博士論文

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Studies on the minimal log discrepancy

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Preface

Birational geometry is a part of algebraic geometry and its goal is to classify algebraic varieties up to birational equivalence. The minimal model program was proposed in 1980's and is a way to find a "simplest" variety in each birational equivalent class. The conjectures on flips—the existence of flips and the termination of flips—are necessary for running the program, but they had been widely open till 2000's. One epoch-making result was by Birkar, Cascini, Hacon, and M$^{\mathrm{c}}$Kernan [4]. They proved that the minimal model program works for varieties of general type. Further, they proved the existence of flips. On the other hand, the termination of flips is still open in general.

The minimal log discrepancy (mld for short) is an invariant of singularities and it was introduced by Shokurov, in order to reduce the conjecture of terminations of flips to a local problem about singularities. Recently, this has been a fundamental invariant in the minimal model program. There are two related conjectures on mld's, the ACC (ascending chain condition) conjecture and the LSC (lower semi-continuity) conjecture. Shokurov showed that these two conjectures imply the conjecture of terminations of flips [25].

In this thesis, we focus on the ACC conjecture. For a positive integer d and a subset $I \subset [0, 1]$ which satisfies the descending chain condition, the ACC conjecture states that the set of minimal log discrepancies

$$\{\mathrm{mld}_x(X, \Delta) \mid (X, \Delta) \text{ is a log pair, } \dim X = d,\, \Delta \in I,\, x \in X\}$$

satisfies the ascending chain condition. Here, we consider all divisor Δ whose coefficients belong to I.

The ACC conjecture is known under some conditions (see Fact 1.2.4 for details). It is known for $d \leq 2$ by Alexeev [1] and Shokurov [23]. Further, Kawakita [12] proved that the ACC conjecture is true for smooth varieties and a finite set I. More generally, he proved the finiteness of the set of log discrepancies on a fixed variety. Precisely, he proved that the set

$$\Big\{ \operatorname{mld}_x \big(X, \sum_i d_i D_i\big) \mid \ d_i \in I,\ x \in X \text{ and } D_i \text{ are effective Cartier divisors} \Big\}$$

becomes finite for a fixed $\mathbb{Q}$-Gorenstein normal variety X and a finite set I. The purpose of this thesis is to generalize this results to the family of the varieties with fixed Gorenstein index (Theorem 2.1.1). Further, as a corollary, we prove the ACC conjecture for three-dimensional canonical pairs (Corollary 2.1.2).

Theorem 2.1.1 is proved by induction on $\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}(I \cup \{1\})$, the dimension of the $\mathbb{Q}$-vector space generated by $I \cup \{1\}$. In the inductive step, we need Theorem 2.1.4, which is about perturbation of an irrational coefficient of log canonical pairs. This theorem is a generalization of the rationality theorem of accumulation points of log canonical thresholds proved by Hacon, M$^{\text{c}}$Kernan, and Xu [10, Theorem 1.11].

In Chapter 1, we define the minimal log discrepancy and we state the ACC conjecture and the LSC conjecture. Further we list some known results around these two conjectures.

In Chapter 2, we prove the main theorem. First, in Section 2.1, we list the statements of the main theorem and its corollaries. Further, in Section 2.2, we list some known propositions which are necessary for the proof of the main theorem. In Section 2.3, we study the accumulation points of log canonical thresholds, and prove Theorem 2.3.8. The essential idea of proof is due to the paper [10]. In Section 2.4, we prove Theorem 2.1.4. In Section 2.5, we prove the main theorem (Theorem 2.1.1) and the corollaries.

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1 Introduction

Notation and convention

Throughout this thesis, we work over the field of complex numbers $\mathbb{C}$.

- For an $\mathbb{R}$-divisor D and a subset $I \subset \mathbb{R}$, we write $D \in I$ when all the non-zero coefficients of D belong to I.
- For an $\mathbb{R}$-ideal sheaf $\mathfrak{A} = \prod \mathfrak{a}_i^{r_i}$ and a subset $I \subset \mathbb{R}$, we write $\mathfrak{A} \in I$ when all the non-zero coefficients r_i of $\mathfrak{A}$ belong to I.
- For a subset $I \subset \mathbb{R}$, we say that I satisfies *the ascending chain condition* (resp. *the descending chain condition*) when there is no infinite increasing (resp. decreasing) sequence $a_i \in I$. *ACC* (resp. *DCC*) stands for the ascending chain condition (resp. the descending chain condition).

1.1 Minimal log discrepancies

We recall some notations in the theory of singularities in the minimal model program. For more details we refer the reader [17].

A *log pair* (X, Δ) is a normal variety X and an effective $\mathbb{R}$-divisor Δ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. If X is $\mathbb{Q}$-Gorenstein, we sometimes identify X with the log pair $(X, 0)$.

An $\mathbb{R}$-*ideal sheaf* on X is a formal product $\mathfrak{a}_1^{r_1} \cdots \mathfrak{a}_s^{r_s}$, where $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$ are ideal sheaves on X and $r_1, \ldots, r_s$ are positive real numbers. For a log pair (X, Δ) and an $\mathbb{R}$-ideal sheaf $\mathfrak{a}$, we call $(X, \Delta, \mathfrak{a})$ a *log triple*. When $\Delta = 0$ (resp. $\mathfrak{A} = \mathcal{O}_X$), we sometimes drop Δ (resp. $\mathfrak{A}$) and write $(X, \mathfrak{a})$ (resp. (X, Δ)).

For a proper birational morphism $f : X' \to X$ from a normal variety X' and a prime divisor E on X', the *log discrepancy* of $(X, \Delta, \mathfrak{a})$ at E is defined as

$$a_E(X, \Delta, \mathfrak{a}) := 1 + \mathrm{coeff}_E(K_{X'} - f^*(K_X + \Delta)) - \mathrm{ord}_E \mathfrak{a},$$

where $\mathrm{ord}_E \mathfrak{a} := \sum_{i=1}^{s} r_i \mathrm{ord}_E \mathfrak{a}_i$. The image $f(E)$ is called the *center of E on X*, and we denote it by $\mathrm{c}_X(E)$. For a closed subset Z of X, the *minimal log discrepancy* (*mld* for short) over Z is defined as

$$\mathrm{mld}_Z(X, \Delta, \mathfrak{a}) := \inf_{\mathrm{c}_X(E) \subset Z} a_E(X, \Delta, \mathfrak{a}).$$

In the above definition, the infimum is taken over all prime divisors E on X' with the center $\mathrm{c}_X(E) \subset Z$, where X' is a higher birational model of X, that is, X' is the source of some proper birational morphism $X' \to X$.

Remark 1.1.1. It is known that $\mathrm{mld}_Z(X, \Delta, \mathfrak{a})$ is in $\mathbb{R}_{\geq 0} \cup \{-\infty\}$ and that if $\mathrm{mld}_Z(X, \Delta, \mathfrak{a}) \geq 0$, then the infimum on the right hand side in the definition is actually the minimum.

Remark 1.1.2. Let D_i be effective Weil divisors on X, and $\mathfrak{a}_i := \mathcal{O}_X(-D_i)$ the corresponding ideal sheaves. When X is $\mathbb{Q}$-Gorenstein and D_i are Cartier divisors, we can identify $(X, \sum r_i D_i)$ and $(X, \prod \mathfrak{a}_i^{r_i})$. Indeed, for any divisor E over X, we have $a_E(X, \sum r_i D_i) = a_E(X, \prod \mathfrak{a}_i^{r_i})$.

For simplicity of notation, we write $\mathrm{mld}_x(X, \Delta, \mathfrak{a})$ instead of $\mathrm{mld}_{\{x\}}(X, \Delta, \mathfrak{a})$ for a closed point x of X, and write $\mathrm{mld}(X, \Delta, \mathfrak{a})$ instead of $\mathrm{mld}_X(X, \Delta, \mathfrak{a})$.

We say that the pair $(X, \Delta, \mathfrak{a})$ is *log canonical* (*lc* for short) if $\mathrm{mld}(X, \Delta, \mathfrak{a}) \geq 0$. Further, we say that the pair $(X, \Delta, \mathfrak{a})$ is *Kawamata log terminal* (*klt* for short) if $\mathrm{mld}(X, \Delta, \mathfrak{a}) > 0$. When E is a divisor over X such that $a_E(X, \Delta, \mathfrak{a}) \leq 0$, the center $\mathrm{c}_X(E)$ is called a *non-klt center*.

We say that the pair $(X, \Delta, \mathfrak{a})$ is *canonical* (resp. *terminal*) if $a_E(X, \Delta, \mathfrak{a}) \geq 1$ (resp. > 1) for any exceptional divisor E over X.

1.2 Conjectures – the ACC conjecture and the LSC conjecture

The ACC (ascending chain condition) conjecture was proposed by Shokurov [24, Conjecture 4.2]. This conjecture states that the set of the minimal log discrepancies satisfies the ACC when we assume that the coefficients of the boundary divisor belong to a fixed DCC set.

Conjecture 1.2.1 (ACC conjecture [24, Conjecture 4.2])**.** *Fix $d \in \mathbb{Z}_{>0}$ and a subset $I \subset [0,1]$ which satisfies the DCC. Then the following set*

$$A(d, I) := \{\mathrm{mld}_x(X, \Delta) \mid (X, \Delta) \textit{ is a log pair}, \dim X = d, \Delta \in I, x \in X\}$$

satisfies the ACC, where x is a closed point of X.

The LSC (lower semi-continuity) conjecture was proposed by Ambro [2].

Conjecture 1.2.2 (LSC conjecture [2])**.** *For a fixed log pair (X, Δ), the function*

$$|X| \to \mathbb{R}_{\geq 0} \cup \{-\infty\}; \quad x \mapsto \mathrm{mld}_x(X, \Delta)$$

is lower semi-continuous.

These two conjectures are motivated by the conjecture of termination of flips. Actually, Shokurov proved that these two conjectures imply the termination of flips [25].

Theorem 1.2.3 (Shokurov [25])**.** *The ACC conjecture and the LSC conjecture imply the termination of flips. More precisely, the ACC conjecture for an arbitrary finite set I and the LSC conjecture imply the termination of flips.*

By this reason, we are mainly interested in the case when I is a finite set.

Fact 1.2.4. The ACC conjecture is known in each of the following cases.

(1) In the case when $d = 2$ [1], [23].

(2) For toric pairs [3].

(3) If $I \subset [0,1]$ is a finite set, then the set

$$\begin{aligned} A_{\mathrm{sm}}(d, I) := \{\mathrm{mld}_x(X, \Delta) \mid (X, \Delta) &\text{ is a log pair with } X \text{ smooth}, \\ &\dim X = d, \Delta \in I, x \in X\} \end{aligned}$$

is a finite set [12].

(4) If $I \subset [0,1]$ is a finite set and X is a $\mathbb{Q}$-Gorenstein normal variety, then the set

$$\begin{aligned} A'(X, I) := \Big\{ \mathrm{mld}_x \Big(X, \sum_i d_i D_i\Big) \mid d_i \in I, x \in X & \\ \text{and } D_i \text{ are effective Cartier divisors} &\Big\} \end{aligned}$$

is a finite set [12].

(5) If $I \subset [0,1]$ satisfies the DCC, then $A_{\mathrm{sm}}(3, I) \cap [1,3]$ satisfies the ACC [13].

Fact 1.2.5. The LSC conjecture is known in each of the following cases.

(1) In the case when $\dim X \leq 3$ [2].

(2) For toric pairs [2].

(3) In the case when X is smooth [7].

(4) In the case when X is a locally complete intersection variety [6].

(5) In the case when X has only quotient singularities [22].

Both the ACC conjecture and the LSC conjecture imply the boundedness of the minimal log discrepancy (BDD Conjecture). Hence, the following BDD (boundedness) conjecture is much weaker than these two conjectures, but still not known.

Conjecture 1.2.6 (BDD conjecture)**.** *For fixed $d \in \mathbb{Z}_{>0}$, there exists a real number $a(d)$ such that $\mathrm{mld}(X) \leq a(d)$ holds for any $\mathbb{Q}$-Gorenstein d-dimensional normal variety X.*

The BDD conjecture is known only for $d \leq 3$ [20]. In arbitrary dimensions, the conjecture is known for the family of varieties with bounded multiplicity [11].

The main results of this thesis are the following.

Theorem 1.2.7 (= Corollary 2.1.2)**.** *Fix $d \in \mathbb{Z}_{>0}$, $r \in \mathbb{Z}_{>0}$ and a finite subset $I \subset [0,1]$. Then the following set*

$$\begin{aligned} A'(d,r,I) := \Big\{ \mathrm{mld}_x\big(X, \sum_i d_i D_i\big) \;\Big|\; & d_i \in I,\ x \in X, \\ & X \textit{ is a normal variety of dimension } d, \\ & rK_X \textit{ is Cartier and } D_i \textit{ are effective Cartier divisors} \Big\} \end{aligned}$$

is discrete in $\mathbb{R}$.

This is a generalization of Fact 1.2.4 (4). As a corollary, we prove the ACC for three-dimensional canonical pairs.

Theorem 1.2.8 (= Corollary 2.1.3)**.** *If $I \subset [0,1]$ is a finite subset, the following set*

$$\{\mathrm{mld}_x(X,\Delta) \mid (X,\Delta) \textit{ is a canonical pair, } \dim X = 3,\ \Delta \in I,\ x \in X\},$$

denoted by $A_{can}(3,I)$, satisfies the ACC. Further, 1 is the only accumulation point of $A_{can}(3,I)$.

2

On minimal log discrepancies on varieties with fixed Gorenstein index

2.1 Statements of the main theorem and the corollaries

Kawakita [12] proved that the ACC conjecture is true for a fixed variety X and a finite set I. More generally, he proved the discreteness of the set of log discrepancies for log triples (see Section 1.1 for the definition)

$$\{a_E(X, \Delta, \mathfrak{a}) \mid (X, \Delta, \mathfrak{a}) \text{ is lc, } \mathfrak{a} \in I,\ E \in \mathcal{D}_X\}$$

when the pair (X, Δ) is fixed and I is a finite set. Here, we denoted by $\mathcal{D}_X$ the set of all divisors over X. Further, $\mathfrak{a} = \prod \mathfrak{a}_i^{r_i}$ is an $\mathbb{R}$-ideal sheaf with coefficients r_i in I. The purpose of this chapter is to generalize this result to the family of the varieties with fixed Gorenstein index.

Theorem 2.1.1. *Fix $d \in \mathbb{Z}_{>0}$, $r \in \mathbb{Z}_{>0}$ and a finite subset $I \subset [0, +\infty)$. Then the following set*

$$B(d, r, I) := \{a_E(X, \mathfrak{a}) \mid (X, \mathfrak{a}) \in P(d, r),\ \mathfrak{a} \in I,\ E \in \mathcal{D}_X\} \subset [0, +\infty)$$

is discrete in $\mathbb{R}$. Here we denote by $P(d, r)$ the set of all d-dimensional lc pairs $(X, \mathfrak{a})$ such that rK_X is a Cartier divisor.

Since $\mathrm{mld}_x(X, \mathfrak{a}) = a_E(X, \mathfrak{a})$ holds for some $E \in \mathcal{D}_X$, we get the following Corollary.

Corollary 2.1.2. *Fix $d \in \mathbb{Z}_{>0}$, $r \in \mathbb{Z}_{>0}$ and a finite subset $I \subset [0, +\infty)$. Then the following set*

$$A'(d, r, I) := \{\mathrm{mld}_x(X, \mathfrak{a}) \mid X \in P(d, r),\ \mathfrak{a} \in I,\ x \in X\} \subset [0, +\infty)$$

is discrete in $\mathbb{R}$. Here we denote by $P(d, r)$ the set of all d-dimensional lc pairs $(X, \mathfrak{a})$ such that rK_X is a Cartier divisor.

Corollary 2.1.2 does not imply the finiteness of $A'(d, r, I)$, because we do not know the boundedness of $A'(d, r, I)$. Hence Corollary 2.1.2 shows the finiteness of $A'(d, r, I)$ modulo the BDD conjecture (Conjecture 1.2.6).

As a corollary of Corollary 2.1.2, we can prove the ACC for three-dimensional canonical pairs.

Corollary 2.1.3. *If $I \subset [0, 1]$ is a finite subset, the following set*

$$\{\mathrm{mld}_x(X, \Delta) \mid (X, \Delta) \text{ is a canonical pair, } \dim X = 3,\ \Delta \in I,\ x \in X\},$$

denoted by $A_{can}(3, I)$, satisfies the ACC. Further, 1 is the only accumulation point of $A_{can}(3, I)$.

Theorem 2.1.1 is proved by induction on $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(I \cup \{1\})$, the dimension of the $\mathbb{Q}$-vector space generated by $I \cup \{1\}$. In the inductive step, we need the following theorem, which is about a perturbation of an irrational coefficient of lc pairs.

Theorem 2.1.4. *Fix $d \in \mathbb{Z}_{>0}$. Let $r_1, \ldots, r_{c'}$ be positive real numbers and let $r_0 = 1$. Assume that $r_0, \ldots, r_{c'}$ are $\mathbb{Q}$-linearly independent. Let $s_1, \ldots, s_c : \mathbb{R}^{c'+1} \to \mathbb{R}$ be $\mathbb{Q}$-linear functions from $\mathbb{R}^{c'+1}$ to $\mathbb{R}$. Assume that $s_i(r_0, \ldots, r_{c'}) \in \mathbb{R}_{\geq 0}$ for each i. Then there exists a positive real number $\epsilon > 0$ such that the following holds: For any $\mathbb{Q}$-Gorenstein normal variety X of dimension d and $\mathbb{Q}$-Cartier effective Weil divisors $D_1, \ldots, D_c$ on X, if $(X, \sum_{1 \leq i \leq c} s_i(r_0, \ldots, r_{c'}) D_i)$ is lc, then $(X, \sum_{1 \leq i \leq c} s_i(r_0, \ldots, r_{c'-1}, t) D_i)$ is also lc for any t satisfying $|t - r_{c'}| \leq \epsilon$.*

Remark 2.1.5. The positive real number ϵ in Theorem 2.1.4 does not depend on X, but depends only on d, $r_1, \ldots, r_{c'}$, and $s_1, \ldots, s_c$.

Kawakita [12] proved this theorem for a fixed variety X using a method of generic limit, and prove the discreteness of log discrepancies for fixed X. When $c' = 1$ and each s_i satisfies $s_i(\mathbb{R}^2_{\geq 0}) \subset \mathbb{R}_{\geq 0}$, this theorem just states the rationality of accumulation points of log canonical thresholds proved by Hacon, M^cKernan, and Xu [10, Theorem 1.11]. Actually, the proof of Theorem 2.1.4 heavily depends on their argument. We also note that the rationality of accumulation points of log canonical thresholds on smooth varieties was proved by Kollár [16, Theorem 7] and by de Fernex and Mustaţă [5, Corollary 1.4] using a method of generic limit.

2.2 Preliminaries

In this subsection, we list some known results on extractions of divisors and on the ACC properties of the log canonical threshold. They are used in Section 2.3.

We can extract a divisor whose log discrepancy is at most one.

Theorem 2.2.1. *Let (X, Δ) be a klt pair, and let E be a divisor over X such that $a_E(X, \Delta) \leq 1$. Then there exists a projective birational morphism $\pi : Y \to X$ such that Y is $\mathbb{Q}$-factorial and the only exceptional divisor is E.*

Proof. This is the special case of [4, Corollary 1.4.3]. □

When (X, Δ) is lc, we can find a modification which is dlt. We call a log pair (X, Δ) *divisorial log terminal* (*dlt* for short) when there exists a log resolution $f : Y \to X$ such that $a_E(X, \Delta) > 0$ for any f-exceptional divisor E on Y.

Theorem 2.2.2 (dlt modification)**.** *Let (X, Δ) be a lc pair. Then there exists a projective birational morphism $f : Y \to X$ with the following properties:*

- *Y is $\mathbb{Q}$-factorial.*
- *(Y, Δ_Y) is dlt, where we define Δ_Y as $K_Y + \Delta_Y = f^*(K_X + \Delta)$.*
- *$a_E(X, \Delta) = 0$ for every f-exceptional divisor E.*

Proof. See [8, Theorem 10.4] for instance. □

In Section 2.3, we need the following ACC properties proved by Hacon, M^cKernan, and Xu [10].

Theorem 2.2.3 (Hacon, M^cKernan, Xu [10, Theorem 1.4])**.** *Fix $d \in \mathbb{Z}_{>0}$ and a subset $I \subset [0, 1]$ satisfying the DCC.*

Then there is a finite subset $I_0 \subset I$ with the following property: If (X, Δ) is a log pair such that

- (X, Δ) *is lc,* $\dim X = d$*,* $\Delta \in I$*, and*
- *there exists a non-klt center* $Z \subset X$ *which is contained in every component of* Δ*,*

then $\Delta \in I_0$*.*

Theorem 2.2.4 (Hacon, M^cKernan, Xu [10, Theorem 1.5])**.** *Fix* $d \in \mathbb{Z}_{>0}$ *and a subset* $I \subset [0, 1]$ *satisfying the DCC.*

Then there is a finite subset $I_0 \subset I$ *with the following property: If* (X, Δ) *is a projective log pair such that*

- (X, Δ) *is lc,* $\dim X = d$*,* $\Delta \in I$*, and*
- $K_X + \Delta \equiv 0$*,*

then $\Delta \in I_0$*.*

2.3 Accumulation points of log canonical thresholds

The goal of this section is to prove Corollary 2.3.9. It is a generalization of [10, Theorem 1.11] and necessary for the proof of Theorem 2.1.4.

Usually, the log canonical threshold is defined as follows: for a lc pair (X, Δ) and a $\mathbb{Q}$-Cartier $\mathbb{Z}$-Weil effective divisor M,

$$\mathrm{LCT}(\Delta; M) := \sup\{c \in \mathbb{R}_{\geq 0} \mid (X, \Delta + cM) \text{ is lc}\}.$$

However, for the proof of Theorem 2.1.4, we need to treat the case when M is not effective. According to this reason, we introduce the new threshold set $\mathfrak{L}_d(I)$. It no longer satisfies the ACC, but we can prove the rationality of the accumulation points (Corollary 2.3.9).

Corollary 2.3.9 easily follows from Theorem 2.3.6 and Theorem 2.3.8. They are proved in essentially the same way of the proof of Proposition 11.5 and Proposition 11.7 in [10]. For the reader's convenience, we follow the proof of Proposition 11.5 and Proposition 11.7 in [10], and use as same notations as possible.

First, we introduce some notations. For a subset $I \subset [0, +\infty)$, we define I_+ as follows:

$$I_+ := \{0\} \cup \Big\{ \sum_{1 \leq i \leq l} r_i \mid l \in \mathbb{Z}_{>0}, r_1, \ldots, r_l \in I \Big\}.$$

This becomes a discrete set if I is discrete. When D_i are finitely many distinct prime divisors and $d_i(t) : \mathbb{R} \to \mathbb{R}$ are $\mathbb{R}$-linear functions, then we call the formal finite sum $\sum_i d_i(t)D_i$ a *linear functional divisor*.

Definition 2.3.1 ($\mathcal{D}_c(I)$)**.** Fix $c \in \mathbb{R}_{\geq 0}$ and a subset $I \subset [0, +\infty)$. For a linear functional divisor $\Delta(t) = \sum_i d_i(t)D_i$, we write $\Delta(t) \in \mathcal{D}_c(I)$ when the following conditions are satisfied:

- Each $d_i(t)$ is equal to 1 or the form of $\frac{m-1+f+kt}{m}$, where $m \in \mathbb{Z}_{>0}$, $f \in I_+$, and $k \in \mathbb{Z}$.
- Further, $f + kt$ above can be written as $f + kt = \sum_j (f_j + k_j t)$, where $f_j \in I \cup \{0\}$, $k_j \in \mathbb{Z}$, and $f_j + k_j c \geq 0$ hold for each j.

Further, by abuse of notation, we also write $d_i(t) \in \mathcal{D}_c(I)$ if $d_i(t)$ satisfies the above conditions.

The form of the coefficient $d_i(t)$ is preserved by adjunction.

Lemma 2.3.2. *Fix $c \in \mathbb{R}_{\geq 0}$ and a subset $I \subset [0, 1]$. Let X be a $\mathbb{Q}$-factorial normal variety and $\Delta(t) = \sum_{0 \leq i \leq c} d_i(t)D_i$ be a linear functional divisor on X. Assume the following conditions:*

- *$\Delta(t) \in \mathcal{D}_c(I)$, and $(X, \Delta(c))$ is lc.*
- *$d_0(t) = 1$, and $d_i(c) > 0$ for each i.*

Let S^n be the normalization of $S := D_0$. Define a linear functional divisor $\Delta_{S^n}(t)$ on S^n by adjunction:

$$(K_X + \Delta(t))|_{S^n} = K_{S^n} + \Delta_{S^n}(t).$$

Then, $\Delta_{S^n}(t) \in \mathcal{D}_c(I)$ holds.

Proof. The statement follows from [18, Proposition 16.6]. We give a sketch of proof.

Let $p \in S$ be a codimension one point of S.

Suppose that (X, D_0) is not plt at p. Then $p \notin \operatorname{Supp} D_i$ for any $i \geq 1$ and $\operatorname{coeff}_p \operatorname{Diff}_{S^n}(0) = 0$ or 1 [18, Proposition 16.6.1-2]. Hence, we have $\operatorname{coeff}_p \Delta_{S^n}(t) = 0$ or 1 for any t.

Suppose that (X, D_0) is plt at p. Then $\operatorname{coeff}_p \operatorname{Diff}_{S^n}(0) = \frac{m-1}{m}$ holds for some $m \in \mathbb{Z}_{>0}$, and mD becomes Cartier at p for any Weil divisor D [18, Proposition 16.6.3]. Hence, $\operatorname{coeff}_p \Delta_{S^n}(t)$ is the form of

$$\frac{m-1}{m} + \frac{1}{m} \sum_j \frac{n_j - 1 + f_j + k_j t}{n_j},$$

where $\frac{n_j - 1 + f_j + k_j t}{n_j}$ is the form as in the definition of $\mathcal{D}_c(I)$. We can prove that such form also satisfies the condition in the definition of $\mathcal{D}_c(I)$ by easy calculation (cf. [21, Lemma 4.4]). $\square$

We define $\mathfrak{L}_d(I)$, the set of all log canonical thresholds derived from coefficients I.

Definition 2.3.3 ($\mathfrak{L}_d(I)$)**.** Let $d \in \mathbb{Z}_{>0}$ and let $I \subset [0, +\infty)$ be a subset. We define $\mathfrak{L}_d(I) \subset \mathbb{R}_{\geq 0}$ as follows: $c \in \mathfrak{L}_d(I)$ if and only if there exist a $\mathbb{Q}$-Gorenstein normal varieties X, and a linear functional divisor $\Delta(t)$ with the following conditions:

- $\dim X \leq d$, $\Delta(t) \in \mathcal{D}_c(I)$,
- $\Delta(a)$ is $\mathbb{R}$-Cartier for any $a \in \mathbb{R}$,
- $(X, \Delta(c))$ is lc, and
- $(X, \Delta(c + \epsilon))$ is not lc for any $\epsilon > 0$, or $(X, \Delta(c - \epsilon))$ is not lc for any $\epsilon > 0$.

Remark 2.3.4. When we say that (X, Δ) is a lc pair, we assume that Δ is effective. Therefore, we say that (X, Δ) is not lc when Δ is not effective.

Further, we define $\mathfrak{G}_d(I)$, the set of all numerically trivial thresholds derived from coefficients I.

Definition 2.3.5 ($\mathfrak{G}_d(I)$)**.** Let $d \in \mathbb{Z}_{>0}$ and let $I \subset [0, +\infty)$ be a subset. We define $\mathfrak{G}_d(I) \subset \mathbb{R}_{\geq 0}$ as follows: $c \in \mathfrak{G}_d(I)$ if and only if there exist a $\mathbb{Q}$-factorial normal projective variety X, and a linear functional divisor $\Delta(t)$ with the following conditions:

- $\dim X \leq d$, $\Delta(t) \in \mathcal{D}_c(I)$,
- $(X, \Delta(c))$ is lc, and $K_X + \Delta(c) \equiv 0$.

- $K_X + \Delta(c') \not\equiv 0$ for some $c' \neq c$ (equivalently for all $c' \neq c$).

By the following theorem, we can reduce a local problem to a global problem.

Theorem 2.3.6. *Let $d \geq 2$ and $I \subset [0, +\infty)$ be a subset. Then, $\mathfrak{L}_d(I) \subset \mathfrak{G}_{d-1}(I)$ holds.*

Lemma 2.3.7. *Let $c \in \mathbb{R}_{\geq 0}$ and $I \subset [0, +\infty)$ be a subset. Suppose that there exists an $\mathbb{R}$-linear function $d(t) : \mathbb{R} \to \mathbb{R}$ with the following conditions:*

- *$d(t) \in \mathcal{D}_c(I)$, and $d(t)$ is not a constant function.*
- *$d(c) = 0$ or 1.*

Then $c \in \mathfrak{G}_d(I)$ for any $d \geq 1$. Especially, $\frac{f}{k} \in \mathfrak{G}_d(I)$ holds for any $d \geq 1$, $f \in I \cup \{0\}$, and $k \in \mathbb{Z}_{>0}$.

Proof. We can easily construct on a curve. $\square$

Proof of Theorem 2.3.6. Let $c \in \mathfrak{L}_d(I)$, and let $(X, \Delta(t))$ be as in Definition 2.3.3. Assume that $(X, \Delta(c + \epsilon))$ is not lc for any $\epsilon > 0$ (the same proof works in the other case). We may write $\Delta(t) = \sum_i d_i(t) D_i$ with distinct prime divisors D_i. By Lemma 2.3.7, we may assume that $d_i(c) > 0$ for any i. Then $\Delta(c + \epsilon) \geq 0$ holds for sufficiently small $\epsilon > 0$.

Let $f : Y \to X$ be a dlt modification (Theorem 2.2.2) of $(X, \Delta(c))$. Then Y is $\mathbb{Q}$-factorial and we can write

$$K_Y + T + \Delta'(c) = f^*(K_X + \Delta(c)),$$

where $\Delta'(t)$ is the strict transform of $\Delta(t)$, and T is the sum of the exceptional divisors. Since the pair $(Y, T + \Delta'(c))$ is dlt, there exists a divisor E on Y such that

$$a_E(X, \Delta(c)) = 0, \quad a_E(X, \Delta(c + \epsilon)) < 0$$

for any $\epsilon > 0$. If E is not f-exceptional, then $d_i(c) = 1$ holds for some $d_i(t)$ which is not identically one. In this case $c \in \mathfrak{G}_{d-1}(I)$ by Lemma 2.3.7.

In what follows, we assume that E is f-exceptional and so a component of $\operatorname{Supp} T$. By adjunction, we can define a linear functional divisor $\Delta_E(t)$ on E such that

$$(K_Y + T + \Delta'(t))|_E = K_E + \Delta_E(t).$$

Here, $\Delta_E(t) \in \mathcal{D}_c(I)$ holds by Lemma 2.3.2.

Let F be a general fiber of $E \to f(E)$. Define $\Delta_F(t)$ as

$$(K_E + \Delta_E(t))|_F = K_F + \Delta_F(t).$$

Then $(F, \Delta_F(t))$ satisfies

- $\dim F \le d-1$, F is projective,
- $\Delta_F(t) \in \mathcal{D}_c(I)$,
- $K_F + \Delta_F(c) = f^*(K_X + \Delta(c))|_F \equiv 0$, and
- $(F, \Delta_F(c))$ is lc.

Hence $(F, \Delta_F(t))$ satisfies all conditions in Definition 2.3.5 except for $K_F + \Delta_F(c') \not\equiv 0$ for some c'.

We may write $\Delta(t) = \Delta + tM$ with an $\mathbb{R}$-divisor Δ and a $\mathbb{Q}$-divisor M. Write $M = M_+ - M_-$, where $M_+ \ge 0$ and $M_- \ge 0$ have no common components. Since $a_E(X, \Delta + (c+\epsilon)M) < a_E(X, \Delta + cM) = 0$, it follows that $\operatorname{ord}_E M_+ > \operatorname{ord}_E M_- \ge 0$. Possibly replacing E by other component of T, we may assume that

$$\operatorname{ord}_E M_- \cdot \operatorname{ord}_{E_j} M_+ \le \operatorname{ord}_{E_j} M_- \cdot \operatorname{ord}_E M_+$$

for any component $E_j \subset \operatorname{Supp} T$. We may take $\epsilon_1 \ge \epsilon_2 > 0$ such that $a_E(X, \Delta + (c+\epsilon_1)M - \epsilon_2 M_+) = 0$. Note that

$$\epsilon_1(\operatorname{ord}_E M_+ - \operatorname{ord}_E M_-) = \epsilon_2 \operatorname{ord}_E M_+$$

holds. Then we have

$$\begin{aligned}
0 &\equiv f^*(K_X + \Delta + (c+\epsilon_1)M - \epsilon_2 M_+)|_F \\
&= (K_Y + T + U + \Delta' + (c+\epsilon_1)M' - \epsilon_2 M'_+)|_F \\
&= K_F + \Delta_F(c+\epsilon_1) + U|_F - \epsilon_2 M'_+|_F,
\end{aligned}$$

where we set

$$U = \sum_j (\epsilon_1 \operatorname{ord}_{E_j} M - \epsilon_2 \operatorname{ord}_{E_j} M_+) E_j.$$

Note that

$$\begin{aligned}
&\epsilon_1 \operatorname{ord}_{E_j} M - \epsilon_2 \operatorname{ord}_{E_j} M_+ \\
=\ &\frac{\epsilon_1}{\operatorname{ord}_E M_+}(\operatorname{ord}_E M_- \cdot \operatorname{ord}_{E_j} M_+ - \operatorname{ord}_{E_j} M_- \cdot \operatorname{ord}_E M_+) \\
\le\ &0.
\end{aligned}$$

Therefore $K_F+\Delta_F(c+\epsilon_1) \equiv \epsilon_2 M'_+|_F - U|_F \geq \epsilon_2 M'_+|_F$. Since $\mathrm{ord}_E\, M_+ > 0$, it follows that $f(E) \subset \mathrm{Supp}\, M_+$ and so $M'_+|_F > 0$. Therefore $K_F + \Delta_F(c+\epsilon_1)$ is not numerically trivial. □

Theorem 2.3.8. *Let $d \geq 2$ and let $I \subset [0, +\infty)$ be a finite subset. The accumulation points of $\mathfrak{G}_d(I)$ are contained in $\mathfrak{G}_{d-1}(I)$.*

As a corollary, we can prove the rationality of the accumulation points of $\mathfrak{L}_d(I)$.

Corollary 2.3.9. *Let $d \in \mathbb{Z}_{>0}$ and let $I \subset [0, +\infty)$ be a finite subset. The accumulation points of $\mathfrak{L}_d(I)$ are contained in $\mathrm{Span}_{\mathbb{Q}}(I \cup \{1\})$, where we denote by $\mathrm{Span}_{\mathbb{Q}}(I \cup \{1\}) \subset \mathbb{R}$ the $\mathbb{Q}$-vector space spanned by the elements of I and 1.*

We prove a stronger statement (cf. [10, Proposition 11.7]).

Proposition 2.3.10. *Let $d \geq 2$ and let $I \subset [0, +\infty)$ be a finite subset. Further, let $c \in \mathbb{R}_{\geq 0}$.*

Suppose that for each $i \in \mathbb{Z}_{>0}$, there exist $c_i \in \mathbb{R}_{\geq 0}$, a $\mathbb{Q}$-factorial normal projective variety X_i, and a linear functional divisor $\Delta_i(t)$ on X_i with the following conditions:

- *The sequence c_i is increasing or decreasing. Further, c_i is accumulating to c.*
- *$\dim X_i \leq d$ for each i.*
- *$\Delta_i(t)$ can be written as $\Delta_i(t) = A_i + B_i(t)$, where the coefficients of A_i are approaching one, and $B_i(t) \in \mathcal{D}_{c_i}(I)$.*
- *$(X_i, \Delta_i(c_i))$ is lc, and $K_{X_i} + \Delta_i(c_i) \equiv 0$.*
- *$K_{X_i} + \Delta_i(c'_i) \not\equiv 0$ for some $c'_i \neq c_i$.*

Then, $c \in \mathfrak{G}_{d-1}(I)$ holds.

Remark 2.3.11. If $c_i \in \mathfrak{G}_d(I)$, then c_i satisfies the above conditions. Hence, Theorem 2.3.8 follows from Proposition 2.3.10.

In the proof of Proposition 2.3.10, we reduce to the case when X_i has Picard number one, and apply the following lemma from [10].

Lemma 2.3.12 ([10, Lemma 11.6])**.** *Let (X, Δ) be a projective $\mathbb{Q}$-factorial lc pair of dimension d and of Picard number one. Assume that $K_X + \Delta \equiv 0$. If the coefficients of Δ are at least $\delta > 0$, then Δ has at most $\frac{d+1}{\delta}$ components.*

Proof of Proposition 2.3.10. Possibly replacing A_i and $B_i(t)$, we may assume that the coefficient of $B_i(t)$ is not identically one. We may write $B_i(t) = \sum_l d_{il}(t) D_{il}$ as in Definition 2.3.1.

By Lemma 2.3.7, we may assume that $(I \cup \{0\}) \cap c\mathbb{Z}_{>0} = \emptyset$. Then we may assume the following conditions on $B_i(t)$.

Lemma 2.3.13. *We may assume the following conditions:*

(1) When we write $d_{il}(t) = \frac{m-1+f+kt}{m}$ as in Definition 2.3.1, f and k have only finitely many possibilities.

(2) $d_{il}(c_i)$ are bounded from zero, and $d_{il}(c_i) < 1$ for any i, l.

(3) $d_{il}(c) > 0$.

(4) The set $\{d_{il}(c) \mid i, l\}$ satisfies the DCC.

Proof. Since $(I \cup \{0\}) \cap c\mathbb{Z}_{>0} = \emptyset$, possibly passing to a tail of the sequence, we may assume that there exist $k' \in \mathbb{Z}_{>0}$ and $\epsilon \in \mathbb{R}_{>0}$ such that for any $f_j \in I \cup \{0\}$, $k_j \in \mathbb{Z}$, and i,

- $f_j + k_j c_i \geq 0$ implies $f_j + k_j c_i \geq \epsilon$ and $k_j \geq -k'$ unless $f_j = k_j = 0$.

Here, we note that I is a finite set.

Let $d_{il}(t) = \frac{m-1+f+kt}{m}$ be a coefficient of $B_i(t)$. By assumption, $f + kt$ above can be written as $f + kt = \sum_j (f_j + k_j t)$, where $f_j \in I$, $k_j \in \mathbb{Z}$, and $f_j + k_j c_i \geq 0$ hold for each j.

Note that $f + kc_i \leq 1$ by the log canonicity. Since $f_j + k_j c_i \geq 0$ implies $f_j + k_j c_i \geq \epsilon$ and $k_j \geq -k'$, it follows that k is bounded from below. Since $c_i \geq \epsilon$, it follows that k is also bounded from above. As the set I_+ is discrete, f has also only finitely many possibilities. Therefore (1) follows.

By (1), it follows that $d_{il}(c_i) \geq \min\{\frac{1}{2}, \epsilon\}$. Hence $d_{il}(c_i)$ are bounded from zero. Since c_i are distinct, by (1), possibly passing to a subsequence, we may assume that $d_{il}(c_i) \neq 1$ (hence $d_{il}(c_i) < 1$) holds for any i, l. Thus, (2) follows.

(3) follows from (2) and (4) follows from (1). $\square$

By Lemma 2.3.13 (2), possibly passing to a tail of the sequence, we may assume that A_i and $B_i(t)$ have no common components, and that $\lfloor A_i \rfloor = \lfloor A_i + B_i(c_i) \rfloor$.

In our setting, the following claim is important and allow the same argument in [10] to work.

Claim 2.3.14. *We may assume that $(X_i, \lceil A_i \rceil + B_i(c))$ is lc for any i.*

Proof. We may write $\Delta_i(t) = A_i + M_i + t(N_i^+ - N_i^-)$, where $N_i^+ \geq 0$ and $N_i^- \geq 0$ have no common components. $(X_i, A_i + M_i + c_i(N_i^+ - N_i^-))$ is lc by the assumption.

First suppose $c_i < c$. Note that $A_i + M_i + c_i N_i^+ - cN_i^- \geq 0$ (Lemma 2.3.13 (3)). Hence $(X_i, A_i + M_i + c_i N_i^+ - cN_i^-)$ is also lc. Here, the coefficients of $M_i - cN_i^-$ satisfy the DCC (Lemma 2.3.13 (4)), and the coefficient of A_i and the sequence c_i are increasing. Hence by Theorem 2.2.3, possibly passing to a tail of the sequence, we may assume that $(X_i, \lceil A_i \rceil + M_i + cN_i^+ - cN_i^-)$ is lc.

Suppose $c_i > c$. Then $(X_i, A_i + M_i + cN_i^+ - c_i N_i^-)$ is lc. Here, the coefficients of $M_i + cN_i^+$ satisfy the DCC (Lemma 2.3.13 (4)), and the coefficient of A_i and the sequence $-c_i$ are increasing. Hence by Theorem 2.2.3, possibly passing to a tail of the sequence, we may assume that $(X_i, \lceil A_i \rceil + M_i + cN_i^+ - cN_i^-)$ is lc. □

Set $a_i := \mathrm{mld}(X_i, \Delta_i(c_i)) \geq 0$. Possibly passing to a subsequence, it is sufficient to treat the following two cases:

(A) a_i is bounded away from zero.

(B) a_i approaches zero.

<u>**Case B**</u> We treat the case when a_i approaches zero from above.

<u>**STEP B-1**</u> We reduce to the case when $A_i \neq 0$ and $(X_i, \Delta_i(c_i))$ is dlt.

We may assume $a_i \leq 1$ for any i. Take an extraction $\pi_i : X_i' \to X_i$ of a divisor E_i computing $\mathrm{mld}(X_i, \Delta_i(c_i)) = a_i$ (Theorem 2.2.1 and 2.2.2). Then we may write

$$K_{X_i'} + (1 - a_i)E_i + T_i + \Delta_i'(c_i) = \pi_i^*(K_{X_i} + \Delta_i(c_i)),$$

where T_i is the sum of exceptional divisors (Note that $T_i = 0$ when $a_i > 0$) and $\Delta_i'(t)$ is the strict transform of $\Delta_i(t)$. Then $\big(X_i', (1 - a_i)E_i + T_i + \Delta_i'(t)\big)$ satisfies the following conditions:

- We may write $(1-a_i)E_i+T_i+\Delta'_i(t) = A'_i+B'_i(t)$ with all the conditions in Proposition 2.3.10.

- $\lfloor A'_i \rfloor = \lfloor A'_i + B'_i(c_i) \rfloor$ and $A'_i \neq 0$.

- $\big(X'_i, (1-a_i)E_i + T_i + \Delta'_i(c_i)\big)$ is dlt.

Hence, we may replace $(X_i, \Delta_i(t))$ by $\big(X'_i, (1-a_i)E_i + T_i + \Delta'_i(t)\big)$.

STEP B-2 We are done if there exists a component $S_i \subset \operatorname{Supp}\lfloor A_i \rfloor$ such that $(K_{X_i} + \Delta_i(c'_i))|_{S_i} \not\equiv 0$.

Suppose that there exists a component $S_i \subset \operatorname{Supp}\lfloor A_i \rfloor$ such that $(K_{X_i} + \Delta_i(c'_i))|_{S_i} \not\equiv 0$. By adjunction, we can define $\Delta_{S_i}(t)$ as follows:

$$(K_{X_i} + \Delta_i(t))|_{S_i} = K_{S_i} + \Delta_{S_i}(t).$$

Then $(S_i, \Delta_{S_i}(t))$ satisfies the following conditions:

- $\dim S_i \leq d-1$.

- $K_{S_i} + \Delta_{S_i}(c'_i) \not\equiv 0$ by the assumption.

- $(S_i, \Delta_{S_i}(t))$ satisfies the other conditions in Proposition 2.3.10.

Hence, we may replace $(X_i, \Delta_i(t))$ by $(S_i, \Delta_{S_i}(t))$. By induction on d, it follows that $c \in \mathfrak{G}_{d-2}(I) \subset \mathfrak{G}_{d-1}(I)$.

STEP B-3 We are done if $f_i : X_i \to Z_i$ is a Mori fiber space with $\dim Z_i > 0$, and $\operatorname{Supp} A_i$ dominates Z_i.

Let F_i be the general fiber of f_i. We may define $\Delta_{F_i}(t)$ as follows:

$$(K_{X_i} + \Delta_i(t))|_{F_i} = K_{F_i} + \Delta_{F_i}(t).$$

Then $(F_i, \Delta_{F_i}(t))$ satisfies all conditions in Proposition 2.3.10 except for $K_{F_i}+\Delta_{F_i}(c'_i) \not\equiv 0$. Hence, if $K_{F_i}+\Delta_{F_i}(c'_i) \not\equiv 0$ for some c'_i, then $c \in \mathfrak{G}_{d-2}(I) \subset \mathfrak{G}_{d-1}(I)$ by induction on d.

Suppose that $K_{F_i} + \Delta_{F_i}(c'_i) \equiv 0$, and so $K_{F_i} + \Delta_{F_i}(c) \equiv 0$. We may write $\Delta_{F_i}(t) = A'_i + B'_i(t)$ with the conditions in Proposition 2.3.10. Note that $(F_i, \Delta_{F_i}(c))$ is lc by the same reason as Claim 2.3.14. Since the coefficients of $B'_i(c)$ satisfies the DCC (Lemma 2.3.13 (4)), it follows that $\lfloor A'_i \rfloor = A'_i$ by Theorem 2.2.4. Therefore, there exists a component $S_i \subset \operatorname{Supp}\lfloor A_i \rfloor$ such that $f(S_i) = Z_i$. Since $(K_{X_i} + \Delta_i(c))|_{F_i} \equiv 0$, $K_{X_i} + \Delta_i(c)$ is linearly equivalent to the pulled back of an $\mathbb{R}$-divisor D_i on Z_i. As $K_{X_i}+\Delta_i(c)\not\equiv 0$, it follows that $D_i\not\equiv 0$, and so $(K_{X_i}+\Delta_i(c))|_{S_i}\not\equiv 0$. Therefore we are done by STEP B-2.

STEP B-4 We finish the case when $(X_i,\Delta_i(c_i))$ is not klt (equivalently $\lfloor A_i\rfloor\neq 0$ by STEP B-1).

Suppose that $(X_i,\Delta_i(c_i))$ is not klt. Then $\lfloor A_i\rfloor\neq 0$. We run a $(K_{X_i}+\Delta_i(c_i)-\lfloor A_i\rfloor)$-MMP. Since $K_{X_i}+\Delta_i(c_i)-\lfloor A_i\rfloor\equiv-\lfloor A_i\rfloor$ is not pseudo-effective, a $(K_{X_i}+\Delta_i(c_i)-\lfloor A_i\rfloor)$-MMP terminates and ends with a Mori fiber space by [4, Corollary 1.3.3].

Let $f_i:X_i\dashrightarrow X_i'$ be a step of the MMP. First suppose that f_i is birational. We write

$$A_i':=f_{i*}A_i,\quad B_i'(t):=f_{i*}B_i(t),\quad \Delta_i'(t)=A_i'+B_i'(t).$$

Then, $(X_i',\Delta_i'(t))$ satisfies all conditions in Proposition 2.3.10 except for $K_{X_i'}+\Delta_i'(c_i')\not\equiv 0$.

Assume that $K_{X_i'}+\Delta_i'(c_i')\equiv 0$ (Hence, f_i is a divisorial contraction). Set $D_i:=K_{X_i}+\Delta_i(c_i')\not\equiv 0$. We may write $D_i-f_i^*f_{i*}D_i=aE$, where E is the f_i-exceptional divisor, and $a\in\mathbb{R}$. Since $D_i\not\equiv 0$ and $f_{i*}D_i\equiv 0$, we have $aE\not\equiv 0$. As f_i is $\lfloor A_i\rfloor$-positive, there exists a component $T_i\subset\operatorname{Supp}\lfloor A_i\rfloor$ such that $E|_{T_i}\not\equiv 0$. Therefore, we are done by STEP B-2.

Hence, we may assume $K_{X_i'}+\Delta_i'(c_i')\not\equiv 0$, and replace $(X_i,\Delta_i(t))$ by $(X_i',\Delta_i'(t))$ and continue the MMP. The MMP must terminate with a Mori fiber space $f_i:X_i\to Z_i$. If $\dim Z_i=0$, then the Picard number of X_i is one. Therefore $(K_{X_i}+\Delta_i(c_i'))|_{T_i}\not\equiv 0$ for any component $T_i\subset\operatorname{Supp}\lfloor A_i\rfloor$, and we are done by STEP B-2. Suppose $\dim Z_i>0$. Since f_i is $\lfloor A_i\rfloor$-positive, $\lfloor A_i\rfloor$ dominates Z_i and we are done by STEP B-3.

STEP B-5 In what follows, we assume that $(X_i,\Delta(c_i))$ is klt. We reduce to the case when X_i has Picard number one.

We run a $(K_{X_i}+B_i(c_i))$-MMP. Since $(K_{X_i}+B_i(c_i))\equiv -A_i$ is not pseudo-effective, a $(K_{X_i}+B_i(c_i))$-MMP terminates and ends with a Mori fiber space by [4, Corollary 1.3.3].

Let $f_i:X_i\dashrightarrow X_i'$ be a step of the MMP. First suppose that f_i is birational. We write

$$A_i':=f_{i*}A_i,\quad B_i'(t):=f_{i*}B_i(t),\quad \Delta_i'(t)=A_i'+B_i'(t).$$

Then, $(X_i',\Delta_i'(t))$ satisfies all conditions in Proposition 2.3.10 except for $K_{X_i'}+\Delta_i'(c_i')\not\equiv 0$. We prove $K_{X_i'}+\Delta_i'(c_i')\not\equiv 0$.

Suppose $K_{X'_i} + \Delta'_i(c'_i) \equiv 0$. It implies that $K_{X'_i} + \Delta'_i(c) \equiv 0$. Note that $(X'_i, \Delta'_i(c))$ is lc by Claim 2.3.14. Further the coefficients of $B'_i(c)$ satisfies the DCC (Lemma 2.3.13 (4)), and the coefficient of A_i are approaching 1 and $\lfloor A_i \rfloor = 0$. It contradicts Theorem 2.2.4.

Since $K_{X'_i} + \Delta'_i(c'_i) \not\equiv 0$, we may replace $(X_i, \Delta_i(t))$ by $(X'_i, \Delta'_i(t))$ and continue the MMP. Then the MMP must terminate and ends with a Mori fiber space $X_i \to Z_i$. If $\dim Z_i = 0$, the Picard number of X_i is one. Suppose $\dim Z_i > 0$. Since f_i is A_i-positive, $\operatorname{Supp} A_i$ dominates Z_i, and we are done by STEP B-3.

STEP B-6 We finish the case B.

Claim 2.3.15. *We may assume that $K_{X_i} + A_i + B_i(c)$ is not ample for any i.*

Proof. Assume that $K_{X_i} + A_i + B_i(c)$ is ample. We may write

$$A_i + B_i(t) = M_i + t(N_i^+ - N_i^-),$$

where $N_i^+ \geq 0$ and $N_i^- \geq 0$ have no common components. Further we may write

$$N_i^+ \equiv n_i^+ H_i, \quad N_i^- \equiv n_i^- H_i$$

with some ample divisor H_i and $n_i^+, n_i^- \in \mathbb{Q}_{\geq 0}$.

First suppose $c_i > c$. Then $K_{X_i} + A_i + B_i(c) \equiv (c - c_i)(N_i^+ - N_i^-)$ is ample, and so $n_i^+ < n_i^-$. Then we have

$$K_{X_i} + M_i + cN_i^+ - \left(c_i - (c_i - c)\frac{n_i^+}{n_i^-}\right)N_i^- \equiv K_{X_i} + M_i + c_i(N_i^+ - N_i^-) \equiv 0.$$

Here, we have $c < c_i - (c_i - c)\frac{n_i^+}{n_i^-} < c_i$, and so

$$0 \leq M_i + cN_i^+ - \big(c_i - (c_i - c)\frac{n_i^+}{n_i^-}\big)N_i^- \leq M_i + c(N_i^+ - N_i^-).$$

Since $(X_i, M_i + c(N_i^+ - N_i^-))$ is lc by Claim 2.3.14, the new pair $\big(X_i, M_i + cN_i^+ - \big(c_i - (c_i - c)\frac{n_i^+}{n_i^-}\big)N_i^-\big)$ is also lc, but it contradicts Lemma 2.3.13 (4) and Theorem 2.2.4.

Suppose $c_i < c$. Then we have $n_i^+ > n_i^-$, and

$$K_{X_i} + M_i + \left(c_i + (c - c_i)\frac{n_i^-}{n_i^+}\right)N_i^+ - cN_i^- \equiv K_{X_i} + M_i + c_i(N_i^+ - N_i^-) \equiv 0.$$

Here, we have $c_i < c_i + (c - c_i)\frac{n_i^-}{n_i^+} < c$, and

$$0 \le M_i + \left(c_i + (c - c_i)\frac{n_i^-}{n_i^+}\right)N_i^+ - cN_i^- \le M_i + c(N_i^+ - N_i^-).$$

Note that the first inequality follows from Lemma 2.3.13 (3). Since $(X_i, M_i + c(N_i^+ - N_i^-))$ is lc by Claim 2.3.14, the new pair $\big(X_i, M_i + \big(c_i + (c - c_i)\frac{n_i^-}{n_i^+}\big)N_i^+ - cN_i^-\big)$ is also lc, but it contradicts Lemma 2.3.13 (4) and Theorem 2.2.4. $\square$

First suppose that $(X_i, \lceil A_i \rceil + B_i(c_i))$ is not lc. Note that $(X_i, \lceil A_i \rceil + B_i(c))$ is lc by Claim 2.3.14. Set

$$d_i := \begin{cases} \sup\{t \in [c, c_i) \mid (X_i, \lceil A_i \rceil + B_i(t)) \text{ is lc}\} & \text{when } c < c_i, \\ \inf\{t \in (c_i, c] \mid (X_i, \lceil A_i \rceil + B_i(t)) \text{ is lc}\} & \text{when } c_i < c. \end{cases}$$

Then $d_i \in \mathfrak{L}_d(I) \subset \mathfrak{G}_{d-1}(I)$, and $\lim d_i = \lim c_i = c$. Therefore we are done by induction on d.

Thus we may assume that $(X_i, \lceil A_i \rceil + B_i(c_i))$ is lc. Set $e_i, f_i \in \mathbb{R}$ as

$$K_{X_i} + \lceil A_i \rceil + B_i(e_i) \equiv 0, \quad K_{X_i} + f_i\lceil A_i \rceil + B_i(c) \equiv 0.$$

Since $B_i(c_i) - B_i(c)$ is ample (Claim 2.3.15) and $K_{X_i} + A_i + B_i(c_i) \equiv 0$, there are only two cases:

- $e_i \ge c > c_i$ or $e_i \le c < c_i$, or
- $c \ge e_i \ge c_i$ or $c \le e_i \le c_i$.

First suppose that $e_i \ge c > c_i$ or $e_i \le c < c_i$. Then $K_{X_i} + \lceil A_i \rceil + B_i(c)$ is ample, and so $f_i < 1$. Further, Since $K_{X_i} + A_i + B_i(c)$ is not ample, and the coefficients of A_i are approaching one, it follows that $\lim f_i = 1$. Therefore, the set of coefficients of $f_i\lceil A_i \rceil + B_i(c)$ satisfies the DCC (Lemma 2.3.13 (4)), which contradicts Theorem 2.2.4.

Next suppose that $c \ge e_i \ge c_i$ or $c \le e_i \le c_i$. Thus, we have $\lim e_i = \lim c_i = c$. Suppose $c \ge e_i \ge c_i$ (the other case can be proved in the same way), we may assume that $e_i < e_{i+1}$ for all i or $e_i = c$ for some i. In the former case, as the sequence e_i is accumulating to c, we may replace $(X_i, \Delta_i(t))$ by $(X_i, \lceil A_i \rceil + B_i(t))$. Remark that $(X_i, \lceil A_i \rceil + B_i(e_i))$ is lc, because both $(X_i, \lceil A_i \rceil + B_i(c_i))$ and $(X_i, \lceil A_i \rceil + B_i(c))$ are lc. Note that the Picard number of X_i is one. Hence for any component of $S_i \subset \operatorname{Supp}\lceil A_i \rceil$, we have

$(K_{X_i} + \lceil A_i \rceil + B_i(c'_i))|_{S_i} \not\equiv 0$ for some c'_i. Therefore we are done by STEP B-2. In the latter case, $c = e_i \in \mathfrak{G}_{d-1}(I)$ by adjunction.

<u>**Case A**</u> We treat the case when a_i is bounded away from zero.

In this case, it follows that $A_i = 0$ and $(X_i, B_i(c_i))$ is klt.

<u>**STEP A-1**</u> We reduce to the case when X_i has Picard number one.

Since $K_{X_i} + B_i(c_i) \equiv 0$ and $K_{X_i} + B_i(c'_i) \not\equiv 0$ for some c'_i, we can take $\epsilon \in \mathbb{R}$ such that $K_{X_i} + B_i(c_i + \epsilon)$ is klt (Lemma 2.3.13 (2)) and not pseudo-effective. We run a $(K_{X_i} + B_i(c_i + \epsilon))$-MMP. As $K_{X_i} + B_i(c_i + \epsilon)$ is not pseudo-effective, a $(K_{X_i} + B_i(c_i + \epsilon))$-MMP terminates and ends with a Mori fiber space [4, Corollary 1.3.3].

Let $f_i : X_i \dashrightarrow X'_i$ be a step of the MMP. First suppose that f_i is birational. We write

$$B'_i(t) := f_{i*}B_i(t), \quad \Delta'_i(t) := B'_i(t).$$

Then, $(X'_i, \Delta'_i(t))$ satisfies all conditions in Proposition 2.3.10 except for $K_{X'_i} + \Delta'_i(c'_i) \not\equiv 0$. We prove $K_{X'_i} + \Delta'_i(c'_i) \not\equiv 0$.

Suppose $K_{X'_i} + \Delta'_i(c_i + \epsilon) \equiv 0$ (hence, f_i is a divisorial contraction). We denote $D := K_{X_i} + \Delta_i(c_i + \epsilon)$, then we may write

$$D \equiv D - f_i^* f_{i*} D = aE,$$

where E is the exceptional divisor and $a \in \mathbb{R}$. Since D is not pseudo-effective, it follows that $a < 0$. It contradicts the fact that f_i is D-negative.

Since $K_{X'_i} + \Delta'_i(c'_i) \not\equiv 0$, we may replace $(X_i, \Delta_i(t))$ by $(X'_i, \Delta'_i(t))$, and continue the MMP. The MMP must terminate with a Mori fiber space $f_i : X_i \to Z_i$. If $\dim Z_i = 0$, then the Picard number of X_i is one. Suppose $\dim Z_i > 0$. Let F_i be the general fiber of f_i. Set $\Delta_{F_i}(t)$ by adjunction:

$$(K_{X_i} + \Delta_i(t))|_{F_i} = K_{F_i} + \Delta_{F_i}(t).$$

Since f_i is $(K_{X_i} + \Delta_i(c_i + \epsilon))$-negative, $(K_{X_i} + \Delta_i(c_i + \epsilon))|_{F_i} \not\equiv 0$. Then $(F_i, \Delta_{F_i}(t))$ satisfies the conditions in Proposition 2.3.10. Since $\dim F_i \leq d - 1$, we are done by induction on d.

<u>**STEP A-1'**</u> Since X_i has Picard number one, by Lemma 2.3.12 and Lemma 2.3.13 (2), the number of components of $B_i(t)$ are bounded. Hence, possibly passing to a subsequence, we may assume that the number of components of $B_i(t)$ are fixed. Since a_i are bounded away from zero, the coefficients of $B_i(t)$

have only finitely many possibilities (Lemma 2.3.13 (1)). Therefore, possibly passing to a subsequence, we may assume that the coefficients of $B_i(t)$ are fixed and of the form

$$\frac{m-1+f+kt}{m},$$

where $m \in \mathbb{Z}_{>0}$, $f \in I_+$, and $k \in \mathbb{Z}$. Here, m, f, and k depend on the component but not on i.

Set

$$h_i^+ := \sup\{t \geq c_i \mid (X_i, B_i(t)) \text{ is lc}\} \geq c_i,$$
$$h_i^- := \inf\{t \leq c_i \mid (X_i, B_i(t)) \text{ is lc}\} \leq c_i.$$

Since h_i^+ and h_i^- are bounded, possibly passing to a subsequence, we may assume that the limits $h^+ = \lim h_i^+$, $h^- = \lim h_i^-$ exist.

STEP A-2 We finish the case when $c \geq h^+$ or $c \leq h^-$.

In this case, we have $h^+ = \lim h_i^+ = c$ or $h^- = \lim h_i^- = c$. Since $h_i^+, h_i^- \in \mathfrak{L}_d(I) \subset \mathfrak{G}_{d-1}(I)$, we are done by induction on d.

STEP A-3 In what follows, we assume that $c < h^+$ and $c > h^-$. Let

$$d_i^+ = \frac{c_i + h_i^+}{2}, \quad d_i^- = \frac{c_i + h_i^-}{2}, \quad d^+ = \frac{c + h^+}{2}, \quad d^- = \frac{c + h^-}{2}.$$

Then $d^+ > c$ and $d^- < c$. Further, we may assume $d^+ > c_i$ and $d^- < c_i$ possibly passing to a tail of the sequence. Note that the following hold:

- If $c_i > c$, then $K_{X_i} + B_i(d^+)$ is ample.
- If $c_i < c$, then $K_{X_i} + B_i(d^-)$ is ample.

This is because, $K_{X_i} + B_i(c)$ is not ample by the same reason as Claim 2.3.15.

In this step, we prove that the following hold:

- If $c_i > c$, then $\operatorname{vol}(X_i, K_{X_i} + B_i(d^+))$ is unbounded.
- If $c_i < c$, then $\operatorname{vol}(X_i, K_{X_i} + B_i(d^-))$ is unbounded.

Suppose that $c_i > c$ and $\operatorname{vol}(X_i, K_{X_i} + B_i(d^+))$ is bounded from above (the other case can be proved in the same way). Since the coefficients of $(X_i, B_i(d^+))$ are fixed, there exists $m \in \mathbb{Z}_{>0}$ such that $\phi_{m(K_{X_i}+B_i(d^+))}$ is birational for all i by [10, Theorem 1.3]. But then, by [9, Lemma 2.4.2], $\{(X_i, B_i(d^+)) \mid i \in \mathbb{Z}_{>0}\}$ is log birationally bounded since $\operatorname{vol}(X_i, K_{X_i} +$

$B_i(d^+))$ is bounded by the assumption. Note that the coefficients of $B_i(d_i^+)$ are bounded from below and $\mathrm{mld}(X_i, B_i(d_i^+))$ is also bounded from below:

$$\mathrm{mld}(X_i, B_i(d_i^+)) \geq \frac{\mathrm{mld}(X_i, B_i(h_i^+)) + \mathrm{mld}(X_i, B_i(c_i))}{2} = \frac{a_i}{2}.$$

Hence by [10, Theorem 1.6], $\{(X_i, B_i(d^+)) \mid i \in \mathbb{Z}_{>0}\}$ turns out to be a bounded family.

Thus, we may take an ample Cartier divisor H_i on X_i such that

$$T_i \cdot H_i^{\dim X_i - 1}, \quad K_{X_i} \cdot H_i^{\dim X_i - 1}$$

are bounded, where T_i is any component of $B_i(t)$. Hence we may assume that these intersection numbers are independent of i possibly passing to a subsequence. We may write $B_i(t) = M_i + tN_i$. As the coefficients of B_i are independent of i, it follows that $M_i \cdot H_i^{\dim X_i - 1}$ and $N_i \cdot H_i^{\dim X_i - 1}$ are also constant. Since

$$0 = (K_{X_i} + B_i(c_i)) \cdot H_i^{\dim X_i - 1} = (K_{X_i} + M_i + c_i N_i) \cdot H_i^{\dim X_i - 1},$$

it follows that c_i is also constant, a contradiction. Remark that $N_i \cdot H_i^{\dim X_i - 1} \neq 0$ holds since $N_i \not\equiv 0$.

STEP A-4 By STEP A-3, the following hold:

- If $c_i > c$, then $K_{X_i} + B_i(d^+)$ is ample and $\mathrm{vol}(X_i, K_{X_i} + B_i(d^+))$ is unbounded.
- If $c_i < c$, then $K_{X_i} + B_i(d^-)$ is ample and $\mathrm{vol}(X_i, K_{X_i} + B_i(d^-))$ is unbounded.

Suppose $c_i > c$ (the other case can be proved in the same way). Note that $K_{X_i} + B_i(d^+) \equiv B_i(d^+) - B_i(c_i)$. Then, by Lemma 3.2.2 and Lemma 3.2.3 in [10], possibly passing to a tail of the sequence, we may find $g_i < c_i$ and an $\mathbb{R}$-divisor Θ_i with the following conditions:

- $0 \leq \Theta_i \sim_{\mathbb{R}} B_i(c_i) - B_i(g_i)$.
- $B_i(g_i) \geq 0$ (cf. Lemma 2.3.13 (3)).
- $\lim g_i = c$.
- $(X_i, B_i(g_i) + \Theta_i)$ has a unique non-klt place.

Let $\phi : Y_i \to X_i$ be a dlt modification of $(X_i, B_i(g_i) + \Theta_i)$. Then we may write

$$K_{Y_i} + B'_i(g_i) + \Theta'_i + S_i = \phi^*(K_{X_i} + B_i(g_i) + \Theta_i),$$

where S_i is the unique exceptional divisor, and $B'_i(t)$ and Θ'_i are the strict transform of $B_i(t)$ and Θ_i. We may also write

$$K_{Y_i} + B'_i(c_i) + s_i S_i = \phi^*(K_{X_i} + B_i(c_i))$$

with $s_i < 1$ as $(X_i, B_i(c_i))$ is klt.

Claim 2.3.16. *We may assume that S_i is ample and $K_{Y_i} + B'_i(l_i) + S_i \equiv 0$ for some $l_i \in [g_i, c_i)$.*

First we assume this claim and finish the proof.

Suppose that $(Y_i, B'_i(c_i) + S_i)$ is not lc. Note that $(Y_i, B'_i(g_i) + S_i)$ is lc. Set

$$k_i := \sup\{t \in [g_i, c_i) \mid (Y_i, B'_i(t) + S_i) \text{ is lc}\}.$$

Then $k_i \in \mathfrak{L}_d(I) \subset \mathfrak{G}_{d-1}(I)$, and $\lim k_i = c$. Therefore we are done by induction on d.

Thus, we may assume that $(Y_i, B'_i(c_i) + S_i)$ is lc. By adjunction, we can define $B''_i(t)$ as follows:

$$(K_{Y_i} + B'_i(t) + S_i)|_{S_i} = K_{S_i} + B''_i(t).$$

Since $(Y_i, B'_i(c_i) + S_i)$ is lc, it follows that $B''_i(t) \in \mathcal{D}_{c_i}(I)$ by Lemma 2.3.2. Further $(S_i, B''_i(c_i))$ and $(S_i, B''_i(g_i))$ are lc. By Claim 2.3.16, it follows that $K_{S_i} + B''_i(l_i) \equiv 0$ and $K_{S_i} + B''_i(c_i) \not\equiv 0$. Therefore $l_i \in \mathfrak{G}_{d-1}(I)$. Since $\lim l_i = c$, we are done by induction on d.

Proof of Claim 2.3.16. We run a $(K_{Y_i}+B'_i(g_i)+\Theta'_i)$-MMP. Since $(Y_i, B'_i(g_i)+\Theta'_i)$ is klt and $K_{Y_i} + B'_i(g_i) + \Theta'_i \equiv -S_i$ is not pseudo-effective, a $(K_{Y_i} + B'_i(g_i) + \Theta'_i)$-MMP $f_i : Y_i \dashrightarrow W_i$ terminates and ends with a Mori fiber space $\pi_i : W_i \to Z_i$ by [4, Corollary 1.3.3].

Let F_i be the general fiber of π_i and let $B'''_i(t)$, Θ'''_i and S'''_i be the restriction of $f_{i*}B'_i(t)$, $f_{i*}\Theta'_i$ and $f_{i*}S_i$ to F_i. Note that $S'''_i \neq 0$ since every step of this MMP is S_i-positive. Further $B'''_i(t)$, Θ'''_i and S'''_i are multiples of the same ample divisor. Therefore S'''_i is ample. Since

$$K_{F_i} + B'''_i(g_i) + \Theta'''_i + S'''_i \equiv 0, \text{ and } \quad K_{F_i} + B'''_i(c_i) + s_i S'''_i \equiv 0,$$

we may find

$$K_{F_i} + B_i'''(l_i) + S_i''' \equiv 0$$

for some $l_i \in [g_i, c_i)$. Therefore, we can apply the same argument above after replacing $(Y_i, B_i'(t) + S_i)$ by $(F_i, B_i'''(t) + S_i''')$. $\square$

$\square$

Proof of Corollary 2.3.9. Since $\mathfrak{G}_d(I) \subset \mathrm{Span}_{\mathbb{Q}}(I \cup \{1\})$, the statement follows from Theorem 2.3.6 and Theorem 2.3.8. $\square$

2.4 Perturbation of irrational coefficients of log canonical pairs

The goal of this section is to prove Theorem 2.1.4. The ideal setting is treated as Theorem 2.4.1.

Proof of Theorem 2.1.4. We may write the $\mathbb{Q}$-linear functions s_i as

$$s_i(x_0, \ldots, x_{c'}) = \sum_{0 \le j \le c'} q_{ij} x_j$$

with $q_{ij} \in \mathbb{Q}$. Since $s_i(r_0, \ldots, r_{c'}) \in \mathbb{R}_{\ge 0}$ and $r_0, \ldots, r_{c'}$ are $\mathbb{Q}$-linearly independent, we can take $t^-, t^+ \in \mathbb{Q}$ with the following conditions:

- $t^- < r_{c'} < t^+$, and
- $s_i(r_0, \ldots, r_{c'-1}, t) \in \mathbb{R}_{\ge 0}$ holds for any t satisfying $t^- \le t \le t^+$.

Suppose that the statement does not hold. Then there exist $\mathbb{Q}$-Gorenstein varieties $X^{(l)}$ ($l \in \mathbb{Z}_{>0}$) of dimension d and $\mathbb{Q}$-Cartier effective Weil divisors $D_0^{(l)}, \ldots, D_c^{(l)}$ on $X^{(l)}$ such that the following holds:

- $\left(X^{(l)}, \sum_{1 \le i \le c} s_i(r_0, \ldots, r_{c'}) D_i^{(l)}\right)$ is lc, and
- $\lim h_l^+ = r_{c'}$ or $\lim h_l^- = r_{c'}$,

where we set

$$h_l^+ := \sup \left\{ t \ge r_{c'} \;\middle|\; \left(X^{(l)}, \sum_{1 \le i \le c} s_i(r_0, \ldots, r_{c'-1}, t) D_i^{(l)}\right) \text{ is lc} \right\},$$
$$h_l^- := \inf \left\{ t \le r_{c'} \;\middle|\; \left(X^{(l)}, \sum_{1 \le i \le c} s_i(r_0, \ldots, r_{c'-1}, t) D_i^{(l)}\right) \text{ is lc} \right\}.$$

Suppose that $\lim h_l^- = r_{c'}$ (the other case can be proved in the same way). We may assume that $t^- \le h_l^- \le r_{c'}$. Note that

$$\sum_{1\le i\le c} s_i(r_0,\dots,r_{c'-1},t)D_i^{(l)} = \sum_{1\le i\le c} s_i(r_0,\dots,r_{c'-1},t^-)D_i^{(l)} + (t-t^-)\sum_{1\le i\le c} q_{ic'}D_i^{(l)}.$$

Let

$$I := \{s_i(r_0,\dots,r_{c'-1},t^-) \mid 1\le i\le c\}.$$

This becomes a finite set. Take $m \in \mathbb{Z}_{>0}$ such that $mq_{ic'} \in \mathbb{Z}$ holds for any i. Then $\frac{h_l^- - t^-}{m} \in \mathfrak{L}_d(I)$. Hence, by Corollary 2.3.9, it follows that

$$\frac{r_{c'}-t^-}{m} \in \mathrm{Span}_{\mathbb{Q}}(I\cup\{1\}) \subset \mathrm{Span}_{\mathbb{Q}}(r_0,\dots,r_{c'-1}).$$

It contradicts the $\mathbb{Q}$-linearly independence of $r_0,\dots,r_{c'}$. $\square$

The case of the pair with ideal sheaves can be also proved.

Theorem 2.4.1. *Fix $d \in \mathbb{Z}_{>0}$. Let $r_1,\dots,r_{c'}$ be positive real numbers and let $r_0 = 1$. Assume that $r_0,\dots,r_{c'}$ are $\mathbb{Q}$-linearly independent. Let $s_1,\dots,s_c : \mathbb{R}^{c'+1} \to \mathbb{R}$ be $\mathbb{Q}$-linear functions from $\mathbb{R}^{c'+1}$ to $\mathbb{R}$. Assume that $s_i(r_0,\dots,r_{c'}) \in \mathbb{R}_{\ge 0}$ for each i. Then there exists a positive real number $\epsilon > 0$ with the following conditions:*

- *$s_i(r_0,\dots,r_{c'-1},t) \ge 0$ holds for any t satisfying $|t-r_{c'}| \le \epsilon$.*
- *For any $\mathbb{Q}$-Gorenstein normal variety X of dimension d and ideal sheaves $\mathfrak{a}_1,\dots,\mathfrak{a}_c$, if $(X,\prod_{1\le i\le c}\mathfrak{a}_i^{s_i(r_0,\dots,r_{c'})})$ is lc, then $(X,\prod_{1\le i\le c}\mathfrak{a}_i^{s_i(r_0,\dots,r_{c'-1},t)})$ is also lc for any t satisfying $|t-r_{c'}|\le\epsilon$.*

This theorem follows from Theorem 2.1.4 by the following lemma (cf. [19, Proposition 9.2.28]).

Lemma 2.4.2. *Fix $l \in \mathbb{Z}_{>0}$. Let X be a $\mathbb{Q}$-Gorenstein normal affine variety, and let $\mathfrak{a}_1,\dots,\mathfrak{a}_c$ be ideal sheaves on X. Fix general elements $f_{i1},\dots,f_{il} \in \mathfrak{a}_i$ for each i, and let $D_{ij} = \mathrm{div}(f_{ij}) \ge 0$ be the corresponding Cartier divisors. Set $D_i := \sum_{1\le j\le l} D_{ij}$.*

Then the following holds for any positive real numbers $r_1,\dots,r_c \le l$ at most l: the pair $(X,\prod_{1\le i\le c}\mathfrak{a}_i^{r_i})$ is lc if and only if the pair $(X,\frac{1}{l}\sum_{1\le i\le c} r_iD_i)$ is lc.

Definition 2.4.3. Let X be an affine variety and $\mathfrak{a}$ an ideal sheaf. Fix generators $g_1, \dots, g_c \in \mathfrak{a}$. Then, a *general element* of $\mathfrak{a}$ is a general $\mathbb{C}$-linear combination of g_i.

Proof of Lemma 2.4.2. Let $\mathfrak{b}_i$ be an ideal sheaf generated by $\prod_{1\le j\le l} f_{ij}$. Then the pair $(X, \frac{1}{l}\sum_{1\le i\le c} r_i D_i)$ is corresponding to the pair $(X, \prod_{1\le i\le c} \mathfrak{b}_i^{r_i/l})$.

Since $\mathfrak{b}_i \subset \mathfrak{a}_i^l$, it easily follows that the log canonicity of $(X, \prod_{1\le i\le c} \mathfrak{b}_i^{r_i/l})$ implies the log canonicity of $(X, \prod_{1\le i\le c} \mathfrak{a}_i^{r_i})$.

Suppose that $(X, \prod_{1\le i\le c} \mathfrak{a}_i^{r_i})$ is lc. Let $Y \to X$ be a log resolution of $(X, \prod_{1\le i\le c} \mathfrak{a}_i^{r_i})$. Then we may write $\mathfrak{a}_i\mathcal{O}_Y = \mathcal{O}_Y(-E_i)$ with some Cartier divisor E_i. Since $\mathfrak{b}_i \subset \mathfrak{a}_i^l$, we may write $\mathfrak{b}_i\mathcal{O}_Y = \mathfrak{c}_i\mathcal{O}_Y(-lE_i)$ with some ideal sheaf $\mathfrak{c}_i \subset \mathcal{O}_Y$. Let e_i be a local generator of $\mathcal{O}_Y(-E_i)$. Then $\mathfrak{c}_i$ is generated by $\prod_{1\le j\le l} g_{ij}$, where we set $g_{ij} := f_{ij}e_i^{-1} \in \mathcal{O}_Y$. As $f_{i1}, \dots, f_{il}$ are general elements of $\mathfrak{a}_i$, the elements $g_{i1}, \dots, g_{il}$ become general elements of $\mathcal{O}_Y$. Therefore $Y \to X$ is also a log resolution of $(X, \prod_{1\le i\le c} \mathfrak{b}_i^{r_i/l})$. Since $\mathrm{ord}_{g_{ij}} \mathfrak{b}_i^{r_i/l} = r_i/l \le 1$, it follows that $(X, \prod_{1\le i\le c} \mathfrak{b}_i^{r_i/l})$ is also lc. □

2.5 Proof of main theorem and corollaries

Theorem 2.1.1 can be proved by the induction on $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(I \cup \{1\})$. The same argument essentially appears in [12].

Proof of Theorem 2.1.1. It is sufficient to prove the case when $1 \in I$. Let $r_0 = 1, r_1, \dots, r_c$ be all the elements of I. Set $c'+1 := \dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(1, r_1, \dots, r_c)$. Possibly rearranging the indices, we may assume that $r_0, \dots, r_{c'}$ are $\mathbb{Q}$-linearly independent. We may write $r_i = \sum_{0\le j\le c'} q_{ij} r_j$ with $q_{ij} \in \mathbb{Q}$.

We prove by induction on c'. If $c' = 0$, we can take $n \in \mathbb{Z}_{>0}$ such that $I \subset \frac{1}{n}\mathbb{Z}$ and $\frac{1}{r} \in \frac{1}{n}\mathbb{Z}$. Then $B(d, r, I) \subset \frac{1}{n}\mathbb{Z}$ and $B(d, r, I)$ turns out to be discrete.

Set $\mathbb{Q}$-linear functions $s_0, \dots, s_c$ as follows:

$$s_i : \mathbb{R}^{c'+1} \to \mathbb{R}; \quad s_i(x_0, \dots, x_{c'}) = \sum_{0\le j\le c'} q_{ij} x_j.$$

Take $\epsilon > 0$ as in Theorem 2.4.1. We fix $t^+, t^- \in \mathbb{Q}$ such that

$$t^+ \in (r_{c'}, r_{c'} + \epsilon] \cap \mathbb{Q}, \quad t^- \in [r_{c'} - \epsilon, r_{c'}) \cap \mathbb{Q}.$$

We define $r_0^+,\dots,r_c^+$ and $r_0^-,\dots,r_c^-$ as

$$r_i^+ = s_i(r_0,\dots,r_{c'-1},t^+),\quad r_i^- = s_i(r_0,\dots,r_{c'-1},t^-).$$

Further, we set $I' := \{r_0^+,\dots,r_c^+,r_0^-,\dots,r_c^-\}$. Then $\dim_{\mathbb{Q}}\operatorname{Span}_{\mathbb{Q}}(I') = c'$, and so $B(d,r,I')$ is discrete by induction.

Let $(X,\prod_{0\le i\le c}\mathfrak{a}_i^{r_i}) \in P(d,r)$, and let E be a divisor over X. Since $(X,\prod_{0\le i\le c}\mathfrak{a}_i^{r_i})$ is lc, $(X,\prod_{0\le i\le c}\mathfrak{a}_i^{r_i^*})$ is also lc for each $*\in\{+,-\}$. Hence we have

$$\begin{aligned}0 &\le a_E(X,\prod_{0\le i\le c}\mathfrak{a}_i^{r_i^*})\\ &= a_E(X,\prod_{0\le i\le c}\mathfrak{a}_i^{r_i}) - (r_{c'}^* - r_{c'})\sum_{0\le i\le c} q_{ic'}\operatorname{ord}_E\mathfrak{a}_i.\end{aligned}$$

Therefore, either of the following holds:

- $0 \le \sum_{0\le i\le c} q_{ic'}\operatorname{ord}_E\mathfrak{a}_i \le \epsilon_+^{-1}a_E(X,\prod_{0\le i\le c}\mathfrak{a}_i^{r_i})$, or
- $-\epsilon_-^{-1}a_E(X,\prod_{0\le i\le c}\mathfrak{a}_i^{r_i}) \le \sum_{0\le i\le c} q_{ic'}\operatorname{ord}_E\mathfrak{a}_i \le 0$,

where we set $\epsilon_+ := r_{c'}^+ - r_{c'}$ and $\epsilon_- := r_{c'} - r_{c'}^-$.

It is sufficient to show the discreteness of $B(d,r,I)\cap[0,a]$ for any $a\in\mathbb{R}_{>0}$. Take $n\in\mathbb{Z}_{>0}$ such that $q_{ic'}\in\frac{1}{n}\mathbb{Z}$ holds for any i. Then, it is sufficient to prove that $B(d,r,I)\cap[0,a]$ is contained in

$$\begin{aligned}&\left\{b+\epsilon_+e \;\middle|\; b\in B(d,r,I'), e\in\frac{1}{n}\mathbb{Z}\cap[0,\epsilon_+^{-1}a]\right\}\\ &\qquad\cup\left\{b-\epsilon_-e \;\middle|\; b\in B(d,r,I'), e\in\frac{1}{n}\mathbb{Z}\cap[-\epsilon_-^{-1}a,0]\right\}.\end{aligned}$$

In fact, this set becomes discrete because $B(d,r,I')$ is discrete, and both $\frac{1}{n}\mathbb{Z}\cap[0,\epsilon_+^{-1}a]$ and $\frac{1}{n}\mathbb{Z}\cap[-\epsilon_-^{-1}a,0]$ are finite.

Let $(X,\prod_{0\le i\le c}\mathfrak{a}_i^{r_i}) \in P(d,t)$, and E a divisor over X. Assume that $a_E(X,\prod_{0\le i\le c}\mathfrak{a}_i^{r_i}) \in [0,a]$ holds. Further, suppose $\sum_{0\le i\le c} q_{ic'}\operatorname{ord}_E\mathfrak{a}_i \ge 0$ (the other case can be proved in the same way). Then, we have

$$a_E(X,\prod_{0\le i\le c}\mathfrak{a}_i^{r_i}) = a_E(X,\prod_{0\le i\le c}\mathfrak{a}_i^{r_i^+}) + (r_{c'}^+ - r_{c'})\sum_{0\le i\le c} q_{ic'}\operatorname{ord}_E\mathfrak{a}_i.$$

Here, we have

- $a_E(X,\prod_{0\le i\le c}\mathfrak{a}_i^{r_i^+}) \in B(d,r,I')$,

- $r^+_{c'} - r_{c'} = \epsilon_+$, and
- $\sum_{0\le i\le c} q_{ic'} \operatorname{ord}_E \mathfrak{a}_i \in \frac{1}{n}\mathbb{Z} \cap [0, \epsilon_+^{-1} a]$.

We complete the proof. $\square$

Proof of Corollary 2.1.3. Note that $A_{\mathrm{can}}(3, I) \subset [1, 3]$ holds (cf. [15], [20]). We prove that for any $a > 1$, the set

$$A_{\mathrm{can}}(3, I) \cap [a, +\infty)$$

is a finite set.

By the classification of three-dimensional $\mathbb{Q}$-factorial terminal singularities (see [15], [20]), the minimal log discrepancy of a three-dimensional terminal singularity is equal to $1 + 1/r$ ($r \in \mathbb{Z}_{>0}$) or 3. In the case when $\mathrm{mld}_x(X) = 3$, the Gorenstein index of X at x is 1. If $\mathrm{mld}_x(X) = 1 + 1/r$, the Gorenstein index of X at x is r. Further, by [14, Corollary 5.2], if X has Gorenstein index r at $x \in X$, then rD is Cartier at x for any Weil divisor D.

Let (X, Δ) be a three-dimensional canonical pair satisfying $\Delta \in I$ and $\mathrm{mld}_x(X, \Delta) \ge a$. By [4, Corollary 1.4.3], there exists a projective morphism $f : Y \to X$ with the following properties:

- Y is a $\mathbb{Q}$-factorial terminal variety.
- $f^*(K_X + \Delta) = K_Y + \Delta_Y$ holds, where Δ_Y is the strict transform on Y of Δ (note that (X, Δ) is canonical).

Take a divisor E over X such that $\mathrm{mld}_x(X, \Delta) = a_E(X, \Delta)$ and $\mathrm{c}_X(E) = \{x\}$.

Suppose $\dim \mathrm{c}_Y(E) = 0$. Then $\mathrm{mld}_x(X, \Delta) = \mathrm{mld}_y(Y, \Delta_Y)$ holds, where $\{y\} := \mathrm{c}_Y(E)$. Since $\mathrm{mld}_y(Y) \ge \mathrm{mld}_y(Y, \Delta_Y) \ge a$ holds, the Gorenstein index of Y at y is at most $\lfloor \frac{1}{a-1} \rfloor$. Let l be the Gorenstein index of Y at y. Since lD is Cartier at y for any Weil divisor D on Y, it follows that $\mathrm{mld}_y(Y, \Delta_Y) \in A'(3, l, \frac{1}{l} I)$ (see Remark 1.1.2), where we set

$$\frac{1}{l} I := \{ f l^{-1} \mid f \in I \}.$$

Therefore we have

$$\mathrm{mld}_x(X, \Delta) \in \bigcup_{l \le \lfloor \frac{1}{a-1} \rfloor} A'\Bigl(3, l, \frac{1}{l} I\Bigr),$$

and the right hand side is a finite set by Corollary 2.1.2.

Suppose $\dim \mathrm{c}_Y(E) = 1$. Then, by [2, Proposition 2.1],

$$\mathrm{mld}_y(Y, \Delta_Y) = 1 + \mathrm{mld}_x(X, \Delta)$$

holds for some $y \in \mathrm{c}_Y(E)$. Since $\mathrm{mld}_y(Y) \geq 1 + a > 2$, it follows that Y has Gorenstein index 1. Hence,

$$\mathrm{mld}_y(Y, \Delta_Y) \in A'(3, 1, I).$$

Therefore, we have

$$\mathrm{mld}_x(X, \Delta) \in -1 + A'(3, 1, I),$$

and the right hand side is a finite set by Corollary 2.1.2.

Suppose $\dim \mathrm{c}_Y(E) = 2$. Then E is a divisor on Y, and we have

$$\mathrm{mld}_x(X, \Delta) = 1 - \mathrm{coeff}_E \Delta_Y.$$

Therefore, we have

$$\mathrm{mld}_x(X, \Delta) \in 1 - I,$$

and the right hand side is a finite set. $\square$

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