博士論文

論文題目: Studies on singular Hermitian metrics
with minimal singularities
on numerically effective line bundles

(数値的半正な正則直線束の
極小特異エルミート計量に関する研究)

氏名: 小池 貴之

Studies on singular Hermitian metrics with minimal singularities on numerically effective line bundles

Takayuki Koike

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Preface

The purpose of this thesis is to study minimal singular metrics on a holomorphic line bundle, which are metrics with the mildest singularities among singular Hermitian metrics whose local weights are plurisubharmonic. Minimal singular metrics were introduced by Demailly, Peternell and Schneider [DPS00] as a weak analytic analogue of the Zariski decomposition. It is known that there exists a minimal singular metric on every pseudo-effective line bundle if the manifold is compact. The singularity of minimal singular metrics is one of the most important data for studying complex analytic properties of a holomorphic line bundle. For example, when the line bundle is big, some vanishing theorems for higher cohomology groups are stated by using multiplier ideal sheaves, which are determined from the singularities of minimal singular metrics (Nadel type vanishing theorems). However, only a few studies has been done on the concrete expression of minimal singular metrics. In this thesis, we study the concrete expression of minimal singular metrics and multiplier ideal sheaves of them especially in some algebro-geometrically important situations.

First, in §2, we give a concrete expression of a minimal singular metric on a big line bundle on a compact Kähler manifold which is the total space of a toric bundle over a complex torus. In this class of manifolds, Nakayama [Na] constructed examples which have line bundles admitting no Zariski decomposition even after modifications. As an application, we discuss the Zariski closedness of non-nef loci.

From §3, we will focus on nef line bundles. A holomorphic line bundle L on a projective manifold X is called *nef* if the intersection number $(L.C)$ is non-negative for each closed curve $C \subset X$. It follows directly from the definition that L is nef if L is semi-positive; i.e. if L admits a smooth Hermitian metric with semi-positive curvature. However the nefness of L does not imply the semi-positivity of L. Indeed, Demailly, Peternell and Schneider [DPS94, 1.7] gave an example of a holomorphic line bundle L on a projective surface X such that L is nef however minimal singular metrics of L actually have the singularities along a curve of X, which is the stable base locus of L. In the subsequent chapters, we will study relations between the singularities of minimal singular metrics of a nef line bundle L on a projective manifold X and the complex structure of a neighborhood of the stable base locus of L in X.

One main conclusion of our general result in §3 is the existence of smooth Hermitian metrics with semi-positive curvatures on the so-called Zariski's example of a line bundle L defined over the blow-up X of $\mathbb{P}^2$ at some twelve points: In this example, the stable base locus S of L is a smooth elliptic curve. This L is nef and big, however has a pathological property that $S \subset \mathrm{Bs}\,|L^{\otimes m}|$ holds for all $m \geq 1$, $|L^{\otimes m} \otimes \mathcal{O}_X(-S)|$ is globally generated for all $m \geq 1$, and that the section ring $\bigoplus_{m \geq 0} H^0(X, L^{\otimes m})$ of L is not finitely generated. More generally, we show the following theorem.

Theorem 1 (a variant of Corollary 3.3.4). *Let X be a smooth projective surface, $D \subset X$ be a smooth compact curve, L be a nef line bindle on X with $(L.C) = 0$. Assume that $L \otimes \mathcal{O}_X(-D)$ admits a smooth Hermitian metrics with semi-positive curvature, and that $(D^2) < \min\{0, 4-4g\}$ holds, where g is the genus of D. Then L admits a smooth Hermitian metric with semi-positive curvature.*

We generalize this result to the higher dimensional case when the stable base locus of a line bundle is a smooth hypersurface with a holomorphic tubular neighborhood as follows:

Theorem 2 (=Theorem 3.1.2). *Let X be a smooth projective variety, D a smooth hypersurface of X, L a pseudo-effective line bundle over X, and let $h_{\min}$ be a minimal singular metric of L. Assume that $L \otimes \mathcal{O}_X(-D)$ is semi-positive, $\mathcal{O}_X(-D)|_D$ is ample, $\mathcal{O}_D(-K_D - D|_D)$ is nef and big, and that D has a holomorphic tubular neighborhood (i.e. an open neighborhood in X which is biholomorphic to an open neighborhood of the zero section in the normal bundle $N_{D/X}$). Then $h_{\min}|_D \not\equiv \infty$ holds if and only if $L|_D$ is pseudo-effective, moreover in this case $h_{\min}|_D$ is a minimal singular metric of $L|_D$.*

In §4 and §5, we study applications of Ueda's theory [U] on a study of singular Hermitian metrics of a nef line bundle L. Ueda's theory is a theory on a flatness criterion around a smooth hypersurface of a certain type of topologically trivial holomorphic line bundles. Let S be a smooth compact hypersurface of a complex manifold X with the topologically trivial normal bundle $N_{S/X}$. Then $\mathcal{O}_X(S)$ is topologically trivial on a tubular neighborhood of S in X. For such a pair (S, X), Ueda defined an obstruction class $u_n(S, X) \in H^1(S, N_{S/X}^{-n})$ for each $n \geq 1$, which enjoys the property that $\mathcal{O}_X(S)$ is not flat around S if there exists an integer $n \geq 1$ such that $u_n(S, X) \neq 0$. By using these obstruction classes, Ueda classified the pair (S, X) into two cases: the *finite type* case and the *infinite type* case. The pair (S, X) is said to be *of finite type* n, when the m-th obstruction class $u_m(S, X)$ is trivial for each $m < n$ and it holds that $u_n(S, X) \neq 0$. The pair (S, X) is said to be *of infinite type*, when $u_n(S, X) = 0$ holds for each integer n. He studied the complex structure of a neighborhood V of S in X and the growth of psh functions on $V \setminus S$ when the pair (S, X) is of finite type. He also gave a sufficient condition for $\mathcal{O}_X(S)$ to be flat around S when the pair (S, X) is of infinite type.

In §4, we determine a minimal singular metric of a nef line bundle $\mathcal{O}_X(S)$ on a smooth surface X when the pair (S, X) is of finite type as an application of Ueda's theorem [U, 2].

THEOREM 3 (=Theorem 4.1.1). *Let X be a smooth complex surface and $S \subset X$ be an embedded smooth compact complex curve. Assume $(S^2) = 0$ and the pair (S, X) is of finite type. Then the singular Hermitian metric $|f_S|^{-2}$ on $\mathcal{O}_X(S)$ has minimal singularities, where $f_S \in H^0(X, \mathcal{O}_X(S))$ is a section whose zero divisor is S. Especially, $\mathcal{O}_X(S)$ is nef, however it admits no smooth Hermitian metric with semi-positive curvature.*

By using this result, we give new examples of (strictly) nef line bundles which admit no smooth Hermitian metric with semi-positive curvature. Here we say that a line bundle L on an algebraic variety X is *strictly nef* if the intersection number $(L.C)$ is positive for each compact curve C.

In §5, we will propose a codimension two analogue of Ueda's theory. Namely we shall describe a sufficient condition for the line bundle $\mathcal{O}_X(S)$ to be flat on a neighborhood of C in X, where S is a smooth hypersurface of a complex manifold X and C is a compact smooth hypersurface of S. As an application, we give a sufficient condition for the anti-canonical bundle of the blow-up of the three dimensional projective space at 8 points to admit a smooth Hermitian metric with semi-positive curvature as follows:

THEOREM 4 (=Corollary 5.1.2). *Let $C_0 \subset \mathbb{P}^3$ be a complete intersection of two quadric surfaces of $\mathbb{P}^3$ and let $p_1, p_2, \dots, p_8 \in C_0$ be 8 points different from each other. Assume that the line bundle $\mathcal{O}_{\mathbb{P}^3}(-2)|_{C_0} \otimes \mathcal{O}_{C_0}(p_1 + p_2 + \cdots + p_8)$*

is a torsion element of $\mathrm{Pic}^0(C_0)$*, or an element of*

$$\mathcal{E}_1(C_0) := \{E \in \mathrm{Pic}^0(C_0) \mid \exists \alpha \in \mathbb{R}_{>0} \text{ s.t. } \forall n \in \mathbb{Z}_{>0},\ d(\mathcal{O}_C, E^n) \geq (2n)^{-\alpha}\},$$

where d *is the Euclidean distance in* $\mathrm{Pic}^0(C_0)$*. Then the anti-canonical bundle of the blow-up of* $\mathbb{P}^3$ *at* $\{p_j\}_{j=1}^8$ *admits a smooth Hermitian metric with semi-positive curvature.*

Note that, when $\mathcal{O}_{\mathbb{P}^3}(-2)|_{C_0} \otimes \mathcal{O}_{C_0}(p_1+p_2+\cdots+p_8)$ is a torsion element of $\mathrm{Pic}^0(C_0)$, the anti-canonical line bundle of the blow-up of $\mathbb{P}^3$ at $\{p_j\}_{j=1}^8$ is semi-ample. When $\mathcal{O}_{\mathbb{P}^3}(-2)|_{C_0} \otimes \mathcal{O}_{C_0}(p_1+p_2+\cdots+p_8)$ is a non-torsion element of $\mathrm{Pic}^0(C_0)$, the stable base locus of the anti-canonical line bundle is the strict transform of C_0 and thus it is not semi-ample (however it is nef). Note also that the set $\mathcal{E}_1(C_0)$ is the union of a countable number of nowhere dense closed subsets of $\mathrm{Pic}^0(C_0)$, however the Lebesgue measure of $\mathrm{Pic}^0(C_0) \setminus \mathcal{E}_1(C_0)$ is zero.

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Takayuki Koike

1

Basic materials: minimal singular metrics and Ueda's theory

In this chapter, we collect the basic notations, definitions and results on minimal singular metrics on holomorphic line bundles and Ueda's theory that are used in the subsequent chapters.

1.1 Minimal singular metrics and the Zariski decomposition

Let X be a complex manifold and L be a line bundle on X (Throughout this thesis, "line bundle" always stands for a holomorphic line bundle). Let h be a singular Hermitian metric of L (for the definition of the singular Hermitian metric, see [D2, 3.12]). Then, for each local trivialization of L on an open set of X, "the inner product" defined by h can be written as

$$\langle \xi, \eta \rangle_z = e^{-\psi(z)} \xi \overline{\eta}$$

where z is a point in the open set, ξ and η are points in $\mathbb{C}$, which we regard as the z-fiber of L, and ψ is a locally integrable function defined on the open set, which we call the *local weight* of h. Here we remark that it is known that the local currents $\sqrt{-1}\partial\overline{\partial}\psi$ glue together to define the curvature current associated to h. We denote it by $\sqrt{-1}\Theta_h$.

Next, let us recall how to compare the singularity of two psh functions.

Definition 1.1.1. ([DPS00, 1.4]) Let φ and ψ be psh functions defined on a neighborhood of $x \in X$. We say ψ *is less singular than* φ and write $\varphi \prec_{\text{sing}} \psi$ at x when there exists a constant C such that the inequality

$\varphi \leq \psi + C$ holds for each point sufficiently near to x. We denote $\varphi \sim_{\text{sing}} \psi$ at x if $\varphi \prec_{\text{sing}} \psi$ and $\varphi \succ_{\text{sing}} \psi$ holds at x.

By using this notation, we can define the minimal singular metric as follows.

DEFINITION 1.1.2. Let $h_{\min}$ be a singular hermitian metric on L which satisfies $\Theta_{h_{\min}} \geq 0$. We call $h_{\min}$ a minimal singular metric if $\psi \prec_{\text{sing}} \varphi_{\min}$ holds at any point $x \in X$ for all singular hermitian metric h satisfying $\Theta_h \geq 0$, where $\varphi_{\min}$ and ψ stand for the local weight functions of $h_{\min}$ and h, respectively, with respect to a local trivialization of L around the point $x \in X$.

Minimal singular metrics were introduced in [DPS00, 1.4] as a (weak) analytic analogue of the Zariski decomposition. It is known that there exists a minimal singular metric on every pseudo-effective line bundle if the manifold is compact. This fact is proved by considering the upper semi-continuous regularization of the supremum of the all appropriately normalized ψ's, where ψ is as in Definition 1.1.2 (see [DPS00, 1.5] for details). We remark that, though the minimal singular metric is not unique, but it is unique up to the relation $\sim_{\text{sing}}$ if it exists.

Let L be a big line bundle and $h_{\min}$ be a minimal singular metric on L. We denote by $N(L)$ the negative part $\sum_{\Gamma\,:\,\text{prime divisor}} \nu(\varphi_{\min}, \Gamma)\Gamma$ of L in the sense of the divisorial Zariski decomposition [Bo], where $\varphi_{\min}$ is the local weight of $h_{\min}$ and $\nu(\varphi_{\min}, \Gamma)$ is the Lelong number of $\varphi_{\min}$ at the divisor Γ. We say that L admits a Zariski decomposition if the positive part $P(L) := c_1(L \otimes \mathcal{O}_X(L))$ is nef class. We here remark that this definition of the Zariski-decomposability coincides with Nakayama's algebraic one [Na].

1.2 Ueda's theory

Let X be a complex manifold and L be a holomorphic line bundle on X. We say that L is a flat line bundle if each of the transition functions $t_{jk} \in \Gamma(U_{jk}, \mathcal{O}^*_{U_{jk}})$ of L can be taken as a complex constant with modulus 1, where $\{U_j\}$ is a suitable open covering of X and $U_{jk} := U_j \cap U_k$. This condition is equivalent to the condition that L can be regarded as an element of $H^1(X, U(1))$ ($\subset H^1(X, \mathcal{O}^*_X)$), where $U(1) := \{z \in \mathbb{C} \mid |z| = 1\}$. It can be said directly from the definition that a flat line bundle admits a metric which can locally be regarded as a constant function by using a suitable local frame of the line bundle (we call such a metric a flat metric). Since the (Chern) curvature tensor of a flat metric is 0, it can be said that the first Chern class of a flat line bundle is trivial, and thus a flat line bundle is topologically trivial. When the manifold X is compact and Kähler, it is known that the converse also holds (see [U, §1] for example). However, in general, a topologically trivial holomorphic line bundle need not admit the flat structure.

Now let us start reviewing Ueda's theory along [U, §2] for a complex manifold X and a smooth hypersurface $S \subset X$ whose normal bundle $N_{S/X}$ is flat. Take a sufficiently fine open covering $\{V_j\}$ of S. From the assumption, $N_{S/X} = \{(V_{jk}, t_{jk})\}$ holds in $H^1(S, U(1))$ for some constants $t_{jk} \in U(1)$, where $V_{jk} := V_j \cap V_k$. Let W be a sufficiently small tubular neighborhood of S in X and $\{W_j\}$ be a sufficiently fine open covering of W. Without loss of generality, we may assume that the index sets of $\{V_j\}$ and $\{W_j\}$ coincide and $W_j \cap S = V_j$ holds. We choose local coordinates (w_j, z_j) of W_j satisfying the conditions that z_j is a coordinate of V_j, $\{w_j = 0\} = V_j$ holds on W_j, and that $(w_j/w_k)|_{V_{jk}} \equiv t_{jk}$ holds on V_{jk} for all j and k. Let n be a positive integer. We say that a system $\{(W_j, w_j)\}$ is *of order* n if $\mathrm{mult}_{V_{jk}}(t_{jk}w_k - w_j) \geq n+1$ holds on each W_{jk}. When there exists a system $\{(W_j, w_j)\}$ of order n, the Taylor expansion of $t_{jk}w_k$ for the variable w_j on W_{jk} around $w_j = 0$ can be written in the form

$$t_{jk}w_k = w_j + f_{jk}^{(n+1)}(z_k) \cdot w_j^{n+1} + O(w_j^{n+2})$$

for some holomorphic function $f_{jk}^{(n+1)}$ defined on V_{jk}. Here we remark that, for all $m > n$, a system $\{(W_j, w_j)\}$ of order m is also a system of order n and in this case the function $f_{jk}^{(m+1)}$ is the constant function 0. It is known that $\{(V_{jk}, f_{jk}^{(n+1)})\}$ satisfies the cocycle condition ([U, p. 588], see also the proof of Proposition 5.3.2 here).

Definition 1.2.1. Suppose that there exists a system of order n. Then the cohomology class

$$u_n(S, X) := \{(V_{jk}, f_{jk}^{(n+1)})\} \in H^1(S, N_{S/X}^{-n})$$

is called *the n-th Ueda class* of the pair (S, X).

The n-th Ueda class does not depend on the choice of local coordinates system up to non-zero constant multiples ([Ne, 1.3], see also the proof of Proposition 5.3.4 here). It is known that $u_n(S, X) = 0$ if and only if there exists a system of order $n+1$. Thus only one phenomenon of the following occurs.

- There exists an integer $n \in \mathbb{Z}_{>0}$ such that $u_m(S, X)$ can be defined only when $m \leq n$, $u_m(S, X) = 0$ holds for all $m < n$, and $u_n(S, X) \neq 0$ holds.

- For every integer $n \in \mathbb{Z}_{>0}$, $u_n(S, X)$ can be defined and it is equal to zero.

DEFINITION 1.2.2 ([U, p. 589]). We denote "type $(S, X) = n$" and say that the pair (S, X) is *of finite type* when $u_m(S, X)$ can be defined only when $m \leq n$, $u_m(S, X) = 0$ holds for all $m < n$, and $u_n(S, X) \neq 0$ holds. In the other case, we denote "type $(S, X) = \infty$" and say that the pair (S, X) is *of infinite type*.

Fix an invariant distance d of $\mathrm{Pic}^0(S)$ (i.e. d is a distance of $\mathrm{Pic}^0(S)$ such that, for each E_1, E_2, and $G \in \mathrm{Pic}^0(S)$, $d(E_1^{-1}, E_2^{-1}) = d(E_1, E_2)$ and $d(E_1 \otimes G, E_2 \otimes G) = d(E_1, E_2)$ hold). By using this distance d, define

$$\mathcal{E}_1(S) := \{E \in \mathrm{Pic}^0(S) \mid \exists \alpha \in \mathbb{R}_{>0} \text{ s.t. } \forall n \in \mathbb{Z}_{>0},\ d(\mathcal{O}_C, E^n) \geq (2n)^{-\alpha}\}.$$

Clearly this definition of the set $\mathcal{E}_1(S)$ does not depend on the choice of an invariant distance d. Note that the Lebesgue measure of $\mathrm{Pic}^0(S) \setminus \mathcal{E}_1(S)$ is zero, however $\mathcal{E}_1(S)$ is the union of countably many nowhere dense closed subsets of $\mathrm{Pic}^0(S)$. Ueda showed the following theorems.

THEOREM 1.2.3 ([U, Theorem 2]). *Let V be a neighborhood of S in X and Ψ be a psh function on $V \setminus S$. Assume that X is a surface,* type $(S, X) = n$ *holds for some integer n, and that there exists a positive real number $a < n$ such that $\Psi(p) = o(\mathrm{dist}\,(p, S)^{-a})$ as p approaches S, where* dist (p, S) *is the Euclidean distance between p and S. Then there exists a neighborhood V_0 of S such that Ψ is constant on $V_0 \setminus S$.*

THEOREM 1.2.4 ([U, Theorem 3]). *Let X be a complex manifold and S be a smooth compact Kähler hypersurface of X. Assume that $N_{S/X} \in \mathcal{E}_0(S) \cup \mathcal{E}_1(S)$ and* type $(S, X) = \infty$. *Then there exists a neighborhood V of S in X such that the line bundle $\mathcal{O}_V(S)$ is flat.*

REMARK 1.2.5. In [U], the above Theorem 1.2.4 is stated only for the case where X is a surface. However, the proof of [U, Theorem 3] does not depends on the dimensions of S and X, thus we obtain the above Theorem 1.2.4.

2

Minimal singular metrics of a line bundle admitting no Zariski decomposition

2.1 Introduction

In this chapter, we consider the positivity of a big holomorphic line bundle over a compact Kähler complex manifold. Especially, we are interested in the information related to the obstruction to the nef-ness of the line bundle. Our main result is the explicit construction of a minimal singular metric, or a singular hermitian metric on L with minimal singularities, of a big line bundle L when the manifold X is the total space of a smooth projective toric bundle over a complex torus (Theorem 2.4.7).

In order to state our main theorem in general form, we have to define some terminology. So in this section, we introduce our result only when (X, L) is a Nakayama example ([Na, IV §2.6]), which is one of the most important examples when we study the obstruction to the nef-ness of the line bundle, since it admits no Zariski decomposition even after modifications. Let E_1 be a sufficiently general smooth elliptic curve such as $\mathbb{C}/(\mathbb{Z}+(\pi+\sqrt{-1})\mathbb{Z})$, E_2 a copy of E_1, and z_j a coordinate of E_j for $j = 1, 2$. Let us fix an integer $a > 1$, points $p_1 \in E_1, p_2 \in E_2$, and define the three line bundles $L_j (j = 0, 1, 2)$ over $V = E_1 \times E_2$ by

$$\begin{aligned} L_0 &= \mathcal{O}_V(2F_1 - 4F_2 + 2\Delta), \\ L_1 &= \mathcal{O}_V((a-1)F_1 + (a-1)F_2 + (a+2)\Delta), \\ L_2 &= \mathcal{O}_V((a+3)F_1 + (a-3)F_2 + a\Delta), \end{aligned}$$

where F_1 stands for the prime divisor $\{p_1\} \times E_2 \subset V$, F_2 stands for the prime divisor $E_1 \times \{p_2\} \subset V$, and Δ stands for the prime divisor $\{(x, y) \in E \times E \mid$

$x = y\}$. Then there exists a hermitian metric h_j over L_j whose curvature tensor $\Theta_{h_j} \in c_1(L_j)$ is a harmonic form and each h_j can be denoted as $h_j(\xi, \eta)_{(z_1,z_2)} = e^{-\varphi_j(z_1,z_2)}\xi\overline{\eta}$, where

$$\begin{aligned}
\varphi_0(z_1, z_2) &= (z_1, z_2)\begin{pmatrix} 4 & -2 \\ -2 & -2 \end{pmatrix}\overline{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}} \\
\varphi_1(z_1, z_2) &= (z_1, z_2)\begin{pmatrix} 2a+1 & -(a+2) \\ -(a+2) & 2a+1 \end{pmatrix}\overline{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}} \\
\varphi_2(z_1, z_2) &= (z_1, z_2)\begin{pmatrix} 2a+3 & -a \\ -a & 2a-3 \end{pmatrix}\overline{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}},
\end{aligned}$$

on each small open subset U of V with appropriate local trivialization s^j of L_j on U. Let us define the variety X as the total space of a $\mathbb{P}^2$-bundle $\pi\colon \mathbb{P}(L_0 \oplus L_1 \oplus L_2) \to V$ over V and $L = \mathcal{O}_{\mathbb{P}(L_0\oplus L_1\oplus L_2)}(1)$. Let U be a sufficiently small open set of V. We use the function

$$\begin{aligned}
([x_0; x_1; x_2], z_1, z_2) &\mapsto [x_0 s_0(z_1, z_2); x_1 s_1(z_1, z_2); x_2 s_2(z_1, z_2)] \\
&\in (\mathbb{C}s^0(z_1, z_2) \oplus \mathbb{C}s^1(z_1, z_2) \oplus \mathbb{C}s^2(z_1, z_2))^*/\mathbb{C}^* \\
&= \pi^{-1}(z_1, z_2)
\end{aligned}$$

as a coordinates system on $\pi^{-1}(U)$, where s_j is a dual section of s^j. Using these coordinates, our main result applied to this example can be stated as follows:

Theorem 2.1.1. *Let (X, L) be the above example, which is introduced by Nakayama [Na] and admits no Zariski decomposition even after modifications. There is a minimal singular metric $h_{\min}$ on L whose local weight function ψ is continuous on $X \setminus \mathbb{P}(L_0)$ and is written as*

$$\psi = \log \max_{(\alpha,\beta)\in H} (|x_1|^{2\alpha} \cdot |x_2|^{2\beta}) + O(1)$$

at each point in $\mathbb{P}(L_0)$ with local coordinates $(x_1, x_2, z_1, z_2) = ([1; x_1; x_2], z_1, z_2)$, where $H = \{(\alpha, \beta) \in \mathbb{R}^2 \mid \alpha, \beta \geq 0,\ a^2(\alpha+\beta)^2 = (1-\alpha)^2 + (1-\beta)^2\}$.

This expression enables us to compute the multiplier ideal sheaf $\mathcal{J}(h^t_{\min})$ for each positive number t, whose stalk at $x_0 \in X$ is defined by

$$\mathcal{J}(h^t_{\min})_{x_0} = \{f \in \mathcal{O}_{X,x} \mid |f|^2 e^{-t\varphi_{\min}} \text{ is integrable around } x_0\},$$

where $\varphi_{\min}$ is the local weight function of $h_{\min}$ around x_0.

Corollary 2.1.2. *$\mathcal{J}(h_{\min}^t)$ is trivial at any point in $X \setminus \mathbb{P}(L_0)$. For a point $x_0 \in \mathbb{P}(L_0)$, the stalk $\mathcal{J}(h_{\min})_{x_0}$ of the multiplier ideal sheaf is the ideal of $\mathcal{O}_{X,x_0}$ which is generated by the polynomials*

$$\{x_1^p x_2^q \mid (p+1, q+1) \in \mathrm{Int}(S_t) \cap \mathbb{Z}^2\},$$

where we denote by S_t the set $\{(t\alpha, t\beta) \in \mathbb{R}^2 \mid \alpha, \beta \geq 0,\ a^2(\alpha+\beta)^2 \geq (1-\alpha)^2 + (1-\beta)^2\}$ (For the shape of S_t in this case, see Figure 2.1).

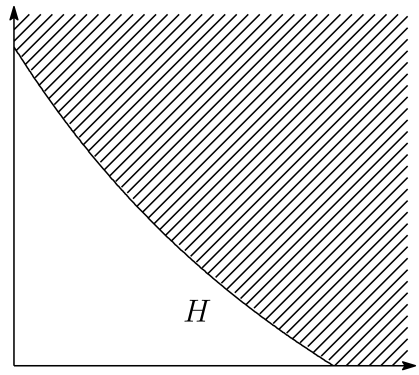

Figure 2.1: The shaded area of this figure represents the set S_1. The set S_t is the set of points $p \in \mathbb{R}^2$ which satisfies $\frac{p}{t} \in S_1$.

According to [Na], this (X, L) is an example which admits no Zariski decomposition even after modifications. So, it can be expected in this case that the behavior of this multiplier ideal sheaf is different from the algebraic cases. Indeed, the set of jumping numbers $\mathrm{Jump}(\psi; x_0)$ for a point x in $\mathbb{P}(L_0)$ (see [ELSV, Section 5] for definition) can be written as follows in this case;

$$\mathrm{Jump}(\psi; x_0) = \left\{ \frac{p + \sqrt{2p^2a^2 - q^2}}{2} \,\middle|\, p, q \in \mathbb{Z},\ 0 \leq q < p,\ p - q \equiv 0 \pmod 2 \right\},$$

which is the set of the largest roots of the quadratic equations $4T^2 - 4pT + (1-2a^2)p^2 + q^2 = 0$ of T, where integers p and q satisfy the above conditions. This set has different properties from algebraic multiplier ideal sheaves. For example, it seems difficult to expect the "periodicity" property, and does not have the "rationality" property in this case (For these property, see [ELSV, 1.12] or Remark 2.6.3 below). Especially, the singularity exponent $c_{x_0}(\psi)$, which is the minimum number in the set of all jumping numbers, satisfies

$$c_{x_0}(\psi) = \sqrt{2}a + 1,$$

and it is clearly irrational.

More generally, we give a concrete expression of a minimal singular metric on a big line bundle L on the total space of such a toric bundle, see Theorem 2.4.7. As an application, we discuss Zariski closedness of the non-nef locus $\mathrm{NNef}(L)$ of L, see Corollary 2.5.5.

The organization of this chapter is as follows. Let X be the total space of a smooth projective toric bundle over a complex torus, and L be a big line bundle over X. In §2.2, we recall some facts and notations related to complex tori and toric bundles. In §2.3, we fix a way to coordinate X, and study how modifications of X or zeros of holomorphic sections of L can be treated by using this coordinates system. In §2.4, we construct a singular hermitian metric $\{e^{-\psi_\sigma}\}$ of L and show it is a minimal singular metric. In §2.5, we study some properties related to the positivity of L, as applications of the result in §2.4. Here we introduce how to calculate the Kiselman numbers and the Lelong numbers of minimal singular metrics, and study the non-nef locus of L and multiplier ideal sheaves associated to minimal singular metrics. In Section §2.6, we introduce three examples for (X, L), all of which is based on the example introduced in [Na], and apply our result to them.

2.2 Preliminaries on complex tori and toric bundles

2.2.1 Complex tori

Here, let us recall some fundamental terminologies related to complex tori. Let $\Lambda \subset \mathbb{C}^d$ be a lattice. We denote $\mathbb{C}^d/\Lambda$ by V and the natural map $\mathbb{C}^d \to V$ by p.

Proposition 2.2.1. *([BL, Chapter 3]) Following four propositions hold for above d, V, and Λ as above. Here, let us denote by $\mathbb{H}_d$ the set of all hermitian matrices of size $d \times d$ with $\mathbb{C}$-coefficients.*
(1) *There exists an injective $\mathbb{R}$-linear map $\mathrm{NS}(V) \otimes \mathbb{R} \to \mathbb{H}_d$.*
(2) *By this linear map, $\mathrm{NS}(V)$ is identified with the space $\{H \in \mathbb{H}_d \mid \forall \lambda, \mu \in \Lambda, \mathrm{Im}\,(\lambda H \bar{\mu}) \in \mathbb{Z}\}$.*
(3) *By this linear map, the nef cone $\mathrm{Nef}(V) \subset \mathrm{NS}(V)$ is identified with*

$$\{H \in \mathbb{H}_d \mid H \geq 0 \text{ and } H \text{ is an element of the image of the set } \mathrm{NS}(V) \otimes \mathbb{R}\}.$$

(4) *Let $c_1(E)$ be identified with $H_E \in \mathbb{H}_d$ by this linear map for a line bundle E on V. Fix a metric h_E of E whose curvature form is a harmonic form*

with respect to the Euclidean metric (such h_E always exists and is unique up to scale). Here we fix a point of V and denote by $z = (z_1, z_2, \dots, z_d)$ the local coordinates of V around the point induced by the map p and the usual coordinates of $\mathbb{C}^d$. Then, there exists a canonically determined local frame e of E on the neighborhood of the point such that, with respect to this local trivialization, the local weight function φ_E of h_E can be written as

$$\varphi_E(z_1, z_2, \dots, z_d) = (z_1, z_2, \dots, z_d) H_E \overline{\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_d \end{pmatrix}}.$$

2.2.2 Toric bundles

Here, we review fundamental terminology related to toric bundles. We follow [Na, IV] basically. Let us denote by V a base complex manifold. For simplicity, we restrict ourselves to the case where V is a complex torus. Let N be a free $\mathbb{Z}$-module of rank n, and M be the dual module $\mathrm{Hom}(N, \mathbb{Z})$. We denote by $e_1, e_2, \dots, e_n$ generators of N,and by $e^1, e^2, \dots, e^n$ the dual generators of M. We write $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ for $N \otimes \mathbb{R}$ and $M \otimes \mathbb{R}$, respectively. We fix a group homomorphism

$$\mathcal{L} \colon M \to \mathrm{Pic}(V)$$

and a fan Σ of N, and construct a toric bundle $\pi \colon \mathbb{T}_N(\Sigma, \mathcal{L}) \to V$. We assume the fan Σ is smooth projective, which means that the fan is defined by a smooth full-dimensional lattice polytope. Under this assumption, the toric variety $\mathbb{T}_N(\Sigma)$ is a smooth projective variety. We denote by $\mathcal{L}^m \in \mathrm{Pic}(V)$ the image of $m \in M$. For simplicity, we also denote by $\mathcal{L}^m$ the image of $m \in M_{\mathbb{R}}$ with respect to the linear map

$$\mathcal{L} \otimes \mathbb{R} \colon M_{\mathbb{R}} \to \mathrm{Pic}(V) \otimes \mathbb{R}.$$

Definition 2.2.2. For $\sigma \in \Sigma$, we define the affine toric bundle $\pi \colon \mathbb{T}_N(\sigma, \mathcal{L}) \to V$ by

$$\mathbb{T}_N(\sigma, \mathcal{L}) = \mathrm{Spec}_V \bigoplus_{m \in \sigma^\vee \cap M} \mathcal{L}^m$$

with the canonical morphism to V, and the toric bundle $\pi \colon \mathbb{T}_N(\Sigma, \mathcal{L}) \to V$ by gluing $\{\mathbb{T}_N(\sigma, \mathcal{L}) \to V\}_{\sigma \in \Sigma}$ in the natural way.

For each cone $\sigma \in \Sigma$, there exists a corresponding $\mathbb{T} := \mathrm{Hom}\,(M, \mathbb{C}^*)$-orbit $\mathbb{O}_\sigma(\mathcal{L})$ as the case of toric varieties. Let us denote by $\mathbb{V}(\sigma, \mathcal{L})$ the closure of

$\mathbb{O}_\sigma(\mathcal{L})$ as the subset of $\mathbb{T}_N(\Sigma, \mathcal{L})$. Just as the case of toric varieties, the codimension of $\mathbb{V}(\sigma, \mathcal{L})$ coincides with the dimension of σ. In particular, for each 1-dimensional $\sigma \in \Sigma$, $\mathbb{V}(\sigma, \mathcal{L})$ is a prime divisor of $\mathbb{T}_N(\Sigma, \mathcal{L})$.

DEFINITION 2.2.3. We denote by $\mathrm{Ver}(\Sigma)$ the set of the whole primitive generators $v \in N$ of one-dimensional cones of Σ. For $v \in \mathrm{Ver}(\Sigma)$, we denote by Γ_v the prime divisor $\mathbb{V}(\mathbb{R}_{\geq 0} v, \mathcal{L})$. Let us set

$$\mathrm{PL}_N(\Sigma, \mathbb{Z}) = \{h\colon N_\mathbb{R} \to \mathbb{R} \mid \text{for each } \sigma \in \Sigma,\ h|_\sigma \text{ is linear, and } h(N) \subset \mathbb{Z}\}.$$

For $h \in \mathrm{PL}_N(\Sigma, \mathbb{Z})$, we define the divisor D_h by

$$D_h = \sum_{v \in \mathrm{Ver}(\Sigma)} (-h(v))\Gamma_v.$$

It is known that any line bundle over $\mathbb{T}_N(\Sigma, \mathcal{L})$ can be written by adding a divisor of the form D_g to the pull-back of a line bundle over V ([Na, 2.3]).

EXAMPLE 2.2.4. The cone $\{0\}$ is always an element of the fan Σ. Here we consider the affine toric bundle $\mathbb{T}_N(\{0\}, \mathcal{L})$. Fix a metric on $\mathcal{L}^{e^j}$ whose curvature form is a harmonic form with respect to the Euclidean metric for each j. Let U be a sufficiently small open set in V and $z \mapsto s^j(z)$ be such a local trivialization of $\mathcal{L}^{e^j}$ on U as in Proposition 2.2.1, and $z \mapsto s_j(z)$ be the dual frame of the local frame $z \mapsto s^j(z)$ for $j = 1, 2, \ldots, n$. It can be easily checked that the frame $z \mapsto s_j(z)$ is also such a section of $\mathcal{L}^{-e^j} = (\mathcal{L}^{e^j})^{-1}$ as in Proposition 2.2.1. Here,

$$\begin{aligned} & \mathbb{T}_N(\{0\}, \mathcal{L})|_{\{z\}} \\ = \;& \mathrm{Spec}\, \mathbb{C}[s^1(z), s^2(z), \ldots, s^n(z), (s^1)^{-1}(z), (s^2)^{-1}(z), \ldots, (s^n)^{-1}(z)] \\ = \;& \prod_{j=1}^n \mathbb{C}^* \cdot s_j(z) \end{aligned}$$

for $z \in U$. Thus, it follows that the affine toric bundle $\mathbb{T}_N(\{0\}, \mathcal{L})$ can be considered as the $(\mathbb{C}^*)^n$-bundle on V of which the system $\{s_j\}_j$ works as a local trivialization on U.

EXAMPLE 2.2.5. Second example is a case where $n = 2$. Let L_0, L_1, L_2 be line bundles over V. Let $\mathcal{L}$ be a map defined by $e^j \mapsto L_j \otimes L_0^{-1}$ $(j = 1, 2)$ and Σ be the fan generated by the three cones

$$\sigma_1 = \mathrm{Cone}\{e_1, e_2\},\ \sigma_2 = \mathrm{Cone}\{e_2, -(e_1+e_2)\},\ \text{and } \sigma_3 = \mathrm{Cone}\{-(e_1+e_2), e_1\}.$$

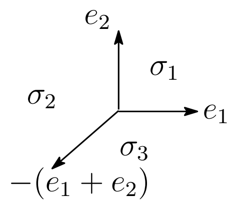

Figure 2.2: Σ.

Fix a metric on $\mathcal{L}^{e^j}$ whose curvature form is a harmonic form with respect to the Euclidean metric for each j. Let U be a sufficiently small open set in V and $z \mapsto s_1(z), z \mapsto s_2(z)$ be such local trivializations of $(L_1 \otimes L_0^{-1})^{-1}, (L_2 \otimes L_0^{-1})^{-1}$ of U as in Proposition 2.2.1, respectively, and s^j be the dual of s_j for $j = 1, 2$. Here,

$$\begin{aligned} \mathbb{T}_N(\sigma_1, \mathcal{L})|_{\{z\}} &= \operatorname{Spec} \mathbb{C}[s^1(z), s^2(z)], \\ \mathbb{T}_N(\sigma_2, \mathcal{L})|_{\{z\}} &= \operatorname{Spec} \mathbb{C}[(s^1(z))^{-1} s^2(z), (s^1(z))^{-1}], \\ \mathbb{T}_N(\sigma_3, \mathcal{L})|_{\{z\}} &= \operatorname{Spec} \mathbb{C}[(s^2(z))^{-1}, s^1(z)(s^2(z))^{-1}], \end{aligned}$$

for $z \in U$. Using this expressions, we can calculate that

$$\mathbb{T}_N(\Sigma, \mathcal{L}) = \mathbb{P}(\mathcal{O}_V \oplus (L_1 \otimes L_0^{-1}) \oplus (L_2 \otimes L_0^{-1})) \cong \mathbb{P}(L_0 \oplus L_1 \oplus L_2).$$

In this case, $\mathrm{Ver}(\Sigma)$ is the set consisting of the following three elements; $v_0 = -(e_1 + e_2)$, $v_1 = e_1$, and $v_2 = e_2$. Let us define $h \in \mathrm{PL}_N(\Sigma, \mathbb{Z})$ by $v_0 \mapsto -1, v_1 \mapsto 0$, and $v_2 \mapsto 0$. Then the line bundle $L = \mathcal{O}_{\mathbb{P}(L_0 \oplus L_1 \oplus L_2)}(1)$ can be written as

$$L \cong \pi^* L_0 \otimes \mathcal{O}_X(D_h).$$

2.3 Toric bundles over complex tori

2.3.1 Holomorphic sections and local coordinates

Let V be a smooth projective variety and Σ be the fan defined by a smooth full-dimensional lattice polytope of M just as in the previous section. We denote by X the total space of the toric bundle $\pi\colon \mathbb{T}_N(\Sigma, \mathcal{L}) \to V$. Here we consider holomorphic sections of a line bundle L over X. According to ([Na, 2.3]), without loss of generality, we may assume $L = \pi^* L_0 \otimes \mathcal{O}_X(D_h)$, where L_0 is a holomorphic line bundle over V, and h is an element of $\mathrm{PL}_N(\Sigma, \mathbb{Z})$.

DEFINITION 2.3.1. We denote by $\Box_h$ the set $\{m \in M_{\mathbb{R}} \mid \forall x \in N_{\mathbb{R}}, \langle m, x\rangle \geq h(x)\}$, and by $\Box_{\mathrm{Nef}}(L_0, h)$ the set $\{m \in \Box_h \mid L_0 \otimes \mathcal{L}^m \text{ is nef}\}$ for a line bundle L_0 over V and an element $h \in \mathrm{PL}_N(\Sigma, \mathbb{Z})$.

Since $\Box_h$ is a bounded closed convex set, we clearly obtain the following lemma.

LEMMA 2.3.2. *$\Box_{\mathrm{Nef}}(L_0, h)$ is a bounded closed convex subset of $M_{\mathbb{R}}$.*

DEFINITION 2.3.3. Here we use notations in Example 2.2.4. For $m \in M$, we define the meromorphic section χ^m of $\pi^*\mathcal{L}^{-m}$ on $\mathbb{T}_N(\Sigma, \mathcal{L})$ by

$$(x_j \cdot s_j(z))_j \longmapsto \prod_{j=1}^{n}(x_j \cdot s_j(z))^{m_j} = (x_1)^{m_1} \cdot (x_2)^{m_2} \cdots (x_n)^{m_n} \cdot \left(\prod_{j=1}^{n}(s^j)^{-m_j}\right)(z)$$

on $\mathbb{T}_N(\{0\}, \mathcal{L})|_U$, where $m_j = \langle m, e_j\rangle$.

$\mathbb{T}_N(\{0\}, \mathcal{L})$, which we considered in Example 2.2.4, is always a dense subset of $\mathbb{T}_N(\Sigma, \mathcal{L})$. In the case of toric varieties, or the case that V is the "0-dimensional complex torus", regular functions on $\mathbb{T}_N(\Sigma, \mathcal{L})$ can be regarded as meromorphic functions on $\mathbb{T}_N(\{0\}, \mathcal{L})$. There is an analogue of this fact in the general setting.

PROPOSITION 2.3.4. *([Na, 2.3, 2.4]) The line bundle $L = \pi^*L_0 \otimes \mathcal{O}_X(D_h)$ is pseudo-effective if and only if the set $\Box_{\mathrm{Nef}}(L_0, h)$ is non-empty. In this case, we obtain the equation*

$$H^0(X, L) = \bigoplus_{m \in \Box_{\mathrm{Nef}}(L_0, h) \cap M} \chi^m \cdot \pi^* H^0(V, L_0 \otimes \mathcal{L}^m).$$

In the following, we assume that V is a complex torus.

OBSERVATION 2.3.5. Here we rewrite the meromorphic function $\chi^m \cdot \pi^* f$ in Proposition 2.3.4 by using notations in Example 2.2.4. Let U be a sufficiently small open set in V and $z \mapsto s^0(z)$ be such a local trivialization of L_0 on U as in Proposition 2.2.1. Under the local trivialization $z \mapsto \left(s^0 \cdot \prod_{j=1}^{n} s^j\right)(z)$ of $L_0 \otimes \mathcal{L}^m$, we may assume f is written as

$$f|_U(z) = \eta(z) \cdot \left(s^0 \cdot \prod_{j=1}^{n}(s^j)^{\langle m, e_j\rangle}\right)(z)$$

on U for some holomorphic function η on U. Since

$$\chi^m \cdot \pi^* f((x_j \cdot s_j(z))_j) = \chi^m((x_j \cdot s_j(z))_j) \cdot f(z) = \left(\prod_{j=1}^{n} (x_j)^{\langle m, e_j \rangle}\right) \eta(z) \cdot s^0(z)$$

holds, it can be checked that $\chi^m \cdot \pi^* f$ is a meromorphic section of $\pi^* L_0$, indeed. Moreover we can check that it is an element of $H^0(X, L) = H^0(X, \pi^* L_0 \otimes \mathcal{O}_X(D_h))$, since m is an element of $\square_h$.

In Observation 2.3.5, we calculated $\chi^m \cdot \pi^* f$ as a meromorphic section of $\pi^* L_0$. We can rewrite it as a holomorphic section of $\pi^* L_0 \otimes \mathcal{O}_X(D_h)$ by using following *canonical local coordinates.*

Definition 2.3.6. Let σ be an element of $\Sigma_{\max} := \{\sigma \in \Sigma \mid \dim \sigma = n\}$. Since the fan Σ is smooth, there exists $v_1, v_2, \ldots, v_n \in \mathrm{Ver}(\Sigma)$ such that $\sigma = \mathrm{Cone}\{v_1, v_2, \ldots, v_n\}$ and $v_1, v_2, \ldots, v_n$ generates N. We call such $v_1, v_2, \ldots, v_n$ N-minimal generators of σ.

Let $v^1, v^2, \ldots, v^n$ be the dual generators of $v_1, v_2, \ldots, v_n$. Then the dual cone of σ can be written as $\sigma^\vee = \mathrm{Cone}\{v^1, v^2, \ldots, v^n\}$. Fix a metric h_{v^j} of $\mathcal{L}^{v^j}$ whose curvature form is a harmonic form with respect to the Euclidean metric for each j. Let U be a sufficiently small open set in V. Let us fix such a local trivializations $z \mapsto t^j(z)$ of $\mathcal{L}^{v^j}$ on U as in Proposition 2.2.1, and the dual section t_j of t^j for $j = 1, 2, \ldots, n$. Using these notations, we can calculate

$$\mathbb{T}_N(\sigma, \mathcal{L})|_{\{z\}} = \mathrm{Spec} \left. \bigoplus_{a_1, a_2, \ldots, a_n \geq 0} \mathcal{L}^{\sum_j a_j v^j} \right|_{\{z\}} = \mathrm{Spec}\, \mathbb{C}[t^1(z), t^2(z), \ldots, t^n(z)]$$

for $z \in U$. So, it turns out that $\mathbb{T}_N(\sigma, \mathcal{L})$ is a $\mathbb{C}^n$-bundle which $t_1, t_2, \ldots, t_n$ gives a local trivialization on U. So, we can regard the map

$$(x_1, x_2, \ldots, x_n, z) \longmapsto (x_j \cdot t_j(z))_j \in \mathbb{T}_N(\sigma, \mathcal{L})|_{\{z\}}$$

as a local coordinates system on $\mathbb{T}_N(\sigma, \mathcal{L})|_U$. We call this local coordinate system the canonical one of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to the N-minimal generator $v_1, v_2, \ldots, v_n$ of σ.

As it is clear from the definition, the canonical coordinates system of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to the N-minimal generator $v_1, v_2, \ldots, v_n$ of σ depends on the choice of the metrics $\{h_{v^j}\}_j$. In the following, we fix basis $e^1, e^2, \ldots, e^n$

of M and a metric h_{e^j} of $\mathcal{L}^{e^j}$ whose curvature form is a harmonic form with respect to the Euclidean metric for each j, and we always choose the metric $h_{e^1}^{\otimes a_1^j} \otimes h_{e^2}^{\otimes a_2^j} \otimes \cdots \otimes h_{e^n}^{\otimes a_n^j}$ for h_{v^j}, where $v^j = \sum_k a_k^j e^k$. By using this metric, we can say that the canonical coordinates system of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to the N-minimal generator $v_1, v_2, \ldots, v_n$ is uniquely determined.

Remark 2.3.7. Let $v_1, v_2, \ldots, v_n$ be N-minimal generators of σ, and $(x_1, x_2, \ldots, x_n, z)$ be the canonical coordinates system of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to $v_1, v_2, \ldots, v_n$. Then, $\{x_j = 0\} = \Gamma_{v_j}$ holds for $j = 1, 2, \ldots, n$ on $\mathbb{T}_N(\sigma, \mathcal{L})|_U$.

Definition 2.3.8. For $\sigma \in \Sigma_{\max}$, we denote by $m_\sigma \in M$ the point which satisfies $h(w) = \langle m_\sigma, w\rangle$ for all $w \in \sigma$. We call $\{m_\sigma\}_\sigma$ the Cartier data of D_h.

Observation 2.3.9. Let σ be an element of $\Sigma_{\max}$, $v_1, v_2, \ldots, v_n$ be N-minimal generators of σ, and $(x_1, x_2, \ldots, x_n, z)$ be the canonical coordinates system of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to $v_1, v_2, \ldots, v_n$. In $\mathbb{T}_N(\sigma, \mathcal{L})|_U$, the map

$$(x_1, x_2, \ldots, x_n, z) \mapsto \prod_{j=1}^{n} (x_j)^{\langle m_\sigma, v_j\rangle}$$

gives a local trivialization of $\mathcal{O}_X(D_h)$, where $\{m_\sigma\}_\sigma$ is the Cartier data of D_h. So, by using notations in Observation 2.3.5,

$$(x_1, x_2, \ldots, x_n, z) \longmapsto \left(\prod_{j=1}^{n} (x_j)^{\langle m_\sigma, v_j\rangle}\right) \cdot s^0(z)$$

gives a local trivialization of L. Under this trivialization, $\chi^m \cdot \pi^* f \in H^0(X, L)$ can be regarded as the holomorphic function

$$(x_1, x_2, \ldots, x_n, z) \longmapsto \left(\prod_{j=1}^{n} (x_j)^{\langle m - m_\sigma, v_j\rangle}\right) \cdot \eta(z)$$

on $\mathbb{T}_N(\sigma, \mathcal{L})|_U$.

The projective line $\mathbb{P}^1 = \{[z; w]\}$ can be regarded as the union of two disks $\{[z; 1] \mid |z| \leq 1\}$ and $\{[1; w] \mid |w| \leq 1\}$ with radius 1. The following proposition is an analogy of this fact.

PROPOSITION 2.3.10. *Let U be a sufficiently small open set in V, z_0 be a point in U, σ be an element of $\Sigma_{\max}$, $v_1, v_2, \ldots, v_n$ be N-minimal generators of σ, and $(x_1, x_2, \ldots, x_n, z)$ be the canonical coordinates system of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to $v_1, v_2, \ldots, v_n$. We set*

$$K_{\sigma,z_0} = \{(x_1, x_2, \ldots, x_n, z_0) \in \mathbb{T}_N(\sigma, \mathcal{L}) \mid \forall j \in \{1, 2, \ldots, n\}, |x_j| \leq 1\}.$$

Then,

$$\bigcup_{\sigma \in \Sigma_{\max}} K_{\sigma,z_0} = \pi^{-1}(z_0)$$

holds.

PROOF. Since $\overline{\mathbb{T}_N(\{0\}, \mathcal{L})|_{\{z_0\}}} = \pi^{-1}(z_0)$, it is sufficient to show that

$$\bigcup_{\sigma \in \Sigma_{\max}} K_{\sigma,z_0} \supset \mathbb{T}_N(\{0\}, \mathcal{L})|_{\{z_0\}}.$$

Let us fix a point $y_0 \in \mathbb{T}_N(\{0\}, \mathcal{L})|_{\{z_0\}}$ and an element $\tau \in \Sigma_{\max}$. Let $u_1, u_2, \ldots, u_n$ be N-minimal generators of τ, and $(y_1, y_2, \ldots, y_n, z)$ be the canonical coordinates system of $\mathbb{T}_N(\tau, \mathcal{L})|_U$ associated to $u_1, u_2, \ldots, u_n$. In this coordinates system, assume y_0 is written as $((y_0)_1, (y_0)_2, \ldots, (y_0)_n, z_0)$. Since $y_0 \in \mathbb{T}_N(\{0\}, \mathcal{L})$, it turns out that $(y_0)_j \neq 0$ for all j. Thus, $w_0 = -\sum_{j=1}^{n} \log|(y_0)_j| \cdot u_j$ defines a point of $N_{\mathbb{R}}$. Since Σ is complete, there exists an element $\sigma \in \Sigma_{\max}$ such that $n_0 \in \sigma$. Let $v_1, v_2, \ldots, v_n$ be N-minimal generators of σ, and $(x_1, x_2, \ldots, x_n, z)$ be the canonical coordinates system of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to $v_1, v_2, \ldots, v_n$. In this coordinates system, y_0 can be written as

$$y_0 = \left(\left(\prod_{k=1}^{n} ((y_0)_k)^{\langle v^j, u_k \rangle} \right)_j, z_0 \right),$$

where $v^1, v^2, \ldots, v^n$ is the dual basis of $v_1, v_2, \ldots, v_n$. On the other hands, w_0 can be rewritten as

$$\begin{aligned} w_0 &= -\sum_{k=1}^{n} \log|(y_0)_k| \cdot u_k \\ &= -\sum_{k=1}^{n} \sum_{j=1}^{n} \log|(y_0)_k| \langle v^j, u_k \rangle \cdot v_j \\ &= -\sum_{j=1}^{n} \log\left| \prod_{k=1}^{n} ((y_0)_k)^{\langle v^j, u_k \rangle} \right| \cdot v_j. \end{aligned}$$

Since we have chosen σ as the condition $n_0 \in \sigma$ holds, $-\log|\prod_{k=1}^{n}((y_0)_k)^{\langle v^j, u_k \rangle}| \geq 0$, or $|\prod_{k=1}^{n}((y_0)_k)^{\langle v^j, u_k \rangle}| \leq 1$ holds for all $j \in \{1, 2, \ldots, n\}$. We thus obtain $y_0 \in K_{\sigma, z_0}$, which proves the proposition. □

2.3.2 Modifications

Let Σ be a smooth projective fan of the n-dimensional lattice N. Here we fix a smooth subdivision fan $\tilde{\Sigma}$ of Σ, and consider a toric bundle $\tilde{X} = \mathbb{T}_N(\tilde{\Sigma}, \mathcal{L})$ and the canonical morphism $\mu\colon \tilde{X} \to X$. As in the case of toric varieties, $\mu\colon \tilde{X} \to X$ is a proper modification of X. From this section, we use letters with subscripts such as $v_1, v_2, \ldots, v_n$ for generators of N, and we denote the dual generators by the same letters with superscripts, such as $v^1, v^2, \ldots, v^n$, throughout this chapter.

First of all, we obtain the following result by simple computations.

Lemma 2.3.11. *Let $\sigma \in \Sigma_{\max}, \tilde{\sigma} \in \tilde{\Sigma}_{\max}$ be cones such that $\tilde{\sigma} \subset \sigma$, $v_1, v_2, \ldots, v_n$ be N-minimal generators of σ, and $\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n$ be N-minimal generators of $\tilde{\sigma}$. We denote by $(x_1, x_2, \ldots, x_n, z)$ and $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n, z)$ the canonical coordinates systems of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ and $\mathbb{T}_N(\tilde{\sigma}, \mathcal{L})|_U$, respectively. In these coordinates, the morphism $\mu\colon \tilde{X} \to X$ can be written as*

$$\mu(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n, z) = \left(\left(\prod_{k=1}^{n} (\tilde{x}_k)^{\langle v^j, \tilde{v}_k \rangle} \right)_j , \ z \right).$$

Lemma 2.3.11 immediately implies the following corollary.

Corollary 2.3.12. *For $j \in \{1, 2, \ldots, n\}$, there exists a subset $J_{v_j} \subset \{1, 2, \ldots, n\}$ such that $\mu^* \Gamma_{v_j} = \bigcup_{k \in J_{v_j}} \{\tilde{x}_k = 0\}$ in $\mathbb{T}_N(\tilde{\sigma}, \mathcal{L})|_U$.*

Remark 2.3.13. For Corollary 2.3.12, the set J_{v_j} can be written as

$$J_{v_j} = \{k \in \{1, 2, \ldots, n\} \mid \langle v^j, \tilde{v}_k \rangle \neq 0\}.$$

For $\sigma \in \Sigma_{\max}$, we define the set $\tilde{\Sigma}_\sigma$ by $\tilde{\Sigma}_\sigma := \{\tilde{\sigma} \in \tilde{\Sigma} \mid \tilde{\sigma} \subset \sigma\}$, and we denote by $(\tilde{\Sigma}_\sigma)_{\max}$ the set $\{\tilde{\sigma} \in \tilde{\Sigma}_\sigma \mid \dim \tilde{\sigma} = n\}$. By using the expression of μ in Lemma 2.3.11, we can get the following lemma.

Lemma 2.3.14. *Fix a point $z_0 \in U$, a set $I \subset \{1, 2, \ldots, n\}$, and a cone $\sigma \in \Sigma_{\max}$. Denote by W_{I, σ, z_0} the set*

$$\{(x_1, x_2, \ldots, x_n, z_0) \in \mathbb{T}_N(\sigma, \mathcal{L}) \mid \forall j \in I, |x_j| \leq 1, \ \forall j \in \{1, 2, \ldots, n\}, x_j \neq 0\},$$

and by $W_{I,\tilde{\sigma},z_0}$ *the subset*

$$\{(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n, z_0) \mid \forall k \in \cup_{j \in I} J_{v_j}, |\tilde{x}_k| \leq 1,\ \forall j \in \{1, 2, \ldots, n\}, \tilde{x}_j \neq 0\}$$

of $\mathbb{T}_N(\tilde{\sigma}, \mathcal{L})$ *for each* $\tilde{\sigma} \in (\tilde{\Sigma}_\sigma)_{\max}$*. Then,*

$$\mu \left(\bigcup_{\tilde{\sigma} \in (\tilde{\Sigma}_\sigma)_{\max}} W_{I,\tilde{\sigma},z_0} \right) = W_{I,\sigma,z_0}$$

holds.

This lemma can be proved in the almost same way as those used in Lemma 2.3.10. Applying this lemma with $I = \{1, 2, \ldots, n\}$, we obtain the next corollary.

Corollary 2.3.15. *Here we use notations in Lemma 2.3.14. Denote by* K_σ *the set*

$$\{(x_1, x_2, \ldots, x_n, z) \in \mathbb{T}_N(\sigma, \mathcal{L})|_{\overline{U}} \mid \forall j \in \{1, 2, \ldots, n\}, |x_j| \leq 1\}$$

and by $K_{\tilde{\sigma}}$ *the set*

$$\{(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n, z) \in \mathbb{T}_N(\tilde{\sigma}, \mathcal{L})|_{\overline{U}} \mid \forall j \in \{1, 2, \ldots, n\}, |\tilde{x}_j| \leq 1\}$$

for each n*-dimensional cone* $\tilde{\sigma} \in \tilde{\Sigma}_\sigma$*. Then,*

$$\mu \left(\bigcup_{\tilde{\sigma} \in (\tilde{\Sigma}_\sigma)_{\max}} K_{\tilde{\sigma}} \right) = K_\sigma$$

holds.

2.3.3 Convex subsets of M

Let Σ be a smooth projective fan of the n-dimensional lattice N, $\sigma \in \Sigma$ be a n-dimensional cone, $v_1, v_2, \ldots, v_n$ be N-minimal generators of σ, and $(x_1, x_2, \ldots, x_n, z)$ be the canonical coordinates system of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to $v_1, v_2, \ldots, v_n$, where U is a sufficiently small open set in V.

Definition 2.3.16. For $A \subset \sigma^\vee$, we denote by $\overline{\overline{A}}$ the set

$$\{m \in \sigma^\vee \mid \forall w \in \sigma, \min_{m' \in A} \langle m', w \rangle \leq \langle m, w \rangle\}.$$

When $A = \emptyset$, we formally regards $\overline{\overline{\emptyset}}$ as $\sigma^\vee$.

Definition 2.3.17. Let m_σ be an element of the Cartier data D_h which is associated to σ. We denote by $S(L_0,h)_\sigma$ the subset $\overline{\overline{\{m-m_\sigma \mid m\in\Box_{\mathrm{Nef}}(L_0,h)\}}}$ $\subset \sigma^\vee$.

Remark 2.3.18. In $\prod_{j\in I}\{|x_j|<1\}\times\prod_{j\notin I}\{x_j\in\mathbb{C}\}\times U$,

$$\max_{m\in S(L_0,h)_\sigma}\prod_{j\in I}|x_j|^{2\langle m,v_j\rangle}=\max_{m\in\Box_{\mathrm{Nef}}(L_0,h)}\prod_{j\in I}|x_j|^{2\langle m-m_\sigma,v_j\rangle}$$

for any $I\subset\{1,2,\ldots,n\}$, where m_σ is an element of the Cartier data D_h which is associated to σ. $\Box$

Definition 2.3.19. For a point $((x_0)_1,(x_0)_2,\ldots,(x_0)_n,z_0)\in\mathbb{T}_N(\sigma,\mathcal{L})|_U$, let us denote by I the set $\{j\in\{1,2,\ldots,n\}\mid x_0^j=0\}$. We define the set $P(f_1,f_2,\ldots,f_l)_{((x_0)_1,(x_0)_2,\ldots,(x_0)_n,z_0)}$ for $f_1,f_2,\ldots,f_l\in\mathcal{O}_{((x_0)_1,(x_0)_2,\ldots,(x_0)_n,z_0)}$ as follows. Let

$$f_\nu(x_1,x_2,\ldots,x_n)=\sum_{\alpha\geq 0}(x_I)^\alpha A_{\nu,\alpha}(x_{I^c},z),$$

be the Taylor expansion of each f_ν ($\nu=1,2,\ldots,l$) around the point $((x_0)_1,(x_0)_2,\ldots,(x_0)_n,z_0)$ for variables $\{x_j\}_{j\in I}$, where $\alpha=(a_j)_{j\in I}$ is a multi-index, the signature "$(x_I)^\alpha$" stands for $\prod_{j\in I}(x_j)^{a_j}$, and $A_{\nu,\alpha}$ is the germ of a holomorphic function with $(n-\#I+d)$-variables $(x_{I^c},z)=((x_j)_{j\notin I},z)$. We define $P(f_1,f_2,\ldots,f_l)_{((x_0)_1,(x_0)_2,\ldots,(x_0)_n,z_0)}$ by

$$P(f_1,f_2,\ldots,f_l)_{((x_0)_1,(x_0)_2,\ldots,(x_0)_n,z_0)}=\overline{\overline{\bigcup_{\nu=1}^{l}\left\{\sum_{j\in I}a_j\cdot v^j\,\middle|\,A_{\nu,(a_j)_j}\not\equiv 0\right\}}}\subset\sigma^\vee.$$

Remark 2.3.20. Here, we use notations in Definition 2.3.19. Set

$$P_\sigma=P(f_1,f_2,\ldots,f_l)_{(0,0,\ldots,0,z_0)}$$

for $(0,0,\ldots,0,z_0)\in\mathbb{T}_N(\sigma,\mathcal{L})|_U$. Let $\tilde{\Sigma}$ be a smooth complete fan which is a subdivision of Σ, $\tilde{\sigma}\in\tilde{\Sigma}_{\max}$ be a cone such that $\tilde{\sigma}\subset\sigma$, $\tilde{v}_1,\tilde{v}_2,\ldots,\tilde{v}_n$ be N-minimal generators of $\tilde{\sigma}$, and $(\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_n,z)$ be the canonical coordinates system of $\mathbb{T}_N(\tilde{\sigma},\mathcal{L})|_U$ associated to $\tilde{v}_1,\tilde{v}_2,\ldots,\tilde{v}_n$. For the point $(0,0,\ldots,0,z_0)$, let us set

$$P_{\tilde{\sigma}}=P(\mu^*f_1,\mu^*f_2,\ldots,\mu^*f_l)_{(0,0,\ldots,0,z_0)},$$

and assume that f_ν is expanded as

$$f_\nu(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n, z) = \sum_{(a_j)_j \geq 0} \prod_{j=1}^{n} (x_j)^{a_j} A_{\nu,(a_j)_j}(z)$$

around $(0, 0, \ldots, 0, z_0)$. Then, by Lemma 2.3.11, $\mu^* f_\nu$ can be written as

$$\mu^* f_\nu(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n, z) = \sum_{(a_j)_j \geq 0} \prod_{k=1}^{n} (\tilde{x}_k)^{\sum_{j=1}^{n} a_j \langle v^j, \tilde{v}_k \rangle} A_{\nu,(a_j)_j}(z)$$

around $(0, 0, \ldots, 0, z_0)$. Thus, it follows that the following two sets are same;

$$\bigcup_{\nu=1}^{l} \left\{ \sum_{j=1}^{n} a_j \cdot v^j \,\middle|\, A_{\nu,(a_j)_j} \not\equiv 0 \right\} = \bigcup_{\nu=1}^{l} \left\{ \sum_{j,k=1}^{n} a_j \langle v^j, \tilde{v}_k \rangle \cdot \tilde{v}^k \,\middle|\, A_{\nu,(a_j)_j} \not\equiv 0 \right\}.$$

However, since the two signature $\overline{\overline{\cdot}}$ appeared in the definition of P_σ and $P_{\tilde{\sigma}}$ are different from each other, we can not say nothing more than $P_\sigma \subset P_{\tilde{\sigma}}$ in general.

Remark 2.3.21. Here, we use notations in Definition 2.3.19. We remark that
$P(f_1, f_2, \ldots, f_l)_{((x_0)_1,(x_0)_2,\ldots,(x_0)_n,z_0)}$ is finitely generated in the following sense; There exists a finite subset

$$\{m_1, m_2, \ldots, m_l\} \subset P(f_1, f_2, \ldots, f_l)_{((x_0)_1,(x_0)_2,\ldots,(x_0)_n,z_0)} \cap \bigoplus_{j=1}^{n} \mathbb{Z}_{\geq 0} v^j$$

of the lattice such that the equation

$$P(f_1, f_2, \ldots, f_l)_{((x_0)_1,(x_0)_2,\ldots,(x_0)_n,z_0)} = \overline{\overline{\{m_1, m_2, \ldots, m_l\}}}$$

holds. More generally, for any subset $A \subset \bigoplus_{j=1}^{n} \mathbb{Z}_{\geq 0} v^j$, there exists a finite subset

$$\{m_1, m_2, \ldots, m_l\} \subset \overline{\overline{A}} \cap \bigoplus_{j=1}^{n} \mathbb{Z}_{\geq 0} v^j$$

of lattice points such that the equation $\overline{\overline{A}} = \overline{\overline{\{m_1, m_2, \ldots, m_l\}}}$ holds.

Lemma 2.3.22. *For each finite set $A \subset \bigoplus_{j=1}^{n} \mathbb{Q}_{\geq 0} v^j$ of rational points, there exists a smooth complete cone $\tilde{\Sigma}$ which satisfies the following two conditions (i) and (ii). (i) $\tilde{\Sigma}$ is a subdivision of Σ. (ii) For all n-dimensional cone $\tilde{\sigma} \in \tilde{\Sigma}$ satisfying $\tilde{\sigma} \subset \sigma$, there exists an element $m_0 \in A$ such that $\min_{m \in \overline{\overline{A}}} \langle m, w \rangle = \langle m_0, w \rangle$ holds for all $w \in \tilde{\sigma}$, where $\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n$ is N-minimal generators of $\tilde{\sigma}$.*

Proof. Let $\tilde{\Sigma}$ be a fan which is made by cutting all cones of Σ by the all hyperplanes

$$\{w \in N_{\mathbb{R}} \mid \langle m_j, w \rangle = \langle m_k, w \rangle\} \ (m_j, m_k \in A)$$

of $N_{\mathbb{R}}$. Since $A \subset \bigoplus_{j=1}^{n} \mathbb{Q}_{\geq 0} v^j$, each cone of $\tilde{\Sigma}$ is rational. Moreover, for all n-dimensional cone of $\tilde{\Sigma}$ satisfying $\tilde{\sigma} \subset \sigma$, there exists an element $m_{\tilde{\sigma}} \in A$ such that $\min_{m \in \overline{A}} \langle m, w \rangle = \langle m_{\tilde{\sigma}}, w \rangle$ holds for all $w \in \tilde{\sigma}$. Let $\tilde{\Sigma}'$ be a smooth fan which is a subdivision of $\tilde{\Sigma}$. This fan $\tilde{\Sigma}'$ is what we desired. □

2.4 Construction of minimal singular metrics

Here, we use notations in the previous section. In this section, we construct a minimal singular metric on the big line bundle $L = \pi^* L_0 \otimes \mathcal{O}_X(D_h)$ over the total space of a toric bundle $X = \mathbb{T}_N(\Sigma, \mathcal{L})$ over a complex torus V, where Σ is a smooth projective fan in a n-dimensional fan N. According to Proposition 2.3.4, it is clear that the set $\square_{\mathrm{Nef}}(L_0, h) = \square_{\mathrm{Nef}}(L_0, h)$ is not empty in this setting.

First of all, we define the singular hermitian metric $e^{-\psi_{\sigma,m}}$ for each $m \in \square_{\mathrm{Nef}}(L_0, h)$.

Definition 2.4.1. Let m be an element of $\square_{\mathrm{Nef}}(L_0, h)$, σ be an element of $\Sigma_{\max}$, $v_1, v_2, \ldots, v_n$ be N-minimal generators of σ, and $\{m_\sigma\}_\sigma$ be the Cartier data of D_h. Here, we define the plurisubharmonic function $\psi_{\sigma,m}$ on $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ by

$$\psi_{\sigma,m}(x_1, x_2, \ldots, x_n, z) = \log \left(\prod_{j=1}^{n} |x_j|^{2\langle m - m_\sigma, v_j \rangle} \right) + \varphi_{L_0 \otimes \mathcal{L}^m}(z),$$

where U is a sufficiently small open set in V and $\varphi_{L_0 \otimes \mathcal{L}^m} = \varphi_{L_0} + \sum_{j=1}^{n} \langle m, v_j \rangle \varphi_{\mathcal{L}^{v_j}}$. For the definition of φ_{L_0} and $\varphi_{\mathcal{L}^{v_j}}$, see Proposition 2.2.1. And here, we formally regard 0^0 as 1.

Remark 2.4.2. In Definition 2.4.1, the first term of the defining equation of $\psi_{\sigma,m}$ is clearly plurisubharmonic. According to Proposition 2.2.1, the second term is also turned out to be plurisubharmonic. Thus $\psi_{\sigma,m}$ is also a plurisubharmonic function, indeed.

REMARK 2.4.3. The functions $\{e^{-\psi_{\sigma,m}}\}_{\sigma\in\Sigma_{\max}}$ glue together to give a singular hermitian metric on L. Here, we explain this fact when m is a rational point of $M_{\mathbb{R}}$ for simplicity.

Let ν be a natural number such that $\nu m \in M$. By Observation 2.3.5, $\nu\psi_{\sigma,m}$ can be rewritten as

$$\nu\psi_{\sigma,m} = \log|\chi^{\nu m}|^2 + \nu\varphi_{L_0\otimes\mathcal{L}^m}.$$

Since $\chi^{\nu m}$ can be regarded as a meromorphic section of the line bundle $\mathcal{O}_X(D_{\nu h})\otimes\pi^*\mathcal{L}^{-\nu m}$, the first term of the right hand side of the above equation is turned out to be a local weight of a singular hermitian metric which is defined globally on $\mathcal{O}_X(D_{\nu h})\otimes\pi^*\mathcal{L}^{-\nu m}$. Since the second term is also a local weight of the hermitian metric globally defined on $\pi^*(L_0^\nu\otimes\mathcal{L}^{\nu m})$, the sum $\nu\psi_{\sigma,m}$ is a local weight of a singular hermitian metric globally defined on $\nu L = \pi^*L_0^\nu\otimes\mathcal{O}_X(D_{\nu h})$.

This explanation also makes sense in the general case, by considering formally with $\mathbb{R}$-line bundles.

DEFINITION 2.4.4. We define the plurisubharmonic function ψ_σ on $\mathbb{T}_N(\sigma,\mathcal{L})|_U$ by

$$\psi_\sigma(x_1,x_2,\ldots,x_n,z) = \max_{m\in\square_{\mathrm{Nef}}(L_0,h)}\psi_{\sigma,m}(x_1,x_2,\ldots,x_n,z)$$

for a sufficiently small open set U of V and $\sigma\in\Sigma_{\max}$.

REMARK 2.4.5. Since each $\psi_{\sigma,m}$ is plurisubharmonic, it is clear that the upper envelope

$$(x_1,x_2,\ldots,x_n,z)\longmapsto \limsup_{(\xi^1,\xi^2,\ldots,\xi^n,\zeta)\to(x_1,x_2,\ldots,x_n,z)}\psi_\sigma(\xi^1,\xi^2,\ldots,\xi^n,\zeta)$$

of ψ_σ is a plurisubharmonic function. Now let us consider the function

$$((x_1,x_2,\ldots,x_n,z),m)\longmapsto e^{\psi_{\sigma,m}(x_1,x_2,\ldots,x_n,z)} = \left(\prod_{j=1}^n|x_j|^{2\langle m-m_\sigma,v_j\rangle}\right)\cdot e^{\varphi_{L_0\otimes\mathcal{L}^m}(z)}.$$

This function is a continuous function defined on $\mathbb{T}_N(\sigma,\mathcal{L})|_U\times\square_{\mathrm{Nef}}(L_0,h)$. Since $\square_{\mathrm{Nef}}(L_0,h)$ is compact (Lemma 2.3.2), the function

$$((x_1,x_2,\ldots,x_n,z),m)\longmapsto e^{\psi_\sigma(x_1,x_2,\ldots,x_n,z)} = \max_{m\in\square_{\mathrm{Nef}}(L_0,h)} e^{\psi_{\sigma,m}(x_1,x_2,\ldots,x_n,z)},$$

is also continuous. Therefore, ψ_σ itself is also a plurisubharmonic function.

Remark 2.4.6. Remark 2.4.3 yields that $\{e^{-\psi_\sigma}\}_{\sigma\in\Sigma_{\max}}$ glue together to give a singular hermitian metric on L whose curvature current is semi-positive.

Theorem 2.4.7. *Assume that L is a big line bundle, then the singular hermitian metric $e^{-\psi_\sigma}$ of L is a minimal singular metric.*

From now on, we will prepare for the proof of Theorem 2.4.7. Let $\sigma \in \Sigma$ be a n-dimensional cone, $v_1, v_2, \ldots, v_n$ be N-minimal generators of σ, and $(x_1, x_2, \ldots, x_n, z)$ be the canonical coordinates system of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to $v_1, v_2, \ldots, v_n$, where U is a sufficiently small open set in V. We use these notations throughout this section.

Lemma 2.4.8. *Let us fix a point $((x_0)_1, (x_0)_2, \ldots, (x_0)_n, z_0) \in \mathbb{T}_N(\sigma, \mathcal{L})|_U$, and denote by I the set $\{j \in \{1, 2, \ldots, n\} \mid x_0^j = 0\}$. Then, there exist constants C_1 and C_2 such that*

$$\max_{m\in\square_{\mathrm{Nef}}(L_0,h)} \log \prod_{j\in I} |x_j|^{2\langle m-m_\sigma, v_j\rangle} + C_1 \le \psi_\sigma \le \max_{m\in\square_{\mathrm{Nef}}(L_0,h)} \log \prod_{j\in I} |x_j|^{2\langle m-m_\sigma, v_j\rangle} + C_2$$

holds on $\prod_{j\in I}^n \{|x_j| \le 1\} \times \prod_{j\notin I} \{|x_j - x_0^j| \le \delta_j\} \times \overline{U}$, where $\{\delta_j\}_{j\notin I}$ is a system of sufficiently small positive numbers such that $0 \notin \{|x_j - x_0^j| \le \delta_j\}$ for all $j \notin I$, and m_σ is the element of the Cartier data of D_h which is associated to σ.

Proof. The function

$$(m, (x_j)_{j\notin I}, z) \longmapsto \log \prod_{j\notin I} |x_j|^{2\langle m-m_\sigma, v_j\rangle} + \varphi_{L_0\otimes\mathcal{L}^m}(z)$$

defined on $\square_{\mathrm{Nef}}(L_0, h) \times \prod_{j\notin I} \{|x_j - x_0^j| \le \delta_j\} \times \overline{U}$ is continuous. According to Lemma 2.3.2, $\square_{\mathrm{Nef}}(L_0, h) \times \prod_{j\notin I} \{|x_j - x_0^j| \le \delta_j\} \times \overline{U}$ is compact, which yields that this function has both the maximum value and the minimum value, which we denote by C_1 and C_2 respectively. Therefore, the inequality

$$\log \prod_{j\in I} |x_j|^{2\langle m-m_\sigma, v_j\rangle} + C_1 \le \psi_{\sigma,m} \le \log \prod_{j\in I} |x_j|^{2\langle m-m_\sigma, v_j\rangle} + C_2$$

follows, which proves the lemma. □

As we have assumed that L is big thus in particular pseudo-effective, there must be a minimal singular metric on L. We fix one of these and denote it by $h_{\min}$.

Lemma 2.4.9. *Let σ be an element of $\Sigma_{\max}$, and we denote the weight function of $h_{\min}$ around $\mathbb{T}_N(\sigma, \mathcal{L})|_{\overline{U}}$ with respect to the local trivialization of L as in Observation 2.3.5 by $\varphi_{\min,\sigma}$. Then, there exists a constant C_σ such that*

$$\varphi_{\min,\sigma} \leq \psi_\sigma + C_\sigma$$

holds on the set

$$K_\sigma = \{(x_1, x_2, \ldots, x_n, z) \in \mathbb{T}_N(\sigma, \mathcal{L})|_{\overline{U}} \mid \forall j \in \{1, 2, \ldots, n\}, |x_j| \leq 1\}.$$

Proof. Let us denote by m_σ the element of the Cartier data of D_h associated to σ. Applying Lemma 2.4.8 with $I = \{1, 2, \ldots, n\}$, it follows that there exists a constant C such that

$$\max_{m \in \square_{\mathrm{Nef}}(L_0,h)} \log \prod_{j=1}^{n} |x_j|^{2\langle m - m_\sigma, v_j \rangle} \leq \psi_\sigma + C$$

holds on K_σ.

Thus here, we compare $\varphi_{\min,\sigma}$ with $\max_{m \in \square_{\mathrm{Nef}}(L_0,h)} \log \prod_{j=1}^{n} |x_j|^{2\langle m - m_\sigma, v_j \rangle}$.

We choose an infinite subsequence $\{\nu\} \subset \mathbb{N}$ and a finite subset $\{f_j^{(\nu)}\}_{1 \leq j \leq N_\nu}$ of $H^0(X, \nu L)$ for each ν satisfying the following condition; The function

$$\varphi_\nu = \frac{1}{\nu} \log \sum_{j=1}^{N_\nu} |f_j^{(\nu)}|^2$$

converges pointwise to $\varphi_{\min,\sigma}$ on X except a subset of measure 0 as $\nu \to \infty$, and the maximum value M_{φ_ν} of φ_ν on K_σ also converges to $M_{\varphi_{\min,\sigma}} = \max_{K_\sigma} \varphi_{\min,\sigma}$ as $\nu \to \infty$. The existence of these functions can be immediately shown by applying [D2, Theorem (13.21)] regarding φ in the theorem as $(1 - \frac{1}{k})\varphi_{\min} + \frac{1}{k}\varphi_+$ for each natural number k, where φ_+ is the local weight of a singular hermitian metric h_+ on L which satisfies $\Theta_{h_+} \geq \varepsilon\omega$ for some positive number ε and a Kähler metric ω on X.

Then, according to the next Lemma 2.4.10, an inequality

$$\varphi_\nu \leq \max_{m \in \square_{\mathrm{Nef}}(L_0,h)} \log \prod_{j=1}^{n} |x_j|^{2\langle m - m_\sigma, v_j \rangle} + M_{\varphi_\nu}$$

holds on K_σ. Considering this inequality as $\nu \to \infty$, we obtain

$$\varphi_{\min,\sigma} \leq \max_{m \in \square_{\mathrm{Nef}}(L_0,h)} \log \prod_{j=1}^{n} |x_j|^{2\langle m - m_\sigma, v_j \rangle} + M_{\varphi_{\min,\sigma}}$$

on K_σ except the subset of measure 0. Since the both hand sides are plurisubharmonic, this inequality holds on whole K_σ.

According to the above argument, we obtain the inequality

$$\varphi_{\min,\sigma} \leq \psi_\sigma + C + M_{\varphi_{\min,\sigma}}$$

on K_σ, which proves the lemma. □

LEMMA 2.4.10. *Here we use notations in the proof of Lemma 2.4.9. The inequality*

$$\varphi_\nu \leq \max_{m \in \square_{\mathrm{Nef}}(L_0,h)} \log \prod_{j=1}^{n} |x_j|^{2\langle m - m_\sigma, v_j \rangle} + M_{\varphi_\nu}$$

holds on K_σ.

PROOF. Let $P(\varphi_\nu)_\sigma := \frac{1}{\nu} P(f_1^{(\nu)}, f_2^{(\nu)}, \dots, f_{N_\nu}^{(\nu)})_{(0,0,\dots,0,z_0)}$. According to Proposition 2.3.4 and Observation 2.3.9, $\nu P(\varphi_\nu)_\sigma$ is a subset of $S(L_0^\nu, \nu h)_\sigma$. Since $\square_{\mathrm{Nef}}(L_0^\nu, \nu h) = \nu \square_{\mathrm{Nef}}(L_0, h)$ holds, it turns out that $S(L_0^\nu, \nu h)_\sigma = \nu S(L_0, h)_\sigma$, thus we obtain

$$P(\varphi_\nu)_\sigma \subset S(L_0, h)_\sigma.$$

Therefore, according to Remark 2.3.18, it is sufficient to show the inequality

$$\varphi_\nu \leq \max_{m \in P(\varphi_\nu)_\sigma} \log \prod_{j=1}^{n} |x_j|^{2\langle m, v_j \rangle} + M_{\varphi_\nu}$$

on K_σ.

According to Remark 2.3.21, there exists a finite subset A of $P(\varphi_\nu)$ whose elements are rational and which satisfies $P(\varphi_\nu) = \overline{\overline{A}}$. For this set A, we fix such a subdivision $\tilde{\Sigma}$ of Σ as in Lemma 2.3.22. In the following, we use notations we used in Section 4.2. According to Corollary 2.3.15, it is sufficient to show that

$$\mu^* \varphi_\nu \leq \mu^* \left(\max_{m \in P(\varphi_\nu)_\sigma} \log \prod_{j=1}^{n} |x_j|^{2\langle m, v_j \rangle} \right) + M_{\varphi_\nu}$$

on $K_{\tilde{\sigma}} = \{(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n, z) \in \mathbb{T}_N(\tilde{\sigma}, \mathcal{L})|_{\overline{U}} \mid \forall j \in \{1, 2, \dots, n\}, |\tilde{x}_j| \leq 1\}$ for each $\tilde{\sigma} \in (\tilde{\Sigma}_\sigma)_{\max}$.

Since

$$\log \prod_{j=1}^{n} |\mu^* x_j|^{2\langle m, v_j \rangle} = \log \prod_{j=1}^{n} \prod_{k=1}^{n} |\tilde{x}_k|^{2\langle m, v_j \rangle \langle v^j, \tilde{v}_k \rangle} = \sum_{k=1}^{n} \langle m, \tilde{v}_k \rangle \log |\tilde{x}_k|^2$$

holds, we obtain

$$\mu^* \left(\max_{m \in P(\varphi_\nu)_\sigma} \log \prod_{j=1}^{n} |x_j|^{2\langle m, v_j \rangle} \right) = \max_{m \in P(\varphi_\nu)_\sigma} \sum_{j=1}^{n} \langle m, \tilde{v}_j \rangle \log |\tilde{x}_j|^2.$$

As $\log |\tilde{x}_j|^2 \leq 0$ holds for all j on $K_{\tilde{\sigma}}$, the equation we desire can be rewritten as

$$\mu^* \varphi_\nu \leq \log \prod_{j=1}^{n} |\tilde{x}_j|^{2\langle m_0, \tilde{v}_j \rangle} + M_{\varphi_\nu},$$

where $m_0 \in P(\varphi_\nu)_\sigma$ is such an element as in Lemma 2.3.22.

Let $P(\varphi_\nu)_{\tilde{\sigma}} := \frac{1}{\nu} P(\mu^* f_1^{(\nu)}, \mu^* f_2^{(\nu)}, \ldots, \mu^* f_{N_\nu}^{(\nu)})_{(0,0,\ldots,0,z_0)}$. According to Remark 2.3.20, and since both $P(\varphi_\nu)_{\tilde{\sigma}}$ and $P(\varphi_\nu)_\sigma$ are generated by the same set, it turns out that $\mu^* f_j^{(\nu)}$ can be divided by the function $\prod_{k=1}^{n} (x_k)^{\langle \nu m_0, \tilde{v}_k \rangle}$ for all $j \in \{1, 2, \ldots, N_\nu\}$. Denoting the quotient by $g_j^{(\nu)}$, the function $\mu^* \varphi_\nu - \log \prod_{j \in I} |\tilde{x}_j|^{2\langle m_0, \tilde{v}_j \rangle}$ can be rewritten as

$$\mu^* \varphi_\nu - \log \prod_{j=1}^{n} |\tilde{x}_j|^{2\langle m_0, \tilde{v}_j \rangle} = \frac{1}{\nu} \log \sum_{j=1}^{N_\nu} |g_j^{(\nu)}|^2.$$

Thus, this function is a plurisubharmonic function on $K_{\tilde{\sigma}}$, and it has the maximum value on $K_{\tilde{\sigma}}$, which we denote by $M_{\varphi_\nu, \tilde{\sigma}}$. Then, since

$$\mu^* \varphi_\nu \leq \log \prod_{j=1}^{n} |\tilde{x}_j|^{2\langle m_0, \tilde{v}_j \rangle} + M_{\varphi_\nu, \tilde{\sigma}}$$

holds on $K_{\tilde{\sigma}}$. Therefore, it remains to prove that $M_{\varphi_\nu, \tilde{\sigma}} \leq M_{\varphi_\nu}$.

Assume that the plurisubharmonic function $\mu^* \varphi_\nu - \log \prod_{j \in I} |\tilde{x}_j|^{2\langle m_0, \tilde{v}_j \rangle}$ has the maximum value at the point $((\tilde{x}_0)_1, (\tilde{x}_0)_2, \ldots, (\tilde{x}_0)_n, z_0) \in K_{\tilde{\sigma}}$. We may assume $|(\tilde{x}_0)_j| = 1$ for all j after we change the point $((\tilde{x}_0)_1, (\tilde{x}_0)_2, \ldots, (\tilde{x}_0)_n, z_0)$ $\in K_{\tilde{\sigma}}$ if necessary. It is because, in the case when $|(\tilde{x}_0)_1| < 1$ for example, by considering the plurisubharmonic function

$$\tilde{x}_1 \mapsto \mu^* \varphi_\nu(\tilde{x}_1, (\tilde{x}_0)_2, (\tilde{x}_0)_3, \ldots, (\tilde{x}_0)_n, z_0) - \log \left(|\tilde{x}_1|^{2\langle m_0, \tilde{v}_1 \rangle} \cdot \prod_{j=2}^{n} |(\tilde{x}_0)_j|^{2\langle m_0, \tilde{v}_j \rangle} \right)$$

defined on $\{|\tilde{x}_1| < 1\}$, the value of the function above must constantly be $M_{\varphi_\nu, \tilde{\sigma}}$.

Then, we can calculate that

$$\begin{aligned} M_{\varphi_\nu,\tilde{\sigma}} &= \mu^*\varphi_\nu((\tilde{x}_0)_1,(\tilde{x}_0)_2,\dots,(\tilde{x}_0)_n,z_0) - \log\prod_{j=1}^{n}|(\tilde{x}_0)_j|^{2\langle m_0,\tilde{v}_j\rangle} \\ &= \varphi_\nu(\mu((\tilde{x}_0)_1,(\tilde{x}_0)_2,\dots,(\tilde{x}_0)_n,z_0)). \end{aligned}$$

Since $\mu((\tilde{x}_0)_1,(\tilde{x}_0)_2,\dots,(\tilde{x}_0)_n,z_0)\in K_\sigma$, the value is at most M_{φ_ν}. □

Proof of Proposition 2.4.7. Let us denote by h the singular hermitian metric defined by $\{e^{-\psi_\sigma}\}_\sigma$, and by h_∞ a smooth hermitian metric on L. Then, there exist upper semi-continuous functions $\varphi'_{\min}$ and ψ' on X such that

$$h_{\min} = h_\infty e^{-\varphi'_{\min}},\ \ h = h_\infty e^{-\psi'}$$

hold. Here, it is sufficient to prove that there exists a constant C such that

$$\varphi'_{\min} \le \psi' + C$$

holds on $\pi^{-1}(\overline{U})\subset X$.

According to Lemma 2.4.9, for each $\sigma\in\Sigma_{\max}$, there exists a constant C_σ such that

$$\varphi'_{\min} \le \psi' + C_\sigma$$

holds on the set

$$K_\sigma = \{(x_1,x_2,\dots,x_n,z)\in\mathbb{T}_N(\Sigma,\mathcal{L})|_{\overline{U}} \mid \forall j\in\{1,2,\dots,n\}, |x_j|\le 1\}.$$

Thus, according to Lemma 2.3.10,

$$\varphi'_{\min} \le \psi' + C$$

holds on $\pi^{-1}(\overline{U})\subset X$, where $C=\max_{\sigma\in\Sigma_{\max}} C_\sigma$. □

2.5 Properties related to the singularities of minimal singular metrics

2.5.1 Kiselman numbers and Lelong numbers of minimal singular metrics and Non-nef loci

Let X be a smooth projective variety and L be a holomorphic line bundle over X. According to [Bo, 3.6], the next proposition follows.

Proposition 2.5.1. *If L is big, then the non-nef locus* $\mathrm{NNef}(L)$ *of L can be written as*

$$\mathrm{NNef}(L) = \{x \in X \mid \nu(\varphi_{\min}, x) > 0\},$$

where $e^{-\varphi_{\min}}$ is a minimal singular metric on L.

According to this proposition, we can specify the non-nef locus of a big line bundle by calculating the Lelong number of a minimal singular metric. It can be done, actually, in our setting.

Proposition 2.5.2. *Let X be the total space of a toric bundle $\mathbb{T}_N(\Sigma, \mathcal{L})$ over a complex torus and $L = \pi^* L_0 \otimes \mathcal{O}_X(D_h)$ be a big line bundle over X, where Σ is a smooth projective fan in a n-dimensional lattice N. The Kiselman number*

$$\nu^K_{\zeta,w}(\varphi_{\min}, x_0) = \sup\left\{ t \geq 0 \,\middle|\, \varphi_{\min} \leq t \log \sum_{j=1}^{n+d} |\zeta_j|^{2w_j} + O(1) \text{ around } x_0 \right\}$$

associated to the coordinates system

$$\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{n+d}) = (x_1, x_2, \ldots, x_n, z_1, z_2, \ldots, z_d)$$

and $w = (w_j) \in \bigoplus_{j\in I} \mathbb{R}_{>0}$ of a minimal singular metric $e^{-\varphi_{\min}}$ at a point $x_0 = ((x_0)_1, (x_0)_2, \ldots, (x_0)_n, z_0) \in \mathbb{T}_N(\sigma, \mathcal{L})$ (see [BFJ, Section 5.2] for the definition) can be calculated by using notations in the previous section that

$$\nu^K_{\zeta,w}(\varphi_{\min}, x_0) = \min_{m \in S(L_0,h)_\sigma} \left\langle m, \sum_{j\in I} \frac{v_j}{w_j} \right\rangle,$$

where we denote by I the set $\{j \mid x_0^j = 0\}$ and by $(x_1, x_2, \ldots, x_n, z_1, z_2, \ldots, z_d)$ the canonical coordinates system of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to N-minimal generators $v_1, v_2, \ldots, v_n$ of σ. Especially, the Lelong number at x_0 can be calculated that

$$\nu(\varphi_{\min}, x_0) = \min_{m \in S(L_0,h)_\sigma} \sum_{j\in I} \langle m, v_j \rangle.$$

Corollary 2.5.3. *Let X, L be as that of the previous proposition. The following conditions are equivalent.*

1. $\varphi_{\min}(x_0) (= \psi_\sigma(x_0)) = -\infty$.

2. ψ_σ *is not continuous at* x_0.

3. $\nu(\varphi_{\min}, x_0)(= \nu(\psi_\sigma, x_0)) > 0$.

especially,

$$\varphi_{\min}^{-1}(-\infty) = \mathrm{Pole}(\varphi_{\min})$$

holds, where we denote by $\mathrm{Pole}(\varphi_{\min})$ *the set* $\{x \in X \mid \nu(\varphi_{\min}, x) > 0\}$.

The next proposition is also obtained easily by Theorem 2.4.7.

PROPOSITION 2.5.4. *Let* X, L *be as that of Proposition 2.5.2. Then,* $\mathrm{Pole}(\varphi_{\min})$ *is a Zariski closed set.*

According to these argument, we obtain the following corollary.

COROLLARY 2.5.5. *Let* X *be the total space of a toric bundle* $\mathbb{T}_N(\Sigma, \mathcal{L})$ *over a complex torus and* $L = \pi^* L_0 \otimes \mathcal{O}_X(D_h)$ *be a big line bundle over* X*, where* Σ *is a smooth projective fan. Then, the set* $\mathrm{NNef}(L)$ *is a Zariski closed subset of* X*.*

2.5.2 Multiplier ideal sheaves

Let Σ be a smooth projective fan of a n-dimensional lattice N. Fix N-minimal generators $v_1, v_2, \ldots, v_n$ of $\sigma \in \Sigma_{\max}$. Let $(x_1, x_2, \ldots, x_n, z)$ be the canonical coordinates system of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to $v_1, v_2, \ldots, v_n$, where U is a sufficiently small open set in V. In this section, we consider the condition

$$f \in \mathcal{J}(h_{\min}^t)_{((x_0)_1, (x_0)_2, \ldots, (x_0)_n, z_0)},$$

where $((x_0)_1, (x_0)_2, \ldots, (x_0)_n, z_0)$ is a point of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$, f is an element of $\mathcal{O}_{X,((x_0)_1,(x_0)_2,\ldots,(x_0)_n,z)} \setminus \{0\}$, t is a positive real number, and $h_{\min}$ is a minimal singular metric on L. In the following, we also denote by $\mathcal{J}(t\varphi_{\min})$ the multiplier ideal sheaf $\mathcal{J}(h_{\min}^t)$ by using the local weight function $\varphi_{\min}$ of the singular hermitian metric $h_{\min}$.

Let $I := \{j \in \{1, 2, \ldots, n\} \mid (x_0)_j = 0\}$. For this set I, let us denote the expansion appeared in Definition 2.3.19 by

$$f(x_1, x_2, \ldots, x_n, z) = \sum_{m \in \mathrm{Pr}^I(\sigma^\vee \cap M)} \prod_{j \in I} (x_j)^{\langle m, v_j \rangle} A_m(x_{I^c}, z),$$

where the map Pr^I is the projection from $M_\mathbb{R}$ to $\mathrm{Span}_\mathbb{R}\{v^j\}_{j \in I}$. As the dual version of this map, we denote the projection from $N_\mathbb{R}$ to $\mathrm{Span}_\mathbb{R}\{v_j\}_{j \in I}$ by Pr_I in the following. Fix a set $A \subset P(f)_((x_0)_1, (x_0)_2, \ldots, (x_0)_n, z_0)$ of lattice points such that

$$P(f)_{((x_0)_1,(x_0)_2,\ldots,(x_0)_n,z_0)} = \overline{\overline{A}}$$

holds.

Corollary 2.5.6. *The followings are equivalent.*

(1) $f \in \mathcal{J}(t\varphi_{\min})_{((x_0)_1,(x_0)_2,\dots,(x_0)_n,z_0)}$.

(2) $\min_{m\in tS(L_0,h)_\sigma}\langle m,w\rangle < \langle m_0+\sum_{j\in I} v^j, w\rangle$ *for all* $m_0 \in A$ *and* $w \in \mathrm{Pr}_I(\sigma)\setminus\{0\}$.

Corollary 2.5.6 immediately follows from Theorem 2.4.7 and the result of Guenancia [Gu] referring to the way to compute the multiplier ideal sheaves associated to "toric plurisubharmonic functions", which can be regarded as a generalization of the famous Howald's result ([Ho, Theorem 11]) in algebraic setting.

According to Corollary 2.5.6, [DEL, 1.10, 1.11], and [Laz2, 11.2.12 (ii)], we obtain next corollary.

Corollary 2.5.7. *Let X be the total space of a smooth projective toric bundle over a complex torus, D a big divisor on X, and $e^{-\varphi_{\min}}$ be a minimal singular metric on the line bundle $\mathcal{O}_X(D)$.*
(1) *If $f \in \mathcal{J}(t\varphi_{\min})_{x_0}$ at the point x_0, then $f \in \mathcal{J}((1+\varepsilon)t\varphi_{\min})_{x_0}$ holds for sufficiently small positive number ε and any positive real number t. Especially, since the sheaf $\mathcal{J}(t\varphi_{\min})$ is coherent, it follows that*

$$\mathcal{J}(t\varphi_{\min}) = \mathcal{J}_+(t\varphi_{\min}).$$

(2) *Let P be a nef big divisor on X, then*

$$H^j(X, \mathcal{O}_X(K_X+P+L)\otimes\mathcal{J}(\varphi_{\min})) = 0$$

holds for all $j>0$.

2.6 Some examples

In this section, we will introduce three examples for X and L in the previous sections. We construct them as $\mathbb{P}^2$-bundles over abelian surfaces, by following [Na, CHAPTER IV §2.6] basically. In this section, we use notations in Example 2.2.5.

As a preparation, we first recall a useful lemma to see L is big.

Lemma 2.6.1. *In the setting of Example 2.2.5, L is big if and only if there exists a triple (a,b,c) of nonnegative integers such that $L_0^a\otimes L_1^b\otimes L_2^c$ is ample line bundle over V.*

This lemma can be easily shown by applying the result known by Cutkosky ([Laz1, Lemma 2.3.2]) and the fact that the ample cones of complex tori coincide with these big cones.

Let E be a sufficiently general smooth elliptic curve and o be a point of E. For example, you can choose $\mathbb{C}/(\mathbb{Z}+(\pi+\sqrt{-1})\mathbb{Z})$ for E. Let

$$V = E \times E.$$

It is known that the rank of the Neron-Severi group $\mathrm{NS}(V)$ of V is three and this group is generated by the following three classes ([Laz1, Chapter 1.5.B]).

- $f_1 = c_1(\mathcal{O}_V(F_1))$, where F_1 stands for the prime divisor $\{o\} \times E \subset V$.
- $f_2 = c_1(\mathcal{O}_V(F_2))$, where F_2 stands for the prime divisor $E \times \{o\} \subset V$.
- $\delta = c_1(\mathcal{O}_V(\Delta))$, where Δ stands for the prime divisor $\{(x, y) \in E \times E \mid x = y\}$.

By using these three classes, the nef cone $\mathrm{Nef}(V)$ of V can be written as

$$\mathrm{Nef}(V) = \{af_1 + bf_2 + c\delta \mid a, b, c \in \mathbb{R},\ ab + bc + ca \geq 0,\ a + b + c \geq 0\}.$$

In order to obtain more useful expression of $\mathrm{Nef}(V)$, let us define the other basis of $\mathrm{NS}(V) \otimes \mathbb{R}$ by

$$l_1 = \frac{1}{6}(f_1 + f_2 - 2\delta),\ l_2 = \frac{1}{6}(-\sqrt{3}f_1 + \sqrt{3}f_2),\ \text{and } l_3 = \frac{1}{6}(f_1 + f_2 + \delta).$$

By using these classes, $\mathrm{Nef}(V)$ can be written as

$$\mathrm{Nef}(V) = \{al_1 + bl_2 + cl_3 \mid c^2 \geq a^2 + b^2,\ c \geq 0\}.$$

This expression of $\mathrm{Nef}(V)$ makes it easy to judge the nef-ness of line bundles.

Example 2.6.2. The first example is an example which admits a Zariski decomposition after appropriate modifications. Let us fix two positive integers $u < v$. Let $L_0 := \mathcal{O}_V(-uF_1 - uF_2 - u\Delta), L_1 := \mathcal{O}_V((u+v)F_1 + (u+v)F_2 + (-2u+v)\Delta)$, and $L_2 := \mathcal{O}_V((-u+v)F_1 + (-u+v)F_2 + (2u+v)\Delta)$. Then $c_1(L_0) = -6ul_3,\ c_1(L_1) = 6(ul_1 + vl_3)$, and $c_1(L_2) = 6(-ul_1 + vl_3)$ hold. These expressions make it clear that the line bundle $L_1 \otimes L_2$ is ample and, according to Lemma 2.6.1, that L is a big line bundle in this case.

The set $\square_{\mathrm{Nef}}(L_0, h)$ in this setting is rational polyhedral. More precisely, $\square_{\mathrm{Nef}}(L_0, h)$ is the convex closure of the five points $e^1,\ e^2,\ \frac{u}{v}e^2,\ \frac{u}{2(u+v)}e^1 + \frac{u}{2(u+v)}e^2,\ \frac{u}{v}e^1$ in $M_{\mathbb{R}}$. So, by applying Theorem 2.4.7, it immediately turns out that the weight of a minimal singular metric ψ_{σ_j} satisfies $\psi_{\sigma_j} \sim_{\mathrm{sing}} 1$ at any points of X except for the locus $\mathbb{P}(L_0)$, and

$$\begin{aligned}\psi_{\sigma_1}(x_1, x_2, z) &\sim_{\mathrm{sing}} \frac{u}{2v(u+v)} \log \max\{|x_1|^{2(2u+2v)}, |x_2|^{2(2u+2v)}, |x_1|^{2v}|x_2|^{2v}\} \\ &\sim_{\mathrm{sing}} \frac{u}{2v(u+v)} \log \left(|x_1|^{2(2u+2v)} + |x_2|^{2(2u+2v)} + |x_1|^{2v}|x_2|^{2v}\right)\end{aligned}$$

at a point $(0, 0, z_0) \in \mathbb{P}(L_0)$. Therefore, it follows that the non-nef locus $\mathrm{NNef}(L)$ is a Zariski closed subset $\mathbb{P}(L_0)$ of X.

According to [Na, 2.5], the fact that $\square_{\mathrm{Nef}}(L_0, h)$ is a rational polyhedral yields that L admits a Zariski decomposition after appropriate proper modifications. Especially, when u and v can be written as

$$u = 1,\ v = 2n - 2$$

for some integer $n > 1$, (X, L) is an example which admits a Zariski decomposition just after the n-time blow-up centered at the non-nef locus of the pull-back of L. It can be also checked out by using the above expression of the minimal singular metric on L.

According to the above expression of $\square_{\mathrm{Nef}}(L_0, h)$, the result of Corollary 2.5.6 can be rewritten as follows. First, it is clear that $\mathcal{J}(h_{\min}^t)$ is trivial at any point in $X \setminus \mathbb{P}(L_0)$. Next, for a point $x_0 \in \mathbb{P}(L_0)$, the stalk of $\mathcal{J}(h_{\min})_{x_0}$ of the multiplier ideal sheaf at x_0 is the ideal of $\mathcal{O}_{X,x_0}$ which is generated by the system of the polynomials

$$\{x_1^p x_2^q \mid (p+1, q+1) \in \mathrm{Int}(S_t) \cap \mathbb{Z}^2\},$$

where we denote by $\mathrm{Int}(S_t)$ the interior of the set

$$S_t = \{(\langle tm, e_1\rangle, \langle tm, e_2\rangle) \in \mathbb{R}^2 \mid m \in S(L_0, h)_{\sigma_1}\}.$$

For the detail shape of S_t, see Figure 2.3.

The set of the whole jumping numbers $\mathrm{Jump}(\psi_{\sigma_1}; x_0)$ at a point $x_0 \in \mathbb{P}(L_0)$ can be written as $\mathrm{Jump}(\psi_{\sigma_1}; x_0) = \left\{2p + (p+q)\frac{v}{u} \,\middle|\, p, q \in \mathbb{Z},\ 1 \le p \le q\right\}$, and the singularity exponent $c_{x_0}(\psi_{\sigma_1})$, which is the least number in $\mathrm{Jump}(\psi_{\sigma_1}; x_0)$, satisfies $c_{x_0}(\psi_{\sigma_1}) = 2\left(1 + \frac{v}{u}\right)$.

Remark 2.6.3. In Example 2.6.2, the behavior of the multiplier ideal sheaf $\mathcal{J}(\psi_{\sigma_1})$ around a point of $\mathbb{P}(L_0)$ coincides with that of the (algebraic)

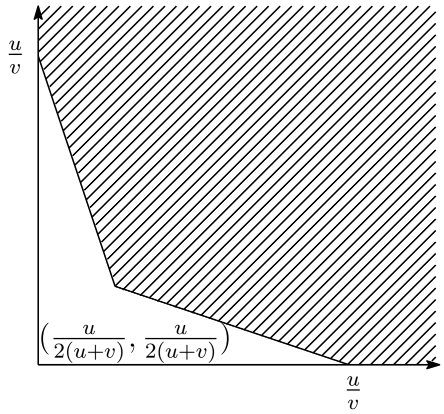

Figure 2.3: The shaded area of this figure represents the set S_1. The set S_t is the set of points $p \in \mathbb{R}^2$ which satisfies $\frac{p}{t} \in S_1$.

multiplier ideal sheaf $\mathcal{J}(\mathfrak{a}^c)$, where $\mathfrak{a}$ is an ideal generated by $(x_1^{2(u+v)}, x_2^{2(u+v)}, x_1^v x_2^v)$ and c is the rational number $\frac{u}{2v(u+v)}$.

This means that the analytic multiplier ideal sheaf $\mathcal{J}(\psi_{\sigma_1})_{x_0}$ has properties same as algebraic multiplier ideal sheaves. For example, it is known that, related to the algebraic multiplier ideal sheaf $\mathcal{J}(\mathfrak{a}^c)$, the set of the whole jumping numbers $\mathrm{Jump}(\mathfrak{a}; x_0)$ is a discrete subset of the set of rational numbers $\mathbb{Q}$, and has the property so-called "periodicity" in a sufficiently big parts of this set (see [ELSV, 1.12] for details). Indeed, it can be easily checked that $\mathrm{Jump}(\psi_{\sigma_1}; x_0)$ is a discrete subset of $\mathbb{Q}$, and has a "period" $c^{-1} = 2v(1 + \frac{v}{u})$.

Example 2.6.4. Second example is the example found out by Nakayama ([Na]), which admits no Zariski decomposition even after modifications.

Let us fix an integer $a > 1$ and set $L_0 := \mathcal{O}_V(2F_1 - 4F_2 + 2\Delta), L_1 := \mathcal{O}_V((a-1)F_1 + (a-1)F_2 + (a+2)\Delta)$, and $L_2 := \mathcal{O}_V((a+3)F_1 + (a-3)F_2 + a\Delta)$. Then $c_1(L_0) = -6(l_1 + \sqrt{3}l_2)$, $c_1(L_1) = 6(-l_1 + al_3)$, and $c_1(L_2) = 6(-\sqrt{3}l_2 + al_3)$ hold. By these expressions, it turns out that the line bundles L_1 and L_2 are ample and, according to Lemma 2.6.1, that L is also a big line bundle in this case. For this example, see Section 1.

Example 2.6.5. Finally, we introduce an example which can be proved that admits no Zariski decomposition even after modifications in the almost same way to the case of previous Nakayama example, however whose minimal singular metric can be expressed more easily.

Let $L_0 := \mathcal{O}_V(4F_1 + 4F_2 + \Delta), L_1 := \mathcal{O}_V$, and $L_2 := \mathcal{O}_V(-F_1 + 9F_2 + \Delta)$.

Then $c_1(L_0) = 6(l_1 + 3l_3)$, $c_1(L_1) = 0$, and $c_1(L_2) = 6l_1 + 10\sqrt{3}l_2 + 18l_3$ hold. By this expression, it turns out that the line bundle L_0 is ample and, from Lemma 2.6.1, that L is also a big line bundle in this case.

The set $\square_{\mathrm{Nef}}(L_0, h)$ in this setting is not rational, but is polyhedral. More precisely, $\square_{\mathrm{Nef}}(L_0, h)$ is the convex closure of the three points 0, e^1, and $\frac{2\sqrt{6}}{5}e^2$ in $M_{\mathbb{R}}$. So, applying theorem 2.4.7, it immediately turns out that the weight of a minimal singular metric ψ_{σ_j} satisfies $\psi_{\sigma_j} \sim_{\mathrm{sing}} 1$ at any points of X except for the locus $\mathbb{P}(L_2)$, and

$$\begin{aligned}\psi_{\sigma_3}(x_1, x_2, z) &\sim_{\mathrm{sing}} \log\max\{|x_0|^{2\alpha}, |x_1|^2\}\\ &\sim_{\mathrm{sing}} \log\left(|x_0|^{2\alpha} + |x_1|^2\right)\end{aligned}$$

at a point $(0, 0, z_0) \in \mathbb{P}(L_2)$, where we denote by α the positive irrational number $1 - \frac{2\sqrt{6}}{5}$.

According to the above expression of $\square_{\mathrm{Nef}}(L_0, h)$, the result of Corollary 2.5.6 can be rewritten as follows. First, it is clear that $\mathcal{J}(h^t_{\min})$ is trivial at any point in $X \setminus \mathbb{P}(L_2)$. Next, for a point $x_0 \in \mathbb{P}(L_2)$, the stalk $\mathcal{J}(h_{\min})_{x_0}$ of the multiplier ideal sheaf at x_0 is the ideal of $\mathcal{O}_{X,x_0}$ which is generated by the polynomials

$$\{x_1^p x_2^q \mid (p+1, q+1) \in \mathrm{Int}(S_t) \cap \mathbb{Z}^2\},$$

where we denote by S_t the set $\{(\langle tm, e_1\rangle, \langle tm, e_2\rangle) \in \mathbb{R}^2 \mid m \in S(L_0, h)_{\sigma_3}\}$. For the detail shape of S_t in this case, see Figure 2.4.

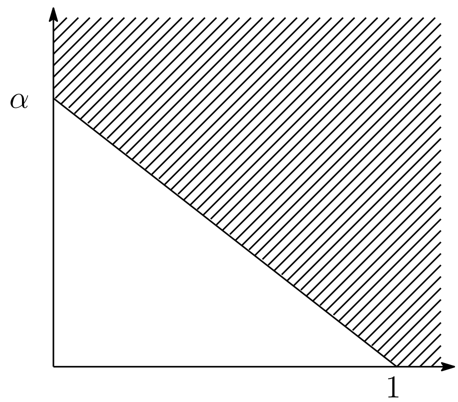

Figure 2.4: The shaded area of this figure represents the set S_1. The set S_t is the set of points $p \in \mathbb{R}^2$ which satisfies $\frac{p}{t} \in S_1$.

Let x_0 be a point in $\mathbb{P}(L_2)$. In this case, $\mathrm{Jump}(\psi_{\sigma_3}; x_0)$ can be calculated that $\mathrm{Jump}(\psi_{\sigma_3}; x_0) = \mathbb{Z}_{>0} \oplus \frac{1}{\alpha} \cdot \mathbb{Z}_{>0}$, and the singularity exponent can be calculated that $c_{x_0}(\psi_{\sigma_1}) = 1 + \frac{1}{\alpha}$, which is not rational, too. It can easily be proved by using ([Na, 2.11]) that L admits no Zariski decomposition even after modifications in this settings.

3

On minimal singular metrics of certain class of line bundles whose section ring is not finitely generated

3.1 Introduction

Our interest in this chapter is a regularity of a minimal singular metric of a line bundle. One main conclusion of our general result in this chapter is the existence of smooth Hermitian metrics with semi-positive curvatures on the so-called Zariski's example ([Laz1, 2.3.A]).

THEOREM 3.1.1 (Example 3.4.3). *Let $C \subset \mathbb{P}^2$ be a smooth elliptic curve, $\pi\colon X \to \mathbb{P}^2$ the blowing-up at general twelve points $p_1, p_2, \dots, p_{12} \in C$, H the pulled back divisor of a line in $\mathbb{P}^2$, and let D be the strict transform of C. Then the line bundle $L = \mathcal{O}_X(H + D)$ is semi-positive (i.e. L admits a smooth Hermitian metric with semi-positive curvature).*

This L is nef and big, however has a pathological property that $D \subset \mathrm{Bs}\,|L^{\otimes m}|$ holds for all $m \geq 1$, $|L^{\otimes m} \otimes \mathcal{O}_X(-D)|$ is globally generated for all $m \geq 1$, and that the section ring $\bigoplus_{m \geq 0} H^0(X, L^{\otimes m})$ of L is not finitely generated. When the twelve points $p_1, p_2, \dots, p_{12} \in C$ is special, the line bundle L is semi-ample and thus it is semi-positive. The main theorem is as follows.

THEOREM 3.1.2. *Let X be a smooth projective variety, D a smooth hypersurface of X, L a pseudo-effective line bundle over X, and let $h_{\min}$ be a minimal singular metric of L. Assume that $L \otimes \mathcal{O}_X(-D)$ is semi-positive, $\mathcal{O}_X(-D)|_D$ is ample, $\mathcal{O}_D(-K_D - D|_D)$ is nef and big, and that D has a* *holomorphic tubular neighborhood (i.e. an open neighborhood in X which is biholomorphic to an open neighborhood of the zero section in the normal bundle $N_{D/X}$). Then $h_{\min}|_D \not\equiv \infty$ holds if and only if $L|_D$ is pseudo-effective, moreover in this case $h_{\min}|_D$ is a minimal singular metric of $L|_D$.*

One of the typical cases of the situations in Theorem 3.1.2 is when X is a surface and the self-intersection number (D^2) is (sufficiently) negative. It is followed by a special case of Grauert's theorem [Gr, Satz 7]: A smooth compact complex curve D with genus g embedded in a complex surface X has a holomorphic tubular neighborhood if $(D^2) < \min\{0, 4-4g\}$ holds. Thus, we can apply our main theorem to Zariski's example to obtain Theorem 3.1.1, and we also can show the existence of a smooth Hermitian metric with semi-positive curvature for the same type examples introduced by Mumford ([Laz1, 2.3.A], or Example 3.4.3 here). When $(L \otimes \mathcal{O}_X(-D))|_D$ is ample, we can write down more concretely a minimal singular metric of L around D by using equilibrium metrics, which are special minimal singular metrics, of $\mathbb{R}$-line bundles $(L \otimes \mathcal{O}_X(-tD))|_D$ for $0 \leq t \leq 1$ (see Theorem 3.2.2 and Remark 3.3.3).

Another application of Theorem 3.1.2 we can expect is a concrete description of minimal singular metrics of a pseudo-effective line bundle which is not big. It is because, it follows from next Theorem 3.1.3, which is a version of Theorem 3.1.2, that we can apply Bergman kernel construction argument even when L is not big but merely pseudo-effective. In more detail, the line bundle $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$ in Theorem 3.1.3 is big if we chose A as an ample line bundle. Thus we can use Bergman kernel construction argument for this line bundle and we can study minimal singular metrics of L itself by restricting argument.

THEOREM 3.1.3. *Let X be a smooth projective variety, A a semi-ample line bundle on X, and let L be a pseudo-effective line bundle on X. Then the restriction of a minimal singular metric of $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$ on $\mathbb{P}(A \oplus L)$ to the divisor $\mathbb{P}(L) \subset \mathbb{P}(A \oplus L)$ corresponding to the projection $A \oplus L \to L$ gives a minimal singular metric of L on X via the natural identification $(\mathbb{P}(L), \mathcal{O}_{\mathbb{P}(L)}(1)) \cong (X, L)$ (see Remark 3.2.3).*

We can prove Theorem 3.1.3 directly by constructing an appropriate singular metric of $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$ from minimal singular metrics of A and L. In the proof of Theorem 3.1.2, we use the assumption on the existence of holomorphic tubular neighborhoods to reduce the situation in Theorem 3.1.2 to that in Theorem 3.1.3. Since L in Theorem 3.1.2 admits a singular Hermitian metric which is smooth on $X \setminus D$ and may be singular along D, all we have to do is to modify this metric around D. We will replace this metric on the tubular neighborhood of D by the metric constructed in the situation of Theorem 3.1.3.

The organization of this chapter is as follows. In §3.2, we treat the case when X has a suitable $\mathbb{P}^1$-bundle structure and L is the relative hyperplane bundle. In §3.3, we prove Theorem 3.1.3 and Theorem 3.1.2. Finally we give some examples in §3.4.

3.2 The $\mathbb{P}^1$-bundle case

In this section, we treat the case when X has a suitable $\mathbb{P}^1$-bundle structure and L is the relative hyperplane bundle. Here we give a minimal singular metric of L concretely by using equilibrium metrics of $\mathbb{R}$-line bundles of the base space of X. First we define the equilibrium metrics for smooth Hermitian metrics on pseudo-effective line bundles.

Definition 3.2.1. Let X be a smooth projective variety, L a pseudo-effective line bundle over X, and let $h = e^{-\varphi}$ be a smooth Hermitian metric on L. We denote by h_e the *equilibrium metric*, whose local weight function φ_e is defined by

$$\varphi_e = \varphi + \sup{}^*\{\psi\colon X \to \mathbb{R} \cup \{-\infty\} \mid \psi \text{ is a } \varphi\text{-psh function}, \ \psi \leq 0\},$$

where $\sup^*$ stands for the upper semi-continuous regularization of the supremum.

Equilibrium metrics are minimal singular metrics ([DPS00, 1.5]). Using this notion, we prove the following theorem.

Theorem 3.2.2. *Let X be a smooth projective variety, A an ample line bundle on X, and let L be a pseudo-effective line bundle on X. Let $h_L = e^{-\varphi_L}$ be a smooth Hermitian metric of L and let $h_A = e^{-\varphi_A}$ be a smooth Hermitian metric of A satisfying $dd^c\varphi_A > 0$. Fix a local coordinate system by $(z, x) \mapsto [zs_A^*(x) + s_L^*(x)] \in \mathbb{P}(A \oplus L)$, where s_A^* and s_L^* are local trivializations of A^{-1} and L^{-1}, respectively. Then the metric of the relative hyperplane line bundle $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$ on $\mathbb{P}(A \oplus L)$ defined by the local weights*

$$\widetilde{\varphi}(z, x) = \log \max_{t\in[0,1]} |z|^{2t} e^{(t\varphi_A+(1-t)\varphi_L)_e(x)}$$

is a minimal singular metric, where $(t\varphi_A + (1-t)\varphi_L)_e$ is the local weight of the equilibrium metric associated to $h_A^t h_L^{1-t}$, which is a smooth Hermitian metric of the "$\mathbb{R}$-line bundle $A^{\otimes t} \otimes L^{\otimes(1-t)}$".

We denote by $\widetilde{X}$ the variety $\mathbb{P}(A \oplus L)$, by $\pi\colon \widetilde{X} \to X$ the canonical projection mapping, and by $\widetilde{L}$ the relative plane line bundle $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$ on $\widetilde{X}$.

Remark 3.2.3. Let us denote by X' the subset $\mathbb{P}(L)$ of $\widetilde{X}$, and X'' the subset $\mathbb{P}(A)$. Then $\mathcal{O}_{\widetilde{X}}(X') = \widetilde{L} \otimes \pi^* A^{-1}$ and $\widetilde{L}|_{X'} = \pi^* L|_{X'}$ hold as equalities of line bundles on $\widetilde{X}$ and X', respectively. Therefore we can regard the restriction of a metric of $\widetilde{L}$ to X' as a metric of L, and by regarding $X' \subset \widetilde{X}$ as $\mathbb{P}(\mathcal{O}_X) \subset \mathbb{P}(\mathcal{O}_X \oplus (L^{-1} \otimes A))$, we can identify $\widetilde{X} \setminus X''$ and X' with the total space of the normal bundle $N_{X'/\widetilde{X}}$, which is isomorphic to the bundle $L \otimes A^{-1}$ via π, and its zero-section.

We also remark here that $\widetilde{L}$ is big if A is ample (see [Laz1, 2.3.2]).

From now on, we prove Theorem 3.2.2. Here we denote by U the domain of definition of s_A^*, s_L^* and x. We also use the smooth Hermitian metric $\widetilde{h}_\infty = e^{-\widetilde{\varphi}_\infty}$ of $\widetilde{L}$, whose local weight is defined as $\widetilde{\varphi}_\infty(z,x) = \log\left(|z|^2 e^{\varphi_A(x)} + e^{\varphi_L(x)}\right)$. To prove Theorem 3.2.2, it is sufficient to show the following two propositions.

Proposition 3.2.4 (Plurisubharmonicity of $\widetilde{\varphi}$). *The function*

$$\widetilde{\varphi}(z,x) = \log \max_{t\in[0,1]} |z|^{2t} e^{(t\varphi_A + (1-t)\varphi_L)_e(x)}$$

is plurisubharmonic and $\{e^{-\widetilde{\varphi}}\}$ *glue up to define a singular Hermitian metric of* $\widetilde{L}$.

Proposition 3.2.5 (Minimal singularity of $\widetilde{\varphi}$). *There is a constant* C *such that* $(\widetilde{\varphi}_\infty)_e \leq \widetilde{\varphi} + C$ *holds.*

3.2.1 Plurisubharmonicity of the weight function $\widetilde{\varphi}$

Here we prove Proposition 3.2.4. Since $\log |z|^{2t} e^{(t\varphi_A + (1-t)\varphi_L)_e(x)}$ is plurisubharmonic and a local weight of a singular Hermitian metric of $\widetilde{L}$ for each $t \in [0,1]$, it is sufficient to show that $\widetilde{\varphi}$ is upper semi-continuous, and it follows immediately from Lemma 3.2.7.

Let us denote by ψ_t the function

$$\sup{}^*\{\psi\colon X \to \mathbb{R} \cup \{-\infty\} \mid \psi \text{ is a } (t\varphi_A + (1-t)\varphi_L)\text{-psh function}, \psi \leq 0\}.$$

For proving Lemma 3.2.7, we need the following lemma.

LEMMA 3.2.6.
(1) *The sequence* $\{\frac{\psi_t}{1-t}\}_{t\in[0,1]}$ *is monotonically increasing with respect to* t.
(2) *For all* $t \in [0,1)$, $\lim_{s\downarrow t}\frac{\psi_s}{1-s} = \frac{\psi_t}{1-t}$ *holds.*

PROOF. (1) Let $t \le s$ be elements of $[0,1]$. Since

$$s\varphi_A + (1-s)\varphi_L + \frac{1-s}{1-t}\psi_t = \frac{1-s}{1-t}(t\varphi_A + (1-t)\varphi_L)_e + \frac{s-t}{1-t}\varphi_A$$

holds, $\frac{1-s}{1-t}\psi_t$ is a $(s\varphi_A + (1-s)\varphi_L)$-psh function. As $\frac{1-s}{1-t}\psi_t \le 0$, $\frac{1-s}{1-t}\psi_t \le \psi_s$ holds.

(2) According to (1), it is sufficient to show that $\lim_{s\downarrow t}\frac{\psi_s}{1-s} \le \frac{\psi_t}{1-t}$ holds. Since the sequence $\frac{s}{1-s}\varphi_A + \varphi_L + \frac{\psi_s}{1-s}$ $(= \frac{1}{1-s}(s\varphi_A + (1-s)\varphi_L)_e)$ is monotonically increasing psh functions, the limit $\frac{t}{1-t}\varphi_A + \varphi_L + \lim_{s\downarrow t}\frac{\psi_s}{1-s}$ is also psh. As $\lim_{s\downarrow t}\frac{\psi_s}{1-s} \le 0$, $(1-t)\lim_{s\downarrow t}\frac{\psi_s}{1-s} \le \psi_t$ holds. □

LEMMA 3.2.7. *The function* $F\colon \mathbb{C}\times U \times [0,1] \to \mathbb{R}\cup\{-\infty\}$ *defined by* $F(z,x,t) = (t\varphi_A + (1-t)\varphi_L)_e(x) + t\log|z|^2$ *is upper semi-continuous.*

PROOF. Let us set the function $H\colon U\times[0,1] \to \mathbb{R}\cup\{-\infty\}$ as $H(x,t) = \frac{\psi_t(x)}{1-t}$. Since $F(z,x,t)$ is a sum of upper semi-continuous functions and $(1-t)H(x,t)$, it is sufficient to show that H is upper semi-continuous. Let us fix an element $(x_0,t_0) \in U\times[0,1)$ and sufficiently small positive number ε. Then, by Lemma 3.2.6 (1),

$$\limsup_{(x,t)\mapsto(x_0,t_0)} H(x,t) = \lim_{r\downarrow 0}\sup_{\substack{|x-x_0|<r\\|t-t_0|<r}} H(x,t) \le \lim_{r\downarrow 0}\sup_{|x-x_0|<r} H(x,t_0+\varepsilon)$$

holds. As $H(-,t_0+\varepsilon) = \frac{\psi_{t_0+\varepsilon}}{1-(t_0+\varepsilon)}$ is upper semi-continuous, we obtain an inequality $\limsup_{(x,t)\mapsto(x_0,t_0)} H(x,t) \le H(x_0,t_0+\varepsilon)$. By Lemma 3.2.6 (2), we can show that the equality $\limsup_{(x,t)\mapsto(x_0,t_0)} H(x,t) \le H(x_0,t_0)$ holds. We can also show this inequality when $t_0 = 1$ by the same argument, and this shows the lemma. □

3.2.2 Minimal singularity of the weight function $\widetilde{\varphi}$

Next we prove Proposition 3.2.5. Let us fix a (sufficiently positive) Kähler metric ω of X and define

$$\widetilde{\omega} = \pi^*\omega + dd^c\log(|z|^2e^{\varphi_A} + e^{\varphi_L}) - \frac{|z|^2e^{\varphi_A}\pi^*dd^c\varphi_A + e^{\varphi_L}\pi^*dd^c\varphi_L}{|z|^2e^{\varphi_A} + e^{\varphi_L}},$$

where $dd^c = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}$. This $\widetilde{\omega}$ defines a global smooth $(1,1)$-form on $\widetilde{X}$, since $dd^c \log(|z|^2 e^{\varphi_A} + e^{\varphi_L})$ is the curvature form of the smooth Hermitian metric of $\widetilde{L}$ associated to the Finsler metric of $A \oplus L$ induced from h_A and h_L, and both of the coefficients $|z|^2 e^{\varphi_A}/(|z|^2 e^{\varphi_A} + e^{\varphi_L})$ of $\pi^* dd^c \varphi_A$ and $e^{\varphi_L}/(|z|^2 e^{\varphi_A} + e^{\varphi_L})$ of $\pi^* dd^c \varphi_L$ glue up to define $\mathbb{R}$-valued functions defined on whole $\widetilde{X}$. It is because, the values $|z|^2 e^{\varphi_A}$, e^{φ_L}, and $|z|^2 e^{\varphi_A} + e^{\varphi_L}$ can be regarded as the norms of the points $(z,0)$, $(0,1)$, and $(z,1)$ of a fiber of the vector bundle $A^{-1} \oplus L^{-1}$, respectively, computed by using the metric induced from h_A and h_L. Thus ratios of these values define genuine functions on the whole of $\widetilde{X}$.

LEMMA 3.2.8. *The form $\widetilde{\omega}$ and the measures $dV_\omega = \frac{\omega^n}{n!}$ of X and $dV_{\widetilde{\omega}} = \frac{\widetilde{\omega}^{n+1}}{(n+1)!}$ of $\widetilde{X}$ satisfy the following properties when ω is sufficiently positive.*

1. *$\widetilde{\omega}$ is a smooth strictly positive $(1,1)$-form on $\widetilde{X}$.*
2. *For all $x \in X$, $\int_{z \in \pi^{-1}(x)} \widetilde{\omega}|_{\pi^{-1}(x)} = 1$ holds.*
3. *For all $\mathbb{R}$-valued measurable function F on $\widetilde{X}$, the equation*
$$\int_{(z,x)\in\widetilde{X}} F(z,x)\, dV_{\widetilde{\omega}} = \int_{x\in X} \left(\int_{z\in\pi^{-1}(x)} F(z,x)\, dV_{\widetilde{\omega}}|_{\pi^{-1}(x)} \right) dV_\omega$$
holds.
4. *Moreover, when F depends only on x and $|z|$, an equation*
$$\int_{(z,x)\in\widetilde{X}} F(z,x)\, dV_{\widetilde{\omega}} = \int_{x\in X} \left(\int_0^\infty \frac{2rG(r,x)e^{\varphi_A(x)+\varphi_L(x)}}{(r^2 e^{\varphi_A(x)} + e^{\varphi_L(x)})^2}\, dr \right) dV_\omega$$
holds, where G is the function such that $G(|z|,x) = F(z,x)$ holds.

PROOF. By straightforward computations, we can obtain the formula
$$\widetilde{\omega} = \pi^*\omega + C(\eta \wedge \overline{\eta} + dz \wedge \overline{\eta} + \eta \wedge d\overline{z} + dz \wedge d\overline{z}),$$
where $C = \frac{\sqrt{-1}}{2\pi} \frac{e^{\varphi_A+\varphi_L}}{(|z|^2 e^{\varphi_A} + e^{\varphi_L})^2}$ and $\eta = z\partial(\varphi_A - \varphi_L)$. From this formula, it is shown that $\widetilde{\omega}^{n+1} = (n+1)C dz \wedge d\overline{z} \wedge (\pi^*\omega)^n$ holds, which shows the lemma. □

We also use the following lemma, which can be proved by straightforward computations.

LEMMA 3.2.9.

$$-\log\int_{z\in\pi^{-1}(x)}|z|^{2t}e^{-\widetilde{\varphi}_\infty(z,x)}\,\widetilde{\omega}|_{\pi^{-1}(x)}=t\varphi_A(x)+(1-t)\varphi_L(x)-\log\frac{\Gamma(1+t)\Gamma(2-t)}{4}$$

holds for all $t\in[0,1]$ and $x\in X$, where Γ stands for the Gamma function.

The following lemma can be shown by using the approximation theorem [D2, 13.21].

LEMMA 3.2.10. *Let Y be a smooth projective variety, dV_Y a smooth volume form of Y, M a pseudo-effective line bundle over Y, and let $h_M=e^{-\psi_\infty}$ be a smooth Hermitian metric of M. Fix points $y_0,y_1,\dots y_N\in Y$ and local coordinates systems around each y_j such that $\bigcup_{j=0}^N\{y\mid |y-y_j|<\frac{1}{\sqrt{\pi}}\}=Y$ holds. Let $h_{M,1}=e^{-\psi_1}$ and $h_{M,2}=e^{-\psi_2}$ be singular Hermitian metrics given by*

$$\begin{aligned}\psi_1&=\psi_\infty+\sup{}^*\left\{\frac{1}{m}\log|f|^2_{h_M^m}\,\middle|\,m\in\mathbb{N},f\in H^0(Y,mM),\log|f|^2_{h_M^m}\le 0\right\},\\ \psi_2&=\psi_\infty+\sup{}^*\left\{\frac{1}{m}\log|f|^2_{h_M^m}\,\middle|\,m\in\mathbb{N},f\in H^0(Y,mM),\int_Y|f|^2_{h_M^m}dV_Y\le 1\right\}.\end{aligned}$$

Let $C'=C_1'+C_2'$ with $C_1'=\max_j\left(\max_{|y-y_j|\le\frac{2}{\sqrt{\pi}}}\psi_\infty(y)-\min_{|y-y_j|\le\frac{2}{\sqrt{\pi}}}\psi_\infty(y)\right)$ and $C_2'=\log\max_j\max_{|y-y_j|\le\frac{2}{\sqrt{\pi}}}\frac{d\lambda}{dV_Y}$, where $d\lambda$ is the Euclidean measure. Then an inequality $\psi_2-C'\le\psi_1\le(\psi_\infty)_e$ holds. Moreover, if M is big, then an inequality $\psi_2-C'\le\psi_1\le(\psi_\infty)_e\le\psi_2$ holds.

Proof of Proposition 3.2.5. (1) We fix points $x_0,x_1,\dots x_N\in X$ and local coordinates systems around each x_j such that $\bigcup_{j=0}^N\{x\mid |x-x_j|<\frac{1}{\sqrt{\pi}}\}=X$ holds. Let us denote by $\varphi_{\infty,t}$ the weight of the "smooth Hermitian metric" $t\varphi_A+(1-t)\varphi_L-\log\frac{\Gamma(1+t)\Gamma(2-t)}{4}$ of $tA+(1-t)L$. We let $C=C_1+C_2+\log 2$ with

$$\begin{aligned}C_1&=\max_j\left(\max_{\substack{|x-x_j|\le\frac{2}{\sqrt{\pi}}\\ t\in[0,1]}}\widetilde{\varphi}_{\infty,t}(x)-\min_{\substack{|x-x_j|\le\frac{2}{\sqrt{\pi}}\\ t\in[0,1]}}\widetilde{\varphi}_{\infty,t}(x)\right),\\ C_2&=\log\max_j\max_{|x-x_j|\le\frac{2}{\sqrt{\pi}}}\frac{d\lambda}{dV_\omega}.\end{aligned}$$

Since $(\varphi_{\infty,t})_e=(t\varphi_A+(1-t)\varphi_L)_e-\log\frac{\Gamma(1+t)\Gamma(2-t)}{4}$ holds, it is sufficient to show that

$$(\widetilde{\varphi}_\infty)_e\le\log\max_{t\in[0,1]}|z|^{2t}e^{(\varphi_{\infty,t})_e(x)}+C$$

holds. According to the last part of Lemma 3.2.10, this is reduced to show that for each $F \in H^0(\widetilde{X}, m\widetilde{L})$ such that $\int_{\widetilde{X}} |F|^2 e^{-m\widetilde{\varphi}_\infty} dV_{\widetilde{\omega}} \leq 1$, an inequality

$$\frac{1}{m} \log |F|^2 \leq \log \max_{t \in [0,1]} |z|^{2t} e^{(\varphi_{\infty,t})_e(x)} + C$$

holds.

(2) We show the last inequality. The holomorphic section $F(z,x)$ can be expanded as $F(z,x) = \sum_{\ell=0}^m z^\ell f_\ell(x)$ with $f_\ell \in H^0(X, \ell A + (m-\ell)L)$. We first show that an inequality

$$\int_{\widetilde{X}} |z^\ell f_\ell|^2 e^{-m\widetilde{\varphi}_\infty} dV_{\widetilde{\omega}} \leq 1 \qquad (*)$$

holds for $\ell = 1, 2, \ldots, m$. For proving $(*)$, we use an inequality

$$|f_\ell(x)|^2 = \left| \frac{1}{\ell!} \frac{\partial^\ell}{\partial z^\ell} F(0,x) \right|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|F(re^{\sqrt{-1}\theta}, x)|^2}{r^{2\ell}} d\theta$$

for each positive number r. We denote by $\widetilde{\varphi}_\infty(r,x)$ the function such that $\widetilde{\varphi}_\infty(|z|,x) = \widetilde{\varphi}_\infty(z,x)$ holds. By multiplying the metric terms and integrating these with r, we obtain the following inequality.

$$\begin{aligned} & \int_0^\infty \frac{2r(r^{2\ell}|f_\ell(x)|^2 e^{-m\widetilde{\varphi}_\infty(r,x)}) e^{\varphi_A(x)+\varphi_L(x)}}{(r^2 e^{\varphi_A(x)} + e^{\varphi_L(x)})^2} dr \\ \leq \; & \frac{1}{2\pi} \int_0^\infty \left(\int_0^{2\pi} \frac{2r(|F(re^{\sqrt{-1}\theta}, x)|^2 e^{-m\widetilde{\varphi}_\infty(r,x)}) e^{\varphi_A(x)+\varphi_L(x)}}{(r^2 e^{\varphi_A(x)} + e^{\varphi_L(x)})^2} d\theta \right) dr \\ = \; & \int_{z \in \pi^{-1}(x)} |F(z,x)|^2 e^{-m\widetilde{\varphi}(z,x)} dV_{\widetilde{\omega}}|_{\pi^{-1}(x)}. \end{aligned}$$

This inequality and Lemma 3.2.8 (2), (3), (4) implies the inequality $(*)$.

Then, by Lemma 3.2.8 (3),

$$\begin{aligned} 1 \;\geq\; & \int_{(z,x) \in \widetilde{X}} |z^\ell f_\ell(x)|^2 e^{-m\widetilde{\varphi}_\infty(z,x)} dV_{\widetilde{\omega}} \\ = \; & \int_{x \in X} |f_\ell(x)|^2 \left(\int_{z \in \pi^{-1}(x)} |z^\ell|^2 e^{-m\widetilde{\varphi}_\infty(z,x)} dV_{\widetilde{\omega}}|_{\pi^{-1}(x)} \right) dV_\omega \\ = \; & \int_{x \in X} |f_\ell(x)|^2 \left(\left(\int_{z \in \pi^{-1}(x)} \left(|z|^{2\frac{\ell}{m}} e^{-\widetilde{\varphi}_\infty(z,x)} \right)^m dV_{\widetilde{\omega}}|_{\pi^{-1}(x)} \right)^{\frac{1}{m}} \right. \\ & \qquad \left. \cdot \left(\int_{z \in \pi^{-1}(x)} 1^{\frac{m}{m-1}} dV_{\widetilde{\omega}}|_{\pi^{-1}(x)} \right)^{\frac{m-1}{m}} \right)^m dV_\omega \\ \geq \; & \int_{x \in X} |f_\ell(x)|^2 \left(\int_{z \in \pi^{-1}(x)} |z|^{2\frac{\ell}{m}} e^{-\widetilde{\varphi}_\infty(z,x)} \right)^m dV_\omega \end{aligned}$$

holds (Here we used Hölder's inequality). Therefore, by Lemma 3.2.9, $\int_{x\in X} |f_\ell(x)|^2 e^{-m\varphi_{\infty,t}(x)}\, dV_\omega \leq 1$ holds for $t = \frac{\ell}{m}$. Then by Lemma 3.2.10, we obtain an inequality $\frac{1}{m}\log|f_\ell|^2 \leq (\varphi_{\infty,t})_e + C_1 + C_2$. Thus

$$\begin{aligned}
& \frac{1}{m}\log|F(z,x)|^2 \\
\leq\ & \frac{1}{m}\log\sum_{l=0}^{m}|z^\ell f_\ell(x)|^2 \\
\leq\ & \frac{1}{m}\log\Big((m+1)\max_{\ell}|z^\ell f_\ell(x)|^2\Big) \\
=\ & \frac{1}{m}\log(m+1) + \log\max_{0\leq\ell\leq m}|z|^{2\frac{\ell}{m}}|f_\ell(x)|^{\frac{2}{m}} \\
\leq\ & \log 2 + \log\max_{t\in[0,1]}|z|^{2t}e^{(\varphi_{\infty,t})_e(x)+C_1+C_2} \\
=\ & \log\max_{t\in[0,1]}|z|^{2t}e^{(\varphi_{\infty,t})_e(x)} + C
\end{aligned}$$

holds. □

3.3 Proof of the main theorems in §3

In this section, we prove Theorem 3.1.2 and Theorem 3.1.3. We first prove Theorem 3.1.3.

Proof of Theorem 3.1.3. As Theorem 3.1.3 immediately follows from Theorem 3.2.2 when A is ample, we show the theorem when A is merely semi-ample (Although we can not obtain a concrete description of a minimal singular metric of $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$ as in Theorem 3.2.2 in this case, we can show the theorem). Let $\widetilde{h}_{\min}$ be a minimal singular metric of $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$, h_A be a smooth Hermitian metric of A with semi-positive curvature, and h_L be a minimal singular metric of L. Let us consider the singular Hermitian metric $\widetilde{h}$ with local weight function $\log(|z|^2 e^{\varphi_A(x)} + e^{\varphi_L(x)})$ where (z,x) is the coordinates just as in Theorem 3.2.2 and φ_A, φ_L is the local weight of h_A, h_L, respectively. Since $\widetilde{h}_{\min}$ is less singular than $\widetilde{h}$ and $\widetilde{h}|_{\mathbb{P}(L)} = h_L$ holds, there exists a positive constant C such that

$$\widetilde{h}_{\min}|_{\mathbb{P}(L)} \leq C\cdot\widetilde{h}|_{\mathbb{P}(L)} = C\cdot h_L$$

holds, which shows that the metric $\widetilde{h}_{\min}|_{\mathbb{P}(L)}$ has minimal singularity. □

Proof of Theorem 3.1.2. Let X be a smooth projective variety, D a 1-codimensional smooth subvariety of X, and let L be a pseudo-effective line bundle over X. We assume that $A = L \otimes \mathcal{O}_X(-D)$ is semi-positive and that there is an open neighborhood U of $D \subset X$ biholomorphic to an open neighborhood U' of the zero section of the normal bundle $N_{D/X}$. Here we may assume that $U' = \{\xi \in N_{D/X} \mid |\xi|_{h_{X/D}} < \varepsilon_0\}$ for some smooth Hermitian metric $h_{X/D}$ with negative curvature of $N_{D/X}$ and a positive number ε_0.

Since $L|_D$ has no singular Hermitian metric of with psh local weights (which is not identically equal to $-\infty$) when $L|_D$ is not pseudo-effective, all we have to do is showing the existence of a singular Hermitian metric of L with psh local weights which is an extension of a minimal singular metric of $L|_D$ assuming $L|_D$ is pseudo-effective but not big. We set X' as the total space $\pi\colon \mathbb{P}(L|_D \oplus A|_D) \to D$ and L' as the relative hyperplane bundle $\mathcal{O}_{\mathbb{P}(L|_D \oplus A|_D)}(1)$. Let us fix a minimal singular metric $h_{L'} = e^{-\varphi_{L'}}$ of L'. We set V' as the subset $\{\xi \in N_{D/X} \mid |\xi|_{h_{X/D}} < \frac{\varepsilon_0}{2}\}$. By Remark 3.2.3, we can regard U' and V' be neighborhoods of $D' = \mathbb{P}(L|_D) \subset X'$. From the assumption, the natural biholomorphic mapping $\pi|_{D'}\colon D' \to D$ extends to a biholomorphic mapping $f\colon U' \to U$. We denote by V the set $f(V') \subset U$.

By Proposition 3.3.1 (2) below, there exists a line bundle F on U' which admits a flat structure and $f^*(L|_U) \cong L'|_{U'} \otimes F$ holds. We fix a flat metric $h_F = e^{-\varphi_F}$ of F. By choosing appropriate local trivialization, we may assume $\varphi_F \equiv 0$. Thus we can regard $(f^{-1})^*\varphi_{L'}$ as the local weight function of the singular Hermitian metric $(f^{-1})^* h_{L'} h_F$ of $L|_U$. To show the theorem, according to Theorem 3.1.3, it is sufficient to construct a singular Hermitian metric $e^{-\varphi_L}$ of L with $dd^c\varphi_L \geq 0$ and $\varphi_L|_V = (f^{-1})^*\varphi_{L'}|_{V'}$ holds. Let $h_A = e^{-\varphi_A}$ be a smooth Hermitian metric of A with $dd^c\varphi_A \geq 0$ and let $f_D \in H^0(X, \mathcal{O}_X(D))$ be a section which vanishes only on D. Without loss of generality, we may assume $\varphi_A \geq 0, (f^{-1})^*\varphi_{L'} \leq -1$ holds on each fixed open set $W_j (j = 1, 2, \ldots, N)$ covering the whole U, and $\log |f_D|^2 \geq -1$ holds on each intersection $W_j \cap (\overline{U} \setminus V)$. We define φ_L as the function $\max\{\varphi_A + \log |f_D|^2, (f^{-1})^*\varphi_{L'}\}$ on each $W_j \cap U$. Since $\varphi_L = \varphi_A + \log |f_D|^2$ holds on each intersection $W_j \cap (\overline{U} \setminus V)$, $e^{-\varphi_L}$ on U and $e^{-(\varphi_A + \log |f_D|^2)}$ on $X \setminus V$ glue up to define a new singular Hermitian metric of L, which proves the theorem. □

Proposition 3.3.1.
(1) *(a version of Rossi's theorem) The natural map* $H^1(U', \mathcal{O}_{U'}) \to H^1(U', \mathcal{O}_{U'}/I_{D'}^n)$ *is injective for some* $n \geq 1$*, where* $I_{D'}$ *the defining ideal*

sheaf of $D \subset U$.
(2) There is a line bundle E on D' such that $c_1(E) = 0$ and $f^*(L|_U) \cong (L' \otimes \pi^* E)|_{U'}$ hold.
(3) The groups $\mathrm{Pic}\,(U)$ and $\mathrm{Pic}\,(D)$ are isomorphic.

Proof of Proposition 3.3.1. (1) We intrinsically use Rossi's theorem [R, Theorem 3]. Here we remark that, from the assumption that $\mathcal{O}_X(-D)|_D$ is ample, U' is a strongly pseudoconvex domain. Thus, from Rossi's theorem, it turns out that there exists an ideal sheaf $J \subset \mathcal{O}_{U'}$ satisfying the condition that (i) $V(J) \subset D' \cup \{p_1, p_2, \cdots .p_l\}$ for some finitely many points $p_1, p_2, \cdots, p_l \in U' \setminus D'$, where $V(J) \subset U'$ stands for the zero set of the ideal sheaf J, and that (ii) the natural map $H^1(U', \mathcal{O}_{U'}) \to H^1(U', \mathcal{O}_{U'}/J)$ is injective. Here we remark that $H^1(U', \mathcal{O}_{U'}/J) = H^1(D', \mathcal{O}_{U'}/J)$ holds. It is because the condition (i) and the fact that the first sheaf cohomology vanishes on the zero-dimensional sets $p_1, p_2, \cdots, p_l$.

Let us denote by $I_{D'}$ the defining ideal sheaf of D', by I_{p_j} the defining ideal sheaf of p_j for $1 \leq j \leq l$, and by $\hat{J}$ the ideal sheaf $I_{p_1} I_{p_2} \cdots I_{p_l} I_{D'}$. By Hilbert's Nullstellensatz, there exists an integer n such that $\hat{J}^n \subset J$ holds. Thus the natural map $H^1(U', \mathcal{O}_{U'}) \to H^1(U', \mathcal{O}_{U'}/J)$ is decomposed into the composition of two natural maps $H^1(U', \mathcal{O}_{U'}) \to H^1(U', \mathcal{O}_{U'}/\hat{J}^n)$ and $H^1(U', \mathcal{O}_{U'}/\hat{J}^n) \to H^1(U', \mathcal{O}_{U'}/J)$. From the condition (ii), it turns out that the map $H^1(U', \mathcal{O}_{U'}) \to H^1(U', \mathcal{O}_{U'}/\hat{J}^n)$ is also injective, and since $H^1(U', \mathcal{O}_{U'}/\hat{J}^n) = H^1(D', \mathcal{O}_{U'}/\hat{J}^n) = H^1(D', \mathcal{O}_{U'}/I_{D'}^n)$ holds, this proves the first assertion.

(2) The projection $\pi\colon U' \to D$ and the injection $i\colon D' \to U'$ induce the maps $\pi^*\colon H^1(D', \mathcal{O}_{D'}) \to H^1(U', \mathcal{O}_{U'})$ and $i^*\colon H^1(U', \mathcal{O}_{U'}) \to H^1(D', \mathcal{O}_{D'})$, respectively. Since $\pi \circ i = \mathrm{id}_{D'}$, π^* is injective.

$$
\begin{array}{ccccc}
H^1(U', \mathcal{O}_{U'}) & \xrightarrow{\alpha} & H^1(U', \mathcal{O}_{U'}^*) & \xrightarrow{\delta} & H^2(U', \mathbb{Z}) \\
\uparrow{\scriptstyle \pi^*} & \circlearrowleft & \uparrow{\scriptstyle \pi^*} & & \\
H^1(D', \mathcal{O}_{D'}) & \xrightarrow{\beta} & H^1(D', \mathcal{O}_{D'}^*) & &
\end{array}
$$

We first check that $f^*(L|_U) \otimes L'|_{U'}^{-1}$ is topologically trivial line bundle. Indeed, $(f^*(L|_U) \otimes L'|_{U'}^{-1})|_{D'}$ is the trivial bundle and $i \circ \pi$ is homotopic to $\mathrm{id}_{U'}$. Thus we conclude that $\delta(f^*(L|_U) \otimes L'|_{U'}^{-1}) = 0$ and we can take an element $\xi \in H^1(U', \mathcal{O}_{U'})$ satisfying $\alpha(\xi) = f^*(L|_U) \otimes L'|_{U'}^{-1}$. When ξ lies in the image of π^*, we can take an element $\eta \in H^1(D', \mathcal{O}_{D'})$ such that $\pi^*(\eta) = \xi$

holds. In this case, $f^*(L|_U) \otimes L'|_{U'}^{-1} = \pi^*\beta(\eta)$ holds and since $\beta(\eta)$ is a flat line bundle, $f^*(L|_U) \otimes L'|_{U'}^{-1}$ is also a flat line bundle.

Thus all we have to do is showing that the inequality $\dim H^1(U', \mathcal{O}_{U'}) \leq \dim H^1(D', \mathcal{O}_{D'})$ holds. Let us consider the short exact sequence $0 \to I_{D'}^l/I_{D'}^{l+1} \to \mathcal{O}_{U'}/I_{D'}^{l+1} \to \mathcal{O}_{U'}/I_{D'}^l \to 0$ for $l \geq 1$. Then it follows that the natural map $H^1(U', \mathcal{O}_{U'}/I_{D'}^{l+1}) \to H^1(U', \mathcal{O}_{U'}/I_{D'}^l)$ is injective. It is because $H^1(U', I_{D'}^l/I_{D'}^{l+1}) = H^1(U', I_{D'}^l \otimes (\mathcal{O}_{U'}/I_{D'})) = H^1(D', \mathcal{O}_{D'}(-lD'|_{D'}))$ vanishes for each $l \geq 1$, since $\mathcal{O}_{D'}(-K_{D'} - lD'|_{D'}) = \mathcal{O}_{D'}(-K_{D'} - D'|_{D'}) \otimes \mathcal{O}_{D'}(-(l-1)D'|_{D'})$ is nef and big from the assumption. From this combined with the injection in Lemma 3.3.1 (1), it holds that the natural map $H^1(U', \mathcal{O}_{U'}) \to H^1(U', \mathcal{O}_{U'}/I_{D'}) = H^1(D', \mathcal{O}_{D'})$ is injective, and thus we obtain the inequality $\dim H^1(U', \mathcal{O}_{U'}) \leq \dim H^1(D', \mathcal{O}_{D'})$.

(3) By the same argument in the proof of Lemma 3.3.1 (2), it can be shown that the restriction map $\mathrm{Pic}\,(U') \to \mathrm{Pic}\,(D')$ is the inverse map of $\pi^*\colon \mathrm{Pic}\,(D') \to \mathrm{Pic}\,(U')$. □

Remark 3.3.2. When $\mathcal{O}(K_D)$ is semi-negative, we can prove $H^1(U', \mathcal{O}_{U'}) \cong H^1(D', \mathcal{O}_{D'})$ more shortly. Let us consider the short exact sequence $0 \to I_{D'} \to \mathcal{O}_{U'} \to \mathcal{O}_{U'}/I_{D'} \to 0$ and the induced exact sequence $H^1(U', I_{D'}) \to H^1(U', \mathcal{O}_{U'}) \to H^1(D', \mathcal{O}_{D'}) \to H^2(U', I_{D'})$. By the assumption that $\mathcal{O}_D(-K_D) = \mathcal{O}_U(-K_U - D)|_D$ is semi-positive and by Ohsawa's theorem [O, 4.5], it follows that the cohomology group $H^p(U', I_{D'})$ vanishes for all $p > 0$. Thus $H^1(U', \mathcal{O}_{U'}) \cong H^1(D', \mathcal{O}_{D'})$ holds.

Remark 3.3.3. In the above proof of Theorem 3.1.2, we compared the singular Hermitian metric of L with that of L' around the tubular neighborhoods of the divisors. By using this technique, it turns out to be clear that the metric $e^{-\varphi_L}$ we constructed above is a minimal singular metric. Moreover, $\varphi_{L'}$ in the above proof of Theorem 3.1.2 can be taken as in Theorem 3.2.2 when $A|_D$ is ample, and thus we can conclude that the minimal singular metric we constructed has just the same form as the metric in Theorem 3.2.2 around D (up to smooth harmonic function). This means that we here determined a minimal singular metric of L around D by only using equilibrium metrics of $tA|_D + (1-t)L|_D$ for $0 \leq t \leq 1$ in the above proof in this case.

When L in Theorem 3.1.2 satisfies that $L|_D$ is semi-positive, we can say that L is also semi-positive.

Corollary 3.3.4. *Let X, D, L be those in Theorem 3.1.2. When $L|_D$ is semi-positive, L is also semi-positive.*

Proof. We use notations in the proof of 3.1.2. By the proof of Theorem 3.1.3, it is clear that we can choose smooth $h_{L'}$ when $L|_D$ is semi-positive. We define φ_L as the function $M(\varphi_A + \log|f_D|^2, (f^{-1})^*\varphi_{L'})$ (instead of $\max\{\varphi_A + \log|f_D|^2, (f^{-1})^*\varphi_{L'}\}$) on each $W_j \cap U$, where M is a regularized max function (see [D1, §5.E] for the definition). Then $\{e^{-\varphi_L}\}$ glues up to define a smooth Hermitian metric of L with semi-positive curvature. □

We here remark that the idea to use a regularized max function instead of the function "max" is pointed out by Prof. Shin-ichi Matsumura.

3.4 Some examples

3.4.1 Nef and big line bundles with no locally bounded minimal singular metrics

One can obtain the following corollary immediately from Theorem 3.1.3.

Corollary 3.4.1. *Let X be a smooth projective variety, L a nef line bundle over X and let A be an ample line bundle over X. Then a minimal singular metric of L is locally bounded if and only if a minimal singular metric of $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$ over $\mathbb{P}(A \oplus L)$ is locally bounded.*

We remark that the line bundle $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$ above is nef and big ([Laz1, 2.3.2]).

Example 3.4.2. Let (X, L) be these in Example 1.7 of [DPS94], which are defined as the relative hyperplane bundle on $X = \mathbb{P}(E)$, where E is a vector bundle defined over an elliptic curve C given by the non-spitting extension $0 \to \mathcal{O}_C \to E \to \mathcal{O}_C \to 0$. In this example, L is nef, not big, and possesses no locally-bounded minimal singular metric. Then we can conclude that the nef and big line bundle $\mathcal{O}_{\mathbb{P}(A\oplus L)}(1)$ defined on $\mathbb{P}(L \oplus A)$ for some ample line bundle A on X also has no locally-bounded minimal singular metric. We remark that the similar example is introduced in [BEGZ, 5.4], [F, 5.2].

3.4.2 Zariski's and Mumford's examples

We can apply Theorem 3.1.2 to Zariski's and Mumford's examples [Laz1, 2.3.A].

Example 3.4.3. Let $C \subset \mathbb{P}^2$ be a smooth elliptic curve and let $p_1, p_2, \ldots, p_{12} \in C$ be twelve general points. We define X as the blow up of $\mathbb{P}^2$ at these twelve points. We denote by H the pulled back divisor of X of a line in $\mathbb{P}^2$ and by D the strict transform of C. In this case, since $(D^2) = 9 - 12 = -3$ and the genus $g(D) = 1$, we can apply Grauert's theorem [Gr, Satz 7] (see §1 here) to see that $X, L = \mathcal{O}_X(H+D)$, and D satisfy the condition of Theorem 3.1.2. Moreover, for $L|_D$ is semi-positive, we can apply Corollary 3.3.4. Thus L is semi-positive.

There is a generalization of this Zariski's example pointed out by Mumford (see also [Laz1, 2.3.1]). Let X be a smooth projective surface, A a very ample divisor on X, and let $D \subset X$ be a curve with $(D^2) < 0$ holds and the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(D)$ is injective. We denote by a, b the positive number $(A.D), -(D^2)$, respectively. Then the line bundle $L = \mathcal{O}_X(bA + aD)$ is nef, big, satisfying $D \subset \mathrm{Bs}\,|L^{\otimes m}|$ for all $m \geq 1$, and there exists a positive integer p_0 such that $|L^{\otimes m} \otimes \mathcal{O}_X(-p_0 D)|$ is generated by global sections for all $m \geq 1$. These X, L, and D satisfy the condition of Corollary 3.3.4 also in this situation when D is smooth, b is sufficiently large, and $p_0 = 1$. Thus, such L is semi-positive, too.

4

On the minimality of canonically attached singular Hermitian metrics on certain nef line bundles

4.1 Introduction

In this chapter, we mainly consider a topologically trivial line bundle on a surface which is defined by a smooth embedded curve with a neighborhood of non-trivial complex structure (rigorously speaking, when the curve is *of finite type* in the sense of [U, p. 589]. See Definition 1.2.2). The goal of this chapter is to determine a minimal singular metric of such a line bundle. The main theorem is as follows.

THEOREM 4.1.1. *Let X be a smooth complex surface and $C \subset X$ be an embedded smooth compact complex curve. Assume $(C^2) = 0$ and the pair (C, X) is of finite type. Then the singular Hermitian metric $|f_C|^{-2}$ on $\mathcal{O}_X(C)$ has minimal singularities, where $f_C \in H^0(X, \mathcal{O}_X(C))$ is a section whose zero divisor is C. Especially, $\mathcal{O}_X(C)$ is nef, however it admits no smooth Hermitian metric with semi-positive curvature.*

We can apply Theorem 4.1.1 to a line bundle defined by the section of a certain ruled surface. We are interested in such a situation since it includes the example of Demailly, Peternell, and Schneider [DPS94, 1.7], which is constructed as a nef line bundle which admits no smooth Hermitian metric with semi-positive curvature. We determine a minimal singular metric of such a line bundle, and in particular give a generalization of their result.

COROLLARY 4.1.2 (Example 4.2.2). *Let C be a smooth projective curve and L be a topologically trivial line bundle on C such that $H^1(C, L) \neq 0$. For* *example, this condition is satisfied for all topologically trivial line bundle when* C *is a curve of genus greater than* 1. *Take a non-zero element* $\xi \in H^1(C, L)$ *and let* E *be the rank-two vector bundle on* C *which appears in the non-splitting exact sequence* $0 \to L \to E \to \mathcal{O}_C \to 0$ *corresponding to the class* ξ. *Denote by* X *a ruled surface* $\mathbb{P}(E)$ *over* C *and by* D *the section of the map* $\mathbb{P}(E) \to C$. *Then a singular Hermitian metric* $|f_D|^{-2}$ *on* $\mathcal{O}_X(D)$ *has minimal singularities, where* $f_D \in H^0(X, \mathcal{O}_X(D))$ *is a section whose zero divisor is* D. *Especially the line bundle* $\mathcal{O}_X(D)$ *is nef, however it admits no smooth Hermitian metric with semi-positive curvature.*

The example in [DPS94, 1.7] is a special case of Example 4.2.2, in which C is an elliptic curve and $L = \mathcal{O}_C$. We remark that they give a minimal singular metric of $\mathcal{O}_X(D)$ for this case by determining all singular Hermitian metrics on this line bundle with semi-positive curvature. Our proof is based on a completely different idea.

We are also interested in minimal singular metrics of strictly nef and non semi-ample line bundles, where we say a line bundle L on a variety X is *strictly nef* if the intersection number $(L.C)$ is positive for all compact curve $C \subset X$. The following examples are strictly nef and non semi-ample line bundles. Example 4.1.3 (1) is so-called Mumford's example, which admits a smooth Hermitian metric with semi-positive curvature. Example 4.1.3 (2) is given by Fujino. He proposed a question whether this example admits a smooth Hermitian metric with semi-positive curvature [F, 5.10].

Example 4.1.3.

(1) ([Ha, 10.6]) Let $\widetilde{C}$ be a smooth compact curve of genus greater than 1. Mumford showed that there exists a rank-two vector bundle F on $\widetilde{C}$ such that its degree is equal to zero and its symmetric powers $S^m(F)$ are stable for all $m \geq 1$. Let Y be the total space of a projective space bundle $\mathbb{P}(F)$ and L_Y be the relative hyperplane bundle $\mathcal{O}_Y(1)$. Then L_Y is a strictly nef and non semi-ample line bundle. In this situation, we can construct a smooth Hermitian metric h_{L_Y} on L_Y with semi-positive curvature as an induced metric from the unitary-flat metric of the stable bundle F (see Remark 4.3.2).

(2) (A variant of [F, 5.9].) Fix a smooth compact curve $\widetilde{C}$ of genus greater than 1. Since $H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}}) \neq 0$, we can take a non-zero element $\xi \in H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}})$. Let E be the rank-two vector bundle on $\widetilde{C}$ which appears in the non-splitting exact sequence $0 \to \mathcal{O}_{\widetilde{C}} \to E \to \mathcal{O}_{\widetilde{C}} \to 0$ corresponding to the class ξ. Denote by $\widetilde{X}$ the ruled surface $\mathbb{P}(E)$ over $\widetilde{C}$, by $\widetilde{D}$ the section of the map $\mathbb{P}(E) \to \widetilde{C}$, and by $\widetilde{Y}$ the fiber product $\widetilde{X} \times_{\widetilde{C}} Y$, where Y is that in Example

4.1.3 (1).

$$\begin{array}{ccc} \widetilde{Y} = \widetilde{X} \times_{\widetilde{C}} Y & \xrightarrow{p_1} & \widetilde{X} = \mathbb{P}(E) \\ \Big\downarrow{\scriptstyle p_2} & & \Big\downarrow \\ Y & \longrightarrow & \widetilde{C} \end{array}$$

In this situation, it can be shown that the line bundle $\widetilde{L} := \mathcal{O}_{\widetilde{Y}}(\widetilde{D} \times_{\widetilde{C}} Y) \otimes p_2^* L_Y$ is a strictly nef and non semi-ample line bundle, where $p_2 \colon \widetilde{Y} \to Y$ is the second projection and L_Y is that in Example 4.1.3 (1).

We also determine a minimal singular metric of $\widetilde{L}$ in Example 4.1.3 (2), and in particular give the answer to the above Fujino's question.

Corollary 4.1.4. *Let $\widetilde{L} = \mathcal{O}_{\widetilde{Y}}(\widetilde{D} \times_{\widetilde{C}} Y) \otimes p_2^* L_Y$ be that in Example 4.1.3 (2), f a global section of $\mathcal{O}_{\widetilde{Y}}(\widetilde{D} \times_{\widetilde{C}} Y)$ whose zero divisor is $\widetilde{D} \times_{\widetilde{C}} Y$, and h_{L_Y} be a smooth Hermitian metric on L_Y with semi-positive curvature (see Remark 4.3.2 for the existence of such a metric on L_Y). Then the metric $|f|^{-2} \otimes (p_2^* h_{L_Y})$ is a minimal singular metric of $\widetilde{L}$. In particular, $\widetilde{L}$ is* not *semi-positive.*

We will prove this corollary for more general situation (see Theorem 4.3.1).

The organization of this chapter is as follows. In §4.2, we prove Theorem 4.1.1, and apply it on some examples. In §4.3, we prove Corollary 4.1.4 in more general form (Theorem 4.3.1).

4.2 Proof of the main theorem in §4 and some examples

Here we prove Theorem 4.1.1 as an application of Ueda's Theorem 1.2.3.

Lemma 4.2.1. *Let X be a smooth complex surface, $C \subset X$ an embedded smooth compact complex curve, $f_C \in H^0(X, \mathcal{O}_X(C))$ a section whose zero divisor is C, and h be a singular Hermitian metric of $\mathcal{O}_X(C)$ with semi-positive curvature. Assume $(C^2) = 0$, $\mathrm{type}\,(C, X) < \infty$, and that the local weight functions of h are less singular than the psh function $\log |f_C|^2$. Then there exists a positive number M such that $h = M|f_C|^{-2}$ holds on a neighborhood of C in X.*

Proof. Let us fix a sufficiently small neighborhood V of C in X and consider a function $\Psi := -\log|f_C|_h^2$ defined on $V \setminus C$. In the following, we sometimes restrict ourselves to a sufficiently small open neighborhood of each point of V. On the open neighborhood, we fix a local trivialization of $\mathcal{O}_X(C)$, and by using a local weight function φ of h, we denote $|f_C|_h^2$ locally by $|f_C|^2 e^{-\varphi}$, where we regard f_C as a locally defined holomorphic function corresponding to the section f_C.

First we check that this function is psh. By using the above notation, it is clear that the equation $\Psi = \varphi - \log|f_C|^2$ holds locally. Since $\log|f_C|^2$ is harmonic on $V \setminus C$ and φ is psh by assumption, we can conclude that Ψ is a psh function.

Next, we evaluate the divergence of Ψ near C. Since $\log|f_C|^2 \prec_{\text{sing}} \varphi$, Ψ is bounded from below. Thus all we have to do is to evaluate the divergence of Ψ to $+\infty$. We use local coordinates system (z, x) of V such that x is a local coordinate of C and the equation $\{z = 0\} = C$ holds locally. Without loss of generality, we may assume $f_C(z, x) = z$ holds on this locus. For φ is a psh function, it is locally bounded from above. Thus

$$\Psi(z, x) = \varphi(z, x) - \log|z|^2 \leq C - \log|z|^2 = o(1/|z|^{1/2}) \text{ as } |z| \to 0.$$

holds. Therefore we can apply Theorem 1.2.3 and thus, after making the neighborhood V smaller, we can conclude that Ψ is a constant map, which shows the lemma. □

Proof of Theorem 4.1.1. Let h be a singular Hermitian metric of $\mathcal{O}_X(C)$ with semi-positive curvature. We denote by $\widetilde{h}$ the new metric $\min\{h, |f_C|^{-2}\}$. Clearly we obtain that the local weight function of $\widetilde{h}$ is $\max\{\log|f_C|^2, \varphi\}$, where φ is the local weight function of h. Since this function is the maximum of two psh functions, $\widetilde{h}$ also has the semi-positive curvature. For $\log|f_C|^2 \prec_{\text{sing}} \max\{\log|f_C|^2, \varphi\}$, we can apply Lemma 4.2.1 and thus we obtain that $\log|f_C|^2 \sim_{\text{sing}} \max\{\log|f_C|^2, \varphi\} \succ_{\text{sing}} \varphi$ holds. □

In the rest of this section, we give some examples of nef but not semi-positive line bundles over smooth projective varieties as applications of Theorem 4.1.1.

Example 4.2.2. Let C be a smooth projective curve of positive genus and E be the rank-2 vector bundle on C which appears in the exact sequence $0 \to L \to E \to \mathcal{O}_C \to 0$, where L is a topologically trivial line bundle on C

such that $H^1(C, L) \neq 0$. Let us define that X is the ruled surface $\mathbb{P}(E)$ over C and $D := \mathbb{P}(\mathcal{O}_C) \subset \mathbb{P}(E)$ be the section of the map $\mathbb{P}(E) \to C$. We denote by $\pi\colon X \to C$ the canonical surjection. From simple computations, we can conclude that the nomal bundle $N_{D/X}$ is isomorphic to $(\pi|_D)^* L^{-1}$ and thus it is topologically trivial. Therefore we can define Ueda classes for the pair (D, X). In our case, we obtain that the first Ueda class $u_1(D, X) \in H^1(D, N_{D/X}^{-1})$ coincides with cohomology class $\{E\} \in \mathrm{Ext}^1(\mathcal{O}_C, L) = H^1(C, L)$ via the isomorphism $\pi|_D\colon D \to C$ (see [Ne, Proposition 7.6]). Thus the first Ueda class $u_1(D, X)$ vanishes if and only if $E = \mathcal{O}_C \oplus L$, and in this case, the line bundle $\mathcal{O}_X(D)$ is semi-positive.

Assume that the exact sequence $0 \to L \to E \to \mathcal{O}_C \to 0$ is non-splitting. In this case, the first Ueda class $u_1(D, X)$ does not vanish and thus $\mathrm{type}\,(D, X) = 1 < \infty$ and we can apply Theorem 4.1.1. As a result, we can conclude that $|f_D|^{-2}$ defines a minimal singular metric of $\mathcal{O}_X(D)$, where $f_D \in H^0(X, \mathcal{O}_X(D))$ is a section whose zero divisor is D.

When C is a smooth elliptic curve and $L = \mathcal{O}_C$, the minimal singularity of the above singular Hermitian metric $|f_D|^{-2}$ is known by Demailly, Peternell, and Schneider [DPS94, 1.7]. We also remark that, in this case, they more precisely show that for every singular Hermitian metric h of $\mathcal{O}_X(D)$ with semi-positive curvature, there exists a positive constant M such that $h = M|f_D|^{-2}$ holds.

Example 4.2.3. Let C be a smooth compact curve of genus 2 and Y be its Jacobian. Fix two points $p, q \in C$ conjugate to each other by the hyperelliptic involution. We denote by X the blow-up of Y at p and q and by D the the strict transformation of C. According to Neeman [Ne, 10.5], $\mathrm{type}\,(D, X) = 1 < \infty$ holds for this D and X. Therefore we can apply Theorem 4.1.1 and conclude that $|f_D|^{-2}$ defines a minimal singular metric of $\mathcal{O}_X(D)$, where $f_D \in H^0(X, \mathcal{O}_X(D))$ is a section whose zero divisor is D.

4.3 Strictly nef line bundle which admits no smooth Hermitian metric with semi-positive curvature

In this section, we prove the following

Theorem 4.3.1. *Let $\widetilde{C}, Y$, and L_Y be those in Example 4.1.3 (1), $\widetilde{X} \to \widetilde{C}$ be a smooth holomorphic fiber bundle over $\widetilde{C}$ ($\widetilde{X}$ need not to be that in Example 4.1.3 (2) and also need not to be compact) and $L_{\widetilde{X}}$ is a pseudo-effective line bundle on $\widetilde{X}$. Denote by $\widetilde{Y}$ the fiber product $\widetilde{X} \times_{\widetilde{C}} Y$ and by $\widetilde{L}$ the line bundle $p_1^* L_{\widetilde{X}} \otimes p_2^* L_Y$, where $p_1 \colon \widetilde{Y} \to \widetilde{X}$ and $p_2 \colon \widetilde{Y} \to Y$ is the first and second projection, respectively. Assume that $L_{\widetilde{X}}$ admits a minimal singular metric $h_{L_{\widetilde{X}}}$. Then the metric $(p_1^* h_{L_{\widetilde{X}}}) \otimes (p_2^* h_{L_Y})$ is a minimal singular metric of $\widetilde{L}$, where h_{L_Y} is that in Example 4.1.3 (1) (see Remark 4.3.2).*

Here a "smooth holomorphic fiber bundle over $\widetilde{C}$" means a holomorphic map from smooth variety onto $\widetilde{C}$ which satisfies that, for each point $x \in \widetilde{C}$, there exists an open neighborhood $\widetilde{U} \subset \widetilde{C}$ of x such that the restriction to the preimage of $\widetilde{U}$ can be regarded as the natural projection from the product space of $\widetilde{U}$ and some manifold to $\widetilde{U}$. We remark that, if $L_{\widetilde{X}}$ in the above theorem is nef, then $\widetilde{L}$ is strictly nef. From this Theorem 4.3.1 and Corollary 4.1.2, we can deduce Corollary 4.1.4.

We first review on the metric of Mumford's example 4.1.3 (1).

Remark 4.3.2. The line bundle L_Y in Example 4.1.3 (1) admits a smooth Hermitian metric with semi-positive curvature. Indeed, we can construct such a metric as follows. First we remark that, by the Narasimhan-Seshadri theorem [NS], there exists a representation $\rho \colon \pi_1(\widetilde{C}) \to U(2)$ such that the dual F^* of F is isomorphic to the quotient $(\mathbb{H} \times \mathbb{C}^2)/\pi_1(\widetilde{C})$, where $\pi_1(\widetilde{C})$ acts on $\mathbb{H} \times \mathbb{C}^2$ via ρ. Here we denote by $\mathbb{H}$ an upper half-plane $\{x \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. Thus the fiber-wise Euclidean metric induces a flat metric h_{F^*} of F^* and, on every open set $U \subset \widetilde{C}$, we can choose nice local frame (s_1^*, s_2^*) of F^* such that

$$|s_1^*|^2_{h_{F^*}} \equiv 1, \ |s_2^*|^2_{h_{F^*}} \equiv 1, \ \langle s_1^*, s_2^* \rangle_{h_{F^*}} \equiv 0$$

holds on U. We use the function

$$(w, x) \mapsto [s_1^*(x) + w s_2^*(x)] \in \mathbb{P}(F)$$

as a local coordinates system of $\mathbb{P}(F)$. Then the fiber-wise Fubini-Study metric defines a smooth Hermitian metric h_{L_Y}, whose curvature tensor can be computed as

$$\sqrt{-1}\Theta_{h_{L_Y}} = \sqrt{-1}\partial\overline{\partial} \log |s_1^*(x) + w s_2^*(x)|^2_{F^*} = \frac{\sqrt{-1} dw \wedge d\overline{w}}{(1 + |w|^2)^2}.$$

Therefore, this smooth Hermitian metric h_{L_Y} clearly has a semi-positive curvature tensor.

Proof of Theorem 4.3.1. We fix a singular Hermitian metric $h_{\widetilde{L}}$ of $\widetilde{L}$ with semi-positive curvature and show the existence of a constant $M_{\widetilde{W}}$ for each sufficiently small open set $\widetilde{W} \subset \widetilde{Y}$ such that $(p_1^* h_{L_{\widetilde{X}}}) \otimes (p_2^* h_{L_Y}) \leq M_{\widetilde{W}} h_{\widetilde{L}}$ holds on $\widetilde{W}$.

Let us fix a smooth Hermitian metric h_∞ of $L_{\widetilde{X}}$. We here remark that the tensor product $(p_1^* h_\infty) \otimes (p_2^* h_{L_Y})$ (or $p_1^* h_\infty p_2^* h_{L_Y}$, for simplicity) defines a smooth Hermitian metric of $\widetilde{L}$. Therefore, there exists a quasi-psh function $\chi\colon \widetilde{Y} \to \mathbb{R}$ such that

$$h_{\widetilde{L}} = p_1^* h_\infty p_2^* h_{L_Y} e^{-\chi}$$

holds. Let us describe our situations in the words of local weight functions. Here we use a local coordinates system (z, w, x) of $\widetilde{Y}$, where x is a local coordinate of an open set $U \subset \widetilde{C}$, (z, x) is a local coordinates system of $\widetilde{X}$ where z is a fiber coordinate, and (w, x) is a local coordinates system of Y just as in Remark 4.3.2. Let φ_∞ be the local weight function of h_∞ and $\varphi_{\widetilde{L}}$ be the local weight function of $h_{\widetilde{L}}$. Then the equation

$$\varphi_{\widetilde{L}}(z, w, x) = \varphi_\infty(z, x) + \log(1 + |w|^2) + \chi(z, w, x)$$

holds. Since $\sqrt{-1}\partial\overline{\partial}\varphi_{\widetilde{L}} \geq 0$ holds from the assumption, the inequality

$$\sqrt{-1}\partial_{z,x}\overline{\partial}_{z,x}\varphi_{\widetilde{L}}(z, w_0, x) \geq 0$$

also holds for each point $w_0 \in p^{-1}(U)$, where $p\colon Y \to \widetilde{C}$ is the natural projection and $\partial_{z,x}$ (resp. $\overline{\partial}_{z,x}$) is the operator ∂ (resp. $\overline{\partial}$) with w_0 fixed. Thus, it can be said that the function $\chi|_{\widetilde{\pi}^{-1}(U)\times\{w_0\}}\colon \widetilde{\pi}^{-1}(U) \to \mathbb{R}; (z, x) \mapsto \chi(z, w_0, x)$ is a φ_∞-psh function (i.e. $\sqrt{-1}\partial_{z,x}\overline{\partial}_{z,x}(\chi|_{\widetilde{\pi}^{-1}(U)\times\{w_0\}} + \varphi_\infty) \geq 0$ holds) on $\widetilde{\pi}^{-1}(U)$, where $\widetilde{\pi}\colon \widetilde{X} \to \widetilde{C}$ is the natural surjection. Therefore the function

$$\widetilde{\chi}(z, x) := \max_{w \in p^{-1}(x)} \chi(z, w, x)$$

is also a φ_∞-psh function on $\widetilde{\pi}^{-1}(U)$ (Since χ is quasi-psh and thus it is locally bounded from above, we can use the same argument as the proof of [DPS00, 1.5]). As this function $\widetilde{\chi}$ can be regarded as a global function on $\widetilde{X}$, it is clear that the singular Hermitian metric $h_2 := h_\infty e^{-\widetilde{\chi}}$ of $L_{\widetilde{X}}$ is well-defined singular Hermitian metric with semi-positive curvature. Therefore, from the assumption, there exists a positive constant M_W for each sufficiently small open set $W \subset \widetilde{X}$ such that $M_W h_\infty e^{-\widetilde{\chi}} \geq h_{L_{\widetilde{X}}}$ holds on W. Thus we obtain an inequality

$$p_1^* h_{L_{\widetilde{X}}} p_2^* h_{L_Y} \leq M_W p_1^*(h_\infty e^{-\widetilde{\chi}}) p_2^* h_{L_Y} \leq M_W p_1^* h_\infty p_2^* h_{L_Y} e^{-\chi} = M_W h_{\widetilde{L}}$$

holds on $p_1^{-1}(W)$ and therefore we can conclude that the metric $p_1^* h_{L_{\widetilde{X}}} p_2^* h_{L_Y}$ is a minimal singular metric of $\widetilde{L}$. □

5
Toward a higher codimensional Ueda theory

5.1 Introduction

In this chapter, we propose a codimension two analogue of Ueda's theory. Namely we shall describe a sufficient condition for the line bundle $\mathcal{O}_X(S)$ to be flat on a neighborhood of C in X, where S is a smooth hypersurface of a complex manifold X and C is a compact smooth hypersurface of S.

Let X be a complex manifold, S a smooth hypersurface of X, and C be a smooth compact hypersurface of S. Assume that the normal bundle $N_{S/X}$ is flat around C. For such a triple (C, S, X), we define a new obstruction class $u_{n,m}(C, S, X) \in H^1(C, N_{S/X}|_C^{-n} \otimes N_{C/S}^{-m})$ for each $n \geq 1, m \geq 0$ as a codimension two version of Ueda's obstruction classes (see §3 for the definition of $u_{n,m}(C, S, X)$). These new obstruction classes enjoy the property that $\mathcal{O}_X(S)$ is not flat around C if there exists a pair of integers $n \geq 1$ and $m \geq 0$ such that $u_{n,m}(C, S, X) \neq 0$. By using these obstruction classes, we can describe a sufficient condition for $\mathcal{O}_X(S)$ to be flat around C as follows.

THEOREM 5.1.1. *Let X be a complex manifold, S a smooth hypersurface of X, and C be a smooth compact Kähler hypersurface of S such that $N_{S/X}|_V$ is flat, where V is a sufficiently small neighborhood of C in S. Assume one of the following three conditions holds: (i) $N_{C/S} \in \mathcal{E}_0(C)$ and $N_{S/X}|_C \in \mathcal{E}_0(C)$, (ii) $N_{C/S} = N_{S/X}|_C \in \mathcal{E}_1(C)$, (iii) $N_{S/X}|_C \in \mathcal{E}_0(C)$ and there exists a strongly 1-convex neighborhood V of C in S such that C is the maximal compact analytic subset of V. Further assume that $u_{n,m}(C, S, X) = 0$ holds for all $n \geq 1, m \geq 0$. Then there exists a neighborhood W of C in X such that $\mathcal{O}_X(S)|_W$ is flat.*

Note that, when C is a curve, the condition (*iii*) above is equivalent to the condition that $N_{C/S}$ is negative and $N_{S/X}|_C \in \mathcal{E}_0(C)$ (see §2.2 here for the details). We will prove Theorem 5.1.1 by considering a codimension two analogue of the argument used in the proof of Ueda's theorem [U, Theorem 3].

Let us explain our motivation here. Let X be a smooth projective manifold and S be a hypersuface of X such that the line bundle $\mathcal{O}_X(S)$ is nef. Our original interest is the existence (or non-existence) of smooth Hermitian metrics on $\mathcal{O}_X(S)$ with semi-positive curvature. When the base locus $C := \mathbb{B}(S)$ of the linear system $|S|$ is a hypersurface of X, we gave some sufficient conditions for $\mathcal{O}_X(S)$ to (or not to) admit a smooth Hermitian metric with semi-positive curvature by considering some flatness criteria for $\mathcal{O}_X(S)$ around C in §3 and §4 (Note that §4 is essentially based on Ueda's theory). Especially, in Corollary 3.3.4, we showed that $\mathcal{O}_X(S)$ admits a smooth Hermitian metric with semi-positive curvature when $\mathcal{O}_X(S)$ is flat around C. Now let us consider the case where the codimension of C is greater than one. In this case, we can also apply the same argument as in Corollary 3.3.4 when $\mathcal{O}_X(S)$ is flat around C (see the proof of Corollary 5.1.2 in §5.2 here). Thus we need a flatness criterion for $\mathcal{O}_X(S)$ around C, which is the motivation for considering the situation as in Theorem 5.1.1.

One of the most important applications of Ueda's theory to algebro-geometric situations is on the semi-positivity of the anti-canonical bundle of the blow-up of $\mathbb{P}^2$ at 9 points ([Br] and [U], see also §5 here). As an analogue, the following corollary follows from Theorem 5.1.1.

COROLLARY 5.1.2. *Let $C_0 \subset \mathbb{P}^3$ be a complete intersection of two quadric surfaces of $\mathbb{P}^3$ and let $p_1, p_2, \dots, p_8 \in C_0$ be 8 points different from each other. Assume $\mathcal{O}_{\mathbb{P}^3}(-2)|_{C_0} \otimes \mathcal{O}_{C_0}(p_1+p_2+\cdots+p_8) \in \mathcal{E}_1(C_0)$. Then the anti-canonical bundle of the blow-up of $\mathbb{P}^3$ at $\{p_j\}_{j=1}^8$ is not semi-ample, however admits a smooth Hermitian metric with semi-positive curvature.*

Note that, when $\mathcal{O}_{\mathbb{P}^3}(-2)|_{C_0} \otimes \mathcal{O}_{C_0}(p_1 + p_2 + \cdots + p_8) \in \mathcal{E}_0(C_0)$, the anti-canonical line bundle of the blow-up of $\mathbb{P}^3$ at $\{p_j\}_{j=1}^8$ is semi-ample, and thus it admits a smooth Hermitian metric with semi-positive curvature. When $\mathcal{O}_{\mathbb{P}^3}(-2)|_{C_0} \otimes \mathcal{O}_{C_0}(p_1 + p_2 + \cdots + p_8) \notin \mathcal{E}_0(C_0)$, the stable base locus of the anti-canonical line bundle is the strict transform of C_0 and thus it is not semi-ample (however it is nef). Note also that it is shown by Lesieutre and Ottem that the anti-canonical bundle $-K_X$ of the blow-up of $\mathbb{P}^3$ at very general 8 points is a nef line bundle such that the set of curves C with $-K_X.C = 0$ is countably infinite [LO], and thus it gives an affirmative answer to the question of Totaro [T]. For the details of this example, see §5 here.

The organization of this chapter is as follows. In §5.2, we will give an explanation on some fundamental results which will be needed in the proof of Theorem 5.1.1. In §5.3, the obstruction class $u_{n,m}(C, S, X)$ will be defined for each $n \geq 1, m \geq 0$, and some fundamental properties of them will be shown. In §5.4, Theorem 5.1.1 will be proven. In §5.5, some examples will be given and Corollary 5.1.2 will be proven.

5.2 Some fundamental results

In this subsection, we give some preliminary results needed in the proof of Theorem 5.1.1. We first give the definition of the exceptionality in the sense of Grauert.

Definition 5.2.1 ([Gr], see also [CM, 2.6]). Let C be a compact connected subvariety of a complex manifold S. We say that C is an *exceptional subvariety of S in the sense of Grauert* if there exists a strongly strongly 1-convex neighborhood V of C in S such that C is the maximal compact analytic subset of V.

When S is a smooth surface and C is a smooth curve embedded in S, it is known that C is an exceptional subvariety of S in the sense of Grauert if and only if the normal bundle $N_{C/S}$ is negative ([Lau, 4.9], see also the last of Chapter 2 of [CM]). In order to deal with the situation of (*iii*) in Theorem 5.1.1, we use the following form of Rossi's theorem.

Proposition 5.2.2 (a version of Rossi's theorem [R, 3]). *Let C be a compact connected subvariety of a complex manifold S, and V be a strongly pseudoconvex neighborhood of C in S such that C is the maximal compact analytic subset of V. Then for each coherent sheaf $\mathcal{S}$ on V, there exists an integer $N(\mathcal{S})$ such that the natural map $H^1(V, \mathcal{S}) \to H^1(V, \mathcal{S} \otimes \mathcal{O}_V/I_C^n)$ is injective for each integer $n \geq N(\mathcal{S})$, where I_C is the defining ideal sheaf of $C \subset V$.*

Proposition 5.2.2 can be shown by the same argument as in the proof of Proposition 3.3.1 (1). In the proof of Theorem 5.1.1, we will use the following Lemma 5.2.3, which is a variant of [KS, Lemma 2] (see also [U, Lemma 3]).

Lemma 5.2.3. *Let C be a compact complex manifold embedded in a complex manifold S. Fix a sufficiently small connected neighborhood V of C in S and a sufficiently fine open covering $\{V_j\}$ of V which consists of a finite* *number of open sets. Fix also be a relatively compact open domain $V_0 \subset V$ which contains C. For each flat line bundle E on V, there exists a positive constant $K = K(E)$ depending only on E such that, for each* 1*-cocycle $\alpha = \{(V_{jk}, \alpha_{jk})\}$ of E which can be realized as the coboundary of some* 0*-cochain, there exists a* 0*-cochain $\beta = \{(V_j, \beta_j)\}$ of E such that α is equal to the coboundary $\delta(\beta)$ of β and the inequality*

$$\max_j \sup_{V_0 \cap V_j} |\beta_j| \leq K \cdot \max_{jk} \sup_{V_0 \cap V_{jk}} |\alpha_{jk}|$$

holds.

Whereas [KS, Lemma 2] ([U, Lemma 3]) is on the existence of such a constant $K = K(E)$ as in Lemma 5.2.3 for a flat line bundle E on a compact complex manifold, Lemma 5.2.3 is on the existence of $K = K(E)$ for E defined on a neighborhood of a compact complex manifold, which is not compact. However, since V can be covered by finitely many sufficiently fine open subsets, Lemma 5.2.3 can be showed by the same argument as in [KS, §6]. As the proof of Theorem 1.2.4 in [U], the proof of Theorem 5.1.1 is also based on the following lemmata.

Lemma 5.2.4 ([U, Lemma 4]). *Let C be a compact complex manifold. Fix a sufficiently fine open covering $\{U_j\}$ of C which consists of a finite number of open sets. Then there exists a positive constant K such that, for each flat line bundle E on C, the following condition holds: For each* 1*-cocycle $\alpha = \{(U_{jk}, \alpha_{jk})\}$ such that the element of $H^1(C, E)$ defined by α is the trivial one, there exists a* 0*-cochain $\beta = \{(U_j, \beta_j)\}$ of E such that $\delta(\beta) = \alpha$ and*

$$d(\mathcal{O}_C, E) \cdot \max_j \sup_{U_j} |\beta_j| \leq K \cdot \max_{jk} \sup_{U_{jk}} |\alpha_{jk}|$$

hold.

Lemma 5.2.5 ([S], see also [U, Lemma 5]). *Let $\{\varepsilon_\lambda\}_{\lambda \geq 1}$ be a series of positive numbers satisfying the following two conditions: the condition that there exists a positive number α such that, for each λ, $\varepsilon_\lambda < (2\lambda)^\alpha$ holds, and the condition that $\varepsilon_{\nu-\mu}^{-1} \leq \varepsilon_\nu^{-1} + \varepsilon_\mu^{-1}$ holds for each $\nu > \mu$. Then for each power series*

$$f(Z) = \sum_{\lambda=2}^{\infty} a_\lambda Z^\lambda$$

with a positive radius of convergence, the formal power series $Z+\sum_{\lambda=2}^{\infty} c_\lambda Z^\lambda$ *uniquely determined by*

$$\sum_{\lambda=2}^{\infty} \varepsilon_{\lambda-1}^{-1} c_\lambda Z^\lambda = f\left(Z+\sum_{\lambda=2}^{\infty} c_\lambda Z^\lambda\right)$$

has a positive radius of convergence.

5.3 The definition and some basic properties of the class $u_{n,m}(C,S,X)$

5.3.1 The definition of the class $u_{n,m}(C,S,X)$ and $\mathrm{type}\,(C,S,X)$

Let X be a complex manifold, S a smooth hypersurface of X, and C be a smooth compact Kähler hypersurface of S such that $N_{S/X}|_V$ is flat, where V is a sufficiently small neighborhood of C in S. Fix a sufficiently small tubular neighborhood W of C such that $W\cap S=V$. Fix also a sufficiently fine open covering $\{U_j\}$ of C, $\{V_j\}$ of V, and W_j of W such that $W_j\cap S=V_j$ and $V_j\cap C=U_j$ hold. Let x_j be a coordinates system of U_j, z_j a defining holomorphic function of U_j in V_j, and w_j be a defining holomorphic function of V_j in W_j. Extending x_j and z_j to W_j in a holomorphic way, we use a system (x_j,z_j,w_j) as a coordinates system of W_j. Let t_{jk} be a transition function of $N_{S/X}|_V$ on V_{jk}: i.e. $N_{S/X}|_V=\{(V_{jk},t_{jk})\}$. As $N_{S/X}|_V$ is flat, t_{jk} can be selected as a constant function on V_{jk} with modulus 1. From the same argument as in [U, §2], it can be said that, without loss of generality, we may assume $(w_j/w_k)|_{V_{jk}}\equiv t_{jk}$ holds. Just from the same reason, we may also assume that $(z_j/z_k)|_{U_{jk}}\equiv s_{jk}$ holds, where s_{jk} is a transition function of $N_{C/S}$ on U_{jk}.

Definition 5.3.1. Let $n\geq 1$ and $m\geq 0$ be integers. A system $\{(W_j,w_j)\}$ is called *of order* (n,m) if, for each j and k, the function $t_{jk}w_k-w_j$ on W_{jk} satisfies that $\mathrm{mult}_{V_{jk}}(t_{jk}w_k-w_j)\geq n+1$ and that $\mathrm{mult}_{U_{jk}}((t_{jk}w_k-w_j)/w_j^{n+1})|_{V_{jk}}\geq m$. This condition is equivalent to the following condition: the coefficient of $z_j^\mu w_j^\nu$ in the Taylor expansion of the function $t_{jk}w_k-w_j$ in the variables z_j and w_j around U_{jk} is equal to zero if $(\nu,\mu)\leq(n,m)$ holds in the lexicographical order, namely $(a,b)\leq(a',b')\overset{\mathrm{def}}{\Leftrightarrow} a<a'$ or "$a=a'$ and $b\leq b'$".

Assume that our system $\{(W_j,w_j)\}$ is of order (n,m). Then the function $t_{jk}w_k$ can be expanded in the variable w_j as follows:

$$t_{jk}w_k=w_j+f_{jk}^{(n+1)}(x_j,z_j)\cdot w_j^{n+1}+f_{jk}^{(n+2)}(x_j,z_j)\cdot w_j^{n+2}+\cdots. \qquad (5.1)$$

Let

$$f_{jk}^{(n+1)}(x_j, z_j) = g_{jk}^{(n+1,m)}(x_j) \cdot z_j^m + g_{jk}^{(n+1,m+1)}(x_j) \cdot z_j^{m+1} + \cdots$$

be the expansion of $f_{jk}^{(n+1)}(x_j, z_j)$ in the variable z_j.

PROPOSITION 5.3.2. *In the above setting, a system* $\{(U_{jk}, g_{jk}^{(n+1,m)})\}$ *enjoys the cocycle condition for the line bundle* $N_{S/X}|_C^{-n} \otimes N_{C/X}^{-m}$.

PROOF. From the equation (5.1),

$$\begin{aligned} t_{jk}^{-n} w_k^{-n} &= (w_j + f_{jk}^{(n+1)}(x_j, z_j) \cdot w_j^{n+1} + f_{jk}^{(n+2)}(x_j, z_j) \cdot w_j^{n+2} + \cdots)^{-n} \\ &= w_j^{-n}(1 + f_{jk}^{(n+1)}(x_j, z_j) \cdot w_j^n + f_{jk}^{(n+2)}(x_j, z_j) \cdot w_j^{n+1} + \cdots)^{-n} \\ &= w_j^{-n}(1 - n f_{jk}^{(n+1)}(x_j, z_j) \cdot w_j^n + O(w_j^{n+1})) \\ &= w_j^{-n} - n f_{jk}^{(n+1)}(x_j, z_j) + O(w_j) \end{aligned}$$

holds. Thus we obtain

$$\begin{aligned} \frac{1}{n}(w_j^{-n} - t_{jk}^{-n} w_k^{-n})|_{V_{jk}} &= f_{jk}^{(n+1)}(x_j, z_j) \\ &= g_{jk}^{(n+1,m)}(x_j) \cdot z_j^m + g_{jk}^{(n+1,m+1)}(x_j) \cdot z_j^{m+1} + \cdots . \end{aligned}$$

Therefore we can conclude that $\{(V_{jk}, f_{jk}^{(n+1)})\}$ satisfies the cocycle condition for the line bundle $N_{S/X}|_V^{-n}$:

$$f_{jk}^{(n+1)} + t_{jk}^{-n} f_{kl}^{(n+1)} + t_{jl}^{-n} f_{lj}^{(n+1)} \equiv 0.$$

Regarding it as an equation of the local sections of $\mathcal{O}_V/\mathcal{O}_V(-(m+1)C)$, it follows that

$$g_{jk}^{(n+1,m)} + t_{jk}^{-n} s_{jk}^{-m} g_{kl}^{(n+1,m)} + t_{jl}^{-n} s_{jl}^{-m} g_{lj}^{(n+1,m)} \equiv 0,$$

which shows the proposition. □

DEFINITION 5.3.3. Let $\{(W_j, w_j)\}$ be a system of order (n, m). we denote by $u_{n,m}(C, S, X)$ the element of $H^1(C, N_{S/X}|_C^{-n} \otimes N_{C/S}^{-m})$ defined by the 1-cocycle $\{(U_{jk}, g_{jk}^{(n+1,m)})\}$ and call it *the* (n, m)*-th Ueda class* of the triple (C, S, X).

Proposition 5.3.4. *The above definition of the (n, m)-th Ueda class $u_{n,m}(C, S, X)$ of the triple (C, S, X) is independent of the choice of a system of order (n, m) up to non-zero constant multiples. Especially it can be said that the condition "$u_{n,m}(C, S, X) = 0$" makes sense whenever there exits a system of order (n, m).*

Proof. Take another system $\{(W_j, \widetilde{w}_j)\}$ of order (n, m). Since both $\widetilde{w}_j$ and w_j are the defining function of V_j in W_j, there exists a nowhere vanishing holomorphic function e_j defined on W_j such that $\widetilde{w}_j = e_j w_j$. As $(\widetilde{w}_j/\widetilde{w}_k)|_{U_{jk}} = (w_j/w_k)|_{U_{jk}} (\equiv t_{jk})$, it holds that $e_j|_{U_{jk}} = e_k|_{U_{jk}}$ and thus $\{U_j, e_j|_{U_j}\}$ glues up to define a holomorphic function defined on the whole C. Therefore we can conclude that there exists a non-zero constant M such that $e_j|_{U_j} \equiv M$ holds for all j. From this fact and the equation (5.2), it follows that the new (n, m)-th Ueda class defined by using the system $\{(W_j, \widetilde{w}_j)\}$ is just M^{-n} times the original one defined by using $\{(W_j, w_j)\}$. □

Definition 5.3.5. The (n, m)-th Ueda class $u_{n,m}(C, S, X)$ of the triple (C, S, X) is said to be *well-defined* if there exits a system of order (n, m).

From the definition, it is clear that, if $u_{n,m}(C, S, X)$ is well-defined, then $u_{\nu,\mu}(C, S, X)$ is also well-defined and is equal to zero for each (ν, μ) less than (n, m) in the lexicographical order. The following proposition is on the converse of it.

Proposition 5.3.6. *Let $n \geq 1$ and $m \geq 0$ be integers. Assume one of the following three conditions holds: (i′) $N_{C/S} \in \mathcal{E}_0(C)$, (ii) $N_{C/S} = N_{S/X}|_C \in \mathcal{E}_1(C)$, (iii′) C is an exceptional subvariety of S in the sense of Grauert. Further assume that $u_{\nu,\mu}(C, S, X)$ is well-defined and is equal to zero for each (ν, μ) less than (n, m) in the lexicographical order. Then $u_{n,m}(C, S, X)$ is also well-defined.*

The strategy of the proof of Proposition 5.3.6 is almost the same as that of Theorem 1.2.4 in [U]. We will prove it in the next subsection. By using Proposition 5.3.6, we can define type (C, S, X) as follows.

Definition 5.3.7. Assume one of the following three conditions holds: (i′) $N_{C/S} \in \mathcal{E}_0(C)$, (ii) $N_{C/S} = N_{S/X}|_C \in \mathcal{E}_1(C)$, (iii′) C is an exceptional subvariety of S in the sense of Grauert. We denote by "type (C, S, X)" the maximum in the lexicographical order of the set of all pairs (n, m) such that the (n, m)-th Ueda class $u_{n,m}(C, S, X)$ of the triple (C, S, X) is well-defined:

i.e. type (C, S, X) is the pair (n, m) such that $u_{n,m}(C, S, X)$ is well-defined and non-trivial. We write "type $(C, S, X) = \infty$" if there is no such pair (n, m) (i.e. if $u_{n,m}(C, S, X)$ is well-defined and $u_{n,m}(C, S, X) = 0$ holds for each $n \geq 1$ and $m \geq 0$).

5.3.2 The well-definedness of the type

Here we prove Proposition 5.3.6. Proposition 5.3.6 directly follows from the following Lemmata 5.3.8, 5.3.9.

LEMMA 5.3.8. *Let $n \geq 1$ and $m \geq 0$ be integers. Assume that $u_{n,m}(C, S, X)$ is well-defined and is equal to zero. Then $u_{n,m+1}(C, S, X)$ is also well-defined.*

LEMMA 5.3.9. *Let $n \geq 1$ be an integer. Assume one of the following three conditions holds: (i') $N_{C/S} \in \mathcal{E}_0(C)$, (ii) $N_{C/S} = N_{S/X}|_C \in \mathcal{E}_1(C)$, (iii') C is an exceptional subvariety of S in the sense of Grauert. Further assume that $u_{n,m}(C, S, X)$ is well-defined and is equal to zero for each $m \geq 0$. Then $u_{n+1,0}(C, S, X)$ is also well-defined.*

We will prove these lemmata in the following.

Proof of Lemma 5.3.8

Fix a system $\{(W_j, w_j)\}$ of order (n, m). From the assumption and the equation (5.2), there exists a system $\{(U_j, F_j(x_j))\}$ of holomorphic functions such that

$$(w_j^{-n} - t_{jk}^{-n} w_k^{-n})|_{V_{jk}} = F_j z_j^m - t_{jk}^{-n} s_{jk}^{-m} F_k z_k^m + O(z_j^{m+1})$$

holds on each V_{jk}. Define

$$\widetilde{w}_j := w_j \cdot (1 - F_j \cdot z_j^m \cdot w_j^n)^{-\frac{1}{n}}$$

by shrinking V and W if necessary. Since

$$\widetilde{w}_j = w_j \cdot \left(1 + \frac{1}{n} F_j \cdot z_j^m \cdot w_j^n + \cdots\right) = w_j + O(w_j^{n+2})$$

holds, it is clear that our new system $\{\widetilde{w}_j\}$ is also of order (n, m). As $\widetilde{w}_j^{-n} = w_j^{-n} - F_j z_j^m$ holds, we obtain that

$$\frac{1}{n}(\widetilde{w}_j^{-n} - t_{jk}^{-n} \widetilde{w}_k^{-n})|_{V_{jk}} = O(z_j^{m+1}),$$

which means that the system $\{\widetilde{w}_j\}$ is of order $(n, m+1)$. $\square$

Proof of Lemma 5.3.9 (i')

We will show Lemma 5.3.9 when the condition (i') $N_{C/S} \in \mathcal{E}_0(C)$ holds. Fix a system $\{(W_j, w_j)\}$ of order $(n, 0)$. We will construct a new system $\{(W_j, \widetilde{w}_j)\}$ of order $(n+1, 0)$ by solving a functional equation

$$w_j = u_j + F_j(x_j, z_j) \cdot u_j^{n+1} \tag{5.2}$$

with an uknown function u_j on each W_j after choosing a system of suitable holomorphic functions $\{(V_{jk}, F_j(x_j, z_j))\}$. We will define the function $F_j(x_j, z_j)$ by inductively defining the coefficient $\{G_j^{(m)}(x_j)\}$ of the variable z_j^m in the expansion of $F_j(x_j, z_j)$:

$$F_j(x_j, z_j) = \sum_{m=0}^{\infty} G_j^{(m)}(x_j) \cdot z_j^m.$$

Fix a positive constant $K(N_{S/X}|_C^{-n} \otimes N_{C/S}^{-m})$ as in [U, Lemma 3] for each $m \geq 0$ and set $K := \max_m K(N_{S/X}|_C^{-n} \otimes N_{C/S}^{-m})$ (here the condition (i') is needed for the existence of the maximum). Let

$$t_{jk}w_k - w_j = f_{jk}^{(n+1)}(x_j, z_j) \cdot w_j^{n+1} + O(w_j^{n+2}),\ f_{jk}^{(n+1)}(x_j, z_j) = \sum_{m=0}^{\infty} g_{jk}^{(n+1,m)}(x_j) \cdot z_j^m \tag{5.3}$$

be the expansion of $t_{jk}w_k - w_j$ on W_{jk}. Let M and R be sufficiently large positive number such that $\sup_{V_{jk}} |f_{jk}^{(n+1)}(x_j, z_j)| < M$ and $\{|z_j| < 2R^{-1}, |w_j| < 2R^{-1}\} \subset W_j$ hold for each j, k. Note that, from Cauchy's inequality, it holds that $|g_{jk}^{(n+1,m)}| < MR^m$. Consider the function

$$A(X) := \frac{KM}{1 - (K+1)RX} = KM + KM(K+1)RX + KM(K+1)^2R^2X^2 + \cdots$$

and denote by A_m the coefficient of X^m in the Taylor expansion of $A(X)$ at $X = 0$.

Lemma 5.3.10. *There exists a system of functions* $\{G_j^{(\mu)}(x_j)\}_{\mu=0}^{\infty}$ *for each* j *satisfying the following conditions. Let*

$$\begin{aligned} G_j^{(\mu)}(x_j) \cdot z_j^\mu &= G_j^{(\mu)}(x_j(x_k, z_k, w_k)) \cdot z_j^\mu = G_j^{(\mu)}(x_j(x_k, z_k, 0)) \cdot z_j^\mu + O(w_k) \\ &= G_j^{(\mu)}(x_j(x_k, 0, 0)) \cdot s_{jk}^\mu z_k^\mu + \sum_{\nu=1}^{\infty} G_{jk,\nu}^{(\mu)}(x_k) \cdot z_k^{\nu+\mu} + O(w_k) \end{aligned}$$

be the expansion of $(G_j^{(\mu)}(x_j)\cdot z_j^\mu)|_{W_{jk}}$ *in the variable* z_k *and* w_k *by regarding* $G_j^{(\mu)}$ *as a function defined on* W_j *which does not depend on* z_j *and* w_j. *Denote by* $h_{1jk,\mu}(x_j)$ *the coefficient of* z_j^μ *in the Taylor expansion of* $\sum_{\mu=0}^\infty\sum_{\nu=1}^\infty G_{kj,\nu}^{(\mu)}(x_j)\cdot z_j^{\nu+\mu}$ *at* $z_j=0$ *for each* μ. *Then the coboundary* $\delta\{(U_j, G_j^{(\mu)})\}$ *is equal to* $\{(U_{jk}, g_{jk}^{(n+1,\mu)} - t_{jk}^{-n}h_{1jk,\mu})\}$, *and* $\max_j\sup_{U_j}\left|G_j^{(\mu)}\right|\leq A_\mu$ *for each* $\mu\geq 0$.

Proof. First we construct $\{G_j^{(0)}\}$ for each j. It follows from the definition that the system $\{(U_{jk}, g_{jk}^{(n+1,0)})\}$ defines the cohomology class $u_{n,0}(C,S,X)$, which is trivial from the assumption of Lemma 5.3.9. Thus there exists a 1-cochain $\{(U_j, G_j^{(0)})\}$ of $N_{S/X}|_C^{-n}$ such that

$$\delta\{(U_j, G_j^{(0)})\} = \{(U_{jk}, g_{jk}^{(n+1,0)})\},\ \max_j\sup_{U_j}\left|G_j^{(0)}\right|\leq K\cdot\max_{jk}\sup_{U_{jk}}\left|g_{jk}^{(n+1,0)}\right|$$

hold. Since $h_{1jk,0}(x_j)\equiv 0$ and $K\cdot|g_{jk}^{(n+1,0)}|\leq KM=A_0$ hold, $\{G_j^{(0)}\}$ is what we wanted.

Next we will construct $\{G_j^{(m)}\}$ for each j assuming that there exists $\{G_j^{(\mu)}\}$ for each j and $\mu\leq m-1$ such that the coboundary $\delta\{(U_j, G_j^{(\mu)})\}$ is equal to $\{(U_{jk}, g_{jk}^{(n+1,\mu)} - t_{jk}^{-n}h_{1jk,\mu})\}$ and $\max_j\sup_{U_j}\left|G_j^{(\mu)}\right|\leq A_\mu$ hold. Let $\widetilde{w}_j$ be the solution of the functional equation

$$w_j = u_j + \left(\sum_{\mu=0}^{m-1} G_j^{(\mu)}(x_j)\cdot z_j^\mu\right)\cdot u_j^{n+1} \tag{5.4}$$

with an unknown function u_j (the existence of the solution is follows from the implicit function theorem). From the functional equation (5.4), the function $(t_{jk}w_k)|_{W_{jk}}$ can be expanded as follows:

$$\begin{aligned}
&t_{jk}w_k\\
=\ &\left(\sum_{\mu=0}^{m-1} G_k^{(\mu)}(x_k)\cdot z_k^\mu\right)\cdot t_{jk}\widetilde{w}_k^{n+1} + t_{jk}\widetilde{w}_k\\
=\ &\left(\sum_{\mu=0}^{m-1}\left(G_k^{(\mu)}(x_k(x_j,0,0))\cdot s_{jk}^{-\mu}z_j^\mu + \sum_{\nu=1}^\infty G_{kj,\nu}^{(\mu)}(x_j)\cdot z_j^{\nu+\mu}\right)\right)\cdot t_{jk}\widetilde{w}_k^{n+1}\\
&+t_{jk}\widetilde{w}_k + O(\widetilde{w}_k^{n+2})
\end{aligned}$$

$$= \left(\sum_{\mu=0}^{m-1}\left(G_k^{(\mu)}(x_k(x_j,0,0))\cdot s_{jk}^{-\mu}+h_{1jk,\mu}\right)\cdot z_j^\mu + h_{1jk,m}\cdot z_j^m + O(z_j^{m+1})\right)$$
$$\cdot t_{jk}\widetilde{w}_k^{n+1} + t_{jk}\widetilde{w}_k + O(\widetilde{w}_k^{n+2})$$
$$= \left(\sum_{\mu=0}^{m-1}\left(G_k^{(\mu)}(x_k(x_j,0,0))\cdot s_{jk}^{-\mu}+h_{1jk,\mu}\right)\cdot z_j^\mu + h_{1jk,m}\cdot z_j^m + O(z_j^{m+1})\right)$$
$$\cdot t_{jk}^{-n}\widetilde{w}_j^{n+1} + t_{jk}\widetilde{w}_k + O(\widetilde{w}_j^{n+2}).$$

Here we used the fact that $h_{1jk,\mu}$ depends only on $\{G_j^{(\mu')}\}_{\mu'=0}^{\mu-1}$. On the other hand, by using the equation (5.3), the function $(t_{jk}w_k)|_{W_{jk}}$ can also be expanded as follows:

$$t_{jk}w_k$$
$$= w_j + \left(\sum_{\mu=0}^{\infty} g_{jk}^{(n+1,\mu)}(x_j)\cdot z_j^\mu\right)\cdot w_j^{n+1} + O(w_j^{n+2})$$
$$= \left(\sum_{\mu=0}^{m-1}\left(G_j^{(\mu)}(x_j)+g_{jk}^{(n+1,\mu)}(x_j)\right)\cdot z_j^\mu + g_{jk}^{(n+1,m)}(x_j)\cdot z_j^m + O(z_j^{m+1})\right)\cdot \widetilde{w}_j^{n+1}$$
$$+\widetilde{w}_j + O(\widetilde{w}_j^{n+2}).$$

Thus we obtain

$$t_{jk}\widetilde{w}_k - \widetilde{w}_j$$
$$= -\left(\sum_{\mu=0}^{m-1}\left(G_k^{(\mu)}(x_k(x_j,0,0))\cdot s_{jk}^{-\mu}+h_{1jk,\mu}\right)\cdot z_j^\mu + h_{1jk,m}\cdot z_j^m\right)\cdot t_{jk}^{-n}\widetilde{w}_j^{n+1}$$
$$+\left(\sum_{\mu=0}^{m-1}\left(G_j^{(\mu)}(x_j)+g_{jk}^{(n+1,\mu)}(x_j)\right)\cdot z_j^\mu + g_{jk}^{(n+1,m)}(x_j)\cdot z_j^m\right)\cdot \widetilde{w}_j^{n+1}$$
$$+O(z_j^{m+1})\cdot\widetilde{w}_j^{n+1}$$
$$+O(\widetilde{w}_j^{n+2}).$$

Therefore, from the assumption of the induction, we can conclude that

$$t_{jk}\widetilde{w}_k - \widetilde{w}_j$$
$$= \left(\left(g_{jk}^{(n+1,m)} - t_{jk}^{-n}h_{1jk,m}\right)\cdot z_j^m + O(z_j^{m+1})\right)\cdot\widetilde{w}_j^{n+1} + O(\widetilde{w}_j^{n+2})$$

and thus it follows that the system $\{(U_{jk}, g_{jk}^{(n+1,m)} - t_{jk}^{-n}h_{1jk,m})\}$ is a 1-cocycle which defines the (n,m)-th Ueda class, which is the trivial one from the assumption of Lemma 5.3.9. Hence it follows from [U, Lemma 3] that there exists a 0-cochain $\{(U_j, G_j^{(m)})\}$ such that $\delta\{(U_j, G_j^{(m)})\} = \{(U_{jk}, g_{jk}^{(n+1,m)} - t_{jk}^{-n} h_{1jk,m})\}$ and

$$\max_j \sup_{U_j} \left| G_j^{(m)} \right| \leq K \cdot \max_{jk} \sup_{U_{jk}} \left| g_{jk}^{(n+1,m)} - t_{jk}^{-n} h_{1jk,m} \right|$$

hold. From the definition of $h_{1jk,m}$, we obtain the inequality

$$|h_{1jk,m}| \leq \text{the coefficient of } z_j^m \text{ in the expansion of } \sum_{\mu=0}^{m-1} \sum_{\nu=1}^{\infty} \left| G_{kj,\nu}^{(\mu)}(x_j) \right| \cdot z_j^{\nu+\mu}.$$

As it follows from the assumption of the induction and Cauchy's inequality (see Remark 5.3.11) that

$$\left| G_{kj,\nu}^{(\mu)}(x_j) \right| \leq \left| G_k^{(\mu)}(x_k) \right| \cdot R^\nu \leq A_\mu R^\nu \tag{5.5}$$

for each $\mu < m$, we obtain the inequality

$$\begin{aligned} |h_{1jk,m}| &\leq \text{the coefficient of } z_j^m \text{ in the expansion of } \sum_{\mu=0}^{m-1} \sum_{\nu=1}^{\infty} A_\mu R^\nu \cdot z_j^{\nu+\mu} \\ &= \text{the coefficient of } z_j^m \text{ in the expansion of } \left(\sum_{\mu=0}^{m-1} A_\mu z_j^\mu \right) \left(\sum_{\nu=1}^{\infty} R^\nu z_j^\nu \right) \\ &\leq \text{the coefficient of } X^m \text{ in the expansion of } \frac{RXA(X)}{1-RX}. \end{aligned}$$

Since

$$\left| g_{jk}^{(n+1,m)} \right| \leq MR^m = \text{the coefficient of } X^m \text{ in the expansion of } \frac{M}{1-RX}$$

holds, it follows that

$$\begin{aligned} \left| G_j^{(m)} \right| &\leq K \cdot \max_{jk} \sup_{U_{jk}} \left| g_{jk}^{(n+1,m)} - t_{jk}^{-n} h_{1jk,m} \right| \\ &\leq \text{the coefficient of } X^m \text{ in the expansion of } \frac{M + A(X)RX}{1-RX}. \end{aligned}$$

As A is a solution of the functional equation

$$\frac{K(M + A(X)RX)}{1-RX} = A(X),$$

the inequality $\left| G_j^{(m)} \right| \leq A_m$ holds. □

REMARK 5.3.11. Strictly speaking, we have to enlarge M and R in the proof of Lemma 5.3.10, which is because we used the non-trivial assumption that

$$\{(x_j, z_j, w_j) \in W_j \mid x_j \in U_{jk}, |z_j| \leq R^{-1}, w_j = 0\} \subset W_j \cap W_k$$

by stealth in the proof of the equation (5.5). However, this difficulty can be avoided by using a new open covering $\{U_j^*\}$ of C such that each U_j^* is a relatively compact subset of U_j (see [U, p. 599] for the details). After replacing R with a sufficiently large number determined by this open covering $\{U_j^*\}$ and M with $2M$, Lemma 5.3.10 can be proven.

Let $\{G_j^{(m)}\}$ be that in Lemma 5.3.10 and consider the function $F_j(x_j, z_j) = \sum_{m=0}^{\infty} G_j^{(m)}(x_j) \cdot z_j^m$. From Lemma 5.3.10, it can be said that F_j is a holomorphic function around C. Let $\widetilde{w}_j$ be the solution of the functional equation (5.2). Then, it can be showed by just the same argument as in the proof of Lemma 5.3.10, that $\{(W_j.\widetilde{w}_j)\}$ is a system of order $(n+1, 0)$, which shows Lemma 5.3.9 when (i') holds. □

Proof of Lemma 5.3.9 *(ii)*

Here we prove Lemma 5.3.9 in the case where the condition (ii) $N_{C/S} = N_{S/X}|_C \in \mathcal{E}_1(C)$ holds. We consider the functional equation (5.2) also in this case, and so what we have to do is determining each coefficients $\{G_j^{(\mu)}\}$ of F_j just as in Lemma 5.3.10. However, in this case, the sequence of constants $\{K(N_{S/X}|_C^{-n} \otimes N_{C/S}^{-m})\}_{m=0}^{\infty}$ as in [U, Lemma 3] need not be bounded from above. So now we will use Lemma 5.2.4 instead of [U, Lemma 3].

Set $N := N_{S/X}|_C = N_{C/S}$ and $\varepsilon_n := d(\mathcal{O}_C, N^{-n})^{-1}$. Let K be the constant as in Lemma 5.2.4. By just the same argument as in the proof of Lemma 5.3.10, we can inductively define the functions $G_j^{(m)}$ such that $\delta\{(U_j, G_j^{(m)})\} = \{(U_{jk}, g_{jk}^{(n+1,m)} - t_{jk}^{-n} h_{1jk,m})\}$ and

$$\max_j \sup_{U_j} \left| G_j^{(m)} \right| \leq \varepsilon_{n+m} K \cdot \max_{jk} \sup_{U_{jk}} \left| g_{jk}^{(n+1,m)} - t_{jk}^{-n} h_{1jk,m} \right|$$

hold for each m and j, k. Thus all we have to do is showing that the function $F_j = \sum_{m=0}^{\infty} G_j^{(m)} z_j^m$ is actually a holomorphic function around C. For this purpose, we will prove that there exists a power series $B(X) = \sum_{m=0}^{\infty} B_m X^m$ with a positive radius of convergence such that $B_0 = KM$ and

$$\sum_{\mu=1}^{\infty} \varepsilon_{n+\mu}^{-1} B_\mu X^\mu = \frac{K(M + B(X))RX}{1 - RX} \tag{5.6}$$

holds, where the constants M and R are that in the proof of Lemma 5.3.9 when (i') holds. Note that the power series B is uniquely determined as a formal power series, since all of the coefficients B_m are inductively determined by the above equation (5.6). By using the equation (5.6), it is also shown by the induction that each coefficient B_m is non-negative real number. From now on, we will prove that this formal power series $B(X)$ has a positive radius of convergence. Consider a formal power series $D(X) = X + \sum_{\lambda=2}^{\infty} D_\lambda X^\lambda$ defined by

$$\sum_{\lambda=2}^{\infty} \varepsilon_{\lambda-1}^{-1} D_\mu X^\lambda = \frac{KRD(X)^2(1+MD(X)^n)}{1-RD(X)}.$$

Just as the above argument on $B(X)$, this power series $D(X)$ is also uniquely determined as a formal power series with non-negative real coefficients.

Claim 5.3.12. *$B_\mu \leq D_{\mu+n+1}$ holds for each $\mu \geq 0$.*

Proof. Consider the power series $\widetilde{B}(X) := X + B(X) \cdot X^{n+1}$. Denote by $\widetilde{B}_\lambda$ the coefficient of X^λ in the expansion of $\widetilde{B}(X)$. We will prove the inequality $\widetilde{B}_\lambda \leq D_\lambda$ for each λ by induction. First, it is clear that this inequality holds for $\lambda = 1, 2, \dots, n$. Next, let us assume that $\widetilde{B}_\lambda \leq D_\lambda$ holds for $\lambda < \ell$ and prove that $\widetilde{B}_\ell \leq D_\ell$. It follows from the equation (5.6) that

$$\sum_{\lambda=1}^{\infty} \varepsilon_{\lambda-1}^{-1} \widetilde{B}_\lambda X^\lambda = \frac{K(MX^{n+1} + \widetilde{B}(X) - X)RX}{1-RX}.$$

Thus all we have to do is showing

$$\begin{aligned} &\text{the coefficient of } X^\ell \text{ in the expansion of } \frac{K(MX^{n+1} + \widetilde{B}(X) - X)RX}{1-RX} \\ \leq\ &\text{the coefficient of } X^\ell \text{ in the expansion of } \frac{KRD(X)^2(1+MD(X)^n)}{1-RD(X)}. \end{aligned}$$

Since the left (resp. right) hand side of the above inequality depends only on $\{\widetilde{B}_\lambda\}_{\lambda<\ell}$ (resp. $\{D_\lambda\}_{\lambda<\ell}$), it follows from the assumption of the induction that it is sufficient to show the inequality

$$\begin{aligned} &\text{the coefficient of } X^\ell \text{ in the expansion of } \frac{K(MX^{n+1} + \widetilde{B}(X) - X)RX}{1-RX} \\ \leq\ &\text{the coefficient of } X^\ell \text{ in the expansion of } \frac{KR\widetilde{B}(X)^2(1+M\widetilde{B}(X)^n)}{1-R\widetilde{B}(X)}, \end{aligned}$$

which is easily obtained from the fact that the coefficients of $\widetilde{B}(X) - X$ and X are less than or equal to that of $\widetilde{B}(X)$. □

According to Claim 5.3.12, it is sufficient for proving Lemma 5.3.9 to show that the formal power series $D(X)$ has a positive radius of convergence. We show this by using Lemma 5.2.5. Note that our $\{\varepsilon_\lambda\}_{\lambda\geq 1}$ enjoys the two conditions in Lemma 5.2.5 (here we used the assumption (ii), see also [U, p. 603]). Thus we can apply Lemma 5.2.5 on

$$f(Z) := \frac{KRZ^2(1+MZ^n)}{1-RZ}$$

and conclude that $D(X)$ has a positive radius of convergence. Thus $B(X)$ and F_j also have a positive radius of convergence. Therefore we can construct a system of order $(n+1,0)$ by solving the functional equation (5.2), which completes the proof of Lemma 5.3.9 in the case where the condition (ii) holds. □

Proof of Lemma 5.3.9 (iii')

Finally we prove Lemma 5.3.9 in the case where the condition (iii') holds. In this case, we can apply 5.2.2 and thus we may assume that the neighborhood V satisfies the following property: there exists an integer m such that the natural map $H^1(V, N_{S/X}|_V^{-n}) \to H^1(V, N_{S/X}|_V^{-n} \otimes \mathcal{O}_V/I_C^m)$ is injective, where I_C is the defining ideal sheaf of $C \subset V$. Fix a system $\{(W_j, w_j)\}$ of order $(n, m+1)$. It clearly follows from the equation (5.2) that the cohomology class $[\{(V_j, (w_j^{-n} - t_{jk}^{-n}w_k^{-n})|_{V_{jk}})\}]_m \in H^1(V, N_{S/X}|_V^{-n} \otimes \mathcal{O}_V/I_C^m)$ induced from $[\{(V_j, (w_j^{-n} - t_{jk}^{-n}w_k^{-n})|_{V_{jk}})\}] \in H^1(V, N_{S/X}|_V^{-n})$ is the trivial one. Thus we can conclude that the cohomology class $[\{(V_j, (w_j^{-n} - t_{jk}^{-n}w_k^{-n})|_{V_{jk}})\}] \in H^1(V, N_{S/X}|_V^{-n})$ itself is also the trivial one. Therefore there exists a system $\{(V_j, F_j(x_j, z_j))\}$ of holomorphic functions such that

$$(w_j^{-n} - t_{jk}^{-n}w_k^{-n})|_{V_{jk}} = F_j - t_{jk}^{-n}F_k$$

holds on each V_{jk}. Define

$$\widetilde{w}_j := w_j \cdot (1 - F_j \cdot w_j^n)^{-\frac{1}{n}}$$

by shrinking V and W if necessary. Since

$$\widetilde{w}_j = w_j \cdot \left(1 + \frac{1}{n}F_j \cdot w_j^n + \cdots\right) = w_j + O(w_j^{n+2})$$

holds, it is clear that our new system $\{\widetilde{w}_j\}$ is also of order $(n, 0)$. As $\widetilde{w}_j^{-n} = w_j^{-n} - F_j$ holds, we obtain that

$$\frac{1}{n}(\widetilde{w}_j^{-n} - t_{jk}^{-n}\widetilde{w}_k^{-n})|_{V_{jk}} \equiv 0,$$

which means that the system $\{\widetilde{w}_j\}$ is of order $(n+1, 0)$. □

5.4 Proof of the main theorem in §5

In this section we prove Theorem 5.1.1. The strategy of the proof is almost the same as that of Lemma 5.3.9: i.e. fixing a system $\{(W_j, w_j)\}$ of order $(1, 0)$, we will construct a new system $\{(W_j, \widetilde{w}_j)\}$ by solving the functional equation

$$w_j = u_j + \sum_{\nu=2}^{\infty} F_j^{(\nu)}(x_j, z_j) \cdot u_j^{\nu} \tag{5.7}$$

after choosing a suitable system of holomorphic functions $\{(V_j, F_j^{(\nu)}(x_j, z_j))\}$. Let

$$F_j^{(\nu)}(x_j, z_j) = \sum_{\mu=0} G_j^{(\nu,\mu)}(x_j) \cdot z_j^{\mu}$$

be the expansion of $F_j^{(\nu)}(x_j, z_j)$. □

5.4.1 Proof of the main theorem in §5 in the case where the condition (i) holds

First we prove Theorem 5.1.1 in the case where the condition (i) $N_{C/S}, N_{S/X}|_C \in \mathcal{E}_0(C)$ holds. Let $K_{n,m} = K(N_{S/X}|_C^{-n} \otimes N_{C/S}^{-m})$ be the constant as in [U, Lemma 3] and set $K := \max_{n,m} K_{n,m}$ (here the condition (i) is needed). Let

$$t_{jk}w_k - w_j = \sum_{\nu=2}^{\infty} f_{jk}^{(\nu)}(x_j, z_j) \cdot w_j^{\nu} = \sum_{\nu=2}^{\infty} \sum_{\mu=0}^{\infty} g_{jk}^{(\nu,\mu)}(x_j) \cdot z_j^{\mu} \cdot w_j^{\nu} \tag{5.8}$$

be the expansions of $t_{jk}w_k - w_j$. Let M and R be sufficiently large positive number such that $\sup_{W_{jk}} |t_{jk}w_k - w_j| < M$ and $\{|z_j| < 2R^{-1}, |w_j| < 2R^{-1}\} \subset W_j$ hold for each j, k. Note that, from Cauchy's inequality, it holds that $|g_{jk}^{(n,m)}| < MR^{n+m+1}$. Consider the solution $A(X, Y)$ of the functional equation

$$A(X, Y) - X = \frac{RK}{1 - RY} \left((A(X, Y) - X)Y + \frac{(1 + MR)A(X, Y)^2}{1 - RA(X, Y)} \right) \tag{5.9}$$

and denote by $A_{n,m}$ the coefficient of X^nY^m in the Taylor expansion of $A(X)$ at $X = Y = 0$. Though the functional equation (5.9) has two solutions, the solution A is uniquely determined by the condition that $A(X, Y) = X + O(X^2)$. Note that $A_{n,m}$ is a non-negative real number for each n and m.

LEMMA 5.4.1. *There exists a system of functions* $\{G_j^{(n,m)}(x_j)\}_{n\geq 2, m\geq 0}$ *for each* j *satisfying the following conditions. Let*

$$\begin{aligned} G_j^{(n,m)}(x_j)\cdot z_j^m &= G_j^{(n,m)}(x_j(x_k,z_k,w_k))\cdot z_j(x_k,z_k,w_k)^m \\ &= G_j^{(n,m)}(x_j(x_k,0,0))\cdot s_{jk}^m z_k^m \\ &\quad +\sum_{q=1}^{\infty} G_{jk,0,q}^{(n,m)}(x_k)\cdot z_k^{m+q} + \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} G_{jk,p,q}^{(n,m)}(x_k)\cdot z_k^{m+q} w_k^p \end{aligned}$$

be the expansion of $(G_j^{(n,m)} z_j^m)|_{W_{jk}}$ *in the variable* z_k *and* w_k *by regarding* $G_j^{(n,m)} z_j^m$ *as a function defined on* W_j *which does not depend on* w_j*. Denote by* $h_{1jk,n,m}$ *the coefficient of* z_j^m *in the expansion of*

$$\sum_{\mu=0}^{m-1}\sum_{q=1}^{\infty} G_{kj,0,q}^{(n,\mu)}(x_j)\cdot z_j^{\mu+q},$$

by $h_{2jk,n,m}$ *the coefficient of* $z_j^m u_j^n$ *in the expansion of*

$$\sum_{\nu=2}^{n-1}\sum_{\mu=0}^{m}\sum_{p=1}^{\infty}\sum_{q=0}^{\infty} G_{kj,p,q}^{(\nu,\mu)}(x_j)\cdot z_j^{\mu+q} u_j^{\nu}\cdot\left(u_j+\sum_{a=2}^{n-1}\sum_{b=0}^{m} G_j^{(a,b)}(x_j)\cdot z_j^b u_j^a\right)^p,$$

and by $h_{3jk,n,m}$ *the coefficient of* $z_j^m u_j^n$ *in the expansion of*

$$\sum_{\nu=2}^{\infty}\sum_{\mu=0}^{\infty} g_{jk}^{(\nu,\mu)}(x_j)\cdot z_j^{\mu}\cdot\left(u_j+\sum_{p=2}^{n-1}\sum_{q=0}^{m} G_j^{(p,q)}(x_j)\cdot z_j^q u_j^p\right)^{\nu} - \sum_{\nu=2}^{n}\sum_{\mu=0}^{m} g_{jk}^{(\nu,\mu)}(x_j)\cdot z_j^{\mu} u_j^{\nu}.$$

Then the coboundary $\delta\{(U_j, G_j^{(n,m)})\}$ *is equal to*

$$\{(U_{jk}, g_{jk}^{(n,m)} - t_{jk}^{-n+1} h_{1jk,n,m} - t_{jk}^{-n+1} h_{2jk,n,m} + h_{3jk,n,m})\},$$

and $\max_j \sup_{U_j}\left|G_j^{(n,m)}\right| \leq A_{n,m}$ *for each* $n\geq 1, m\geq 0$*.*

PROOF. It immediately follows from the definition that $\{(U_{jk}, g_{jk}^{(2,0)})\}$ is a 1-cocycle which defines the cohomology class $u_{1,0}(C,S,X)$, which is equal to $0\in H^1(C, N_{S/X}|_C^{-1})$ from the assumption. Thus there exists a 1-cochain $\{(U_j, G_j^{(2,0)})\}$ of $N_{S/X}|_C^{-n}$ such that

$$\delta\{(U_j, G_j^{(2,0)})\} = \{(U_{jk}, g_{jk}^{(2,0)})\},\ \max_j \sup_{U_j}\left|G_j^{(2,0)}\right| \leq K\cdot\max_{jk}\sup_{U_{jk}}\left|g_{jk}^{(2,0)}\right|$$

hold. As $h_{1jk,2,0} = h_{2jk,2,0} = h_{3jk,2,0} = 0$ and $K \cdot |g_{jk}^{(2,0)}| \leq KMR^2 \leq RK(1 + MR) = A_{2,0}$, $\{(G_j^{(2,0)})\}$ is what we wanted.

Fix (n, m) and assume that there exists a system $\{G_j^{(\nu,\mu)}\}$ as in Lemma 5.3.10 for each (ν, μ) less than (n, m) in the lexicographical order. From now on, we will construct $\{G_j^{(n,m)}\}$. For simplicity, we assume that $m > 0$ (the construction of $\{G_j^{(n+1,0)}\}$ is just the same as the following construction formally replaced (n, m) with (n, ∞)). Denote by $\widetilde{w}_j$ the solution of the functional equation

$$w_j = u_j + \sum_{\nu=2}^{n-1} F_j^{(\nu)}(x_j, z_j) \cdot u_j^\nu + \left(\sum_{\mu=0}^{m-1} G_j^{(n,\mu)}(x_j) \cdot z_j^\mu \right) \cdot u_j^n \tag{5.10}$$

with an unknown function u_j.

First we will show that our new system $\{(W_j, \widetilde{w}_j)\}$ is of order $(n-1, 0)$. Note that $\{(W_j, \widetilde{w}_j)\}$ is clearly of order $(1, 0)$. Thus all we have to do is to show that $\{(W_j, \widetilde{w}_j)\}$ is of order $(a-1, 0)$ assuming that $\{(W_j, \widetilde{w}_j)\}$ is of order $(a-2, 0)$ for each $a \leq n$.

From the functional equation (5.10), the function $(t_{jk}w_k)|_{W_{jk}}$ can be expanded as follows:

$$\begin{aligned}
& t_{jk}w_k \\
= \; & t_{jk}\widetilde{w}_k + \sum_{\nu=2}^{a-1} \sum_{\mu=0}^{\infty} G_k^{(\nu,\mu)}(x_k) \cdot z_k^\mu \cdot t_{jk}\widetilde{w}_k^\nu + O(\widetilde{w}_k^a) \\
= \; & t_{jk}\widetilde{w}_k + \sum_{\nu=2}^{a-1} \sum_{\mu=0}^{\infty} G_k^{(\nu,\mu)}(x_k(x_j, 0, 0)) \cdot t_{jk}s_{jk}^{-\mu} z_j^{-\mu} \widetilde{w}_k^\nu \\
& + \sum_{\nu=2}^{a-1} \sum_{\mu=0}^{\infty} \sum_{q=1}^{\infty} G_{kj,0,q}^{(n,\mu)}(x_j) \cdot z_j^{\mu+q} t_{jk}\widetilde{w}_k^\nu \\
& + \sum_{\nu=2}^{a-1} \sum_{\mu=0}^{\infty} \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} G_{kj,p,q}^{(\nu,\mu)}(x_j) \cdot z_j^{\mu+q} w_j^p \cdot t_{jk}\widetilde{w}_k^\nu + O(\widetilde{w}_k^a) \\
= \; & t_{jk}\widetilde{w}_k + \sum_{\nu=2}^{a-1} \sum_{\mu=0}^{\infty} G_k^{(\nu,\mu)}(x_k(x_j, 0, 0)) \cdot t_{jk}^{-\nu+1} s_{jk}^{-\mu} z_j^{-\mu} \widetilde{w}_j^\nu \\
& + \sum_{\nu=2}^{a-1} \sum_{\mu=0}^{\infty} \sum_{q=1}^{\infty} G_{kj,0,q}^{(n,\mu)}(x_j) \cdot z_j^{\mu+q} t_{jk}\widetilde{w}_k^\nu
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\nu=2}^{a-1}\sum_{\mu=0}^{\infty}\sum_{p=1}^{\infty}\sum_{q=0}^{\infty} G_{kj,p,q}^{(\nu,\mu)}(x_j)\cdot z_j^{\mu+q}w_j^p\cdot t_{jk}^{-\nu+1}\widetilde{w}_j^{\nu}+O(\widetilde{w}_k^a)\\
=\ & t_{jk}\widetilde{w}_k+\sum_{\nu=2}^{a-1}\sum_{\mu=0}^{\infty} G_k^{(\nu,\mu)}(x_k(x_j,0,0))\cdot t_{jk}^{-\nu+1}s_{jk}^{-\mu}z_j^{-\mu}\widetilde{w}_j^{\nu}\\
&+\sum_{\nu=2}^{a-1}\sum_{\mu=0}^{\infty} h_{1jk,\nu,\mu}t_{jk}^{-\nu+1}z_j^{\mu}\widetilde{w}_j^{\nu}\\
&+\sum_{\nu=0}^{a-1}\sum_{\mu=0}^{\infty} t_{jk}^{-\nu+1}h_{2jk,\nu,\mu}z_j^{\mu}\widetilde{w}_j^{\nu}+O(\widetilde{w}_k^a).
\end{aligned}
$$

Here we used the fact that $h_{1jk,\nu,\mu}$ and $h_{2jk,\nu,\mu}$ depend only on $\{G_j^{(p,q)}\}_{(p,q)<(\nu,\mu)}$. Note that we also used the fact that $\widetilde{w}_k^{\nu}=t_{jk}^{-\nu}\widetilde{w}_j^{\nu}+O(\widetilde{w}_j^a)$ holds for each $\nu\geq 2$, which directly follows from the assumption that $\{(W_j,\widetilde{w}_j)\}$ is of order $(a-2,0)$. On the other hand, the function $(t_{jk}w_k)|_{W_{jk}}$ can also be expanded as follows:

$$
\begin{aligned}
& t_{jk}w_k \qquad\qquad (5.11)\\
=\ & w_j+\sum_{\nu=2}^{n-1} f_{jk}^{(\nu)}(x_j,z_j)\cdot w_j^{\nu}+\left(\sum_{\mu=0}^{m} g_{jk}^{(n,\mu)}(x_j)\cdot z_j^{\mu}+O(z_j^{m+1})\right)\cdot w_j^n+O(w_j^{n+1})\\
=\ & w_j+\sum_{\nu=2}^{n-1} f_{jk}^{(\nu)}(x_j,z_j)\cdot\left(\widetilde{w}_j+\sum_{\nu=2}^{n-1}F_j^{(\nu)}(x_j,z_j)\cdot\widetilde{w}_j^{\nu}+\left(\sum_{\mu=0}^{m-1}G_j^{(n,\mu)}(x_j)\cdot z_j^{\mu}\right)\cdot\widetilde{w}_j^n\right)^{\nu}\\
&+\left(\sum_{\mu=0}^{m} g_{jk}^{(n,\mu)}(x_j)\cdot z_j^{\mu}+O(z_j^{m+1})\right)\cdot\widetilde{w}_j^n+O(\widetilde{w}_j^{n+1})\\
=\ & w_j+\sum_{\nu=2}^{n-1}\left(f_{jk}^{(\nu)}(x_j,z_j)+\sum_{\mu=0}^{\infty}h_{3jk,\nu,\mu}z_j^{\mu}\right)\cdot\widetilde{w}_j^{\nu}\\
&+\left(\sum_{\mu=0}^{m}h_{3jk,n,\mu}z_j^{\mu}+\sum_{\mu=0}^{m} g_{jk}^{(n,\mu)}(x_j)\cdot z_j^{\mu}+O(z_j^{m+1})\right)\cdot\widetilde{w}_j^n+O(\widetilde{w}_j^{n+1})\\
=\ & \widetilde{w}_j+\sum_{\nu=2}^{n-1}\left(F_j^{(\nu)}(x_j,z_j)+f_{jk}^{(\nu)}(x_j,z_j)+\sum_{\mu=0}^{\infty}h_{3jk,\nu,\mu}z_j^{\mu}\right)\cdot\widetilde{w}_j^{\nu}\\
&+\left(\sum_{\mu=0}^{m-1}\left(G_j^{(n,\mu)}(x_j)+g_{jk}^{(n,\mu)}(x_j)+h_{3jk,n,\mu}\right)\cdot z_j^{\mu}\right)\cdot\widetilde{w}_j^n\\
&+\left(\left(g_{jk}^{(n,m)}(x_j)+h_{3jk,n,m}\right)\cdot z_j^m+O(z_j^{m+1})\right)\cdot\widetilde{w}_j^n+O(\widetilde{w}_j^{n+1}).
\end{aligned}
$$

Here we used the fact that $h_{3jk,\nu,\mu}$ depends only on $\{G_j^{(p,q)}\}_{(p,q)<(\nu,\mu)}$. From these two expansions, it follows that

$$\begin{aligned} t_{jk}\widetilde{w}_k - \widetilde{w}_j &= \sum_{\nu=2}^{a-1}\sum_{\mu=0}^{\infty}\Big(-t_{jk}^{-\nu+1}s_{jk}^{-\mu}G_k^{(\nu,\mu)}(x_k(x_j,0,0)) - t_{jk}^{-\nu+1}h_{1jk,\nu,\mu} \\ &\quad -t_{jk}^{-\nu+1}h_{2jk,\nu,\mu} + G_j^{(\nu,\mu)}(x_j) + g_{jk}^{(\nu,\mu)}(x_j) + h_{3jk,\nu,\mu}\Big)\cdot z_j^{\mu}\widetilde{w}_j^{\nu} + O(\widetilde{w}_j^{a}). \end{aligned}$$

As it follows from the assumption of the induction that the coboundary $\delta\{(U_j, G_j^{(\nu,\mu)})\}$ is equal to

$$\{(U_{jk}, g_{jk}^{(\nu,\mu)} - t_{jk}^{-\nu+1}h_{1jk,\nu,\mu} - t_{jk}^{-\nu+1}h_{2jk,\nu,\mu} + h_{3jk,\nu,\mu})\},$$

for $\nu \leq a(< n)$ and $\nu \geq 0$, we can conclude that $t_{jk}\widetilde{w}_k - \widetilde{w}_j = O(\widetilde{w}_j^a)$, which means that the system $\{\widetilde{w}_j\}$ is of order $(a-1, 0)$. Therefore, it can be said that the system $\{\widetilde{w}_j\}$ is of order $(n-1, 0)$.

Using this fact and the functional equation (5.10), let us consider again the expansion of function $(t_{jk}w_k)|_{W_{jk}}$:

$$\begin{aligned} & t_{jk}w_k \\ =\ & t_{jk}\widetilde{w}_k + \sum_{\nu=2}^{n-1}\sum_{\mu=0}^{\infty} G_k^{(\nu,\mu)}(x_k)\cdot z_k^{\mu}\cdot t_{jk}\widetilde{w}_k^{\nu} + \left(\sum_{\mu=0}^{m-1} G_k^{(n,\mu)}(x_k)\cdot z_k^{\mu}\right)\cdot t_{jk}\widetilde{w}_k^{n} \\ =\ & t_{jk}\widetilde{w}_k + \sum_{\nu=2}^{n-1}\sum_{\mu=0}^{\infty} G_k^{(\nu,\mu)}(x_k(x_j,0,0))\cdot t_{jk}s_{jk}^{-\mu}z_j^{-\mu}\widetilde{w}_k^{\nu} \\ & + \sum_{\nu=2}^{n-1}\sum_{\mu=0}^{\infty} h_{1jk,\nu,\mu}t_{jk}z_j^{\mu}\widetilde{w}_k^{\nu} \\ & + \sum_{\nu=2}^{n-1}\sum_{\mu=0}^{\infty}\sum_{p=1}^{\infty}\sum_{q=0}^{\infty} G_{kj,p,q}^{(\nu,\mu)}(x_j)\cdot z_j^{\mu+q}w_j^{p}\cdot t_{jk}\widetilde{w}_k^{\nu} \\ & + \left(\sum_{\mu=0}^{m-1}\Big(G_k^{(n,\mu)}(x_k(x_j,0,0))\cdot s_{jk}^{-\mu} + h_{1jk,n,\mu}z_j^{\mu}\Big)\cdot z_j^{\mu}\right)\cdot t_{jk}\widetilde{w}_k^{n} \\ & + \big(h_{1jk,n,m}z_j^{m} + O(z_j^{m+1})\big)\cdot t_{jk}\widetilde{w}_k^{n} + O(\widetilde{w}_k^{n+1}) \\ =\ & t_{jk}\widetilde{w}_j + \sum_{\nu=2}^{n-1}\sum_{\mu=0}^{\infty} G_k^{(\nu,\mu)}(x_k(x_j,0,0))\cdot t_{jk}^{-\nu+1}s_{jk}^{-\mu}z_j^{-\mu}\widetilde{w}_j^{\nu} \\ & + \sum_{\nu=2}^{n-1}\sum_{\mu=0}^{\infty} h_{1jk,\nu,\mu}t_{jk}^{-\nu+1}z_j^{\mu}\widetilde{w}_j^{\nu} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{\nu=2}^{n-1}\sum_{\mu=0}^{\infty}\sum_{p=1}^{\infty}\sum_{q=0}^{\infty} G^{(\nu,\mu)}_{kj,p,q}(x_j)\cdot z_j^{\mu+q}w_j^p\cdot t_{jk}^{-\nu+1}\widetilde{w}_j^{\nu}\\
&+\left(\sum_{\mu=0}^{m-1}\left(G^{(n,\mu)}_k(x_k(x_j,0,0))\cdot s_{jk}^{-\mu}+h_{1jk,n,\mu}z_j^{\mu}\right)\cdot z_j^{\mu}\right)\cdot t_{jk}^{-\nu+1}\widetilde{w}_j^{n}\\
&+\left(h_{1jk,n,m}z_j^m+O(z_j^{m+1})\right)\cdot t_{jk}^{-\nu+1}\widetilde{w}_j^{n}+O(\widetilde{w}_j^{n+1})\\
=\ & t_{jk}\widetilde{w}_j+\sum_{\nu=2}^{n-1}\sum_{\mu=0}^{\infty} G^{(\nu,\mu)}_k(x_k(x_j,0,0))\cdot t_{jk}^{-\nu+1}s_{jk}^{-\mu}z_j^{-\mu}\widetilde{w}_j^{\nu}\\
&+\sum_{\nu=2}^{n-1}\sum_{\mu=0}^{\infty} h_{1jk,\nu,\mu}t_{jk}^{-\nu+1}z_j^{\mu}\widetilde{w}_j^{\nu}\\
&+\sum_{\nu=0}^{n}\left(\sum_{\mu=0}^{\infty} t_{jk}^{-\nu+1}h_{2jk,\nu,\mu}z_j^{\mu}\right)\cdot\widetilde{w}_j^{\nu}\\
&+\left(\sum_{\mu=0}^{m-1}\left(G^{(n,\mu)}_k(x_k(x_j,0,0))\cdot s_{jk}^{-\mu}+h_{1jk,n,\mu}z_j^{\mu}\right)\cdot z_j^{\mu}\right)\cdot t_{jk}^{-\nu+1}\widetilde{w}_j^{n}\\
&+\left(h_{1jk,n,m}z_j^m+O(z_j^{m+1})\right)\cdot t_{jk}^{-\nu+1}\widetilde{w}_j^{n}+O(\widetilde{w}_j^{n+1}).
\end{aligned}
$$

Comparing this expansion with the previous expansion (5.11), we obtain that

$$
\begin{aligned}
t_{jk}\widetilde{w}_k-\widetilde{w}_j=&\left(-t_{jk}^{-n+1}h_{1jk,n,m}-t_{jk}^{-n+1}h_{2jk,n,m}+g^{(n,m)}_{jk}(x_j)+h_{3jk,n,m}\right)\cdot z_j^m\widetilde{w}_j^n\\
&+O(z_j^{m+1})\cdot\widetilde{w}_j^n+O(\widetilde{w}_j^{n+1}).
\end{aligned}
$$

Note that here we again used the assumption of the induction on the coboundary
$\delta\{(U_j,G^{(\nu,\mu)}_j)\}$. Now it can be said that the system

$$
\{(U_{jk},g^{(n,m)}_{jk}(x_j)-t_{jk}^{-n+1}h_{1jk,n,m}-t_{jk}^{-n+1}h_{2jk,n,m}+h_{3jk,n,m})\}
$$

is a 1-cocycle which defines the $(n-1,m)$-th Ueda class, which is the trivial one from the assumption of Theorem 5.1.1. Therefore it follows from [U, Lemma 3] that there exists a 0-cochain $\{(U_j,G^{(n,m)}_j)\}$ such that $\delta\{(U_j,G^{(n,m)}_j)\}$ coincides with the above 1-cocycle and

$$
\begin{aligned}
&\max_j\sup_{U_j}\left|G^{(n,m)}_j\right|\\
\leq\ & K\cdot\max_{jk}\sup_{U_{jk}}\left|g^{(n,m)}_{jk}(x_j)-t_{jk}^{-n+1}h_{1jk,n,m}-t_{jk}^{-n+1}h_{2jk,n,m}+h_{3jk,n,m}\right|
\end{aligned}
$$

holds. From the definition of $h_{1jk,n,m}$, $h_{2jk,n,m}$, and $h_{3jk,n,m}$ (and the assumption of the induction), we obtain the following inequalities:

$$
\begin{aligned}
|h_{1jk,n,m}| &\leq \text{ the coefficient of } z_j^m \text{ in the expansion of } \sum_{\mu=0}^{m-1}\sum_{q=1}^{\infty} A_{n,\mu}R^q \cdot z_j^{\mu+q} \\
&\leq \text{ the coefficient of } X^nY^m \text{ in the expansion of } \frac{RY(A(X,Y)-X)}{1-RY},
\end{aligned}
$$

$$
\begin{aligned}
|h_{2jk,n,m}| &\leq \text{ the coefficient of } u_j^n z_j^m \text{in the expansion of} \\
&\quad \sum_{\nu=2}^{n-1}\sum_{\mu=0}^{m}\sum_{p=1}^{\infty}\sum_{q=0}^{\infty} A_{\nu,\mu}R^{p+q}z_j^{\mu+q}u_j^{\nu} \cdot A(u_j,z_j)^p \\
&\leq \text{ the coefficient of } X^nY^m \text{in the expansion of} \\
&\quad A(X,Y)\sum_{p=1}^{\infty} R^pA(X,Y)^p \sum_{q=0}^{\infty} R^qY^q \\
&= \text{ the coefficient of } X^nY^m \text{in the expansion of} \\
&\quad \frac{RA(X,Y)^2}{(1-RY)(1-RA(X,Y))},
\end{aligned}
$$

$$
\begin{aligned}
|g_{jk}^{(n,m)} + h_{3jk,n,m}| &\leq \text{ the coefficient of } u_j^n z_j^m \text{in the expansion of} \\
&\quad \sum_{\nu=2}^{\infty}\sum_{\mu=0}^{\infty} MR^{\nu+\mu}z_j^{\mu}A(u_j,z_j)^{\nu} \\
&\leq \text{ the coefficient of } X^nY^m \text{in the expansion of} \\
&\quad M\sum_{\nu=2}^{\infty} R^{\nu}A(X,Y)^{\nu} \sum_{\mu=0}^{\infty} R^{\mu}Y^{\mu} \\
&= \text{ the coefficient of } X^nY^m \text{in the expansion of} \\
&\quad \frac{MR^2A(X,Y)^2}{(1-RY)(1-RA(X,Y))}.
\end{aligned}
$$

Note that, strictly speaking, here we have to modify the constants R and M (see Remark 5.3.11). From the above three inequalities, it follows that $\max_j \sup_{U_j} |G_j^{(n,m)}|$ is less than or equal to the coefficient of X^nY^m in the expansion of

$$
\frac{RK}{1-RY}\left((A(X,Y)-X)Y + \frac{(1+MR)A(X,Y)^2}{1-RA(X,Y)}\right),
$$

which is equal to $A_{n,m}$ from the functional equation (5.9). □

Let $\{G_j^{(n,m)}\}$ be that in Lemma 5.4.1 and consider the function $F_j^{(n)} = \sum_{m=0}^{\infty} G_j^{(n,m)} z_j^m$. From Lemma 5.4.1, it can be said that $\sum_{n=2}^{\infty} F_j^{(n)} u_j^n$ is a holomorphic function around C. Let $\widetilde{w}_j$ be the solution of the functional equation (5.7). Then, it follows from the same argument as in the proof of Lemma 5.4.1 that $t_{jk}\widetilde{w}_k - \widetilde{w}_j \equiv 0$ holds for each j and k, which shows Theorem 5.1.1 when (i) holds.

5.4.2 Proof of the main theorem in §5 in the case where the condition (ii) holds

Next we prove Theorem 5.1.1 in the case where the condition (ii) holds. By considering the proof when (i) holds with replacing the functional equation (5.9) with

$$\sum_{\nu=2}^{\infty}\sum_{\mu=0}^{\infty} \varepsilon_{\nu+\mu-1}^{-1} A_{\nu,\mu} X^{\nu} Y^{\mu} = \frac{RK}{1-RY}\left((A(X,Y)-X)Y + \frac{(1+MR)A(X,Y)^2}{1-RA(X,Y)}\right) \tag{5.12}$$

and using Lemma 5.2.4 instead of [U, Lemma 3], it can be said that the proof of Theorem 5.1.1 in this case is reduced to show that the formal power series solution $A(X,Y) = X + O(X^2)$ of the functional equation (5.12) is a holomorphic function around $X = Y = 0$. Note that, by enlarging K, we can replace the functional equation (5.12) with

$$\sum_{\nu=2}^{\infty}\sum_{\mu=0}^{\infty} \varepsilon_{\nu+\mu-1}^{-1} A_{\nu,\mu} X^{\nu} Y^{\mu} = \frac{K}{1-RY}\left((A(X,Y)-X)Y + \frac{A(X,Y)^2}{1-RA(X,Y)}\right). \tag{5.13}$$

From now on, we will prove that the formal solution $A(X,Y) = X+O(X^2)$ of the functional equation (5.14) is a holomorphic function around $X = Y = 0$.

Note that, if the formal power series $A(X,X)$ has a positive radius of convergence, then $A(X,Y)$ is a holomorphic function around $X = Y = 0$. This fact immediately follows from the fact that each coefficient A_{nm} of $A(X,Y)$ is a non-negative real number and thus the inequality

$$A_{n,m} \leq \sum_{0\leq \nu,\mu,\nu+\mu=n+m} A_{\nu,\mu} = \widetilde{A}_{n+m} := \text{the coefficient of } X^{n+m} \text{ in the expansion of } A(X,X)$$

holds. Therefore, all we have to do is to show that $A(X,X)$ has a positive radius of convergence. For this purpose, consider a formal power series $B(X) = X + \sum_{n=2}^{\infty} B_n X^n$ defined by

$$\sum_{n=2}^{\infty} \varepsilon_{n-1}^{-1} B_n X^n = \frac{2KB(X)^2}{(1-RB(X))^2}.$$

As it follows from Lemma 5.2.5 that $B(X)$ has a positive radius of convergence, it is sufficient to show that the inequality $\widetilde{A}_n \leq B_n$ holds for each $n \geq 2$. We will prove it by induction. Since $\widetilde{A}_2 = \varepsilon_1 K$ and $B_2 = 2\varepsilon_1 K$ hold, it is clear that the inequality $\widetilde{A}_n \leq B_n$ holds for $n = 2$. Next, let us assume that $\widetilde{A}_\nu \leq B_\nu$ holds for each $\nu < n$. For it follows from the equation (5.14) that

$$\sum_{n=2}^{\infty} \varepsilon_{n-1}^{-1} \widetilde{A}_n X^n = \frac{K}{1-RX}\left((A(X,X)-X)X + \frac{A(X,X)^2}{1-RA(X,X)}\right),$$

by using the assumption of the induction, it is reduced to showing the inequality

$$\begin{aligned} & \text{the coefficient of } X^n \text{ in the expansion of} \\ & \frac{K}{1-RX}\left((B(X)-X)X + \frac{B(X)^2}{1-RB(X)}\right) \\ \leq \ & \text{the coefficient of } X^n \text{ in the expansion of } \frac{2KB(X)^2}{(1-RB(X))^2}, \end{aligned}$$

which is easily obtained from the fact that the coefficients of $B(X) - X$ and X are less than or equal to that of $B(X)$. Now we have proven that $A(X,X)$ has a positive radius of convergence. Therefore we can construct a system $\{(W_j, \widetilde{w}_j)\}$ with $t_{jk}\widetilde{w}_k \equiv \widetilde{w}_j$ by solving the functional equation (5.7), which completes the proof of Theorem 5.1.1 in the case where the condition (ii) holds.

5.4.3 Proof of the main theorem in §5 in the case where the condition (iii) holds

Finally we prove Theorem 5.1.1 in the case where the condition (iii) holds. In this case, we can apply 5.2.2 and thus we may assume that the neighborhood V satisfies the following property: for each $n \geq 1$, there exists an integer $N(N_{S/X}|_V^{-n})$ such that the natural map $H^1(V, N_{S/X}|_V^{-n}) \to H^1(V, N_{S/X}|_V^{-n} \otimes \mathcal{O}_V/I_C^m)$ is injective for each integers $n \geq 1$ and $m \geq N(N_{S/X}|_V^{-n})$, where I_C is the defining ideal sheaf of $C \subset V$. Fix a relatively compact open domain $V_0 \subset V$ which contains C.

Fix a system $\{(W_j, w_j)\}$ of order $(1,0)$ and consider again the functional equation (5.7). Let M and R be sufficiently large positive number as in the proof of Theorem 5.1.1 in the case where (i) (or (ii)) holds. Fix a positive real number K and consider the solution $A(X)$ of the functional equation

$$A(X) - X = \frac{KR(1+MR)A(X)^2}{1-RA(X)} \tag{5.14}$$

and denote by A_n the coefficient of X^n in the Taylor expansion of $A(X)$ at $X = 0$. Though the functional equation (5.14) has two solutions, the solution A is uniquely determined by the condition that $A(X) = X + O(X^2)$. Note that A_n is a non-negative real number for each n.

LEMMA 5.4.2. *When K is sufficiently large, there exists a system of functions $\{F_j^{(n)}(x_j, z_j)\}_{n\geq 2}$ for each j satisfying the following conditions. Let*

$$\begin{aligned} F_j^{(n)}(x_j, z_j) &= F_j^{(n)}(x_j(x_k, z_k, w_k), z_j(x_k, z_k, w_k)) \\ &= F_j^{(n)}(x_j(x_k, z_k, 0), z_j(x_k, z_k, 0)) + \sum_{p=1}^{\infty} F_{jk,p}^{(n)}(x_k, z_k) \cdot w_k^p \end{aligned}$$

be the expansion of $F_j^{(n)}|_{W_{jk}}$ in the variable w_k by regarding $F_j^{(n)}$ as a function defined on W_j which does not depend on w_j. Denote by $h_{2jk,n}$ the coefficient of u_j^n in the expansion of

$$\sum_{\nu=2}^{n-1}\sum_{p=1}^{\infty} F_{kj,p}^{(\nu)}(x_j, z_j) \cdot u_j^\nu \cdot \left(u_j + \sum_{a=2}^{n-1} F_j^{(a)}(x_j, z_j) \cdot u_j^a\right)^p$$

and by $h_{3jk,n}$ the coefficient of $z_j^m u_j^n$ in the expansion of

$$\sum_{\nu=2}^{\infty} f_{jk}^{(\nu)}(x_j, z_j) \cdot \left(u_j + \sum_{p=2}^{n-1} F_j^{(p)}(x_j, z_k) \cdot u_j^p\right)^\nu - \sum_{\nu=2}^{n} f_{jk}^{(\nu)}(x_j, z_j) \cdot u_j^\nu,$$

where $f_{jk}^{(\nu)}$ is that in the expansion (5.8). Then the coboundary $\delta\{(V_j, F_j^{(n)})\}$ is equal to

$$\{(V_{jk}, f_{jk}^{(n)} - t_{jk}^{-n+1} h_{2jk,n} + h_{3jk,n})\},$$

and $\max_j \sup_{V_0\cap V_j} \left|F_j^{(n)}\right| \leq A_n$ *for each* $n \geq 1$.

PROOF. Set $F_j^{(1)}(x_j, z_j) := 1$. We construct the system $\{F_j^{(\nu)}\}$ inductively. Assume that there exists a system $\{F_j^{(\nu)}\}$ as in Lemma 5.4.2 for each $\nu < n$. Let $\widetilde{w}_j$ be the solution of the new functional equation

$$w_j = \sum_{\nu=1}^{n-1} F_j^{(\nu)}(x_j, z_j) \cdot u_j^\nu.$$

From just the same argument as in the proof of Theorem 1.2.4 in [U], it follows that the new system $\{(W_j, \widetilde{w}_j)\}$ is of order $(n-1, 0)$ and

$$\frac{1}{n}(\widetilde{w}_j^{-(n-1)} - t_{jk}^{-(n-1)}\widetilde{w}_k^{-(n-1)})|_{V_{jk}} = f_{jk}^{(n)} - t_{jk}^{-n+1}h_{2jk,n} + h_{3jk,n}$$

holds. From the assumption of Theorem 5.1.1, it holds for each integer m that the induced cohomology class $[\{(V_{jk}, f_{jk}^{(n)} - t_{jk}^{-n+1}h_{2jk,n} + h_{3jk,n})\}]_m \in H^1(V, N_{S/X}|_V^{-(n-1)} \otimes \mathcal{O}_V/I_C^m)$ is the trivial one. By considering this fact especially for $m = N(N_{S/X}|_V^{-(n-1)})$, we can conclude that the cohomology class $[\{(V_{jk}, f_{jk}^{(n)} - t_{jk}^{-n+1}h_{2jk,n} + h_{3jk,n})\}] \in H^1(V, N_{S/X}|_V^{-(n-1)})$ itself is also the trivial one. Thus Lemma 5.4.2 follows from just the same argument as in the proof of Theorem 1.2.4 in [U] by using Lemma 5.2.3 instead of [U, Lemma 3]. □

Let $\{F_j^{(n)}\}$ be that in Lemma 5.4.2. Then, it can be said that $\sum_{n=2}^{\infty} F_j^{(n)} u_j^n$ is a holomorphic function around V_0. Let $\widetilde{w}_j$ be the solution of the functional equation (5.7). Then, it follows from the same argument as in the proof of Theorem 1.2.4 in [U] that $t_{jk}\widetilde{w}_k - \widetilde{w}_j \equiv 0$ holds for each j and k, which shows Theorem 5.1.1 when (iii) holds.

5.5 Some examples and proof of Corollary 5.1.2

5.5.1 $\mathbb{P}^1$-bundle examples

Example 5.5.1. Let S_0 be a complex manifold and $C_0 \subset S_0$ be a smooth compact Kähler hypersurface with topologically trivial normal bundle. Assume that the line bundle $\mathcal{O}_{S_0}(C_0)$ is flat around C_0 (it follows from Theorem 1.2.4 that it is sufficient to assume that $N_{C_0/S_0} \in \mathcal{E}_0(C_0) \cup \mathcal{E}_1(C_0)$ and type $(C_0, S_0) = \infty$). Fix an extension of the trivial line bundle $\mathcal{O}_{S_0}$ by $\mathcal{O}_{S_0}(C_0)$:

$$0 \to \mathcal{O}_{S_0}(C_0) \to E \to \mathcal{O}_{S_0} \to 0 : \text{exact}. \tag{5.15}$$

Define $X := \mathbb{P}(E)$. Denote by S the section of $X \to S_0$ and by C the submanifold of S isomorphic to C_0 via the natural isomorphism $S_0 \to S$. Fix a sufficiently fine open covering $\{V_j\}_j$ of a sufficiently small neighborhood V of C_0 in S_0. Then, by using a local frame t_j of $\mathcal{O}_{S_0}(C_0)$ on V_j, a coordinate x_j of $V_j \cap C_0$, and a defining function z_j of $V_j \cap C_0$ in V_j, we can define a local coordinates system

$$(x_j, z_j, w_j) := [(1, w_j \cdot t_j(x_j, z_j))] \in (E^*|_{V_j} \setminus (0\text{-section}))/\mathbb{C}^* \subset X$$

of X. Note that we can choose $\{(V_j, t_j)\}$ such that $t_{jk} := t_k/t_j \in U(1)$ holds for each j and k by shrinking V if necessary. Then simple computation shows that there exists a holomorphic function p_{jk} on V_{jk} such that

$$w_j = \frac{t_{jk}^{-1} \cdot w_k}{1 + p_{jk}(x_j, z_j) \cdot w_k} \tag{5.16}$$

holds. Note that the system $\{(V_{jk}, p_{jk})\}$ is a 1-cocycle of $\mathcal{O}_{S_0}(C_0)$ and the class $[\{(V_{jk}, p_{jk})\}] \in H^1(V, \mathcal{O}_{S_0}(C_0))$ coincides with the restriction of the extension class of the exact sequence (5.15). It immediately follows from (5.16) that $dw_j = t_{jk}^{-1} dw_k$ holds, and thus we obtain $N_{S/X}^{-1} = \mathcal{O}_S(C)$ (i.e. $N_{S/X}|_C = N_{C/X}^{-1}$). Moreover, by expanding (5.16) in the variable w_k, we obtain

$$t_{jk} w_j = w_k - p_{jk}(x_j, z_j) w_k^2 + O(w_k^3).$$

Thus it can be said that, if the exact sequence (5.15) does not split around C_0, then there exists an integer m such that type $(C, S, X) = (1, m)$. Moreover, the equation

$$u_{1,\mu}(C, S, X) = \{(U_{jk}, -(p_{kj}/z_j^{\mu})|_{U_{jk}})\} \in H^1(C, N_{S/X}|_C^{-1} \otimes N_{C/S}^{-\mu})$$

holds for each integer μ less than $\min_{jk} \mathrm{mult}_{U_{jk}} p_{jk}$, where $U_{jk} := V_{jk} \cap C_0$.

On the other hand, when (5.15) splits, we may assume $p_{jk} \equiv 0$ for each j, k and thus we obtain $t_{jk} w_j = w_k$ for each j, k, which prove that type $(C, S, X) = \infty$.

Example 5.5.2. Let $C_1, C_2 \subset \mathbb{P}^2$ be two smooth elliptic curves which intersects at 9 points $p_1, p_2, \ldots, p_9$ transversally. Denote by S_0 the blow-up of $\mathbb{P}^2$ at these points $p_1, p_2, \ldots, p_9$, and by C_0 the strict transform of C_1. Note that S_0 has a structure of an elliptic surface and C_0 is a fiber of S_0, and thus $\mathcal{O}_{S_0}(C_0)$ is trivial around C_0. Take an extension E of $\mathcal{O}_{S_0}$ by $\mathcal{O}_{S_0}(C_0)$ as follows. First consider the short exact sequence

$$0 \to \mathcal{O}_{S_0} \to \mathcal{O}_{S_0}(C_0) \to i^*\mathcal{O}_{C_0} \to 0$$

obtained from the fact that $\mathcal{O}_{S_0}(C_0)|_{C_0} = \mathcal{O}_{C_0}$, where $i\colon C_0 \to S_0$ is the inclusion. Let

$$H^1(S_0, \mathcal{O}_{S_0}(C_0)) \to H^1(C_0, \mathcal{O}_{C_0}) \to H^2(S_0, \mathcal{O}_{S_0}) = 0$$

be the sequence induced from the above short exact sequence. From this exact sequence, it follows that there exists a non-trivial element $\xi \in H^1(S_0, \mathcal{O}_{S_0}(C_0))$. Let

$$0 \to \mathcal{O}_{S_0}(2C_0) \to E \to \mathcal{O}_{S_0} \to 0$$

be an extension corresponds to the class $f_{C_0} \cdot \xi \in H^1(S_0, \mathcal{O}_{S_0}(2C_0)) = \mathrm{Ext}^1(\mathcal{O}_{S_0}, \mathcal{O}_{S_0}(2C_0))$, where $f_{C_0} \in H^0(S_0, \mathcal{O}_{S_0})$ is a canonical section. Define $X := \mathbb{P}(E)$. Denote by S the section of $X \to S_0$ and by $C \subset S$ the curve corresponds to C_0 via $S_0 \to S$. Then it follows from the same argument as in Example 5.5.1 that $u_{1,0}(C, S, X) = 0$ and $u_{1,1}(C, S, X) = \xi \neq 0$, and thus it holds that type $(C, S, X) = (1, 1)$.

5.5.2 On the blow-up of $\mathbb{P}^3$ at 8 points

Corollary 5.1.2 can be regarded as an analogue of an application of Ueda's theory on the blow-up of $\mathbb{P}^2$ at 9 points. First we review this result on the blow-up of $\mathbb{P}^2$ at 9 points. Let $p_1, p_2, \dots p_9$ be general 9 points of $\mathbb{P}^2$. Then there exists a unique elliptic curve $C_0 \subset \mathbb{P}^2$ passing through all of these points. Assume that this curve C_0 is smooth. Denote by N the line bundle $\mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}(-p_1 - p_2 - \cdots - p_9)$. Clearly it can be said that N is an element of $\mathrm{Pic}^0(C_0)$ and thus N is flat, since C_0 is Kähler. When N is a torsion element, the canonical bundle K_X^{-1} of the blow-up X of $\mathbb{P}^2$ at $\{p_1, p_2 \dots, p_9\}$ is semi-ample. Especially, in this case, K_X^{-1} is semi-positive: i.e. K_X^{-1} admits a smooth Hermitian metric with semi-positive curvature. On the other hand, when N is a non-torsion element, K_X^{-1} is not semi-ample. In this case, Brunella showed that K_X^{-1} is semi-positive if and only if C has a pseudoflat neighborhood in X, where C is the strict transform of C_0 [Br]. As this condition holds if the line bunlde $\mathcal{O}_X(C)$ is flat around C, it follows from this Brunella's theorem and Theorem 1.2.4 that K_X^{-1} is semi-positive if $N \in \mathcal{E}_1(C_0)$.

	N : torsion	N : non-torsion
the base locus $\mathbb{B}(K_X^{-1})$	$\emptyset$ or C	C
semi-ampleness	semi-ample	not semi-ample
Iitaka dimension	1	0

Table 5.1: Properties of the anti-canonical bundle of the blow-up X of $\mathbb{P}^2$ at 9 points

Now let us start considering the blow-up of $\mathbb{P}^3$ at 8 points. Fix general 8 points $p_1, p_2, \dots, p_8$ of $\mathbb{P}^3$. Then there exits a 1-dimensional family of quadric surfaces of $\mathbb{P}^3$ passing through these points $p_1, p_2, \dots, p_8$. Fix such quadric surfaces Q_0 and Q_∞ of $\mathbb{P}^3$. Assume that Q_0 and Q_∞ intersect each other transversally along $C_0 := Q_0 \cap Q_\infty$. Note that C_0 is a smooth elliptic curve and $\mathcal{O}_{Q_0}(C_0) = K_{Q_0}^{-1}$ holds in this case. Denote by X the blow-up of $\mathbb{P}^3$ at $\{p_1, p_2 \dots, p_8\}$, by C the strict transform of C_0, and by S_0 (resp. S_∞) the strict transform of Q_0 (resp. Q_∞). Note that $K_X^{-1} = \mathcal{O}_X(2S_0) = \mathcal{O}_X(2S_\infty)$, $N_{S_0/X} = \mathcal{O}_{S_0}(C)$, and that N_{C/S_0} is isomorphic to $N := \mathcal{O}_{\mathbb{P}^3}(2)|_{C_0} \otimes \mathcal{O}(-p_1 - p_2 - \cdots - p_8)$ via the natural isomorphism $C \to C_0$. When $N \in \mathrm{Pic}^0(C)$ is a torsion element, K_X^{-1} is semi-ample, and thus it is semi-positive. On the other hand, when N is a non-torsion element, the base locus $\mathbb{B}(K_X^{-m})$ is equal to C for each $m \geq 1$ and thus K_X^{-1} is not semi-ample.

	N : torsion	N : non-torsion
the base locus $\mathbb{B}(K_X^{-1})$	$\emptyset$ or C	C
semi-ampleness	semi-ample	not semi-ample
Iitaka dimension	2	1

Table 5.2: Properties of the anti-canonical bundle of the blow-up X of $\mathbb{P}^3$ at 8 points

Proof of Corollary 5.1.2. We apply Theorem 5.1.1 on the triple (C, S_0, X). For this purpose, we show that $N_{S_0/X} = \mathcal{O}_X(S_0)|_{S_0} = \mathcal{O}_X(S_\infty)|_{S_0} = \mathcal{O}_{S_0}(C)$ is flat on a neighborhood of C in S_0 by applying Ueda's theory on the pair (C, S_0). From the assumption, N_{C/S_0} is an element of $\mathcal{E}_1(C)$. Note that $H^1(C, N_{C/S_0}^{-n}) = 0$ holds for $n \geq 1$, since C is an elliptic curve and N_{C/S_0} is non-torsion. Thus $u_n(C, S_0) = 0$ holds for all $n \geq 1$ and then it follows from Theorem 1.2.4 that $N_{S_0/X} = \mathcal{O}_{S_0}(C)$ is flat on a neighborhood of C in S_0.

As the triple (C, S_0, X) enjoys the condition (ii) in Theorem 5.1.1 and it follows from just the same argument on the cohomology classes as above that $u_{n,m}(C, S_0, X) = 0$, we can apply Theorem 5.1.1 to conclude that $\mathcal{O}_X(S_0)$ is flat on a neighborhood W of C in X. As a conclusion, it can be said there exists a flat metric h_1 on the line bundle $K_X^{-1}|_W$.

Let $f_0 \in H^0(X, S_0)$ and $f_\infty \in H^0(X, S_\infty)$ be canonical sections. Denote by h_2 the singular Hermitian metric defined by two sections $f_0^2, f_\infty^2 \in H^0(X, K_X^{-1})$: $h_2 = (|f_0|^2 + |f_\infty|^2)^{-1}$. Clearly h_2 has a semi-positive curvature current on the whole X and a singularity along C. We can construct a smooth Hermitian metric on K_X^{-1} with semi-positive curvature by taking the "regularized minimum" of two metrics $M \cdot h_1$ and h_2, where M is a sufficiently large real number (see the proof of Corollary 3.3.4 for the precise meaning of the "regularized minimum" of the singular Hermitian metrics). □

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