博士論文

論文題目:
Stable presentation length of 3-manifold groups
(三次元多様体の基本群の安定表示長)

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STABLE PRESENTATION LENGTH OF 3-MANIFOLD GROUPS

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Abstract

We introduce the stable presentation length of a finitely presented group. The stable presentation length of the fundamental group of a 3-manifold can be considered as an analogue of the simplicial volume. We show that the stable presentation length have some additive properties like the simplicial volume, and the simplicial volume of a closed 3-manifold is bounded from above and below by constant multiples of the stable presentation length of its fundamental group.

Contents

1 Introduction

Mostow-Prasad rigidity [Mos73, Pra73] states that a finite volume hyperbolic 3-manifold is determined by its fundamental group. In particular, the volume of a hyperbolic 3-manifold is a topological invariant. Gromov [Gro82] introduced the simplicial volume of a manifold, and showed some fundamental properties of the simplicial volume. For example, the simplicial volume of a hyperbolic manifold is proportional to its volume as a Riemannian manifold. The simplicial volume of a manifold with an amenable fundamental group vanishes. Furthermore, the simplicial volume have additivity for a decomposition along manifolds with

amenable fundamental groups. Therefore, the geometrization theorem proved by Perelman [Per02, Per03] implies that the simplicial volume of a closed 3-manifold is equal to the sum of the simplicial volumes of hyperbolic pieces after the prime decomposition and the JSJ decomposition. Since the decompositions for fundamental group of a 3-manifold correspond to the decompositions of the 3-manifold along essential surfaces, the simplicial volume of a closed 3-manifold is uniquely determined by its fundamental group. In order to consider a direct relation between the simplicial volume of a 3-manifold and its fundamental group, we will introduce the stable presentation length of a finitely presented group.

Milnor and Thurston [MT77] considered some characteristic numbers of manifolds, where "characteristic" means multiplicativity for the finite sheeted coverings, i.e. an invariant C of manifolds is a characteristic number if it holds that $C(N) = d \cdot C(M)$ for any d-sheeted covering $N \to M$. For example, the Euler characteristic and the simplicial volume are characteristic numbers. We say such an invariant is *volume-like* instead of a characteristic number in order to indicate similarity to the volume. Milnor and Thurston introduced the following volume-like invariant of a manifold, which is called the stable Δ-complexity by Francaviglia , Frigerio and Martelli [FFM12]. Δ-*complexity* $\sigma(M)$ of a closed 3-manifold M is the minimal number of simplices in a triangulation of M. Δ-complexity is not volume-like, but an upper volume in the sense of Reznikov [Rez96], i.e. it holds that $\sigma(N) \leq d \cdot \sigma(M)$ for any d-sheeted covering $N \to M$. Then a natural way gives a volume-like invariant defined by

$$\sigma_\infty(M) = \inf_{N \to M} \frac{\sigma(M)}{\deg(N \to M)},$$

where the infimum is taken among the finite sheeted coverings of M. $\sigma_\infty(M)$ is called the stable Δ-complexity of M.

While the stable Δ-complexity is hard to handle, the simplicial volume following it can work similarly and has more application. Thus the stable Δ-complexity became something obsolete, but recently Francaviglia, Frigerio and Martelli [FFM12] brought a further development. They introduced the stable complexity of a 3-manifold. The *complexity* $c(M)$ of 3-manifold M is the minimal number of vertices in a simple spine for M. Matveev [Mat90, Theorem 5] showed that the complexity of M is equal to its Δ-complexity if M is irreducible and not $S^3, \mathbb{RP}$ or the lens space $L(3,1)$. In particular, the two complexities of M coincide if M is a hyperbolic 3-manifold. The stable complexity $c_\infty(M)$ is defined in the same way as the stable Δ-complexity. Francaviglia, Frigerio and Martelli showed that the stable complexity has same additivity as the simplicial volume of 3-manifold, and therefore $c_\infty(M)$ is the sum of the ones of hyperbolic pieces after the geometrization. Moreover, the stable complexity of 3-manifold is bounded from above and below by constant multiples of the simplicial volume. This is implied from the fact that the stable Δ-complexity of a hyperbolic 3-manifold is so.

Delzant [Del96] introduced a complexity $T(G)$ of a finitely presented group G. We call it the presentation length according to Agol and Liu [AL12]. Delzant also introduced a relative version of presentation length, and he gave an estimate of presentation length for a decomposition of group. There are some applications for the presentation length of the fundamental group of a 3-manifold. Cooper [Coo99] gave an estimate for the volume of a hyperbolic 3-manifold by

the presentation length. White [Whi01] gave an estimate for the diameter of a closed hyperbolic 3-manifold by the presentation length. Agol and Liu [AL12] solved Simon conjecture by using presentation length.

Delzant and Potyagailo [DP13] remarked that the volume of hyperbolic 3-manifold is not bounded from below by a constant multiple of the presentation length. They considered a relative presentation length for a thick part of a hyperbolic 3-manifold, and showed that the volume is bounded from above and below by constant multiples of this relative presentation length. We will introduce the stable presentation length $T_\infty(G)$ instead of this.

The presentation length is an upper volume. Hence we can define the stabilization of the presentation length. We will show the stable presentation length of a 3-manifold has additivity like the simplicial volume and the stable complexity.

Theorem 1.1. • *For finitely presented groups G_1 and G_2, it holds that*

$$T_\infty(G_1 * G_2) = T_\infty(G_1) + T_\infty(G_2).$$

• *Let M be an irreducible 3-manifold. Suppose $M = M_1 \cup \cdots \cup M_h$ is the JSJ decomposition. $M_1, \ldots, M_h$ are compact 3-manifolds with incompressible torus boundary. Then*

$$T_\infty(M) = T_\infty(M_1) + \cdots + T_\infty(M_h).$$

Francaviglia, Frigerio and Martelli gave a problem whether the simplicial volume and the stable complexity coincide, which they call the 3-dimensional Ehrenpreis conjecture. They showed the simplicial volume and the stable Δ-complexity of a higher dimensional hyperbolic manifold cannot coincide [FFM12, Theorem 2.1]. We conjecture the stable presentation length for a 3-manifold is half of the stable complexity.

Conjecture 1.2. *For a finite volume hyperbolic 3-manifold M, it holds that*

$$T_\infty(M) = \frac{1}{2}\sigma_\infty(M).$$

This conjecture seems more likely than the 3-dimensional Ehrenpreis conjecture. We expect the stable presentation length is useful for approaching the 3-dimensional Ehrenpreis conjecture.

Organization of the thesis

In Section 2, we review the definition and elementary properties of the presentation length.

In Section 3, we define the stable presentation length as a volume-like invariant of a finitely presented group.

In Section 4, we consider the stable presentation length of a hyperbolic 3-manifold. For a 3-manifold with boundary M, it is natural to consider its presentation length relative to the fundamental groups of the boundary component. we show that the stable presentation length of the hyperbolic 3-manifold relative to the cusp subgroups coincides the non-relative stable presentation length. In fact, we show a more general result.

Theorem 1.3. *Let G be a finitely presented group, and let $C_1, \dots, C_l$ be free abelian subgroups of G whose ranks are at least two. Suppose G is residually finite. Then it holds that $T_\infty(G; C_1, \dots, C_l) = T_\infty(G)$.*

This result is the most technical part in this thesis. The simplicial volume has a similar property [LS09, Theorem 1.5]. Namely, We can consider two versions of simplicial volume of a manifold M with boundary. One is the seminorm of the relative fundamental class, and another is for the open manifold $\mathrm{int}\,M$. They coincide if the fundamental groups of the boundary components are amenable.

Furthermore, we show that the stable presentation length of a hyperbolic 3-manifold is bounded by constant multiples of the volume and the stable complexity.

Theorem 1.4. *There exists a constant $K > 0$ such that the following holds. If M be a hyperbolic 3-manifold, then*

$$K \cdot T_\infty(M) \leq \mathrm{vol}(M) \leq \pi \cdot T_\infty(M).$$

In Section 5, we show additivity of the stable presentation length. We give a proof as with the proof for the stable complexity by using Delzant's estimate [Del96, Theorem II] and Theorem 1.3. We also show that the stable presentation length of a Seifert 3-manifold vanishes. These result implies that the stable presentation length of a closed 3-manifold is equal to the sum of the stable presentation lengths of hyperbolic pieces after the geometrization.

In Appendix, we give examples of stable presentation length. The stable presentation lengths of the surface groups are the only example of non-zero explicit value of stable presentation length in this thesis. We also give examples for fundamental groups of some hyperbolic 3-manifolds. Those examples support Conjecture 1.2.

Acknowledgements

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2 Preliminaries for presentation length

We review the definition of presentation length and some elementary facts. See
Delzant [Del96] for details.

Definition 2.1. Let G be a finitely presented group. We define the *presentation
length* $T(G)$ of G by

$$T(G) = \min_{\mathcal{P}} \sum_{i=1}^{m} \max\{0, |r_i| - 2\},$$

where we take the minimum among the presentations such as
$\mathcal{P} = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ of G, and let $|r_i|$ denote the word length of r_i.

We associate the *presentation complex* P to a presentation
$\mathcal{P} = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ of G. P is the 2-dimensional cell complex consisting
of a single 0-cell, 1-cells and 2-cells corresponding to the generators and relators.
Then $\pi_1(P)$ is isomorphic to G. By dividing a k-gon of a presentation complex
into $k - 2$ triangles, $T(G)$ can be realized by a *triangular presentation* of G, i.e.
a presentation $\langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ in which each word length $|r_i|$ is equal to 2
or 3. If G has no 2-torsion, we can assume $|r_i| = 3$. From now on, a presentation
complex is always assumed to be triangular, i.e. each of its 2-cells is a triangle
or a bigon. $T(G)$ is the minimal number of triangles in a presentation complex
for G.

Delzant [Del96] also introduced a relative version of the presentation length.
We need this in order to estimate the presentation length under a decomposition
of group.

Definition 2.2. Let G be a finitely presented group. Suppose that $C_1, \ldots, C_l$
are subgroups of G. A *(relative) presentation complex* P for $(G; C_1, \ldots, C_l)$ is
a 2-dimensional cell complex satisfying the following conditions:

- P consists of triangles, bigons, edges and l vertices marked with $C_1, \ldots, C_l$.

- P is an orbihedron in the sense of Haefliger [Hae91], with isotropies $C_1, \ldots, C_l$
  on the vertices.

- The fundamental group $\pi_1^{\mathrm{orb}}(P)$ of P as an orbihedron is isomorphic to
  G. This isomorphism makes the isotropies $C_1, \ldots, C_l$ be the subgroups of
  G up to conjugacy.

We define the *relative presentation length* $T(G; C_1, \ldots, C_l)$ as the minimal num-
ber of triangles in a relative presentation complexes for $(G; C_1, \ldots, C_l)$. We say
that a presentation complex P is *minimal* if P realizes the presentation length.

Our definition requires that the isotropy is only on the vertices, but this
is not essential. Indeed, if isotropy of a 2-complex is on edges or 2-cells, we
can construct a presentation complex by replacing edges with bigons. We can
consider only the conjugacy classes of $C_1, \ldots, C_l < G$. By definition, we have
$T(G; \{1\}) = T(G)$. We can allow a presentation complex for G to have more
than one vertex, namely, $T(G; \{1\}, \ldots, \{1\}) = T(G; \{1\})$. This follows by con-
tracting vertices of a presentation complex along edges, without changing the
fundamental group. More generally, the following holds.

Proposition 2.3. [Del96, Lemma I.1.3] *For a finitely presented group G and its subgroups $C, C', C_1, \ldots, C_l$, suppose that C' is contained in a conjugate of C. Then*

$$T(G; C, C', C_1, \ldots, C_l) = T(G; C, C_1, \ldots, C_l).$$

The relative presentation length is finite in a usual case though it was not declared. The construction in the proof will be used for the proof of Theorem 4.2.

Proposition 2.4. *Let G be a finitely presented group. Suppose that $C_1, \ldots, C_l$ are finitely generated subgroups of G. Then $T(G; C_1, \ldots, C_l) < \infty$.*

Proof. Take a presentation complex P for G. Let $y_{i1}, \ldots, y_{ik_i}$ be generators of C_i for $a \leq i \leq l$. There exist simplicial paths $a_{i1}, \ldots, a_{ik_i}$ in P corresponding to $y_{i1}, \ldots, y_{ik_i}$. We construct a complex P' by attaching cones of $a_{i1}, \ldots, a_{ik_i}$ to P (Figure 1). Put isotropy C_i on the vertex of the i-th cone. Then P' is a finite presentation complex for $(G; C_1, \ldots, C_l, \{1\})$. $\square$
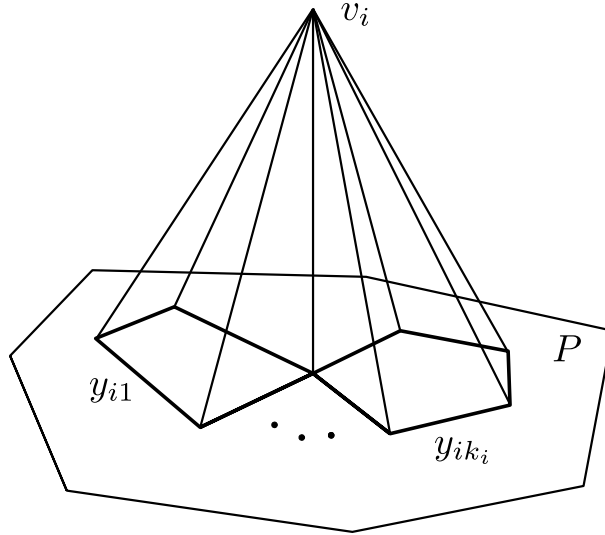

Figure 1: Construction of a relative presentation complex

Delzant [Del96] show how the presentation length behaves under a decomposition into a graph of groups. A *graph of groups* $\mathcal{G}$ in the sense of Serre [Ser80] is a collection of the following data:

- An underlying connected graph Γ, consisting a vertex set V, an edge set E and maps $o_\pm \colon E \to V$ from edges to their end points.

- Vertex groups $\{G_v\}$ and edge groups $\{C_e\}$ for $v \in V$ and $e \in E$.

- Injections $\{\iota_\pm \colon C_e \hookrightarrow G_{o_\pm(e)}\}$ for $e \in E$.

$\mathcal{G}$ induces the fundamental group $\pi_1(\mathcal{G})$. A graph of spaces $\mathcal{X}$ corresponding to $\mathcal{G}$ is a collection of CW-complexes $\{X_v\}, \{X_e\}$ and π_1-injective maps $\{i_\pm \colon X_e \hookrightarrow$

$X_{o_\pm(e)}\}$, where $\pi_1(X_v) = G_v, \pi_1(X_e) = C_e$ and $i_\pm$ induces $\iota_\pm$. We construct a space

$$X_{\mathcal{X}} = \left(\coprod_{v \in V} X_v \sqcup \coprod_{e \in E}(X_e \times [-1, 1])\right)/\sim,$$

where the gluing relation is that $(x, \pm 1) \sim i_\pm(x)$ for $x \in X_e$. Then $\pi_1(\mathcal{G}) = \pi_1(X_{\mathcal{X}})$. For a given group G, we say that $\mathcal{G}$ is a decomposition of G if $G \cong \pi_1(\mathcal{G})$.

Let $\mathcal{G}$ be a decomposition of a group G. Suppose that $G_1, \ldots, G_n$ are the vertex groups of $\mathcal{G}$ and $C_1, \ldots, C_l$ are the edge groups of $\mathcal{G}$. We construct presentation complexes P_i for $(G_i; C_{i1}, \ldots, C_{il_i})$, where C_{ij} for $1 \le j \le l_i$ are the edge groups corresponding to the edges such that the i-th vertex is its end point. We can construct a presentation complex P for $(G; C_1, \ldots, C_l)$ by gluing $P_1, \ldots, P_n$ along their vertices. Then the number of the triangles of P is the sum of the ones of P_i. Therefore we have the following proposition.

Proposition 2.5. [Del96, Lemma I.1.4] *Let G, C_i and C_{ij} be as above. Then*

$$T(G; C_1, \ldots, C_l) \le \sum_{i=1}^{n} T(G_i; \{C_{ij}\}_{1 \le j \le l_i}).$$

We need to consider a "good" decomposition in order to estimate the presentation length from below.

Definition 2.6. Let $\mathcal{G}$ be a decomposition of G, and let $C_1, \ldots, C_l$ be the edge subgroups of $\mathcal{G}$. A subgroup C of G is *rigid* if it satisfies the following condition: If G acts a tree T without inversion and C contains a nontrivial stabilizer of an edge of T, C fixes a vertex of T. $\mathcal{G}$ is *rigid* if every edge group of $\mathcal{G}$ is rigid.

Let C_{ij} be as above. $\mathcal{G}$ is *reduced* if there is no decomposition $\mathcal{G}'$ of G_i such that C_{ij} is a vertex group of $\mathcal{G}'$, for any G_i and C_{ij}.

Under the above preparation, we can state the following highly nontrivial fact.

Theorem 2.7. [Del96, Theorem II] *Let $\mathcal{G}, G_i$ and C_{ij} be as Proposition 2.5. Suppose that $\mathcal{G}$ is rigid and reduced. Then*

$$T(G) \ge \sum_{i=1}^{n} T(G_i; \{C_{ij}\}_{1 \le j \le l_i}).$$

Since a free product decomposition of a group is rigid and reduced, we have the following theorem.

Corollary 2.8. [Del96, Corollary I] *Let $G = A * B$ be a free product of finitely presented groups. Then $T(G) = T(A) + T(B)$.*

We will mainly consider the fundamental group of a 3-manifold. A decomposition of the fundamental group of a 3-manifold corresponds to a decomposition of the 3-manifold along an essential surface. Then a component of the decomposed manifold corresponds to a vertex group, and a component of the essential surface corresponds to an edge group. We can apply Theorem 2.7 in this case.

Proposition 2.9. [Del96, Proposition I.6.1] *Let $\mathcal{G}$ be a decomposition of the fundamental group of an irreducible 3-manifold M. Suppose that $\mathcal{G}$ corresponds to a decomposition of M along an essential surface. Then $\mathcal{G}$ is rigid and reduced.*

3 Definition of stable presentation length

The (relative) presentation length is an upper volume, i.e. it has the following sub-multiplicative property.

Proposition 3.1. *For a finitely presented group G, let H be a finite index subgroup of G. Let $d = [G : H]$ denote the index of H in G. Suppose that $C_1, \ldots, C_l$ are subgroups of G. Then*

$$T(H; \{gC_ig^{-1} \cap H\}_{1 \leq i \leq l, g \in G}) \leq d \cdot T(G; C_1, \ldots, C_l).$$

In particular, $T(H) \leq d \cdot T(G)$.

We remark that $\{gC_ig^{-1} \cap H\}_{1 \leq i \leq l, g \in G}$ is a finite family of subgroups up to conjugate in H, since H is a finite index subgroup of G.

Proof. Let P be a minimal presentation complex for $(G; C_1, \ldots, C_l)$. There exists a d-sheeted covering $\widetilde{P}$ of P as an orbihedron which corresponds to $H \leq G$. Then the isotropies on the vertices of $\widetilde{P}$ are $\{gC_ig^{-1} \cap H\}_{1 \leq i \leq l, g \in G}$. Therefore $\widetilde{P}$ a presentation complex for $(H; \{gC_ig^{-1} \cap H\}_{1 \leq i \leq l, g \in G})$ with $d \cdot T(G; C_1, \ldots, C_l)$ triangles. $\square$

Proposition 3.1 leads the definition of stable presentation length as an analogue of the stable complexity by Francaviglia , Frigerio and Martelli [FFM12]. Stable presentation length is a "volume-like" invariant, i.e. it is multiplicative for finite index subgroups.

Definition 3.2. We define the *stable presentation length $T_\infty(G)$* of a finitely presented group G by

$$T_\infty(G) = \inf_{H \leq G} \frac{T(H)}{[G : H]},$$

where we take the infimum among all the finite index subgroups H. Furthermore, suppose that $C_1, \ldots, C_l$ are subgroups of G. We define the *(relative) stable presentation length* as

$$T_\infty(G; C_1, \ldots, C_l) = \inf_{H \leq G} \frac{T(H; \{gC_ig^{-1} \cap H\}_{1 \leq i \leq l, g \in G})}{[G : H]}.$$

Proposition 3.3. *Let G, H, d and $C_1, \ldots, C_l$ be as Proposition 3.1. Then*

$$T_\infty(H; \{gC_ig^{-1} \cap H\}_{1 \leq i \leq l, g \in G}) = d \cdot T_\infty(G; C_1, \ldots, C_l).$$

In particular, $T_\infty(H) = d \cdot T_\infty(G)$.

Proof. Take a finite index subgroup G' of G. Then $H' = G' \cap H$ is also a finite index subgroup of G. We have

$$T(H'; \{gC_ig^{-1} \cap H'\}_{1 \leq i \leq l, g \in G}) \leq [G' : H']T(G'; \{gC_ig^{-1} \cap G'\}_{1 \leq i \leq l, g \in G})$$

by Proposition 3.1. Hence we can calculate $T_\infty(G; C_1, \ldots, C_l)$ by taking the infimum for only the subgroups of H. Therefore

$$\begin{aligned}
T_\infty(H; \{gC_ig^{-1} \cap H\}_{1 \leq i \leq l, g \in G}) &= \inf_{H' \leq H} \frac{T(H'; \{gC_ig^{-1} \cap H'\}_{1 \leq i \leq l, g \in G})}{[H : H']} \\
&= d \cdot \inf_{H' \leq H} \frac{T(H'; \{gC_ig^{-1} \cap H'\}_{1 \leq i \leq l, g \in G})}{[G : H']} \\
&= d \cdot T_\infty(G; C_1, \ldots, C_l).
\end{aligned}$$

$\square$

4   Stable presentation length for hyperbolic 3-manifolds

We consider the stable presentation length of the fundamental group of a compact 3-manifold M. We write

$$T(M) = T(\pi_1(M)), \quad T_\infty(M) = T_\infty(\pi_1(M)),$$
$$T(M; \partial M) = T(\pi_1(M); \pi_1(S_1), \dots, \pi_1(S_l)),$$
$$T_\infty(M; \partial M) = T_\infty(\pi_1(M); \pi_1(S_1), \dots, \pi_1(S_l)),$$

where $S_1, \dots, S_l$ are the components of ∂M. We call them the (relative, stable) presentation length of M respectively.

If M is a 3-manifold with boundary, we can also consider the relative presentation length $T(M; \partial M)$. For instance, let M be a finite volume cusped hyperbolic 3-manifold. We consider M as a compact 3-manifold with boundary. The interior of M admits a hyperbolic metric. Let $S_1, \dots, S_l$ be the components of ∂M. The 2-skeleton of an ideal triangulation of M (i.e. a cell decomposition of the space obtained by smashing each boundary component of M to a point such that every 3-cell is tetrahedron and its vertices are the points from boundary components of M) can be regarded as a relative presentation complex of $(\pi_1(M); \pi_1(S_1), \dots, \pi_1(S_l))$. We show that this relative stable presentation length coincides with the non-relative stable presentation length.

Theorem 4.1. *For a finite volume hyperbolic 3-manifold M, it holds that $T_\infty(M; \partial M) = T_\infty(M)$.*

More generally, we show the following theorem. Since $\pi_1(M)$ is linear for a hyperbolic 3-manifold M, it is residually finite [Hem87].

Theorem 4.2. *Let G be a finitely presented group, and let $C_1, \dots, C_l$ be free abelian subgroups of G whose ranks are at least two. Suppose G is residually finite. Then it holds that $T_\infty(G; C_1, \dots, C_l) = T_\infty(G)$.*

We remark that it is necessary to suppose the rank of C_i is at least two. The inequality does not hold for the case of Theorem A.2, since $T_\infty(\pi_1(\Sigma_{g,b})) = 0$.

For an integer $p > 1$, the *p-characteristic covering* of torus T^2 is the covering which corresponds to the subgroup $p\mathbb{Z} \times p\mathbb{Z} < \mathbb{Z} \times \mathbb{Z} \cong \pi_1(T^2)$. A *p-characteristic covering* of M is a finite covering whose restriction on each cusp is a union of p-characteristic coverings of torus. A hyperbolic 3-manifold M admits p-characteristic coverings for arbitrarily large p [Hem87, Lemma 4.1]. We can use them for a proof of Theorem 4.1. In general, however, a residually finite group G with $C_1, \dots, C_l$ may not have such subgroups. Nonetheless, we can take a nearly orthogonal basis of subgroup of C_i with respect to a basis of C_i $(1 \le i \le l)$.

A *lattice* in $\mathbb{R}^n$ is a discrete subgroup of $\mathbb{R}^n$ which spans $\mathbb{R}^n$. A lattice in $\mathbb{R}^n$ has a nearly orthogonal basis in the following sense. Such a basis is called a *reduced basis*. We refer to Cassels [Cas71, Ch.VIII.5.2] for a proof. Lenstra, Lenstra and Lovász [LLL82] gave a polynomial time algorithm to find a reduce basis. We will use the following lemma with a 1-norm on $\mathbb{R}^n$.

Lemma 4.3. *Given a norm $\| \cdot \|$ in $\mathbb{R}^n$, there is a constant ϵ_n such that the following holds. If Λ is a lattice in $\mathbb{R}^n$, then there is a reduced basis $(v_1, \ldots, v_n)$ of Λ such that*

$$d(\Lambda) \geq \epsilon_n \|v_1\| \cdots \|v_n\|,$$

where $d(\Lambda)$ is the covolume of Λ, which is the determinant of the matrix whose columns are v_i's.

proof of Theorem 4.2. For simplicity, we assume $l = 1$ and write $C = C_1$ and $r = \text{rank}(C) \geq 2$. We first show that $T_\infty(G; C) \leq T_\infty(G)$. It is sufficient to show that $T_\infty(G; C) \leq T(G)$.

Take a minimal presentation complex P for G. Let $\alpha_1, \ldots, \alpha_r$ be simplicial paths in P representing generators $x_1, \ldots, x_r$ of C. Let a_i denote the length of α_i for $1 \leq i \leq r$.

Suppose that H is a finite index normal subgroup of G. Let d denote the index of $H < G$. Let $\widetilde{P}$ be the covering of P corresponding to H. Let $\{C'_1, \ldots, C'_m\}$ be subgroups of H representing the conjugacy classes of $\{gCg^{-1} \cap H\}_{g \in G}$. C'_i can be regarded as a finite index subgroup of C by the natural inclusion $\iota_i \colon C'_i \hookrightarrow C$. Since H is normal in G, all the images of ι_i's coincide and have index d/m in C. We regard $C \cong \mathbb{Z}^r$ as a lattice in $\mathbb{R}^r$ and put the 1-norm $\| \cdot \|$ in $\mathbb{R}^r$ with respect to the basis $(x_1/a_1, \ldots, x_r/a_r)$.

We construct a presentation complex $\widetilde{P}'$ for $(H; \{C'_i\}_{1 \leq i \leq m})$ by attaching 2-cells to $\widetilde{P}$. We take a reduced basis $(y_1, \ldots, y_r)$ of $\iota_i(C'_i)$ as in Lemma 4.3. Let $\beta_{i1}, \ldots, \beta_{ir}$ be paths in $\widetilde{P}$ representing $y_1, \ldots, y_r \in \iota_i(C'_i)$ such that the length of β_{ij} is $\|y_j\|$. We obtain a presentation complex $\widetilde{P}'$ by attaching cones of β_{ij}'s as in the proof of Proposition 2.4. The number of the triangles of $\widetilde{P}'$ is

$$d \cdot T(G) + m(\|y_1\| + \cdots + \|y_r\|).$$

It holds that $d/m \geq \epsilon_r \|y_1\| \cdots \|y_r\|$ by Lemma 4.3. Hence

$$T_\infty(G; C) \leq \frac{T(H; \{C'_i\}_{1 \leq i \leq m})}{d} \leq T(G) + \frac{\|y_1\| + \cdots + \|y_r\|}{\epsilon_r \|y_1\| \cdots \|y_r\|}$$

Since G is residually finite, there is a normal subgroup H of G such that every $\|y_j\|$ for $1 \leq j \leq r$ is arbitrarily large. We have supposed that $r \geq 2$. Therefore we obtain $T_\infty(G; C) \leq T(G)$.

Conversely we show that $T_\infty(G) \leq T(G; C)$. Take a minimal presentation complex Q for $(G; C)$. We construct a presentation complex for G by truncating a neighborhood of the vertex of Q (Figure 2) and attaching 2-cells. Let Q' be the truncated complex. Let Γ be the sectional graph of the truncation in Q'. Attaching edges to Γ if necessary, we may assume that Γ is connected and the natural map from $\pi_1(\Gamma)$ to C is surjective. We contract vertices of Q' along edges of Γ to obtain a 2-complex Q''. We obtain a bouquet Γ' in Q'' from Γ. Then we have the natural surjection $p \colon \pi_1(\Gamma') \to C$. Attaching more edges to Γ' if necessary, we may assume that there are edges $\gamma_1, \ldots, \gamma_r$ such that the images of the elements $[\gamma_1], \ldots, [\gamma_r] \in \pi_1(\Gamma')$ forms a basis of C. Let $\gamma'_1, \ldots, \gamma'_s$ be the other edges of Γ'. Write $z_j = p[\gamma_j]$ and $z'_k = p[\gamma'_k]$ for $1 \leq j \leq r$ and $1 \leq k \leq s$. z'_k can be presented as a product of z_j's, and let b_k denote its word length. We obtain a presentation complex Q''' for G by attaching triangles to Q'' along Γ', where $r(r-1)$ attached triangles correspond to the commutators

$[z_i, z_j] = z_i z_j z_i^{-1} z_j^{-1}$ $(1 \leq i, j \leq r)$ and at most $b_1 + \cdots + b_s - s$ attached triangles correspond to the presentation of z'_k by z_j's. Let K denote the union of Γ' and the attached triangles.
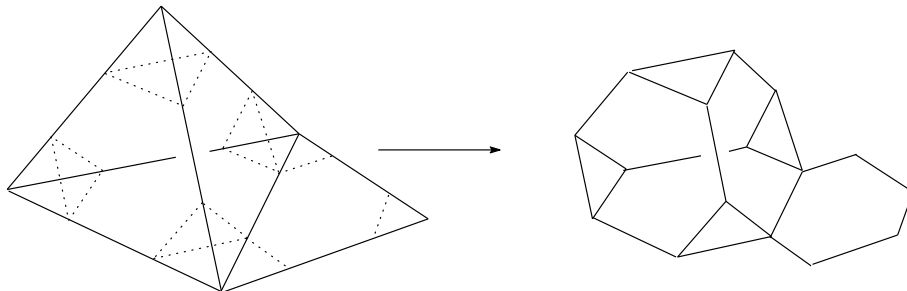

Figure 2: Truncation of the presentation complex Q

Suppose that H is a finite index normal subgroup of G. d and $\{C'_1, \ldots, C'_m\}$ are as above. Let $\widetilde{Q}$ be the covering of Q'' corresponding to H. Let $\widetilde{K}_1, \ldots, \widetilde{K}_m$ be the components of covering of K in $\widetilde{Q}$ corresponding to $\{C'_1, \ldots, C'_m\}$. Each covering $\widetilde{K}_i \to K$ has degree d/m. In order to construct a presentation complex $\widetilde{Q}'$ for H, we contract simplices of $\widetilde{K}_i \subset \widetilde{Q}$ in the following manner.

We describe the way of contraction on the universal covering of K. We regard $\pi_1(K) = C$ as a lattice in $\mathbb{R}^r$. Take a reduced basis of $\pi_1(\widetilde{K}_i)(< \pi_1(K))$. Let F be the fundamental domain of $\pi_1(\widetilde{K}_i)$ defined by this reduced basis. We contract simplices in the interior of F into a point.

We give an example in Figure 3. Suppose $z_1 = (1, 0), Z_2 = (0, 1)$ and $z'_1 = (2, 1)$. The 2-complex K consists of three triangles corresponding to the commutator $[z_1, z_2]$ and $z'_1 = z_1^2 z_2$. Now let $((3, -1), (1, 4))$ be taken as a basis of a lattice $\pi_1(\widetilde{K}_i)$. Then we contract 15 triangles whose projection is in the interior of F.

This construction does not change the fundamental group of $\widetilde{Q}$. (If $r \geq 3$, this construction may change the homotopy type of $\widetilde{Q}$.) Thus we obtain a presentation complex $\widetilde{Q}'$ for H.
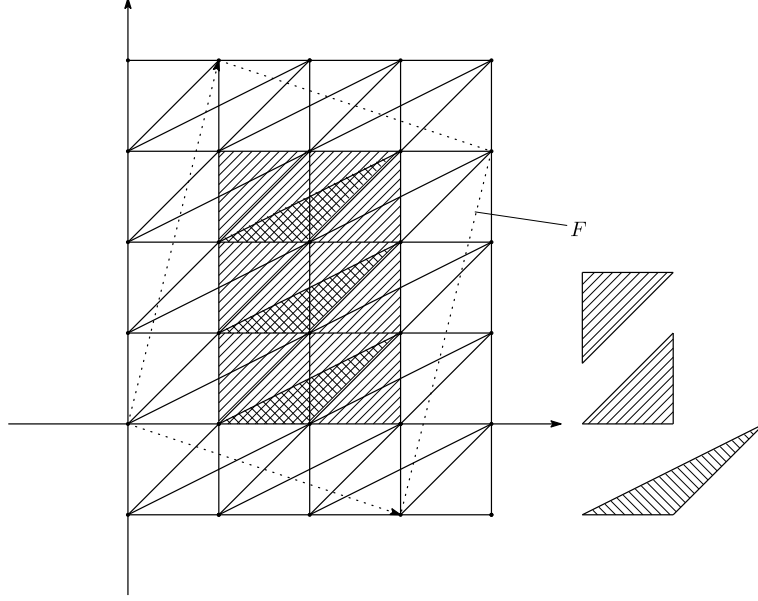
Figure 3: Contraction of simplices in F

The number of the triangles of $\widetilde{Q}'$ is at most

$$d \cdot T(G; C) + m(e + f),$$

where

$$e = e_{11} + \cdots + e_{1r} + e_{21} + \cdots + e_{2s},$$
$$f = f_1 + f_{21} + \cdots + f_{2s},$$

and e_{1j} and e_{2k} are the numbers of the edges of $\widetilde{Q}'$ which derive from γ_j and γ'_k, f_1 is the number of the triangles of $\widetilde{Q}'$ which derive from ones corresponding to the commutators $[x_i, x_j]$, and f_{2k} is the number of the triangles of $\widetilde{Q}'$ which derive from ones corresponding to the presentation of z'_k by z_j's. $d \cdot T(G; C) + me$ triangles of $\widetilde{Q}'$ derive from the hexagons of Q' and mf triangles of $\widetilde{Q}'$ derive from the triangles of K.

If the edges and triangles are not contracted by the above construction, They are near the boundary of F in the above picture. Hence there exists a constant $\delta_r > 0$ such that the followings hold:

$$e_{1j} \le \delta_r \mathrm{vol}(\partial F), \qquad\qquad e_{2k} \le b_k \delta_r \mathrm{vol}(\partial F),$$
$$f_1 \le r(r-1)\delta_r \mathrm{vol}(\partial F), \qquad\qquad f_{2k} \le (b_k - 1)e_{2k},$$

where $\mathrm{vol}(\partial F)$ is the surface area of F with respect to the standard Euclidean metric of $\mathbb{R}^r$. Therefore

$$T_\infty(G) \le \frac{T(H)}{d} \le T(G; C) + \frac{m}{d}(e + f)$$
$$\le T(G; C) + (r^2 + \sum_{k=1}^{s} b_k^2)\delta_r \cdot \frac{\mathrm{vol}(\partial F)}{\mathrm{vol}(F)}.$$

13

Since G is residually finite and F is defined by a reduced basis, there is a normal subgroup H of G such that $\mathrm{vol}(\partial F)/\mathrm{vol}(F)$ is arbitrarily small. $\qquad\square$

Cooper [Coo99] showed that $\mathrm{vol}(M) < \pi \cdot T(M)$ for a closed hyperbolic 3-manifold M. The isoperimetric inequality by Agol and Liu [AL12, Lemma 4.4] implies that this inequality also holds for a cusped hyperbolic 3-manifold. Delzant and Potyagailo [DP13] remarked that a converse inequality does not hold, namely, the infimum of $\mathrm{vol}(M)/T(M)$ for the hyperbolic 3-manifolds is zero. Indeed, hyperbolic Dehn surgery [Thu80, Ch. 4 and 6] gives infinitely many hyperbolic manifolds whose presentation length are divergent while their volumes are bounded. Delzant and Potyagailo used a relative presentation length $T(\pi_1(M); \mathcal{E})$ to bound the volume from below, where $\mathcal{E}$ consists of the elementary subgroups of $\pi_1(M)$ whose translation length are less than a Margulis number. They also showed that $\mathrm{vol}(M) \leq \pi \cdot T(\pi_1(M); \mathcal{E})$ [DP13, Theorem B]. In particular $\mathrm{vol}(M) \leq \pi \cdot T(M; \partial M)$. We use the stable presentation length to bound the volume instead of $T(\pi_1(M); \mathcal{E})$. Cooper's inequality immediately implies that $\mathrm{vol}(M) \leq \pi \cdot T_\infty(M)$. A converse estimate holds for the stable presentation length.

Proposition 4.4. *The infimum of $\mathrm{vol}(M)/T_\infty(M)$ for the hyperbolic 3-manifolds is positive.*

In order to show this, we mention a connection between the presentation length and the complexity of a 3-manifold. For a closed 3-manifold M, the Δ-complexity (or Kneser complexity) $\sigma(M)$ is defined as the minimal number of tetrahedra over the triangulations of M. $\sigma(M)$ is also defined for a cusped finite volume hyperbolic 3-manifold M by ideal triangulations. The complexity $c(M)$ by Matveev [Mat90] is the minimal number of vertices over the simple spines of M. It holds that $\sigma(M) = c(M)$ if M is irreducible and not $S^3, \mathbb{R}P^3$ or the lens space $L(3,1)$, in particular, if M is a hyperbolic 3-manifold [Mat90, Theorem 5].

Francaviglia, Frigerio and Martelli [FFM12] introduced stable complexities $\sigma_\infty(M)$ and $c_\infty(M)$ of 3-manifold M. They are defined as $\inf \sigma(\widetilde{M})/d$ and $\inf c(\widetilde{M})/d$ by taking the infimum among all the finite coverings $\widetilde{M}$ of M, where d is the degree of the covering. It holds that $\sigma_\infty(M) = c_\infty(M)$ if M is a hyperbolic 3-manifold. $c_\infty(M)$ vanishes for a Seifert 3-manifold M, and c_∞ has additivity for the prime decomposition and the JSJ decomposition.

Proposition 4.5. *For a closed 3-manifold M, it holds that $T(M) \leq \sigma(M) + 1$.*

Proof. We take a minimal triangulation of M. Consider the 2-skeleton P_0 of this triangulation. P_0 has $2\sigma(M)$ triangles. Since a 2-complex P in M has a fundamental group isomorphic to $\pi_1(M)$ as long as $M \setminus P$ consists of 3-balls, We can remove $(\sigma(M) - 1)$ triangles from P_0 without changing the fundamental group. Therefore we obtain a presentation complex for $\pi_1(M)$ with $(\sigma(M) + 1)$ triangles. $\qquad\square$

Proposition 4.6. *For a cusped finite volume hyperbolic 3-manifold M, it holds that $T(M) \leq \sigma(M) + 3$.*

Proof. We take a minimal ideal triangulation of M. Consider the dual spine P_0 of this triangulation. P_0 has $\sigma(M)$ 2-cells, $2\sigma(M)$ edges and $\sigma(M)$ vertices. This $\sigma(M)$ 2-cells can be decomposed into $4\sigma(M)$ triangles. We contract $(\sigma(M)-1)$ vertices along edges. Since every edge of P_0 is incident on three triangles, we obtain a presentation complex of $\pi_1(M)$ with $(\sigma(M)+3)$ triangles. $\qquad\square$

Since the fundamental group of 3-manifold is residually finite [Hem87], M admits arbitrarily large finite covering if $\pi_1(M)$ is infinite. This implies the following corollary.

Corollary 4.7. *If M is a closed 3-manifold or a finite volume hyperbolic 3-manifold, it holds that $T_\infty(M) \le \sigma_\infty(M)$.*

The stable complexity of a hyperbolic 3-manifold is bounded from above and below by constant multiples of its volume. For a finite volume hyperbolic 3-manifold M, it holds that $\mathrm{vol}(M) \le V_3\sigma(M)$, where V_3 is the volume of ideal regular tetrahedron, which is the maximum of the volumes of geodesic tetrahedra in the hyperbolic 3-space. This implies that $\mathrm{vol}(M) \le V_3\sigma_\infty(M)$. Conversely, there exists a constant $K > 0$ such that $\sigma_\infty(M) \le K \cdot \mathrm{vol}(M)$ holds for any hyperbolic manifold M. This follows from the fact by Jørgensen and Thurston that a thick part of a hyperbolic 3-manifold can be decomposed by uniformly thick tetrahedra. Proofs of this fact are given by Francaviglia, Frigerio and Martelli [FFM12, Proposition 1.5] in the case M is closed, and by Breslin [Bre09] and Kobayashi and Rieck [KR11] otherwise. Proposition 4.4 follows from this inequality and Corollary 4.7.

We conjecture an equality between the stable presentation length and the stable complexity.

Conjecture 4.8. *For a finite volume hyperbolic 3-manifold M, it holds that*

$$T_\infty(M) = \frac{1}{2}\sigma_\infty(M).$$

We give some examples supporting that $T_\infty(M) \le \sigma_\infty(M)/2$ in Appendix.

It holds that $T(M) \ge \sigma(M)/2$ if a minimal (relative) presentation complex for $\pi_1(M)$ injects to M. This is because M can be decomposed into $2T(M)$ tetrahedra.

If Conjecture 4.8 holds, $T_\infty(M) = (1/2V_3)\mathrm{vol}(M)$ for a hyperbolic 3-manifold M which is commensurable with the figure-eight knot complement M_1. Indeed, $\sigma_\infty(M_1) = 2$ since M_1 can be decomposed into two ideal regular tetrahedra. Conjecture 4.8 implies a best possible refinement of Cooper's inequality $\mathrm{vol}(M) < 2V_3 \cdot T(M)$.

5 Additivity of stable presentation length

We will show additivity of the stable presentation length of 3-manifold groups in the same manner as the simplicial volume. The proofs of Theorem 5.1 and 5.3 are similar. Let G be a finitely presented group and let $\{G_i\}$ be decomposed groups of G. We will construct a presentation complex for a finite index subgroup of G by gluing finite coverings of presentation complexes for G_i. This implies an inequality between $T_\infty(G)$ and $\sum_i T_\infty(G_i)$. In order to show the

converse inequality, we will obtain presentation complexes for finite index subgroups of G_i's by decomposing a finite covering of a presentation complex for G.

We first show additivity for a free product. This holds for any finitely presented group.

Theorem 5.1. *For finitely presented groups G_1 and G_2, it holds that*

$$T_\infty(G_1 * G_2) = T_\infty(G_1) + T_\infty(G_2).$$

Proof. We will use additivity of presentation length for a free product in Corollary 2.8. Write $G = G_1 * G_2$. We first show that $T_\infty(G) \leq T_\infty(G_1) + T_\infty(G_2)$. For $i = 1, 2$, let P_i be presentation complexes for G_i. Take d_i-index subgroups H_i of G_i. Let $\widetilde{P_i}$ denote the coverings of P_i corresponding to H_i. Since each $\widetilde{P_i}$ has d_i vertices, we can glue d_2 copies of $\widetilde{P_1}$ and d_1 copies of $\widetilde{P_2}$ along the vertices to obtain a $d_1 d_2$-sheet covering $\widetilde{P}$ of $P_1 \vee P_2$. The wedge sum $P_1 \vee P_2$ is a presentation complex for G. Then $\pi_1(\widetilde{P})$ is isomorphic to a free product $H_1^{*d_2} * H_2^{*d_1} * F_k$, where F_k is a free group. Corollary 2.8 implies that $T(\pi_1(\widetilde{P})) = d_2 \cdot T(H_1) + d_1 \cdot T(H_2)$. Therefore

$$T_\infty(G) \leq \frac{T(\pi_1(\widetilde{P}))}{d_1 d_2} = \frac{T(H_1)}{d_1} + \frac{T(H_2)}{d_2}.$$

Since we took H_1 and H_2 arbitrarily, we obtain that $T_\infty(G) \leq T_\infty(G_1) + T_\infty(G_2)$.

Conversely, we show that $T_\infty(G_1) + T_\infty(G_2) \leq T_\infty(G)$. Let P_i be as above. $P = P_1 \vee P_2$ is a presentation complex for G. Take a d-index subgroup H of G. Let $\widetilde{P}$ denote the covering of P corresponding to H. $\widetilde{P}$ is homotopic to

$$P_{11} \vee \cdots \vee P_{1m} \vee P_{21} \vee \cdots \vee P_{2n} \vee S^1 \vee \cdots \vee S^1,$$

where P_{ij} is a covering of P_i. Let d_{ij} be the degree of the covering $P_{ij} \to P_i$. Then $\sum_{j=1}^m d_{1j} = \sum_{j=1}^n d_{2j} = d$. Since $H = \pi_1(\widetilde{P})$ is isomorphic to

$$\pi_1(P_{11}) * \cdots * \pi_1(P_{1m}) * \pi_1(P_{21}) * \cdots * \pi_1(P_{2n}) * F_k,$$

Corollary 2.8 and Proposition 3.3 implies that

$$
\begin{aligned}
T(H) =& T(\pi_1(P_{11})) + \cdots + T(\pi_1(P_{1m})) + T(\pi_1(P_{21})) + \cdots + T(\pi_1(P_{2n})) \\
\geq& d_{11} \cdot T_\infty(\pi_1(P_1)) + \cdots + d_{1m} \cdot T_\infty(\pi_1(P_1)) \\
& + d_{21} \cdot T_\infty(\pi_1(P_2)) + \cdots + d_{2n} \cdot T_\infty(\pi_1(P_2)) \\
=& d \cdot T_\infty(G_1) + d \cdot T_\infty(G_2).
\end{aligned}
$$

Therefore $T_\infty(G_1) + T_\infty(G_2) \leq \dfrac{T(H)}{d}$. Since we took H arbitrarily, we obtain that $T_\infty(G_1) + T_\infty(G_2) \leq T_\infty(G)$. $\qquad\square$

Before we show additivity for the JSJ decomposition, we show that the stable presentation length for a Seifert 3-manifold vanishes.

Theorem 5.2. *For a compact Seifert 3-manifold M,*

$$T_\infty(M) = T_\infty(M; \partial M) = 0.$$

16

Proof. Since a Seifert 3-manifold can be regarded as an S^1-bundle over an 2-orbifold, M is covered by an S^1-bundle over a surface. Hence we can assume M is an S^1-bundle over a compact surface.

If M has boundary, M is a product of S^1 and a surface. Then M admits a d-sheeted covering homeomorphic to M for any $d \leq 1$. This implies that $T_\infty(M) = T_\infty(M; \partial M) = 0$ by Proposition 3.3.

We consider an S^1-bundle over a closed surface Σ_g of genus g. Homeomorphic class of an S^1-bundle over Σ_g is determined by the Euler number e. Let $M(\Sigma_g, e)$ denote the S^1-bundle over Σ_g of the Euler number e. Since $\pi_1(M(S^2, e))$ is finite or isomorphic to $\mathbb{Z}$, we have $T_\infty(M(S^2, e)) = 0$. Suppose $g \geq 1$. $\pi_1(M(\Sigma_g, e))$ has a presentation

$$\langle x_1, y_1, \ldots, x_g, y_g, z | [x_1, y_1] \ldots [x_g, y_g] z^e, [x_i, z], [y_i, z] \quad (1 \leq i \leq g) \rangle,$$

where x_i, y_i's are corresponding to generators of the fundamental group of the base surface and z is a generator of the fundamental group of the ordinary fiber, and $[x, y]$ denotes the commutator $xyx^{-1}y^{-1}$. Therefore

$$T(\pi_1(M(\Sigma_g, e))) \leq 8g + |e| - 2.$$

For any integer $d \geq 1$, $M(\Sigma_g, e)$ admits $M(\Sigma_{g'}, de)$ as a d-sheeted covering along the base space, where $g' = d(g - 1) + 1$. Furthermore, $M(\Sigma_{g'}, de)$ admits $M(\Sigma_{g'}, e)$ as a d-sheeted covering along the fiber direction. Thus we obtain a d^2-sheeted covering $M(\Sigma_{g'}, e) \to M(\Sigma_g, e)$. Hence

$$T_\infty(\pi_1(M(\Sigma_g, e))) \leq \frac{T(\pi_1(M(\Sigma_{g'}, e)))}{d^2} \leq \frac{8(d(g-1) + 1) + |e| - 2}{d^2}.$$

The right hand side converges to zero when d increases. $\qquad\square$

Finally we show additivity for the JSJ decomposition.

Theorem 5.3. *Let M be an irreducible 3-manifold. Suppose $M = M_1 \cup \cdots \cup M_h$ is the JSJ decomposition. $M_1, \ldots, M_h$ are compact 3-manifolds with incompressible torus boundary. Then*

$$T_\infty(M) = T_\infty(M_1) + \cdots + T_\infty(M_h).$$

Proof. We remark that the fundamental group of a compact 3-manifold is residually finite by Hempel [Hem87] and the geometrization.

We first show that $T_\infty(M) \leq T_\infty(M_1) + \cdots + T_\infty(M_h)$. Take d_i-sheet coverings $f_i \colon \widetilde{M_i} \to M_i$ for $1 \leq i \leq h$. Then there exists an integer p independent of i and coverings $g_i \colon N_i \to \widetilde{M_i}$ such that $f_i \circ g_i \colon N_i \to M_i$ is a p-characteristic covering, i.e. the restriction of the covering on each component of ∂M_i is the covering corresponding to $p\mathbb{Z} \times p\mathbb{Z} < \mathbb{Z} \times \mathbb{Z}$ [FFM12, Proposition 4.7]. We can glue copies N_{ij} of N_i along boundary to obtain a d-sheeted covering $f \colon N \to M$. Then $f^{-1}(M_i) = N_{11} \cup \cdots \cup N_{il_i}$. Each copy $g_{ij} \colon N_{ij} \to \widetilde{M_i}$ of g_i is a $d/l_i d_i$-sheeted covering. $N = \bigcup_{i,j} N_{ij}$ is the JSJ decomposition. Therefore we obtain that

$$T(\pi_1(N); \{\pi_1(\partial N_{ij})\}) \leq \sum_{i,j} T(N_{ij}; \partial N_{ij})$$

$$\leq \sum_{i,j} \frac{d}{l_i d_i} T(\widetilde{M_i}; \partial \widetilde{M_i}) = \sum_i \frac{d}{d_i} T(\widetilde{M_i}; \partial \widetilde{M_i})$$

17

by Proposition 2.5. Hence

$$T_\infty(\pi_1(M); \{\pi_1(\partial M_i)\}) \leq \frac{T(\pi_1(N); \{\pi_1(\partial N_{ij})\})}{d} \leq \sum_i \frac{T(\widetilde{M_i}; \partial \widetilde{M_i})}{d_i}.$$

Since we took $\widetilde{M_i}$ arbitrarily, we obtain that

$$T_\infty(\pi_1(M); \{\pi_1(\partial M_i)\}) \leq \sum_i T_\infty(\widetilde{M_i}; \partial \widetilde{M_i}).$$

Furthermore, $T_\infty(M) = T_\infty(\pi_1(M); \{\pi_1(\partial M_i)\})$ and $T_\infty(\widetilde{M_i}) = T_\infty(\widetilde{M_i}; \partial \widetilde{M_i})$ by Theorem 4.2.

Conversely, we show that $T_\infty(M_1) + \cdots + T_\infty(M_h) \leq T_\infty(M)$. Take a d-sheet covering $p \colon \widetilde{M} \to M$. Then the components M_{ij} of $p^{-1}(M_i)$ are the components of the JSJ decomposition of $\widetilde{M}$. Let d_{ij} denote the degree of the covering $M_{ij} \to M_i$. Then $\sum_j d_{ij} = d$. We have

$$\sum_j T(M_{ij}; \partial M_{ij}) \geq \sum_j d_{ij} \cdot T_\infty(M_i; \partial M_i) = d \cdot T_\infty(M_i; \partial M_i)$$

by definition. Theorem 2.7 implies that

$$\sum_{i,j} T(M_{ij}; \partial M_{ij}) \leq T(\widetilde{M}).$$

Therefore it holds that

$$\sum_i T_\infty(M_i; \partial M_i) \leq \frac{T(\widetilde{M})}{d}.$$

Since we took $\widetilde{M}$ arbitrarily, we obtain that

$$\sum_i T_\infty(M_i; \partial M_i) \leq T_\infty(M).$$

Furthermore, $T_\infty(M_i) = T_\infty(M_i; \partial M_i)$ by Theorem 4.2. $\qquad\square$

Corollary 5.4. *There exists a constant $K > 0$ such that the following holds. For a closed 3-manifold M, it holds that*

$$K \cdot T_\infty(M) \leq \|M\| \leq \frac{\pi}{V_3} T_\infty(M),$$

where $\|M\|$ is the simplicial volume of M and V_3 is the volume of an ideal regular tetrahedron.

Proof. We can assume that M is orientable by taking the double covering. Let $M = M_1 \# \ldots \# M_n$ be the prime decomposition. Each connected summand M_i is irreducible or homeomorphic to $S^1 \times S^2$. Let $M_i = M_{i1} \cup \cdots \cup M_{ih_i}$ be the JSJ decomposition if M_i is irreducible. The geometrization implies that each JSJ component M_{ij} is Seifert fibered or hyperbolic. Let $N_1, \ldots, N_m$ denote the hyperbolic components among M_{ij}. Then

$$\|M\| = 1/V_3(\mathrm{vol}(N_1) + \cdots + \mathrm{vol}(N_m))$$

18

by additivity and proportionality of simplicial volume [Gro82]. Now we have that

$$T_\infty(M) = T_\infty(N_1) + \cdots + T_\infty(N_m)$$

by Theorem 5.1, Theorem 5.2 and Theorem 5.3. Therefore we are reduced to proving for hyperbolic 3-manifolds. A hyperbolic 3-manifold M satisfies the above inequalities by Cooper's inequality and Proposition 4.4. $\qquad\square$

A Examples of stable presentation length

A.1 Surface groups

We calculate the explicit value of the stable presentation length of a surface group, which coincides with the simplicial volume of the surface.

Theorem A.1. *Let Σ_g is the closed orientable surface of genus $g \geq 1$. Then*

$$T_\infty(\pi_1(\Sigma_g)) = 4g - 4 = -2\chi(\Sigma_g).$$

Proof. If $g = 1$, $\pi_1(\Sigma_g) \cong \mathbb{Z} \times \mathbb{Z}$ has a finite index proper subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Then $T_\infty(\pi_1(\Sigma_g)) = 0$ by Proposition 3.3.

Suppose that $g \geq 2$. Since there is a presentation

$$\pi_1(\Sigma_g) = \langle x_1, y_1, \ldots, x_g, y_g | [x_1, y_1] \cdots [x_g, y_g] \rangle,$$

we have $T(\pi_1(\Sigma_g)) \leq 4g - 2$. In order to estimate from below, take a minimal presentation complex P for $\pi_1(\Sigma_g)$. We put a hyperbolic metric on Σ_g. There exists a map $f \colon P \to \Sigma_g$ inducing an isomorphism between their fundamental groups. We can take f which maps every 2-cell of P to a geodesic triangle in Σ_g.

We claim that f is surjective. If f is not surjective, there is a point p in $\Sigma_g - f(P)$. Then f induces an injection from $\pi_1(\Sigma_g)$ to $\pi_1(\Sigma_g - \{p\})$. Since $\pi_1(\Sigma_g - \{p\})$ is a free group and $\pi_1(\Sigma_g)$ is not a free group , we have a contradiction. Now area$(\Sigma_g) = (4g - 4)\pi$ and the area of a geodesic triangle in Σ_g is smaller than π. Hence we obtain $(4g - 4)\pi < \pi \cdot T(\pi_1(\Sigma_g))$.

We finally compute $T_\infty(\pi_1(\Sigma_g))$. Since $\Sigma_{d(g-1)+1}$ covers Σ_g with degree d, $T_\infty(\pi_1(\Sigma_g)) \leq \frac{1}{d}T(\pi_1(\Sigma_{d(g-1)+1})) \leq \frac{1}{d}(4(d(g-1)+1) - 2)$. Hence we obtain that $T_\infty(\pi_1(\Sigma_g)) \leq 4g - 4$ by $d \to \infty$. Conversely, $4g - 4 < \frac{1}{d}T(\pi_1(\Sigma_{d(g-1)+1}))$ for any $d \geq 1$ implies that $4g - 4 \leq T_\infty(\pi_1(\Sigma_g))$. $\qquad\square$

Theorem A.2. *Let $\Sigma_{g,b}$ is the compact orientable surface of genus g whose boundary components are $S_1, \ldots, S_b$. Suppose that $b > 0$ and $2g - 2 + b > 0$. Then*

$$T_\infty(\pi_1(\Sigma_{g,b}); \pi_1(S_1), \ldots, \pi_1(S_b)) = T(\pi_1(\Sigma_{g,b}); \pi_1(S_1), \ldots, \pi_1(S_b))$$
$$= 4g - 4 + 2b = -2\chi(\Sigma_{g,b}).$$

Proof. $\Sigma_{g,b}$ admits a hyperbolic metric with cusps $S_1, \ldots, S_b$. An ideal triangulation of this hyperbolic surface gives a presentation complex for $(\pi_1(\Sigma_{g,b}); \pi_1(S_1), \ldots, \pi_1(S_b))$, which consists of $4g - 4 + 2b$ triangles. Therefore $T(\pi_1(\Sigma_{g,b}); \pi_1(S_1), \ldots, \pi_1(S_b)) \leq 4g - 4 + 2b$.

In order to obtain the converse inequality, we put a hyperbolic metric with geodesic boundary on $\Sigma_{g,b}$. Take a minimal presentation complex P for $(\pi_1(\Sigma_{g,b}); \pi_1(S_1), \ldots, \pi_1(S_b))$. Let P' be the complex obtained by truncating P. There is a continuous map $f\colon P' \to \Sigma_{g,b}$ such that f sends the truncated section $\partial P'$ of P' to the corresponding boundary components and f induces an isomorphism between their fundamental groups. Then f induces a map $Df\colon DP' \to D\Sigma_{g,b}$ between their doubles. Since Df induces an isomorphism between the fundamental groups, Df is surjective by the proof of Theorem A.1. Therefore f is also surjective. After straightening f relatively to the boundary, the 2-cells of P' map to right-angled hexagons, whose areas are equal to π. Then

$$(4g - 4 + 2b)\pi = \text{area}(\Sigma_{g,b}) \leq \pi \cdot T(\pi_1(\Sigma_{g,b}); \pi_1(S_1), \ldots, \pi_1(S_b)).$$

Now we have $T(\pi_1(\Sigma_{g,b}); \pi_1(S_1), \ldots, \pi_1(S_b)) = 4g - 4 + 2b$. Since these values are already volume-like, their stable presentation lengths coincide with their presentation lengths. $\qquad\square$

A.2   Bianchi groups

We consider the stable presentation lengths of Bianchi groups $PSL(2, \mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers in the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, namely,

$$\mathcal{O}_d = \begin{cases} \mathbb{Z}[\frac{1+\sqrt{-d}}{2}] & \text{if} -d \equiv 1 \mod 4 \\ \mathbb{Z}[\sqrt{-d}] & \text{if} -d \equiv 2, 3 \mod 4. \end{cases}$$

It is known that the fundamental group of every finite volume cusped arithmetic hyperbolic 3-manifold is commensurable with a Bianchi group ([NR92, Proposition 4.1]). We give an upper bound of stable presentation lengths of some arithmetic link components by constructing explicit presentations of their fundamental groups. We consider them as links in $T^2 \times [0, 1]$ in order to take coverings efficiently.

A.2.1 $d = 3$ (Figure-eight knot complement)

The figure-eight knot complement M_1 is obtained from two ideal regular tetrahedra. Hence $\text{vol}(M_1) = 2V_3 = 2.0298...$ and $\sigma(M_1) = \sigma_\infty(M_1) = 2$. $\pi_1(M_1)$ is an index 12 subgroup of $PSL(2, \mathcal{O}_3)$.

Proposition A.3.
$$T_\infty(M_1) \leq 1.$$

Proof. We consider a link in $T^2 \times [0, 1]$ constructed by gluing of the piece in Figure 4 along faces of top and bottom, left and right. Let $M_{1,1}$ denote the complement of this link. $M_{1,1}$ can be decomposed into four ideal regular hexagonal pyramids (Figure 5). Since a union of two ideal regular hexagonal pyramids can be decomposed into six ideal regular tetrahedra, $M_{1,1}$ is obtained from 12 ideal regular tetrahedra. Hence $T_\infty(M_{1,1}) = 6T_\infty(M_1)$.

Let $M_{m,n}$ be the mn-sheeted covering of $M_{1,1}$ which is the m-sheeted covering along s and the n-sheeted covering along t as Figure 6. We obtain an explicit presentation of $\pi_1(M_{m,n})$. The generators are

$$x_{ij}, y_{ij}, z_{ij}, w_{ij}, a_{ij}, b_{ij}, x_{m+1,j}, y_{m+1,j}, x_{i,n+1}, z_{i,n+1}, x_{m+1,n+1}, s, t,$$

and the relators are

$$a_{ij} = y_{ij}x_{ij}, \qquad a_{ij} = z_{ij}y_{ij}, \qquad a_{ij} = w_{ij}z_{ij},$$

$$b_{ij} = z_{i,j+1}w_{ij}, \qquad b_{ij} = w_{ij}x_{i+1,j+1}, \qquad b_{ij} = x_{i+1,j+1}y_{i+1,j},$$

$$x_{m+1,j} = sx_{1,j}s^{-1}, \quad x_{m+1,n+1} = sx_{1,n+1}s^{-1}, \quad y_{m+1,j} = sy_{1,j}s^{-1},$$

$$x_{i,n+1} = tx_{i,1}t^{-1}, \quad x_{m+1,n+1} = tx_{m+1,1}t^{-1}, \quad z_{i,n+1} = tz_{i,1}t^{-1}, \quad st = ts,$$

for $1 \le i \le m, 1 \le j \le n$. Therefore

$$T_\infty(M_{1,1}) \le \inf_{m,n} \frac{T(M_{m,n})}{mn} \le \inf_{m,n} \frac{6mn + 4m + 4n + 6}{mn} = 6.$$

$\square$
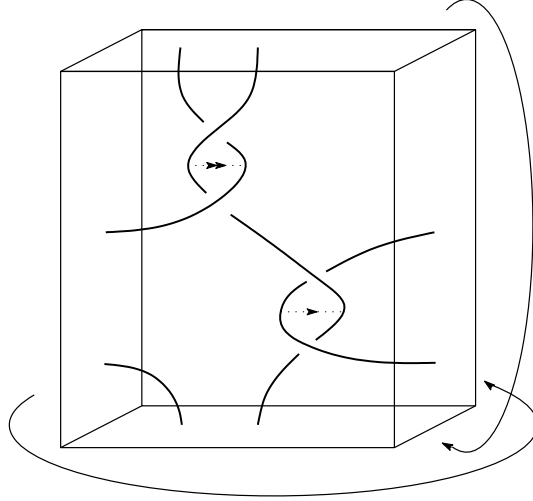
Figure 4: $M_{1,1}$
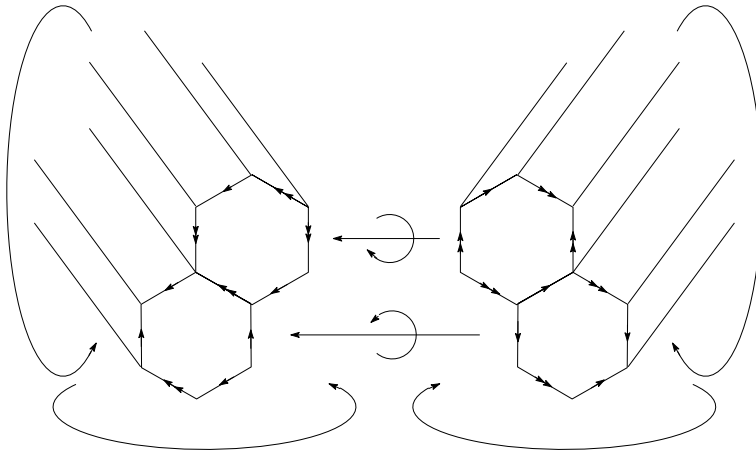
Figure 5: a decomposition of $M_{1,1}$
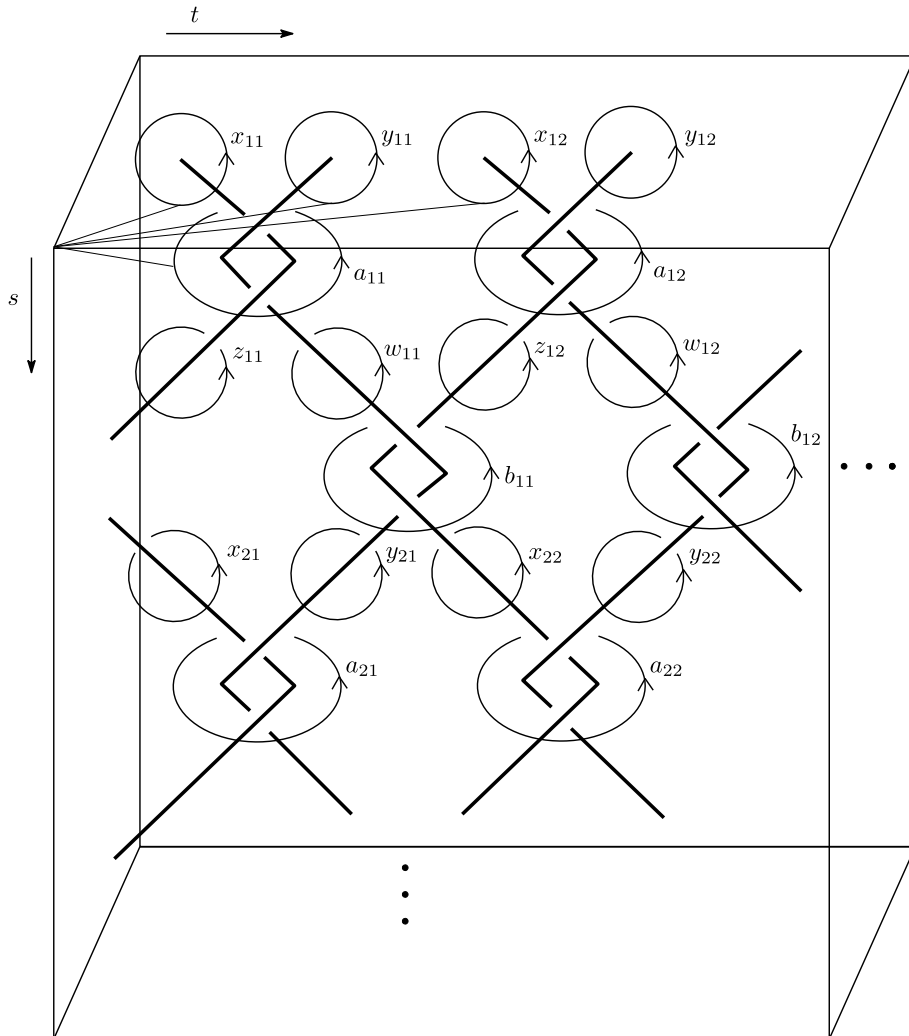
Figure 6: generators of $\pi_1(M_{m,n})$

We give another proof of Proposition A.3. $M_{1,1}$ has four cusps S_0, S_1, S_2, S_3, where S_0 and S_1 are the boundary component of $T^2 \times [0,1]$. We construct a fundamental domain X of $M_{1,1}$ as a union of 12 ideal regular tetrahedra such that S_0 corresponds to a single vertex v of X (Figure 7). Then we obtain a presentation complex for $(\pi_1(M_{1,1}); \pi_1(S_1), \pi_1(S_2), \pi_1(S_3))$ from the triangles in ∂X which do not contain v. Hence $T(\pi_1(M_{1,1}); \pi_1(S_1), \pi_1(S_2), \pi_1(S_3)) \le 6$. Theorem 4.2 implies that

$$T_\infty(M_{1,1}) = T_\infty(\pi_1(M_{1,1}); \pi_1(S_1), \pi_1(S_2), \pi_1(S_3)) \le 6.$$
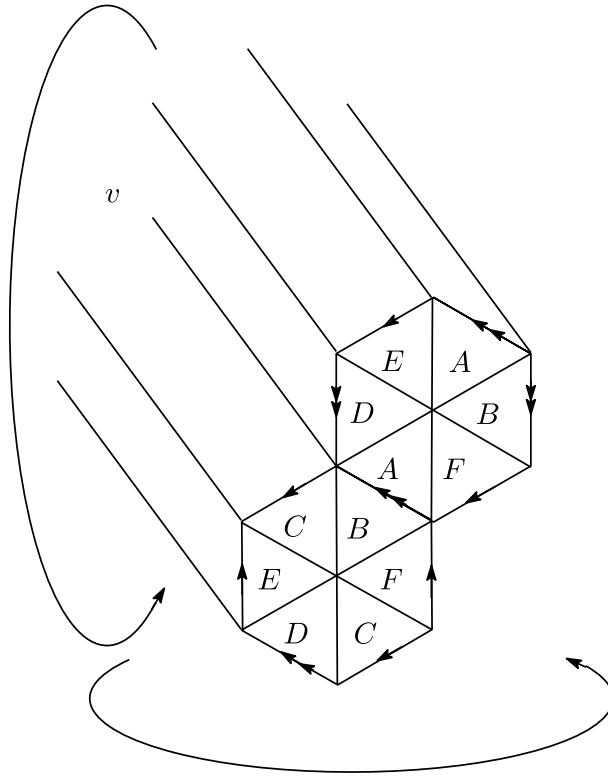
Figure 7: a fundamental domain X

A.2.2 $d = 1$ (Whitehead link complement)

The Whitehead link complement M_2 is obtained from one ideal regular octahedron. Since $\mathrm{vol}(M_2) = 3.6638...$, $\sigma(M_2) = 4$ and $3.6 < \sigma_\infty(M_2) \le 4$. It is unknown whether $\sigma_\infty(M_2) = 4$ or not. $\pi_1(M_2)$ is an index 12 subgroup of $PSL(2, \mathcal{O}_1)$.

Proposition A.4.
$$T_\infty(M_2) \le 2.$$

Proof. As with the above proposition, we consider a link in $T^2 \times [0, 1]$ (Figure 8). Let M_2' denote the complement of this link. M_2' can be decomposed into four ideal regular square pyramids (Figure 9). Since a union of two ideal regular square pyramids is an ideal regular octahedron, M_2' is obtained from two ideal regular octahedra. Hence $T_\infty(M_2') = 2T_\infty(M_2)$.

We obtain an explicit presentation of $\pi_1(M_2')$. The generators are

$$x_{11}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, a, b, s, t,$$

and the relators are

$$a = y_{11}x_{11}, \qquad a = x_{22}y_{11}, \qquad b = y_{12}x_{22}, \qquad b = x_{22}y_{21},$$
$$x_{21} = sx_{11}s^{-1}, \quad y_{21} = sy_{11}s^{-1}, \quad y_{12} = ty_{11}t^{-1}, \quad x_{22} = tx_{21}t^{-1}, \quad st = ts.$$

23

After we take large coverings along $T^2 \times [0,1]$ as with Proposition A.3, the relators which does not contain s or t contribute an estimate of the stable presentation length. Therefore $T_\infty(M_2') \leq 4$. □
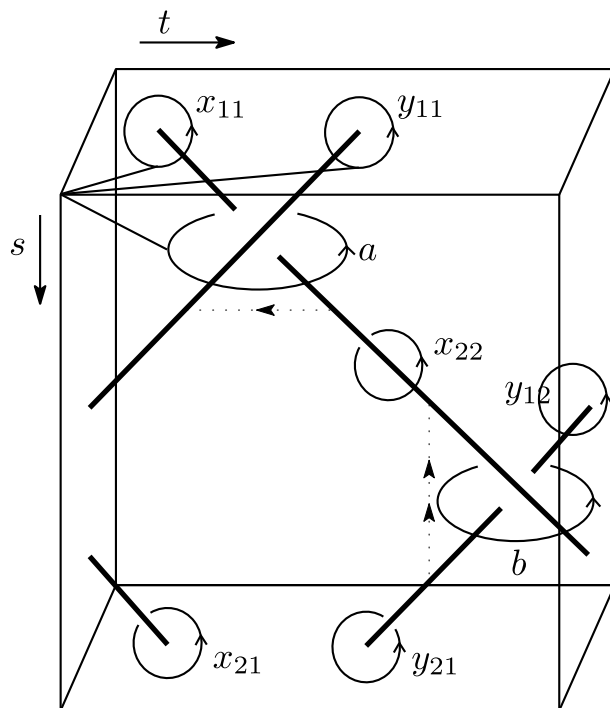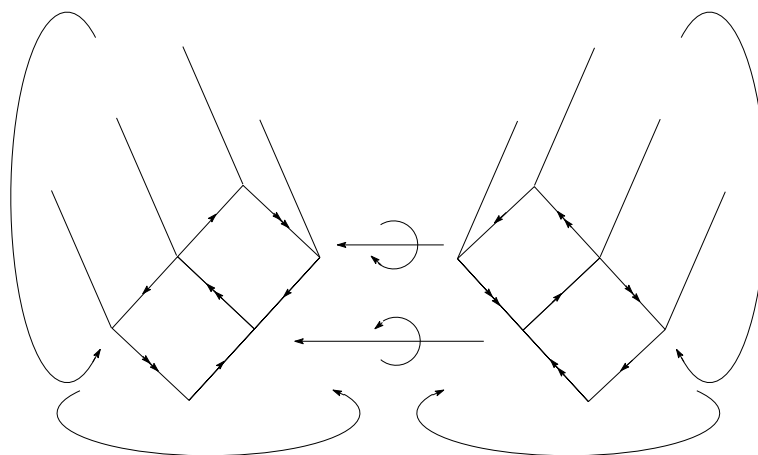

Figure 8: M_2

Figure 9: a decomposition of M_2

We can prove that $T_\infty(M_2') \leq 4$ by constructing a relative presentation complex as with Proposition A.3.

24

A.2.3 $d = 7$ (Magic manifold)

The alternating 3-chain link complement M_3 is called the Magic manifold (Figure 10). M_3 is obtained from two ideal regular triangular prism. Since $\mathrm{vol}(M_3) = 5.3334...$, $\sigma(M_3) = 6$ and $5.2 < \sigma_\infty(M_3) \leq 6$. $\pi_1(M_3)$ is an index 6 subgroup of $PSL(2, \mathcal{O}_7)$ ([Thu80, Ch.6, Example 6.8.2]).

Proposition A.5.
$$T_\infty(M_3) \leq 3.$$

Proof. We can consider M_3 as the complement of a link in $T^2 \times [0, 1]$ (Figure 11).

We obtain an explicit presentation of $\pi_1(M_3)$. The generators are

$$x_{11}, x_{21}, y_{11}, y_{12}, a, s, t,$$

and the relators are

$$a = y_{12}x_{11}, \qquad a = x_{11}x_{21}, \qquad a = x_{21}y_{11},$$
$$x_{21} = sx_{11}s^{-1}, \qquad y_{12} = ty_{11}t^{-1}, \qquad st = ts.$$

After we take large coverings along $T^2 \times [0, 1]$ as with the above propositions, the relators which does not contain s or t contribute an estimate of the stable presentation length. Therefore $T_\infty(M_3) \leq 3$. $\qquad \square$
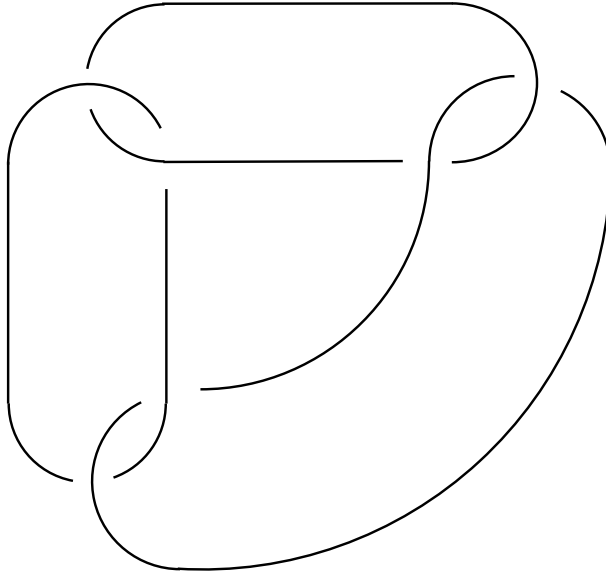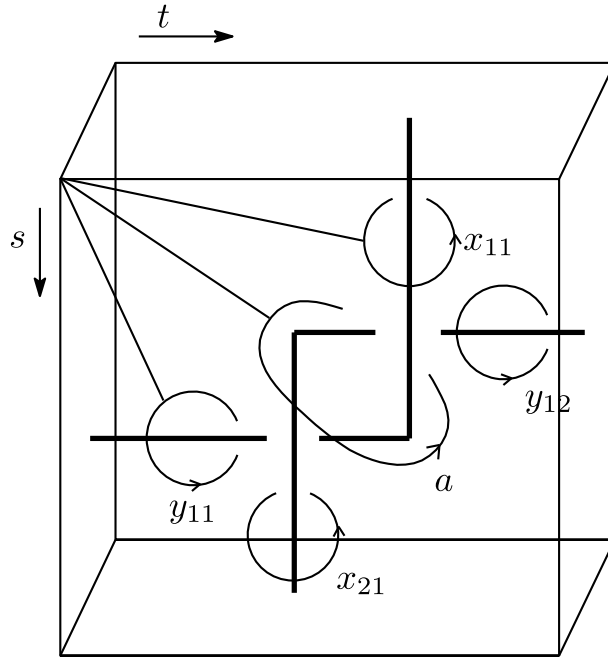

Figure 10: the alternating 3-chain link

Figure 11: M_3

A.2.4 $d = 2$

Let M_4 denote the complement of the link in Figure 12. M_4 is obtained from one ideal regular cuboctahedron ([Thu80, Ch.6, Example 6.8.10]). Since $\mathrm{vol}(M_4) =$ 12.0460... and a cuboctahedron can be decomposed into 14 tetrahedra compatible to a decomposition of M_4, $12 \leq \sigma(M_4) \leq 14$ and $11.8 < \sigma_\infty(M_3) \leq 14$. $\pi_1(M_4)$ is an index 6 subgroup of $PSL(2, \mathcal{O}_2)$.

Proposition A.6.
$$T_\infty(M_4) \leq 7.$$

Proof. We can consider M_4 as the complement of a link in $T^2 \times [0, 1]$ (Figure 13).

We obtain an explicit presentation of $\pi_1(M_4)$. The generators are

$$x_{11}, x_{21}, y_{11}, y_{21}, y_{20}, z_{11}, z_{12}, w, u, a, b, c, s, t,$$

and the relators are

$$
\begin{array}{llll}
a = y_{11}x_{11}, & a = wy_{11}, & a = uw, & b = y_{20}z_{11}, \\
c = z_{12}u, & c = ax_{21}, & c = x_{21}b, & \\
x_{21} = sx_{11}s^{-1}, & y_{21} = sy_{11}s^{-1}, & z_{12} = tz_{11}t^{-1}, & y_{21} = ty_{20}t^{-1}, \quad st = ts.
\end{array}
$$

After we take large coverings along $T^2 \times [0, 1]$ as with the above propositions, the relators which does not contain s or t contribute an estimate of the stable presentation length. Therefore $T_\infty(M_4) \leq 7$. □
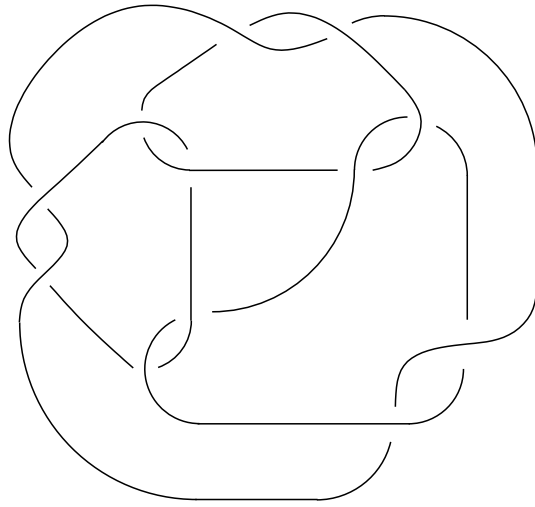
26

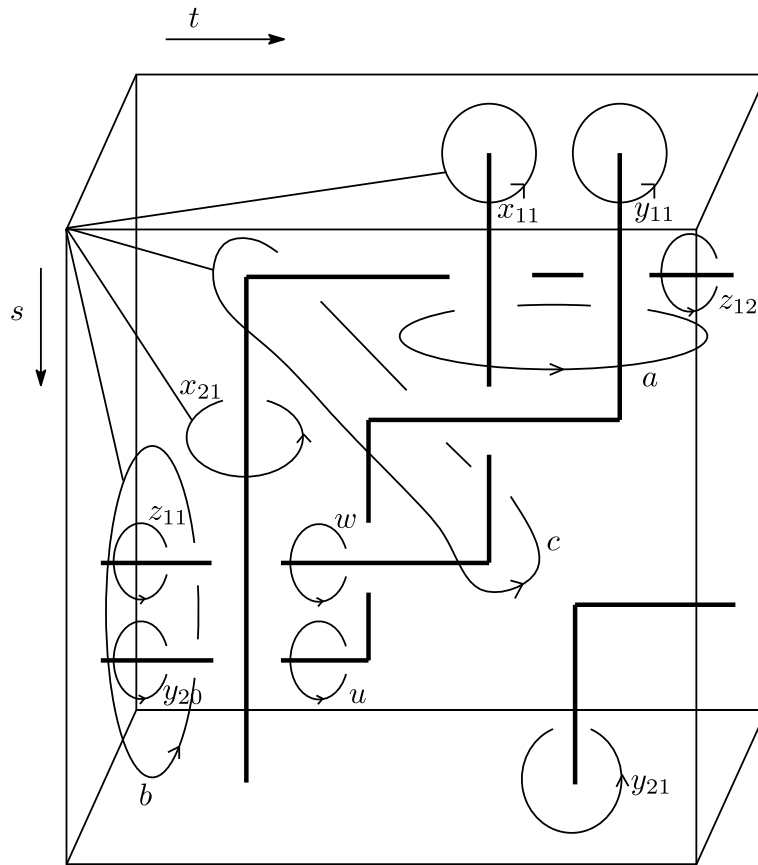Figure 12: a link whose complement is M_4

Figure 13: M_4

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