博士論文(要約)

論文題目　On Boundedness of Volumes and Birationality in Birational Geometry
(双有理幾何学における体積と双有理性の有界性について)

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This is a summary of my Ph.D. thesis. Throughout this thesis, we work over the field of complex numbers $\mathbb{C}$.

The aim of birational geometry is to classify all varieties up to birational equivalence. According to Minimal Model Program, minimal varieties and Fano varieties with mild singularities form fundamental classes in birational geometry. To understand these special classes of varieties, it is very natural and interesting to prove some boundedness results. The goal of this thesis is to collect my recent works in birational geometry centered around the theme of boundedness.

Chapter 1 contains a brief summary of the motivations, main problems, histories, and main results on boundedness of volumes and birationality.

Chapter 2 provides basic knowledge on volumes, Hirzebruch surfaces, non-klt centers, connectedness lemma, rational map defined by a Weil divisor, Reid's Riemann–Roch formula, and so on. Basic lemmas are also provided to support the following chapters.

Chapter 3 focuses on the boundedness of anti-canonical volumes. We prove Weak Borisov–Alexeev–Borisov Conjecture in dimension three which states that the anti-canonical volume of an ϵ-klt log Fano pair of dimension three is bounded from above. As a corollary, we give a different proof of boundedness of log Fano threefolds of fixed index.

Chapters 4 and 5 are devoted to the boundedness of birationality.

In Chapter 4, we investigate the pluri-anti-canonical linear systems of weak $\mathbb{Q}$-Fano 3-folds. We prove that, for a $\mathbb{Q}$-Fano 3-fold X, $|-mK_X|$ gives a birational map for $m \geq 39$, and for a weak $\mathbb{Q}$-Fano 3-fold X, $|-mK_X|$ gives a birational map for $m \geq 97$. We also consider the generic finiteness and prove that for a $\mathbb{Q}$-Fano 3-fold X, $|-mK_X|$ gives a generically finite map for $m \geq 28$. Plenty of examples are provided for discussing the optimality of these results.

In Chapter 5, we investigate minimal 3-fold X with numerically trivial canonical divisor and a nef and big Weil divisor L on X. We prove that $|mL|$ and $|K_X + mL|$ give birational maps for $m \geq 17$.

Chapters 3 and 5 are based on my preprints [Jiang14b, Jiang14a]. Chapter 4 is based on a joint work with Meng Chen [CJ14].

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1 Boundedness of anti-canonical volumes

Definition 1.1. A *pair* (X, Δ) consists of a normal projective variety X and an effective $\mathbb{Q}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. (X, Δ) is called a *log Fano pair* (resp. *weak log Fano pair*) if $-(K_X + \Delta)$ is ample (resp. nef and big). If $\dim X = 2$, we will use *del Pezzo* instead of Fano.

Definition 1.2. Let (X, Δ) be a pair. Let $f : Y \to X$ be a log resolution of (X, Δ), write

$$K_Y = f^*(K_X + \Delta) + \sum a_i F_i,$$

where F_i is a prime divisor. The coefficient a_i is called the *discrepancy* of F_i with respect to (X, Δ), and denoted by $a_{F_i}(X, \Delta)$. For some $\epsilon \in [0, 1]$, the pair (X, Δ) is called

(a) *ϵ-kawamata log terminal* (*ϵ-klt*, for short) if $a_i > -1 + \epsilon$ for all i;
(b) *ϵ-log canonical* (*ϵ-lc*, for short) if $a_i \geq -1 + \epsilon$ for all i;
(c) *terminal* if $a_i > 0$ for all f-exceptional divisors F_i.

Note that 0-klt (resp. 0-lc) is just klt (resp. lc) in the usual sense.

Definition 1.3. A variety X is *of ϵ-Fano type* if there exists an effective $\mathbb{Q}$-divisor Δ such that (X, Δ) is an ϵ-klt log Fano pair.

We are mainly interested in the boundedness of ϵ-Fano type varieties.

Definition 1.4. A collection of varieties $\{X_\lambda\}_{\lambda \in \Lambda}$ is said to be *bounded* if there exists $h : \mathcal{X} \to S$ a morphism of finite type of Neotherian schemes such that for each X_λ, $X_\lambda \simeq \mathcal{X}_s$ for some $s \in S$.

Our motivation is the following BAB Conjecture due to A. Borisov, L. Borisov, and V. Alexeev.

Conjecture 1.5 (BAB Conjecture)**.** *Fix $0 < \epsilon < 1$, an integer $n > 0$.*
Then the set of all n-dimensional ϵ-Fano type varieties is bounded.

BAB Conjecture is one of the most important conjecture in birational geometry and it is related to the termination of flips. As the approach to this conjecture, we are interested in the following much weak conjecture for anti-canonical volumes which is a consequence of BAB Conjecture.

Conjecture 1.6 (Weak BAB Conjecture)**.** *Fix $0 < \epsilon < 1$, an integer $n > 0$.*
Then there exists a number $M(n, \epsilon)$ depending only on n and ϵ with the following property:
If (X, Δ) is an n-dimensional ϵ-klt log Fano pair, then

$$\mathrm{Vol}(-(K_X + \Delta)) = (-(K_X + \Delta))^n \leq M(n, \epsilon).$$

Further, if K_X is $\mathbb{Q}$-Cartier, then

$$\mathrm{Vol}(-K_X) \leq M(n, \epsilon).$$

BAB Conjecture was proved in dimension two by Alexeev [Ale94] with a simplified argument by Alexeev–Mori [AM04]. In dimension three or higher dimension, BAB Conjecture is still open. There are only some partial boundedness results. For example, we have boundedness of smooth Fano manifolds by Kollár–Miyaoka–Mori [KoMiMo92], that of terminal $\mathbb{Q}$-Fano $\mathbb{Q}$-factorial threefolds of Picard number one by

Kawamata [Kaw92a], that of canonical $\mathbb{Q}$-Fano threefolds by Kollár–Miyaoka–Mori–Takagi [KMMT00], and that of toric varieties by Borisov–Borisov [BB92].

Weak BAB Conjecture in dimension two was treated by Alexeev [Ale94], Alexeev–Mori [AM04], and Lai [Lai12]. Recently, the author [Jiang13] gave an optimal value for the number $M(2, \epsilon)$. For Weak BAB Conjecture in dimension three assuming that Picard number of X is one, an effective value of $M(3, \epsilon)$ was announced by Lai [Lai12]. For general case of dimension three and higher, Weak BAB Conjecture is still open.

As the main theorem in Chapter 3, we prove Weak BAB Conjecture in dimension three.

Theorem 1.7. *Weak BAB Conjecture holds for* $n = 3$.

As a consequence, we get a different proof of a result on the boundedness of log Fano varieties of fixed index in dimension three which was conjectured by Batyrev, and proved by A. Borisov [Bor01] in dimension three and Hacon–M^{c}Kernan–Xu [HMX14, Corollary 1.8] in arbitrary dimension.

Corollary 1.8. *Fix a positive integer* r*.*

Let $\mathcal{D}$ *be the set of all normal projective varieties* X*, where* $\dim X = 3$*,* K_X *is* $\mathbb{Q}$*-Cartier, and there exists an effective* $\mathbb{Q}$*-divisor* Δ *such that* (X, Δ) *is klt and* $-r(K_X + \Delta)$ *is Cartier and ample.*

Then $\mathcal{D}$ *forms a bounded family.*

1.1 Description of the proof

Firstly, we give an approach to Weak BAB Conjecture via Mori fiber spaces.

Definition 1.9. A projective morphism $X \to T$ between normal varieties is called a *Mori fiber space* if the following conditions hold:

(i) X is $\mathbb{Q}$-factorial with terminal singularities.
(ii) f is a *contraction*, i.e. $f_*\mathcal{O}_X = \mathcal{O}_T$.
(iii) $-K_X$ is ample over T.
(iv) $\rho(X/T) = 1$.
(v) $\dim X > \dim T$.

At this time, we say that X is with a *Mori fiber structure.*

We raise the following conjecture for Mori fiber spaces.

Conjecture 1.10 (Weak BAB Conjecture for Mori fiber spaces)**.** *Fix* $0 < \epsilon < 1$*, an integer* $n > 0$*.*

Then there exists a number $M(n, \epsilon)$ *depending only on* n *and* ϵ *with the following property:*

If X *is an* n*-dimensional* ϵ*-Fano type variety with a Mori fiber structure, then*

$$\mathrm{Vol}(-K_X) \leq M(n, \epsilon).$$

We prove the following theorem by using Minimal Model Program.

Theorem 1.11. *Weak BAB Conjecture holds for fixed* ϵ *and* n *if and only if Weak BAB Conjecture for Mori fiber spaces holds for fixed* ϵ *an* n*.*

By Theorem 1.11, to consider the boundedness of anti-canonical volumes of log Fano pairs, we only need to consider the ones with better singularities ($\mathbb{Q}$-factorial terminal singularities) and with additional structures (Mori fiber structures). This is the advantage of this theorem. In dimension two, this theorem appears as a crucial step to get the optimal value of $M(2,\epsilon)$ (c.f. [Jiang13]).

Restricting our interest to dimension three, we prove the following theorem.

Theorem 1.12. *Weak BAB Conjecture for Mori fiber spaces holds for $n = 3$.*

Theorem 1.7 follows from Theorems 1.11 and 1.12 directly.

To prove Theorem 1.12 , we need to consider ϵ-Fano type 3-fold X with a Mori fiber structure $X \to T$. There are 3 cases:

(1) $\dim T = 0$, X is a $\mathbb{Q}$-factorial terminal $\mathbb{Q}$-Fano 3-folds with $\rho = 1$;
(2) $\dim T = 1$, $X \to T \simeq \mathbb{P}^1$ is a *del Pezzo fibration*, i.e. a general fiber is a smooth del Pezzo surface;
(3) $\dim T = 2$, $X \to T$ is a *conic bundle*, i.e. a general fiber is a smooth rational curve.

The second statement is implied by the following fact: if (X, Δ) is a klt log Fano pair, then X is rationally connected (see [Zha06, Theorem 1]), in particular, for any surjective morphism $X \to T$ to a normal curve, $T \simeq \mathbb{P}^1$.

In Case (1), X is bounded by Kawamata [Kaw92a], and the optimal bound of $\mathrm{Vol}(-K_X) = (-K_X)^3$ is 64 due to the classification on smooth Fano 3-folds of Iskovskikh and Mori–Mukai and by Namikawa's result [Nam97] (Gorenstein case) and Prokhorov [Pro07] (non-Gorenstein case).

We will mainly treat Cases (2) and (3).

One basic idea is to construct singular pairs which is not klt along fibers of $X \to T$. Then by Connectedness Lemma, we may find a non-klt center intersecting with the fibers. Finally by restricting on a general fiber, we get the bound after some arguments on lower dimensional varieties. But several difficulties arise here.

In Case (3), the difficulty arises in the construction of singular pair because we need to avoid components which are vertical over T. To do this, we need a good understanding of the singularities and boundedness of the surface T, which was done by several papers as [Ale94], [MP08], and [Bir14].

In Case (2), the difficulty arises in the last step. After restricting on a general fiber, we need to bound the (generalized) log canonical thresholds on surfaces. So we are done by proving the following (generalized) Ambro's conjecture in dimension two.

Definition 1.13. Let (X, B) be a lc pair and $D \geq 0$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. The *log canonical threshold* of D with respect to (X, B) is

$$\mathrm{lct}(X, B; D) = \sup\{t \in \mathbb{Q} \mid (X, B + tD) \text{ is lc}\}.$$

For the use of this thesis, we need to consider the case when D is not effective. Let G be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor satisfying $G + B \geq 0$, The *generalized log canonical threshold* of G with respect to (X, B) is

$$\mathrm{glct}(X, B; G) = \sup\{t \in [0, 1] \cap \mathbb{Q} \mid (X, B + tG) \text{ is lc}\}.$$

Conjecture 1.14 (Ambro's conjecture). *Fix $0 < \epsilon < 1$ and integer $n > 0$.*

Then there exists a number $\mu(n, \epsilon) > 0$ depending only on n and ϵ with the following property:

If (Y, B) is an ϵ-klt log Fano pair of dimension n, then

$$\inf\{\mathrm{lct}(Y, B; D) \mid D \sim_{\mathbb{Q}} -(K_Y + B), D \geq 0\} \geq \mu(n, \epsilon).$$

Note that we do not assume any special conditions on the coefficients of B. The left-hand side of the inequality is called *α-invariant* of (Y, B) which generalizes the concept of α-invariant of Tian for Fano manifolds in differential geometry (see [CMG14, CS08, Tian87]). Recently Ambro [Amb14] announced a proof of this conjecture assuming that (Y, B) is a toric pair where an explicit sharp number $\mu(n, \epsilon)$ was given. For the use of this paper, we need a stronger version of this conjecture where D may not be effective.

Conjecture 1.15 (generalized Ambro's conjecture)**.** *Fix $0 < \epsilon < 1$ and integer $n > 0$.*

Then there exists a number $\mu(n, \epsilon) > 0$ depending only on n and ϵ with the following property:

If (Y, B) is an ϵ-klt weak log Fano pair of dimension n and Y has at worst terminal singularities, then

$$\inf\{\mathrm{glct}(Y, B; G) \mid G \sim_{\mathbb{Q}} -(K_Y + B), G + B \geq 0\} \geq \mu(n, \epsilon).$$

Note that Conjecture 1.14 follows from Conjecture 1.15 easily after taking terminalization of (Y, B).

We prove the conjecture in dimension two by following some ideas in the proof of BAB Conjecture in dimension two ([Ale94, AM04]). But it seems that this conjecture does not follow from BAB Conjecture trivially.

Theorem 1.16. *Conjecture 1.15 holds for $n = 2$.*

For the proof of Corollary 1.8, we basically follow the idea in [Bor01] to bound the Hilbert polynomials by [KoMa83].

2 Boundedness of birationality

Definition 2.1. A normal projective variety X is called a *weak $\mathbb{Q}$-Fano variety* if X has at worst $\mathbb{Q}$-factorial terminal singularities and the anti-canonical divisor $-K_X$ is nef and big. A weak $\mathbb{Q}$-Fano variety is said to be *$\mathbb{Q}$-Fano* if $-K_X$ is $\mathbb{Q}$-ample and the Picard number $\rho(X) = 1$.

Definition 2.2. A normal projective variety X is said to be *minimal* if X has at worst $\mathbb{Q}$-factorial terminal singularities and the canonical divisor K_X is nef.

According to Minimal Model Program, $\mathbb{Q}$-Fano varieties and minimal varieties form fundamental classes in birational geometry.

Given an n-dimensional normal projective variety X with mild singularities and a big Weil divisor L on X, we are interested in the geometry of the rational map $\Phi_{|mL|}$ defined by the linear system $|mL|$. By definition, $\Phi_{|mL|}$ is birational onto its image when m is sufficiently large. Therefore it is interesting to find such a practical number $m(n)$, depending only on $\dim X$, which stably guarantees the birationality of $\Phi_{|mL|}$. In fact, the following three special cases are the most interesting:

(i) K_X is nef and big, $L = K_X$;

(ii) $K_X \equiv 0$, L is an arbitrary nef and big Weil divisor;

(iii) $-K_X$ is nef and big, $L = -K_X$.

It is an interesting exercise to deal the case X being a smooth curve or surface.

Theorem 2.3 (c.f. Bombieri [Bom73], Reider [Reider88])**.** *Let S be a smooth surface.*

(i) If K_S is nef and big, then $|mK_S|$ gives a birational map for $m \geq 5$;

(ii) If $K_S \equiv 0$, then $|mL|$ gives a birational map for $m \geq 3$ and L an arbitrary nef and big divisor;

(iii) If $-K_S$ is nef and big, then $|-mK_S|$ gives a birational map for $m \geq 3$.

For a 3-fold X, when X is smooth, these cases were treated by Matsuki [Mat86], Ando [Ando87], Fukuda [Fuk91], Oguiso [Ogu91], and many others, and we have the following known results.

Theorem 2.4 (Matsuki [Mat86], Fukuda [Fuk91]). *Let X be a smooth 3-fold.*

(i) If K_X is nef and big, then $|mK_X|$ gives a birational map for $m \geq 6$;

(ii) If $K_X \equiv 0$, then $|mL|$ gives a birational map for $m \geq 6$ and L an arbitrary nef and big divisor;

(iii) If $-K_X$ is nef and big, then $|-mK_X|$ gives a birational map for $m \geq 4$.

When X is a 3-fold with $\mathbb{Q}$-factorial terminal singularities, Case (i) was systematically treated by J. A. Chen and M. Chen [CC10a, CC10b, CC13].

Theorem 2.5 (Chen–Chen [CC13]). *Let X be a minimal 3-fold of general type (i.e K_X is nef and big), then $|mK_X|$ gives a birational map for $m \geq 61$.*

We are going to treat Cases (ii) and (iii) systematically.

2.1 $\mathbb{Q}$-Fano threefolds

In Chapter 4, for a weak $\mathbb{Q}$-Fano 3-fold X, the *anti-m-canonical map* φ_{-m} is the rational map defined by the linear system $|-mK_X|$. Such a number m_3 that stably guarantees the birationality of φ_{-m_3} exists due to the boundedness of $\mathbb{Q}$-Fano 3-folds, which was proved by Kawamata [Kaw92a], and the boundedness of weak $\mathbb{Q}$-Fano 3-folds proved by Kollár–Miyaoka–Mori–Takagi [KMMT00]. It is natural to consider the following problem.

Problem 2.6. *Find the optimal constant c such that φ_{-m} is birational onto its image for all $m \geq c$ and for all (weak) $\mathbb{Q}$-Fano 3-folds.*

The following example tells us that $c \geq 33$.

Example 2.7 ([IF00, List 16.6, No.95]). The general weighted hypersurface $X_{33} \subset \mathbb{P}(1, 5, 6, 22, 33)$ is a $\mathbb{Q}$-Fano 3-fold. It is clear that φ_{-m} is birational onto its image for $m \geq 33$, but φ_{-32} fails to be birational.

It is worthwhile to compare the birational geometry induced from $|mK|$ on varieties of general type with the geometry induced from $|-mK|$ on (weak) $\mathbb{Q}$-Fano varieties. An obvious feature on Fano varieties is that the behavior of φ_{-m} is not necessarily birationally invariant. For example, consider degree 2 (rational) del Pezzo surface S_2 and $\mathbb{P}^2$, $|-K_{\mathbb{P}^2}|$ gives a birational map but $|-K_{S_2}|$ does not. This causes difficulties in studying Problem 2.6. In fact, even if in dimension 3, there is no known practical upper bound for c in written records.

When X is smooth, we may take $c = 4$ according to Ando [Ando87] and Fukuda [Fuk91]. When X has terminal singularities, Problem 2.6 was treated by M. Chen in [Chen11], where an effective upper bound of c in terms of the Gorenstein index of X is proved (cf. [Chen11, Theorem 1.1]). Since, however, the Gorenstein index of a weak $\mathbb{Q}$-Fano 3-fold can be as large as "840", the number "$3 \times 840 + 10 = 2530$" obtained in [Chen11, Theorem 1.1] is far from being optimal. It turns out that Problem 2.6 is closely related to the following problem (cf. [Chen11, Theorem 4.5]).

Problem 2.8. *Given a (weak) $\mathbb{Q}$-Fano 3-fold X, can one find the least positive integer $\delta_1 = \delta_1(X)$ such* *that* $\dim \overline{\varphi_{-\delta_1}(X)} > 1$*?*

Problem 2.8 is parallel to the following question on 3-folds of general type:

> *Let Y be a 3-fold of general type on which $|nK_Y|$ is composed with a pencil of surfaces for some fixed integer $n > 0$. Can one find an integer m (bounded from above by a function in terms of n) so that $|mK_Y|$ is not composed with a pencil any more?*

This question was solved by Kollár [Kol86] who proved that one may take $m \leq 11n + 5$. The result is a direct application of the semi-positivity of $f_*\omega^l_{Y/B}$ since, modulo birational equivalence, one may assume that there is a fibration $f : Y \longrightarrow B$ onto a curve B. As far as we know, there is still no known analogy of Kollár's method in treating $\mathbb{Q}$-Fano varieties.

Firstly, we shall prove the following theorem.

Theorem 2.9. *Let X be a $\mathbb{Q}$-Fano 3-fold. Then there exists an integer $n_1 \leq 10$ such that* $\dim \overline{\varphi_{-n_1}(X)} > 1$.

Theorem 2.9 is close to be optimal due to the following example.

Example 2.10 ([IF00, List 16.7, No.85])**.** Consider the general codimension 2 weighted complete intersection $X := X_{24,30} \subset \mathbb{P}(1, 8, 9, 10, 12, 15)$ which is a $\mathbb{Q}$-Fano 3-fold. Then $\dim \overline{\varphi_{-9}(X)} > 1$ while $\dim \overline{\varphi_{-8}(X)} = 1$ since $h^0(-8K_X) = 2$.

In fact, theoretically, there are only 4 possible weighted baskets for which we need to take $n_1 = 10$. Theorem 2.9 allows us to prove the following result.

Theorem 2.11. *Let X be a $\mathbb{Q}$-Fano 3-fold. Then φ_{-m} is birational onto its image for all $m \geq 39$.*

In particular, as a by-product we have the following corollary which is optimal.

Corollary 2.12. *Let X be a $\mathbb{Q}$-Fano 3-fold.*

(i) If $h^0(-K_X) \geq 3$, then φ_{-m} is birational onto its image for all $m \geq 6$;
(ii) If $h^0(-K_X) = 2$, then φ_{-m} is birational onto its image for all $m \geq 21$.

The optimality is shown by the general weighted hypersurfaces $X_{12} \subset \mathbb{P}(1, 1, 1, 4, 6)$ and $X_{42} \subset \mathbb{P}(1, 1, 6, 14, 21)$ ([IF00, List 16.6, No.14, No.88]).

T. Sano suggested that we can consider the generic finiteness of φ_{-m}, and we get the following result.

Theorem 2.13. *Let X be a $\mathbb{Q}$-Fano 3-fold. Then φ_{-m} is generically finite onto its image for all $m \geq 28$.*

Note that in Example 2.7, φ_{-22} is generically finite onto its image but φ_{-21} is not.

A key point in proving Theorem 2.9 is that we have $\rho(X) = 1$, which is not the case for arbitrary weak $\mathbb{Q}$-Fano 3-folds. Therefore we should study weak $\mathbb{Q}$-Fano 3-folds in an alternative way. Our result is as follows.

Theorem 2.14. *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Then* $\dim \overline{\varphi_{-n_2}(X)} > 1$ *for all $n_2 \geq 71$.*

Theorem 2.14 allows us to study the birationality.

Theorem 2.15. *Let X be a weak $\mathbb{Q}$-Fano 3-fold. Then φ_{-m} is birational onto its image for all $m \geq 97$.*

Also we can prove similar result on generic finiteness, but it seems not so interesting.

2.2 Minimal 3-folds with $K \equiv 0$

In Chapter 5, for a minimal 3-fold X with $K_X \equiv 0$ and an arbitrary nef and big Weil divisor L on X, we are interested in the rational map $\Phi_{|mL|}$ defined by the linear system $|mL|$. If X is smooth, then $|mL|$ gives a birational map for $m \geq 6$ by Fukuda [Fuk91]. If X is with Gorenstein terminal singularities and $q(X) := h^1(\mathcal{O}_X) = 0$, then $|mL|$ gives a birational map for $m \geq 5$ by Oguiso–Peternell [OP95].

The motivation of Chapter 5 is to systematically study the birational geometry of minimal 3-fold with $K \equiv 0$. For an arbitrary nef and big Weil divisor L on X, we investigate the birationality of the linear system $|mL|$. For special interest, we also investigate the birationality of the adjoint linear system $|K_X + mL|$.

The difficulty arises from the singularities of X, and the assumption that L is only a Weil divisor. If we assume that L is Cartier, then the problem becomes relatively easy and can be treated by the method of Fukuda [Fuk91] using Reider's theorem [Reider88]. On the other hand, fortunately, the singularities of minimal 3-folds with $K \equiv 0$ is not so complicated due to Kawamata [Kaw86] and Morrison [Mor86], and this makes it possible to deal with the birationality problem.

We prove the following theorem.

Theorem 2.16. *Let X be a minimal 3-fold with $K_X \equiv 0$ and a nef and big Weil divisor L. Then $|mL|$ and $|K_X + mL|$ give birational maps for all $m \geq 17$.*

In fact, we prove a more general theorem.

Theorem 2.17. *Let X be a minimal 3-fold with $K_X \equiv 0$, a nef and big Weil divisor L, and a Weil divisor $T \equiv 0$. Then $|K_X + mL + T|$ gives a birational map for all $m \geq 17$.*

Moreover, by Log Minimal Model Program, the assumption that L is nef can be weaken. We say that a divisor D *has no stable base components* if $|mD|$ has no base components for sufficiently divisible m.

Theorem 2.18. *Let X be a minimal 3-fold with $K_X \equiv 0$, a big Weil divisor L without stable base components, and a Weil divisor $T \equiv 0$. Then $|K_X + mL + T|$ gives a birational map for all $m \geq 17$. In particular, $|mL|$ and $|K_X + mL|$ give birational maps for all $m \geq 17$.*

As a by-product, we prove a direct generalization of Fukuda [Fuk91] and Oguiso–Peternell [OP95] which is optimal by the general weighted hypersurface $X_{10} \subset \mathbb{P}(1,1,1,2,5)$.

Theorem 2.19. *Let X be a minimal Gorenstein 3-fold with $K_X \equiv 0$, a nef and big Weil divisor L, and a Weil divisor $T \equiv 0$. Then $|K_X + mL + T|$ gives a birational map for all $m \geq 5$.*

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