博士論文

Monte Carlo Methods for Non linear Problems in Mathematical Finance
(数理ファイナンスにおける非線形問題のモンテカルロ法による数値計算)

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第1章　概論

1.1　序

本論文では数理ファイナンスに現れる非線形問題に対するモンテカルロ法を用いた数値計算法について論じる．

数理ファイナンスに現れる非線形問題として代表的ものはアメリカン (バミューダン)・オプション価格付けである．アメリカン (バミューダン)・オプションとは満期までの任意の時点（予め定められた複数時点）で早期権利行使可能なオプションである．権利行使が満期時点のみで可能なヨーロピアン・オプション価格は，線形偏微分方程式の解になり，Black-Scholes モデルなどの基本的なモデルの場合は解析解を持つ．また，解析解を持たない場合も Feynmann-Kac の定理により，満期でのペイオフの期待値になるため，モンテカルロ法による数値計算手法が直接適用可能である．

一方，早期行使可能なオプション価格は自由境界問題と呼ばれる非線形偏微分方程式の解になり，ほとんどの場合解析解は存在せず，以下で説明するようにモンテカルロ法による数値計算も適用が難しい．

実務上はアメリカン・オプションと呼ばれるものでも，日次の行使判定日を持つバミューダン・オプションであることが多く，以下ではバミューダン・オプションに焦点を当てて説明する．バミューダン・オプション価格を求めることはベルマン原理により，行使時点ごとに満期時点から順に継続価値と呼ばれる一時点先の Option 価格の条件付期待値と，行使価値の大きい値を繰り返し求める Daynamic Programming に帰着される．ここで，継続価値と行使価値の大小比較という非線形性があるため，満期から順に後退的に継続価値を計算していく必要があ，ラティス法や偏微分方程式の有限差分法は適用可能であるが，原資産のパスを時間の流れに沿って前進的に発生させるモンテカルロ法を直接用いることはできない．

一方で，モンテカルロ法はアルゴリズムが比較的簡明であることと多次元問題の際に利用しやすく汎用性が高いため，経路依存性がある商品，多資産を参照する商品，原資産のモデルが多次元の確率過程で表される商品等は，モンテカルロ法以外の手法でその価格を評価することは難しいとされている．このため複雑なスキームを持つデリバティブ取引が増加するにつれデリバティブの価格評価にモンテカルロ法が用いられることが多くなってきている．また，仮に全てのデリバティブの価格評価をモンテカルロ法で統一できれば，各商品の価格を計算する方法を単一の枠組みで整理・運用することができるため，バミューダン・オプション価格をモンテカルロ法を用いて評価するメリットは大きい．

こうした需要に応えるため，バミューダン・オプション価格をモンテカルロ法を用いて評価することを可能にした新しい計算手法が開発された．モンテカルロ法を用いてバミューダン・オプション価格を評価する際には，原資産パスの各状態における条件付期待値を近似計算することになるが，計算効率の面から条件付期待値を state の関数として近似することが必要になる．代表的な手法として Longstaff and Schwartz[13] による Least Square Regression 法を挙げることができる．これは条件付期待値を事前に選択した幾つかの関数で回帰する近

似手法を用いており，計算効率の面で優れている．また，実証研究でペイオフを決める関数の形が複雑とはならない範囲では十分な精度を獲得できることが確認されていることを踏まえて，実務でも広く用いられている．しかし，回帰に基づく手法の欠点として，計算の精度が事前に選択した回帰関数の形に依存するため，選択した関数で条件付期待値がうまく近似できる場合には精度の高い計算結果が得られるが，そうでない場合には無視できない計算誤差が生じうる点を挙げることができる．こうした回帰による欠点を回避し，より計算の精度を重視した手法として，Broadie and Glasserman[2]，Glasserman[7] は Sotchastic Mesh 法を提案した. この方法は条件付き期待値をランダムな関数として近似する方法である．ただし，この手法は条件付期待値を計算する際に原資産の推移密度関数を必要とするため，適用できるモデルが限られてしまうという制約を持つ．このように, バミューダン・オプション価格のモンテカルロ法を用いた計算手法に関しては様々な方法が提案されているが，決定的な方法は確立されておらず，現在でも活発な研究対象となっている．

数理ファイナンスに現れる非線形問題の中でも近年注目を集めているのが，CVA(Credit Valuation Adjustment）の計算手法である．CVA は OTC(店頭) デリバティブ取引におけるカウンターパーティ・リスクを管理するための概念であり金融危機以降，大手金融機関を中心に広く活用されている．CVA とは取引相手が将来デフォルトした際に受ける損失の期待値として算出されるが，それは取引相手と結んでいる契約全体の合計価値の正の部分であるため，CVA の価格計算は非線形な問題となる．

本論文の構成は以下の通りである．まず第 1 章で問題の背景や既存研究，そして本論文で得られた結果を概観する．そして第 2 章において Hörmander 型の拡散過程に対し，Stochastic Mesh 法を用いたバミューダン・オプション価格の近似がモンテカルロ法のシミュレーション回数を増やすことで，真の価値に収束することを証明する．また，re-simulation と呼ばれる方法に Stochastic Mesh 法を適用した場合を考察し，モンテカルロ法のパスに対する収束のオーダーに関する評価を行う．第 3 章では Least Square Regression 法の評価を行ったうえで，評価に現れる各項と，回帰に用いる関数系の関係についての関係から，どのような関数系を用いるのがよいかについて議論する．第 4 章では Stochastic Mesh 法を適用した CVA のモンテカルロ法による 2 つの近似手法について紹介する．そして，2 つの近似手法に関する収束を証明し，そのオーダーについて論じる．

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1.2 バミューダン・オプションの価格評価

本節ではバミューダン・オプション価格評価理論について解説する．$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqq 0}, P)$ をフィルター付き確率空間とし，N 次元の時間一様な x を出発点とする拡散過程 $X(t;x)$ を考える．そして，x_0 を固定して $X(t,x_0)$ を基底過程 (Underlying Process)[1] とする．そして，線形作用素の族 $\{P_t, t \geqq 0\}$ を次のように定義する．

$$(P_t f)(x) = E[f(X(t,x))].$$

ここで，x は時刻 0 における値である．$0 = t_0 < t_1 < \cdots < t_n = T$ を行使可能時刻とし $g : [0,T] \times \mathbf{R}^N \to \mathbf{R}$, を時刻 t でオプションを行使した時のペイオフ関数とする．この時,

[1]Underlying Process はしばしば原資産価格過程と翻訳されるが，資産価格だけではなく，確率ボラティリティモデルなどの場合，そのボラティリティなども $X(t,x_0)$ を用いて表されるため，ここでは基底過程と呼ぶ.

時刻 t_i におけるバミューダン・オプション価値 $v_i(x)$ は次のように表される.

$$v_i(x) = \sup\{E\left[g(\tau, X(\tau - t_i, x))\right]; \tau \in \mathcal{S}_i\}, \tag{1.1}$$
$$\mathcal{S}_i = \{\tau : \Omega \to \{t_i, \ldots, t_n\}; \{\mathcal{F}_t\}\text{-停止時刻}\}. \tag{1.2}$$

バミューダン・オプション時刻 t_i での価値 v_i は以下の Dynamic Programming により満期の価格から順に決めていくことができる.

$$\begin{cases} v_n = g(t_n, \cdot), \\ v_i = g(t_i, \cdot) \vee c_i, \quad i = 1, \ldots, n-1. \end{cases} \tag{1.3}$$

ここで $c_i(x), i = 0, \ldots, n$ は次のように定められる.

$$\begin{cases} c_n = g(t_n, \cdot), \\ c_i = P_{t_{i+1}-t_i} v_{i+1}, \quad i = 1, \ldots, n-1. \end{cases} \tag{1.4}$$

また，バミューダン・オプション価格を定める (1.1) 式で sup を実現する最適行使時刻 τ_i は,

$$\tau_i = \min\{t_k; k = i, \ldots, n, c_k(X(t_k, x)) \leqq g(t_k, X(t_i, x))\}, \tag{1.5}$$

となることが知られている.

1.3 バミューダン・オプション価格のモンテカルロ法による数値計算

モンテカルロ法は確率変数 $X \in L^1(\Omega, P)$ に対し，X と同分布を持つ独立な確率変数列 $\{X_\ell\}_{\ell=1}^L$ を用いて，X の期待値 $E[X]$ を近似する方法である．各 X_ℓ を乱数を用いたシミュレーションで発生させ，その実現値 $X_\ell(\omega)$ の平均値

$$\bar{X} = \frac{1}{L}\sum_{\ell=1}^{L} X_\ell(\omega)$$

を期待値の近似値とする方法である．

バミューダン・オプション価格を数値計算するために，Dynamic Programming において c_i の条件付期待値をモンテカルロ法で求めることを考える．すなわち，時刻 t_i において，x_i から出発する $\{X_\ell(t_{i+1}, x_i)\}_{\ell=1}^L$ を用意し，

$$\hat{c}_i(x_i) = \frac{1}{L}\sum_{\ell=1}^{L} v_{i+1}(X_\ell(t_{i+1}, x_i))$$

を計算する．ただし，

$$v_{i+1}(X_\ell(t_{i+1}, x_i)) = g(t_{i+1}, X_\ell(t_{i+1}, X_\ell(t_{i+1}, x_i))) \vee c_{i+1}(t_{i+1}, X_\ell(t_{i+1}, x_i))$$

であり，各 ℓ ごとに新たに $X_\ell(t_{i+1}, x_i)$ から出発する $\{X'_k(t_{i+2}, X_\ell(t_{i+1}, x_i))\}_{k=1}^L$ を用意し，$c_{i+1}(X_\ell(t_{i+1}, x_i))$ をモンテカルロ法で近似計算することになる．そのため，全体で L^N オーダーの計算が必要となり，短時間で精度良い計算をおこなうことは難しい．

そこで，バミューダン・オプションをモンテカルロ法で計算する場合，c_i を関数として近似することが重要になる．すなわち f という関数に対して $\{X_\ell(\cdot))\}_{\ell=1}^L$ というシミュレーションパスを用いて，$P_{t-s}f, t > s$ を如何に上手く近似するかがアルゴリズムの鍵となっている．

また，c_i の関数としての近似 $\hat{c}_i$ が $i = 0, \ldots, n$ に対して得られたとしよう．この時，バミューダン・オプション価格 $v_0(x_0)$ の近似として自然に $\hat{v}_0(x_0) = \hat{c}_0(x_0) \vee g(0, x_0)$ を考えることが出来る．

一方，バミューダン・オプション最適行使時刻が (1.5) 式で与えられることから，τ_i の近似 $\hat{\tau}_i$ を次のように与えることができる．

$$\hat{\tau}_i = \min\{t_k; k = i, \ldots, N, \hat{c}_k(X(t_k, x_0)) \leqq g(t_k, X(t_k, x_0))\}, i = 1, \ldots, n.$$

そして，バミューダン・オプション価格 v_0 の近似 $\bar{v}_0$ を

$$\bar{v}_0 = \frac{1}{L}\sum_{\ell=1}^{L} g(\hat{\tau}_1, X_\ell(\hat{\tau}_1, x_0)) \vee g(0, x_0) \tag{1.6}$$

とする．この時，$\hat{c}_i$ は行使時刻を求めるためだけに使われ, その値は直接には $\bar{v}_0(x_0)$ に影響を与えないことに注意されたい．(1.5) 式によれば, $\hat{c}_i$ が c_i とどんなに離れていようとも, 行使価格との大小関係さえ等しければ $\hat{c}_i$ の近似による誤差は一切生じない．

そのため，$\bar{v}_0(x_0)$ は $\hat{v}_0(x_0)$ よりも良い近似になっていると期待できる．本論文ではこの方法を re-simulation の方法と呼ぶ．

以下では，具体的な $\hat{c}_i$ の関数としての近似方法について述べる．

1.3.1 Stochastic Mesh 法

(1) 設定

$(E, \mathcal{B})$ を可測空間とし，E 上の可測関数の集合を $m(E)$ とする．本節では $X(t, x)$ は Hölmander 型の拡散過程であるとし，推移密度 $p(t, x, y), t > 0, x, y \in E$ が存在するとし，$\nu_t(dx) = p(t, x_0, dx)$ とする．また，$X(t, x), t \geqq 0$ と同分布で独立な確率過程を $X_\ell(t), t \geqq 0, \ell = 1, 2, \ldots$ とし，filtration $\{\mathcal{F}_t^\infty\}_{t \geqq 0}$ を

$$\mathcal{F}_t^\infty = \sigma\{X_\ell(s), s \leqq t, \ell = 1, 2, \ldots\}$$

とする．Broadie and Glasserman が考案した Stochastic Mesh 法では $P_{T-t}f$ の近似として，次のようなランダムな作用素を導入する．

$$(Q_{s,t}^{(L)} f)(x) = \frac{1}{L}\sum_{\ell=1}^{L} \frac{p(t-s, x, X_\ell(t)) f(X_\ell(t))}{q_{s,t}^{(L)}(X_\ell(t))}, \qquad x \in E,\ f \in m(E).$$

ここで,

$$q_{s,t}^{(L)}(y, \omega) = \frac{1}{L}\sum_{\ell=1}^{L} p(t-s, X_\ell(s, \omega), y), \qquad y \in E,\ \omega \in \Omega,$$

である．$(Q_{s,t}^{(L)} f)$ は次のような性質を持つ．

$$E^P[(Q_{s,t}^{(L)} f)(x) | \mathcal{F}_s^{(\infty)}] = (P_{t-s} f)(x), \qquad \nu_s - a.e. x \in E.$$

(2) Stochastic Mesh 法のアルゴリズム

Broadie and Glasserman [2] による Stochastic Mesh 法を用いたバミューダン・オプション価格計算のアルゴリズムを示す.

Step 1 : $\{X_\ell(t_1),\ldots,X_\ell(t_n)\}_{\ell=1}^{L_0}$ というシミュレーション・パスを用意する.
Step 2 : $\hat{c}_n(x)=g(t_n,x)$ とし，そして $\hat{c}_{i+1}$ まで決まったとする. この時 $\hat{v}_{i+1}(x)=\hat{c}_{i+1}\vee g(t_{i+1},x)$ とする. そして $\hat{c}_i$ を

$$\hat{c}_i(x)=(Q^{(L)}_{t_i,t_{i+1}}\hat{v}_{i+1})(x)$$

により定める.
これを $i=0$ まで繰り返すことにより，$\hat{v}_0(x_0)$ を得る.
また，各 $t_i, i=0,\ldots,n$ に対し，$\hat{c}_i$ を関数として得ているので, re-simulation により $\hat{c}_0'(x_0)$ を求めることも出来る.

(3) 収束に関する先行研究

Stochastic Mesh 法の収束に関しては次の，Avramidis-Matzinger [1] による次の結果がある.

Theorem 1.1 ある $C>0$ が存在し，任意の $k,\ell,m=1,\ldots,L, k\neq\ell$ に対し，以下が成り立つとする.

$$\max_{i=0,\ldots,n}E[\frac{p(t_i,X_k(t_i),X_\ell(t_{i+1}))^4}{p(t_i,X_m(t_i),X_\ell(t_{i+1}))^4}\max_{j=i+1,\ldots,n}g(t_j,X_\ell(t_j))^4]\leqq\frac{C}{8},$$

$$\max_{i=0,\ldots,n}E[\frac{p(t_i,X_k(t_i),X_k(t_{i+1}))^4}{p(t_i,X_m(t_i),X_k(t_{i+1}))^4}\max_{j=i+1,\ldots,n}g(t_j,X_k(t_j))^4]<\infty,$$

$$\max_{i=0,\ldots,n}E[\frac{p(t_i,X_k(t_i),X_\ell(t_{i+1}))^4}{p(t_i,X_m(t_i),X_\ell(t_{i+1}))^4}]<\infty,$$

$$\max_{i=0,\ldots,n}E[\frac{p(t_i,X_k(t_i),X_k(t_{i+1}))^4}{p(t_i,X_m(t_i),X_k(t_{i+1}))^4}]<\infty,$$

この時，任意の $\delta>0$ と $0<\gamma<1/4$ に対し，次が成り立つ.

$$P\left(|\hat{v}_0(x_0)-v_0(x_0)|>\left(1+\frac{\delta}{L^\gamma}\right)^n-1\right)\leqq\frac{6Cn}{\delta^4L^{1-4\gamma}}+O(L^{-2+4\gamma}).$$

1.3.2 第 2 章の結果

本節では以下の設定で考える．$N,d\geqq 1$ とし，$W_0=\{w\in C([0,\infty);\mathbf{R}^d);\ w(0)=0\}$ とする．$\mathcal{F}$ を W_0 上の Borel algebra とし，μ を $(W_0,\mathcal{F})$ 上の Wiener measure とする．また，$\{(B^1(t),\ldots,B^d(t);t\in[0,\infty)\}$ を d-次元 Brown 運動とし，$B^0(t)=t, t\in[0,\infty)$ とする．そして，$V_0,V_1,\ldots,V_d\in C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$ とし，(UFG) 条件をみたすとする．そして，$X(t,x)$, $t\in[0,\infty)$, $x\in\mathbf{R}^N$, を次の Stratonovich 型の確率微分方程式の解とする.

$$X(t,x)=x+\sum_{i=0}^d\int_0^t V_i(X(s,x))\circ dB^i(s). \tag{1.7}$$

次に，バミューダン・オプションに関する設定を行う．$T > 0$, とし， $g : [0,T] \times \mathbf{R}^N \to \mathbf{R}$ を $\sup\{(1+|x|)^{-1}|g(t,x)|;\ x \in \mathbf{R}^N, t \in [0,T]\} < \infty$ を満たす連続関数とする．そして，$n \geqq 1$, $0 = t_0 < t_1 < \ldots < t_n = T$, に対し $c_{t_k,t_{k+1},\ldots,t_n} : E \to \mathbf{R}$, と $\tilde{c}^{(L)}_{t_k,t_{k+1},\ldots,t_n} : E \times \Omega \to \mathbf{R}$, $k = n, n-1, \ldots, 0$, $L \geqq 1$, をそれぞれ帰納的に次のように定める．
$c_{t_n}(x) = \tilde{c}^{(L)}_{t_n}(x) = g(T,x)$, $x \in E$,

$$c_{t_k,t_{k+1},\ldots,t_n}(x) = \int_E p(t_{k+1}-t_k, x, y)(g(t_{k+1},y) \vee c_{t_{k+1},\ldots,t_n}(y))dy,$$

$$\tilde{c}^{(L)}_{t_k,t_{k+1},\ldots,t_n}(x) = Q^{(L)}_{t_k,t_{k+1}}(g(t_{k+1},\cdot) \vee \tilde{c}^{(L)}_{t_{k+1},\ldots,t_n}(\cdot))(x)$$

この時，次の定理が得られる．

Theorem 1.2 $n(L) \geqq 1$, $0 = t_0^{(L)} < t_1^{(L)} < \ldots < t_{n(L)}^{(L)} = T$ とする．また，

$$L^{-(1-\varepsilon)/2} \sum_{k=1}^{n(L)} (t_k^{(L)} - t_{k-1}^{(L)})^{-(N+1)\ell_0/4} \to 0,$$

となる $\varepsilon > 0$ が存在するとする．このとき，次の定理が得られる．

$$E[|\tilde{c}^{(L)}_{t_0^{(L)},t_1^{(L)},\ldots,t_{n(L)}^{(L)}}(x_0) - c_{t_0^{(L)},t_1^{(L)},\ldots,t_{n(L)}^{(L)}}(x_0)|^2] \to 0, \qquad L \to \infty.$$

Theorem 1.1 は $\hat{v}_0(x_0)$ が $v_0(x_0)$ に $L \to \infty$ のもとで確率収束することを述べているが，Theorem 1.2 は L^2 収束することを主張している．また，行使の間隔は L の大きさに伴い限りなく小さくできることを主張している．

次に，re-simulation に関する主張を述べる．$n \geqq 1$, とし $0 = T_0 < T_1 < \ldots < T_n = T$ を固定して考える．各 $\omega \in \Omega$ に対し stopping time $\hat{\tau}_{L,\omega} W_0 \to \{T_1, \ldots, T_n\}$ を次のように与える．

$$\hat{\tau}_{L,\omega} = \min\{T_k;\ k = 1,2,\ldots,n,\ \tilde{c}^L_{T_k,T_{k+1},\ldots,T_n}(X(T_k,x_0),\omega) \leqq g(T_k, X(T_k,x_0))\}.$$

そして $\hat{c} : \Omega \to \mathbf{R}$ を

$$\hat{c}(\omega) = E^\mu[g(\hat{\tau}_{L,\omega}, X(\hat{\tau}_{L,\omega}, x_0)]$$

とする．このとき以下が成り立つ．

Theorem 1.3 $\gamma \in (0,1]$ とし，次を仮定する．

$$\sum_{k=1}^n \mu(|c_{T_k,T_{k+1},\ldots,T_n}(X(T_k,x_0)) - g(T_k, X(T_k,x_0))| < \varepsilon) = O(\varepsilon^\gamma),\ \ as\ \varepsilon \downarrow 0.$$

このとき，任意の $\alpha \in (1/2, (1+\gamma)/(2+\gamma))$, に対しある $\Omega_L \in \mathcal{F}$, $L \geqq 1$, と $C > 0$ が存在し，$P(\Omega_L) \to 1$, $L \to \infty$, かつ任意の $\omega \in \Omega_L$ と $L \geqq 1$ に対し

$$|\hat{c}(\omega) - c_{T_0,T_1,\ldots,T_n}| \leqq CL^{-\alpha}$$

が成り立つ．

1.3.3 Least Square Regression 法

Least Square Regression 法では事前に選択した関数 $e_1,\cdots,e_K$ を用いて $P_{T-t}f$ を次のような形で近似する．

$$h=\sum_{k=1}^{K}a_k e_k, \tag{1.8}$$

ここで，$a_1,\ldots,a_K\in\mathbf{R}$ は

$$\{\frac{1}{L}\sum_{\ell=1}^{L}(h(X_\ell(t))-f(X_\ell(T)))^2\}^{1/2}$$

を最小にするように決める．

(1) Longstaff-Schwartz のアルゴリズム

Longstaff-Schwartz [13] による Least Square Regreesion 法を用いたバミューダン・オプション価格計算のアルゴリズムを示す．
Step 0 : $e_1,\cdots,e_K$ を選択する．
Step 1 : $\{X_\ell(t_1),\ldots,X_\ell(t_n)\}_{\ell=1}^{L}$ という原資産のパスを発生させる．
Step 2 : $\hat{c}_n(x)=g(t_n,x)$ とし，そして $\hat{c}_{i+1}$ まで決まったとする．この時

$$\hat{\tau}_{i+1,\ell}^{(K)}=\inf\{t\in\{t_{i+1},\ldots,t_n\};g(X_\ell(t))\geq\hat{c}(t,X_\ell(t))\},$$

とする．そして,$a_{i,k},k=1,\ldots,K$ を

$$\frac{1}{L}\sum_{\ell=1}^{L}\left|g(\hat{\tau}_{i+1,\ell},X_\ell(\hat{\tau}_{i+1,l}^{(K)}))-\sum_{k=1}^{K}\alpha_{i,k}e_k(X_\ell(t_i))\right|^2,$$

を最小化するように決める．これにより $\hat{c}_i$ が決まる．この操作を $i=1$ まで繰り返す．
Step 3 : モンテカルロ・シミュレーション用のパス $\{\tilde{X}_\ell(t_1),\ldots,\tilde{X}_\ell(t_n)\}_{\ell=1}^{L_0}$ を発生させ,re-simulation の方法で $\hat{v}_0(x_0)$ を計算する．

$$\hat{v}_0(x_0)=\left(\frac{1}{L_0}\sum_{\ell=1}^{L_0}g(\hat{\tau}_{1,l}^{(K)},\tilde{X}_\ell(\hat{\tau}_{1,\ell}^{(K)}))\right)\vee g(t_0,x_0), \tag{1.9}$$

(2) 収束に関する先行研究

Least Square Regression 法の収束に関する先行研究として Clement-Lamberton-Protter [4] による結果について述べる．$P_j^{(K)}$ を $L^2(\Omega,\mathcal{F},P)$ から $\{e_1(X(t_j)),\ldots,e_K(X(t_j))\}$ で張られるベクトル空間への直行射影とする．そして，$\tilde{\tau}_j^{(K)}$ を次のように定義する．

$$\begin{cases}\tilde{\tau}_n^{(K)}=t_n,\\ \tilde{\tau}_j^{(K)}=t_j 1_{\{g(t_j,X(t_j,x_0))\geqq P_j^{(K)}g(\tilde{\tau}_{j+1}^{(K)},X(\tilde{\tau}_{j+1}^{(K)},x_0))\}}+\tilde{\tau}_{j+1}^{(K)}1_{\{g(t_j,X(t_j,x_0))<P_j^{(K)}g(\tilde{\tau}_{j+1}^{(K)},X(\tilde{\tau}_{j+1}^{(K)},x_0))\}}\end{cases}$$

このとき，以下が成り立つ．

Theorem 1.4 $\{e_k\}_{k=1}^{\infty}$ を E 上の可測関数の列で, 各 $j=1,\ldots,n$ に対し，$\{e_k(X(t_j))\}_{k=1}^{\infty}$ は $L^2(\sigma(X(t_j)))$ の完全系になっているとする. このとき，

$$\lim_{K\to\infty} E[g(\tilde{\tau}_0^{(K)}, X(\tilde{\tau}_0^{(K)}))] = v_0, \qquad in\ L^2.$$

次に，(1) のアルゴリズムで，特にモンテカルロ·シミュレーション用のパス $\{\tilde{X}_\ell(t_1),\ldots,\tilde{X}_\ell(t_N)\}_{\ell=1}^{L_0}$ を回帰に用いたパス $\{X_\ell(t_1),\ldots,X_\ell(t_N)\}_{\ell=1}^{L}$ と同じものとする．このとき，次の定理が成り立つ．

Theorem 1.5 $P(\hat{c}_i(X(t_i,x)) = c_i(X(t_i,x))) = 0, i=1,\ldots,n.$ と仮定する．このとき，次が成り立つ．

$$\lim_{L\to\infty}\frac{1}{L}\sum_{\ell=1}^{L} g(\hat{\tau}_{i,\ell}^{(K)}, X(\hat{\tau}_{i,\ell}^{(K)}, x_0)) \vee g(0,x_0) = E[g(\tilde{\tau}_i^{(K)}, X(\tilde{\tau}_i^{(K)}, x_0)) \vee g(0,x_0)], \qquad a.s.$$

(3)　第 3 章の結果

ここでは 1.3.1 節 (1) の設定に加え，次の設定を行う．各 $i=1,2,\ldots,n$ に対し，$g_i = g(t_i,\cdot) \in L^4(E; d\nu_{t_i})$ とし，さらに v_i, c_i を (1.3) 式で定義されるものとする．また，簡単のため $P_{t_{i+1}-t_i} = P_i$, $\nu_{t_i} = \nu_i$ とし，さらに $X_\ell(t_i)$ を $X_i^{(\ell)}$ と表記する．

$\mathcal{V}$ を $m(E)$ の有限次元線形部分空間の集合とする．$(E,\mathcal{B})$ 上の任意の確率測度 ν に対し $\mathcal{V}(\nu)$ を $\mathcal{V}$ の部分集合で，任意の $V \in \mathcal{V}(\nu)$ に対し次の条件を満たすものとする.
(1) $g \in V$, に対し，$\int_E g(x)^4 \nu(dx) < \infty$.
(2) $g \in V$ と $g(x) = 0\ \nu - a.e.x$, ならば $g \equiv 0$.
そして $\lambda_0(V,\nu)$ and $\lambda_1(V,\nu)$ を次で定義する.

$$\lambda_0(V,\nu) = \sup\{\frac{\int_E g(x)^4\nu(dx)}{(\int_E g(x)^2\nu(dx))^2};\ g \in V \setminus \{0\}\}$$

$$\lambda_1(V;\nu) = \inf\{\int_E (\sum_{r=1}^{\dim V} e_r(x)^2)^2 \nu(dx);\ \{e_r\}_{r=1}^{\dim V} \text{ is an orthonormal basis of } V \text{ as a subspace of } L^2(E;d\nu)\ \}.$$

この時，次が成り立つ.

$$\lambda_1(V;\nu) \leqq (\dim V)^2 \lambda_0(V;\nu) \text{ and } \lambda_0(V;\nu) \leqq \lambda_1(V;\nu).$$

任意の $i=0,1,\ldots,n-1$ と $L \geqq 1$ に対し $D_i^{(L)} : m(E)\times m(E)\times\Omega \to [0,\infty)$ を次のように定義する.

$$D_i^{(L)}(g,f)(\omega) = (\frac{1}{L}\sum_{\ell=1}^{L}(g(X_i^{(\ell)}(\omega) - f(X_{i+1}^{(\ell)}(\omega))^2)^{1/2}, \qquad g,f \in m(E).$$

任意の $i=1,\ldots,n-1$ に対し，$V_i^{(k)}$, $k=1,2,\ldots$ を $\mathcal{V}(\nu_i)$ の真に増大するベクトル空間の列で，$\bigcup_{k=1}^{\infty} V_i^{(k)}$ が $L^2(E;d\nu_i)$ の中で稠密であるものとする．

そして $g_i^{(L)}:\Omega\to V_i^{(L)}$, $i=0,1,\ldots,n-1$, $L=1,2,\ldots$ が次を満たすと仮定する．

$$\begin{aligned} &D_{i-1}(g_{i-1}^{(L)}(\omega), g_i^{(L)}(\omega)\vee f_i)(\omega) \\ &=\inf\{D_{i-1}(h, g_i^{(L)}(\omega)\vee f_i);\ h\in V_i^{(L)}(\omega)\} \end{aligned} \tag{1.10}$$

for $i=1,2,\ldots,n$. ここで $g_n^{(L)}=g_n$ である．

また，$g_i^{(L)}, i=1,2,\ldots,n$ は常に存在することは示すことができる．

以上の設定のもとで，次の定理が成り立つ．

Theorem 1.6 $i=1,\ldots,n-1$ に対し，$\lambda_1(V_i^{(L)};\nu_i)/L\to 0,\ as\ L\to\infty$ とする．このとき，ある $\Omega_L\in\mathcal{F}$, $L=1,2,\ldots,$ と確率変数 Z_L, $L=1,2,\ldots,$ が存在し，以下を満たす．

$$\begin{aligned} &P(\Omega_L)\to 1,\ as\ L\to\infty, \\ &|v_0-g_0^{(L)}(\omega)|\leqq Z_L(\omega), \qquad L\geqq 1,\ \omega\in\Omega_L, \\ &E[Z_L^2,\Omega_L]^{1/2}\to 0,\ as\ L\to\infty. \end{aligned}$$

さらに，以下が成り立つ．

$$\begin{aligned} &E[Z_L^2,\Omega_L]^{1/2} \\ &\leqq 6\sum_{i=1}^{n-1}\frac{1}{L^{1/2}}\lambda_1(V_i^{(L)},\nu_i)^{1/4}(1+\lambda_0(V_i^{(L)},\nu_i))^{1/4}||c_i||_{L^4(E;d\nu_i)} \\ &\qquad +5\sum_{i=1}^{n-1}||c_i-\pi_{i,V_i^{(L)}}c_i||_{L^2(E;d\nu_i)}. \end{aligned}$$

ここで $\pi_{i,V_i^{(L)}}$ は $L^2(E,d\nu_i)$ から $V_i^{(L)}$ への直交射影である．

1.4 Credit Valuation Adjustment (CVA)

2007-2008 の金融危機により，カウンターパーティ・リスク管理手法に関する注目が急速に高まり，Credit Valuation Adjustment (CVA）と呼ばれる価格調整を勘案した取引価格の設定等が行われるようになった．カウンターパーティがデフォルトした際に被る損失をエクスポージャーと呼び，時刻 t におけるエクスポージャーはカウンターパーティと結んでいる契約全体のポートフォリオのデフォルトを考慮しない場合の価値 $\tilde{V}_0(t)$ の正の部分 $\tilde{V}_0(t)\vee 0$ と与えられる．そして，CVA はカウンターパーティーに対する将来に渡るエクスポージャーの期待値として次のように表される．

$$\mathrm{CVA}=E[L(\tau)D(0,\tau)1_{\{\tau<T\}}(\tilde{V}_0(\tau)\vee 0)] \tag{1.11}$$

ここで，L はデフォルト時損失率，D を割引率とする．(1.11) はカウンターパーティのデフォルトのみを考慮した CVA だが，CVA を勘案した価格で取引を行う場合は，自社のデフォルトも考慮しなければ，互いの提示する価格が一致せず，取引が成立しない．そこで，(1.11) を特に Unilateral CVA と呼び，自社のデフォルトによる調整項，すなわちカウンターパーティから見た Unilateral CVA を Debt Value Adjustment(DVA) と呼び，取引の際には次の Bilateral CVA を調整項として利用する．

$$\text{Bilateral CVA}=\text{Unilateral CVA}-\text{DVA}.$$

1.4.1 Duffie-Huang の結果

CVA の由来は Duffie-Huang [5] に始まるとされる．Duffie-Huang [5] は自社とカウンターパーティー双方のデフォルトリスクを加味した場合のデリバティブの取引価格が満たすべき方程式を導出し, その一次近似として現在の CVA の定義に相当するものが現れることを示した．以下にその概要を示す．
r を short rate process として，Q をそれに付随する Equivalent Martingale Measure とする．$i=1$ を自社，$i=2$ をカウンターパーティを表すインデックスとし，L_i をそれぞれがデフォルトした際の損失率，λ_i をそれぞれの hazard rate process とする．また，$s_i(t)=(1-L_i(t))\lambda_i(t)$ とし，R を次のように定義する．

$$R(v,t)=r(t)+s_1(t)1_{\{v<0\}}+s_2(t)1_{\{v\geqq 0\}}.$$

そして満期 T で $g(X(T))$ というペイオフを持つヨーロピアン・オプション，デフォルトを加味した価値過程 V はいくつかの仮定の下，次の方程式を満たす．

$$V(t)=E^Q\left[e^{-\int_t^s R(u,V(u))du}g(X(T))|\mathcal{F}_t\right],\quad t\leqq T. \tag{1.12}$$

(1.12) 式を解いて $V(t)$ を求めることは，次の Backward Stochastic Equation (BSDE) を解くことと同等である．

$$dV(t)=R(t,V(t))V(t)dt+Z(t)dB(t),\quad V(T)=g(X(T)). \tag{1.13}$$

BSDE の解を求めることは難しいため，$V(t)$ を求めるのではなく，デフォルトスプレッドに関する一次近似を考える．

(1.12) で与えられる V をデフォルトスプレッド $\eta=s_2-s_1$ の汎関数として見て $V(t;\eta)$ と表す．ここでそして η に関する Gateaux 微分 $\nabla V(t;\eta)$ を次を満たす確率過程として定義する．

$$\lim_{\varepsilon\to 0,}\sup_t\left|\nabla V(t;\eta)-\frac{V(t;\varepsilon\eta)-V(t;0)}{\varepsilon}\right|=0.$$

このとき，次の命題が成り立つ．

Proposition 1.7 任意の可予測過程 η に関して，*Gateaux* 微分 $\nabla V(t;\eta)$ が存在し，次で与えられる．

$$\nabla V(t;\eta)=-E^Q[\int_t^T e^{-\int_t^s R(V(u;0),u)du}(V(s;0)\vee 0)\eta(s)ds|\mathcal{F}_t]. \tag{1.14}$$

1.4.2 CVA の計算手法

ここでは CVA の数値計算手法について述べる．CVA の数値計算においては，次の 2 点の難しさがある．
(A)CVA はポートフォリオ単位で計算されるものであるため，参照する資産の数が多く，多次元の数値計算が必要であること
(B) エクスポージャーの定義が価値の正の部分であるという非線形性を持つこと
CVA の計算では (A) の理由により通常モンテカルロ法による計算が行われる．そして, Cesari et al [3] では $V(t)$ の計算に Least Square Regeression 法を用いることを提唱しており，実務的にもそのように計算されることが多い．

1.4.3　第 4 章の結果

本節では Stochastic Mesh 法を用いた CVA の計算法と，その収束性について議論する．本節では以下の設定を行う．
$X^{(m)}(t), m = 0, 1, \ldots, M,$ を $\mathbf{R}^{N_m}$-値の原資産価格過程とし，$X(t) = (X^{(0)}(t), \ldots, X^{(M)}(t))$ とする．$X^{(0)}(t)$ は金利などのマクロファクターを表し, 個々のデリバティブの満期 $T_k, k = 1, \ldots, K$ におけるペイオフは次のように与えられるとする．

$$\sum_{m=1}^{M} \tilde{F}_{m,k}(X^{(0)}(T_k), X^{(m)}(T_k)).$$

このとき，$\tilde{V}_0$ は次のように表される．

$$\tilde{V}_0(t) = E[\sum_{m=1}^{M} \sum_{k; T_k \geqq t} D(t, T_k) F_{m,k}(X^{(0)}(T_k), X^{(m)}(T_k)) | \mathcal{F}_t].$$

そして，CVA は次のように表される．

$$\begin{aligned} \mathrm{CVA} &= E[\int_0^T L(t) \exp(-\int_0^t \lambda(s) ds) \lambda(t) D(0, t)(\tilde{V}_0(t) \vee 0) dt] \\ &= E[\int_0^T L(t) \exp(-\int_0^t \lambda(s) ds) \lambda(t)(V_0(t) \vee 0) dt], \end{aligned} \tag{1.15}$$

ここで

$$V_0(t) = E[\sum_{m=1}^{M} \sum_{k; T_k \geqq t} F_{m,k}(X^{(0)}(T_k), X^{(m)}(T_k)) | \mathcal{F}_t],$$

かつ $F_{m,k}$ はディスカウントされたペイオフ

$$F_{m,k}(X^{(0)}(T_k), X^{(m)}(T_k)) = D(0, T_k) \tilde{F}_{m,k}(X^{(0)}(T_k), X^{(m)}(T_k))$$

とした．
　ここで，$L(t) \exp(-\int_0^t \lambda(s) ds) \lambda(t)$ を時刻 t と $X(t)$ の関数で表されると仮定し，$g(t, X(t))$ とおく．このとき， CVA は次のようになる．

$$\mathrm{CVA} = E[\int_0^T g(t, X(t))(V_0(t) \vee 0) dt]. \tag{1.16}$$

　$M \geqq 1$, $N_m \geqq 1, m = 1, \cdots, M$ とする．そして $N = N_0 + \cdots + N_M$ かつ $\tilde{N}_m = N_0 + N_m$, $\tilde{N} = \max_{m=1\ldots,M} \tilde{N}_m$ とする．
$W_0 = \{w \in C([0, \infty); \mathbf{R}^d);\ w(0) = 0\}$, $\mathcal{F}$ を W_0 上の Borel algebra とし，μ を $(W_0, \mathcal{F})$ の Wiener measure とする．$\{(B^1(t), \ldots, B^d(t); t \in [0, \infty)\}$ を d-次元 Brownian 運動とし，$B^0(t) = t$, $t \in [0, \infty)$ とする．
Let $V_i^{(0)} \in C_b^\infty(\mathbf{R}^{N_0}; \mathbf{R}^{N_0})$, $V_i^{(m)} \in C_b^\infty(\mathbf{R}^{N_0} \times \mathbf{R}^{N_m}; \mathbf{R}^{N_m}), i = 0, \cdots, d, m = 1, \cdots, M.$
そして次の Stratonovich 型の確率微分方程式を考える．

$$X^{(0)}(t, x_0) = x_0 + \sum_{i=0}^{d} \int_0^t V_i^{(0)}(X^{(0)}(s, x_0)) \circ dB_i(s), \tag{1.17}$$

$$X^{(m)}(t,\tilde{x}_m)=\tilde{x}_m+\sum_{i=1}^{d}\int_0^t V_i^{(m)}(X^{(0)}(s,x_0),X^{(m)}(s,\tilde{x}_m))\circ dB_i(s), \tag{1.18}$$

for $x_m\in\mathbf{R}^{N_m},\tilde{x}_m=(x_0,x_m)\in\mathbf{R}^{N_0}\times\mathbf{R}^{N_m},m=1,\ldots,M$.
さらに，$\tilde{X}^{(m)}(t,\tilde{x}_m)=(X^{(0)}(t,x_0),X^{(m)}(t,x_m))$ とし，$\tilde{V}_i^{(m)}\in C_b^\infty(\mathbf{R}^{N_0}\times\mathbf{R}^{N_k};\mathbf{R}^{N_0}\times\mathbf{R}^{N_k}),i=0,\cdots,d,\ m=1,\cdots,M$ を次のように定義する．

$$\tilde{V}_i^{(m)}(\tilde{x}_m)=\begin{pmatrix}V_i^{(0)}(x_0)\\ V_i^{(m)}(\tilde{x}_m)\end{pmatrix},$$

このとき，$\tilde{X}^{(m)}(t,\tilde{x}_m)$ は次の Stratonovich 型の確率微分方程式の解になる．

$$\tilde{X}^{(m)}(t,\tilde{x}_m)=\tilde{x}_m+\sum_{i=0}^{d}\int_0^t \tilde{V}_i^{(m)}(\tilde{X}^{(m)}(t,\tilde{x}_m))\circ dB_i(s). \tag{1.19}$$

また，$X(t,x),x\in\mathbf{R}^N$ も次の Stratonovich 型の確率微分方程式を満たす．

$$X(t,x)=x+\sum_{i=0}^{d}\int_0^t V_i(X(s,x))\circ dB^i(s), \tag{1.20}$$

ここで $V_i,i=1,\ldots,d$ は次のように与えられるものとした．

$$V_i(x)=\begin{pmatrix}V_i^{(0)}(x_0)\\ V_i^{(1)}(\tilde{x}_1)\\ \vdots\\ V_i^{(M)}(\tilde{x}_M)\end{pmatrix}.$$

ここで，各 m に対し，Vector fields $V^{(m)}{}_i,i=1,\ldots,d$ は (UFG) 条件をを満たすとする．そして E_m を Section 4.2 の (4.11) で定義されるものとする．このとき $\tilde{x}_m\in E_m$, であれば， $\tilde{X}^{(m)}(t,\tilde{x}_m)$ の μ に対する分布は滑らかな密度関数 $p^{(m)}(t,x_m,\cdot):\mathbf{R}^{\tilde{N}_m}\to[0,\infty)$ for $t>0,m=1,\ldots,M$ を持つことが示される．
$x^*=(x_0^*,\ldots,x_M^*)\in\mathbf{R}^N$. とし，(1.20) で与えられる $X(t,x^*)$ 原資産として考える．

$$\tilde{x}_m^*=(x_0^*,x_m^*)\in E_m,m=1,\ldots,M$$

とする．
$\hat{\mathcal{D}}(\mathbf{R}^n)$ を $\mathbf{R}^n$, 上の関数の空間で次で与えられるものとする．

$$\hat{\mathcal{D}}(\mathbf{R}^n)=\{f\in C^2(\mathbf{R}^n);\|\frac{\partial^\alpha f}{\partial x^\alpha}\|_\infty<\infty,\text{for } 1\leqq|\alpha|\leqq 2\},$$

ここで $\|f\|_\infty=\sup\{|f(x)|;x\in\mathbf{R}^n\}$ である．
$Lip(\mathbf{R}^n)$ を $\mathbf{R}^n$, 上の Lipshitz 連続な関数の空間とし，

$$\|f\|_{Lip}=\sup_{x,y\in\mathbf{R}^n,x\neq y}\frac{|f(x)-f(y)|}{|x-y|},\quad f\in Lip(\mathbf{R}^n)$$

とする．Let $\mathcal{M}(\mathbf{R}^n)$ を $Lip(\mathbf{R}^n)$ の部分空間で $\{f\vee g;f,g\in\mathcal{D}(\mathbf{R}^n)\}$ で張られるものとする．

$P_t : Lip(\mathbf{R}^N) \to Lip(\mathbf{R}^N)$ を

$$(P_t f)(x) = E^{\mu}[f(X(t,x))], f \in Lip(\mathbf{R}^N),$$

とし， $P_t^{(m)} : Lip(\mathbf{R}^{\tilde{N}_m}) \to Lip(\mathbf{R}^{\tilde{N}_m}), m = 1, \ldots, M,$ を

$$(P_t^{(m)} f)(\tilde{x}_m) = E^{\mu}[f(\tilde{X}^{(m)}(t, \tilde{x}_m))], f \in Lip(\mathbf{R}^{\tilde{N}_m}).$$

とする．

$g : [0,T] \times \mathbf{R}^N \to [0,\infty)$, を次の条件をみたすものとする．
(1) $g(t,x)$ は t に関して微分可能で，ある n_1, と $C_1 > 0$ が存在し，次をみたす．

$$\sup_{t \in [0,T]} |\frac{\partial}{\partial t} g(t,x)| \leqq C_1(1 + |x|^{n_1}), \quad x \in \mathbf{R}^N.$$

(2) $g(t,x)$ は x に関して 2 階微分可能で, ある n_2, と $C_2 > 0$, が存在し，任意の multi index $|\alpha| \leqq 2$ に対し次を満たす．

$$\sup_{t \in [0,T]} |\frac{\partial^{\alpha}}{\partial x^{\alpha}} g(t,x)| \leqq C_2(1 + |x|^{n_2}), \quad x \in \mathbf{R}^N.$$

以上の設定のもと，c_0 を次のように定義する．

$$c_0 = E^{\mu}[\int_0^T \{g(t, X(t,x^*)) E^{\mu}[\sum_{m=1}^{M} \sum_{k:T_k \geqq t} F_{m,k}(\tilde{X}^{(m)}(T_k, \tilde{x}_m^*)) | \mathcal{F}_t] \vee 0\} dt]. \qquad (1.21)$$

続いて c_0 の近似計算に関する設定について述べる．
Let $(\Omega, \mathcal{F}, P)$ を確率空間とし， $X_\ell : [0,\infty) \times \Omega \to \mathbf{R}^N$, $\ell = 1, 2, \ldots,$ を連続な確率過程で，全ての $\ell = 1, 2, \ldots$ に対し，$X_\ell(\cdot)$ の $C([0,\infty); \mathbf{R}^N)$ 上の P に関する分布は $X(\cdot, \tilde{x}_0)$ と同じであり，$\sigma\{X_\ell(t);\ t \geqq 0\}$, $\ell = 1, 2, \ldots,$ が独立であるものとする．
$\pi_m : \mathbf{R}^N \to \mathbf{R}^{\tilde{N}_m}, m = 1, \ldots, M,$ を $\pi_m(x) = \tilde{x}_m = (x_0, x_m)$, とし， $\varepsilon_0 > 0$ を $\varepsilon_0 = \min_{1 \leqq k \leqq K}(T_k - T_{k-1})$ とする．そして，$f \in Lip(\mathbf{R}^{\tilde{N}_m})$ に対して Stochastic mesh operator $Q_{t,T_k,\varepsilon}^{(m)} = Q_{t,T_k,\varepsilon}^{(m,L,\omega)}, 0 \leqq t \leqq T, 0 < \varepsilon < \varepsilon_0,$ を次のように定義する．

$$(Q_{t,T_k,\varepsilon}^{(m,L,\omega)} f)(\tilde{x}_m) = \begin{cases} \frac{1}{L} \sum_{\ell=1}^{L} \frac{f(X_\ell^{(m)}(T_k)) p^{(m)}(T_k - t, \tilde{x}_m, \pi_m(X_\ell(T_k)))}{q_{t,T_k}^{(m,L,\omega)}(\pi_m(X_\ell(T_k)))}, & 0 \leqq t < T_k - \varepsilon, \\ f(\tilde{x}_m), \quad T_k - \varepsilon \leqq t \leqq T_k, \\ 0, \quad T_k < t \leqq T. \end{cases}$$

$$\text{where} \quad q_{t,T_k}^{(m,L,\omega)}(\tilde{y}_m) = \frac{1}{L} \sum_{\ell=1}^{L} p^{(m)}(T_k - t, \pi_m(X_\ell(t)), \tilde{y}_m)).$$

Π を分割 $\Delta = \{t_0, t_1, \ldots, t_n\}$ で $0 = t_0 < t_1 < \ldots < t_n = T$ かつ $\{T_k; k = 1, \ldots, K\} \subset \Delta$ となるもの全体の集合とする．また，$|\Delta| = \max_{i=1,\ldots,n}(t_{i+1} - t_i)$ とする．

そして，c_0 の近似 $\hat{c}_i = \hat{c}_i(\varepsilon, \Delta, L) : \Omega \to \mathbf{R}, i = 1, 2$ を次のように定義する．

$$
\begin{aligned}
&\hat{c}_1(\varepsilon, \Delta, L)(\omega) \\
=&\frac{1}{L}\sum_{\ell=1}^{L}\sum_{i=0}^{n-1}(t_{i+1}-t_i)g(t_i, X_\ell(t_i))(\sum_{m=1}^{M}\sum_{k:T_k \geqq t_{i+1}}(Q_{t_i,T_k,\varepsilon}^{(m,L,\omega)}F_{m,k})(\pi_k(X_\ell(t_i)))\vee 0), \qquad (1.22)\\
&\hat{c}_2(\varepsilon, \Delta, L)(\omega) \\
=&\sum_{i=0}^{n-1}(t_{i+1}-t_i)E^{\mu}[g(t_i, X(t_i, x^*))(\sum_{m=1}^{M}\sum_{k:T_k \geqq t_{i+1}}F_{m,k}(\pi_k X(T_k, x^*))) \\
&\qquad \times 1_{\{\sum_{m=1}^{M}\sum_{k:T_k \geqq t_{i+1}}(Q_{t_i,T_k,\varepsilon}^{(m,L,\omega)}F_{m,k})(\pi_k X(t_i,x^*)))\geqq 0\}}]. \qquad (1.23)
\end{aligned}
$$

$\hat{c}_1$ はエクスポージャーの計算を Stochastic Mesh 作用素で置き換えたものであり，さらに Stochastic Mesh 作用素を計算するパスと Stochastic Mesh 作用素を使って再度期待値計算を行うパスは同じものを使っているところに特徴がある．一方，$\hat{c}_2$ では Stochastic 作用素は，値そのものが使われるのではなく，将来のポートフォリオ価値の正負の判定のみに用いられる．そのため， Stochastic Mesh 作用素の誤差がどんなに大きくても，正負の判定さえ間違っていなければ，この近似作用素による誤差は生じないため精緻な近似になることが期待される．この点はバミューダン・オプションにおける re-simulation の方法と同様の考え方である．

$\hat{c}_1$ の評価に関して，次の定理が成り立つ．

Theorem 1.8 $\alpha_0 = (1+\delta)(\tilde{N}+1)\ell_0/4 \vee 1$ とする．$\{\varepsilon_L\}_{L=1}^{\infty} \subset (0, \varepsilon_0)$ を $\varepsilon_L \leqq C_0 L^{-\frac{1+\delta}{2(1+\alpha_0)}}$ となる $C_0 > 0$ が存在するような列とする．このとき全ての $L \geqq 1$ と $\Delta \in \Pi$. に対し，次を満たす $C_1 \in (0, \infty)$ が存在する．

$$E^P[|\hat{c}_1(\varepsilon_L, \Delta, L) - c_0|] \leqq C_1(L^{-\frac{1}{1+\alpha_0}} + |\Delta|)$$

$\hat{c}_2$ の評価に関して，次の定理が成り立つ．

Theorem 1.9 $\alpha_1 = (1+\delta)(\tilde{N}+1)\ell_0/2 \vee 1,$ とする．$\{\varepsilon_L\}_{L=1}^{\infty} \subset (0, \varepsilon_0)$ を $\varepsilon_L \leqq C_0 L^{-\frac{1+\delta}{2\alpha_1+1}}$ を満たす $C_0 > 0$ が存在するような列とする．さらに，$\gamma \in (0, 1]$ と $C_\gamma > 0$ が存在し，すべての $\theta \in (0, 1]$ に対し，次を満たすとする．

$$\sup_{\Delta}\sum_{i=0}^{n-1}(t_{i+1}-t_i))\mu(|\sum_{m=1}^{M}\sum_{k:T_k \geqq t_{i+1}}(P_{T_k-t_i}^{(m)}F_{m,k})(\pi_m X(t_i, x^*)))| \leqq \theta) \leqq C_\gamma \theta^\gamma.$$

このとき $C_1 \in (0, \infty)$ と $\tilde{\Omega}(L) \in \mathcal{F}$, $L \geqq 1$, が存在し，全ての $L \geqq 1$, と $\Delta \in \Pi$ に対し，次が成り立つ．

$$P(\tilde{\Omega}(L)) \to 1, \ L \to \infty,$$

$$1_{\tilde{\Omega}(L)}|\hat{c}_2(\varepsilon_L, \Delta, L) - c_0| \leqq C_1(L^{-(\frac{1}{2}+\frac{(1-\delta)}{2\alpha_1+1})\frac{1+\gamma}{2+\gamma}} + |\Delta|)$$

Remark 1.10 $\tilde{\Omega}'(L)$ を次のように与える．

$$\tilde{\Omega}'(L) = \left\{\omega \in \Omega; |\hat{c}_1(\varepsilon_L, \Delta, L) - c_0| \leqq C L^{-\frac{1-\delta}{1+\alpha_0}}\right\}.$$

このとき *Theorem 1.8* により次を得る.

$$P(\tilde{\Omega}'(L)) \to 1, L \to \infty,$$

$$1_{\tilde{\Omega}(L)}|\hat{c}_1(\varepsilon_L, \Delta, L) - c_0| \leqq CL^{-\frac{1-\delta}{1+\alpha_0}}.$$

よって，*Theorem 1.9* で得られる $\hat{c}_2$ の評価は $\hat{c}_1$ の評価より良くなっているといえる.

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第2章 Stochastic mesh methods for Hörmander type diffusion processes

In the present paper the authors discuss the efficiency of stochastic mesh methods introduced by Broadie and Glasserman [5]. The authors apply stochastic mesh methods to certain type of Hörmander type diffusion processes and show the following. (1) If one carefully takes partitions, the estimated price of American option converges to the real price with probability one. (2) One can obtain better estimates by re-simulation methods discussed in Belomestny [4], although the order is not so sharp as his result.

2.1 Introduction

Stochastic mesh methods were introduced by Broadie and Glasserman [5], and Avramidis and Hyden [1] and Avramidis and Matzinger[2] proved the efficiency of them in some cases (see [11] also). Also, Belomestny [4] showed in Bermuda options that once we have estimated functions for the so-called continuation values, we have a better estimated value if we construct a pre-optimal stopping time by using these estimated functions and estimate the expectation of pay-off functionals based on this stopping time by re-simulation.

In the present paper, we consider the efficiency of stochastic mesh methods and re-simulation in the case that we apply them to Hörmander type diffusion processes.

Let $N, d \geqq 1$. Let $W_0 = \{w \in C([0,\infty); \mathbf{R}^d);\ w(0) = 0\}$, $\mathcal{F}$ be the Borel algebra over W_0 and μ be the Wiener measure on $(W_0, \mathcal{F})$. Let $B^i : [0,\infty) \times W_0 \to \mathbf{R}$, $i = 1, \ldots, d$, be given by $B^i(t,w) = w^i(t)$, $(t,w) \in [0,\infty) \times W_0$. Then $\{(B^1(t), \ldots, B^d(t); t \in [0,\infty)\}$ is a d-dimensional Brownian motion. Let $B^0(t) = t$, $t \in [0,\infty)$. Let $V_0, V_1, \ldots, V_d \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$. Here $C_b^\infty(\mathbf{R}^N; \mathbf{R}^n)$ denotes the space of $\mathbf{R}^n$-valued smooth functions defined in $\mathbf{R}^N$ whose derivatives of any order are bounded. We regard elements in $C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$ as vector fields on $\mathbf{R}^N$.

Now let $X(t,x)$, $t \in [0,\infty)$, $x \in \mathbf{R}^N$, be the solution to the Stratonovich stochastic integral equation

$$X(t,x) = x + \sum_{i=0}^{d} \int_0^t V_i(X(s,x)) \circ dB^i(s). \tag{2.1}$$

Then there is a unique solution to this equation. Moreover we may assume that $X(t,x)$ is continuous in t and smooth in x and $X(t,\cdot) : \mathbf{R}^N \to \mathbf{R}^N$, $t \in [0,\infty)$, is a diffeomorphism with probability one.

Let $\mathcal{A} = \{\emptyset\} \cup \bigcup_{k=1}^{\infty} \{0, 1, \ldots, d\}^k$ and for $\alpha \in \mathcal{A}$, let $|\alpha| = 0$ if $\alpha = \emptyset$, let $|\alpha| = k$ if $\alpha = (\alpha^1, \ldots, \alpha^k) \in \{0, 1, \ldots, d\}^k$, and let $\| \alpha \| = |\alpha| + \text{card}\{1 \leqq i \leqq |\alpha|;\ \alpha^i = 0\}$. Let $\mathcal{A}^*$ and $\mathcal{A}^{**}$ denote $\mathcal{A} \setminus \{\emptyset\}$ and $\mathcal{A} \setminus \{\emptyset, 0\}$, respectively. Also, for each $m \geqq 1$, $\mathcal{A}^{**}_{\leqq m}$, $\{\alpha \in \mathcal{A}^{**};\ \| \alpha \| \leqq m\}$.

We define vector fields $V_{[\alpha]}$, $\alpha \in \mathcal{A}$, inductively by

$$V_{[\emptyset]} = 0, \qquad V_{[i]} = V_i, \quad i = 0, 1, \ldots, d,$$

$$V_{[\alpha * i]} = [V_{[\alpha]}, V_i], \qquad i = 0, 1, \ldots, d.$$

Here $\alpha * i = (\alpha^1, \ldots, \alpha^k, i)$ for $\alpha = (\alpha^1, \ldots, \alpha^k)$ and $i = 0, 1, \ldots, d$.

We say that a system $\{V_i; i = 0, 1, \ldots, d\}$ of vector fields satisfies the following condition (UFG).

(UFG) There are an integer ℓ_0 and $\varphi_{\alpha,\beta} \in C_b^{\infty}(\mathbf{R}^N)$, $\alpha \in \mathcal{A}^{**}$, $\beta \in \mathcal{A}^{**}_{\leqq \ell_0}$, satisfying the following.

$$V_{[\alpha]} = \sum_{\beta \in \mathcal{A}^{**}_{\leqq \ell_0}} \varphi_{\alpha,\beta} V_{[\beta]}, \qquad \alpha \in \mathcal{A}^{**}.$$

Let $A(x) = (A^{ij}(x))_{i,j=1,\ldots,N}$, $t > 0$ $x \in \mathbf{R}^N$ be a $N \times N$ symmetric matrix given by

$$A^{ij}(x) = \sum_{\alpha \in \mathcal{A}^{**}_{\leqq \ell_0}} V^i_{[\alpha]}(x) V^j_{[\alpha]}(x), \qquad i, j = 1, \ldots, N.$$

Let $h(x) = \det A(x), x \in \mathbf{R}^N$ and $E = \{x \in \mathbf{R}^N;\ h(x) > 0\}$. By Kusuoka-Stroock [15], we see that if $x \in E$, the distribution law of $X(t, x)$ under μ has a smooth density function $p(t, x, \cdot) : \mathbf{R}^N \to [0, \infty)$ for $t > 0$. Moreover, we will show in that $\int_E p(t, x, y) dy = 1$, $x \in E$.

Now let $x_0 \in E$ and fix it throughout this paper. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $X_\ell : [0, \infty) \times \Omega \to \mathbf{R}^N$, $\ell = 1, 2, \ldots$, be continuous stochastic processes such that the probability laws on $C([0, \infty); \mathbf{R}^N)$ of $X_\ell(\cdot)$ under P and of $X(\cdot, x_0)$ under μ are the same for all $\ell = 1, 2, \ldots$, and that $\sigma\{X_\ell(t);\ t \geqq 0\}$, $\ell = 1, 2, \ldots$, are independent.

Let $q^{(L)}_{s,t} : E \times \Omega \to [0, \infty)$, $t > s \geqq 0$, $L \geqq 1$, be given by

$$q^{(L)}_{s,t}(y, \omega) = \frac{1}{L} \sum_{\ell=1}^{L} p(t - s, X_\ell(s, \omega), y), \qquad y \in E,\ \omega \in \Omega,$$

Let $m(E)$ denote the space of measurable functions on E.

We define a random linear operator $Q^{(L)}_{s,t}$, $t > s \geqq 0$, $L \geqq 1$, defined in $m(E)$ by

$$(Q^{(L)}_{s,t} f)(x) = \frac{1}{L} \sum_{\ell=1}^{L} \frac{p(t - s, x, X_\ell(t)) f(X_\ell(t))}{q^{(L)}_{s,t}(X_\ell(t))}, \qquad x \in E,\ f \in m(E).$$

Now let $T > 0$, and $g : [0, T] \times \mathbf{R}^N \to \mathbf{R}$ be a continuous function with $\sup\{(1 + |x|)^{-1} |g(t, x)|;\ x \in \mathbf{R}^N, t \in [0, T]\} < \infty$. For any $n \geqq 1$, and $0 = t_0 < t_1 < \ldots < t_n = T$, we define $c_{t_k, t_{k+1}, \ldots, t_n} : E \to \mathbf{R}$, and $\tilde{c}^{(L)}_{t_k, t_{k+1}, \ldots, t_n} : E \times \Omega \to \mathbf{R}$, $k = n, n-1, \ldots, 0$, $L \geqq 1$, inductively by $c_{t_n}(x) = \tilde{c}^{(L)}_{t_n}(x) = g(T, x)$, $x \in E$, and

$$c_{t_k, t_{k+1}, \ldots, t_n}(x) = \int_E p(t_{k+1} - t_k, x, y)(g(t_{k+1}, y) \vee c_{t_{k+1}, \ldots, t_n}(y)) dy,$$

and

$$\tilde{c}^{(L)}_{t_k,t_{k+1},\ldots,t_n}(x) = Q^{(L)}_{t_k,t_{k+1}}(g(t_{k+1},\cdot) \vee \tilde{c}^{(L)}_{t_{k+1},\ldots,t_n}(\cdot))(x)$$

for $x \in E$ and $k = n-1,\ldots,0$.

Then we will show the following.

Theorem 2.1 *Suppose that* $n(L) \geqq 1$, $0 = t^{(L)}_0 < t^{(L)}_1 < \ldots < t^{(L)}_{n(L)} = T$. *If there is an* $\varepsilon > 0$ *such that*

$$L^{-(1-\varepsilon)/2} \sum_{k=1}^{n(L)} (t^{(L)}_k - t^{(L)}_{k-1})^{-(N+1)\ell_0/4} \to 0,$$

then

$$E[|\tilde{c}^{(L)}_{t^{(L)}_0,t^{(L)}_1,\ldots,t^{(L)}_{n(L)}}(x_0) - c_{t^{(L)}_0,t^{(L)}_1,\ldots,t^{(L)}_{n(L)}}(x_0)|^2] \to 0, \qquad L \to \infty.$$

Let $n \geqq 1$, and $0 = T_0 < T_1 < \ldots < T_n = T$ and fix them. For each $\omega \in \Omega$, let $\hat{\tau}_{L,\omega} W_0 \to \{T_1,\ldots,T_n\}$ be a stopping time given by

$$\hat{\tau}_{L,\omega} = \min\{T_k;\ k = 1,2,\ldots,n,\ \tilde{c}^L_{T_k,T_{k+1},\ldots,T_n}(X(T_k,x_0),\omega) \leqq g(T_k,X(T_k,x_0))\}.$$

Let $\hat{c} : \Omega \to \mathbf{R}$ be given by

$$\hat{c}(\omega) = E^\mu[g(\hat{\tau}_{L,\omega}, X(\hat{\tau}_{L,\omega},x_0)].$$

Then we have the following.

Theorem 2.2 *Suppose that* $\gamma \in (0,1]$. *If*

$$\sum_{k=1}^{n} \mu(|c_{T_k,T_{k+1},\ldots,T_n}(X(T_k,x_0)) - g(T_k,X(T_k,x_0))| < \varepsilon) = O(\varepsilon^\gamma), \ \text{as } \varepsilon \downarrow 0,$$

then for any $\alpha \in (1/2,(1+\gamma)/(2+\gamma))$, *there are* $\Omega_L \in \mathcal{F}$, $L \geqq 1$, *and* $C > 0$ *such that* $P(\Omega_L) \to 1$, $L \to \infty$, *and*

$$|\hat{c}(\omega) - c_{T_0,T_1,\ldots,T_n}| \leqq CL^{-\alpha} \ \text{for any } \omega \in \Omega_L \text{ and } L \geqq 1.$$

2.2 The basic property of Hörmander diffusion processes

Let $J : [0,\infty) \times \mathbf{R}^N \times W_0 \to \mathbf{R}^N \otimes \mathbf{R}^N$, $J(t,x) = (J^i_j(t,x))_{i,j=1,\ldots,N}$ be given by

$$J^i_j(t,x) = \frac{\partial}{\partial x^j} X^i(t,x)$$

Then it has been shown in [13] Section 2 that there are $b^\beta_\alpha : [0,\infty) \times \mathbf{R}^N \times W_0 \to \mathbf{R}$, $\alpha,\beta \in \mathcal{A}^{**}_{\leqq \ell_0}$, such that

$$V_{[\alpha]}(x) = \sum_{\beta \in \mathcal{A}^{**}_{\leqq \ell_0}} b^\beta_\alpha(t,x) J(t,x)^{-1} V_{[\beta]}(X(t,x)), \qquad \alpha \in \mathcal{A}^{**}_{\leqq \ell_0},$$

and

$$\sup_{x \in \mathbf{R}^N, t \in [0,T]} E^\mu[|b^\beta_\alpha(t,x)|^p] < \infty \qquad \alpha,\beta \in \mathcal{A}^{**}_{\leqq \ell_0},\ T > 0,\ p \geqq 1.$$

So we see that for any $\xi \in \mathbf{R}^N$,

$$(A(x)\xi, \xi) = \sum_{\alpha \in \mathcal{A}^{**}_{\leqq \ell_0}} (V_{[\alpha]}(x), \xi)^2$$

$$\leqq \sum_{\alpha \in \mathcal{A}^{**}_{\leqq \ell_0}} (\sum_{\beta \in \mathcal{A}^{**}_{\leqq \ell_0}} b^{\beta}_{\alpha}(t,x)^2)(\sum_{\beta \in \mathcal{A}^{**}_{\leqq \ell_0}} (J(t,x)^{-1} V_{[\beta]}(X(t,x)), \xi)^2)$$

$$= (\sum_{\alpha \in \mathcal{A}^{**}_{\leqq \ell_0}} \sum_{\beta \in \mathcal{A}^{**}_{\leqq \ell_0}} b^{\beta}_{\alpha}(t,x)^2)(J(t,x)A(X(t,x))^t J(t,x)\xi, \xi)$$

Therefore we see that

$$h(x) \leqq (\sum_{\alpha \in \mathcal{A}^{**}_{\leqq \ell_0}} \sum_{\beta \in \mathcal{A}^{**}_{\leqq \ell_0}} b^{\beta}_{\alpha}(t,x)^2)^N det(J(t,x))^2 h(X(t,x)). \tag{2.2}$$

Then we have the following.

Proposition 2.3 (1) *$\mu(X(t,x) \in E) = 1$ for any $x \in E$ and $t > 0$. In particular, $p(t,x,y) = 0$, $y \in \mathbf{R}^N \setminus E$, $x \in E$.*
(2) *For any $p > 1$ and $T > 0$, there exists a $C > 0$ such that*

$$E^{\mu}[h(X(t,x))^{-p}] \leqq Ch(x)^{-p}, \qquad x \in E, \ t \in [0,T].$$

(3) *For any $n, m \geqq 0$, $p \in (1,\infty)$, and $T > 0$, there exists a $C > 0$ such that*

$$||h(X(t,x))^{-m}||_{W^{n,p}} \leqq Ch(x)^{-(n+m)} \qquad x \in E, \ t \in [0,T].$$

Here $||\cdot||_{W^{n,p}}$ is the norm of a Soblev space $W^{n,p}$ (c.f. Shigekawa [19]).

Proof. The assertions (1) and (2) are easy consequence of Equation (2.2). Note that

$$D(h^{-m}(X(t,x))) = -mh^{-(m+1)}(X(t,x)))D(h(X(t,x))).$$

Thus we easily obatain the assertion (3) by induction. ∎

By Kusuoka-Stroock [15], we have the following.

Proposition 2.4 *Let $\delta_0 > 0$ be given by*

$$\delta_0 = (3N(\sup_{x \in \mathbf{R}^N} \sum_{k=1}^{d} |V_k(x)|^2))^{-1}$$

Then we have the following.
(1) *For any $T > 0$,*

$$\sup_{t \in (0,T],\, x \in \mathbf{R}^N} E^{\mu}[\exp(\frac{2\delta_0}{t}|X(t,x) - x|^2)] < \infty.$$

(2) *For any $T > 0$, $n \geqq 1$, and $p \in (1,\infty)$,*

$$\sup_{t \in (0,T],\, x \in \mathbf{R}^N} t^{n/2} || \exp(\frac{\delta_0}{t}|X(t,x) - x|^2)||_{W^{n,p}} < \infty.$$

Proposition 2.5 *For any* $\gamma \in \mathbf{Z}^N_{\geqq 0}$, *there are* $g_{\gamma,\alpha_1,\dots,\alpha_k} \in C_b^\infty(\mathbf{R}^N)$, $k = 1,\dots,|\gamma|$, $\alpha_i \in \mathcal{A}^{**}_{\leqq \ell_0}$, $i = 1,\dots,k$, *such that*

$$h(x)^{|\gamma|}\frac{\partial^{|\gamma|}}{\partial x^\gamma}f(x) = \sum_{k=1}^{|\gamma|}\sum_{\alpha_1,\dots,\alpha_k\in\mathcal{A}^{**}_{\leqq \ell_0}} g_{\gamma,\alpha_1,\dots,\alpha_k}(x)(V_{[\alpha_1]}\cdots V_{[\alpha_k]}f)(x), \qquad x \in \mathbf{R}^N$$

for any $f \in C_b^\infty(\mathbf{R}^N)$,

Proof. Let $\tilde{A}(x) = (\tilde{A}_{ij}(x))_{i,j=1,\dots,N}$ be the cofactor matrix of the matrix $A(x)$ for $x \in \mathbf{R}^N$. Also, let $c_{\alpha,i}(x)$, $x \in \mathbf{R}^N$, $\alpha \in \mathcal{A}^{**}_{\leqq \ell_0}$, $i = 1,\dots,N$, be given by

$$c_{\alpha,i}(x) = \sum_{j=1}^N \tilde{A}_{ij}(x)V^j_{[\alpha]}(x).$$

Then we see that $h, c_{\alpha,i} \in C_b^\infty(\mathbf{R}^N)$, and

$$\sum_{\alpha\in\mathcal{A}^{**}_{\leqq \ell_0}} c_{\alpha,i}(x)(V_{[\alpha]}f)(x) = h(x)\frac{\partial f}{\partial x^i}(x), \qquad i = 1,\dots,N.$$

So we have the assertion for the case that $|\gamma| = 1$. Since

$$h(x)^{|\gamma|+1}(x)\frac{\partial}{\partial x^i}\frac{\partial^{|\gamma|}}{\partial x^\gamma}f(x)$$

$$= h(x)\frac{\partial}{\partial x^i}(h^{|\gamma|}\frac{\partial^{|\gamma|}}{\partial x^\gamma}f)(x) - |\gamma|\frac{\partial h}{\partial x^i}(x)h^{|\gamma|}(x)\frac{\partial^{|\gamma|}}{\partial x^\gamma}f(x),$$

we have our assertion by induction. ∎

Now we have the following lemma.

Lemma 2.6 *For any* $t > 0$, $x \in E$ *and* $\gamma_0, \gamma_1 \in \mathbf{Z}^N_{\geqq 0}$, *there are* $k_{\gamma_0,\gamma_1}(t,x) \in W^{\infty,\infty-}$ *such that*

$$\int_{\mathbf{R}^N}\partial_x^{\gamma_0}\partial_y^{\gamma_1}p(t,x,y)f(y)dy = E^\mu[h(X(t,x))^{-2(|\gamma_0|+|\gamma_1|)\ell_0}f(X(t,x))k_{\gamma_0,\gamma_1}(t,x)], \quad f \in C_0^\infty(\mathbf{R}^N),$$

and

$$\sup_{t\in(0,T],x\in E} t^{(|\gamma_0|+|\gamma_1|)\ell_0/2}||k_{\gamma_0,\gamma_1}(t,x)||_{W^{n,p}} < \infty, \qquad T > 0,\ n \in \mathbf{N},\ p \in (1,\infty).$$

Here $\partial_x^\gamma = \partial^{|\gamma|}/\partial x^\gamma$ *and* $\partial_y^\gamma = \partial^{|\gamma|}/\partial y^\gamma$.

Proof. First, by the argument in Shigekawa [19] we see that for $\gamma \in \mathbf{Z}^N_{\geqq 0}$, there are $J_{\gamma,\beta}(t,x) \in W^{\infty,\infty-}$, $t \geqq 0$, $x \in \mathbf{R}^N$, $\beta \in \mathbf{Z}^N_{\geqq 0}$, $|\beta| \leqq |\gamma|$, such that

$$\partial_x^\gamma(f(X(t,x)) = \sum_{\beta\in\mathbf{Z}^N_{\geqq 0},\ |\beta|\leqq|\gamma|}(\partial_x^\beta f)(X(t,x))J_{\gamma,\beta}(t,x),$$

and

$$\sup_{t\in(0,T],x\in\mathbf{R}^N} ||J_{\gamma,\beta}(t,x)||_{W^{n,p}} < \infty, \qquad T > 0,\ n \in \mathbf{N},\ p \in (1,\infty).$$

Then we have for any $x \in E$ and $f \in C_0^\infty(\mathbf{R}^N)$,

$$\int_{\mathbf{R}^N} \partial_x^{\gamma_0}\partial_y^{\gamma_1} p(t,x,y) f(y) dy$$

$$= (-1)^{|\gamma_1|} \int_{\mathbf{R}^N} \partial_x^{\gamma_0} p(t,x,y)(\partial_y^{\gamma_1} f)(y) dy$$

$$= (-1)^{|\gamma_1|} \partial_x^{\gamma_0} E^\mu[(\partial_y^{\gamma_1} f)(X(t,x))]$$

$$= (-1)^{|\gamma_1|} \sum_{\beta\in\mathbf{Z}^N_{\geqq 0},\ |\beta|\leqq|\gamma_0|} E^\mu[(\partial_x^{\gamma_1+\beta} f)(X(t,x)) J_{\gamma_0,\beta}(t,x)]$$

$$= (-1)^{|\gamma_1|} \sum_{\beta\in\mathbf{Z}^N_{\geqq 0},\ |\beta|\leqq|\gamma_0|} \sum_{k=0}^{|\gamma_1|+|\beta|} \sum_{\alpha_1,\ldots,\alpha_k\in\mathcal{A}^{**}_{\leqq \ell_0}} E^\mu[h(X(t,x))^{-(|\gamma_1|+|\beta|)} g_{\gamma_1+\beta,\alpha_1,\ldots,\alpha_k}(X(t,x))$$
$$\times J_{\gamma_0,\beta}(t,x)(V_{[\alpha_1]}\cdots V_{[\alpha_k]}f)(X(t,x))].$$

So by the integration parts formula in [13] Lemma 8 and by Propositions 2.3 and 3.3, we have our assertion. ∎

Proposition 2.7 *For any $t > 0$, $x \in E$ and $\gamma_0,\gamma_1 \in \mathbf{Z}^N_{\geqq 0}$,*

$$\partial_x^{\gamma_0}\partial_y^{\gamma_1} p(t,x,y) = 0 \ \ a.e. y \in \mathbf{R}^N \setminus E.$$

Moreover, for any $\gamma_0,\gamma_1 \in \mathbf{Z}^N_{\geqq 0}$, $p \in (1,\infty)$, $T > 0$, and $m \in \mathbf{Z}$ with $m \leqq 2(|\gamma_0| + |\gamma_1|)$,

$$\sup\{t^{(|\gamma_0|+\gamma_1|)\ell_0/2} h(x)^{2(|\gamma_0|+|\gamma_1|)\ell_0-m}(\int_E h(y)^{pm}\exp(\frac{p\delta_0}{t}|y-x|^2)\frac{|\partial_x^{\gamma_0}\partial_y^{\gamma_1}p(t,x,y)|^p}{p(t,x,y)^{p-1}}dy)^{1/p};$$
$$t \in (0,T],\ x \in E\} < \infty.$$

Proof. Let

$$\varphi_{t,x}(y) = \exp(\frac{\delta_0}{t}|y-x|^2), \qquad x,y \in \mathbf{R}^N,\ t > 0.$$

Then we have for any $\varepsilon > 0$, $f \in C_0^\infty(\mathbf{R}^N)$ and $x \in E$

$$\int_{\mathbf{R}^N} \partial_x^{\gamma_0}\partial_y^{\gamma_1} p(t,x,y) f(y)(\varepsilon + h(y))^m \varphi_{t,x}(y) dy$$

$$= E^\mu[h(X(t,x))^{-2(|\gamma_0|+|\gamma_1|)\ell_0} f(X(t,x))(\varepsilon + h(X(t,x)))^m \varphi_{t,x}(X(t,x))) k_{\gamma_0,\gamma_1}(t,x)]$$

By Propositions 2.3 and 3.2, we see that

$$\int_{\mathbf{R}^N} \partial_x^{\gamma_0}\partial_y^{\gamma_1} p(t,x,y) f(y) h(y)^m \varphi_{t,x}(y) dy$$

$$= E^\mu[h(X(t,x))^{m-2(|\gamma_0|+|\gamma_1|)\ell_0} f(X(t,x))\varphi_{t,x}(X(t,x))) k_{\gamma_0,\gamma_1}(t,x)].$$

Let $k'(t,x) = h(X(t,x))^{m-2(|\gamma_0|+|\gamma_1|)\ell_0}\varphi_{t,x}(X(t,x))k_{\gamma_0,\gamma_1}(t,x)$. Then we see that

$$\sup_{t\in(0,T],x\in E} t^{(|\gamma_0|+|\gamma_1|)\ell_0/2} h(x)^{2(|\gamma_0|+|\gamma_1|)\ell_0-m} E^\mu[|k'(t,x)|^p]^{1/p} < \infty, \qquad T > 0,\ p \in (1,\infty).$$

Note that there is a Borel function $\tilde{k}(t,x) : \mathbf{R}^N \to \mathbf{R}$, $t \in (0,T]$, $x \in E$, such that

$$E^\mu[k'(t,x)|\sigma\{X(t,x))\}] = \tilde{k}(t,x)(X(t,x)), \qquad t \in (0,T],\ x \in E.$$

Then we have

$$\int_{\mathbf{R}^N} \partial_x^{\gamma_0}\partial_y^{\gamma_1} p(t,x,y) f(y) h(y)^m \varphi_{t,x}(y) dy$$

$$= E^\mu[k'(t,x)f(X(t,x))] = E^\mu[\tilde{k}(t,x)(X(t,x))f(X(t,x))] = \int_{\mathbf{R}^N} f(y)\tilde{k}(t,x)(y)p(t,x,y)dy,$$

for any $f \in C_0^\infty(\mathbf{R}^N)$. This implies that $\partial_x^{\gamma_0}\partial_y^{\gamma_1} p(t,x,y)h(y)^m\varphi_{t,x}(y) = \tilde{k}(t,x)(y)p(t,x,y)$ $a.e.y$, $t \geqq 0$, $x \in E$. Therefore letting $m = 0$, we have the first assertion. Since

$$\int_E h(y)^{pm}\varphi_{t,x}(y)^p \frac{|\partial_y^\gamma p(t,x,y)|^p}{p(t,x,y)^{p-1}} dy = \int_E |\tilde{k}(t,x,y)|^p p(t,x,y) dy$$

$$= E^\mu[|\tilde{k}(t,x)(X(t,x))|^p] \leqq E^\mu[|k'(t,x)|^p],$$

we have our assertion. ∎

Proposition 2.8 *For any $T > 0$, there is a $C > 0$ such that*

$$p(t,x,y) \leqq Ct^{-(N+1)\ell_0/2} h(x)^{-2(N+1)\ell_0} \exp(-\frac{2\delta_0}{t}|y-x|^2), \qquad t \in (0,T],\ x,y \in E$$

and

$$p(t,x,y) \leqq Ct^{-(N+1)\ell_0/2} h(y)^{-2(N+1)\ell_0} \exp(-\frac{2\delta_0}{t}|y-x|^2), \qquad t \in (0,T],\ x,y \in E.$$

In particular, for any $T > 0$ and $m \geqq 1$, there is a $C > 0$ such that

$$p(t,x,y) \leqq Ct^{-(N+1)\ell_0/2} h(x)^{-2(N+1)\ell_0} (1+|x|^2)^m (1+|y|^2)^{-m}, \qquad t \in (0,T],\ x,y \in E$$

Proof. Let C_0

$$= \sup\{t^{\ell_0/2}h(x)^2(\int_E \exp(\frac{2(N+1)\delta_0}{t}|y-x|^2)\frac{|\partial_{y^i} p(t,x,y)|^{N+1}}{p(t,x,y)^N}dy)^{1/(N+1)}; t \in (0,T],\ x \in E, \varepsilon > 0\}.$$

Let

$$\rho_\varepsilon(t,x,y) = (p(t,x,y) + \varepsilon\exp(-(1+\frac{2\delta_0}{t})|y-x|^2))^{1/(N+1)}.$$

Then we see that

$$(\int_{\mathbf{R}^N} \exp(\frac{2\delta_0}{t}|y-x|^2)|\frac{\partial}{\partial y^i}\rho_\varepsilon(t,x,y)|^{N+1}dy)^{1/(N+1)}$$

$$= (N+1)^{-1}(\int_{\mathbf{R}^N} \exp(\frac{2\delta_0}{t}|y-x|^2)\frac{|\partial_{y^i}(p(t,x,y)+\varepsilon\exp(-(1+\frac{2\delta_0}{t})|y-x|^2))|^{N+1}}{(p(t,x,y)+\varepsilon\exp(-(1+\frac{2\delta_0}{t})|y-x|^2))^N}dy)^{1/(N+1)}$$

$$\leqq (\int_{\mathbf{R}^N} \exp(\frac{2\delta_0}{t}|y-x|^2)\frac{|\partial_{y^i}p(t,x,y)|^{N+1}}{(p(t,x,y)+\varepsilon\exp(-(1+\frac{2\delta_0}{t})|y-x|^2))^N}dy)^{1/(N+1)}$$

$$+(\int_{\mathbf{R}^N} \exp(\frac{2\delta_0}{t}|y-x|^2)\frac{|\partial_{y^i}(\varepsilon\exp(-(1+\frac{2\delta_0}{t})|y-x|^2))|^{N+1}}{(p(t,x,y)+\varepsilon\exp(-(1+\frac{2\delta_0}{t})|y-x|^2))^N}dy)^{1/(N+1)}$$

$$\leqq C_0 t^{-\ell_0/2} h(x)^{-2} + (\varepsilon \int_{\mathbf{R}^N} (2|y^i - x^i|)^{N+1} (1+\frac{1}{t})^{N+1} \exp(-|y-x|^2)) dy)^{1/(N+1)}.$$

Also, we have

$$(\int_{\mathbf{R}^N} \exp(\frac{2\delta_0}{t}|y-x|^2)\rho_\varepsilon(t,x,y)^{N+1} dy)^{1/(N+1)}$$

$$= (\int_{\mathbf{R}^N} \exp(\frac{2\delta_0}{t}|y-x|^2)(p(t,x,y) + \varepsilon \exp(-(1+\frac{2\delta_0}{t})|y-x|^2)) dy)^{1/(N+1)}$$

$$= (E^\mu[\exp(\frac{2\delta_0}{t}|X(t,x)-x|^2)] + \pi^N \varepsilon)^{1/(N+1)},$$

and

$$(\int_{\mathbf{R}^N} (|\partial_{y_i}(\exp(\frac{2\delta_0}{(N+1)t}|y-x|^2))|\rho_\varepsilon(t,x,y))^{N+1} dy)^{1/(N+1)}$$

$$= (\int_{\mathbf{R}^N} (\frac{4\delta_0|y_i - x_i|}{t})^{N+1} \exp(\frac{2\delta_0}{(N+1)t}|y-x|^2)(p(t,x,y)+\varepsilon \exp(-(1+\frac{2\delta_0}{t})|y-x|^2)) dy)^{1/(N+1)}$$

$$\leqq \frac{4\delta_0}{t} E^\mu[|X(t,x)-x|^{N+1} \exp(\frac{2\delta_0}{t}|X(t,x)-x|^2)]^{1/(N+1)} + \varepsilon \frac{4\delta_0}{t} (\int_{\mathbf{R}^N} |y_i|^{N+1} \exp(-|y|^2)) dy)^{1/(N+1)}.$$

Then by Sobolev's inequality, we see that there is a constant $C > 0$ such that

$$\sup_{y \in \mathbf{R}^N} (\exp(\frac{2\delta_0}{t}|y-x|^2)((p(t,x,y) + \varepsilon \exp(-|y|^2)))^{1/(N+1)}$$

$$\leqq C(C_0 t^{-\ell_0/2} h(x)^{-2\ell_0} + C t^{-1/2} + C\varepsilon(1+\frac{1}{t})).$$

So letting $\varepsilon \downarrow 0$, we have our first assertion.

Let

$$\tilde{\rho}_\varepsilon(t,x,y) = (p(t,x,y)h(y)^{2(N+1)\ell_0} + \varepsilon \exp(-(1+\frac{2\delta_0}{t})|y-x|^2))^{1/(N+1)}.$$

Then similarly we can show that

$$\int_{\mathbf{R}^N} (\exp(\frac{2\delta_0}{(N+1)t}|y-x|^2)\tilde{\rho}_\varepsilon(t,x,y))^{N+1}$$

$$+ \sum_{i=1}^N |\partial_{y_i}(\exp(\frac{2\delta_0}{(N+1)t}|y-x|^2)\tilde{\rho}_\varepsilon(t,x,y))|^{N+1}) dy \leqq C t^{-\ell_0/2}, \qquad t \in (0,T],\ x \in E.$$

So we have our second assertion.

Finally note that

$$|\log(1+|x|^2) - \log(1+|y|^2)| \leqq |\int_{|y|}^{|x|} \frac{2t}{1+t^2} dt| \leqq |x-y| \leqq \frac{1}{\varepsilon} + \varepsilon|x-y|^2, \qquad x,y \in \mathbf{R}^N,\ \varepsilon > 0.$$

So we have the final assertion. ∎

Proposition 2.9 *Let $\delta \in (0,1/N)$, $\alpha, \beta \in \mathbf{Z}^N_{\geqq 0}$ and $T > 0$. Then there are $C > 0$ and $q > 0$ such that*

$$|\partial_x^\alpha \partial_y^\beta p(t,x,y)| \leqq C t^{-(|\alpha|+|\beta|+1)\ell_0/2} h(x)^{-2(|\alpha|+|\beta|+1)\ell_0} p(t,x,y)^{1-\delta}, \qquad x,y \in E,\ t \in (0,T],$$

and

$$|\partial_x^\alpha \partial_y^\beta p(t,x,y)| \leqq C t^{-(|\alpha|+|\beta|+1)\ell_0/2} h(y)^{-2(|\alpha|+|\beta|+1)\ell_0} p(t,x,y)^{1-\delta}, \qquad x,y \in E,\ t \in (0,T].$$

Proof. Let $p = 1/\delta > N$, and let

$$\rho_\varepsilon(t,x,y) = \frac{\partial_x^\alpha \partial_y^\beta p(t,x,y)}{(p(t,x,y)+\varepsilon)^{1-\delta}}$$

for $\varepsilon > 0$. Then we see by Proposition 3.5 that there is a $C_1 > 0$ such that

$$(\int_{\mathbf{R}^N} |\rho_\varepsilon(t,x,y)|^p dy)^{1/p} = (\int_{\mathbf{R}^N} \frac{|\partial_x^\alpha \partial_y^\beta p(t,x,y)|^p}{(p(t,x,y)+\varepsilon)^{p-1}} dy)^{1/p}$$
$$\leqq C_1 t^{-(|\alpha|+|\beta|)\ell_0/2} h(y)^{-2(|\alpha|+|\beta|)\ell_0}, \qquad \varepsilon > 0,\ t \in (0,T],\ x \in E.$$

Also, we have

$$(\int_{\mathbf{R}^N} |\partial_{y^i} \rho_\varepsilon(t,x,y)|^p dy)^{1/p}$$
$$\leqq (\int_{\mathbf{R}^N} \frac{|\partial_x^\alpha \partial_y^\beta \partial_{y^i} p(t,x,y)|^p}{(p(t,x,y)+\varepsilon)^{p-1}} dy)^{1/p}$$
$$+(1-\delta)(\int_{\mathbf{R}^N} \frac{|\partial_x^\alpha \partial_y^\beta p(t,x,y)|^{2p}}{(p(t,x,y)+\varepsilon)^{2p-1}} dy)^{1/(2p)} (\int_{\mathbf{R}^N} \frac{|\partial_{y_i} p(t,x,y)|^{2p}}{(p(t,x,y)+\varepsilon)^{2p-1}} dy)^{1/(2p)}.$$

So we see by Proposition 3.5 that there is a $C_2 > 0$ such that

$$(\int_{\mathbf{R}^N} |\partial_{y_i} \rho_\varepsilon(t,x,y)|^p dy)^{1/p}$$
$$\leqq C_2 t^{-(|\alpha|+|\beta|+1)\ell_0/2} h(y)^{-2(|\alpha|+|\beta|+1)\ell_0}, \qquad \varepsilon > 0,\ t \in (0,T],\ x \in E.$$

So by Sobolev's inequality, we see that there is a $C_3 > 0$ such that

$$\sup_{y \in \mathbf{R}^N} |\rho_\varepsilon(t,x,y)|$$
$$\leqq C_3 t^{-(|\alpha|+|\beta|+1)\ell_0/2} h(x)^{-2(|\alpha|+|\beta|+1)}, \qquad \varepsilon > 0,\ t \in (0,T],\ x \in E.$$

Letting $\varepsilon \downarrow 0$, we have the first assertion.

Let

$$\tilde\rho_\varepsilon(t,x,y) = \frac{\partial_x^\alpha \partial_y^\beta p(t,x,y)}{(p(t,x,y)+\varepsilon)^{1-\delta}} h(y)^{2(|\alpha|+|\beta|+1)}$$

for $\varepsilon > 0$. Then a similar argument implies that there is a $C_4 > 0$ such that

$$\sup_{y \in \mathbf{R}^N} |\tilde\rho_\varepsilon(t,x,y)|$$
$$\leqq C_4 t^{-(|\alpha|+|\beta|+1)\ell_0/2}, \qquad \varepsilon > 0,\ t \in (0,T],\ x \in E.$$

So we have the second assertion. ∎

Proposition 2.10 *Let $m \geqq 0$, $\alpha, \beta \in \mathbf{Z}_{\geqq 0}^N$, $p \in [1,\infty)$, $\delta \in (0,1)$ and $T > 0$. Then there is a $C > 0$ such that*

$$\int_{\mathbf{R}^N} |\partial_t^m \partial_x^\alpha \partial_y^\beta p(t-s,x,y)|^p p(s,x_0,x) dx$$
$$\leqq C(t-s)^{p(|\alpha|+|\beta|+2m+2)\ell_0/2} p(t,x_0,y)^{1-\delta}$$

for any $t \in (0,T]$, $s \in [0,t)$, $y \in \mathbf{R}^N$.

Proof. First note that

$$\partial_t p(t,x,y) = L_x p(t,x,y), \qquad \text{where } L = \frac{1}{2}\sum_{k=1}^{d} V_k^2 + V_0.$$

So it is sufficient to prove the case $m = 0$.

Let $r = 1/(1-\delta)$. Since $p > 1-\delta$, we see by Propositions 3.6and 3.7, that there are $C > 0$ and $b > 0$ such that

$$|\partial_x^\alpha \partial_y^\beta p(t-s,x,y)|^p \leqq C(t-s)^{p(|\alpha|+|\beta|+2)\ell_0/2} h(x)^{-b} p(t-s,x,y)^{1-\delta},$$

for any $t \in (0,T],\ s \in [0,t),\ x \in E,\ y \in \mathbf{R}^N$. So we see that

$$\int_{\mathbf{R}^N} |\partial_x^\alpha \partial_y^\beta p(t-s,x,y)|^p p(s,x_0,x)dx$$

$$\leqq C(t-s)^{p(|\alpha|+|\beta|+2)\ell_0/2} \int_{\mathbf{R}^N} h(z)^{-b} p(t-s,z,y)^{1/r} p(s,x_0,z)dz$$

$$\leqq C(t-s)^{p(|\alpha|+|\beta|+2)\ell_0/2} (\int_{\mathbf{R}^N} (h(z)^{-b/\delta} p(s,x_0,z)dz)^\delta (\int_{\mathbf{R}^N} p(t-s,z,y)p(s,x_0,z)dz)^{1-\delta}.$$

Since

$$\int_{\mathbf{R}^N} p(t-s,z,y)p(s,x_0,z)dz = p(t,x_0,y),$$

we have our assertion. ∎

Proposition 2.11 *Let $a \in (0,1]$, and $b \in (0,a)$. Then we have*

$$\int_{\mathbf{R}^N} p(s,x_0,x)^a p(t-s,x,y)^b \phi(x)dx \leqq p(t,x_0,y)^b (\int_E dx p(s,x_0,x)^{(a-b)/(1-b)} \phi(x)^{1/(1-b)})^{1-b}$$

for any $t > s \geqq 0$, and non-negative measurable function $\phi : E \to [0,\infty)$.

Proof. Let $\delta = (a-b)/(1-b)$, $p = 1/b$, and $q = 1/(1-b)$. Then we see that $1-\delta = (1-a)/(1-b)$ and $a - \delta = b(1-a)/(1-b)$, and so we have

$$\int_{\mathbf{R}^N} p(s,x_0,x)^a p(t-s,x,y)^b \phi(x)dx = \int_{\mathbf{R}^N} p(s,x_0,x)^\delta p(s,x_0,x)^{(1-\delta)/p} p(t-s,x,y)^{1/p} \phi(x)dx$$

$$\leqq (\int_E p(s,x_0,x)^\delta p(s,x_0,x)^{1-\delta} p(t-s,x,y)dx)^{1/p} (\int_E p(s,x_0,x)^\delta \phi(x)^q dx)^{1/q}$$

$$= p(s,x_0,y)^b (\int_E p(s,x_0,x)^{(a-b)/(1-b)} \phi(x)^{1/(1-b)} dx)^{1-b}.$$

This proves our assertion. ∎

Proposition 2.12 *Let $p \geqq 1$, $m \geqq 1$. $\alpha, \beta \in \mathbf{Z}_{\geqq 0}^N$, $T > 0$, $a \in (0,1/p]$ and $b \in (a-1/N, a)$. Then there is a $C > 0$ such that*

$$\int_{\mathbf{R}^N} |\partial_x^\alpha (p(s,x_0,x)^a)|^p |\partial_x^\beta p(t-s,x,y)|^p dx$$

$$\leqq C s^{-p(|\alpha|+1)\ell_0/2} (t-s)^{-p(|\beta|+2)\ell_0/2} p(t,x_0,y)^{pb} (1+|y|^2)^{-m}$$

for any $y \in E$ and $s,t \in (0,T]$ with $s < t$.

Proof. Let $\delta = (a-b)/2 < 1/N$. Note that $\partial_x^\alpha(p(s,x_0,x)^a)$ is a linear combination of $a(a-1)\cdots(a-m+1)p(s,x_0,x)^{a-m}\partial_x^{\alpha_1}p(s,x_0,x)\cdots\partial_x^{\alpha_m}p(s,x_0,x)$, $m=1,\dots,|\alpha|$, $\alpha_k \in \mathbf{Z}_{\geqq 0}$, $|\alpha_k| \geqq 1$, $k=1,\dots,m$, $\alpha_1+\cdots+\alpha_m=\alpha$.

Then by Propositions 3.7, we see that there is a $C_1>0$ such that

$$|\partial_x^\alpha(p(s,x_0,x)^a)||\partial_x^\beta p(t-s,x,y)|$$
$$\leqq C_1 s^{-(|\alpha|+1)\ell_0/2}(t-s)^{-(|\beta|+1)\ell_0/2}h(x)^{-2(|\beta|+1)\ell_0}p(s,x_0,x)^{a-\delta}p(t-s,x,y)^{1-\delta}$$

for any $a\in(0,1/p]$, $b\in(a-1/N,a)$, $x,y\in E$ and $s,t\in[0,T]$ with $s<t$. By Propositions 3.6, we see that there is a $C_2>0$ such that

$$p(t-s,x,y)^{1-\delta-b} \leqq C_2(t-s)^{-\ell_0/2}h(x)^{-2(N+1)\ell_0}(1+|x|^2)^m(1+|y|^2)^{-(1-\delta-b)m}$$

for any $x,y\in E$ and $s,t\in[0,T]$ with $s<t$. So we have

$$|\partial_x^\alpha(p(s,x_0,x)^a)||\partial_x^\beta p(t-s,x,y)|$$
$$\leqq C_1C_2 s^{-(|\alpha|+1)\ell_0/2}(t-s)^{-(|\beta|+2)\ell_0/2}h(x)^{-2(|\beta|+N+2)\ell_0}p(s,x_0,x)^{a-\delta}p(t-s,x,y)^b$$
$$\times(1+|x|^2)^m(1+|y|^2)^{-(1-(a+b)/2)m}$$

Note that $pb<p(a-\delta)<1$, and so we have

$$\int_E (h(x)^{-2(|\beta|+N+2)\ell_0}p(s,x_0,x)^{a-\delta}p(t-s,x,y)^b(1+|x|^2)^m)^p dx$$
$$=\int_E p(s,x_0,x)^{p(a-\delta-b)/(1-pb)}p(s,x_0,y)^{pb(1-p(a-\delta))/(1-pb)}p(t-s,x,y)^{pb}$$
$$\times(h(x)^{-2p(|\beta|+N+2)\ell_0}(1+|x|^2)^{mp}\,dx$$
$$\leqq (\int_E p(s,x_0,x)^{p(a-\delta-b)/(1-pb)}p(s,x_0,y)^{(1-p(a-\delta))/(1-pb)}p(t-s,x,y)\,dx)^{pb}$$
$$\times(\int_E p(s,x_0,x)^{p(a-\delta-b)/(1-pb)}h(x)^{-p(|\beta|+N+2)/(1-pb)}(1+|x|^2)^{mp/(1-pb)}\,dx)^{1-pb}$$
$$=p(t,x_0,y)^{pb}(\int_E(1+|x|^2)^{-N}p(s,x_0,x)^{p(a-\delta-b)/(1-pb)}h(x)^{-2p(|\beta|+N+2)/(1-pb)}$$
$$\times(1+|x|^2)^{mp/(1-pb)+N}\,dx)^{1-pb}$$
$$\leqq p(t,x_0,y)^{pb}(\int_E(1+|x|^2)^{-N}p(s,x_0,x)h(x)^{-p(|\beta|+N+2)/\ell_0(p(a-\delta-b))}$$
$$\times(1+|x|^2)^{(mp+N(1-pb))/(p(a-\delta-b))}\,dx)^{p(a-\delta-b)}(\int_E(1+|x|^2)^{-N}\,dx)^{(1-p(a-\delta-b))/(1-pb)}$$
$$=p(t,x_0,y)^{pb}(\int_E(1+|x|^2)^{-N}\,dx)^{(1-p\delta)/(1-pb)}$$
$$\times E^\mu[h(X(s,x_0))^{-(|\beta|+N+2)\ell_0/\delta}(1+|X(s,x_0)|^2)^{(mp+N(1-pb))/(p\delta)}]^{p\delta}.$$

So by Proposition 2.3, we have our assertion. ∎

2.3 Stochastic mesh and random norms

Let $\mathcal{F}_t^{(L)}$, $t \geqq 0$, $L = 0, 1, \dots, \infty$ be sub σ-algebra of $\mathcal{F}$ given by

$$\mathcal{F}_t^{(L)} = \sigma\{X_\ell(s);\ s \in [0,t],\ \ell = 1, 2, \dots .L\},$$

and

$$\mathcal{F}_t^{(\infty)} = \sigma\{X_\ell(s);\ s \in [0,t],\ \ell = 1, 2, \dots\}.$$

Let ν_t, $t \geqq 0$, be the probability law of $X(t, x_0)$ under μ. Then we see that ν_0 is the probability measure concentrated in x_0, and $\nu_t(dx) = p(t, x_0, x)dx$, $t > 0$.

Then for any $t > s \geqq 0$, we can define a linear contraction map $P_{s,t} : L^1(E; d\nu_t) \to L^1(E; d\nu_s)$ by

$$(P_{s,t}f)(x) = \int_E p(t-s, x, y)f(y)dy, \qquad x \in E,\ f \in L^1(E; d\nu_t).$$

Proposition 2.13 *Let $t > s \geqq 0$, $\alpha \in \mathbf{Z}^N_{\geqq 0}$ and bounded measurable function $f : E \to \mathbf{R}$. Then we have*

$$E[\partial_x^\alpha (Q_{s,t}^{(L)} f)(x) | \mathcal{F}_s^{(\infty)}] = \partial_x^\alpha (P_{s,t}f)(x), \qquad \nu_s - a.e.x.$$

and

$$E[|\partial_x^\alpha (Q_{s,t}^{(L)} f)(x) - \partial_x^\alpha (P_{s,t}f)(x)|^2 | \mathcal{F}_s^{(\infty)}] \leqq \frac{1}{L} \int_E \frac{(\partial_x^\alpha p(t-s, x, y))^2 |f(y)|^2}{q_{s,t}^{(L)}(y)}\, dy.$$

Proof. Note that

$$E[\partial_x^\alpha (Q_{s,t}^{(L)} f)(x) | \mathcal{F}_s^{(\infty)}] = \frac{1}{L} \sum_{\ell=1}^{L} \int_E \frac{\partial_x^\alpha p(t-s, x, y) f(y)}{q_{s,t}^{(L)}(y)} p(t-s, X_\ell(s), y)\, dy$$

$$= \int_E \partial_x^\alpha p(t-s, x, y) f(y)\, dy = \partial_x^\alpha (P_{s,t}f)(x).$$

This implies the first assertion.

Let

$$m_\ell = \frac{1}{L} \int_E \frac{\partial_x^\alpha p(t-s, x, y) f(y)}{q_{s,t}^{(L)}(y)} p(t-s, X_\ell(s), y)\, dy$$

and

$$d_\ell = \frac{1}{L} \frac{\partial_x^\alpha p(t-s, x, X_\ell(t)) f(X_\ell(t))}{q_{s,t}^{(L)}(X_\ell(t))} - m_\ell$$

for $\ell = 1, \dots, L$. Then we see that

$$E[d_\ell | \mathcal{F}_s^{(\infty)} \vee \mathcal{F}_t^{(\ell-1)}] = 0, \qquad \ell = 1, \dots, L.$$

Here we let $\mathcal{F}_t^{(0)} = \{\emptyset, \Omega\}$. Moreover, we have

$$\sum_{\ell=1}^{L} d_\ell = \partial_x^\alpha (Q_{s,t}^{(L)} f)(x) - \partial_x^\alpha (P_{s,t}f)(x)$$

So we see that

$$E[|\partial_x^\alpha(Q_{s,t}^{(L)}f)(x) - \partial_x^\alpha(P_{s,t}f)(x)|^2|\mathcal{F}_s^{(\infty)}] \leqq E[(\sum_{\ell=1}^L |d_\ell|^2)|\mathcal{F}_s^{(\infty)}]$$

$$\leqq \sum_{\ell=1}^L E[(\frac{1}{L}\frac{\partial_x^\alpha p(t-s,x,X_\ell(t))f(X_\ell(t))}{q_{s,t}^{(L)}(X_\ell(t))})^2|\mathcal{F}_s^{(\infty)}]$$

$$\leqq \frac{1}{L^2}\sum_{\ell=1}^L \int_E \frac{(\partial_x^\alpha p(t-s,x,y))^2|f(y)|^2}{q_{s,t}^{(L)}(y)^2} p(t-s,X_\ell(s),y)\,dy$$

$$= \frac{1}{L}\int_E \frac{(\partial_x^\alpha p(t-s,x,y))^2|f(y)|^2}{q_{s,t}^{(L)}(y)}\,dy.$$

So we have the second assertion. ∎

Now let $M_t^{(L)} : m(E)\times\Omega \to \mathbf{R}$, and $N_t^{(L)} : m(E)\times\Omega \to [0,\infty)$, $t \geqq 0$, $L \geqq 1$, be random functionals given by

$$M_t^{(L)}(f) = M_t^{(L)}(f;\omega) = \frac{1}{L}\sum_{\ell=1}^L f(X_\ell(t)), \qquad f \in m(E),$$

and

$$N_t^{(L)}(f) = N_t^{(L)}(f;\omega) = M_t^{(L)}(|f|) = \frac{1}{L}\sum_{\ell=1}^L |f(X_\ell(t))|, \qquad f \in m(E).$$

Then we see that $M_t^{(L)}$ is a linear function and $N_t^{(L)}$ is a semi-norm in $m(E)$.

Proposition 2.14 *Let $t > s \geqq 0$ and $L \geqq 1$ (1) For any $f \in m(E)$,*

$$M_s^{(L)}(Q_{s,t}^{(L)}f) = M_t(f).$$

(2) *For any $f \in m(E)$*

$$N_s^{(L)}(Q_{s,t}^{(L)}f) \leqq N_t(f).$$

Proof. Suppose that $f \in m(E)$. Then we have

$$M_s^{(L)}(Q_{s,t}^{(L)}f) = \frac{1}{L}\sum_{\ell=1}^L \frac{1}{L}\sum_{k=1}^L \frac{p(t-s,X_\ell(s),X_k(t))f(X_k(t))}{q_{s,t}^{(L)}(X_k(t))}$$

$$= \frac{1}{L}\sum_{k=1}^L(\frac{1}{L}\sum_{\ell=1}^L \frac{p(t-s,X_\ell(s),X_k(t))f(X_k(t))}{q_{s,t}^{(L)}(X_k(t))}) = M_t(f).$$

So we have the assertion (1).

The second assertion is an easy consequence of the assertion (1). ∎

Proposition 2.15 (1) *Let $T > 0$ and $m \geqq 1$. Then there is a $C > 0$ such that*

$$\frac{1}{L}\sum_{\ell=1}^{L} E[((Q_{s,t}^{(L)} f)(X_\ell(s)) - (P_{s,t}f)(X_\ell(s))^2|\mathcal{F}_s^{(\infty)}]$$

$$\leqq \frac{C}{L}(t-s)^{-(N+1)\ell_0/2} \max_{\ell=1\dots,L} h(X_\ell(s))^{-2(N+1)}(1+|X_\ell(s)|^2)^m) \int_E f(y)^2(1+|y|^2)^{-m}\, dy \qquad a.s.$$

for any $L \geqq 1$ and $s, t \in [0, T]$ with $s < t$.

In particular,

$$E[N_s^{(L)}(Q_{s,t}^{(L)} f - P_{s,t}f)^2]$$

$$\leqq \frac{C}{L}(t-s)^{-(N+1)\ell_0/2} E[\max_{\ell=1\dots,L} h(X_\ell(s))^{-2(N+1)}(1+|X_\ell(s)|^2)^m] \int_E f(y)^2(1+|y|^2)^{-m}\, dy$$

for any $L \geqq 1$ and $s, t \in [0, T]$ with $s < t$.

(2) *For any $\varepsilon > 0$ and $T > 0$,*

$$\overline{\lim_{L\to\infty}} L^{-\varepsilon} \sup_{s\in[0,T]} E[\max_{\ell=1\dots,L} h(X_\ell(s))^{-2(N+1)}(1+|X_\ell(s)|^2)^m] = 0$$

Proof. By Proposition 3.11, we see that

$$\frac{1}{L}\sum_{\ell=1}^{L} E[((Q_{s,t}^{(L)} f)(X_\ell(s)) - (P_{s,t}f)(X_\ell(s))^2|\mathcal{F}_s^{(\infty)}]$$

$$\leqq \frac{1}{L^2}\sum_{\ell=1}^{L}\int_E \frac{p(t-s, X_\ell(s), y)^2 f(y)^2}{q_{s,t}^{(L)}(y)} dy$$

$$\leqq \frac{1}{L^2}\sum_{\ell=1}^{L}\int_E (\max_{\ell'=1,\dots,L} p(t-s, X_{\ell'}(s), y)) \frac{p(t-s, X_\ell(s), y) f(y)^2}{q_{s,t}^{(L)}(y)} dy$$

$$= \frac{1}{L}\int_E (\max_{\ell=1,\dots,L} p(t-s, X_\ell(s), y)) f(y)^2 dy.$$

Then by Proposition 3.6 we have the assertion (1).

Let $\varepsilon > 0$. Let us take $p > 1/\varepsilon$. Then we have

$$E[\max_{\ell=1\dots,L} h(X_\ell(s))^{-2(N+1)}(1+|X_\ell(s)|^2)^m]$$

$$\leqq E[(\sum_{\ell=1}^{L}(h(X_\ell(s))^{-2(N+1)}(1+|X_\ell(s)|^2)^m)^p)^{1/p}]$$

$$\leqq E[\sum_{\ell=1}^{L}(h(X_\ell(s))^{-2(N+1)}(1+|X_\ell(s)|^2)^m))^p]^{1/p}$$

$$= L^{1/p} E^\mu[h(X(s,x_0))^{-2p(N+1)}(1+|X(s,x_0)|^2)^{mp}]^{1/p}$$

$$\leqq L^{1/p} E^\mu[h(X(s,x_0))^{-4p(N+1)}]^{1/(2p)} E^\mu[(1+|X(s,x_0)|^2)^{2pm}]^{1/(2p)}$$

So we have the assertion (2) by Proposition 2.3.

2.4 Application 1

Let $r \geqq 0$, and let $\mathcal{B}_r$ be the set of Borel measrurable functions $f : \mathbf{R}^N \to \mathbf{R}$ such that $\sup_{x\in\mathbf{R}^N}(1+|x|^2)^{-r/2}|f(x)| < \infty$.

Then we see that $Q^{(L)}_{s,t}$ and $P_{s,t}$, $t > s \geqq 0$, can be regarded as linear operators on $\mathcal{B}_r$.

Now let $\phi_{s,t} : \mathbf{R}^n \times \mathbf{R}$, $s,t \in [0,\infty)$, $s < t$, be measurable functions. We assume that there is a $\lambda \geqq 0$, such that

$$|\phi_{s,t}(x,y) - \phi_{s,t}(x,z)| \leqq \exp(\lambda(t-s))|y-z|, \qquad x \in \mathbf{R}^N,\ y,z \in \mathbf{R},\ t > s \geqq 0.$$

Also, we assume that $\phi_{s,t}(\cdot,0) \in \mathcal{B}_r$, $t > s \geqq 0$.

Let us define a nonlinear operator $\Phi_{s,t} : \mathcal{B}_r \to \mathcal{B}_r$, $s,t \in [0,\infty)$, $s < t$, by

$$(\Phi_{s,t}f)(x) = \phi_{s,t}(x, f(x)), \qquad x \in E,\ f \in \mathcal{B}_r.$$

Then we have

$$N^{(L)}_s(\Phi_{s,t}f - \Phi_{s,t}g) \leqq \exp(\lambda(t-s))N^{(L)}_s(f-g)$$

for any $f,g \in \mathcal{B}_r$.

Let us define operators $\tilde{Q}^{(L)}_{s,t}$ and $\tilde{P}_{s,t}$ on $\mathcal{B}_r$ by $\tilde{Q}^{(L)}_{s,t} = \Phi_{s,t} \circ Q^{(L)}_{s,t}$ and $\tilde{P}_{s,t} = \Phi_{s,t} \circ P_{s,t}$.

Then we have the following easily from Propositions 2.14 and 2.15.

Proposition 2.16 (1)

$$N^{(L)}_s(\tilde{Q}^{(L)}_{s,t}f - \tilde{Q}^{(L)}_{s,t}g) \leqq \exp(\lambda(t-s))N^{(L)}_t(f-g)$$

for any $f,g \in \mathcal{B}_r$.
(2) *Let* $T > 0$ *and* $m \geqq 1$. *Then there is a* $C > 0$ *such that*

$$E[N^{(L)}_s(\tilde{Q}^{(L)}_{s,t}f - \tilde{P}_{s,t}f)^2]$$
$$\leqq \frac{C}{L}a(L)\exp(2\lambda(t-s))(t-s)^{-(N+1)\ell_0/2}\int_E f(y)^2(1+|y|^2)^{-(r+N)}\,dy$$

for any $L \geqq 1$ *and* $s,t \in [0,T]$ *with* $s < t$. *Here*

$$a(L) = \sup_{s\in[0,T]} E[\max_{\ell=1\dots,L} h(X_\ell(s))^{-2(N+1)}(1+|X_\ell(s)|^2)^m]$$

Note that by Proposition 2.15(2), we see that for any $\delta > 0$,

$$L^{-\delta}a(L) \to 0, \qquad L \to \infty.$$

So we have the following.

Theorem 2.17 *For* $T > 0$, *there is a* $C > 0$ *satisfying the following. For any* $n \geqq 1$, *and* $0 = t_0 < t_1 < \cdots < t_n \leqq T$,

$$E[|(\tilde{Q}^{(L)}_{t_0,t_1}\cdots\tilde{Q}^{(L)}_{t_{n-1},t_n}f)(x_0) - (\tilde{P}_{t_0,t_1}\cdots\tilde{P}_{t_{n-1},t_n}f)(x_0)|^2]^{1/2}$$
$$\leqq \frac{C}{L^{1/2}}a(L)^{1/2}\exp(\lambda t_n)\sum_{k=1}^n(t_k - t_{k-1})^{-(N+1)\ell_0/4}$$
$$(\int_E(\tilde{P}_{t_k,t_{k+1}}\cdots\tilde{P}_{t_{n-1},t_n}f)(y)^2(1+|y|^2)^{-(r+N)}\,dy)^{1/2}$$

Proof. Note that

$$|(\tilde{Q}^{(L)}_{t_0,t_1}\cdots\tilde{Q}^{(L)}_{t_{n-1},t_n}f)(x_0)-(\tilde{P}_{t_0,t_1}\cdots\tilde{P}_{t_{n-1},t_n}f)(x_0)|$$
$$=N_0^{(L)}((\tilde{Q}^{(L)}_{t_0,t_1}\cdots\tilde{Q}^{(L)}_{t_{n-1},t_n}f)-(\tilde{P}_{t_0,t_1}\cdots\tilde{P}_{t_{n-1},t_n}f))$$
$$\leqq\sum_{k=1}^n N_0^{(L)}((\tilde{Q}^{(L)}_{t_0,t_1}\cdots\tilde{Q}^{(L)}_{t_{k-1},t_k}\tilde{P}_{t_k,t_{k+1}}\cdots\tilde{P}_{t_{n-1},t_n}f)-(\tilde{Q}^{(L)}_{t_0,t_1}\cdots\tilde{Q}^{(L)}_{t_{k-2},t_{k-1}}\tilde{P}_{t_{k-1},t_k}\cdots\tilde{P}_{t_{n-1},t_n}f))$$
$$\leqq\sum_{k=1}^n\exp(\lambda t_{k-1})N^{(L)}_{t_{k-1}}(\tilde{Q}^{(L)}_{t_{k-1},t_k}\tilde{P}_{t_k,t_{k+1}}\cdots\tilde{P}_{t_{n-1},t_n}f)-(\tilde{P}_{t_{k-1},t_k}\cdots\tilde{P}_{t_{n-1},t_n}f)).$$

Also, we have by Propostion 2.16

$$E[N^{(L)}_{t_{k-1}}(\tilde{Q}^{(L)}_{t_{k-1},t_k}\tilde{P}_{t_k,t_{k+1}}\cdots\tilde{P}_{t_{n-1},t_n}f)-(\tilde{P}_{t_{k-1},t_k}\cdots\tilde{P}_{t_{n-1},t_n}f))^2]^{1/2}$$
$$\leqq\frac{C^{1/2}}{L^{1/2}}a(L)^{1/2}\exp(\lambda(t_k-t_{k-1}))(t_k-t_{k-1})^{-(N+1)\ell_0/4}$$
$$\times(\int_E(\tilde{P}_{t_k,t_{k+1}}\cdots\tilde{P}_{t_{n-1},t_n}f)(y)^2(1+|y|^2)^{-(r+N)}\,dy)^{1/2}.$$

These imply our theorem. ∎

Now we apply the above theorem to American option. Let $g:[0,T]\times\mathbf{R}^n\to\mathbf{R}$ be a continuous function such that there are $r\geqq 1$ and $C_1>0$ such that $|g(t,x)|\leqq C_1(1+|x|^2)^{r/2}$, $t\in[0,T]$, $x\in\mathbf{R}^n$. Let $\phi_{s,t}(x,y)=g(s,x)\vee y$, for $x\in\mathbf{R}^n$, $y\in\mathbf{R}$, and $s,t\in[0,T]$ with $s<t$. Then we have $\phi_{s,t}(x,y)-\phi_{s,t}(x,z)|\leqq|y-z|$. It is easy to see that there is a $a\geqq 0$ such that

$$E[\sup_{t\in[0,T]}(1+|X(t,x)|^2)^{r/2}]\leqq\exp(aT)(1+|x|^2)^{r/2},\qquad x\in\mathbf{R}^n.$$

So we see that

$$\sup_{x\in\mathbf{R}^n}(1+|x|^2)^{-r/2}|\tilde{P}_{s,t}f(x)|\leqq\exp C_1\vee\exp(a(t-s))\sup_{x\in\mathbf{R}^n}(1+|x|^2)^{-r/2}|f(x)|,\qquad f\in\mathcal{B}_r.$$

Then we see that

$$(\int_E(\tilde{P}_{t_k,t_{k+1}}\cdots\tilde{P}_{t_{n-1},t_n}g(t_n,\cdot))(y)^2(1+|y|^2)^{-(r+N)}\,dy)^{1/2}\leqq C_1\exp(a(t_n-t_k))\int_E(1+|y|^2)^{-N}\,dy)^{1/2}.$$

So we have by Theorem 2.17, we see that there is a $C_2>0$ such that

$$E[|(\tilde{Q}^{(L)}_{t_0,t_1}\cdots\tilde{Q}^{(L)}_{t_{n-1},t_n}g(t_n,\cdot)(x_0)-(\tilde{P}_{t_0,t_1}\cdots\tilde{P}_{t_{n-1},t_n}g(t_n,\cdot))(x_0)|^2]^{1/2}$$
$$\leqq\frac{C_2}{L^{1/2}}a(L)^{1/2}\sum_{k=1}^n(t_k-t_{k-1})^{-(N+1)\ell_0/4}$$

for any $n\geqq 1$, and $0=t_0<t_1<\cdots<t_n\leqq T$. So if we take $n_L\geqq 1$ and $0=t_0^{(L)}<t_1^{(L)}<\cdots<t_n^{(L)}=T$ for each $L\geqq 1$, and there is a $\delta_0,\delta_1>0$, with $\delta_0<\delta_1<1/2$ such that

$$\lim_{L\to\infty}L^{-\delta_0}\sum_{k=1}^{n_L}(t_k^{(L)}-t_{k-1}^{(L)})^{-(N+1)\ell_0/4}=0,$$

then we see that

$$L^{-(1-\delta_1)/2}|(\tilde{Q}^{(L)}_{t_0^{(L)},t_1^{(L)}}\cdots\tilde{Q}^{(L)}_{t^{(L)}_{n_L-1},t^{(L)}_{n_L}}g(T,\cdot)(x_0)-(\tilde{P}_{t_0^{(L)},t_1^{(L)}}\cdots\tilde{P}_{t^{(L)}_{n_L-1},t^{(L)}_{n_L}}g(T,\cdot))(x_0)|\to 0$$

in probability.

2.5 Preparations for estimates of functions

Proposition 2.18 *Let Z_k, $k = 1, 2 \dots$ be independent integrable random variables.*
(1) *For any $p \geqq 1$, there is a $C > 0$ only depend on p such that*

$$E[|\sum_{k=1}^{n}(Z_k - E[Z_k])|^{2p}] \leqq C(E[(\sum_{k=1}^{n} Z_k^2)^p] + (\sum_{k=1}^{n} |E[Z_k]|)^{2p}), \ n \geqq 1.$$

(2) *For any $p \geqq 1$, there is a $C > 0$ only depend on p such that*

$$E[|\sum_{k=1}^{n} Z_k|^{2p}] \leqq C(E[(\sum_{k=1}^{n} Z_k^2)^p] + (\sum_{k=1}^{n} |E[Z_k]|)^{2p}), \ n \geqq 1.$$

(3) *For any $m \in \mathbf{N}$, there is a $C > 0$ only depend on m such that*

$$E[|\sum_{k=1}^{n} Z_k^2|^{2^m}] \leqq C \sum_{r=1}^{m+1} (\sum_{k=1}^{n} E[Z_k^{2^r}])^{2^{m+1-r}} \ n \geqq 1.$$

Proof. (1) If $\sum_{k=1}^{n} E[|Z_k|^{2p}] = \infty$, the right hand side is infinity, and so the inequality is valid. So we assume that $\sum_{k=1}^{n} E[|Z_k|^{2p}] < \infty$. Then by Burkholder's inequality we have

$$E[|\sum_{k=1}^{n}(Z_k - E[Z_k])|^{2p}] \leqq C_{2p} E[(\sum_{k=1}^{n}(Z_k - E[Z_k])^2)^p].$$

Since we have

$$E[(\sum_{k=1}^{n}(Z_k - E[Z_k])^2)^p] \leqq 2^p E[(\sum_{k=1}^{n}(Z_k^2 + E[Z_k]^2))^p]$$

$$\leqq 2^{2p} E[(\sum_{k=1}^{n} Z_k^2)^p] + 2^{2p}(\sum_{k=1}^{n} E[Z_k]^2)^p \leqq 2^{2p} E[(\sum_{k=1}^{n} Z_k^2)^p] + 2^{2p}(\sum_{k=1}^{n} |E[Z_k]|)^{2p},$$

we have our assertion.
(2) Note that

$$E[|\sum_{k=1}^{n} Z_k|^{2p}] = E[|\sum_{k=1}^{n}((Z_k - E[Z_k]) + E[Z_k])|^{2p}]$$

$$\leqq 2^{2p}(E[|\sum_{k=1}^{n}(Z_k - E[Z_k]|^{2p}] + |\sum_{k=1}^{n} E[Z_k]|^{2p}).$$

So we have our assertion by the assertion (1).

We can show the assertion (3) easily by induction and the assertion (2). ∎

Proposition 2.19 *For any $m \geqq 1$, $j \geqq 0$, $\alpha \in \mathbf{Z}^N_{\geqq 0}$, $\delta \in (0, 1)$, and $T > 0$, there is a $C > 0$ such that*

$$E[\sup_{s \in [0, t-\varepsilon]} |(\frac{1}{L} \sum_{\ell=1}^{L} \partial_t^j \partial_y^\alpha p(t - s, X_\ell(s), y)) - \partial_t^j \partial_y^\alpha p(t, x_0, y)|^{2^{m+1}}]$$

$$\leqq C \varepsilon^{-2^m (j+|\alpha|+3)\ell_0} L^{-2^m} L p(t, x_0, y)^{1-\delta} (L^{-1} + p(t, x_0, y)^{1-\delta})^{2^m},$$

for any $y \in \mathbf{R}^N$, $L \geqq 1$, $t \in (0, T]$, $\varepsilon \in (0, t)$.

Proof. Let us note that

$$\frac{\partial}{\partial t}\partial_t^j\partial_y^\alpha p(t,x,y) = L_x\partial_t^j\partial_y^\alpha p(t,x,y), \qquad t>0,\ x\in E,\ y\in \mathbf{R}^N,$$

where

$$L_x = \frac{1}{2}\sum_{k=1}^d V_k^2 + V_0.$$

So we see that $\partial_t^j\partial_y^\alpha p(t-s, X_\ell(s), y)$, $s\in[0,t)$, $h>0$, is a martingale, and

$$\langle \partial_t^j\partial_y^\alpha p(t-\cdot, X_\ell(\cdot), y)\rangle_s$$

$$= \sum_{k=1}^d \int_0^s |\partial_t^j\partial_y^\alpha V_{k,x} p(t-r, X_\ell(r), y)|^2 dr.$$

So we have by Burkholder's inequality and Proposition 2.18 (3),

$$E[\sup_{s\in[0,t-\varepsilon]} |\sum_{\ell=1}^L (\partial_t^j\partial_y^\alpha p(t-s, X_\ell(s), y) - \partial_t^j\partial_y^\alpha p(t,x_0,y))|^{2^{m+1}}]$$

$$\leqq C_{2^{m+1}} E[(\sum_{\ell=1}^L\sum_{k=1}^d \int_0^{t-\varepsilon} |\partial_t^j\partial_y^\alpha V_{k,x} p(t-s, X_\ell(s), y)|^2 ds)^{2^m}]$$

$$\leqq C_{2^{m+1}} d^{2^m} \sum_{k=1}^d E[(\sum_{\ell=1}^L \int_0^{t-\varepsilon} |\partial_t^j\partial_y^\alpha V_{k,x} p(t-s, X_\ell(s), y)|^2 ds)^{2^m}]$$

$$\leqq C\sum_{k=1}^d\sum_{r=0}^m (\sum_{\ell=1}^L E[(\int_0^{t-\varepsilon} |\partial_t^j\partial_y^\alpha V_{k,x} p(t-s, X_\ell(s), y)|^2 ds)^{2^r}])^{2^{m-r}}$$

$$\leqq C\sum_{k=1}^d\sum_{r=0}^m t^{2^m-2^{m-r}} (\sum_{\ell=1}^L E[(\int_0^{t-\varepsilon} |\partial_t^j\partial_y^\alpha V_{k,x} p(t-s, X_\ell(s), y)|^{2^{r+1}} ds])^{2^{m-r}}$$

$$= C\sum_{k=1}^d\sum_{r=0}^m t^{2^m-2^{m-r}} L^{2^{m-r}} (\int_0^{t-\varepsilon} (\int_{\mathbf{R}^N} |\partial_t^j\partial_y^\alpha V_{k,x} p(t-s, z, y)|^{2^{r+1}} p(s,x_0,z)dz)ds)^{2^{m-r}}.$$

Then by Proposition 3.8, we have

$$E[\sup_{s\in[0,t-\varepsilon]} |\sum_{\ell=1}^L (\partial_t^j\partial_y^\alpha p(t-s, X_\ell(s), y) - \partial_t^j\partial_y^\alpha p(t,x_0,y))|^{2^{m+1}}]$$

$$\leqq C' t^{2^m} \varepsilon^{-2^m(j+|\alpha|+3)\ell_0} \sum_{r=0}^m L^{2^{m-r}} p(t,x_0,y)^{2^{m-r}(1-\delta)}.$$

$$\leqq C' t^{2^m} \varepsilon^{-2^m(j+|\alpha|+3)\ell_0} L^{2^m} L p(t,x_0,y)^{1-\delta} (L^{-1} + p(t,x_0,y)^{1-\delta})^{2^m}.$$

This implies our assertion. ∎

Proposition 2.20 *For any $\delta\in(0,1/2)$, $T>0$ and $p\in[2,\infty)$, there is a $C>0$ such that*

$$E[(\sup_{y\in\mathbf{R}^N}\sup_{t\in[\varepsilon,T], s\in[0,t-\varepsilon]} (\frac{|q_{s,t}^{(L)}(y) - p(t,x_0,y)|}{(L^{-1/(1-\delta)} + p(t,x_0,y))^{(1-\delta)/2}})^p]^{1/p} \leqq C\varepsilon^{-5\ell_0} L^{-1/2+1/p}, \quad L\geqq 1,\ \varepsilon\in(0,1).$$

Proof. Let us take an $m \geqq 1$ such that $p + N < 2^m$. Note that

$$L^{-1} + p(t, x_0, y)^{1-\delta} \leqq 2(L^{-1/(1-\delta)} + p(t, x_0, y))^{1-\delta}.$$

Let

$$\rho_L(s,t,y) = \frac{q_{s,t}^{(L)}(y) - p(t,x_0,y)}{(L^{-1/(1-\delta)} + p(t,x_0,y))^{(1-\delta)/2}}, \qquad 0 \leqq s < t \leqq T,\ y \in \mathbf{R}^N.$$

We see by Proposition 3.7, we see that for any $a > 0$, $j \geqq 0$, and $\alpha \in \mathbf{Z}^N_{\geqq 0}$, there is a $C > 0$ such that

$$(L^{-1/(1-\delta)} + p(t,x_0,y))^{-a+2j+|\alpha|} |\partial_t^j \partial_y^\alpha ((L^{-1/(1-\delta)} + p(t,x_0,y))^{-a})|$$

$$\leqq C t^{-(2j+|\alpha|)\ell_0} p(t,x_0,y)^{-\delta}, \qquad y \in \mathbf{R}^N,\ t \in (0,T].$$

So we see that by Proposition 2.18, for any $a > 0$, $j = 0, 1$, and $\alpha \in \mathbf{Z}^N_{\geqq 0}$ with $|\alpha| \leqq 1$, there is a $C > 0$ such that

$$E[\sup_{s\in[0,t-\varepsilon]} |\partial_t^j \partial_y^\alpha \rho_L(s,t,y)|^{2^{m+1}}]$$

$$\leqq C\varepsilon^{-2^{m+1}4\ell_0} L^{-2^m} L p(t,x_0,y)^{1-2\delta}, \qquad y \in \mathbf{R}^N,\ L \geqq 1,\ \varepsilon \in (0,1),\ t \in (\varepsilon, T].$$

Therefore we see that

$$E[\int_{\mathbf{R}^N} dy \sup_{s\in[0,t-\varepsilon]} |\partial_t^j \partial_y^\alpha \rho_L(s,t,y)|^{2^{m+1}}]$$

$$\leqq C\varepsilon^{-2^{m+3}\ell_0} L^{-2^m} L \int_{\mathbf{R}^N} p(t,x_0,y)^{1-2\delta} dy, \qquad L \geqq 1,\ \varepsilon \in (0,1),\ t \in (\varepsilon, T].$$

Note by Proposition 3.6 that there is a $C > 0$ such that

$$\int_{\mathbf{R}^N} p(t,x_0,y)^{1-2\delta} dy \leqq C t^{(N+1)\ell_0\delta}, \qquad t \in (0,T].$$

Also, note that

$$\partial_y^\alpha \rho_L(s,t,y) = \partial_y^\alpha \rho_L(s,T,y) - \int_t^T \partial_r \partial_y^\alpha \rho_L(s,r,y) dr,$$

and so we see that

$$\sup_{t\in[\varepsilon,T], s\in[0,t-\varepsilon]} \int_{\mathbf{R}^N} |\partial_y^\alpha \rho_L(s,t,y)|^{2^m} dy$$

$$\leqq 2^{m+1} \int_{\mathbf{R}^N} dy \sup_{s\in[0,T-\varepsilon]} |\partial_y^\alpha \rho_L(s,T,y)|^{2^m}$$

$$+ 2^{m+1}(T+1)^{2^m} \int_t^T dr \int_{\mathbf{R}^N} dy \sup_{s\in[0,r-\varepsilon)} |\partial_y^\alpha \rho_L(s,r,y)|^{2^m}.$$

Then by Sobolev's inequality, we see that there is a $C > 0$ such that

$$E[\sup_{y\in\mathbf{R}^N} \sup_{t\in[\varepsilon,T], s\in[0,t-\varepsilon]} |\rho_L(s,t,y)|^{2^{m+1}}]^{1/2^{m+1}} \leqq C\varepsilon^{-(4+(N+1)/2^{m+1})\ell_0} L^{-1/2+1/2^{m+1}},$$

$L \geqq 1,\ \varepsilon \in (0,1)$. This implies our assertion. ∎

Let

$$Z_L(s,t;\delta) = \sup_{y \in \mathbf{R}^N} \frac{|q_{s,t}^{(L)}(y) - p(t,x_0,y)|}{(L^{-1/(1-\delta)} + p(t,x_0,y))^{(1-\delta)/2}}, \qquad t>0,\ s \in [0,t)$$

and

$$\tilde{Z}_L(\varepsilon,\delta,T) = \sup_{t \in [\varepsilon,T], s \in [0,t-\varepsilon]} Z_L(s,t;\delta)$$

for $T>0$, $\varepsilon \in (0,T]$, and $\delta \in (0,1)$. Note that $Z_L(s,t;\delta)$ is $\mathcal{F}_s^{(\infty)}$-measurable.

Then we have the following.

Proposition 2.21 (1) *Let $T>0$, $\varepsilon \in (0,T]$, and $\delta \in (0,1)$. Then for any $p>1$, there is a $C>0$ such that*

$$E[(L^{(1-\delta^2)/2}\tilde{Z}_L(\varepsilon,\delta,T))^p]^{1/p} \leqq C\varepsilon^{-5\ell_0}L^{-p\delta^2/2+1/p}, \qquad L \geqq 1.$$

(2)*Let $\delta \in (0,1)$, $t>0$, and $s \in (0,t)$. If $L^{(1-\delta^2)/2}Z_L(s,t;\delta) \leqq 1/4$, and $p(t,x_0,y) \geqq L^{-(1-\delta)}$, then*

$$\frac{1}{2} \leqq \frac{q_{s,t}^{(L)}(y)}{p(t,x_0,y)} \leqq 2, \qquad t \in (\varepsilon,T],\ s \in [0,t-\varepsilon]..$$

Proof. The assertion (1) is an immediate consequence of Proposition 2.20. Note that

$$|q_{s,t}^{(L)}(y) - p(t,x_0,y)| \leqq Z_L(s,t;\delta)(L^{-1/(1-\delta)} + p(t,x_0,y))^{(1-\delta)/2}$$

. for any $y \in \mathbf{R}^N$, $t \in [\varepsilon,T]$ and $s \in [0,t-\varepsilon]$.

If $p(t,x_0,y) \geqq L^{-(1-\delta)}$, we have

$$|\frac{q_{s,t}^{(L)}(y)}{p(t,x_0,y)} - 1| \leqq Z_L(s,t;\delta)(L^{-1/(1-\delta)}p(t,x_0,y)^{-1} + 1)^{(1-\delta)/2}p(t,x_0,y)^{-(1+\delta)/2}$$

$$\leqq Z_L(s,t;\delta)(L^{-1/(1-\delta)}L^{1-\delta} + 1)^{(1-\delta)/2}L^{(1-\delta^2)/2} \leqq 2L^{(1-\delta^2)/2}Z_L(s,t;\delta).$$

This implies our second assertion. ∎

Proposition 2.22 *Let $T>0$, and $\delta \in (0,1)$. Let $B_L(s,t) \in \mathcal{F}$, $L \geqq 1$, be given by*

$$B_L(s,t) = \{\omega \in \Omega :\ L^{(1-\delta^2)/2}Z_L(s,t;\delta) \leqq 1/4\}, \qquad t > ands \in (0,t),$$

and $\varphi_{t,L} : E \to \{0,1\}$, $t \in (0,T]$, $L \geqq 1$, be given by

$$\varphi_{t,L} = 1_{\{y \in E; p(t,x_0,y) > L^{-(1-\delta)}\}}, \qquad t>0.$$

(1) *Let $a \in (1/(2N), 1/2)$, $b \in (a - 1/(2N), a)$, and $m \geqq 1$. Then there is a $C>0$ such that*

$$1_{B_L(s,t)}E[(\sup_{x \in E} p(s,x_0,x)^a|(Q_{s,t}^{(L)}(\varphi_{t,L}f))(x) - (P_{s,t}(\varphi_{t,L}f))(x)|)^2|\mathcal{F}_s^{(\infty)}]$$

$$\leqq \frac{C}{L}s^{-(N+2)\ell_0}(t-s)^{-(N+2)\ell_0}\int_E p(t,x_0,y)^{-1+2b}(1+|y|^2)^{-m}\varphi_{t,L}(y)f(y)^2dy \qquad a.s.$$

for $t \in (0,T]$, $s \in (0,t)$, $L \geqq 1$, *and any bounded measurable function* f *defined in* E.
(2) *Let* $a \in (0,1/2)$, *and* $m \geqq 1$. *Then there is a* $C > 0$ *such that*

$$1_{B_L(s,t)} E[(\sup_{x\in E} p(s,x_0,x)^{1/2-\delta/4} |Q_{s,t}^{(L)}(\varphi_{t,L} f))(x) - (P_{s,t}(\varphi_{t,L} f))(x)|)^2 | \mathcal{F}_s^{(\infty)}]$$

$$\leqq \frac{C}{L^{1-\delta}} s^{-(N+2)\ell_0} (t-s)^{-(N+2)\ell_0} \int_E (1+|y|^2)^{-m} \varphi_{t,L}(y) f(y)^2 dy \qquad a.s.$$

for $t \in (0,T]$, $s \in (0,t)$, $L \geqq 1$, *and any bounded measurable function* f *defined in* E.
(3) *Let* $a \in (0,1/2)$ *and* $b \in (a - 1/(2N), a)$. *Then there is a* $C > 0$ *such that*

$$1_{B_L(s,t)} E[(\sup_{x\in E} p(s,x_0,x)^a |Q_{s,t}^{(L)}(\varphi_{t,L} p(t,x_0,\cdot)^{-b}))(x) - (P_{s,t}(\varphi_{t,L} p(t,x_0,\cdot)^{-b}))(x)|)^2 | \mathcal{F}_s^{(\infty)}]$$

$$\leqq \frac{C}{L^\delta} s^{-(N+2)\ell_0} (t-s)^{-(N+2)\ell_0} \qquad a.s.$$

for $t \in (0,T]$, $s \in (0,t)$, $L \geqq 1$.

Proof. Note that for $\alpha \in \mathbf{Z}^N_{\geqq 0}$

$$1_{B_L(s,t)} E[|\partial_x^\alpha (p(s,x_0,x)^a (Q_{s,t}^{(L)}(\varphi_{t,L} f))(x) - (P_{s,t}(\varphi_{t,L} f))(x)))|^2 \mathcal{F}_s^{(\infty)}]$$

$$\leqq \frac{1}{L} 1_{B_L(s,t)} \int_E \frac{|\partial_x^\alpha (p(s,x_0,x)^a p(t-s,x,y))|^2}{q_{s,t}^{(L)}(y)} \varphi_{t,L(y)} f(y)^2 \, dy$$

$$\leqq \frac{2}{L} 1_{B_L(s,t)} \int_E |\partial_x^\alpha (p(s,x_0,x)^a p(t-s,x,y))|^2 p(t,x_0,y)^{-1} \varphi_{t,L(y)} f(y)^2 \, dy$$

So we have by Proposition 3.10 there is a $C > 0$ such that

$$1_{B_L(s,t)} E[\int_{\mathbf{R}^N} dx \, |\partial_x^\alpha (p(s,x_0,x)^a (Q_{s,t}^{(L)}(\varphi_{t,L} f))(x) - (P_{s,t}(\varphi_{t,L} f))(x)))|^2 \mathcal{F}_s^{(\infty)}]$$

$$\leqq \frac{C}{L} s^{-(N+2)\ell_0} (t-s)^{-(N+2)\ell_0} \int_E p(t,x_0,y)^{-1+2b} (1+|y|^2)^{-m} \varphi_{t,L}(y) f(y)^2 dy.$$

This and Sobolev's inequality imply the assertion (1).

In the assertion (1), if $a = 1 - \delta/4$ and $b > 1/2 - \delta/2$, then we have

$$p(t,x_0,y)^{-1+2b} \varphi_{t,L}(y) \leqq L^{-\delta}.$$

This implies the assertion (2).

In the assertion (1), if $m = N+1$ and $f = p(t,x_0,\cdot)^{-b}$ then we have

$$\int_E p(t,x_0,y)^{-1+2b} (1+|y|^2)^{-m} \varphi_{t,L}(y) f(y)^2 dy \leqq L^{1-\delta} \int_{\mathbf{R}^N} (1+|y|^2)^{-(N+1)} dy$$

This implies the assertion (3). ∎

Similarly by using Proposition 3.10, we have the following.

Proposition 2.23 *Let* $a \in (1/(2N), 1/2)$ *and* $b \in (a - 1/(2N), a)$. *Then there is a* $C > 0$ *such that*

$$\sup_{x\in E} p(s,x_0,x)^a |(P_{s,t} f)(x)|$$
$$\leqq C s^{-(N+2)\ell_0/2} (t-s)^{-(N+3)\ell_0/2} \sup_{y\in E} p(t,x_0,y)^b |f(y)|$$

for $t \in (0,T]$, $s \in (0,t)$, *and any bounded measurable function* f *defined in* E.

2.6 Application to Bermuda type problem

Let us think of the situation in Section 4. Then we have the following.

Theorem 2.24 *Let $0 = T_0 < T_1 < \ldots < T_n < T$, $\delta \in (0, 1/2)$, and $f \in \mathcal{B}_r$, for some $r \geqq 0$. Then there are $C > 0$, $\Omega^L \in \mathcal{F}$, $L \geqq 1$, and measurable functions $d_{m,i}^{(L)} : E \times \Omega \to [0, \infty)$, $m = 1, \ldots, n-1$, $i = 1, 2$, $L \geqq 1$, such that*

$$\lim_{L\to\infty} L^p(1 - P(\Omega^L)) = 0, \qquad p \in (1, \infty),$$

$$1_{\Omega^L}|(\tilde{Q}^{(L)}_{T_m,T_{m+1}} \cdots \tilde{Q}^{(L)}_{T_{n-1},T_n} f)(x) - (\tilde{P}_{T_m,T_{m+1}} \cdots \tilde{P}_{T_{n-1},T_n} f)(x)|$$
$$\leqq d^{(L)}_{m,1}(x) + d^{(L)}_{m,2}(x), \qquad x \in E, \ m = 1, \ldots, n-1, \ L \geqq 1$$

and

$$E[\int_E d^{(L)}_{m,1}(x) p(T_m, x_0, x) dx] \leqq CL^{-(1-\delta)^2}$$
$$E[\int_E d^{(L)}_{m,2}(x)^2 p(T_m, x_0, x) dx] \leqq CL^{-(1-\delta)}$$

for any $L \geqq 1$, $m = 1, \ldots, n-1$.

Proof. Note that for $f, g \in \mathcal{B}_{r'}$

$$|(\tilde{Q}^{(L)}_{s,t} f)(x) - (\tilde{Q}^{(L)}_{s,t} g)(x)|$$
$$= |\phi_{s,t}(x, (Q^{(L)}_{s,t} f)(x)) - \phi_{s,t}(x, (Q^{(L)}_{s,t} g)(x))|$$
$$\leqq \exp(\lambda(t-s))(Q^{(L)}_{s,t}(|f-g|))(x)$$

So we see that

$$|(\tilde{Q}^{(L)}_{T_m,T_{m+1}} \cdots \tilde{Q}^{(L)}_{T_{k-1},T_k} f)(x) - (\tilde{Q}^{(L)}_{T_m,T_{m+1}} \cdots \tilde{Q}^{(L)}_{T_{k-1},T_k} g)(x)|$$
$$\leqq \exp(\lambda(T_k - T_m))(Q_{T_m,T_{m+1}} \cdots Q^{(L)}_{T_{k-1}.T_k}(|f-g|))(x)$$

Similary we have

$$|(\tilde{Q}^{(L)}_{s,t} f)(x) - (\tilde{P}_{s,t} g)(x)| \leqq \exp(\lambda(t-s))|(Q^{(L)}_{s,t} f)(x) - (P_{s,t} g)(x)|$$

Let us take a_k, $k = 0, 1, \ldots, n$ such taht $1/2 > a_0 > a_1 > \ldots > a_n > 1/2 - \delta$. Also, let

$$c_m(x) = (\tilde{P}_{T_m,T_{m+1}} \cdots \tilde{P}_{T_{n-1},T_n} f)(x).$$

Note that

$$|(\tilde{Q}^{(L)}_{T_m,T_{m+1}} \cdots \tilde{Q}^{(L)}_{T_{n-1},T_n} f)(x) - (\tilde{P}_{T_m,T_{m+1}} \cdots \tilde{P}_{T_{n-1},T_n} f)(x)|$$

$$\leqq \sum_{k=1}^{n-m} |(\tilde{Q}^{(L)}_{T_m,T_{m+1}} \cdots \tilde{Q}^{(L)}_{T_{m+k-1},T_{m+k}} \tilde{P}_{T_{m+k},T_{m+k+1}} \cdots \tilde{P}_{T_{n-1},T_n} f)(x)$$
$$-(\tilde{Q}^{(L)}_{T_m,T_{m+1}} \cdots \tilde{Q}^{(L)}_{T_{m+k-2},T_{m+k-1}} \tilde{P}_{T_{m+k-1},T_{m+k}} \cdots \tilde{P}_{T_{n-1},T_n} f)(x)|$$

$$\leqq \exp(\lambda T)\sum_{k=1}^{n-m}(Q^{(L)}_{T_m,T_{m+1}}\cdots Q^{(L)}_{T_{m+k-2},T_{m+k-1}}(|Q^{(L)}_{T_{m+k-1},T_{m+k}}c_{m+k}-P_{T_{m+k-1},T_{m+k}}c_{m+k}|))(x)$$

Let

$$R_k = 1_{B_L(T_{k-1},T_k)}\sup_{x\in E} p(T_{k-1},x_0,x)^{a_{k-1}}(|Q^{(L)}_{T_{k-1},T_k}(\varphi_{T_k,L}c_k) - P_{T_{k-1},T_k}(\varphi_{T_k,L}c_k)|)(x),$$

$$Z_k$$

$$= 1_{B_L(T_{k-1},T_k)}\sup_{x\in E} p(T_{k-1},x_0,x)^{a_{k-1}}(|Q^{(L)}_{T_{k-1},T_k}(\varphi_{T_k,L}p(T_k,x_0,\cdot)^{-a_k}) - P_{T_{k-1},T_k}(\varphi_{T_k,L}p(T_k,x_0,\cdot)^{-a_k})|)(x),$$

and

$$D_k = \sup_{x\in E} p(T_{k-1},x_0,x)^{a_{k-1}}(P_{T_{k-1},T_k}(\varphi_{T_k,L}p(T_k,x_0,\cdot)^{-a_k}))(x) < \infty, \qquad k=1,\ldots,n.$$

Then R_k and Z_k are $\mathcal{F}^{(\infty)}_{T_k}$-measurable for $k=1,\ldots,n$, and by Proposition 2.22 we see that there is a $C>0$ such that

$$E[R_k^2|\mathcal{F}^{(\infty)}_{T_{k-1}}] \leqq CL^{-1}, \qquad E[Z_k^2|\mathcal{F}^{(\infty)}_{T_{k-1}}] \leqq CL^{-(1-\delta)}$$

for any $L \geqq 1$, and $k=1,\ldots,n$. So inductively we have

$$E[R_k^2(\prod_{i=\ell+1}^{k}(Z_i+D_i)^2)|\mathcal{F}^{(\infty)}_{T_\ell}] \leqq 2^{k-\ell}C^{k+1-\ell}L^{-1}\prod_{i=\ell+1}^{k}(D_i^2 + CL^{-(1-\delta)})$$

for any $L \geqq 1$, and $1 \leqq \ell \leqq k \leqq n$. Let $\Omega^L = \bigcap_{k=1}^n B_L(T_{k-1},T_k)$. Then we have

$$1_{\Omega^L}Q^{(L)}_{T_{k-1},T_k}(p(T_k,x_0,\cdot)^{-a_k})(x)$$

$$= 1_{\Omega^L}Q^{(L)}_{T_{k-1},T_k}(\varphi_{T_k,L}p(T_k,x_0,\cdot)^{-a_k})(x) + Q^{(L)}_{T_{k-1},T_k}((1-\varphi_{T_k,L})p(T_k,x_0,\cdot)^{-a_k})(x)$$

$$\leqq 1_{\Omega^L}(Z_k+D_k)p(T_{k-1},x_0,x)^{-a_{k-1}} + 1_{\Omega^L}Q^{(L)}_{T_{k-1},T_k}((1-\varphi_{T_k,L})p(T_k,x_0,\cdot)^{-a_k})(x)$$

Therefore we have

$$1_{\Omega^L}(Q^{(L)}_{T_m,T_{m+1}}\cdots Q^{(L)}_{T_{m+k-2},T_{m+k-1}}(|Q^{(L)}_{T_{m+k-1},T_{m+k}}c_{m+k} - P_{T_{m+k-1},T_{m+k}}c_{m+k}|)(x)$$

$$\leqq 1_{\Omega^L}R_{m+k}(Q^{(L)}_{T_m,T_{m+1}}\cdots Q^{(L)}_{T_{m+k-2},T_{m+k-1}}p(T_{m+k-1},x_0,\cdot)^{-a_{m+k-1}})(x)$$

$$+1_{\Omega^L}(Q^{(L)}_{T_m,T_{m+1}}\cdots Q^{(L)}_{T_{m+k-2},T_{m+k-1}}(Q^{(L)}_{T_{m+k-1},T_{m+k}}((1-\varphi_{T_{m+k},L})|c_{m+k}|)$$

$$+P_{T_{m+k-1},T_{m+k}}((1-\varphi_{T_{m+k},L})|c_{m+k}|)))(x)$$

$$\leqq \tilde{d}_{m,2}(x) + \tilde{d}_{m.1}(x),$$

where

$$\tilde{d}^{(L)}_{m,2}(x) = R_{m+k}(\prod_{i=1}^{k}(Z_{m+i}+D_{m+i}))p(T_m,x_0,x)^{-a_m})$$

and

$$\tilde{d}^{(L)}_{m,1}(x)$$

$$= \sum_{\ell=1}^{k} R_{m+k}(\prod_{i=m+\ell+1}^{m+k}(Z_i+D_i))(Q^{(L)}_{T_m,T_{m+1}}\cdots Q^{(L)}_{T_{m+\ell-2},T_{m+\ell-1}}((1-\varphi_{T_{m+\ell-1},L})p(T_{m+\ell-1},x_0,\cdot)^{a_{m+\ell-1}}))(x))$$

$$+(Q^{(L)}_{T_m,T_{m+1}}\cdots Q^{(L)}_{T_{m+k-2},T_{m+k-1}}(Q^{(L)}_{T_{m+k-1},T_{m+k}}((1-\varphi_{T_{m+k},L})|c_{m+k}|)$$

$$+P_{T_{m+k-1},T_{m+k}}((1-\varphi_{T_{m+k},L})|c_{m+k}|)))(x)$$

Note that

$$E[R_{m+k}(\prod_{i=m+\ell+1}^{m+k}(Z_i+D_i))(Q^{(L)}_{T_m,T_{m+1}}\cdots Q^{(L)}_{T_{m+\ell-2},T_{m+\ell-1}}((1-\varphi_{T_{m+\ell-1},L})p(T_{m+\ell-1},x_0,\cdot)^{a_{m+\ell-1}})(x))]$$

$$= E[E[R_{m+k}(\prod_{i=m+\ell+1}^{m+k}(Z_i+D_i))|\mathcal{F}^{(\infty)}_{T_{m+\ell}}]$$

$$\times(Q^{(L)}_{T_m,T_{m+1}}\cdots Q^{(L)}_{T_{m+\ell-2},T_{m+\ell-1}}((1-\varphi_{T_{m+\ell-1},L})p(T_{m+\ell-1},x_0,\cdot)^{a_{m+\ell-1}})(x))]$$

$$\leqq (2^k C^{k+1} L^{-1}\prod_{i=m+\ell+1}^{m+k}(D_i^2+CL^{-(1-\delta)}))^{1/2}$$

$$\times E[Q^{(L)}_{T_m,T_{m+1}}\cdots Q^{(L)}_{T_{m+\ell-2},T_{m+\ell-1}}((1-\varphi_{T_{m+\ell-1},L})p(T_{m+\ell-1},x_0,\cdot)^{a_{m+\ell-1}})(x))]$$

$$= (2^k C^{k+1} L^{-1}\prod_{i=m+\ell+1}^{m+k}(D_i^2+CL^{-(1-\delta)}))^{1/2} P_{T_m,T_{m+\ell-1}}((1-\varphi_{T_{m+\ell-1},L})p(T_{m+\ell-1},x_0,\cdot)^{a_{m+\ell-1}})(x).$$

Note that for $a \geqq 0$

$$\int_E P_{T_m,T_{m+\ell-1}}((1-\varphi_{T_{m+\ell-1},L})p(T_{m+\ell-1},x_0,\cdot)^{a})(x)p(T_m,x_0,x)dx$$

$$= \int_E 1_{\{p(T_{m+\ell-1},x_0,x)\leqq L^{-(1-\delta)}\}} p(T_{m+\ell-1},x_0,x)^{1+a}dx$$

$$\leqq L^{-(1-\delta)^2}\int_E p(T_{m+\ell-1},x_0,x)^{\delta+a}dx.$$

Then we have our assertion. ∎

2.7 re-simulation

We think of application to pricing Bermuda derivatives.

Let $r \geqq 1$ and let $g : [0,T]\times \mathbf{R}^N \to \mathbf{R}$ be a continuous function such that

$$\sup_{x\in\mathbf{R}^N,\, t\in[0,T]} (1+|x|^2)^{-r/2}|g(t,x)| < \infty.$$

Let $\phi_{s,t}(x,y) = g(s,x)\vee y$, $0 \leqq s < t \leqq T$, $x \in \mathbf{R}^N$ and $y \in \mathbf{R}$. Let $0 = T_0 < T_1 < \ldots < T_n < T$, and let $c_m : E \to \mathbf{R}$, $m = 0,1,\ldots,n$, be given by

$$c_m(x) = (\tilde{P}_{T_m,T_{m+1}}\cdots\tilde{P}_{T_{n-1},T_n} g(T_n,\cdot))(x),\ m \leqq n-1,\ \text{and}\ c_n(x) = g(T_n.x).$$

Now let $\tilde{c}_m : E \to \mathbf{R}$, $m = 1, \dots, n-1$, be given and let $\tilde{c}_n = g(T_n, \cdot)$. We regard $\tilde{c}_m$ as estimators of c_m, $m = 1, \dots, n$.

Let us think of the SDE in Introduction. Let $\tau : W_0 \to \{T_1, \dots, T_n\}$ and $\tilde{\tau} : W_0 \to \{T_1, \dots, T_n\}$ be stopping times given by

$$\tau = \min\{T_k;\ c_k(X(T_k, x_0)) \leqq g(T_k, X(T_k, x_0),\ k = 1, \dots, n\}$$

and

$$\tilde{\tau} = \min\{T_k;\ \tilde{c}_k(X(T_k, x_0)) \leqq g(T_k, X(T_k, x_0),\ k = 1, \dots, n\}$$

Let $\bar{c}_m$, $m = 0, \dots, n$, be given by inductively, $\bar{c}_n = g(T_n, \cdot)$, and

$$\bar{c}_{m-1} = P_{T_{m-1}, T_m}(g(T_m, \cdot)1_{\{\tilde{c}_m \leqq g(T_m, \cdot)\}} + \bar{c}_m 1_{\{\tilde{c}_m > g(T_m, \cdot)\}}), \quad m = n, n-1, \dots, 1.$$

Then we have the following.

Proposition 2.25 (1) *For* $m = 0, 1, \dots, n-1$,

$$E^\mu[g(\tau, X(\tau, x_0)|\mathcal{B}_{T_m}]1_{\{\tau \geqq T_{m+1}\}} = c_m(X(T_m, x_0))1_{\{\tau \geqq T_{m+1}\}} \ a.s.$$

and

$$E^\mu[g(\tilde{\tau}, X(\tilde{\tau}, x_0)|\mathcal{B}_{T_m}]1_{\{\tilde{\tau} \geqq T_{m+1}\}} = \bar{c}_m(X(T_m, x_0))1_{\{\tilde{\tau} \geqq T_{m+1}\}} \ a.s.$$

Here $\mathcal{B}_t = \sigma\{B^i(s);\ s \leqq t,\ i = 1, \dots, d\}$.
(2) *For* $m = 0, 1, \dots, n-1$, *and* $x \in E$,

$$0 \leqq c_m(x) - \bar{c}_m(x) \leqq P_{T_m, T_{m+1}}(|c_{m+1} - \tilde{c}_{m+1}|)(x) + P_{T_m, T_{m+1}}(1_{\{\tilde{c}_{m+1} > g_{m+1}\}}(c_{m+1} - \bar{c}_{m+1}))(x).$$

In particular,

$$0 \leqq c_m(x) - \bar{c}_m(x) \leqq \sum_{k=m+1}^{n} P_{T_m, T_k}(|c_k - \tilde{c}_k|)(x), \qquad m = 0, 1, \dots, n.$$

Proof. Since we have

$$\begin{aligned} &E^\mu[g(\tilde{\tau}, X(\tilde{\tau}, x_0)|\mathcal{B}_{T_{m-1}}]1_{\{\tilde{\tau} \geqq T_m\}} \\ &= E^\mu[E^\mu[g(\tilde{\tau}, X(\tilde{\tau}, x_0)1_{\{\tilde{\tau} \geqq T_{m+1}\}}|\mathcal{B}_{T_m}] + g(T_m, X(T_m, x_0))1_{\{\tilde{\tau} = T_m\}}|\mathcal{B}_{T_{m-1}}], \end{aligned}$$

we can easily obtain the assertion (1) by induction.

Note that

$$\begin{aligned} &c_m - \bar{c}_m \\ &= P_{T_m, T_{m+1}}(1_{\{\tilde{c}_{m+1} \leqq g(T_{m+1}, \cdot)\}}((g(T_{m+1}, \cdot) \vee c_{m+1}) - g(T_{m+1}, \cdot))) \\ &\quad + P_{T_m, T_{m+1}}(1_{\{\tilde{c}_{m+1} > g(T_{m+1}, \cdot)\}}((g(T_{m+1}, \cdot) \vee c_{m+1}) - \bar{c}_{m+1})) \\ &= P_{T_m, T_{m+1}}(1_{\{\tilde{c}_{m+1} \leqq g(T_{m+1}, \cdot)\}}((g(T_{m+1}, \cdot) \vee c_{m+1}) - (g(T_{m+1}, \cdot) \vee \tilde{c}_{m+1}))) \\ &+ P_{T_m, T_{m+1}}(1_{\{\tilde{c}_{m+1} > g(T_{m+1}, \cdot)\}}((g(T_{m+1}, \cdot) - c_{m+1}) \vee 0) - ((g(T_{m+1}, \cdot) - \tilde{c}_{m+1}) \vee 0) + c_{m+1} - \bar{c}_{m+1})) \\ &\leqq P_{T_m, T_{m+1}}(|c_{m+1} - \tilde{c}_{m+1}|) + P_{T_m, T_{m+1}}(1_{\{\tilde{c}_{m+1} > g(T_{m+1}, \cdot)\}}(c_{m+1}) - \bar{c}_{m+1})) \end{aligned}$$

This implies the first inequlity of the assertion (2). The second inequality follows from this by induction. ∎

Proposition 2.26

$$c_0(x_0) - \bar{c}_0(x_0)$$

$$\leqq \sum_{k=1}^{n} \int_E (|\tilde{c}_k - \bar{c}_k| + |c_k - \bar{c}_k|)(x) 1_{\{|\tilde{c}_k - c_k| + |c_k - \bar{c}_k| \geqq \varepsilon\}}(x) + \varepsilon 1_{\{|g(T_k,\cdot) - c_k| < \varepsilon\}} p(T_k, x_0, x) dx$$

for any $\varepsilon > 0$.

Proof. Note that

$$c_0(x_0) - \bar{c}_0(x_0) = E^{\mu}[g(\tau, X(\tau, x_0)) - g(\tilde{\tau}, X(\tilde{\tau}, x_0))]$$

$$= E^{\mu}[g(\tau, X(\tau, x_0)) - g(\tilde{\tau}, X(\tilde{\tau}, x_0)), \tau > \tilde{\tau}] + E^{\mu}[g(\tau, X(\tau, x_0)) - g(\tilde{\tau}, X(\tilde{\tau}, x_0)), \tau < \tilde{\tau}]$$

$$= E^{\mu}[E^{\mu}[g(\tau, X(\tau, x_0))|\mathcal{B}_{\tilde{\tau}}] - g(\tilde{\tau}, X(\tilde{\tau}, x_0), \tau > \tilde{\tau}]$$

$$+ E^{\mu}[g(\tau, X(\tau, x_0) - E^{\mu}[g(\tilde{\tau}, X(\tilde{\tau}, x_0))|\mathcal{B}_{\tau}], \tau < \tilde{\tau}]$$

$$= \sum_{k=1}^{n-1} (E^{\mu}[c_k(X(T_k, x_0)) - g(T_k, X(T_k, x_0)), \tau > T_k, \tilde{\tau} = T_k]$$

$$+ E^{\mu}[g(T_k, X(k, x_0)) - \bar{c}_k(X(T_k, x_0)), \tau = T_k, T_k < \tilde{\tau}])$$

$$\leqq \sum_{k=1}^{n-1} (E^{\mu}[(c_k(X(T_k, x_0)) - g(T_k, X(T_k, x_0)) 1_{\{\tilde{c}_k \leqq g(T_k,\cdot) < c_k\}}(X(T_k, x_0))]$$

$$+ E^{\mu}[((g(T_k, X(k, x_0)) - \bar{c}_k(X(T_k, x_0)))) \vee 0) 1_{\{c_k \leqq g(T_k,\cdot) < \tilde{c}_k\}}(X(T_k, x_0))]).$$

For any $\varepsilon > 0$, we see that

$$(c_k - g(T_k, \cdot)) 1_{\{\tilde{c}_k \leqq g(T_k,\cdot) < c_k\}}$$

$$\leqq \varepsilon 1_{\{g(T_k,\cdot) < c_k \leqq g(T_k,\cdot) + \varepsilon\}} + (c_k - g(T_k, \cdot)) 1_{\{\tilde{c}_k \leqq g(T_k,\cdot) < c_k\}} 1_{\{g_k + \varepsilon < c_k\}}$$

$$\leqq \varepsilon 1_{\{g(T_k,\cdot) < (T_k,\cdot) + \varepsilon\}} + (c_k - \tilde{c}_k) 1_{\{c_k - \tilde{c}_k > \varepsilon\}} 1_{\{\tilde{c}_k \leqq g(T_k,\cdot) < c_k\}},$$

and

$$((g(T_k, \cdot) - \bar{c}_k) \vee 0) 1_{\{c_k \leqq g(T_k,\cdot) < \tilde{c}_k\}}$$

$$\leqq ((g(T_k, \cdot) - \bar{c}_k) \vee 0) 1_{\{c_k \leqq g(T_k,\cdot) < \tilde{c}_k\}} 1_{\{|\tilde{c}_k - c_k| + |c_k - \bar{c}_k| \geqq \varepsilon\}} + ((g(T_k, \cdot) - \bar{c}_k) \vee 0) 1_{\{c_k \leqq g(T_k,\cdot) < \tilde{c}_k\}} 1_{\{|\tilde{c}_k - c_k| + |c_k - \bar{c}_k| <}$$

$$\leqq (|\tilde{c}_k - c_k| + |c_k - \bar{c}_k|) 1_{\{|\tilde{c}_k - c_k| + |c_k - \bar{c}_k| \geqq \varepsilon\}} 1_{\{c_k \leqq g(T_k,\cdot) < \tilde{c}_k\}} + \varepsilon 1_{\{c_k \leqq g(T_k,\cdot) < c_k + \varepsilon\}}$$

So we have our assertion. ∎

Now we have the following.

Lemma 2.27 *Let* $d_{m,i} : E \to [0, \infty)$. $m = 1, \ldots, n$, $i = 1, 2$, *be measurable functions. Assume that* $|\tilde{c}_m - c_m| \leqq d_{m,1} + d_{m,2}$, $m = 1, \ldots, n$. *Then we have the following.*

$$c_0(x_0) - \bar{c}_0(x_0)$$

$$\leqq n \sum_{k=1}^{n} \int_E d_{k,1}(x) p(T_k, x_0, x) dx + n (\sum_{k=1}^{n} (\int_E d_{k,2}(x)^2 p(T_k, x_0, x) dx)^{1/2})$$

$$\times(\varepsilon^{-1/2}(\sum_{k=1}^{n}\int_E d_{k,1}(x)p(T_k,x_0,x)dx)+\varepsilon^{-1}(\sum_{k=1}^{n}(\int_E d_{k,2}(x)^2p(T_k,x_0,x)dx)^{1/2}))$$
$$+2\varepsilon\sum_{k=1}^{n}\int_E 1_{\{|g(T_k,\cdot)-c_k|<2\varepsilon\}}p(T_k,x_0,x)dx$$

for any $\varepsilon>0$.

Proof. Let

$$\tilde{d}_{m,i}(x)=\sum_{k=m}^{n}(P_{T_m,T_k}d_{k,i})(x), m=1,\ldots,n.$$

Then by Proposition 2.25, we have

$$|\tilde{c}_m(x)-c_m(x)|+|\bar{c}_m(x)-c_m(x)|\leqq \tilde{d}_{m,1}(x)+\tilde{d}_{m,2}(x).$$

Note that

$$\int_E(\tilde{d}_{m,1}(x)+\tilde{d}_{m,2}(x))1_{\{\tilde{d}_{m,1}(x)+\tilde{d}_{m,2}(x)\geqq 2\varepsilon\}}(x)p(T_m,x_0,x)dx$$
$$\leqq \int_E\tilde{d}_{m,1}(x)p(T_m,x_0,x)dx+(\int_E\tilde{d}_{m,2}(x)^2p(T_m,x_0,x)dx)^{1/2}((\int_E 1_{\{\tilde{d}_{m,1}(x)\geqq\varepsilon\}}(x)p(T_m,x_0,x)dx)^{1/2}$$
$$+(\int_E 1_{\{\tilde{d}_{m,2}(x)\geqq\varepsilon\}}(x)p(T_m,x_0,x)dx)^{1/2})$$
$$\leqq \int_E\tilde{d}_{m,1}(x)p(T_m,x_0,x)dx+(\int_E\tilde{d}_{m,2}(x)^2p(T_m,x_0,x)dx)^{1/2}(\varepsilon^{-1/2}(\int_E\tilde{d}_{m,1}(x)p(T_m,x_0,x)dx)^{1/2}$$
$$+\varepsilon^{-1}(\int_E\tilde{d}_{m,2}(x)^2p(T_m,x_0,x)dx)^{1/2})$$

Also, note that

$$\int_E\tilde{d}_{m,1}(x)p(T_m,x_0,x)dx\leqq\sum_{k=m}^{n}\int_E d_{k,1}(x)p(T_k,x_0,x)dx,$$

and

$$(\int_E\tilde{d}_{m,2}(x)^2p(T_m,x_0,x)dx)^{1/2}\leqq\sum_{k=m}^{n}(\int_E d_{k,2}(x)^2p(T_k,x_0,x)dx)^{1/2}.$$

This and Proposition 2.26 imply our assertion. ∎

Now we apply this Lemma and the results in the previous section to a Bermuda derivative.

Let $\phi_{s,t}(x,y)=g(s,x)\vee y$, $0\leqq s<t\leqq T$, $x\in\mathbf{R}^N$ and $y\in\mathbf{R}$. Let $\tilde{c}_m:E\to\mathbf{R}$, $m=1,\ldots,n-1$, be given by

$$\tilde{c}_m(x)=(\tilde{Q}^{(L)}_{T_m,T_{m+1}}\cdots\tilde{Q}^{(L)}_{T_{n-1},T_n}g(T_n,\cdot))(x).$$

Then by Theorem 2.24, we see that for any $\delta\in(0,1/2)$, there are $\Omega'_L\in\mathcal{F}$, $L\geqq 1$, $C>0$ and measurable functions $d^{(L)}_{m,i}:E\times\Omega\to[0,\infty)$, $m=1,\ldots,n-1$, $i=1,2$, $L\geqq 1$, such that

$$\lim_{L\to\infty}P(\Omega'_L)=1,$$

$$|\tilde{c}_m(x) - c_m(x)| \leqq d_{m,1}(x) + d_{m,2}(x), \qquad x \in E,\ \omega \in \Omega'_L,\ m = 1,\ldots,n-1,\ L \geqq 1$$

and

$$E[\int_E d_{m,1}(x)p(T_m, x_0, x)dx] \leqq CL^{-(1-\delta)^2} \qquad m = 1,\ldots,n-1,\ L \geqq 1,$$

and

$$E[\int_E d_{m,2}(x)^2 p(T_m, x_0, x)dx] \leqq CL^{-(1-\delta)^2} \qquad m = 1,\ldots,n-1,\ L \geqq 1.$$

Let

$$\Omega''_L = \{\omega \in \Omega;\ \int_E d_{m,1}(x)p(T_m, x_0, x)dx \geqq L^{-(1-\delta)^3} \text{ or } \int_E d_{m,2}(x)^2 p(T_m, x_0, x)dx \geqq L^{-(1-\delta)^3}\}.$$

Then we see that

$$P(\Omega \setminus \Omega''_L) \leqq 2CL^{-(1-\delta)^2\delta}, \qquad L \geqq 1.$$

Let $\Omega_L = \Omega'_L \cap \Omega''_L$, $L \geqq 1$. Then we see that $P(\Omega_L) \to 1$, $l \to \infty$. So if we use these $\tilde{c}_m(x)$, $m = 1,\ldots,n-1$, as estimators and use the re-simulation method, we have

$$\begin{aligned} &c_0(x_0) - \bar{c}_0(x_0) \\ &\leqq n^2 L^{-(1-\delta)^3} + n^3 L^{-(1-\delta)^3/2}(\varepsilon^{-1/2}L^{-(1-\delta)^3/2} + \varepsilon^{-1}L^{-(1-\delta)^3/2}) \\ &\quad + \varepsilon \sum_{k=1}^{n} \int_E 1_{\{|g(T_k,\cdot)-c_k|<\varepsilon\}} p(T_k, x_0, x)dx \end{aligned}$$

for any $\varepsilon > 0$, $\omega \in \Omega_L$, and $L \geqq 1$. Suppose that

$$\sum_{k=1}^{n-1} \int_E 1_{\{|g(T_k,\cdot)-c_k|<\varepsilon\}} p(T_k, x_0, x)dx = O(\varepsilon^\gamma), \quad \varepsilon \downarrow 0,$$

for some $\gamma \in (0,1]$. Then letting $\varepsilon = L^{-(1-\delta)^3/(2+\gamma)}$, we see that $c_0(x_0) - \bar{c}_0(x_0) = O(L^{-(1-\delta)^3(1+\gamma)/(2+\gamma)})$ as $L \to \infty$.

Since δ is arbitrary, this proves Theorem 3.18.

第3章　Least Square Regression methods for Bermudan Derivatives and systems of functions

Least square regression methods are Monte Carlo methods to solve non-linear problems related to Markov processes and are widely used in practice. In these methods, first we choose a system of functions to approximate value functions. So one of questions on these methods is what kinds of systems of functions one has to take to get a good approximation. In the present paper, we will discuss on this problem.

3.1 Introduction

Least square regression methods are Monte Carlo methods to solve non-linear problems related to Markov processes. These methods were introduced by Longstaff-Schwartz [18] and Tsitsiklis-Van Roy[20] and are widely used in practice. There are many works related to this methods. Concerning the applications for pricing Bermudan derivatives, the convergence to a real price was proved by Clement-Lamberton-Protter [7] and rate of convergence was studied by Belomestny [4]. In these methods, first we choose a system of functions to approximate value functions. So one of questions on these methods is what kinds of systems of functions one has to take to get a good approximation. In the present paper, we will discuss on this problem. Related topics have been discussed by Gobet-Lemor-Warin [10] and Bally-Pagés [3].

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $M \geqq 1$, and $\{\mathcal{G}_m\}_{m=0}^M$ be a filtration on $(\Omega, \mathcal{F}, P)$. Let $(E, \mathcal{B})$ a measurable space and $m(E)$ be the set of Borel measurable functions on E. Let $p_m : E \times \mathcal{B} \to [0,1]$, $m = 0, \ldots, M-1$, be such that $p_m(x, \cdot) : \mathcal{B} \to [0,1]$ is a probability measure on E for any $x \in E$, and $p_m(\cdot, A) : E \to [0,1]$ is $\mathcal{B}$-measurable for any $A \in \mathcal{B}$. Let $x_0 \in E$ and fix it throughout. Let $X : \{0, 1, \ldots, M\} \times \Omega \to E$ be an E-valued process such that $X_0 = x_0$, $X_m : \Omega \to E$ is $\mathcal{G}_m$-measurable, $m = 0, \ldots, M$, and

$$P(X_{m+1} \in A | \mathcal{G}_m) = p_m(X_m, A) \ a.s. \qquad A \in \mathcal{B},\ m = 0, \ldots, M-1.$$

So X is a Markov process starting from x_0 whose transition probability is given by $p_m(x, dy)$.

Let ν_m, $m = 1, \ldots, M$, be the probability law of X_m, $m = 0, 1, \ldots, M$. Then ν_0 is the probability measure concentrated in x_0, and

$$\nu_{m+1}(A) = \int_E p_m(x, A) \nu_m(dx), \qquad y \in E, m = 0, 1, \ldots, M-1.$$

Let $P_m : L^2(E; d\nu_{m+1}) \to L^2(E; d\nu_m)$, $m = 0, 1, \dots, M-1$, be a linear operator given by

$$(P_m f)(x) = \int_E p_m(x, dy) f(y), \qquad f \in L^2(E; d\nu_{m+1}).$$

Now let $f_m \in L^4(E; d\nu_m)$, $m = 1, 2, \dots, M$. We define $\tilde{f}_m$, $\tilde{f}^*_m \in L^4(E; d\nu_m)$, $m = 0, 1, 2, \dots, M$, inductively by the following.

$$\tilde{f}_M = f_M,$$

and

$$\tilde{f}^*_m = \tilde{f}_m \vee f_m, \quad \tilde{f}_{m-1} = P_m(\tilde{f}_m \vee f_m), \qquad m = M, M-1, \dots, 1.$$

Then it is well-known that

$$\tilde{f}_0 = \sup\{E[f_\tau(X_\tau)];\ \ \tau \text{ is a } \{\mathcal{G}_m\}_{m=0}^M\text{-stopping time with } \tau \in \{1, 2, \dots, M\}\ a.s.\}.$$

$\tilde{f}_0$ is the price of a Bermudan derivative for which exercisable times are $1, \dots, M$, and pay-off at each time is $f_m(X_m)$, $m = 1, \dots, M$. Our concern is to compute $\tilde{f}_0$ numerically.

Let $\mathcal{V}$ denote the set of finite dimensional vector subspaces of $m(E)$. For any probability measure ν on $(E, \mathcal{B})$, let $\mathcal{V}(\nu)$ denote the subset of $\mathcal{V}$ such that $V \in \mathcal{V}(\nu)$, if and only if V satisfies the following two conditions.
(1) If $g \in V$, then $\int_E g(x)^4 \nu(dx) < \infty$.
(2) If $g \in V$ and $g(x) = 0$ $\nu - a.e.x$, then $g \equiv 0$.

For any probability measure ν on $(E, \mathcal{B})$ and $V \in \mathcal{V}(\nu)$, we define $\lambda_0(V, \nu)$ and $\lambda_1(V, \nu)$ by the following.

$$\lambda_0(V, \nu) = \sup\{\frac{\int_E g(x)^4 \nu(dx)}{(\int_E g(x)^2 \nu(dx))^2};\ g \in V \setminus \{0\}\}$$

$$\lambda_1(V; \nu) = \inf\{\int_E (\sum_{r=1}^{\dim V} e_r(x)^2)^2 \nu(dx);\ \{e_r\}_{r=1}^{\dim V} \text{ is an orthonormal basis of } V \text{ as a subspace of } L^2(E; d\nu)\ \}.$$

We will show in Proposition 3.4 that

$$\lambda_1(V; \nu) \leqq (\dim V)^2 \lambda_0(V; \nu) \text{ and } \lambda_0(V; \nu) \leqq \lambda_1(V; \nu).$$

Now let $(X_0^{(\ell)}, X_1^{(\ell)}, \dots, X_M^{(\ell)})$, $\ell = 1, 2, \dots$, be independent identically distributed E^{M+1}-valued random variables such that the law of $(X_0^\ell, X_1^\ell, \dots, X_M^\ell)$, $\ell = 1, 2, \dots$, is the same as the law of $(X_0, X_1, \dots, X_M)$ under P.

For any $m = 0, 1, \dots, M-1$, and $L \geqq 1$, we define $D_m^{(L)} : m(E) \times m(E) \times \Omega \to [0, \infty)$ by

$$D_m^{(L)}(g, f)(\omega) = (\frac{1}{L} \sum_{\ell=1}^{L} (g(X_m^{(\ell)}(\omega) - f(X_{m+1}^{(\ell)}(\omega))^2)^{1/2}, \qquad g, f \in m(E).$$

Let $V_m^{(k)}$, $k = 1, 2, \dots$, be a sequence of strictly increasing vector spaces in $\mathcal{V}(\nu_m)$ such that $\bigcup_{k=1}^\infty V_m^{(k)}$ is dense in $L^2(E; d\nu_m)$ for $m = 1, \dots, M-1$.

Now we assume that $g_m^{(L)} : \Omega \to V_m^{(L)}$, $m = 0, 1, \dots, M-1$, $L = 1, 2, \dots$, satisfy the following.

$$\begin{aligned} &D_{m-1}(g_{m-1}^{(L)}(\omega), g_m^{(L)}(\omega) \vee f_m)(\omega) \\ &= \inf\{D_{m-1}(h, g_m^{(L)}(\omega) \vee f_m);\ h \in V_m^{(L)}(\omega)\} \end{aligned} \tag{3.1}$$

for $m = 1, 2, \dots, M$. Here we let $g_M^{(L)} = f_M$.

We will show that such $g_m^{(L)}$'s always exist.

Then we will prove the following.

Theorem 3.1 *Suppose that $\lambda_1(V_m^{(L)}; \nu_m)/L \to 0$, as $L \to \infty$ for $m = 1, \dots, M-1$. Then there are $\Omega_L \in \mathcal{F}$, $L = 1, 2, \dots$, and random variables Z_L, $L = 1, 2, \dots$, such that*

$$P(\Omega_L) \to 1, \text{ as } L \to \infty,$$

$$|\tilde{f}_0 - g_0^{(L)}(\omega)| \leqq Z_L(\omega), \qquad L \geqq 1,\ \omega \in \Omega_L,$$

and

$$E[Z_L^2, \Omega_L]^{1/2} \to 0, \text{ as } L \to \infty.$$

Moreover, we have

$$\begin{aligned} &E[Z_L^2, \Omega_L]^{1/2} \\ &\leqq 6 \sum_{m=1}^{M-1} \frac{1}{L^{1/2}} \lambda_1(V_m^{(L)}, \nu_m)^{1/4} (1 + \lambda_0(V_m^{(L)}, \nu_m))^{1/4} ||P_m \tilde{f}_{m+1}^*||_{L^4(E; d\nu_m)} \\ &\quad + 5 \sum_{m=1}^{M-1} ||P_m \tilde{f}_{m+1}^* - \pi_{m, V_m^{(L)}} P_m \tilde{f}_{m+1}^*||_{L^2(E; d\nu_m)}. \end{aligned}$$

Here $\pi_{m, V_m^{(L)}}$ is the orthogonal projection in $L^2(E, d\nu_m)$ onto $V_m^{(L)}$, $m = 1, \dots, M$.

So roughly speaking, $g_0^{(L)} \to f_0$ in probability as $L \to \infty$ in a certain rate.

It is obvious that $\lambda_0(V; \nu_m) \geqq 1$ and $\lambda_1(V; \nu_m) \geqq \dim V$ for any $V \in \mathcal{V}_m$, $m = 1, 2, \dots, M$. So the above theorem raises the following question. Can one estimate $\lambda_0(V; \nu)$ and $||P_m \tilde{f}_{m+1}^* - \pi_{m,V} P_m \tilde{f}_{m+1}^*||_{L^2(E; d\nu_m)}$ for $V \in \mathcal{V}(\nu_m)$? If we can do it, we may find a sequence $V_m^{(k)} \in \mathcal{V}(\nu_m)$ such that the convergence rate is good.

We give an estimate when an underlying process is a 1-dimensional Brownian motion and V is a space of polynomials in Section 6. Also, we introduce a random systems of piece-wise polynomials in Section 8, and we give some estimates when an underlying process is a Hörmander type diffusion process as discussed in [14]. As far as we judge from these estimates, a usual polynomial system is not good, and such a random system of piece-wise polynomials is better.

3.2 Preliminary results

Let $\mathcal{P}_f(E \times E)$ be the set of probability measures on $(E \times E, \mathcal{B} \times \mathcal{B})$ whose supports are finite subsets of $E \times E$. Let $\pi_i : E \times E$, $i = 1, 2$, be natural projections given by $\pi_1(x, y) = x$, $\pi_2(x, y) = y$, $x, y \in E$. For any $\rho \in \mathcal{P}_f(E \times E)$, let $S(\cdot, *; \rho) : m(E) \times m(E) \to \mathbf{R}$ be given by

$$S(g, f; \rho) = \int_{E \times E} (g(x) - f(y))^2 \rho(dx, dy), \qquad g, f \in m(E). \tag{3.2}$$

Then we have the following.

Proposition 3.2 *Let $\rho \in \mathcal{P}_f(E \times E)$. For any $f \in m(E)$ and $V \in \mathcal{V}$, let*

$$s_*(f; V, \rho) = \inf\{S(g, f; \rho);\ g \in V\}$$

and

$$\Gamma(f; V, \rho) = \{g \in V;\ S(g, f; \rho) = s_*(f, V, \rho)\}.$$

Then we have the following.
(1) *$\Gamma(f; V, \rho)$ is not empty for any $f \in m(E)$ and $V \in \mathcal{V}$.*
(2) *Let $V \in \mathcal{V}$. If $f \in m(E)$ and $g \in \Gamma(f; V, \rho)$, then*

$$\int_{E \times E} h(x)(f(y) - g(x))\rho(dx, dy) = 0 \text{ for any } h \in V.$$

Moreover, if $f_1, f_2 \in m(E)$, $g_i \in \Gamma(f_i; V, \rho)$, $i = 1, 2$, then

$$S(g_1 - g_2, 0; \rho) \leqq S(0, f_1 - f_2; \rho).$$

(3) *If $f \in m(E)$, $g \in \Gamma(f; V, \rho)$ and $\tilde{g} \in V$, then*

$$S(g - \tilde{g}, 0; \rho)^{1/2} = \sup\{|\int_{E \times E} h(x)(f(y) - \tilde{g}(x))\rho(dx, dy)|;\ h \in V,\ S(h, 0; \rho) = 1\}.$$

Proof. (1) It is easy to see that

$$S(g, f; \rho) \geqq S(0, f; \rho) + S(g, 0; \rho) - 2S(g, 0; \rho)^{1/2} S(0, f; \rho)^{1/2}, \qquad g \in V.$$

Let $V_0 = \{g \in V;\ S(g, 0; \rho) = 0\} = \{g \in V :\ g(x) = 0 \text{ for } \rho\text{ -a.e. } (x, y) \in E \times E\}$. Then it is easy to see that V_0 is a vector subspace of V. So there is a vector subspace V_1 of V such that $V_0 + V_1 = V$ and $V_0 \cap V_1 = \{0\}$. It is easy to see that $g \in V_1 \to S(g, f; A)$ is a continuous function from V_1 to $[0, \infty)$ and that $S(g, f; A) \to \infty$ as $g \to \infty$ in V_1. So we see that there is a minimum point $g_0 \in V_1$. Note that $S(g + h, f; \rho) = S(g, f; \rho)$ for any $g \in V$ and $h \in V_0$. Therefore we see that $S(g_0, f; \rho) = s_*(f; V, \rho)$ and that $\Gamma(f; V, \rho)$ is not empty.
(2) Let $g \in \Gamma(f; V, \rho)$. The first assertion is obvious, since

$$0 = \frac{d}{dt} S(g + th, f; \rho)|_{t=0} = \int_{E \times E} h(x)(f(y) - g(x))\rho(dx, dy)$$

for any $h \in V$.

Let $f_i \in m(E)$, $g_i \in \Gamma(f_i; V, \rho)$, $i = 1, 2$. Then we have

$$S(g_1 - g_2, f_1 - f_2; \rho)$$

$$= -S(g_1 - g_2, 0; \rho) + S(0, f_1 - f_2; \rho)$$

$$-2 \int_{E\times E} (g_1(x) - g_2(x))(f_1(y) - g_1(x) - (f_2(y) - g_2(x)))\rho(dx, dy).$$

By the first assertion, we see that

$$S(0, f_1 - f_2; \rho) = S(g_1 - g_2, f_1 - f_2; \rho) + S(g_1 - g_2, 0; \rho).$$

So we have the second assertion.
(3) Let $g \in \Gamma(f; V, \rho)$ and $\tilde{g} \in V$. Then we have

$$S(\tilde{g} + h, f; \rho)$$

$$= S(\tilde{g}, f; \rho) + S(h, 0; \rho) - 2 \int_{E\times E} h(x)(f(y) - \tilde{g}(x))\rho(dx, dy).$$

Let

$$c = \sup\{\int_{E\times E} h(x)(f(y) - \tilde{g}(x))\rho(dx, dy);\ h \in V,\ S(h, 0; \rho) = 1\} \geqq 0.$$

Then we see that

$$s_*(f; V, \rho) = S(\tilde{g}, f; \rho) + \inf_{t\geqq 0}(t^2 - 2tc) = S(\tilde{g}, f; \rho) - c^2.$$

Also, we have by Assertion (2)

$$S(\tilde{g}, f, \rho) = S(g + (\tilde{g} - g), f; \rho) = S(g, f; \rho) + S(\tilde{g} - g, 0 : \rho)$$

$$= s_*(f; A, V) + S(\tilde{g} - g, 0 : \rho).$$

So we see that $c^2 = S(\tilde{g} - g, 0 : \rho)$. This implies our assertion. ∎

For any $m = 1, 2, \ldots, M$, $V \in \mathcal{V}(\nu_m)$, and $\rho \in \mathcal{P}_f(E \times E)$, let

$$\delta_m(V; \rho) = \sup\{|S(h, 0; \rho) - 1|; h \in V,\ \int_E h(x)^2 \nu_m(dx) = 1\}.$$

Then we have the following.

Proposition 3.3 *Let $m = 1, 2, \ldots, M$, $V \in \mathcal{V}(\nu_m)$, and $\rho \in \mathcal{P}_f(E \times E)$. Let $\{e_k;\ k = 1, \ldots, \dim\ V\}$ be an orthonormal basis of V. Here we regard V as a Hilbert subspace of $L^2(E, \mathcal{B}(E), d\nu_m)$, and so we have*

$$\int_E e_i(x)e_j(x)\nu_m(dx) = \delta_{ij}, \qquad i, j = 1, \ldots, \dim V.$$

Let A be a $(\dim V) \times (\dim V)$-symmetric matrix valued function defined in E given by

$$A(x) = (A_{ij}(x))_{i,j=1}^{\dim V} = (e_i(x)e_j(x))_{i,j=1}^{\dim V}, \qquad x \in E.$$

Then $\delta_m(V;\rho)$ *is equal to the operator norm of the* $\dim V \times \dim V$*-symmetric matrix* $\bar{A} - I$*. Here* I *is the identity matrix and* $\bar{A} = (\bar{A}_{ij})_{i,j=1}^{\dim V}$*, where*

$$\bar{A}_{ij} = \int_E e_i(x)e_j(x)\rho(dx,dy), \qquad i,j = 1,\ldots,\dim V.$$

In particular,

$$\delta_m(V;\rho)^2 \leqq \sum_{i,j=1}^{\dim V} (\int_E (e_i(x)e_j(x) - \delta_{ij})\rho(dx,dy))^2.$$

Proof. It is easy to see that

$$\delta_m(V;\rho) = \sup\{|S(\sum_{i=1}^{\dim V} a_i e_i, 0;\rho) - 1|;\ \sum_{i=1}^{\dim V} a_i^2 = 1\}$$

$$= \sup\{|\sum_{i,j=1}^{\dim V} a_i a_j(\bar{A}_{ij} - \delta_{ij})|;\ \sum_{i=1}^{\dim V} a_i^2 = 1\}.$$

Since $\bar{A} - I$ is symmetric, we see our assertion. ∎

Proposition 3.4 *For any probability measure* ν *on* $(E,\mathcal{B})$*, and* $V \in \mathcal{V}(\nu)$*,*

$$\lambda_1(V,\nu) \leqq (\dim V)^2 \lambda_0(V,\nu)$$

and

$$\lambda_0(V,\nu) \leqq \lambda_1(V,\nu).$$

Proof. Let $\{e_r\}_{r=1}^{\dim V}$ be an orthonormal basis of V. Then we see that

$$\int_E (\sum_{r=1}^{\dim V} e_r(x)^2)^2 \nu(dx) \leqq \int_E (\dim V)(\sum_{r=1}^{\dim V} e_r(x)^4)\nu(dx) \leqq (\dim V)^2 \lambda_0(V,\nu).$$

So we have the first assertion.

Let $g \in V$. Then we have

$$\int_E g(x)^4 \nu(dx) = \int_E (\sum_{r=1}^{\dim V} (g,e_r)_{L^2(d\nu)} e_r(x))^4 \nu(dx)$$

$$\leqq \int_E (\sum_{r=1}^{\dim V} (g,e_r)_{L^2(d\nu)}^2)^2 (\sum_{r=1}^{\dim V} e_r(x)^2)^2 \nu(dx).$$

Note that

$$\sum_{r=1}^{\dim V} (g,e_r)_{L^2(d\nu)}^2 = \int_E g(x)^2 \nu(dx).$$

So we have the second assertion. ∎

3.3 random measures

For $m = 1, \ldots, M$, and $L \geqq 1$, let $\rho_m^{(L)}$ be a random probability measure belonging to $\mathcal{P}_f(E \times E)$ given by

$$\rho_m^{(L)}(A) = \frac{1}{L}\#\{\ell \in \{1, \ldots, L\};\ (X_{m-1}^{(\ell)}, X_m^{(\ell)}) \in A\}, \qquad A \in \mathcal{B} \times \mathcal{B}.$$

For any $m = 0, 1, \ldots, M-1$, and $L \geqq 1$, we define $N_m^{(L)} : m(E) \times \Omega \to [0, \infty)$ by

$$N_m^{(L)}(f)(\omega) = (\frac{1}{L}\sum_{\ell=1}^{L} f(X_m^{(\ell)}(\omega))^2)^{1/2}.$$

Then we see that

$$N_{m-1}^{(L)}(g) = S(g, 0;\ \rho_m^{(L)}), \qquad g \in m(E),\ m = 1, \ldots, M.$$

Then we have the following.

Proposition 3.5 *Let $m = 1, \ldots, M-1$, $L \geqq 1$, and $V \in \mathcal{V}(\nu_m)$. Then we have the following.*
(1) *If $\delta_m(V; \rho_m^{(L)}) \leqq 1/2$, then*

$$\frac{1}{2}N_{m-1}^{(L)}(g)^2 \leqq \int_E g(x)^2 \nu_m(dx) \leqq 2N_{m-1}^{(L)}(g)^2, \qquad g \in V.$$

(2)

$$E[\delta_m(V; \rho_m^{(L)})^2] \leqq \frac{1}{L}\lambda_1(V, \nu_m).$$

In particular, we have

$$P(\delta_m(V; \rho_m^{(L)}) > \frac{1}{2}) \leqq \frac{4}{L}\lambda_1(V, \nu_m).$$

Proof. (1) Suppose that $\delta_m(V; \rho_m^{(L)}) \leqq 1/2$. If $h \in V$ and $\int_E h(x)^2 \nu_m(dx) = 1$, then from the definition we have

$$\frac{1}{2} \leqq N_{m-1}^{(L)}(h)^2 \leqq 2.$$

So we have our assertion.
(2) Let $\{e_r\}_{r=1}^{\dim V}$ be an orthonormal basis of V. It is easy to see that

$$E[\delta_m(V; \rho_m^{(L)})^2] \leqq \sum_{r,r'=1}^{\dim V} E[(\frac{1}{L}\sum_{\ell=1}^{L}(e_r(X_m^\ell)e_{r'}(X_m^\ell) - \delta_{r,r'}))^2]$$

$$= \frac{1}{L}\sum_{r,r'=1}^{\dim V}\int_E (e_r(x)e_{r'}(x) - \delta_{r,r'})^2 \nu_m(dx) \leqq \frac{1}{L}\sum_{r,r'=1}^{\dim V}\int_E e_r(x)^2 e_{r'}(x)^2 \nu_m(dx).$$

$$= \frac{1}{L}\int_E (\sum_{r=1}^{\dim V} e_r(x)^2)^2 \nu_m(dx).$$

So we have the first part of our assertion . The second part is an easy consequence of Chebyshev's inequality. ∎

For any $m = 1, 2, \dots, M-1$, and $V \in \mathcal{V}(\nu_m)$, let $\hat{\Gamma}_{m,V} : m(E) \times \mathcal{P}_f(E \times E) \to V$ be defined by the following. $g = \hat{\Gamma}_{m,V}(f, \rho)$, $f \in m(E)$, $\rho \in \mathcal{P}_f(E \times E)$, if $g \in \Gamma(f, V; \rho)$ and

$$\int_E g(x)^2 \nu_m(dx) = \inf\{\int_E \tilde{g}(x)^2 \nu_m(dx);\ \tilde{g} \in \Gamma(f, V; \rho)\}.$$

$\hat{\Gamma}_{m,V}$ is well-defined by Proposition 3.2 and the definition of $\mathcal{V}(\nu_m)$.

Let $F : E \times \Omega \to \mathbf{R}$ be $\mathcal{B} \times \mathcal{F}$-measurable function. Then it is easy to see that the mapping $\omega \in \Omega \to s_*(F(\cdot, \omega), V, \rho_m^{(L)}(\omega))$ is $\mathcal{F}$-measurable. So we see that the mapping $\omega \in \Omega \to \hat{\Gamma}_{m,V}(F(\cdot, \omega), \rho_m^{(L)}(\omega))$ is also $\mathcal{F}$-measurable (see Castaing [6] for example).

For $V \in \mathcal{V}(\nu_m)$, $m = 1, \dots, M$, let $\pi_{m,V} : L^2(E; d\nu_m) \to V$ be the orthogonal projection onto V.

Then we have the following.

Proposition 3.6 *Let $m = 1, \dots, M-1$, and $L \geqq 1$. Then for $V \in \mathcal{V}_m$ and $f \in L^4(E, \mathcal{B}(E), d\nu_{m+1})$, we have*

$$E[N_m^{(L)}(\pi_{m,V} P_m f - \hat{\Gamma}_{m,V}(f; \rho_m^{(L)}))^2, \delta_m(V, \rho_m^{(L)}) \leqq \frac{1}{2}]$$

$$\leqq \frac{8}{L}(\lambda_1(V, \nu)(1 + \lambda_0(V, \nu)))^{1/2}(\int_E f(y)^4 \nu_{m+1}(dy))^{1/2}.$$

Proof. Let $g = \pi_{m,V} P_m f$, and $\{e_r\}_{r=1}^{\dim V}$ be an orthonormal basis of V. Note that

$$E[e_r(X_m^1)(f(X_{m+1}^1) - g(X_m^1))] = \int_{E \times E} e_r(x)(f(y) - g(x)) \nu_m(dx) p_m(x, dy)$$

$$= \int_E e_r(x)(P_m f(x) - g(x)) \nu_m(dx) = 0, \qquad r = 1, \dots, \dim V.$$

By Proposition 3.2(3) we see that

$$E[N_m^{(L)}(g - \hat{\Gamma}_{m,V}(f; \rho_m^{(L)}))^2, \delta_m(V, \rho_m^{(L)}) \leqq \frac{1}{2}]$$

$$\leqq 2E[\sup\{|\int_{E \times E} h(x)(f(y) - g(x)) \rho_{m+1}^{(L)}(dx, dy)|^2;\ h \in V,\ \int_E h(x)^2 \nu_m(dx) = 1\}]$$

$$= 2E[\sup\{|\sum_{r=1}^{\dim V} a_r \int_{E \times E} e_r(x)(f(y) - g(x)) \rho_{m+1}^{(L)}(dx, dy)|^2;\ \sum_{r=1}^{\dim V} a_r^2 = 1\}]$$

$$= 2E[\sum_{r=1}^{\dim V} (\int_{E \times E} e_r(x)(f(y) - g(x)) \rho_{m+1}^{(L)}(dx, dy))^2]$$

$$= 2 \sum_{r=1}^{\dim V} E[(\frac{1}{L} \sum_{\ell=1}^{L} e_r(X_m^\ell)(f(X_{m+1}^\ell) - g(X_m^\ell)))^2]$$

$$= \frac{2}{L} \sum_{r=1}^{\dim V} E[e_r(X_m^1)^2 (f(X_{m+1}^1) - g(X_m^1))^2]$$

$$= \frac{2}{L} \sum_{r=1}^{\dim V} \int_{E \times E} e_r(x)^2 (f(y) - g(x))^2 \nu_m(dx) p_m(x, dy)$$

$$\leqq \frac{2}{L}(\int_E(\sum_{r=1}^{\dim V} e_r(x)^2)^2\nu_m(dx))^{1/2}(\int_{E\times E}(f(y)-g(x))^4\nu_m(dx)p_m(x,dy))^{1/2}.$$

Note that

$$\int_{E\times E}(f(y)-g(x))^4\nu_m(dx)p_m(x,dy) \leqq 16\int_{E\times E}(f(y)^4+g(x)^4)\nu_m(dx)p_m(x,dy)$$

$$= 16(\int_E f(y)^4\nu_{m+1}(dy)+\int_E g(x)^4\nu_m(dx)).$$

By Proposition 3.4, we see that

$$\int_E g(x)^4\nu_m(dx) \leqq \lambda_0(V,\nu_m)(\int_E (P_m f)(x)^2\nu_m(dx))^2 \leqq \lambda_0(V,\nu_m)\int_E f(y)^4\nu_{m+1}(dy).$$

So we have our assertion . ∎

The following is obvious.

Proposition 3.7 *Let $m=1,\ldots,M$, and $L\geqq 1$. Then for any $f\in L^2(E,\mathcal{B}(E),d\nu_m)$, we have*

$$E[N_m^{(L)}(f)^2] = \int_E f(x)^2\nu_m(dx).$$

3.4 Proof of Theorem 3.1

Now let us think of the setting in Introduction. Let $\phi_m : E\times \mathbf{R}\to\mathbf{R}$, $m=1,\ldots,M$, be given by

$$\phi_m(x,z) = f_m(x)\vee z, \qquad x\in E,\ z\in\mathbf{R},\ m=1,2,\ldots,M.$$

Then we see that

$$|\phi_m(x,z_1)-\phi_m(x,z_2)| \leqq |z_1-z_2|, \qquad x\in E,\ z_1,z_2\in\mathbf{R},\ m=1,\ldots,M.$$

Note that

$$\tilde f_m^*(x) = \phi_m(x,\tilde f_m(x)) \text{ and } \tilde f_{m-1} = P_{m-1}\tilde f_m^*, \qquad m=1,\ldots,M.$$

Remind that $V_m^{(L)}\in\mathcal{V}(\nu_m)$, $L\geqq 1$, $m=1,\ldots,M$. Let us take $g_m^{(L)}:\Omega\to V_m^{(L)}$, $m=M,\ldots,0$, such that

$$g_M^{(L)}(\omega) = f_M,$$

$$g_m^{(L)}(\omega)\in\Gamma(\phi_{m+1}(\cdot,g_{m+1}^{(L)}(\omega)(\cdot)),V_m^{(L)};\ \rho_m^L(\omega)), \qquad m=M-1,\ldots,0.$$

Then we see that Equation (3.1) is satisfied. Let $\tilde Z_m^{(L)}$, $m=0,1,\ldots,M-1$, be given by

$$\tilde Z_m^{(L)}$$

$$= N_m^{(L)}(P_m\tilde f_{m+1}^* - \pi_{m,V_m^{(L)}}P_m\tilde f_{m+1}^*) + N_m^{(L)}(\pi_{m,V_m^{(L)}}P_m\tilde f_{m+1}^* - \hat\Gamma_{m,V_{m,L}}(\tilde f_{m+1}^*;\ \rho_m^L)).$$

Also, let $Z_m^{(L)}$, $m = 0, 1, \ldots, M-1$, be given by

$$Z_0^{(L)} = \sum_{k=0}^{M-1} \tilde{Z}_k^{(L)},$$

and

$$Z_m^{(L)}$$
$$= ||\tilde{f}_m - \pi_{m,V_m^{(L)}} \tilde{f}_m||_{L^2(E,d\mu_m)} + 2N_m(\tilde{f}_m - \pi_{m,V_m^{(L)}} \tilde{f}_m, \omega) + 2\sum_{k=m}^{M-1} \tilde{Z}_k^{(L)},$$

$m = 1, \ldots, M-1$. Finally, let

$$\Omega_L = \bigcap_{m=1}^{M-1} \{\delta_m(V_m^{(L)}; \rho_m^{(L)}) \leqq \frac{1}{2}\}.$$

Then we have the following.

Proposition 3.8 (1) $|\tilde{f}_0 - g_0^{(L)}(\omega)| \leqq Z_0^{(L)}$.
(2) *For any* $\omega \in \Omega^{(L)}$,

$$||\tilde{f}_m^* - (g_m^{(L)}(\omega) \vee f_m)||_{L^2(E;d\nu_m)} \leqq ||\tilde{f}_m - g_m^{(L)}(\omega)||_{L^2(E;d\nu_m)} \leqq Z_m^{(L)}, \quad m = 1, \ldots, M.$$

(3)

$$P(\Omega \setminus \Omega_L) \leqq \sum_{k=1}^{M-1} \frac{4}{L} \lambda_1(V_k^{(L)}, \nu_k),$$

and

$$E[|Z_m^{(L)}|^2, \Omega_L]^{1/2}$$
$$\leqq 6\sum_{k=1}^{M-1} \{(\frac{1}{L}\lambda_1(V_k^{(L)}, \nu_k)^{1/2}(1 + \lambda_0(V_k^{(L)}, \nu_k))^{1/2}\}^{1/2} ||P_k \tilde{f}_{k+1}^*||_{L^4(E;d\nu_k)}$$
$$+5\sum_{k=1}^{M-1} ||P_k \tilde{f}_{k+1}^* - \pi_{k,V_k^{(L)}} P_k \tilde{f}_{k+1}^*||_{L^2(E;d\nu_k)}, \qquad m = 0, 1, \ldots, M-1.$$

Proof. Note that

$$N_m^{(L)}(\tilde{f}_m - g_m^{(L)}(\omega), \omega)$$
$$\leqq N_m^{(L)}(P_m \tilde{f}_{m+1}^* - \hat{\Gamma}_{m,V_{m,L}}(\tilde{f}_{m+1}^*;\ \rho_m^L), \omega) + N_m^{(L)}(\hat{\Gamma}_{m,V_{m,L}}(\tilde{f}_{m+1}^*;\ \rho_m^L)) - g_m^{(L)}(\omega), \omega).$$

By Proposition 3.2(2), we have

$$N_m^{(L)}(\hat{\Gamma}_{m,V_{m,L}}(\tilde{f}_{m+1}^*;\ \rho_m^L)) - g_m^{(L)}(\omega), \omega)$$
$$\leqq N_{m+1}^{(L)}(\phi_{m+1}(\cdot, \tilde{f}_{m+1}(\cdot)) - \phi_{m+1}(\cdot, g_{m+1}^{(L)}(\omega)(\cdot)), \omega)$$
$$\leqq N_{m+1}^{(L)}(\tilde{f}_{m+1} - g_{m+1}^{(L)}(\omega)(\cdot), \omega).$$

So we see that

$$N_m^{(L)}(\tilde{f}_m - g_m^{(L)}(\omega), \omega) \leqq \sum_{k=m}^{M-1} N_k^{(L)}(P_k \tilde{f}_{k+1}^* - \hat{\Gamma}_{k,V_{k,L}}(\tilde{f}_{k+1}^*;\ \rho_k^L), \omega).$$

Then we have

$$N_m^{(L)}(\tilde{f}_m - g_m^{(L)}(\omega), \omega) \leqq \sum_{k=m}^{M-1} \tilde{Z}_k^{(L)}.$$

In particular,

$$|\tilde{f}_0 - g_0^{(L)}(\omega)| \leqq \sum_{k=0}^{M-1} \tilde{Z}_k^{(L)} = Z_0^{(L)}.$$

This implies Assertion (1).

Also, we see that if $\omega \in \Omega_L$, then

$$\begin{aligned}
&||\tilde{f}_m - g_m^{(L)}(\omega)||_{L^2(E,d\mu_m)} \\
&\leqq ||\tilde{f}_m - \pi_{m,V_m^{(L)}}\tilde{f}_m||_{L^2(E,d\mu_m)} + ||\pi_{m,V_m^{(L)}}\tilde{f}_m - g_m^{(L)}(\omega)||_{L^2(E,d\mu_m)} \\
&\leqq ||\tilde{f}_m - \pi_{m,V_m^{(L)}}\tilde{f}_m||_{L^2(E,d\mu_m)} + N_m^{(L)}(\pi_{m,V_m^{(L)}}\tilde{f}_m - g_m^{(L)}(\omega), \omega) \\
&\leqq ||\tilde{f}_m - \pi_{m,V_m^{(L)}}\tilde{f}_m||_{L^2(E,d\mu_m)}| + 2N_m(\tilde{f}_m - \pi_{m,V_m^{(L)}}\tilde{f}_m, \omega) + 2N_m^{(L)}(\tilde{f}_m - g_m^{(L)}(\omega), \omega).
\end{aligned}$$

This implies Assertion (2).

The first assertion of (3) is obvious from Propositions 3.5. By Propositions 3.6 and 3.7, we have

$$\begin{aligned}
&E[(\tilde{Z}_m^{(L)})^2, \Omega_L]^{1/2} \\
&\leqq ||\tilde{f}_m - \pi_{m,V_m^{(L)}}P_m\tilde{f}_m||_{L^2(E,d\mu_m)} \\
&\quad +3(\frac{1}{L}(\lambda_1(V,\nu)(1+\lambda_0(V,\nu))^{1/2})^{1/2}||\tilde{f}_{m+1}^*||_{L^4(E;d\nu_{m+1})}.
\end{aligned}$$

So we have the second assertion of (3). ∎

Theorem 3.1 follows from Proposition 3.8 immediately.

The following is an easy consequence of Proposition 3.8.

Proposition 3.9 *Asuume that* $\lambda_1(V_m^{(L)};\nu_m)/L \to 0$, $L \to \infty$, $m = 1, \ldots, M-1$. *Let* $\delta \in (0,1)$, *and let*

$$d_L = \sum_{m=0}^{M-1} E[(Z_m^{(L)})^2, \Omega_L]^{1/2}, \qquad L \geqq 1,$$

and let $\tilde{\Omega}_L^\delta \in \mathcal{F}$, $L \geqq 1$, *be given by*

$$\tilde{\Omega}_L^\delta = \Omega_L \cap \bigcap_{m=1}^{M-1} \{Z_m^{(L)} \leqq d_L^{1-\delta}\}.$$

Then $d_L \to 0$, *and* $P(\tilde{\Omega}_L^\delta) \to 1$, $L \to \infty$. *Also, we have*

$$||\tilde{f}_m - g_m(\omega)||_{L^2(E;d\nu_m)} \leqq d_L^{1-\delta}, \qquad m = 1\ldots, M,\ \omega \in \tilde{\Omega}_L^\delta,\ L \geqq 1.$$

3.5 re-simulation

Let us be back to the situation in Introduction. Let $h_m \in L^2(E; d\nu_m)$, $m = 1, \ldots, M$, with $h_M = f_M$. Let σ a stopping time given by $\sigma = \min\{k = 0, 1, \ldots, M;\ f_k(X_k) \geqq h_k(X_k)\}$, and let

$$c_0 = c_0(\{h_m\}_{m=1}^{M-1}) = E[f_\sigma(X_\sigma)].$$

Then we have the following.

Proposition 3.10 *Let $\beta \geqq 0$. Assume that there is a $C_0 > 0$ such that*

$$\nu_m(\{|f_m - \tilde{f}_m| \leqq \varepsilon\}) \leqq C_0 \varepsilon^\beta, \qquad \varepsilon > 0,\ m = 1, 2, \ldots, M.$$

Then we have

$$|\tilde{f}_0 - c_0| \leqq (C_0 + 1) \sum_{m=1}^{M-1} ||\tilde{f}_m - h_m||_{L^2(E; d\nu_m)}^{1+\beta/(2+\beta)}.$$

Proof. Let $\hat{h}_m$, $m = M, M-1, \ldots, 0$, be inductively given by

$$\hat{h}_M = f_M = h_M,$$

$$\hat{h}_{m-1} = P_{m-1}(1_{\{f_m \geqq h_m\}} f_m + 1_{\{f_m < h_m\}} \hat{h}_m), \qquad m = M, M-1, \ldots, 1.$$

Then we see that $c_0 = \hat{h}_0$.

Note that

$$\tilde{f}_{m-1} = P_{m-1}(1_{\{f_m \geqq \tilde{f}_m\}} f_m + 1_{\{f_m < \tilde{f}_m\}} \tilde{f}_m), \qquad m = M, M-1, \ldots, 1.$$

Therefore we have

$$\begin{aligned} &\tilde{f}_{m-1} - \hat{h}_{m-1} \\ &= P_{m-1}(1_{\{f_m < \tilde{f}_m \wedge h_m\}}(\tilde{f}_m - \hat{h}_m) + 1_{\{h_m \leqq f_m < \tilde{f}_m\}}(\tilde{f}_m - f_m) + 1_{\{\tilde{f}_m \leqq f_m < h_m\}}(f_m - \hat{h}_m)) \\ &= P_{m-1}(1_{\{f_m < h_m\}}(\tilde{f}_m - \hat{h}_m) + 1_{\{h_m \leqq f_m < \tilde{f}_m\}}(\tilde{f}_m - f_m) + 1_{\{\tilde{f}_m \leqq f_m < h_m\}}(f_m - \tilde{f}_m)), \end{aligned}$$

and so we see that

$$\begin{aligned} &|\tilde{f}_{m-1} - \hat{h}_{m-1}| \\ &\leqq P_{m-1}(|\tilde{f}_m - \hat{h}_m|) + P_{m-1}(1_{\{|f_m - \tilde{f}_m| \leqq |\tilde{f}_m - h_m|\}}|f_m - \tilde{f}_m|) \\ &\leqq P_{m-1}(|\tilde{f}_m - \hat{h}_m|) + P_{m-1}(1_{\{|f_m - \tilde{f}_m| \leqq \varepsilon\}}|f_m - \tilde{f}_m|) + P_{m-1}(1_{\{\varepsilon < |\tilde{f}_m - h_m|\}}|\tilde{f}_m - h_m|) \end{aligned}$$

So we have

$$\begin{aligned} &||\tilde{f}_{m-1} - \hat{h}_{m-1}||_{L^1(E; d\nu_{m-1})} \\ &\leqq ||\tilde{f}_m - \hat{h}_m||_{L^1(E; d\nu_m)} + \varepsilon \nu_m(\{|f_m - \tilde{f}_m| \leqq \varepsilon\}) + \varepsilon^{-1}||\tilde{f}_m - h_m||_{L^2(E; d\nu_{m-1})}^2 \\ &\leqq ||\tilde{f}_m - \hat{h}_m||_{L^1(E; d\nu_m)} + C_0 \varepsilon^{1+\beta} + \varepsilon^{-1}||\tilde{f}_m - h_m||_{L^2(E; d\nu_m)}^2 \end{aligned}$$

So letting

$$\varepsilon = ||\tilde{f}_m - h_m||_{L^2(E; d\nu_m)}^{2/(2+\beta)},$$

we have

$$||\tilde{f}_{m-1} - \hat{h}_{m-1}||_{L^1(E;d\nu_{m-1})} \leqq ||\tilde{f}_m - \hat{h}_m||_{L^1(E;d\nu_m)} + (C_0+1)||\tilde{f}_m - h_m||^{1+\beta/(2+\beta)}_{L^2(E;d\nu_m)}.$$

Since $\tilde{f}_M = \hat{h}_M = h_M = f_M$, we have our assertion. ∎

Now let $\tilde{X}^n = (\tilde{X}^n_0, \tilde{X}^n_1, \dots, \tilde{X}^n_M)$, $n = 1, 2, \dots$, be independent identically distributed E^{M+1}-valued random variables whose distribution is the same as $(X_0, X_1, \dots, X_M)$ under P. We assume that $\sigma\{X_m;\ m = 0, 1, \dots, M\}$, $\sigma\{X^\ell_m,\ m = 0, 1, \dots, M,\ \ell \geqq 1\}$ and $\sigma\{\tilde{X}^n_m;\ m = 0, 1, \dots, M,\ n \geqq\}$ are independent. Let $g^{(L)}_m(\omega) \in V^{(L)}_m$, $m, L \geqq 1$, as in Introduction. Let

$$\tau_n(\omega) = \min\{m \geqq 0; g_m(\omega)(\tilde{X}^n_m(\omega)) \geqq f_m(\tilde{X}^n_m(\omega))\}, \qquad n \geqq 1,$$

and let

$$\tilde{c}^n_0(\omega) = \frac{1}{n}\sum_{k=1}^n f_{\tau_k(\omega)}(\tilde{X}^k_{\tau_k(\omega)}(\omega))$$

Then by law of large number, we have

$$\tilde{c}^n_0(\omega) \to c_0(\{g^{(L)}_m(\omega)\}^{M-1}_{m=1}) \ \ a.s., \qquad n \to \infty.$$

By Proposition 3.8, we see that

$$|\tilde{f}_0 - g^{(L)}_0(\omega)| \leqq d_L, \qquad \omega \in \Omega_L.$$

But Propositions 3.9 and 3.10 imply that

$$|\tilde{f}_0 - c_0(\{g_m(\omega)\}^{M-1}_{m=1})| \leqq C d_L^{(1-\delta)(1+\beta/(2+\beta))}, \qquad \omega \in \tilde{\Omega}^\delta_L,$$

even though β is unknown. So $\tilde{c}^n_0(\omega)$ can be a better estimator of $\tilde{f}_0$.

3.6 Brownian motion Case

From now on, we try to give estimates for $\lambda_0(V, \nu)$ and $||P_m\tilde{f}^*_{m+1} - \pi_{m,V}P_m\tilde{f}^*_{m+1}||$ for some examples.

Let $\{B_t;\ t \geqq 0\}$ be a standard Brownian motion and $T > 0$. Now let V_n, $n \geqq 1$, be the space of polynomials of degree less than or equal to n. Let P_t, $t \geqq 0$, be the diffusion operators for the standard Brownian motion, i.e.,

$$(P_tg)(x) = (\frac{1}{2\pi t})^{1/2}\int_{\mathbf{R}} g(y)\exp(-\frac{(x-y)^2}{2t})dy, \qquad g \in m(\mathbf{R}).$$

Let ν be a probability law of B_T. So we have

$$\nu(dx) = \frac{1}{\sqrt{2\pi T}}\exp(-\frac{x^2}{2T})dx.$$

Then we have the following.

Proposition 3.11 *We have*

$$\lim_{n\to\infty}\frac{1}{n}\log\lambda_0(V_n,\nu)=\lim_{n\to\infty}\frac{1}{n}\log\lambda_1(V_n,\nu)=\log 9.$$

Also, let $f:\mathbf{R}\to[0,\infty)$ be given by $f(x)=x\vee 0$, $x\in\mathbf{R}$. Then there is a $C_0>0$ such that

$$||P_tf-\pi_nP_tf||_{L^2(d\nu)}\geqq C_0n^{-3/4}(1+t/T)^{-n/2},\qquad n\geqq 1.$$

Here π_n is the orthogonal projection in $L^2(\mathbf{R},d\nu)$ onto V_n.

Proof. Let

$$H_n(x;v)=\exp(\frac{x^2}{2v})\frac{d^n}{dx^n}\exp(-\frac{x^2}{2v}),\qquad x\in\mathbf{R}^N,\ v>0,\ n\geqq 0.$$

Then we have

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x;v)=\exp(\frac{x^2}{2v})\exp(-\frac{(x+t)^2}{2v})=\exp(-\frac{xt}{v}-\frac{t^2}{2v}),$$

and

$$\sum_{n,m=0}^{\infty}\frac{t^n}{n!}H_n(x;v)\frac{s^m}{n!}H_m(x;v)=\exp(-\frac{x(t+s)}{v}-\frac{t^2+s^2}{2v}).$$

So we have

$$\sum_{n,m=0}^{\infty}\frac{t^n}{n!}\frac{s^m}{n!}\int_{\mathbf{R}}H_n(x;T)H_m(x;T)\nu(dx)$$

$$=\exp(\frac{(t+s)^2}{2T}-\frac{t^2+s^2}{2T})=\exp(\frac{ts}{T})=\sum_{n=0}^{\infty}\frac{t^ns^n}{n!T^n},$$

and

$$\int_{\mathbf{R}}H_n(x;T)H_m(x;T)\nu(dx)=\delta_{nm}\frac{n!}{T^n}.$$

So we see that $e_n(x;T)=(\frac{T^n}{n!})^{1/2}H_n(x;T)$, $n=1,2,\ldots$, is an orthonormal basis in $L^2(\mathbf{R},d\nu)$.

Note that

$$\sum_{n_1,n_2,n_3,n_4=0}^{\infty}(\prod_{i=1}^{4}\frac{t_i^{n_i}}{n_i!})\prod_{i=1}^{4}H_{n_i}(x;v)=\exp(-\frac{x(\sum_{i=1}^4 t_i)}{v}-\frac{\sum_{i=1}^4 t_i^2}{2v}).$$

and so

$$\sum_{n_1,n_2,n_3,n_4=0}^{\infty}\prod_{i=1}^{4}\frac{t_i^{n_i}}{n_i!}\int_{\mathbf{R}}\prod_{i=1}^{4}H_{n_i}(x;T)\nu(dx)$$

$$=\exp(\frac{(\sum_{i=1}^4 t_i)^2}{2T}-\frac{\sum_{i=1}^4 t_i^2}{2T})=\exp(\frac{1}{T}\sum_{1\leqq i<j\leqq 4}t_it_j).$$

So we have

$$\int_{\mathbf{R}}H_n(x;T)^4\nu(dx)=\frac{1}{(2n)!}\frac{d^n}{dt_4^n}\cdots\frac{d^n}{dt_1^n}(\frac{1}{T^{2n}}(\sum_{1\leqq i<j\leqq 4}t_it_j)^{2n}))|_{t_1=\cdots=t_4=0}.$$

Note that

$$\sum_{1\leqq i<j\leqq 4} t_i t_j = t_1(t_2+t_3+t_4)+t_2(t_3+t_4)+t_3t_4$$

and so we have

$$\frac{d^n}{dt_1^n}((\sum_{1\leqq i<j\leqq 4} t_i t_j)^{2n})|_{t_1=0} = \frac{(2n)!}{n!}(t_2+t_3+t_4)^n(t_2(t_3+t_4)+t_3t_4)^n,$$

$$\frac{d^n}{dt_2^n}\frac{d^n}{dt_1^n}((\sum_{1\leqq i<j\leqq 4} t_i t_j)^{2n})|_{t_1=t_2=0}$$

$$= \frac{(2n)!}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{n!}{k!}(t_3+t_4)^k\frac{n!}{(n-k)!}(t_3+t_4)^k(t_3t_4)^{n-k}$$

$$= (2n)!\sum_{k=0}^{n}\binom{n}{k}^2(t_3+t_4)^{2k}(t_3t_4)^{n-k}.$$

So we have

$$\frac{d^n}{dt_4^n}\cdots\frac{d^n}{dt_1^n}(\sum_{1\leqq i<j\leqq 4} t_i t_j)^{2n})|_{t_1=\cdots=t_4=0} = (2n)!(n!)^2\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}.$$

Therefore we see that

$$\int_{\mathbf{R}} e_n(x;T)^4\nu(dx) = (\frac{T^n}{n!})^2\int_{\mathbf{R}} H_n(x;T)^4\nu(dx) = \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}.$$

Let

$$a_n = \log(\frac{n!}{n^{n-1/2}e^{-n}}), \qquad n \geqq 0.$$

Then it is well known that $\{a_n\}_{n=1}^{\infty}$ is bounded.

Since

$$\log(n!) = n\log n - n - \frac{1}{2}\log n + a_n,$$

we have

$$\frac{1}{n}\log(\binom{n}{k}^2\binom{2k}{k}) = 2\frac{1}{n}\log\binom{n}{k} + \frac{1}{n}\log\binom{2k}{k}$$

$$= 2h(\frac{k}{n}) + \frac{1}{n}(-\log n + \log(n-k) + \log k + 2a_n - 2a_{n-k} - 2a_k)$$

$$+\frac{2k}{n}\log 2 + \frac{1}{n}(-\frac{1}{2}\log(2k) + \log k + a_{2k} - 2a_{2k}),$$

where

$$h(x) = -(x\log x + (1-x)\log(1-x)), \qquad x\in[0,1].$$

Also, we have

$$\max_{k=0,1,\ldots,n}\frac{1}{n}\log(\binom{n}{k}^2\binom{2k}{k}) \leqq \frac{1}{n}\log(\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k})$$

$$\leqq \max_{k=0,1,\ldots,n}\frac{1}{n}\log(\binom{n}{k}^2\binom{2k}{k}) + \frac{1}{n}\log(n+1).$$

So we have

$$\frac{1}{n}\log(\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}) \to \max_{x\in[0,1]}(2h(x)+2x\log 2) = \log 9, \qquad n\to\infty.$$

Therefore we have by Proposition 3.4

$$\frac{1}{n}\log(\int_{\mathbf{R}} e_n(x;T)^4\nu(dx)) \to \log 9, \qquad n\to\infty.$$

Since

$$\int_{\mathbf{R}} e_n(x;T)^4\nu(dx) \leqq \lambda_0(V_n,\nu)$$

and

$$\lambda_0(V_n,\nu) \leqq \lambda_1(V_n,\nu) \leqq (n+1)\sum_{k=0}^{n}\int_{\mathbf{R}} e_k(x;T)^4\nu(dx)$$

$$\leqq (n+1)^2 \max_{k=0,\ldots,n}\int_{\mathbf{R}} e_k(x;T)^4\nu(dx),$$

we see that

$$\lim_{n\to\infty}\frac{1}{n}\log\lambda_0(V_n,\nu) = \lim_{n\to\infty}\frac{1}{n}\log\lambda_1(V_n,\nu) = \log 9.$$

Note that $\frac{d^2}{dx^2}f(x) = \delta(x)$. So we have

$$\frac{d^2}{dx^2}(P_tf)(x) = \frac{1}{\sqrt{2\pi t}}\exp(-\frac{x^2}{2t}).$$

Then we have

$$\sum_{n=0}^{\infty}\frac{s^n}{n!}\int_{\mathbf{R}} H_{n+2}(x;T)(P_tf)(x)\nu(dx)$$

$$= \sum_{n=0}^{\infty}\frac{s^n}{n!}\frac{1}{\sqrt{2\pi T}}\int_{\mathbf{R}}\frac{d^{n+2}}{dx^{n+2}}(\exp(-\frac{x^2}{2T}))(P_tf)(x)dx$$

$$= \sum_{n=0}^{\infty}\frac{s^n}{n!}\frac{1}{\sqrt{2\pi T}}\int_{\mathbf{R}}\frac{d^{n}}{dx^{n}}(\exp(-\frac{x^2}{2T}))\frac{d^2}{dx^2}(P_tf)(x)dx$$

$$= \frac{1}{\sqrt{2\pi t}}\frac{1}{\sqrt{2\pi T}}\int_{\mathbf{R}}\exp(-\frac{sx}{T}-\frac{s^2}{2T})\exp(-\frac{x^2}{2T})\exp(-\frac{x^2}{2t})dx$$

$$= \frac{1}{\sqrt{2\pi(T+t)}}\exp(\frac{tTs^2}{2T^2(T+t)}-\frac{s^2}{2T}) = \frac{1}{\sqrt{2\pi(T+t)}}\exp(-\frac{s^2}{2(T+t)}).$$

So we have

$$\int_{\mathbf{R}} H_{2m+2}(x;T)(P_tf)(x)\nu(dx) == \frac{1}{\sqrt{2\pi(T+t)}}\frac{(2m)!}{m!}(-\frac{1}{2(T+t)})^m$$

and so

$$\int_{\mathbf{R}} e_{2m+2}(x;T)(P_tf)(x)\nu(dx)$$

$$= (\frac{T^{2m+2}}{(2m+2)!})^{1/2}\frac{1}{\sqrt{2\pi(T+t)}}\frac{(2m)!}{m!}(-\frac{1}{2(T+t)})^m$$

$$= \frac{1}{\sqrt{2\pi(T+t)}}\left(\frac{1}{(2m+1)(2m+2)}\right)^{1/2}$$

$$\times T\frac{(2m)^m e^{-m}(2m)^{-1/4}\exp(a_{2m}/2)}{2^m m^m e^{-m} m^{-1/2}\exp(a_m)}(-1)^m(1+\frac{t}{T})^{-m}.$$

So we see that

$$\lim_{m\to\infty} m^{3/4}(1+\frac{t}{T})^m \left|\int_{\mathbf{R}} e_{2m+2}(x;T)P_t f(x)\nu(dx)\right|$$

exists and is positive. Since we see that

$$\left|\int_{\mathbf{R}} e_{2m+2}(x;T)P_t f(x)\nu(dx)\right|^2 \leqq ||P_t f - \pi_{2m}P_t f||^2_{L^2(d\nu)},$$

we have our assertion.

3.7 A remark on Hörmander type diffusion processes

Let $N, d \geqq 1$. Let $W_0 = \{w \in C([0,\infty);\mathbf{R}^d);\ w(0) = 0\}$, $\mathcal{F}$ be the Borel algebra over W_0 and μ be the Wiener measure on $(W_0, \mathcal{F})$. Let $B^i : [0,\infty) \times W_0 \to \mathbf{R}$, $i = 1,\dots,d$, be given by $B^i(t,w) = w^i(t)$, $(t,w) \in [0,\infty) \times W_0$. Then $\{(B^1(t),\dots,B^d(t); t \in [0,\infty)\}$ is a d-dimensional Brownian motion under μ. Let $B^0(t) = t$, $t \in [0,\infty)$. Let $V_0, V_1, \dots, V_d \in C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$. Here $C_b^\infty(\mathbf{R}^N;\mathbf{R}^n)$ denotes the space of $\mathbf{R}^n$-valued smooth functions defined in $\mathbf{R}^N$ whose derivatives of any order are bounded. We regard elements in $C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$ as vector fields on $\mathbf{R}^N$.

Now let $X(t,x)$, $t \in [0,\infty)$, $x \in \mathbf{R}^N$, be the solution to the Stratonovich stochastic integral equation

$$X(t,x) = x + \sum_{i=0}^d \int_0^t V_i(X(s,x)) \circ dB^i(s). \tag{3.3}$$

Then there is a unique solution to this equation. Moreover we may assume that $X(t,x)$ is continuous in t and smooth in x and $X(t,\cdot) : \mathbf{R}^N \to \mathbf{R}^N$, $t \in [0,\infty)$, is a diffeomorphism with probability one.

Let $\mathcal{A} = \{\emptyset\} \cup \bigcup_{k=1}^\infty \{0,1,\dots,d\}^k$ and for $\alpha \in \mathcal{A}$, let $|\alpha| = 0$ if $\alpha = \emptyset$, let $|\alpha| = k$ if $\alpha = (\alpha^1,\dots,\alpha^k) \in \{0,1,\dots,d\}^k$, and let $\parallel \alpha \parallel = |\alpha| + \mathrm{card}\{1 \leqq i \leqq |\alpha|;\ \alpha^i = 0\}$. Let $\mathcal{A}^*$ and $\mathcal{A}^{**}$ denote $\mathcal{A} \setminus \{\emptyset\}$ and $\mathcal{A} \setminus \{\emptyset, 0\}$, respectively. Also, for each $m \geqq 1$, $\mathcal{A}^{**}_{\leqq m}$, $\{\alpha \in \mathcal{A}^{**};\ \parallel \alpha \parallel \leqq m\}$.

We define vector fields $V_{[\alpha]}$, $\alpha \in \mathcal{A}$, inductively by

$$V_{[\emptyset]} = 0, \qquad V_{[i]} = V_i, \quad i = 0,1,\dots,d,$$

$$V_{[\alpha * i]} = [V_{[\alpha]}, V_i], \qquad i = 0,1,\dots,d.$$

Here $\alpha * i = (\alpha^1,\dots,\alpha^k, i)$ for $\alpha = (\alpha^1,\dots,\alpha^k)$ and $i = 0,1,\dots,d$.

We say that a system $\{V_i; i = 0,1,\dots,d\}$ of vector fields satisfies the following condition (UFG).

(UFG) There are an integer ℓ_0 and $\varphi_{\alpha,\beta} \in C_b^\infty(\mathbf{R}^N)$, $\alpha \in \mathcal{A}^{**}$, $\beta \in \mathcal{A}^{**}_{\leqq \ell_0}$, satisfying the following.

$$V_{[\alpha]} = \sum_{\beta \in \mathcal{A}^{**}_{\leqq \ell_0}} \varphi_{\alpha,\beta} V_{[\beta]}, \qquad \alpha \in \mathcal{A}^{**}.$$

Let $A(x) = (A^{ij}(x))_{i,j=1,\dots,N}$, $t > 0$, $x \in \mathbf{R}^N$, be a $N \times N$ symmetric matrix given by

$$A^{ij}(x) = \sum_{\alpha \in \mathcal{A}^{**}_{\leqq \ell_0}} V^i_{[\alpha]}(x) V^j_{[\alpha]}(x), \qquad i,j = 1,\dots,N.$$

Let $h(x) = \det A(x), x \in \mathbf{R}^N$, and $E = \{x \in \mathbf{R}^N;\ h(x) > 0\}$. By Kusuoka-Stroock [15], we see that if $x \in E$, the distribution law of $X(t,x)$ under μ has a smooth density function $p(t,x,\cdot) : \mathbf{R}^N \to [0,\infty)$ for $t > 0$.

By Kusuoka-Morimoto [14] Propositions 3, 8 and 9, we see the following.

Proposition 3.12 *For any $p > 1$ and $T > 0$, there is a $C \in (0,\infty)$ such that*

$$\int_E p(t,x,y) h(y)^{-p} dy \leqq C h(x)^{-p}, \qquad x \in E,\ t \in (0,T].$$

Proposition 3.13 *For any $T > 0$, there are $C \in (0,\infty)$ and $\delta_0 > 0$ such that*

$$p(t,x,y) \leqq C t^{-(N+1)\ell_0/2} h(x)^{-2(N+1)\ell_0} \exp(-\frac{2\delta_0}{t}|y-x|^2)$$

and

$$p(t,x,y) \leqq C t^{-(N+1)\ell_0/2} h(y)^{-2(N+1)\ell_0} \exp(-\frac{2\delta_0}{t}|y-x|^2)$$

for any $x, y \in E$, and $t \in (0,T]$.

Proposition 3.14 *Let $\delta \in (0,1/N)$, $\alpha, \beta \in \mathbf{Z}^N_{\geqq 0}$ and $T > 0$. Then there are $C \in (0,\infty)$ such that*

$$|\partial_x^\alpha \partial_y^\beta p(t,x,y)| \leqq C t^{-(|\alpha|+|\beta|+1)\ell_0/2} h(x)^{-2(|\alpha|+|\beta|+1)\ell_0} p(t,x,y)^{1-\delta}$$

and

$$|\partial_x^\alpha \partial_y^\beta p(t,x,y)| \leqq C t^{-(|\alpha|+|\beta|+1)\ell_0/2} h(y)^{-2(|\alpha|+|\beta|+1)\ell_0} p(t,x,y)^{1-\delta}$$

for any $x, y \in E$, and $t \in (0,T]$.

Then we have the following.

Proposition 3.15 *For any $m \geqq 1$ and $T > 0$, there is a $C \in (0,\infty)$ such that*

$$p(t,x,y) \leqq C t^{-N\ell_0} h(x)^{-(4N\ell_0+m+1)} h(y)^m, \qquad x, y \in E,\ t \in (0,T].$$

Proof. Note that for any $\varepsilon > 0$ we have

$$|\frac{\partial}{\partial y_i}(p(t,x,y)(\varepsilon + h(y))^{-m})|^{N+1}$$

$$\leqq 2^{N+1} |\frac{\partial}{\partial y_i} p(t,x,y))|^{N+1} (\varepsilon + h(y))^{-m(N+1)}$$

$$+2^{N+1}m^{N+1}p(t,x,y)^{N+1}(\varepsilon+h(y))^{-(m+1)(N+1)}|\frac{\partial h}{\partial y_i}(y)|^{N+1}.$$

By Proposition 3.12 and 3.13, we see that

$$\sup\{t^{N(N+1)\ell_0/2}h(x)^{2N(N+1)\ell_0+m(N+1)}\int_{\mathbf{R}^N}|p(t,x,y)(\varepsilon+h(y))^{-m}|^{N+1}dy;$$
$$t\in[0,T],\ x\in E,\ \varepsilon>0\}<\infty.$$

Also letting $\delta=1/(N+1)$ in Proposition 3.14, we see by Proposition 3.12 that

$$\sup\{t^{(N-1)(N+1)\ell_0}h(x)^{4(N-1)(N+1)\ell_0+(m+1)(N+1)}$$
$$\times\sum_{i=1}^N\int_{\mathbf{R}^N}|\frac{\partial}{\partial y_i}(p(t,x,y)(\varepsilon+h(y))^{-m}))|^{N+1}dy;$$
$$t\in[0,T],\ x\in E,\ \varepsilon>0\}<\infty.$$

These and Sobolev's inequality imply that there is a $C>0$ such that

$$t^{N\ell_0}h(x)^{4N\ell_0+m+1}p(t,x,y)(\varepsilon+h(y))^{-m}\leqq C,$$

for any $x\in E$, $y\in\mathbf{R}^N$, $t\in(0,T]$, and $\varepsilon>0$. This proves our assertion. ∎

Let P_t, $t\geqq 0$, be a diffusion operator defined in $C_b^\infty(\mathbf{R}^N)$ given by

$$(P_tf)(x)=E[f(X(t,x))],\qquad f\in C_b^\infty(\mathbf{R}^N).$$

Then we see that

$$(P_tf)(x)=\int_E p(t,x,y)f(y)dy,\qquad x\in E.$$

Then we have the following.

Proposition 3.16 *For any $T>0$ and $\alpha\in\mathbf{Z}^N_{\geqq 0}$, there is a $C\in(0,\infty)$ such that*

$$|\frac{\partial^\alpha}{\partial x^\alpha}(P_tf)(x)|\leqq Ct^{-(|\alpha|+N+2)\ell_0/2}h(x)^{-2(|\alpha|+N+2)\ell_0}(P_t(|f|^2)(x))^{1/2}$$

for any $t\in(0,T]$, $x\in E$ and $f\in C_b^\infty(\mathbf{R}^N)$.

Proof. By Proposition 3.14, we see that there is a $C_1\in(0,\infty)$ such that for any $f\in C_b^\infty(\mathbf{R}^N)$

$$|\frac{\partial^\alpha}{\partial x^\alpha}(P_tf)(x)|\leqq\int_E|\frac{\partial^\alpha p}{\partial x^\alpha}(t,x,y)||f(y)|dy$$
$$\leqq C_1t^{-(|\alpha|+1)\ell_0/2}h(x)^{-2(|\alpha|+1)\ell_0}\int_E p(t,x,y)^{2N/(2N+1)}|f(y)|dy$$
$$\leqq C_1t^{-(|\alpha|+1)\ell_0/2}h(x)^{-2(|\alpha|+1)\ell_0}|\int_E f(y)^2p(t,x,y)dy|^{1/2}$$
$$\times|\int_E p(t,x,y)^{(2N-1)/(4N+2)}dy|^{1/2}.$$

By Proposition 3.13, we see that there is a $C_2>0$ such that

$$\int_E p(t,x,y)^{(2N-1)/(4N+2)}dy\leqq C_2t^{-(N+1)\ell_0/4}h(x)^{-(N+1)\ell_0},\qquad x\in E,t\in(0,T].$$

So we have our assertion. ∎

The following is an easy consequence of Proposition 3.14.

Proposition 3.17 *For any $\beta \in (0,1/N)$ and $T > 0$, there is a $C > 0$ such that*

$$|\frac{\partial}{\partial y^i}(p(t,x,y)^\beta)| \leqq Ct^{-\ell_0}h(x)^{-4\ell_0}, \qquad x \in E,\ t \in (0,T].$$

3.8 A random system of piece-wise polynomials

Let ν be a probability measure on $\mathbf{R}^N$.

For any $m \geqq 2$, let

$$D^{(m)}_{\vec{k}} = \prod_{i=1}^N [\frac{(2(k_i-1)-m)}{m}\log m, \frac{(2k_i - m)}{m}\log m),$$

for $\vec{k} = (k_1,\dots,k_N) \in \{1,\dots,m\}^N$. Let $\mathcal{D}_m = \{D^{(m)}_{\vec{k}};\ \vec{k} \in \{1,\dots,m\}^N\}$. Then we have $\bigcup \mathcal{D}_m = [-\log m, \log m)^N$.

Let $X_1, X_2, \dots,$ i.i.d. random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ whose distributions are ν. Let $\mathcal{D}_{m,n}(\omega)$, $m,n \geqq$, $\omega \in \Omega$, be a random sub-family of $\mathcal{D}_m$ given by

$$\mathcal{D}_{m,n}(\omega) = \{D \in \mathcal{D}_m;\ \text{there is a } k \in \{1,\dots,n\} \text{ such that } X_k(\omega) \in D\ \}.$$

Let $\mathcal{P}_r$, $r = 0,1,2,\dots,$ be the set of polynomials on $\mathbf{R}^N$ of degree less than or equal to r. Now let $V_{n,m,r}(\omega)$, $m,n \geqq 2$, $r \geqq 0$, $\omega \in \Omega$, be a finite dimensional vector subspace of $m(\mathbf{R}^N)$ hulled by $f1_D$, $f \in \mathcal{P}_r$, $D \in \mathcal{D}_{m,n}(\omega)$. It is obvious that dim $V_{n,m,r}(\omega) \leqq N^m(N+1)^r$.

Now let us use the notation in the previous section. Let $X(t,x)$, $t \in [0,\infty)$, $x \in \mathbf{R}^N$, be the solution to the SDE (4.6) and we assume the (UFG) condition holds. Let $x_0 \in \mathbf{R}^N$ such that $h(x_0) > 0$, and so $x_0 \in E$. Let $T_0 > 0$ and $\rho(x) = p(T_0, x_0, x)$, $x \in \mathbf{R}^N$. We think of the case that $\nu(dx) = \rho(x)dx$.

Then we have the following.

Theorem 3.18 *Let $r \geqq 0$, $\delta > 0$, $\gamma > 0$, and $T > 0$, and let n_m, $m =, 2, \dots,$ be integers satisfying $m^{N+\gamma} \leqq n_m < 2m^{N+\gamma}$. Then there are $\Omega_m \in \mathcal{F}$, $m = 1,2,\dots,$ and $C \in (0,\infty)$ satisfying the following.*

(1) *$P(\Omega_m) \to 1$, $m \to \infty$.*

(2) *For any $\omega \in \Omega_m$,*

$$\inf_{x \in D} \rho(x) \geqq \frac{1}{2} \sup_{x\in D} \rho(x),$$

and

$$\nu(D) \geqq C^{-1} m^{-(2N+\gamma+\delta)}$$

for any $D \in \mathcal{D}_{m,n_m}(\omega)$ and $m \geqq 2$.

(3) *For any $\omega \in \Omega_m$, $\lambda_0(V_{m,n_m,r}, \nu) \leqq Cm^{2N+\gamma+\delta}$.*

(4) *For any $\omega \in \Omega_m$, $f \in C^\infty_b(\mathbf{R}^N)$ and $t \in (0,T]$,*

$$||P_t f - \pi_{V_{m,n_m,r}} P_t f||_{L^2(d\nu)}$$

$$\leqq C(t^{-(r+2N+3)\ell_0} m^{-(r+1)+\delta} + m^{-\gamma/4+\delta})(\int_{\mathbf{R}^N} f(y)^4 p(T_0+t, x_0, y)dy)^{1/4}.$$

Here $\pi_{V_{m,n,r}}$ is the orthogonal projection in $L^2(E; d\nu)$ onto $V_{m,n,r}(\omega)$.

We make some preparations to prove Theorem 3.18.

Proposition 3.19 *For any $r \geqq 0$, there is a $C_r > 0$ such that*

$$(\int_{(-\varepsilon,\varepsilon)^N} f(y)^4\, dy)^{1/4} \leqq C_r \varepsilon^{-N/4} (\int_{(-\varepsilon,\varepsilon)^N} f(y)^2\, dy)^{1/2}$$

for any $\varepsilon > 0$ and $f \in \mathcal{P}_r$.

Proof. Let us fix $n \geqq 0$. Since $\mathcal{P}_r$ is a finite dimensional vector space, any norms on $\mathcal{P}_r$ are equivalent. So we see that there is a $C_r > 0$ such that

$$(\int_{(-1,1)^N} |f(x)|^4\, dx)^{1/4} \leqq C_r (\int_{(-1,1)^N} |f(x)|^2\, dx)^{1/2}, \qquad f \in \mathcal{P}_r.$$

Then we see that

$$(\int_{(-\varepsilon,\varepsilon)^N} f(x)^4\, dx)^{1/4} = \varepsilon^{N/4} (\int_{(-1,1)^N} f(\varepsilon x)^4\, dx)^{1/4}$$

$$\leqq C_r \varepsilon^{N/4} (\int_{(-1,1)^N} f(\varepsilon x)^2\, dx)^{1/2} = C_r \varepsilon^{-N/4} (\int_{(-\varepsilon,\varepsilon)^N} f(x)^2\, dx)^{1/2}.$$

This implies our assertion. ∎

For any Borel subset A in $\mathbf{R}^N$ and n, let $N_n(A)$ be $N_n(A) = \sum_{k=1}^n 1_A(X_i)$.

Let $\gamma > 0$ and $\delta \in (0, \gamma/2)$, and fix them. Let $\gamma_0 = N + \gamma - \delta/3$ and $\gamma_1 = 2N + \gamma + \delta/3$. Now let $\mathcal{D}_m^{(0)}$ and $\mathcal{D}_m^{(1)}$ be subsets of $\mathcal{D}_m$, $m \geqq 1$, given by

$$\mathcal{D}_m^{(0)} = \{D \in \mathcal{D}_m;\ \nu(D) \geqq m^{-\gamma_0}\},$$

and

$$\mathcal{D}_m^{(1)} = \{D \in \mathcal{D}_m;\ \nu(D) \geqq m^{-\gamma_1}\}.$$

Then it is obvious that $\mathcal{D}_m^{(0)} \subset \mathcal{D}_m^{(1)}$.

Then we have the following.

Proposition 3.20 (1) *Let $\Omega_{0,m,n}$, $m \geqq 2$, $n \geqq 1$, be the set of $\omega \in \Omega$ such that $\mathcal{D}_m^{(0)} \subset \mathcal{D}_{m,n}(\omega)$. Then we have*

$$P(\Omega \setminus \Omega_{0,m,n}) \leqq m^N \exp(-nm^{-(N+\gamma)} m^{\delta/3}), \qquad n \geqq 1,\ m \geqq 2.$$

(2) *Let $\Omega_{1,m,n}$, $m \geqq 2$, $n \geqq 1$, be the set of $\omega \in \Omega$ such that $\mathcal{D}_{m,n}(\omega) \subset D_m^{(1)}$. Then there is an $m_1 \geqq 1$ such that*

$$P(\Omega \setminus \Omega_{1,m,n}) \leqq (2\log 2) nm^{-(N+\gamma)} m^{-\delta/3} \qquad n \geqq 1,\ m \geqq m_1.$$

Proof. Since $(1 - 1/x)^x$, $x \in (1, \infty)$ is increasing in x, we see that

$$\frac{1}{4} \leqq (1 - \frac{1}{x})^x \leqq e^{-1}, \qquad x \geqq 2.$$

For $D \in \mathcal{D}_m$ we have

$$P(N_n(D) = 0) = (1 - \nu(D))^n = ((1 - \nu(D))^{1/\nu(D)})^{n\nu(D)}.$$

Thus we see that

$$P(N_n(D) = 0) \leqq \exp(-n\nu(D))$$

for any $D \in \mathcal{D}_m$, and

$$2^{-2n\nu(D)} \leqq P(N_n(D) = 0)$$

for any $D \in \mathcal{D}_m$ with $\nu(D) \in [0, 1/2]$. So we see that for any $D \in \mathcal{D}_m$ with $\nu(D) \in [0, 1/2]$,

$$P(N_n(D) \geqq 1) \leqq 1 - \exp(-(2\log 2)n\nu(D)) \leqq (2\log 2)n\nu(D).$$

Note that

$$\nu(D) \leqq (2m^{-1}\log m)^N \sup_{x \in \mathbf{R}^N} \rho(x).$$

So there is an $m_1 \geqq 1$ such that $\nu(D) \leqq 1/2$ for $D \in \mathcal{D}_m$, $m \geqq m_1$.

Therefore we see that

$$P(\Omega \setminus \Omega_{0,m,n}) \leqq \sum_{D \in \mathcal{D}_m^{(0)}} P(N_n(D) = 0) \leqq m^N \exp(-nm^{-\gamma_0}),$$

and

$$P(\Omega \setminus \Omega_{1,m,n}) \leqq \sum_{D \in \mathcal{D}_m \setminus \mathcal{D}_m^{(0)}} P(N_n(D) \geqq 1) \leqq (2\log 2)nm^{N-\gamma_1} \qquad m \geqq m_1.$$

So we have our assertions. ∎

Proposition 3.21 *There is an $m_2 \geqq 1$ satisfying the following.*
If $D \in \mathcal{D}_m^{(1)}$, then

$$\inf_{x \in D} \rho(x) \geqq \frac{1}{2} \sup_{x \in D} \rho(x) \geqq m^{-(N+\gamma+2\delta/3)}, \qquad m \geqq m_2.$$

Proof. Assume that $D \in \mathcal{D}_m^{(1)}$. Let $x_1 \in \bar{D}$ be a maximal point of $\rho(x)$, $x \in \bar{D}$. Then we see that $\rho(x_1) \geqq (2\log m)^N m^{-N-\gamma-\delta/3}$. Appliing Proposition 3.17 for $\beta = 1/(2(N+\gamma+\delta/3)) > 0$, we see that there is a $C_0 > 0$ such that

$$|\rho(x)^\beta - \rho(y)^\beta| \leqq C_0|x - y|, \qquad x, y \in \mathbf{R}^N.$$

So we see that

$$|\rho(x)^\beta - \rho(x_1)^\beta| \leqq C_0 \frac{2N\log m}{m}. \qquad x \in D,$$

and so

$$\rho(x)^\beta \geqq \rho(x_1)^\beta - C_0 \frac{2N\log m}{m}$$

$$\geqq (\frac{1}{2}\rho(x_1))^\beta + (1 - 2^{-\beta})(2\log m)^{-N\beta} m^{-1/2} - C_0 \frac{2N\log m}{m}$$

So we see that if m is sufficiently large

$$\inf_{x \in D} \rho(x) \geqq \frac{1}{2} \sup_{x \in D} \rho(x) \geqq m^{-(N+\gamma+2\delta/3)}.$$

Thus we have our assertion.

Proposition 3.22 *There is an* $m_3 \geqq 1$ *satisfying the following. If* $\omega \in \Omega_{1,n,m}$ *and* $m \geqq m_3$*, then*

$$\lambda_0(V_{m,n,r}(\omega);\nu) \leqq m^{2N+\gamma+\delta}.$$

Proof. Let $m_2 \geqq 1$ be as in Proposition 3.21. Suppose that $\omega \in \Omega_{1,n,m}$ and $m \geqq m_2$. Then $\mathcal{D}_{m,n}(\omega) \subset \mathcal{D}_m^{(1)}$.

Let $f \in V_{m,n,r}(\omega)$. Then there are $f_D \in \mathcal{P}_r$, $D \in \mathcal{D}_{m.n}(\omega)$, such that

$$f = \sum_{D\in\mathcal{D}_{m.n}(\omega)} f_D 1_D.$$

Then we see that

$$\int_{\mathbf{R}^N} f(x)^4\nu(dx) = \sum_{D\in\mathcal{D}_{m.n}(\omega)} \int_D f_D(x)^4\nu(dx) \leqq \sum_{D\in\mathcal{D}_{m.n}(\omega)} \sup_{x\in D}\rho(x)\int_D f_D(x)^4 dx$$

$$\leqq 2\sum_{D\in\mathcal{D}_{m.n}(\omega)} \inf_{x\in D}\rho(x) C_r^4(2m^{-1}\log m)^{-N}(\int_D f_D(x)^2dx)^2$$

$$\leqq 2\sum_{D\in\mathcal{D}_{m.n}(\omega)} \frac{1}{\inf_{x\in D}\rho(x)} C_r^4(2m^{-1}\log m)^{-N}(\int_D f_D(x)^2\nu(dx))^2$$

$$\leqq m^{2N+\gamma+\delta}(2^{N+1}C_r^4 m^{-\delta/3}(\log m)^{-N})(\int_{\mathbf{R}^N} f(x)^2\nu(dx))^2.$$

This implies our assertion. ∎

Proposition 3.23 *For any* $r \geqq 0$*, there is a* $C \in (0,\infty)$ *satisfying the following.*

$$\inf\{(\int_{(-\varepsilon,\varepsilon)^N} |f(x)-g(x)|^2dx)^{1/2};\ g\in\mathcal{P}_r\}$$

$$\leqq C\varepsilon^{r+1}\sum_{\alpha\in\mathbf{Z}^N_{\geqq 0}, r+1\leqq|\alpha|\leqq r+N+1} (\int_{(-\varepsilon,\varepsilon)^N} |\frac{\partial^\alpha f}{\partial x^\alpha}(x)|^2dx)^{1/2}$$

for any $f \in C^\infty(\mathbf{R}^N)$ *and* $\varepsilon \in (0,1]$.

Proof. By Sobolev's inequality, we see that there is a $C_0 >$ such that

$$\sup_{x\in(-1,1)^N}|f(x)| \leqq C_0\sum_{\alpha\in\mathbf{Z}^N_{\geqq 0},|\alpha|\leqq N}(\int_{(-1,1)^N}|\frac{\partial^\alpha f}{\partial x^\alpha}(x)|^2dx)^{1/2}, \qquad f\in C^\infty(\mathbf{R}^N).$$

So we see that

$$\sup_{x\in(-\varepsilon,\varepsilon)^N}|f(x)| \leqq C_0\sum_{\alpha\in\mathbf{Z}^N_{\geqq 0},|\alpha|\leqq N}|\int_{(-1,1)^N}|\frac{\partial^\alpha}{\partial x^\alpha}(f(\varepsilon x))|^2dx)^{1/2}$$

$$\leqq C_0\sum_{\alpha\in\mathbf{Z}^N_{\geqq 0},|\alpha|\leqq N}\varepsilon^{|\alpha|-N/2}(\int_{(-\varepsilon,\varepsilon)^N}|\frac{\partial^\alpha f}{\partial x^\alpha}(x)|^2dx)^{1/2}.$$

For any $f \in C^\infty(\mathbf{R}^N)$,

$$|f(x) - \sum_{\alpha \in \mathbf{Z}^N_{\geqq 0}, |\alpha| \leqq r} \frac{1}{\alpha!}\frac{\partial^\alpha f}{\partial x^\alpha}(0)x^\alpha| \leqq \int_0^t \frac{(1-t)^r}{r!}|\frac{d^{r+1}}{dt^{r+1}}f(tx)|dt$$

$$\leqq |x|^{r+1} \sum_{\alpha \in \mathbf{Z}^N_{\geqq 0}, |\alpha| = r+1} \sup_{t \in [0,1]} |\frac{\partial^\alpha f}{\partial x^\alpha}(tx)|,$$

and so we have

$$\inf\{(\int_{(-\varepsilon,\varepsilon)^N} |f(x) - g(x)|^2 dx)^{1/2};\ g \in \mathcal{P}_r\}$$

$$\leqq (2N\varepsilon)^{r+1+N/2} \sum_{\alpha \in \mathbf{Z}^N_{\geqq 0}, |\alpha| = r+1} \sup_{x \in (-\varepsilon,\varepsilon)^N} |\frac{\partial^\alpha f}{\partial x^\alpha}(x)|$$

$$\leqq \varepsilon^{r+1} C_0 (2N)^{r+1+N/2} \sum_{\alpha,\beta \in \mathbf{Z}^N_{\geqq 0}, |\alpha| = r+1, |\beta| \leqq N} (\int_{((-\varepsilon,\varepsilon)^N} |\frac{\partial^{\alpha+\beta} f}{\partial x^{\alpha+\beta}}(x)|^2 dx)^{1/2}.$$

This implies our assertion ∎

Proposition 3.24 *For any $T > 0$ there is an $m_4 \geqq 1$ such that for any $D \in \mathcal{D}^{(1)}_m$, $m \geqq m_4$,*

$$\inf\{\int_D |P_t f(x) - g(x)|^2 \nu(dx);\ g \in \mathcal{P}_r\}$$

$$\leqq m^{-2(r+1)+2\delta/3} t^{-(r+2N+3)\ell_0/2} \int_D P_t(|f|^2)(x)\nu(dx), \qquad t \in (0,T],\ f \in C^\infty_b(\mathbf{R}^N).$$

Proof. Let $m_2 \geqq 1$ be as in Proposition 3.21. Then

$$\rho(x) \geqq m^{-(N+\gamma+2\delta/3)}, \qquad x \in D,\ D \in \mathcal{D}^{(1)}_m$$

for any $m \geqq m_2$. By Proposition 3.15, there is a $C_0 > 0$ such that

$$h(x) \geqq C_0 m^{-\delta/(8(r+2N+3)\ell_0)}, \qquad x \in D,\ D \in \mathcal{D}^{(1)}_m,\ m \geqq m_2.$$

Then by Proposition 3.16 we see that there is a $C_1 > 0$ such that

$$\sum_{\alpha \in \mathbf{Z}^N_{\geqq 0}, r+1 \leqq |\alpha| \leqq N+r+1} |\frac{\partial^\alpha}{\partial x^\alpha} P_t f(x)| \leqq C_1 m^{\delta/4} t^{-(r+2N+3)\ell_0/2} (P_t(|f|^2)(x))^{1/2},$$

for any $x \in D$, $D \in \mathcal{D}^{(1)}_m$, $m \geqq m_2$, and $f \in C^\infty_b(\mathbf{R}^N)$. Then by Propositions 3.23 we see that there is a $C_2 > 0$ such that for $D \in \mathcal{D}^{(1)}_m$, $m \geqq m_2$,

$$\inf\{(\int_D |P_t f(x) - g(x)|^2 \nu(dx))^{1/2};\ g \in \mathcal{P}_r\}$$

$$\leqq (\sup_{x \in D} \rho(x))^{1/2} \inf\{(\int_D |P_t f(x) - g(x)|^2 dx)^{1/2};\ g \in \mathcal{P}_r\}$$

$$\leqq 2(\inf_{x\in D}\rho(x))^{1/2}C_2(2m^{-1}\log m)^{r+1}\sum_{\alpha\in\mathbf{Z}^N_{\geqq 0},r+1\leqq|\alpha|\leqq r+N}(\int_D|\frac{\partial^\alpha}{\partial x^\alpha}P_tf(x)|^2dx)^{1/2}$$

$$\leqq 2C_2(2m^{-1}\log m)^{r+1}C_1m^{\delta/4}t^{-(r+2N+3)\ell_0/2}(\int_D(P_t(|f|^2))(x)\nu(dx))^{1/2}.$$

So we have our assertion. ∎

Proposition 3.25 *Let* $A_{0,m}=\bigcup\mathcal{D}^{(0)}_m$*. Then there is an* $m_5\geqq 1$ *such that*

$$\nu(\mathbf{R}^N\setminus A_{0,m})\leqq m^{-\gamma+\delta},\qquad m\geqq m_5.$$

Proof. We see by Proposition 3.13 that

$$\nu(\mathbf{R}^N\setminus A_{0,m})=\nu([-\log m,\log m)^N\setminus A_{0,m})+\nu(\mathbf{R}^N\setminus[-\log m,\log m)^N)$$

$$=\sum_{D\in\mathcal{D}_m\setminus\mathcal{D}^{(0)}_m}\nu(D)+\int_{\mathbf{R}^N\setminus[-\log m,\log m)^N)}p(T_0,x_0,x)dx$$

$$\leqq m^{N-\gamma_0}+CT_0^{-(N+1)\ell_0/2}h(x_0)^{-2(N+1)\ell_0}\int_{\mathbf{R}^N\setminus[-\log m,\log m)^N)}\exp(-\frac{2\delta_0|x-x_0|^2}{T_0})dx.$$

This implies our assertion. ∎

Proposition 3.26 *Let* $r\geqq 0$*, and* $T>0$*. There is an* $m_6\geqq 2$ *satisfying the following. For any* $\omega\in\Omega_{0,m,n}$*,* $m\geqq m_6$*,* $n\geqq 1$*,*

$$||P_tf-\pi_{V_{m,n,r}}P_tf||_{L^2(d\nu)}$$

$$\leqq(t^{-(r+2N+3)\ell_0/2}m^{-(r+1)+\delta/2}+m^{-\gamma/4+\delta/2})(\int_{\mathbf{R}^N}f(y)^4p(T_0+t,x_0,y)dy)^{1/4}$$

for any $t\in(0,T]$*, and* $f\in C^\infty_b(\mathbf{R}^N)$*.*

Proof. Let $m_4,m_5\geqq 2$ be as in Propositions 3.24 and 3.25. Let $\omega\in\Omega_{0,m,n}$, and $m\geqq m_4\vee m_5$. Then we see that $\mathcal{D}_{m,n}(\omega)\supset\mathcal{D}^{(0)}_m$ and so we see that

$$\inf\{\int_{\mathbf{R}^N}|P_tf(x)-g(x)|^2\nu(dx);\ g\in\mathcal{P}_r\}$$

$$=\sum_{D\in\mathcal{D}_{m,n}(\omega)}\inf\{\int_D|(P_tf)(x)-g(x)|^2\nu(dx);\ g\in\mathcal{P}_r\}+\int_{\mathbf{R}^N\setminus\bigcup\mathcal{D}_{m,n}(\omega)}|P_tf(x)|^2\nu(dx)$$

$$\leqq\sum_{D\in\mathcal{D}_{m,n}(\omega)}m^{-2(r+1)+2\delta/3}t^{-(r+2N+3)\ell_0}\int_D P_t(|f|^2)(x)\nu(dx)$$

$$+\nu(\mathbf{R}^N\setminus A_{0,m})^{1/2}(\int_{\mathbf{R}^N}|P_tf(x)|^4\nu(dx))^{1/2}$$

$$\leqq m^{-2(r+1)+2\delta/3}t^{-(r+2N+3)\ell_0}\int_{\mathbf{R}^N}f(y)^2p(T_0+t,x_0,y)dy$$

$$+m^{-(\gamma-\delta)/2}(\int_{\mathbf{R}^N}f(y)^4p(T_0+t,x_0,y)dy)^{1/2}.$$

So this and Proposition 3.25 imply our assertion. ∎

Now we have Theorem 3.18 from Propositions 3.20, 3.21, 3.22 and 3.26, letting $\Omega_m=\Omega_{0,m,n_m}\cap\Omega_{1,m,n_m}$.

第4章 Application of Stochastic Mesh Method to Approximation of CVA

In this paper, we give two estimations of CVA by Monte Carlo simulation. Stochastic Mesh method is applied explicitly to one method and implicitly to the other. We show both of the estimations converge to the true CVA value. We also discuss the convergence speed.

4.1 Introduction

The credit valuation adjustment (CVA) is, by definition, the difference between the risk-free portfolio value and the true portfolio value that takes into account default risk of the counterparty. In other words, CVA is the market value of counterparty credit risk. After the financial crisis in 2007-2008, it has been widely recognized that even major financial institutions may default. Therefore, the market participants has become fully aware of counterparty credit risk. In order to reflect the counterparty credit risk in the price of over-the-counter (OTC) derivative transactions, CVA is widely used in the financial institutions today.

Although Duffie and Huang [8] has already introduced the basic idea of CVA in 1990's, several people reconsidered the theory of CVA related to collateralized derivatives (cf. [9]) and also efficient numerical calculation methods appeared(cf. [16]).

There are two approaches to measuring CVA: unilateral and bilateral (cf. [12]). Under the unilateral approach, it is assumed that the the own company is default-free. CVA measured in this way is the current market value of future losses due to the counterparty's potential default. The problem with unilateral CVA is that both the own company and the counterparty require a premium for the credit risk they are bearing and can never agree on the fair value of the trades in the portfolio. Therefore, we have to consider not only the market value of the counterparty's default risk, but also the company's own counterparty credit risk called debit value adjustment (DVA) in order to calculate the correct fair value. Bilateral CVA (it is calculated by netting unilateral CVA and DVA) takes into account the possibility of both the counterparty default and the own default. It is thus symmetric between the own company and the counterparty, and results in an objective fair value calculation.

Mathematically, unilateral CVA and DVA are calculated in the same way, and bilateral CVA is the difference of them. So we focus on the calculation of unilateral CVA in this paper.

CVA is measured at the counterparty level and there are many assets in the portfolio generally. Therefore, we have to be involved in the high dimensional numerical problem to obtain the value of CVA. This is the reason why CVA calculation is difficult. On the other hand, each payoff usually depends only on a few assets. We will focus on this property and suggest an efficient calculation methods of CVA in the present paper.

Let us consider the portfolio consist of the contracts on one counterparty. Let $X^{(m)}(t), m = 0, 1, \ldots, M$, be $\mathbf{R}^{N_m}$-valued stochastic processes. We think that $X(t) = (X^{(0)}(t), \ldots, X^{(M)}(t))$ is an underlying process. We consider the model that the macro factor is determined by $X^{(0)}(t)$, and the payoff of each derivative at maturity $T_k, k = 1, \ldots, K$, is the form of

$$\sum_{m=1}^{M} \tilde{F}_{m,k}(X^{(0)}(T_k), X^{(m)}(T_k)).$$

Let $T = T_K$ be the final maturity of the all contracts in the portfolio. Let τ be the default time of the counterparty, $\lambda(t)$ be the its hazard rate process, $L(t)$ be the process of loss when default takes place at time t, and $D(t, T)$ be the discount factor process from t to T. We assume that $D(0, t)$ is the function of $X^{(0)}(t)$ and that $L(t), \lambda(t)$ and $\exp(-\int_0^t \lambda(s)ds)$ are the function of $X(t)$.

Let $V_0(t)$ be the total value of all contracts in the portfolio at time t under the assumption that counterparty is default free. Then $V_0(t)$ is given by

$$V_0(t) = E[\sum_{m=1}^{M} \sum_{k; T_k \geqq t} D(t, T_k) F_{m,k}(X^{(0)}(T_k), X^{(m)}(T_k)) | \mathcal{F}_t],$$

where E denotes the expectation with respect to the risk neutral measure. Then unilateral CVA on this portfolio is the restructuring cost when the counterparty defaults. So unilateral CVA is given by

$$\begin{aligned} \mathrm{CVA} &= E[L(\tau) D(0, \tau) 1_{\{\tau < T\}} (\tilde{V}_0(\tau) \vee 0)] \\ &= E[\int_0^T L(t) \exp(-\int_0^t \lambda(s)ds) \lambda(t) D(0, t) (\tilde{V}_0(t) \vee 0) dt] \\ &= E[\int_0^T L(t) \exp(-\int_0^t \lambda(s)ds) \lambda(t) (V_0(t) \vee 0) dt], \end{aligned} \tag{4.1}$$

where

$$V_0(t) = E[\sum_{m=1}^{M} \sum_{k; T_k \geqq t} F_{m,k}(X^{(0)}(T_k), X^{(m)}(T_k)) | \mathcal{F}_t],$$

and $F_{m,k}$ is a function such as

$$F_{m,k}(X^{(0)}(T_k), X^{(m)}(T_k)) = D(0, T_k) \tilde{F}_{m,k}(X^{(0)}(T_k), X^{(m)}(T_k)).$$

Since $L(t) \exp(-\int_0^t \lambda(s)ds) \lambda(t)$ is a function of $X(t)$, we denote it by $g(t, X(t))$. Then CVA is given by the following form.

$$\mathrm{CVA} = E[\int_0^T g(t, X(t)) (E[\sum_{m=1}^{M} \sum_{k; T_k \geqq t} F_{m,k}(X^{(0)}(T_k), X^{(m)}(T_k)) | \mathcal{F}_t] \vee 0) dt]. \tag{4.2}$$

Now we prepare the mathematical setting. Let $M \geqq 1$ be fixed, $N_m \geqq 1, m = 1, \cdots, M$, $N = N_0 + \cdots + N_M$ and $\tilde{N}_m = N_0 + N_m$, $\tilde{N} = \max_{m=1\ldots,M} \tilde{N}_m$.
Let $W_0 = \{w \in C([0,\infty); \mathbf{R}^d);\ w(0) = 0\}$, $\mathcal{F}$ be the Borel algebra over W_0 and μ be the Wiener measure on $(W_0, \mathcal{F})$. Let $B^i : [0,\infty) \times W_0 \to \mathbf{R}$, $i = 1, \ldots, d$, be given by $B^i(t,w) = w^i(t)$, $(t,w) \in [0,\infty) \times W_0$. Then $\{(B^1(t), \ldots, B^d(t); t \in [0,\infty)\}$ is a d-dimensional Brownian motion. Let $B^0(t) = t$, $t \in [0,\infty)$.
Let $V_i^{(0)} \in C_b^\infty(\mathbf{R}^{N_0}; \mathbf{R}^{N_0})$, $V_i^{(m)} \in C_b^\infty(\mathbf{R}^{N_0} \times \mathbf{R}^{N_m}; \mathbf{R}^{N_m}), i = 0, \cdots, d, m = 1, \cdots, M$. Here $C_b^\infty(\mathbf{R}^m; \mathbf{R}^n)$ denotes the space of $\mathbf{R}^n$-valued smooth functions defined in $\mathbf{R}^m$ whose derivatives of any order are bounded. We regard elements in $C_b^\infty(\mathbf{R}^n; \mathbf{R}^n)$ as vector fields on $\mathbf{R}^n$.
Now let us consider the following Stratonovich stochastic differential equations.

$$X^{(0)}(t,x_0) = x_0 + \sum_{i=0}^{d} \int_0^t V_i^{(0)}(X^{(0)}(s,x_0)) \circ dB_i(s), \tag{4.3}$$

$$X^{(m)}(t,\tilde{x}_m) = x_m + \sum_{i=1}^{d} \int_0^t V_i^{(m)}(X^{(0)}(s,x_0), X^{(m)}(s,\tilde{x}_m)) \circ dB_i(s), \tag{4.4}$$

where $x_m \in \mathbf{R}^{N_m}, \tilde{x}_m = (x_0, x_m) \in \mathbf{R}^{N_0} \times \mathbf{R}^{N_m}, m = 1, \ldots, M$.
Let $\tilde{X}^{(m)}(t,\tilde{x}_m) = (X^{(0)}(t,x_0), X^{(m)}(t,\tilde{x}_m))$ and $\tilde{V}_i^{(m)} \in C_b^\infty(\mathbf{R}^{N_0} \times \mathbf{R}^{N_k}; \mathbf{R}^{N_0} \times \mathbf{R}^{N_k}), i = 0, \cdots, d,\ m = 1, \cdots, M$ be

$$\tilde{V}_i^{(m)}(\tilde{x}_m) = \begin{pmatrix} V_i^{(0)}(x_0) \\ V_i^{(m)}(\tilde{x}_m) \end{pmatrix}.$$

Then we have

$$\tilde{X}^{(m)}(t,\tilde{x}_m) = \tilde{x}_m + \sum_{i=0}^{d} \int_0^t \tilde{V}_i^{(m)}(\tilde{X}^{(m)}(t,\tilde{x}_m)) \circ dB_i(s). \tag{4.5}$$

There is a unique solution $\tilde{X}^{(m)}(t,\tilde{x}_m)$ to this equation. Then $X(t,x), x \in \mathbf{R}^N$ also satisfies the solution to the following Stratonovich stochastic differential equation.

$$X(t,x) = x + \sum_{i=0}^{d} \int_0^t V_i(X(s,x)) \circ dB^i(s), \tag{4.6}$$

where $V_i, i = 1, \ldots, d$ is

$$V_i(x) = \begin{pmatrix} V_i^{(0)}(x_0) \\ V_i^{(1)}(\tilde{x}_1) \\ \vdots \\ V_i^{(M)}(\tilde{x}_M) \end{pmatrix}.$$

We assume that vector fields $V_i, i = 1, \ldots, d$, satisfy condition (UFG) stated in the section 4.2. Let E_m be defined by (4.11) in Section 4.2. By [14], if $\tilde{x}_m \in E_m$, the distribution law of $\tilde{X}^{(m)}(t,\tilde{x}_m)$ under μ has a smooth density function $p^{(m)}(t,\tilde{x_m},\cdot)$:

$\mathbf{R}^{\tilde{N}_m} \to [0,\infty)$ for $t > 0, m = 1,\dots,M$.
Let $x^* = (x_0^*,\dots,x_M^*) \in \mathbf{R}^N$. We assume that the underlying asset process is $X(t) = X(t,x^*)$. We also assume that

$$\tilde{x}_m^* = (x_0^*, x_m^*) \in E_m, m = 1,\dots,M.$$

Let $\hat{\mathcal{D}}(\mathbf{R}^n)$ denotes the space of functions on $\mathbf{R}^n$ given by

$$\hat{\mathcal{D}}(\mathbf{R}^n) = \{f \in C^2(\mathbf{R}^n); \|\frac{\partial^\alpha f}{\partial x^\alpha}\|_\infty < \infty, \text{for } 1 \leqq |\alpha| \leqq 2\},$$

where $\|f\|_\infty = \sup\{|f(x)|; x \in \mathbf{R}^n\}$.
$Lip(\mathbf{R}^n)$ denotes the space of Lipschitz continuous functions on $\mathbf{R}^n$, and we define a semi-norm $\|\cdot\|_{Lip}$ on $Lip(\mathbf{R}^n)$ by

$$\|f\|_{Lip} = \sup_{x,y\in\mathbf{R}^n, x\neq y} \frac{|f(x)-f(y)|}{|x-y|}, \quad f \in Lip(\mathbf{R}^n).$$

Let $\mathcal{M}(\mathbf{R}^n)$ be the linear subspace of $Lip(\mathbf{R}^n)$ spanned by $\{f \vee g; f, g \in \mathcal{D}(\mathbf{R}^n)\}$.

We define linear operators $P_t : Lip(\mathbf{R}^N) \to Lip(\mathbf{R}^N), t \geqq 0$, by

$$(P_t f)(x) = E^\mu[f(X(t,x))], \quad f \in Lip(\mathbf{R}^N),$$

and $P_t^{(m)} : Lip(\mathbf{R}^{\tilde{N}_m}) \to Lip(\mathbf{R}^{\tilde{N}_m}), t \geqq 0, m = 1,\dots,M$, by

$$(P_t^{(m)} f)(\tilde{x}_m) = E^\mu[f(\tilde{X}^{(m)}(t,\tilde{x}_m))], \quad f \in Lip(\mathbf{R}^{\tilde{N}_m}).$$

We remind that $L(t)\exp(-\int_0^t \lambda(s)ds)\lambda(t)$ is represented by

$$L(t)\exp(-\int_0^t \lambda(s)ds)\lambda(t) = g(t, X(t,x^*)).$$

We assume that $g : [0,T] \times \mathbf{R}^N \to [0,\infty)$ satisfies the following two conditions.
(1) $g(t,x)$ is differentiable in t and there is an integer n_1, and a constant $C_1 > 0$ such that

$$\sup_{t\in[0,T]} |\frac{\partial}{\partial t} g(t,x)| \leqq C_1(1 + |x|^{n_1}), \quad x \in \mathbf{R}^N.$$

(2) $g(t,x)$ is 2-times coninuously differentiable in x and there is an integer n_2, and a constant $C_2 > 0$ such that

$$\sup_{t\in[0,T]} |\frac{\partial^\alpha}{\partial x^\alpha} g(t,x)| \leqq C_2(1 + |x|^{n_2}), \quad x \in \mathbf{R}^N$$

for any multi index $|\alpha| \leqq 2$.
We assume that a discounted payoff functions $F_{m,k}, m = 1,\dots,M, k = 1,\dots,K$ in equation (4.2) belong to $\mathcal{M}(\mathbf{R}^{\tilde{N}_m})$. Under the assumptions above, CVA c_0 is given by

$$c_0 = E^\mu[\int_0^T \{g(t,X(t,x^*))E^\mu[\sum_{m=1}^M \sum_{k:T_k\geqq t} F_{m,k}(\tilde{X}^{(m)}(T_k,\tilde{x}_m^*))|\mathcal{F}_t] \vee 0\}dt]. \tag{4.7}$$

We will introduce numerical calculation methods by Monte Carlo simulation for c_0.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $X_\ell : [0,\infty)\times\Omega \to \mathbf{R}^N$, $\ell = 1, 2, \dots$, be continuous stochastic processes such that each probability law on $C([0,\infty); \mathbf{R}^N)$ of $X_\ell(\cdot)$ under P is the same as that of $X(\cdot, x^*)$ for all $\ell = 1, 2, \dots$, and that $\sigma\{X_\ell(t);\ t \geqq 0\}$, $\ell = 1, 2, \dots$, are independent.
Let us define projections $\pi_m : \mathbf{R}^N \to \mathbf{R}^{\tilde{N}_m}, m = 1, \dots, M$, by $\pi_m(x) = \tilde{x}_m = (x_0, x_m)$, and define $\varepsilon_0 > 0$ by $\varepsilon_0 = \min_{1\leqq k\leqq K}(T_k - T_{k-1})$. We define random linear operators (stochastic mesh operators) $Q^{(m)}_{t,T_k,\varepsilon} = Q^{(m,L,\omega)}_{t,T_k,\varepsilon}, 0 \leqq t \leqq T, 0 < \varepsilon < \varepsilon_0$, on $Lip(\mathbf{R}^{\tilde{N}_m})$ by

$$(Q^{(m,L,\omega)}_{t,T_k,\varepsilon} f)(\tilde{x}_m) = \begin{cases} \frac{1}{L}\sum_{\ell=1}^{L} \frac{f(X^{(m)}_\ell(T_k))p^{(m)}(T_k-t,\tilde{x}_m,\pi_m(X_\ell(T_k)))}{q^{(m,L,\omega)}_{t,T_k}(\pi_m(X_\ell(T_k)))}, & 0 \leqq t < T_k - \varepsilon, \\ f(\tilde{x}_m), \quad T_k - \varepsilon \leqq t \leqq T_k, \\ 0, \quad T_k < t \leqq T. \end{cases}$$

$$\text{where} \quad q^{(m,L,\omega)}_{t,T_k}(\tilde{y}_m) = \frac{1}{L}\sum_{\ell=1}^{L} p^{(m)}(T_k - t, \pi_m(X_\ell(t)), \tilde{y}_m)).$$

Let Π denotes the set of partitions $\Delta = \{t_0, t_1, \dots, t_n\}$ such that $0 = t_0 < t_1 < \dots < t_n = T$ and that $\{T_k; k = 1, \dots, K\} \subset \Delta$. Let $|\Delta| = \max_{i=1,\dots,n}(t_{i+1} - t_i)$. We define estimators $\hat{c}_i = \hat{c}_i(\varepsilon, \Delta, L) : \Omega \to \mathbf{R}, i = 1, 2$, in the following.

$$\begin{aligned} &\hat{c}_1(\varepsilon, \Delta, L)(\omega) \\ =&\frac{1}{L}\sum_{\ell=1}^{L}\sum_{i=0}^{n-1}(t_{i+1} - t_i)g(t_i, X_\ell(t_i))(\sum_{m=1}^{M}\sum_{k:T_k\geqq t_{i+1}}(Q^{(m,L,\omega)}_{t_i,T_k,\varepsilon}F_{m,k})(\pi_k(X_\ell(t_i))) \vee 0), \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} &\hat{c}_2(\varepsilon, \Delta, L)(\omega) \\ =&\sum_{i=0}^{n-1}(t_{i+1} - t_i)E^\mu[g(t_i, X(t_i, x^*))(\sum_{m=1}^{M}\sum_{k:T_k\geqq t_{i+1}} F_{m,k}(\pi_k X(T_k, x^*))) \\ &\qquad \times 1_{\{\sum_{m=1}^{M}\sum_{k:T_k\geqq t_{i+1}}(Q^{(m,L,\omega)}_{t_i,T_k,\varepsilon}F_{m,k})(\pi_k X(t_i,x^*)))\geqq 0\}}]. \end{aligned} \tag{4.9}$$

Our main results are following.

Theorem 4.1 *Let $\alpha_0 = (1 + \delta)(\tilde{N} + 1)\ell_0/4 \vee 1$. Let $\{\varepsilon_L\}_{L=1}^\infty \subset (0, \varepsilon_0)$ be a sequence and suppose that there is a constant $C_0 \in (0,\infty)$ such that $\varepsilon_L \leqq C_0 L^{-\frac{1+\delta}{2(1+\alpha_0)}}, L \geqq 1$. Then there exists a constant $C_1 \in (0,\infty)$ such that*

$$E^P[|\hat{c}_1(\varepsilon_L, \Delta, L) - c_0|] \leqq C_1(L^{-\frac{1}{1+\alpha_0}} + |\Delta|)$$

for any $L \geqq 1$ and $\Delta \in \Pi$.

Theorem 4.2 *Let $\alpha_1 = (1 + \delta)(\tilde{N} + 1)\ell_0/2 \vee 1$, and let $\{\varepsilon_L\}_{L=1}^\infty \subset (0, \varepsilon_0)$ be a sequence such that there is a constant $C_0 \in (0,\infty)$, such that $\varepsilon_L \leqq C_0 L^{-\frac{1+\delta}{2\alpha_1+1}}, L \geqq 1$. Suppose that* *there are constants* $\gamma \in (0,1]$ *and* $C_\gamma \in (0,\infty)$ *such that*

$$\sup_{\Delta} \sum_{i=0}^{n-1} (t_{i+1}-t_i))\mu(|\sum_{m=1}^{M} \sum_{k:T_k \geqq t_{i+1}} (P^{(m)}_{T_k-t_i} F_{m,k})(\pi_m X(t_i,x^*)))| \leqq \theta) \leqq C_\gamma \theta^\gamma$$

for all $\theta \in (0,1]$.
Then then there exists a constant $C_1 \in (0,\infty)$ *and* $\tilde{\Omega}(L) \in \mathcal{F}$, $L \geqq 1$, *such that*

$$P(\tilde{\Omega}(L)) \to 1,\ L \to \infty,$$

and

$$1_{\tilde{\Omega}(L)} |\hat{c}_2(\varepsilon_L, \Delta, L) - c_0| \leqq C_1 (L^{-(\frac{1}{2}+\frac{(1-\delta)}{2\alpha_1+1})\frac{1+\gamma}{2+\gamma}} + |\Delta|)$$

for any $L \geqq 1$, *and* $\Delta \in \Pi$.

Remark 4.3 *Let* $\tilde{\Omega}'(L)$ *be*

$$\tilde{\Omega}'(L) = \left\{ \omega \in \Omega; |\hat{c}_1(\varepsilon_L, \Delta, L) - c_0| \leqq C L^{-\frac{1-\delta}{1+\alpha_0}} \right\}.$$

Then by Theorem 4.1, we see that

$$P(\tilde{\Omega}'(L)) \to 1, L \to \infty,$$

and

$$1_{\tilde{\Omega}(L)} |\hat{c}_1(\varepsilon_L, \Delta, L) - c_0| \leqq C L^{-\frac{1-\delta}{1+\alpha_0}}.$$

Theorem 4.2 shows that the estimation of $\hat{c}_2$ *may be better than* $\hat{c}_1$.

We can compute the estimators $\hat{c}_i, i = 1,2$, practically in the following way. First, we generate a family of independent paths

$$\mathbf{X}_1 = \{X_\ell(t); 0 \leqq t \leqq T, \ell = 1,2,\ldots,L\}.$$

Next, by using $\mathbf{X}_1$, we compute

$$(Q^{(m,L,\omega)}_{t_i,T_k,\varepsilon} F_{m,k})(\pi_k(X_m(t_i))), \text{for every } k \text{ such that } T_k > t_i.$$

Then our estimator $\tilde{c}_1$ is

$$\tilde{c}_1 = \frac{1}{L} \sum_{\ell=1}^{L} \sum_{i=0}^{n-1} g(t_i, X_\ell(t_i)) (\sum_{k:T_k \geqq t_{i+1}} (Q^{(m,L,\omega)}_{t_i,T_k,\varepsilon} F_{m,k})(\pi_k(X_\ell(t_i))) \vee 0)(t_{i+1}-t_i).$$

We used the same paths for Monte Carlo simulation and construction of Stochastic mesh operator.

For $\tilde{c}_2$, we generate another independent family of independent paths

$$\mathbf{X}_2 = \{X'_m(t); 0 \leqq t \leqq T, m = 1,2,\ldots,M\},$$

and we compute

$$\tilde{c}_2 = \frac{1}{M}\sum_{m=1}^{M}\sum_{i=0}^{n-1} g(t_i, X'_m(t_i))\{\sum_{k:T_k \geqq t_{i+1}} F_{m,k}(X'^{(k)}_m(T_k))\}$$
$$\times 1_{\{\sum_{k:T_k \geqq t_{i+1}}(Q^{(m,L,\omega)}_{t,T,\varepsilon}F_{m,k})(\pi_k(X'_m(t_i))))\geqq 0\}}(t_{i+1}-t_i).$$

In the above computations of $\tilde{c}_1$ and $\tilde{c}_2$, we use the values of $X_\ell(t_i), t_i \in \Delta$, only. As for the computation of $\tilde{c}_2$, we do not use $Q^{(m,L,\omega)}_{t,T,\varepsilon}F_{m,k}$ explicitly. We use $Q^{(m,L,\omega)}_{t,T,\varepsilon}F_{m,k}$ only to judge whether $(P^{(k)}_{T_k-t}F_{m,k})(\pi_k(X'_m(t_i)))) > 0$ or not. So the approximation has no error when the signs of $(Q^{(m,L,\omega)}_{t,T,\varepsilon}F_{m,k})(\pi_k X(t)))$ and $(P^{(k)}_{T_k-t}F_{m,k})(\pi_k(X'_m(t_i)))$ are the same, even if there are large differences between them.

4.2 Structure of Vector Fields

Let $\mathcal{A} = \bigcup_{k=1}^{\infty}\{0,1,\ldots,d\}^k$ and $\mathcal{A}^* = \mathcal{A}\setminus\{0\}$. For $\alpha \in \mathcal{A}$, Let $|\alpha| = k$ if $\alpha = (\alpha^1,\ldots,\alpha^k) \in \{0,1,\ldots,d\}^k$, and let $\parallel \alpha \parallel = |\alpha| + \mathrm{card}\{1 \leqq i \leqq |\alpha|;\ \alpha^i = 0\}$. Also, for each $m \geqq 1$, $\mathcal{A}^*_{\leqq m} = \{\alpha \in \mathcal{A}^*;\ \parallel \alpha \parallel \leqq m\}$.

We define vector fields $V_{[\alpha]}$, $\alpha \in \mathcal{A}$, inductively by

$$V_{[i]} = V_i, \quad i = 0,1,\ldots,d,$$

$$V_{[\alpha * i]} = [V_{[\alpha]}, V_i], \qquad i = 0,1,\ldots,d.$$

Here $\alpha * i = (\alpha^1,\ldots,\alpha^k, i)$ for $\alpha = (\alpha^1,\ldots,\alpha^k)$ and $i = 0,1,\ldots,d$.

We assume that a system $\{V_i; i = 0,1,\ldots,d\}$ of vector fields satisfies the following condition (UFG).

(UFG) There is an integer ℓ_0 and there are functions $\varphi_{\alpha,\beta} \in C^\infty_b(\mathbf{R}^N)$, $\alpha \in \mathcal{A}^*$, $\beta \in \mathcal{A}^*_{\leqq \ell_0}$, satisfying the following.

$$V_{[\alpha]} = \sum_{\beta \in \mathcal{A}^*_{\leqq \ell_0}} \varphi_{\alpha,\beta} V_{[\beta]}, \qquad \alpha \in \mathcal{A}^*.$$

Proposition 4.4 *A system $\{\tilde{V}^{(m)}_i; i = 0,1,\ldots,d\}$ of vector fields also satisfies the (UFG) condition.*

Proof. We prove following by induction on $|\alpha|$.

$$V_{[\alpha]}(f \circ \pi_m) = (\tilde{V}^{(m)}_{[\alpha]} f) \circ \pi_m, \quad f \in C^\infty_b(\mathbf{R}^{\tilde{N}_m}), \tag{4.10}$$

for any $\alpha \in \mathcal{A}$ and $m = 1,\ldots,M$.

It is trivial in the case of $|\alpha| = 1$. By the assumption for induction,

$$V_{[\alpha * i]}(f \circ \pi_m) = (V_{[\alpha]}V_i - V_i V_{[\alpha]})(f \circ \pi_m)$$

$$= V_{[\alpha]}((\tilde{V}^{(m)}_i f) \circ \pi_m) - V_i((\tilde{V}^{(m)}_{[\alpha]} f) \circ \pi_m)$$

$$= (\tilde{V}^{(m)}_{[\alpha]}(\tilde{V}^{(m)}_i f)) \circ \pi_m - (\tilde{V}^{(m)}_i(\tilde{V}^{(m)}_{[\alpha]} f)) \circ \pi_m.$$

So we have (4.10). From (UFG) condition, we have

$$V_{[\alpha]}(f \circ \pi_m) = \sum_{\beta \in \mathcal{A}^*_{\leqq \ell_0}} \varphi_{\alpha,\beta} V_{[\beta]}(f \circ \pi_m)$$

$$= \sum_{\beta \in \mathcal{A}^*_{\leqq \ell_0}} \varphi_{\alpha,\beta} (\tilde{V}^{(m)}_{[\beta]} f) \circ \pi_m.$$

Let $j_m : \mathbf{R}^{\tilde{N}_m} \to \mathbf{R}^N$ be

$$j_m(\tilde{x}_m) = (x_0, 0 \dots, 0, x_m, 0, \dots, 0).$$

Then

$$\tilde{V}^{(m)}_{[\alpha]} f = (V^{(m)}_{[\alpha]} f) \circ \pi_m \circ j_m$$

$$= (\sum_{\beta \in \mathcal{A}^*_{\leqq \ell_0}} \varphi_{\alpha,\beta} (\tilde{V}^{(m)}_{[\beta]} f) \circ \pi_m) \circ j_m = \sum_{\beta \in \mathcal{A}^*_{\leqq \ell_0}} (\varphi_{\alpha,\beta} \circ j_m) \tilde{V}^{(m)}_{[\beta]} f.$$

So we have our assertion. ∎

Let $A_m(\tilde{x}_m) = (A^{ij}_m(\tilde{x}_m))_{i,j=1,\dots,\tilde{N}_m}$, $t > 0$, $\tilde{x}_m \in \mathbf{R}^{\tilde{N}_m}$, be a $\tilde{N}_m \times \tilde{N}_m$ symmetric matrix given by

$$A^{ij}_m(\tilde{x}_m) = \sum_{\alpha \in \mathcal{A}^*_{m,\leqq \ell_0}} \tilde{V}^i_{m,[\alpha]}(\tilde{x}_m) \tilde{V}^j_{m,[\alpha]}(\tilde{x}_m), \qquad i, j = 1, \dots, \tilde{N}_m.$$

Let $h_m(\tilde{x}_m) = \det A_m(\tilde{x}_m), \tilde{x}_m \in \mathbf{R}^{\tilde{N}_m}$ and

$$E_m = \{\tilde{x}_m \in \mathbf{R}^{\tilde{N}_m};\ h_m(\tilde{x}_m) > 0\}. \tag{4.11}$$

By [15], we see that if $\tilde{x}_m \in E_m$, the distribution law of $\tilde{X}^{(m)}(t, \tilde{x}_m)$ under μ has a smooth density function $p^{(m)}(t, \tilde{x}_m, \cdot) : \mathbf{R}^{\tilde{N}_m} \to [0, \infty)$ for $t > 0$. Moreover, by [14] we see that $\int_{E_m} p^{(m)}(t, \tilde{x}_m, \tilde{y}_m) dy = 1$, $\tilde{x}_m \in E_m$. We have $p_m(t, \tilde{x}_m, y) = 0, y \in E^c_m$ by [14].

4.3 Prepareations

In this section, we use the notation in [13]. We have the following Lemma similarly to the proof of [13] Lemma 8 (3).

Lemma 4.5 *For any* $\Phi \in \mathbf{D}^1_{\infty-}, \alpha \in \mathcal{A}^*_{\leqq \ell_0}$, *let*

$$(D^{(\beta)}\Phi)(t, x) = (D\Phi(t, x), k^\beta(t, x))_H$$

and

$$\Phi_\alpha(t,x) = \sum_{\beta\in\mathcal{A}^*_{\leqq\ell_0}} t^{-\|\alpha\|/2}\{-D^{(\beta)}\Phi(t,x)M^{-1}_{\alpha\beta}(t,x)$$
$$-(\sum_{\gamma_1\gamma_2\in\mathcal{A}^*_{\leqq\ell_0}} \Phi(t,x)M^{-1}_{\alpha\gamma_1}(t,x))D^{(\beta)}M^{\gamma_1\gamma_2}(t,x)M^{-1}_{\gamma_1\beta}(t,x)$$
$$+\Phi(t,x)M^{-1}_{\alpha\beta}(t,x))D^*k^\beta(t,x)\}, \quad t>0, x\in\mathbf{R}^N.$$

Then

$$E^\mu[\Phi(t,x)(V_{[\alpha]}f)(X(t,x))] = t^{-\|\alpha\|/2}E^\mu[\Phi_\alpha(t,x)f(X(t,x))],$$

and

$$\sup_{t\in[0,T],x\in\mathbf{R}^N,p\in(1,\infty)} E[|\Phi_\alpha(t,x)|^p] < \infty.$$

Let φ be a smooth function such that

$$\varphi(z) = \begin{cases} 1, z\geqq 1 \\ 0, z<0, \end{cases} \tag{4.12}$$

$$\varphi'(z)\geqq 0. \tag{4.13}$$

Let $\varphi_m(z) = \varphi(mz)$ and $\bar{\varphi}$ be

$$\bar{\varphi}_m(z) = \int_0^z \varphi_m(z')dz'. \tag{4.14}$$

Then for any $z\in\mathbf{R}$,

$$\bar{\varphi}_m(z)\to z\vee 0, \quad m\to\infty.$$

Lemma 4.6 *If* $\Phi\in\mathbf{D}^1_{\infty-}$, *then* $|\Phi|\in\mathbf{D}^1_{\infty-}$.

Proof. Let $\bar{\psi}_m(z) = \bar{\varphi}_m(z)+\bar{\varphi}_m(-z)$. Then for any $z\in\mathbf{R}$,

$$\bar{\psi}_m(z)\to|z|, \quad m\to\infty,$$

and $|\bar{\psi}'_m(z)|\leqq 1$. We have

$$D(\bar{\psi}_m(\Phi(t,x))) = \bar{\psi}'_m(\Phi(t,x))D\Phi(t,x)$$
$$=(\varphi_m(\Phi(t,x))-\varphi_m(-\Phi(t,x)))D\Phi(t,x), \quad m\geqq 1.$$

Then $\{D(\bar{\psi}_m(\Phi(t,x)))\}_{m=1}^\infty$ is a Cauchy sequence in $L^p(W_0,\mathcal{L}^1_{(2)}(H;\mathbf{R})), p>1$, because

$$\|D(\bar{\psi}_m(\Phi(t,x)))-D(\bar{\psi}_n(\Phi(t,x)))\|_H \leqq 1_{\{|\Phi(t,x)|\in[0,1/m]\}}\|D\Phi(t,x)\|_H, \quad n\geqq m\geqq 1.$$

Because $D:\mathbf{D}^1_p\to L^p(W_0,\mathcal{L}^1_{(2)}(H;\mathbf{R}))$ is a closed operator, we have $|\Phi(t,x)|\in\mathbf{D}^1_p$, for any $p>1$. So we have the assertion. ∎

Let us denote $\|\nabla F\|_\infty = \sup_{x\in\mathbf{R}^N}|(\frac{\partial F}{\partial x_1}(x),\ldots,\frac{\partial F}{\partial x_N}(x))|, \quad F\in C^\infty(\mathbf{R}^N)$.

Lemma 4.7 *Let $T > 0$. Then there exists a $C > 0$ such that*

$$E[|g(t, X(t, x^*))(P_{T-t}F)(X(t, x^*)) \vee 0 - g(s, X(s, x^*)(P_{T-s}F)(X(s, x^*)) \vee 0|]$$
$$\leqq C\|\nabla F\|_\infty \int_s^t (r^{-1/2} + (T-r)^{-1/2})dr,$$

for any $F \in C_b^\infty(\mathbf{R}^N)$ and any $0 < s < t < T$.

Proof. Let $\{M(t)\}_{0 \leqq t \leqq T}$ be

$$M(t) = E^\mu[F(X(T, x^*))|\mathcal{F}_t] = (P_{T-t}F)(X(t, x^*)).$$

$\{M(t)\}_{0 \leqq t \leqq T}$ is a $\{\mathcal{F}_t\}_{t \geqq 0}$-martingale. Let $Y(t) = g(t, X(t, x^*)), 0 \leqq t \leqq T$.

Let

$$L_t = \frac{\partial}{\partial t} + V_0 + \frac{1}{2}\sum_{i=1}^d V_i^2.$$

By Itô formula,

$$\begin{aligned} Y(t)\bar{\varphi}_m(M(t)) = Y(s)\bar{\varphi}_m(M(s)) &+ \int_s^t Y(r)\varphi_m(M(r))dM(r) \\ &+ \frac{1}{2}\int_s^t Y(r)\varphi_m'(M(r))d\langle M\rangle(r) \\ &+ \int_s^t \bar{\varphi}_m(M(r))dY(r) + \int_s^t d\langle Y, \bar{\varphi}_m(M)\rangle(r). \end{aligned}$$

Note that,

$$M(t) = M(s) + \sum_{j=1}^d \int_s^t V_j(P_{T-r}F)(X(r, x^*))dB^j(r),$$

$$\langle M\rangle(t) = \langle M\rangle(s) + \sum_{j=1}^d \int_s^t (V_j(P_{T-r}F)(X(r, x^*)))^2 dr,$$

$$Y(t) = Y(s) + \sum_{j=1}^d \int_s^t (V_j g)(X(r, x^*))dB^j(r) + \int_s^t (L_t g)(X(r, x^*))dr,$$

and

$$\langle Y, \bar{\varphi}_m(M)\rangle(t) = \langle Y, \bar{\varphi}_m(M)\rangle(s) + \sum_{j=1}^d \int_s^t (V_j g)(X(r, x^*))(V_j(P_{T-r}F))(X(r, x^*))dr.$$

So we have

$$E^\mu[|Y(t)\bar{\varphi}_m(M(t)) - Y(s)\bar{\varphi}_m(M(s))|]$$
$$= \frac{1}{2}\sum_{j=1}^d \int_s^t E^\mu[|Y(r)\varphi_m'((P_{T-r}F)(X(r, x^*)))\,(V_j(P_{T-r}F)(X(r, x^*)))^2\,|]dr$$
$$+ \int_s^t E^\mu[|\bar{\varphi}_m(M(r))(L_t g)(X(r, x^*))|]dr + \sum_{j=1}^d \int_s^t E^\mu[|(V_j g)(V_j(P_{T-r}F))(X(r, x^*))|]dr.$$

Now by the definition of φ_m and g, we have,

$$\int_s^t E^\mu[|\bar{\varphi}_m(M(r))(L_t g)(X(r,x^*))|]dr \leqq \int_s^t E^\mu[|M(r)|^2]^{1/2} E^\mu[|(L_t g)(X(r,x^*))|^2]^{1/2} dr$$

$$\leqq \sup_{t\in[0,T]} E^\mu[|(L_t g)(X(r,x^*))|^2]^{1/2} \int_s^t E^\mu[|M(r)|^2]^{1/2} dr.$$

By Burkholder's inequality,

$$\int_s^t E^\mu[|M(r)|^2]^{1/2} dr \leqq E^\mu[\langle M\rangle_t]^{1/2}(t-s) \leqq \sup_{r\in(s,t)} \|V_j(P_{T-r}F)\|_\infty (t-s).$$

On the other hand, we have,

$$\sum_{j=1}^d \int_s^t E^\mu[|(V_j g)(X(r,x^*))(V_j(P_{T-r}F))(X(r,x^*))|]dr$$

$$\leqq \|V_j(P_{T-r}F)\|_\infty \sum_{j=1}^d \int_s^t E^\mu[|(V_j g)(X(r,x^*))|]dr$$

$$\leqq \sum_{j=1}^d \sup_{r\in(s,t)} \|V_j(P_{T-r}F)\|_\infty (t-s),$$

for any $F \in C_b^\infty(\mathbf{R}^N)$ and any $0 < s < t < T$.
On the other hand

$$\varphi_m'\left((P_{T-r}F)(x^*)\right)\left(V_j(P_{T-r}F)(x^*)\right)^2$$

$$= \left(V_j(\varphi_m \circ (P_{T-r}F))\right)(x^*)\left(V_j(P_{T-r}F)\right)(x^*)$$

$$= V_j\left(\varphi_m \circ (P_{T-r}F)(x^*) V_j(P_{T-r}F)(x^*)\right) - \varphi_m \circ (P_{T-r}F)(x^*) V_j^2(P_{T-r}F)(x^*).$$

Notice that $\varphi_m' \geqq 0$, we have

$$E^\mu[|Y(r)\varphi_m'\left((P_{T-r}F)(X(r,x^*))\right)\left(V_j(P_{T-r}F)(X(r,x^*))\right)^2|]$$

$$= E^\mu[|Y(r)|\varphi_m'\left((P_{T-r}F)(X(r,x^*))\right)\left(V_j(P_{T-r}F)(X(r,x^*))\right)^2]$$

$$= I_{1,j}(r,f) - I_{2,j}(r,f),$$

where

$$I_{1,j}(r,F) = E^\mu[|g(r,X(r,x^*))| V_j\left(\varphi_m \circ (P_{T-r}F) V_j(P_{T-r}F)\right)(X(r,x^*))],$$

$$I_{2,j}(r,F) = E^\mu[|g(r,X(r,x^*))| \varphi_m \circ (P_{T-r}F)(X(r,x^*)) V_j^2(P_{T-r}F)(X(r,x^*))].$$

Let $\Phi_g(r,x) = |g(r,X(r,x^*))|$. Then by Lemma 4.6, $\Phi_g \in \mathbf{D}_p^1$. Let $\Phi_{g,i}(r,x), i = 1,\ldots,N$ be defined by the formula of Lemma 4.5. Then we have

$$I_{1,j}(r,F) = r^{-1/2} E^\mu[\Phi_{g,j}(r,x)\varphi_m \circ (P_{T-r}F)(X(r,x^*)) V_j(P_{T-r}F)(X(r,x^*))],$$

and

$$\sup_{t\in[0,T],x\in\mathbf{R}^N} E^\mu[|\Phi_{g,i}(t,x)|^p] < \infty.$$

Then there exists a constant $C > 0$ such that

$$|I_{1,j}(r,F)| \leqq Cr^{-1/2}\|V_j(P_{T-r}F)\|_\infty.$$

Also we have

$$|I_{2,j}(r,F)| \leqq CE^\mu[|g(r,X(r,x^*))|]\|V_j^2(P_{T-r}F)\|_\infty,$$

for any $F \in C_b^\infty(\mathbf{R}^N)$ and any $0 < r < T$.

Let vector field V_j be represented by $V_j = \sum_{i=1}^N v_j^i(x)\frac{\partial}{\partial x_i}$. Then we have

$$V_j(P_{T-r}F)(x) = \sum_{i=1}^N\sum_{k=1}^N v_j^i(x)(T_{\Phi_{k,i}}(T-r)\frac{\partial F}{\partial x_i})(x),$$

where $\Phi_{k,i}(t,x) = \frac{\partial X^k(t,x)}{\partial x_i}$ and

$$(T_{\Phi_{k,i}}(t)F)(x) = E^\mu[\Phi_{k,i}(t,x)F(X(t,x))].$$

Moreover, we have

$$V_j^2(P_{T-r}F)(x) = \sum_{i=1}^N\sum_{k=1}^N (V_jv_j^i(x)(T_{\Phi_{k,i}}(T-r)\frac{\partial F}{\partial x_i})(x) + v_j^i(x)(V_jT_{\Phi_{k,i}}(T-r)\frac{\partial F}{\partial x_i})(x)).$$

Then by Corollary 9 of [13], since $\Phi_{k,i} \in \mathcal{K}_0$ and there is a constant $C > 0$ such that

$$\|V_j(P_{T-r}F)\|_\infty \leqq C\|\nabla F\|_\infty,$$

and

$$\|V_j^2(P_{T-r}F)\|_\infty \leqq C(T-r)^{-1/2}\|\nabla F\|_\infty,$$

for any $F \in C_b^\infty(\mathbf{R}^N), j = 1,\ldots,d$, and any $0 < r < T$.

So we have

$$E[|g(t,X(t,x^*))\bar{\varphi}_m((P_{T-t}F)(X(t,x))) - g(s,X(s,x^*))\bar{\varphi}_m((P_{T-s}F)(X(s,x)))|]$$
$$\leqq C\|\nabla F\|_\infty\int_s^t (r^{-1/2} + (T-r)^{-1/2})dr.$$

Letting $m \to \infty$, we have our assertion. ∎

Corollary 4.8 *Let $T > 0$. There exists a constant $C > 0$ such that*

$$E[|g(t,X(t,x^*))(P_{T-t}F)(X(t,x^*)) \vee 0 - g(s,X(s,x^*))(P_{T-s}F)(X(s,x^*)) \vee 0|]$$
$$\leqq C\|F\|_{Lip}\int_s^t (r^{-1/2} + (T-r)^{-1/2})dr,$$

for any $F \in Lip(\mathbf{R}^N)$ and any $0 < s < t < T$.

Proof. For $F \in Lip(\mathbf{R}^N)$, there exists $F_m \in C_b^\infty(\mathbf{R}^N), m = 1,2,\cdots$, such that $\|\nabla F_m\| \leqq \|F\|_{Lip}$ and $F_m(x) \to F(x)$, for any $x \in \mathbf{R}^N$. So we obtain the result from Lemma 4.7. ∎

Lemma 4.9 *Let $m = 1, \dots, M$, and $T > 0$. There exists a constant $C > 0$ such that*

$$\begin{aligned} &E^\mu[|g(t, X(t, x^*))(P^{(m)}_{T-t}h)(\tilde{X}^{(m)}(t, \tilde{x}^*_m)) - h(\tilde{X}^{(m)}(t, \tilde{x}^*_m))|] \\ \leqq &C(\|\nabla h\|_\infty + \|\nabla^2 h\|_\infty)(T - t), \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} &E^\mu[|g(t, X(t, x))(P^{(m)}_{T-t}(h \vee 0))(\tilde{X}^{(m)}(t, \tilde{x}^*_m)) - (h \vee 0)(\tilde{X}^{(m)}(t, \tilde{x}^*_m))|] \\ \leqq &C(\|\nabla h\|_\infty + \|\nabla^2 h\|_\infty)(T - t). \end{aligned} \tag{4.16}$$

for any $h \in C^\infty_b(\mathbf{R}^{\tilde{N}_m}), t \in [0, T)$.

Proof. (4.15) follows from Itô's formula. So we show (4.16).
Let $\bar{\varphi}_k, k = 1, \dots,$ are as defined in (4.14). Let

$$\tilde{L}_m = \tilde{V}^{(m)}_0 + \frac{1}{2}\sum_{i=1}^d (\tilde{V}^{(m)}_i)^2.$$

By Itô's formula

$$\begin{aligned} &\bar{\varphi}_k(h(\tilde{X}^{(m)}(T, \tilde{x}^*_m)) - \bar{\varphi}_k(h(\tilde{X}^{(m)}(t, \tilde{x}^*_m)) \\ = &\int_t^T \varphi_k(h(\tilde{X}^{(m)}(s, \tilde{x}^*_m))(\tilde{V}^{(m)}_i h)(\tilde{X}^{(m)}(s, \tilde{x}^*_m))dB^{m,i}(s) \\ &+ \int_t^T \varphi_k(h(\tilde{X}^{(m)}(s, \tilde{x}^*_m))(\tilde{L}_m h)(\tilde{X}^{(m)}(s, \tilde{x}^*_m))ds \\ &+\frac{1}{2}\int_t^T \varphi'_k(h(\tilde{X}^{(m)}(s, \tilde{x}^*_m))\sum_{i=1}^{\tilde{d}_m}((\tilde{V}^{(m)}_i h)(\tilde{X}^{(m)}(s, \tilde{x}^*_m)))^2 ds \end{aligned}$$

So we have

$$\begin{aligned} &E^\mu[\bar{\varphi}_k(h(\tilde{X}^{(m)}(T, \tilde{x}^*_m))|\tilde{\mathcal{F}}^{(m)}_t] - \bar{\varphi}_k(h(\tilde{X}^{(m)}(t, \tilde{x}^*_m)) \\ = &\int_t^T E^\mu[\varphi_k(h(\tilde{X}^{(m)}(s, \tilde{x}^*_m))(\tilde{L}_m h)(\tilde{X}^{(m)}(s, \tilde{x}^*_m))|\tilde{\mathcal{F}}^{(m)}_t]ds \\ &+\frac{1}{2}\sum_{i=1}^{\tilde{d}_m}\int_t^T E[\varphi'_k(h(\tilde{X}^{(m)}(s, \tilde{x}^*_m)))((\tilde{V}^{(m)}_i h)(\tilde{X}^{(m)}(s, \tilde{x}^*_m)))^2|\tilde{\mathcal{F}}^{(m)}_t]ds \end{aligned}$$

Notice that $\varphi'_k \geqq 0$, then we have

$$\begin{aligned} &E[|g(t, X(t, x^*))E^\mu[\bar{\varphi}_k(h(\tilde{X}^{(m)}(T, \tilde{x}^*_m))|\tilde{\mathcal{F}}^{(m)}_t] - \bar{\varphi}_k(h(\tilde{X}^{(m)}(t, \tilde{x}^*_m))|] \\ &\leqq E[|g(t, X(t, x^*))|]\|\tilde{L}_m h\|_\infty(T - t) \\ &+\frac{1}{2}\sum_{i=1}^{\tilde{d}_m}\int_t^T E[|g(t, X(t, x^*))|\varphi'_k(h(\tilde{X}^{(m)}(s, \tilde{x}^*_m))(\tilde{V}^{(m)}_i h(\tilde{X}^{(m)}(s, \tilde{x}^*_m)))^2]ds. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\varphi'_k(h(\tilde{x}^*_m))(\tilde{V}^{(m)}_i h(\tilde{x}^*_m))^2 \\ = &\tilde{V}^{(m)}_i(\varphi_k \circ h)(\tilde{x}^*_m)\left(\tilde{V}^{(m)}_i h\right)(\tilde{x}^*_m) \end{aligned}$$

$$= \tilde{V}_i^{(m)} \left((\varphi_k \circ h)(\tilde{x}_m^*) \tilde{V}_i^{(m)} h(\tilde{x}_m^*) \right) - (\varphi_k \circ h)(\tilde{x}_m^*)(\tilde{V}_i^{(m)})^2 h(\tilde{x}_m^*).$$

Let $\Phi_g(t,x) = |g(t, X(t,x^*))|$ and $\Phi_{g,i}(t,x), i = 1, \dots, N$ be defined by the formula of Lemma 4.5. Then it follows that

$$\|E^\mu[\Phi_g(t,x)\tilde{V}_i^{(m)} \left((\varphi_k \circ h(\tilde{X}^{(m)}(s,\tilde{x}_m^*)))) \tilde{V}_i^{(m)} h(\tilde{X}^{(m)}(s,\tilde{x}_m^*))) \right)]\|_\infty$$

$$\leqq Cs^{-1/2} E^\mu[|\Phi_{g,1}(t,x)|] \|(\varphi_k \circ h)\tilde{V}_i^{(m)} h\|_\infty \leqq Cs^{-1/2} \|\tilde{V}_i^{(m)} h\|_\infty,$$

and

$$\|E^\mu[\Phi_g(t,x)(\varphi_k \circ h)(\tilde{X}^{(m)}(s,\tilde{x}_m^*))(\tilde{V}_i^{(m)})^2 h(\tilde{X}^{(m)}(s,\tilde{x}_m^*))]\|_\infty \leqq C\|(\tilde{V}_i^{(m)})^2 h\|_\infty.$$

So we have

$$\frac{1}{2}\sum_{i=1}^{\tilde{d}_m} \int_t^T E[g(t,X(t,x))\varphi_k'(h(\tilde{X}^{(m)}(s,\tilde{x}_m^*))(\tilde{V}_i^{(m)} h(\tilde{X}^{(m)}(s,\tilde{x}_m^*)))^2]ds$$

$$\leqq \int_t^T C'(\|\nabla h\|_\infty + \|\nabla^2 h\|_\infty)(1 + s^{-1/2})ds$$

$$\leqq C'(\|\nabla h\|_\infty + \|\nabla^2 h\|_\infty)(T-t)(1 + (T^{1/2} + t^{1/2})^{-1}).$$

Letting $k \to \infty$, we have the assertion. ∎

Corollary 4.10 *Let $m = 1, \dots, M, T > 0$ and $F \in \mathcal{M}(\mathbf{R}^{\tilde{N}_m})$. There exists a constant $C > 0$ such that*

$$E^\mu[|g(t,X(t,x^*))(P_{T-t}^{(m)}F)(\pi_m X(t,x^*)) - F(\pi_m X(t,x^*))|] \leqq C(T-t), \tag{4.17}$$

for any $t \in [0,T)$.

Proof. Notice that $\pi_m X(t,x) = \tilde{X}^{(m)}(t,\tilde{x}_m^*)$ and Lemma 4.9 is valid for $h \in \hat{\mathcal{D}}(\mathbf{R}^{\tilde{N}_m})$. On the other hand, for $F \in \mathcal{M}(\mathbf{R}^{\tilde{N}_m})$, we have the expression that

$$F = \sum_{k=1}^{K_F} a_k (f_k \vee g_k) = \sum_{k=1}^{K_F} a_k \left((f_k - g_k) \vee 0 + g_k \right),$$

$a_k \in \mathbf{R}, f_k, g_k \in \hat{\mathcal{D}}(\mathbf{R}^{\tilde{N}_m}), k = 1, \dots, K_F$. So our assertion follows from Lemma 4.9. ∎

4.4 Discretization

Let $c_\Delta, \Delta \in \Pi$, be given by

$$c_\Delta = \sum_{i=0}^{n-1} (t_{i+1} - t_i) E^\mu[g(t_i, X(t_i,x^*))\{\sum_{m=1}^{M} \sum_{k;T_k \geqq t_{i+1}}^{K} (P_{T_k - t_i}^{(m)} F_{m,k})(\pi_m X(t_i,x^*)))\} \vee 0].$$

Let $i_{(k)}, k = 1, \ldots, K$, be such that $T_k = t_{i_{(k)}}$. Then we have c_Δ is as follows.

$$c_\Delta = \sum_{k=1}^{K} \sum_{i=i_{(k-1)}}^{i_{(k)}-1} (t_{i+1} - t_i) E^\mu[g(t_i, X(t_i, x^*))\{\sum_{m=1}^{M} \sum_{k'=k}^{K} (P^{(m)}_{T_k - t_i} F_{m,k'})(\pi_m X(t_i, x^*))\} \vee 0]$$

Let $\mathcal{F}^{(\infty)}_t$, $t \geqq 0$, be sub σ-algebra of $\mathcal{F}$ given by

$$\mathcal{F}^{(\infty)}_t = \sigma\{X_\ell(s);\ s \in [0, t],\ \ell = 1, 2, \ldots\}.$$

Proposition 4.11 *There exists a constant $C > 0$ such that*

$$|c_0 - c_\Delta| \leqq C|\Delta|, \quad \Delta \in \Pi$$

.

Proof. Let

$$\tilde{F}_k(x) = \sum_{m=1}^{M} \sum_{k'=k}^{K} (P^{(m)}_{T_{k'} - T_k} F_{m,k'})(\pi_m x), k = 1, \ldots, K.$$

Then by Lemma 4.7, there is a constant $C > 0$ such that

$$\begin{aligned}
&|c_0 - c_\Delta| \\
\leqq& \sum_{k=1}^{K} \sum_{i=i_{(k-1)}}^{i_{(k)}-1} \int_{t_i}^{t_{i+1}} |E^\mu[g(t, X(t, x^*)) \left((P_{T_k - t}\tilde{F}_k)(X(t, x^*)) \vee 0\right)] \\
&- E^\mu[g(t_i, X(t_i, x^*))(P_{T_k - t_i}\tilde{F}_k)(X(t_i, x^*)) \vee 0]|dt \\
\leqq& C \sum_{k=1}^{K} \sum_{i=i_{(k-1)}}^{i_{(k)}-1} \int_{t_i}^{t_{i+1}} dt \int_{t_i}^{t} (r^{-1/2} + (T_k - r)^{-1/2}) dr \\
\leqq& C|\Delta| \sum_{k=1}^{K} \sum_{i=i_{(k-1)}}^{i_{(k)}-1} \int_{t_i}^{t_{i+1}} (r^{-1/2} + (T_k - r)^{-1/2}) dr.
\end{aligned}$$

So we have

$$|c_0 - c_\Delta| \leqq C|\Delta| \sum_{k=1}^{K} \int_{T_{k-1}}^{T_k} (r^{-1/2} + (T_k - r)^{-1/2}) dr.$$

So the assertion follows. ∎

4.5 Property of Stochastic Mesh Operator

To estimate the stochastic mesh operator, we use the following estimation of transition kernel $p^{(m)}(t, \tilde{x}_m, x)$ obtained by Proposition 8 of [14].

Proposition 4.12 *Let $\delta_0^{(m)}$ be given by*

$$\delta_0^{(m)} = (3\tilde{N}_m(\sup_{x\in\mathbf{R}^{\tilde{N}_m}} \sum_{i=1}^d |\tilde{V}_i^{(m)}(x)|^2))^{-1},$$

then for any $T > 0$, and $m = 1, \dots, M$, there is a $C > 0$ such that

$$p^{(m)}(t,x,y) \leqq Ct^{-(\tilde{N}_m+1)\ell_0/2} h_m(x)^{-2(\tilde{N}_m+1)\ell_0} \exp(-\frac{2\delta_0^{(m)}}{t}|y-x|^2), \qquad t \in (0,T],\ x,y \in E_m,$$

and

$$p^{(m)}(t,x,y) \leqq Ct^{-(\tilde{N}_m+1)\ell_0/2} h_m(y)^{-2(\tilde{N}_m+1)\ell_0} \exp(-\frac{2\delta_0^{(m)}}{t}|y-x|^2), \qquad t \in (0,T],\ x,y \in E_m.$$

In particular, for any $T > 0, m = 1, \dots, M$, and $q \geqq 1$, there is a $C > 0$ such that

$$p^{(m)}(t,x,y) \leqq Ct^{-(\tilde{N}_m+1)\ell_0/2} h_m(x)^{-2(\tilde{N}_m+1)\ell_0} (1+|x|^2)^q(1+|y|^2)^{-q}, \qquad t \in (0,T],\ x,y \in E_m.$$

Let $\nu_t^{(m)}(dx) = p^{(m)}(t, \tilde{x}_m^*, x)dx$. From Proposition 13 , 21 and Proposition 15 (1) of [14] , we have the followings.

Proposition 4.13 *Let $t > 0, f \in L^2(E_m; d\nu_t^{(m)})$ and $t > s \geqq 0$. Then we have*

$$E^P[(Q_{s,t}^{(m,L,\omega)}f)(x)|\mathcal{F}_s^{(\infty)}] = (P_{s,t}^{(m)}f)(x), \qquad \nu_s^{(m)} - a.e.x \in E_m.$$

and

$$E^P[|(Q_{s,t}^{(m,L,\omega)}f)(x) - (P_{s,t}^{(m)}f)(x)|^2|\mathcal{F}_s^{(\infty)}] \leqq \frac{1}{L}\int_{E_m} \frac{p^{(m)}(t-s,x,y)^2|f(y)|^2}{q_{s,t}^{(m,L,\omega)}(y)}\, dy.$$

Proposition 4.14 *Let $\delta \in (0,1)$ then there exists a $C > 0$ such that*

$$\left(\frac{1}{L}\sum_{\ell=1}^L E^P[|(Q_{t,T_k,\varepsilon}^{(m)}f)(\pi_m(X_\ell(t))) - (P_{t,T_k}^{(m)}f)(\pi_m(X_\ell(t))|^2]\right)^{1/2}$$
$$\leqq CL^{-(1-\delta)/2}(T_k - t)^{-(1+\delta)(\tilde{N}+1)\ell_0/4}(\int_{E_m} f(y)^2(1+|y|^2)^{-\tilde{N}_m}dy)^{1/2}.$$

for any $\varepsilon > 0$ any $m = 1, \dots, M$, and any $f \in Lip(\mathbf{R}^{\tilde{N}_m})$.

Proposition 4.15 *Let*

$$Z_L^{(m,k)}(t;\delta) = \sup_{y\in\mathbf{R}^{\tilde{N}_m}} \frac{|q_{t,T_k}^{(m,L,\omega)}(y) - p^{(m)}(T_k,\tilde{x}^*,y)|}{(L^{-1/(1-\delta)} + p^{(m)}(T_k,\tilde{x}^*,y))^{(1-\delta)/2}},$$

$$\tilde{Z}_L^{(m,k)}(t;\delta) = \sup_{s\in[0,t]} Z_L^{(m,k)}(s;\delta).$$

Then we have following.
(1) *For any $\delta \in (0,1)$, and $p > 1$, there is a $C_{p,\delta} > 0$ such that*

$$E^P[(L^{(1-\delta^2)/2}\tilde{Z}_L^{(m,k)}(T_k - \varepsilon;\delta))^p]^{1/p} \leqq C_{p,\delta}\varepsilon^{-5l_0}L^{-p\delta^2/2+1/p}$$

for any $\varepsilon \in (0, T_k], k = 1, \dots, K,$ *and* $L \geqq 1$.

(2) *Let* $\delta \in (0,1), t \in (0, T_k)$ *and* $\varepsilon \in (0,T)$. *If* $L^{(1-\delta^2)/2}\tilde{Z}_L^{(m,k)}(t;\delta) \leqq 1/4$, *and* $p^{(m)}(T_k, x_0, y) \geqq L^{-(1-\delta)}$, *then*

$$\frac{1}{2} \leqq \frac{q_{t,T_k}^{(m,L,\omega)}(y)}{p^{(m)}(T_k, \tilde{x}^*, y)} \leqq 2,$$

for any $t \in (0, T_k - \varepsilon]$, $\quad k = 1, \dots, K.$ *and* $L \geqq 1$.

Now we introduce the following sets and functions. Let $B^{(m,k)}(t,\delta,L) \in \mathcal{F}$, $\varphi_{m,k,L}, m = 1, \dots, M, k = 1, \dots, K$, be given by

$$B^{(m,k)}(t,\delta,L) = \{\omega \in \Omega; L^{(1-\delta^2)/2}\tilde{Z}_L^{(m,k)}(t;\delta) \leqq 1/4\},$$

and

$$\varphi_{m,k,L}(y) = 1_{\{y \in E_m; p^{(m)}(T_k, x_0, y) > L^{-(1-\delta)}\}}.$$

Let $d_{i,\varepsilon,L}^{(m,k)} : [0,T] \times E \times \Omega \to [0,\infty), i = 1,2,3$, be the measurable functions given by

$$d_{1,\varepsilon,L}^{(m,k)}(t,x) = |(Q_{t,T_k,\varepsilon}^{(m,L,\omega)}(1-\varphi_{m,k,L})F_{m,k})(\pi_m(x)) - (P_{T_k-t}^{(m)}(1-\varphi_{m,k,L})F_{m,k})(\pi_m(x))|1_{[0,T_k-\varepsilon)}(t),$$

$$d_{2,\varepsilon,L}^{(m,k)}(t,x) = 1_{B^{(m,k)}(T_k-\varepsilon,\delta,L)}|(Q_{t,T_k,\varepsilon}^{(m,L,\omega)}\varphi_k F_{m,k})(\pi_m(x)) - (P_{T_k-t}^{(m)}\varphi_{m,k,L}F_{m,k})(\pi_m(x))|1_{[0,T_k-\varepsilon)}(t),$$

$$\begin{aligned} d_{3,\varepsilon,L}^{(m,k)}(t,x) &= |(Q_{t,T_k,\varepsilon}^{(m,L,\omega)}F_{m,k})(\pi_m(x)) - (P_{T_k-t}^{(m)}F_{m,k})(\pi_m(x))|1_{[T_k-\varepsilon,T_k)}(t) \\ &= |F_{m,k}(\pi_m(X(T_k,x^*))) - (P_{T_k-t}^{(m)}F_{m,k})(\pi_m(x))|1_{[T_k-\varepsilon,T_k)}(t), \quad k = 1, \cdots, K. \end{aligned}$$

Let $p(t,x,dy)$ be the transition kernel of $X(t,x)$.

Proposition 4.16 *Let* $\delta \in (0,1)$. *Then there exists a constant* $C > 0$ *such that*

$$\int_E E^P[d_{1,\varepsilon,L}^{(m,k)}(t,x)]|g(t,x)|p(t,x^*,dx) \leqq CL^{-(1-\delta)^3}1_{[0,T_k-\varepsilon)}(t), \tag{4.18}$$

$$\begin{aligned} &(\int_{E_m} E^P[\ d_{2,\varepsilon,L}^{(m,k)}(t,x)^2]p(t,x^*,dx))^{1/2} \\ \leqq &CL^{-(1-\delta)/2}(T_k-t)^{-(1+\delta)(\tilde{N}+1)\ell_0/4}1_{[0,T_k-\varepsilon)}(t), \end{aligned} \tag{4.19}$$

and

$$\int_{E_m} d_{3,\varepsilon,L}^{(m,k)}(t,x)|g(t,x)|p(t,x^*,dx) \leqq C(T_k-t)1_{[T_k-\varepsilon,T_k)}(t). \tag{4.20}$$

for any $\varepsilon \in (0,\varepsilon_0), t \in (0,T_k], L \geqq 1$, $m = 1, \dots, M$, *and* $k = 1, \dots, K$.

Proof. Equation (4.20) follows from Lemma 4.9. So we will show (4.18) and (4.19). Note that if $t \geqq T_k - \varepsilon$, both side of (4.18) and (4.19) are 0. So we will consider the case $t < T_k - \varepsilon$. By Proposition 4.13, we have

$$\begin{aligned} &\int_E E^P[d_{1,\varepsilon,L}^{(m,k)}(t,\pi_m(x))]|g(t,x)|p(t,x^*,dx) \\ = &\int_E E^P[|(Q_{t,T_k,\varepsilon}^{(m)}(1-\varphi_{m,k,L})F_{m,k})(x) - (P_{T_k-t}^{(m)}(1-\varphi_{m,k,L})F_{m,k})(\pi_m(x))|]|g(t,x)|p(t,x^*,dx) \\ \leqq &\int_E (E^P[(Q_{t,T_k,\varepsilon}^{(m)}|(1-\varphi_{m,k,L})F_{m,k}|)(\pi_m(x))] \\ &\qquad + (P_{T_k-t}^{(m)}|(1-\varphi_{m,k,L})F_{m,k}|)(\pi_m(x)))|g(t,x)|p(t,x^*,dx) \\ \leqq &2\int_E (P_{T_k-t}^{(m)}(1-\varphi_{m,k,L})|F_{m,k}|)(\pi_m(x))g(t,x)p(t,x^*,dx). \end{aligned}$$

Using Hölder's inequality for $p = \frac{1}{\delta}, q = \frac{1}{1-\delta}$,

$$\int_E (P^{(m)}_{T_k-t}(1-\varphi_{m,k,L})|F_{m,k}|)(\pi_m(x))|g(t,x)|p(t,x^*,dx)$$

$$\leqq \{\int_E (P^{(m)}_{T_k-t}(1-\varphi_{m,k,L})|F_{m,k}|)(\pi_m(x))^{1/(1-\delta)}p(t,x^*,dx)\}^{1-\delta}$$

$$\times\{\int_E |g(t,x)|^{1/\delta}p(t,x^*,dx)\}^{\delta}$$

$$\leqq \{\int_{E_m} (1-\varphi_{m,k,L}(\tilde{y}_m))^{1/(1-\delta)}|F_{m,k}(\tilde{y}_m)|^{1/(1-\delta)}p^{(m)}(T_k,\pi_m(x^*),\tilde{y}_m)d\tilde{y}_m\}^{1-\delta}$$

$$\times\{\int_E |g(t,x)|^{1/\delta}p(t,x^*,dx)\}^{\delta}$$

$$\leqq L^{-(1-\delta)^3}\{\int_{E_m} |F_{m,k}(\pi_m(y))|^{1/(1-\delta)}p^{(m)}(T_k,\pi_m(x^*),\tilde{y}_m)^{\delta}d\tilde{y}_m\}^{1-\delta}$$

$$\times\{\int_E |g(t,x)|^{1/\delta}p(t,x^*,dx)\}^{\delta}.$$

We used $(1-\varphi_{m,k,L}(\tilde{y}_m))^{1/(1-\delta)}p^{(m)}(T_k,\pi_m(x^*),\tilde{y}_m)^{(1-\delta)} \leqq L^{-(1-\delta)^2}$ in the last inequality. So we have Equation (4.18).

Next we will show Equation (4.19) . Noting that from $B^{(m,k)}(T_k-\varepsilon,\delta,L) \subset B^{(m,k)}(t,\delta,L), t \in [0,T_k-\varepsilon), k=1,\dots,K$, and $L \geqq 1$,

$$d^{(m,k)}_{2,\varepsilon,L}(t,x) \leqq 1_{B^{(m,k)}(t,\delta,L)}|(Q^{(m)}_{t,T_k,\varepsilon}\varphi_{m,k,L}F_{m,k})(\pi_m(x)) - (P^{(m)}_{T_k-t}\varphi_{m,k,L}F_{m,k})(\pi_m(x))|.$$

Since Propsition 4.13, $1_{B^{(m,k)}(t,\delta,L)}q^{(m,L,\omega)}_{t,T_k}(\tilde{y}_m)^{-1} \leqq 2p^{(m)}(T_k,\pi_m(x^*),\tilde{y}_m)^{-1}$. And by Proposition 4.15, we have

$$1_{B^{(m,k)}(t,\delta,L)}E^P[|(Q^{(m)}_{t,T_k,\varepsilon}\varphi_{m,k,L}F_{m,k})(\tilde{x}_m) - (P^{(m)}_{T_k-t}\varphi_{m,k,L}F_{m,k})(\tilde{x}_m)|^2|\mathcal{F}_t]$$
$$\leqq 1_{B^{(m,k)}(t,\delta,L)}\frac{1}{L}\int_{E_m}\frac{|\varphi_{m,k,L}F_{m,k}(\tilde{y}_m)|^2p^{(m)}(T_k-t,\tilde{x}_m,\tilde{y}_m)^2}{q^{(m,L,\omega)}_{t,T_k}(\tilde{y}_m)}d\tilde{y}_m$$
$$\leqq \frac{2}{L}\int_{E_m}\frac{|\varphi_{m,k,L}F_{m,k}(\tilde{y}_m)|^2}{p^{(m)}(T_k,\pi_m(x^*),\tilde{y}_m)}p^{(m)}(T_k-t,\tilde{x}_m,\tilde{y}_m)^2dy.$$

Then we have

$$(\int_E E^P[\ d^{(m,k)}_{2,\varepsilon,L}(t,x)^2]p(t,x^*,dx))^{1/2}$$
$$\leqq(\int_E E^P[1_{B^{(m,k)}(t,\delta,L)}E^P[|(Q^{(m)}_{t,T_k,\varepsilon}\varphi_{m,k,L}F_{m,k})(x) - (P^{(m)}_{T_k-t}\varphi_{m,k,L}F_{m,k})(x)|^2|\mathcal{F}_t]]p(t,x^*,dx))^{1/2}$$
$$\leqq(\frac{2}{L}\int_{E_m}\frac{|\varphi_{m,k,L}F_{m,k}(y)|^2}{p^{(m)}(T_k,\pi_m(x^*),\tilde{y}_m)}(\int_E p^{(m)}(T_k-t,\tilde{x}_m,\tilde{y}_m)^{(1-\delta)+(1+\delta)}p(t,x^*,dx))d\tilde{y}_m)^{1/2}$$
$$\leqq(\frac{2}{L}\int_{E_m}\frac{|\varphi_{m,k,L}F_{m,k}(y)|^2}{p^{(m)}(T_k,\pi_m(x^*),\tilde{y}_m)^{\delta}}(\int_{E_m} p^{(m)}(t,\pi_m(x^*),\tilde{x}_m)p^{(m)}(T_k-t,\tilde{x}_m,\tilde{y}_m)^{(1+\delta)/\delta}d\tilde{x}_m)^{\delta}d\tilde{y}_m)^{1/2}.$$

Let $q \geqq \tilde{N}$. From Lemma 4.12, there exists a constant $C > 0$ such that

$$p^{(m)}(T_k-t,\tilde{x}_m,\tilde{y}_m) \leqq C(T_k-t)^{-(\tilde{N}_m+1)\ell_0/2}h_m(\tilde{x}_m)^{-(\tilde{N}_m+1)\ell_0}(1+|\tilde{x}_m|^2)^q(1+|\tilde{y}_m|^2)^{-q}.$$

We set C_1 as

$$C_1=\sup_{t\in[0,T]}\max_{\substack{m=1,\dots,M,\\ k=1,\dots,K}}(\int_{E_m}h_m(x)^{-(\tilde{N}_m+1)\ell_0(1+\delta)/\delta}(1+|x|^2)^{q(1+\delta)/\delta}p^{(m)}(t,\tilde{x}^*_m,x)dx)^{\delta/2}$$
$$\times(\int_E|g(t,x)|dx)^{1/2}.$$

C_1 is bounded by Proposition 3 of [14]. Then since $\varphi_{m,k,L}(y)p^{(m)}(T_k,\tilde{x}^*_m,y)^{-\delta}\leqq L^\delta$, we have

$$\int_E E^P[\ d^{(m,k)}_{2,\varepsilon,L}(t,x)^2]^{1/2}|g(t,x)|p(t,x^*,dx)$$
$$\leqq\frac{C_1}{L}\int_{E_m}p^{(m)}(T_k,\tilde{x}^*_m,\tilde{y}_m)^{-\delta}|\varphi_{m,k,L}F_{m,k}(\tilde{y}_m)|^2(1+|\tilde{y}_m|^2)^{-q(1+\delta)}d\tilde{y}_m(T_k-t)^{-(1+\delta)(N+1)\ell_0/2},$$
$$\leqq C_1L^{-(1-\delta)}(T_k-t)^{-(1+\delta)(\tilde{N}_m+1)l_0/2}\int_{E_m}|F_{m,k}(\tilde{y}_m)|^2(1+|\tilde{y}_m|^2)^{-q(1+\delta)}d\tilde{y}_m.$$

Since $q\geqq\tilde{N}$, and $F_{m,k}$ is Lipschitz continuous,

$$\int_{\mathbf{R}^{\tilde{N}_m}}|F_{m,k}(\tilde{y}_m)|^2(1+|\tilde{y}_m|^2)^{-q(1+\delta)}d\tilde{y}_m<\infty.$$

So we have the assertion. ∎

Let $a,b,\alpha,\beta\geqq 0$, and $a_i,b_i,\alpha_i,\beta_i\geqq 0, i=1,2$. Let $\phi^{(k)}(t,\varepsilon;a,\alpha,b,\beta)$ and $\hat{e}(\varepsilon,\gamma), t\in[0,T_k)$ be

$$\phi^{(k)}(t,\varepsilon;a,\alpha,b,\beta)=a(T_k-t)^{-\alpha}1_{[0,T_k-\varepsilon)}(t)+b(T_k-t)^{\beta}1_{[T_k-\varepsilon,T_k)}(t),$$
$$\hat{e}(\varepsilon,\gamma)=\begin{cases}\varepsilon^{-(\gamma-1)}) & ,\gamma>1,\\ \log\varepsilon & ,\gamma=1,\\ 1 & ,0\leqq\gamma<1.\end{cases}$$

Proposition 4.17 *There exists a constant $C>0$ such that*

$$\sum_{i=0}^{n-1}(t_{i+1}-t_i)\sum_{k;T_k\geqq t_{i+1}}\phi^{(k)}(t_i,\varepsilon;a,\alpha,b,\beta)\leqq C(a\hat{e}(\varepsilon,\alpha)+b\varepsilon^{\beta+1}),\qquad(4.21)$$

and,

$$\sum_{i=0}^{n-1}(t_{i+1}-t_i)(\sum_{k_1;T_{k_1}\geqq t_{i+1}}\phi^{(k)}(t_i,\varepsilon;a_1,\alpha_1,b_1,\beta_1))(\sum_{k_1;T_{k_1}\geqq t_{i+1}}\phi^{(k)}(t_i,\varepsilon;a_2,\alpha_2,b_2,\beta_2))$$
$$\leqq C\left(a_1a_2\hat{e}(\varepsilon,\alpha_1+\alpha_2)+a_1b_2\varepsilon^{\beta 2+1}+a_2b_1\varepsilon^{\beta_1+1}+b_1b_2\varepsilon^{\beta_1+\beta 2+1}\right)\qquad(4.22)$$

for any $\varepsilon>0$.

Proof. Let us take $i_{(k)}$ as $t_{i_{(k)}}=T_k, k=1,\dots,K$. If $t_i\in[T_{k-1},T_k]$ and $k'>k$ then $T_{k'}-t_i>\varepsilon$. So notice that

$$1_{[T_{k'}-\varepsilon,T_{k'})}(t_i)=0,\quad\text{for } i_{(k-1)}\leqq i\leqq i_{(k)}-1, k'>k.\qquad(4.23)$$

So we have

$$\sum_{i=0}^{n-1}(t_{i+1}-t_i)\sum_{k;T_k\geqq t_{i+1}}\phi^{(k)}(t_i,\varepsilon;a,\alpha,b,\beta)=\sum_{k=1}^{K}\sum_{i=i_{(k-1)}}^{i_{(k)}-1}(t_{i+1}-t_i)\sum_{k'=k}^{K}\phi^{(k')}(t_i,\varepsilon;a,\alpha,b,\beta)$$

$$=\sum_{k=1}^{K}\sum_{i=i_{(k-1)}}^{i_{(k)}-1}(t_{i+1}-t_i)\sum_{k'=k}^{K}\Big(a(T_{k'}-t_i)^{-\alpha}1_{[0,T_k-\varepsilon)}(t_i)+b(T_{k'}-t_i)^{\beta}1_{[T_{k'}-\varepsilon,T_{k'})}(t_i)\Big)$$

$$\leqq\sum_{k=1}^{K}\left(\sum_{k'=k}^{K}\int_{T_{k'-1}}^{T_{k'}-\varepsilon}a(T_{k'}-t)^{-\alpha}dt+\sum_{i=i_{(k-1)}}^{i_{(k)}-1}b1_{[T_k-\varepsilon,T_k)}(t_i)(t_{i+1}-t_i)(T_k-t_i)^{\beta}\right),$$

because $(T_{k'}-t_i)^{-\alpha}\leqq(T_{k'}-t)^{-\alpha}$ for $t_i\leqq t\leqq t_{i+1}$.
On the other hand,

$$\sum_{k=1}^{K}\sum_{k'=k}^{K}\int_{T_{k'-1}}^{T_{k'}-\varepsilon}(T_{k'}-t)^{-\alpha}dt\leqq K^2\hat{e}(\varepsilon,\alpha),$$

and

$$\sum_{k=1}^{K}\sum_{i=i_{(k-1)}}^{i_{(k)}-1}1_{[T_k-\varepsilon,T_k)}(t_i)(t_{i+1}-t_i)(T_k-t_i)\leqq K\varepsilon^2,$$

So we have Equation (4.21).
Next we show Equation (4.22).

$$\sum_{i=0}^{n-1}(t_{i+1}-t_i)(\sum_{k_1;T_{k_1}\geqq t_{i+1}}\phi^{(k_1)}(t_i,\varepsilon;a_1,\alpha_1,b_1,\beta_1))(\sum_{k_1;T_{k_1}\geqq t_{i+1}}\phi^{(k_2)}(t_i,\varepsilon;a_2,\alpha_2,b_2,\beta_2))$$

$$\leqq\sum_{k=1}^{K}\sum_{i=i_{(k-1)}}^{i_{(k)}-1}(t_{i+1}-t_i)\sum_{j=1}^{4}I_{i,j}^{(k)},$$

where

$$I_{i,1}^{(k)}=\sum_{k_1,k_2=k}^{K}a_1a_2(T_{k_1}-t_i)^{-\alpha_1}(T_{k_2}-t_i)^{-\alpha_2}1_{[0,T_{k_1}-\varepsilon)}(t_i)1_{[0,T_{k_2}-\varepsilon)}(t_i),$$

$$I_{i,2}^{(k)}=\sum_{k_1,k_2=k}^{K}a_1b_2(T_{k_1}-t_i)^{-\alpha_1}(T_{k_2}-t_i)^{\beta_2}1_{[0,T_{k_1}-\varepsilon)}(t_i)1_{[T_{k_2}-\varepsilon,T_{k_2})}(t_i),$$

$$I_{i,3}^{(k)}=\sum_{k_1,k_2=k}^{K}a_2b_1(T_{k_2}-t_i)^{-\alpha_2}(T_{k_1}-t_i)^{\beta_1}1_{[0,T_{k_2}-\varepsilon)}(t_i)1_{[T_{k_1}-\varepsilon,T_{k_1})}(t_i),$$

$$I_{i,4}^{(k)}=\sum_{k_1,k_2=k}^{K}b_1b_2(T_{k_1}-t_i)^{\beta_1}(T_{k_2}-t_i)^{\beta_2}1_{[T_{k_1}-\varepsilon,T_{k_1})}(t_i)1_{[T_{k_2}-\varepsilon,T_{k_2})}(t_i).$$

Note that (4.23) and

$$1_{[0,T_{k_1}-\varepsilon)}(t_i)1_{[T_k-\varepsilon,T_k)}(t_i)=\begin{cases}0,k_1\leqq k\\1_{[T_k-\varepsilon,T_k)}(t_i),k_1>k,\end{cases}$$

we have for $i \in \{i_{(k-1)}, \ldots, i_{(k)-1}\}$,

$$I_{i,2}^{(k)} = \sum_{k_1=k+1}^{K} a_1 b_2 (T_{k_1} - t_i)^{-\alpha_1} (T_k - t_i)^{\beta_2} 1_{[T_k-\varepsilon, T_k)}(t_i)$$

$$\leqq K(T_{k+1} - T_k)^{-\alpha_1} a_1 b_2 (T_k - t_i)^{\beta_2} 1_{[T_k-\varepsilon,T_k)}(t_i).$$

We have the followings similarly.

$$\sum_{i=0}^{n-1}(t_{i+1} - t_i) I_{i,1}^{(k)} \leqq a_1 a_2 \sum_{k_1,k_2=k}^{K} \int_{t_{i_{(k-1)}} \wedge (T_{k_1} \wedge T_{k_2} - \varepsilon)}^{t_{i_{(k)-1}} \wedge (T_{k_1} \wedge T_{k_2} - \varepsilon)} (T_{k_1} \wedge T_{k_2} - t)^{-(\alpha_1+\alpha_2)} dt,$$

$$\sum_{i=0}^{n-1}(t_{i+1} - t_i) I_{i,2}^{(k)} \leqq C a_1 b_2 \varepsilon^{\beta_2+1},$$

$$\sum_{i=0}^{n-1}(t_{i+1} - t_i) I_{i,3}^{(k)} \leqq C a_2 b_1 \varepsilon^{\beta_1+1},$$

$$\sum_{i=0}^{n-1}(t_{i+1} - t_i) I_{i,4}^{(k)} \leqq b_1 b_2 \varepsilon^{\beta_1+\beta_2+1}.$$

So we obtain (4.22). ∎

4.6 Proof of Theorem 4.1 and Theorem 4.2

Theorem 4.18 *There exists a constant $C > 0$ such that*

$$E^P[|\hat{c}_1(\varepsilon_L, \Delta, L) - c_\Delta|] \leqq C\left(L^{-(1-\delta)/2} \hat{e}\left(\varepsilon, (1+\delta)(\tilde{N}+1)\ell_0/4\right) + \varepsilon^2\right), L \geqq 1.$$

Proof.

$$E^P[|\hat{c}_1(\varepsilon_L, \Delta, L) - c_\Delta|]$$

$$\leqq \frac{1}{L}\sum_{\ell=1}^{L}\sum_{i=0}^{n-1}(t_{i+1}-t_i)\sum_{m=1}^{M}\sum_{k:T_k \geqq t_{i+1}} |E^P[g(t_i, X(t_i,x))$$

$$((Q_{t_i,T_k,\varepsilon}^{(m,L,\omega)} F_{m,k})(\pi_k(X_\ell(t_i)) - (P_{T_k-t_i}^{(k)} F_{m,k})(\pi_k(X_\ell(t_i)))]|.$$

Then by Schwartz's inequality ,

$$\frac{1}{L}\sum_{\ell=1}^{L} |E^P[g(t_i, X(t_i,x))(Q_{t_i,T_k,\varepsilon}^{(m,L,\omega)} F_{m,k})(\pi_k(X_\ell(t_i)) - (P_{T_k-t_i}^{(k)} F_{m,k})(\pi_k(X_\ell(t_i))]|$$

$$\leqq \frac{1}{L}\sum_{\ell=1}^{L} E^P[|g(t_i, X(t_i,x))||(Q_{t_i,T_k,\varepsilon}^{(m,L,\omega)} F_{m,k})(\pi_k(X_\ell(t)) - (P_{T_k-t_i}^{(k)} F_{m,k})(\pi_k(X_\ell(t))|1_{[0,T_k-\varepsilon)}]$$

$$+C(T_k - t)1_{[T_k-\varepsilon,T_k)}$$

$$\leqq \left(\frac{1}{L}\sum_{\ell=1}^{L} E^P[|(Q_{t_i,T_k,\varepsilon}^{(m,L,\omega)} F_{m,k})(\pi_k(X_\ell(t)) - (P_{T_k-t_i}^{(k)} F_{m,k})(\pi_k(X_\ell(t))|^2]\right)^{1/2} 1_{[0,T_k-\varepsilon)}$$

$$\times E[g(t_i, X(t_i,x))^2]^{1/2} + C(T_k - t)1_{[T_k-\varepsilon,T_k)}$$

By Proposition 4.14

$$E^P[|\hat{c}_1(\varepsilon_L, \Delta, L) - c_\Delta|]$$

$$\leqq C\sum_{i=0}^{n-1}(t_{i+1}-t_i)\sum_{k:T_k\geqq t_{i+1}}\phi^{(k)}(t_i,\varepsilon; L^{-(1-\delta)/2}, (1+\delta)(\tilde{N}+1)\ell_0/4, 1, 1).$$

By Proposition4.17, we have the assertion. ∎

Lemma 4.19 *Let $a, b \in \mathbf{R}$ and $c, \theta > 0$. Then we have*

$$c|a||1_{\{b\geqq 0\}} - 1_{\{a\geqq 0\}}| \leqq c|b-a|1_{\{|b-a|\geqq\theta\}} + c\theta 1_{\{|a|<\theta\}}$$

Proof. If $|a| > |a-b|$, then

$$1_{\{b\geqq 0\}} - 1_{\{a\geqq 0\}} = 0.$$

So we see that

$$|a|(|1_{\{b\geqq 0\}} - 1_{\{a\geqq 0\}}|$$

$$\leqq |a|1_{\{|a|\leqq|a-b|\}}$$

$$\leqq |a-b|1_{\{|a-b|\geqq\theta\}} + |a|1_{\{|a|<\theta\}}.$$

∎

Theorem 4.20 *Let $\delta \in (0,1), p > 1$. Suppose that there is $\gamma \in (0,1]$ and $C_\gamma > 0$ such that*

$$\sup_\Delta \sum_{i=0}^{n-1}(t_{i+1}-t_i))\mu(|\sum_{m=1}^{M}\sum_{k:T_k\geqq t_{i+1}}(P^{(m)}_{T_k-t_i}F_{m,k})(\pi_m X(t_i,x^*)))| \leqq \theta)$$

$$\leqq C_\gamma\theta^\gamma, \theta \in (0,1).$$

Then there exists a constant $C > 0$, $\tilde{\Omega}(L,\varepsilon) \in \mathcal{F}$, such that

$$P(\tilde{\Omega}(L,\varepsilon)^c) \leqq C\left(\left(L^{-(1-\delta)^2/2}\left(L^{-(1-\delta)^2/2}\hat{e}\left(\varepsilon,(1-\delta^2)(\tilde{N}+1)\ell_0/2\right)+\varepsilon^{3(1-\delta)/2}\right)\right)^{\frac{\delta(1+\gamma)}{2+\gamma}}\right),$$

and

$$1_{\tilde{\Omega}(L,\varepsilon)}|\hat{c}_2(\varepsilon_L, \Delta, L) - c_\Delta|$$

$$\leqq C\left(L^{-(1-\delta)^2/2}\left(L^{-(1-\delta)^2/2}\hat{e}\left(\varepsilon,(1-\delta^2)(\tilde{N}+1)\ell_0/2\right)+\varepsilon^{3(1-\delta)/2}\right)\right)^{\frac{(1-\delta)(1+\gamma)}{2+\gamma}},$$

for $L \geqq 1$.

Proof. In this proof, we denote $X(t,x^*)$ by $X(t)$ for simplicity. Let

$$\tilde{B}_\varepsilon = \bigcap_{m=1}^{M}\bigcap_{k=1}^{K} B^{(m,k)}(T_k-\varepsilon,\delta,L).$$

Let $F_{P,i}:\mathbf{R}^N \to \mathbf{R}$ be given by

$$F_{P,i}(x) = \sum_{\substack{m=1,\dots,M \\ k:T_k \geqq t_{i+1}}} (P^{(m)}_{T_k - t_i} F_{m,k})(\pi_m x),$$

and let $F_{Q,i}:\mathbf{R}^N \to \mathbf{R}$ be given by

$$F_{Q,i}(x) = \sum_{\substack{m=1,\dots,M \\ k:T_k \geqq t_{i+1}}} (Q^{(m)}_{t_i,T_k,\varepsilon} F_{m,k})(\pi_m x).$$

Then

$$1_{\tilde{B}_\varepsilon}|\hat{c}_2(\varepsilon_L,\Delta,L) - c_\Delta|$$

$$\leqq 1_{\tilde{B}_\varepsilon} \sum_{i=0}^{n-1} (t_{i+1}-t_i)|E^\mu[|g(t_i,X(t_i))| \sum_{m=1}^{M} \sum_{k:T_k \geqq t_{i+1}} F_{m,k}(\pi_k(X(T_k)))(1_{\{F_{Q,i}(X(t_i)\geqq 0\}} - 1_{\{F_{P,i}(X(t_i)\geqq 0\}})]|,$$

$$\leqq 1_{\tilde{B}_\varepsilon} \sum_{i=0}^{n-1} (t_{i+1}-t_i)E^\mu[|g(t_i,X(t_i))||F_{P,i}(X(t_i)||1_{\{F_{Q,i}(X(t_i)\geqq 0\}} - 1_{\{F_{P,i}(X(t_i)\geqq 0\}}|],$$

since $|g(t_i,X(t_i))|(1_{\{F_{Q,i}(X(t_i)\geqq 0\}} - 1_{\{F_{P,i}(X(t_i)\geqq 0\}})$ is $\mathcal{F}_t$ measurable.
Applying Lemma 4.19 to $a = F_{P,i}(X(t_i), b = F_{Q,i}(X(t_i)$, and $c = |g(t_i,X(t_i))|$, we have

$$1_{\tilde{B}_\varepsilon}|\hat{c}_2(\varepsilon_L,\Delta,L) - c_\Delta| \leqq I_1 + I_2,$$

where

$$I_1 = 1_{\tilde{B}_\varepsilon} \sum_{i=0}^{n-1} (t_{i+1}-t_i)E^\mu[|g(t_i,X(t_i))||F_{Q,i}(X(t_i) - F_{P,i}(X(t_i)|1_{\{|F_{Q,i}(X(t_i) - F_{P,i}(X(t_i)|>\theta\}}],$$

$$I_2 = 1_{\tilde{B}_\varepsilon} \sum_{i=0}^{n-1} (t_{i+1}-t_i)\theta E^\mu[|g(t_i,X(t_i))||1_{\{F_{P,i}(X(t_i)|\leqq \theta\}}].$$

By Hölder's inequality,

$$I_2 \leqq \theta E^\mu[|g(t_i,X(t_i))|^{1/\delta}]^\delta \mu(|F_{P,i}(X(t_i)| \leqq \theta)^{1-\delta} \leqq C\theta\mu(|F_{P,i}(X(t_i)| \leqq \theta).$$

From assumption, we have

$$I_2 \leqq C\theta^{(\gamma+1)(1-\delta)}.$$

Next we will estimate I_1.

$$|F_{Q,i}(X(t_i)) - F_{P,i}(X(t_i))| \leqq \sum_{m=1}^{M} \sum_{k:T_k \geqq t_{i+1}} (d^{(m,k)}_{1,\varepsilon,L}(t_i,X(t_i)) + d^{(m,k)}_{2,\varepsilon,L}(t_i,X(t_i)) + d^{(m,k)}_{3,\varepsilon,L}(t_i,X(t_i)))$$

$$I_1 \leqq 1_{\tilde{B}_\varepsilon} \sum_{i=0}^{n-1} (t_{i+1}-t_i)E^\mu[|g(t_i,X(t_i))|$$

$$\sum_{m=1}^{M} \sum_{k:T_k \geqq t_{i+1}} \left(d^{(m,k)}_{1,\varepsilon,L}(t_i,\pi_m(X(t_i))) + d^{(m,k)}_{2,\varepsilon,L}(t_i,\pi_m(X(t_i) + d^{(m,k)}_{3,\varepsilon,L}(t_i,\pi_m(X(t_i)\right)$$

$$\times 1_{\{\sum_{m=1}^M \sum_{k:T_k \geqq t_{i+1}} (d^{(m,k)}_{1,\varepsilon,L}(t_i,\pi_m(X(t_i))) + d^{(m,k)}_{2,\varepsilon,L}(t_i,\pi_m(X(t_i)) + d^{(m,k)}_{3,\varepsilon,L})(t_i,\pi_m(X(t_i))) > \theta\}}].$$

I_1 is dominated by

$$I_1 \leqq I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4},$$

where

$$I_{1,1} = \sum_{i=0}^{n-1}(t_{i+1}-t_i)\sum_{m=1}^{M}\sum_{k:T_k \geqq t_{i+1}} E^{\mu}[|g(t_i,X(t_i))|d_{1,\varepsilon,L}^{(m,k)}(t_i,\pi_m(X(t_i)))],$$

$$I_{1,2} = \sum_{i=0}^{n-1}(t_{i+1}-t_i)E^{\mu}[\sum_{m=1}^{M}\sum_{k:T_k \geqq t_{i+1}} |g(t_i,X(t_i))|d_{2,\varepsilon,L}^{(m,k)}(t_i,\pi_m(X(t_i)))$$
$$\times 1_{\{\sum_{m=1}^{M}\sum_{k:T_k\geqq t_{i+1}}(d_{1,\varepsilon,L}^{(m,k)}(t_i,\pi_k(X(t_i)+d_{3,\varepsilon,L}^{(m,k)}(t_i,\pi_m(X(t_i))>\theta/2\}}],$$

$$I_{1,3} = \sum_{i=0}^{n-1}(t_{i+1}-t_i)E^{\mu}[\sum_{m=1}^{M}\sum_{k:T_k \geqq t_{i+1}} |g(t_i,X(t_i))|d_{2,\varepsilon,L}^{(m,k)}(t_i,\pi_m(X(t_i)))$$
$$\times 1_{\{\sum_{m=1}^{M}\sum_{k:T_k\geqq t_{i+1}} d_{2,\varepsilon,L}^{(m,k)}(t_i,\pi_m(X(t_i)>\theta/2\}}].$$

$$I_{1,4} = \sum_{i=0}^{n-1}(t_{i+1}-t_i)E^{\mu}[\sum_{m=1}^{M}\sum_{k:T_k \geqq t_{i+1}} |g(t_i,X(t_i))|d_{3,\varepsilon,L}^{(m,k)}(t_i,\pi_m(X(t_i)))]$$

From Proposition 4.16,

$$E^P[I_{1,1}] = \sum_{i=0}^{n-1}(t_{i+1}-t_i)\sum_{\substack{m=1,\ldots,M,\\ k;T_k\geqq t_{i+1}}}\int_E E^P[d_{1,\varepsilon,L}^{(m,k)}(t,\tilde{x}_m)]|g(t_i,x)|p(t_i,x^*,x)dx$$
$$\leqq C\sum_{i=0}^{n-1}(t_{i+1}-t_i)\sum_{k;T_k\geqq t_{i+1}}\phi^{(k)}(t_i,\varepsilon;L^{-(1-\delta)^3},0,0,0)$$
$$\leqq C\hat{e}\left(\varepsilon, L^{-(1-\delta)^3}\right).$$

Next, we will estimate $I_{1,2}$. By Hölder's inequality

$$I_{1,2} \leqq \sum_{i=0}^{n-1}(t_{i+1}-t_i)E^{\mu}[(\sum_{m=1}^{M}\sum_{k:T_k\geqq t_{i+1}} d_{2,\varepsilon,L}^{(m,k)}(t_i,\pi_m(X(t_i)))^2)]^{1/2}$$

$$\times E^{\mu}[|g(t_i,X(t_i))|^{2/\delta}]^{\delta/2}E^{\mu}[1_{\{\sum_{m=1}^{M}\sum_{k:T_k\geqq t_{i+1}}(d_{1,\varepsilon,L}^{(m,k)}(t_i,\pi_m(X(t_i)+d_{3,\varepsilon,L}^{(m,k)}(t_i,\pi_m(X(t_i))>\theta/2\}}]^{(1-\delta)/2},$$

$$\leqq C\sum_{i=0}^{n-1}(t_{i+1}-t_i)E^{\mu}[\sum_{m=1}^{M}\sum_{k:T_k\geqq t_{i+1}} d_{2,\varepsilon,L}^{(m,k)}(t_i,\pi_m(X(t_i)))^2]^{1/2}$$

$$\times(\frac{2}{\theta}E^{\mu}[\sum_{m=1}^{M}\sum_{k:T_k\geqq t_{i+1}}(d_{1,\varepsilon,L}^{(m,k)}(t_i,\pi_m(X(t_i)+d_{3,\varepsilon,L}^{(m,k)}(t_i,\pi_m(X(t_i))])^{(1-\delta)/2}.$$

So we have

$$E^P[I_{1,2}] \leqq C\sqrt{\frac{2}{\theta}}\sum_{i=0}^{n-1}(t_{i+1}-t_i)E^P[\sum_{\substack{m=1,\dots,M,\\ k:T_k\geqq t_{i+1}}}\int_{E_m} d_{2,\varepsilon,L}^{(m,k)}(t_i,x)^2 p^{(m)}(t_i,\tilde{x}_m^*,x)dx]^{1/2}$$

$$\times E^P[\sum_{\substack{m=1,\dots,M,\\ k:T_k\geqq t_{i+1}}}\int_{E_m}(d_{1,\varepsilon,L}^{(m,k)}(t_i,x)+d_{3,\varepsilon,L}^{(m,k)}(t_i,x))p^{(m)}(t_i,\tilde{x}_m^*,x)dx]^{(1-\delta)/2}$$

$$\leqq C\theta^{-(1-\delta)/2}\sum_{i=0}^{n-1}(t_{i+1}-t_i)\left(\sum_{k:T_k\geqq t_{i+1}}\phi^{(k)}(t_i,\varepsilon;L^{-(1-\delta)^2/2},(1-\delta^2)(\tilde{N}+1)\ell_0/4,0,0)\right)$$

$$\times\left(\sum_{k:T_k\geqq t_{i+1}}\phi^{(k)}(t_i,\varepsilon;L^{-(1-\delta)^4/2},0,1,(1-\delta)/2)\right).$$

By Proposition 4.17,

$$E^P[I_{1,2}] \leqq C\theta^{-1/2}\left(L^{-(1-\delta)^4}\hat{e}\left(\varepsilon,(1-\delta^2)(\tilde{N}+1)\ell_0/4\right)+L^{-(1-\delta)^2/2}\varepsilon^{3(1-\delta)/2}\right).$$

Similarly, we have

$$E^P[I_{1,3}] \leqq C\theta^{-(1-\delta)/2}\sum_{i=0}^{n-1}(t_{i+1}-t_i)E[\sum_{\substack{m=1,\dots,M,\\ k:T_k\geqq t_{i+1}}}\int_{E_m} d_{2,\varepsilon,L}^{(k)}(t_i,x)^2 p^{(k)}(t_i,\tilde{x}_k^*,x)dx]^{(1-\delta)}$$

$$\leqq C\theta^{-(1-\delta)/2}\sum_{i=0}^{n-1}(t_{i+1}-t_i)(\sum_{k:T_k\geqq t_{i+1}}\phi^{(k)}(t_i,\varepsilon;L^{-(1-\delta)^2/2},(1-\delta^2)(\tilde{N}+1)\ell_0/4,0,0))^2$$

$$\leqq C\theta^{-1}L^{-(1-\delta)^2}\hat{e}\left(\varepsilon,(1-\delta^2)(\tilde{N}+1)\ell_0/2\right).$$

It follows easily that

$$E^P[I_{1,4}] \leqq C\varepsilon^2.$$

Notice that

$$\theta^{-(1-\delta)/2}\hat{e}(\varepsilon,(1-\delta^2)(\tilde{N}+1)\ell_0/4) \leqq \theta^{-1}\hat{e}\left(\varepsilon,(1-\delta^2)(\tilde{N}+1)\ell_0/2\right),$$

we have

$$E^P[I] \leqq C\left(\theta^{\gamma+1}+\theta^{-1}L^{-(1-\delta)^2/2}\left(L^{-(1-\delta)^2/2}\hat{e}\left(\varepsilon,(1-\delta^2)(\tilde{N}+1)\ell_0/2\right)+\varepsilon^{3(1-\delta)/2}\right)\right).$$

In particular if we take $\theta=\theta_L$ as

$$\theta_L = O\left(L^{-(1-\delta)^2/2}\left(L^{-(1-\delta)^2/2}\hat{e}\left(\varepsilon,(1-\delta^2)(\tilde{N}+1)\ell_0/2\right)+\varepsilon^{3(1-\delta)/2}\right)\right)^{\frac{1}{2+\gamma}},$$

then we have

$$E^P[I] \leqq C\left(L^{-(1-\delta)^2/2}\left(L^{-(1-\delta)^2/2}\hat{e}\left(\varepsilon,(1-\delta^2)(\tilde{N}+1)\ell_0/2\right)+\varepsilon^{3(1-\delta)/2}\right)\right)^{(1+\gamma)/(2+\gamma)}.$$

Let $\tilde{\Omega}(L,\varepsilon)$ be

$$\tilde{\Omega}(L,\varepsilon) = \tilde{B}_\varepsilon \cap \{\omega \in \Omega;$$
$$I \leqq C\left(\left(L^{-(1-\delta)^2/2}\left(L^{-(1-\delta)^2/2}\hat{e}\left(\varepsilon,(1-\delta)^2(\tilde{N}+1)\ell_0/2\right)+\varepsilon^{3(1-\delta)/2}\right)\right)\right)^{\frac{(1-\delta)(1+\gamma)}{2+\gamma}}\}.$$

From Proposition 4.15, we have

$$P(\tilde{\Omega}(L,\varepsilon)^c) \leqq C\left(\left(L^{-(1-\delta)^2/2}\left(L^{-(1-\delta)^2/2}\hat{e}\left(\varepsilon,(1-\delta^2)(\tilde{N}+1)\ell_0/2\right)+\varepsilon^{3(1-\delta)/2}\right)\right)^{\frac{\delta(1+\gamma)}{2+\gamma}}\right.$$
$$\left. +(\varepsilon^{-5\ell_0}L^{-p\delta^2/2+1/p})^p\right),$$

and

$$1_{\tilde{\Omega}(L,\varepsilon)}|\hat{c}_2(\varepsilon_L,\Delta,L)-c_\Delta|$$
$$\leqq C\left(L^{-(1-\delta)^2/2}\left(L^{-(1-\delta)^2/2}\hat{e}\left(\varepsilon,(1-\delta^2)(\tilde{N}+1)\ell_0/2\right)+\varepsilon^{3(1-\delta)/2}\right)\right)^{\frac{(1-\delta)(1+\gamma)}{2+\gamma}}.$$

∎

Corollary 4.21 *Theorem 4.1 and Theorem 4.2 follow from the Theorem 4.18 and Theorem 4.19.*

4.7 Numerical Example

Let $\{B(t); t \geqq 0\}$ be 1 dimensional Brownian motion. Let $t_i = i/n, i = 0,\ldots,n$. Let c be

$$c = E[\int_0^1 (E[B(1)|\mathcal{F}_t] \vee 0)\ dt] = \frac{2}{3\sqrt{2\pi}}.$$

Let c_Δ be the discretization of c, such that

$$c_\Delta = \sum_{i=0}^{n-1}(t_{i+1}-t_i)E[B(t_i)\vee 0].$$

We approximate c as Remark 4.3, where $F(x) = x$. Let $\mathbf{X}_1 = \{X_\ell(t_i); i = 0,1,\ldots,n\}_{\ell=1}^L$ be i.i.d sample paths of $\{B(t_i); i = 0,1,\ldots,n\}$. We compute $Q_{t,T,\varepsilon}^{(L,\omega)}$ and $\hat{c}_1$ by using of paths $\mathbf{X}_1$.

$$\hat{c}_1 = \frac{1}{L}\sum_{\ell=1}^{L}\sum_{i=0}^{n-1}(Q_{t_i,T}^{(L,\omega)}F)((X_\ell(t_i)))\vee 0)(t_{i+1}-t_i).$$

Let $\mathbf{X}_2 = \{X'_\ell(t_i); i = 0,1,\ldots,n\}_{\ell=1}^L$ be another i.i.d sample paths of $\{B(t_i); i = 0,1,\ldots,n\}$. We compute $\hat{c}_2$ by

$$\hat{c}_2 = \frac{1}{L_0}\sum_{\ell_0=1}^{L_0}\sum_{i=0}^{n-1}\{F(X'_{\ell_0}(T))\}1_{\{(Q_{t,T}^{(L,\omega)}F)(X'_{\ell_0}(t_i))\geqq 0\}}(t_{i+1}-t_i).$$

We have $c \approx 0.2659615203$. When we take $n = 100$, we have $c_\Delta \approx 0.2638855365$. We also take $L_0 = 10000$ and $L = 100, 200, 400, 800, 1600, 3200, 6400$. We replicate 100 estimators of $|\hat{c}_i - c_\Delta|, i = 1, 2$ for each L. Let "average i" denote the average and "standard dviation i" denote the unbiased standard deviation of these 100 estimators for $i = 1, 2$. We show the result in Table 4.1. Table 4.1 display the numerical result of $|\hat{c}_i - c|$, and unbiased standard deviation of $c_i, i = 1, 2$. We see in Figure 4.1 that both $|\hat{c}_1 - c_\Delta|$ and $|\hat{c}_2 - c_\Delta|$ approach 0 as $L \to \infty$, but $|\hat{c}_2 - c|$ is more stable than $|\hat{c}_1 - c|$.

L	average 1	average 2	standard deviation 1	standard deviation 2
100	0.0453447368	0.0053580746	0.0334099140	0.0039872730
200	0.0287991842	0.0052000712	0.0244301557	0.0038576326
400	0.0216224492	0.0051810512	0.0168528548	0.0039305129
800	0.0161228222	0.0051754299	0.0114840865	0.0039276461
1600	0.0117769017	0.0051955081	0.0092734158	0.0039546398
3200	0.0082335939	0.0052074140	0.0071782226	0.0039613689
6400	0.0060464529	0.0052099324	0.0050607508	0.0039677394

表 4.1: Average and Standard Deviation of Absolute Error

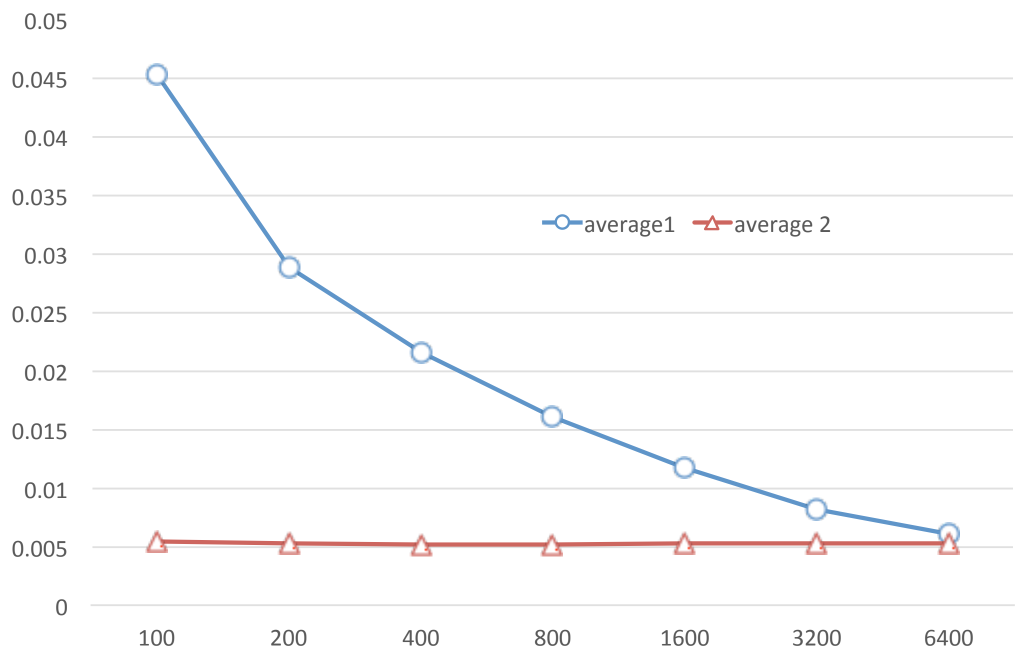

図 4.1: Average of Absolute Error

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