad \forall \mu \in M. \qquad (2.4.37c)$$

Substituting $v = u - u_\epsilon$ into (2.4.36a)$-$(2.4.37a), we have

$$\begin{aligned} a(u-u_\epsilon, u-u_\epsilon) + \frac{1}{4}[a_1(u-u_\epsilon, u+u_\epsilon, u-u_\epsilon) \\ - a_1(u-u_\epsilon, u-u_\epsilon, u+u_\epsilon)] + c(\lambda-\lambda_\epsilon, u_n - u_{\epsilon n}) = 0. \end{aligned}$$

Noticing $u_n = 0$ and $u_{\epsilon n} = \epsilon\lambda_\epsilon$, we derive

$$\begin{aligned} c(\lambda-\lambda_\epsilon, u_n - u_{\epsilon n}) = -\epsilon c(\lambda-\lambda_\epsilon, \lambda_\epsilon) \\ =\epsilon c(\lambda-\lambda_\epsilon, \lambda-\lambda_\epsilon) - \epsilon c(\lambda-\lambda_\epsilon, \lambda). \end{aligned} \qquad (2.4.38)$$

It is proved in Remark 2.4.1, Theorem 2.4.1, that u and u_ϵ satisfy

$$\|u\|_{H^1}, \|u_\epsilon\|_{H^1} \leq \|f\|_{V'}/\alpha. \qquad (2.4.39)$$

Therefore, we have

$$\begin{aligned}&(\alpha - \|a_1\|\|f\|_\Omega/\alpha)\|u - u_\epsilon\|_{1,\Omega}^2 + \epsilon c(\lambda - \lambda_\epsilon, \lambda - \lambda_\epsilon)\\ \le&\epsilon c(\lambda - \lambda_\epsilon, \lambda) \le \frac{\epsilon}{2}\|\lambda - \lambda_\epsilon\|_{L^2(\Gamma)}^2 + \frac{\epsilon}{2}\|\lambda\|_{L^2(\Gamma)}^2.\end{aligned} \tag{2.4.40}$$

Under the assumption $\alpha^2 > \|a_1\|\|f\|_\Omega$, we obtain,

$$\|u - u_\epsilon\|_{H^1} + \sqrt{\epsilon}\|\lambda - \lambda_\epsilon\|_{L^2(\Gamma)} \le C\sqrt{\epsilon}\|\lambda\|_{L^2(\Gamma)}.$$

Using *inf-sup condition of* b (3.2.7) and (2.4.39), we conclude

$$\|p - \mathring{p}_\epsilon\|_{L^2} \le C\|u_\epsilon - u\|_{H^1}. \tag{2.4.41}$$

The proof is completed. □

Theorem 2.4.3. *Let* $\tau_n(u,p) \in H^{1/2}(\Gamma)$*, and with the same assumption of Theorem 2.4.2, then we have*

$$\|u - u_\epsilon\|_{H^1} + \|p - \mathring{p}_\epsilon\|_{L^2} \le C\epsilon(\|\tau_n(u,p)\|_{H^{1/2}(\Gamma)} + \|f\|_{L^2}). \tag{2.4.42}$$

Proof. Instead of using (2.4.38), we derive

$$\begin{aligned}&c(\lambda - \lambda_\epsilon, u_n - u_{\epsilon n}) = c(\lambda - \lambda_\epsilon + k_\epsilon, u_n - u_{\epsilon n}) = -\epsilon c(\lambda - \lambda_\epsilon + k_\epsilon, \lambda_\epsilon)\\ =&\epsilon c(\lambda - \lambda_\epsilon + k_\epsilon, \lambda - \lambda_\epsilon + k_\epsilon) - \epsilon c(\lambda - \lambda_\epsilon + k_\epsilon, \lambda + k_\epsilon),\end{aligned} \tag{2.4.43}$$

and obtain

$$\begin{aligned}&(\alpha - \|a_1\|\|f\|_{V'}/\alpha)\|u - u_\epsilon\|_{H^1}^2 + \epsilon c(\lambda - \lambda_\epsilon + k_\epsilon, \lambda - \lambda_\epsilon + k_\epsilon)\\ \le&\epsilon c(\lambda - \lambda_\epsilon + k_\epsilon, \lambda + k_\epsilon) \le \epsilon\|\lambda - \lambda_\epsilon + k_\epsilon\|_{M'}\|\lambda + k_\epsilon\|_M.\end{aligned} \tag{2.4.44}$$

If we show

$$\|\lambda - \lambda_\epsilon + k_\epsilon\|_{M'} \le C\|u - u_\epsilon\|_{H^1}, \tag{2.4.45}$$

then with the assumption $\lambda \in H^{1/2}(\Gamma) = M$, we can derive the error estimate

$$\|u - u_\epsilon\|_{H^1} \le C\epsilon(\|\lambda\|_\Lambda + k_\epsilon), \tag{2.4.46}$$

where k_ϵ is bounded independent of ϵ(Theorem 2.4.1). $\|p - \mathring{p}_\epsilon\|_{L^2} \le C\epsilon$ follows from (2.4.41) and (2.4.46). Therefore, we are only left to prove (2.4.45). Since

$$\begin{aligned}&-c(\lambda - \lambda_\epsilon + k_\epsilon, v_n)\\ =&a(u - u_\epsilon, v) + b(v, p - \mathring{p}_\epsilon) + \frac{1}{2}[a_1(u - u_\epsilon, u, v)\\ &\quad + a_1(u_\epsilon, u - u_\epsilon, v) + a_1(u_\epsilon - u, v, u_\epsilon) + a_1(u, v, u_\epsilon - u)]\\ \le&C(1 + \|u\|_{H^1} + \|u_\epsilon\|_{H^1})(\|u - u_\epsilon\|_{H^1} + \|p - \mathring{p}_\epsilon\|_{L^2})\|v\|_{H^1}.\end{aligned}$$

From (2.4.39), (2.4.41) and the *inf-sup condition* of c (3.2.8), we obtain (2.4.45). $\square$

Remark 2.4.6. In above, we show the error estimates of penalty scheme (2.4.6). For penalty scheme (2.4.5), under the assumption that u_ϵ with $\|u_\epsilon\|_{1,\Omega} \leq \frac{3\|f\|_\Omega}{2\alpha}$ and $\alpha^2 > \frac{3\|a_1\|\|f\|_\Omega}{2}$, then we can obtain the same error estimates as (2.4.35) and (2.4.42).

2.4.5 The finite element method to $(\mathbf{NS}'_\epsilon)$

Finite element penalty scheme.

We adopt the same notation of Section 2.2.3. For simplicity, we only consider the $P1b/P1$ approximation. Setting

$$a_{1h}(u_h, v_h, w_h) = \int_{\Omega_h} (u_h \cdot \nabla v_h) w_h \, dx, \quad \forall u_h, v_h, w_h \in V_h.$$

the finite element approximation to penalty problem (2.4.6) reads as: find $(u_h, p_h) \in V_h \times Q_h$ such that,

$$\begin{aligned} a_h(u_h, v_h) + \frac{1}{2}[a_{1h}(u_h, u_h, v_h) - a_{1h}(u_h, v, u_h)] \\ + b_h(v_h, p_h) + \frac{1}{\epsilon} c_h(u_h \cdot n_h, v_h \cdot n_h) = (\tilde{f}, v_h)_{\Omega_h}, \quad \forall v_h \in X_h, \end{aligned} \tag{2.4.47a}$$

$$b_h(u_h, q_h) = 0, \quad \forall q_h \in M_h, \tag{2.4.47b}$$

Theorem 2.4.4. *There exists a solution $(u_h, p_h) \in V_h \times Q_h$ to (2.2.36) with c_h defined by both (2.2.30) and (2.2.31), and the solution satisfies*

$$\|u_h\|_{H^1(\Omega_h)} + \|\mathring{p}_h\|_{L^2(\Omega_h)} + \sqrt{\epsilon}\|u_h \cdot n_h\|_{c_h} \leq C\|\tilde{f}\|_{L^2(\Omega_h)}, \tag{2.4.48}$$

where $p_h = \mathring{p}_h + k_h$, $\mathring{p}_h \in \mathring{Q}_h$, $k_h = \int_{\Omega_h} p_h dx / |\Omega_h|$, and

$$|k_h| \leq C\left(\|\tilde{f}\|_{L^2(\Omega_h)} + \|u_h\|_{H^1(\Omega_h)} + \|u_h\|^2_{H^1(\Omega_h)} + \frac{h}{\epsilon}\right). \tag{2.4.49}$$

Moreover, if $\alpha_1^2 > \|a_{1h}\|\|\tilde{f}\|_{L^2(\Omega_h)}$, then the solution is unique.

Proof. The proof is similar to that of Theorem 2.2.6. $\square$

With a similar argument to Proposition 2.2.1, we have the consistency error estimates of the stationary Navier-Stokes equations.

Proposition 2.4.4. *Let (u,p) and (u_h,p_h) be solutions of (2.4.1) and (2.4.47), respectively. Set $\lambda = -\tau_n(u,p)$, $\lambda_h = \frac{1}{\epsilon}u_h \cdot n_h$. We assume $f \in L^2(\Omega)$, and $(u,p) \in H^2(\Omega)^d \times H^1(\Omega)$, and the same assumption of Lemma 2.2.5. For any $v_h \in V_h$, we set the consistency error*

$$\begin{aligned} E(v_h) :=& a_h(\tilde{u}-u_h, v_h) + \frac{1}{2}[a_{1h}(\tilde{u}-u_h,\tilde{u},v_h) + a_{1h}(u_h,\tilde{u}-u_h,v_h) \\ &- a_{1h}(\tilde{u}-u_h,v_h,\tilde{u}) - a_{1h}(\tilde{u},v_h,\tilde{u}-u_h)] \\ &+ b_h(v_h,\tilde{p}-p_h) + c_h(v_h\cdot n_h, \tilde{\lambda}-\lambda_h), \end{aligned}$$

where $(\tilde{u},\tilde{p})$ is the extension(Lemma 2.2.1) of (u,p) onto $\tilde{\Omega} = \Omega\cup\Omega_h$. Then, we have

$$|E(v_h)| \le Ch\|v_h\|_{H^1(\Omega_h)}. \tag{2.4.50}$$

Error estimates

Theorem 2.4.5. *c_h is defined by (2.2.30). Let (u,p) and (u_h,p_h) be the unique solutions of (2.4.1) and (2.4.47), respectively. Assuming $f \in L^2(\Omega)$, $(u,p) \in H^2(\Omega)^d \times H^1(\Omega)$, and $\alpha_1^2 > \|a_{1h}\|\|\tilde{f}\|_{L^2(\Omega_h)}$, we have*

$$\|\tilde{u}-u_h\|_{H^1(\Omega_h)} + \|\tilde{p}-p_h\|_{L^2(\Omega_h)} \le C(\sqrt{h}+\sqrt{\epsilon}+h/\sqrt{\epsilon}). \tag{2.4.51}$$

Theorem 2.4.6. *Let (u,p) and (u_h,p_h) be solutions of (2.4.1) and (2.4.47), respectively. We assume $f \in L^2(\Omega)$, $(u,p) \in W^{2,\infty}(\Omega)^d \times W^{1,\infty}(\Omega)$, and $\alpha_1^2 > \|a_{1h}\|\|\tilde{f}\|_{L^2(\Omega_h)}$. We also assume $(\tilde{u},\tilde{p})$, the extension of (u,p), satisfy the condition of Lemma 2.2.6, then we have*

$$\|\tilde{u}-u_h\|_{H^1(\Omega_h)} + \|\tilde{p}-p_h\|_{L^2(\Omega_h)} \le C(h+\sqrt{\epsilon}+h^2/\sqrt{\epsilon}), \quad \text{for } d=2, \tag{2.4.52}$$

$$\|\tilde{u}-u_h\|_{H^1(\Omega_h)} + \|\tilde{p}-p_h\|_{L^2(\Omega_h)} \le C(\sqrt{h}+\sqrt{\epsilon}+h/\sqrt{\epsilon}), \quad \text{for } d=3. \tag{2.4.53}$$

We skip the detailed proof of Theorem 2.4.5 and 2.4.6, which are similar to the argument of Theorem 2.2.7 and 2.2.8, respectively.

The numerical experiment

Set $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$. We consider the equation (2.4.1) with exact solution $u = (10x^3y^2, -10x^2y^3)^T$, $p = 10x^2y^2$.

$$\|u\|_{L^2} \simeq 1.11, \quad \|u\|_{H^1} \simeq 6.88.$$

Here $\tau_T(u) \neq 0$, therefore we add $\int_\Gamma \tau_T(u)v_T ds$ to the RHS of variational forms (2.4.5),(2.4.6), and $\int_{\Gamma_h} \tau_T(u)v_{hT} ds$ to (2.2.36).

Newton's method is applied to solve the nonlinear equation(see Sect. 3.2.1(ii)). We test two penalty schemes (2.4.5),(2.4.6) for $P1b/P1$ elements. We compare two implement methods of penalty term(nonreduced-integration scheme (2.2.30) and reduced-integration scheme (2.2.31)), with different choices of ϵ and h($\epsilon \simeq h$ and $\epsilon \simeq h^2$).

From Figure 2.4.1 and 2.4.2, the numerical experiments show the H^1 norm error $\|u-u_h\|_{1,\Omega_h}$ is $O(h)$ for both fine and reduced-integration schemes((2.2.30) and (2.2.31)). Moreover, the L^2 norm error $\|u - u_h\|_{\Omega_h}$ seems to be $O(h^2)$ for reduced-integration scheme with $\epsilon \simeq h^2$. However, the nonreduced-integration fails when $\epsilon \simeq h^2$(or $\epsilon \ll h$), which coincides with our error estimates(Theorem 2.4.5). (The numerical experiments are implemented with software FeniCS).

Notice: In Figure 2.4.2, line $\epsilon \sim h^2, \|\cdot\|_{L^2}$ overlaps with line $y = 2x$; and line $\epsilon \sim h^2, \|\cdot\|_{H^1}$ overlaps with line $\epsilon \sim h, \|\cdot\|_{H^1}$.

Remark

This chapter is based on [24, 50, 51]

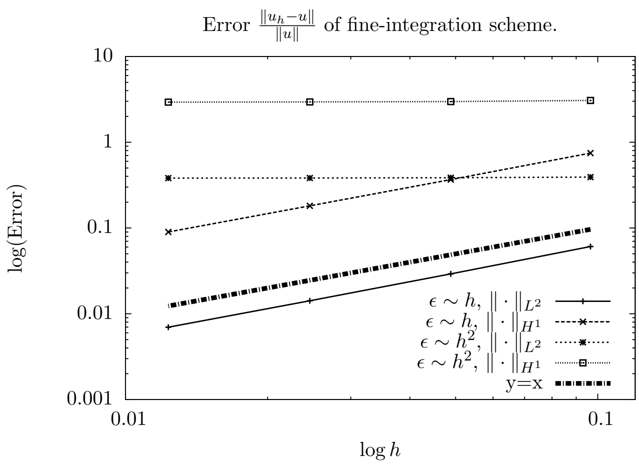

Figure 2.4.1: penalty scheme (2.4.6): nonreduced-integration (2.2.30)

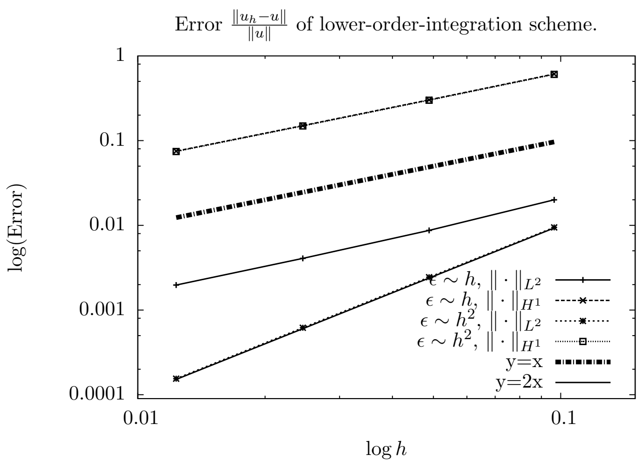

Figure 2.4.2: penalty scheme (2.4.6): reduced-integration (2.2.31)

Chapter 3

The Stokes/Navier-Stokes equations with a unilateral boundary condition of Signorini's type and its penalty method

3.1 Introduction

In this chapter, we consider the Navier-Stokes equations with a unilateral boundary condition of Signorini's type (the inequality boundary condition), and show the application of penalty method to the inequality boundary condition.

Our motivation lies to propose a suitable outflow boundary condition for the Navier-Stokes equations modeling the blood flow in arteries. The outflow boundary condition plays very important role to the solutions governing the blood flow in the large arteries (cf. [17]). Usually, the prescribed constant pressure, traction or velocity are applied to the outflow boundary condition. In many realistic cases, the pressure, traction or velocity on the outflow boundary cannot be prescribed, due to the unknown flow distribution in the modeled domain. In numerical simulation, the free-traction outflow boundary condition is frequently used, which requires no addition implementation of the outflow boundary condition in computation. However, the energy inequality of velocity is not satisfied under the free-traction boundary condition, which may cause the outflow instabilities or "blow-up" of solution in numerical simulation.

We introduce the model problem. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded domain. The boundary $\partial\Omega$ is composed of S (inflow boundary), C (the wall) and Γ (outflow boundary) (see Figure 3.1.1); those S, C and Γ are assumed to be smooth surfaces. In particular, S and Γ are smooth domains in $\mathbb{R}^{d-1}$. That is, S and Γ are line segments ($d = 2$) and flat surfaces ($d = 3$). Then, for $t \in (0, T]$, $T > 0$, we consider the Navier-Stokes equations in Ω,

$$u_t + (u \cdot \nabla)u = \nabla \cdot \sigma(u, p) + f, \quad \text{in } \Omega, \tag{3.1.1a}$$
$$\nabla \cdot u = 0, \quad \text{in } \Omega, \tag{3.1.1b}$$
$$u|_S = b, \tag{3.1.1c}$$
$$u|_C = 0, \tag{3.1.1d}$$
$$u(x, 0) = u_0, \text{ on } \Omega, \tag{3.1.1e}$$

where $\sigma(u, p)$ is the stress tensor defined by (2.1.2). Force f and initial velocity u_0 are given functions. On the wall C we impose the homogeneous Dirichlet boundary condition (3.1.1d). On the inflow boundary S, we give the Dirichlet boundary condition $u|_S = b(t, x)$, where we assume

$$\beta(t) := -\int_S b_n \, ds > 0, \quad \forall t \in [0, T],$$

and $u_0 = b(0)$ on S, $u_0 = 0$ on C.

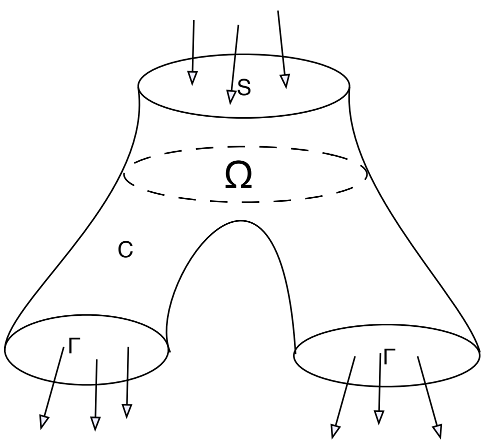

Figure 3.1.1: Ω, S, Γ and C.

If we impose the free-traction boundary condition

$$\tau(u, p) = 0 \quad \text{on } \Gamma,$$

where $\tau(u,p)$ is traction vector defined by (2.1.3), then we cannot obtain the energy inequality such as

$$\|u(T)\|_{L^2}^2 + \int_0^T \|\mathcal{E}(u)\|_{L^2}^2 dt \leq C.$$

Here C is some constant dependent on some norms of f, u_0 and b.

To tackle this problem, various types of artificial outflow boundary condition are proposed. In [7, Chapter VII], [10, 11], the authors introduce and analysis the nonlinear boundary condition

$$\tau(u,p) = -\frac{1}{2}[u_n]_-(u-g) + \tau(g,\pi) \text{ on } \Gamma,$$

where $[w]_\pm = \max\{0, \pm w\}$ and (g,π) is some reference flow defined below by (3.2.2). Under this boundary condition, one can show the energy inequality. In [4, Y. Bazilevs et al.], a regularized traction vector

$$\tilde{\tau}(u,p) = \tau(u,p) - \rho[u_n]_- u$$

is introduced, and they consider the resistance boundary condition

$$\tilde{\tau}_n(u,p) + R\int_\Gamma u_n \, ds + p_0 = 0, \quad \tilde{\tau}_T(u,p) = 0 \text{ on } \Gamma.$$

This boundary condition also satisfies the energy inequality.

These approaches are verified to be important for the overall stability of the computations. However, a certain relation between u and $\tau(u,p)$ on Γ is assumed in order to ensure the energy inequality. Here, we propose another approach. We pose the following unilateral boundary condition of Signorini's type:

$$\left\{ \begin{array}{l} u_n \geq 0, \\ \tau_n(u,p) \geq 0, \;\; u_n \tau_n(u,p) = 0, \;\; \tau_T(u) = 0 \end{array} \right. \text{ on } \Gamma. \tag{3.1.2}$$

(3.1.2) guarantees the energy inequality to the Navier-Stokes problems (3.1.1).

In this chapter, we study the well-posedness of (3.1.1) under the outflow boundary condition (3.1.2) (cf. Theorem 3.3.1, Proposition 3.3.1, 3.3.2.). Since the Signorini's boundary condition leads to a variational inequality for weak form, which is not easy to solve by numerical method. For that purpose, we introduce the penalty method to approximate the variational inequality by variational equation. The well-posedness of penalty problem is also been investigated (cf. Theorem 3.4.1, Proposition 3.4.1, 3.4.2.).

To apply this model problem in numerical simulation, we have to study the error estimates of penalty method and the finite element method to the model problem. As a first step, we consider a simple case of stationary Stokes equations with Signorini's boundary condition (3.1.2). In Section 3.6, We examine not only the well-posedness of Stokes problem and its penalty problem, but also we obtain the error estimates of penalty method.

3.2 The energy inequality and the variational inequality

Reference flow.

To describe the energy inequality, we take a reference flow (g, π).

In view of $\beta(t) = -\int_S b_n(t)\ ds > 0$, for any $t \in [0, T]$, there exists some $g_0(x) \in C_0^\infty(\Gamma)^n$, with

$$\int_\Gamma g_0 \cdot n\ ds = 1, \quad g_0 \cdot n \geq 0. \tag{3.2.1}$$

We set the reference flow (g, π) such that, for all $t \in [0, T]$,

$$-\nabla \cdot \sigma(g, \pi) = 0, \quad \nabla \cdot g = 0, \quad \text{in } \Omega, \tag{3.2.2a}$$

$$g = b \text{ on } S, \quad g = 0 \text{ on } C, \quad g = g_0(x)\beta(t) \text{ on } \Gamma. \tag{3.2.2b}$$

And we find (u, p) of the form

$$u = U + g, \qquad p = P + \pi.$$

Assume $u_0 = g(0)$ on $\partial\Omega$, then we have $U_0 = u_0 - g \in H_0^1(\Omega)^d$. It is equivalent to consider the problem of (U, P), denoted as (**NS**). For all $t \in (0, T)$, (U, P) satisfies

$$U_t + ((U + g) \cdot \nabla)U + (U \cdot \nabla)g - \nabla \cdot \sigma(U, P) = F, \quad \text{in } \Omega, \tag{3.2.3a}$$

$$\nabla \cdot U = 0, \quad \text{in } \Omega, \tag{3.2.3b}$$

$$U = 0, \quad \text{on } S \cup C, \tag{3.2.3c}$$

$$U_n + g_n \geq 0, \quad \tau_n(U + g, P + \pi) \geq 0, \quad \text{on } \Gamma, \tag{3.2.3d}$$

$$(U_n + g_n)\tau_n(U + g, P + \pi) = 0, \quad \tau_T(U) = -\tau_T(g), \quad \text{on } \Gamma, \tag{3.2.3e}$$

$$U(x, 0) = U_0, \quad \text{on } \Omega. \tag{3.2.3f}$$

where $F = f - g_t - (g \cdot \nabla)g$, $U_0 = u_0 - g(0)$.

Theorem 3.2.1 (Energy inequality)**.** *If (U, P) is a smooth solution of (3.2.3), then we have*

$$\sup_{0 \le t \le T} \|U(t)\|_{L^2}^2 + 2\nu \int_0^T \|\mathcal{E}(U)\|_{L^2}^2 \, dt \le C. \tag{3.2.4}$$

The proof of Theorem 3.2.1 is presented later. Let us set some function spaces and bilinear forms, and write the variational form of (**NS**). The following settings are slightly different to Chapter. 2.

Function spaces.

- $V = \{v \in H^1(\Omega)^d \mid v = 0 \text{ on } C \cap S\}, \quad V^\sigma = V \cap \{v \mid \nabla \cdot v = 0\}$.
- $V_0 = H_0^1(\Omega)^d, \quad V_0^\sigma = V_0 \cap \{v \mid \nabla \cdot v = 0\}$.
- $K = \{v \in V \mid v_n + g_n \ge 0 \text{ on } \Gamma\}, \quad K^\sigma = K \cap \{v \mid \nabla \cdot v = 0\}$.
- $Q = L^2(\Omega), \quad \mathring{Q} = L_0^2(\Omega) := \{v \in Q \mid \int_\Omega v dx = 0\}$.
- $M = \begin{cases} H^{\frac{1}{2}}(\Gamma) & \text{if } \overline{\Gamma} \cap \overline{C} = \emptyset, \\ H_{00}^{\frac{1}{2}}(\Gamma) & \text{if } \overline{\Gamma} \cap \overline{C} \ne \emptyset. \end{cases}$
- We denote X' as the dual space of Banach space X. For example, $M' = H^{-\frac{1}{2}}(\Gamma)$.

Bilinear and trilinear forms.

$$a(u, v) = 2\nu \int_\Omega \mathcal{E}(u) : \mathcal{E}(v) \, dx, \quad \forall u, v \in H^1(\Omega)^d, \tag{3.2.5a}$$

$$a_1(u, v, w) = \int_\Omega (u \cdot \nabla) v w \, dx, \quad \forall u, v, w \in H^1(\Omega)^d, \tag{3.2.5b}$$

$$b(v, p) = -\int_\Omega (\nabla \cdot v) p \, dx, \quad \forall v \in H^1(\Omega)^d, \; p \in L^2(\Omega), \tag{3.2.5c}$$

$$[\lambda, \eta] = \text{ the duality paring between } M \text{ and } M', \tag{3.2.5d}$$

$$[[\lambda, \eta]] = \text{ the duality paring between } M^d \text{ and } (M^d)', \tag{3.2.5e}$$

Korn's inequality and *inf-sup* conditions.(cf. [7, 27, 43])

(1) Korn's inequality: there exists a constant $\alpha > 0$, such that,

$$a(v,v) \geq \alpha \|v\|_{H^1}^2, \quad \forall v \in V. \tag{3.2.6}$$

(2) *inf-sup* conditions: there exists constants $\gamma_1, \gamma_2 > 0$, such that,

$$\inf_{q \in \mathring{Q} \backslash \{0\}} \sup_{v \in V_0 \backslash \{0\}} \frac{b(v,q)}{\|v\|_{H^1} \|q\|_{L^2}} \geq \gamma_1, \tag{3.2.7}$$

$$\inf_{\eta \in M' \backslash \{0\}} \sup_{v \in V \backslash \{0\}} \frac{[\eta, v_n]}{\|v\|_{H^1} \|\eta\|_{M'}} \geq \gamma_2. \tag{3.2.8}$$

Lemma 3.2.1. *For all $u, v, w \in H^1(\Omega)^d$, we have, when $d = 2$,*

$$\begin{aligned} |a_1(u,v,w)| &\leq C \|u\|_{L^4} \|v\|_{H^1} \|w\|_{L^4} \\ &\leq C \|u\|_{\Omega}^{\frac{1}{2}} \|u\|_{H^1}^{\frac{1}{2}} \|v\|_{H^1} \|w\|_{L^2}^{\frac{1}{2}} \|w\|_{H^1}^{\frac{1}{2}}. \end{aligned} \tag{3.2.9}$$

When $d = 3$, we have,

$$\begin{aligned} a_1(u,v,w) &\leq C \|u\|_{L^3} \|v\|_{H^1} \|w\|_{L^6} \\ &\leq C \|u\|_{L^2}^{\frac{1}{2}} \|u\|_{H^1}^{\frac{1}{2}} \|v\|_{H^1} \|w\|_{H^1}. \end{aligned} \tag{3.2.10}$$

Moreover, for all $u, v \in V^\sigma$, $d = 2, 3$, we have,

$$\begin{aligned} a_1(u,v,v) &= \frac{1}{2} \int_\Gamma u_n |v|^2 \, ds \\ &\leq \|u_n\|_{L^2(\Gamma)} \|v\|_{L^4}^2 \leq c_1 \|u_n\|_{L^2(\Gamma)} \|v\|_{H^1}^2. \end{aligned} \tag{3.2.11}$$

Proof. It follows form Sobolev's embedding theorem and the trace theorem. □

Remark 3.2.1. Applying Young's inequality and Lemma 3.2.1, for any $\eta_0 > 0$, when $d = 2$, we have,

$$\begin{aligned} |a_1(u,v,u)| &\leq C \|u\|_{L^2} \|u\|_{H^1} \|v\|_{H^1} \\ &\leq \eta_0 \|u\|_{H^1}^2 + C \eta_0^{-1} \|u\|_{H^1}^2 \|v\|_{H^1}^2. \end{aligned} \tag{3.2.12}$$

When $d = 3$,

$$\begin{aligned} |a_1(u,v,u)| &\leq C \|u\|_{L^2}^{\frac{1}{2}} \|u\|_{H^1}^{\frac{3}{2}} \|v\|_{H^1} \\ &\leq \eta_0 \|u\|_{H^1}^2 + C \eta_0^{-3} \|u\|_{H^1}^2 \|v\|_{H^1}^4. \end{aligned} \tag{3.2.13}$$

3.2.1 The re-definition of traction vectors

For $(U, P) \in V \times Q$, we cannot define $\tau(U, P)$ as a function on Γ. However, if (U, P) is smooth and satisfies (3.2.3a), it also satisfies

$$\int_\Gamma \tau(U,P) \cdot v \, d\Gamma = (U_t, v) + a(U, v) + a_1(U + g, U, v) + a_1(U, g, v) + b(v, P) - (F, v) \qquad (\forall v \in V), \quad (3.2.14)$$

where $\tau(U, p)$ is understood as a usual function on Γ.

Based on this identity, we re-define the traction vector $\tau(U, P)$ as a *functional* over M^d for $(U, P) \in V \times Q$. We recall the following result (cf. [20] for $M = H_{00}^{1/2}(\Gamma)$ and [29] for $M = H^{1/2}(\Gamma)$).

Lemma 3.2.2. *There exists an extension operator $E : M^d \to V$ such that $E\eta = \eta$ on Γ and $\|E\eta\|_V \le C\|\eta\|_{M^d}$ for all $\eta \in M^d$. Conversely, for any $w \in V$, we have $\eta = w|_\Gamma \in M^d$ and $\|\eta\|_{M^d} \le C\|w\|_V$.*

As a consequence, we obtain an extension operator $E_n : M \to V$; for any $\eta \in M$,

$$(E_n\eta)_n = \eta \ , (E_n\eta)_T = 0 \ \text{ on } \Gamma, \qquad \|E_n\eta\|_V \le C\|\eta\|_M.$$

Now we propose the re-definition of $\tau(U, P)$ as follows:

$$[[\tau(U,P), \eta]] = (U_t, w_\eta) + a(U, w_\eta) + a_1(U + g, U, w_\eta) + a_1(U, g, w_\eta) + b(w_\eta, P) - (F, w_\eta) \qquad (\eta \in M^d), \quad (3.2.15)$$

where $w_\eta = E\eta \in V$. Actually, the right-hand side of (3.2.15) does not depend on the way of extension; Hence, this definition is well-defined. Similarly, we re-define as

$$[[\tau_T(U), \eta]] = (U_t, w_\eta) + a(U, w_\eta) + a_1(U + g, U, w_\eta) + a_1(U, g, w_\eta) + b(w_\eta, P) - (F, w_\eta) \qquad (\eta \in M^d \text{ with } \eta_n = 0;\ w_\eta = E\eta) \quad (3.2.16)$$

and

$$[\tau_n(U,P), \eta] = (U_t, w_\eta) + a(U, w_\eta) + a_1(U + g, U, w_\eta) + a_1(U, g, w_\eta) + b(w_\eta, P) - (F, w_\eta) \qquad (\eta \in M;\ w_\eta = E_n\eta). \quad (3.2.17)$$

Then, we deduce an expression

$$[[\tau(U,P), \eta]] = [\tau_n(U,P), \eta_n] + [[\tau_T(U), \eta_T]] \qquad (\eta \in M^d). \qquad (3.2.18)$$

On the other hand, we will assume that $\tau(g,\pi) \in H^1(0,T;L^2(\Gamma)^d)$ (see, (A1) below) so that we have

$$[[\tau(g,\pi),\eta]] = \int_\Gamma \tau(g,\pi)\cdot\eta \, d\Gamma \quad (\eta \in M^d).$$

3.2.2 Variational form of (NS).

(**NSE**): For a.e. $t \in (0,T)$, find $(U(t),P(t)) \in V \times Q$, with $U_t \in V$, such that

$$\begin{aligned} &(U_t,v) + a(U,v) + a_1(U+g,U,v) + a_1(U,g,v) \\ &\quad + b(v,P) = (F,v) \quad \forall v \in V_0, \end{aligned} \tag{3.2.19a}$$

$$b(U,q) = 0, \quad \forall q \in Q, \tag{3.2.19b}$$

$$U = 0, \quad \text{on } (S \cup C), \tag{3.2.19c}$$

$$U_n + g_n \geq 0, \text{ on } \Gamma, \tag{3.2.19d}$$

$$[\tau_n(U+g,P+\pi),\eta] \geq 0, \quad \forall \eta \in M, \ \eta \geq 0, \tag{3.2.19e}$$

$$[\tau_n(U+g,P+\pi),(U_n+g_n)] = 0, \tag{3.2.19f}$$

$$[[\tau_T(U)+\tau_T(g),\eta]] = 0, \quad \forall \eta \in M, \tag{3.2.19g}$$

$$U(x,0) = U_0, \quad \text{on } \Omega. \tag{3.2.19h}$$

Proof of Theorem 3.2.1(Energy inequality). Suppose that (U,P) is a smooth solution of (3.2.19), multiplying U to (3.2.3a), it yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|U(t)\|_{L^2}^2 + 2\nu\int_\Omega |\mathcal{E}(U)|^2 dx + \int_\Omega ((U+g)\cdot\nabla)U\cdot U dx \\ &\qquad = -\int_\Omega (U\cdot\nabla)g\cdot U dx + \int_\Omega F\cdot U dx. \end{aligned} \tag{3.2.20}$$

Applying Lemma 3.2.1 and Remark 3.2.1, we have, for any $\eta_0 > 0$,

$$\int_\Omega |(U\cdot\nabla g)U| dx \leq \begin{cases} \eta_0\|U\|_{H^1}^2 + C\eta_0^{-1}\|U\|_{L^2}^2\|g\|_{H^1}^2, & \text{for } d=2, \\ \eta_0\|U\|_{H^1}^2 C + \eta_0^{-3}\|g\|_{H^1}^4\|U\|_{L^2}^2, & \text{for } d=3, \end{cases}$$

$$\int_\Omega |F\cdot U| dx \leq C\|F\|_{(H^1(\Omega)^d)'}\|U\|_{H^1} \leq \eta_0\|U\|_{H^1}^2 + C\eta_0^{-1}\|F\|_{(H^1(\Omega)^d)'}^2.$$

In view of $U_n + g_n \geq 0$ on Γ, and

$$\int_\Omega ((U+g)\cdot\nabla)U\cdot U dx = \frac{1}{2}\int_\Gamma (U_n+g_n)|U|^2 ds \geq 0,$$

from (3.2.20), we see that, for any $\eta_0 > 0$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|U(t)\|_{L^2}^2 + 2\nu\|\mathcal{E}(U)\|_{L^2}^2 - 2\eta_0\|U\|_{H^1}^2 \\ &\leq \begin{cases} C\eta_0^{-1}\|U\|_{L^2}^2\|g\|_{H^1}^2 + C\eta_0^{-1}\|F\|_{(H^1(\Omega)^d)'}^2, & \text{for } d=2, \\ C\eta_0^{-3}\|g\|_{H^1}^4\|U\|_{L^2}^2 + C\eta_0^{-1}\|F\|_{(H^1(\Omega)^d)'}^2, & \text{for } d=3. \end{cases} \end{aligned} \tag{3.2.21}$$

From Korn's inequality,

$$\int_\Omega |\mathcal{E}(U)|^2 dx \geq \alpha\|U\|_{H^1}^2, \qquad \text{for some } \alpha > 0,$$

and for sufficiently small η_0, such that

$$\nu\alpha - 2\eta_0 > 0,$$

applying Gronwall's inequality to (3.2.21), it yields (3.2.4). □

(**NSE**) can be written into a variational inequality, denoted as (**NSI**).

(**NSI**): For a.e. $t \in (0,T)$, find $(U(t), P(t)) \in K \times Q$, with $U_t \in V$, such that

$$\begin{aligned} &(U_t, v-U) + a(U, v-U) + a_1(U+g, U, v-U) + a_1(U, g, v-U) \\ &\quad + b(v-U, P) \geq (F, v-U) - [[\tau(g,\pi), v-U]] \quad \forall v \in K, \end{aligned} \tag{3.2.22a}$$

$$b(U,q) = 0, \quad \forall q \in Q, \tag{3.2.22b}$$

$$U(x,0) = U_0, \quad \text{on } \Omega. \tag{3.2.22c}$$

Definition 3.2.1. We say that (U,P) is a solution of (**NSE**) if and only if

$$U \in L^\infty(0,T;V), \quad U' \in L^2(0,T;V) \cap L^\infty(0,T;L^2),$$

$$P \in L^\infty(0,T;Q),$$

and (U,P) satisfies (3.2.19).

Definition 3.2.2. We say that (U,P) is a solution of (**NSI**) if and only if

$$U \in L^\infty(0,T;K), \quad U' \in L^2(0,T;V) \cap L^\infty(0,T;L^2),$$

$$P \in L^\infty(0,T;Q),$$

and (U,P) satisfies (3.2.22).

Theorem 3.2.2. (**NSE**) *is equivalent to* (**NSI**). *Thus, a solution of* (**NSE**) *solves* (**NSI**) *and the converse is also true.*

Proof. First, letting (U,P) be a solution of (NSE), we show (U,P) satisfies (NSI). Let $v \in K$ be arbitrary. Since $v - U \in V$, we see from (3.2.15)

$$\begin{aligned}(U_t, v-U) + a(U, v-U) + a_1(U+g, U, v-U) + a_1(U, g, v-U)\\ + b(v-U, P) - [[\tau(U,P), v-U]] = (F, v-U).\end{aligned}$$

Thus,

$$\begin{aligned}(U_t, v-U) + a(U, v-U) + a_1(U+g, U, v-U) + a_1(U, g, v-U)\\ + b(v-U, P) - [[\tau(U,P) + \tau(g,\pi), v-U]] = (F, v-U).\end{aligned}$$

Since $v_n + g_n \geq 0$ a.e. Γ, by using (3.2.18), (3.2.19e) and (3.2.19f)

$$\begin{aligned}&[[\tau(U,P) + \tau(g,\pi), v-U]]\\ &= [\tau_n(U,P) + \tau_n(g,\pi), v_n - U_n] + [[\tau_T(U) + \tau_T(g), v_T - U_T]]\\ &= [\tau_n(U,P) + \tau_n(g,\pi), v_n + g_n] - [\tau_n(U,P) + \tau_n(g,\pi), U_n + g_n] \geq 0.\end{aligned}$$

Hence, (U,P) solves (NSI).

Conversely, letting (U,P) be a solution to (NSI), we show (U,P) satisfies (NSE).

For any $\phi \in V_0$, substituting $v = U \pm \phi \in K$ into (3.2.22a), we immediately obtain (3.2.19a).

Let $\varphi \in V$ with $\varphi_n = 0$ on Γ be arbitrary. Substituting $v = U \pm \varphi \in K$ into (3.2.22a), we have

$$\begin{aligned}(U_t, \varphi) + a(U, \varphi) + a_1(U+g, U, \varphi) + a_1(U, g, \varphi)\\ + b(\varphi, P) = (F, \varphi) - [[\tau_T(g), \varphi_T]].\end{aligned}$$

This, together with (3.2.16), implies (3.2.19g). Let $w \in V$ with $w_n \geq 0$ on Γ be arbitrary. Substituting $v = w + U \in K$ into (3.2.22a), we have (3.2.19e).

Finally, substituting $v = -g \in K$ and $v = 2U + g \in K$ into (3.2.22a), we deduce

$$\begin{aligned}(U_t, U+g) + a(U, U+g) + a_1(U+g, U, U+g) + a_1(U, g, U+g)\\ + b(U+g, P) = (F, U+g) - [\tau(g,\pi), U+g].\end{aligned} \tag{3.2.23}$$

This, together with (3.2.15), gives (3.2.19f).

□

3.3 The well-posedness of (NSI)

We are concerned with the class of solutions of Ladyzhenskaya type(cf. [27]), that is to find (u, p) satsifying,

$$u \in L^\infty(0,T;V^\sigma), \quad u_t \in L^2(0,T;V^\sigma) \cap L^\infty(0,T;L^2(\Omega)^d),$$
$$p \in L^\infty(0,T;Q).$$

Assumptions.

(A1) $f \in H^1(0,T;L^2(\Omega)^d)$, $\tau(g,\pi)|_\Gamma \in H^1(0,T;L^2(\Gamma)^d)$.

(A2) $g \in H^2(0,T;L^2(\Omega)^d) \cap L^\infty(0,T;V^\sigma)$. $g' \in L^2(0,T;V^\sigma)$.

(A3) $g_n \geq 0$ on Γ, $\int_\Gamma g_n \, ds = -\int_S b_n \, ds = \beta(t) \geq \beta_0 > 0$. $\beta(t) \in C^2(0,T)$.

(A4) $U_0 \in V_0^\sigma \cap H^2(\Omega)^d$, satisfying

$$-(\nu \Delta U_0, v) = a(U_0, v) + \int_\Gamma \tau(g,\pi)(0) v \, ds, \quad \forall v \in V^\sigma. \tag{3.3.1}$$

Remark 3.3.1. (A1), (A2) $\Rightarrow F \in H^1(0,T;L^2(\Omega)^d)$.

Theorem 3.3.1. *Under the assumptions (A1)-(A4), when $d = 2$, there exists a unique solution (U, P) to (3.2.22) for any $T \in (0, \infty)$, that is*

$$U \in L^\infty(0,T;V^\sigma), \quad U_t \in L^2(0,T;V^\sigma) \cap L^\infty(0,T;L^2(\Omega)^d), \tag{3.3.2}$$

$$P \in L^\infty(0,T;Q). \tag{3.3.3}$$

When $d = 3$, the same conclusion holds for a smaller time interval $(0, \mathring{T}]$.

($\mathbf{NSI}^\sigma$): For a.e. $t \in (0,T)$, find $U \in K^\sigma$, with $U_t \in V^\sigma$, such that

$$\begin{aligned}(U_t, v-U) + a(U, v-U) + a_1(U+g, U, v-U) + a_1(U, g, v-U)& \\ \geq (F, v-U) - [\tau(g,\pi), v-U] \quad \forall v \in K^\sigma,&\end{aligned} \tag{3.3.4a}$$

$$U(x,0) = U_0, \quad \text{on } \Omega. \tag{3.3.4b}$$

Proposition 3.3.1 (Existence of P)**.** *Let U be the solution to (3.3.4) satisfying (3.3.2), then there exists a unique $P \in L^\infty(0,T;Q)$, such that (U, P) is the solution to (3.2.22).*

Proof. (Existence) Let $\phi \in V_0 \cap V^\sigma$ be arbitrary. Substitution $v = \phi + U \in K^\sigma$ into (3.3.4) yields

$$(U_t, \phi) + a(U, \phi) + a_1(U + g, U, \phi) + a_1(U, g, \phi) = (F, \phi).$$

Then, there exists a unique $\mathring{P} \in \mathring{Q}$ (cf. [36, Lemma IV.1.4.3]) such that, for a.e. $t \in (0, T)$,

$$(U', \phi) + a(U, \phi) + a_1(U + g, U, \phi) + a_1(U, g, \phi) + b(v, \mathring{P}) = (F, \phi) \quad \forall \phi \in V_0 \tag{3.3.5}$$

and

$$\|\mathring{P}\|_{L^2} \le C(\|U'\|_{L^2} + \|U\|_{H^1} + \|F\|_{L^2} + \|(U+g)\cdot\nabla U\|_{L^2} + \|U\cdot\nabla g\|_{L^2}). \tag{3.3.6}$$

We will show that there exists $k \in L^\infty(0, T)$ such that $(U, \mathring{P} + k)$ solves (NS-E).

First, by virtue of (3.3.5), (3.2.19a) is satisfied for $P = \mathring{P} + k$ with any $k \in L^\infty(0, T)$.

Recall that (3.2.18) and (3.3.4a) give

$$\begin{aligned}[][\tau_T(U), v_T - U_T]] + [\tau_n(U, \mathring{P} + k), v_n - U_n] \\ \ge -[[\tau_T(g), v_T - U_T]] - [\tau_n(g, \pi), v_n - U_n] \quad \forall v \in K^\sigma. \end{aligned} \tag{3.3.7}$$

Let $\psi \in C_0^\infty(\Gamma)^d$ be a function such that supp $\psi \subset \Gamma$ and $\psi_n = 0$. Then, since $\int_\Gamma \psi_n \, d\Gamma = 0$, there is a function $w \in V$ such that $w|_\Gamma = \psi$, $\nabla \cdot w = 0$ and $\|w\|_V \le C\|\psi\|_{M^d}$. Substituting $v = U \pm w \in K^\sigma$ into (3.3.7), we have

$$[[\tau_T(U), \psi_T]] = [\tau_T(g), \psi_T].$$

By the density, this implies (3.2.19g). Moreover, since (3.3.7) is valid for an arbitrary $k \in L^\infty(0, T)$, we have

$$[\tau_n(U, \mathring{P}) + \tau_n(g, \pi), v_n + g_n] \ge [\tau_n(U, \mathring{P}) + \tau_n(g, \pi), U_n + g_n] \quad \forall v \in K^\sigma. \tag{3.3.8}$$

At this stage, we set

$$\gamma = \gamma(t) = \frac{1}{\beta}[\tau_n(U + g, \mathring{P} + \pi), U_n + g_n] \tag{3.3.9}$$

and take $k = \gamma$.

Then, noting $\int_\Gamma U_n \; d\Gamma = 0$ by $\nabla \cdot U = 0$ in Ω and $U|_{S\cup C} = 0$, we can calculate as

$$\begin{aligned}[\tau_n(U, \mathring{P} + \gamma) + \tau_n(g, \pi), U_n + g_n] &= [\tau_n(U, \mathring{P}) + \tau_n(g, \pi), U_n + g_n] - \gamma \int_\Gamma g_n \; d\Gamma \\ &= [\tau_n(U, \mathring{P}) + \tau_n(g, \pi), U_n + g_n] - \gamma\beta \\ &= 0;\end{aligned}$$

which implies (3.2.19e).

For the time being, we admit

$$\gamma = \inf_{\eta \in Y} [\tau_n(U + g, \mathring{P} + \pi), \eta], \tag{3.3.10}$$

where

$$Y = \left\{ \eta \in M \mid \eta \geq 0, \; \eta \not\equiv 0, \; \int_\Gamma \eta \; d\Gamma = 1 \right\}.$$

For $\xi \in M$ with $\xi \geq 0$ and $\xi \not\equiv 0$, we have, by setting $m = \int_\Gamma \xi \; d\Gamma \neq 0$,

$$\begin{aligned}[\tau_n(U, \mathring{P} + \gamma) + \tau_(g, \pi), \xi] &= [\tau_n(U, \mathring{P}) + \tau_n(g, \pi), \xi] - \gamma m \\ &= m[\tau_n(U, \mathring{P}) + \tau_n(g, \pi), \xi/m] - \gamma m \\ &\geq m\gamma - \gamma m = 0.\end{aligned}$$

Hence, we get (3.2.19e).

It remains to verify (3.3.10). Let $\eta \in Y$ be arbitrary and set $\tilde{\eta} = \beta\eta - g_n \in M$. Since $\int_\Gamma \tilde{\eta} \; d\Gamma = 0$, there exists $\tilde{v} \in V^\sigma$ such that $\tilde{v}_n|_\Gamma = \tilde{\eta}$. Then, the function $\tilde{v}$ satisfies that $\tilde{v}_n + g_n = \beta\eta \geq 0$ on Γ. Thus, $\tilde{v} \in K^\sigma$. Consequently, we have by (3.3.8)

$$\begin{aligned}[\tau_n(U, \mathring{P}) + \tau_n(g, \pi), \eta] &= \left[\tau_n(U, \mathring{P}) + \tau_n(g, \pi), \frac{\tilde{\eta} + g_n}{\beta}\right] \\ &= \left[\tau_n(U, \mathring{P}) + \tau_n(g, \pi), \frac{\tilde{v}_n + g_n}{\beta}\right] \\ &\geq \frac{1}{\beta}[\tau_n(U, \mathring{P}) + \tau_n(g, \pi), U_n + g_n] = \gamma;\end{aligned}$$

which yields (3.3.10).

(Regularity) According to the expression (3.3.9) and the definition (3.2.17), we deduce, for a.e. $t \in (0, T)$,

$$|\gamma| \leq C_1,$$

where $C_1 = C_1(t)$ denotes a positive function in $L^\infty(0, T)$ which depends only on $\|U_t\|$, $\|U\|_1$, $\|F\|$ and $\|g\|_1$. This, together with (3.3.6), gives $P \in L^\infty(0, T; Q)$.
(Uniqueness) Suppose that there is another pressure P'. Since $\mathring{P}$ and k are unique, we have

$$P' + k' = \mathring{P}, \quad k' \equiv -\frac{1}{|\Omega|} \int_\Omega P' \, dx = k.$$

Hence, $P = P'$. $\square$

Proposition 3.3.2 (Uniqueness). *If (U_1, P_1) and (U_2, P_2) are two strong solutions to (3.2.22), then $(U_1, P_1) = (U_2, P_2)$.*

Proof. From Proposition 3.3.1, we know that P is uniquely determined by U; therefore, we only need to show the uniqueness of U.

Suppose U_1, U_2 are two strong solutions to (3.2.22). Let $w = U_1 - U_2$. From (3.2.22), we have

$$\begin{aligned}(U_1', U_2 - U_1) &+ a(U_1, U_2 - U_1) + a_1(U_1 + g, U_1, U_2 - U_1) \\ &+ a_1(U_1, g, U_2 - U_1) \geq (F, U_2 - U_1) - [\tau(g, \pi), U_2 - U_1],\end{aligned} \tag{3.3.11}$$

$$\begin{aligned}(U_2', U_1 - U_2) &+ a(U_2, U_1 - U_2) + a_1(U_2 + g, U_2, U_1 - U_2) \\ &+ a_1(U_2, g, U_1 - U_2) \geq (F, U_1 - U_2) - [\tau(g, \pi), U_1 - U_2].\end{aligned} \tag{3.3.12}$$

From (3.3.11) and (3.3.11), we obtain

$$(w', w) + a(w, w) + a_1(U_2 + g, w, w) \leq -a_1(w, U_1 + g, w). \tag{3.3.13}$$

In view of Korn's inequality (3.2.6), Lemma 3.2.1, Remark 3.2.1 and

$$a_1(U_2 + g, w, w) = \frac{1}{2} \int_\Gamma \underbrace{(U_2 \cdot n + g_n)}_{\geq 0} |w|^2 \, ds \geq 0,$$

we have

$$\begin{aligned}&\frac{1}{2}\|w(t)\|_{L^2}^2 + \alpha \|w(t)\|_{H^1}^2 \\ &\leq \begin{cases} \eta_0 \|w\|_{H^1}^2 + C\eta_0^{-1} \|U_1 + g\|_{H^1}^2 \|w\|_{L^2}^2, & \text{for } d = 2, \\ \eta_0 \|w\|_{H^1}^2 + C\eta_0^{-3} \|U_1 + g\|_{H^1}^4 \|w\|_{L^2}^2, & \text{for } d = 3, \end{cases}\end{aligned} \tag{3.3.14}$$

Let η_0 be sufficiently small such that, $\alpha - \eta_0 > \alpha/2$, then from Gronwall's inequality, we have, for all $t \in (0, T]$,

$$\|w(t)\|_{L^2}^2 + \alpha \int_0^t \|w(1)\|_{H^1}^2 \leq C e^{C_{\eta_0} t \|U_1 + g\|_{L^\infty(0,t;V)}} \|w(0)\|_{L^2}^2. \tag{3.3.15}$$

Since $w(0) = U_1(0) - U_2(0) = 0$, we conclude $U_1 = U_2$. $\square$

3.4 Penalty method

We introduce a penalty problem to (**NS**), denoted as ($\mathbf{NS}_\epsilon$). Let $0 < \epsilon \ll 1$. ($\mathbf{NS}_\epsilon$) reads as: for a.e. $t \in (0,T)$, find $(U_\epsilon, P_\epsilon) \in V \times Q$, with $U'_\epsilon \in V$, such that,

$$U'_\epsilon + (U_\epsilon + g\cdot\nabla)U_\epsilon + (U_\epsilon\cdot\nabla)g - \nabla\cdot\sigma(U_\epsilon, P_\epsilon) = F, \quad \text{in } \Omega \tag{3.4.1a}$$

$$\nabla\cdot U_\epsilon = 0, \quad \text{in } \Omega, \tag{3.4.1b}$$

$$U_\epsilon = 0, \quad \text{on } S\cup C, \tag{3.4.1c}$$

$$\tau_n(U_\epsilon + g, P_\epsilon + \pi) = \frac{1}{\epsilon}[U_{\epsilon n} + g_n]_-, \quad \tau_T(U_\epsilon) = -\tau_T(g), \quad \text{on } \Gamma \tag{3.4.1d}$$

$$U_\epsilon(x,0) = u_0 - g(0), \quad \text{on } \Omega, \tag{3.4.1e}$$

where $[v]_- = v - [v]_+$, $[v]_+ = \max\{0, v\}$. We write the variational form of ($\mathbf{NS}_\epsilon$), denoted as ($\mathbf{NS}_\epsilon\mathbf{E}$).

($\mathbf{NS}_\epsilon\mathbf{E}$): For a.e. $t \in (0,T)$, find $(U_\epsilon, P_\epsilon) \in V \times Q$, with $U'_\epsilon \in V$, such that

$$\begin{aligned}(U'_\epsilon, v) &+ a(U_\epsilon, v) + a_1(U_\epsilon + g, U_\epsilon, v) + a_1(U_\epsilon, g, v) + b(v, P_\epsilon)\\ &- \frac{1}{\epsilon}\int_\Gamma [U_{\epsilon n} + g_n]_- v_n \, ds = (F, v) - \int_\Gamma \tau(g,\pi)v \quad \forall v \in V,\end{aligned} \tag{3.4.2a}$$

$$b(U_\epsilon, q) = 0, \quad \forall q \in Q, \tag{3.4.2b}$$

$$U_\epsilon(x,0) = U_0, \quad \text{on } \Omega. \tag{3.4.2c}$$

Well-posedness of penalty problem

Theorem 3.4.1. *Under the assumptions (A1)-(A4), when $d = 2$, there exists a unique strong solution (U_ϵ, P_ϵ) to (3.4.2) for any $T \in (0,\infty)$, that is*

$$U_\epsilon \in L^\infty(0,T;V^\sigma), \quad U'_\epsilon \in L^2(0,T;V^\sigma) \cap L^\infty(0,T;L^2(\Omega)^d), \tag{3.4.3}$$

$$P_\epsilon \in L^\infty(0,T;Q). \tag{3.4.4}$$

When $d = 3$, the same conclusion holds for a smaller time interval $(0,T']$.

($\mathbf{NS}_\epsilon\mathbf{E}^\sigma$): For a.e. $t \in (0,T)$, find $U_\epsilon \in V^\sigma$, with $U'_\epsilon \in V^\sigma$, $t \in (0,T)$, such that

$$\begin{aligned}(U'_\epsilon, v) &+ a(U_\epsilon, v) + a_1(U_\epsilon + g, U_\epsilon, v) + a_1(U_\epsilon, g, v)\\ &- \frac{1}{\epsilon}\int_\Gamma [U_{\epsilon n} + g_n]_- v_n \, ds = (F, v) - \int_\Gamma \tau(g,\pi)v \quad \forall v \in V^\sigma,\end{aligned} \tag{3.4.5a}$$

$$U_\epsilon(x,0) = U_0, \quad \text{on } \Omega. \tag{3.4.5b}$$

Lemma 3.4.1. *Let U_ϵ be the strong solution to (3.4.5), that is U_ϵ satisfies (3.4.3), then we have*

$$\|[U_{\epsilon n} + g_n]_-\|_{L^2(\Gamma)} \leq C\sqrt{\epsilon}. \tag{3.4.6}$$

Proof. Substituting $v = U_\epsilon$ into (3.4.5), it yields

$$\begin{aligned} -\frac{1}{\epsilon}\int_\Gamma [U_{\epsilon n} + g_n]_- U_{\epsilon n}\ ds = (F, U_\epsilon) - \int_\Gamma \tau(g,\pi) U_\epsilon\ ds - (U'_\epsilon, U_\epsilon) \\ - a(U_\epsilon, U_\epsilon) + a_1(U_\epsilon + g, U_\epsilon, U_\epsilon) + a_1(U_\epsilon, g, U_\epsilon). \end{aligned} \tag{3.4.7}$$

Since $g_n \geq 0$, we see that

$$\begin{aligned} LHS &= -\frac{1}{\epsilon}\int_\Gamma [U_{\epsilon n} + g_n]_-(U_{\epsilon n} + g_n - g_n)\ ds \\ &= \frac{1}{\epsilon}\int_\Gamma |[U_{\epsilon n} + g_n]_-|^2\ ds + \frac{1}{\epsilon}\int_\Gamma [U_{\epsilon n} + g_n]_- g_n\ ds \\ &\geq \frac{1}{\epsilon}\|[U_{\epsilon n} + g_n]_-\|^2_{L^2(\Gamma)}. \end{aligned}$$

In view of U_ϵ satisfies (3.4.3), the RHS of (3.4.7) is bounded. And we have

$$\epsilon^{-1}\|[U_{\epsilon n} + g_n]_-\|^2_{L^2(\Gamma)} \leq C.$$

□

Proposition 3.4.1 (Existence of P_ϵ)**.** *Let U_ϵ be the strong solution to (3.4.5) satisfying (3.4.3), then there exists a unique $P_\epsilon \in L^\infty(0,T;Q)$, such that (U_ϵ, P_ϵ) is the solution to (3.4.2).*

Proof. From (3.4.5), there exists a unique $\mathring{P}_\epsilon \in \mathring{Q}$ (cf. [36, Lemma IV.1.4.3]) such that

$$\begin{aligned} (U'_\epsilon, v) + a(U_\epsilon, v) + a_1(U_\epsilon + g, U_\epsilon, v) + a_1(U_\epsilon, g, v) \\ + b(v, \mathring{P}_\epsilon) = (F, v) \qquad \forall v \in V_0 \end{aligned} \tag{3.4.8}$$

and

$$\|\mathring{P}_\epsilon\|_{L^2} \leq C(\|U'_\epsilon\|_{L^2} + \|U_\epsilon\|_{H^1} + \|(U_\epsilon + g)\cdot\nabla U_\epsilon\|_{L^2} + \|U_\epsilon \cdot \nabla g\|_{L^2} + \|F\|_{L^2}).$$

We write $C_1 = C_1(t)$ to express a positive function in $L^\infty(0,T)$ which depends only on $\|U'_\epsilon\|_{L^2}$, $\|U_\epsilon\|_{H^1}$, $\|F\|_{L^2}$ and $\|g\|_{H^1}$. Thus, we have

$$\|\mathring{P}_\epsilon\|_{L^2} \leq C_1. \tag{3.4.9}$$

We will show that there is $k_\epsilon \in L^\infty(0,\infty)$ such that (U_ϵ, P_ϵ) with $P_\epsilon = \mathring{P}_\epsilon + k_\epsilon$ is a solution of (NS$_\epsilon$-E).

Recalling (3.2.17) and using (3.4.5a), we have

$$\begin{aligned}[\tau_n(U_\epsilon, P_\epsilon), v_n] &= (U_{\epsilon,t}, v) + a(U_\epsilon, v) + a_1(U_\epsilon + g, U_\epsilon, v) \\ &\qquad + a_1(U_\epsilon, g, v) + b(v, P_\epsilon) - (F, v) \\ &= \frac{1}{\epsilon}\int_\Gamma [U_{\epsilon n} + g_n]_- v_n - [[\tau_n(g,\pi), v_n]] \quad (v \in V^\sigma,\ v_T|_\Gamma = 0).\end{aligned}$$

Hence,

$$\left[\tau_n(U_\epsilon, P_\epsilon) + \tau_n(g,\pi) - \epsilon^{-1}[\tau_n(U_\epsilon, P_\epsilon), v_n], \eta\right] = 0 \qquad (\eta \in M^\sigma), \quad (3.4.10)$$

where

$$M^\sigma = \left\{\eta \in M \mid \int_\Gamma \eta \, d\Gamma = 0\right\}.$$

Now we introduce

$$Z = \left\{\phi \in C_0^\infty(\Gamma) \mid \int_\Gamma \phi = 1\right\}$$

and take (and fix below) $\phi \in Z$. Then, for any $v \in V$, $\hat{\eta} = v_n - \alpha\phi$ with $\alpha = \int_\Gamma v_n \, d\Gamma$ belongs to M_0. Therefore, by (3.4.10),

$$\begin{aligned}&[\tau_n(U_\epsilon, P_\epsilon) + \tau_n(g,\pi) - \epsilon^{-1}[U_{\epsilon n} + g_n]_-, v_n] \\ &\quad= [\tau_n(U_\epsilon, P_\epsilon) + \tau_n(g,\pi) - \epsilon^{-1}[U_{\epsilon n} + g_n]_-, v_n - \alpha\phi] \\ &\qquad\quad + [\tau_n(U_\epsilon, P_\epsilon) + \tau_n(g,\pi) - \epsilon^{-1}[U_{\epsilon n} + g_n]_-, \alpha\phi] \\ &\quad= \alpha[\tau_n(U_\epsilon, P_\epsilon) + \tau_n(g,\pi) - \epsilon^{-1}[U_{\epsilon n} + g_n]_-, \phi] \qquad (v \in V).\end{aligned}$$

Now, since

$$\begin{aligned}&[\tau_n(U_\epsilon, P_\epsilon) + \tau_n(g,\pi) - \epsilon^{-1}[U_{\epsilon n} + g_n]_-, \phi] \\ &\qquad\qquad = [\tau_n(U_\epsilon, \mathring{P}_\epsilon) + \tau_n(g,\pi) - \epsilon^{-1}[U_{\epsilon n} + g_n]_-, \phi] - k_\epsilon,\end{aligned}$$

choosing

$$k_\epsilon = [\tau_n(U_\epsilon, \mathring{P}_\epsilon) + \tau_n(g,\pi) - \epsilon^{-1}[U_{\epsilon n} + g_n]_-, \phi] \qquad (3.4.11)$$

we obtain

$$[\tau_n(U_\epsilon, P_\epsilon) + \tau_n(g,\pi) - \epsilon^{-1}[U_{\epsilon n} + g_n]_-, v_n] = 0 \quad (v \in V);$$

which, together with (3.2.17), implies (3.4.2a).

It should be checked that k_ϵ defined as (3.4.11) actually independent of $\phi \in Z$ and it represents a function only of t. We let $\phi, \phi' \in Z$ with $\phi \not\equiv \phi'$. Then $\eta = \phi - \phi' \in M^\sigma$. Hence, by (3.4.10),

$$[\tau_n(U_\epsilon, P_\epsilon) + \tau_n(g,\pi) - \epsilon^{-1}[U_{\epsilon n} + g_n]_-, \phi] \\ = [\tau_n(U_\epsilon, P_\epsilon) + \tau_n(g,\pi) - \epsilon^{-1}[U_{\epsilon n} + g_n]_-, \phi'],$$

which means that k_ϵ does not depend on the choice of $\phi \in Z$.

Finally, in view of (3.4.11), (3.2.17) and (3.4.6), we get

$$|k_\epsilon| \leq C_1.$$

Combining this with (3.4.9), we conclude $P_\epsilon \in L^\infty(0,T;Q)$. □

Proposition 3.4.2 (Uniqueness)**.** *If $(U_{\epsilon 1}, P_{\epsilon 1})$ and $(U_{\epsilon 2}, P_{\epsilon 2})$ are two strong solutions to (3.4.2), then $(U_{\epsilon 1}, P_{\epsilon 1}) = (U_{\epsilon 2}, P_{\epsilon 2})$.*

Proof. Since P_ϵ is uniquely determined by U_ϵ from Proposition 3.4.1, we show $U_{\epsilon 1} = U_{\epsilon 2}$. Let $w = U_\epsilon^1 - U_\epsilon$, from (3.4.2), we have, for any $v \in V^\sigma$,

$$\begin{aligned}(w', v) + a(w, v) + a_1(U_{\epsilon 1} + g, U_{\epsilon 1}, v) - a_1(U_{\epsilon 2} + g, U_{\epsilon 2}, v) \\ + a_1(w, g, v) - \frac{1}{\epsilon}\int_\Gamma ([U_{\epsilon 1}\cdot n + g_n]_- - [U_{\epsilon 2}\cdot n + g_n]_-) v_n \, ds = 0.\end{aligned} \tag{3.4.12}$$

Substituting $v = w$ into (3.4.12), it yields

$$\begin{aligned}(w', w) + a(w, w) - \frac{1}{\epsilon}\int_\Gamma ([U_{\epsilon 1}\cdot n + g_n]_- - [U_{\epsilon 2}\cdot n + g_n]_-) w_n \, ds \\ + a_1(U_{\epsilon 2} + g, w, w) = -a_1(w, U_{\epsilon 1} + g, w).\end{aligned} \tag{3.4.13}$$

We show that

$$\begin{aligned}&- \int_\Gamma ([U_{\epsilon 1}\cdot n + g_n]_- - [U_{\epsilon 2}\cdot n + g_n]_-) w_n \, ds \\ =& - \int_\Gamma ([U_{\epsilon 1}\cdot n + g_n]_- - [U_{\epsilon 2}\cdot n + g_n]_-)(U_{\epsilon 1}\cdot n + g_n - (U_{\epsilon 2}\cdot n + g_n)) \, ds \\ =& \int_\Gamma |[U_{\epsilon 1}\cdot n + g_n]_- - [U_{\epsilon 2}\cdot n + g_n]_-|^2 \, ds \\ &+ \int_\Gamma ([U_{\epsilon 1}\cdot n + g_n]_-[U_{\epsilon 2}\cdot n + g_n]_+ + [U_{\epsilon 1}\cdot n + g_n]_+[U_{\epsilon 2}\cdot n + g_n]_-) \, ds \\ \geq& 0.\end{aligned} \tag{3.4.14}$$

$$
\begin{aligned}
&a(w,w)+a_1(U_{\epsilon 2}+g,w,w)\\
\geq&\alpha\|w\|_{H^1}^2+\frac{1}{2}\int_\Gamma (U_{\epsilon 2}\cdot n+g_n)|w|^2\ ds\\
=&\alpha\|w\|_{H^1}^2+\frac{1}{2}\int_\Gamma ([U_{\epsilon 2}\cdot n+g_n]_+-[U_{\epsilon 2}\cdot n+g_n]_-)|w|^2\ ds\\
\geq&(\alpha-c_1\|[U_{\epsilon 2}\cdot n+g_n]_-\|_{L^2(\Gamma)})\|w\|_{H^1}^2. \quad (\because \text{Lemma 3.2.1.})
\end{aligned}
\tag{3.4.15}
$$

In view of Lemma 3.4.1, we have $\|[U_{\epsilon 2}\cdot n+g_n]_-\|_{L^2(\Gamma)} \leq C\epsilon$. For sufficiently small ϵ, such that $\alpha-c_1\|[U_{\epsilon 2}\cdot n+g_n]_-\|_{L^2(\Gamma)} \geq \alpha/2$, following from (3.4.13), (3.4.14), and (3.4.15), we have, for arbitrary $\eta_0>0$,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2+\frac{\alpha}{2}\|w\|_{H^1}\leq -a_1(w,U_{\epsilon 1}+g,w)\\
\leq&\begin{cases}\eta_0\|w\|_{H^1}^2+C\eta_0^{-1}\|U_{\epsilon 1}+g\|_{H^1}^2\|w\|_{L^2}^2, \text{ for } d=2,\\ \eta_0\|w\|_{H^1}^2+C\eta^{-3}\|U_{\epsilon 1}+g\|_{H^1}^4\|w\|_{L^2}^2, \text{ for } d=3.\end{cases}
\end{aligned}
\tag{3.4.16}
$$

Setting $\eta=\alpha/4$, from (3.4.16) and Gronwall's inequality, it yields, for any $t\in(0,T]$,

$$
\|w(t)\|_{L^2}^2+\int_0^t \|w\|_{H^1}^2 \leq Ce^{Ct\|U_{\epsilon 1}+g\|_{L^\infty(0,t;V)}}\|w(0)\|_{L^2}^2.
$$

Since $w(0)=U_{\epsilon 1}(0)-U_{\epsilon 2}(0)=0$, we conclude $U_{\epsilon 1}=U_{\epsilon 2}$. □

3.5 The completion the proof of Theorem 3.3.1 and 3.4.1

Let (U,P) be the solution to (3.2.3), we set

$$
\tilde{U}=\frac{U}{\beta(t)},\quad \tilde{P}=\frac{P}{\beta(t)},\quad \tilde{\pi}=\frac{\pi}{\beta(t)},\quad \tilde{f}=\frac{f}{\beta(t)},\quad \tilde{g}=\frac{g}{\beta(t)}.
$$

$(\tilde{U}, \tilde{P})$ satisfies, for all $t \in (0,T)$,

$$\begin{aligned} &\tilde{U}' + \frac{\beta'(t)}{\beta(t)}\tilde{U} + \beta(t)((\tilde{U} + \tilde{g}) \cdot \nabla)\tilde{U} + \beta(t)(\tilde{U} \cdot \nabla)\tilde{g} \\ &\quad - \nabla \cdot \sigma(\tilde{U}, \tilde{P}) = \tilde{F}, \quad \text{in } \Omega, \end{aligned} \tag{3.5.1a}$$

$$\nabla \cdot \tilde{U} = 0, \quad \text{in } \Omega, \tag{3.5.1b}$$

$$\tilde{U} = 0, \quad \text{on } S \cup C, \tag{3.5.1c}$$

$$\tilde{U}_n + \tilde{g}_n \geq 0, \quad \tau_n(\tilde{U} + \tilde{g}, \tilde{P} + \tilde{\pi}) \geq 0, \quad \text{on } \Gamma, \tag{3.5.1d}$$

$$(\tilde{U}_n + \tilde{g}_n)\tau_n(U + \tilde{g}, \tilde{P} + \tilde{\pi}) = 0, \quad \tau_T(\tilde{U}) = -\tau_T(\tilde{g}), \quad \text{on } \Gamma, \tag{3.5.1e}$$

$$\tilde{U}(x,0) = \tilde{U}_0, \quad \text{on } \Omega. \tag{3.5.1f}$$

where $\tilde{U}_0 = \frac{U_0}{\beta(0)}$, and $\tilde{F} = \tilde{f} - \tilde{g}' - \frac{\beta'(t)}{\beta(t)}\tilde{g} - \beta(t)(\tilde{g} \cdot \nabla)\tilde{g} = F/\beta(t)$.

To study the well-posedness of U, it is equivalent to consider $\tilde{U}$ of (3.5.1). Setting

$$\tilde{K} = \{v \in V \mid v_n + \tilde{g}_n \geq 0 \text{ on } \Gamma\}, \quad \tilde{K}^\sigma = \tilde{K} \cap V^\sigma$$

We give the variational inequality of $\tilde{U}$.

$(\widetilde{\mathbf{NSI}}^\sigma)$. For a.e. $t \in (0,T)$, find $\tilde{U} \in \tilde{K}^\sigma$, with $\tilde{U}_t \in V^\sigma$, such that

$$\begin{aligned} &(\tilde{U}', v - \tilde{U}) + \frac{\beta'(t)}{\beta(t)}(\tilde{U}, v - \tilde{U}) + a(\tilde{U}, v - \tilde{U}) \\ &\quad + \beta(t)a_1(\tilde{U} + \tilde{g}, \tilde{U}, v - \tilde{U}) + \beta(t)a_1(\tilde{U}, \tilde{g}, v - \tilde{U}) \\ &\geq (\tilde{F}, v - \tilde{U}) - [\tau(\tilde{g}, \tilde{\pi}), v - \tilde{U}] \quad \forall v \in \tilde{K}^\sigma, \end{aligned} \tag{3.5.2a}$$

$$\tilde{U}(x,0) = \tilde{U}_0, \quad \text{on } \Omega. \tag{3.5.2b}$$

We write the penalty problem to $(\widetilde{\mathbf{NSI}}^\sigma)$, denoted as $(\widetilde{\mathbf{NSI}}_\epsilon^\sigma)$.

$(\widetilde{\mathbf{NS}_\epsilon\mathbf{E}}^\sigma)$. For a.e. $t \in (0,T)$, find $\tilde{U}_\epsilon^\sigma \in V^\sigma$, with $\tilde{U}_t \in V^\sigma$, $t \in (0,T)$, such that

$$\begin{aligned} &(\tilde{U}_\epsilon', v) + \frac{\beta'(t)}{\beta(t)}(\tilde{U}_\epsilon, v) + a(\tilde{U}_\epsilon, v) + \beta(t)a_1(\tilde{U}_\epsilon + \tilde{g}, \tilde{U}_\epsilon, v) \\ &\quad + \beta(t)a_1(\tilde{U}_\epsilon, \tilde{g}, v) - \frac{1}{\epsilon}\int_\Gamma [\tilde{U}_{\epsilon n} + \tilde{g}_n]_- v_n \, ds \\ &= (\tilde{F}, v) - [\tau(\tilde{g}, \tilde{\pi}), v] \quad \forall v \in V^\sigma, \end{aligned} \tag{3.5.3a}$$

$$\tilde{U}(x,0) = \tilde{U}_0, \quad \text{on } \Omega. \tag{3.5.3b}$$

We see that, for U_ϵ the solution to $(\mathbf{NS}_\epsilon\mathbf{E}^\sigma)$, $\tilde{U}_\epsilon = U_\epsilon/\beta(t)$. We consider the well-posedness of (3.5.3). We shall apply the Galerkin's approximation method to construct the smooth approximation solutions (we need C^2 with respect to t). However, for arbitrary $w(x), g(x) \in H^{\frac{1}{2}}_{00}(\Gamma)$, with $g(x) \geq 0$, $\int_\Gamma w(x)ds = 0$, It is not obvious that $\int_\Gamma [c(t)w(x) + g(x)]_- w(x)ds$ is C^1 with respect to t. Therefore, we introduce a regularization of $[\cdot]_-$. For any δ with $0 < \delta \ll 1$, we set

$$\rho_\delta(s) = \begin{cases} 0, & \text{for } s \geq 0, \\ \sqrt{s^2 + \delta^2} - \delta, & \text{for } s \leq 0. \end{cases} \tag{3.5.4}$$

We have $\rho_\delta(s) \in C^1(\mathbb{R})$, and

$$\frac{d}{ds}\rho_\delta(s) = \begin{cases} 0, & \text{for } s \geq 0, \\ \dfrac{s}{\sqrt{s^2+\delta^2}}, & \text{for } s \leq 0. \end{cases} \quad \frac{d^2}{ds^2}\rho_\delta(s) = \begin{cases} 0, & \text{for } s > 0, \\ \dfrac{\delta^2}{(s^2+\delta^2)^{\frac{3}{2}}}, & \text{for } s < 0. \end{cases} \tag{3.5.5}$$

Then we introduce the regularization problem to the penalty problem $(\widetilde{\mathbf{NS}_\epsilon\mathbf{E}}^\sigma)$, denoted as $(\widetilde{\mathbf{NS}_\epsilon\mathbf{E}}^\sigma_\delta)$.

$(\widetilde{\mathbf{NS}_\epsilon\mathbf{E}}^\sigma_\delta)$ For a.e. $t \in [0,T]$, find $\tilde{U}_{\epsilon\delta}(t) \in V^\sigma$, with $\tilde{U}'_{\epsilon\delta}(t) \in V^\sigma$, such that

$$\begin{aligned} &(\tilde{U}'_{\epsilon\delta}, v) + \frac{\beta'(t)}{\beta(t)}(\tilde{U}_{\epsilon\delta}, v) + a(\tilde{U}_{\epsilon\delta}, v) + \beta(t)a_1(\tilde{U}_{\epsilon\delta} + \tilde{g}, \tilde{U}_{\epsilon\delta}, v) \\ &\quad + \beta(t)a_1(\tilde{U}_{\epsilon\delta}, \tilde{g}, v) - \frac{1}{\epsilon}\int_\Gamma \rho_\delta(\tilde{U}_{\epsilon\delta n} + \tilde{g}_n)v_n \, d\Gamma \\ &= (\tilde{F}, v) - [[\tau(\tilde{g}, \tilde{\pi}), v]] \quad \forall v \in V^\sigma, \end{aligned} \tag{3.5.6a}$$

$$\tilde{U}_{\epsilon\delta}(x,0) = \tilde{U}_0, \quad \text{on } \Omega. \tag{3.5.6b}$$

Here, we propose the regularization problem $(\widetilde{\mathbf{NS}_\epsilon\mathbf{E}}^\sigma_\delta)$ to study the well-posedness of penalty problem $(\widetilde{\mathbf{NS}_\epsilon\mathbf{E}}^\sigma)$. We have to mention that $(\widetilde{\mathbf{NS}_\epsilon\mathbf{E}}^\sigma_\delta)$ is more valuable for practical use than $(\widetilde{\mathbf{NS}_\epsilon\mathbf{E}}^\sigma)$, because to exactly compute the integration such as $\int_\Gamma [c(t)w(x) + g(x)]_- w(x)ds$ is not easy. And we recommend to use the regularization in numerical computation.

We show the well-posedness of $(\widetilde{\mathbf{NS}_\epsilon\mathbf{E}}^\sigma_\delta)$. To do so, we construct approximate solutions by Galerkin's method. Let $\{w_k\}_{k=1}^\infty \subset V^\sigma$ be the linear independent elements. $w_1 = \tilde{U}_0$ and $\overline{\cup_{m=1}^\infty \text{span}\{w_k\}_{k=1}^m}$ is dense in V^σ. We write the Galerkin's approximation problems for $m \in \mathbb{N}$.

$(\widetilde{\mathbf{NS}_\epsilon\mathbf{E}}^\sigma_{\delta\mathbf{m}})$. Find $\tilde{U}_{\epsilon\delta m} = \sum_{k=1}^m c_{\epsilon\delta k}(t)w_k$, where $c_{\epsilon\delta k} \in C^2([0,T])$, such that, $\tilde{U}_{\epsilon\delta m}(0) = U_0$, and for all $k = 1, \ldots, m$,

$$\begin{aligned}
&(\tilde{U}'_{\epsilon\delta m}, w_k) + \frac{\beta'(t)}{\beta(t)}(\tilde{U}_{\epsilon\delta m}, \tilde{U}_{\epsilon\delta m}) + a(\tilde{U}_{\epsilon\delta m}, w_k) \\
&\quad + \beta(t) a_1(\tilde{U}_{\epsilon\delta m} + \tilde{g}, \tilde{U}_{\epsilon\delta m}, w_k) + \beta(t) a_1(\tilde{U}_{\epsilon\delta m}, \tilde{g}, w_k) \\
&\quad - \frac{1}{\epsilon}\int_\Gamma \rho_\delta(\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n) w_{kn}\, d\Gamma = (\tilde{F}, w_k) - [[\tau(\tilde{g}, \tilde{\pi}), w_k]],
\end{aligned} \tag{3.5.7}$$

where $\tilde{U}_{\epsilon\delta m}(0) = \tilde{U}_0$, $\tilde{U}_{\epsilon\delta mn} = \tilde{U}_{\epsilon m} \cdot n$, and $w_{kn} = w_k \cdot n$.

Remark 3.5.1 (The existence of $c_{\epsilon\delta k} \in C^2$)**.** To make the argument rigorous, we have to replace $\tilde{F}$ and $\tau(\tilde{g}, \tilde{\pi})$ by $\tilde{F}_m$ and $\tau(\tilde{g}_m, \tilde{\pi}_m)$ in (3.5.7), respectively, where

$$\tilde{F}_m \in C^1([0,T]; L^2(\Omega)^d), \quad \tau(\tilde{g}_m, \tilde{\pi}_m) \in C^1([0,T]; L^2(\Gamma)^d),$$

and as $m \to \infty$

$$\tilde{F}_m \to \tilde{F} \text{ in } H^1([0,T]; L^2(\Omega)^d), \quad \tau(\tilde{g}_m, \tilde{\pi}_m) \to \tau(\tilde{g}, \tilde{\pi}) \text{ in } H^1([0,T]; L^2(\Gamma)^d).$$

Since $C^1([0,T])$ is dense in $H^1((0,T))$, the existence of such $\tilde{F}_m$ and $\tau(\tilde{g}_m, \tilde{\pi}_m)$ is obvious. Hence, to make the notation simple, let us admit that

$$\tilde{F} = \tilde{F}_m, \quad \tau(\tilde{g}, \tilde{\pi}) = \tau(\tilde{g}_m, \tilde{\pi}_m)$$

in (3.5.7), which does not effect the argument in this section. Now, we see that (3.5.7) can be written into the system of ordinary equations:

$$\mathbf{B}_m \mathbf{c}'_{\epsilon\delta m}(t) = \mathbf{G}(t, \mathbf{c}_{\epsilon\delta m}(t)),$$

where $\mathbf{B}_m \in \mathbb{R}^{m\times m}$,

$$\mathbf{c}_{\epsilon\delta m} = (c_{\epsilon\delta 1}, \ldots, c_{\epsilon\delta m})^T,$$

and $\mathbf{G}(t, \mathbf{c}_{\epsilon\delta m})$ is C^1 with respect to t and $\mathbf{c}_{\epsilon\delta m}$, because $\rho_\delta(s)$ is C^1 with respect to s, and $\tilde{F}, \tau(\tilde{g}, \tilde{\pi})$ are C^1 with respect to t. Therefore, we conclude the existence of $c_{\epsilon\delta k} \in C^2([0,T])$ for $k = 1, \ldots, m$.

Lemma 3.5.1. *Let (A1)–(A4) be valid, $\delta \le C\epsilon$ and ϵ be sufficiently small.*

(1) When $d = 2$, for any $T \in (0, \infty)$, there exists a unique solution $\tilde{U}_{\epsilon\delta m}$ to (3.5.7), such that

$$\|\tilde{U}_{\epsilon\delta m}\|^2_{L^\infty(0,T;L^2(\Omega)^d)} + \|\tilde{U}_{\epsilon\delta m}\|^2_{L^2(0,T;V^\sigma)} \le C, \tag{3.5.8a}$$

$$\|\tilde{U}_{\epsilon\delta m}\|^2_{L^\infty(0,T;V^\sigma)} + \epsilon^{-1}\|[\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n]_-\|^2_{L^\infty(0,T;L^2(\Gamma))} \le C, \tag{3.5.8b}$$

$$\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^\infty(0,T;L^2(\Omega)^d)} + \|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2(0,T;V^\sigma)} \le C. \tag{3.5.8c}$$

(2) When $d = 3$, the same conclusion holds for a small time interval $(0, \mathring{T}]$.

Proof. Multiplying (3.5.7) with $c_{\epsilon\delta k}(t)$ and taking the summation of k, it yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\tilde{U}_{\epsilon\delta m}\|_{L^2}^2 + \frac{\beta'(t)}{\beta(t)}\|\tilde{U}_{\epsilon\delta m}\|_{L^2}^2 + \alpha\|\tilde{U}_{\epsilon\delta m}\|_{H^1}^2 \\ + \beta(t)a_1(\tilde{U}_{\epsilon\delta m} + \tilde{g}, \tilde{U}_{\epsilon\delta m}, \tilde{U}_{\epsilon\delta m}) + \beta(t)a_1(\tilde{U}_{\epsilon\delta m}, \tilde{g}, \tilde{U}_{\epsilon\delta m}) \\ - \frac{1}{\epsilon}\int_\Gamma \rho_\delta(\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n)\tilde{U}_{\epsilon\delta mn}\, ds \le (F, U_{\epsilon m}) - [[\tau(g,\pi), U_{\epsilon m}]].\end{aligned} \tag{3.5.9}$$

We see that

$$\begin{aligned}&-\rho_\delta(\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n)\tilde{U}_{\epsilon\delta mn} = \rho_\delta(\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n)(\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n - \tilde{g}_n) \\ =&\rho_\delta([\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n]_-)[\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n]_- + \rho_\delta(\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n)\tilde{g}_n \ge 0.\end{aligned} \tag{3.5.10}$$

$$\begin{aligned}&\beta(t)a_1(\tilde{U}_{\epsilon\delta m} + \tilde{g}, \tilde{U}_{\epsilon\delta m}, \tilde{U}_{\epsilon\delta m}) = \frac{\beta(t)}{2}\int_\Gamma (\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n)|\tilde{U}_{\epsilon\delta m}|^2\, d\Gamma \\ =&\frac{\beta(t)}{2}\int_\Gamma [\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n]_+|\tilde{U}_{\epsilon\delta m}|^2\, d\Gamma - \frac{\beta(t)}{2}\int_\Gamma [\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n]_-|\tilde{U}_{\epsilon\delta m}|^2\, d\Gamma. \\ \ge& - C_1\|[\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n]_-\|_{L^2(\Gamma)}\|\tilde{U}_{\epsilon\delta m}\|_{H^1}^2. \quad (\because \text{ Lemma 3.2.1.})\end{aligned} \tag{3.5.11}$$

Applying Lemma 3.2.1 and Remark 3.2.1, we have, for arbitrary $\eta_0 > 0$,

$$\begin{aligned}&|\beta(t)a_1(\tilde{U}_{\epsilon\delta m}, \tilde{g}, \tilde{U}_{\epsilon\delta m})| \\ &\le \begin{cases} \eta_0\|\tilde{U}_{\epsilon\delta m}\|_{H^1}^2 + C\eta_0^{-1}\|\tilde{g}\|_{H^1}^2\|\tilde{U}_{\epsilon\delta m}\|_{L^2}^2, & \text{for } d = 2, \\ \eta_0\|\tilde{U}_{\epsilon\delta m}\|_{H^1}^2 + C\eta_0^{-3}\|\tilde{g}\|_{H^1}^4\|\tilde{U}_{\epsilon\delta m}\|_{L^2}^2, & \text{for } d = 3. \end{cases}\end{aligned} \tag{3.5.12}$$

$$\left|(\tilde{F}, \tilde{U}_{\epsilon\delta m}) - [[\tau(\tilde{g},\tilde{\pi}), \tilde{U}_{\epsilon\delta m}]]\right| \le \eta_0\|\tilde{U}_{\epsilon\delta m}\|_{H^1}^2 + C\eta_0^{-1}(\|\tilde{F}\|_{L^2}^2 + \|\tau(\tilde{g},\tilde{\pi})\|_{L^2(\Gamma)}^2). \tag{3.5.13}$$

From (3.5.9) to (3.5.13), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\tilde{U}_{\epsilon m}\|_{L^2}^2 + \tilde{\alpha}\|\tilde{U}_{\epsilon\delta m}\|_{H^1}^2 + \frac{1}{\epsilon}[\rho_\delta(\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n), [\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n]_-] \\ \le C\eta_0^{-1}(\|\tilde{F}\|_{L^2}^2 + \|\tau(\tilde{g},\tilde{\pi})\|_{L^2(\Gamma)}^2) + C_{\eta_0,\tilde{g}}\|\tilde{U}_{\epsilon m}\|_{L^2}^2,\end{aligned} \tag{3.5.14}$$

where $\tilde{\alpha} := \alpha - 2\eta_0 - c_1\|[\tilde{U}_{\epsilon\delta mn} + g_n]_-\|_{L^2(\Gamma)}$, $C_{\eta_0,g} = C\eta_0^{-1}\|\tilde{g}\|_{H^1}^2 + C_\beta$ for $d = 2$, $C_{\eta_0,g} = C\eta_0^{-3}\|\tilde{g}\|_{H^1}^4 + C_\beta$ for $d = 3$, and $C_\beta = \max_{t\in[0,T]}\frac{|\beta'(t)|}{\beta(t)}$.

Let $\eta_0 = \alpha/8$. Since $\tilde{U}_{\epsilon\delta mn}(0)+\tilde{g}_n(0) = \tilde{U}_0+\tilde{g}_n \geq 0$, we have $\|[\tilde{U}_{\epsilon\delta mn}(0)+\tilde{g}_n(0)]_-\|_{L^2(\Gamma)} = 0$. Let T_1 be the maximum time such that, for all $t \in [0, T_1]$,

$$c_1\|[\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n]_-\|_{L^2(\Gamma)} \leq \alpha/4, \tag{3.5.15}$$

we have

$$\tilde{\alpha} = \alpha - 2\eta_0 - c_1\|[\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n]_-\|_{L^2(\Gamma)} \geq \alpha/2, \quad \forall t \in [0, T_1].$$

Applying Gronwall's inequality to (3.5.14), we obtain, for any $t \in [0, T_1]$,

$$\begin{aligned}
&\|\tilde{U}_{\epsilon\delta m}(t)\|_{L^2}^2 + \alpha \int_0^t \|\tilde{U}_{\epsilon\delta m}(s)\|_{H^1}^2 \\
&\quad + \frac{1}{\epsilon}\int_0^t \int_\Gamma \rho_\delta([\tilde{U}_{\epsilon\delta mn}(s) + \tilde{g}_n]_-)[\tilde{U}_{\epsilon\delta mn}(s) + \tilde{g}_n]_- d\Gamma \\
\leq & C(\|\tilde{F}\|_{L^2(0,t;L^2(\Omega)^d)}^2 + \|\tau(\tilde{g}, \tilde{\pi})\|_{L^2(0,t;L^2(\Gamma)^d)}^2 + \|\tilde{U}_0\|_{L^2}^2),
\end{aligned} \tag{3.5.16}$$

which proves

$$\begin{aligned}
&\|\tilde{U}_{\epsilon\delta m}\|_{L^\infty(0,T_1;L^2(\Omega)^d)}^2 + \|\tilde{U}_{\epsilon\delta m}\|_{L^2(0,T_1;V^\sigma)}^2 \\
&\quad + \epsilon^{-1}\int_0^{T_1} \int_\Gamma \rho_\delta([\tilde{U}_{\epsilon\delta mn}(s) + \tilde{g}_n]_-)[\tilde{U}_{\epsilon\delta mn}(s) + \tilde{g}_n]_- d\Gamma dt \leq C.
\end{aligned} \tag{3.5.17}$$

(3.5.17) implies

$$\begin{aligned}
&\epsilon^{-1}\int_0^{T_1} \int_\Gamma \frac{|[\tilde{U}_{\epsilon\delta mn}(s) + \tilde{g}_n]_-|^3}{\sqrt{([\tilde{U}_{\epsilon\delta mn}(s) + \tilde{g}_n]_-)^2 + \delta^2}} d\Gamma dt \\
=&\epsilon^{-1}\int_0^{T_1} \int_\Gamma \rho_\delta([\tilde{U}_{\epsilon\delta mn}(s) + \tilde{g}_n]_-)[\tilde{U}_{\epsilon\delta mn}(s) + \tilde{g}_n]_- d\Gamma dt \\
&+ \epsilon^{-1}\int_0^{T_1} \int_\Gamma (\delta - \frac{\delta^2}{\sqrt{([\tilde{U}_{\epsilon\delta mn}(s) + \tilde{g}_n]_-)^2 + \delta^2}})[\tilde{U}_{\epsilon\delta mn}(s) + \tilde{g}_n]_- d\Gamma dt \\
\leq & C + C\frac{\delta}{\epsilon} \leq C \quad (\because \delta \leq C\epsilon).
\end{aligned} \tag{3.5.18}$$

Differentiating (3.5.7) with respect to t, it yields

$$
\begin{aligned}
&(\tilde U''_{\epsilon\delta m}, w_k) + \left(\frac{\beta'(t)}{\beta(t)}\right)'(\tilde U_{\epsilon\delta m}, w_k) + \frac{\beta'(t)}{\beta(t)}(\tilde U'_{\epsilon\delta m}, w_k) + a(\tilde U'_{\epsilon\delta m}, w_k)\\
&\quad + \beta'(t)a_1(\tilde U_{\epsilon\delta m} + \tilde g, \tilde U_{\epsilon\delta m}, w_k) + \beta(t)a_1(\tilde U'_{\epsilon\delta m} + \tilde g, \tilde U_{\epsilon\delta m}, w_k)\\
&\quad + \beta(t)a_1(\tilde U_{\epsilon\delta m} + \tilde g, \tilde U'_{\epsilon\delta m}, w_k) + \beta'(t)a_1(\tilde U_{\epsilon\delta m}, \tilde g, w_k)\\
&\quad + \beta(t)a_1(\tilde U'_{\epsilon\delta m}, \tilde g, w_k) + \beta(t)a_1(\tilde U_{\epsilon\delta m}, \tilde g', w_k)\\
&\quad - \frac{1}{\epsilon}\int_\Gamma (\rho_\delta(\tilde U_{\epsilon\delta mn} + \tilde g_n))' w_{kn}\, ds = (\tilde F', w_k) - [[\tau(\tilde g', \pi'), w_k]].
\end{aligned}
\tag{3.5.19}
$$

Multiplying (3.5.19) with $c'_{\epsilon\delta k}(t)$ and taking the summation of k, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\tilde U'_{\epsilon\delta m}\|^2_{L^2} + \alpha\|\tilde U'_{\epsilon\delta m}\|^2_{H^1} + \beta(t)a_1(\tilde U_{\epsilon\delta m} + \tilde g, \tilde U'_{\epsilon\delta m}, \tilde U'_{\epsilon\delta m})\\
&\quad - \frac{1}{\epsilon}\int_\Gamma (\rho_\delta(\tilde U_{\epsilon\delta mn} + \tilde g_n))' \tilde U'_{\epsilon\delta mn}\, ds\\
&\le -\left(\frac{\beta'(t)}{\beta(t)}\right)'(\tilde U_{\epsilon\delta m}, \tilde U'_{\epsilon\delta m}) - \frac{\beta'(t)}{\beta(t)}\|\tilde U'_{\epsilon\delta m}\|^2_{L^2}\\
&\quad - \beta'(t)a_1(\tilde U_{\epsilon\delta m} + \tilde g, \tilde U_{\epsilon\delta m}, \tilde U'_{\epsilon\delta m}) - \beta(t)a_1(\tilde U'_{\epsilon\delta m} + \tilde g', \tilde U_{\epsilon\delta m}, \tilde U'_{\epsilon\delta m})\\
&\quad - \beta'(t)a_1(\tilde U_{\epsilon\delta m}, \tilde g, \tilde U_{\epsilon\delta m}) - \beta(t)a_1(\tilde U'_{\epsilon\delta m}, \tilde g, \tilde U'_{\epsilon\delta m})\\
&\quad - \beta(t)a_1(\tilde U_{\epsilon\delta m}, \tilde g', \tilde U'_{\epsilon\delta m}) + (\tilde F', \tilde U'_{\epsilon\delta m}) - [[\tau(\tilde g', \tilde\pi'), \tilde U'_{\epsilon\delta m}]].
\end{aligned}
\tag{3.5.20}
$$

The same to (3.5.11), we have

$$
\beta(t)a_1(\tilde U_{\epsilon\delta m}+\tilde g, \tilde U'_{\epsilon\delta m}, \tilde U'_{\epsilon\delta m}) \ge -C_1\|[\tilde U_{\epsilon\delta mn}+\tilde g_n]_-\|_{L^2(\Gamma)}\|\tilde U'_{\epsilon\delta m}\|^2_{H^1}. \tag{3.5.21}
$$

From (3.2.2b), we see that $\tilde g = g_0(x)$ on Γ, and $\tilde g'_n = 0$ on Γ. Therefore,

$$
\begin{aligned}
&-\int_\Gamma (\rho_\delta(U_{\epsilon\delta mn} + g_n))' \tilde U'_{\epsilon\delta mn}\, d\Gamma\\
&= -\int_\Gamma (\rho_\delta(U_{\epsilon\delta mn} + g_n))' (\tilde U_{\epsilon\delta mn} + \tilde g_n)'\, d\Gamma\\
&= \int_\Gamma \frac{[U_{\epsilon\delta mn} + g_n]_-}{\sqrt{(U_{\epsilon\delta mn} + g_n)^2 + \delta^2}}|(\tilde U_{\epsilon\delta mn} + \tilde g_n)'|^2\, d\Gamma \ge 0.
\end{aligned}
\tag{3.5.22}
$$

In view of (3.5.17), we have, for all $t \in [0, T_1]$,

$$
\left|\left(\frac{\beta'(t)}{\beta(t)}\right)'(\tilde U_{\epsilon\delta m}, \tilde U'_{\epsilon\delta m}) + \frac{\beta'(t)}{\beta(t)}\|\tilde U'_{\epsilon\delta m}\|^2_{L^2}\right| \le C\|\tilde U'_{\epsilon\delta m}\|^2_{L^2} + C. \tag{3.5.23}
$$

The same to (3.5.13), for arbitrary $\eta_0 > 0$,

$$
\begin{aligned}
&\left|(\tilde{F}', \tilde{U}'_{\epsilon\delta m}) - [[\tau(\tilde{g}', \tilde{\pi}'), \tilde{U}'_{\epsilon\delta m}]]\right| \\
\leq &\eta_0 \|\tilde{U}'_{\epsilon\delta m}\|^2_{H^1} + C\eta_0^{-1}(\|\tilde{F}'\|^2_{L^2} + \|\tau(\tilde{g}', \tilde{\pi}')\|^2_{L^2(\Gamma)}).
\end{aligned} \tag{3.5.24}
$$

(1) First, let us consider the case of $d = 2$. Applying Lemma 3.2.1, Remark 3.2.1 and (3.5.17), we have, for arbitrary $\eta_0 > 0$,

$$
\begin{aligned}
&\left|\beta'(t)a_1(\tilde{U}_{\epsilon\delta m} + \tilde{g}, \tilde{U}_{\epsilon\delta m}, \tilde{U}'_{\epsilon\delta m})\right| \leq C\|\tilde{U}_{\epsilon\delta m} + \tilde{g}\|_{L^4}\|\tilde{U}_{\epsilon\delta m}\|_{H^1}\|\tilde{U}'_{\epsilon\delta m}\|_{L^4} \\
\leq &C\|\tilde{U}_{\epsilon\delta m} + \tilde{g}\|^{1/2}_{L^2}\|\tilde{U}_{\epsilon\delta m} + \tilde{g}\|^{1/2}_{H^1}\|\tilde{U}_{\epsilon\delta m}\|_{H^1}\|\tilde{U}'_{\epsilon\delta m}\|^{1/2}_{L^2}\|\tilde{U}'_{\epsilon\delta m}\|^{1/2}_{H^1} \\
\leq &\eta_0\|\tilde{U}'_{\epsilon\delta m}\|^2_{H^1} + C\eta_0^{-1/3}(\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2}\|\tilde{U}_{\epsilon\delta m} + \tilde{g}\|^2_{H^1} + \|\tilde{U}_{\epsilon\delta m}\|^2_{H^1}),
\end{aligned} \tag{3.5.25}
$$

$$
\begin{aligned}
&\left|\beta(t)a_1(\tilde{U}'_{\epsilon\delta m} + \tilde{g}', \tilde{U}_{\epsilon\delta m}, \tilde{U}'_{\epsilon\delta m})\right| \leq C\|\tilde{U}'_{\epsilon\delta m} + \tilde{g}'\|_{L^4}\|\tilde{U}_{\epsilon\delta m}\|_{H^1}\|\tilde{U}'_{\epsilon\delta m}\|_{L^4} \\
\leq &C\|\tilde{U}'_{\epsilon\delta m}\|_{L^2}\|\tilde{U}_{\epsilon\delta m}\|_{H^1}\|\tilde{U}'_{\epsilon\delta m}\|_{H^1} \\
&\quad + C\|\tilde{g}'\|^{1/2}_{L^2}\|\tilde{g}'\|^{1/2}_{H^1}\|\tilde{U}_{\epsilon\delta m}\|_{H^1}\|\tilde{U}'_{\epsilon\delta m}\|^{1/2}_{L^2}\|\tilde{U}'_{\epsilon\delta m}\|^{1/2}_{H^1} \\
\leq &\eta_0\|\tilde{U}'_{\epsilon\delta m}\|^2_{H^1} + C\eta_0^{-1}\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2}\|\tilde{U}_{\epsilon\delta m}\|^2_{H^1} \\
&\quad + C\eta_0^{-1/3}(\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2}\|\tilde{g}'\|^2_{H^1} + \|\tilde{U}_{\epsilon\delta m}\|^2_{H^1}),
\end{aligned} \tag{3.5.26}
$$

$$
\begin{aligned}
&\left|\beta'(t)a_1(\tilde{U}_{\epsilon\delta m}, \tilde{g}, \tilde{U}'_{\epsilon\delta m})\right| \\
\leq &\eta_0\|\tilde{U}'_{\epsilon\delta m}\|^2_{H^1} + C\eta_0^{-1/3}(\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2}\|\tilde{U}_{\epsilon\delta m}\|^2_{H^1} + \|\tilde{g}\|^2_{H^1}),
\end{aligned} \tag{3.5.27}
$$

$$
\left|\beta(t)a_1(\tilde{U}'_{\epsilon\delta m}, \tilde{g}, \tilde{U}'_{\epsilon\delta m})\right| \leq \eta_0\|\tilde{U}'_{\epsilon\delta m}\|^2_{H^1} + C\eta_0^{-1}\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2}\|\tilde{g}\|^2_{H^1}, \tag{3.5.28}
$$

$$
\begin{aligned}
&\left|\beta(t)a_1(\tilde{U}_{\epsilon\delta m}, \tilde{g}', \tilde{U}'_{\epsilon\delta m})\right| \\
\leq &\delta\|\tilde{U}'_{\epsilon\delta m}\|^2_{H^1} + C\delta^{-1/3}(\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2}\|\tilde{U}_{\epsilon\delta m}\|^2_{H^1} + \|\tilde{g}'\|^2_{H^1}).
\end{aligned} \tag{3.5.29}
$$

From (3.5.20) to (3.5.29), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\tilde{U}'_{\epsilon\delta m}\|^2_{\Omega} + \hat{\alpha}\|\tilde{U}'_{\epsilon\delta m}\|^2_{1,\Omega} \\
\leq &C(\|\tilde{g}\|^2_{H^1} + \|\tilde{g}'\|^2_{H^1} + \|\tilde{U}_{\epsilon\delta m}\|^2_{H^1})\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2} \\
&\quad + C_\delta(\|\tilde{F}\|^2_{L^2} + \|\tau(\tilde{g}, \tilde{\pi})\|^2_{L^2(\Gamma)}) + C_\delta(\|\tilde{g}'\|^2_{H^1} + \|\tilde{U}_{\epsilon\delta m}\|^2_{H^1}),
\end{aligned} \tag{3.5.30}
$$

where $\hat{\alpha} := \alpha - 6\delta - C_1\|[\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n]_-\|_{L^2(\Gamma)}$.

Let $\delta = \alpha/12$. From (3.5.15), we see that

$$
\hat{\alpha} = \alpha - 6\eta_0 - C_1\|[\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n]_-\|_{L^2(\Gamma)} \geq \alpha/2, \quad \forall t \in [0, T_1].
$$

Applying Gronwall's inequality to (3.5.30), it yields,

$$\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^\infty(0,T_1;L^2(\Omega)^d)} + \|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2(0,T_1;V^\sigma)} \le C + \|\tilde{U}'_{\epsilon\delta m}(0)\|^2_{L^2}. \quad (3.5.31)$$

To show the boundedness of $\|\tilde{U}'_{\epsilon\delta m}(0)\|^2_\Omega$, we multiply $c'_{\epsilon m}(t)$ to (3.5.7), add the resulting equations, and make $t = 0$, then it yields

$$\begin{aligned}
&\|\tilde{U}'_{\epsilon\delta m}(0)\|^2_{L^2} + a(\tilde{U}_0, \tilde{U}'_{\epsilon\delta m}(0)) - [[\tau(\tilde{g}, \tilde{\pi})(0), \tilde{U}'_{\epsilon\delta m}(0)]] \\
&\quad - \frac{1}{\epsilon}\int_\Gamma \rho_\delta(\tilde{U}_0 + \tilde{g}_n(0))\tilde{U}'_{\epsilon\delta mn}(0)\ ds \\
=& - \frac{\beta'(t)}{\beta(t)}(\tilde{U}_0, \tilde{U}'_{\epsilon\delta m}(0)) - \beta(t)a_1(\tilde{U}_0 + \tilde{g}(0), \tilde{U}_0, \tilde{U}'_{\epsilon\delta m}(0)) \\
&\quad - \beta(t)a_1(\tilde{U}_0, \tilde{g}(0), \tilde{U}'_{\epsilon\delta m}(0)) + (\tilde{F}(0), \tilde{U}'_{\epsilon m}(0)).
\end{aligned} \quad (3.5.32)$$

Since $[\tilde{U}_0 + \tilde{g}_n(0)]_- = 0$ and (A4)(3.3.1), we have

$$\begin{aligned}
&\|\tilde{U}'_{\epsilon\delta m}(0)\|^2_{L^2} \le |a(\tilde{U}_0, \tilde{U}'_{\epsilon\delta m}(0))| + |(\Delta\tilde{U}_0, \tilde{U}'_{\epsilon\delta m}(0))| \\
&\quad + \left|\frac{\beta'(t)}{\beta(t)}(\tilde{U}_0, \tilde{U}'_{\epsilon\delta m}(0))\right| + \left|\beta(t)a_1(\tilde{U}_0 + \tilde{g}(0), \tilde{U}_0, \tilde{U}'_{\epsilon\delta m}(0))\right| \\
&\quad + \left|\beta(t)a_1(\tilde{U}_0, \tilde{g}(0), \tilde{U}'_{\epsilon\delta m}(0))\right| + \left|(\tilde{F}(0), \tilde{U}'_{\epsilon\delta m}(0))\right| \\
\le& C\Big(\|\tilde{U}_0\|_{L^2} + \|\tilde{U}_0\|_{H^2} + \|\tilde{U}_0 + \tilde{g}(0)\|_{L^\infty}\|\tilde{U}_0\|_{H^1} \\
&\quad + \|\tilde{U}_0\|_{L^\infty}\|\tilde{g}(0)\|_{H^1} + \|\tilde{F}(0)\|_{L^2}\Big)\|\tilde{U}'_{\epsilon\delta m}(0)\|_{L^2},
\end{aligned} \quad (3.5.33)$$

which shows $\|\tilde{U}'_{\epsilon\delta m}(0)\|_{L^2} \le C$. Furthermore, from (3.5.32), we prove

$$\begin{aligned}
&\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^\infty(0,T_1;L^2(\Omega)^d)} + \|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2(0,T_1;V^\sigma)} \\
&\quad + \epsilon^{-1}\int_0^{T_1}\int_\Gamma \frac{[\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n]_-}{\sqrt{(U_{\epsilon\delta mn} + g_n)^2 + \delta^2}}|(\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n)'|^2 d\Gamma dt \le C.
\end{aligned} \quad (3.5.34)$$

Multiplying $c'_{\epsilon\delta m}(t)$ to (3.5.7) and taking the summation $w.r.t$ k, it gives

$$\begin{aligned}
&\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2} + \frac{1}{2}\frac{d}{dt}a(\tilde{U}_{\epsilon\delta m}, \tilde{U}_{\epsilon\delta m}) - \frac{1}{\epsilon}\int_\Gamma \rho_\delta(\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n)\tilde{U}'_{\epsilon\delta mn}\ d\Gamma \\
=& - \frac{\beta'(t)}{\beta(t)}(\tilde{U}_{\epsilon\delta m}, \tilde{U}'_{\epsilon\delta m}) - \beta(t)a_1(\tilde{U}_{\epsilon\delta m} + \tilde{g}, \tilde{U}_{\epsilon\delta m}, \tilde{U}'_{\epsilon\delta m}) \\
&\quad - \beta(t)a_1(\tilde{U}_{\epsilon\delta m}, \tilde{g}, \tilde{U}'_{\epsilon\delta m}) + (\tilde{F}, \tilde{U}'_{\epsilon\delta m}) + [\tau(\tilde{g}, \tilde{\pi}), \tilde{U}'_{\epsilon\delta m}] =: RHS.
\end{aligned} \quad (3.5.35)$$

Since $\tilde{g}' = 0$ on Γ, we have

$$\begin{aligned}
&-\int_0^{T_1}\int_\Gamma \rho_\delta(\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n)\tilde{U}'_{\epsilon\delta mn}\ d\Gamma dt \\
=&-\int_0^{T_1}\int_\Gamma \rho_\delta(\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n)(\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n)'\ d\Gamma dt \\
=&\int_0^{T_1}\int_\Gamma -\frac{d}{dt}(\rho_\delta(\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n)(\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n)) \\
&\quad+\int_0^{T_1}\int_\Gamma -\frac{d}{dt}(\rho_\delta(\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n))'(\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n)d\Gamma dt =: I_1+I_2.
\end{aligned} \tag{3.5.36}$$

In view of $(\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n)(0)\geq 0$, we get

$$\begin{aligned}
I_1 =&[\rho_\delta(\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n)(T_1),[\tilde{U}_{\epsilon\delta mn}(T_1)+\tilde{g}_n(T_1)]_-]-0 \\
=&\|[\tilde{U}_{\epsilon\delta mn}(T_1)+\tilde{g}_n(T_1)]_-\|^2_{L^2(\Gamma)}+\int_\Gamma[\tilde{U}_{\epsilon\delta mn}(T_1)+\tilde{g}_n(T_1)]_- \\
&\cdot(\rho_\delta(\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n)(T_1)-[\tilde{U}_{\epsilon\delta mn}(T_1)+\tilde{g}_n(T_1)]_-)d\Gamma \\
\geq&\|[\tilde{U}_{\epsilon\delta mn}(T_1)+\tilde{g}_n(T_1)]_-\|^2_{L^2(\Gamma)}-C\delta \quad (\because\ |\rho_\delta(s)-[s]_-|\leq\delta).
\end{aligned} \tag{3.5.37}$$

$$\begin{aligned}
\frac{1}{\epsilon}|I_2| =&\int_0^{T_1}\int_\Gamma \frac{|[\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n]_-|^2}{\sqrt{[\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n]_-|^2+\delta^2}}|(\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n)'|d\Gamma dt \\
\leq&\frac{1}{\epsilon}\left(\int_0^{T_1}\int_\Gamma \frac{[\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n]_-}{\sqrt{[\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n]_-|^2+\delta^2}}|(\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n)'|^2\right)^{1/2} \\
&\cdot\left(\int_0^{T_1}\int_\Gamma \frac{|[\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n]_-|^3}{\sqrt{[\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n]_-|^2+\delta^2}}\right)^{1/2} \\
\leq&C+C\frac{\delta}{\epsilon}\leq C \qquad (\because\ (3.5.18),(3.5.34)).
\end{aligned} \tag{3.5.38}$$

In view of (3.5.17) and (3.5.34), we have

$$\begin{aligned}
RHS \leq& C(\|\tilde{g}\|^2_{H^1}+\|\tilde{U}_{\epsilon m}\|^2_{H^1})\|\tilde{U}_{\epsilon m}\|^2_{H^1} \\
&+C(\|\tilde{U}'_{\epsilon m}\|^2_{H^1}+\|\tilde{U}_{\epsilon m}\|^2_{H^1}+\|\tilde{F}\|^2_{L^2}+\|\tau(\tilde{g},\tilde{\pi})\|^2_{L^2}).
\end{aligned} \tag{3.5.39}$$

From (3.5.35), (3.5.36)-(3.5.38), (3.5.39), and recalling that we assume $\delta\leq$

$C\epsilon$, we have, for all $t \in [0, T_1]$,

$$\begin{aligned}
&\int_0^t \|\tilde{U}'_{\epsilon\delta m}(s)\|^2_{L^2} ds + \frac{1}{2} a(\tilde{U}_{\epsilon\delta m}(t), \tilde{U}_{\epsilon\delta m}(t)) + \|[\tilde{U}_{\epsilon\delta mn}(t) + \tilde{g}_n(t)]_-\|^2_{L^2(\Gamma)} \\
\leq & C \int_0^t \|\tilde{U}_{\epsilon\delta m}(s)\|^2_{H^1} + C + C\frac{\delta}{\epsilon} + C\delta \leq Ca(\tilde{U}_{\epsilon\delta m}, \tilde{U}_{\epsilon m}) + C.
\end{aligned} \tag{3.5.40}$$

Applying Gronwall's inequality to (3.5.40), it yields,

$$\|\tilde{U}_{\epsilon\delta m}\|^2_{L^\infty(0,T_1;V^\sigma)} + \epsilon^{-1}\|[\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n]_-\|^2_{L^\infty(0,T_1;L^2(\Gamma))} \leq C. \tag{3.5.41}$$

In view of (3.5.41), for sufficiently small ϵ,

$$\|[\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n]_-\|_{L^2(\Gamma)} \leq C\sqrt{\epsilon} \ll 1, \quad \forall t \in [0, T_1].$$

Hence, there exists $T_2 > T_1$, such that (3.5.15) is satisfied for all $t \in [0, T_2]$. Furthermore, we can replace T_1 in (3.5.17), (3.5.34) and (3.5.41) by T_2.

Once again, for sufficiently small ϵ,

$$\|[\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n]_-\|_{L^2(\Gamma)} \leq C\sqrt{\epsilon} \ll 1, \quad \forall t \in [0, T_2].$$

There exists $T_3 > T_2$, such that (3.5.15) is satisfied for all $t \in [0, T_3]$. We can continue this process for sufficiently small ϵ, till we reach some $T_k > T$, for any $T \in (0, \infty)$, and (3.5.17), (3.5.34) and (3.5.41) are satisfied with T_1 replaced by T_k. Hence, we proved (3.5.8) when $d = 2$.

(2) When $d = 3$, the discussion before (3.5.25) and the observation for $\|\tilde{U}'_{\epsilon\delta m}(0)\|_{L^2}$(see (3.5.33)) are the same to the case of $d = 2$. The estimates from (3.5.35) to (3.5.41) can also be applied to the case of $d = 3$. What changes from the case $d = 2$ is the estimates of $\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^\infty(0,\mathring{T}_1;L^2(\Omega)^d)}$, $\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2(0,\mathring{T}_1;V)}$.

In place of (3.5.25)-(3.5.29), we derive, for arbitrary $\eta_0 > 0$,

$$\begin{aligned}
&\left|\beta'(t) a_1(\tilde{U}_{\epsilon\delta m} + \tilde{g}, \tilde{U}_{\epsilon\delta m}, \tilde{U}'_{\epsilon\delta m})\right| \leq C\|\tilde{U}_{\epsilon\delta m} + \tilde{g}\|_{L^6}\|\tilde{U}_{\epsilon\delta m}\|_{H^1}\|\tilde{U}'_{\epsilon\delta m}\|_{L^3} \\
\leq & C\|\tilde{U}_{\epsilon\delta m} + \tilde{g}\|_{H^1}\|\tilde{U}_{\epsilon\delta m}\|_{H^1}\|\tilde{U}'_{\epsilon\delta m}\|^{1/2}_{L^2}\|\tilde{U}'_{\epsilon\delta m}\|^{1/2}_{H^1} \\
\leq & \eta_0\|\tilde{U}'_{\epsilon\delta m}\|^2_{H^1}\|\tilde{U}_{\epsilon\delta m}\|^2_{H^1} + C\eta_0^{-1/3}\|\tilde{U}'_{\epsilon\delta m}\|^{2/3}_{L^2}\|\tilde{U}_{\epsilon\delta m} + \tilde{g}\|^{4/3}_{H^1}\|\tilde{U}_{\epsilon\delta m}\|^{2/3}_{H^1} \\
\leq & \eta_0\|\tilde{U}'_{\epsilon\delta m}\|^2_{H^1}\|\tilde{U}_{\epsilon\delta m}\|^2_{H^1} + C\eta_0^{-1/3}\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2}\|\tilde{U}_{\epsilon\delta m}\|^2_{H^1} \\
& \quad + C\eta_0^{-1/3}\|\tilde{U}_{\epsilon\delta m} + \tilde{g}\|^2_{H^1},
\end{aligned} \tag{3.5.42}$$

$$\left|\beta(t)a_1(\tilde{U}'_{\epsilon\delta m}+\tilde{g}',\tilde{U}_{\epsilon\delta m},\tilde{U}'_{\epsilon\delta m})\right| \le C\|\tilde{U}'_{\epsilon\delta m}+\tilde{g}'\|_{L^6}\|\tilde{U}_{\epsilon\delta m}\|_{H^1}\|\tilde{U}'_{\epsilon\delta m}\|_{L^3}$$
$$\begin{aligned}&\le \eta_0\|\tilde{U}'_{\epsilon\delta m}\|^2_{H^1}(\|\tilde{U}_{\epsilon\delta m}\|_{H^1}+\|\tilde{U}_{\epsilon\delta m}\|^2_{H^1})+C\eta_0^{-3}\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2}\|\tilde{U}_{\epsilon\delta m}\|_{H^1}\\&\quad+C\eta_0^{-1/3}\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2}\|\tilde{U}_{\epsilon\delta m}\|^2_{H^1}+C\eta_0^{-1/3}\|\tilde{g}'\|^2_{H^1},\end{aligned} \tag{3.5.43}$$

$$\left|\beta'(t)a_1(\tilde{U}_{\epsilon\delta m},\tilde{g},\tilde{U}'_{\epsilon\delta m})\right| \le \eta_0\|\tilde{U}'_{\epsilon\delta m}\|^2_{H^1}\|\tilde{g}\|^2_{H^1}+C\eta_0^{-1}\|\tilde{U}_{\epsilon\delta m}\|^2_{H^1}, \tag{3.5.44}$$

$$\left|\beta(t)a_1(\tilde{U}'_{\epsilon\delta m},\tilde{g},\tilde{U}'_{\epsilon\delta m})\right| \le \eta_0\|\tilde{U}'_{\epsilon\delta m}\|^2_{H^1}+C\eta_0^{-3}\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2}\|\tilde{g}\|^4_{H^1}, \tag{3.5.45}$$

$$\left|\beta(t)a_1(\tilde{U}_{\epsilon\delta m},\tilde{g}',\tilde{U}'_{\epsilon\delta m})\right| \le \eta_0\|\tilde{U}'_{\epsilon\delta m}\|^2_{H^1}\|\tilde{U}_{\epsilon\delta m}\|^2_{H^1}+C\eta_0^{-1}\|\tilde{g}'\|^2_{H^1}. \tag{3.5.46}$$

Hence, in place of (3.5.30), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2}+\bar{\alpha}\|\tilde{U}'_{\epsilon\delta m}\|^2_{H^1}\\&\qquad+\epsilon^{-1}\int_\Gamma \frac{[\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n]_-}{\sqrt{(U_{\epsilon\delta mn}+g_n)^2+\delta^2}}|(\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n)'|^2d\Gamma\\&\le C(\|\tilde{g}\|^4_{H^1}+\|\tilde{g}\|^2_{H^1}+\|\tilde{U}_{\epsilon\delta m}\|^2_{H^1})\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2}\\&\quad+C_\delta(\|\tilde{F}\|^2_{L^2}+\|\tau(\tilde{g},\tilde{\pi})\|^2_{L^2(\Gamma)})+C_\delta(\|\tilde{g}'\|^2_{H^1}+\|\tilde{U}_{\epsilon\delta m}\|^2_{H^1}),\end{aligned} \tag{3.5.47}$$

where

$$\bar{\alpha}:=\alpha-2\eta_0-4\eta_0\|\tilde{U}_{\epsilon\delta m}\|^2_{H^1}-\eta_0\|\tilde{U}_{\epsilon m}\|_{H^1}-C_1\|[\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n]_-\|_{L^2(\Gamma)}.$$

We choose η_0 satisfying $2\eta_0+4\eta_0\|\tilde{U}_0\|^2_{H^1}+\eta_0\|\tilde{U}_0\|_{H^1}\le\alpha/12$. Let $\hat{T}_1$ be the maximum value of t such that $2\eta_0+4\eta_0\|\tilde{U}_0(t)\|^2_{H^1}+\eta_0\|\tilde{U}_0(t)\|_{H^1}\le\alpha/4$. Let $\mathring{T}_1=\min(\hat{T}_1,T_1)$, then we have, for all $t\in[0,\mathring{T}_1]$,

$$\bar{\alpha}:=\alpha-2\eta_0-4\eta_0\|\tilde{U}_{\epsilon\delta m}\|^2_{H^1}-\eta_0\|\tilde{U}_{\epsilon\delta m}\|_{H^1}-C_1\|[\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n]_-\|_{L^2(\Gamma)}\ge\alpha/2.$$

Applying Gronwall's inequality to (3.5.47), we obtain

$$\begin{aligned}&\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^\infty(0,\mathring{T}_1;L^2(\Omega)^d)}+\|\tilde{U}'_{\epsilon\delta m}\|^2_{L^2(0,\mathring{T}_1;V^\sigma)}\\&\quad+\epsilon^{-1}\int_0^{\mathring{T}_1}\int_\Gamma \frac{[\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n]_-}{\sqrt{(U_{\epsilon\delta mn}+g_n)^2+\delta^2}}|(\tilde{U}_{\epsilon\delta mn}+\tilde{g}_n)'|^2d\Gamma dt\le C.\end{aligned} \tag{3.5.48}$$

Therefore, we show (3.5.8) holds for a small time interval $[0,\mathring{T}]$ when $d=3$. □

Lemma 3.5.2. *Under the assumptions of Lemma 3.5.1, when* $d = 2$*, for any* $T \in (0, \infty)$ *and sufficiently small* ϵ*, there exists a solution* $\tilde{U}_{\epsilon\delta}$ *to* $(\widetilde{\mathbf{NS}_\epsilon \mathbf{E}^\sigma_\delta})$*, such that*

$$\|\tilde{U}_{\epsilon\delta}\|_{L^\infty(0,T;V^\sigma)} + \epsilon^{-1/2}\|[\tilde{U}_{\epsilon\delta} + \tilde{g}_n]_-\|_{L^\infty(0,T;L^2(\Gamma))} \leq C, \tag{3.5.49a}$$

$$\|\tilde{U}'_{\epsilon\delta}\|_{L^\infty(0,T;L^2(\Omega)^d)} + \|\tilde{U}'_{\epsilon\delta}\|_{L^2(0,T;V^\sigma)} \leq C. \tag{3.5.49b}$$

When $d = 3$*, the same conclusion holds for a smaller time interval* $(0, \mathring{T})$*.*

Proof. The proof below is valid for both $d = 2, 3$, except that when $d = 3$, we have to replace T by $\mathring{T}$. As a consequence of Proposition 3.5.1, there exists some $\bar{U}_{\epsilon\delta}$ and a subsequence of $\{\tilde{U}_{\epsilon\delta m}\}_{m=1}^\infty$, such that $\bar{U}_{\epsilon\delta} \in L^\infty(0, T; V^\sigma)$, $\bar{U}'_{\epsilon\delta} \in L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; V\sigma)$, and as $m \to \infty$,

$$\tilde{U}_{\epsilon\delta m} \to \bar{U}_{\epsilon\delta}, \quad \text{weakly* in } L^\infty(0, T; V^\sigma), \tag{3.5.50a}$$

$$[\tilde{U}_{\epsilon\delta m} + g_n]_- \to [\bar{U}_{\epsilon\delta} + g_n]_- \text{ weakly* in } L^\infty(0, T; L^2(\Gamma)), \tag{3.5.50b}$$

$$\tilde{U}'_{\epsilon\delta m} \to \bar{U}'_{\epsilon\delta}, \quad \text{weakly* in } L^\infty(0, T; L^2(\Omega)^d), \tag{3.5.50c}$$

$$\tilde{U}'_{\epsilon\delta m} \to \bar{U}'_{\epsilon\delta}, \quad \text{weakly in } L^2(0, T; V^\sigma). \tag{3.5.50d}$$

We show $\bar{U}_{\epsilon\delta}$ is the solution to (3.5.6). Multiplying (3.5.7) with any $\phi \in C_0^\infty(0, T)$, and integrating over $(0, T)$, it yields, for all $k = 1, 2, \ldots, m$,

$$\begin{aligned}
\int_0^T \phi(t) \Bigg\{ &(\tilde{U}'_{\epsilon\delta m}, w_k) + \frac{\beta'(t)}{\beta(t)}(\tilde{U}_{\epsilon\delta m}, \tilde{U}_{\epsilon\delta m}) + a(\tilde{U}_{\epsilon\delta m}, w_k) \\
&+ \beta(t) a_1(\tilde{U}_{\epsilon\delta m} + \tilde{g}, \tilde{U}_{\epsilon\delta m}, w_k) + \beta(t) a_1(\tilde{U}_{\epsilon\delta m}, \tilde{g}, w_k) \\
&- \frac{1}{\epsilon}\int_\Gamma rho_\delta(\tilde{U}_{\epsilon\delta mn} + \tilde{g}_n) w_{kn}\, ds - (\tilde{F}, w_k) + [[\tau(\tilde{g}, \tilde{\pi}), w_k]] \Bigg\}\, dt = 0.
\end{aligned} \tag{3.5.51}$$

It follows from [6, 43] that the embedding

$$\{w \mid w \in L^2(0, T; V),\ w' \in L^2(0, T; L^2(\Omega)^d)\} \hookrightarrow L^2(0, T; L^4(\Omega)^d)$$

is compact. Hence $\tilde{U}_{\epsilon\delta m} \to \bar{U}_{\epsilon\delta}$ strongly in $L^2(0, T; L^4(\Omega)^d)$. Since the trace mapping $H^1(0, T; V) \to L^2(0, T; L^2(\Gamma)^d)$ is compact, we have

$$\tilde{U}_{\epsilon\delta mn} \to \bar{U}_{\epsilon\delta n}, \text{ strongly in } L^2(0, T; L^2(\Gamma)).$$

Therefore, $\tilde{U}_{\epsilon\delta mn} \to \bar{U}_{\epsilon\delta n}$ a.e. on Γ. $\rho_\delta(\cdot)$ is continuous, so that $\rho_\delta(\tilde{U}_{\epsilon mn} + \tilde{g}_n) \to rho_\delta(\bar{U}_{\epsilon\delta n} + \tilde{g}_n)$ a.e. on Γ.

Let $m \to \infty$, we obtain, for all $k \in \mathbb{N}$,

$$\begin{aligned}\int_0^T \phi(t) \Big\{ &(\bar{U}'_{\epsilon\delta}, w_k) + \frac{\beta'(t)}{\beta(t)}(\bar{U}_{\epsilon\delta}, w_k) + a(\bar{U}_{\epsilon\delta}, w_k) \\ &+\beta(t) a_1(\bar{U}_{\epsilon\delta} + \bar{g}, \bar{U}_{\epsilon\delta}, w_k) + \beta(t) a_1(\bar{U}_{\epsilon\delta}, \bar{g}, w_k) \\ &-\frac{1}{\epsilon}\int_\Gamma \rho_\delta(\bar{U}_{\epsilon\delta n} + \tilde{g}_n) w_{kn}\, ds - (\tilde{F}, w_k) + [[\tau(\tilde{g}, \tilde{\pi}), w_k]] \Big\}\, dt = 0.\end{aligned} \tag{3.5.52}$$

Since $\overline{\cup_{m=1}^{\infty} \text{span}\{w_k\}_{k=1}^m}$ is dense in V^σ, we can replace the test function w_k of (3.5.52) by any $v \in V^\sigma$. And we proved $\bar{U}_{\epsilon\delta} = \tilde{U}_{\epsilon\delta}$ is the solution to (3.5.6) satisfying (3.5.49). □

Lemma 3.5.3. *Under the assumptions of Lemma 3.5.2, when $d = 2$, for any $T \in (0, \infty)$ and sufficiently small ϵ, there exists a solution $\tilde{U}_\epsilon$ to $(\widetilde{\mathbf{NS}_\epsilon \mathbf{E}}^\sigma)$, such that*

$$\|\tilde{U}_\epsilon\|_{L^\infty(0,T;V^\sigma)} + \epsilon^{-1/2} \|[\tilde{U}_\epsilon + \tilde{g}_n]_-\|_{L^\infty(0,T;L^2(\Gamma))} \le C, \tag{3.5.53a}$$

$$\|\tilde{U}'_\epsilon\|_{L^\infty(0,T;L^2(\Omega)^d)} + \|\tilde{U}'_\epsilon\|_{L^2(0,T;V^\sigma)} \le C. \tag{3.5.53b}$$

When $d = 3$, the same conclusion holds for a smaller time interval $(0, \mathring{T})$.

Proof. The proof below is valid for both $d = 2, 3$, except that when $d = 3$, we have to replace T by $\mathring{T}$. As a consequence of Proposition 3.5.2, there exists some $\bar{U}_\epsilon$ and a subsequence of $\{\tilde{U}_{\epsilon\delta_i}\}_{i=1}^\infty$, with $\lim_{i\to\infty} \delta_i = 0$ such that $\bar{U}_\epsilon \in L^\infty(0, T; V^\sigma)$, $\bar{U}'_\epsilon \in L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; V\sigma)$, and as $i \to \infty$, $\delta_i \to 0$,

$$\tilde{U}_{\epsilon\delta_i} \to \bar{U}_\epsilon, \quad \text{weakly* in } L^\infty(0, T; V^\sigma), \tag{3.5.54a}$$

$$\rho_{\delta_i}(\tilde{U}_{\epsilon\delta_i} + g_n) \to [\bar{U}_\epsilon + g_n]_- \text{ weakly* in } L^\infty(0, T; L^2(\Gamma)), \tag{3.5.54b}$$

$$\tilde{U}'_{\epsilon\delta_i} \to \bar{U}'_\epsilon, \quad \text{weakly* in } L^\infty(0, T; L^2(\Omega)^d), \tag{3.5.54c}$$

$$\tilde{U}'_{\epsilon\delta_i} \to \bar{U}'_\epsilon, \quad \text{weakly in } L^2(0, T; V^\sigma). \tag{3.5.54d}$$

It is not difficult to verify that $\bar{U}_\epsilon$ is the solution to (3.5.3). And we proved $\bar{U}_\epsilon = \tilde{U}_\epsilon$ is the solution to (3.5.3) satisfying (3.5.53). □

Proposition 3.5.1. *Under the assumptions of Proposition 3.5.1, when $d = 2$, for any $T \in (0, \infty)$, there exists a solution $\tilde{U}$ to $(\widetilde{\mathbf{NSI}}^\sigma)$, such that*

$$\|\tilde{U}\|_{L^\infty(0,T;V^\sigma)} \le C, \tag{3.5.55a}$$

$$\|\tilde{U}'\|_{L^\infty(0,T;L^2(\Omega)^d)} + \|\tilde{U}'\|_{L^2(0,T;V^\sigma)} \le C. \tag{3.5.55b}$$

When $d = 3$, the same conclusion holds for a smaller time interval $(0, \mathring{T})$.

Proof. The proof is valid for both $d = 2, 3$, except we replace T by $\mathring{T}$ for the case $d = 3$.

In view of Proposition 3.5.3, we have, for sufficiently small ϵ, $\|\tilde{U}_\epsilon\|_{L^\infty(0,T;V^\sigma)}$, $\|\tilde{U}'_\epsilon\|_{L^\infty(0,T;L^2(\Omega)^d)}$ and $\|\tilde{U}'_\epsilon\|_{L^2(0,T;V^\sigma)}$ are bounded independent of ϵ, and $\|[\tilde{U}_\epsilon + \tilde{g}_n]_-\|_{L^\infty(0,T;L^2(\Gamma))} \leq C\sqrt{\epsilon}$.

There exists a subsequence $\epsilon_i \to 0$, and $\bar{U}$ such that $\bar{U} \in L^\infty(0, T; V^\sigma)$, $\bar{U}' \in L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; V\sigma)$, and as $\epsilon \to 0$,

$$\tilde{U}_\epsilon \to \bar{U}, \quad \text{weakly* in } L^\infty(0, T; V^\sigma), \quad \text{weakly in } L^2(0, T; V^\sigma), \tag{3.5.56a}$$

$$[\tilde{U}_{\epsilon n} + \tilde{g}_n]_- \to 0, \ \text{weakly* in } L^\infty(0, T; L^2(\Gamma)), \tag{3.5.56b}$$

$$\tilde{U}'_\epsilon \to \bar{U}', \quad \text{weakly* in } L^\infty(0, T; L^2(\Omega)^d), \tag{3.5.56c}$$

$$\tilde{U}'_\epsilon \to \bar{U}', \quad \text{weakly in } L^2(0, T; V^\sigma). \tag{3.5.56d}$$

The same to the proof of Proposition 3.5.3, we have

$$\tilde{U}_\epsilon \to \bar{U}, \quad \text{strongly in } L^4(0, T; L^2(\Omega)^2), \tag{3.5.57a}$$

$$\tilde{U}_{\epsilon n} \to \bar{U}_n, \quad \text{strongly in } L^2(0, T; L^2(\Omega)^2), \tag{3.5.57b}$$

$$[\tilde{U}_\epsilon + \tilde{g}_n]_- \to [\bar{U}_n + \tilde{g}_n]_- \text{ a.e. on } \Gamma. \tag{3.5.57c}$$

Hence, $[\bar{U}_n + \tilde{g}_n]_- = 0$ a.e. on Γ, $\bar{U} \in \tilde{K}^\sigma$, and

$$\int_0^T a(\bar{U}, \bar{U})dt \leq \underline{\lim}_{\epsilon\to 0} \int_0^T a(\tilde{U}_\epsilon, \tilde{U}_\epsilon)dt.$$

For arbitrary $v \in \tilde{K}^\sigma$, from (3.5.3), we have,

$$\begin{aligned}
&(\tilde{U}'_\epsilon, v - \tilde{U}_\epsilon) + \frac{\beta'(t)}{\beta(t)}(\tilde{U}_\epsilon, v - \tilde{U}_\epsilon) + a(\tilde{U}_\epsilon, v - \tilde{U}_\epsilon) \\
&\quad + \beta(t)a_1(\tilde{U}_\epsilon, \tilde{g}, v - \tilde{U}_\epsilon) + \beta(t)a_1(\tilde{U}_\epsilon + \tilde{g}, \tilde{U}_\epsilon, v - \tilde{U}_\epsilon) \\
&\quad - \frac{1}{\epsilon}\int_\Gamma [\tilde{U}_{\epsilon n} + \tilde{g}_n]_-(v_n - \tilde{U}_{\epsilon n})ds \\
&- (\tilde{F}, v - \tilde{U}_\epsilon) - [[\tau(\tilde{g}, \tilde{\pi}), v - \tilde{U}_\epsilon]] = 0,
\end{aligned} \tag{3.5.58a}$$

$$\tilde{U}(x, 0) = \tilde{U}_0, \qquad \text{on } \Omega. \tag{3.5.58b}$$

In view of

$$\begin{aligned}
&- [\tilde{U}_{\epsilon n} + \tilde{g}_n]_-(v_n - \tilde{U}_{\epsilon n}) = -[\tilde{U}_{\epsilon n} + \tilde{g}_n]_-[v_n + \tilde{g}_n - (\tilde{U}_{\epsilon n} + \tilde{g}_n)] \\
=& - [\tilde{U}_{\epsilon n} + \tilde{g}_n]_-(v_n + \tilde{g}_n) - |[\tilde{U}_{\epsilon n} + \tilde{g}_n]_-|^2 \\
\leq& 0 \quad (\forall v \in \tilde{K}),
\end{aligned} \tag{3.5.59}$$

we have, for all $t \in [0,T]$,

$$\begin{aligned}\int_0^t \Big\{ &(\tilde{U}_\epsilon', v-\tilde{U}_\epsilon) + (\beta'(t)/\beta(t))(\tilde{U}_\epsilon, v-\tilde{U}_\epsilon) + a(\tilde{U}_\epsilon, v-\tilde{U}_\epsilon) \\ &+\beta(t)a_1(\tilde{U}_\epsilon, \tilde{g}, v-\tilde{U}_\epsilon) + \beta(t)a_1(\tilde{U}_\epsilon+\tilde{g}, \tilde{U}_\epsilon, v-\tilde{U}_\epsilon) \\ &-(\tilde{F}, v-\tilde{U}_\epsilon) - [[\tau(\tilde{g},\tilde{\pi}), v-\tilde{U}_\epsilon]] \Big\} \geq 0,\end{aligned} \tag{3.5.60}$$

Therefore, taking the lower limit $\underline{\lim}_{\epsilon\to 0}$ to (3.5.60), we obtian

$$\begin{aligned}\int_0^t \big\{ &(\bar{U}', v-\bar{U}) + (\beta'(t)/\beta(t))(\bar{U}, v-\bar{U}) + a(\bar{U}, v-\bar{U}) \\ &+\beta(t)a_1(\bar{U}, \tilde{g}, v-\bar{U}) + \beta(t)a_1(\bar{U}+\tilde{g}, \bar{U}, v-\bar{U}) \\ &-(\tilde{F}, v-\bar{U}) - [[\tau(\tilde{g},\tilde{\pi}), v-\bar{U}]] \Big\} \geq 0,\end{aligned} \tag{3.5.61}$$

Follows from Lebesgue differentiation theorem(cf. [15]), we have $\bar{U} = \tilde{U}$ is the solution to (3.5.2) for a.e. $t \in [0,T]$. □

Since $U = \tilde{U}\beta(t)$ and $U_\epsilon = \tilde{U}_\epsilon\beta(t)$, in view of Proposition 3.5.1 and 3.5.3, we obtain the well-posedness of U and U_ϵ.

Proposition 3.5.2. *Under the assumptions (A1)(A2)(A3)(A4), when $d = 2$, for any $T \in (0,\infty)$, there exists a solution U to $(\mathbf{NSI}^\sigma)$, such that*

$$\|U\|_{L^\infty(0,T;V^\sigma)} \leq C, \tag{3.5.62a}$$

$$\|U'\|_{L^\infty(0,T;L^2(\Omega)^d)} + \|U'\|_{L^2(0,T;V^\sigma)} \leq C. \tag{3.5.62b}$$

When $d = 3$, the same conclusion holds for a smaller time interval $(0, \mathring{T})$.

Proposition 3.5.3. *Under the assumptions (A1)(A2)(A3)(A4), when $d = 2$, for any $T \in (0,\infty)$ and sufficiently small ϵ, there exists a solution U_ϵ to $(\mathbf{NS}_\epsilon\mathbf{E}^\sigma)$, such that*

$$\|U_\epsilon\|_{L^\infty(0,T;V^\sigma)} + \epsilon^{-1/2}\|[U_\epsilon + g_n]_-\|_{L^\infty(0,T;L^2(\Gamma))} \leq C, \tag{3.5.63a}$$

$$\|U_\epsilon'\|_{L^\infty(0,T;L^2(\Omega)^d)} + \|U_\epsilon'\|_{L^2(0,T;V^\sigma)} \leq C. \tag{3.5.63b}$$

When $d = 3$, the same conclusion holds for a smaller time interval $(0, \mathring{T})$.

Proof of Theorem 3.3.1. It follows from Proposition 3.5.2, 3.3.1 and 3.3.2. □

Proof of Theorem 3.4.1. It follows from Proposition 3.5.3, 3.4.1, and 3.4.2. □

3.6 The Stokes problem with a unilateral boundary condition of Signorini's type

From now on, we consider the Stokes equations with unilateral boundary condition of Signorini's type.

Find a velocity u and a pressure p such that

$$
\begin{aligned}
& -\nu\Delta u + \nabla p = f, \quad \nabla\cdot u = 0, && \text{in } \Omega, && (3.6.1a)\\
& u = b, && \text{on } S, && (3.6.1b)\\
& u = 0, && \text{on } C, && (3.6.1c)\\
& u_n \geq 0, \quad \tau_n(u,p) \geq 0, && \text{on } \Gamma, && (3.6.1d)\\
& u_n\tau_n(u,p) = 0, \quad \tau_T(u) = 0, && \text{on } \Gamma. && (3.6.1e)
\end{aligned}
$$

Remark 3.6.1. The Signorini's problem has been considered in [25] with a traction boundary condition on a portion of Γ, i.e. there exists $\Gamma_0 \subset \Gamma$, $|\Gamma_0| > 0$, such that, $\tau(u,p) = H(x)$ on Γ_0, which leads to an essentially different argument.

We set the reference flow (g,π) satisfying

$$\nabla\cdot\sigma(g,\pi) = 0, \quad \nabla\cdot g = 0, \quad \text{in } \Omega,$$

$$g|_C = 0, \quad g|_S = b.$$

And we assume that

$$\beta := \int_\Gamma g_n = -\int_S b_n \geq 0.$$

We assume that $f \in L^2(\Omega)^d$ and $\tau(g,\pi) \in M'$.

Setting $(U,P) = (u-g, p-\pi)$, our target problem becomes the following equations.

(**S**) Find a velocity U and a pressure P such that

$$
\begin{aligned}
& -\nu\Delta U + \nabla P = f, \quad \nabla\cdot U = 0, && \text{in } \Omega, && (3.6.2a)\\
& U = 0 && \text{on } S\cup C, && (3.6.2b)\\
& U_n + g_n \geq 0, \quad \tau_n(U,P) + \tau_n(g,\pi) \geq 0, && \text{on } \Gamma && (3.6.2c)\\
& (U_n + g_n)(\tau_n(U,P) + \tau_n(g,\pi)) = 0, && \text{on } \Gamma && (3.6.2d)\\
& \tau_T(U) + \tau_T(g) = 0 && \text{on } \Gamma && (3.6.2e)
\end{aligned}
$$

Weak formulation of (S).

We interpret (**S**) as follows.

(**S′**)Find $(u, p) \in V \times Q$ s.t.

$$a(U,\varphi) + b(P,\varphi) = \int_\Omega f \cdot \varphi \, dx \qquad (\forall \varphi \in H_0^1(\Omega)^d), \tag{3.6.3a}$$

$$b(q,U) = 0 \qquad (\forall q \in Q), \tag{3.6.3b}$$

$$U_n + g_n \geq 0 \qquad \text{a.e. on } \Gamma, \tag{3.6.3c}$$

$$[\tau_n(U,P) + \tau_n(g,\pi), \eta] \geq 0 \qquad (\forall \eta \in M,\ \eta \geq 0), \tag{3.6.3d}$$

$$[\tau_n(U,P) + \tau_n(g,\pi), U_n + g_n] = 0 \qquad (\forall \eta \in M,\ \eta \geq 0), \tag{3.6.3e}$$

$$[[\tau_T(U) + \tau_T(g), \eta]] = 0 \qquad (\forall \eta \in M^d,\ \eta_n = 0). \tag{3.6.3f}$$

Formulation by a variational inequality

(VI) Find $(U, P) \in K \times Q$ s.t.

$$a(U, v - U) + b(v - U, p) \geq \langle F, v - U \rangle \qquad (\forall v \in K), \tag{3.6.4a}$$

$$b(q,U) = 0 \qquad (\forall q \in Q), \tag{3.6.4b}$$

where $F : V \to V'$ is defined as

$$\langle F, v \rangle = \langle F, v \rangle_{V',V} = \int_\Omega f \cdot v \, dx - [[\tau(g,\pi), v]]. \tag{3.6.5}$$

Theorem 3.6.1. (**VI**) $\Leftrightarrow$ (**S′**) .

Proof. The argument is similar to Theorem 3.2.2. □

Theorem 3.6.2. *There exists a unique solution $(U, P) \in K \times Q$ of (VI).*

Proof. Since a is a coercive bilinear form in $V^\sigma \times V^\sigma$ by Korn's inequality, we can apply Stampacchia's theorem (cf. [9, Theorem 5.6]) to conclude that there exists a unique $U \in K^\sigma$ satisfying

$$a(U, v - U) \geq \langle F, v - U \rangle \qquad (\forall v \in K^\sigma). \tag{3.6.6}$$

Taking $v = U \pm \varphi$ with $\varphi \in H^1_{0,\sigma}(\Omega)$ in (3.6.6), we deduce

$$a(U,\varphi) = \int_\Omega f \cdot \varphi \, dx, \quad (\forall \varphi \in H_0^1(\Omega)^d \cap V^\sigma). \tag{3.6.7}$$

Hence, according to the *inf-sup condition* of b, there exists $\mathring{P} \in L^2_0(\Omega)$ satisfying

$$(\mathring{P}, \nabla \cdot v) = a(U, v) - \int_\Omega f \cdot v \, dx \qquad (\forall v \in H^1_0(\Omega)^d).$$

Thus we obtain $(U, \mathring{P}) \in K \times L^2_0(\Omega)$ satisfying

$$a(U, v) + b(\mathring{P}, v) = \int_\Omega f \cdot v \, dx \qquad (\forall v \in H^1_0(\Omega)^d). \tag{3.6.8}$$

Setting

$$l \equiv \inf_{\eta \in Y} [\tau_n(u, \hat{p}) + h_n, \eta] = \frac{[\tau_n(u, \hat{p}) + h_n, u_n + g_n]}{\beta}, \tag{3.6.9}$$

where

$$Y = \left\{ \eta \in M \mid \eta \geq 0, \ \eta \not\equiv 0, \ \int_\Gamma \eta = 1 \right\}.$$

With a similar argument to the proof of Theorem 3.3.1, it is not difficult to verify that (U, P) is the solution of (**VI**) where $P = \mathring{P} + l$ □

3.6.1 Penalty method for the Stokes problem

We introduce $\rho : V \to V'$ by setting

$$\langle \rho(U), v \rangle = - \int_\Gamma [U_n + g_n]_- v_n \, ds, \tag{3.6.10}$$

where $[w]_\pm = \max\{0, \pm w\}$ and $w = [w]_+ - [w]_-$.

Lemma 3.6.1. *(i) ρ is a bounded, monotone and hemicontinuous operator from V to V'.*

(ii) $K = \{v \in V \mid \rho(v) = 0\}$.

Proof. We show (i).

1. (boundness) By using the trace theorem, we have

$$\begin{aligned} \langle \rho(U), v \rangle &\leq \int_\Gamma [U_n + g_n]_- |v_n| \, ds \\ &\leq \|[U_n + g_n]_-\|_{L^2(\Gamma)} \|v_n\|_{L^2(\Gamma)} \\ &\leq (\|U_n\|_{L^2(\Gamma)} + \|g_n\|_{L^2(\Gamma)}) \|v_n\|_{L^2(\Gamma)} \\ &\leq (\|U\|_V + \|g_n\|_{L^2(\Gamma)}) \|v\|_V \end{aligned}$$

for $U, v \in V$. Hence,

$$\|\rho(U)\|_{V'} \leq \|u\|_V + \|g_n\|_{L^2(\Gamma)}.$$

2. (monotonicity) For U, v, we have

$$\begin{aligned}
&\langle \rho(U) - \rho(v), u - v\rangle = \langle \rho(U), U - v\rangle - \langle \rho(v), U - v\rangle \\
=& -\int_\Gamma [U_n + g_n]_-(U_n - v_n) + \int_\Gamma [v_n + g_n]_-(U_n - v_n) - \\
=& -\int_\Gamma ([U_n + g_n]_- - [v_n + g_n]_-)(U_n + g_n - (v_n + g_n)) \\
=& \int_\Gamma ([U_n + g_n]_- - [v_n + g_n]_-)(U_n + g_n - (v_n + g_n)) \\
=& \|[U_n + g_n]_- - [v_n + g_n]_-\|^2_{L^2(\Gamma)} \\
&- \int_\Gamma ([U_n + g_n]_- - [v_n + g_n]_-)([U_n + g_n]_+ - [v_n + g_n]_+) \\
\geq& \int_\Gamma [U_n + g_n]_-[v_n + g_n]_+ + \int_\Gamma [v_n + g_n]_-[U_n + g_n]_+ \\
\geq& 0.
\end{aligned}$$

3. (hemicontinuity) Let $U, v, w \in U$ and consider a real-valued function

$$\eta(\lambda) = \langle \rho(U + \lambda v), w\rangle = \int_\Gamma [U_n + \lambda v_n]_- w_n \quad (\lambda \in \mathbb{R}).$$

   This is a continuous function, since the function $[\cdot]_-$ is continuous.

(ii) It is obvious. □

Penalty problem of (S)

Let $0 < \epsilon \ll 1$. We give the penalty problem to (**S**).

($\mathbf{S}_\epsilon$) Find $(U_\epsilon, P_\epsilon) \in V \times Q$ such that

$$a(U_\epsilon, v) + b(P_\epsilon, v) + \frac{1}{\epsilon}\langle \rho(U_\epsilon), v\rangle = \langle F, v\rangle \qquad (\forall v \in V), \qquad (3.6.11a)$$

$$b(q, U_\epsilon) = 0 \qquad (\forall q \in Q). \qquad (3.6.11b)$$

($\mathbf{S}_\epsilon^\sigma$) Find $U_\epsilon \in V^\sigma$ such that

$$a(U_\epsilon, v) + \frac{1}{\epsilon}\langle \rho(U_\epsilon), v\rangle = \langle F, v\rangle \qquad (\forall v \in V^\sigma). \qquad (3.6.12)$$

Theorem 3.6.3. *There exists a unique solution U_ϵ of $(\mathbf{S}_\epsilon^\sigma)$ and it satisfies*

$$\|U_\epsilon\|_V \leq C(\|F\|_{V'} + \|g_n\|_M), \tag{3.6.13}$$

$$\|\rho(u_\epsilon)\|_{M'} = \sup_{\eta\in M} \frac{\langle \rho(u_\epsilon), \eta\rangle}{\|\eta\|_M} \leq C\epsilon(\|F\|_{V'} + \|g_n\|_M). \tag{3.6.14}$$

Theorem 3.6.4. *There exists a unique solution (U_ϵ, U_ϵ) of $(\mathbf{S}_\epsilon)$.*

Proof of Theorem 3.6.3

We will make use of

Lemma 3.6.2 (Theorem 2.1 of [28])**.** *Let X be a separable reflexive Banach space and let $T : X \to X'$ be a (possibly nonlinear) operator satisfying the following conditions:*

1. *(boundness) There exist $C, C', m > 0$ s.t. $\|Tu\|_{X'} \leq C\|u\|_X^m + C'$ for all $u \in X$;*
2. *(monotonicity) $\langle Tu - Tv, u - v\rangle \geq 0$ for all $u, v \in X$;*
3. *(hemicontinuity) For any $u, v, w \in X$, the function $\lambda \mapsto \langle A(u + \lambda v), w\rangle$ is continuous on $\mathbb{R}$;*
4. *(coerciveness) $\dfrac{\langle Tu, u\rangle}{\|u\|_X} \to \infty$ as $\|u\|_X \to \infty$.*

Then, for any $\varphi \in X'$, there exists a unique $u \in X$ such that $Tu = \varphi$. Furthermore, if T is strictly monotone:

$$\langle Tu - Tv, u - v\rangle > 0 \quad (\forall u, v \in X, u \neq v),$$

then the solution is unique.

Proof of Theorems 3.6.3. We consider a nonlinear operator $A_\epsilon : V \to V'$ by setting

$$A_\epsilon v = Av + \frac{1}{\epsilon}\rho(v) \qquad (v \in V),$$

where $A : V \to V'$ is a linear operator defined as $\langle Au, v\rangle = a(u, v)$ for $u, v \in V$. We verify that the restriction $A_\epsilon|_{V^\sigma}$ of A_ϵ satisfies the conditions in Lemma 3.6.2. Below we write $A_\epsilon = A_\epsilon|_{V^\sigma}$, and we use Lemma 3.6.1 (i).

1. (boundness)

$$\begin{aligned} |\langle A_\epsilon u, v\rangle| &\leq |\langle Au, v\rangle| + \frac{1}{\epsilon}|\langle \rho(u), v\rangle| \\ &\leq \|a\| \cdot \|u\|_V \|v\|_V + \frac{1}{\epsilon}(\|u\|_V + \|g_n\|_{L^2(\Gamma)})\|v\| \end{aligned}$$

for $u, v \in V$. Hence,

$$\|A_\epsilon u\|_{(V^\sigma)'} \leq \|A_\epsilon u\|_V \leq \left(\|a\| + \frac{1}{\epsilon}\right)\|u\|_V + \frac{1}{\epsilon}\|g_n\|_{L^2(\Gamma)} \qquad (u \in V^\sigma).$$

2. (strictly monotonicity) By virtue of Korn's inequality,

$$\begin{aligned} &\langle A_\epsilon u - A_\epsilon v, u - v\rangle = \langle A_\epsilon u, u - v\rangle - \langle A_\epsilon v, u - v\rangle \\ =&\langle Au, u - v\rangle + \frac{1}{\epsilon}\langle \rho(u), u - v\rangle - \langle Av, u - v\rangle - \frac{1}{\epsilon}\langle \rho(v), u - v\rangle \\ =&\langle A(u - v), u - v\rangle + \frac{1}{\epsilon}\langle \rho(u) - \rho(v), u - v\rangle \\ =&a(u - v, u - v) + \frac{1}{\epsilon}\langle \rho(u) - \rho(v), u - v\rangle \\ =&C_{\mathrm{K}}\|u - v\|_V^2 - \frac{1}{\epsilon}\langle \rho(u - v), u - v\rangle \\ >&0 \end{aligned}$$

for $u, v \in V$, $u \neq v$.

3. (hemicontinuity) Let $u, v, w \in V$ and consider a real-valued function

$$\eta(\lambda) = \langle A_\epsilon(u + \lambda v), w\rangle = a(u + \lambda v, w) + \frac{1}{\epsilon}\langle \rho(u + \lambda v), w\rangle \quad (\lambda \in \mathbb{R}).$$

This is a continuous function, since $a(\cdot, w)$ is continuous and $\rho(\cdot)$ is hemicontinuous.

4. (Coerciveness) For $u \in V$, we have

$$\begin{aligned}
&\langle \rho(u), u\rangle = -\int_\Gamma [u_n + g_n]_- u_n \; ds \\
&= -\int_\Gamma [u_n + g_n]_- \left([u_n + g_n]_+ - [u_n + g_n]_- - [g_n]_+ + [g_n]_-\right) \; d\Gamma \\
&\geq -\int_\Gamma [u_n + g_n]_- [g_n]_- \; ds \\
&\geq -\|[u_n + g_n]_-\|_{L^2(\Gamma)} \|[g_n]_-\|_{L^2(\Gamma)} \\
&\geq -\|u_n + g_n\|_{L^2(\Gamma)} \|g_n\|_{L^2(\Gamma)} \\
&\geq -\left(\|u\|_V + \|g_n\|_{L^2(\Gamma)}\right) \|g_n\|_{L^2(\Gamma)}.
\end{aligned} \tag{3.6.15}$$

Hence,

$$\frac{\langle Au + \frac{1}{\epsilon}\rho(u), u\rangle}{\|u\|_V} \geq C_{\mathrm{K}} \|u\|_V - \frac{\left(\|u\|_V + \|g_n\|_{L^2(\Gamma)}\right)}{\epsilon \|u\|_V} \|g_n\|_{L^2(\Gamma)}$$

This gives

$$\frac{\langle Au + \frac{1}{\epsilon}\rho(u), u\rangle}{\|u\|_V} \to \infty \quad \text{as} \quad \|u\|_V \to \infty.$$

As a consequence, we can apply Lemma 3.6.2 to conclude that there exists a unique $u_\epsilon \in V^\sigma$ satisfying $A_\epsilon u_\epsilon = F_0$, where $F_0 \in (V^\sigma)'$ is the restriction of $F \in V'$. Thus, we have proved a unique existence of the solution $u_\epsilon \in V^\sigma$ of $(\mathbf{S}_\epsilon^\sigma)$.

Next, we derive (3.6.13) and (3.6.14). To this end, we recall $\beta = \int_\Gamma g_n > 0$. First, we set

$$\eta = g_n - \beta\phi,$$

where $\phi \in C_0^\infty(\Gamma)$ is a function satisfying $\phi \geq 0$ and $\int_\Gamma \phi = 1$ and below we fix it. We have $\eta \in M$ and $\int_\Gamma \eta = 0$. Hence, there exists an extension $w \in V_0$ of η satisfying $\|w\|_V \leq C\|\eta\|_M \leq C\|g_n\|_M$ and $w_n|_\Gamma = \eta$.

Substituting $v = U_\epsilon + w \in V_0$ into (3.6.11), we have

$$a(U_\epsilon, U_\epsilon + w) - \frac{1}{\epsilon}\int_\Gamma [U_{\epsilon n} + g_n]_-(U_{\epsilon n} + g_n - \beta\phi) = \langle F, U_\epsilon + w\rangle.$$

Noticing that

$$U_{\epsilon n} + g_n - \beta\phi \leq U_{\epsilon n} + g_n,$$

which guarantees

$$-\frac{1}{\epsilon}\int_\Gamma [U_{\epsilon n}+g_n]_-(U_{\epsilon n}+g_n-\beta\phi) \geq \frac{1}{\epsilon}\|[U_{\epsilon n}+g_n-\beta\phi]_-\|^2_{L^2(\Gamma)} \geq 0.$$

Hence we have

$$a(U_\epsilon, U_\epsilon + w) \leq \langle F, U_\epsilon + w\rangle.$$

From this, we can deduce

$$\|U_\epsilon\|_V \leq C(\|F\|_{(V^\sigma)'} + \|g_n\|_M) \leq C(\|F\|_{V'} + \|g_n\|_M)$$

and

$$\|[U_{\epsilon n}+g_n-\beta\phi]_-\|_{L^2(\Gamma)} \leq C\sqrt{\epsilon}(\|F\|_{V'} + \|g_n\|_M).$$

Further, equation (3.6.11) implies

$$\langle \rho(U_\epsilon), v\rangle = \epsilon\langle F, v\rangle - \epsilon a(U_\epsilon, v) \qquad (\forall v \in V^\sigma),$$

so we have

$$\begin{aligned}\|\rho(U_\epsilon)\|_{M'} &= \sup_{v\in V_0,\ v\neq 0} \frac{\langle \rho(U_\epsilon), v\rangle}{\|v\|_V} \\ &= \epsilon \sup_{v\in V_0,\ v\neq 0} \frac{\langle F, v\rangle - a(U_\epsilon, v)}{\|v\|_V} \\ &\leq C\epsilon(\|F\|_{V'} + \|u_\epsilon\|_V).\end{aligned}$$

This completes the proof. □

Proof of Theorem 3.6.4. From Theorem 3.6.3, we know that there exists a unique solution $U_\epsilon \in V_0$ of $(\mathbf{S}^\sigma_\epsilon)$. Then, by the standard theory, there exists the associating pressure $\mathring{P}_\epsilon \in= L^2_0(\Omega)$ of the velocity U_ϵ;

$$a(U_\epsilon, v) + b(\mathring{P}_\epsilon, v) = \int_\Omega f\cdot v \qquad (v \in H^1_0(\Omega)^d).$$

For any $\phi \in C^\infty_0(\Gamma)$ with $\int_\Gamma \phi = 1$, we set

$$l_\epsilon = \int_\Gamma (\tau_n(U_\epsilon, \mathring{P}_\epsilon) + \tau_n(g,\pi) - \epsilon^{-1}[U_{\epsilon n}]_-)\phi\ ds. \tag{3.6.16}$$

We see that l_ϵ is a constant independent of ϕ. It is not difficult to verify that (U, P_ϵ) is a solution of $(\mathbf{S}_\epsilon)$, where

$$P_\epsilon = \mathring{P}_\epsilon + l_\epsilon.$$

□

3.6.2 Error estimate of penalty method

Theorem 3.6.5. *Let (U, P) and (U_ϵ, P_ϵ) be the unique solutions of* **(S)** *and* $(\mathbf{S}_\epsilon)$*, respectively. Then, we have*

$$\|U - U_\epsilon\|_V + \|\mathring{P} - \mathring{P}_\epsilon\|_Q \leq C\sqrt{\epsilon}\|\tau_n(U, P)\|_{M'}, \tag{3.6.17}$$

where $\mathring{P}$ and $\mathring{P}_\epsilon$ are defined by

$$\mathring{P} = P - l, \quad \mathring{P}_\epsilon = P_\epsilon - l_\epsilon, \quad l = \frac{1}{|\Omega|}\int_\Omega P, \quad l_\epsilon = \frac{1}{|\Omega|}\int_\Omega P_\epsilon. \tag{3.6.18}$$

Proof. Recall (U, P) satisfies for any $v \in V$,

$$a(U, v) + b(P, v) - [\tau_n(U, P) + \tau_n(g, \pi), v_n] = \int_\Omega f \cdot v \; dx - \int_\Gamma \tau(g, \pi) v \; ds.$$

Together with (3.6.11), it implies that for all $v \in V$,

$$\begin{aligned} &a(U - U_\epsilon, v) + b(P - P_\epsilon, v) \\ &\qquad = \int_\Gamma (\tau_n(U, P) + \tau_n(g, \pi) - \epsilon^{-1}[U_{\epsilon n} + g_n]_-) v_n \; ds, \end{aligned} \tag{3.6.19}$$

and for any $v \in V^\sigma$,

$$a(U - U_\epsilon, v) = \int_\Gamma (\tau_n(U, P) + \tau_n(g, \pi) - \epsilon^{-1}[U_{\epsilon n} + g_n]_-) v_n \; ds. \tag{3.6.20}$$

Now we take $v = U - U_\epsilon \in V^\sigma$ and obtain

$$\begin{aligned} &a(U - U_\epsilon, U - U_\epsilon) = [\tau_n(U, P) + \tau_n(g, \pi) - \epsilon^{-1}[U_{\epsilon n} + g_n]_-, U_n - U_{\epsilon n}] \\ =& \underbrace{[\tau_n(U, P) + \tau_n(g, \pi), U_n - U_{\epsilon n}]}_{=I_1} \underbrace{- \epsilon^{-1}[U_{\epsilon n} + g_n]_-, U_n - U_{\epsilon n}]}_{=I_2}. \end{aligned}$$

We calculate as

$$\begin{aligned} I_1 =& \underbrace{[\tau_n(U, P) + \tau_n(g, \pi), U_n + g_n]}_{=0} - [\tau_n(U, P) + \tau_n(g, \pi), U_{\epsilon,n} + g_n] \\ =& - [\tau_n(U, P) + \tau_n(g, \pi), [U_{\epsilon,n} + g_n]_+ - [U_{\epsilon,n} + g_n]_-] \\ \leq& \epsilon[\tau_n(U, P) + \tau_n(g, \pi), \epsilon^{-1}[U_{\epsilon n} + g_n]_-], \end{aligned}$$

and

$$\begin{aligned} I_2 =& -[\epsilon^{-1}[U_{\epsilon n} + g_n]_-, U_n + g_n] + [\epsilon^{-1}[U_{\epsilon n} + g_n]_-, U_{\epsilon n} + g_n] \\ \leq& - \frac{1}{\epsilon}\int_\Gamma [U_{\epsilon n} + g_n]_- [U_{\epsilon n} + g_n]_- ds \\ =& - \epsilon \int_\Gamma (\epsilon^{-1}[U_{\epsilon n} + g_n]_-)^2 \; ds \end{aligned}$$

As a result, we get,

$$\begin{aligned} a(U-U_\epsilon, U-U_\epsilon) \leq& \epsilon \int_\Gamma (\tau_n(U,P)+\tau_n(g,\pi))\epsilon^{-1}[U_{\epsilon n}+g_n]_-] \ ds \\ & - \epsilon \int_\Gamma (\epsilon^{-1}[U_{\epsilon n}+g_n]_-)^2 \ ds, \end{aligned} \tag{3.6.21}$$

which implies

$$\|U-U_\epsilon\|_V \leq C\sqrt{\epsilon}\|\tau_n(U,P)+\tau_n(g,\pi)\|_{L^2(\Gamma)}.$$

We proceed to the pressure part. We have

$$\begin{aligned} a(U-U_\epsilon, v)+b(\mathring{P}-\mathring{P}_\epsilon, v) = & \ a(U-U_\epsilon, v)+b(P-P_\epsilon, v) \\ = & \quad 0 \qquad (\forall v \in H_0^1(\Omega)^d). \end{aligned}$$

We apply the *inf-sup condition* of b, and conclude

$$\|\mathring{P}-\mathring{P}_\epsilon\|_Q \leq C\|U-U_\epsilon\|_V \leq C\sqrt{\epsilon}\|\tau_n(U,P)+\tau_n(g,\pi)\|_{L^2(\Gamma)},$$

which completes the proof. □

Theorem 3.6.6. *Let* (U,P) *and* (U_ϵ, P_ϵ) *be the unique solutions of* $(\mathbf{S})$ *and* $(\mathbf{S}_\epsilon)$*, respectively. Further, assume that*

$$g_n \in C(\overline{\Gamma}), \ \tau_n(g,\pi) \in H^{1/2}(\Gamma), \tag{3.6.22}$$

$$U, U_\epsilon \in H^2(\Omega)^d, \quad P, P_\epsilon \in H^1(\Omega), \tag{3.6.23}$$

$$\|U_n - U_{\epsilon n}\|_{L^\infty(\Gamma)} \to 0 \quad \text{as } \epsilon \downarrow 0. \tag{3.6.24}$$

Then, we have as $\epsilon \downarrow 0$

$$\|U-U_\epsilon\|_V + \|P-P_\epsilon\|_Q \leq C\epsilon\|\tau_n(U,P)+\tau_n(g,\pi)\|_M. \tag{3.6.25}$$

Remark 3.6.2. If $\overline{\Gamma} \cap \overline{C} = \emptyset$ (say, Γ is a smooth closed surface), we can deduce

$$U, U_\epsilon \in H^2(\Omega)^d, \quad P, P_\epsilon \in H^1(\Omega), \quad \|U-U_\epsilon\|_{H^2} + \|P-P_\epsilon\|_{H^1} \to 0 \ (\epsilon \downarrow 0).$$

by the standard manner using local coordinates and difference quotients (cf. [34] etc.). Thus, (3.6.23) and (3.6.24) actually take place if data are smooth.

Proof of Theorem 3.6.6. Set

$$\lambda_\epsilon = \tau_n(U,P) + \tau_n(g,\pi) - \epsilon^{-1}[u_{\epsilon n} + g_n]_-.$$

Recall that (cf. Proof of Theorem 3.6.5)

$$a(U - U_\epsilon, v) + b(P - P_\epsilon, v) = [\lambda_\epsilon, v_n] \qquad (\forall v \in V). \tag{3.6.26}$$

This implies

$$a(U - U_\epsilon, v) + b(\mathring{P} - \mathring{P}_\epsilon, v) = [\lambda_\epsilon - l + l_\epsilon, v_n] \qquad (\forall v \in V), \tag{3.6.27}$$

From the *inf-sup condition* of b, we have

$$\begin{aligned}\|\mathring{P} - \mathring{P}_\epsilon\|_Q &\le \frac{1}{\beta_2} \sup_{v \in H_0^1(\Omega)^d} \frac{-b(\mathring{P} - \mathring{P}_\epsilon, v)}{\|v\|_V} \\ &\le \frac{1}{\beta_2} \sup_{v \in H_0^1(\Omega)^d} \frac{|a(U - U_\epsilon, v)|}{\|v\|_V} \le C\|U - U_\epsilon\|_V.\end{aligned} \tag{3.6.28}$$

On the other hand, by the *inf-sup condition* of c,

$$\begin{aligned}\|\lambda_\epsilon - l + l_\epsilon\|_{M'} &\le \sup_{v \in V} \frac{[\lambda_\epsilon - l + l_\epsilon, v_n]}{\|v\|_V} \\ &\le \sup_{v \in V} \frac{|a(U - U_\epsilon, v)| + |b(\mathring{P} - \mathring{P}_\epsilon, v)|}{\|v\|_V} \\ &\le C\|u - u_\epsilon\|_V.\end{aligned} \tag{3.6.29}$$

Thanks to (3.6.23), we have

$$\tau_n(U,P) + \tau_n(g,\pi) \in M = H^{1/2}(\Gamma), \quad U|_\Gamma, U_\epsilon|_\Gamma \in C(\overline{\Gamma})^d. \tag{3.6.30}$$

Since $U_n + g_n \ge 0$ a.e. on Γ and $\int_\Gamma g_n > 0$ (and U_n, g_n are continuous), there exists a subset (with the positive area) $\omega \subset \Gamma$ such that $U_n + g_n > 0$ on ω. According to (3.6.2d), $\tau_n(U,P) + \tau_n(g,\pi) = 0$ on ω. Then, in view of (3.6.24), there exist $\epsilon_1 > 0$ and $\omega' \subset \omega$ with $|\omega'| > 0$ such that $U_{\epsilon n} + g_n > 0$ on ω' if $\epsilon \in (0, \epsilon_1]$. Consequently, $\epsilon^{-1}[U_{\epsilon n} + g_n]_- = 0$ on ω'. Hence, $\lambda_\epsilon = 0$ on ω'.

At this stage, we take $\eta \in C^\infty(\Gamma)$ such that supp $\eta \subset \omega'$, $\eta \ge 0$ on ω' and $\int_\Gamma \eta = 1$, and the extension of η into V is denoted by $v_\eta = E_n\eta \in V$.

Substituting $v = v_\eta$ into (3.6.27), we have

$$\begin{aligned}|[\lambda_\epsilon - l + l_\epsilon, \eta]| &\le |a(U - U_\epsilon, v_\eta)| + |b(\mathring{P} - \mathring{P}_\epsilon, v)| \\ &\le C\|U - U_\epsilon\|_V + C\|\mathring{P} - \mathring{P}_\epsilon\|_Q \le C\|U - U_\epsilon\|_V,\end{aligned}$$

where C denotes a positive constant depending on η. On the other hand,

$$\begin{aligned} |\lambda_\epsilon - l + l_\epsilon| &= \left|\int_{\omega'} (\lambda_\epsilon - l + l_\epsilon)\eta\right| \\ &= \left|\int_{\omega'} (l_\epsilon - l)\eta\right| \\ &= \left|(l_\epsilon - l)\int_{\omega'} \eta\right| = |l_\epsilon - l|. \end{aligned}$$

Hence,

$$|l_\epsilon - l| \le C\|U - U_\epsilon\|_V.$$

This, together with (3.6.29), gives

$$\begin{aligned} \|\lambda_\epsilon\|_{M'} &\le \|\lambda_\epsilon - l + l_\epsilon\|_{M'} + |k_\epsilon - k| \\ &\le C\|U - U_\epsilon\|_V. \end{aligned} \tag{3.6.31}$$

Recall that, from (3.6.21), we deduce

$$\alpha\|U - U_\epsilon\|_V^2 \le \epsilon\|\lambda_\epsilon\|_{M'}\|\tau_n(u,p) + \tau_n(g,\pi)\|_M. \tag{3.6.32}$$

Applying (3.6.29) to (3.6.32), it yields

$$\alpha\|U - U_\epsilon\|_V^2 \le \epsilon C\|U - U_\epsilon\|_V\|\tau_n(U,P) + \tau_n(g,\pi)\|_M,$$

which completes the proof. □

Remark

The chapter is based on [52, 38]

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