博士論文

論文題目

Visible actions of reductive algebraic groups on complex algebraic varieties

（簡約代数群の複素代数多様体への可視的作用について）

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Dedicated to my grandfathers Kesao Tanaka and Morio Saeki, grandmother Iku Saeki and friend Yohei Sato with wishes for good health of my grandmother Setsuko Tanaka and greatuncle Shin-nosuke Saeki.

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Contents

Chapter 1

Introduction

We study visible actions on complex algebraic varieties, and the main result is a classification of visible actions on generalized flag varieties.

Definition 1.0.1 (Kobayashi [Ko2])**.** We say a holomorphic action of a Lie group G on a complex manifold X is strongly visible if the following two conditions are satisfied:

1. There exists a real submanifold S (called a "slice") such that

$$X' \;:=\; G \cdot S \text{ is an open subset of } X.$$

2. There exists an anti-holomorphic diffeomorphism σ of X' such that

$$\begin{aligned} \sigma|_S &= \mathrm{id}_S, \\ \sigma(G \cdot x) &= G \cdot x \text{ for any } x \in X'. \end{aligned}$$

In the above setting, we say the action of G on X is S-visible. This terminology will be used also if S is just a subset of X.

Definition 1.0.2 (Kobayashi [Ko2])**.** We say a holomorphic action of a Lie group G on a complex manifold X is previsible if the condition (1) of Definition 1.0.1 is satisfied for a totally real submanifold S of X.

The notion of visible actions on complex manifolds was introduced by T. Kobayashi [Ko2] with the aim of uniform treatment of multiplicity-free representations of Lie groups.

Definition 1.0.3. We say a unitary representation V of a locally compact group G is multiplicity-free if the ring $\mathrm{End}_G(V)$ of intertwining operators on V is commutative.

There are various kinds of multiplicity-free representations (c.f. [BR, HU, Ka, VK]), and for the proof of the multiplicity-freeness property of representations, typical approaches are the following: verifying the existence of an open orbit of a Borel subgroup; using a combinatorial method (computing or estimating coefficients of the character of a representation). These two approaches work very well for (the direct sum of) finite dimensional representations, but it would be hard to apply them to the infinite dimensional representations with continuous spectra. A new approach has been introduced by Kobayashi, namely, the propagation theorem of the multiplicity-freeness property under visible actions:

Fact 1.0.4 (Kobayashi [Ko3]). *Let G be a Lie group and $\mathcal{W}$ a G-equivariant Hermitian holomorphic vector bundle on a connected complex manifold X. Let V be a unitary representation of G. If the following conditions from* (0) *to* (3) *are satisfied, then V is multiplicity-free as a representation of G.*

(0) *There exists a continuous and injective G-intertwining operator from V to the space $\mathcal{O}(X, \mathcal{W})$ of holomorphic sections of $\mathcal{W}$.*

(1) *The action of G on X is S-visible. That is, there exist a subset $S \subset X$ and an anti-holomorphic diffeomorphism σ of X' satisfying the conditions given in Definition 1.0.1. Further, there exists an automorphism $\hat{\sigma}$ of G such that $\sigma(g \cdot x) = \hat{\sigma}(g) \cdot \sigma(x)$ for any $g \in G$ and $x \in X'$.*

(2) *For any $x \in S$, the fiber $\mathcal{W}_x$ at x decomposes as the multiplicity-free sum of irreducible unitary representations of the isotropy subgroup G_x. Let $\mathcal{W}_x = \bigoplus_{1 \le i \le n(x)} \mathcal{W}_x^{(i)}$ denote the irreducible decomposition of $\mathcal{W}_x$.*

(3) *σ lifts to an anti-holomorphic automorphism $\tilde{\sigma}$ of $\mathcal{W}$ and satisfies $\tilde{\sigma}(\mathcal{W}_x^{(i)}) = \mathcal{W}_x^{(i)}$ for each $x \in S$ $(1 \le i \le n(x))$.*

The advantage of this new approach is that not only finite dimensional cases but also infinite dimensional (both discrete and continuous spectra) cases can be applied by this method. Indeed, we can see in the statement of the above theorem that we do not need to assume

G is compact, reductive,

V is of finite-dimensional, discretely decomposable, or

X is compact.

In the following, we quote a few examples of applications of Fact 1.0.4 from [Ko2]. The first example is an infinite dimensional unitary representation with only continuous spectrum.

Example 1.0.5. Let G be a semisimple Lie group and K a maximal compact subgroup of G. Then it is well-known that the space $L^2(G/K)$ of square integrable functions on the Riemannian symmetric space G/K is multiplicity-free (see [Wo] for example). We can also prove the multiplicity-freeness property by combining Fact 1.0.4 with the following facts.

- The G-action on the complexification $G_{\mathbb{C}}/K_{\mathbb{C}}$ is strongly visible by Kobayashi [Ko2].
- Let U be the complex crown of G/K, which was introduced by Akhiezer and Gindikin [AG]. Then there exists a G-embedding $L^2(G/K) \hookrightarrow \mathcal{O}(U)$ by Krötz and Stanton [KS].

Next, we give an example of a multiplicity-free representation arising from a visible action of a semisimple Lie group on a Hermitian symmetric space.

Example 1.0.6. Let G be a simple Lie group of Hermitian type, K a maximal compact subgroup and H a symmetric subgroup of G, i.e., H is an open subgroup of the τ-fixed points subgroup G^τ for an involution τ of G. Let π be a unitary highest weight representation of the scalar type of G. Then the restriction of π to H is multiplicity-free [Ko5] by Fact 1.0.4 combined with the following facts.

- π can be realized in the space $\mathcal{O}(G/K, \mathcal{L})$ of holomorphic sections of a G-equivariant holomorphic line bundle $\mathcal{L}$ on the Hermitian symmetric space G/K.
- The H-action on G/K is strongly visible by Kobayashi [Ko5] by the Cartan decomposition $G = HAK$ in the symmetric setting (see Flensted-Jensen [Fl1], Hoogenboom [Ho] and Matsuki [Ma2, Ma3]).

As the last example, we show a multiplicity-free representation of a non-reductive Lie group.

Example 1.0.7. Let G, K and π as in Example 1.0.6. Let N be a maximal unipotent subgroup of G. Then the restriction of π to N is multiplicity-free by Fact 1.0.4 combined with the facts that π can be realized in $\mathcal{O}(G/K, \mathcal{L})$ for a G-equivariant holomorphic line bundle $\mathcal{L}$ on G/K, and that the action of N on G/K is strongly visible by Kobayashi [Ko2] by the Iwasawa decomposition $G = NAK$.

As these examples show, we can obtain multiplicity-free representations from a visible action of a Lie group. Therefore it would be natural to try to find, or even classify, visible actions. In the following, we exhibit preceding results on a classification problem of visible actions. We firstly state a result on visible actions on symmetric spaces.

Fact 1.0.8 (Kobayashi [Ko5])**.** *Let (G, K) be a Hermitian symmetric pair and (G, H) a symmetric pair. Then H acts on the Hermitian symmetric space G/K strongly visibly.*

The next result concerns the visibility of linear actions. Let $G_{\mathbb{C}}$ be a connected complex reductive algebraic group and V a finite-dimensional representation of $G_{\mathbb{C}}$.

Definition 1.0.9. We say V is a linear multiplicity-free space of $G_{\mathbb{C}}$ if the space $\mathbb{C}[V]$ of polynomials on V is multiplicity-free as a representation of $G_{\mathbb{C}}$.

Fact 1.0.10 (Sasaki [Sa1, Sa4])**.** *Let V be a linear multiplicity-free space of $G_{\mathbb{C}}$. Then a compact real form U of $G_{\mathbb{C}}$ acts on V strongly visibly.*

Remark 1.0.11. We note that if U acts on a representation V of $G_{\mathbb{C}}$ strongly visibly, then V is a linear multiplicity-free space of $G_{\mathbb{C}}$ by Fact 1.0.4.

A linear multiplicity-free space is a special case of smooth affine spherical varieties. Let $G_{\mathbb{C}}$ be a connected complex reductive algebraic group and X a connected complex algebraic $G_{\mathbb{C}}$-variety.

Definition 1.0.12. We say X is a spherical variety of $G_{\mathbb{C}}$ if a Borel subgroup B of $G_{\mathbb{C}}$ has an open orbit on X.

A typical example of spherical varieties is a complex symmetric space (e.g. $G_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$ and $X = \mathrm{GL}(n, \mathbb{C})/(\mathrm{GL}(m, \mathbb{C}) \times \mathrm{GL}(n-m, \mathbb{C}))$). The third result deals with visible actions on affine homogeneous spherical varieties.

Fact 1.0.13 (Sasaki [Sa2, Sa3, Sa5])**.** *Let $G_{\mathbb{C}}/H_{\mathbb{C}}$ be one of the following affine homogeneous spherical varieties:*

$$\begin{aligned}
&\mathrm{SL}(m+n, \mathbb{C})/(\mathrm{SL}(m, \mathbb{C}) \times \mathrm{SL}(n, \mathbb{C}))\ (m \neq n),\\
&\mathrm{Spin}(4n+2, \mathbb{C})/\mathrm{SL}(2n+1, \mathbb{C}),\\
&\mathrm{SL}(2n+1, \mathbb{C})/\mathrm{Sp}(n, \mathbb{C}),\\
&\mathrm{E}_6(\mathbb{C})/\mathrm{Spin}(10, \mathbb{C}),\\
&\mathrm{SO}(8, \mathbb{C})/\mathrm{G}_2(\mathbb{C}).
\end{aligned}$$

Then the action of a compact real form U of $G_{\mathbb{C}}$ on $G_{\mathbb{C}}/H_{\mathbb{C}}$ is strongly visible.

Lastly we state a classification result on visible actions on generalized flag varieties of type A, which is the prototype of the main result of this paper. Let $G = \mathrm{U}(n)$ and L, H Levi subgroups of G. Kobayashi [Ko4] classified the triple (G, H, L) such that the following actions are strongly visible (we denote by $\Delta(G)$ the diagonal subgroup of $G \times G$).

$$L \curvearrowright G/H, \; H \curvearrowright G/L, \; \Delta(G) \curvearrowright (G \times G)/(H \times L).$$

In fact, all the above three actions are strongly visible if and only if at least one of those is strongly visible [Ko2]. The visibility of the three actions on generalized flag varieties was proved by giving a generalized Cartan decomposition:

Definition 1.0.14. Let G be a connected compact Lie group, T a maximal torus and H, L Levi subgroups of G, which contain T. We take a Chevalley–Weyl involution σ of G with respect to T. If the multiplication mapping

$$L \times B \times H \to G$$

is surjective for a subset B of the σ-fixed points subgroup G^{σ}, then we say the decomposition $G = LBH$ is a generalized Cartan decomposition.

Definition 1.0.15. An involution σ of a compact Lie group G is said to be a Chevalley–Weyl involution if there exists a maximal torus T of G such that $\sigma(t) = t^{-1}$ for any $t \in T$.

The definition of a generalized Cartan decomposition comes from that of a visible action. Let us explain. We retain the setting of Definition 1.0.14. Suppose that $G = LBH$ holds for some $B \subset G^{\sigma}$. Since σ acts on generalized flag varieties

$$G/H, \qquad G/L, \qquad (G \times G)/(H \times L)$$

as anti-holomorphic diffeomorphisms, we can obtain three strongly visible actions.

$$L \curvearrowright G/H, \quad H \curvearrowright G/L, \quad \Delta(G) \curvearrowright (G \times G)/(H \times L).$$

Furthermore, we can obtain three multiplicity-free theorems by using Fact 1.0.4.

$$\mathrm{ind}_H^G \chi_H|_L, \quad \mathrm{ind}_L^G \chi_L|_H, \quad \mathrm{ind}_H^G \chi_H \otimes \mathrm{ind}_L^G \chi_L.$$

Here $\mathrm{ind}_H^G \chi_H$ and $\mathrm{ind}_L^G \chi_L$ denote the holomorphically induced representations from unitary characters χ_H and χ_L of H and L, respectively. As we saw, one generalized Cartan decomposition leads us to three strongly visible actions, and three multiplicity-free theorems (Kobayashi's triunity principle [Ko1]).

As the name indicates, the decomposition $G = LBH$ can be regarded as a generalization of the Cartan decomposition. Under the assumption that both (G, H) and (G, L) are symmetric pairs, the decomposition theorem of the form $G = LBH$ or its variants has been well-established: $G = KAK$ with K compact by É. Cartan, $G = KAH$ with G, H non-compact and K compact by Flensted-Jensen [Fl1], $G = KAH$ with G compact by Hoogenboom [Ho], and the double coset decomposition $L\backslash G/H$ by Matsuki [Ma2, Ma3]. We note that in our setting the subgroups L and H of G are not necessarily symmetric.

1.1 Main result 1: Classification of visible triples

The theorem below gives a classification of generalized Cartan decompositions (Definition 1.0.14).

Theorem 1.1.1 ([Ta2, Ta3, Ta4, Ta5])**.** *Let G be a connected compact simple Lie group, T a maximal torus, Π a simple system and L_1, L_2 Levi subgroups of G, whose simple systems are given by proper subsets Π_1, Π_2 of Π. Let σ be a Chevalley–Weyl involution of G with respect to T. Then the triples (G, L_1, L_2) listed below exhaust all the triples such that the multiplication mapping*

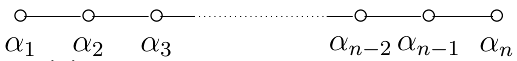
$$L_1 \times B \times L_2 \to G$$

is surjective for a subset B of G^σ.

Remark 1.1.2. For the type A simple Lie groups (or $G = \mathrm{U}(n)$), this theorem was proved by Kobayashi [Ko4].

In the following, we specify only the types of simple Lie groups G since our classification is independent of coverings, and list pairs (Π_1, Π_2) of proper subsets of Π instead of pairs (L_1, L_2) of Levi subgroups of G. Also, we put $(\Pi_j)^c := \Pi \setminus \Pi_j$ $(j = 1, 2)$.

Classification for type $\mathbf{A}_n$ [Ko4]



Hermitian type:

I. $(\Pi_1)^c = \{\alpha_i\}$, $(\Pi_2)^c = \{\alpha_j\}$.

Non-Hermitian type:

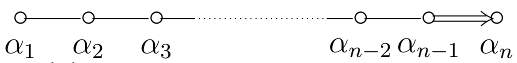
I. $(\Pi_1)^c = \{\alpha_i, \alpha_j\}$, $(\Pi_2)^c = \{\alpha_k\}$, $\min_{p=i,j}\{p, n+1-p\} = 1$ or $i = j \pm 1$.

II. $(\Pi_1)^c = \{\alpha_i, \alpha_j\}$, $(\Pi_2)^c = \{\alpha_k\}$, $\min\{k, n+1-k\} = 2$.

III. $(\Pi_1)^c = \{\alpha_l\}$, Π_2: arbitrary, $l = 1$ or n.

Here i, j, k satisfy $1 \leq i, j, k \leq n$.

Classification for type $\mathbf{B}_n$



Hermitian type:

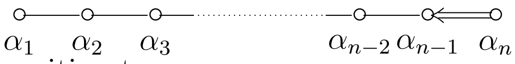
I. $(\Pi_1)^c = \{\alpha_1\}$, $(\Pi_2)^c = \{\alpha_1\}$.

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_n\}$, $(\Pi_2)^c = \{\alpha_n\}$.

II. $(\Pi_1)^c = \{\alpha_1\}$, $(\Pi_2)^c = \{\alpha_i\}$, $2 \leq i \leq n$.

Classification for type $\mathbf{C}_n$



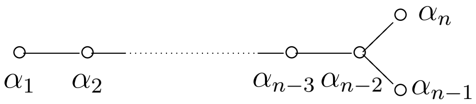
Hermitian type:

I. $(\Pi_1)^c = \{\alpha_n\}$, $(\Pi_2)^c = \{\alpha_n\}$.

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_1\}$, $(\Pi_2)^c = \{\alpha_i\}$, $1 \leq i \leq n$.

Classification for type $\mathbf{D}_n$



Hermitian type:

I. $(\Pi_1)^c = \{\alpha_i\}$, $(\Pi_2)^c = \{\alpha_j\}$, $i, j \in \{1, n-1, n\}$.

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_1\}$, $(\Pi_2)^c = \{\alpha_j\}$, $j \neq 1, n-1, n$.

II. $(\Pi_1)^c = \{\alpha_i\}$, $(\Pi_2)^c = \{\alpha_j\}$, $i \in \{n-1, n\}$, $j \in \{2, 3\}$.

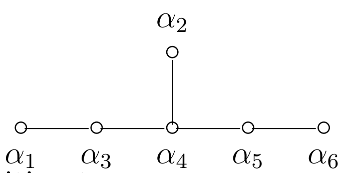
III. $(\Pi_1)^c = \{\alpha_i\}$, $(\Pi_2)^c = \{\alpha_j, \alpha_k\}$, $i \in \{n-1, n\}$, $j, k \in \{1, n-1, n\}$.

IV. $(\Pi_1)^c = \{\alpha_i\}$, $(\Pi_2)^c = \{\alpha_1, \alpha_2\}$, $i \in \{n-1, n\}$.

V. $(\Pi_1)^c = \{\alpha_1\}$, $(\Pi_2)^c = \{\alpha_j, \alpha_k\}$, j or $k \in \{n-1, n\}$.

VI. $(\Pi_1)^c = \{\alpha_i\}$, $(\Pi_2)^c = \{\alpha_2, \alpha_j\}$, $n = 4$, $(i, j) = (3, 4)$ or $(4, 3)$.

Classification for type $\mathbf{E}_6$



Hermitian type:

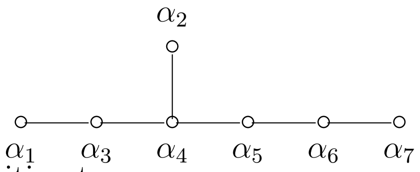
I. $(\Pi_1)^c = \{\alpha_i\}$, $(\Pi_2)^c = \{\alpha_j\}$, $i, j \in \{1, 6\}$.

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_i\}$, $(\Pi_2)^c = \{\alpha_1, \alpha_6\}$, $i = 1$ or 6.

II. $(\Pi_1)^c = \{\alpha_i\}$, $(\Pi_2)^c = \{\alpha_j\}$, $i = 1$ or 6, $j \neq 1, 4, 6$.

Classification for type $\mathbf{E}_7$



Hermitian type:

I. $(\Pi_1)^c = \{\alpha_7\}$, $(\Pi_2)^c = \{\alpha_7\}$.

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_7\}$, $(\Pi_2)^c = \{\alpha_i\}$, $i = 1$ or 2.

Classification for type $\mathbf{E}_8$, $\mathbf{F}_4$, $\mathbf{G}_2$

There is no pair (Π_1, Π_2) of proper subsets of Π such that $G = L_1 G^\sigma L_2$ holds.

For the proof of sufficiency of Theorem 1.1.1, we use the herringbone stitch method introduced by Kobayashi [Ko4], which reduces unknown decompositions to the known decomposition in the symmetric case. This method enables us to obtain a generalized Cartan decomposition $G = L_1 B L_2$ with $B \subset G^\sigma$ (Definition 1.0.14). For the proof of necessity in the classical case, we prove that $G \neq L_1 G^\sigma L_2$ for any pair (Π_1, Π_2) which is not in the list in Theorem 1.1.1 by using invariant theory for quivers associated to Levi subgroups. For the proof in the exceptional case, we use Fact 1.0.4 and Stembridge's classification of multiplicity-free tensor product representations ([St2]). See Chapters 2–5 for the details.

1.2 Main result 2: Classification of visible actions on generalized flag varieties

As we explained before, one generalized Cartan decomposition (Definition 1.0.14) leads us to three strongly visible actions. The following corollary shows that the converse is also true in our setting. Therefore we can obtain a classification of visible actions on generalized flag varieties from Theorem 1.1.1.

Corollary 1.2.1 ([Ta1])**.** *We retain the setting of Theorem 1.1.1. We denote by $G_{\mathbb{C}}$ and $(L_j)_{\mathbb{C}}$ the complexifications of G and L_j, respectively ($j = 1, 2$). We let P_j be a parabolic subgroup of $G_{\mathbb{C}}$ with Levi subgroup $(L_j)_{\mathbb{C}}$, and put $\mathcal{P}_j = G_{\mathbb{C}}/P_j$ ($j = 1, 2$). Then the following eleven conditions are equivalent.*

(i) The multiplication mapping $L_1 \times G^\sigma \times L_2 \to G$ is surjective.

(ii) The natural action $L_1 \curvearrowright \mathcal{P}_2$ is strongly visible.

(iii) The natural action $L_2 \curvearrowright \mathcal{P}_1$ is strongly visible.

(iv) The diagonal action $\Delta(G) \curvearrowright \mathcal{P}_1 \times \mathcal{P}_2$ is strongly visible.

(v) Any irreducible representation of G, which belongs to $\mathcal{P}_2$-series is multiplicity-free when restricted to L_1.

(vi) Any irreducible representation of G, which belongs to $\mathcal{P}_1$-series is multiplicity-free when restricted to L_2.

(vii) The tensor product of arbitrary two irreducible representations π_1 and π_2 of G, which belong to $\mathcal{P}_1$ and $\mathcal{P}_2$-series, respectively, is multiplicity-free.

(viii) $\mathcal{P}_2$ is a spherical variety of $(L_1)_{\mathbb{C}}$.

(ix) $\mathcal{P}_1$ is a spherical variety of $(L_2)_{\mathbb{C}}$.

(x) $\mathcal{P}_1 \times \mathcal{P}_2$ is a spherical variety of $\Delta(G_{\mathbb{C}})$.

(xi) The pair (Π_1, Π_2) is one of the entries listed in Theorem 1.1.1 up to switch of the factors.

Here an irreducible representation of G is in $\mathcal{P}_j$-series if it is a holomorphically induced representation from a unitary character of the Levi subgroup L_j $(j = 1, 2)$.

Proof. * We prove that Theorem 1.1.1 implies this corollary. The strategy of the proof is summarized in the below diagram.

$$
\begin{array}{ccccccl}
 & & \text{(vii)} & & & & \cdots\cdots \text{multiplicity-free} \\
 & & \Updownarrow & & & & \\
 & & \text{(xi)} & & & & \cdots\cdots \text{classification of } (L_1, L_2) \\
 & & \Updownarrow & & & & \\
 & & \text{(i)} & & & & \cdots\cdots \text{Cartan decomposition} \\
 & \swarrow & \Downarrow & \searrow & & & \\
\text{(ii)} & & \text{(iv)} & & \text{(iii)} & & \cdots\cdots \text{visible action} \\
\Downarrow & & \Downarrow & & \Downarrow & & \\
\text{(v)} & \Leftrightarrow & \text{(vii)} & \Leftrightarrow & \text{(vi)} & & \cdots\cdots \text{multiplicity-free} \\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \\
\text{(viii)} & & \text{(x)} & & \text{(ix)} & & \cdots\cdots \text{spherical action.}
\end{array}
$$

The implication (vii) $\Rightarrow$ (xi) can be verified by comparing Stembridge's classification [St2] with Theorem 1.1.1. The converse implication (xi) $\Rightarrow$ (vii) follows from Fact 1.0.4. The equivalence (xi) $\Leftrightarrow$ (i) is Theorem 1.1.1. The implications (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv) are the triunity of visibility ([Ko1]). Each of the three implications (ii) $\Rightarrow$ (v), (iii) $\Rightarrow$ (vi) and (iv) $\Rightarrow$ (vii) is followed by Fact 1.0.4. As in the proof of [Ko2, Corollary 15], we see that a result of Vinberg and Kimel'fel'd [VK, Corollary 1] implies the three equivalences (v) $\Leftrightarrow$ (viii), (vi) $\Leftrightarrow$ (ix) and (vii) $\Leftrightarrow$ (x). The equivalence (v) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (vi) on the multiplicity-freeness property of representations follows from a result of Stembridge [St2, Corollary 2.5]. This completes the proof of the corollary. □

Remark 1.2.2. For the type A simple Lie groups (or $G = \mathrm{U}(n)$), this corollary was proved by Kobayashi [Ko2].

*This proof for Corollary 1.2.1 is quoted from [Ta1].

Remark 1.2.3. Littelmann [Li] classified for any simple algebraic group G over any algebraically closed field of characteristic zero, all the pairs of maximal parabolic subgroups P_ω and $P_{\omega'}$ corresponding to fundamental weights ω and ω', respectively, such that the tensor product representation $V_{n\omega} \otimes V_{m\omega'}$ decomposes multiplicity-freely for arbitrary non-negative integers n and m. Moreover, he found the branching rules of $V_{n\omega} \otimes V_{m\omega'}$ and the restriction of $V_{n\omega}$ to the maximal Levi subgroup $L_{\omega'}$ of $P_{\omega'}$ for any pair (ω, ω') that admits a $\Delta(G)$-spherical action on $G/P_\omega \times G/P_{\omega'}$.

Remark 1.2.4. Stembridge [St2] gave a complete list of a pair (μ, ν) of highest weights such that the corresponding tensor product representation $V_\mu \otimes V_\nu$ is multiplicity-free for any complex simple Lie algebra $\mathfrak{g}$. His method was combinatorial and not based on the Borel–Weil realization of irreducible representations. He also classified all the pairs $(\mu, \mathfrak{l})$ of highest weights μ and Levi subalgebras $\mathfrak{l}$ of $\mathfrak{g}$ with the restrictions $V_\mu|_{\mathfrak{l}}$ multiplicity-free.

1.3 Main result 3: Seeds and visible actions for the orthogonal group

As we mentioned in Remark 1.2.4, finite dimensional multiplicity-free tensor product representations were classified by Stembridge [St2]. By using the notion of visible actions on complex manifolds, we would be able to, and indeed can in the types A, B and D cases, understand his classification more deeply. By Fact 1.0.4, we can reduce complicated multiplicity-free theorems to a pair of data:

visible actions on complex manifolds, and

much simpler multiplicity-free representations (*seeds* of multiplicity-free representations).

For the type A simple Lie groups, Kobayashi found the following *seeds* of multiplicity-free representations that combined with visible actions can produce all the cases of the pair of two representations (V_1, V_2) of $\mathrm{U}(n)$ such that $V_1 \otimes V_2$ is multiplicity-free [Ko1].

- One-dimensional representations.
- $(\mathrm{U}(n) \downarrow \mathbb{T}^n)$ The restriction of an alternating tensor product representation $\Lambda^k(\mathbb{C}^n)$.
- $(\mathrm{U}(n) \downarrow \mathbb{T}^n)$ The restriction of a symmetric tensor product representation $S^k(\mathbb{C}^n)$.
- $(\mathrm{U}(n) \downarrow \mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \mathrm{U}(n_3))$ The restriction of an irreducible representation $V_{2\omega_k}$ $(n = n_1 + n_2 + n_3)$.

Here V_λ denotes an irreducible representation of $\mathrm{U}(n)$ with highest weight λ and $\{\omega_k\}_{1 \le k \le n-1}$ is the set of fundamental weights of $\mathrm{U}(n)$. On the other hand, he classified in [Ko4] visible actions on generalized flag varieties of type A as listed in Theorem 1.1.1. By combining the above *seeds* of multiplicity-free representations with the visible actions and using his triunity principle, Kobayashi constructed all the multiplicity-free tensor product representations of $\mathrm{U}(n)$ [Ko1]. In this paper we construct all the multiplicity-free tensor product representations for $\mathrm{SO}(N)$ and its covering group $\mathrm{Spin}(N)$ by following Kobayashi's argument for $\mathrm{U}(n)$. In our case, visible actions come from triples (G, L_1, L_2) for $G = \mathrm{Spin}(N)$

listed in Theorem 1.1.1 as in the case of the type A groups. On the other hand, seeds of multiplicity-free tensor product representations arise only from one-dimensional representations, alternating tensor product representations and spin representations. These are exhibited in Proposition 1.3.1. We can see how to combine those visible actions and seeds to obtain multiplicity-free tensor product representations in Theorem 1.3.2.

We denote by $\Pi = \{\alpha_i\}_{1 \leq i \leq [N/2]}$ (see Theorem 1.1.1 for the labeling of the Dynkin diagrams) a simple system of the root system of $G = \mathrm{Spin}(N)$ with respect to its maximal torus T, and by $\{H_i\}_{1 \leq i \leq [N/2]}$ the dual basis of Π. We define a subgroup M of $\mathrm{Spin}(2n+1)$ as follows.

$$M := \left\{\exp\left(\sqrt{-1}m\pi H_1\right)\right\}_{1 \leq m \leq 4} \cdot \mathrm{Spin}(2n-1), \tag{1.3.1}$$

where exp denotes the exponential mapping, and the simple system of $\mathrm{Spin}(2n-1)$ is given by $\{\alpha_k \in \Pi : 2 \leq k \leq n\}$.

Proposition 1.3.1. *We denote by* $\mathbf{1}$, $\mathbb{C}^N$ *and* Spin_N *for the one-dimensional trivial representation, the natural representation and the spin representation of* $\mathrm{Spin}(N)$, *respectively. Then the following hold.*

(1) *One-dimensional representations are multiplicity-free.*

(2) $\mathbf{1}$, $\mathbb{C}^N$ *and* Spin_N *are multiplicity-free as representations of a maximal torus* T *of* $\mathrm{Spin}(N)$.

(3) $\Lambda^i(\mathbb{C}^N)$ *is multiplicity-free as a representation of a maximal Levi subgroup* $\mathrm{U}(j) \times \mathrm{SO}(N-2j)$ *of* $\mathrm{SO}(N)$ (*when* N *is even and* $i = N/2$, *we replace* $\Lambda^{N/2}(\mathbb{C}^N)$ *by its* $\mathrm{SO}(N)$*-irreducible constituent whose highest weight is* $2\omega_{N/2-1}$ *or* $2\omega_{N/2}$) *if the following condition* (3-1) *or* (3-2) *is satisfied* ($1 \leq i, j \leq [N/2]$).

 (3-1) N *is odd.*

 (3-2) N *is even, and* i, j *satisfy* $i + j \leq N/2$, $j = N/2$ *or* $i = N/2$.

(4) Spin_N *is multiplicity-free as a representation of* M, *where* N *is odd and* M *as in* (1.3.1).

The theorem below gives a geometric construction of all the multiplicity-free tensor product representations for the orthogonal group. For a realization of irreducible representations of a compact Lie group, we use the Borel–Weil theory. Namely, we realize an irreducible representation of a compact Lie group G as the space $\mathcal{O}(G/L, \mathcal{W})$ of holomorphic sections of a vector bundle $\mathcal{W}$ on a generalized flag variety G/L, which is associated to an irreducible representation W of a Levi subgroup L of G.

Theorem 1.3.2. *We let* $G = \mathrm{Spin}(N)$. *For any two irreducible representations* V_{λ_1} *and* V_{λ_2} *of* G *such that* $V_{\lambda_1} \otimes V_{\lambda_2}$ *is multiplicity-free, there exists a pair of*

- *a generalized flag variety* $(G \times G)/(L_1 \times L_2)$ *with a strongly visible* $\Delta(G)$*-action, and*

- *irreducible representations* (*a seed given in* Proposition 1.3.1) W_1 *and* W_2 *of* L_1 *and* L_2, *respectively,* *such that* $V_{\lambda_k} \simeq \mathcal{O}(G/L_k, \mathcal{W}_k)$ *as* G*-modules* $(k = 1, 2)$.

The correspondence between the data (L_k, W_k) *of visible actions and seeds, and the highest weights* λ_k *of* V_{λ_k} $(k = 1, 2)$ *is given as in* Tables 1.3.1–1.3.4 *below. In the tables,* $\mathbb{C}_\lambda$ *denotes a one-dimensional representation with weight* λ, T *a maximal torus of* G *and* L_λ *a Levi subgroup of* G*, whose simple system is given by* $\{\alpha_l \in \Pi : \langle \lambda, \check{\alpha}_l \rangle = 0\}$ *where* $\check{\alpha}_l$ *is the coroot of* α_l $(1 \leq l \leq [N/2])$.

Table 1.3.1: Line bundle type

L_1	L_2	W_1	W_2	N	λ_1	λ_2
L_{λ_1}	L_{λ_2}	$\mathbb{C}_{\lambda_1}$	$\mathbb{C}_{\lambda_2}$	$2n+1$	$s\omega_1$	$t\omega_j$
					$s\omega_n$	$t\omega_n$
				$2n$	$s\omega_1$	$t\omega_j + u\omega_{n-\delta}$
					$s\omega_{n-\delta}$	$t\omega_3$, $t\omega_1 + u\omega_2$, $t\omega_1 + u\omega_{n-\delta'}$ or $t\omega_{n-1} + u\omega_n$
				8	$s\omega_{5-\epsilon}$	$t\omega_2 + u\omega_{2+\epsilon}$

$1 \leq j \leq n$, $s, t, u \in \mathbb{N}$, $\delta = 0$ or 1, $\delta' = 0$ or 1 and $\epsilon = 1$ or 2.

Table 1.3.2: Weight multiplicity-free type

L_1	L_2	W_1	W_2	N	λ_1	λ_2
G	T	V_{λ_1}	$\mathbb{C}_{\lambda_2}$	$2n+1$	0, ω_1 or ω_n	arbitrary
				$2n$	0, ω_1, ω_{n-1} or ω_n	arbitrary

Table 1.3.3: Alternating tensor product type

L_1	L_2	W_1	W_2	N	λ_1	λ_2	Condition
G	L_{λ_2}	V_{λ_1}	$\mathbb{C}_{\lambda_2}$	$2n+1$	ω_i or $2\omega_n$	$t\omega_j$	
				$2n$	ω_i	$t\omega_j$	$i + j \leq n$
					ω_i	$t\omega_{n-\delta}$	
					$2\omega_{n-\delta}$	$t\omega_j$	

$1 \leq i, j \leq n$, $t \in \mathbb{N}$ and $\delta = 0$ or 1.

Table 1.3.4: Spin type

L_1	L_2	W_1	W_2	N	λ_1	λ_2
L_{λ_1}	L_{ω_j}	$\mathbb{C}_{\lambda_1}$	$\mathbb{C}_{(1/2+t)\omega_j} \boxtimes \mathrm{Spin}_{N-2j}$	$2n+1$	$s\omega_1$	$\omega_n + t\omega_j$

$1 \leq j \leq n-1$ and $s, t \in \mathbb{N}$.

See Chapter 6 for the proof of Theorem 1.3.2. By virtue of Fact 1.0.4 and the triunity principle [Ko1], we obtain the following corollary. This corollary was proved by Stembridge [St2] by a combinatorial method.

Corollary 1.3.3. *We retain the notation of Theorem 1.3.2. For the data* $(L_1, L_2, N, \lambda_1, \lambda_2)$ *of each row in* Tables 1.3.1–1.3.4*, the representations* V_{λ_1} *and* V_{λ_2} *of* G *decompose multiplicity-freely when restricted to the subgroups* L_2 *and* L_1 *of* G*, respectively.*

So far we have considered visible actions of Levi subgroups on generalized flag varieties. For a general spherical variety, we have the following result on the visibility of actions of compact Lie groups. Let U be a compact real form of a connected complex reductive algebraic group $G_{\mathbb{C}}$, and X a $G_{\mathbb{C}}$-spherical variety. We denote by θ the Cartan involution of $G_{\mathbb{C}}$, which corresponds to U, and by ν a Chevalley–Weyl involution of $G_{\mathbb{C}}$ (i.e., ν is an involution of $G_{\mathbb{C}}$, which satisfies $\nu(t) = t^{-1}$ for any element $t \in T_{\mathbb{C}}$ for some maximal torus $T_{\mathbb{C}}$), which preserves U. We put $\iota = \theta \circ \nu$.

Theorem 1.3.4. *Assume that there exists a real structure μ on a $G_{\mathbb{C}}$-spherical variety X compatible with ι and that the μ-fixed points subset X^{μ} is non-empty. Then a compact real form U acts on X strongly visibly.*

Here by a real structure on a complex manifold Z we mean an anti-holomorphic involution $\eta : Z \to Z$ [Ak, AC]. Also for a real structure η on a complex manifold Z with an action of a group K, we say η is compatible with an automorphism ϕ of K if η satisfies $\eta(kz) = \phi(k)\eta(z)$ for any $k \in K$ and $z \in Z$. Combining Theorem 1.3.4 with Akhiezer's result [Ak] on the existence of compatible real structures on Stein manifolds, we obtain

Corollary 1.3.5. *Let $(G_{\mathbb{C}}, V)$ be a linear multiplicity-free space. Then a compact real form U acts on V strongly visibly.*

Corollary 1.3.6. *Let X be a smooth affine $G_{\mathbb{C}}$-spherical variety. Then a compact real form U acts on X strongly visibly.*

Here a typical example of smooth affine spherical varieties is a complex symmetric space. On the other hand, we have the principal affine space $G_{\mathbb{C}}/N$ (N is a maximal unipotent subgroup) as an example of non-affine smooth spherical varieties. We remark that Corollary 1.3.5 was earlier proved by Sasaki (Fact 1.0.10) by constructing slices explicitly. By combining Theorem 1.3.4 with Akhiezer and Cupit-Foutou's result [AC], we also have

Corollary 1.3.7. *Let X be a $G_{\mathbb{C}}$-wonderful variety. Then a compact real form U acts on X strongly visibly.*

Definition 1.3.8. A $G_{\mathbb{C}}$-variety X is said to be wonderful if

- X is smooth and projective,
- $G_{\mathbb{C}}$ has an open orbit on X, whose complement is a union of finitely many smooth prime divisors X_i ($i \in I$) with normal crossings, and
- the closure of any $G_{\mathbb{C}}$-orbit on X is given as a partial intersection of X_i ($i \in I$).

To prove the visibility of actions of non-compact reductive groups on complex manifolds, we use the following extension of a result of Matsuki [Ma2, Ma3]. Let L and H be reductive subgroups of a connected real semisimple algebraic group G such that both $G_{\mathbb{C}}/L_{\mathbb{C}}$ and $G_{\mathbb{C}}/H_{\mathbb{C}}$ are $G_{\mathbb{C}}$-spherical varieties.

Theorem 1.3.9. *There exist finitely many abelian subspaces $\mathfrak{j}_i$ of $\mathfrak{g}$ and elements x_i of G ($i = 1, \ldots, m$) such that $\bigcup_{i=1}^{m} LC_iH$ contains an open dense subset of G, where $C_i = \exp(\mathfrak{j}_i)x_i$.*

We use this decomposition to show the previsibility of actions of non-compact reductive groups.

Theorem 1.3.10. *Let X be a $G_{\mathbb{C}}$-spherical variety and G a real form of inner type of $G_{\mathbb{C}}$. Then G acts on X previsibly.*

Here a real reductive Lie group is said to be of inner type if its Lie algebra has a compact Cartan subalgebra.

See Chapter 7 for the proofs of Theorems 1.3.4 and 1.3.10, and Corollaries 1.3.5–1.3.7. The proof of Theorem 1.3.9 is given in Chapter 9.

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Organization of this thesis

Each of the following chapters could be read independently of the other chapters. We give a classification of generalized Cartan decompositions for compact Lie groups (Definition 1.0.14) of the types B, C, D and of the exceptional type in Chapters 2, 3, 4 and 5, respectively. Those classification results combined with [Ko4] show Theorem 1.1.1. In Chapter 6, we deal with seeds for multiplicity-free representations of the types B and D groups, and prove Theorem 1.3.2. In Chapter 7, we prove the visibility of actions of reductive groups on spherical varieties. In Chapter 8, we give another proof for the existence of generalized Cartan decompositions for compact Lie groups (Definition 1.0.14), which is useful for an explicit calculation. A KAK-decomposition for Gelfand pairs is also given there. In Chapter 9, we deal with a double coset decomposition of a real reductive group with respect to reductive spherical subgroups and prove Theorem 1.3.9.

Chapter 2

Visible actions on flag varieties of type B and a generalization of the Cartan decomposition

2.1 Introduction for Chapter 2

Let G be a connected compact simple Lie group of type B and σ a Chevalley–Weyl involution of G. The aim of this section is to classify all the pairs (L, H) of Levi subgroups of G such that $G = LG^{\sigma}H$ holds. The motivation for considering this kind of decomposition comes from the theory of *visible actions* on complex manifolds introduced by T. Kobayashi ([Ko2]), and $G = LG^{\sigma}H$ can be interpreted as a generalization of the Cartan decomposition to the non-symmetric setting. (We refer the reader to [He1], [Ho], [Ma2] and [Ko4] and references therein for some aspects of the Cartan decomposition from geometric and group theoretic viewpoints.)

A generalization of the Cartan decomposition for symmetric pairs has been used in various contexts including analysis on symmetric spaces, however, there was no analogous result for non-symmetric cases before Kobayashi's paper [Ko4]. Motivated by visible actions on complex manifolds ([Ko1], [Ko2]), he completely determined the pairs of Levi subgroups

$$(L, H) = (\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_k), \mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_l))$$

of the unitary group $G = \mathrm{U}(n)$ such that the multiplication mapping $L \times \mathrm{O}(n) \times H \to G$ is surjective. Furthermore he developed a method to find a suitable subset B of $\mathrm{O}(n)$ which gives the following decomposition (a generalized Cartan decomposition, see [Ko4]):

$$G = LBH.$$

On the other hand, A. Sasaki studied recently visible actions in the setting where (G, H) is a pair of *complex* reductive Lie groups, and gave a generalization of the Cartan decomposition $G = LBH$ ([Sa2], [Sa3]).

Back to the decomposition theory [Ko4], we consider the following problems:

The contents of this section are taken from [Ta2].

Let G be a connected compact Lie group, $\mathfrak{t}$ a Cartan subalgebra, and σ a Chevalley–Weyl involution of G with respect to $\mathfrak{t}$. (Here, we recall that an involutive automorphism μ of a connected compact Lie group K is said to be a Chevalley–Weyl involution if there is a maximal torus T of K such that $\mu(t) = t^{-1}$ for every $t \in T$. For instance, an involution $\sigma(g) = \bar{g}$ defines a Chevalley–Weyl involution of $G = \mathrm{U}(n)$ with the standard maximal torus, and $G^\sigma = \{g \in \mathrm{U}(n) : \bar{g} = g\} \simeq \mathrm{O}(n)$.)

1) Classify all the pairs of Levi subgroups L and H with respect to $\mathfrak{t}$ such that the multiplication mapping $\psi : L \times G^\sigma \times H \to G$ is surjective.

2) Find a "good" representative $B \subset G^\sigma$ such that $G = LBH$ in the case ψ is surjective.

We call such a decomposition $G = LBH$ a *generalized Cartan decomposition*. Here we note that the role of the subgroups H and L is symmetric.

In the present chapter, we solve the above problems for connected compact simple Lie groups G of type B. In order to state the results, we label the Dynkin diagram of type B_n as follows. For a subset Π' of the set Π of simple roots, we denote by $\mathrm{L}_{\Pi'}$ the Levi

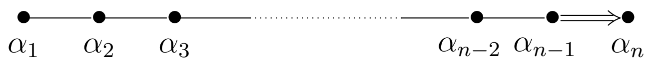


Figure 2.1.1: Dynkin diagram of type B_n

subgroup whose root system is generated by Π'. For example, $\mathrm{L}_\varnothing$ is a maximal torus of G and $\mathrm{L}_{\{\alpha_p\}^c} = \mathrm{U}(p) \times \mathrm{SO}(2(n-p)+1)$ for $G = \mathrm{SO}(2n+1)$ $(1 \leq p \leq n)$. Here $(\Pi')^c$ denotes the complement $\Pi \setminus \Pi'$.

Theorem 2.1.1. *Let G be a connected compact simple Lie group of type B_n, σ a Chevalley–Weyl involution, Π', Π'' proper subsets of the simple system Π, and $\mathrm{L}_{\Pi'}$, $\mathrm{L}_{\Pi''}$ the corresponding Levi subgroups. Then the following two conditions on $\{\Pi', \Pi''\}$ are equivalent.*
(i). $G = \mathrm{L}_{\Pi'}\, G^\sigma\, \mathrm{L}_{\Pi''}$.
(ii). *One of the following conditions holds up to switch of the factors Π' and Π'' :*

$$\begin{array}{lll} \text{Case I.} & (\Pi')^c = \{\alpha_n\}, & (\Pi'')^c = \{\alpha_n\}. \\ \text{Case II.} & (\Pi')^c = \{\alpha_1\}, & (\Pi'')^c = \{\alpha_j\}, \quad 1 \leq j \leq n. \end{array}$$

We note that the pair $(G, \mathrm{L}_{\Pi'})$ forms a symmetric pair if and only if $(\Pi')^c = \{\alpha_1\}$, and that $G/\mathrm{L}_{\Pi'} = G/\mathrm{L}_{\Pi''}$ is a (non-symmetric) spherical variety in Case I (c.f. [Kr]).

Theorem 2.1.1 implies that $G = LG^\sigma H$ holds if and only if (G, L, H) satisfies one of the following two conditions: Case I both H and L are maximal and of type A, or Case II (G, H) is symmetric and L comes from a maximal parabolic subgroup up to switch of H and L. In each case, we give a generalized Cartan decomposition $G = LBH$ explicitly with $\dim B = \operatorname{rank} G$ in Case I and $\dim B = 2$ or 3 in Case II. This is stated in Propositions 2.3.2 and 2.3.3.

Application to representation theory. A generalized Cartan decomposition $G = LBH$ implies that the subgroup L acts on G/H in a (strongly) visible fashion, and likewise H on G/L, and G on $(G \times G)/(L \times H)$. Then Kobayashi's theory leads us to three multiplicity-free theorems (*triunity* à la [Ko1]):

$$\begin{array}{ll} \text{Restriction } G \downarrow L & : \mathrm{Ind}_H^G(\mathbb{C}_\lambda)|_L, \\ \text{Restriction } G \downarrow H & : \mathrm{Ind}_L^G(\mathbb{C}_\lambda)|_H, \\ \text{Tensor product} & : \mathrm{Ind}_H^G(\mathbb{C}_\lambda) \otimes \mathrm{Ind}_L^G(\mathbb{C}_\mu). \end{array}$$

Here $\mathrm{Ind}_H^G(\mathbb{C}_\lambda)$ denotes a holomorphically induced representation of G from a character $\mathbb{C}_\lambda$ of H by the Borel–Weil theorem. See [Ko1], [Ko2], [Ko3] for the general theory on the application of visible actions (including the vector bundle setting), and Corollaries 2.5.4 and 2.5.5 for type B groups.

Organization. Chapter 2 is organized as follows. In Section 2.2, we give a matrix realization of the orthogonal group $G = \mathrm{SO}(2n+1)$ and its subgroups which are used in Sections 2.3 and 2.4. In Section 2.3, we prove that (ii) implies (i). Furthermore, we find explicitly a slice B that gives a generalized Cartan decomposition $G = \mathrm{L}_{\Pi'}\, B\, \mathrm{L}_{\Pi''}$. The converse implication on (ii) $\Rightarrow$ (i) is proved in Section 2.4 by using the invariant theory for quivers. An application to multiplicity-free representations is discussed in Section 2.5.

2.2 Matrix realization

The surjectivity of $\psi : L \times G^\sigma \times H \to G$ is independent of the coverings and the choice of Cartan subalgebras and Chevalley–Weyl involutions. Thus, we may and do work with the orthogonal group $\mathrm{SO}(2n+1)$ and a fixed pair of a Cartan subalgebra and a Chevalley–Weyl involution in Sections 2.2, 2.3 and 2.4.

Throughout this chapter, we realize $G = \mathrm{SO}(2n+1)$ as a matrix group as follows.

$$G := \{g \in SL(2n+1, \mathbb{C}) : {}^t g J_{2n+1} g = J_{2n+1},\ {}^t\overline{g} g = I_{2n+1}\}, \tag{2.2.1}$$

where ${}^t g$ denotes the transpose of g, and J_m is defined by

$$J_m := \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & O & & & \\ & & \iddots & & \\ & & & O & \\ 1 & & & & \end{pmatrix} \in \mathrm{GL}(m, \mathbb{R}).$$

Then, the corresponding Lie algebra $\mathfrak{g} = \mathfrak{so}(2n+1)$ of G forms

$$\mathfrak{g} := \{X \in \mathfrak{sl}(2n+1, \mathbb{C}) : {}^t X J_{2n+1} + J_{2n+1} X = O,\ {}^t\overline{X} + X = O\}. \tag{2.2.2}$$

We take a Cartan subalgebra $\mathfrak{t}$ and a involution σ of G as follows.

$$\mathfrak{t} := \bigoplus_{1 \le i \le n} \mathbb{R}\sqrt{-1} H_i, \tag{2.2.3}$$

$$\sigma : G \to G, \quad g \mapsto \bar{g}, \tag{2.2.4}$$

where $H_i := \mathrm{E}_{i,i} - \mathrm{E}_{2n+2-i,2n+2-i}$, and $\bar{g}$ denotes the complex conjugation of $g \in G$. The differential of σ is denoted by the same letter. Then, σ is a Chevalley–Weyl involution of G with respect to $\mathfrak{t}$.

We let $\{\varepsilon_i\}_{1 \le i \le n} \subset (\mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C})^*$ be the dual basis of $\{H_i\}_{1 \le i \le n}$. Then we define a simple system $\Pi := \{\alpha_1, ..., \alpha_n\}$ by

$$\alpha_1 := \varepsilon_1 - \varepsilon_2,\ \ldots,\ \alpha_{n-1} := \varepsilon_{n-1} - \varepsilon_n,\ \alpha_n := \varepsilon_n.$$

Let $n = n_1 + \cdots + n_k$ be a partition of n with $n_1, ..., n_{k-1} > 0$ and $n_k \geq 0$. We put

$$s_i := \sum_{1 \leq p \leq i} n_p \quad (1 \leq i \leq k-1),$$
$$\Pi' := \Pi \setminus \{\alpha_{s_i} \in \Pi : 1 \leq i \leq k-1\},$$

and denote by $\mathrm{L}_{\Pi'}$ the Levi subgroup whose root system is generated by Π'. In the matrix realization, $\mathrm{L}_{\Pi'}$ takes the form:

$$\mathrm{L}_{\Pi'} = \left\{ \begin{pmatrix} A_1 &&&&&& \\ & \ddots &&&&& \\ && A_{k-1} &&&& \\ &&& B &&& \\ &&&& J_{n_{k-1}} \overline{A_{k-1}} J_{n_{k-1}} && \\ &&&&& \ddots & \\ &&&&&& J_{n_1} \overline{A_1} J_{n_1} \end{pmatrix} : \begin{array}{l} A_i \in U(n_i) \ (1 \leq i \leq k-1), \\ B \in \mathrm{SO}(2n_k+1) \end{array} \right\} \tag{2.2.5}$$
$$\simeq \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k+1).$$

Here, we note that the pair $(G, \mathrm{L}_{\Pi'})$ forms a symmetric pair if and only if $(\Pi')^c = \Pi \setminus \Pi' = \{\alpha_1\}$, and that $G/\mathrm{L}_{\{\alpha_n\}^c}$ is a weakly symmetric space in the sense of Selberg. For a later purpose, we give explicitly an involution τ_1 and an automorphism μ satisfying $\mu^4 = \mathrm{id}$ of which the connected component of fixed point subgroups are $\mathrm{L}_{\{\alpha_1\}^c}$ and $\mathrm{L}_{\{\alpha_n\}^c}$ respectively.

$$\mathrm{L}_{\{\alpha_1\}^c} = (G^{\tau_1})_0, \qquad \tau_1 : G \to G, \qquad g \mapsto I_{1,2(n-1)+1,1}\, g\, I_{1,2(n-1)+1,1}, \tag{2.2.6}$$
$$\mathrm{L}_{\{\alpha_n\}^c} = G^{\mu}, \qquad \mu : G \to G, \qquad g \mapsto I_{\sqrt{-1}}\, g\, I_{\sqrt{-1}}, \tag{2.2.7}$$

where K_0 denotes the connected component of K containing the identity element for a Lie group K, and $I_{1,2(n-1)+1,1}$, $I_{\sqrt{-1}}$ are defined by

$$I_{1,2(n-1)+1,1} := \mathrm{diag}(-1, \overbrace{1, \ldots, 1}^{2(n-1)+1}, -1),$$
$$I_{\sqrt{-1}} := \mathrm{diag}(\overbrace{\sqrt{-1}, \ldots, \sqrt{-1}}^{n}, 1, \overbrace{-\sqrt{-1}, \ldots, -\sqrt{-1}}^{n}).$$

To obtain a generalized Cartan decomposition by the herringbone stitch method, we will use an involutive automorphism τ_p of G $(1 \leq p \leq n)$ given by

$$\tau_p : G \to G, \qquad g \mapsto I_{p,2(n-p)+1,p}\, g\, I_{p,2(n-p)+1,p}, \tag{2.2.8}$$

where $I_{p,2(n-p)+1,p} := \mathrm{diag}(\overbrace{-1, \ldots, -1}^{p}, \overbrace{1, \ldots, 1}^{2(n-p)+1}, \overbrace{-1, \ldots, -1}^{p})$. Then $(G^{\tau_p})_0$ is given by

$$\mathrm{SO}(2p) \times \mathrm{SO}(2n-2p+1) = \tag{2.2.9}$$

$$\left\{ \left(\begin{array}{c|c|c} A & & B \\ \hline & S & \\ \hline C & & D \end{array} \right) : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SO}(2p),\ S \in \mathrm{SO}(2n-2p+1) \right\}.$$

2.3 Generalized Cartan decomposition

In this section, we give a proof of the implication (ii) $\Rightarrow$ (i) in Theorem 2.1.1. The idea is to use the herringbone stitch method that reduces unknown decompositions for non-symmetric pairs to the known Cartan decomposition for symmetric pairs.

2.3.1 Cartan decomposition for symmetric pairs.

In this subsection we recall a well-known fact on the Cartan decomposition for the symmetric case ([Ho, Theorem 6.10], [Ma3, Theorem 1]).

Fact 2.3.1. *Let K be a connected compact Lie group with Lie algebra $\mathfrak{k}$ and two involutions τ, τ' ($\tau^2 = (\tau')^2 = \mathrm{id}$). Let H and L be subgroups of K such that*

$$(K^\tau)_0 \subset L \subset K^\tau \quad \textit{and} \quad (K^{\tau'})_0 \subset H \subset K^{\tau'}.$$

We take a maximal abelian subspace $\mathfrak{b}$ in

$$\mathfrak{k}^{-\tau,-\tau'} := \{X \in \mathfrak{k} : \tau(X) = \tau'(X) = -X\},$$

and write B for the connected abelian subgroup with Lie algebra $\mathfrak{b}$. Suppose that $\tau\tau'$ is semisimple on the center $\mathfrak{z}$ of $\mathfrak{k}$. Then,

$$K = LBH.$$

2.3.2 Decomposition for Case I

This subsection is devoted to showing the following proposition.

Proposition 2.3.2 (generalized Cartan decomposition for Case I)**.** *Let $G = \mathrm{SO}(2n+1)$ and $(\Pi')^c = (\Pi'')^c = \{\alpha_n\}$. Then we have $G = \mathrm{L}_{\Pi'} \exp(\mathfrak{a} \oplus \mathfrak{q})\, \mathrm{L}_{\Pi''}$ where $\mathfrak{a}$ and $\mathfrak{q}$ are defined by*

$$\mathfrak{a} := \bigoplus_{i=1}^{[\frac{n}{2}]} \mathbb{R}(\mathrm{E}_{2i-1,2n-2i+2} - \mathrm{E}_{2i,2n-2i+3} - \mathrm{E}_{2n-2i+2,2i-1} + \mathrm{E}_{2n-2i+3,2i}), \tag{2.3.1}$$

$$\mathfrak{q} := \bigoplus_{i=1}^{[\frac{n+1}{2}]} \mathbb{R}(\mathrm{E}_{2i-1,n+1} - \mathrm{E}_{n+1,2n+3-2i} - \mathrm{E}_{n+1,2i-1} + \mathrm{E}_{2n+3-2i,n+1}). \tag{2.3.2}$$

Proof. Since an automorphism μ of $\mathfrak{g}$ is an involution of $\mathfrak{g}^{\mu^2}$ (see (2.2.7) for the definition of μ) and $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{g}^{-\mu}$, we have

$$\mathfrak{g} = \mathfrak{g}^\mu \oplus \left(\bigcup_{g \in G^\mu} \mathrm{Ad}(g)\mathfrak{a} \right) \oplus \mathfrak{g}^{-\mu^2}. \tag{2.3.3}$$

Let $Z_{\mathfrak{g}^\mu}(\mathfrak{a})$ denote the centralizer of $\mathfrak{a}$ in $\mathfrak{g}^\mu$. Then $M := \exp(Z_{\mathfrak{g}^\mu}(\mathfrak{a}))$ is given by

$$M = \begin{cases} \mathrm{SU}(2)^m & (n = 2m), \\ \mathrm{SU}(2)^m \times \mathrm{U}(1) & (n = 2m+1). \end{cases} \tag{2.3.4}$$

By using this block diagonal matrix group M, we rewrite the third factor $\mathfrak{g}^{-\mu^2}$ of the decomposition (2.3.3) as follows.

$$\mathfrak{g}^{-\mu^2} = \bigcup_{g \in M} \mathrm{Ad}(g)\mathfrak{q}. \tag{2.3.5}$$

We omit details since this can be verified by a simple matrix computation. The equation (2.3.5) leads us to the following.

$$\left(\bigcup_{g \in G^\mu} \mathrm{Ad}(g)\mathfrak{a} \right) \oplus \mathfrak{g}^{-\mu^2} = \bigcup_{g \in G^\mu} \mathrm{Ad}(g)(\mathfrak{a} \oplus \mathfrak{q}). \tag{2.3.6}$$

Let us verify (2.3.6). It is clear that the left-hand-side contains the right-hand-side. We show the converse inclusive relation. From (2.3.5), for any $l \in G^\mu$, $X \in \mathfrak{a}$ and $Z \in \mathfrak{g}^{-\mu^2}$ there exist $h \in M$ and $Y \in \mathfrak{q}$ satisfying

$$\mathrm{Ad}(h)Y = \mathrm{Ad}(l)^{-1}Z.$$

Then we have

$$\begin{aligned} \mathrm{Ad}(l)X + Z &= \mathrm{Ad}(l)(\mathrm{Ad}(h)X) + \mathrm{Ad}(lh)(Y) \\ &= \mathrm{Ad}(lh)(X + Y). \end{aligned}$$

Thus $\mathrm{Ad}(l)X + Z$ belongs to $\bigcup_{g \in G^\mu} \mathrm{Ad}(g)(\mathfrak{a} \oplus \mathfrak{q})$, and we have shown (2.3.6).

We are ready to give a generalized Cartan decomposition for Case I. We continue the decomposition (2.3.3) as follows.

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}^\mu \oplus \left(\bigcup_{g \in G^\mu} \mathrm{Ad}(g)\mathfrak{a} \right) \oplus \mathfrak{g}^{-\mu^2} \\ &= \mathfrak{g}^\mu \oplus \left(\bigcup_{g \in G^\mu} \mathrm{Ad}(g)(\mathfrak{a} \oplus \mathfrak{q}) \right) && \text{by (2.3.6).} \end{aligned}$$

Hence we can find that the exponential mapping

$$\exp : \bigcup_{g \in G^\mu} \mathrm{Ad}(g)(\mathfrak{a} \oplus \mathfrak{q}) \to G/G^\mu$$

is surjective ([He1]). Consequently we have

$$\begin{aligned} G &= \exp \left(\bigcup_{g \in G^\mu} \mathrm{Ad}(g)(\mathfrak{a} \oplus \mathfrak{q}) \right) G^\mu \\ &= G^\mu \exp(\mathfrak{a} \oplus \mathfrak{q}) G^\mu \\ &= \mathrm{L}_{\Pi'} \exp(\mathfrak{a} \oplus \mathfrak{q})\, \mathrm{L}_{\Pi''}\,. \end{aligned}$$

$\square$

2.3.3 Decomposition for Case II

The aim of this subsection is to show the following proposition.

Proposition 2.3.3 (generalized Cartan decomposition for Case II)**.** *Let* $G = \mathrm{SO}(2n+1)$, $(\Pi')^c = \{\alpha_1\}$ *and* $(\Pi'')^c = \{\alpha_j\}$ $(1 \leq j \leq n)$. *We define abelian subspaces* $\mathfrak{b}_1$ *and* $\mathfrak{b}_2$ *of* $\mathfrak{g}$ *by*

$$\begin{aligned} \mathfrak{b}_1 :=\ & \mathbb{R}(\mathrm{E}_{1,j+1} - \mathrm{E}_{2n-j+1,2n+1} - \mathrm{E}_{j+1,1} + \mathrm{E}_{2n+1,2n-j+1}) && (2.3.7)\\ & + \mathbb{R}(\mathrm{E}_{1,2n-j+1} - \mathrm{E}_{j+1,2n+1} - \mathrm{E}_{2n-j+1,1} + \mathrm{E}_{2n+1,j+1}), \\ \mathfrak{b}_2 :=\ & \mathbb{R}(\mathrm{E}_{1,2n-j+2} - \mathrm{E}_{j,2n+1} - \mathrm{E}_{2n-j+2,1} + \mathrm{E}_{2n+1,j}). && (2.3.8) \end{aligned}$$

Then we have $G = \mathrm{L}_{\Pi'} \exp(\mathfrak{b}_1) \exp(\mathfrak{b}_2)\, \mathrm{L}_{\Pi''}$.

Proof. We put $L = \mathrm{L}_{\Pi'}$, $H = \mathrm{L}_{\Pi''}$ for simplicity. Let us take a symmetric subgroup $G'G'' = (G^{\tau_j})_0$ containing H where G' and G'' are given by $G' := \mathrm{SO}(2j) \times I_{2n-2j+1}$ and $G'' := I_{2j} \times \mathrm{SO}(2n-2j+1)$. In light that $\mathfrak{b}_1$ is a maximal abelian subspace of $\mathfrak{g}^{-\tau_1,-\tau_j}$, we can see from Fact 2.3.1 that

$$G = L \exp(\mathfrak{b}_1) G'G''. \tag{2.3.9}$$

We take a symmetric subgroup $(G')^{\mu} = \mathrm{U}(j) \times I_{2n-2j+1}$ of G'. We again use Fact 2.3.1 as follows.

$$G' = (G')^{\tau_1}_0 \exp(\mathfrak{b}_2)(G')^{\mu}. \tag{2.3.10}$$

Further, the equality (2.3.10) can be rewritten as

$$G' = (G')^{\tau_1}_{ss} \exp(\mathfrak{b}_2)(G')^{\mu}, \tag{2.3.11}$$

where $(G')^{\tau_1}_{ss}$ denotes the analytic subgroup of $(G')^{\tau_1}$ with Lie algebra the semisimple part of the Lie algebra of $(G')^{\tau_1}$. Then we continue the decomposition (2.3.9) as follows.

$$\begin{aligned} G &= L \exp(\mathfrak{b}_1) G'G'' && \text{by (2.3.9)} \\ &= L \exp(\mathfrak{b}_1)((G')^{\tau_1}_{ss} \exp(\mathfrak{b}_2)(G')^{\mu})G'' && \text{by (2.3.11)} \\ &= L (G')^{\tau_1}_{ss} \exp(\mathfrak{b}_1) \exp(\mathfrak{b}_2)(G')^{\mu}G'' && \text{by } (G')^{\tau_1}_{ss} \subset Z_G(\mathfrak{b}_1) \\ &= L \exp(\mathfrak{b}_1) \exp(\mathfrak{b}_2) H && \text{by } (G')^{\mu}G'' = H. \end{aligned}$$

□

Here is a herringbone stitch which we have used for $L\backslash G/H$ in Case II.

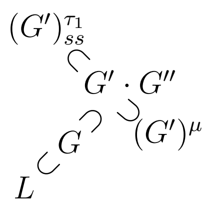


Figure 2.3.1: Herringbone stitch for Case II

2.4 Application of invariant theory of quivers

This section aims for proving the implication (i) $\Rightarrow$ (ii) in Theorem 2.1.1. We shall use invariants of quivers for the proof as in [Ko4]. This section could be read independently of Section 2.3 which gives a proof on the opposite implication (ii) $\Rightarrow$ (i) in Theorem 2.1.1.

2.4.1 Invariants of quivers

In the sequel, the proofs of Lemmas 2.4.1, 2.4.2 and 2.4.3 are essentially the same as [Ko4, Lemmas 6.1, 6.2 and 6.3] respectively. So, we give necessary changes and precise statements, but omit the proof.

Let $\sigma : \mathrm{M}(N,\mathbb{C}) \to \mathrm{M}(N,\mathbb{C})$ be the complex conjugation with respect to $\mathrm{M}(N,\mathbb{R})$.

Lemma 2.4.1. (c. f. *[Ko4, Lemma 6.1]*) *Let $G \subset \mathrm{GL}(N,\mathbb{C})$ be a σ-stable subgroup, $R \in \mathrm{M}(N,\mathbb{R})$, and L a subgroup of G. If there exists $g \in G$ such that*

$$\mathrm{Ad}(L)(\mathrm{Ad}(g)R) \cap \mathrm{M}(N,\mathbb{R}) = \varnothing, \tag{2.4.1}$$

then $G \neq LG^\sigma G_R$. Here $G_R := \{h \in G : hRh^{-1} = R\}$.

We return to the case $G = \mathrm{SO}(2n+1)$. We fix a partition $n = n_1 + \cdots + n_k$ with $n_i > 0$ $(1 \le i \le n-1)$, $n_k \ge 0$, and a positive integer $r \ge 2$. We consider the following loop:

$$i_0 \to i_1 \to \cdots \to i_r, \quad i_s \in \{1,\dots,2k-1\},\ i_0 = i_r,\ i_{s-1} \neq i_s\ (1 \le s \le r).$$

Correspondingly to this loop, we define a non-linear mapping

$$A_{i_0\cdots i_r} : \mathrm{M}(2n+1,\mathbb{C}) \to \begin{cases} \mathrm{M}(n_{i_0},\mathbb{C}) & (i_0 = i_r \neq k), \\ \mathrm{M}(2n_{i_0}+1,\mathbb{C}) & (i_0 = i_r = k), \end{cases}$$

as follows: let $P \in \mathrm{M}(2n,\mathbb{C})$, and we write P as $(P_{ij})_{1\le i,j\le 2k-1}$ in the block matrix form corresponding to the partition $2n+1 = n_1 + \cdots + n_{k-1} + (2n_k+1) + n_{k-1} + \cdots + n_1$ such that

$$P_{ij} \in \begin{cases} \mathrm{M}(n_i,n_j;\mathbb{C}) & (i,j \neq k), \\ \mathrm{M}(2n_k+1,n_j;\mathbb{C}) & (i = k, j \neq k), \\ \mathrm{M}(n_i,2n_k+1;\mathbb{C}) & (i \neq k, j = k), \\ \mathrm{M}(2n_k+1,\mathbb{C}) & (i = j = k), \end{cases} \tag{2.4.2}$$

where $n_{2k-i} := n_i$ $(1 \le i \le k)$. Then we define $(\tilde{P})_{1\le i,j\le 2k-1}$ and $A_{i_0\cdots i_k}(P)$ by

$$\tilde{P}_{ij} := \begin{cases} P_{ij} & (i+j \le 2k), \\ J_{n_i}{}^tP_{2k-j,2k-i}J_{n_j} & (i+j > 2k,\ i,j \neq k), \\ J_{2n_k+1}{}^tP_{2k-j,k}J_{n_j} & (i = k, j > k), \\ J_{n_i}{}^tP_{k,2k-i}J_{2n_k+1} & (i > k, j = k), \end{cases}$$

and

$$A_{i_0\cdots i_r}(P) := \tilde{P}_{i_0i_1}\tilde{P}_{i_1i_2}\cdots\tilde{P}_{i_{r-1}i_r}.$$

The point here is that for any $l = (l_1, \ldots, l_{k-1}, l_k) \in L := \mathrm{U}(n_1)\times\cdots\times\mathrm{U}(n_{k-1})\times\mathrm{SO}(2n_k+1)$ (see (2.2.5) in Section 2.2 for the realization as a matrix), the following equality holds.

$$(\widetilde{\mathrm{Ad}(l)P})_{ij} = l_i \tilde{P}_{ij} l_j^{-1}. \tag{2.4.3}$$

We omit details since (2.4.3) can be verified by a simple matrix computation. This equality leads us to the following lemma (c. f. [Ko4, Lemma 6.2]):

Lemma 2.4.2. *If there exists a loop $i_0 \to i_1 \to \cdots \to i_r$ such that at least one of the coefficients of the characteristic polynomial $\det(\lambda I_{n_{i_0}} - A_{i_0\cdots i_r}(P))$ is not real, then*

$$\mathrm{Ad}(L)P \cap \mathrm{M}(2n, \mathbb{R}) = \varnothing.$$

Combining Lemma 2.4.1 with Lemma 2.4.2, we obtain the next lemma (c. f. [Ko4, Lemma 6.3]):

Lemma 2.4.3. *Let $n = n_1+\cdots+n_k$ be a partition with $n_i > 0$ $(1 \le i \le n-1)$, $n_k \ge 0$, and $L = \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k + 1)$ the corresponding Levi subgroup of $\mathrm{SO}(2n + 1)$. Let us suppose that R is a block diagonal matrix* :

$$R := \begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_{2k-1} \end{pmatrix},$$

where $R_s, R_{2k-s} \in \mathrm{M}(n_s, \mathbb{R})$ $(1 \le s \le k-1)$, $R_k \in \mathrm{M}(2n_k + 1, \mathbb{R})$.

If there exist $X \in \mathfrak{so}(2n + 1)$ and a loop $i_0 \to \cdots \to i_r$ such that

$$\det(\lambda I_{n_{i_0}} - A_{i_0\cdots i_r}([X, R])) \notin \mathbb{R}[\lambda],$$

then the multiplication map $L\times G^\sigma \times G_R \to G$ is not surjective. Here, $[X, R] := XR-RX$.

We shall use Lemma 2.4.3 in each of the subsequent Propositions 2.4.4, 2.4.5 and 2.4.6.

2.4.2 Necessary conditions for $G = LG^\sigma H$

Throughout this subsection, we set $G = \mathrm{SO}(2n + 1)$ and $(L, H) =$

$$(\mathrm{U}(n_1)\times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k + 1),\ \mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_{l-1}) \times \mathrm{SO}(2m_l + 1)),$$

where $n = n_1 + \cdots + n_k = m_1 + \cdots + m_l$ with $n_i, m_j > 0$ $(1 \le i \le k-1,\ 1 \le j \le l-1)$, and $n_k, m_l \ge 0$. We give necessary conditions on (L, H) under which $G = LG^\sigma H$ holds. We divide the proof into three cases (Propositions 2.4.4–2.4.6).

Proposition 2.4.4. $G \ne LG^\sigma H$ *if* $k = 3,\ l = 2,\ m_1 = 1.$

Proposition 2.4.5. $G \ne LG^\sigma H$ *if* $k = l = 2,\ n_1, m_1 \ge 2,\ n_2, m_2 \ne 0.$

Proposition 2.4.6. $G \ne LG^\sigma H$ *if* $k = l = 2,\ n_1 \ge 2,\ n_2 \ne 0,\ m_2 = 0.$

In the following proofs of Propositions 2.4.4, 2.4.5 and 2.4.6, all entries in the blank space are zero in any matrix.

Proof of Proposition 2.4.4. Let $1 \to 3 \to 5 \to 2 \to 1$ be a loop, and R a diagonal matrix $R = \mathrm{diag}(1, 0, \dots, 0, -1)$ of size $(2n+1) \times (2n+1)$. Then, G_R coincides with H. Let us fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \le i,j \le 5} \in \mathfrak{so}(2n+1)$ in the block matrix form corresponding to the partition $2n+1 = n_1 + n_2 + (2n_3+1) + n_2 + n_1$ as (2.4.2):

$$X_{13} := \mathrm{E}_{1,n_3+1} = \begin{pmatrix} O & \begin{matrix} 1 \\ \\ 0 \end{matrix} & O \end{pmatrix} \in \mathrm{M}(n_1, 2n_3+1; \mathbb{C}),$$

$$X_{41} := \mathrm{E}_{n_2,1} = \begin{pmatrix} & O \\ 1 & \end{pmatrix}, \quad X_{21} := u\,\mathrm{E}_{1,1} = \begin{pmatrix} u & \\ & O \end{pmatrix} \in \mathrm{M}(n_2, n_1; \mathbb{C}).$$

We define the block entries X_{11}, X_{15}, X_{22}, X_{23}, X_{24}, X_{32}, X_{33}, X_{34}, X_{42}, X_{43}, X_{44}, X_{51} and X_{55} to be zero matrices. The remaining block entries are automatically determined by the definition (2.2.2) of $\mathfrak{so}(2n+1)$. Then $Q := [X, R]$ has the following block entries:

$$Q_{13} = -\mathrm{E}_{1,n_3+1}, \; Q_{41} = \mathrm{E}_{n_2,1}, \; Q_{21} = u\,\mathrm{E}_{1,1}.$$

By a simple matrix computation, we have (here we recall $k = 3$)

$$A_{13521}(Q) = Q_{13} J_{2n_3+1} {}^t\!Q_{13} J_{n_1} J_{n_1} {}^t\!Q_{41} J_{n_2} Q_{21} = u\,\mathrm{E}_{1,1} \in \mathrm{M}(n_1, \mathbb{C}).$$

Therefore we obtain

$$\det(\lambda I_{n_1} - A_{13521}(Q)) = \lambda^{n_1} - u\lambda^{n_1 - 1} \notin \mathbb{R}[\lambda] \;\text{ if } u \notin \mathbb{R}.$$

By Lemma 2.4.3, we have shown $G \ne LG^\sigma H$. □

Proof of Proposition 2.4.5. We may and do assume $m_1 \ge n_1 \ge 2$ without loss of generality. Let $1 \to 2 \to 3 \to 1$ be a loop, and $R \in \mathrm{M}(2n+1, \mathbb{R})$ a diagonal matrix with the following entries:

$$R := \mathrm{diag}(\overbrace{1, \dots, 1}^{m_1}, \overbrace{2, \dots, 2}^{2m_2+1}, \overbrace{-1, \dots, -1}^{m_1}).$$

Then, $G_R = H$. We fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \le i,j \le 3} \in \mathfrak{so}(2n+1)$ in the block matrix form corresponding to the partition $2n+1 = n_1 + (2n_2+1) + n_1$ as (2.4.2):

$$\begin{aligned} X_{12} :&= \mathrm{E}_{1,n_2} + u\,\mathrm{E}_{1,n_2+1} + \mathrm{E}_{n_1,n_2+1} + \mathrm{E}_{n_1,n_2+2} \\ &= \begin{pmatrix} O & 1 \; u \; 0 & O \\ & O & \\ O & 0 \; 1 \; 1 & O \end{pmatrix} \in \mathrm{M}(n_1, 2n_2+1; \mathbb{C}), \end{aligned}$$

$$\begin{aligned} X_{31} :&= -\mathrm{E}_{1,1} + \mathrm{E}_{n_1,n_1} \\ &= \begin{pmatrix} -1 & & \\ & O & \\ & & 1 \end{pmatrix} \in \mathrm{M}(n_1, \mathbb{C}). \end{aligned}$$

We define the block entries X_{11}, X_{22} and X_{33} to be zero matrices. The remaining block entries of X are determined automatically by (2.2.2). Then $Q := [X, R]$ has the following block entries.

$$Q_{12} = \mathrm{E}_{1,n_2} + u\,\mathrm{E}_{1,n_2+1} + \mathrm{E}_{n_1,n_2+1} + \mathrm{E}_{n_1,n_2+2},\ Q_{31} = -2\,\mathrm{E}_{1,1} + 2\,\mathrm{E}_{n_1,n_1}.$$

A simple matrix computation shows (here we recall $k = 2$)

$$\begin{aligned} A_{1231}(Q) &= Q_{12} J_{2n_2+1} {}^tQ_{12} J_{n_1} Q_{31} \\ &= -2(1+u)\,\mathrm{E}_{1,1} + 2u^2\,\mathrm{E}_{1,n_1} - 2\,\mathrm{E}_{n_1,1} + 2(1+u)\,\mathrm{E}_{n_1,n_1} \in \mathrm{M}(n_1, \mathbb{C}). \end{aligned}$$

Consequently we obtain

$$\det(\lambda I_{n_1} - A_{1231}(Q)) = \lambda^{n_1} - 4(1+2u)\lambda^{n_1-2} \notin \mathbb{R}[\lambda] \quad \text{if} \ \ u \notin \mathbb{R}.$$

By using Lemma 2.4.3, we have $G \neq LG^\sigma H$. □

Proof of Proposition 2.4.6. We consider the loop $1 \to 2 \to 1 \to 3 \to 2 \to 1$, and a diagonal matrix $R \in \mathrm{M}(2n+1, \mathbb{R})$ with the following entries:

$$R := \mathrm{diag}(\overbrace{1, \dots, 1}^{n-1}, -1, 0, 1, \overbrace{-1, \dots, -1}^{n-1}).$$

Then G_R is conjugate to H by an element of G^σ. We fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \leq i,j \leq 3} \in \mathfrak{so}(2n+1)$ in the block matrix form corresponding to the partition $2n+1 = n_1 + (2n_2+1) + n_1$ as (2.4.2):

$$\begin{aligned} X_{12} :&= \mathrm{E}_{1,n_2} + \mathrm{E}_{1,n_2+1} - \mathrm{E}_{n_1,n_2+1} \\ &= \begin{pmatrix} O & 1\ 1\ 0 & O \\ & O & \\ O & 0\ {-1}\ 0 & O \end{pmatrix} \in \mathrm{M}(n_1, 2n_2+1; \mathbb{C}), \end{aligned}$$

$$\begin{aligned} X_{13} :&= u\,\mathrm{E}_{1,1} - u\,\mathrm{E}_{n_1,n_1} \\ &= \begin{pmatrix} u & & \\ & O & \\ & & -u \end{pmatrix} \in \mathrm{M}(n_1, \mathbb{C}). \end{aligned}$$

We define the block entries X_{11}, X_{22} and X_{33} to be zero matrices. The remaining block entries are automatically determined by (2.2.2). Then, $Q := [X, R]$ has the following block entries.

$$\begin{aligned} Q_{12} &= -2\,\mathrm{E}_{1,n_2} - \mathrm{E}_{1,n_2+1} + \mathrm{E}_{n_1,n_2+1}, & Q_{13} &= -2u\,\mathrm{E}_{1,1} + 2u\,\mathrm{E}_{n_1,n_1}, \\ Q_{21} &= -2\,\mathrm{E}_{n_2,1} - \mathrm{E}_{n_2+1,1} + \mathrm{E}_{n_2+1,n_1}. \end{aligned}$$

By a simple matrix computation, we have (here we recall $k = 2$)

$$A_{121321}(Q) = Q_{12} Q_{21} Q_{13} J_{n_1} {}^tQ_{21} J_{2n_2+1} Q_{21} = 8u\,\mathrm{E}_{1,1} - 8u\,\mathrm{E}_{1,n_1} \in \mathrm{M}(n_1, \mathbb{C}),$$

and thus

$$\det(\lambda I_n - A_{121321}(Q)) = \lambda^{n_1} - 8u\lambda^{n_1-1} \notin \mathbb{R}[\lambda] \text{ if } u \notin \mathbb{R}.$$

By Lemma 2.4.3, we have shown $G \neq LG^\sigma H$. □

2.4.3 Completion of the proof of Theorem 2.1.1

We complete the proof of the implication (i) $\Rightarrow$ (ii) in Theorem 2.1.1 (Proposition 2.4.7) by using Propositions 2.4.4–2.4.6. We recall that for a given partition $n = n_1 + \cdots + n_k$ with $n_1, ..., n_{k-1} > 0$ and $n_k \geq 0$, we have the corresponding Levi subgroup $\mathrm{L}_{\Pi'} = \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k + 1)$ of $\mathrm{SO}(2n + 1)$, which is associated to the subset

$$\Pi' := \Pi \setminus \left\{ \alpha_i \in \Pi : i = \sum_{s=1}^{j} n_s,\ 1 \leq j \leq k - 1 \right\}$$

of the simple system Π (see Diagram 2.1.1 for the label of the Dynkin diagram).

Proposition 2.4.7. *Let G be the special orthogonal group $\mathrm{SO}(2n+1)$, σ a Chevalley –Weyl involution, Π', Π'' subsets of Π, and $\mathrm{L}_{\Pi'}$, $\mathrm{L}_{\Pi''}$ the corresponding Levi subgroups. Then we have*

$$G \neq \mathrm{L}_{\Pi'}\, G^{\sigma}\, \mathrm{L}_{\Pi''}, \tag{2.4.4}$$

if one of the following conditions up to switch of Π' and Π'' is satisfied $(1 \leq i, j, k \leq n)$:

(I). *Either $(\Pi')^c$ or $(\Pi'')^c$ contains more than one element.*
(II). $(\Pi')^c = \{\alpha_i\},\ (\Pi'')^c = \{\alpha_j\}$ *and* $i, j \notin \{1, n\}$.
(III). $(\Pi')^c = \{\alpha_i\},\ (\Pi'')^c = \{\alpha_n\}$ *and* $i \notin \{1, n\}$.

Proof. Let $(\mathrm{L}_{\Pi'}, \mathrm{L}_{\Pi''}) =$

$$(\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k + 1),\ \mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_{l-1}) \times \mathrm{SO}(2m_l + 1)).$$

First, let us show the condition (I) implies (2.4.4). Without loss of generality, we may and do assume that $n_1 \geq \cdots \geq n_{k-1}$, $m_1 \geq \cdots \geq m_{l-1}$ and that $(\Pi')^c$ contains more than one element since the role of Π' and Π'' is symmetric.

Case (I–1): $m_1 = 1$. Since L and H are contained in $\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \mathrm{SO}(2(n_3 + \cdots + n_k) + 1)$ and $\mathrm{U}(1) \times \mathrm{SO}(2(m_2 + \cdots + m_l) + 1)$ respectively, we can see that (2.4.4) holds by Proposition 2.4.4.

Case (I–2): $m_1 \geq 2,\ n_k \neq 0$. Since L and H are contained in $\mathrm{U}(n_1 + n_2) \times \mathrm{SO}(2(n_3 + \cdots + n_k) + 1)$ and $\mathrm{U}(m_1) \times \mathrm{SO}(2(m_2 + \cdots + m_l) + 1)$ with $m_1 \geq 2$ respectively, we can find that (2.4.4) holds by using Propositions 2.4.5 and 2.4.6.

Case (I–3): $m_1 \geq 2,\ n_k = 0$. In this case n_1 is greater than one, and thus (2.4.4) follows from Propositions 2.4.5 and 2.4.6. Here, we note that L and H are contained in $\mathrm{U}(n_1) \times \mathrm{SO}(2(n_2 + \cdots + n_k) + 1)$ with $n_2 \neq 0$ and $\mathrm{U}(m_1) \times \mathrm{SO}(2(m_2 + \cdots + m_l) + 1)$ respectively.

Next, let us treat the conditions (II) and (III). Then, we can immediately find that each of the conditions (II) and (III) implies (2.4.4) by using Propositions 2.4.5 and 2.4.6 respectively.

Therefore we have finished the proof. $\square$

By Propositions 2.3.2, 2.3.3 and 2.4.7, we have finished the proof of Theorem 2.1.1.

2.5 Application of visible actions to representation theory

As an application of Theorem 2.1.1, we obtain some multiplicity-free theorems by using Kobayashi's theory of visible actions. Here we recall the definition ([Ko2]).

Definition 2.5.1. We say a biholomorphic action of a Lie group G on a complex manifold D is *strongly visible* if the following two conditions are satisfied:

1. There exists a real submanifold S such that (we call S a "slice")
$$D' := G \cdot S \text{ is an open subset of } D.$$
2. There exists an antiholomorphic diffeomorphism σ of D' such that
$$\begin{aligned} \sigma|_S &= \mathrm{id}_S, \\ \sigma(G \cdot x) &= G \cdot x \quad \text{for any } x \in S. \end{aligned}$$

Definition 2.5.2. In the above setting, we say the action of G on D is S-visible. This terminology will be used also if S is just a subset of D.

Let G be a compact Lie group and L, H its Levi subgroups. Then G/L, G/H and $(G \times G)/(L \times H)$ are complex manifolds. If the triple (G, L, H) satisfies $G = LG^\sigma H$, the following three group-actions are all strongly visible:

$$\begin{aligned} L &\curvearrowright G/H, \\ H &\curvearrowright G/L, \\ \Delta(G) &\curvearrowright (G \times G)/(L \times H). \end{aligned}$$

Here, $\Delta(G)$ is defined by $\Delta(G) := \{(x, y) \in G \times G : x = y\}$. The following fact ([Ko3, Theorem 4.3]) constructs a family of multiplicity-free representations from visible actions.

Fact 2.5.3. *Let G be a Lie group and $\mathcal{V}$ a G-equivariant Hermitian holomorphic vector bundle on a connected complex manifold D. If the following three conditions from* (1) *to* (3) *are satisfied, then any unitary representation that can be embedded in the vector space $\mathcal{O}(D, \mathcal{V})$ of holomorphic sections of $\mathcal{V}$ decomposes multiplicity-freely:*

1. *The action of G on D is S-visible. That is, there exists a subset $S \subset D$ satisfying the conditions given in Definition* 2.5.1*. Further, there exists an automorphism $\hat{\sigma}$ of G such that $\sigma(g \cdot x) = \hat{\sigma}(g) \cdot \sigma(x)$ for any $g \in G$ and $x \in D'$.*
2. *For any $x \in S$, the fiber $\mathcal{V}_x$ at x decomposes as the multiplicity free sum of irreducible unitary representations of the isotropy subgroup G_x. Let $\mathcal{V}_x = \bigoplus_{1 \le i \le n(x)} \mathcal{V}_x^{(i)}$ denote the irreducible decomposition of $\mathcal{V}_x$.*
3. *σ lifts to an antiholomorphic automorphism $\tilde{\sigma}$ of $\mathcal{V}$ and satisfies $\tilde{\sigma}(\mathcal{V}_x^{(i)}) = \mathcal{V}_x^{(i)}$ $(1 \le i \le n(x))$ for each $x \in S$.*

We return to the case where $G = \mathrm{SO}(2n+1)$. The fundamental weights $\omega_1, ..., \omega_n$ with respect to the simple roots $\alpha_1, ..., \alpha_n$ are given as follows (see Diagram 2.1.1 for the label of the Dynkin diagram).

$$\omega_i = \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-1} + \frac{1}{2}\alpha_n) \ \ (1 \leq i \leq n).$$

By using the Borel–Weil theory together with Fact 2.5.3 and our generalized Cartan decompositions, we obtain the following two corollaries of Theorem 2.1.1.

Corollary 2.5.4. *If the pair (L, λ) is an entry in the* Table *2.5.1, then the restriction $\pi_\lambda|_L$ of the irreducible representation π_λ of* $\mathrm{SO}(2n+1)$ *with highest weight λ to L decomposes multiplicity-freely. Here, $1 \leq i \leq n$ and a is an arbitrary non-negative integer.*

Table 2.5.1: Restriction

Levi subgroup L	highest weight λ
$\mathrm{U}(n)$	$a\omega_n$
$\mathrm{U}(1) \times \mathrm{SO}(2n-1)$	$a\omega_i$
$\mathrm{U}(i) \times \mathrm{SO}(2n-2i+1)$	$a\omega_1$

Corollary 2.5.5. *The tensor product representation $\pi_{a\omega_1} \otimes \pi_{b\omega_i}$ decomposes as a multiplicity-free sum of irreducible representations of* $\mathrm{SO}(2n+1)$ *for $1 \leq i \leq n$ and arbitrary non-negative integers a, b. Likewise, the tensor product $\pi_{a\omega_n} \otimes \pi_{b\omega_n}$ is also multiplicity-free for any $a, b \in \mathbb{N}$.*

Remark 2.5.6. The above representations have been known to be multiplicity-free by P. Littelmann ([Li2]) by checking the sphericity of the product of flag varieties associated to maximal parabolic subgroups and J. R. Stembridge ([St2]) by a combinatorial method using the Weyl character to analyze the tensor product multiplicities. Our approach is different from these two methods, and uses the notion of visible actions.

We have listed an application of Fact 2.5.3 only for the line bundle case. Let us give a simple example of that in the vector bundle setting. Let G be the spin group $\mathrm{Spin}(2n+1)$ and T a maximal torus of G. We let π_λ denote any irreducible representation of G with highest weight λ and π_{ω_n} as above. Since π_{ω_n} is weight multiplicity-free, i.e. π_{ω_n} decomposes multiplicity-freely as a representation of T, we can apply Fact 2.5.3 to the tensor product representation of π_λ and π_{ω_n} by setting $\mathcal{V} := G \times_T (\mathbb{C}_\lambda \otimes \pi_{\omega_n})$, $D := G/T$, $S := \{o\}$, and then conclude that $\pi_\lambda \otimes \pi_{\omega_n}$ is multiplicity-free as a representation of G (the irreducible decomposition may be thought of a Pierri rule for a type B group). Here, we note that $\mathcal{V}_x$ and G_x for $x = o$ are given by $\mathbb{C}_\lambda \otimes \pi_{\omega_n}$ and T respectively in this setting. Further applications of Theorem 2.1.1 and Fact 2.5.3 to representation theory are discussed in Chapter 6.

Chapter 3

Visible actions on flag varieties of type C and a generalization of the Cartan decomposition

3.1 Introduction for Chapter 3

In this chapter, we classify all the pairs of Levi subgroups (L, H) of a connected compact simple Lie group G of type C such that $G = LG^\sigma H$ holds, where σ is a Chevalley–Weyl involution of G. The motivation for considering this kind of decomposition is the theory of *visible actions* on complex manifolds introduced by Kobayashi [Ko2], and $G = LG^\sigma H$ can be interpreted as a generalization of the Cartan decomposition to the non-symmetric setting. (We refer the reader to [He1], [Ho], [Ma2] and [Ko4] and references therein for some aspects of the Cartan decomposition from geometric and group theoretic viewpoints.)

A generalization of the Cartan decomposition for symmetric pairs has been used in various contexts including analysis on symmetric spaces. However, there was no analogous result for non-symmetric cases before Kobayashi's paper [Ko4]. Motivated by visible actions on complex manifolds [Ko1], [Ko2], he completely determined the pairs of Levi subgroups

$$(L, H) = (U(n_1) \times \cdots \times U(n_k), U(m_1) \times \cdots \times U(m_l))$$

of the unitary group $G = U(n)$ such that the multiplication mapping $L \times O(n) \times H \to G$ is surjective. Furthermore, he developed a method to find a suitable subset B of $O(n)$ which gives the following decomposition (a generalized Cartan decomposition, see [Ko4]):

$$G = LBH.$$

On the other hand, Sasaki has been studying recently visible actions on a homogeneous space G/H in the setting where both G and H are *complex* reductive Lie groups, and in his papers [Sa2], [Sa3], he gave a generalization of the Cartan decomposition $G = LBH$.

Back to the decomposition theory [Ko4], we consider the following problems:
Let G be a connected compact Lie group, T a maximal torus, and σ a Chevalley–Weyl

The contents of this section are taken from [Ta3].

involution of G with respect to T. Here, we recall that an involutive automorphism μ of a connected compact Lie group K is said to be a Chevalley–Weyl involution if there exists a maximal torus T of K such that $\mu(t) = t^{-1}$ for every $t \in T$. For example, $\sigma(g) = \bar{g}$ defines a Chevalley–Weyl involution of $G = U(n)$ with respect to the maximal torus consisting of diagonal matrices, and the fixed point subgroup G^σ is given by $G^\sigma = O(n)$.

1) Classify all the pairs of Levi subgroups L and H with respect to $\mathfrak{t}$ such that the multiplication mapping $\psi : L \times G^\sigma \times H \to G$ is surjective.

2) Find a "good" representative $B \subset G^\sigma$ such that $G = LBH$ in the case ψ is surjective. We call such a decomposition $G = LBH$ a *generalized Cartan decomposition*. Here we note that the roles of the subgroups H and L are symmetric.

We solve the problems for type C groups. In order to state the results, we label the Dynkin diagram of type C_n as follows: For a subset Π' of the set Π of simple roots, we

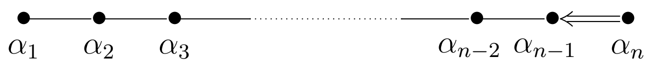


Figure 3.1.1: Dynkin diagram of type C_n

denote by $\mathrm{L}_{\Pi'}$ the Levi subgroup whose root system is generated by Π'. For example, $\mathrm{L}_\varnothing$ is a maximal torus of G and $\mathrm{L}_{\{\alpha_p\}^c} = U(p) \times Sp(n-p)$ for $G = Sp(n)$ $(1 \leq p \leq n)$. Here, we set $Sp(0) := \{1\}$ for the convenience, and $(\Pi')^c$ denotes the complement $\Pi \setminus \Pi'$.

Theorem 3.1.1. *Let G be a connected compact simple Lie group of type C_n, σ a Chevalley–Weyl involution, Π', Π'' two proper subsets of Π, and $\mathrm{L}_{\Pi'}$, $\mathrm{L}_{\Pi''}$ the corresponding Levi subgroups. Then the following two conditions on $\{\Pi', \Pi''\}$ are equivalent.*

(i). *$G = \mathrm{L}_{\Pi'}\, G^\sigma\, \mathrm{L}_{\Pi''}$.*

(ii). *One of the following conditions holds up to switch of the factors Π' and Π'' :*

$$\begin{array}{ll} \text{Case I.} & (\Pi')^c = \{\alpha_n\}, \quad (\Pi'')^c = \{\alpha_n\}. \\ \text{Case II.} & (\Pi')^c = \{\alpha_1\}, \quad (\Pi'')^c = \{\alpha_i\}, \quad 1 \leq i \leq n. \end{array}$$

In the case where G is simply connected, i.e., $G = Sp(n)$, Theorem 3.1.1 means that the pairs $(\mathrm{L}_{\Pi'}, \mathrm{L}_{\Pi''})$ satisfying (i) are classified as follows:

$$\begin{array}{ll} \text{Case I.} & (\mathrm{L}_{\Pi'}, \mathrm{L}_{\Pi''}) = (U(n), U(n)). \\ \text{Case II.} & (\mathrm{L}_{\Pi'}, \mathrm{L}_{\Pi''}) = (U(1) \times Sp(n-1), U(i) \times Sp(n-i)), \quad 1 \leq i \leq n. \end{array}$$

In each of the two cases in Theorem 3.1.1, we give a generalized Cartan decomposition $G = LBH$ explicitly with $B \subset G^\sigma$. In Case I, B is an abelian subgroup of dimension n, and in Case II, B is given by $B = T \cdot T' \cdot T'' = \{xyz \in G \ ; \ x \in T, y \in T', z \in T''\}$ or $B = T' \cdot T'' = \{yz \in G \ ; \ y \in T', z \in T''\}$, where T, T' and T'' denote one-dimensional abelian subgroups. This is stated in Propositions 3.3.2 and 3.3.3. Here, we note that B is no longer a subgroup in Case II.

A generalized Cartan decomposition $G = LBH$ implies that the subgroup L acts on G/H in a (strongly) visible fashion, and likewise H on G/L, and G on $(G \times G)/(L \times H)$. Then Kobayashi's theory leads us to three multiplicity-free theorems (*triunity* in [Ko1]):

$$\begin{array}{ll} \text{Restriction } G \downarrow L & : \mathrm{Ind}_H^G(\mathbb{C}_\lambda)|_L, \\ \text{Restriction } G \downarrow H & : \mathrm{Ind}_L^G(\mathbb{C}_\lambda)|_H, \\ \text{Tensor product} & : \mathrm{Ind}_H^G(\mathbb{C}_\lambda) \otimes \mathrm{Ind}_L^G(\mathbb{C}_\mu). \end{array}$$

Here $\mathrm{Ind}_H^G(\mathbb{C}_\lambda)$ denotes a holomorphically induced representation of G from a character $\mathbb{C}_\lambda$ of H by the Borel-Weil theorem. See [Ko1], [Ko2] and [Ko3] for the general theory on the application of visible actions (including the vector bundle setting), and Corollaries 3.5.5 and 3.5.6 in this section for type C groups.

Organization. In Section 3.2, we give a matrix realization of the symplectic group $G = Sp(n)$ and its subgroups which are used in Sections 3.3 and 3.4. In Section 3.3, we prove that (ii) implies (i). Furthermore, we find explicitly a slice B that gives a generalized Cartan decomposition $G = \mathrm{L}_{\Pi'}\, B\, \mathrm{L}_{\Pi''}$. The converse implication (ii) $\Rightarrow$ (i) is proved in Section 3.4 by using the invariant theory for quivers. An application to multiplicity-free representations is discussed in Section 3.5.

3.2 Matrix realization

The surjectivity of $\psi : L \times G^\sigma \times H \to G$ depends on neither the coverings of the group G nor the choice of Cartan subalgebras and Chevalley–Weyl involutions. Thus, we may and do work with the symplectic group $Sp(n)$, and a fixed pair of a Cartan subalgebra and a Chevalley–Weyl involution as below.

Throughout this chapter, we realize $G = Sp(n)$ as the matrix group

$$G := \{g \in SL(2n, \mathbb{C})\ ;\ {}^t g J_n g = J_n,\ {}^t\bar{g} g = I_{2n}\}, \tag{3.2.1}$$

where ${}^t g$ denotes the transpose of g, and J_n is defined by

$$J_n := \begin{pmatrix} \overbrace{}^{n} & \overbrace{}^{n} \\ & & & & & 1 \\ & O & & & \cdots & \\ & & & 1 & & \\ & & -1 & & & \\ & \cdots & & & O & \\ -1 & & & & & \end{pmatrix} \in GL(2n, \mathbb{R}).$$

Then, the corresponding Lie algebra $\mathfrak{g}$ of G is given by

$$\mathfrak{g} := \{X \in \mathfrak{sl}(2n, \mathbb{C})\ ;\ {}^t X J_n + J_n X = O,\ {}^t\overline{X} + X = O\}.$$

We take a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and a involution σ as

$$\mathfrak{t} = \bigoplus_{1 \le i \le n} \mathbb{R}\sqrt{-1} H_i, \tag{3.2.2}$$

$$\sigma : G \to G, \quad g \mapsto \bar{g}, \tag{3.2.3}$$

where $H_i := \mathrm{E}_{i,i} - \mathrm{E}_{2n+1-i,2n+1-i}$, and $\bar{g}$ denotes the complex conjugation of $g \in G$. The differential of σ is denoted by the same letter. Then, σ is a Chevalley–Weyl involution of G with respect to $\mathfrak{t}$. We let $\{\varepsilon_i\}_{1 \le i \le n} \subset (\mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C})^*$ be the dual basis of $\{H_i\}_{1 \le i \le n}$. Then we define a set of simple roots $\Pi := \{\alpha_1, \ldots, \alpha_n\}$ by

$$\alpha_1 := \varepsilon_1 - \varepsilon_2,\ \ldots,\ \alpha_{n-1} := \varepsilon_{n-1} - \varepsilon_n,\ \alpha_n := 2\varepsilon_n.$$

Let $n = n_1 + \cdots + n_k$ be a partition of n with $n_1, \ldots, n_{k-1} > 0$ and $n_k \geq 0$. We put

$$s_i := \sum_{1 \leq p \leq i} n_p \quad (1 \leq i \leq k-1),$$
$$\Pi' := \Pi \setminus \{\alpha_{s_i} \in \Pi \; ; \; 1 \leq i \leq k-1\},$$

and denote by $\mathrm{L}_{\Pi'}$ the Levi subgroup whose root system is generated by Π'. In the matrix realization, $\mathrm{L}_{\Pi'}$ takes the form:

$$\mathrm{L}_{\Pi'} = \left\{ \begin{pmatrix} A_1 & & & & & & \\ & \ddots & & & & & \\ & & A_{k-1} & & & & \\ & & & B & & & \\ & & & & J'_{n_{k-1}} \overline{A_{k-1}} J'_{n_{k-1}} & & \\ & & & & & \ddots & \\ & & & & & & J'_{n_1} \overline{A_1} J'_{n_1} \end{pmatrix} \; ; \; \begin{array}{l} A_i \in U(n_i) \; (1 \leq i \leq k-1), \\ B \in Sp(n_k) \end{array} \right\} \tag{3.2.4}$$
$$\simeq U(n_1) \times \cdots \times U(n_{k-1}) \times Sp(n_k).$$

Here, all entries in the blank space are zero, and J'_m is defined by

$$J'_m := \begin{pmatrix} & & & 1 \\ & O & 1 & \\ & \iddots & & \\ 1 & & O & \end{pmatrix} \in GL(m, \mathbb{R}).$$

We note that the pair $(G, \mathrm{L}_{\Pi'})$ forms a symmetric pair if and only if $(\Pi')^c = \{\alpha_n\}$. For a later purpose, we give an explicit involution μ of G of which the fixed point subgroup is $\mathrm{L}_{\{\alpha_n\}^c}$. We set

$$\mathrm{L}_{\{\alpha_n\}^c} = G^\mu, \qquad \mu : G \to G, \qquad g \mapsto I_{n,n} g I_{n,n}, \tag{3.2.5}$$

where $I_{n,n}$ is defined by $I_{n,n} := \mathrm{diag}(\underbrace{1, \ldots, 1}_{n}, \underbrace{-1, \ldots, -1}_{n})$.

To obtain a generalized Cartan decomposition by the herringbone stitch method [Ko4], we will use the following symmetric subgroup ($1 \leq i \leq n$).

$$Sp(i) \times Sp(n-i) = \tag{3.2.6}$$
$$\left\{ \left(\begin{array}{c|c|c} A & & B \\ \hline & S & \\ \hline C & & D \end{array} \right) ; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(i), \; S \in Sp(n-i) \right\}.$$

3.3 Generalized Cartan decomposition

The aim of this section is to prove the implication (ii) ⇒ (i) in Theorem 3.1.1. The main tool for our proof is the herringbone stitch method that reduces unknown decompositions for non-symmetric pairs to the known Cartan decomposition for symmetric pairs.

3.3.1 Symmetric case (Decomposition for Case I)

In this subsection we recall a well-known fact on the Cartan decomposition for the symmetric case [Ho, Theorem 6.10], [Ma3, Theorem 1], and give a generalized Cartan decomposition for Case I.

Fact 3.3.1. *Let K be a connected compact Lie group with Lie algebra $\mathfrak{k}$ and two involutions τ, τ' ($\tau^2 = (\tau')^2 = \mathrm{id}$). Let H and L be subgroups of K such that*

$$(K^{\tau})_0 \subset L \subset K^{\tau} \quad \textit{and} \quad (K^{\tau'})_0 \subset H \subset K^{\tau'}.$$

Here F_0 denotes the connected component of F containing the identity element for a Lie group F. We take a maximal abelian subspace $\mathfrak{b}$ in

$$\mathfrak{k}^{-\tau,-\tau'} := \{X \in \mathfrak{k} \,;\, \tau(X) = \tau'(X) = -X\}$$

and write B for the connected abelian subgroup with Lie algebra $\mathfrak{b}$.

Suppose that $\tau\tau'$ is semisimple on the center $\mathfrak{z}$ of $\mathfrak{k}$. Then we have

$$K = LBH.$$

We shall apply Fact 3.3.1 to Case I. Let us set

$$G = Sp(n), \quad (\Pi')^c = \{\alpha_n\}. \tag{3.3.1}$$

(See Diagram 3.1.1 for the label of the Dynkin diagram.) Then, $(G, \mathrm{L}_{\Pi'})$ is a symmetric pair with μ the corresponding involution (see (3.2.5) for the definition of μ). We take a maximal abelian subspace $\mathfrak{b}$ of $\mathfrak{g}^{-\mu}$ as

$$\mathfrak{b} := \bigoplus_{1\leq i\leq n} \mathbb{R}(\mathrm{E}_{i,2n+1-i} - \mathrm{E}_{2n+1-i,i}). \tag{3.3.2}$$

We note that $\mathfrak{b}$ is fixed by the Chevalley–Weyl involution σ. Using Fact 3.3.1, we obtain the following proposition.

Proposition 3.3.2 (Generalized Cartan decomposition for Case I)**.** *Let G, $\mathrm{L}_{\Pi'}$ be as in* (3.3.1) *and $B := \exp(\mathfrak{b})$ for $\mathfrak{b}$ as in* (3.3.2)*. Then we have*

$$G = \mathrm{L}_{\Pi'}\, B\, \mathrm{L}_{\Pi'}\,.$$

3.3.2 Decomposition for Case II

This subsection is devoted to showing the following proposition.

Proposition 3.3.3 (Generalized Cartan decomposition for Case II)**.** *Let G be the symplectic group $Sp(n)$, and $(\Pi')^c = \{\alpha_1\}$, $(\Pi'')^c = \{\alpha_i\}$ $(1 \leq i \leq n)$. We define an abelian subgroup B' and a subset B'' by*

$$B' := \begin{cases} \exp(\mathbb{R}(\mathrm{E}_{1,i+1} - \mathrm{E}_{2n-i,2n} - \mathrm{E}_{i+1,1} + \mathrm{E}_{2n,2n-i})) & (1 \leq i < n), \\ I_{2n} & (i = n), \end{cases} \tag{3.3.3}$$

$$B'' := \exp(\mathbb{R}X)\exp(\mathbb{R}Y) \tag{3.3.4}$$

for $X := \mathrm{E}_{1,2n+1-i} + \mathrm{E}_{i,2n} - \mathrm{E}_{2n+1-i,1} - \mathrm{E}_{2n,i}$ and $Y := \mathrm{E}_{1,2n} - \mathrm{E}_{2n,1}$. Then we have $G = \mathrm{L}_{\Pi'}\, B' B''\, \mathrm{L}_{\Pi''}$.

We put $L := \mathrm{L}_{\{\alpha_1\}^c}$, $H := \mathrm{L}_{\{\alpha_i\}^c}$ for simplicity. To prove Proposition 3.3.3, we shall show three lemmas. First, we consider the double coset decomposition of G by L and a symmetric subgroup $G'G'' = Sp(i) \times Sp(n-i)$ containing H, where G' and G'' are given by $G' := Sp(i) \times I_{2n-2i}$ and $G'' := I_{2i} \times Sp(n-i)$ (see (3.2.6) for the realization of $G'G''$).

Lemma 3.3.4. *The equality $G = LB'G'G''$ holds.*

Proof. If $i = n$, both (G, L) and $(G, G'G'')$ are symmetric and thus the lemma is followed by Fact 3.3.1. Let us suppose $i \neq n$. We identify G/L with $\mathbb{C}P^{2n-1}$ in the natural way (which is induced from the natural action of G on $\mathbb{C}^{2n}$). For any $x \in \mathbb{C}P^{2n-1}$, since S^{4i-1} and $S^{4n-4i-1}$ admit transitive actions of G' and G'', respectively, there exist $g = g'g'' \in G'G''$ and $\theta \in \mathbb{R}$ such that

$$g \cdot x = [\cos\theta : 0 : \cdots : 0 : \overset{\overset{i+1}{\smile}}{\sin\theta} : 0 : \cdots : 0] \in B' \cdot \overline{e_1},$$

where $\overline{e_1} := [1 : 0 : \cdots : 0] \in \mathbb{C}P^{2n-1}$. Thus we obtain $G = LB'G'G''$. $\square$

Next, we consider the double coset decomposition of G' by $(G')^\mu$ and $L \cap G'$.

Lemma 3.3.5. *The equality $G' = (G')^\mu (B'')^{-1} (L \cap G')$ holds, where $(B'')^{-1}$ is defined by $(B'')^{-1} := \{b^{-1} \,;\, b \in B''\}$.*

Proof. Let us identify $G'/(L \cap G')$ with $\mathbb{C}P^{2i-1}$ in the natural way by taking I_{2n-2i} away from $G' = Sp(i) \times I_{2n-2i}$. For any $z \in \mathbb{C}P^{2i-1}$, we write $z = [z' : z'']$ where both z' and z'' have i entries. Since $(G')^\mu \simeq U(i)$ acts on S^{2i-1} transitively, there exists $g \in (G')^\mu$ such that

$$g \cdot z = [\|z'\| : \overbrace{0 : \cdots : 0}^{i-1} : w]$$

for some w with i entries, where $\|\cdot\|$ denotes the usual Euclidean norm. We write $w = [w' : re^{\sqrt{-1}\theta}]$ where $r, \theta \in \mathbb{R}$, and w' has $(i-1)$ entries. Then there is $g' \in ((L \cap G')_{ss})^\mu$ satisfying

$$g' \cdot (gz) = [\|z'\| : \overbrace{0 : \cdots : 0}^{i-1} : \|w'\| : \overbrace{0 : \cdots : 0}^{i-2} : re^{\sqrt{-1}\theta}]$$

since $((L \cap G')_{ss})^\mu \simeq U(i-1)$ acts on S^{2i-3} transitively. Here, $(L \cap G')_{ss}$ denotes the analytic subgroup of $L \cap G'$ whose Lie algebra is the semisimple part of the Lie algebra of $L \cap G'$. Let us set

$$t := \mathrm{diag}(e^{\sqrt{-1}\theta/2}, \overbrace{0, \ldots, 0}^{i-2}, e^{-\sqrt{-1}\theta/2}, e^{\sqrt{-1}\theta/2}, \overbrace{0, \ldots, 0}^{i-2}, e^{-\sqrt{-1}\theta/2}) \in (G')^\mu.$$

We then obtain

$$\begin{aligned} t \cdot (g'gz) &= [\|z'\|e^{\sqrt{-1}\theta/2} : 0 : \cdots : 0 : \|w'\|e^{\sqrt{-1}\theta/2} : 0 : \cdots : 0 : re^{\sqrt{-1}\theta/2}] \\ &= [\|z'\| : 0 : \cdots : 0 : \|w'\| : 0 : \cdots : 0 : r] \in (B'')^{-1} \cdot \overline{e_1}, \end{aligned}$$

where $\overline{e_1} = [1 : 0 : \cdots : 0]$. Therefore we have $G' = (G')^\mu (B'')^{-1} (L \cap G')$. $\square$

Noting that the centralizer of B' in $L \cap G'$ is the subgroup $(L \cap G')_{ss} \simeq Sp(i-1)$ of codimension 1, we introduce the subgroup, which also centralizes B', by

$$\hat{L} := A \cdot (L \cap G')_{ss} (= \{a \cdot x \in G \; ; \; a \in A, \; x \in (L \cap G')_{ss}\}),$$

where A is defined by $A := \exp(\mathbb{R}\sqrt{-1}(\mathrm{E}_{1,1} + \mathrm{E}_{i+1,i+1} - \mathrm{E}_{2n-i,2n-i} - \mathrm{E}_{2n,2n}))$. By using Lemma 3.3.5, we obtain a decomposition of $G'G''$ by $\hat{L}$ and H.

Lemma 3.3.6. *The equality $G'G'' = \hat{L}B''H$ holds.*

Proof. By Lemma 3.3.5, we have

$$\begin{aligned} G'G'' &= ((L \cap G')B''(G')^{\mu})G'' \\ &= (L \cap G')B''H \qquad\qquad \text{by } H = (G')^{\mu}G''. \end{aligned} \tag{3.3.5}$$

Further, $(L \cap G')B''H$ coincides with $\hat{L}B''H$:

$$(L \cap G')B''H = \hat{L}B''H. \tag{3.3.6}$$

Let us verify the equality (3.3.6). We define $A' = \exp(\mathbb{R}\sqrt{-1}(\mathrm{E}_{i+1,i+1} - \mathrm{E}_{2n-i,2n-i}))$ an abelian subgroup of H. Since A' centralizes both B'' and $L \cap G'$, and since any element of A can be written in terms of elements of the center of $L \cap G'$ and A', the equality $(L \cap G')B''H = (L \cap G')B''(A'H) = A'(L \cap G')B''H$ shows that $(L \cap G')B''H$ contains $\hat{L}B''H$. Conversely, since AA' contains the analytic subgroup of $L \cap G'$, which corresponds to the center of the Lie algebra of $L \cap G'$, the equality $\hat{L}B''H = A(L \cap G')_{ss}B''(A'H) = AA'(L \cap G')_{ss}B''H$ shows $\hat{L}B''H \supset (L \cap G')B''H$. Therefore we have the equality (3.3.6). By the two equalities (3.3.6) and (3.3.5), the lemma follows. $\square$

We are ready to give a proof of a generalized Cartan decomposition by using the herringbone stitch method [Ko4].

PROOF OF PROPOSITION 3.3.3. By using Lemmas 3.3.4 and 3.3.6, we have

$$\begin{aligned} G &= LB'G'G'' && \text{by Lemma 3.3.4} \\ &= LB'(\hat{L}B''H) && \text{by Lemma 3.3.6} \\ &= LB'B''H. \end{aligned}$$

This completes the proof of the proposition. $\square$

Here is a herringbone stitch which we have used for $L\backslash G/H$ in Case II. Now we have

$$\begin{array}{ccccc}
L \cap G' & & & & \\
 & \subset & & & \\
 & & G' \cdot G'' & & \\
 & \supset & & \supset & \\
 & G & & & (G')^{\mu} \\
\subset & & & & \\
L & & & &
\end{array}$$

Figure 3.3.1: Herringbone stitch for Case II

finished the proof of the implication (ii) $\Rightarrow$ (i) in Theorem 3.1.1 since the abelian group B in Proposition 3.3.2 and subsets B', B'' in Proposition 3.3.3 are contained in G^{σ}.

3.4 Application of invariant theory for quivers

The aim of this section is to prove that (i) implies (ii) in Theorem 3.1.1. For the proof, we use invariants of quivers as in [Ko4]. This section could be read independently of Section 3.3 which gives a proof on the opposite implication (ii) $\Rightarrow$ (i).

3.4.1 Invariants of quivers

In the following, Lemmas 3.4.1, 3.4.2 and 3.4.3 are parallel to [Ko4, Lemmas 6.1, 6.2 and 6.3] respectively, and their proofs are essentially the same as that in [Ko4]. So, we give necessary changes and precise statements, but omit the proof.

Let $\sigma : \mathrm{M}(N, \mathbb{C}) \to \mathrm{M}(N, \mathbb{C})$ be the complex conjugation with respect to $\mathrm{M}(N, \mathbb{R})$.

Lemma 3.4.1. (c.f. [Ko4, Lemma 6.1]) *Let $G \subset GL(N, \mathbb{C})$ be a σ-stable subgroup, $R \in \mathrm{M}(N, \mathbb{R})$ and L a subgroup of G. If there exists $g \in G$ such that*

$$\mathrm{Ad}(L)(\mathrm{Ad}(g)R) \cap \mathrm{M}(N, \mathbb{R}) = \varnothing, \tag{3.4.1}$$

then $G \neq L G^\sigma G_R$. Here $G_R := \{h \in G \; ; \; hRh^{-1} = R\}$.

We return to the case $G = Sp(n)$. We fix a partition $n = n_1 + \cdots + n_k$ with $n_i > 0$ $(1 \leq i \leq k-1)$, $n_k \geq 0$ and a positive integer $r \geq 2$. We consider the following loop:

$$i_0 \to i_1 \to \cdots \to i_r, \quad i_s \in \{1, \dots, 2k-1\}, \; i_0 = i_r, \; i_{s-1} \neq i_s \; (1 \leq s \leq r).$$

Correspondingly to this loop, we define a non-linear mapping

$$A_{i_0 \cdots i_r} : \mathrm{M}(2n, \mathbb{C}) \to \begin{cases} \mathrm{M}(n_{i_0}, \mathbb{C}) & (i_0 = i_r \neq k), \\ \mathrm{M}(2n_{i_0}, \mathbb{C}) & (i_0 = i_r = k) \end{cases}$$

as follows: Let $P \in \mathrm{M}(2n, \mathbb{C})$, and we write P as $(P_{ij})_{1 \leq i,j \leq 2k-1}$ in the block matrix form corresponding to the partition $2n = n_1 + \cdots + n_{k-1} + 2n_k + n_{k-1} + \cdots + n_1$ of $2n$ such that

$$P_{ij} \in \begin{cases} \mathrm{M}(n_i, n_j; \mathbb{C}) & (i, j \neq k), \\ \mathrm{M}(2n_k, n_j; \mathbb{C}) & (i = k, j \neq k), \\ \mathrm{M}(n_i, 2n_k; \mathbb{C}) & (i \neq k, j = k), \\ \mathrm{M}(2n_k, \mathbb{C}) & (i = j = k), \end{cases} \tag{3.4.2}$$

where $n_{2k-i} := n_i$ $(1 \leq i \leq k)$. Then we define $(\tilde{P}_{ij})_{1 \leq i,j \leq 2k-1}$ and $A_{i_0 \cdots i_r}(P)$ by

$$\tilde{P}_{ij} := \begin{cases} P_{ij} & (i + j \leq 2k), \\ J'_{n_i} {}^t P_{2k-j, 2k-i} J'_{n_j} & (i + j > 2k, \; i, j \neq k), \\ J_{n_k} {}^t P_{2k-j, k} J'_{n_j} & (i = k, j > k), \\ J'_{n_i} {}^t P_{k, 2k-i} J_{n_k} & (i > k, j = k), \end{cases}$$

and

$$A_{i_0 \cdots i_r}(P) := \tilde{P}_{i_0 i_1} \tilde{P}_{i_1 i_2} \cdots \tilde{P}_{i_{r-1} i_r}.$$

For any $\ell \in L := U(n_1)\times\cdots\times U(n_{k-1})\times Sp(n_k)$ (see (3.2.4) in Section 3.2 for the realization as a matrix), a direct computation shows

$$(\widetilde{\mathrm{Ad}(\ell)P})_{ij} = \ell_i \tilde{P}_{ij}\ell_j^{-1} \quad (1 \leq i, j \leq 2k-1), \tag{3.4.3}$$

where ℓ_s $(1 \leq s \leq 2k-1)$ denotes the (s,s)-th block entry of ℓ. The equality (3.4.3) leads us to the following lemma (cf. [Ko4, Lemma 6.2]).

Lemma 3.4.2. *If there exists a loop $i_0 \to i_1 \to \cdots \to i_r$ such that at least one of the coefficients of the characteristic polynomial $\det(\lambda I_{n_{i_0}} - A_{i_0\cdots i_r}(P))$ is not real, then*

$$\mathrm{Ad}(L)P \cap \mathrm{M}(2n, \mathbb{R}) = \varnothing.$$

By Lemmas 3.4.1 and 3.4.2, we can obtain the next lemma (cf. [Ko4, Lemma 6.3]).

Lemma 3.4.3. *Let $n = n_1+\cdots+n_k$ be a partition and $L := U(n_1)\times\cdots\times U(n_{k-1})\times Sp(n_k)$ a Levi subgroup of $Sp(n)$. We define a block diagonal matrix R by*

$$R := \begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_{2k-1} \end{pmatrix},$$

where $R_s, R_{2k-s} \in \mathrm{M}(n_s, \mathbb{R})$ $(1 \leq s \leq k-1)$, $R_k \in \mathrm{M}(2n_k, \mathbb{R})$ (the last condition makes sense when $n_k \neq 0$). If there exist $X \in \mathfrak{sp}(n)$ and a loop $i_0 \to \cdots \to i_r$ such that

$$\det(\lambda I_{n_{i_0}} - A_{i_0\cdots i_r}([X, R])) \notin \mathbb{R}[\lambda],$$

then the multiplication map $L \times G^\sigma \times G_R \to G$ is not surjective Here, $[X, R] := XR - RX$.

We shall repeatedly use this lemma in the next subsection.

3.4.2 Necessary conditions for $G = LG^\sigma H$

Throughout this subsection, we set

$$(G, L, H) = (Sp(n), U(n_1)\times\cdots\times U(n_{k-1})\times Sp(n_k), U(m_1)\times\cdots\times U(m_{l-1})\times Sp(m_l)),$$

where $n = n_1+\cdots+n_k = m_1+\cdots+m_l$ with $n_i, m_j > 0$ $(1 \leq i \leq k-1, 1 \leq j \leq l-1)$ and $n_k, m_l \geq 0$. We give necessary conditions on (L, H) under which $G = LG^\sigma H$ holds. We divide the proof into three cases (Propositions 3.4.4 through 3.4.6).

Proposition 3.4.4. *$G \neq LG^\sigma H$ if $k = 3,\ l = 2,\ m_1 = 1$.*

Proposition 3.4.5. *$G \neq LG^\sigma H$ if $k = l = 2,\ n_1, m_1 \geq 2,\ n_2, m_2 \neq 0$.*

Proposition 3.4.6. *$G \neq LG^\sigma H$ if $k = l = 2,\ n_1 \geq 2,\ n_2 \neq 0,\ m_2 = 0$.*

PROOF OF PROPOSITION 3.4.4. Let $1 \to 5 \to 2 \to 1$ be a loop. We define a diagonal matrix R by $R := \mathrm{diag}(1, 0, \ldots, 0, -1) \in \mathrm{M}(2n, \mathbb{R})$. Then, the centralizer G_R coincides with H. We fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \leq i,j \leq 5} \in \mathfrak{sp}(n)$ in the block matrix form corresponding to the partition $2n = n_1 + n_2 + 2n_3 + n_2 + n_1$ as (3.4.2):

$$X_{15} := u\,\mathrm{E}_{1,n_1} \in \mathrm{M}(n_1, \mathbb{C}),\ X_{41} := \mathrm{E}_{n_2,1} \in \mathrm{M}(n_2, n_1; \mathbb{C}),\ X_{21} := \mathrm{E}_{1,1} \in \mathrm{M}(n_1, n_2; \mathbb{C}).$$

We define the block entries X_{11}, X_{13}, X_{22}, X_{23}, X_{24}, X_{31}, X_{32}, X_{33}, X_{34}, X_{35}, X_{42}, X_{43}, X_{44}, X_{53} and X_{55} to be zero matrices. The remaining block entries are automatically determined by the definition (3.2.1) of $G = Sp(n)$. Then, $Q := [X, R]$ has the following block entries:

$$Q_{15} = -2u\,\mathrm{E}_{1,n_1} \in \mathrm{M}(n_1, \mathbb{C}),\ Q_{41} = \mathrm{E}_{n_2,1} \in \mathrm{M}(n_2, n_1; \mathbb{C}),\ Q_{21} = \mathrm{E}_{1,1} \in \mathrm{M}(n_1, n_2; \mathbb{C}).$$

By a simple matrix computation, we have (here, we recall $k = 3$)

$$A_{1521}(Q) = Q_{15} J'_{n_1} {}^t Q_{41} J'_{n_2} Q_{21} = -2u\,\mathrm{E}_{1,1} \in \mathrm{M}(n_1, \mathbb{C}).$$

Hence we obtain

$$\det(\lambda I_{n_1} - A_{1521}(Q)) = \lambda^{n_1} + 2u\lambda^{n_1 - 1} \notin \mathbb{R}[\lambda] \text{ if } u \notin \mathbb{R}.$$

By Lemma 3.4.3, we have shown $G \neq L G^\sigma H$. $\square$

PROOF OF PROPOSITION 3.4.5. We may and do assume $m_1 \geq n_1$ without loss of generality since the roles of L and H are symmetric. Let $1 \to 2 \to 3 \to 1$ be a loop, and $R = \mathrm{diag}(r_1, \ldots, r_{2n}) \in \mathrm{M}(2n, \mathbb{R})$ a diagonal matrix with the following entries:

$$R := \mathrm{diag}(\overbrace{1, \ldots, 1}^{m_1}, \overbrace{2, \ldots, 2}^{2m_2}, \overbrace{-1, \ldots, -1}^{m_1}).$$

Then, we have $G_R = H$. We fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \leq i,j \leq 3} \in \mathfrak{sp}(n)$ in the block matrix form corresponding to the partition $2n = n_1 + 2n_2 + n_1$ as (3.4.2):

$$X_{12} := \mathrm{E}_{1,n_2} + \mathrm{E}_{n_1,n_2+1} \in \mathrm{M}(n_1, 2n_2; \mathbb{C}),\ X_{31} := -\mathrm{E}_{1,n_1} + u\,\mathrm{E}_{n_1,1} \in \mathrm{M}(n_1, \mathbb{C}).$$

We define the block entries X_{11}, X_{22} and X_{33} to be zero matrices. The remaining block entries of X are determined automatically by (3.2.1). Then the block entries of $Q := [X, R]$ are given by

$$Q_{12} = \mathrm{E}_{1,n_2} + \mathrm{E}_{n_1,n_2+1} \in \mathrm{M}(n_1, 2n_2; \mathbb{C}),\ Q_{31} = -2\,\mathrm{E}_{1,n_1} + 2u\,\mathrm{E}_{n_1,1} \in \mathrm{M}(n_1, \mathbb{C}).$$

A simple matrix computation shows (here, we recall $k = 2$)

$$A_{1231}(Q) = Q_{12} J_{n_2} {}^t Q_{12} J'_{n_1} Q_{31} = -2\,\mathrm{E}_{1,n_1} - 2u\,\mathrm{E}_{n_1,1} \in \mathrm{M}(n_1, \mathbb{C}),$$

and thus we have

$$\det(\lambda I_{n_1} - A_{1231}(Q)) = \lambda^{n_1} - 4u\lambda^{n_1 - 2} \notin \mathbb{R}[\lambda] \quad \text{if} \quad u \notin \mathbb{R}.$$

By using Lemma 3.4.3, we obtain $G \neq L G^\sigma H$. $\square$

Proof of Proposition 3.4.6. Let $1 \to 2 \to 3 \to 1$ be a loop, and R a diagonal matrix

$$R := \operatorname{diag}(\overbrace{1,\dots,1}^{n_1-1}, -1, -1, \overbrace{1,\dots,1}^{n_2-1}, \overbrace{-1,\dots,-1}^{n_2-1}, 1, 1, \overbrace{-1,\dots,-1}^{n_1-1}).$$

Then, G_R is conjugate to H by an element of G^σ. We fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1\le i,j\le 3} \in \mathfrak{sp}(n)$ in the block matrix form corresponding to the partition $2n = n_1 + 2n_2 + n_1$ as (3.4.2):

$$X_{12} := u\,\mathrm{E}_{1,1} + \mathrm{E}_{n_1,2n_2} \in \mathrm{M}(n_1, 2n_2; \mathbb{C}),\ X_{31} := -\,\mathrm{E}_{1,n_1} - \mathrm{E}_{n_1,1} \in \mathrm{M}(n_1, \mathbb{C}).$$

We define the block entries X_{11}, X_{22} and X_{33} to be zero matrices. The remaining block entries of X are determined automatically by (3.2.1). Then $Q := [X, R]$ has the block entries

$$Q_{12} = -2u\,\mathrm{E}_{1,1} + 2\,\mathrm{E}_{n_1,2n_2} \in \mathrm{M}(n_1, 2n_2; \mathbb{C}),\ Q_{31} = 2\,\mathrm{E}_{1,n_1} - 2\,\mathrm{E}_{n_1,1} \in \mathrm{M}(n_1, \mathbb{C}).$$

By a simple matrix computation, we have (here, we recall $k = 2$)

$$A_{1231}(Q) = Q_{12} J_{n_2} {}^t Q_{12} J'_{n_1} Q_{31} = -8u\,\mathrm{E}_{1,n_1} - 8u\,\mathrm{E}_{n_1,1} \in \mathrm{M}(n_1, \mathbb{C}).$$

Consequently we obtain

$$\det(\lambda I_{n_1} - A_{1231}(Q)) = \lambda^{n_1} - 64u^2\lambda^{n_1-2} \notin \mathbb{R}[\lambda] \quad \text{if } u^2 \notin \mathbb{R}.$$

From Lemma 3.4.3, we have $G \neq L G^\sigma H$. $\square$

3.4.3 Completion of the proof of Theorem 3.1.1

We complete the proof of the implication (i) $\Rightarrow$ (ii) in Theorem 3.1.1 (Proposition 3.4.7) by using Propositions 3.4.4 through 3.4.6. We recall that for a given partition $n = n_1 + \cdots + n_k$ with $n_1, \dots, n_{k-1} > 0$ and $n_k \ge 0$, we have the corresponding Levi subgroup $\mathrm{L}_{\Pi'} = U(n_1) \times \cdots \times U(n_{k-1}) \times Sp(n_k)$ of $Sp(n)$, which is associated to the subset

$$\Pi' := \Pi \setminus \left\{ \alpha_i \in \Pi \ ;\ i = \sum_{s=1}^{j} n_s,\ 1 \le j \le k-1 \right\}$$

of the set of simple roots Π (see Diagram 3.1.1 for the label of the Dynkin diagram).

Proposition 3.4.7. *Let G be the symplectic group $Sp(n)$, σ a Chevalley–Weyl involution, Π',Π'' subsets of the set of simple roots Π, and $\mathrm{L}_{\Pi'}$, $\mathrm{L}_{\Pi''}$ the corresponding Levi subgroups. Then we have*

$$G \neq \mathrm{L}_{\Pi'}\, G^\sigma\, \mathrm{L}_{\Pi''} \tag{3.4.4}$$

if one of the following conditions up to switch of Π' and Π'' is satisfied $(1 \le i, j, k \le n)$:

(I) *Either $(\Pi')^c$ or $(\Pi'')^c$ contains more than one element.*
(II) *$(\Pi')^c = \{\alpha_i\}$, $(\Pi'')^c = \{\alpha_j\}$ and $i, j \notin \{1, n\}$.*
(III) *$(\Pi')^c = \{\alpha_i\}$, $(\Pi'')^c = \{\alpha_n\}$ and $i \notin \{1, n\}$.*

Proof. Let

$$(\mathrm{L}_{\Pi'},\ \mathrm{L}_{\Pi''}) = (U(n_1)\times\cdots\times U(n_{k-1})\times Sp(n_k),\ U(m_1)\times\cdots\times U(m_{l-1})\times Sp(m_l)).$$

First, let us show the condition (I) implies (3.4.4). Without loss of generality, we may and do assume that $n_1 \geq \cdots \geq n_{k-1}$ and $m_1 \geq \cdots \geq m_{l-1}$, and that $(\Pi')^c$ contains more than one element, that is, $k \geq 3$ since the roles of Π' and Π'' are symmetric.

Case (I–1): $m_1 = 1$. Since $\mathrm{L}_{\Pi'}$ and $\mathrm{L}_{\Pi''}$ are contained in $U(n_1)\times U(n_2)\times Sp(n_3+\cdots+n_k)$ and $U(1)\times Sp(m_2+\cdots+m_l)$, respectively, we can see that (3.4.4) holds by Proposition 3.4.4.

Case (I–2): $m_1 \geq 2,\ n_k \neq 0$. Since $\mathrm{L}_{\Pi'}$ and $\mathrm{L}_{\Pi''}$ are contained in $U(n_1+n_2)\times Sp(n_3+\cdots+n_k)$ and $U(m_1)\times Sp(m_2+\cdots+m_l)$, respectively with $n_1+n_2 \geq 2$ and $m_1 \geq 2$, we can find that (3.4.4) holds by using Propositions 3.4.6 and 3.4.5.

Case (I–3): $m_1 \geq 2,\ n_k = 0$. In this case $n_1 \geq 2$ and thus (3.4.4) follows from Propositions 3.4.6 and 3.4.5. Here, we note that $\mathrm{L}_{\Pi'}$ and $\mathrm{L}_{\Pi''}$ are contained in $U(n_1)\times Sp(n_2+\cdots+n_k)$ and $U(m_1)\times Sp(m_2+\cdots+m_l)$, respectively with $n_2 \neq 0$.

Next, let us treat the conditions (II) and (III). Then, we can immediately find that each of the conditions (II) and (III) implies (3.4.4) by Propositions 3.4.5 and 3.4.6, respectively.

Therefore we have finished the proof of the proposition. □

3.5 Application of visible actions to representation theory

As an application of Theorem 3.1.1, we obtain some multiplicity-free theorems by using Kobayashi's theory of visible actions. Here we recall the definition [Ko3, Definition 4.1].

Definition 3.5.1. We say a biholomorphic action of a Lie group G on a complex manifold D is *strongly visible* if the following two conditions are satisfied:

1. There exists a real submanifold S such that (we call S a "slice")

$$D' := G\cdot S \text{ is an open subset of } D.$$

2. There exists an antiholomorphic diffeomorphism σ of D' such that

$$\begin{aligned} \sigma|_S &= \mathrm{id}_S, \\ \sigma(G\cdot x) &= G\cdot x \quad \text{for any } x\in D'. \end{aligned}$$

Definition 3.5.2. In the above setting, we say the action of G on D is S-visible. This terminology will be used also if S is just a subset of D.

Let G be a compact Lie group and L, H its Levi subgroups. Then $G/L,\ G/H$ and $(G\times G)/(L\times H)$ are complex manifolds. If the triple (G,L,H) satisfies $G = LG^\sigma H$, the following three group-actions are all strongly visible:

$$\begin{aligned} L &\curvearrowright G/H, \\ H &\curvearrowright G/L, \\ \Delta(G) &\curvearrowright (G\times G)/(L\times H). \end{aligned}$$

Here, $\Delta(G)$ is defined by $\Delta(G) := \{(x,y)\in G\times G\ ;\ x=y\}$. The following fact [Ko3, Theorem 4.3] constructs a family of multiplicity-free representations from visible actions.

Fact 3.5.3. *Let G be a Lie group and $\mathcal{V}$ a G-equivariant Hermitian holomorphic vector bundle on a connected complex manifold D. If the following three conditions from (1) to (3) are satisfied, then any unitary representation that can be embedded in the vector space $\mathcal{O}(D, \mathcal{V})$ of holomorphic sections of $\mathcal{V}$ decomposes multiplicity-freely.*

1. *The action of G on D is S-visible. That is, there exist a subset $S \subset D$ and an anti-holomorphic diffeomorphism σ of D' satisfying the conditions given in Definition* 3.5.1. *Further, there exists an automorphism $\hat{\sigma}$ of G such that $\sigma(g \cdot x) = \hat{\sigma}(g) \cdot \sigma(x)$ for any $g \in G$ and $x \in D'$.*

2. *For any $x \in S$, the fiber $\mathcal{V}_x$ at x decomposes as the multiplicity free sum of irreducible unitary representations of the isotropy subgroup G_x. Let $\mathcal{V}_x = \bigoplus_{1 \le i \le n(x)} \mathcal{V}_x^{(i)}$ denote the irreducible decomposition of $\mathcal{V}_x$.*

3. *σ lifts to an antiholomorphic automorphism $\tilde{\sigma}$ of $\mathcal{V}$ and satisfies $\tilde{\sigma}(\mathcal{V}_x^{(i)}) = \mathcal{V}_x^{(i)}$ $(1 \le i \le n(x))$ for each $x \in S$.*

We return to the case where $G = Sp(n)$. The fundamental weights $\omega_1, ..., \omega_n$ with respect to the simple roots $\alpha_1, \dots, \alpha_n$ are given as follows (see Diagram 3.1.1 for the label of the Dynkin diagram).

$$\omega_i = \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-1} + \frac{1}{2}\alpha_n) \ \ (1 \le i \le n).$$

In the sequel, we denote by π_λ an irreducible representation of $Sp(n)$ with highest weight $\lambda = \sum_{i=1}^n c_i \omega_i$ with $c_1, \dots, c_n \in \mathbb{N}$. By using the Borel-Weil theory together with Fact 3.5.3 and our generalized Cartan decompositions, we can give a geometric proof of the multiplicity-freeness property of some representations.

Example 3.5.4. The tensor product $\pi_{a\omega_n} \otimes \pi_{b\omega_n + c\omega_i}$ is multiplicity-free for any i $(1 \le i \le n)$ and for arbitrary non-negative integers a, $b \in \mathbb{N}$, and $c = 0$ or 1.

To see this example, we apply Fact 3.5.3 to $\pi_{a\omega_n} \otimes \pi_{b\omega_n + c\omega_i}$ by setting $\mathcal{V} := (Sp(n) \times Sp(n)) \times_{(U(n) \times U(n))} (\mathbb{C}_{\omega_n} \otimes \Lambda^i)$, $D := (Sp(n) \times Sp(n))/(U(n) \times U(n))$, $S := B \cdot o \times \{o\}$ and $G := \Delta(Sp(n))$, where Λ^i is the representation of $U(n)$ on the i-th alternating tensor product of $\mathbb{C}^n$, B is as in Proposition 3.3.2 and o denotes the identity coset. In this situation, G_x contains $M := \{\mathrm{diag}(\varepsilon_1, \dots, \varepsilon_n) \in U(n); \varepsilon_j = \pm 1 \ (1 \le j \le n)\}$, and $\mathcal{V}_x$ is given by $\mathbb{C}_{\omega_n} \otimes \Lambda^i$ for any $x \in S$. Since Λ^i is multiplicity-free as a representation of the subgroup M of $U(n)$, we find that $\pi_{a\omega_n} \otimes \pi_{b\omega_n + c\omega_i}$ decomposes multiplicity-freely as a representation of $Sp(n)$ by Fact 3.5.3. On the other hand, it follows from Stembridge [St2] that $\pi_{a\omega_n} \otimes \pi_{b\omega_n + c\omega_i}$ is not multiplicity-free if c is greater than one.

In the following, we confine ourselves to the line bundle case and give applications of Theorem 3.1.1 and Fact 3.5.3.

Corollary 3.5.5. *If the pair (L, λ) is an entry in the Table 3.5.1, then the restriction $\pi_\lambda|_L$ of the irreducible representation π_λ of $Sp(n)$ with highest weight λ to L decomposes multiplicity-freely. Here, $1 \le i \le n$ and a is an arbitrary non-negative integer.*

Corollary 3.5.6. *The tensor product representation $\pi_{a\omega_1} \otimes \pi_{b\omega_i}$ decomposes multiplicity freely for any non-negative integers a, b. Likewise, the tensor product $\pi_{a\omega_n} \otimes \pi_{b\omega_n}$ is multiplicity-free for any $a, b \in \mathbb{N}$.*

Table 3.5.1: Restriction

Levi subgroup L	highest weight λ
$\mathrm{U}(n)$	$a\omega_n$
$\mathrm{U}(1) \times \mathrm{Sp}(n-1)$	$a\omega_i$
$\mathrm{U}(i) \times \mathrm{Sp}(n-i)$	$a\omega_1$

The above representations have been known to be multiplicity-free by Littelmann [Li2] by checking the sphericity and Stembridge [St2] by a combinatorial method using character formulas. Our approach uses visible actions and is different from these two methods. We hope that further applications of Theorem 3.1.1 and Fact 3.5.3 to representation theory will be discussed in a future paper.

Chapter 4

Visible actions on flag varieties of type D and a generalization of the Cartan decomposition

4.1 Introduction for Chapter 4

The aim of this chapter is to classify all the pairs of Levi subgroups (L, H) of connected compact simple Lie groups of type D with the following property: $G = LG^\sigma H$ where σ is a Chevalley–Weyl involution of G (Definition 4.2.1). The motivation for considering this kind of decomposition is the theory of *visible actions* on complex manifolds introduced by T. Kobayashi ([Ko2]), and the decomposition $G = LG^\sigma H$ serves as a basis to generalize the Cartan decomposition to the non-symmetric setting. (We refer to [He1], [Ho], [Ma2] and [Ko4] and references therein for some aspects of the Cartan decomposition from geometric and group theoretic viewpoints.)

A generalization of the Cartan decomposition for symmetric pairs has been used in various contexts including analysis on symmetric spaces, however, there were no analogous results for non-symmetric cases before Kobayashi's paper [Ko4]. Motivated by visible actions on complex manifolds ([Ko1], [Ko2]), he completely determined the pairs of Levi subgroups

$$(L, H) = (\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_k), \mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_l))$$

of the unitary group $G = \mathrm{U}(n)$ such that the multiplication mapping $L \times \mathrm{O}(n) \times H \to G$ is surjective. Further he developed a method to find a suitable subset B of $\mathrm{O}(n)$ which gives the following decomposition (a generalized Cartan decomposition, see [Ko4]):

$$G = LBH.$$

In view of this decomposition theory, we consider the following problems: Let G be a connected compact Lie group, $\mathfrak{t}$ a Cartan subalgebra, and σ a Chevalley–Weyl involution of G with respect to $\mathfrak{t}$.

The contents of this section are taken from [Ta4].

1) Classify all the pairs of Levi subgroups L and H with respect to $\mathfrak{t}$ such that the multiplication map $\psi : L \times G^\sigma \times H \to G$ is surjective.

2) Find a "good" representative $B \subset G^\sigma$ such that $G = LBH$ in the case ψ is surjective.

We call such a decomposition $G = LBH$ a *generalized Cartan decomposition*. Here we note that the role of the subgroups H and L is symmetric.

The surjectivity of ψ implies that the subgroup L acts on the flag variety G/H in a (strongly) visible fashion (see Definition 4.5.1). At the same time the H-action on G/L, and the diagonal G-action on $(G \times G)/(L \times H)$ are strongly visible. Then Kobayashi's theory leads us to three multiplicity-free theorems (*triunity* à la [Ko1]):

$$\begin{array}{ll} \text{Restriction } G \downarrow L & : \mathrm{Ind}_H^G(\mathbb{C}_\lambda)|_L, \\ \text{Restriction } G \downarrow H & : \mathrm{Ind}_L^G(\mathbb{C}_\lambda)|_H, \\ \text{Tensor product} & : \mathrm{Ind}_H^G(\mathbb{C}_\lambda) \otimes \mathrm{Ind}_L^G(\mathbb{C}_\mu). \end{array}$$

Here $\mathrm{Ind}_H^G(\mathbb{C}_\lambda)$ denotes a holomorphically induced representation of G from a character $\mathbb{C}_\lambda$ of H by the Borel–Weil theorem. See [Ko1], [Ko2], [Ko3] for the general theory on the application of visible actions (including the vector bundle setting), and also Section 4.5 for the compact simple Lie groups of type D.

We solve the aforementioned problems for connected compact simple Lie groups G of type D. That is, we give a complete list of the pairs of Levi subgroups that admit generalized Cartan decompositions, by using the herringbone stitch method that Kobayashi introduced in [Ko4].

In order to state the results, we label the Dynkin diagram of type D_n as follows: For

Figure 4.1.1: Dynkin diagram of type D_n

a subset Π' of the set Π of simple roots, we denote by $\mathrm{L}_{\Pi'}$ the Levi subgroup whose root system is generated by Π'. For example, $\mathrm{L}_\varnothing$ is a maximal torus of G and $\mathrm{L}_{\{\alpha_p\}^c} = \mathrm{U}(p) \times \mathrm{SO}(2(n-p))$ for $G = \mathrm{SO}(2n)$ $(1 \le p \le n-2)$. Here $(\Pi')^c$ denotes $\Pi \setminus \Pi'$.

Theorem 4.1.1. *Let G be a connected compact simple Lie group of type D_n $(n \ge 4)$, σ a Chevalley–Weyl involution, Π', Π'' two proper subsets of Π, and $\mathrm{L}_{\Pi'}$, $\mathrm{L}_{\Pi''}$ the corresponding Levi subgroups. Then the following two conditions on $\{\Pi', \Pi''\}$ are equivalent.*

(i). $G = \mathrm{L}_{\Pi'}\, G^\sigma\, \mathrm{L}_{\Pi''}$.

(ii). *One of the conditions below holds up to switch of the factors Π' and Π'' :*

$$\begin{array}{llll} \text{I.} & (\Pi')^c = \{\alpha_i\}, & (\Pi'')^c = \{\alpha_j\}, & i \in \{n-1, n\},\ j \in \{1, 2, 3, n-1, n\}, \\ \text{II.} & (\Pi')^c = \{\alpha_i\}, & (\Pi'')^c \subset \{\alpha_j, \alpha_k\}, & i \in \{n-1, n\},\ j, k \in \{1, n-1, n\}, \\ \text{III.} & (\Pi')^c = \{\alpha_i\}, & (\Pi'')^c \subset \{\alpha_j, \alpha_k\}, & i \in \{n-1, n\},\ j, k \in \{1, 2\}, \\ \text{IV.} & (\Pi')^c = \{\alpha_1\}, & (\Pi'')^c \subset \{\alpha_j, \alpha_k\}, & \textit{either } j \textit{ or } k \in \{n-1, n\}, \\ \text{V.} & (\Pi')^c = \{\alpha_i\}, & (\Pi'')^c \subset \{\alpha_2, \alpha_j\}, & n = 4,\ (i, j) = (3, 4) \textit{ or } (4, 3). \end{array}$$

Here $G^\phi := \{g \in G : \phi(g) = g\}$ for an automorphism ϕ of G. We did not intend to make the above cases I–V be exclusive, that is, there is a small overlap among Cases I, II and III.

As a corollary, we obtain three multiplicity-free theorems for type D groups (see Corollary 4.5.4 for the restriction to Levi subgroups and Corollary 4.5.5 for the tensor product representations).

Organization. In Section 4.2, we see that Theorem 4.1.1 is reduced to the standard Levi subgroups of a matrix group $G = \mathrm{SO}(2n)$ without any loss of generality. In Section 4.3, we prove that (ii) implies (i). Furthermore, we find explicitly a slice B that gives a generalized Cartan decomposition $G = \mathrm{L}_{\Pi'}\, B\, \mathrm{L}_{\Pi''}$. The converse implication on (ii) $\Rightarrow$ (i) is proved in Section 4.4 by using the invariant theory for quivers. An application to multiplicity-free representations is discussed in Section 4.5.

4.2 Reduction and Matrix realization

4.2.1 Reduction

In this subsection, we show that the surjectivity of $\psi : L \times G^\sigma \times H \to G$ depends on neither the coverings of the group G nor the choice of Cartan subalgebras and Chevalley–Weyl involutions. This consideration reduces a proof of Theorem 4.1.1 to the case $G = \mathrm{SO}(2n)$.

We firstly recall the definition of a Chevalley–Weyl involution of a connected compact Lie group, and then we show the independence of the coverings.

Definition 4.2.1. Let G be a connected compact Lie group and σ an involution of G. We call σ a Chevalley–Weyl involution if there exists a maximal torus T of G such that $\sigma(t) = t^{-1}$ for every $t \in T$.

Proposition 4.2.2. *Let G be a connected compact semisimple Lie group, $\tilde{G}$ its universal covering group, $\phi : \tilde{G} \to G$ the covering homomorphism, and σ (resp. $\tilde{\sigma}$) a Chevalley–Weyl involution with respect to a maximal torus T (resp. $\tilde{T}$) of G (resp. $\tilde{G}$) such that the following diagram commutes.*

$$\begin{array}{ccc} G & \xrightarrow{\sigma} & G \\ {\scriptstyle\phi}\big\uparrow & & {\scriptstyle\phi}\big\uparrow \\ \tilde{G} & \xrightarrow{\tilde{\sigma}} & \tilde{G} \end{array}$$

Then for any subsets Π', Π'' of the set of simple roots Π of the Lie algebra $\mathfrak{g}$ of G, $G = \mathrm{L}_{\Pi'}\, G^\sigma\, \mathrm{L}_{\Pi''}$ holds if and only if $\tilde{G} = \tilde{\mathrm{L}}_{\phi^*\Pi'} \tilde{G}^{\tilde{\sigma}} \tilde{\mathrm{L}}_{\phi^*\Pi''}$ does. Here, ϕ^* denotes the natural induced map from ϕ, $\mathrm{L}_{\Pi'}$ (resp. $\mathrm{L}_{\Pi''}$) the Levi subgroup of G whose root system is generated by Π' (resp. Π''), and $\tilde{\mathrm{L}}_{\phi^*\Pi'}$ (resp. $\tilde{\mathrm{L}}_{\phi^*\Pi''}$) the Levi subgroup of $\tilde{G}$ whose root system is generated by $\phi^*\Pi'$ (resp. $\phi^*\Pi''$).

Proof. Let $Z_{\tilde{G}}$ denote the center of $\tilde{G}$. Assume $G = \mathrm{L}_{\Pi'}\, G^\sigma\, \mathrm{L}_{\Pi''}$. Since $G^\sigma \subset \phi(\tilde{T} \cdot \tilde{G}^{\tilde{\sigma}})$, we have $\phi(\tilde{\mathrm{L}}_{\phi^*\Pi'} \tilde{G}^{\tilde{\sigma}} \tilde{\mathrm{L}}_{\phi^*\Pi''}) = \mathrm{L}_{\Pi'}\, G^\sigma\, \mathrm{L}_{\Pi''} = G$. Then we obtain $\tilde{G} = Z_{\tilde{G}} \cdot (\tilde{\mathrm{L}}_{\phi^*\Pi'} \tilde{G}^{\tilde{\sigma}} \tilde{\mathrm{L}}_{\phi^*\Pi''}) = \tilde{\mathrm{L}}_{\phi^*\Pi'} \tilde{G}^{\tilde{\sigma}} \tilde{\mathrm{L}}_{\phi^*\Pi''}$.

Conversely, assume $\tilde{G} = \tilde{\mathrm{L}}_{\phi^*\Pi'} \tilde{G}^{\tilde{\sigma}} \tilde{\mathrm{L}}_{\phi^*\Pi''}$. Then we have $G = \mathrm{L}_{\Pi'}\, \phi(\tilde{G}^{\tilde{\sigma}})\, \mathrm{L}_{\Pi''}$ because ϕ is surjective. Since $\phi(\tilde{G}^{\tilde{\sigma}}) \subset G^\sigma$, we obtain $G = \mathrm{L}_{\Pi'}\, G^\sigma\, \mathrm{L}_{\Pi''}$. $\square$

Further, we can see that Theorem 4.1.1 is independent of the choice of Cartan subalgebras and Chevalley–Weyl involutions because any two Cartan subalgebras are conjugate to each other by an inner automorphism, and any two Chevalley–Weyl involutions of the same Cartan subalgebra $\mathfrak{t}$ are conjugate to each other by the adjoint action of $\exp(\mathfrak{t})$ (see

[Wo]). For these reasons, we may and do work with the matrix group $\mathrm{SO}(2n)$, and fix a Cartan subalgebra and a Chevalley–Weyl involution as in the next subsection.

4.2.2 Matrix realization

Throughout this chapter, we realize $G = \mathrm{SO}(2n)$ as a matrix group as follows:

$$G := \{g \in \mathrm{SL}(2n, \mathbb{C}) : {}^t g J_{2n} g = J_{2n},\ {}^t \overline{g} g = I_{2n}\}, \tag{4.2.1}$$

where J_m is defined by

$$J_m := \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & O & & & \\ & & \iddots & & \\ & & & O & \\ 1 & & & & \end{pmatrix} \in \mathrm{GL}(m, \mathbb{R}).$$

Then, the corresponding Lie algebra of G forms

$$\mathfrak{g} := \{X \in \mathfrak{sl}(2n, \mathbb{C}) : {}^t X J_{2n} + J_{2n} X = O,\ {}^t \overline{X} + X = O\}.$$

We take a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ as diagonal matrices:

$$\mathfrak{t} = \bigoplus_{1 \le i \le n} \mathbb{R}\sqrt{-1} H_i,$$

where $H_i := \mathrm{E}_{i,i} - \mathrm{E}_{2n+1-i, 2n+1-i}$.

We define

$$\sigma : G \to G, \quad g \mapsto \bar{g}, \tag{4.2.2}$$

where $\bar{g}$ denotes the complex conjugate of $g \in G$. The differential of σ is denoted by the same letter. This involutive automorphism σ is a Chevalley–Weyl involution with respect to $\mathfrak{t}$.

We let $\{\varepsilon_i\}_{1 \le i \le n} \subset (\mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C})^*$ be the dual basis of $\{H_i\}_{1 \le i \le n}$. Then we define a set of simple roots $\Pi := \{\alpha_1, ..., \alpha_n\}$ by

$$\alpha_i := \varepsilon_i - \varepsilon_{i+1}\ (1 \le i \le n-1),\ \alpha_n := \varepsilon_{n-1} + \varepsilon_n.$$

Let $n = n_1 + \cdots + n_k$ be a partition of n with $n_1, ..., n_{k-1} > 0$ and $n_k \ge 0$. We put

$$s_i := \sum_{1 \le p \le i} n_p \quad (1 \le i \le k-1),$$
$$\Pi' := \Pi \setminus \{\alpha_{s_i} \in \Pi : 1 \le i \le k-1\},$$

and denote by $\mathrm{L}_{\Pi'}$ the Levi subgroup whose root system is generated by Π'. In the matrix realization, $\mathrm{L}_{\Pi'}$ takes the form:

$$\mathrm{L}_{\Pi'} = \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k) = \tag{4.2.3}$$

$$\left\{ \begin{pmatrix} A_1 &&&&&& \\ & \ddots &&&&& \\ && A_{k\text{-}1} &&&& \\ &&& B &&& \\ &&&& J_{n_{k-1}} \overline{A_{k\text{-}1}} J_{n_{k-1}}^{-1} && \\ &&&&& \ddots & \\ &&&&&& J_{n_1} \overline{A_1} J_{n_1}^{-1} \end{pmatrix} : A_i \in \mathrm{U}(n_i), B \in \mathrm{SO}(2n_k) \right\}.$$

Here, we note that the pair $(G, \mathrm{L}_{\Pi'})$ forms a symmetric pair if and only if $\Pi \setminus \Pi' = \{\alpha_1\}$, $\{\alpha_n\}$ or $\{\alpha_{n-1}\}$. For a later purpose, we give explicit involutions τ_1, μ and μ^ξ of G of which the connected component of fixed point subgroups are $\mathrm{L}_{\{\alpha_1\}^c}$, $\mathrm{L}_{\{\alpha_n\}^c}$ and $\mathrm{L}_{\{\alpha_{n-1}\}^c}$.

$$\mathrm{L}_{\{\alpha_1\}^c} = (G^{\tau_1})_0, \qquad \tau_1 : G \to G, \qquad g \mapsto I_{1,2(n-1),1} g I_{1,2(n-1),1}, \tag{4.2.4}$$

$$\mathrm{L}_{\{\alpha_n\}^c} = G^{\mu}, \qquad \mu : G \to G, \qquad g \mapsto I_{n,n} g I_{n,n}, \tag{4.2.5}$$

$$\mathrm{L}_{\{\alpha_{n-1}\}^c} = G^{\mu^\xi}, \qquad \mu^\xi = \xi \circ \mu \circ \xi : G \to G \qquad \text{(see (4.2.7))}, \tag{4.2.6}$$

where K_0 denotes the connected component of K containing the identity element for a Lie group K, and $I_{1,2(n-1),1}$ and $I_{n,n}$ are defined by $I_{1,2(n-1),1} := \mathrm{diag}(1, -1, \dots, -1, 1)$ and $I_{n,n} := \mathrm{diag}(\underbrace{1, \dots, 1}_{n}, \underbrace{-1, \dots, -1}_{n})$.

The Dynkin diagram of type D_n has an outer automorphism of order two, which switches α_{n-1} and α_n. This outer automorphism is induced from the following involution of G.

$$\xi : G \to G, \qquad x \to g_\xi x g_\xi^{-1}, \tag{4.2.7}$$

$$\text{where } g_\xi := \begin{array}{c} \begin{array}{cccc} & & n \; \check{} & n{+}1 \; \check{} \end{array} \\ \begin{pmatrix} 1 &&&&&&& \\ & \ddots &&&&&& \\ && 1 &&&&& \\ &&& 0 & 1 &&& \\ &&& 1 & 0 &&& \\ &&&&& 1 && \\ &&&&&& \ddots & \\ &&&&&&& 1 \end{pmatrix} \end{array}.$$

Note that for any $\Pi' \subset \Pi$, $\xi(\mathrm{L}_{\Pi'})$ coincides with $\mathrm{L}_{\Pi''}$ where Π'' is obtained from Π' by replacing α_{n-1} with α_n. We also note that ξ preserves G^σ.

To obtain a generalized Cartan decomposition by the herringbone stitch method ([Ko4]), we will use an involutive automorphism τ_p of G $(1 \leq p \leq n-1)$ given by

$$\tau_p : G \to G, \qquad g \mapsto I_{p,2(n-p),p} g I_{p,2(n-p),g} \tag{4.2.8}$$

where $I_{p,2(n-p),p} := \mathrm{diag}(\underbrace{1, \dots, 1}_{p}, \underbrace{-1, \dots, -1}_{2(n-p)}, \underbrace{1, \dots, 1}_{p})$. Then the connected component of the fixed point subgroup G^{τ_p} is given by

$$\mathrm{SO}(2p) \times \mathrm{SO}(2(n-p)) := \tag{4.2.9}$$

$$\left\{ \left(\begin{array}{c|c|c} A & & B \\ \hline & S & \\ \hline C & & D \end{array} \right) : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SO}(2p),\ S \in \mathrm{SO}(2(n-p)) \right\}.$$

4.3 Generalized Cartan decomposition

In this section, we give a proof of the implication (ii) $\Rightarrow$ (i) in Theorem 4.1.1. The idea is to use the herringbone stitch method that reduces unknown decompositions for non-symmetric pairs to the known Cartan decomposition for symmetric pairs. For this, we divide the proof to six cases.

4.3.1 Decomposition for the symmetric case (Case $\mathrm{I}-1$)

In this subsection we recall a well-known fact on the Cartan decomposition for the symmetric case ([Ho, Theorem 6.10], [Ma3, Theorem 1]) and deal with Case I with $i, j \in \{n-1, n\}$.

Fact 4.3.1. *Let K be a connected compact Lie group with Lie algebra $\mathfrak{k}$ and two involutions τ, τ' ($\tau^2 = (\tau')^2 = \mathrm{id}$). Let H and H' be subgroups of K such that*

$$(K^\tau)_0 \subset H \subset K^\tau \quad \textit{and} \quad (K^{\tau'})_0 \subset H' \subset K^{\tau'}.$$

We take a maximal abelian subspace $\mathfrak{b}$ in

$$\mathfrak{k}^{-\tau,-\tau'} := \{X \in \mathfrak{k} : \tau(X) = \tau'(X) = -X\}$$

and write B for the connected abelian subgroup with Lie algebra $\mathfrak{b}$.

Suppose that $\tau\tau'$ is semisimple on the center $\mathfrak{z}$ of $\mathfrak{k}$. Then we have

$$K = HBH'.$$

We shall apply Fact 4.3.1 to Case I with $i, j \in \{n-1, n\}$ in Theorem 4.1.1. Let

$$(\Pi')^c = \Pi \setminus \Pi' = \{\alpha_n\}, \quad (\Pi'')^c = \Pi \setminus \Pi'' = \{\alpha_{n-1}\}. \tag{4.3.1}$$

(See Diagram 4.1.1 for the label of the Dynkin diagram.) Then, both $(G, \mathrm{L}_{\Pi'})$ and $(G, \mathrm{L}_{\Pi''})$ are symmetric pairs with μ and $\mu^\xi = \xi \circ \mu \circ \xi$ the corresponding involutions respectively (see (4.2.5) and (4.2.7) for the definitions of μ and ξ). We take maximal abelian subspaces $\mathfrak{b} \subset \mathfrak{g}^{-\mu}$ and $\mathfrak{b}' \subset \mathfrak{g}^{-\mu,-\mu^\xi}$ as follows:

$$\begin{aligned} \mathfrak{b} &:= \bigoplus_{1 \le i \le [\frac{n}{2}]} \mathbb{R}(\mathrm{E}_{2i-1,2n-2i+1} - \mathrm{E}_{2i,2n-2i+2} - \mathrm{E}_{2n-2i+1,2i-1} + \mathrm{E}_{2n-2i+2,2i}), \\ \mathfrak{b}' &:= \mathfrak{b} \cap \xi(\mathfrak{b}). \end{aligned} \tag{4.3.2}$$

We note that both $\mathfrak{b}$ and $\mathfrak{b}'$ are contained in $\mathfrak{g}^\sigma$ where σ is the complex conjugation (4.2.2). Using Fact 4.3.1, we obtain the following proposition.

Proposition 4.3.2. (*Generalized Cartan decomposition.*) *Let $G = \mathrm{SO}(2n)$ and $\mathrm{L}_{\Pi'}$, $\mathrm{L}_{\Pi''}$ be as in* (4.3.1), *and define $B := \exp(\mathfrak{b})$, $B' := \exp(\mathfrak{b}')$ for $\mathfrak{b}$, $\mathfrak{b}'$ as in* (4.3.2). *Then we have the following three decompositions of G.*

$$\begin{aligned} G &= \mathrm{L}_{\Pi'}\, B\, \mathrm{L}_{\Pi'} \\ &= \mathrm{L}_{\Pi''}\, \xi(B)\, \mathrm{L}_{\Pi''} \\ &= \mathrm{L}_{\Pi'}\, B'\, \mathrm{L}_{\Pi''}\,. \end{aligned}$$

4.3.2 Decomposition for Case $\mathrm{I}-2$

In this subsection, we deal with the following case:

$$(\Pi')^c = \{\alpha_i\},\ (\Pi'')^c = \{\alpha_3\}\ (i = n-1 \text{ or } n).$$

Since ξ switches the role of $n-1$ and n, $G = \mathrm{L}_{\Pi'}\, G^\sigma\, \mathrm{L}_{\Pi''}$ holds for $i = n$ if and only if so does for $i = n-1$ (see (4.2.7) for the definition of ξ). Thus, we may and do assume $i = n$ without loss of generality, and put

$$\begin{aligned} L &:= \mathrm{L}_{\{\alpha_n\}^c} (= \mathrm{U}(n)), \\ H &:= \mathrm{L}_{\{\alpha_3\}^c} (= \mathrm{U}(3) \times \mathrm{SO}(2n-6)), \end{aligned} \tag{4.3.3}$$

for simplicity. We also note that the equality $G = LG^\sigma H$ follows for $n = 4$ from Case II in Theorem 4.1.1. (See Subsection 4.3.3.)

First, let us take a symmetric subgroup $G'G'' = (G^{\tau_6})_0$ containing H where $G' := \mathrm{SO}(6) \times I_{2n-6}$ and $G'' := I_6 \times \mathrm{SO}(2n-6) (\subset H)$ (see (4.2.8) for the definition of τ_6). We define a maximal abelian subspace $\mathfrak{b}'$ of $\mathfrak{g}^{-\tau_6,-\mu}$ by

$$\mathfrak{b}' := \begin{cases} \bigoplus_{1\le j\le 3} \mathbb{R}(\mathrm{E}_{j,n+j} - \mathrm{E}_{n+j,j} - \mathrm{E}_{n+1-j,2n+1-j} + \mathrm{E}_{2n+1-j,n+1-j}) & (n \ge 6), \\ \bigoplus_{1\le j\le 2} \mathbb{R}(\mathrm{E}_{j,n+j} - \mathrm{E}_{n+j,j} - \mathrm{E}_{n+1-j,2n+1-j} + \mathrm{E}_{2n+1-j,n+1-j}) & (n = 5). \end{cases} \tag{4.3.4}$$

Then we give a decomposition of G by using Fact 4.3.1 as follows.

$$G = L\exp(\mathfrak{b}')(G'G''). \tag{4.3.5}$$

Second, we consider the centralizer of $\mathfrak{b}'$. We define an abelian subgroup T'' by $T'' := \exp(\mathfrak{t}'')$ where

$$\mathfrak{t}'' := \begin{cases} \bigoplus_{1\le i\le 3} \mathbb{R}\sqrt{-1}(\mathrm{E}_{i,i} - \mathrm{E}_{2n+1-i,2n+1-i} - \mathrm{E}_{n+1-i,n+1-i} + \mathrm{E}_{n+i,n+i}) & (n \ge 6), \\ \bigoplus_{1\le i\le 2} \mathbb{R}\sqrt{-1}(\mathrm{E}_{i,i} - \mathrm{E}_{11-i,11-i} - \mathrm{E}_{6-i,6-i} + \mathrm{E}_{5+i,5+i}) & \\ \qquad\qquad \oplus\, \mathbb{R}\sqrt{-1}(\mathrm{E}_{3,3} - \mathrm{E}_{8,8}) & (n = 5). \end{cases}$$

A simple matrix computation shows that $\mathfrak{b}'$ commutes with $\mathfrak{t}''$.

Lemma 4.3.3. *$Z_G(\mathfrak{b}') \supset T''$.*

Third, we consider the double coset decomposition of G' by $(G')^\mu$ and a maximal torus $T' := G' \cap \exp(\mathfrak{t})$ of G', which consists of diagonal matrices. For this, we decompose the Lie algebra $\mathfrak{g}'$ of G' as follows.

$$\mathfrak{g}' = (\mathfrak{g}')^\mu \oplus (\mathfrak{g}')^{-\mu}.$$

It is easy to see that $(\mathfrak{g}')^{-\mu}$ is rewritten as

$$(\mathfrak{g}')^{-\mu} = \bigcup_{g\in T'} \mathrm{Ad}(g)(\mathfrak{g}')^{-\mu,\sigma}.$$

Then we can find that the exponential mapping

$$\exp : \bigcup_{g \in T'} \mathrm{Ad}(g)(\mathfrak{g}^{-\mu,\sigma}) \to G'/(G')^{\mu}$$

is surjective. Thus we have

$$G' = T' \exp(\mathfrak{g}^{-\mu,\sigma})(G')^{\mu}. \tag{4.3.6}$$

We are ready to give a proof of a generalized Cartan decomposition for Case I with $(i, j) = (n, 3)$.

Proposition 4.3.4. (*Generalized Cartan decomposition.*) *Let* $G = \mathrm{SO}(2n)$ *and* L, H *be as in* (4.3.3). *We set* $B := \exp(\mathfrak{b}') \exp(\mathfrak{g}^{-\mu,\sigma})$ (*see* (4.3.4) *for the definition of* $\mathfrak{b}'$). *Then we have*

$$G = LBH.$$

Proof. In the following proof, we use the herringbone stitch method introduced by Kobayashi ([Ko4]).

$$\begin{aligned} G &= L \exp(\mathfrak{b}')(G'G'') && \text{by } (4.3.5) \\ &= L \exp(\mathfrak{b}')T' \exp(\mathfrak{g}^{-\mu,\sigma})(G')^{\mu}G'' && \text{by } (4.3.6) \\ &= L \exp(\mathfrak{b}')T' \exp(\mathfrak{g}^{-\mu,\sigma})H && \text{by } (G')^{\mu}G'' = H. \end{aligned} \tag{4.3.7}$$

Since T' and T'' satisfy $T' \exp(\mathfrak{g}^{-\mu,\sigma})H = T'' \exp(\mathfrak{g}^{-\mu,\sigma})H$, we can continue the decomposition (4.3.7) as follows.

$$\begin{aligned} (4.3.7) &= L \exp(\mathfrak{b}')T'' \exp(\mathfrak{g}^{-\mu,\sigma})H \\ &= LT'' \exp(\mathfrak{b}') \exp(\mathfrak{g}^{-\mu,\sigma})H && \text{by Lemma 4.3.3} \\ &= L \exp(\mathfrak{b}') \exp(\mathfrak{g}^{-\mu,\sigma})H \\ &= LBH. \end{aligned}$$

□

Here is a herringbone stitch which we have used for $L\backslash G/H$ in Case I with $i = n, j = 3$.

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & T' \arrow[d, hook] & \\
 & & G' \cdot G'' \arrow[dl, hook] & \\
 & G & & (G')^{\mu} \arrow[ul, hook] \\
L \arrow[ur, hook] & & &
\end{tikzcd}$$

Figure 4.3.1: Herringbone stitch for Case I with $i = n, j = 3$

4.3.3 Decomposition for Case II

In this subsection we deal with the following case:

$$(\Pi')^c = \{\alpha_i\},\ (\Pi'')^c = \{\alpha_j, \alpha_k\}\ (\ i \in \{n-1, n\},\ j \neq k \text{ and } j,k \in \{1, n-1, n\}).$$

Since ξ (see (4.2.7) for the definition of ξ) switches the role of $n-1$ and n, and $\mathrm{L}_{\{\alpha_1,\alpha_n\}^c}$ is conjugate to $\mathrm{L}_{\{\alpha_{n-1},\alpha_n\}^c}$ by an element of G^σ where σ is the complex conjugation (4.2.2), $G = \mathrm{L}_{\Pi'}\, G^\sigma\, \mathrm{L}_{\Pi''}$ holds for $(i,j,k) = (n,1,n)$ if and only if so does for each of the other triples (i,j,k). Thus, we may and do assume $(i,j,k) = (n,1,n)$ without any loss of generality, and put

$$\begin{aligned} L &:= \mathrm{L}_{\{\alpha_n\}^c} (= \mathrm{U}(n)), \\ H &:= \mathrm{L}_{\{\alpha_1,\alpha_n\}^c} (= \mathrm{U}(1) \times \mathrm{U}(n-1)), \end{aligned} \tag{4.3.8}$$

for simplicity. The goal of this subsection is to prove

$$G = L \exp(\mathfrak{b}') D H, \tag{4.3.9}$$

where the subspace $\mathfrak{b}'$ and the subset D are defined by

$$\mathfrak{b}' := \bigoplus_{1 \le i \le [\frac{n}{2}]} \mathbb{R}(\mathrm{E}_{2i-1,2n-2i+1} - \mathrm{E}_{2i,2n-2i+2} - \mathrm{E}_{2n-2i+1,2i-1} + \mathrm{E}_{2n-2i+2,2i}), \tag{4.3.10}$$

$$D := D_1 D_2 \cdots D_{[\frac{n-1}{2}]} \tag{4.3.11}$$

for $D_j := \exp(\mathbb{R}(\mathrm{E}_{2j-1,2j+1} - \mathrm{E}_{2j+1,\,2j-1} - \mathrm{E}_{2n-2j,2n-2j+2} + \mathrm{E}_{2n-2j+2,2n-2j}))$ $(1 \le j \le [\frac{n-1}{2}])$. This subspace $\mathfrak{b}'$ is a maximal abelian subspace of $\mathfrak{g}^{-\mu}$.

As the first step to the goal, we use Proposition 4.3.2 and then obtain

$$G = L \exp(\mathfrak{b}') L. \tag{4.3.12}$$

Second, we consider the centralizer of $\mathfrak{b}'$. We omit details of the proof of the following lemma since it follows from a simple matrix computation.

Lemma 4.3.5. $Z_G(\mathfrak{b}') \supset K := \begin{cases} (\mathrm{SU}(2))^m & (n = 2m), \\ (\mathrm{SU}(2))^m \times \mathrm{U}(1) & (n = 2m+1). \end{cases}$
Here, we realize the subgroup K as block diagonal matrices in G.

Third, we consider the double coset decomposition of L by K and H.

Lemma 4.3.6. $L = KDH$.

Proof. The following proof is due to [Sa3]. Let us identify L/H with $\mathbb{C}P^n$ in the natural way. Here, we note that $D \cdot H/H$ is identified with a subset

$$\left\{ [z_1 : \cdots : z_n] \in \mathbb{C}P^n : z_k \in \mathbb{R}\ (1 \le k \le n) \text{ and } z_{2l} = 0\ \left(1 \le l \le [\frac{n}{2}]\right) \right\}$$

of $\mathbb{C}P^n$. We shall show the equality $K \cdot D \cdot H/H = L/H$ for two cases $n = 2m$ and $n = 2m+1$ separately.

- Case 1: $n = 2m$. Since the SU(2)-action on S^3 is transitive, for any $[z_1 : \cdots : z_{2m}] \in \mathbb{C}P^n$, there exists $g = (g_1, \ldots, g_m) \in K$ such that

$$\begin{aligned} g \cdot [z_1 : \cdots : z_{2m}] =& [g_1 \cdot (z_1 : z_2) : \cdots : g_m \cdot (z_{2m-1} : z_{2m})] \\ =& [(\sqrt{|z_1|^2 + |z_2|^2} : 0) : \cdots : (\sqrt{|z_{2m-1}|^2 + |z_{2m}|^2} : 0)] \\ & \in D \cdot H/H. \end{aligned}$$

  Thus, we obtain $K \cdot D \cdot H/H = L/H$.

- Case 2: $n = 2m + 1$. As similar to the case $n = 2m$, for any $[z_1 : \cdots : z_{2m} : z_{2m+1}] \in \mathbb{C}P^n$, we can find an element $h = (h_1, \ldots, h_m)$ of the commutator subgroup $K_{ss} = [K, K]$ satisfying

$$h \cdot [z_1 : \cdots : z_{2m}] = [(\sqrt{|z_1|^2 + |z_2|^2} : 0) : \cdots : (\sqrt{|z_{2m-1}|^2 + |z_{2m}|^2} : 0)].$$

  We then put $\theta := arg(z_{2m+1})$ and $g := (h, e^{-\sqrt{-1}\theta}) \in K$, and obtain

$$\begin{aligned} & g \cdot [z_1 : \cdots : z_{2m} : z_{2m+1}] = \\ & \quad [(\sqrt{|z_1|^2 + |z_2|^2} : 0) : \cdots : (\sqrt{|z_{2m-1}|^2 + |z_{2m}|^2} : 0) : |z_{2m+1}|] \in D \cdot H/H. \end{aligned}$$

  Hence we have $K \cdot D \cdot H/H = L/H$.

$\square$

We are ready to give a proof of a generalized Cartan decomposition (4.3.9).

Proposition 4.3.7. (*Generalized Cartan decomposition.*) *Let $G = \mathrm{SO}(2n)$ and L, H be as in* (4.3.8)*. We put $B := \exp(\mathfrak{b}')D$ (see* (4.3.10) *and* (4.3.11) *for the definitions of $\mathfrak{b}'$ and D). Then we have $G = LBH$.*

Proof.

$$\begin{aligned} G &= L \exp(\mathfrak{b}') L && \text{by (4.3.12)} \\ &= L \exp(\mathfrak{b}') KDH && \text{by Lemma 4.3.6} \\ &= LK \exp(\mathfrak{b}') DH && \text{by Lemma 4.3.5} \\ &= LBH. \end{aligned}$$

$\square$

Here is a herringbone stitch which we have used for $L \backslash G/H$ in Case II with $(i, j, k) = (n, 1, n)$.

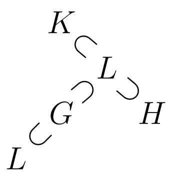


Figure 4.3.2: Herringbone stitch for Case II with $(i, j, k) = (n, 1, n)$

4.3.4 Decomposition for Case Ⅲ

In this subsection we deal with the following case:

$$(\Pi')^c = \{\alpha_i\},\ (\Pi'')^c = \{\alpha_1, \alpha_2\}\ (i = n-1 \text{ or } n).$$

As in the beginning of Subsection 4.3.2, we may and do assume $i = n$ without loss of generality, and put

$$\begin{aligned} L &:= \mathrm{L}_{\{\alpha_n\}^c}(= \mathrm{U}(n)), \\ H &:= \mathrm{L}_{\{\alpha_1,\alpha_2\}^c}(= \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SO}(2n-4)), \end{aligned} \tag{4.3.13}$$

for simplicity. This subsection aims for showing

$$G = L\exp(\mathfrak{b}')\exp(\mathfrak{b}'')H, \tag{4.3.14}$$

where the subspaces $\mathfrak{b}'$ and $\mathfrak{b}''$ are defined by

$$\mathfrak{b}' := \bigoplus_{i=1,2} \mathbb{R}\ (\mathrm{E}_{i,n+i} - \mathrm{E}_{n+i,i} - \mathrm{E}_{n+1-i,2n+1-i} + \mathrm{E}_{2n+1-i,n+1-i}), \tag{4.3.15}$$

$$\begin{aligned} \mathfrak{b}'' := \mathbb{R}&(\mathrm{E}_{1,2} - \mathrm{E}_{2,1} - \mathrm{E}_{2n-1,2n} + \mathrm{E}_{2n,2n-1}) \\ &\oplus \mathbb{R}(\mathrm{E}_{1,2n-1} - \mathrm{E}_{2n-1,1} - \mathrm{E}_{2,2n} + \mathrm{E}_{2n,2}). \end{aligned} \tag{4.3.16}$$

First, we take a symmetric subgroup $(G^{\tau_4})_0 = G'G''$ containing H where $G' := \mathrm{SO}(4) \times I_{2n-4}$ and $G'' := I_4 \times \mathrm{SO}(2n-4) (\subset H)$. In light that $\mathfrak{b}'$ is a maximal abelian subspace of $\mathfrak{g}^{-\tau_4,-\mu}$, we see from Fact 4.3.1 that

$$G = L\exp(\mathfrak{b}')(G'G''). \tag{4.3.17}$$

Next we consider the double coset decomposition of G' by a symmetric subgroup T' defined by $T' := (G')_0^{\tau_1}$. The point here is that T' satisfies $T'G'' = H$. Applying Fact 4.3.1 to $(G', \tau_1|_{G'}, \tau_1|_{G'})$, we have

$$G' = T'\exp(\mathfrak{b}'')T'. \tag{4.3.18}$$

We are ready to give a proof of a generalized Cartan decomposition (4.3.14) by using the herringbone stitch method.

Proposition 4.3.8. (*Generalized Cartan decomposition.*) *Let* $G = \mathrm{SO}(2n)$ *and* L, H *be as in* (4.3.13). *We put* $B := \exp(\mathfrak{b}')\exp(\mathfrak{b}'')$ (*see* (4.3.15) *and* (4.3.16) *for the definitions of* $\mathfrak{b}'$ *and* $\mathfrak{b}''$). *Then we have* $G = LBH$.

Proof.

$$\begin{aligned} G &= L\exp(\mathfrak{b}')(G'G'') && \text{by (4.3.17)} \\ &= L\exp(\mathfrak{b}')(T'\exp(\mathfrak{b}'')T')G'' && \text{by (4.3.18)} \\ &= L\exp(\mathfrak{b}')T'\exp(\mathfrak{b}'')H && \text{by } T'G'' = H. \end{aligned} \tag{4.3.19}$$

We define

$$T'' := \exp\left(\bigoplus_{i=1,2} \mathbb{R}\sqrt{-1}((\mathrm{E}_{i,i} - \mathrm{E}_{2n+1-i,2n+1-i}) - (\mathrm{E}_{n+1-i,n+1-i} - \mathrm{E}_{n+i,n+i}))\right).$$

Then T' and T'' satisfy the following equality:

$$T' \exp(\mathfrak{b}'')H = T'' \exp(\mathfrak{b}'')H,$$

and T'' centralizes $\mathfrak{b}'$. From this, we can continue the decomposition as follows.

$$\begin{aligned}
(4.3.19) &= L \exp(\mathfrak{b}')T'' \exp(\mathfrak{b}'')H \\
&= LT'' \exp(\mathfrak{b}') \exp(\mathfrak{b}'')H \\
&= LBH.
\end{aligned}$$

□

Here is a herringbone stitch which we have used for $L\backslash G/H$ in Case III with $i = n$.

$$\begin{array}{ccccc}
T' & & & & \\
 & \subset & & & \\
 & & G' \cdot G'' & & \\
 & \subset & & \supset & \\
 & G & & & T' \\
\subset & & & & \\
L & & & &
\end{array}$$

Figure 4.3.3: Herringbone stitch for Case III with $i = n$

4.3.5 Decomposition for Case IV

In this subsection we deal with the following case:

$$(\Pi')^c = \{\alpha_1\},\ (\Pi'')^c = \{\alpha_j, \alpha_k\}\ (1 \leq j \leq n \text{ and } k = n-1 \text{ or } n).$$

As in the beginning of Subsection 4.3.2, we may and do assume $k = n$ without loss of generality, and put

$$\begin{aligned}
L &:= \mathrm{L}_{\{\alpha_1\}^c} (= \mathrm{U}(1) \times \mathrm{SO}(2n-2)), \\
H &:= \mathrm{L}_{\{\alpha_j, \alpha_n\}^c} (= \mathrm{U}(j) \times \mathrm{U}(n-j)),
\end{aligned} \tag{4.3.20}$$

for simplicity. The goal of this subsection is to prove

$$G = L \exp(\mathfrak{b}') \exp(\mathfrak{b}'')H, \tag{4.3.21}$$

where the subspaces $\mathfrak{b}'$ and $\mathfrak{b}''$ are defined by

$$\mathfrak{b}' := \bigoplus_{i=1,2} \mathbb{R}(\mathrm{E}_{1,n+i-1} - \mathrm{E}_{n+i-2,1} - \mathrm{E}_{n+2-i,2n} + \mathrm{E}_{2n,n+2-i}), \tag{4.3.22}$$

$$\begin{aligned}
\mathfrak{b}'' := {} & \mathbb{R}(\mathrm{E}_{1,2n+1-j} - \mathrm{E}_{2n+1-j,1} - \mathrm{E}_{j,2n} + \mathrm{E}_{2n,j}) \\
& \oplus \mathbb{R}(\mathrm{E}_{j+1,n+1} - \mathrm{E}_{n+1,j+1} - \mathrm{E}_{n,2n-j} + \mathrm{E}_{2n-j,n}).
\end{aligned} \tag{4.3.23}$$

Then $\mathfrak{b}'$ and $\mathfrak{b}''$ are maximal abelian subspaces of $\mathfrak{g}^{-\tau_1, -\tau_j}$ and $(\mathfrak{g}^{\tau_j})^{-(\tau_1 \tau_{n-1}), -\mu}$ respectively. We apply Fact 4.3.1 to (G, τ_1, τ_j), and then obtain

$$G = L \exp(\mathfrak{b}')(G^{\tau_j})_0. \tag{4.3.24}$$

Next we consider the double coset decomposition of $(G^{\tau_j})_0$ by H and a subgroup L' of $G^{\tau_j} \cap L$ given by

$$
\begin{aligned}
L' : &= I_2 \times \mathrm{SO}(2j-2) \times \mathrm{SO}(2(n-j)-2) \times I_2 \\
&= \left\{ \begin{pmatrix} 1 & & & & & & & 0 \\ & A & & & & & B & \\ & & E & & & F & & \\ & & & 1 & 0 & & & \\ & & & 0 & 1 & & & \\ & & G & & & H & & \\ & C & & & & & D & \\ 0 & & & & & & & 1 \end{pmatrix} : \begin{array}{l} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SO}(2j-2), \\ \\ \begin{pmatrix} E & F \\ G & H \end{pmatrix} \in \mathrm{SO}(2(n-j)-2) \end{array} \right\}.
\end{aligned}
$$

The point here is that L' centralizes $\mathfrak{b}'$. Applying Fact 4.3.1 to $((G^{\tau_j})_0,\ \tau_1\tau_{n-1},\ \mu)$, we get

$$(G^{\tau_j})_0 = (G^{\tau_j,(\tau_1\tau_{n-1})})_0 \exp(\mathfrak{b}'')G^{\tau_j,\mu}. \tag{4.3.25}$$

Further, it is easy to see $(G^{\tau_j,(\tau_1\tau_{n-1})})_0 \exp(\mathfrak{b}'')G^{\tau_j,\mu} = L' \exp(\mathfrak{b}'')\ G^{\tau_j,\mu}$. Thus we have

$$(G^{\tau_j})_0 = L' \exp(\mathfrak{b}'')G^{\tau_j,\mu}. \tag{4.3.26}$$

We are ready to give a proof of a generalized Cartan decomposition (4.3.21) by using the herringbone stitch method.

Proposition 4.3.9. (*Generalized Cartan decomposition.*) *Let* $G = \mathrm{SO}(2n)$ *and* L, H *be as in* (4.3.20), *and put* $B := \exp(\mathfrak{b}')\exp(\mathfrak{b}'')$ (*see* (4.3.22) *and* (4.3.23) *for the definitions of* $\mathfrak{b}'$ *and* $\mathfrak{b}''$). *Then we have* $G = LBH$.

Proof.

$$
\begin{aligned}
G &= L\exp(\mathfrak{b}')(G^{\tau_j})_0 && \text{by (4.3.24)} \\
&= L\exp(\mathfrak{b}')(L'\exp(\mathfrak{b}'')G^{\tau_j,\mu}) && \text{by (4.3.26)} \\
&= L\exp(\mathfrak{b}')(L'\exp(\mathfrak{b}'')H) && \text{by } G^{\tau_j,\mu} = H \\
&= LL'\exp(\mathfrak{b}')\exp(\mathfrak{b}'')H && \text{by } L' \subset Z_G(\mathfrak{b}') \\
&= LBH.
\end{aligned}
$$

□

Here is a herringbone stitch which we have used for $L\backslash G/H$ in Case IV with $k = n$.

$$
\begin{array}{ccccc}
L' & & & & \\
 & \subset & & & \\
 & & (G^{\tau_j})_0 & & \\
 & \supset & & \supset & \\
 & G & & & H \\
\subset & & & & \\
L & & & &
\end{array}
$$

Figure 4.3.4: Herringbone stitch for Case IV with $k = n$

4.3.6 Decomposition for Case V

In this subsection, we deal with the following case for $G = \mathrm{SO}(8)$:

$$(\Pi')^c = \{\alpha_i\},\ (\Pi'')^c = \{\alpha_2, \alpha_j\}\ ((i,j) = (3,4) \text{ or } (4,3)).$$

We may assume $(i,j) = (4,3)$ without any loss of generality since $\xi(\mathrm{L}_{\{\alpha_4\}^c}) = \mathrm{L}_{\{\alpha_3\}^c}$ and $\xi(\mathrm{L}_{\{\alpha_2,\alpha_3\}^c}) = \mathrm{L}_{\{\alpha_2,\alpha_4\}^c}$. For simplicity, we put

$$\begin{aligned} L &:= \mathrm{L}_{\{\alpha_4\}^c} (= \mathrm{U}(4)), \\ H &:= \mathrm{L}_{\{\alpha_2,\alpha_3\}^c} (= \xi(\mathrm{U}(2) \times \mathrm{U}(2))). \end{aligned} \tag{4.3.27}$$

The goal of this subsection is to prove

$$G = L \exp(\mathfrak{a}) \xi(B''B')H, \tag{4.3.28}$$

where the subspace $\mathfrak{a}$ and the subgroups B', B'' are defined by

$$\mathfrak{a} := \mathbb{R}(\mathrm{E}_{1,7} - \mathrm{E}_{2,8} - \mathrm{E}_{7,1} + \mathrm{E}_{8,2}), \tag{4.3.29}$$

$$B' := \exp(\mathbb{R}(\mathrm{E}_{1,4} - \mathrm{E}_{4,1} - \mathrm{E}_{5,8} + \mathrm{E}_{8,5}) \oplus \mathbb{R}(\mathrm{E}_{2,3} - \mathrm{E}_{3,2} - \mathrm{E}_{6,7} + \mathrm{E}_{7,6})), \tag{4.3.30}$$

$$B'' := \exp(\mathbb{R}(\mathrm{E}_{1,3} - \mathrm{E}_{3,1} - \mathrm{E}_{6,8} + \mathrm{E}_{8,6})). \tag{4.3.31}$$

Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{g}^{-\mu,-\mu^\xi}$.

First, we decompose G by using Proposition 4.3.2 as follows.

$$G = L \exp(\mathfrak{a}) \xi(L). \tag{4.3.32}$$

Next, we recall a generalized Cartan decomposition for type A group ([Ko4, Theorem 3.1]). We set $H' := \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1) \subset L$ which is realized as block diagonal matrices and $T := \left\{ \begin{pmatrix} e^{\sqrt{-1}\theta} I_4 & O \\ O & e^{-\sqrt{-1}\theta} I_4 \end{pmatrix} \in L : \theta \in \mathbb{R} \right\}$. Then we have

Lemma 4.3.10. ([Ko4, Theorem 3.1]) $L = (H'T)B''B'\xi(H)$.

Further, we can see that $L = (H'T)B''B'\xi(H) = H'B''B'\xi(H)$ since T is the center of L, and thus we have the following decomposition of $\xi(L)$.

$$\xi(L) = H'\xi(B''B')H. \tag{4.3.33}$$

Here, we note $\xi(H') = H'$.

We are ready to give a proof of a generalized Cartan decomposition (4.3.28).

Proposition 4.3.11. (*Generalized Cartan decomposition.*) *Let* $G = \mathrm{SO}(8)$ *and* L, H *be as in* (4.3.27). *Put* $B := \exp(\mathfrak{a})\xi(B''B')$ (*see* (4.3.29), (4.3.30) *and* (4.3.31) *for the definitions of* $\mathfrak{a}$, B' *and* B''). *Then we have* $G = LBH$.

Proof.

$$\begin{aligned} G &= L \exp(\mathfrak{a}) \xi(L) && \text{by (4.3.32)} \\ &= L \exp(\mathfrak{a})(H'\xi(B''B')H) && \text{by (4.3.33)} \\ &= LH' \exp(\mathfrak{a}) \xi(B''B')H && \text{by } H' \subset Z_G(\mathfrak{a}) \\ &= LBH. \end{aligned}$$

□

Here is a herringbone stitch which we have used for $L\backslash G/H$ in Case V.

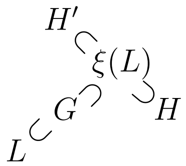


Figure 4.3.5: Herringbone stitch for Case V

4.4 Application of invariant theory

In this section, we prove that (i) implies (ii) in Theorem 4.1.1. The idea of the proof is to use invariants of quivers. Although Lemmas 4.4.1, 4.4.2 and 4.4.3 are parallel to [Ko4, Lemmas 6.1, 6.2 and 6.3] respectively, we give proofs of these lemmas for the sake of completeness. This section could be read independently of Section 4.3 which gives a proof on the opposite implication of (ii) $\Rightarrow$ (i) in Theorem 4.1.1.

4.4.1 Invariants of quivers

Let $\sigma : \mathrm{M}(N, \mathbb{C}) \to \mathrm{M}(N, \mathbb{C})$ be the complex conjugation with respect to $\mathrm{M}(N, \mathbb{R})$.

Lemma 4.4.1. (*c.f.* [Ko4, Lemma 6.1]) *Let $G \subset \mathrm{GL}(N, \mathbb{C})$ be a σ-stable subgroup, $R \in \mathrm{M}(N, \mathbb{R})$, and L a subgroup of G. If there exists $g \in G$ such that*

$$\mathrm{Ad}(L)(\mathrm{Ad}(g)R) \cap \mathrm{M}(N, \mathbb{R}) = \varnothing, \tag{4.4.1}$$

then $G \neq LG^\sigma G_R$. Here $G_R := \{h \in G : hRh^{-1} = R\}$.

Proof. Let us observe that $\mathrm{Ad}(G^\sigma G_R)R = \mathrm{Ad}(G^\sigma)R \subset \mathrm{M}(N, \mathbb{R})$. Then, the condition (4.4.1) implies $\mathrm{Ad}(Lg)R \cap \mathrm{Ad}(G^\sigma G_R)R = \varnothing$, and thus $Lg \cap G^\sigma G_R = \varnothing$. Therefore we have $g \notin LG^\sigma G_R$. $\square$

We return to the case $G = \mathrm{SO}(2n)$. Let $k, r \geq 2$ be integers. We fix a partition $n = n_1 + \cdots + n_k$ of a positive integer n with $n_1, ..., n_{k-1} > 0$ and $n_k \geq 0$, and consider a loop $i_0 \to i_1 \to \cdots \to i_r$ such that

$$i_s \in \begin{cases} \{1, \ldots, 2k-1\} & (n_k \neq 0), \\ \{1, \ldots, k-1, k+1, \ldots, 2k-1\} & (n_k = 0), \end{cases}$$

and $i_0 = i_r,\ i_{s-1} \neq i_s\ (1 \leq s \leq r)$. Correspondingly to this loop, we define a non-linear mapping

$$A_{i_0 \cdots i_r} : \mathrm{M}(2n, \mathbb{C}) \to \begin{cases} \mathrm{M}(n_{i_0}, \mathbb{C}) & (i_0 = i_r \neq k) \\ \mathrm{M}(2n_k, \mathbb{C}) & (i_0 = i_r = k) \end{cases}$$

as follows: Let $P \in \mathrm{M}(2n, \mathbb{C})$, and we write P as $(P_{ij})_{1 \leq i,j \leq 2k-1}$ in the block matrix form corresponding to the partition $2n = n_1 + \cdots + n_{k-1} + 2n_k + n_{k-1} + \cdots + n_1$ of $2n$ such that

$$P_{ij} \in \begin{cases} \mathrm{M}(n_i, n_j; \mathbb{C}) & (i, j \neq k), \\ \mathrm{M}(2n_k, n_j; \mathbb{C}) & (i = k, j \neq k), \\ \mathrm{M}(n_i, 2n_k; \mathbb{C}) & (i \neq k, j = k), \\ \mathrm{M}(2n_k, \mathbb{C}) & (i = j = k), \end{cases} \tag{4.4.2}$$

where $n_{2k-i} := n_i$ $(1 \leq i \leq k)$. We define $\tilde{P}_{ij}$ and $A_{i_0 \cdots i_r}(P)$ by

$$\tilde{P}_{ij} := \begin{cases} P_{ij} & (i+j \leq 2k), \\ J_{n_i} {}^t P_{2k-j,2k-i} J_{n_j} & (i+j > 2k,\ i,j \neq k), \\ J_{2n_k} {}^t P_{2k-j,k} J_{n_j} & (i = k, j > k), \\ J_{n_i} {}^t P_{k,2k-i} J_{2n_k} & (i > k, j = k). \end{cases}$$

$$A_{i_0 \cdots i_r}(P) := \tilde{P}_{i_0 i_1} \tilde{P}_{i_1 i_2} \cdots \tilde{P}_{i_{r-1} i_r}.$$

Then for any $l = (l_1, \ldots, l_{k-1}, l_k) \in L := \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k)$ (see (4.2.3) in Section 4.2 for the realization as matrices), a direct computation shows

$$(\widetilde{\mathrm{Ad}(l)P})_{ij} = l_i \tilde{P}_{ij} l_j^{-1} \tag{4.4.3}$$

where $l_s \in \mathrm{U}(n_s)(1 \leq s \leq k-1)$, $l_k \in \mathrm{SO}(2n_k)$. The equation (4.4.3) leads us to the following lemma (c.f. [Ko4, Lemma 6.2]):

Lemma 4.4.2. *If there exists a loop $i_0 \to i_1 \to \cdots \to i_r$ such that at least one of the coefficients of the characteristic polynomial* $\det(\lambda I_{n_{i_0}} - A_{i_0 \cdots i_r}(P))$ *is not real, then*

$$\mathrm{Ad}(L)P \cap \mathrm{M}(2n, \mathbb{R}) = \varnothing.$$

Proof. From (4.4.3), we can see that the characteristic polynomial of $A_{i_0 \cdots i_r}(P)$ is invariant under the conjugation by L. Therefore if there exists $l \in L$ such that $\mathrm{Ad}(l)P \in \mathrm{M}(2n, \mathbb{R})$ and thus the characteristic polynomial of $A_{i_0 \cdots i_r}(\mathrm{Ad}(l)P)$ is real, then that of $A_{i_0 \cdots i_r}(P)$ is also a real polynomial. By contraposition, our lemma holds. $\square$

By using Lemmas 4.4.1 and 4.4.2, we obtain the next lemma (c.f. [Ko4, Lemma 6.3]):

Lemma 4.4.3. *Let $n = n_1 + \cdots + n_k$ be a partition and $L = \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k)$ the corresponding Levi subgroup of $\mathrm{SO}(2n)$. Let us suppose that R is a block diagonal matrix :*

$$R := \begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_{2k-1} \end{pmatrix},$$

where $R_s, R_{2k-s} \in \mathrm{M}(n_s, \mathbb{R})$ $(1 \leq s \leq k-1)$, and $R_k \in \mathrm{M}(2n_k, \mathbb{R})$ (the last condition makes sense when $n_k \neq 0$).

If there exist $X \in \mathfrak{so}(2n)$ and a loop $i_0 \to \cdots \to i_r$ such that

$$\det(\lambda I_{n_{i_0}} - A_{i_0 \cdots i_r}([X, R])) \notin \mathbb{R}[\lambda],$$

then the multiplication map $L \times G^\sigma \times G_R \to G$ is not surjective. Here, $[X, R] := XR - RX$.

Proof. Let us set $P(\varepsilon) := \mathrm{Ad}(\exp(\varepsilon X))R$. It suffices to show

$$\det(\lambda I_{n_{i_0}} - A_{i_0\cdots i_r}(P(\varepsilon))) \notin \mathbb{R}[\lambda]$$

for some $\varepsilon > 0$. We set $Q := [X, R]$. The matrix $P(\varepsilon)$ depends real analytically on ε, and we have

$$P(\varepsilon) = R + \varepsilon Q + O(\varepsilon^2),$$

as ε tends to 0. In particular, if $i \neq j$ then the (i,j)-block of matrix $P_{ij}(\varepsilon) \in \mathrm{M}(n_i, n_j; \mathbb{C})$ satisfies

$$P_{ij}(\varepsilon) = \varepsilon Q_{ij} + O(\varepsilon^2) \ \text{ as } \varepsilon \text{ tends to } 0.$$

Then, we have

$$\begin{aligned}\det(\lambda I_{n_{i_0}} - A_{i_0\cdots i_r}(P(\varepsilon))) &= \det(\lambda I_{n_{i_0}} - \varepsilon^r \tilde{Q}_{i_0 i_1} \cdots \tilde{Q}_{i_{r-1} i_r} + O(\varepsilon^{r+1})) \\ &= \det(\lambda I_{n_{i_0}} - \varepsilon^r A_{i_0\cdots i_r}(Q) + O(\varepsilon^{r+1})) \\ &= \sum_{s=0}^{n_{i_0}} \lambda^{n_{i_0}-s} \varepsilon^{sr} h_s(\varepsilon), \end{aligned} \tag{4.4.4}$$

where $h_s(\varepsilon)$ $(0 \leq s \leq n_{i_0})$ are real analytic functions of ε such that

$$\det(\lambda I_{n_{i_0}} - A_{i_0\cdots i_r}(Q)) = \sum_{s=0}^{n_{i_0}} \lambda^{n_{i_0}-s} h_s(0).$$

From our assumption, this polynomial has complex coefficients, namely, there exists s such that $h_s(0) \notin \mathbb{R}$. It follows from (4.4.4) that $\det(\lambda I_{n_{i_0}} - A_{i_0\cdots i_r}(P(\varepsilon))) \notin \mathbb{R}[\lambda]$ for any sufficiently small ε. Hence, we have shown the lemma. $\square$

4.4.2 Necessary conditions for $G = LG^\sigma H$

Throughout this subsection, we set $(G, L, H) =$
$(\mathrm{SO}(2n),\ \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k),\ \mathrm{U}(m_1) \times \cdots \mathrm{U}(m_{l-1}) \times \mathrm{SO}(2m_l))$, where $n = n_1 + \cdots + n_k = m_1 + \cdots + m_l$ with $n_i, m_j > 0$ $(1 \leq i \leq k-1,\ 1 \leq j \leq l-1)$ and $n_k, m_l \geq 0$. We give necessary conditions on (L, H) (resp. $(L, \xi(H))$) under which $G = LG^\sigma H$ (resp. $G = LG^\sigma \xi(H)$) holds. We divide the proof into six cases (Propositions 4.4.4–4.4.9).

Proposition 4.4.4. *$G \neq LG^\sigma H$ if one of the following two conditions is satisfied.*

$$k \geq 4,\ m_1 = 1. \tag{4.4.5}$$

$$k \geq 3,\ n_k \neq 0,\ m_1 = 1. \tag{4.4.6}$$

Proposition 4.4.5. *$G \neq LG^\sigma H$ if $n_k, m_l \neq 0,\ n_1, m_1 \geq 2$.*

Proposition 4.4.6. *$G \neq LG^\sigma H$ if $k = 2,\ \ n_1 \geq 4,\ \ n_2 \geq 2,\ m_l = 0$.*

Proposition 4.4.7. *$G \neq LG^\sigma H$ if $k = 3,\ \max\{n_1, n_2\} \geq 2,\ n_3 \neq 0,\ m_l = 0$.*

Proposition 4.4.8. *$G \neq LG^\sigma H$ if $k = 3,\ n_1, n_2 \geq 2,\ n_3 = m_l = 0$.*

Proposition 4.4.9. *$G \neq LG^\sigma \xi(H)$ if $n \geq 5,\ k = 3,\ n_1, n_2 \geq 2,\ n_k = m_l = 0$.*

Proof of Proposition 4.4.4. We note that the following two inclusive relations reduce a proof of Proposition 4.4.4 to the case $k = 3,\ l = 2,\ n_3 \neq 0$ and $m_1 = 1$:

$$L \subset \begin{cases} \mathrm{U}(n_1) \times \mathrm{U}(n_2 + \cdots + n_{k-2}) \times \mathrm{SO}(2(n_{k-1} + n_k)) & (k \geq 4), \\ \mathrm{U}(n_1) \times \mathrm{U}(n_2 + \cdots + n_{k-1}) \times \mathrm{SO}(2n_k) & (k \geq 3, n_k \neq 0), \end{cases}$$
$$H \subset \mathrm{U}(1) \times \mathrm{SO}(2(m_2 + \cdots + m_l)).$$

We shall show that $G \neq LG^\sigma H$ if $k = 3,\ l = 2,\ n_3 \neq 0$ and $m_1 = 1$. Under this condition, (G, L, H) takes the form:

$$(G, L, H) = (\mathrm{SO}(2n), \mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \mathrm{SO}(n_3), \mathrm{U}(1) \times \mathrm{SO}(2n-2)).$$

Let $1 \to 2 \to 5 \to 3 \to 1$ be a loop and define a diagonal matrix R by $R := \mathrm{diag}(1, 0, \ldots, 0, -1)$. Then, G_R coincides with H. We fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \leq i,j \leq 5} \in \mathfrak{so}(2n)$ in the block matrix form corresponding to the partition $2n = n_1 + n_2 + 2n_3 + n_2 + n_1$ as (4.4.2):

$$X_{12} := \begin{pmatrix} & -u \\ O & \end{pmatrix} \in \mathrm{M}(n_1, n_2; \mathbb{C}),\ X_{14} := \begin{pmatrix} -1 & \\ & O \end{pmatrix} \in \mathrm{M}(n_1, n_2; \mathbb{C}),$$

$$X_{31} := \begin{pmatrix} 1 & \\ & O \\ 1 & \end{pmatrix} \in \mathrm{M}(2n_3, n_1; \mathbb{C}).$$

We define the block entries $X_{11},\ X_{15},\ X_{22},\ X_{23},\ X_{24},\ X_{32},\ X_{33},\ X_{34},\ X_{42},\ X_{43},\ X_{44},\ X_{51}$ and X_{55} to be zero matrices. The remaining block entries are automatically determined by the definition (4.2.1) of $G = \mathrm{SO}(2n)$. Then, $Q := [X, R]$ has the following block entries.

$$Q_{12} = \begin{pmatrix} & u \\ O & \end{pmatrix} \in \mathrm{M}(n_1, n_2; \mathbb{C}),\ Q_{14} = \begin{pmatrix} 1 & \\ & O \end{pmatrix} \in \mathrm{M}(n_1, n_2; \mathbb{C}),$$

$$Q_{31} = \begin{pmatrix} 1 & \\ & O \\ 1 & \end{pmatrix} \in \mathrm{M}(2n_3, n_1; \mathbb{C}).$$

By a simple matrix computation, we have

$$A_{12531}(Q) = Q_{12} J_{n_2} {}^t Q_{14} J_{n_1} J_{n_1} {}^t Q_{31} J_{2n_3} Q_{31} = \begin{pmatrix} 2u & \\ & O \end{pmatrix},$$

and thus the characteristic polynomial $\det(\lambda I_{n_1} - A_{12531}(Q)) = \lambda^{n_1} - 2u\lambda^{n_1 - 1}$ is not defined over $\mathbb{R}$ if u is not real. By using Lemma 4.4.3, we obtain $G \neq LG^\sigma H$. $\square$

Proof of Proposition 4.4.5. We can reduce a proof of Proposition 4.4.5 to the case $k = l = 2,\ m_1 \geq n_1 \geq 2$ and $n_2, m_2 \neq 0$ because the following two inclusive relations hold:

- $\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k)$ is contained in
$$\begin{cases} \mathrm{U}(n_1+n_2) \times \mathrm{SO}(2(n_3+\cdots+n_k)) & (k \geq 4), \\ \mathrm{U}(n_1) \times \mathrm{SO}(2(n_2+\cdots+n_k)) & (n_k \neq 0,\ n_1 \geq 2), \end{cases}$$
- $\mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_{l-1}) \times \mathrm{SO}(2m_l) \subset \mathrm{U}(m_1) \times \mathrm{SO}(2(m_2+\cdots+m_l))$.

We shall show $G \neq LG^\sigma H$ in the case $k = l = 2$, $m_1 \geq n_1 \geq 2$ and $n_2, m_2 \neq 0$. Let $1 \to 2 \to 3 \to 1$ be a loop and define a diagonal matrix R by

$$R := \mathrm{diag}(-1, \overbrace{1, \ldots, 1}^{n_1-2}, -1, \overbrace{1, \ldots, 1}^{m_1-n_1}, 0, \ldots, 0, \overbrace{-1, \ldots, -1}^{m_1-n_1}, 1, \overbrace{-1, \ldots, -1}^{n_1-2}, 1).$$

Then, G_R is conjugate to H by an element of G^σ. We fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \leq i,j \leq 3} \in \mathfrak{so}(2n)$ in the block matrix form corresponding to the partition $2n = n_1 + 2n_2 + n_1$ as (4.4.2):

$$X_{12} := \begin{pmatrix} & u\ 0 & \\ O & & O \\ & 0\ 1 & \end{pmatrix} \in \mathrm{M}(n_1, 2n_2; \mathbb{C}),\ X_{31} := \begin{pmatrix} 1 & & \\ & O & \\ & & -1 \end{pmatrix} \in \mathrm{M}(n_1, \mathbb{C}).$$

We define the block entries X_{11}, X_{22} and X_{33} to be zero matrices. The remaining block entries of X are determined automatically by (4.2.1). Then $Q := [X, R]$ has the following block entries.

$$Q_{12} = \begin{pmatrix} & u\ 0 & \\ O & & O \\ & 0\ 1 & \end{pmatrix},\ Q_{31} = \begin{pmatrix} -2 & & \\ & O & \\ & & 2 \end{pmatrix}.$$

A simple matrix computation shows

$$A_{1231}(Q) = Q_{12} J_{2n_2} {}^t Q_{12} J_{n_1} Q_{31} = \begin{pmatrix} -2u & & \\ & O & \\ & & 2u \end{pmatrix},$$

and we find that $\det(\lambda I_{n_1} - A_{1231}(Q)) = \lambda^{n_1} - 4u^2 \lambda^{n_1-2} \notin \mathbb{R}[\lambda]$ if $u^2 \notin \mathbb{R}$. By Lemma 4.4.3, we have proved $G \neq LG^\sigma H$. $\square$

Proof of Proposition 4.4.6. Clearly H is contained in $\mathrm{U}(n)$ under the condition of Proposition 4.4.6. Hence, it is enough to prove the following:

$$G \neq LG^\sigma H \quad \text{if} \quad k = l = 2,\ n_1 \geq 4,\ n_2 \geq 2,\ m_2 = 0. \tag{4.4.7}$$

Let $1 \to 2 \to 3 \to 1$ be a loop and define a diagonal matrix R as follows.

$$R := \mathrm{diag}(\overbrace{1, \ldots, 1}^{n_1-2}, -1, -1, -1, -1, \overbrace{1, \ldots, 1}^{n_2-2}, \overbrace{-1, \ldots, -1}^{n_2-2}, 1, 1, 1, 1, \overbrace{-1, \ldots, -1}^{n_1-2}).$$

Then G_R is conjugate to H by an element of G^σ. Let us fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \leq i,j \leq 3} \in \mathfrak{so}(2n)$ in the block matrix form corresponding to the partition $2n = n_1 +$

$2n_2 + n_1$ as (4.4.2):

$$X_{12} := \begin{pmatrix} u & & & & \\ & 1 & & & \\ & & O & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \ X_{31} := \begin{pmatrix} & & & -1 & \\ & & & & 1 \\ & & O & & \\ -1 & & & & \\ & 1 & & & \end{pmatrix},$$

where $X_{12} \in \mathrm{M}(n_1, 2n_2; \mathbb{C})$, $X_{31} \in \mathrm{M}(n_1, \mathbb{C})$. We define the block entries X_{11}, X_{22} and X_{33} to be zero matrices. The remaining block entries of X are automatically determined by (4.2.1). Then $Q := [X, R]$ has the following block entries:

$$Q_{12} = \begin{pmatrix} -2u & & & & \\ & -2 & & & \\ & & O & & \\ & & & 2 & \\ & & & & 2 \end{pmatrix}, \ Q_{31} = \begin{pmatrix} & & & 2 & \\ & & & & -2 \\ & & O & & \\ -2 & & & & \\ & 2 & & & \end{pmatrix}.$$

By a simple matrix computation, we have

$$A_{1231}(Q) = Q_{12} J_{2n_2} {}^t Q_{12} J_{n_1} Q_{31} = 8 \begin{pmatrix} & & & -u & \\ & & & & 1 \\ & & O & & \\ 1 & & & & \\ & -u & & & \end{pmatrix},$$

and thus $\det(\lambda I_{n_1} - A_{1231}(Q)) = \lambda^{n_1-4}(\lambda^2 + 64u)^2 \notin \mathbb{R}[\lambda]$ if $u \notin \mathbb{R}$. From Lemma 4.4.3, we obtain $G \neq LG^\sigma H$. $\square$

Proof of Proposition 4.4.7. For Proposition 4.4.7, it is enough to show that

$$G \neq LG^\sigma H \quad \text{if} \quad k = 3,\ l = 2,\ n_2 \geq 2,\ n_3 \neq 0,\ m_2 = 0. \tag{4.4.8}$$

Under this condition, (G, L, H) takes the form:

$$(G, L, H) = (\mathrm{SO}(2n), \mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \mathrm{SO}(2n_3), \mathrm{U}(n)) \ (n_2 \geq 2).$$

Let $1 \to 2 \to 3 \to 4 \to 1$ be a loop and define a diagonal matrix R as follows.

$$R := \mathrm{diag}(\overbrace{1, \ldots, 1}^{n_1+n_2-1}, -1, -1, \overbrace{1, \ldots, 1}^{n_3-1}, \overbrace{-1, \ldots, -1}^{n_3-1}, 1, 1, \overbrace{-1, \ldots, -1}^{n_1+n_2-1}).$$

We note that G_R is conjugate to H by an element of G^σ. Let us fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \leq i,j \leq 5} \in \mathfrak{so}(2n)$ in the block matrix form corresponding to the partition $2n = n_1 + n_2 + 2n_3 + n_2 + n_1$ as (4.4.2):

$X_{12} := \begin{pmatrix} & 1 \\ O & \end{pmatrix} \in \mathrm{M}(n_1, n_2; \mathbb{C})$, $X_{23} := \begin{pmatrix} u & & \\ & O & \\ & & 1 \end{pmatrix} \in \mathrm{M}(n_2, 2n_3; \mathbb{C})$, $X_{41} := \begin{pmatrix} & O \\ -1 & \end{pmatrix} \in \mathrm{M}(n_2, n_1; \mathbb{C})$.

We define the block entries X_{11}, X_{13}, X_{15}, X_{22}, X_{24}, X_{31}, X_{33}, X_{35}, X_{42}, X_{44}, X_{51}, X_{53} and

X_{55} to be zero matrices. The remaining block entries of X are automatically determined by 4.2.1. Then, $Q := [X, R]$ has the following block entries:

$$Q_{12} = \begin{pmatrix} & -2 \\ O & \end{pmatrix}, \ Q_{23} = \begin{pmatrix} -2u & & \\ & O & \\ & & 2 \end{pmatrix}, \ Q_{41} = \begin{pmatrix} & O \\ -2 & \end{pmatrix}.$$

A simple matrix computation shows

$$A_{12341}(Q) = Q_{12}Q_{23}J_{2n_3}{}^t\!Q_{23}J_{n_2}Q_{41} = \begin{pmatrix} -16u & \\ & O \end{pmatrix},$$

and thus we obtain $\det(\lambda I_{n_1} - A_{12341}(Q)) = \lambda^{n_1} + 16u\lambda^{n_1-1} \notin \mathbb{R}[\lambda]$ if $u \notin \mathbb{R}$. From Lemma 4.4.3, we have proved $G \neq LG^\sigma H$. □

Proof of Proposition 4.4.8. Under the condition of Proposition 4.4.8, H is contained in $\mathrm{U}(n)$. Hence it is enough to show the following:

$$G \neq LG^\sigma H \text{ if } k = 3,\ l = 2,\ n_1, n_2 \geq 2,\ n_3 = m_2 = 0.$$

Let $1 \to 4 \to 2 \to 5 \to 1$ be a loop and define R by

$$R := \mathrm{diag}(\overbrace{1, \ldots, 1}^{n}, \overbrace{-1, \ldots, -1}^{n}).$$

Then, $G_R = H$. Let us fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \leq i,j \leq 5} \in \mathfrak{so}(2n)$ in the block matrix form corresponding to the partition $2n = n_1 + n_2 + 2n_3 + n_2 + n_1$ as (4.4.2):

$$X_{14} := \begin{pmatrix} -u & & -1 \\ & O & \\ -1 & & -1 \end{pmatrix} \in \mathrm{M}(n_1, n_2; \mathbb{C}),\ X_{42} := \begin{pmatrix} -1 & & \\ & O & \\ & & 1 \end{pmatrix} \in \mathrm{M}(n_2, \mathbb{C}),$$

$$X_{51} := \begin{pmatrix} -1 & & \\ & O & \\ & & 1 \end{pmatrix} \in \mathrm{M}(n_1, \mathbb{C}).$$

We define X_{11}, X_{12}, X_{21}, X_{22}, X_{44}, X_{45}, X_{54} and X_{55} to be zero matrices. The remaining block entries of X are automatically determined by (4.2.1). Here we note that none of the block entries X_{13}, X_{23}, X_{31}, X_{33}, X_{34}, X_{35}, X_{43} and X_{53} exists since $n_3 = 0$. Then $Q := [X, R]$ has the following block entries:

$$Q_{14} = \begin{pmatrix} 2u & & 2 \\ & O & \\ 2 & & 2 \end{pmatrix}, \ Q_{42} = \begin{pmatrix} -2 & & \\ & O & \\ & & 2 \end{pmatrix}, \ Q_{51} = \begin{pmatrix} -2 & & \\ & O & \\ & & 2 \end{pmatrix}.$$

A simple matrix computation shows

$$A_{14251}(Q) = Q_{14}Q_{42}J_{m_2}{}^t\!Q_{14}J_{m_1}Q_{51} = \begin{pmatrix} 16(u-1) & & \\ & O & \\ & & 16(u-1) \end{pmatrix},$$

and thus, $\det(\lambda I_{n_1} - A_{14251}(Q)) = \lambda^{n_1-2}(\lambda - 16(u-1))^2 \notin \mathbb{R}[\lambda]$ if $u \notin \mathbb{R}$. From Lemma 4.4.3, we get $G \neq LG^\sigma H$. □

Proof of Proposition 4.4.9. We may assume $n_2 \geq n_1$. It suffices to show that $G \neq LG^\sigma\xi(H)$ if $k=3,\ l=2,\ n_1 \geq 2,\ n_2 \geq 3$ and $n_3 = m_2 = 0$. Let $1 \to 4 \to 2 \to 5 \to 1$ be a loop and define R by $R := \operatorname{diag}(\overbrace{1,\dots,1}^{n-1}, -1, 1, \overbrace{-1,\dots,-1}^{n-1})$. Then, $G_R = \xi(H)$. Let us fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1\leq i,j\leq 5} \in \mathfrak{so}(2n)$ in the block matrix form corresponding to the partition $2n = n_1 + n_2 + 2n_3 + n_2 + n_1$ as (4.4.2):

$$X_{14} := \begin{pmatrix} 0 & -u & & -1 \\ & & O & \\ 0 & -1 & & -1 \end{pmatrix} \in \mathrm{M}(n_1, n_2; \mathbb{C}),$$

$$X_{42} := \begin{pmatrix} 0 & & 0\,0 \\ -1 & & 0\,0 \\ & O & \\ 0 & & 1\,0 \end{pmatrix} \in \mathrm{M}(n_2, \mathbb{C}),\ X_{51} := \begin{pmatrix} -1 & & \\ & O & \\ & & 1 \end{pmatrix} \in \mathrm{M}(n_1, \mathbb{C}).$$

The remaining block entries of X are defined in the same way as in the proof of Proposition 4.4.8. Then $Q := [X, R]$ has the following block entries.

$$Q_{14} = \begin{pmatrix} 0 & 2u & & 2 \\ & & O & \\ 0 & 2 & & 2 \end{pmatrix},\ Q_{42} = \begin{pmatrix} 0 & & 0\,0 \\ -2 & & 0\,0 \\ & O & \\ 0 & & 2\,0 \end{pmatrix},\ Q_{51} = \begin{pmatrix} -2 & & \\ & O & \\ & & 2 \end{pmatrix}.$$

By a simple matrix computation, we have

$$A_{14251}(Q) = Q_{14}Q_{42}J_{m_2}{}^tQ_{14}J_{m_1}Q_{51} = \begin{pmatrix} 16(u-1) & & \\ & O & \\ & & 16(u-1) \end{pmatrix},$$

and find that $\det(\lambda I_{n_1} - A_{14251}(Q)) = \lambda^{n_1-2}(\lambda - 16(u-1))^2 \notin \mathbb{R}[\lambda]$ if $u \notin \mathbb{R}$. From Lemma 4.4.3, we have proved $G \neq LG^\sigma\xi(H)$. □

4.4.3 Completion of the proof of Theorem 4.1.1

We complete the proof of the implication (i) ⇒ (ii) in Theorem 4.1.1 (Proposition 4.4.10) by using Propositions 4.4.4–4.4.9. For a given partition $n = n_1 + \cdots + n_k$ with $n_1, \dots, n_{k-1} > 0$ and $n_k \geq 0$, we have a Levi subgroup $\mathrm{L}_{\Pi'} = \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k)$ of $\mathrm{SO}(2n)$, which is associated to the subset

$$\Pi' := \Pi \setminus \left\{ \alpha_i \in \Pi : i = \sum_{s=1}^{j} n_s,\ 1 \leq j \leq k-1 \right\}$$

of the set of simple roots Π (see Diagram 4.1.1 for the label of the Dynkin diagram).

Proposition 4.4.10. *Let G be the special orthogonal group $\mathrm{SO}(2n)$ $(n \geq 4)$, σ a Chevalley–Weyl involution, Π',Π'' subsets of the set of simple roots Π, and $\mathrm{L}_{\Pi'}$, $\mathrm{L}_{\Pi''}$ the corresponding Levi subgroups. Then we have*

$$G \neq \mathrm{L}_{\Pi'}\, G^\sigma\, \mathrm{L}_{\Pi''} \tag{4.4.9}$$

if one of the following conditions up to switch of Π' *and* Π'' *is satisfied* $(1 \le i, j, k \le n)$:

(I). *Either* $(\Pi')^c$ *or* $(\Pi'')^c$ *contains more than two elements.*
(II). *Both* $(\Pi')^c$ *and* $(\Pi'')^c$ *contain two elements.*
(III). *Both* $(\Pi')^c$ *and* $(\Pi'')^c$ *contain some simple root other than* $\alpha_1, \alpha_{n-1}, \alpha_n$.
(IV). $\#(\Pi')^c = 2,\ (\Pi'')^c = \{\alpha_i\}, \textit{and } i \notin \{1, n-1, n\}$.
(V). $\#(\Pi')^c = 2,\ (\Pi'')^c = \{\alpha_1\},\ \textit{and } (\Pi')^c \textit{contains neither } \alpha_{n-1} \textit{ nor } \alpha_n$.
(VI). $(\Pi')^c = \{\alpha_i\},\ (\Pi'')^c = \{\alpha_j\},\ i \notin \{1, 2, 3, n-1, n\},\ j \in \{n-1, n\}$.
(VII). $(n \ge 5)\ (\Pi')^c = \{\alpha_i, \alpha_j\},\ (\Pi'')^c = \{\alpha_k\},\ i \ne j,\ k \in \{n-1, n\},$
$\textit{and } (i, j) \ne (1, 2), (1, n-1), (1, n), (n-1, n)$.
(VIII). $(n = 4)\ (\Pi')^c = \{\alpha_i, \alpha_2\},\ (\Pi'')^c = \{\alpha_i\},\ i \in \{3, 4\}$.

Proof. We note the following:

1. The role of $\mathrm{L}_{\Pi'}$ and $\mathrm{L}_{\Pi''}$ is symmetric.

2. $G \ne \mathrm{L}_{\Pi'}\, G^\sigma\, \mathrm{L}_{\Pi''}$ holds if and only if $G \ne \xi(\mathrm{L}_{\Pi'}) G^\sigma \xi(\mathrm{L}_{\Pi''})$ does.

First, we can see that (I) implies (4.4.9) by combining (4.4.5) of Proposition 4.4.4 with Propositions 4.4.5 and 4.4.7. Second, Proposition 4.4.5 implies that (4.4.9) holds under each of the conditions (II), (III) and (IV). Third, we can see the condition (V) implies (4.4.9) by using (4.4.6) of Proposition 4.4.4. Fourth, we can also see that the condition (VI) implies (4.4.9) by Proposition 4.4.6. Fifth, by combining Proposition 4.4.7 with Propositions 4.4.8 and 4.4.9, we can see that (4.4.9) holds under the condition (VII). Finally, it follows from Proposition 4.4.8 that the condition (VIII) implies (4.4.9). □

4.5 Application to representation theory

In this section, we shall see our generalized Cartan decomposition leads to three multiplicity-free representations by using the framework of visible actions ("triunity" à la [Ko1]). The concept of (strongly) visible actions on complex manifolds was introduced by T. Kobayashi. Let us recall the definition ([Ko2]).

Definition 4.5.1. We say a biholomorphic action of a Lie group G on a complex manifold D is *strongly visible* if the following two conditions are satisfied:

1. There exists a real submanifold S such that (we call S a "slice")

$$D' := G \cdot S \text{ is an open subset of } D.$$

2. There exists an antiholomorphic diffeomorphism σ of D' such that

$$\sigma|_S = \mathrm{id}_S,$$
$$\sigma(G \cdot x) = G \cdot x \text{ for any } x \in D'.$$

Definition 4.5.2. In the above setting, we say the action of G on D is S-visible. This terminology will be used also if S is just a subset of D.

Let G be a compact Lie group and L, H its Levi subgroups. Then G/L, G/H and $(G \times G)/(L \times H)$ are complex manifolds. If the triple (G, L, H) satisfies $G = LG^{\sigma}H$, the following three group-actions are all strongly visible:

$$L \curvearrowright G/H$$
$$H \curvearrowright G/L$$
$$\mathrm{diag}(G) \curvearrowright (G \times G)/(L \times H)$$

The following theorem ([Ko3]) leads us to multiplicity-free representations:

Theorem 4.5.3. *Let G be a Lie group and $\mathcal{V}$ a G-equivalent Hermitian holomorphic vector bundle on a connected complex manifold D. If the following three conditions from* (1) *to* (3) *are satisfied, then any unitary representation that can be embedded in the vector space $\mathcal{O}(D, \mathcal{V})$ of holomorphic sections of $\mathcal{V}$ decomposes multiplicity-freely:*

1. *The action of G on D is S-visible. That is, there exists a subset $S \subset D$ satisfying the conditions given in Definition* 4.5.1. *Further, there exists an automorphism $\hat{\sigma}$ of G such that $\sigma(g \cdot x) = \hat{\sigma}(g) \cdot \sigma(x)$ for any $g \in G$ and $x \in D'$.*
2. *For any $x \in S$, the fiber $\mathcal{V}_x$ at x decomposes as the multiplicity free sum of irreducible unitary representations of the isotropy subgroup G_x. Let $\mathcal{V}_x = \bigoplus_{1 \leq i \leq n(x)} \mathcal{V}_x^{(i)}$ denote the irreducible decomposition of $\mathcal{V}_x$.*
3. *σ lifts to an antiholomorphic automorphism $\tilde{\sigma}$ of $\mathcal{V}$ and satisfies $\tilde{\sigma}(\mathcal{V}_x^{(i)}) = \mathcal{V}_x^{(i)}$ $(1 \leq i \leq n(x))$ for each $x \in S$.*

Although our application is limited to finite dimensional representations, it is noteworthy that this theorem works for both compact and non-compact complex manifolds, for both finite and infinite dimensional representations, and for both discrete and continuous spectra. See, for example, [Ko1] and [Ko6]. [Ko1] deals with finite dimensional representations whereas the latter deals with infinite dimensional representations (not necessarily highest weight modules).

Now we return to the case where G is a connected compact Lie group of type D_n. The fundamental weights $\omega_1, ..., \omega_n$ with respect to the set of simple roots $\alpha_1, ..., \alpha_n$ are given as follows (see Diagram 1.1 for the label of the Dynkin diagram).

$$\begin{aligned}
\omega_i &= \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{l-2}), \\
&\quad + \frac{1}{2}i(\alpha_{n-1} + \alpha_n) \ \ (1 \leq i < n-1), \\
\omega_{n-1} &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \frac{1}{2}n\alpha_{n-1} + \frac{1}{2}(n-2)\alpha_n), \\
\omega_n &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \frac{1}{2}(n-2)\alpha_{n-1} + \frac{1}{2}n\alpha_n).
\end{aligned}$$

By using the Borel–Weil theory together with Theorem 4.5.3 and our generalized Cartan decompositions, we obtain the following Corollaries 4.5.4 and 4.5.5.

Corollary 4.5.4. (*Corollary of Theorem* 4.1.1.) *If the pair (L, λ) is an entry in the Table 4.5.1 or 4.5.2, then the restriction $\pi_\lambda|_L$ of the irreducible representation π_λ of G with highest weight λ to L decomposes multiplicity-freely.*

Table 4.5.1: Multiplicity-free restriction

Levi subgroup L	highest weight λ	Levi subgroup L	highest weight λ
$\mathrm{L}_{\{\alpha_l\}^c}$	$a\omega_1$, $a\omega_2$, $a\omega_3$, $a\omega_{n-2}$, $a\omega_l$	$\mathrm{L}_{\{\alpha_n\}^c}$, $\mathrm{L}_{\{\alpha_{n-1}\}^c}$	$a\omega_1 + b\omega_l$, $a\omega_1 + b\omega_2$, $a\omega_{n-2} + b\omega_l$
		$\mathrm{L}_{\{\alpha_1\}^c}$	$a\omega_i + b\omega_l$
$\mathrm{L}_{\{\alpha_1\}^c}$	$a\omega_i$	$\mathrm{L}_{\{\alpha_1,\alpha_2\}^c}$, $\mathrm{L}_{\{\alpha_1,\alpha_n\}^c}$, $\mathrm{L}_{\{\alpha_1,\alpha_{n-1}\}^c}$, $\mathrm{L}_{\{\alpha_{n-1},\alpha_n\}^c}$	$a\omega_l$
$\mathrm{L}_{\{\alpha_2\}^c}$, $\mathrm{L}_{\{\alpha_3\}^c}$	$a\omega_l$		
$\mathrm{L}_{\{\alpha_j\}^c}$	$a\omega_1$	$\mathrm{L}_{\{\alpha_j,\alpha_n\}^c}$, $\mathrm{L}_{\{\alpha_j,\alpha_{n-1}\}^c}$	$a\omega_1$

Table 4.5.2: Multiplicity-free restriction with $n = 4$

L	λ
$\mathrm{L}_{\{\alpha_i\}^c}$	$a\omega_2 + b\omega_j$
$\mathrm{L}_{\{\alpha_2,\alpha_i\}^c}$	$a\omega_j$

Here, $l = n-1$ or n and i, j, a, b are integers satisfying $1 \leq i, j \leq n$ and $0 \leq a, b$. The following Table 4.5.2 is only for $n = 4$ ($(i, j) = (3, 4)$ or $(4, 3)$) :

Corollary 4.5.5. (*Corollary of Theorem* 4.1.1.) *The tensor product representation* $\pi_\lambda \otimes \pi_\mu$ *of any two irreducible representations* π_λ, π_μ *of* G *with highest weights* (λ, μ) *listed in the below* Table *4.5.3 decomposes as a multiplicity-free sum of irreducible representations of* G.

Table 4.5.3: Multiplicity-free tensor product

pair of highest weights (λ, μ)	n	pair of highest weights (λ, μ)
$(a\omega_k,\ b\omega_1)$, $(a\omega_k,\ b\omega_2)$, $(a\omega_k,\ b\omega_3)$, $(a\omega_k,\ b\omega_{n-2})$, $(a\omega_k,\ b\omega_l)$, $(a\omega_1,\ b\omega_i)$	$n \geq 4$	$(a\omega_k,\ b\omega_{n-2} + c\omega_l)$, $(a\omega_k,\ b\omega_1 + c\omega_l)$, $(a\omega_k,\ b\omega_1 + c\omega_2)$, $(a\omega_1,\ b\omega_i + c\omega_l)$
	$n = 4$	$(a\omega_4,\ b\omega_2 + c\omega_3)$, $(a\omega_3,\ b\omega_2 + c\omega_4)$

Here, $k, l \in \{n-1, n\}$, $1 \leq i \leq n$ and a, b, c are arbitrary non-negative integers. We note that the condition (2) of Theorem 4.5.3 is automatically satisfied since the fiber of a holomorphic vector bundle is one-dimensional in the setting of the Borel–Weil Theory. We also note that we can take the complex conjugation as σ in Theorem 4.5.3.

Remark 4.5.6. P. Littelmann ([Li2]) classified all the pairs of maximal parabolic subgroups $(P_\omega, P_{\omega'})$ of any simple Lie group G over any algebraically closed field such that the corresponding tensor products $n\omega \otimes m\omega'$ (n and m are arbitrary non-negative integers) decomposes multiplicity-freely where ω and ω' are fundamental weights. Moreover, he found the branching rules of $n\omega \otimes m\omega'$ and the restriction of $n\omega$ to the maximal Levi subgroup $L_{\omega'}$ of $P_{\omega'}$ for any pair (ω, ω') that admits a G-spherical action on $G/P_\omega \times G/P_{\omega'}$ by using the generalized Littlewood–Richardson rule ([Li1]).

Remark 4.5.7. J. R. Stembridge ([St2]) gave a complete list of pairs of highest weights with the corresponding tensor product representation multiplicity-free. His method is combinatorial. He also classified multiplicity-free restrictions to Levi subalgebras for all complex simple Lie algebras. Our approach has given a geometric proof of a part of his work based on generalized Cartan decompositions.

We have listed an application of Theorem 4.5.3 only for the line bundle case. See Chapter 6 for an application to the vector bundle case.

Chapter 5

Visible actions on flag varieties of exceptional groups and a generalization of the Cartan decomposition

5.1 Introduction for Chapter 5

We give a classification of a pair of Levi subgroups (L, H) of any connected compact exceptional Lie group G such that $G = LG^{\sigma}H$ holds where σ is a Chevalley–Weyl involution of G. This can be interpreted as a generalization of the Cartan decomposition to the non-symmetric setting. (We refer the reader to [Fl2, He1, Ho, Ko4, Ma3] and references therein for some aspects of the Cartan decomposition from geometric and group theoretic viewpoints.) The motivation for considering such kind of decomposition comes from the notion of *visible action* on complex manifolds, which was introduced by T. Kobayashi. It is a geometric condition for the propagation theorem of the multiplicity-freeness property, and classification theory of (strongly) visible action has been recently made in the linear case [Sa1], symmetric case [Ko5] and some other non-symmetric cases [Sa2].

A generalization of the Cartan decomposition for symmetric pairs has been used in various contexts including analysis on symmetric spaces, however, there was no analogous result for non-symmetric cases before Kobayashi's paper [Ko4]. Motivated by visible actions on flag varieties of type A, he introduced a generalization of the Cartan decomposition for the unitary group $\mathrm{U}(n)$ taking the form:

$$G = LBH,$$

where B is a subset of the orthogonal group $\mathrm{O}(n)$. He completely classified a pair of Levi subgroups (L, H) satisfying $\mathrm{U}(n) = L\,\mathrm{O}(n)H$, and gave a *slice* B explicitly for each of such pairs (L, H) by the herringbone stitch method [Ko4].

More generally, we consider the following problem: Let G be a connected compact Lie group, T a maximal torus, and σ a Chevalley–Weyl involution of G with respect to T. (We

The contents of this section are taken from [Ta5].

recall that an involutive automorphism σ of a connected compact Lie group G is said to be a Chevalley–Weyl involution if there exists a maximal torus T of G such that $\sigma(t) = t^{-1}$ for any $t \in T$ [Wo].)

1) Classify all the pairs of Levi subgroups L and H with respect to $\mathfrak{t}$ such that the multiplication map $\psi : L \times G^\sigma \times H \to G$ is surjective.

2) Find a "good" representative $B \subset G^\sigma$ such that $G = LBH$ in the case ψ is surjective.

Here $G^\sigma = \{g \in G : \sigma(g) = g\}$. We call such a decomposition $G = LBH$ a *generalized Cartan decomposition*. We note that the role of H and L is symmetric. The surjectivity of ψ implies that the subgroup L acts on G/H in a (strongly) visible fashion (see Definition 5.6.1). At the same time the H-action on G/L, and the diagonal G-action on $(G\times G)/(L\times H)$ are strongly visible. Then the propagation theorem of multiplicity-freeness property ([Ko3, Theorem 4.3]) leads us to three multiplicity-free theorems (*triunity* à la [Ko1]):

$$\begin{array}{ll} \text{Restriction } G \downarrow L & : \mathrm{Ind}_H^G(\mathbb{C}_\lambda)|_L, \\ \text{Restriction } G \downarrow H & : \mathrm{Ind}_L^G(\mathbb{C}_\lambda)|_H, \\ \text{Tensor product} & : \mathrm{Ind}_H^G(\mathbb{C}_\lambda) \otimes \mathrm{Ind}_L^G(\mathbb{C}_\mu). \end{array}$$

Here $\mathrm{Ind}_H^G(\mathbb{C}_\lambda)$ denotes a holomorphically induced representation of G from a unitary character $\mathbb{C}_\lambda$ of H by the Borel–Weil theorem. See [Ko1, Ko2, Ko3, Ko5, Ko6] for the general theory on the application of visible actions (including the vector bundle setting), and also Section 5.6 for an application of our results to the exceptional groups.

In this chapter we deal with exceptional Lie groups. In order to state the results, we label the Dynkin diagrams of type E_6 and type E_7 following Bourbaki [Bo] (see Figures 5.3.2 and 5.3.3). For a subset Π' of a simple system Π, we denote by $\mathrm{L}_{\Pi'}$ the Levi subgroup whose root system is generated by Π', and by $(\Pi')^c$ for the complement of Π' in Π.

Theorem 5.1.1. *Let G be a connected compact Lie group with an exceptional simple Lie algebra $\mathfrak{g}$, Π a simple system of the root system $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ with respect to a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, and σ a Chevalley–Weyl involution of G with respect to $\mathfrak{t}$. Take proper subsets Π' and Π'' of Π. Then the following two conditions are equivalent.*

(i) $G = \mathrm{L}_{\Pi'}\, G^\sigma\, \mathrm{L}_{\Pi''}$.

(ii) *Up to switch of Π' and Π'', one of the below conditions is satisfied.*

$$\begin{array}{llll} \text{I.} & \mathfrak{g} = \mathfrak{e}_6,\ (\Pi')^c = \{\alpha_i\}, & (\Pi'')^c = \{\alpha_1, \alpha_6\}, & i = 1 \textit{ or } 6. \\ \text{II.} & \mathfrak{g} = \mathfrak{e}_6,\ (\Pi')^c = \{\alpha_i\}, & (\Pi'')^c = \{\alpha_j\}, & i = 1 \textit{ or } 6,\ j \neq 4. \\ \text{III.} & \mathfrak{g} = \mathfrak{e}_7,\ (\Pi')^c = \{\alpha_7\}, & (\Pi'')^c = \{\alpha_i\}, & i = 1,\ 2 \textit{ or } 7. \end{array}$$

In particular, there are no such pair (Π', Π'') for $\mathfrak{g} = \mathfrak{e}_8$, $\mathfrak{f}_4$ or $\mathfrak{g}_2$.

Cases I, II and III amount to

$$\begin{array}{llll} \text{I.} & \mathfrak{g} = \mathfrak{e}_6, & \mathfrak{l}_{\Pi'} = \mathfrak{so}(10) \oplus \mathbb{R}, & \mathfrak{l}_{\Pi''} = \mathfrak{so}(8) \oplus \mathbb{R}^2. \\ \text{II.} & \mathfrak{g} = \mathfrak{e}_6, & \mathfrak{l}_{\Pi'} = \mathfrak{so}(10) \oplus \mathbb{R}, & \mathfrak{l}_{\Pi''} = \mathfrak{so}(10) \oplus \mathbb{R},\ \mathfrak{su}(6) \oplus \mathbb{R} \text{ or} \\ & & & \qquad \mathfrak{su}(2) \oplus \mathfrak{su}(5) \oplus \mathbb{R}. \\ \text{III.} & \mathfrak{g} = \mathfrak{e}_7, & \mathfrak{l}_{\Pi'} = \mathfrak{e}_6 \oplus \mathbb{R}, & \mathfrak{l}_{\Pi''} = \mathfrak{so}(12) \oplus \mathbb{R},\ \mathfrak{su}(7) \oplus \mathbb{R} \text{ or } \mathfrak{e}_6 \oplus \mathbb{R}. \end{array}$$

As a corollary of Theorem 5.1.1, we obtain three kinds of multiplicity-free theorems for representations of exceptional Lie groups (see Corollary 5.6.4 for the restrictions to Levi subgroups and Corollary 5.6.5 for the tensor products). In the course of the proof, we find explicitly a slice B that gives a generalized Cartan decomposition $G = \mathrm{L}_{\Pi'}\, B\, \mathrm{L}_{\Pi''}$ (see Propositions 5.3.2, 5.3.3, 5.3.5, 5.3.6, 5.3.8 and 5.3.10.) by using the herringbone stitch method [Ko4]. The 'slice' B plays an important role in dealing with more delicate cases (vector bundle cases) in the application to representation theory, which is not discussed in this section.

Special feature of exceptional Lie groups. Here we mention some new features in dealing with exceptional groups, which arise both in the proof and in the results:

- (In the proof.) In order to find explicit generalized Cartan decompositions in the exceptional case, our argument relies on the root systems rather than matrix computations that were effectively used in the classical case.
- (In the main results.) For all classical compact groups G, there exist pairs of proper Levi subgroups $\mathrm{L}_{\Pi'}$ and $\mathrm{L}_{\Pi''}$ such that the multiplication mapping $\mathrm{L}_{\Pi'} \times G^{\sigma} \times \mathrm{L}_{\Pi''} \to G$ is surjective [Ko4, Ta1]. However, none of the exceptional compact Lie groups G_2, F_4 or E_8 admits such a pair of proper Levi subgroups. This corresponds to the representation theoretic fact (c.f. [Li2, St2] and Section 5.6) that $\#\mathrm{MF}_f(G, L)$ is finite for any Levi subgroup L of G if and only if G is of type E_8, F_4 or G_2, where $\mathrm{MF}_f(G, L)$ is the set of equivalence classes of finite dimensional irreducible representations π of G such that the restrictions $\pi|_L$ to L are multiplicity-free.

Organization. This chapter is organized as follows. In Section 5.2 we discuss a slice for symmetric pairs. In Section 5.3, we give a proof of the implication (ii) $\Rightarrow$ (i) together with a generalized Cartan decomposition $G = \mathrm{L}_{\Pi'}\, B\, \mathrm{L}_{\Pi''}$ by postponing the proofs of some technical lemmas to Section 5.4. In Section 5.4, we deal with double coset decompositions of classical Lie groups, and complete the proof for the implication (ii) $\Rightarrow$ (i). The converse implication (i) $\Rightarrow$ (ii) is proved in Section 5.5 by using the fact that a strongly visible action gives rise to multiplicity-free representations, and classifications of multiplicity-free tensor product representations by P. Littelmann [Li2] for the maximal parabolic case and J. R. Stembridge [St2] for the general case. An application to multiplicity-free representations is discussed in Section 5.6.

In the following, we denote Lie groups by capital Latin letters and their Lie algebras by corresponding small German letters. Also, for a given real Lie algebra $\mathfrak{g}$, we denote its complexification by $\mathfrak{g}_{\mathbb{C}}$.

5.2 Construction of a slice for the symmetric case

Let $\mathfrak{g}$ be a compact Lie algebra and τ an involution of $\mathfrak{g}$. Then we take a τ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and write $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ where $\mathfrak{t} = \mathfrak{h}^{\tau}$ and $\mathfrak{a} = \mathfrak{h}^{-\tau}$. Here, $\mathfrak{h}^{-\tau}$ is defined by $\mathfrak{h}^{-\tau} := \{X \in \mathfrak{h} : \tau(X) = -X\}$. In this section, we shall see how to construct a maximal abelian subspace of $\mathfrak{g}^{-\tau}$, which is fixed by σ. We begin by the following proposition.

Proposition 5.2.1. *Let us suppose that there exists an automorphism σ of $\mathfrak{g}$, which preserves $\mathfrak{a}$ and acts on $\mathfrak{t}$ as the multiplication by (-1), and that the Cartan subalgebra $\mathfrak{t} \oplus \sqrt{-1}\mathfrak{a}$ of the non-compact dual $\mathfrak{g}^{\tau} \oplus \sqrt{-1}\mathfrak{g}^{-\tau}$ is not maximally non-compact. Then* *for any root vector $X_\beta \in \mathfrak{g}_\mathbb{C}$ of any imaginary non-compact root β, there exists $Z \in \mathfrak{t}$ such that $\mathrm{Ad}(\exp(Z))(X_\beta + \overline{X_\beta})$ is fixed by σ. Here we extend σ to $\mathfrak{g}_\mathbb{C}$ holomorphically, and $\overline{X}$ denotes the conjugate element with respect to $\mathfrak{g}$ for any $X \in \mathfrak{g}_\mathbb{C}$.*

Proof. Since both $\overline{X_\beta}$ and $\sigma(X_\beta)$ belong to the root subspace $\mathfrak{g}_{-\beta}$ of $-\beta$, $\sigma(X_\beta) = e^{\sqrt{-1}\theta}\overline{X_\beta}$ for some $\theta \in \mathbb{R}$. Then we take $Z \in \mathfrak{t}$ satisfying $\beta(Z) = -\sqrt{-1}\theta/2$. (Here we note that β is imaginary.) For this $Z \in \mathfrak{t}$, we have

$$\begin{aligned}
\sigma(\mathrm{Ad}(\exp(Z))(X_\beta + \overline{X_\beta})) &= \sigma(e^{-\frac{\sqrt{-1}\theta}{2}} X_\beta + \overline{e^{\frac{-\sqrt{-1}\theta}{2}} X_\beta}) \\
&= e^{-\frac{\sqrt{-1}\theta}{2}}(e^{\sqrt{-1}\theta}\overline{X_\beta}) + e^{\frac{\sqrt{-1}\theta}{2}}(e^{-\sqrt{-1}\theta}X_\beta) \\
&= e^{\frac{\sqrt{-1}\theta}{2}}\overline{X_\beta} + e^{-\frac{\sqrt{-1}\theta}{2}} X_\beta \\
&= \mathrm{Ad}(\exp(Z))(X_\beta + \overline{X_\beta}).
\end{aligned}$$

□

By using Proposition 5.2.1, we can give a simpler proof of the following result, which was originally proved in [Ko5] by Berger's classification of semisimple symmetric pairs.

Corollary 5.2.2. *Let us suppose that $(\mathfrak{g}, \mathfrak{g}^\tau)$ is a Hermitian symmetric pair and that $\mathfrak{a} = \{0\}$, i.e., $\mathfrak{h} = \mathfrak{t}$. Let σ be a Chevalley–Weyl involution of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Then there exists a maximal abelian subspace of $\mathfrak{g}^{-\tau}$, which is fixed by σ.*

Proof. Consider the non-compact dual $\mathfrak{g}^\tau \oplus \sqrt{-1}\mathfrak{g}^{-\tau}$ of $\mathfrak{g}$ with respect to τ. Since we can construct a maximally non-compact Cartan subalgebra by a succession of the Cayley transforms from $\mathfrak{t}$, the corollary follows from Proposition 5.2.1. Here we note that we can choose strongly orthogonal roots in a succession of the Cayley transforms if $(\mathfrak{g}, \mathfrak{g}^\tau)$ is a Hermitian symmetric pair. □

Remark 5.2.3. From the proofs of Proposition 5.2.1 and Corollary 5.2.2, we can see the following: Retain the setting of the proof of Corollary 5.2.2. We let $\{\beta_1, \dots, \beta_r\}$ be a set of imaginary non-compact roots (with respect to $\mathfrak{t}\otimes\mathbb{C}$), which may be used in a succession of the Cayley transforms for obtaining a σ-fixed maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\tau}$. If some roots (with respect to $\mathfrak{t}\otimes\mathbb{C}$) $\alpha_1, \dots, \alpha_s$ are strongly orthogonal to β_i for any i ($1 \le i \le r$), then a semisimple subalgebra whose set of simple roots is given by $\{\alpha_1, \dots, \alpha_s\}$ centralizes $\mathfrak{a}$.

Remark 5.2.3 may help us to determine a set of simple roots of the centralizer of $\mathfrak{a}$ in the next section.

5.3 Generalized Cartan Decomposition

The aim of this section is to prove that (ii) implies (i) in Theorem 5.1.1 (by postponing some technical lemmas to the next section). The idea is to use the herringbone stitch method [Ko4] that reduces unknown decompositions for non-symmetric pairs to the known Cartan decomposition for symmetric pairs (Fact 5.3.1). We divide the proof to four parts (Subsections 5.3.1–5.3.4).

In the following, $\mathfrak{k}_{ss}$ denotes the semisimple part of $\mathfrak{k}$ and K_{ss} the analytic subgroup of K with Lie algebra $\mathfrak{k}_{ss}$ for a compact Lie group K. Also, we write $G_1 \approx G_2$ if two Lie groups G_1 and G_2 are locally isomorphic.

5.3.1 Decompositions for the symmetric case

In this subsection, we recall a well-known fact ([Ho, Theorem 6.10], [Ma3, Theorem 1]) on the Cartan decomposition for the symmetric case, and deal with Case II with $j = 1$ or 6 and Case III with $i = 7$ in Theorem 5.1.1.

Fact 5.3.1. *Let K be a connected compact Lie group with Lie algebra $\mathfrak{k}$, and τ and τ' two involutions of K. Let H and L be subgroups of K such that*

$$(K^\tau)_0 \subset L \subset K^\tau \quad \textit{and} \quad (K^{\tau'})_0 \subset H \subset K^{\tau'}.$$

Here F_0 denotes the connected component of F containing the identity element for a Lie group F.

We take a maximal abelian subspace $\mathfrak{a}$ in

$$\mathfrak{k}^{-\tau,-\tau'} := \{X \in \mathfrak{k} : \tau(X) = \tau'(X) = -X\}.$$

Suppose that $\tau\tau'$ is semisimple on the center $\mathfrak{z}$ of $\mathfrak{k}$. Then we have

$$K = L \exp(\mathfrak{a}) H.$$

By combining this fact with Corollary 5.2.2, we immediately obtain the following (cf. [Ko5]).

Proposition 5.3.2 (Case II with $j = 1$ or 6 and Case III with $i = 7$)**.** *Let G be a connected compact Lie group, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$, σ a Chevalley–Weyl involution of G with respect to $\mathfrak{t}$ and L, H Levi subgroups of G with respect to a simple system of the root system $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. Suppose that both L and H are Hermitian symmetric subgroups of G. Then we have*

$$G = L \exp(\mathfrak{a}) H$$

where $\mathfrak{a}$ is an abelian subspace of $\mathfrak{g}^\sigma$.

Since the surjectivity of the multiplication mapping $\mathrm{L}_{\Pi'} \times G^\sigma \times \mathrm{L}_{\Pi''} \to G$ depends on neither the coverings of the group G nor the choice of Cartan subalgebras and Chevalley–Weyl involutions, we may and do work with connected simply connected compact exceptional Lie groups, and fix a Cartan subalgebra and a Chevalley–Weyl involution in each of the subsections below.

5.3.2 Decompositions for the type E_6 (non-maximal parabolic type)

In this subsection, we deal with Case I in Theorem 5.1.1.

Proposition 5.3.3 (Case I)**.** *Let G be the connected simply connected compact simple Lie group of type E_6, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$ and σ a Chevalley–Weyl involution of G with respect to $\mathfrak{t}$. Take two subsets Π' and Π'' of a simple system Π as $(\Pi')^c = \{\alpha_i\}$ ($i = 1$ or 6) and $(\Pi'')^c = \{\alpha_1, \alpha_6\}$. (We label the Dynkin diagram following Bourbaki [Bo]. See Figure 5.3.2 in Subsection 5.3.3.) Then we have*

$$G = \mathrm{L}_{\Pi'}\, B\, \mathrm{L}_{\Pi''}$$

for a subset $B \subset G^\sigma$.

Proof. Let us explicitly write the root system Δ and the simple system Π of type E_6 as follows (see Plate V of [Bo]):

$$\begin{aligned}
&\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \\
&\quad = \left\{ \pm\varepsilon_i \pm \varepsilon_j, \frac{1}{2}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 + \sum_{k=1}^{5}(-1)^{\nu_k}\varepsilon_k) : 1 \le i < j \le 5, \sum_{k=1}^{5}\nu_k \text{ is even} \right\}, \\
&\Pi = \{\alpha_i : 1 \le i \le 6\}, \\
&\quad \text{where } \alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8) - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7),\ \alpha_2 = \varepsilon_1 + \varepsilon_2, \\
&\qquad\quad \alpha_3 = \varepsilon_2 - \varepsilon_1,\ \alpha_4 = \varepsilon_3 - \varepsilon_2,\ \alpha_5 = \varepsilon_4 - \varepsilon_3,\ \alpha_6 = \varepsilon_5 - \varepsilon_4.
\end{aligned}$$

We may and do assume that $i = 6$ since $\mathrm{L}_{\{\alpha_1\}^c}$ is conjugate to $\mathrm{L}_{\{\alpha_6\}^c}$ under the action of the Weyl group, and hence that of G^σ ([Kna, Theorem 6.57]). Let τ denote the involution of G, which corresponds to $\mathrm{L}_{\Pi'}$. By using two non-compact imaginary roots $\varepsilon_5 - \varepsilon_4$ and $\varepsilon_5 + \varepsilon_4$ for the Cayley transforms of the compact Cartan subalgebra $\mathfrak{t}$ of the non-compact dual $\mathfrak{g}^\tau \oplus \sqrt{-1}\mathfrak{g}^{-\tau}$ of $\mathfrak{g}$, we obtain a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\tau}$, which is fixed by σ (Corollary 5.2.2). We apply Fact 5.3.1 to (G, τ, τ) as follows.

$$G = G^\tau \exp(\mathfrak{a}) G^\tau. \tag{5.3.1}$$

Since the pair $(\mathfrak{g}, \mathfrak{g}^\tau)$ is Hermitian of non-tube type, there exists $X \in \mathfrak{g}^\tau_{ss}$ such that $\mathbb{R}(Z+X)$ is the center of $Z_{\mathfrak{g}^\tau}(\mathfrak{a})$ where Z is a non-zero element of the center of $\mathfrak{g}^\tau$. Then we have the following lemma on a representative of the double coset decomposition of $G^\tau_{ss}(\approx \mathrm{SO}(10))$ by $\exp(\mathbb{R}X) \cdot M_{ss}$ and $G^\tau_{ss} \cap \mathrm{L}_{\Pi''}(\approx \mathrm{U}(1) \times \mathrm{SO}(8))$, where $M(\approx \mathrm{U}(4))$ is the analytic subgroup of G^τ with Lie algebra $Z_{\mathfrak{g}^\tau}(\mathfrak{a})$.

Lemma 5.3.4. *There exists a subset B' of G^σ such that the multiplication mapping*

$$(\exp(\mathbb{R}X)M_{ss}) \times B' \times (G^\tau_{ss} \cap \mathrm{L}_{\Pi''}) \to G^\tau_{ss}$$

is surjective.

We postpone the proof of this lemma to Lemma 5.4.1 in Section 5.4. The above surjection implies that $G^\tau_{ss} = (\exp(\mathbb{R}X)M_{ss})B'(G^\tau_{ss} \cap \mathrm{L}_{\Pi''})$, and thus we obtain

$$G^\tau = M B' \,\mathrm{L}_{\Pi''}\,. \tag{5.3.2}$$

Then we put $B = \exp(\mathfrak{a})B'$, and substitute (5.3.2) to (5.3.1) as follows.

$$\begin{aligned}
G &= G^\tau \exp(\mathfrak{a})(M B' \,\mathrm{L}_{\Pi''}) \\
&= G^\tau M \exp(\mathfrak{a}) B' \,\mathrm{L}_{\Pi''} \\
&= \mathrm{L}_{\Pi'}\, B \,\mathrm{L}_{\Pi''}\,.
\end{aligned}$$

This completes the proof since $B = \exp(\mathfrak{a})B'$ is contained in G^σ. $\square$

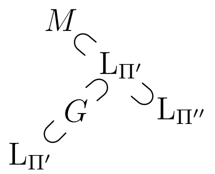


Figure 5.3.1: Herringbone stitch used for $\mathrm{L}_{\Pi'} \backslash G / \mathrm{L}_{\Pi''}$ in Case I.

5.3.3 Decompositions for type E_6 (maximal parabolic type)

In this subsection, we discuss Case II with $j = 2, 3$ or 5 in Theorem 5.1.1. Let G denote the connected simply connected compact simple Lie group of type E_6, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$ and σ a Chevalley–Weyl involution of G with respect to $\mathfrak{t}$. We take a simple system Π of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and two commuting involutions τ and τ' of $\mathfrak{g}_{\mathbb{C}}$, which preserve $\mathfrak{g}$ and correspond to the below Vogan diagrams of type E III and type E II respectively (see Appendix C of [Kna]).

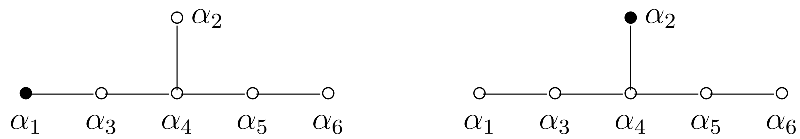


Figure 5.3.2: Vogan diagrams of type E III and type E II.

Then the fixed part of the involution $\tau\tau'$ is given by $\mathfrak{g}^{\tau\tau'} = \mathbb{R} \oplus \mathfrak{so}(10)$. Since the root system $\Delta(\mathfrak{g}_{\mathbb{C}}^{\tau\tau'}, \mathfrak{t}_{\mathbb{C}})$ of $\mathfrak{g}^{\tau\tau'}$ is contained in $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, there exists $\gamma \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ such that $\{\alpha_3, \alpha_4, \alpha_5, \alpha_6, \gamma\}$ gives rise to a simple system of $\mathfrak{g}^{\tau\tau'}$. We may and do assume that γ is connected to α_4. We take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\tau,-\tau'}$ as follows: Let us explicitly write the simple system $\Pi(\mathfrak{g}_{\mathbb{C}}^{\tau\tau'}, \mathfrak{t}_{\mathbb{C}})$ and the root system $\Delta(\mathfrak{g}_{\mathbb{C}}^{\tau\tau'}, \mathfrak{t}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}^{\tau\tau'}$ (see Plate IV of [Bo]).

$$\Pi(\mathfrak{g}_{\mathbb{C}}^{\tau\tau'}, \mathfrak{t}_{\mathbb{C}}) = \{\alpha_i,\ \gamma : 3 \leq i \leq 6\},$$
$$\Delta(\mathfrak{g}_{\mathbb{C}}^{\tau\tau'}, \mathfrak{t}_{\mathbb{C}}) = \{\pm f_i \pm f_j : 1 \leq i < j \leq 5\},$$
where $\alpha_3 = f_4 - f_5,\ \alpha_4 = f_3 - f_4,\ \alpha_5 = f_2 - f_3,\ \alpha_6 = f_1 - f_2,\ \gamma = f_4 + f_5.$

Using two non-compact imaginary roots $f_2 + f_3$ and $f_4 + f_5$ for the Cayley transforms of the compact Cartan subalgebra $\mathfrak{t}$ of the non-compact dual $\mathfrak{g}^{\tau,\tau'} \oplus \sqrt{-1}\mathfrak{g}^{-\tau,-\tau'}$ of $\mathfrak{g}^{\tau\tau'}$, we obtain a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\tau,-\tau'}$, which is fixed by σ (Corollary 5.2.2). Then we consider the centralizer $Z_{\mathfrak{g}^{\tau,\tau'}}(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathfrak{g}^{\tau,\tau'}$. For simplicity, we put $\mathfrak{m} = Z_{\mathfrak{g}^{\tau,\tau'}}(\mathfrak{a})$. We note the following decomposition of $\mathfrak{g}^{\tau\tau'}$.

$$\begin{aligned}
\mathfrak{g}^{\tau\tau'} &= \mathfrak{g}^{\tau,\tau'} \oplus \mathfrak{g}^{-\tau,-\tau'} \\
&= \mathbb{R}K_1 \oplus ((\mathfrak{g}^{\tau\tau'})_{ss})^{\tau} \oplus \mathfrak{g}^{-\tau,-\tau'} \\
&= \mathbb{R}K_1 \oplus \mathbb{R}K_2 \oplus (\mathfrak{g}^{\tau,\tau'})_{ss} \oplus \mathfrak{g}^{-\tau,-\tau'},
\end{aligned}$$

where K_1 is a non-zero element of the center of $\mathfrak{g}^{\tau\tau'}$, and K_2 that of the center of $((\mathfrak{g}^{\tau\tau'})_{ss})^{\tau} = \mathfrak{u}(5)$. Since the pair $((\mathfrak{g}^{\tau\tau'})_{ss},\ ((\mathfrak{g}^{\tau\tau'})_{ss})^{\tau}) = (\mathfrak{so}(10), \mathfrak{u}(5))$ is Hermitian of non-tube type, there exists $K_3 \in (\mathfrak{g}^{\tau,\tau'})_{ss} \cap \mathfrak{t}$ such that $\mathbb{R}(K_2 + K_3) \oplus \mathfrak{m}_{ss}$ gives rise to the centralizer of $\mathfrak{a}$ in $((\mathfrak{g}^{\tau\tau'})_{ss})^{\tau}$. Hence we obtain

$$\mathfrak{m} = \mathbb{R}K_1 \oplus \mathbb{R}(K_2 + K_3) \oplus \mathfrak{m}_{ss}. \tag{5.3.3}$$

The subalgebra $\mathfrak{g}^{\tau'}$ has two simple factors $\mathfrak{su}(2)$ and $\mathfrak{su}(6)$. So, we write $\mathfrak{g}^{\tau'} = \mathfrak{g}' \oplus \mathfrak{g}''$ where $\mathfrak{g}' = \mathfrak{su}(2)$ and $\mathfrak{g}'' = \mathfrak{su}(6)$. Then we decompose $\mathfrak{g}^{\tau'}$ as follows.

$$\begin{aligned}\mathfrak{g}^{\tau'} &= \mathfrak{g}^{\tau',\tau} \oplus \mathfrak{g}^{\tau',-\tau} \\ &= (\mathfrak{g}')^{\tau} \oplus (\mathfrak{g}'')^{\tau} \oplus \mathfrak{g}^{\tau',-\tau} \\ &= \mathbb{R}Z_1 \oplus (\mathfrak{g}'')^{\tau} \oplus \mathfrak{g}^{\tau',-\tau} \\ &= \mathbb{R}Z_1 \oplus \mathbb{R}Z_2 \oplus (\mathfrak{g}^{\tau,\tau'})_{ss} \oplus \mathfrak{g}^{\tau',-\tau}.\end{aligned}$$

Here Z_1 is a non-zero element of $(\mathfrak{g}')^{\tau}$, and Z_2 that of the center of $(\mathfrak{g}'')^{\tau} = \mathfrak{u}(5)$. Then we give generalized Cartan decompositions for Case II with $j = 2$ and $j = 3$ or 5 separately.

Case II with $j = 2$

Proposition 5.3.5 (Case II with $j = 2$). *Let G, $\mathfrak{g}$, $\mathfrak{t}$, σ, Π, τ and τ' be as in the beginning of this subsection. Take two subsets Π' and Π'' of the simple system Π of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ as $(\Pi')^c = \{\alpha_i\}$ and $(\Pi'')^c = \{\alpha_2\}$ where $i = 1$ or 6. Then we have*

$$G = \mathrm{L}_{\Pi'}\, B\, \mathrm{L}_{\Pi''}$$

for a subset $B \subset G^{\sigma}$.

Proof. Let G' and G'' be the analytic subgroups of $G^{\tau'}$ with Lie algebras $\mathfrak{g}' = \mathfrak{su}(2)$ and $\mathfrak{g}'' = \mathfrak{su}(6)$ respectively. We apply Fact 5.3.1 to (G, τ, τ'):

$$\begin{aligned}G &= G^{\tau} \exp(\mathfrak{a}) G^{\tau'} \\ &= G^{\tau} \exp(\mathfrak{a}) G' G''.\end{aligned} \tag{5.3.4}$$

Here $\mathfrak{a}$ is the maximal abelian subspace of $\mathfrak{g}^{-\tau,-\tau'}$, which is constructed in the above. Since $(\mathfrak{g}', \mathbb{R}Z_1)$ is also a symmetric pair, we can again use Fact 5.3.1 as follows.

$$G' = \exp(\mathbb{R}Z_1) \exp(\mathfrak{a}') \exp(\mathbb{R}Z_1), \tag{5.3.5}$$

where $\mathfrak{a}'$ is the σ-fixed one dimensional subspace of $\mathfrak{g}'$. Since the vector space $\mathbb{R}Z_1 \oplus \mathbb{R}Z_2$ coincides with $\mathbb{R}K_1 \oplus \mathbb{R}K_2$, there are real numbers a and b such that $Z_1 = aK_1 + bK_2$. Then we have the following equality.

$$\begin{aligned}&(\exp(\mathbb{R}Z_1) \exp(\mathfrak{a}') \exp(\mathbb{R}Z_1))G'' \\ &\qquad = (\exp(\mathbb{R}(aK_1 + b(K_2 + K_3))) \exp(\mathfrak{a}') \exp(\mathbb{R}Z_1))G''.\end{aligned} \tag{5.3.6}$$

Put $B = \exp(\mathfrak{a}) \exp(\mathfrak{a}')$. By combining (5.3.5) and (5.3.6) with (5.3.4), we obtain

$$\begin{aligned}G &= G^{\tau} \exp(\mathfrak{a}) G' G'' \text{ by (5.3.4)} \\ &= G^{\tau} \exp(\mathfrak{a})(\exp(\mathbb{R}Z_1) \exp(\mathfrak{a}') \exp(\mathbb{R}Z_1))G'' \text{ by (5.3.5)} \\ &= G^{\tau} \exp(\mathfrak{a})(\exp(\mathbb{R}(aK_1 + b(K_2 + K_3))) \exp(\mathfrak{a}') \exp(\mathbb{R}Z_1))G'' \text{ by (5.3.6)} \\ &= G^{\tau}(\exp(\mathbb{R}(aK_1 + b(K_2 + K_3))) \exp(\mathfrak{a}) \exp(\mathfrak{a}') \exp(\mathbb{R}Z_1)G'' \text{ by (5.3.3)} \\ &= G^{\tau} B \exp(\mathbb{R}Z_1) G''.\end{aligned}$$

Since $\exp(\mathbb{R}Z_1)G''$ coincides with $\mathrm{L}_{\Pi''}$ and G^{τ} is conjugate to $\mathrm{L}_{\Pi'}$ under the Weyl group and hence under G^{σ} ([Kna, Theorem 6.57]), we have shown the proposition. ☐

Case II with $j = 3$ or 5

Proposition 5.3.6 (Case II with $j = 3$ or 5). *Let G, $\mathfrak{g}$, $\mathfrak{t}$, σ, Π, τ and τ' be as in the beginning of this subsection. Take two subsets Π' and Π'' of the simple system Π of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ as $(\Pi')^c = \{\alpha_i\}$ and $(\Pi'')^c = \{\alpha_j\}$ where $(i, j) = (1, 3)$, $(1, 5)$, $(6, 3)$ or $(6, 5)$. Then we have*

$$G = \mathrm{L}_{\Pi'}\, B\, \mathrm{L}_{\Pi''}$$

for a subset $B \subset G^\sigma$.

Proof. Retain the notations $G', G'', \mathfrak{a}, Z_1, Z_2, K_1, K_2, a$ and b in the proof of Proposition 5.3.5. We have the following lemma on a representative of the double coset of $G''(\approx \mathrm{SU}(6))$ by $\exp(\mathbb{R}Z_2)M_{ss}(\approx \mathrm{U}(1) \times \mathrm{SU}(2)^2)$ and $(G'')^\tau_0(\approx \mathrm{U}(5))$, where M is the analytic subgroup of $G^{\tau'}$ with Lie algebra $\mathfrak{m} = Z_{\mathfrak{g}^{\tau,\tau'}}(\mathfrak{a})$.

Lemma 5.3.7. *There exists a subset B' of G^σ such that the multiplication mapping*

$$(\exp(\mathbb{R}Z_2)M_{ss}) \times B' \times (G'')^\tau_0 \to G''$$

is surjective.

We postpone the proof of this lemma to Lemma 5.4.2 in Section 5.4. As in the proof of Proposition 5.3.5, there are real numbers c and d such that $Z_2 = cK_1 + dK_2$. Hence we have

$$\begin{aligned}
G'G'' &= G'\left(\exp(\mathbb{R}(cK_1 + dK_2))M_{ss}B'(G'')^\tau_0\right) \text{ by lemma 5.3.7} \\
&= \exp(\mathbb{R}(cK_1 + dK_2))G'M_{ss}B'(G'')^\tau_0 \\
&= \exp\left(\mathbb{R}\left(cK_1 + dK_2 - \frac{d}{b}Z_1\right)\right) G'M_{ss}B'(G'')^\tau_0 \text{ by } Z_1 \in \mathfrak{g}' \\
&= \exp(\mathbb{R}K_1)G'M_{ss}B'(G'')^\tau_0 \text{ by } Z_1 = aK_1 + bK_2. \qquad (5.3.7)
\end{aligned}$$

Here, we note that a direct computation shows $b \neq 0$. Put $B = \exp(\mathfrak{a})B'$. Substituting (5.3.7) to (5.3.4), we obtain

$$\begin{aligned}
G &= G^\tau \exp(\mathfrak{a})G'G'' \text{ by (5.3.4)} \\
&= G^\tau \exp(\mathfrak{a})\left(\exp(\mathbb{R}K_1)G'M_{ss}B'(G'')^\tau_0\right) \text{ by (5.3.7)} \\
&= G^\tau \exp(\mathbb{R}K_1)M_{ss} \exp(\mathfrak{a})B'G'(G'')^\tau_0 \text{ by } \mathbb{R}K_1, \mathfrak{m}_{ss} \subset Z_{\mathfrak{g}}(\mathfrak{a}) \\
&= G^\tau BG'(G'')^\tau_0.
\end{aligned}$$

This completes the proof since G^τ and $G'(G'')^\tau_0$ are conjugate to $\mathrm{L}_{\Pi'}$ and $\mathrm{L}_{\Pi''}$ respectively by elements of G^σ. □

5.3.4 Decompositions for type E_7

In this subsection we discuss Case III with $i = 1$ or 2 in Theorem 5.1.1. Let G denote the connected simply connected compact simple Lie group of type E_7, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$, and σ a Chevalley–Weyl involution of G with respect to $\mathfrak{t}$. We fix a simple system Π of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. (For the labeling of the Dynkin diagram, see Figure 5.3.3 below.) We give the proofs for Case III with $i = 1$ and with $i = 2$ separately.

Case Ⅲ with $i = 2$

Proposition 5.3.8 (Case Ⅲ with $i = 2$)**.** *Let G, Π and σ be as in the beginning of this subsection. Take two subsets Π' and Π'' of Π as $(\Pi')^c = \{\alpha_7\}$ and $(\Pi'')^c = \{\alpha_2\}$. Then we have*

$$G = \mathrm{L}_{\Pi'}\, B\, \mathrm{L}_{\Pi''}$$

for a subset $B \subset G^\sigma$.

Proof. We take two commuting involutions τ and τ' of $\mathfrak{g}_\mathbb{C}$, which preserve $\mathfrak{g}$ and correspond to the below Vogan diagrams of type E VII and type E V respectively (see Appendix C of [Kna]).

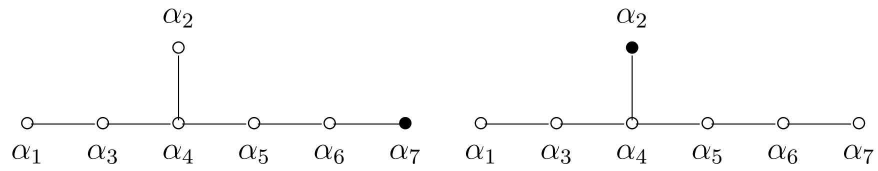


Figure 5.3.3: Vogan diagrams of type E VII and type E V.

Then the fixed part of the involution $\tau\tau'$ is given by $\mathfrak{g}^{\tau\tau'} = \mathfrak{su}(2) \oplus \mathfrak{so}(12)$. Let $\tilde{\alpha}$ denote the smallest root of $\mathfrak{g} = \mathfrak{e}_7$, and $\tilde{\beta}$ that of $\mathfrak{g}^{\tau'} = \mathfrak{su}(8)$. Since the root system $\Delta(\mathfrak{g}_\mathbb{C}^{\tau\tau'}, \mathfrak{t}_\mathbb{C})$ of $\mathfrak{g}^{\tau\tau'}$ is contained in $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, there exists $\gamma \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ such that $\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \tilde{\beta}, \gamma\}$ gives rise to a simple system of $\mathfrak{g}^{\tau\tau'}$. We note that γ is connected to α_3 or α_5. We may and do assume that γ is connected to α_5. Then we take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\tau,-\tau'}$ as follows: Let us explicitly write the simple system $\Pi(\mathfrak{g}_\mathbb{C}^{\tau\tau'}, \mathfrak{t}_\mathbb{C})$ and the root system $\Delta(\mathfrak{g}_\mathbb{C}^{\tau\tau'}, \mathfrak{t}_\mathbb{C})$ of $\mathfrak{g}_\mathbb{C}^{\tau\tau'}$ (see Plate I and Plate IV of [Bo]).

$$\begin{aligned}
&\Pi(\mathfrak{g}_\mathbb{C}^{\tau\tau'}, \mathfrak{t}_\mathbb{C}) = \{\tilde{\beta}\} \cup \{\alpha_i,\ \gamma : 1 \leq i \neq 2 \leq 6\},\\
&\Delta(\mathfrak{g}_\mathbb{C}^{\tau\tau'}, \mathfrak{t}_\mathbb{C}) = \{\pm\tilde{\beta}\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq 6\},\\
&\text{where } \alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_3 = \varepsilon_2 - \varepsilon_3,\ \alpha_4 = \varepsilon_3 - \varepsilon_4,\ \alpha_5 = \varepsilon_4 - \varepsilon_5,\ \alpha_6 = \varepsilon_5 - \varepsilon_6,\\
&\qquad\quad \gamma = \varepsilon_5 + \varepsilon_6.
\end{aligned}$$

By using three non-compact imaginary roots $\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4$ and $\varepsilon_5 + \varepsilon_6$ for the Cayley transforms of the compact Cartan subalgebra $\mathfrak{t}$ of the non-compact dual $\mathfrak{g}^{\tau,\tau'} \oplus \sqrt{-1}\mathfrak{g}^{-\tau,-\tau'}$ of $\mathfrak{g}^{\tau\tau'}$, we obtain a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\tau,-\tau'}$, which is fixed by σ (Corollary 5.2.2). We apply Fact 5.3.1 to (G, τ, τ'):

$$G = G^\tau \exp(\mathfrak{a}) G^{\tau'}. \tag{5.3.8}$$

We define a subgroup $M (\approx \mathrm{SU}(2)^4)$ to be the analytic subgroup of G with Lie algebra $Z_{\mathfrak{g}^{\tau,\tau'}}(\mathfrak{a})$, the centralizer of $\mathfrak{a}$ in $\mathfrak{g}^{\tau,\tau'}$. Then we have the following lemma on the double coset of $G^{\tau'} (\approx \mathrm{SU}(8))$ by $M (\approx \mathrm{SU}(2)^4)$ and $\mathrm{L}_{\Pi''} (\approx \mathrm{U}(7))$.

Lemma 5.3.9. *There exists a subset B' of G^σ such that the multiplication mapping*

$$M \times B' \times \mathrm{L}_{\Pi''} \to G^{\tau'}$$

is surjective.

We postpone the proof of this lemma to Lemma 5.4.3 in Section 5.4. Put $B = \exp(\mathfrak{a})B'$. Combining Lemma 5.3.9 with (5.3.8), we obtain

$$\begin{aligned} G &= G^{\tau} \exp(\mathfrak{a})(MB' \, \mathrm{L}_{\Pi''}) \\ &= G^{\tau} M(\exp(\mathfrak{a})B') \, \mathrm{L}_{\Pi''} \\ &= G^{\tau} B \, \mathrm{L}_{\Pi''} \, . \end{aligned}$$

This completes the proof since $G^{\tau} = \mathrm{L}_{\Pi'}$. □

Case Ⅲ with $i = 1$

Proposition 5.3.10 (Case Ⅲ with $i = 1$)**.** *Let G, Π and σ be as in the beginning of this subsection. Take two subsets Π' and Π'' of the simple system Π as $(\Pi')^c = \{\alpha_7\}$ and $(\Pi'')^c = \{\alpha_1\}$. Then we have*

$$G = \mathrm{L}_{\Pi'} \, B \, \mathrm{L}_{\Pi''}$$

for a subset $B \subset G^{\sigma}$.

Proof. We take two commuting involutions τ and τ' of $\mathfrak{g}_{\mathbb{C}}$, which preserve $\mathfrak{g}$ and correspond to the below Vogan diagrams of type E VII and type E VI respectively (see Appendix C of [Kna]).

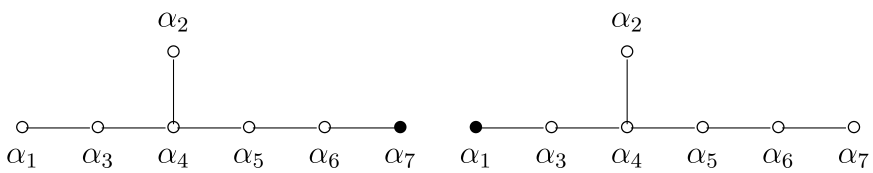


Figure 5.3.4: Vogan diagrams of type E VII and type E VI.

Then the fixed part of the involution $\tau\tau'$ is given by $\mathfrak{g}^{\tau\tau'} = \mathbb{R} \oplus \mathfrak{e}_6$. Let us take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\tau,-\tau'}$, which is fixed by σ as in the proof of Proposition 5.3.3. We also take the σ-fixed one dimensional subspace $\mathfrak{a}'$ of the normal subalgebra $\mathfrak{su}(2)$ of $\mathfrak{g}^{\tau'}$. Put $B = \exp(\mathfrak{a}) \exp(\mathfrak{a}')$. By the same argument as in the proof of Proposition 5.3.5 (we note that $((\mathfrak{g}^{\tau\tau'})_{ss}, ((\mathfrak{g}^{\tau\tau'})_{ss})^{\tau}) = (\mathfrak{e}_6, \mathbb{R} \oplus \mathfrak{so}(10))$ is Hermitian of non-tube type), we obtain a generalized Cartan decomposition for Case Ⅲ with $i = 1$:

$$G = \mathrm{L}_{\Pi'} \, B \, \mathrm{L}_{\Pi''} \, .$$

□

5.4 Completion of the proofs in Section 5.3

We have postponed the proofs of double coset decompositions for some subgroups of the exceptional compact simple Lie groups E_6 and E_7, which were used in the herringbone stitch method in the previous section. This section gives the proofs of Lemmas 5.3.4, 5.3.7 and 5.3.9.

All of the compact Lie groups which appear in this section are of classical type. Thus we work on (non-symmetric) generalized Cartan decompositions in the classical case. However, we have to be careful how they are embedded in exceptional Lie groups.

5.4.1 Proof of Lemma 5.3.4

Retain the setting in the proof of Proposition 5.3.3. We note that simple systems of $(\mathfrak{g}^\tau)_{ss}$, $(\mathfrak{g}^\tau)_{ss} \cap \mathfrak{l}_{\Pi''}$ and $Z_{\mathfrak{g}^\tau}(\mathfrak{a})_{ss}$ are given by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and $\{\alpha_2, \alpha_3, \alpha_4\}$ respectively (Remark 5.2.3), and that $X \in (\mathfrak{g}^\tau)_{ss}$ centralizes $Z_{\mathfrak{g}^\tau}(\mathfrak{a})_{ss}$. Let $\{H_i\}_{i=1}^6 \subset \mathfrak{t}_\mathbb{C}$ denotes the dual basis of $\{\alpha_i\}_{i=1}^6$ with respect to the Killing form. Then a direct computation shows that $\sqrt{-1}H_1$ has a non-zero coefficient in $X = \sum_{1\le i\le 5} a_i\sqrt{-1}H_i$, i.e., $a_1 \neq 0$. Now we find that Lemma 5.3.4 follows from the lemma below.

Lemma 5.4.1. *Let L be a connected compact simple Lie group of type* D_5*, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{l}$ and σ a Chevalley–Weyl involution of L with respect to $\mathfrak{t}$. We label the Dynkin diagram of type* D_5 *as follows:*

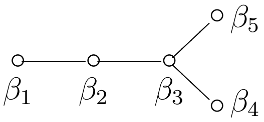


Figure 5.4.1: Dynkin diagram of type D_5.

We take two subsets Φ' and Φ'' of the simple system $\Phi = \{\beta_i : 1 \le i \le 5\}$ of $\mathfrak{l}$ as $(\Phi')^c = \{\beta_1\}$ and $(\Phi'')^c = \{\beta_1, \beta_5\}$, and define a one dimensional abelian subgroup U by $U := \exp(\mathbb{R}(\sum_{i=1}^5 a_i\sqrt{-1}H_i))$ with $a_1 \neq 0$ where $\{H_i\}_{i=1}^5$ denotes the dual basis of $\{\beta_i\}_{i=1}^5$ with respect to the Killing form. Then we have

$$L = U(\mathrm{L}_{\Phi''})_{ss}B'\,\mathrm{L}_{\Phi'},$$

for a subset B' of L^σ.

Proof. It suffices to consider the case where $L = \mathrm{SO}(10)$. We give a matrix realization of L as follows:

$$L = \mathrm{SO}(10) = \{g \in \mathrm{SL}(10, \mathbb{C}) : {}^t g J_{10} g = J_{10},\ {}^t\overline{g} g = I_{10}\},$$

where I_m denotes the identity matrix and J_m is defined by a bilinear form given by

$$\mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}, \quad (x, y) \mapsto {}^t x J_m y := \sum_{i=1}^m x_i y_{m+1-i}.$$

Here x_i and y_i denote the i-th entries in x and y respectively. We take a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{l}$ as diagonal matrices:

$$\mathfrak{t} = \bigoplus_{1\le i\le 5} \mathbb{R}\sqrt{-1}A_i,$$

where $A_i := E_{i,i} - E_{11-i,11-i}$.

We define an involutive automorphism σ of L by

$$\sigma : L \to L, \quad g \mapsto \bar{g}, \tag{5.4.1}$$

where $\bar{g}$ denotes the complex conjugate of $g \in L$. Then σ is a Chevalley–Weyl involution of L with respect to $\mathfrak{t}$. Note that Lemma 5.4.1 is independent of the choice of a Chevalley–Weyl involution since $\mathrm{L}_{\Phi'}$ contains $\exp(\mathfrak{t})$, and both U and $(\mathrm{L}_{\Phi''})_{ss}$ are stable under the conjugation by any element of $\exp(\mathfrak{t})$.

We let $\{\varepsilon_i\}_{1\le i\le 5} \subset (\mathfrak{t}\otimes_{\mathbb{R}}\mathbb{C})^*$ be the dual basis of $\{A_i\}_{1\le i\le 5}$. Then we define a set of simple roots $\Phi := \{\beta_1, ..., \beta_5\}$ by

$$\beta_i := \varepsilon_i - \varepsilon_{i+1}\ (1 \le i \le 4),\ \beta_5 := \varepsilon_4 + \varepsilon_5.$$

Since the sets of the simple roots of $\mathrm{L}_{\Phi''}$ and $\mathrm{L}_{\Phi'}$ are given by $\{\beta_2, \beta_3, \beta_4\}$ and $\{\beta_2, \beta_3, \beta_4, \beta_5\}$ respectively, $\mathrm{L}_{\Phi''}$ and $\mathrm{L}_{\Phi'}$ take the forms:

$$\mathrm{L}_{\Phi''} = \left\{\begin{pmatrix} e^{\sqrt{-1}\theta} & & & \\ & A & & \\ & & J_4\overline{A}J_4 & \\ & & & e^{-\sqrt{-1}\theta}\end{pmatrix} \in \mathrm{SO}(10) : \theta\in\mathbb{R},\ A\in\mathrm{U}(4)\right\},$$

$$\mathrm{L}_{\Phi'} = \left\{\begin{pmatrix} e^{\sqrt{-1}\theta} & & \\ & A & \\ & & e^{-\sqrt{-1}\theta}\end{pmatrix} \in \mathrm{SO}(10) : \theta\in\mathbb{R},\ A\in\mathrm{SO}(8)\right\}.$$

Here, all the entries in the blank space are zero. We give a proof of the lemma by the herringbone stitch method [Ko4]. First, we show that $L = \mathrm{L}_{\Phi''}\, B'\, \mathrm{L}_{\Phi'}$ for a subset B' of L^σ. Next, we prove that $\mathrm{L}_{\Phi''}\, B'\, \mathrm{L}_{\Phi'}$ coincides with $U \cdot (\mathrm{L}_{\Phi''})_{ss} B'\, \mathrm{L}_{\Phi'}$. Then we can see that $L = U\cdot(\mathrm{L}_{\Phi''})_{ss} B'\, \mathrm{L}_{\Phi'}$ holds.

Let us show the first assertion that the group L can be written as $\mathrm{L}_{\Phi''}\, B'\, \mathrm{L}_{\Phi'}$ with $B' \subset L^\sigma$. We define an abelian subgroup B_1 by

$$B_1 := \exp\left(\bigoplus_{i=1,2}\mathbb{R}(E_{1,4+i} - E_{4+i,1} - E_{7-i,10} + E_{10,7-i})\right).$$

Then we have the following decomposition of L by Fact 5.3.1.

$$L = \mathrm{L}_{\Phi'}\, B_1\, \mathrm{L}_{\Phi'}\,. \tag{5.4.2}$$

We define a symmetric subgroup K of $(\mathrm{L}_{\Phi'})_{ss}$ and an abelian subgroup B_2 by

$$\begin{aligned} K &:= \mathrm{SO}(6)\times\mathrm{SO}(2) \\ &= \left\{\begin{pmatrix} 1 & & & & & 0 \\ & A & & & B & \\ & & e^{\sqrt{-1}\theta} & & & \\ & & & e^{-\sqrt{-1}\theta} & & \\ & C & & & D & \\ 0 & & & & & 1\end{pmatrix} \in \mathrm{SO}(10) : \begin{matrix} \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \mathrm{SO}(6), \\ \theta\in\mathbb{R}\end{matrix}\right\}, \\ B_2 &:= \exp(\mathbb{R}(E_{2,6} - E_{6,2} - E_{5,9} + E_{9,5})). \end{aligned}$$

Then we obtain the following decomposition of $\mathrm{L}_{\Phi'}$ by using Fact 5.3.1.

$$\mathrm{L}_{\Phi'} = \mathrm{L}_{\Phi''}\, B_2 K.$$

It is easy to see that K and K_{ss} satisfy $\mathrm{L}_{\Phi''}\, B_2 K = \mathrm{L}_{\Phi''}\, B_2 K_{ss}$. Thus we have

$$\mathrm{L}_{\Phi'} = \mathrm{L}_{\Phi''}\, B_2 K_{ss}. \tag{5.4.3}$$

Let us set $B' := B_2 B_1$. The following is a proof of the first assertion.

$$\begin{aligned} L &= \mathrm{L}_{\Phi'}\, B_1\, \mathrm{L}_{\Phi'} \ \text{ by (5.4.2)} \\ &= (\mathrm{L}_{\Phi''}\, B_2 K_{ss}) B_1\, \mathrm{L}_{\Phi'} \ \text{ by (5.4.3)} \\ &= \mathrm{L}_{\Phi''}\, B_2 B_1 K_{ss}\, \mathrm{L}_{\Phi'} \ \text{ by } K_{ss} \subset Z_L(B_1) \\ &= \mathrm{L}_{\Phi''}\, B'\, \mathrm{L}_{\Phi'}\, . \end{aligned}$$

Then we give a proof of the second assertion, that is, we shall prove that $\mathrm{L}_{\Phi''}\, B'\, \mathrm{L}_{\Phi'}$ coincides with $U \cdot (\mathrm{L}_{\Phi''})_{ss} B'\, \mathrm{L}_{\Phi'}$. We define one dimensional abelian subgroup T_1 by

$$T_1 := \exp(\mathbb{R}\sqrt{-1}(E_{3,3} - E_{8,8})) \subset \mathrm{L}_{\Phi''}\, .$$

Since T_1 centralizes B', $U \cdot (\mathrm{L}_{\Phi''})_{ss} B'\, \mathrm{L}_{\Phi'}$ is equal to $U \cdot ((\mathrm{L}_{\Phi''})_{ss} \cdot T_1) B'\, \mathrm{L}_{\Phi'}$, and hence to $U \cdot ((\mathrm{L}_{\Phi'})_{ss} \cap \mathrm{L}_{\Phi''}) B'\, \mathrm{L}_{\Phi'}$. Further, $U \cdot ((\mathrm{L}_{\Phi'})_{ss} \cap \mathrm{L}_{\Phi''})$ is equal to $\mathrm{L}_{\Phi''}$ because $a_1 \neq 0$ (we recall that $U = \exp(\mathbb{R} \sum_{i=1}^{5} a_i \sqrt{-1} H_i))$. Consequently we have

$$U(\mathrm{L}_{\Phi''})_{ss} B'\, \mathrm{L}_{\Phi'} = U \cdot ((\mathrm{L}_{\Phi'})_{ss} \cap \mathrm{L}_{\Phi''}) B'\, \mathrm{L}_{\Phi'} = \mathrm{L}_{\Phi''}\, B'\, \mathrm{L}_{\Phi'} = L.$$

We have finished the proof. □

5.4.2 Proof of Lemma 5.3.7

Retain the setting of subsection 5.3.3. We note that simple systems of $\mathfrak{g}''$, $(\mathfrak{g}'')^\tau$ and $Z_{\mathfrak{g}^{\tau,\tau'}}(\mathfrak{a})$ are given by $\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$, $\{\alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ and $\{\alpha_3, \alpha_5\}$ respectively (Remark 5.2.3), and that $\mathbb{R}Z_2$ is the center of $(\mathfrak{g}'')^\tau$. Now we can see that Lemma 5.3.7 follows from the lemma below.

Lemma 5.4.2. *Let L be a connected compact simple Lie group of type A_5. We take a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{l}$ and label the Dynkin diagram of $\mathfrak{l}$ as follows*:

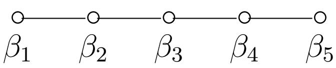


Figure 5.4.2: Dynkin diagram of type A_5.

Let $\mathfrak{k}$ be a Levi subalgebra whose root system is generated by $\{\beta_2, \beta_3, \beta_4, \beta_5\}$. We also define a reductive subalgebra $\mathfrak{m}$ by $\mathfrak{m} := \mathbb{R} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ where a simple system of $\mathfrak{m}$ is given by $\{\beta_2, \beta_4\}$ and the center of $\mathfrak{m}$ coincides with that of $\mathfrak{k}$. Denote by K and M the analytic subgroups of L with Lie algebras $\mathfrak{k}$ and $\mathfrak{m}$ respectively. Then we have

$$L = MB'K$$

for a subset B' of L^σ where σ is a Chevalley–Weyl involution of L with respect to $\mathfrak{t}$.

Proof. It suffices to consider the case where L is simply connected. We realize $L = \mathrm{SU}(6)$ as a matrix group:

$$L = \{g \in \mathrm{SL}(6, \mathbb{C}) : g\,{}^t\overline{g} = I_6\}.$$

Let us take the diagonal matrices consisting of purely imaginary numbers as a Cartan subalgebra $\mathfrak{t}$, and the complex conjugation as a Chevalley–Weyl involution σ of L. Here we note that both K and M are stable under the conjugation by any element of the maximal torus $\exp(\mathfrak{t})$ (independence of the choice of a Chevalley–Weyl involution). We define a simple system Φ of L by $\Phi := \{\varepsilon_i - \varepsilon_{i+1} : 1 \leq i \leq 5\}$ where ε_i are given by $\varepsilon_i : \operatorname{diag}(a_1, \dots, a_6) \mapsto a_i$. The Levi subgroup K and the closed subgroup M take the forms:

$$K = \left\{ \begin{pmatrix} \det(A)^{-1} & \\ & A \end{pmatrix} \in L : A \in \mathrm{U}(5) \right\},$$

$$M = \left\{ \begin{pmatrix} a^{-5} & & & \\ & aA & & \\ & & aB & \\ & & & a \end{pmatrix} \in L : a \in \mathrm{U}(1),\ A, B \in \mathrm{SU}(2) \right\}.$$

Then we define a subset B' of L^σ by $B' := B_1 B_2 B_3$ where

$$B_1 := \exp(\mathbb{R}(E_{1,2} - E_{2,1})),$$
$$B_2 := \exp(\mathbb{R}(E_{1,4} - E_{4,1})),$$
$$B_3 := \exp(\mathbb{R}(E_{1,6} - E_{6,1})).$$

We identify L/K with $\mathbb{C}P^5$ in the natural way. Through the identification, $B' \cdot K/K$ is identified with

$$\{[x_1 : x_2 : 0 : x_3 : 0 : x_4] \in \mathbb{C}P^5 : x_i \in \mathbb{R}\ (1 \leq i \leq 4)\}.$$

For any $z = [z_1 : \cdots : z_6] \in L/K$, we may and do assume that $\arg z_1 + 5 \arg z_6 = 0$. Then there exists $g \in M$ such that

$$g \cdot z = [|z_1| : \sqrt{|z_2|^2 + |z_3|^2} : 0 : \sqrt{|z_4|^2 + |z_5|^2} : 0 : |z_6|] \in B' \cdot K/K.$$

Thus we obtain

$$M \cdot B' \cdot K/K = L/K.$$

□

5.4.3 Proof of Lemma 5.3.9

Retain the setting in the proof of Proposition 5.3.8. Since the set of simple roots of M is given by $\{\alpha_1, \alpha_4, \alpha_6, \tilde{\beta}\}$ (Remark 5.2.3) and that of $\mathrm{L}_{\Pi'}$ by $\{\alpha_2\}^c$, we can see that Lemma 5.3.9 is followed by the lemma below.

Lemma 5.4.3. *Let L be a connected compact simple Lie group of type A_7, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{l}$ and σ a Chevalley–Weyl involution of L with respect to $\mathfrak{t}$. We label the extended Dynkin diagram of $\mathfrak{l}$ as follows (see* Plate I *of [Bo]).*

Define a semisimple subalgebra $\mathfrak{m}$ by $\mathfrak{m} := \mathfrak{su}(2)^4$ whose simple system is given by $\{\beta_2, \beta_4, \beta_6, \tilde{\beta}\}$, and a Levi subalgebra $\mathfrak{k}$ by $\mathfrak{k} := \mathbb{R} \oplus \mathfrak{su}(7)$ whose simple system is given by $\{\beta_1\}^c$. Let M and K denote the analytic subgroups of L with Lie algebras $\mathfrak{m}$ and $\mathfrak{k}$ respectively. Then we have

$$L = MB'K$$

for a subset B' of L^σ.

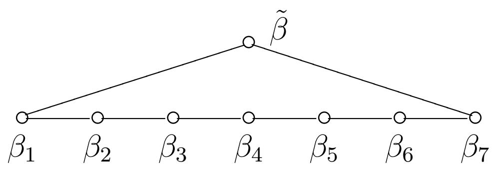


Figure 5.4.3: Extended Dynkin diagram of type A_7.

Proof. It suffices to consider the case where L is simply connected. We realize $L = \mathrm{SU}(8)$ as a matrix group as follows.

$$L = \{g \in \mathrm{SL}(8, \mathbb{C}) : g {}^t\overline{g} = I_8\}.$$

Let us take the diagonal matrices consisting of purely imaginary numbers as a Cartan subalgebra $\mathfrak{t}$, and the complex conjugation as a Chevalley–Weyl involution σ of L. Then we realize $K = \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(7))$ as follows:

$$K = \left\{ \begin{pmatrix} \det(A)^{-1} & \\ & A \end{pmatrix} \in L : A \in \mathrm{U}(7) \right\}.$$

We define a subgroup M' by

$$M' = \left\{ \begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & D_3 & \\ & & & D_4 \end{pmatrix} \in L : D_i \in \mathrm{SU}(2),\ 1 \leq i \leq 4 \right\} = \mathrm{SU}(2)^4,$$

and a subset B' of L^σ by $B' := B_1 B_2 B_3$ where

$$\begin{aligned} B_1 &:= \exp(\mathbb{R}(E_{1,3} - E_{3,1})), \\ B_2 &:= \exp(\mathbb{R}(E_{1,5} - E_{5,1})), \\ B_3 &:= \exp(\mathbb{R}(E_{1,7} - E_{7,1})). \end{aligned}$$

We identify L/K with $\mathbb{C}P^7$ in the natural way. Since $\mathrm{SU}(2)$ acts on S^2 transitively, for any $z = [z_1 : \cdots : z_8] \in L/K$ there exists $m \in M'$ such that

$$\begin{aligned} & m \cdot z = \\ & [\sqrt{|z_1|^2 + |z_2|^2} : 0 : \sqrt{|z_3|^2 + |z_4|^2} : 0 : \sqrt{|z_5|^2 + |z_6|^2} : 0 : \sqrt{|z_7|^2 + |z_8|^2} : 0] \\ & \in B' \cdot K/K. \end{aligned}$$

Thus we obtain

$$M' \cdot B' \cdot K/K = L/K.$$

Since M' is conjugate to M by an element of $L^\sigma = \mathrm{SO}(8)$, the lemma follows. □

Lemmas 5.4.1–5.4.3 complete the proofs in Section 5.3, and therefore we have finished the proof of the implication (ii) $\Rightarrow$ (i).

5.5 Proof of the implication (i) $\Rightarrow$ (ii) of Theorem 5.1.1

In this section, we prove that the list in Theorem 5.1.1 (ii) exhausts all the triple $(\mathfrak{g}, \mathfrak{l}_{\Pi'}, \mathfrak{l}_{\Pi''})$ satisfying the condition (i) in Theorem 5.1.1, and thus complete the proof of the remaining implication (i) $\Rightarrow$ (ii) of Theorem 5.1.1.

In the classical case (see [Ko4] for type A), invariant theory for quivers was used in the proof for the classification of $G = \mathrm{L}_{\Pi'} G^\sigma \mathrm{L}_{\Pi''}$, however, it is not obvious if the method is applicable to exceptional groups. Instead we use the general theory that strongly visible actions give rise to multiplicity-free representations [Ko3], and then apply the classification theorems of multiplicity-free tensor product representations by Littelmann [Li2] for the maximal parabolic case and Stembridge [St2] for the general case.

Proof of the implication (i) $\Rightarrow$ (ii) of Theorem 5.1.1. Let G be a connected simply connected compact simple Lie group, and $G_{\mathbb{C}}$ its complexification. We fix a Cartan subalgebra and a simple system Π of $\mathfrak{g}$, and denote by B the corresponding Borel subgroup of $G_{\mathbb{C}}$. For a given subset Π' of Π, we write $P_{\Pi'} \supset B$ for a parabolic subgroup whose reductive part is given by the complexification of a Levi subgroup $\mathrm{L}_{\Pi'}$ of G (we recall that Π' is a simple system of $\mathrm{L}_{\Pi'}$). Also, we denote by ω_i a fundamental weight of G, which corresponds to a simple root α_i (we label the Dynkin diagrams of type E_6 and type E_7 following Bourbaki [Bo] as in Section 5.3), and by π_λ a finite dimensional irreducible representation with highest weight λ.

We let λ be a unitary character of $\mathrm{L}_{\Pi'}$, and extend it to a holomorphic character of $P_{\Pi'}$. By the Borel–Weil theory, we can realize the contragradient representation π_λ^* of π_λ as the space of holomorphic sections $\mathcal{O}(G_{\mathbb{C}}/P_{\Pi'}, \mathcal{L}_{-\lambda})$ of the line bundle $\mathcal{L}_{-\lambda}$ on $G_{\mathbb{C}}/P_{\Pi'}$.

Let us suppose that the condition (i) holds. Then the diagonal action of G on $G_{\mathbb{C}}/P_{\Pi'} \times G_{\mathbb{C}}/P_{\Pi''}$ is strongly visible, and thus by Fact 5.6.3 below and the Borel–Weil theory,

$$\text{the tensor product representation } \pi_\lambda^* \otimes \pi_\mu^* \text{ is multiplicity-free } \cdots\cdots\diamondsuit$$

where λ and μ are any unitary characters of $\mathrm{L}_{\Pi'}$ and $\mathrm{L}_{\Pi''}$ respectively.

On the other hand, we can extract the following results from the classification theorems [Li2] and [St2] on when $\pi_\lambda \otimes \pi_\mu$ is multiplicity-free for the maximal parabolic case and for the general case, respectively.

Fact 5.5.1. *Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra of type E_6. Let I and J be non-empty subsets of $\{1,2,3,4,5,6\}$. Then the tensor product of $\mu = \sum_{i\in I} m_i\omega_i$ and $\nu = \sum_{j\in J} n_j\omega_j$ is multiplicity-free for arbitrary non-negative integers m_i ($i \in I$) and n_j ($j \in J$) if and only if one of the following conditions holds up to switch of the factors I and J.*

$$\begin{array}{ll} (i) & I = \{1\} \text{ or } \{6\},\ J = \{j\} \text{ with } j \neq 4. \\ (ii) & I = \{1\} \text{ or } \{6\},\ J = \{1,6\}. \end{array}$$

Fact 5.5.2. *Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra of type E_7. Let I and J be non-empty subsets of $\{1,2,3,4,5,6,7\}$. Then the tensor product of $\mu = \sum_{i\in I} m_i\omega_i$ and $\nu = \sum_{j\in J} n_j\omega_j$ is multiplicity-free for arbitrary non-negative integers m_i ($i \in I$) and n_j ($j \in J$) if and only if the following condition holds up to switch of the factors I and J.*

$$(i) \quad I = \{7\},\ J = \{j\} \text{ with } j = 1, 2 \text{ or } 7.$$

Fact 5.5.3. *Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra of type* E_8, F_4 *or* G_2. *Then there is no pair of non-empty subsets I and J of $\{1, ..., r\}$ with $r = \operatorname{rank} \mathfrak{g}$, which satisfies the following:*

The tensor product of $\mu = \sum_{i \in I} m_i \omega_i$ and $\nu = \sum_{j \in J} n_j \omega_j$ is multiplicity-free for arbitrary non-negative integers m_i $(i \in I)$ and n_j $(j \in J)$.

By the comparison of $\diamondsuit$ with Facts 5.5.1, 5.5.2 and 5.5.3, the triple $(\mathfrak{g}, \Pi', \Pi'')$ must be in the list given in Theorem 5.1.1 (ii). Therefore the implication (i) $\Rightarrow$ (ii) holds. $\square$

5.6 Application to representation theory

In this section, we shall see a generalized Cartan decomposition leads to three kinds of multiplicity-free representations by using the framework of visible actions ("triunity" à la [Ko1]). The notion of (strongly) visible actions on complex manifolds was introduced by T. Kobayashi. Let us recall the definition [Ko2]

Definition 5.6.1. We say a biholomorphic action of a Lie group G on a complex manifold D is *strongly visible* if the following two conditions are satisfied:

1. There exists a real submanifold S such that (we call S a "slice")
$$D' := G \cdot S \text{ is an open subset of } D.$$
2. There exists an antiholomorphic diffeomorphism σ of D' such that
$$\begin{aligned} \sigma|_S &= \mathrm{id}_S, \\ \sigma(G \cdot x) &= G \cdot x \ \text{ for any } x \in S. \end{aligned}$$

Definition 5.6.2. In the above setting, we say the action of G on D is S-visible. This terminology will be used also if S is just a subset of D.

Let G be a connected compact Lie group and L, H its Levi subgroups. Then G/L, G/H and $(G \times G)/(L \times H)$ are complex manifolds. If the triple (G, L, H) satisfies $G = LG^{\sigma}H$, the following three group-actions are all strongly visible:

$$\begin{aligned} L &\curvearrowright G/H, \\ H &\curvearrowright G/L, \\ \Delta(G) &\curvearrowright (G \times G)/(L \times H). \end{aligned}$$

Here $\Delta(G)$ is defined by $\Delta(G) := \{(x, y) \in G \times G : x = y\}$. The following fact ([Ko3, Theorem 4.3]) leads us to multiplicity-free representations:

Fact 5.6.3. *Let G be a Lie group and $\mathcal{V}$ a G-equivariant Hermitian holomorphic vector bundle on a connected complex manifold D. If the following three conditions from* (1) *to* (3) *are satisfied, then any unitary representation that can be embedded in the vector space $\mathcal{O}(D, \mathcal{V})$ of holomorphic sections of $\mathcal{V}$ decomposes multiplicity-freely:*

1. *The action of G on D is S-visible. That is, there exist a subset $S \subset D$ and an anti-holomorphic diffeomorphism σ of D' satisfying the conditions given in Definition* 5.6.1*. Further, there exists an automorphism $\hat{\sigma}$ of G such that $\sigma(g \cdot x) = \hat{\sigma}(g) \cdot \sigma(x)$ for any $g \in G$ and $x \in D'$.*

2. *For any $x \in S$, the fiber $\mathcal{V}_x$ at x decomposes as the multiplicity free sum of irreducible unitary representations of the isotropy subgroup G_x. Let $\mathcal{V}_x = \bigoplus_{1 \le i \le n(x)} \mathcal{V}_x^{(i)}$ denote the irreducible decomposition of $\mathcal{V}_x$.*

3. *σ lifts to an antiholomorphic automorphism $\tilde{\sigma}$ of $\mathcal{V}$ and satisfies $\tilde{\sigma}(\mathcal{V}_x^{(i)}) = \mathcal{V}_x^{(i)}$ for any i $(1 \le i \le n(x))$ for each $x \in S$.*

By using the Borel–Weil theory together with Fact 5.6.3 and our generalized Cartan decompositions, we obtain the following two corollaries of Theorem 5.1.1. Let G be a connected compact exceptional simple Lie group and ω_i $(1 \le i \le \operatorname{rank} \mathfrak{g})$ its fundamental weights (we label the Dynkin diagrams following Bourbaki [Bo] as in Section 5.3).

Corollary 5.6.4. *If the triple (G, L, λ) is an entry in* Tables *5.6.1 or 5.6.2, then the restriction $\pi_\lambda|_L$ of the irreducible representation π_λ of G with highest weight λ to L decomposes multiplicity-freely. Here, a and b are arbitrary non-negative integers.*

Table 5.6.1: Maximal parabolic type.

G	L	λ	Conditions.
E_6	$L_{\{\alpha_i\}^c}$	$a\omega_j$	$i = 1$ or 6, $j \neq 4$.
			$i \neq 4$, $j = 1$ or 6.
E_7	$L_{\{\alpha_i\}^c}$	$a\omega_j$	$i = 7$, $j = 1, 2,$ or 7.
			$i = 1, 2$ or 7, $j = 7$.

Table 5.6.2: Non-maximal parabolic type.

G	L	λ	Conditions.
E_6	$L_{\{\alpha_1, \alpha_6\}^c}$	$a\omega_i$	$i = 1$ or 6.
E_6	$L_{\{\alpha_i\}^c}$	$a\omega_1 + b\omega_6$	$i = 1$ or 6.

Corollary 5.6.5. *The tensor product representation $\pi_\lambda \otimes \pi_\mu$ of any two irreducible representations π_λ and π_μ of G with highest weights λ and μ listed in the below* Tables *5.6.3 or 5.6.4 decomposes as a multiplicity-free sum of irreducible representations of G.*
Here, a, b and c are arbitrary non-negative integers.

We note that the condition (2) of Fact 5.6.3 is automatically satisfied since the fiber of a holomorphic vector bundle is one-dimensional in the setting of the Borel–Weil Theory.

Table 5.6.3: Maximal parabolic type.

G	(λ, μ)	Conditions.
E_6	$(a\omega_i, b\omega_j)$	$i = 1$ or 6, $j \neq 4$.
E_7	$(a\omega_i, b\omega_j)$	$i = 7$, $j = 1, 2$, or 7.

Table 5.6.4: Non-maximal parabolic type.

G	(λ, μ)	Conditions.
E_6	$(a\omega_1 + b\omega_6, c\omega_i)$	$i = 1$ or 6.

Remark 5.6.6. Littelmann [Li2] classified for any simple algebraic group G over any algebraically closed field of characteristic zero, all the pairs of maximal parabolic subgroups P_ω and $P_{\omega'}$ corresponding to fundamental weights ω and ω' respectively such that the tensor product representation $\pi_{n\omega} \otimes \pi_{m\omega'}$ decomposes multiplicity-freely for arbitrary non-negative integers n and m. (His classification is exactly Table 5.6.3 and does not include Table 5.6.4 in the exceptional case.) Moreover, he found the branching rules of $\pi_{n\omega} \otimes \pi_{m\omega'}$ and the restriction of $\pi_{n\omega}$ to the maximal Levi subgroup $L_{\omega'}$ of $P_{\omega'}$ for any pair (ω, ω') that admits a G-spherical action on $G/P_\omega \times G/P_{\omega'}$.

Remark 5.6.7. Stembridge [St2] gave a complete list of a pair (μ, ν) of highest weights such that the corresponding tensor product representation $\pi_\mu \otimes \pi_\nu$ is multiplicity-free for any complex simple Lie algebra. His method is combinatorial. He also classified all the pairs $(\mu, \mathfrak{l})$ of highest weights and Levi subalgebras with the restrictions $\pi_\mu|_{\mathfrak{l}}$ to Levi subalgebras multiplicity-free. Our approach has given a geometric proof of a part of his work based on generalized Cartan decompositions.

We hope that further applications of Theorem 5.1.1 and Fact 5.6.3 to representation theory will be discussed in a future paper.

Chapter 6

Visible actions and seeds for the orthogonal group

6.1 Introduction for Chapter 6

Let G be a connected compact Lie group, T a maximal torus, L_j Levi subgroups containing T $(j = 1, 2)$ and σ a Chevalley–Weyl involution of G with respect to T. We state Corollary 1.2.1 again.

Corollary 6.1.1 (Corollary 1.2.1)**.** *We denote by $G_{\mathbb{C}}$ and $(L_j)_{\mathbb{C}}$ the complexifications of G and L_j, respectively $(j = 1, 2)$. We let P_j be a parabolic subgroup of $G_{\mathbb{C}}$ with Levi subgroup $(L_j)_{\mathbb{C}}$, and put $\mathcal{P}_j = G_{\mathbb{C}}/P_j$ $(j = 1, 2)$. Then the following eleven conditions are equivalent.*

(i) The multiplication mapping $L_1 \times G^{\sigma} \times L_2 \to G$ is surjective.

(ii) The natural action $L_1 \curvearrowright \mathcal{P}_2$ is strongly visible.

(iii) The natural action $L_2 \curvearrowright \mathcal{P}_1$ is strongly visible.

(iv) The diagonal action $\Delta(G) \curvearrowright \mathcal{P}_1 \times \mathcal{P}_2$ is strongly visible.

(v) Any irreducible representation of G, which belongs to $\mathcal{P}_2$-series is multiplicity-free when restricted to L_1.

(vi) Any irreducible representation of G, which belongs to $\mathcal{P}_1$-series is multiplicity-free when restricted to L_2.

(vii) The tensor product of arbitrary two irreducible representations π_1 and π_2 of G, which belong to $\mathcal{P}_1$ and $\mathcal{P}_2$-series, respectively, is multiplicity-free.

(viii) $\mathcal{P}_2$ is a spherical variety of $(L_1)_{\mathbb{C}}$.

(ix) $\mathcal{P}_1$ is a spherical variety of $(L_2)_{\mathbb{C}}$.

(x) $\mathcal{P}_1 \times \mathcal{P}_2$ is a spherical variety of $\Delta(G_{\mathbb{C}})$.

(xi) The pair (Π_1, Π_2) is one of the entries listed in Theorem 1.1.1 up to switch of the factors.

Here an irreducible representation of G is in $\mathcal{P}_j$-series if it is a holomorphically induced representation from a unitary character of the Levi subgroup L_j $(j = 1, 2)$.

We also show again the definition of a visible action for the sake of the convenience.

Definition 6.1.2 (Definition 1.0.1)**.** We say a holomorphic action of a Lie group G on a complex manifold X is strongly visible if the following two conditions are satisfied:

1. There exists a real submanifold S (called a "slice") such that

$$X' := G \cdot S \text{ is an open subset of } X.$$

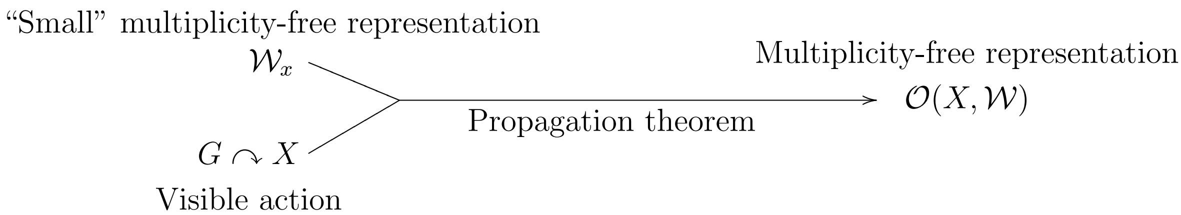
2. There exists an anti-holomorphic diffeomorphism σ of X' such that

$$\begin{aligned} \sigma|_S &= \mathrm{id}_S, \\ \sigma(G \cdot x) &= G \cdot x \text{ for any } x \in X'. \end{aligned}$$

In the above setting, we say the action of G on X is S-visible. This terminology will be used also if S is just a subset of X.

The above corollary says that for a triple (G, L_1, L_2), if the tensor product representation $\mathrm{ind}_{L_1}^G \chi_1 \otimes \mathrm{ind}_{L_2}^G \chi_2$ of holomorphically induced representations $\mathrm{ind}_{L_1}^G \chi_1$ and $\mathrm{ind}_{L_2}^G \chi_2$ is multiplicity-free for any unitary characters χ_1 and χ_2 of L_1 and L_2, respectively, then we have a visible action of G on a generalized flag variety $(G \times G)/(L_1 \times L_2)$. However, in general the multiplicity-freeness property of $\mathrm{ind}_{L_1}^G \chi_1 \otimes \mathrm{ind}_{L_2}^G \chi_2$ depends on the choice of χ_1 and χ_2. To understand such representations whose multiplicity-freeness property depends on the choice of characters from the view point of visible actions, we recall Kobayashi's theory on the propagation of the multiplicity-freeness property under visible actions (Fact 1.0.4). According to Fact 1.0.4 that we state again as Fact 6.3.1 below for the convenience, we can reduce complicated multiplicity-free theorems to a pair of data:

- visible actions on complex manifolds, and
- much simpler multiplicity-free representations (*seeds* of multiplicity-free representations introduced by Kobayashi).



With this picture in mind, we consider the following problem:

> Let G be a Lie group and V a multiplicity-free representation of G. Then find a visible action of G on a connected complex manifold X and a seed $\mathcal{W}_x$ as the isotropy representation (at a generic point x of X) on the fiber of a G-equivariant Hermitian holomorphic vector bundle $\mathcal{W}$ on X such that V can be G-equivariantly embedded into the space $\mathcal{O}(X, \mathcal{W})$ of holomorphic sections of the vector bundle $\mathcal{W}$.

In this chapter, we consider this problem in a special setting. Namely, we deal with multiplicity-free tensor product representations of compact Lie groups. As mentioned in Chapter 1, Kobayashi gave an answer for the unitary group [Ko1]. Following his argument, we treat the case of the orthogonal group, and Theorem 1.3.2 (Theorem 6.3.2 in this chapter) gives an answer for this case. Below, the proof for Theorem 1.3.2 is given after some computations of seeds that are exhibited in Proposition 1.3.1 (Proposition 6.2.1 in this chapter).

We fix some notations. We denote by $\Pi = \{\alpha_i\}_{1\leq i\leq [N/2]}$ a simple system of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ of $G = \mathrm{Spin}(N)$ with respect to its maximal torus T. We denote by $\{H_i\}_{1\leq i\leq [N/2]}$ the dual basis of Π. We define a subgroup M of $\mathrm{Spin}(2n+1)$ as follows.

$$M := \left\{\exp\left(\sqrt{-1}m\pi H_1\right)\right\}_{1\leq m\leq 4} \cdot \mathrm{Spin}(2n-1), \tag{6.1.1}$$

where exp denotes the exponential mapping, and the simple system of $\mathrm{Spin}(2n-1)$ is given by $\{\alpha_k \in \Pi : 2 \leq k \leq n\}$.

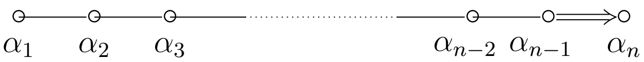


Figure 6.1.1: Dynkin diagram of type B_n

Figure 6.1.2: Dynkin diagram of type D_n

6.2 Proof for Proposition 1.3.1 (computation of seeds)

In this section we give a proof of the following multiplicity-free results.

Proposition 6.2.1 (Proposition 1.3.1)**.** *We denote by* $\mathbf{1}$, $\mathbb{C}^N$ *and* Spin_N *for the one-dimensional trivial representation, the natural representation and the spin representation of* $\mathrm{Spin}(N)$*, respectively. Then the following hold.*

(1) *One-dimensional representations are multiplicity-free.*

(2) $\mathbf{1}$, $\mathbb{C}^N$ *and* Spin_N *are multiplicity-free as representations of a maximal torus* T *of* $\mathrm{Spin}(N)$.

(3) $\Lambda^i(\mathbb{C}^N)$ *is multiplicity-free as a representation of a maximal Levi subgroup* $\mathrm{U}(j) \times \mathrm{SO}(N-2j)$ *of* $\mathrm{SO}(N)$ *(when* N *is even and* $i = N/2$*, we replace* $\Lambda^{N/2}(\mathbb{C}^N)$ *by its* $\mathrm{SO}(N)$*-irreducible constituent whose highest weight is* $2\omega_{N/2-1}$ *or* $2\omega_{N/2}$*) if the following condition* (3-1) *or* (3-2) *is satisfied* $(1 \leq i, j \leq [N/2])$*.*

  (3-1) N *is odd.*

  (3-2) N *is even and* i, j *satisfy*

    (3-2-1) $i + j \leq N/2$,

(3-2-2) $j = N/2$ *or*
(3-2-3) $i = N/2$.

(4) Spin_N *is multiplicity-free as a representation of* M*, where* N *is odd and* M *as in* (6.1.1).

We note that the multiplicity-freeness property in the case (1) is trivial. Also, the case (2) is clear since the one-dimensional trivial representation, the natural representation and the spin representation are known to be weight multiplicity-free. In the case (4), we can see that the restriction of the spin representation Spin_N of $\mathrm{Spin}(N)$ to M decomposes multiplicity-freely as follows.

$$\mathrm{Spin}_N|_M = \chi \boxtimes \mathrm{Spin}_{N-2} \oplus \chi' \boxtimes \mathrm{Spin}_{N-2}. \tag{6.2.1}$$

Here we denote by χ and χ' two non-equivalent faithful one-dimensional representations of the cyclic group $\{\exp(\sqrt{-1}m\pi H_1)\}_{1 \le m \le 4} \simeq \mathbb{Z}/4\mathbb{Z}$. We note that the group M is the quotient of the direct product group

$$M' := \left\{\exp\left(\sqrt{-1}m\pi H_1\right)\right\}_{1 \le m \le 4} \times \mathrm{Spin}(N-2) \tag{6.2.2}$$

by the subgroup $\mathrm{diag}(\mathbb{Z}/2\mathbb{Z}) \simeq \{(1,1),(-1,-1)\}$ of its center $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \simeq \{\pm 1, \pm\sqrt{-1}\} \times \{\pm 1\}$. Hence we regard representations of M as those descended from representations of M'. In the rest of this section, we prove the case (3). More strongly, we give the branching law for the restriction of the alternating tensor product representation $\Lambda^i(\mathbb{C}^N)$ of $GL(N,\mathbb{C})$ to the subgroup $GL(j,\mathbb{C}) \times SO(N-2j,\mathbb{C})$ $(1 \le i,j \le [N/2])$. It is well-known that the restriction of $\Lambda^i(\mathbb{C}^N)$ from $GL(N,\mathbb{C})$ to $O(N,\mathbb{C})$ is irreducible, and that from $GL(N,\mathbb{C})$ to $GL(2j,\mathbb{C}) \times GL(N-2j,\mathbb{C})$ is not irreducible but multiplicity-free. Below we find that $\Lambda^i(\mathbb{C}^N)$ (or its irreducible constituents if N is even and $i = N/2$) is still multiplicity-free when it is further restricted to $GL(j,\mathbb{C}) \times SO(N-2j,\mathbb{C})$ by using the branching law for $\Lambda^i(\mathbb{C}^N)$ under the assumption that i, j and N satisfy at least one of the conditions in Proposition 6.2.1 (3).

We fix some notations. We denote by $V_{a,b}^{GL(N)}$ the finite-dimensional irreducible representation of $GL(N,\mathbb{C})$ with highest weight $(\overbrace{1,\cdots,1}^{a},0,\cdots,0,\overbrace{-1,\cdots,-1}^{b})$ $(a+b \le N)$. In particular, $V_{a,0}^{GL(N)}$ is the a-th alternating tensor product representation $\Lambda^a(\mathbb{C}^N)$, and $V_{0,b}^{GL(N)}$ is the dual representation $\left(\Lambda^b(\mathbb{C}^N)\right)^\vee$ of $\Lambda^b(\mathbb{C}^N)$.

Lemma 6.2.2. (1) *For the symmetric subgroup* $GL(M,\mathbb{C}) \times GL(N,\mathbb{C})$ *of* $GL(M+N,\mathbb{C})$*, we have* $(0 \le l \le M+N)$

$$V_{l,0}^{GL(M+N)}|_{GL(M,\mathbb{C}) \times GL(N,\mathbb{C})} \simeq \bigoplus_{\substack{a+b=l,\\ a \le M,\, b \le N}} V_{a,0}^{GL(M)} \boxtimes V_{b,0}^{GL(N)}. \tag{6.2.3}$$

(2) *We embed* $GL(N,\mathbb{C})$ *into* $GL(2N,\mathbb{C})$ *by the homomorphism defined by* $g \mapsto (g, {}^t g^{-1})$*, which factors* $GL(N,\mathbb{C}) \times GL(N,\mathbb{C})$ *(here* ${}^t g$ *denotes the transpose of* g*). Then we have* $(0 \le l \le 2N)$

$$V_{l,0}^{GL(2N)}|_{GL(N,\mathbb{C})} \simeq \bigoplus_{\substack{a+b \le \min(l,2N-l),\\ a+b \equiv l \bmod 2}} V_{a,b}^{GL(N)}. \tag{6.2.4}$$

(3) *For the natural inclusions* $\mathrm{SO}(N,\mathbb{C}) \subset \mathrm{O}(N,\mathbb{C}) \subset \mathrm{GL}(N,\mathbb{C})$ *of classical groups, we have the following* $(0 \le l, l' \le N)$.

- $V_{l,0}^{\mathrm{GL}(N)}|_{\mathrm{O}(N,\mathbb{C})}$ *is irreducible.*
- $V_{l,0}^{\mathrm{GL}(N)}|_{\mathrm{SO}(N,\mathbb{C})}$ *is irreducible if* $N \neq 2l$, *and decomposes into the sum of two irreducible representations* $\left(V_{\frac{N}{2},0}^{\mathrm{GL}(N)}\right)^{even}$ *and* $\left(V_{\frac{N}{2},0}^{\mathrm{GL}(N)}\right)^{odd}$ *of* $\mathrm{SO}(N,\mathbb{C})$ *if* $N = 2l$.
- $V_{l,0}^{\mathrm{GL}(N)} \simeq V_{l',0}^{\mathrm{GL}(N)}$ *as representations of* $\mathrm{SO}(N,\mathbb{C})$ *if* $l + l' = N$.

Proof. Both (1) and (3) are well-known. We give a proof for (2). By (6.2.3), we have

$$V_{l,0}^{\mathrm{GL}(2N)}|_{\mathrm{GL}(N,\mathbb{C})\times\mathrm{GL}(N,\mathbb{C})} \simeq \bigoplus_{\substack{a+b=l,\\ a,b\le N}} V_{a,0}^{\mathrm{GL}(N)} \boxtimes V_{b,0}^{\mathrm{GL}(N)}. \tag{6.2.5}$$

Using the Littlewood–Richardson rule, we further restrict (6.2.5) to $\mathrm{GL}(N,\mathbb{C})$ that is embedded into $\mathrm{GL}(2N,\mathbb{C})$ by the homomorphism defined by $g \mapsto (g, {}^t g^{-1})$, which factors $\mathrm{GL}(N,\mathbb{C}) \times \mathrm{GL}(N,\mathbb{C})$.

$$\begin{aligned}
(6.2.5)|_{\mathrm{GL}(N,\mathbb{C})} &\simeq \bigoplus_{\substack{a+b=l,\\ a,b\le N}} V_{a,0}^{\mathrm{GL}(N)} \otimes \left(V_{b,0}^{\mathrm{GL}(N)}\right)^{\vee} \\
&\simeq \bigoplus_{\substack{a+b=l,\\ a,b\le N}} V_{a,0}^{\mathrm{GL}(N)} \otimes V_{0,b}^{\mathrm{GL}(N)} \\
&\simeq \bigoplus_{\substack{a+b\le \min(l,2N-l),\\ a+b\equiv l \bmod 2}} V_{a,b}^{\mathrm{GL}(N)}.
\end{aligned}$$

$\square$

Proposition 6.2.3. *We embed* $\mathrm{GL}(j,\mathbb{C}) \times \mathrm{SO}(N-2j,\mathbb{C})$ *into* $\mathrm{GL}(2j,\mathbb{C}) \times \mathrm{GL}(N-2j,\mathbb{C})$ *by the map* $(g,h) \mapsto ((g, {}^t g^{-1}), h)$, *which factors* $(\mathrm{GL}(j,\mathbb{C}) \times \mathrm{GL}(j,\mathbb{C})) \times \mathrm{GL}(N-2j,\mathbb{C})$.

(1) *Suppose that* N *is odd. Then we have* $(1 \le i, j \le [\frac{N}{2}])$

$$V_{i,0}^{\mathrm{GL}(N)}|_{\mathrm{GL}(j,\mathbb{C})\times\mathrm{SO}(N-2j,\mathbb{C})} \simeq \bigoplus_{\max(i+2j-N,0)\le l\le\min(i,2j)} \bigoplus_{\substack{a+b\le\min(l,2j-l),\\ a+b\equiv l \bmod 2}} V_{a,b}^{\mathrm{GL}(j)} \boxtimes V_{i-l,0}^{\mathrm{GL}(N-2j)}|_{\mathrm{SO}(N-2j,\mathbb{C})}. \tag{6.2.6}$$

(2) *Suppose that* N *is even and* $i + j \le \frac{N}{2}$ $(1 \le i, j \le \frac{N}{2})$. *Then we have*

$$V_{i,0}^{\mathrm{GL}(N)}|_{\mathrm{GL}(j,\mathbb{C})\times\mathrm{SO}(N-2j,\mathbb{C})} \simeq \bigoplus_{0\le l\le\min(i,2j)} \bigoplus_{\substack{a+b\le\min(l,2j-l),\\ a+b\equiv l \bmod 2}} V_{a,b}^{\mathrm{GL}(j)} \boxtimes V_{i-l,0}^{\mathrm{GL}(N-2j)}|_{\mathrm{SO}(N-2j,\mathbb{C})}. \tag{6.2.7}$$

(3) *Suppose that N is even. Then we have* $(1 \leq i \leq \frac{N}{2})$

$$V_{i,0}^{\mathrm{GL}(N)}|_{\mathrm{GL}(\frac{N}{2},\mathbb{C})} \simeq \bigoplus_{\substack{a+b \leq i, \\ a+b \equiv i \bmod 2}} V_{a,b}^{\mathrm{GL}(\frac{N}{2})}. \tag{6.2.8}$$

(4) *Suppose that N is even. Then we have* $(1 \leq j \leq \frac{N}{2} - 1)$

$$\begin{aligned} & V_{\frac{N}{2},0}^{\mathrm{GL}(N)}|_{\mathrm{GL}(j,\mathbb{C}) \times \mathrm{SO}(N-2j,\mathbb{C})} \simeq \qquad (6.2.9) \\ & 2 \times \left(\bigoplus_{\max(2j-\frac{N}{2},0) \leq l \leq j-1} \bigoplus_{\substack{a+b \leq l, \\ a+b \equiv l \bmod 2}} V_{a,b}^{\mathrm{GL}(j)} \boxtimes V_{N/2-2j+l,0}^{\mathrm{GL}(N-2j)}|_{\mathrm{SO}(N-2j,\mathbb{C})} \right) \\ & \oplus \bigoplus_{\substack{a+b \leq j, \\ a+b \equiv j \bmod 2}} V_{a,b}^{\mathrm{GL}(j)} \boxtimes \left(\left(V_{N/2-j,0}^{\mathrm{GL}(N-2j)} \right)^{even} \oplus \left(V_{N/2-j,0}^{\mathrm{GL}(N-2j)} \right)^{odd} \right). \end{aligned}$$

Here by $2 \times V$ we mean that a representation V appears twice, that is, the multiplicity of V is two.

Proof. Using the fact that the embedding of $\mathrm{GL}(j,\mathbb{C}) \times \mathrm{SO}(N-2j,\mathbb{C})$ into $\mathrm{GL}(N,\mathbb{C})$ factors the symmetric subgroup $\mathrm{GL}(2j,\mathbb{C}) \times \mathrm{GL}(N-2j,\mathbb{C})$, we have

$$\begin{aligned} & V_{i,0}^{\mathrm{GL}(N)}|_{\mathrm{GL}(j,\mathbb{C}) \times \mathrm{SO}(N-2j,\mathbb{C})} \\ & \overset{(6.2.3)}{\simeq} \bigoplus_{\max(i+2j-N,0) \leq l \leq \min(i,2j)} V_{l,0}^{\mathrm{GL}(2j)}|_{\mathrm{GL}(j,\mathbb{C})} \boxtimes V_{i-l,0}^{\mathrm{GL}(N-2j)}|_{\mathrm{SO}(N-2j,\mathbb{C})} \\ & \overset{(6.2.4)}{\simeq} \bigoplus_{\max(i+2j-N,0) \leq l \leq \min(i,2j)} \bigoplus_{\substack{a+b \leq \min(l, 2j-l), \\ a+b \equiv l \bmod 2}} V_{a,b}^{\mathrm{GL}(j)} \boxtimes V_{i-l,0}^{\mathrm{GL}(N-2j)}|_{\mathrm{SO}(N-2j,\mathbb{C})}. \qquad (6.2.10) \end{aligned}$$

This proves (1). We shall prove (2), (3) and (4) separately by using (6.2.10).

(2) If N is even and $i + j \leq \frac{N}{2}$, then we can rewrite (6.2.10) as follows.

$$(6.2.10) = \bigoplus_{0 \leq l \leq \min(i,2j)} \bigoplus_{\substack{a+b \leq \min(l, 2j-l), \\ a+b \equiv l \bmod 2}} V_{a,b}^{\mathrm{GL}(j)} \boxtimes V_{i-l,0}^{\mathrm{GL}(N-2j)}|_{\mathrm{SO}(N-2j,\mathbb{C})}.$$

(3) By (6.2.10) for the case $j = \frac{N}{2}$, we have

$$V_{i,0}^{\mathrm{GL}(N)}|_{\mathrm{GL}(\frac{N}{2},\mathbb{C})} \simeq \bigoplus_{\substack{a+b \leq i, \\ a+b \equiv i \bmod 2}} V_{a,b}^{\mathrm{GL}(\frac{N}{2})}.$$

(4) By (6.2.10) and Lemma 6.2.2 (3), we have

$$
\begin{aligned}
& V^{\mathrm{GL}(N)}_{\frac{N}{2},0}|_{\mathrm{GL}(j,\mathbb{C})\times\mathrm{SO}(N-2j,\mathbb{C})} \\
\simeq & \bigoplus_{\max(2j-\frac{N}{2},0)\le l\le j-1} \bigoplus_{\substack{a+b\le l,\\ a+b\equiv l \bmod 2}} V^{\mathrm{GL}(j)}_{a,b}\boxtimes V^{\mathrm{GL}(N-2j)}_{N/2-l,0}|_{\mathrm{SO}(N-2j,\mathbb{C})} \\
& \oplus \bigoplus_{\substack{a+b\le j,\\ a+b\equiv j \bmod 2}} V^{\mathrm{GL}(j)}_{a,b}\boxtimes V^{\mathrm{GL}(N-2j)}_{N/2-j,0}|_{\mathrm{SO}(N-2j,\mathbb{C})} \\
& \oplus \bigoplus_{j+1\le l\le \min(\frac{N}{2},2j)} \bigoplus_{\substack{a+b\le 2j-l,\\ a+b\equiv l \bmod 2}} V^{\mathrm{GL}(j)}_{a,b}\boxtimes V^{\mathrm{GL}(N-2j)}_{N/2-l,0}|_{\mathrm{SO}(N-2j,\mathbb{C})} \\
\simeq & \bigoplus_{\max(2j-\frac{N}{2},0)\le l\le j-1} \bigoplus_{\substack{a+b\le l,\\ a+b\equiv l \bmod 2}} V^{\mathrm{GL}(j)}_{a,b}\boxtimes V^{\mathrm{GL}(N-2j)}_{N/2-2j+l,0}|_{\mathrm{SO}(N-2j,\mathbb{C})} \\
& \oplus \bigoplus_{\substack{a+b\le j,\\ a+b\equiv j \bmod 2}} V^{\mathrm{GL}(j)}_{a,b}\boxtimes \left(\left(V^{\mathrm{GL}(N-2j)}_{N/2-j,0}\right)^{even}\oplus\left(V^{\mathrm{GL}(N-2j)}_{N/2-j,0}\right)^{odd}\right) \\
& \oplus \bigoplus_{\max(2j-\frac{N}{2},0)\le l'\le j-1} \bigoplus_{\substack{a+b\le l',\\ a+b\equiv l' \bmod 2}} V^{\mathrm{GL}(j)}_{a,b}\boxtimes V^{\mathrm{GL}(N-2j)}_{N/2-2j+l',0}|_{\mathrm{SO}(N-2j,\mathbb{C})} \\
= & \ 2\times\left(\bigoplus_{\max(2j-\frac{N}{2},0)\le l\le j-1} \bigoplus_{\substack{a+b\le l,\\ a+b\equiv l \bmod 2}} V^{\mathrm{GL}(j)}_{a,b}\boxtimes V^{\mathrm{GL}(N-2j)}_{N/2-2j+l,0}|_{\mathrm{SO}(N-2j,\mathbb{C})}\right) \\
& \oplus \bigoplus_{\substack{a+b\le j,\\ a+b\equiv j \bmod 2}} V^{\mathrm{GL}(j)}_{a,b}\boxtimes \left(\left(V^{\mathrm{GL}(N-2j)}_{N/2-j,0}\right)^{even}\oplus\left(V^{\mathrm{GL}(N-2j)}_{N/2-j,0}\right)^{odd}\right).
\end{aligned}
$$

On the right hand side of the second isomorphism, we have used Lemma 6.2.2 (3), namely, the isomorphism $V^{\mathrm{GL}(N-2j)}_{N/2-l,0}|_{\mathrm{SO}(N-2j,\mathbb{C})}\simeq V^{\mathrm{GL}(N-2j)}_{N/2-2j+l,0}|_{\mathrm{SO}(N-2j,\mathbb{C})}$ for the first sum, and the irreducible decomposition $V^{\mathrm{GL}(N-2j)}_{N/2-j,0}|_{\mathrm{SO}(N-2j,\mathbb{C})}=\left(V^{\mathrm{GL}(N-2j)}_{N/2-j,0}\right)^{even}\oplus\left(V^{\mathrm{GL}(N-2j)}_{N/2-j,0}\right)^{odd}$ for the second sum. Also, we put $l'=2j-l$ in the third sum.

$\square$

Proposition 6.2.4. *We embed* $\mathrm{GL}(j,\mathbb{C})$ *into* $\mathrm{GL}(j,\mathbb{C})\times\mathrm{GL}(j,\mathbb{C})$ *by* $g\mapsto(g,{}^tg^{-1})$ *as in* Lemma 6.2.2 (2). *Then we have the inclusions of classical groups* (*the natural inclusions except for the inclusion* $\mathrm{GL}(j,\mathbb{C})\subset\mathrm{GL}(j,\mathbb{C})\times\mathrm{GL}(j,\mathbb{C})$)

$$
\begin{aligned}
\mathrm{GL}(N,\mathbb{C}) &\supset \mathrm{GL}(2j,\mathbb{C})\times\mathrm{GL}(N-2j,\mathbb{C})\\
&\supset (\mathrm{GL}(j,\mathbb{C})\times\mathrm{GL}(j,\mathbb{C}))\times\mathrm{GL}(N-2j,\mathbb{C})\\
&\supset \mathrm{GL}(j,\mathbb{C})\times\mathrm{GL}(N-2j,\mathbb{C})\\
&\supset \mathrm{GL}(j,\mathbb{C})\times\mathrm{SO}(N-2j,\mathbb{C}).
\end{aligned}
$$

We obtain the following multiplicity-free representations for the bottom group $\mathrm{GL}(j,\mathbb{C})\times\mathrm{SO}(N-2j,\mathbb{C})$.

(1) *Suppose that N is odd. Then the restriction* $V_{i,0}^{\mathrm{GL}(N)}|_{\mathrm{GL}(j,\mathbb{C})\times\mathrm{SO}(N-2j,\mathbb{C})}$ *is multiplicity-free* $(1 \leq i, j \leq [\frac{N}{2}])$.

(2) *Suppose that N is even and $i + j \leq \frac{N}{2}$ $(1 \leq i, j \leq \frac{N}{2})$. Then the restriction* $V_{i,0}^{\mathrm{GL}(N)}|_{\mathrm{GL}(j,\mathbb{C})\times\mathrm{SO}(N-2j,\mathbb{C})}$ *is multiplicity-free.*

(3) *Suppose that N is even. Then the restriction* $V_{i,0}^{\mathrm{GL}(N)}|_{\mathrm{GL}(\frac{N}{2},\mathbb{C})}$ *is multiplicity-free* $(1 \leq i \leq \frac{N}{2})$.

(4) *Suppose that N is even. The restrictions* $\left(V_{\frac{N}{2},0}^{\mathrm{GL}(N)}\right)^{even}|_{\mathrm{GL}(j,\mathbb{C})\times\mathrm{SO}(N-2j,\mathbb{C})}$ *and* $\left(V_{\frac{N}{2},0}^{\mathrm{GL}(N)}\right)^{odd}|_{\mathrm{GL}(j,\mathbb{C})\times\mathrm{SO}(N-2j,\mathbb{C})}$ *are multiplicity-free* $(1 \leq j \leq \frac{N}{2} - 1)$. *Here we regard* $\mathrm{GL}(j,\mathbb{C}) \times \mathrm{SO}(N-2j,\mathbb{C})$ *as a Levi subgroup of* $\mathrm{SO}(N,\mathbb{C})$ *by an inner automorphism of* $\mathrm{GL}(N,\mathbb{C})$.

Proof. (2) and (3) are clear from Proposition 6.2.3 (2) and (3). Suppose N is odd. In (6.2.6), if two representations $V_{i-l,0}^{\mathrm{GL}(N-2j)}$ and $V_{i-l',0}^{\mathrm{GL}(N-2j)}$ are isomorphic as representations of $\mathrm{SO}(N-2j,\mathbb{C})$ for some different l and l', then $l+l' \equiv 1 \bmod 2$ since $(i-l)+(i-l') = N-2j$. This implies that for such l and l', $V_{a,b}^{\mathrm{GL}(j)}$ and $V_{a',b'}^{\mathrm{GL}(j)}$ are not isomorphic as representations of $\mathrm{GL}(j,\mathbb{C})$ if $a + b \equiv l$ and $a' + b' \equiv l' \bmod 2$. This shows (1). Then let us prove (4). By (6.2.9), it is enough to show that the sub-representation

$$\bigoplus_{\max(2j-\frac{N}{2},0)\leq l\leq j-1} \bigoplus_{\substack{a+b\leq l,\\ a+b\equiv l \bmod 2}} V_{a,b}^{\mathrm{GL}(j)} \boxtimes V_{N/2-2j+l,0}^{\mathrm{GL}(N-2j)}|_{\mathrm{SO}(N-2j,\mathbb{C})}$$

is contained in both $\left(V_{\frac{N}{2},0}^{\mathrm{GL}(N)}\right)^{even}|_{\mathrm{GL}(j,\mathbb{C})\times\mathrm{SO}(N-2j,\mathbb{C})}$ and $\left(V_{\frac{N}{2},0}^{\mathrm{GL}(N)}\right)^{odd}|_{\mathrm{GL}(j,\mathbb{C})\times\mathrm{SO}(N-2j,\mathbb{C})}$. This follows from the fact that the direct summand $V_{a,b}^{\mathrm{GL}(j)} \boxtimes V_{N/2-2j+l,0}^{\mathrm{GL}(N-2j)}|_{\mathrm{SO}(N-2j,\mathbb{C})}$ is invariant under the action of the outer automorphism ξ that induces the switching of the two simple roots $\alpha_{N/2-1}$ and $\alpha_{N/2}$ in the Dynkin diagram of type $\mathrm{D}_{N/2}$ for any $l \leq j-1$ as a representation of $\mathrm{GL}(j,\mathbb{C}) \times \mathrm{SO}(N-2j,\mathbb{C})$. Here we note that ξ switches $\left(V_{\frac{N}{2},0}^{\mathrm{GL}(N)}\right)^{even}$ and $\left(V_{\frac{N}{2},0}^{\mathrm{GL}(N)}\right)^{odd}$ in $V_{\frac{N}{2},0}^{\mathrm{GL}(N)}$. $\square$

By Weyl's unitary trick, Proposition 6.2.4 completes the proof for Proposition 6.2.1 (Proposition 1.3.1). The example below gives a counter example for the multiplicity-freeness property of the restriction $V_{i,0}^{\mathrm{GL}(N)}|_{\mathrm{GL}(j,\mathbb{C})\times\mathrm{SO}(N-2j,\mathbb{C})}$.

Example 6.2.5. Let $N = 6$, $i = 2$ and $j = 2$ so that $i \neq \frac{N}{2}$, $j \neq \frac{N}{2}$ and $i+j > \frac{N}{2}$. Then we can see from (6.2.10) that the restriction of the irreducible representation $V_{2,0}^{\mathrm{GL}(6)}$ of $\mathrm{SO}(6,\mathbb{C})$ to $\mathrm{GL}(2,\mathbb{C}) \times \mathrm{SO}(2,\mathbb{C})$ contains two copies of the trivial representation $\mathbf{1}_{\mathrm{GL}(2,\mathbb{C})} \boxtimes \mathbf{1}_{\mathrm{SO}(2,\mathbb{C})}$ of $\mathrm{GL}(2,\mathbb{C}) \times \mathrm{SO}(2,\mathbb{C})$. Hence this restriction is not multiplicity-free.

Remark 6.2.6. The explicit branching law of $\Lambda^i(\mathbb{C}^N)$ as a representation of $\mathrm{GL}(j,\mathbb{C}) \times \mathrm{SO}(N-2j,\mathbb{C})$ (Proposition 6.2.3) also would follow from Koike–Terada's character formulas [KT2] and Okada's branching formulas [Ok].

6.3 Proof for Theorem 1.3.2 (realization of multiplicity-free representations from the view point of visible actions)

We use Fact 1.0.4 for the construction of multiplicity-free representations in Theorem 1.3.2. Here we recall Fact 1.0.4 and Theorem 1.3.2.

Fact 6.3.1 (Fact 1.0.4). *Let G be a Lie group and $\mathcal{W}$ a G-equivariant Hermitian holomorphic vector bundle on a connected complex manifold X. Let V be a unitary representation of G. If the following conditions from* (0) *to* (3) *are satisfied, then V is multiplicity-free as a representation of G.*

(0) *There exists a continuous and injective G-intertwining operator from V to the space $\mathcal{O}(X, \mathcal{W})$ of holomorphic sections of $\mathcal{W}$.*

(1) *The action of G on X is S-visible. That is, there exist a subset $S \subset X$ and an anti-holomorphic diffeomorphism σ of X' satisfying the conditions given in Definition 6.1.2. Further, there exists an automorphism $\hat{\sigma}$ of G such that $\sigma(g \cdot x) = \hat{\sigma}(g) \cdot \sigma(x)$ for any $g \in G$ and $x \in X'$.*

(2) *For any $x \in S$, the fiber $\mathcal{W}_x$ at x decomposes as the multiplicity-free sum of irreducible unitary representations of the isotropy subgroup G_x. Let $\mathcal{W}_x = \bigoplus_{1 \le i \le n(x)} \mathcal{W}_x^{(i)}$ denote the irreducible decomposition of $\mathcal{W}_x$.*

(3) *σ lifts to an anti-holomorphic automorphism $\tilde{\sigma}$ of $\mathcal{W}$ and satisfies $\tilde{\sigma}(\mathcal{W}_x^{(i)}) = \mathcal{W}_x^{(i)}$ for each $x \in S$ $(1 \le i \le n(x))$.*

Theorem 6.3.2 (Theorem 1.3.2). *We let $G = \mathrm{Spin}(N)$. For any two irreducible representations V_{λ_1} and V_{λ_2} of G such that $V_{\lambda_1} \otimes V_{\lambda_2}$ is multiplicity-free, there exists a pair of*

- *a generalized flag variety $(G \times G)/(L_1 \times L_2)$ with a strongly visible $\Delta(G)$-action, and*
- *irreducible representations* (*a seed given in* Proposition 6.2.1) *W_1 and W_2 of L_1 and L_2, respectively,*

such that $V_{\lambda_k} \simeq \mathcal{O}(G/L_k, \mathcal{W}_k)$ as G-modules $(k = 1, 2)$.

The correspondence between the data (L_k, W_k) of visible actions and seeds, and the highest weights λ_k of V_{λ_k} $(k = 1, 2)$ is given as in Tables 6.3.1–6.3.4 *below. In the tables, $\mathbb{C}_\lambda$ denotes a one-dimensional representation with weight λ, T a maximal torus of G and L_λ a Levi subgroup of G, whose simple system is given by $\{\alpha_l \in \Pi : \langle \lambda, \check{\alpha}_l \rangle = 0\}$ where $\check{\alpha}_l$ is the coroot of α_l $(1 \le l \le [N/2])$.*

We take a Chevalley–Weyl involution σ of $G = \mathrm{Spin}(N)$ with respect to T, that is, $\sigma(t) = t^{-1}$ for any $t \in T$ (we note that σ is called a Weyl involution in [Wo]). We prove the multiplicity-freeness property of tensor product representations of G by verifying the conditions from (0) to (3) of Fact 6.3.1. Here, we note the following:

- Since G is compact, any finite-dimensional representation of G is unitary.

Table 6.3.1: Line bundle type

L_1	L_2	W_1	W_2	N	λ_1	λ_2
L_{λ_1}	L_{λ_2}	$\mathbb{C}_{\lambda_1}$	$\mathbb{C}_{\lambda_2}$	$2n+1$	$s\omega_1$	$t\omega_j$
					$s\omega_n$	$t\omega_n$
				$2n$	$s\omega_1$	$t\omega_j + u\omega_{n-\delta}$
					$s\omega_{n-\delta}$	$t\omega_3$, $t\omega_1 + u\omega_2$, $t\omega_1 + u\omega_{n-\delta'}$
						or $t\omega_{n-1} + u\omega_n$
				8	$s\omega_{5-\epsilon}$	$t\omega_2 + u\omega_{2+\epsilon}$

$1 \le j \le n$, $s, t, u \in \mathbb{N}$, $\delta = 0$ or 1, $\delta' = 0$ or 1 and $\epsilon = 1$ or 2.

Table 6.3.2: Weight multiplicity-free type

L_1	L_2	W_1	W_2	N	λ_1	λ_2
G	T	V_{λ_1}	$\mathbb{C}_{\lambda_2}$	$2n+1$	0, ω_1 or ω_n	arbitrary
				$2n$	0, ω_1, ω_{n-1} or ω_n	arbitrary

Table 6.3.3: Alternating tensor product type

L_1	L_2	W_1	W_2	N	λ_1	λ_2	Condition
G	L_{λ_2}	V_{λ_1}	$\mathbb{C}_{\lambda_2}$	$2n+1$	ω_i or $2\omega_n$	$t\omega_j$	
				$2n$	ω_i	$t\omega_j$	$i+j \le n$
					ω_i	$t\omega_{n-\delta}$	
					$2\omega_{n-\delta}$	$t\omega_j$	

$1 \le i, j \le n$, $t \in \mathbb{N}$ and $\delta = 0$ or 1.

Table 6.3.4: Spin type

L_1	L_2	W_1	W_2	N	λ_1	λ_2
L_{λ_1}	L_{ω_j}	$\mathbb{C}_{\lambda_1}$	$\mathbb{C}_{(1/2+t)\omega_j} \boxtimes \mathrm{Spin}_{N-2j}$	$2n+1$	$s\omega_1$	$\omega_n + t\omega_j$

$1 \le j \le n-1$ and $s, t \in \mathbb{N}$.

- The condition (0) is automatically satisfied by the Borel–Weil theory.
- The condition (3) is satisfied by virtue of the properties of a Chevalley–Weyl involution (see the argument in Subsection 6.3 of [Ko3], and the conditions 5.2.4 (a), (b) in loc. cit).

Therefore we only need to check the visibility of the base space (the condition (1)) and the multiplicity-freeness property of the fiber (the condition (2)). Below, we see that the former follows from a classification of visible actions given in Theorem 1.1.1 (Table 6.3.5 in this chapter) and the latter from Proposition 6.2.1. The proof is divided into four cases.

For the convenience, we quote a classification of visible actions for the orthogonal group as Table 6.3.5 from Theorem 1.1.1. In Table 6.3.5, ξ is an outer automorphism of type D groups, which induces the switching of the two simple roots α_{n-1} and α_n in the Dynkin diagram of type D_n.

Table 6.3.5: Classification of visible actions for types B and D

$\mathfrak{g}$	Π_1	Π_2
$\mathfrak{so}(N)$	Π	arbitrary subset of Π
	$\{\alpha_1\}^c$	$\{\alpha_j\}^c$
	$\{\alpha_{[N/2]}\}^c$	$\{\alpha_{[N/2]}\}^c$
$\mathfrak{so}(2n)$	$\{\alpha_1\}^c$	$\{\alpha_j, \alpha_{n-\delta}\}^c$
	$\{\alpha_{n-\delta}\}^c$	$\{\alpha_3\}^c$, $\{\alpha_1, \alpha_2\}^c$, $\{\alpha_2\}^c$, $\{\alpha_1, \alpha_{n-\delta'}\}^c$, $\{\alpha_{n-1}, \alpha_n\}^c$ or $\{\alpha_{n-1}\}^c$
$\mathfrak{so}(8)$	$\{\alpha_{5-\epsilon}\}^c$	$\{\alpha_2, \alpha_{2+\epsilon}\}^c$

$1 \leq j \leq \min(n, [N/2])$, $\delta = 0$ or 1, $\delta' = 0$ or 1 and $\epsilon = 1$ or 2.

6.3.1 Line bundle type

Let $G = \mathrm{Spin}(N)$ and $(L_1, L_2, W_1, W_2, N, \lambda_1, \lambda_2)$ be a member of Table 6.3.1. By the Borel–Weil theory, V_{λ_k} is G-isomorphic to $\mathcal{O}(G/L_k, \mathcal{W}_k)$ $(k = 1, 2)$. Since (G, L_1, L_2) is given as in Table 6.3.1, the diagonal action of G on the base space $(G \times G)/(L_1 \times L_2)$ is strongly visible by [Ta2, Ta4] (Theorem 1.1.1 or Table 6.3.5 in this paper). On the other hand, the multiplicity-freeness property of the fiber is clear since it is of one-dimension (Proposition 6.2.1 (1). Therefore $V_{\lambda_1} \otimes V_{\lambda_2}$ is multiplicity-free by Fact 6.3.1. Furthermore, the proof implies that seeds in this case are one-dimensional representations.

6.3.2 Weight multiplicity-free type

Let $G = \mathrm{Spin}(N)$ and $(L_1, L_2, W_1, W_2, N, \lambda_1, \lambda_2)$ be a member of Table 6.3.2. By the Borel–Weil theory, V_{λ_k} is G-isomorphic to $\mathcal{O}(G/L_k, \mathcal{W}_k)$ $(k = 1, 2)$. In this setting, the visibility of the base space $(G \times G)/(L_1 \times L_2)$ with respect to the diagonal action of G is clear since the action is transitive. Also, the fiber $W_1 \boxtimes W_2$ is multiplicity-free as a representation of the isotropy subgroup $L_2 = T$ by Proposition 6.2.1 (2). We note that $L_1 = G$. Therefore $V_{\lambda_1} \otimes V_{\lambda_2}$ is multiplicity-free. Seeds in this case are the trivial representation, the natural representation and the (half) spin representation restricted to a maximal torus.

6.3.3 Alternating tensor product type

Let $G = \mathrm{Spin}(N)$ and $(L_1, L_2, W_1, W_2, N, \lambda_1, \lambda_2)$ be a member of Table 6.3.3. By the Borel–Weil theory, V_{λ_k} is G-isomorphic to $\mathcal{O}(G/L_k, \mathcal{W}_k)$ $(k = 1, 2)$. In this setting, the visibility of the base space $(G \times G)/(L_1 \times L_2)$ with respect to the diagonal action of G is clear since the action is transitive. Also, the fiber $W_1 \boxtimes W_2$ is multiplicity-free as a representation of the isotropy subgroup L_2 by Proposition 6.2.1 (3). We note that $L_1 = G$. Therefore $V_{\lambda_1} \otimes V_{\lambda_2}$ is multiplicity-free. Seeds in this case are alternating tensor product representations (or their G-irreducible constituents) restricted to maximal Levi subgroups.

6.3.4 Spin type

Let $G = \mathrm{Spin}(N)$ and $(L_1, L_2, W_1, W_2, N, \lambda_1, \lambda_2)$ be a member of Table 6.3.4. Then V_{λ_k} is G-isomorphic to $\mathcal{O}(G/L_k, \mathcal{W}_k)$ $(k = 1, 2)$ by the Borel–Weil theory. The visibility of the base space $(G \times G)/(L_1 \times L_2)$ with respect to the diagonal action of G follows from [Ta2] (Theorem 1.1.1 or Table 6.3.5 in this paper). We recall from [Ta2, Proposition 3.3] that the corresponding "slice" S of the visible action is given by $(B \times \{e\}) \cdot o$, where o denotes the identity coset of $(G \times G)/(L_1 \times L_2)$, and $B := B_1 \cdot B_2$ for the two abelian subgroups B_1 and B_2 defined by

$$\begin{aligned} B_1 &:= \exp\left(\mathbb{R}Z_{\sum_{k=1}^{j}\alpha_k} + \mathbb{R}Z_{(\sum_{k=1}^{j}\alpha_k + 2\sum_{k=j+1}^{n}\alpha_k)}\right), \\ B_2 &:= \exp\left(\mathbb{R}Z_{(\sum_{k=1}^{j-1}\alpha_k + 2\sum_{k=j}^{n}\alpha_k)}\right). \end{aligned}$$

Here we put $Z_\alpha = X_\alpha + \theta X_\alpha$, where X_α is a non-zero root vector of $\mathfrak{g}_\mathbb{C}$, which corresponds to a root $\alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, and denote by θ the Cartan involution of $\mathfrak{g}_\mathbb{C}$ with $\mathfrak{g}$ =Lie(G) its fixed points set. We note that the centralizer of B in the intersection $L_1 \cap L_2 = L_{\omega_1} \cap L_{\omega_j}$ contains the group M_j defined by

$$M_j := \{\exp(\sqrt{-1}m\pi H_{j+1})\}_{1 \le m \le 4} \cdot \mathrm{Spin}(2(n-j)-1), \tag{6.3.1}$$

where the connected component of M_j is isomorphic to $\mathrm{Spin}(2(n-j)-1)$ whose root system is spanned by $\{\alpha_i \in \Pi : j+2 \le i \le n\}$. By Lemma 6.3.3 below, the multiplicity-freeness property of the fiber $W_1 \boxtimes W_2$ as a representation of M_j follows from Proposition 6.2.1 (4). Therefore we can find that the tensor product representation $V_{\lambda_1} \otimes V_{\lambda_2}$ is multiplicity-free. The proof implies that seeds in this case are spin representations (twisted by characters) restricted to twisted products of spinor groups and cyclic groups.

6.3.5 "Seeds" for the spin type in Theorem 6.3.2

We put $a := \exp(\sqrt{-1}\pi H_j)$ and denote by $\langle x \rangle$ the group generated by x. Let χ_a be a character of $\langle a \rangle$ such that the representation $\chi_a \boxtimes \mathrm{Spin}_{2(n-j)+1}$ of $\langle a \rangle \times \mathrm{Spin}(2(n-j)+1)$ descends to the subgroup M_{j-1} (6.3.1) of L_{ω_j}.

Lemma 6.3.3. *The representation $\chi_a \boxtimes \mathrm{Spin}_{2(n-j)+1}$ is multiplicity-free as a representation of M_j $(1 \le j \le n-1)$ (see (6.3.1) for the definition of M_j).*

Proof. We put $b := \exp(\sqrt{-1}\pi(H_{j+1} - H_j))$. Since the group M_j is the quotient of the direct product group $\langle ab \rangle \times \mathrm{Spin}(2(n-j)-1)$ by a subgroup of its center, it suffices to show that the representation $\chi_a \boxtimes \mathrm{Spin}_{2(n-j)+1}$ of $\langle a \rangle \times \mathrm{Spin}(2(n-j)+1)$ decomposes multiplicity-freely as a representation of $\langle ab \rangle \times \mathrm{Spin}(2(n-j)-1)$.

By Proposition 6.2.1 (4), $\mathrm{Spin}_{2(n-j)+1}$ is multiplicity-free as a representation of $\langle b \rangle \times \mathrm{Spin}(2(n-j)-1)$. This implies that $\chi_a \boxtimes \mathrm{Spin}_{2(n-j)+1}$ is multiplicity-free as a representation of $1 \times (\langle b \rangle \times \mathrm{Spin}(2(n-j)-1))$, and thus as that of $\langle ab \rangle \times \mathrm{Spin}(2(n-j)-1)$. This shows the lemma. □

According to Stembridge's classification results [St2] for types B and D cases given below, the above four types stated in Sections 6.3.1, 6.3.2, 6.3.3 and 6.3.4 (which correspond to (i) and (i)′, (ii) and (ii)′, (iii) and (iii)′, and (iv) in Facts 6.3.4 and 6.3.5, respectively) exhaust all the multiplicity-free tensor product representations for the spinor group. Therefore we have completed the proof for Theorem 6.3.2.

Fact 6.3.4 ([St2])**.** *Let λ_1 and λ_2 be the highest weights of irreducible representations V_{λ_1} and V_{λ_2} of* $\mathrm{Spin}(2n+1)$*, respectively. Then the tensor product representation $V_{\lambda_1} \otimes V_{\lambda_2}$ is multiplicity-free if and only if one of the following four conditions* (i), (ii), (iii) *and* (iv) *is satisfied up to switch of the factors λ_1 and λ_2* (*see Figure 6.1.1 for the labeling of the Dynkin diagram*):

(i) $(\lambda_1, \lambda_2) = (s\omega_1, t\omega_j)$ *or* $(s\omega_n, t\omega_n)$ *with* $1 \le j \le n$ *and* $s, t \in \mathbb{N}$.

(ii) $\lambda_1 = 0$, ω_1 *or* ω_n; λ_2 *is arbitrary.*

(iii) $\lambda_1 = \omega_i$ *or* $2\omega_n$; $\lambda_2 = t\omega_j$ *with* $1 \le i, j \le n$ *and* $t \in \mathbb{N}$.

(iv) $\lambda_1 = s\omega_1$; $\lambda_2 = \omega_n + t\omega_j$ *with* $1 \le j \le n$ *and* $s, t \in \mathbb{N}$.

Fact 6.3.5 ([St2]). *Let* λ_1 *and* λ_2 *be the highest weights of irreducible representations* V_{λ_1} *and* V_{λ_2} *of* $\mathrm{Spin}(2n)$*, respectively. Then the tensor product representation* $V_{\lambda_1} \otimes V_{\lambda_2}$ *is multiplicity-free if and only if one of the following three conditions* (i)$'$, (ii)$'$ *and* (iii)$'$ *is satisfied up to switch of the factors* λ_1 *and* λ_2 (*see Figure 6.1.2 for the labeling of the Dynkin diagram*):

(i)$'$ $(\lambda_1, \lambda_2) = (s\omega_1, t\omega_j + u\omega_{n-1})$ *or* $(s\omega_1, t\omega_j + u\omega_n)$ *with* $1 \le j \le n$ *and* $s, t, u \in \mathbb{N}$,
$\lambda_1 = s\omega_{n-1}$ *or* $s\omega_n$; $\lambda_2 = t\omega_3$, $t\omega_1 + u\omega_2$, $t\omega_1 + u\omega_{n-1}$, $t\omega_1 + u\omega_n$ *or* $t\omega_{n-1} + u\omega_n$ *with* $s, t, u \in \mathbb{N}$, *or*
$\lambda_1 = s\omega_{5-\epsilon}$; $\lambda_2 = t\omega_2 + u\omega_{2+\epsilon}$ *with* $n = 4$ *and* $\epsilon = 1$ *or* 2.

(ii)$'$ $\lambda_1 = 0$, ω_1, ω_{n-1} *or* ω_n; λ_2 *is arbitrary.*

(iii)$'$ $\lambda_1 = \kappa\omega_i$; $\lambda_2 = t\omega_j$, *where* $t \in \mathbb{N}$ *and* κ, i, j *satisfy one of the following three conditions.*

- (iii-1)$'$ $\kappa = 1$ *and* $i + j \le n$.
- (iii-2)$'$ $\kappa = 1$, $1 \le i \le n$ *and* $j = n - 1$ *or* n.
- (iii-3)$'$ $\kappa = 2$, $i = n - 1$ *or* n *and* $1 \le j \le n$.

Remark 6.3.6. The pair $(V_{\lambda_1}, V_{\lambda_2})$ corresponding to the spherical double cone associated with a pair of maximal parabolic subgroups of any simple algebraic group over any algebraically closed field of characteristic zero was classified by Littelmann [Li2].

Chapter 7

Visible actions on spherical varieties

7.1 Introduction for Chapter 7

Let X be a connected complex algebraic variety with an action of a connected complex reductive algebraic group G.

Definition 7.1.1. X is said to be spherical if a Borel subgroup of G has an open orbit on X.

We want to prove the visibility of the action of a real form $G_{\mathbb{R}}$ of G on X. Here we say $G_{\mathbb{R}}$ is a real form of G if $G_{\mathbb{R}}$ is a subgroup of G and the complexification of the Lie algebra of $G_{\mathbb{R}}$ coincides with that of G. For the convenience, we recall the definition of a visible action.

Definition 7.1.2 (Kobayashi [Ko2])**.** We say a holomorphic action of a Lie group G on a complex manifold X is strongly visible if the following two conditions are satisfied:

1. There exists a real submanifold S (called a "slice") such that

$$X' := G \cdot S \text{ is an open subset of } X.$$

2. There exists an anti-holomorphic diffeomorphism σ of X' such that

$$\begin{aligned} \sigma|_S &= \mathrm{id}_S, \\ \sigma(G \cdot x) &= G \cdot x \text{ for any } x \in X'. \end{aligned}$$

In the above setting, we say the action of G on X is S-visible. This terminology will be used also if S is just a subset of X.

Since the visibility is a local condition, we may replace X with a unique open G-orbit on X. We denote by σ an anti-holomorphic involution of G, which corresponds to $G_{\mathbb{R}}$, and by H a stabilizer of the open orbit of G on X. In the following, we say a subgroup L of a complex reductive algebraic group G is a spherical subgroup of G if G/L is a G-spherical variety.

7.2 Reduction to the affine homogeneous case

A real reductive Lie group is said to be of inner type if its Lie algebra has a compact Cartan subalgebra. The following is a list of real simple Lie algebras of inner type.

$$\mathfrak{sl}(2,\mathbb{R}), \mathfrak{su}(p,q), \mathfrak{so}(r,s) \text{ (with } r \text{ or } s \text{ even)}, \mathfrak{so}^*(2n), \mathfrak{sp}(2n,\mathbb{R}), \mathfrak{sp}(p,q),$$
$$\text{EII, EIII, EV, EVI, EVII, EVIII, EIX, FI, FII, G.}$$

Lemma 7.2.1. *Assume that $G_\mathbb{R}$ is compact or of inner type. Then there exists a σ-stable Levi subgroup G' of G such that the subset $(\mathrm{Ad}(x)G_\mathbb{R})(\mathrm{Ad}(x)G')H$ contains a non-empty open subset of G for an element x of G, and $(\mathrm{Ad}(x)G' \cap H)$ is a reductive spherical subgroup of $\mathrm{Ad}(x)G'$.*

Proof. Since G/H is a G-spherical variety, there exists a Borel subgroup B of G such that the subset BH is open in G. Assume that H is not reductive. Then the unipotent radical $\mathrm{Rad}_u(H)$ of H is non-trivial, and there exists a proper parabolic subgroup P' of G such that $P' \supset H$ and $\mathrm{Rad}_u(P') \supset \mathrm{Rad}_u(H)$ by Borel and Tits [BT]. Let B' be a Borel subgroup of G, contained in P'. By taking a conjugate of $G_\mathbb{R}$ if necessary, we may assume that the subset $G_\mathbb{R}B'$ is open in G. We take a maximal torus T' of B' such that the intersection $T' \cap G_\mathbb{R}$ is a compact maximal torus of $G_\mathbb{R}$. Then we put $\overline{B'} = \sigma(B')$ and $\overline{P'} = \sigma(P')$. By the Bruhat decomposition, we can write $B = \mathrm{Ad}(p'w_i'\bar{b}')\overline{B'}$ for $p' \in P'$, $\bar{b}' \in \overline{B'}$ and $w_i' \in \{w_1' = e, w_2', \dots, w_r'\}$ = a subset of $N_G(T')$, which gives a complete representatives of the double coset $P'\backslash G/\overline{B'}$. Then we have

$$\mathrm{Ad}(p'w_i'\bar{b}')\overline{\mathfrak{b}'} + \mathfrak{h} = \mathfrak{g}.$$

Since P' contains H, the subset BP' is also open in G. This implies that

$$\mathrm{Ad}(p'w_i'\bar{b}')\overline{\mathfrak{b}'} + \mathfrak{p}' = \mathfrak{g}.$$

Therefore

$$\mathrm{Ad}(w_i')\overline{\mathfrak{b}'} + \mathfrak{p}' = \mathfrak{g}.$$

This equality implies that we can take w_i' as $w_i' = e$ since there is only one open $\overline{B}'$-orbit on $P'\backslash G$. Let us write $\overline{\mathfrak{b}'} = \overline{\mathfrak{u}'} + \mathfrak{b}_{\mathfrak{l}'}$ for a Borel subalgebra $\mathfrak{b}_{\mathfrak{l}'}$ of the Levi subalgebra $\mathfrak{l}' = \mathfrak{p}' \cap \overline{\mathfrak{p}'}$ and the nilpotent radical $\overline{\mathfrak{u}'}$ of $\overline{\mathfrak{p}'}$. We have

$$\mathrm{Ad}(p')\overline{\mathfrak{u}'} + \mathrm{Ad}(p')\mathfrak{b}_{\mathfrak{l}'} + \mathfrak{h} = \mathfrak{g}, \tag{7.2.1}$$
$$\mathrm{Ad}(p')\overline{\mathfrak{u}'} + \mathfrak{p}' = \mathfrak{g}. \tag{7.2.2}$$

Since $\mathrm{Ad}(p')\mathfrak{b}_{\mathfrak{l}'} + \mathfrak{h} \subset \mathfrak{p}'$ and $\dim(\mathrm{Ad}(p')\mathfrak{b}_{\mathfrak{l}'} + \mathfrak{h}) \geq \dim \mathfrak{g} - \dim \mathrm{Ad}(p')\overline{\mathfrak{u}'} = \dim \mathfrak{p}'$, we have $\mathfrak{p}' = \mathrm{Ad}(p')\mathfrak{b}_{\mathfrak{l}'} + \mathfrak{h}$. From this we obtain $\mathrm{Ad}(p')\mathfrak{l}' = \mathrm{Ad}(p')\mathfrak{b}_{\mathfrak{l}'} + \mathfrak{h} \cap \mathrm{Ad}(p')\mathfrak{l}'$. We put $G_1 = \mathrm{Ad}(p')(L')^0$ and $H_1 = (G_1 \cap H)^0$, and find that G_1/H_1 is an G_1-spherical variety. We succeed this procedure to construct a sequence of pairs of a complex reductive algebraic group and a spherical subgroup $(G_0, H_0) = (G, H)$, (G_1, H_1), (G_2, H_2), $\dots$, $(G_k, H_k) = (G', H')$ where G'/H' is affine, that is, the stabilizer H' is reductive. $\square$

The above argument shows that our construction of visible actions of a real form (compact or of inner type) of G on G/H is reduced to the affine homogeneous case by the locality of the visible action.

7.3 Affine homogeneous case

Proposition 7.3.1. *Let $G_{\mathbb{R}}$ be a real form of G and H a reductive spherical subgroup of G. There exist finitely many abelian subspaces $\mathfrak{j}_i$ of $\mathfrak{g}^{-\sigma}$ and elements x_i and y_i of G ($i \in I$) such that $\bigcup_{i\in I} G_{\mathbb{R}} C_i H$ contains an open dense subset of G, where $C_i = x_i \exp(\mathfrak{j}_i) y_i$.*

Proof. We use the induction on the dimension of G. By classification results by Brion [Br], Krämer [Kr], Mikityuk [Mi] and Yakimova [Ya], there exists a complex symmetric subgroup G^τ of G, which contains H except for the two cases that we discuss later if (G, H) is irreducible. Then by Matsuki [Ma2, Ma3], we have $G_{ss} = \bigcup_{i\in I'} G_{\mathbb{R}} C_i' G^\tau$ for a real form $G_{\mathbb{R}}$ of G with the corresponding anti-holomorphic involution σ. Here, the set K_{ss} of semisimple elements of a Lie group K with (not necessarily commutative) two involutions σ and τ is defined by ([Ma2, Ma3])

$$K_{ss} = \{g \in K; \sigma\tau_g = \sigma \operatorname{Ad}(g)\tau \operatorname{Ad}(g)^{-1} \text{ is semisimple}\},$$

and $C_i' = \exp(\mathfrak{a}_i)\exp(\mathfrak{t}_i)t_i$ are the representatives of standard Cartan subsets with $\mathfrak{a} \subset \mathfrak{a}_i \subset \mathfrak{p}$, $\mathfrak{t}_i \subset \mathfrak{t} \subset \mathfrak{k}^{-\sigma} \cap \mathfrak{k}^{-\tau}$ for a maximal abelian subspace $\mathfrak{t}$ of $\mathfrak{k}^{-\sigma} \cap \mathfrak{k}^{-\tau}$ and $\mathfrak{a}_i + \mathfrak{t}_i \subset \mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau_{t_i}}$ with $t_i \in \exp(\mathfrak{t})$. We note that $\mathfrak{a}_i + \mathfrak{t}_i$ is a maximal abelian subspace of $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau_{t_i}}$ and that t_i is not necessarily an element of $\exp(\mathfrak{t}_i)$. We put $L_i = Z_G(\mathfrak{t}_i)$ if $\mathfrak{t}_i \neq 0$ otherwise $L_i = Z_G(\operatorname{Ad}(t_i^{-1})\mathfrak{a}_i)$. Then the subset $\bigcup_{i\in I'} t_i \operatorname{Ad}(t_i^{-1}) G_{\mathbb{R}} L_i H$ contains an open dense subset of G. Here we note that $G^\tau = (L_i \cap G^\tau)H$. We use the induction hypothesis for each L_i:

For abelian subspaces $\mathfrak{j}_j^{L_i}$ of $\mathfrak{l}_i^{-\sigma t_i}$ and elements $x_j^{L_i}$ and $y_j^{L_i}$ of L_i ($j \in J_{L_i}$), an open dense subset of L_i is contained in $\displaystyle\bigcup_{j\in J_{L_i}} (\operatorname{Ad}(t_i^{-1})G_{\mathbb{R}} \cap L_i) C_j^{L_i} (L_i \cap H)$ with $C_j^{L_i} = x_j^{L_i} \exp(\mathfrak{j}_j^{L_i}) y_j^{L_i}$.

Thus the following subset contains an open dense subset of G.

$$\bigcup_{i\in I'} t_i \operatorname{Ad}(t_i^{-1}) G_{\mathbb{R}} \Big(\bigcup_{j\in J_{L_i}} \operatorname{Ad}(t_i^{-1}) G_{\mathbb{R}} \cap L_i) C_j^{L_i} (L_i \cap H)\Big) H = \bigcup_{i\in I'} \bigcup_{j\in J_{L_i}} G_{\mathbb{R}} t_i C_j^{L_i} H.$$

We deal with the two cases mentioned in the above. In the following, we only deal with non-compact real forms $G_{\mathbb{R}}$. For the case where $G_{\mathbb{R}}$ is compact, see Theorem 7.4.4.

- $(G, H) = (\mathrm{Spin}(7, \mathbb{C}), \mathrm{G}_{2,\mathbb{C}})$.

  - $G_{\mathbb{R}} = \mathrm{Spin}_e(3, 4)$.
    We use the isomorphism $\mathrm{Spin}(7, \mathbb{C})/\mathrm{G}_{2,\mathbb{C}} \simeq \mathrm{Spin}(8, \mathbb{C})/\mathrm{Spin}(7, \mathbb{C})$. By [Ma2, Ma3], we have

    $$\begin{aligned}\mathrm{Spin}(8, \mathbb{C})_{ss} = \mathrm{Spin}_e(4, 4) T^1 \mathrm{Spin}(7, \mathbb{C}) &\cup \mathrm{Spin}_e(4, 4) A^1 \mathrm{Spin}(7, \mathbb{C}) \\ &\cup \mathrm{Spin}_e(4, 4) A^1 t \mathrm{Spin}(7, \mathbb{C})\end{aligned}$$

    for a one-dimensional torus T^1 and a one-dimensional split real torus A^1 with t an element of T^1. Since $\mathrm{Spin}_e(4, 4) = \mathrm{Spin}_e(3, 4) \mathrm{SL}(4, \mathbb{R}) = \mathrm{Spin}_e(3, 4) \mathrm{SU}(2, 2)$, we obtain

    $$\begin{aligned}\mathrm{Spin}(8, \mathbb{C})_{ss} = \mathrm{Spin}_e(3, 4) T^1 \mathrm{Spin}(7, \mathbb{C}) &\cup \mathrm{Spin}_e(3, 4) A^1 \mathrm{Spin}(7, \mathbb{C}) \\ &\cup \mathrm{Spin}_e(3, 4) A^1 t \mathrm{Spin}(7, \mathbb{C}).\end{aligned}$$

– $G_{\mathbb{R}} = \mathrm{Spin}_e(2,5)$.
By [Ma2, Ma3], we have

$$\begin{aligned}\mathrm{Spin}(8,\mathbb{C})_{ss} = \mathrm{Spin}_e(2,6)T^1\,\mathrm{Spin}(7,\mathbb{C}) \cup \mathrm{Spin}_e(2,6)A^1\,\mathrm{Spin}(7,\mathbb{C})\\ \cup\, \mathrm{Spin}_e(2,6)A^1 t\,\mathrm{Spin}(7,\mathbb{C})\end{aligned}$$

for a one-dimensional torus T^1 and a one-dimensional split real torus A^1 with t an element of T^1. Since $\mathrm{Spin}_e(2,6) = \mathrm{Spin}_e(2,5)\,\mathrm{SU}(1,3) = \mathrm{Spin}_e(2,5)\,\mathrm{SU}(2,2)$, we obtain

$$\begin{aligned}\mathrm{Spin}(8,\mathbb{C})_{ss} = \mathrm{Spin}_e(2,5)T^1\,\mathrm{Spin}(7,\mathbb{C}) \cup \mathrm{Spin}_e(2,5)A^1\,\mathrm{Spin}(7,\mathbb{C})\\ \cup\, \mathrm{Spin}_e(2,5)A^1 t\,\mathrm{Spin}(7,\mathbb{C}).\end{aligned}$$

– $G_{\mathbb{R}} = \mathrm{Spin}_e(1,6)$.
In the same way as above, we have

$$\begin{aligned}\mathrm{Spin}(8,\mathbb{C})_{ss} = \mathrm{Spin}_e(2,6)T^1\,\mathrm{Spin}(7,\mathbb{C}) \cup \mathrm{Spin}_e(2,6)A^1\,\mathrm{Spin}(7,\mathbb{C})\\ \cup\, \mathrm{Spin}_e(2,6)A^1 t\,\mathrm{Spin}(7,\mathbb{C})\end{aligned}$$

for a one-dimensional torus T^1 and a one-dimensional split real torus A^1 with t an element of T^1. Since $\mathrm{Spin}_e(2,6) = \mathrm{Spin}_e(1,6)\,\mathrm{SU}(1,3) = \mathrm{Spin}_e(1,6)\,\mathrm{SU}(2,2)$, we obtain

$$\begin{aligned}\mathrm{Spin}(8,\mathbb{C})_{ss} = \mathrm{Spin}_e(1,6)T^1\,\mathrm{Spin}(7,\mathbb{C}) \cup \mathrm{Spin}_e(1,6)A^1\,\mathrm{Spin}(7,\mathbb{C})\\ \cup\, \mathrm{Spin}_e(1,6)A^1 t\,\mathrm{Spin}(7,\mathbb{C}).\end{aligned}$$

- $(G,H) = (\mathrm{G}_{2,\mathbb{C}}, \mathrm{SL}(3,\mathbb{C}))$.
We use the isomorphism $\mathrm{G}_{2,\mathbb{C}}/\mathrm{SL}(3,\mathbb{C}) \simeq \mathrm{SO}(7,\mathbb{C})/\mathrm{SO}(6,\mathbb{C})$. By [Ma2, Ma3], we have

$$\mathrm{SO}(7,\mathbb{C})_{ss} = \mathrm{SO}_e(3,4)T^1\,\mathrm{SO}(6,\mathbb{C}) \cup \mathrm{SO}_e(3,4)A^1\,\mathrm{SO}(6,\mathbb{C}) \cup \mathrm{SO}_e(3,4)A^1 t\,\mathrm{SO}(6,\mathbb{C})$$

for a one-dimensional torus T^1 and a one-dimensional split real torus A^1 with t an element of T^1. Since $\mathrm{SO}_e(3,4) = \mathrm{G}_{2(2)}\,\mathrm{SO}_e(2,3) = \mathrm{G}_{2(2)}\,\mathrm{SO}_e(3,2)$, we obtain

$$\mathrm{SO}(7,\mathbb{C})_{ss} = \mathrm{G}_{2(2)}\,T^1\,\mathrm{SO}(6,\mathbb{C}) \cup \mathrm{G}_{2(2)}\,A^1\,\mathrm{SO}(6,\mathbb{C}) \cup \mathrm{G}_{2(2)}\,A^1 t\,\mathrm{SO}(6,\mathbb{C}).$$

□

Here, we note the following: To apply Matsuki's decomposition [Ma2, Ma3], we actually need to put some assumptions on the connected components of G, the semisimplicity of the action of $\sigma\tau$ on $\mathfrak{g}$ and the existence of a "good" Cartan involution [Ma2, Ma3]. Those assumptions are satisfied if for example G is connected and semisimple after taking a conjugate of τ (or σ) if necessary. Let us see how to reduce a general case to the connected semisimple case. Let G be a (not necessarily connected) complex reductive algebraic group, $G_{\mathbb{R}}$ a real form of G and H a reductive spherical subgroup of G. We write $G = \bigcup_k G^0 g_k$ with $\{g_k\}$ a finite subset of G, where G^0 stands for the identity component of G. Suppose that $\bigcup_i (G^0 \cap G_{\mathbb{R}})C_{i_k}(G^0 \cap \mathrm{Ad}(g_k)H)$ contains an open dense subset of G^0. Then $\bigcup_k \bigcup_i (G^0 \cap G_{\mathbb{R}})C_{i_k} g_k (G^0 \cap H)$ contains an open dense subset of G

since $\bigcup_{i_k}(G^0 \cap G_{\mathbb{R}})C_{i_k}g_k(G^0 \cap H) = \bigcup_{i_k}(G^0 \cap G_{\mathbb{R}})C_{i_k}(G^0 \cap \mathrm{Ad}(g_k)H)g_k = (\bigcup_{i_k}(G^0 \cap G_{\mathbb{R}})C_{i_k}(G^0 \cap \mathrm{Ad}(g_k)H))g_k$ contains an open dense subset of $G^0 g_k$. Hence we may assume that G is connected. Let $G_{\mathbb{R}}$ and H be as before. Let $\pi : G \to G/\exp(\mathfrak{z})$ be the projection mapping, where $\mathfrak{z}$ is the center of $\mathfrak{g}$. Then $\pi(H)$ is a spherical subgroup of $\pi(G)$. Suppose that $\bigcup_i \pi(G_{\mathbb{R}})\pi(x_i)\pi(\exp(\mathfrak{j}_i))\pi(y_i)\pi(H)$ contains an open dense subset of $\pi(G)$. We put $C_i = \exp(\mathfrak{j}_i)\exp(\mathfrak{z})$. Then we find that $\bigcup_i G_{\mathbb{R}}C_iH$ contains an open dense subset of G.

7.4 Generalized Cartan decomposition involving maximal compact subgroup

If a real form is compact, we obtain a simple description of the double coset decomposition.

Definition 7.4.1. Let G be a locally compact group and H a compact subgroup of G. We say the pair (G, H) is a Gelfand pair if $L^2(G/H)$ is multiplicity-free as a unitary representation of G.

Let G be a connected complex algebraic group and H a complex reductive subgroup of G. Let $(G_{\mathbb{R}}, H_{\mathbb{R}})$ be a pair of real forms of (G, H), that is, $G_{\mathbb{R}}$ and $H_{\mathbb{R}}$ are real forms of G and H, respectively, and $H_{\mathbb{R}}$ is a subgroup of $G_{\mathbb{R}}$. It is known that $(G_{\mathbb{R}}, H_{\mathbb{R}})$ is a Gelfand pair if and only if $H_{\mathbb{R}}$ is compact and G/H is a G-spherical variety (see [Wo] for example).

Lemma 7.4.2. *Let $\mathfrak{j} = \mathfrak{t} \oplus \mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$. We suppose that $\mathfrak{j}$ is not maximally non-compact and that ν acts as the multiplication by -1 on $\mathfrak{j}$. Then for any root vector $X_\beta \in \mathfrak{g}$ of any imaginary non-compact root β, there exists $Z \in \mathfrak{t}$ such that $\mathrm{Ad}(\exp(Z))(X_\beta + \overline{X_\beta})$ is fixed by ν. Here we extend ν to $\mathfrak{g}$ holomorphically, and $\overline{X}$ denotes the conjugate element with respect to $\mathfrak{g}_{\mathbb{R}}$ for any $X \in \mathfrak{g}$.*

Proof. Since both $\overline{X_\beta}$ and $\nu(X_\beta)$ belong to the root subspace $\mathfrak{g}_{-\beta}$ of $-\beta$, $\nu(X_\beta) = e^{\sqrt{-1}\phi}\overline{X_\beta}$ for some $\phi \in \mathbb{R}$. Then we take $Z \in \mathfrak{t}$ satisfying $\beta(Z) = -\frac{\sqrt{-1}(\phi+\pi)}{2}$. (Here we note that β is imaginary.) For this $Z \in \mathfrak{t}$, we have

$$\begin{aligned}\nu(\mathrm{Ad}(\exp(Z))(X_\beta + \overline{X_\beta})) &= \nu(e^{-\frac{\sqrt{-1}(\phi+\pi)}{2}}X_\beta + e^{\frac{\sqrt{-1}(\phi+\pi)}{2}}\overline{X_\beta}) \\ &= e^{\sqrt{-1}\pi}e^{-\frac{\sqrt{-1}(\phi+\pi)}{2}}X_\beta + e^{-\sqrt{-1}\pi}e^{\frac{\sqrt{-1}(\phi+\pi)}{2}}\overline{X_\beta} \\ &= -\mathrm{Ad}(\exp(Z))(X_\beta + \overline{X_\beta}).\end{aligned}$$

□

Definition 7.4.3. Let G be a real reductive Lie group and ν an involution of G. ν is said to be a Chevalley–Weyl involution if there exists a Cartan subalgebra $\mathfrak{a}$ such that ν acts on $\mathfrak{a}$ as the scalar-multiplication by (-1).

Theorem 7.4.4. *Let $G_{\mathbb{R}}$ be a connected compact Lie group, $(G_{\mathbb{R}}, H_{\mathbb{R}})$ a Gelfand pair and ν a Chevalley–Weyl involution of $G_{\mathbb{R}}$, which preserves $H_{\mathbb{R}}$. We write θ for the Cartan involution of the complexification H, which corresponds to $H_{\mathbb{R}}$, and extend it to the complexification G of $G_{\mathbb{R}}$. Then there exists an abelian subgroup A of the complexification G of $G_{\mathbb{R}}$ such that $G = G_{\mathbb{R}}AH$ holds and both θ and ν act on A as the inverse mapping, where we extend ν holomorphically to G.*

Proof. We may assume that G is semisimple. For the proof, we use the induction on the dimension of G. We firstly assume that there exists a compact symmetric subgroup K of $G_{\mathbb{R}}$, which contains $H_{\mathbb{R}}$. We claim that ν preserves K.

Indeed, let ν' be a Chevalley–Weyl involution of $H_{\mathbb{R}}$ and extend ν' to K and to G. We take an element g of $G_{\mathbb{R}}$ such that $\nu^g = \nu'$. Then we see that $\nu'\nu$ is an inner automorphism by the element $g\nu(g^{-1})$ of $G_{\mathbb{R}}$ and that $\nu'\nu$ preserves $H_{\mathbb{R}}$. This means that $g\nu(g^{-1})$ is an element of $N_{G_{\mathbb{R}}}(H_{\mathbb{R}})$. We claim that $N_{G_{\mathbb{R}}}(H_{\mathbb{R}})$ is contained in K. For the simplicity, we put $G'_{\mathbb{R}} = N_{G_{\mathbb{R}}}(H_{\mathbb{R}})$. It suffices to show that the $\mathfrak{k}^{\perp}$-part of $\mathfrak{g}'_{\mathbb{R}}$ is the zero vector space by the KP-decomposition for Riemannian symmetric pairs. Assume that $\mathfrak{k}^{\perp} \cap \mathfrak{g}'_{\mathbb{R}}$ is non-zero. Then we take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{k}^{\perp} \cap \mathfrak{g}'_{\mathbb{R}}$. By the fact that $(G_{\mathbb{R}}, H_{\mathbb{R}})$ is a Gelfand pair, $\mathfrak{a}$ is contained in the center of $\mathfrak{g}'_{\mathbb{R}}$. Furthermore we have $K = H_{\mathbb{R}} \cdot (Z_{G_{\mathbb{R}}}(\mathfrak{a}) \cap K)$. Then we obtain $K \subset Z_{G_{\mathbb{R}}}(\mathfrak{a})$. This contradicts to the fact that $\mathfrak{g}_{\mathbb{R}}$ is semisimple. Thus $\mathfrak{k}^{\perp} \cap \mathfrak{g}'_{\mathbb{R}}$ is the zero vector space, and $G' = N_{G_{\mathbb{R}}}(H_{\mathbb{R}})$ is contained in K. Then $\nu'\nu$ coincides with $\mathrm{Ad}(k)$ for some element k of K as an automorphism of $\mathfrak{g}_{\mathbb{R}}$. This implies that ν also preserves K.

Then we have the Flensted-Jensen decomposition $G = G_{\mathbb{R}} A_0 K_{\mathbb{C}}$ for an abelian subgroup A_0 on which both θ and ν act as the inverse mapping (Lemma 7.4.2). We apply the induction hypothesis to the pair $(M_0, M_0 \cap H_{\mathbb{R}})$ and obtain $M_{0,\mathbb{C}} = (G_{\mathbb{R}} \cap M_{0,\mathbb{C}}) A_M (M_{0,\mathbb{C}} \cap H)$ for an abelian subgroup A_M. Thus we have $G = G_{\mathbb{R}} A_0 A_M H$.

By a classification of reductive spherical subgroups [Br, Mi, Kr, Ya], we have the following two cases where we can not take K as above and $(G_{\mathbb{R}}, H_{\mathbb{R}})$ is irreducible.

- $(G_{\mathbb{R}}, H_{\mathbb{R}}) = (\mathrm{G}_2, \mathrm{SU}(3))$.
  We take a non-zero element $X \in \mathfrak{h}_{\mathbb{R}}^{\perp}$ such that $\nu(X) = -X$. By the isomorphism $G_{\mathbb{R}} \times_{H_{\mathbb{R}}} \sqrt{-1}\mathfrak{h}^{\perp} \simeq G/H$, we have $G = G_{\mathbb{R}} A H$ for $A = \exp(\sqrt{-1}\mathbb{R}X)$. Here we note that $\mathfrak{h}_{\mathbb{R}}^{\perp} = \bigcup_{h \in H_{\mathbb{R}}} \mathrm{Ad}(h)(\mathbb{R}X)$.

- $(G_{\mathbb{R}}, H_{\mathbb{R}}) = (\mathrm{Spin}(7), \mathrm{G}_2)$.
  Let $\mathfrak{j} = \mathfrak{t} + \mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$, on which ν acts as the inverse mapping, where $\mathfrak{t} \subset \mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{a} \subset \mathfrak{h}_{\mathbb{R}}^{\perp}$. Since we have $\mathfrak{h}_{\mathbb{R}}^{\perp} = \bigcup_{h \in H_{\mathbb{R}}} \mathrm{Ad}(h)\mathfrak{a}$, we obtain $G = G_{\mathbb{R}} A H$ for $A = \exp(\sqrt{-1}\mathfrak{a})$.

□

According to the works of Akhiezer [Ak1] and Akhiezer and Cupit-Foutou [AC], we have compatible real structures on affine spherical varieties and wonderful varieties. Here by a real structure on a complex manifold Z we mean an anti-holomorphic involution $\eta : Z \to Z$. Also for a real structure η on a complex manifold Z with an action of a group K, we say η is compatible with an automorphism ϕ of K if η satisfies $\eta(kz) = \phi(k)\eta(z)$ for any $k \in K$ and $z \in Z$. We expect the existence of strongly visible actions on these varieties. For this purpose, the following theorem would be useful.

Theorem 7.4.5. *Let H be a spherical subgroup of a connected complex reductive algebraic group G and ι an anti-holomorphic involution of G, which defines a split real form of G. Suppose that H is stable under ι. Then a compact real form $G_{\mathbb{R}}$ of G acts on G/H strongly visibly.*

Proof. We write $\iota = \theta \circ \nu$ for a commuting pair (θ, ν) of a Cartan involution θ and a Chevalley–Weyl involution ν of G. Let $G_{\mathbb{R}}$ be the compact real form corresponding to θ.

By the argument in the beginning of this chapter, there exists a Levi subgroup G' of G such that ι preserves G', $(G' \cap H)$ is a reductive spherical subgroup of G' and $G_{\mathbb{R}}G'H$ contains a non-empty open subset of G. Then we apply Theorem 7.4.4. Here we note that ι is unique up to conjugate. $\square$

We note that Theorem 1.3.4 follows from this theorem.

7.5 Applications

7.5.1 Application: Strongly visible action on vector space

We deal with the visibility of linear actions of compact Lie groups [Sa1, Sa4]. Let (G, V) be a linear multiplicity-free space.

Definition 7.5.1. Let G be a connected complex reductive algebraic group and V a finite-dimensional representation of G. We say V is a linear multiplicity-free space of G if the space $\mathbb{C}[V]$ of polynomials on V is multiplicity-free as a representation of G.

We denote by ν and θ a Chevalley–Weyl involution and a Cartan involution of G, respectively. Let $G_{\mathbb{R}}$ be the compact real form of G, which corresponds to θ.

Corollary 7.5.2. *The $G_{\mathbb{R}}$-action on V is strongly visible.*

Proof. By [Ak1], there exists an anti-linear involution μ of V, which is $\iota = \nu \circ \theta$-compatible. We take a $v \in V$ such that $G \cdot v$ is open in V. Since $\mu(G \cdot v)$ is also open and V contains only one open orbit, $G \cdot v$ is μ-stable. We let $v_0 \in V$ be a μ-fixed point and denote by H the stabilizer of the G-action at v_0. Then H is a spherical subgroup of G, which is stable under ι. Thus we can apply Theorem 7.4.5. $\square$

Remark 7.5.3. Sasaki [Sa1, Sa4] not only proves the existence of strongly visible actions but also constructs "slices" explicitly by using the case-by-case argument. To get a precise information on slices that appear in our proof, we need to know the generic stabilizer of a G-action on V.

7.5.2 Application: Strongly visible action on smooth affine spherical variety

Corollary 7.5.4. *Let X be a smooth affine G-spherical variety and $G_{\mathbb{R}}$ a compact real form of G. Then the $G_{\mathbb{R}}$-action on X is strongly visible.*

Proof. By [Ak09], there exists a real structure on X, which is compatible with $\iota = \theta \circ \nu$. The remaining argument is the same as that for the case of linear multiplicity-free spaces. $\square$

A typical example of smooth affine spherical varieties is a complex symmetric space. On the other hand, we have the principal affine space G/N (N is a maximal unipotent subgroup) as an example of non-affine spherical varieties.

7.5.3 Application: Strongly visible action on wonderful variety

Definition 7.5.5. A G-variety X is said to be wonderful if

- X is smooth and projective,
- G has an open orbit on X, whose complement is the union of finitely many smooth prime divisors X_i $(i \in I)$ with normal crossings.
- The closure of any G-orbit on X is given as a partial intersection of X_i $(i \in I)$.

Remark 7.5.6.
- De Concini–Procesi compactification [CP] of a complex symmetric space is wonderful.
- The stabilizer H of a unique open orbit G/H on a G-wonderful variety X is almost self-normalizing, that is, the group $N_G(H)/H$ is finite by Brion and Pauer [BP].

Corollary 7.5.7. *Let X be a wonderful G-variety and $G_{\mathbb{R}}$ a compact real form of G. Then the $G_{\mathbb{R}}$-action on X is strongly visible.*

Proof. We can show this statement by the same argument as in the case of linear multiplicity-free spaces by using the fact that any wonderful variety admits an $\iota = (\theta \circ \nu)$-equivariant real structure by Akhiezer and Cupit-Foutou [AC]. $\square$

7.5.4 Application: Strongly visible action on generalized flag variety

Corollary 7.5.8. *Let P be a parabolic subgroup and H a Levi subgroup of G. If G/P is an H-spherical variety, then a compact real form of H acts on G/P strongly visibly.*

Proof. The same argument as in the case of linear multiplicity-free spaces can be applied since a unique open orbit of H is stable under a Chevalley–Weyl involution. $\square$

7.6 Visible actions of real forms of inner type on spherical varieties

We recall the definition of a previsible action.

Definition 7.6.1 (Definition 1.0.2)**.** Let K be a Lie group and X a complex manifold on which K acts holomorphically. Then we say this action is previsible if there exists a totally real submanifold S of X such that $K \cdot S$ is a non-empty open subset of X.

Let $G_{\mathbb{R}} \subset G$ be a real form of inner type (i.e., the Lie algebra of $G_{\mathbb{R}}$ has a compact Cartan subalgebra).

Theorem 7.6.2. *Let X be a G-spherical variety. Then $G_{\mathbb{R}}$ acts on X previsibly.*

Proof. Use Lemma 7.2.1 and Proposition 7.3.1. $\square$

Chapter 8

A conceptual proof for generalized Cartan decomposition

8.1 Introduction for Chapter 8

We give an abstract proof for the existence of a generalized Cartan decomposition (Definition 8.1.1), which is also useful for an explicit calculation. Let G be a connected semisimple compact Lie group, T its maximal torus and σ a Chevalley–Weyl involution of G with respect to T (that is, σ is an involutive automorphism of G, which satisfies $\sigma(t) = t^{-1}$ for any $t \in T$). Let (L, H) be a pair of Levi subgroups of G with respect to a simple system $\Pi = \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ of the root system $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$.

Definition 8.1.1 (Definition 1.0.14)**.** Let G, H, L and σ as above. If the multiplication mapping

$$L \times B \times H \to G$$

is surjective for a subset B of the σ-fixed points subgroup G^σ, then we say the decomposition $G = LBH$ is a generalized Cartan decomposition.

We extend σ to the complexification $G_{\mathbb{C}}$ of G anti-holomorphically, and let $B = T_{\mathbb{C}}N$ be the Borel subgroup of $G_{\mathbb{C}}$ corresponding to the positive system $\Delta^+ = \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ generated by Π. We denote by θ the Cartan involution of $G_{\mathbb{C}}$ with respect to G. Also, for a Levi subgroup G' of G, we denote by $U^{G_{\mathbb{C}}}_{G'_{\mathbb{C}}}$ the unipotent radical of the parabolic subgroup $P_{G'_{\mathbb{C}}}$ of $G_{\mathbb{C}}$, which contains B and has $G'_{\mathbb{C}}$ as its Levi part. We write $N_{G'_{\mathbb{C}}}$ for the maximal unipotent subgroup $N \cap G'_{\mathbb{C}}$ of $G'_{\mathbb{C}}$. Hence $U^{G_{\mathbb{C}}}_{G_{\mathbb{C}}} = \{e\}$ and $N_{G_{\mathbb{C}}} = N$. We put $\overline{N} = \theta(N)$, $\overline{U}^{G_{\mathbb{C}}}_{G'_{\mathbb{C}}} = \theta(U^{G_{\mathbb{C}}}_{G'_{\mathbb{C}}})$ and $\overline{N}_{G'_{\mathbb{C}}} = \theta(N_{G'_{\mathbb{C}}})$. In the following, we assume that

$$L_{\mathbb{C}} \text{ acts spherically on a generalized flag variety } G/H \simeq G_{\mathbb{C}}/P_{H_{\mathbb{C}}} = G_{\mathbb{C}}/H_{\mathbb{C}}U^{G_{\mathbb{C}}}_{H_{\mathbb{C}}}.$$

That is, a Borel subgroup of $L_{\mathbb{C}}$ has an open orbit on $G_{\mathbb{C}}/P_{H_{\mathbb{C}}}$.

Proposition 8.1.2. *For any $g \in G$, there exist $l \in L$ and $h \in H$ such that $\sigma(g) = lgh$.*

Proof. We write $\overline{U}^{G_{\mathbb{C}}}_{L_{\mathbb{C}}} = (\overline{U}^{G_{\mathbb{C}}}_{L_{\mathbb{C}}} \cap \overline{U}^{G_{\mathbb{C}}}_{H_{\mathbb{C}}}) \cdot \overline{U}^{H_{\mathbb{C}}}_{(L\cap H)_{\mathbb{C}}}$. We note that

$$\begin{aligned}\overline{N} \cdot o &= \overline{U}^{G_{\mathbb{C}}}_{L_{\mathbb{C}}} \cdot \overline{N}_{L_{\mathbb{C}}} \cdot o \\ &\subset \overline{U}^{G_{\mathbb{C}}}_{L_{\mathbb{C}}} \cdot L \cdot o = L \cdot \overline{U}^{G_{\mathbb{C}}}_{L_{\mathbb{C}}} \cdot o = L \cdot (\overline{U}^{G_{\mathbb{C}}}_{L_{\mathbb{C}}} \cap \overline{U}^{G_{\mathbb{C}}}_{H_{\mathbb{C}}}) \cdot \overline{U}^{H_{\mathbb{C}}}_{(H\cap L)_{\mathbb{C}}} \cdot o = L \cdot (\overline{U}^{G_{\mathbb{C}}}_{L_{\mathbb{C}}} \cap \overline{U}^{G_{\mathbb{C}}}_{H_{\mathbb{C}}}) \cdot o.\end{aligned}$$

This shows that $L\cdot(\overline{U}^{G_\mathbb{C}}_{L_\mathbb{C}}\cap\overline{U}^{G_\mathbb{C}}_{H_\mathbb{C}})\cdot o$ contains an open dense subset of $G_\mathbb{C}/P_{H_\mathbb{C}}$. Here o denotes the identity coset of $G_\mathbb{C}/P_{H_\mathbb{C}}$. Further, since $\overline{U}^{L_\mathbb{C}}_{(L\cap H)_\mathbb{C}}\cdot(\overline{U}^{G_\mathbb{C}}_{L_\mathbb{C}}\cap\overline{U}^{G_\mathbb{C}}_{H_\mathbb{C}})\cdot o$ is open in $G_\mathbb{C}/P_{H_\mathbb{C}}$ and stable under the action of the Borel subgroup $T_\mathbb{C}\overline{N}_{L_\mathbb{C}}=T_\mathbb{C}\overline{N}_{(L\cap H)_\mathbb{C}}\cdot\overline{U}^{L_\mathbb{C}}_{(L\cap H)_\mathbb{C}}$ of $L_\mathbb{C}$, there is an open orbit of the Borel subgroup $T_\mathbb{C}\overline{N}_{(L\cap H)_\mathbb{C}}\cdot\overline{U}^{L_\mathbb{C}}_{(L\cap H)_\mathbb{C}}$ on $\overline{U}^{L_\mathbb{C}}_{(L\cap H)_\mathbb{C}}\cdot(\overline{U}^{G_\mathbb{C}}_{L_\mathbb{C}}\cap\overline{U}^{G_\mathbb{C}}_{H_\mathbb{C}})\cdot o$ by the sphericality of the action of $L_\mathbb{C}$. Here we note that the dimension of any orbit of the Borel subgroup $T_\mathbb{C}\overline{N}_{(L\cap H)_\mathbb{C}}\cdot\overline{U}^{L_\mathbb{C}}_{(L\cap H)_\mathbb{C}}$ of $L_\mathbb{C}$ on $\overline{U}^{L_\mathbb{C}}_{(L\cap H)_\mathbb{C}}\cdot(\overline{U}^{G_\mathbb{C}}_{L_\mathbb{C}}\cap\overline{U}^{G_\mathbb{C}}_{H_\mathbb{C}})\cdot o$ is less than or equal to the sum of the dimension of $\overline{U}^{L_\mathbb{C}}_{(L\cap H)_\mathbb{C}}$ and the maximum of the dimensions of the orbits of the Borel subgroup $T_\mathbb{C}\overline{N}_{(L\cap H)_\mathbb{C}}$ of $(L\cap H)_\mathbb{C}$ on $(\overline{U}^{G_\mathbb{C}}_{L_\mathbb{C}}\cap\overline{U}^{G_\mathbb{C}}_{H_\mathbb{C}})$. Therefore any Borel subgroup of $(L\cap H)_\mathbb{C}$ has an open orbit on the Stein manifold $(\overline{U}^{G_\mathbb{C}}_{L_\mathbb{C}}\cap\overline{U}^{G_\mathbb{C}}_{H_\mathbb{C}})$. Then by [Ak1], for any $u\in(\overline{U}^{G_\mathbb{C}}_{L_\mathbb{C}}\cap\overline{U}^{G_\mathbb{C}}_{H_\mathbb{C}})$ there is $m\in(L\cap H)$ such that $\sigma(u)=\mathrm{Ad}(m)u$ since $(\overline{U}^{G_\mathbb{C}}_{L_\mathbb{C}}\cap\overline{U}^{G_\mathbb{C}}_{H_\mathbb{C}})$ is an $(L\cap H)_\mathbb{C}$-spherical Stein manifold. Thus, for any $x\in L\cdot(\overline{U}^{G_\mathbb{C}}_{L_\mathbb{C}}\cap\overline{U}^{G_\mathbb{C}}_{H_\mathbb{C}})\cdot o$, there is $l\in L$ such that $\sigma(x)=l\cdot x$. Hence the subset $\{x\in G_\mathbb{C}/P_{H_\mathbb{C}}:\sigma(x)\in L\cdot x\}$ of $G_\mathbb{C}/P_{H_\mathbb{C}}$ contains an open dense subset of $G_\mathbb{C}/P_{H_\mathbb{C}}$. Since this subset is closed, it coincides with the whole space. □

Theorem 8.1.3. *Let G be a connected compact Lie group, T a maximal torus of G and L, H Levi subgroups of G, which contain T. Let σ be a Chevalley–Weyl involution of G with respect to T. We have a generalized Cartan decomposition $G=LBH$ (Definition 8.1.1), where B is a subset of G^σ.*

Proof. We prove this theorem by the induction on the dimension of G. By a classification of multiplicity-free tensor product representations proved by J. Stembridge[St2], either (G,L) or (G,H) is a Gelfand pair. Let us assume that (G,L) is a Gelfand pair. We take a symmetric subgroup H' of G, which contains H. Then we have a decomposition $G=LA_1A_2H'$ where A_1 and A_2 are abelian subgroups of G (see Lemma 8.2.5 and Remark 8.2.6). By Proposition 8.1.2, for any $a\in A_1$, $t\in A_2$ and $h'\in H'$, there exist $l\in L$ and $h\in H$ such that $\sigma(ath')=at\sigma(h')=lath'h$. By [Ho, Chapter 8], $l\in Z_L(a^4)$. So, if a^4 is a generic element of A_1, $l\in Z_L(A_1)$. Hence $t^{-1}lt\in H'$, and then $l\in Z_L(A_2)$ if we choose a suitable t.

Indeed, let a_1 and a_2 be any generic elements of A_1 and A_2, respectively, and suppose that $la_2a_1h=a_2a_1$ holds for some $h\in H'$ and $l\in L$. Then we have $a_2^{-1}la_2a_1h=a_1$. By [Ho, Chapter 8], this implies $a_2^{-1}la_2\in Z_K(A_1)$ where K is a symmetric subgroup of G, which contains both L and A_2, and used for obtaining the decomposition $G=LA_1A_2H'$ in Lemma 8.2.5. We put $G_1=Z_K(A_1)$, and consider the double coset decomposition $(G_1\cap L)\backslash G_1/(G_1\cap H')$. By the induction on the dimension, $l\in Z_{L\cap G_1}(A_2)$. Hence $l\in Z_L(A_1A_2)$.

Thus we have $\sigma(h')=lh'h$ with $l\in Z_{L\cap H'}(A_1A_2)$. Then by the propagation theorem of the multiplicity-freeness property for visible actions [Ko3], the holomorphically induced representation $\mathrm{Ind}_H^{H'}\chi_H$ of H' is multiplicity-free when restricted to $Z_{L\cap H'}(A_1A_2)$ for any unitary character χ_H of H. Let L' be a Levi subgroup of H', which satisfies $[L',L']\subset Z_{L\cap H'}(A_1A_2)\subset L'$. Then we can see that $L'_\mathbb{C}$ acts on H'/H in the spherical fashion. By the induction hypothesis, we have $H'=L'B'H$ for some subset $B'\subset H'^\sigma$. Also, we have $H'=Z_{L\cap H'}(A_1A_2)B'H$. Indeed, Let z and b be any elements of $\exp(\mathfrak{z}_{\mathfrak{l}'})$ and B', respectively. Here $\mathfrak{z}_{\mathfrak{l}'}$ is the center of the Lie algebra $\mathfrak{l}'$ of L'. There are $l\in Z_{L\cap H'}(A_1A_2)$ and $h\in H$ such that $\sigma(zb)=z^{-1}b=lzbh$, and thus $z^{-2}b=lbh\in Z_{L\cap H'}(A_1A_2)B'H$. We obtain

$$\begin{aligned}
H' &= L'B'H \\
&= Z_{L\cap H'}(A_1A_2)(\exp(\mathfrak{z}_{\mathfrak{l}'})B')H \\
&\subset Z_{L\cap H'}(A_1A_2)(Z_{L\cap H'}(A_1A_2)B'H)H \\
&= Z_{L\cap H'}(A_1A_2)B'H.
\end{aligned}$$

Therefore we have $G = LA_1A_2H' = LA_1A_2Z_{L\cap H'}(A_1A_2)B'H = LA_1A_2B'H$ with $A_1A_2B' \subset G^\sigma$. □

Remark 8.1.4. This theorem combined with the propagation theorem of multiplicity-freeness property under visible actions (Fact 1.0.4) and Stembridge's classification [St2] reproduces the list in Theorem 1.1.1.

8.2 Double coset decomposition with abelian slice

The aim of this section is to give a KAK-decomposition for Gelfand pairs. We use the induction on the dimension of G in order to give the decomposition. We prepare two lemmas. In the following, we say (G, H) is a reductive Gelfand pair if (G, H) is a Gelfand pair and G is a real reductive Lie group.

Let (G, K') be a reductive Gelfand pair with G/K' connected. Let K be a maximal compact subgroup of G, which contains K' and $M_0 \subset K$ the M-part of the Langlands decomposition of a minimal parabolic subgroup $P_0 = M_0A_0N_0$ of G with $A_0 = \exp(\mathfrak{a}_0)$ for a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{q} = \mathfrak{k}^\perp$.

Lemma 8.2.1. $K = M_0K'$.

Proof. Since $G_\mathbb{C}/K'_\mathbb{C}$ is a $G_\mathbb{C}$-spherical variety and since G/K' is totally real of maximal dimension in $G_\mathbb{C}/K'_\mathbb{C}$, we have

$$\mathfrak{g}_\mathbb{C} = \mathrm{Ad}(k)\mathfrak{k}'_\mathbb{C} + \mathfrak{m}_{0,\mathbb{C}} + \mathfrak{a}_{0,\mathbb{C}} + \mathfrak{n}_{0,\mathbb{C}}$$

for some $k \in K$. Thus

$$\mathfrak{g} = \mathrm{Ad}(k)\mathfrak{k}' + \mathfrak{m}_0 + \mathfrak{a}_0 + \mathfrak{n}_0.$$

Therefore K' has an open orbit on $G/M_0A_0N_0 \simeq K/M_0$. Since K' is compact and $G/M_0A_0N_0$ is connected, we have $G = K'M_0A_0N_0$, and thus $K = M_0K'$. □

Lemma 8.2.2. *$(M_0, M_0 \cap K')$ is a Gelfand pair with K/K' connected.*

Proof. Since $G_\mathbb{C}/K'_\mathbb{C}$ is a $G_\mathbb{C}$-spherical variety and since $K_\mathbb{C}A_{0,\mathbb{C}}N_{0,\mathbb{C}}$ is an open dense subset of $G_\mathbb{C}$, there is $k \in K_\mathbb{C}$ such that

$$\mathfrak{g}_\mathbb{C} = \mathrm{Ad}(k)\mathfrak{k}'_\mathbb{C} + \mathfrak{b}_{\mathfrak{m}_{0,\mathbb{C}}} + \mathfrak{a}_{0,\mathbb{C}} + \mathfrak{n}_{0,\mathbb{C}}$$

where $\mathfrak{b}_{\mathfrak{m}_{0,\mathbb{C}}}$ is a Borel subalgebra of $\mathfrak{m}_{0,\mathbb{C}}$. Further, the condition that $k \in K_\mathbb{C}$ satisfies the above equality is Zariski open and thus especially holomorphic, so we can take k from the compact real form $k \in K$. Since $K = M_0K'$ we can further take k to be an element of $m \in M_0$. The fact that $\mathfrak{m}_{0,\mathbb{C}}$ is the centralizer of $\mathfrak{a}_0$ in $\mathfrak{k}_\mathbb{C}$ implies

$$\begin{aligned}
\mathfrak{m}_{0,\mathbb{C}} &= \mathfrak{m}_{0,\mathbb{C}} \cap \mathrm{Ad}(m)\mathfrak{k}'_\mathbb{C} + \mathfrak{b}_{\mathfrak{m}_{0,\mathbb{C}}} \\
&= \mathrm{Ad}(m)(\mathfrak{m}_{0,\mathbb{C}} \cap \mathfrak{k}'_\mathbb{C}) + \mathfrak{b}_{\mathfrak{m}_{0,\mathbb{C}}}.
\end{aligned}$$

Therefore the lemma follows. Here we note that $M_0/M_0 \cap K' \simeq K/K'$ is connected since G/K' is connected and that component groups of G and K are the same by $K \times \mathfrak{p}$ decomposition. $\square$

Proposition 8.2.3. *Let (G, K') be a reductive Gelfand pair with G/K' connected. We have $G = K'A'K'$ for some abelian subgroup A'.*

Proof. We use the induction on the dimension of G. We divide the proof into two cases.

- Assumption: There is a compact symmetric subgroup K of G, which contains K'. By Lemma 8.2.2, the pair $(M_0, M_0 \cap K')$ is a Gelfand pair with $M_0/M_0 \cap K'$ connected, where M_0 is the centralizer $Z_K(\mathfrak{a}_0)$ of a maximal abelian subspace $\mathfrak{a}_0'$ of $\mathfrak{g}^{-\tau}$ in K, and τ denotes the corresponding involution of G. Since $\dim(M_0)$ is smaller than $\dim(G)$, we can use the induction hypothesis, and obtain a decomposition

$$M_0 = (M_0 \cap K')A'_{M_0}(M_0 \cap K')$$

for some abelian subgroup A'_{M_0} of M_0. By using this decomposition of M_0 and by Lemma 8.2.1, we have

$$\begin{aligned}
G &= K \exp(\mathfrak{a}_0) K \\
&= K' M_0 \exp(\mathfrak{a}_0) M_0 K' \\
&= K' M_0 \exp(\mathfrak{a}_0) K' \\
&= K'(M_0 \cap K')A'_{M_0}(M_0 \cap K') \exp(\mathfrak{a}_0) K' \\
&= K' A'_{M_0} \exp(\mathfrak{a}_0) K'.
\end{aligned}$$

We put $A' = A'_{M_0} \exp(\mathfrak{a}_0)$, and the proposition follows.

- Assumption: There is no compact symmetric subgroup of G, which contains K'. By a classification of reductive Gelfand pairs [Br, Kr, Mi, Ya], there are two possibilities if (G, K') is irreducible:

Case 1. $(G, K') = (\mathrm{Spin}(7), \mathrm{G}_2)$.

Case 2. $(G, K') = (\mathrm{G}_2, \mathrm{SU}(3))$.

By [Da, DK], there is a one dimensional subspace $\mathfrak{a}'$ of $(\mathfrak{k}')^\perp$, which satisfies $(\mathfrak{k}')^\perp = \bigcup_{k \in K'} \mathrm{Ad}(k)\mathfrak{a}'$ in each of two cases. Therefore we have

$$G = K'A'K'.$$

Here we put $A' = \exp(\mathfrak{a}')$. This completes the proof. $\square$

Remark 8.2.4. We explain how to reduce a general case to the case where G is connected and semisimple. Let (G, K') be a reductive Gelfand pair with G/K' connected. We denote by $\mathfrak{z}$ the center of $\mathfrak{g}$, and by $\pi : G^0 \to G^0/\exp(\mathfrak{z})$ the quotient mapping, where G^0 is the identity component of G. Then $(\pi(G^0), \pi(G^0 \cap K'))$ is also a Gelfand pair [Ya, Wo]. Suppose that we have $\pi(G^0) = \pi(G^0 \cap K')\pi(A'')\pi(G^0 \cap K')$ for an abelian subgroup A'' of G^0. We put $A' = A'' \exp(\mathfrak{z})$. Then we have $G^0 = (G^0 \cap K')A'(G^0 \cap K')$, and thus $G = K'A'K'$ since G/K' is connected.

Lemma 8.2.5. *Let G be a compact Lie group, (G, L) a Gelfand pair and (G, H) a symmetric pair with the corresponding involution τ satisfying $\tau(L) = L$. Then there is an abelian subgroup A such that $G = LAH$.*

Proof. We prove this lemma by the induction on the dimension of G. Let us suppose that the lemma is true for any G' whose dimension is less than $\dim G$. Then, we firstly assume that there is a symmetric subgroup K of G, which contains L. By the Cartan decomposition in the symmetric setting [Ho, Ma3], there is an abelian subgroup A_1 such that $G = KA_1H$. Since (G, L) is a Gelfand pair, the equality $K = LM$ holds, where M is the centralizer of a maximal abelian subspace of $\mathfrak{k}^\perp$ (Lemma 8.2.1). Further, since M is stable under the involution τ, and since $(M, M \cap L)$ is again a Gelfand pair (Lemma 8.2.2), we can use the induction-hypothesis and obtain $G = L(L \cap MA_2M \cap H)A_1H = LA_2A_1H$ for some abelian subgroup A_2. There are two cases where we cannot apply this argument, so we explicitly give double coset decompositions below. □

Remark 8.2.6. If $\operatorname{rank} H = \operatorname{rank} G$ and G is semisimple, then we can construct the abelian subgroup A in the above as an abelian subgroup that is fixed by a Chevalley–Weyl involution of G by a succession of the Cayley transform combined with the following lemma.

Lemma 8.2.7. *Let $\mathfrak{j}$ be a Cartan subalgebra of $\mathfrak{g}$, which is stable under the involution τ of $\mathfrak{g}$, and $\mathfrak{j} = \mathfrak{t} \oplus \mathfrak{a}$ the eigenspace decomposition as the sum of the $+1$-eigenspace $\mathfrak{t}$ and the -1-eigenspace $\mathfrak{a}$. Suppose that there exists an automorphism σ of $\mathfrak{g}$, which preserves $\mathfrak{a}$ and acts on $\mathfrak{t}$ as the multiplication by (-1), and that $\mathfrak{t} \oplus \sqrt{-1}\mathfrak{a}$ is not maximally non-compact as a Cartan subalgebra of the non-compact dual $\mathfrak{g}^d = \mathfrak{g}^\tau + \sqrt{-1}\mathfrak{g}^{-\tau}$. Then for any root vector $X_\beta \in \mathfrak{g}_\mathbb{C}$ of any imaginary non-compact root β, there exists $Z \in \mathfrak{t}$ such that $\mathrm{Ad}(\exp(Z))(X_\beta + \overline{X_\beta})$ is fixed by σ. Here we extend σ to $\mathfrak{g}_\mathbb{C}$ holomorphically, and $\overline{X}$ denotes the conjugate element with respect to $\mathfrak{g}$ for any $X \in \mathfrak{g}_\mathbb{C}$.*

Proof. Since both $\overline{X_\beta}$ and $\sigma(X_\beta)$ belong to the root subspace $\mathfrak{g}_{-\beta}$ of $-\beta$, $\sigma(X_\beta) = e^{\sqrt{-1}\theta}\overline{X_\beta}$ for some $\theta \in \mathbb{R}$. Then we take $Z \in \mathfrak{t}$ satisfying $\beta(Z) = -\sqrt{-1}\theta/2$. (Here we note that β is imaginary.) For this $Z \in \mathfrak{t}$, we have

$$\begin{aligned}
\sigma(\mathrm{Ad}(\exp(Z))(X_\beta + \overline{X_\beta})) &= \sigma(e^{-\frac{\sqrt{-1}\theta}{2}}X_\beta + \overline{e^{\frac{-\sqrt{-1}\theta}{2}}X_\beta}) \\
&= e^{-\frac{\sqrt{-1}\theta}{2}}(e^{\sqrt{-1}\theta}\overline{X_\beta}) + e^{\frac{\sqrt{-1}\theta}{2}}(e^{-\sqrt{-1}\theta}X_\beta) \\
&= e^{\frac{\sqrt{-1}\theta}{2}}\overline{X_\beta} + e^{-\frac{\sqrt{-1}\theta}{2}}X_\beta \\
&= \mathrm{Ad}(\exp(Z))(X_\beta + \overline{X_\beta}).
\end{aligned}$$

□

8.3 Explicit decompositions for the minimal cases

1 (Non-factorizable case)

- $(G, L, H) = (\mathrm{G}_2, \mathrm{A}_2, \mathrm{A}_1 \times \mathrm{A}_1)$
- $(G, L, H) = (\mathrm{B}_3, \mathrm{G}_2, \mathrm{B}_1 \times \mathrm{D}_2)$

We denote by τ the involution of G, which corresponds to H. Since the pair (G, L) is polar, we have $\mathfrak{l}^\perp = \mathfrak{a} + [\mathfrak{l}, Z]$ for any non-zero element Z of $(\mathfrak{l}^\perp)^{-\tau}$ and the one-dimensional subspace $\mathfrak{a} = \mathbb{R}Z$ [Da]. We note that $[\mathfrak{l}^\tau, Z] \subset (\mathfrak{l}^\perp)^{-\tau}$ and $[\mathfrak{l}^{-\tau}, Z] \subset (\mathfrak{l}^\perp)^\tau$. This implies that $\mathfrak{a} + [\mathfrak{l} \cap \mathfrak{h}, Z] = (\mathfrak{h}^\perp)^{-\tau}$. Because of the facts that the action of $L \cap H$ on $(\mathfrak{l}^\perp)^{-\tau}$ is unitary with respect to the Killing form, that the decomposition $\mathfrak{a} + [\mathfrak{l} \cap \mathfrak{h}, Z] = (\mathfrak{h}^\perp)^{-\tau}$ is a direct sum and that $\mathfrak{a}$ is of one dimension, $L \cap H$ has an open orbit on the unit sphere in $(\mathfrak{h}^\perp)^{-\tau}$. Since $L \cap H$ is compact, this implies that the action is transitive on the unit sphere. Hence $(\mathfrak{h}^\perp)^{-\tau} = \bigcup_{h \in L \cap H} \mathrm{Ad}(h)\mathfrak{a}$. The remaining argument is the same as [Ho, Chapter 6].

2 (Factorizable case)

- $(G, L, H) = (\mathrm{B}_3, \mathrm{G}_2, \mathrm{D}_3)$
- $(G, L, H) = (\mathrm{B}_3, \mathrm{G}_2, T \times \mathrm{B}_2)$

For each of the above two cases, we have $G = HL$ by [On2, Chapter 4].

Remark 8.3.1. For the case 1 (non-factorizable case), we have also the following elementary proof.

- The case $(G, L, H) = (\mathrm{G}_2, \mathbb{Z}/2\mathbb{Z}\backslash(\mathrm{SU}(2) \times \mathrm{SU}(2)), \mathrm{SU}(3))$.
To study the double coset decomposition $\mathbb{Z}/2\mathbb{Z}\backslash(\mathrm{SU}(2) \times \mathrm{SU}(2))\backslash G_2/\mathrm{SU}(3)$, we use an embedding $G_2 \to \mathrm{SO}(7)(\to \mathrm{SO}(8))$. We let $\{\varepsilon_i - \varepsilon_{i+1}, \varepsilon_3 + \varepsilon_4; 1 \leq i \leq 3\}$ be a simple system of $\mathfrak{so}(8, \mathbb{C})$ and σ an involution of $\mathfrak{so}(8, \mathbb{C})$, which induces the switching of the two simple roots $\varepsilon_3 - \varepsilon_4$ and $\varepsilon_3 + \varepsilon_4$. Also, we take an automorphism ϕ of order three, which sends $\varepsilon_1 - \varepsilon_2$, $\varepsilon_3 + \varepsilon_4$ and $\varepsilon_3 - \varepsilon_4$ to $\varepsilon_3 + \varepsilon_4$, $\varepsilon_3 - \varepsilon_4$ and $\varepsilon_1 - \varepsilon_2$, respectively. Then $\mathfrak{so}(8, \mathbb{C})^\sigma$ and $\mathfrak{so}(8, \mathbb{C})^\phi$ are isomorphic to $\mathfrak{so}(7, \mathbb{C})$ and $\mathfrak{g}_2$, respectively. We write $\{f_1 - f_2, f_2 - f_3, f_3\}$ and $\{\alpha_1, \alpha_2\}$ for simple systems for $\mathfrak{so}(7, \mathbb{C})$ and $\mathfrak{g}_2$, respectively. Then we express the root vectors for $\mathfrak{g}_2$ in terms of that for $\mathfrak{so}(7, \mathbb{C})$ as follows.

$$\begin{aligned} X_{\alpha_1} &= X_{f_1 - f_2} + X_{f_3}, \quad X_{\alpha_2} = X_{f_2 - f_3}, \quad X_{\alpha_1 + \alpha_2} = -X_{f_1 - f_3} + X_{f_2}, \\ X_{2\alpha_2 + \alpha_2} &= -X_{f_2 + f_3} - X_{f_1}, \quad X_{3\alpha_1 + \alpha_2} = -X_{f_1 + f_3}, \quad X_{3\alpha_1 + 2\alpha_2} = -X_{f_1 + f_2}, \\ X_{-\alpha_1} &= X_{-f_1 + f_2} + X_{-f_3}, \quad X_{-\alpha_2} = X_{-f_2 + f_3}, \quad X_{-\alpha_1 - \alpha_2} = -X_{-f_1 + f_3} + X_{-f_2}, \\ X_{-2\alpha_1 - \alpha_2} &= -X_{-f_2 - f_3} - X_{-f_1}, \quad X_{-3\alpha_1 - \alpha_2} = X_{-f_1 - f_3}, \quad X_{-3\alpha_1 - 2\alpha_2} = X_{-f_1 - f_2}. \end{aligned}$$

We note the bijection

$$(\mathbb{Z}/2\mathbb{Z}\backslash(\mathrm{SU}(2) \times \mathrm{SU}(2)))\backslash \mathrm{G}_2 / \mathrm{SU}(3) \simeq \mathbb{Z}/2\mathbb{Z}\backslash(\mathrm{SU}(2) \times \mathrm{SU}(2)))\backslash \mathrm{SO}(7)/\mathrm{SO}(6).$$

We can take $\{\alpha_1, 3\alpha_1 + 2\alpha_2\}$ as a simple system of $(\mathbb{Z}/2\mathbb{Z}\backslash(\mathrm{SU}(2) \times \mathrm{SU}(2)))$. For a (one-dimensional) abelian subgroup T^1, we have

$$\mathrm{SO}(7) = (\mathrm{SO}(3) \times \mathrm{SO}(4))T^1\,\mathrm{SO}(6)$$

since both $\mathrm{SO}(3) \times \mathrm{SO}(4)$ and $\mathrm{SO}(6)$ are symmetric subgroups of $\mathrm{SO}(7)$. If we choose a suitable T^1, there exists a subgroup M of $(1 \times \mathrm{SO}(4)) \cap \mathrm{SO}(6)$, which is of type A_1, centralizes T^1 and has $\{X_{f_1 + f_2} + X_{f_1 - f_2}\}$ as the set of its positive root vector. Here we note that we can take $\{X_{f_3}, X_{f_1 + f_2}, X_{f_1 - f_2}\}$ and $\{X_{f_2 - f_3}, X_{f_1 + f_2}, X_{f_1 - f_2}\}$ as the sets of positive root vectors of $\mathrm{SO}(3) \times \mathrm{SO}(4)$ and $\mathrm{SO}(6)$, respectively. Then we can see that the Lie algebras of $(\mathbb{Z}/2\mathbb{Z}\backslash(\mathrm{SU}(2)\times\mathrm{SU}(2)))$ and M generate that of $\mathrm{SO}(3)\times\mathrm{SO}(4)$. Thus we obtain

$$\begin{aligned}\mathrm{SO}(7) =&(\mathrm{SO}(3)\times\mathrm{SO}(4))T^1\,\mathrm{SO}(6)\\ =&((\mathbb{Z}/2\mathbb{Z}\backslash(\mathrm{SU}(2)\times\mathrm{SU}(2)))\cdot M)T^1\,\mathrm{SO}(6)\\ =&(\mathbb{Z}/2\mathbb{Z}\backslash(\mathrm{SU}(2)\times\mathrm{SU}(2)))T^1M\,\mathrm{SO}(6)\\ =&(\mathbb{Z}/2\mathbb{Z}\backslash(\mathrm{SU}(2)\times\mathrm{SU}(2)))T^1\,\mathrm{SO}(6).\end{aligned}$$

- The case $(G,L,H)=(\mathrm{SO}(7),\mathrm{SO}(3)\times\mathrm{SO}(4),\mathrm{G}_2)$.
  We note the factorization $\mathrm{SO}(7)=\mathrm{SO}(6)\,\mathrm{G}_2$. Since the pair $(\mathrm{SO}(6),\mathrm{SU}(3))$ is a Gelfand pair, we have a decomposition $\mathrm{SO}(6)=(\mathrm{SO}(3)\times\mathrm{SO}(3))A\,\mathrm{SU}(3)$ for an abelian subgroup A (of two dimension). Thus we obtain a decomposition $\mathrm{SO}(7)=(\mathrm{SO}(3)\times\mathrm{SO}(3))A\,\mathrm{G}_2$.

Chapter 9

An extension of a result of Matsuki

9.1 Introduction for Chapter 9

In this chapter we consider the double coset decomposition $L\backslash G/H$ of a reductive group G with respect to reductive subgroups L, H. Our main result gives an extension of Matsuki's decomposition concerning symmetric subgroups [Ma2, Ma3]. Here we recall a result of [Ma3]. Let G be a real reductive Lie group with two involutions σ, τ. The set G_{ss} of semisimple elements is defined by

$$G_{ss} = \{g \in G; \sigma\tau_g = \sigma \operatorname{Ad}(g)\tau \operatorname{Ad}(g)^{-1} \text{ is semisimple}\}.$$

Assume that there exists a Cartan involution θ of G, which commutes with both σ and τ, and that $G = G^\sigma G^0 G^\tau$ where G^0 stands for the identity component of G, and G^σ and G^τ denote the σ and τ-fixed points subgroups of G, respectively.

Fact 9.1.1 ([Ma3]). *We have $G_{ss} = \bigcup_{i\in I'} G^\sigma C_i' G^\tau$ where C_i' are the representatives of standard Cartan subsets.*

Here standard Cartan subsets are of the form $C_i' = \exp(\mathfrak{a}_i)\exp(\mathfrak{t}_i)t_i$ with $\mathfrak{a} \subset \mathfrak{a}_i \subset \mathfrak{p}$, $\mathfrak{t}_i \subset \mathfrak{t} \subset \mathfrak{k}^{-\sigma} \cap \mathfrak{k}^{-\tau}$ for a maximal abelian subspace $\mathfrak{t}$ of $\mathfrak{k}^{-\sigma} \cap \mathfrak{k}^{-\tau}$ and $\mathfrak{a}_i + \mathfrak{t}_i \subset \mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau_{t_i}}$ with $t_i \in \exp(\mathfrak{t})$. We note that $\mathfrak{a}_i + \mathfrak{t}_i$ is a maximal abelian subspace of $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau_{t_i}}$ and that t_i is not necessarily an element of $\exp(\mathfrak{t}_i)$. Our aim is to obtain the double coset decomposition of a reductive group with respect to reductive spherical subgroups. We say a subgroup H of a real reductive algebraic group G is a spherical subgroup of G if $G_\mathbb{C}/H_\mathbb{C}$ is a $G_\mathbb{C}$-spherical variety.

Theorem 9.1.2. *Let L and H be reductive spherical subgroups of a connected real semisimple algebraic group G. There exist finitely many abelian subspaces $\mathfrak{j}_i$ of $\mathfrak{g}$ and elements x_i of G $(1 \leq i \leq k)$ such that $\bigcup_{1\leq i\leq k} LC_iH$ contains an open dense subset of G, where $C_i = \exp(\mathfrak{j}_i)x_i$.*

Proof. We use the induction on the dimension of G. For this, we firstly deal with some minimal cases. We only consider the non-complex and non-compact case. (See Sections 8.2 and 8.3 for the compact case. The argument for the complex case is similar to, and even more, simpler than the real case.)
The case $G = \mathrm{G}_{2(2)}$ (split real simple Lie group of type G_2).

- $(L,H) = (\mathrm{SL}(3,\mathbb{R}), \mathrm{SL}(3,\mathbb{R}))$.
We note the following bijection

$$\mathrm{SL}(3,\mathbb{R})\backslash \mathrm{G}_{2(2)} / \mathrm{SL}(3,\mathbb{R}) \simeq \mathrm{SL}(3,\mathbb{R})\backslash \mathrm{SO}_e(3,4)/ \mathrm{SO}_e(3,3).$$

By Matsuki's decomposition [Ma2, Ma3], we have

$$\mathrm{SO}_e(3,4)_{ss} = \mathrm{SO}_e(3,3)T^1 \mathrm{SO}_e(3,3) \cup \mathrm{SO}_e(3,3)A^1 \mathrm{SO}_e(3,3) \cup \mathrm{SO}_e(3,3)A^1 t\, \mathrm{SO}_e(3,3). \tag{9.1.1}$$

Here T^1 is a one-dimensional torus, A^1 a one-dimensional real split torus and t an element of T^1. By the factorization $\mathrm{SO}_e(3,3) = \mathrm{SL}(3,\mathbb{R})\,\mathrm{SO}_e(2,3) = \mathrm{SL}(3,\mathbb{R})\,\mathrm{SO}_e(3,2)$, we obtain

$$\begin{aligned}(9.1.1) &= (\mathrm{SL}(3,\mathbb{R})\,\mathrm{SO}_e(3,2))T^1 \mathrm{SO}_e(3,3) \cup (\mathrm{SL}(3,\mathbb{R})\,\mathrm{SO}_e(2,3))A^1 \mathrm{SO}_e(3,3) \\ &\qquad \cup (\mathrm{SL}(3,\mathbb{R})\,\mathrm{SO}_e(2,3))A^1 t\, \mathrm{SO}_e(3,3) \\ &= \mathrm{SL}(3,\mathbb{R})T^1 \mathrm{SO}_e(3,3) \cup \mathrm{SL}(3,\mathbb{R})A^1 \mathrm{SO}_e(3,3) \cup \mathrm{SL}(3,\mathbb{R})A^1 t\, \mathrm{SO}_e(3,3).\end{aligned}$$

- $(L,H) = (\mathrm{SL}(3,\mathbb{R}), \mathrm{SU}(1,2))$.
We note the following bijection

$$\mathrm{SL}(3,\mathbb{R})\backslash \mathrm{G}_{2(2)} / \mathrm{SU}(1,2) \simeq \mathrm{SL}(3,\mathbb{R})\backslash \mathrm{SO}_e(3,4)/ \mathrm{SO}_e(2,4).$$

By Matsuki's decomposition [Ma2, Ma3], we have

$$\mathrm{SO}_e(3,4)_{ss} = \mathrm{SO}_e(3,3)A^1 \mathrm{SO}_e(2,4). \tag{9.1.2}$$

Here A^1 is a one-dimensional real split torus. By the factorization $\mathrm{SO}_e(3,3) = \mathrm{SL}(3,\mathbb{R})\,\mathrm{SO}_e(2,3)$, we obtain

$$\begin{aligned}(9.1.2) &= (\mathrm{SL}(3,\mathbb{R})\,\mathrm{SO}_e(2,3))A^1 \mathrm{SO}_e(2,4) \\ &= \mathrm{SL}(3,\mathbb{R})A^1 \mathrm{SO}_e(2,4).\end{aligned}$$

- $(L,H) = (\mathrm{SU}(1,2), \mathrm{SU}(1,2))$.
We note the following bijection

$$\mathrm{SU}(1,2)\backslash \mathrm{G}_{2(2)} / \mathrm{SU}(1,2) \simeq \mathrm{SU}(1,2)\backslash \mathrm{SO}_e(3,4)/ \mathrm{SO}_e(2,4).$$

By Matsuki's decomposition [Ma2, Ma3], we have

$$\mathrm{SO}_e(3,4)_{ss} = \mathrm{SO}_e(2,4)T^1 \mathrm{SO}_e(2,4) \cup \mathrm{SO}_e(2,4)A^1 \mathrm{SO}_e(2,4) \cup \mathrm{SO}_e(2,4)A^1 t\, \mathrm{SO}_e(2,4). \tag{9.1.3}$$

Here T^1 is a one-dimensional torus, A^1 a one-dimensional real split torus and t an element of T^1. By the factorization $\mathrm{SO}_e(2,4) = \mathrm{SU}(1,2)\,\mathrm{SO}_e(1,4) = \mathrm{SU}(1,2)\,\mathrm{SO}_e(2,3)$, we obtain

$$\begin{aligned}(9.1.3) &= (\mathrm{SU}(1,2)\,\mathrm{SO}_e(1,4))T^1 \mathrm{SO}_e(2,4) \cup (\mathrm{SU}(1,2)\,\mathrm{SO}_e(2,3))A^1 \mathrm{SO}_e(2,4) \\ &\qquad \cup (\mathrm{SU}(1,2)\,\mathrm{SO}_e(2,3))A^1 t\, \mathrm{SO}_e(2,4) \\ &= \mathrm{SU}(1,2)T^1 \mathrm{SO}_e(2,4) \cup \mathrm{SU}(1,2)A^1 \mathrm{SO}_e(2,4) \cup \mathrm{SU}(1,2)A^1 t\, \mathrm{SO}_e(2,4).\end{aligned}$$

The case $G = \mathrm{SO}_e(3,4)$.

- $$(L,H) = (\mathrm{SO}_e(2,4), \mathrm{G}_{2(2)}),\ (\mathrm{SO}_e(3,3), \mathrm{G}_{2(2)}),\ (\mathrm{SO}(2)\times\mathrm{SO}_e(1,4), \mathrm{G}_{2(2)}),\\ (\mathrm{SO}(2)\times\mathrm{SO}_e(3,2), \mathrm{G}_{2(2)}) \text{ or } (\mathrm{SO}_e(1,1)\times\mathrm{SO}_e(2,3), \mathrm{G}_{2(2)}).$$

  In each of these cases, $G = LH$ holds by [Ak2, On1, On2].

- $(L,H) = (\mathrm{G}_{2(2)}, \mathrm{G}_{2(2)})$.
  We note the following bijection
  $$\mathrm{G}_{2(2)} \backslash \mathrm{SO}_e(3,4) / \mathrm{G}_{2(2)} \simeq \mathrm{G}_{2(2)} \backslash \mathrm{SO}_e(4,4) / \mathrm{Spin}_e(3,4).$$
  By Matsuki's decomposition [Ma2, Ma3], we have
  $$\begin{aligned}\mathrm{SO}_e(4,4)_{ss} = \mathrm{Spin}_e(3,4) T^1 \mathrm{Spin}_e(3,4) \cup \mathrm{Spin}_e(3,4) A^1 \mathrm{Spin}_e(3,4)\\ \cup \mathrm{Spin}_e(3,4) A^1 t \,\mathrm{Spin}_e(3,4).\end{aligned} \tag{9.1.4}$$
  Here T^1 is a one-dimensional torus, A^1 a one-dimensional real split torus and t an element of T^1. By the factorization $\mathrm{Spin}_e(3,4) = \mathrm{G}_{2(2)} \mathrm{Spin}_e(2,4) = \mathrm{G}_{2(2)} \mathrm{Spin}_e(3,3)$, we obtain
  $$\begin{aligned}(9.1.4) &= (\mathrm{G}_{2(2)} \mathrm{Spin}_e(2,4)) T^1 \mathrm{Spin}_e(3,4) \cup (\mathrm{G}_{2(2)} \mathrm{Spin}_e(3,3)) A^1 \mathrm{Spin}_e(3,4)\\ &\qquad \cup (\mathrm{G}_{2(2)} \mathrm{Spin}_e(3,3)) A^1 t \,\mathrm{Spin}_e(3,4)\\ &= \mathrm{G}_{2(2)}\, T^1 \mathrm{Spin}_e(3,4) \cup \mathrm{G}_{2(2)}\, A^1 \mathrm{Spin}_e(3,4) \cup \mathrm{G}_{2(2)}\, A^1 t \,\mathrm{Spin}_e(3,4).\end{aligned}$$

- $(L,H) = (\mathrm{SO}_e(1,2)\times\mathrm{SO}_e(2,2), \mathrm{G}_{2(2)})$.
  We note the following bijection
  $$(\mathrm{SO}_e(1,2)\times\mathrm{SO}_e(2,2)) \backslash \mathrm{SO}_e(3,4) / \mathrm{G}_{2(2)} \simeq (\mathrm{SO}_e(1,2)\times\mathrm{SO}_e(2,2)) \backslash \mathrm{SO}_e(4,4) / \mathrm{Spin}_e(3,4).$$
  By Matsuki's decomposition [Ma2, Ma3], we have
  $$\begin{aligned}\mathrm{SO}_e(4,4)_{ss} =& (\mathrm{SO}_e(2,2)\times\mathrm{SO}_e(2,2)) T^1 \mathrm{Spin}_e(3,4)\\ &\cup (\mathrm{SO}_e(2,2)\times\mathrm{SO}_e(2,2)) A^1 \mathrm{Spin}_e(3,4)\\ &\cup (\mathrm{SO}_e(2,2)\times\mathrm{SO}_e(2,2)) A^1 t \,\mathrm{Spin}_e(3,4).\end{aligned} \tag{9.1.5}$$
  Here T^1 is a one-dimensional torus, A^1 a one-dimensional real split torus and t an element of T^1. Let $M = Z_{\mathrm{Spin}_e(3,4)}(T^1) \cap (\mathrm{SO}_e(2,2)\times\mathrm{SO}_e(2,2))$ and $M' = Z_{\mathrm{Ad}(t)\,\mathrm{Spin}_e(3,4)}(A^1) \cap (\mathrm{SO}_e(2,2)\times\mathrm{SO}_e(2,2))$. Then we have
  $$\begin{aligned}(9.1.5) =& ((\mathrm{SO}_e(1,2)\times\mathrm{SO}_e(2,2)) \cdot M) T^1 \mathrm{Spin}_e(3,4)\\ &\cup ((\mathrm{SO}_e(1,2)\times\mathrm{SO}_e(2,2)) \cdot M') A^1 \mathrm{Spin}_e(3,4)\\ &\cup ((\mathrm{SO}_e(1,2)\times\mathrm{SO}_e(2,2)) \cdot M') A^1 t \,\mathrm{Spin}_e(3,4)\\ =& (\mathrm{SO}_e(1,2)\times\mathrm{SO}_e(2,2)) T^1 \mathrm{Spin}_e(3,4) \cup (\mathrm{SO}_e(1,2)\times\mathrm{SO}_e(2,2)) A^1 \mathrm{Spin}_e(3,4)\\ &\cup (\mathrm{SO}_e(1,2)\times\mathrm{SO}_e(2,2)) A^1 t \,\mathrm{Spin}_e(3,4).\end{aligned}$$

- $(L, H) = (\mathrm{SO}(3) \times \mathrm{SO}_e(1,3), \mathrm{G}_{2(2)})$.
  We note the following bijection
  $$(\mathrm{SO}(3) \times \mathrm{SO}_e(1,3))\backslash \mathrm{SO}_e(3,4)/\,\mathrm{G}_{2(2)} \simeq (\mathrm{SO}(3) \times \mathrm{SO}_e(1,3))\backslash \mathrm{SO}_e(4,4)/\,\mathrm{Spin}_e(3,4).$$
  By Matsuki's decomposition [Ma2, Ma3], we have
  $$\begin{aligned} \mathrm{SO}_e(4,4)_{ss} =& (\mathrm{SO}_e(3,1) \times \mathrm{SO}_e(1,3))T^1 \,\mathrm{Spin}_e(3,4) \\ & \cup (\mathrm{SO}_e(3,1) \times \mathrm{SO}_e(1,3))A^1 \,\mathrm{Spin}_e(3,4) \\ & \cup (\mathrm{SO}_e(3,1) \times \mathrm{SO}_e(1,3))A^1 t \,\mathrm{Spin}_e(3,4). \end{aligned} \tag{9.1.6}$$
  Here T^1 is a one-dimensional torus, A^1 a one-dimensional real split torus and t an element of T^1. Let $M = Z_{\mathrm{Spin}_e(3,4)}(T^1) \cap (\mathrm{SO}_e(3,1) \times \mathrm{SO}_e(1,3))$ and $M' = Z_{\mathrm{Ad}(t)\,\mathrm{Spin}_e(3,4)}(A^1) \cap (\mathrm{SO}_e(3,1) \times \mathrm{SO}_e(1,3))$. Then we have
  $$\begin{aligned} (9.1.6) =& ((\mathrm{SO}(3) \times \mathrm{SO}_e(1,3)) \cdot M)T^1 \,\mathrm{Spin}_e(3,4) \\ & \cup ((\mathrm{SO}(3) \times \mathrm{SO}_e(1,3)) \cdot M')A^1 \,\mathrm{Spin}_e(3,4) \\ & \cup ((\mathrm{SO}(3) \times \mathrm{SO}_e(1,3)) \cdot M')A^1 t \,\mathrm{Spin}_e(3,4) \\ =& (\mathrm{SO}(3) \times \mathrm{SO}_e(1,3))T^1 \,\mathrm{Spin}_e(3,4) \\ & \cup (\mathrm{SO}(3) \times \mathrm{SO}_e(3,1))A^1 \,\mathrm{Spin}_e(3,4) \\ & \cup (\mathrm{SO}(3) \times \mathrm{SO}_e(3,1))A^1 t \,\mathrm{Spin}_e(3,4). \end{aligned}$$

General case.
We use the induction on the dimension of G. By classification results by Brion [Br], Krämer [Kr], Mikityuk [Mi] and Yakimova [Ya], there exist symmetric subgroups G^σ and G^τ of G, which contain L and H, respectively, except for the above two cases. Then by Matsuki's decomposition [Ma2, Ma3], we have $G_{ss} = \bigcup_{i \in I'} G^\sigma C_i' G^\tau$, where $C_i' = \exp(\mathfrak{a}_i)\exp(\mathfrak{t}_i)t_i$ are the representatives of standard Cartan subsets with $\mathfrak{a} \subset \mathfrak{a}_i \subset \mathfrak{p}$, $\mathfrak{t}_i \subset \mathfrak{t} \subset \mathfrak{k}^{-\sigma} \cap \mathfrak{k}^{-\tau}$ for a maximal abelian subspace $\mathfrak{t}$ of $\mathfrak{k}^{-\sigma} \cap \mathfrak{k}^{-\tau}$ and $\mathfrak{a}_i + \mathfrak{t}_i \subset \mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau_{t_i}}$ with $t_i \in \exp(\mathfrak{t})$. Here we note that $\mathfrak{a}_i + \mathfrak{t}_i$ is a maximal abelian subspace of $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau_{t_i}}$ and that t_i is not necessarily an element of $\exp(\mathfrak{t}_i)$. We put $L_i = Z_G(\mathfrak{t}_i)$ if $\mathfrak{t}_i \neq 0$ otherwise $L_i = Z_G(\mathrm{Ad}(t_i^{-1})\mathfrak{a}_i)$. Then the subset $\bigcup_{i \in I'} t_i \,\mathrm{Ad}(t_i^{-1})G^\sigma L_i G^\tau$ contains an open dense subset of G. Here we note that $\mathrm{Ad}(t_i^{-1})G^\sigma = (L_i \cap \mathrm{Ad}(t_i^{-1})G^\sigma))\,\mathrm{Ad}(t_i^{-1})L$ and $G^\tau = (L_i \cap G^\tau)H$. We use the induction hypothesis for each L_i:

For abelian subspaces $\mathfrak{j}_j^{L_i}$ of $\mathfrak{l}_i$ and elements $x_j^{L_i}$ of L_i $(1 \leq j \leq k_i)$, an open dense subset of L_i is contained in $\displaystyle\bigcup_{1 \leq j \leq k_i} (\mathrm{Ad}(t_i^{-1})L \cap L_i)C_j^{L_i}(L_i \cap H)$ with $C_j^{L_i} = \exp(\mathfrak{j}_j^{L_i})x_j^{L_i}$.

Here, since L_i is neither semisimple nor connected in general, we explain how to reduce a general case to the connected semisimple case.

Let G be a (not necessarily connected) real reductive algebraic group, and L and H reductive spherical subgroups of G. We write $G = \bigcup_k G^0 g_k$ with $\{g_k\}$ a finite subset of G, where G^0 stands for the identity component of G. Suppose that $\bigcup_i (G^0 \cap L)C_{i_k}(G^0 \cap \mathrm{Ad}(g_k)H)$ contains an open dense subset of G^0. Then $\bigcup_k \bigcup_i LC_{i_k} g_k H$ contains an open dense subset of G since $\bigcup_{i_k} (G^0 \cap L)C_{i_k} g_k (G^0 \cap H) = \bigcup_{i_k} (G^0 \cap L)C_{i_k}(G^0 \cap \mathrm{Ad}(g_k)H)g_k = (\bigcup_{i_k} (G^0 \cap L)C_{i_k}(G^0 \cap \mathrm{Ad}(g_k)H))g_k$ contains an open dense subset of $G^0 g_k$. Hence we may assume that G is connected. Let L and H be as before. Let $\pi : G \to G/\exp(\mathfrak{z})$ be the projection mapping, where $\mathfrak{z}$ is the center of $\mathfrak{g}$. Suppose that $\bigcup_i \pi(L)\pi(\exp(\mathfrak{j}_i))\pi(x_i)\pi(H)$ contains an open dense subset of $\pi(G)$ for finitely many elements x_i of G and abelian subalgebras $\mathfrak{j}_i$ of $\mathfrak{g}$. We put $C_i = \exp(\mathfrak{j}_i)\exp(\mathfrak{z})$. Then we find that $\bigcup_i LC_iH$ contains an open dense subset of G.

Thus the following subset contains an open dense subset of G.

$$\bigcup_{i\in I'} t_i \operatorname{Ad}(t_i^{-1})L(\bigcup_{j\in J_{L_i}} \operatorname{Ad}(t_i^{-1})L\cap L_i)C_j^{L_i}(L_i\cap H))H = \bigcup_{i\in I'}\bigcup_{j\in J_{L_i}} L\operatorname{Ad}(t_i)C_j^{L_i}t_iH.$$

$\square$

Remark 9.1.3. When a triple (G, G^σ, G^τ) is factorizable, that is, $G = G^\sigma G^\tau$ holds, the induction argument does not work since in that case $\mathfrak{g}^{-\sigma,-\tau}$ is the zero vector space and hence $L_i = G$ for any i in the above notation. However, we can read from a classification of factorizable triples in [Ak2, On1, On2] and a classification of reductive spherical subgroups in [Br, Kr, Mi, Ya] that if L is a reductive spherical subgroup and H is a non-symmetric spherical reductive subgroups of G, contained in symmetric subgroups G^σ and G^τ of G, respectively, and if (G, G^σ, G^τ) is factorizable, then the double coset decomposition is reduced to the case where either $G = G^\sigma H$ and $G^\sigma/(G^\sigma\cap H)$ is a G^σ-spherical variety, or $G = LG^\tau$ and $G^\tau/(G^\tau\cap L)$ is a G^τ-spherical variety. Since G^σ/L and G^τ/H are G^σ- and G^τ-spherical varieties, respectively, we can use the induction hypothesis: $\bigcup_i LC_i(G^\sigma\cap H)$ contains an open dense subset of G^σ, or $\bigcup_j (L\cap G^\tau)C'_jH$ contains an open dense subset of G^τ, and thus find that $\bigcup_i LC_iH$ or $\bigcup_j LC'_jH$ contains an open dense subset of G.

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