博 士 論 文

On the study of front propagation in nonlinear free boundary problems

（非線形自由境界問題における波面の伝播の研究）

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Introduction

In this thesis, we consider a free boundary problem that combines the nonlinear reaction-diffusion equation with the Stefan boundary condition.

$$
\begin{cases}
u_t - u_{xx} = f(u), & t > 0,\ x \in (g(t), h(t)),\\
g'(t) = -\mu u_x(t, g(t)), u(t, g(t)) = 0, & t > 0,\\
h'(t) = -\mu u_x(t, h(t)), u(t, h(t)) = 0 & t > 0,\\
-g(0) = h(0) = h_0,\ u(0, x) = u_0(x), & x \in (-h_0, h_0).
\end{cases}
\tag{0.1}
$$

where $g(t), h(t)$ are the free boundaries, supplemented together with some nonnegative initial datum

$$
u_0 \in \mathscr{X}(h_0) := \left\{ W \in C^2[-h_0, h_0] \ : \ \begin{array}{l} W(-h_0) = W(h_0) = 0,\ W'(-h_0) > 0, \\ W'(h_0) < 0, W(x) > 0 \text{ in } (-h_0, h_0) \end{array} \right\}.
\tag{0.2}
$$

μ is a positive constant while $f \in C^1$. The first equation in (0.1) is the reaction-diffusion equation, which has many applications in physics, chemistry and biology. The second and third lines in (0.1) are the so-called Stefan condition, which is a well-known free boundary condition that appears typically in the melting of ice. The combination of the reaction-diffusion equation and the Stefan condition makes the problem highly unique and different from what has been known before. Indeed the presence of the free boundary in the reaction-diffusion equation also gives rise to various interesting phenomena which I will explain later.

In this thesis we deal with a large class of nonlinearities f. In the special case where $f(u) = u(a - bu)$, the equation in (0.1) reduces to the well-known Fisher-KPP equation, which is a classical mathematical model in population genetics and ecology. For instance, in the celebrated work of J. G. Skellam [35], he established a reaction-diffusion model in order to explain the spreading of muskrats in central Europe. More precisely, he calculated the area of the muskrat territory from a map obtained from earlier field data, took the square foot (which gives the spreading radius) and plotted it against years, and found that the data points lie on a straight line. This means that the spreading radius eventually exhibits a linear growth curve against time. Skellam then derived a KPP type reaction-diffusion model to explain this linear growth. Since then, many researchers started to use a reaction-diffusion model to describe biological invasion. See, for example, [36] and the references therein.

A great deal of previous mathematical investigation on the spreading of population has been based on the diffusive logistic equation over the entire space :

$$
u_t - du_{xx} = u(a - bu),\ t > 0,\ x \in \mathbb{R}.
\tag{0.3}
$$

In the pioneering works of Fisher [20] and Kolmogorov, Petrovsky, and Piskunov [28], the traveling wave solutions have been found for (0.3): for any $|c| \geq c_0 := 2\sqrt{ad}$, there exists a solution $u(t, x) := W(x - ct)$ with the property that $W'(y) < 0$ for $y \in R_1$, $W(-\infty) = a/b$, $W(+\infty) = 0$; no such solution exists if $|c| < c_0$. The number c_0 is called the minimal speed of the traveling waves. It is known that c_0 also coincides with what is called the spreading speed. Here, the spreading speed means the speed of expanding fronts of a solution $u(t, x)$ with compactly supported initial datum $u(0, x)$. In 1975, D. G. Aronson and H. F. Weinberger [2] proved that if the initial value $u(0, x)$ is confined to a compact set, then the following holds for any $\epsilon > 0$

$$
\lim_{t\to\infty, |x|<(c_0-\epsilon)t} u(t, x) = a/b, \quad \lim_{t\to\infty, |x|>(c_0+\epsilon)t} u(t, x) = 0,
\tag{0.4}
$$

These results have been extended to higher dimensions in [3].

As to the asymptotic behavior of the Fisher-KPP equation, in 1937, Kolmogorov et al. [28] proved that if the initial datum is given by $u(0,x) = I_{(-\infty,0)}$, then we have:

$$u(t, m(t) + x) \to U_{c_0}(x), \tag{0.5}$$

where $m(t) = \sup\{x : u(t,x) = 1/2\}$ and U is the unique solution of

$$\begin{cases} -U''(x) - c_0 U'(x) = U(x)(a - bU(x)), \ x \in \mathbb{R}, \\ U(-\infty) = a/b, U(+\infty) = 0. \end{cases} \tag{0.6}$$

K. Uchiyama [37] proved that the property (0.5) holds also for solutions of (0.3) with compactly supported initial datum.

Recently, Yihong Du added the Stefan condition to the Fisher-KPP equation as follows:

$$\begin{cases} u_t - du_{xx} = u(a - bu), & t > 0, \ x \in (g(t), h(t)), \\ g'(t) = -\mu u_x(t, g(t)), u(t, g(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t, h(t)), u(t, h(t)) = 0 & t > 0, \\ -g(0) = h(0) = h_0, \ u(0,x) = u_0(x), & x \in (-h_0, h_0). \end{cases} \tag{0.7}$$

In 2010, Y. Du and Z. Lin [10] proved the existence, uniqueness and regularity of solutions to the equation (0.7). Moreover, they proposed a spreading-vanishing dichotomy. If $h_\infty - g_\infty = \lim_{t\to+\infty}(h(t) - g(t)) \le \pi\sqrt{d/a}$, it happens vanishing, which means

$$h_\infty, g_\infty \text{ are bounded and } \lim_{t\to+\infty} \|u(t,\cdot)\|_{C([g(t),h(t)])} = 0;$$

otherwise it happens spreading, which means

$$h(t), -g(t) \to +\infty \text{ and } u(t,x) \to a/b \text{ locally uniformly in } \mathbb{R} \text{ as } t \to +\infty.$$

However, according to the well-known hair-trigger effect, we know that u, which is the solution of (0.3) with a compactly supported initial datum, must converge to a/b locally uniformly in $\mathbb{R}$. This is a striking difference between (0.3) and (0.7). The phenomenon exhibited by the spreading-vanishing dichotomy seems closer the reality, and is supported by numerous empirical evidences; for example, the introduction of several bird species from Europe to North America in the 1900's was successful only after many initial attempts.

If adding a advection term to (0.3) and (0.7), the difference between them becomes more apparent. For the equation

$$v_t - \beta v_x - dv_{xx} = v(a - bv), \ t > 0, \ x \in \mathbb{R} \tag{0.8}$$

is just a translation of (0.3), it is easy to see that $\|v(t,\cdot)\|_{L^\infty(\mathbb{R})}$ converges to a/b as $t \to +\infty$, where v is a solution of (0.8) with a compactly supported initial datum. In a recent paper [24], the corresponding free boundary problem

$$\begin{cases} v_t - \beta v_x - dv_{xx} = v(a - bv), & t > 0, \ x \in (g(t), h(t)), \\ g'(t) = -\mu v_x(t, g(t)), v(t, g(t)) = 0, & t > 0, \\ h'(t) = -\mu v_x(t, h(t)), v(t, h(t)) = 0 & t > 0, \\ -g(0) = h(0) = h_0, \ v(0,x) = v_0(x), & x \in (-h_0, h_0). \end{cases} \tag{0.9}$$

is considered. [24] shows us that if β is sufficiently large, it must be that h_∞, g_∞ are bounded and

$$\lim_{t\to+\infty} \|v(t,\cdot)\|_{C([g(t),h(t)])} = 0,$$

where (v, g, h) is the solution of (0.9) with any compactly supported initial datum. Here, βv_x can be considered as something like wind which can affect the propagation. We can naturally consider that it is hard for the species to exist if the wind is too strong. As a consequence, the free boundary problem is closer to the reality in some sense.

For more general types of homogeneous f, [16] gives out a sharp estimate about the asymptotic behavior of (u, g, h) when spreading happens. First, let us introduce three classical types of nonlinearities. We say that f is called **monostable**, if $f \in C^1$, and it satisfies

$$f(0) = f(1) = 0, \quad f(u) \begin{cases} > 0 & \text{in } (0,1), \\ < 0 & \text{in } (1,\infty) \end{cases} \tag{0.10}$$

and $f'(0) > 0$, $f'(1) < 0$. We say f is of **bistable type**, if $f \in C^1$ and it satisfies

$$f(0) = f(\theta) = f(1) = 0, \quad f(u) \begin{cases} < 0 & \text{in } (0,\theta), \\ > 0 & \text{in } (\theta,1), \\ < 0 & \text{in } (1,\infty) \end{cases} \tag{0.11}$$

for some $\theta \in (0,1)$, $f'(0) < 0$, $f'(1) < 0$ and

$$\int_0^1 f(s)ds > 0. \tag{0.12}$$

We say f is of **combustion type**, if $f \in C^1$ and it satisfies

$$f(u) = 0 \ \text{ in } [0,\theta], \quad f(u) > 0 \text{ in } (\theta,1), \quad f'(1) < 0, \quad f(u) < 0 \text{ in } (1,\infty) \tag{0.13}$$

for some $\theta \in (0,1)$, and there exists a small $\delta > 0$ such that $f(u)$ is nondecreasing in $(\theta, \theta + \delta)$.

Theorem 1. *Suppose that $f(u)$ is of monostable, bistable or combustion type. Then the problem*

$$\begin{cases} q_{zz} - cq_z + f(q) = 0 & \textit{for } z \in (0,\infty), \\ q(0) = 0, \ \mu q_z(0) = c, \ q(\infty) = 1, \ q(z) > 0 \textit{ for } z > 0. \end{cases} \tag{0.14}$$

has a unique solution pair $(c, q) = (c^*, q_{c^*})$, and $c^* > 0$, $q'_{c^*}(z) > 0$. (u, g, h) is a solution of (0.1), which is obtained in [10]. Moreover, if spreading happens, then we have

$$|h(t) - c^*t - c_1| \to 0, \quad |g(t) + c^*t + c_2| \to 0, \quad h'(t) \to c^*, \ g'(t) \to -c^*$$

and

$$\max_{0 \le x \le h(t)} |u(t,x) - q_{c^*}(h(t) - x)| \to 0, \quad \max_{g(t) \le x \le 0} |u(t,x) - q_{c^*}(x - g(t))| \to 0$$

as $t \to +\infty$.

This thesis has two purposes: 1. to show Theorem 1 ([16]) still holds for heterogeneous f; 2. to consider the propagation of the free boundary in higher dimensional spaces.

In the first half, we will consider the equation

$$\begin{cases} u_t - u_{xx} = f(x,u), & t > 0, \ x \in (g(t), h(t)), \\ g'(t) = -\mu(t, g(t)), u(t, g(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t, h(t)), u(t, h(t)) = 0 & t > 0, \\ u(0,x) = u_0(x), & x \in (g(0), h(0)), \end{cases} \tag{0.15}$$

where $f \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfies the periodicity condition $f(x + L, u) \equiv f(x, u)$ and $f(x, 0) \equiv 0$ for some $L > 0$. We assume that there exists a positive and L-periodic stationary solution $p(x)$ of

$$\begin{cases} p''(x) + f(x, p(x)) = 0, \ x \in \mathbb{R}, \\ p(x) > 0, \ p(x + L) \equiv p(x). \end{cases} \tag{0.16}$$

The function p is also a stationary solution of the following auxiliary equation

$$\begin{cases} \partial_t u(t,x) - \partial_{xx} u(t,x) = f(x, u(t,x)), \ t > 0, x \in \mathbb{R}, \\ u(t,\cdot) \ L\text{-periodic for any } t \in \mathbb{R}. \end{cases} \tag{E_{per}}$$

We now state two main assumptions:

Asumption 1. *There exists a solution* (u, g, h) *of* (0.15) *with compactly supported initial datum* $0 \le u_0(x) < p(x)$ *such that* u *converges locally uniformly to* p, $h(t) \to +\infty$ *and* $g(t) \to -\infty$ *as* $t \to \infty$.

Assumption 2. *There exists no stationary solution* q *with* $0 < q(x) < p(x)$ *that is both isolated from below and stable from below with respect to* (E_{per}).

Here we say a stationary solution q of (E_{per}) is **isolated from below** (reps. **above**) if there exists no sequence of other stationary solutions converging to q from below (reps. above). A stationary solution q is said to be **stable from below** (reps. **above**) with respect to (E_{per}) if it is stable in the L^∞ topology under nonpositive (reps. nonnegative) perturbations. Otherwise, q is called **unstable from below** (reps. **above**).

We note that Assumption 2 holds for a large class of nonlinearities including the following:

Case 1. (Monostable nonlinearity) *There exists no* L*-periodic stationary solution* q *satisfying* $0 < q(x) < p(x)$ *for all* $x \in \mathbb{R}$. *Futhermore,* 0 *is unstable from above.*

Case 2. (Bistable nonlinearity) *The stationary solution* 0 *is stable from above with respect to* (E_{per}), *and* p *is stable from below with respect to* (E_{per}). *Furthermore, all other stationary solutions between* 0 *and* p *are unstable.*

Case 3. (Combustion nonlinearity) *There exists a family of* L*-periodic stationary solutions* $(q_\lambda)_{\lambda \in [0,1]}$, *that forms a continuum in* $L^\infty(\mathbb{R})$ *and satisfies* $0 = q_0 < q_1 < p$. *Furthermore, there exists no stationary solution* q *satisfying* $q_1(x) < q(x) < p(x)$ *for all* $x \in \mathbb{R}$.

Then by following the idea of [18], we can show the existence of the pulsating traveling wave for the right side (U, H), which is a solution of

$$(F_h) \qquad \begin{cases} \partial_t u(t,x) - \partial_{xx} u(t,x) = f(x, u(t,x)), & t > 0,\ -\infty < x < h(t), \\ u(t, h(t)) = 0, & t \ge 0, \\ h'(t) = -\mu \partial_x u(t, h(t)), & t > 0, \\ h(0) = h_0,\ u(0,x) = u_0(x) > 0, & -\infty < x < h_0. \end{cases}$$

and satisfies

$$\begin{cases} U(t, x-L) = U(t+T, x), & t \in \mathbb{R}, x \in (-\infty, H(t+T)), \\ H(t) + L = H(t+T), & t \in \mathbb{R}, \end{cases}$$

for some $T > 0$. Correspondingly, there also exists the pulsating traveling wave for the left side (U_*, H_*).

To give out a sharp estimate on the asymptotic behavior of solutions of (0.15), we still need one more assumption to estimate the middle part of solutions, on which u converges to stable stationary solution p uniformly.

Assumption 3.*The principle eigenvalue of* $\mathcal{L}_0$ *is negative, where* $\mathcal{L}_0$ *is defined by* $\mathcal{L}_0\psi = \psi'' + \partial_u f(x,p)\psi$ *and* ψ *is* L*-periodic. In other words, there exist a function* ψ *and a constant* $\lambda > 0$, *such that*

$$\begin{cases} \psi''(x) + \partial_u f(x, p(x))\psi(x) = -\lambda\psi(x), \\ \psi(x) \equiv \psi(x+L),\ \psi(x) > 0, \end{cases}$$

hold for all $x \in \mathbb{R}$.

Under Assumption 3, we can show that the pulsating traveling for the right side is unique up to time shift. Moreover, we have the following theorem (see Corollary 1.1.12 and Theorem 1.1.15 in Section 1 for details):

Theorem 2.*Let Assumptions 1-3 hold. Then we have the pulsating semi-wave for the right side* (U, H) *and that for the left side* (U_*, H_*). *Let* (u, g, h) *be a solution of* (0.15) *with initial datum* $u_0(x) < p(x)$ *for which* u *converges to* p *locally uniformly in* $\mathbb{R}$ *as* $t \to +\infty$. *Then there exists* *constants T_r and T_l such that as $t \to +\infty$, we have*

$$\|u(t,\cdot) - U(t+T_r,\cdot)\|_{L^\infty([0,\min\{h(t),H(t+T_r)\}])} \to 0, \tag{0.17}$$

$$|h(t) - H(t+T_r)| \to 0 \text{ and } |h'(t) - H'(t+T_r)| \to 0, \tag{0.18}$$

$$\|u(t,\cdot) - U_*(t+T_l,\cdot)\|_{L^\infty([\max\{g(t),H_*(t+T_l)\},0])} \to 0, \tag{0.19}$$

$$|g(t) - H_*(t+T_l)| \to 0 \text{ and } |g'(t) - H_*'(t+T_l)| \to 0. \tag{0.20}$$

In the second half, we are interested in the spreading speed in higher dimensions for homogeneous f. Therefore we will consider the equation

$$\begin{cases} u_t - u_{rr} - \frac{N-1}{r} u_r = f(u), & 0 < r < h(t),\ t > 0, \\ u_r(t,0) = 0,\ u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_r(t,h(t)), & t > 0, \\ h(0) = h_0, \quad u(0,r) = u_0(r), & 0 \le r \le h_0 \end{cases} \tag{0.21}$$

with initial datum u_0 chosen from

$$\mathscr{K}(h_0) := \Big\{ \psi \in C^2([0,h_0]) : \psi'(0) = \psi(h_0) = 0,\ \psi(r) > 0 \text{ in } [0,h_0) \Big\}. \tag{0.22}$$

We assume that spreading happens, namely $\lim_{t\to\infty} h(t) = \infty$, $\lim_{t\to\infty} u(t,|x|) = 1$. For the case of one space dimension ($N = 1$), it has been solved by Theorem 1 ([16]). In the latter half, we consider the case $N \ge 2$ and give out a sharp estimate as follows (see Theorem 2.4.1 in Section 2 for details):

Theorem 3. *Suppose that $f(u)$ is of monostable, bistable or combustion type. Let (u,h) be the solution of (0.21) for which spreading happens. There exists a constant $\hat{h} \in \mathbb{R}^1$ such that*

$$\lim_{t\to\infty} \big\{ h(t) - \big[c^* t - c_N \log t\big] \big\} = \hat{h},\ \lim_{t\to\infty} h'(t) = c^*$$

and

$$\lim_{t\to\infty} \|u(t,\cdot) - q_{c^*}(h(t) - \cdot)\|_{L^\infty([0,h(t)])} = 0,$$

where $c_N = \frac{N-1}{\zeta c^*}$ and $\zeta = 1 + \frac{c^*}{\mu^2 \int_0^\infty q_{c^*}'(z)^2 e^{-c^* z} dz}$.

At the same time, we also obtain a rather clear description of the spreading profile of $u(t,r)$.

Contents

1. **Front propagation in periodic media with free boundaries**

1.1. **Introduciton.**

In this section, we consider a free boundary problem that combines the reaction-diffusion equation with the Stefan boundary condition. Equation

$$(F)\qquad \begin{cases} \partial_t u(t,x) - \partial_{xx}u(t,x) = f(x,u), & g(t) < x < h(t), t > 0, \\ u(t,h(t)) = u(t,g(t)) = 0, & t > 0, \\ h'(t) = -\mu\partial_x u(t,h(t)), & t > 0, \\ g'(t) = -\mu\partial_x u(t,g(t)), & t > 0, \\ h(0) = -g(0) = h_0 > 0,\ u(0,x) = u_0(x), & -h_0 < x < h_0, \end{cases}$$

where $x = g(t)$ and $x = h(t)$ are the moving boundaries to be determined together with $u(t,x)$, supplemented together with some nonnegative initial datum

$$(1.1)\qquad u_0 \in \mathscr{X}(h_0) := \left\{ W \in C^2[-h_0,h_0] \ : \ \begin{array}{l} W(-h_0) = W(h_0) = 0,\ W'(-h_0) > 0, \\ W'(h_0) < 0, W(x) > 0 \text{ in } (-h_0,h_0) \end{array} \right\}.$$

μ is a given positive constant, and the function $f(x,u) \in C^1(\mathbb{R}^2;\mathbb{R})$ satisfies the periodicity condition

$$(1.2)\qquad f(x+L,u) \equiv f(x,u) \text{ and } f(x,0) \equiv 0$$

for some $L > 0$.

Problem (F) with $f(u) = au - bu^2$ was introduced by [10] to describe the spreading of a new or invasive species. The free boundary boundaries $x = g(t)$ and $x = h(t)$ represent the spreading fronts of the population whose density is represented by $u(t,x)$. The results in [10] were extended by [7, 8] to higher dimensions, while the regularity of the free boundary in higher dimensions was recently solved in [15].

Problem (F) with a rather general homogeneous $f(u)$ (of monostable, or bistable, or combustion type) was recently studied by [11]. It shows that problem (F) has a unique solution which is defined for all $t > 0$, and as $t \to \infty$, the interval $(g(t),h(t))$ converges either to a finite interval (g_∞,h_∞), or to $(-\infty,+\infty)$. Moreover, in the former case, $u(t,x) \to 0$ uniformly in x, while in the latter case, $u(t,x) \to 1$ locally uniformly in $x \in (-\infty,+\infty)$ (except for a non-generic transition case when f is of bistable or combustion type). The situation that

$$u \to 0 \text{ and } (g,h) \to (g_\infty,h_\infty)$$

is called the **vanishing** case, and

$$u \to 1 \text{ and } (g,h) \to (-\infty,+\infty)$$

is called the **spreading** case.

Moreover, in the setting of [11], when spreading happens, it is shown in [16] that there exists $c^* > 0$, such that

$$\lim_{t\to\infty}(h(t) - c^*t - H) = 0, \lim_{t\to\infty} h'(t) = c^*,$$
$$\lim_{t\to\infty}(g(t) + c^*t - G) = 0, \lim_{t\to\infty} g'(t) = -c^*,$$

where $H, G \in \mathbb{R}$, and

$$\lim_{t\to\infty} \sup\nolimits_{x\in[0,h(t)]}|u(t,x) - q_{c^*}(h(t)-x)| = 0,$$
$$\lim_{t\to\infty} \sup\nolimits_{x\in[g(t),0]}|u(t,x) - q_{c^*}(x-g(t))| = 0,$$

where (c^*, q_{c^*}) is uniquely determined by (see [16])

$$\begin{cases} c^* q_{c^*}'(x) - q_{c^*}''(x) = f(q_{c^*}(x)),\ x \in (0, +\infty), \\ q_{c^*}(0) = 0,\ q_{c^*}(+\infty) = 1,\ c^* = \mu q_{c^*}'(0). \end{cases}$$

Because q_{c^*} is only defined on the right axis, we call it a **semi-wave**, which is very similar to the traveling wave arising from the corresponding Cauchy problem.

For the special heterogenous case $f(x, u) = a(x)u - b(x)u^2$, with $a(x)$ and $b(x)$ positive L-periodic functions, Du and Liang [9] proved the existence and uniqueness of the pulsating semi-wave which governs the spreading speed of (F) when spreading happens.

Recently in [18], Ducrot, Giletti and Matano introduced the definition of propagating terrace for the corresponding Cauchy problem of (F), namely

$$(E) \qquad \begin{cases} \partial_t u - \partial_{xx} u = f(x, u), & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

They use the propagating terrace to describe the long-time dynamical behavior for very general nonlinearities which may give rise to many different steady states. Such a propagating terrace consists of several different stationary solutions and the pulsating traveling waves connecting them.

The current section has two goals:

(1) We want to extend the results of the propagating terrace to the free boundary problem (F), and compare the differences between the Cauchy problem and (F). In the first half, we will give out the existence of the propagating terrace with a free boundary by borrowing some important ideas from [18], and then show how the Stefan coefficient μ will affect the propagating terrace.

(2) Using the pulsating semi-wave obtained in (1), which covers rather general nonlinearities f, we will give a sharp estimate on the asymptotic behavior of solutions of problem (F) when spreading happens. We will show in particular that, in a certain moving frame, the solution of (F) converges to the pulsating semi-wave as $t \to +\infty$, and in contrast to the Cauchy problem treated in [18], here the convergence does not involve any phase shift.

1.1.1. *Description on f.*

In this subsection, we will introduce some assumptions on f, which will be widely used in the following.

In this section, we always assume that there exists some positive L-periodic stationary solution $p(x)$ of (E):

$$(1.3) \qquad \begin{cases} p''(x) + f(x, p(x)) = 0,\ \forall x \in \mathbb{R}, \\ p(x) > 0,\ p(x + L) \equiv p(x). \end{cases}$$

The function p is also a stationary solution of the following auxiliary equation, the L-periodic counterpart of (E):

$$(E_{per}) \qquad \begin{cases} \partial_t u(t, x) - \partial_{xx} u(t, x) = f(x, u(t, x)),\ t > 0, x \in \mathbb{R}, \\ u(t, \cdot)\ L\text{-periodic for any } t \in \mathbb{R}. \end{cases}$$

It is obvious that any solution of (E_{per}) is also a solution of (E).

We now state our main assumptions. The following two are concerned with the attractiveness of p from below.

Assumption 1.1.1. *There exists an initial datum $\phi_0 \in \mathscr{X}(l_0)$, which satisfies $\phi_0(x) < p(x)$ for all $x \in (-l_0, l_0)$. If $u_0 = \phi_0$, then we have $u(t, x)$ converges to $p(x)$ locally uniformly in $\mathbb{R}$ while $h(t)$ and $-g(t)$ converge to $+\infty$ as $t \to +\infty$, where (u, g, h) is the solution of (F).*

From the view of ecology, p means the stable state for some species in a natural enviroment. It is reasonable for us to assume that in some cases such a stable state is unique.

Assumption 1.1.2. *There exists no L-periodic stationary solution q with $0 < q(x) < p(x)$ that is both isolated from below and stable from below with respect to (E_{per}).*

Let us clarify the notions introduced in this assumption. A stationary solution q of (E_{per}) is said to be **isolated from below** (resp. bf above) if there exists no sequence of other stationary solutions converging to q from below (resp. above). A stationary solution q is said to be **stable from below** (resp. bf above) with respect to (E_{per}) if it is stable in the L^∞ topology under nonpositive (resp. nonnegative) perturbations(see Theorem 8 in [30]).

To give a sharp estimate on the asymptotic behavior of solutions of (F), we still need one more assumption to estimate the middle part of solutions, on which u converges to stable stationary solution p uniformly.

Assumption 1.1.3. *The principle eigenvalue of $\mathcal{L}_0$ is negative, where $\mathcal{L}_0$ is defined by $\mathcal{L}_0\psi = \psi'' + \partial_u f(x,p)\psi$ and ψ is L-periodic. In other words, there exist a function ψ and a constant $\lambda > 0$, such that*

$$\begin{cases} \psi''(x) + \partial_u f(x, p(x))\psi(x) = -\lambda\psi(x), \\ \psi(x) \equiv \psi(x+L), \ \ \psi(x) > 0, \end{cases}$$

hold for all $x \in \mathbb{R}$.

Note that the above assumptions cover a wide variety of nonlinearities, including such standard cases as monostable,bistable or combustion nonlinearities, but also to much more general and complex cases.

A classical example of the bistable nonlinearity is the Allen-Cahn nonlinearity $u(1-u)(u-a(x))$, where $0 < a(x) < 1, \ a(x+L) \equiv a(x)$. An important subclass of the monostable nonlinearity is the KPP type nonlinearity, in which 0 is assumed to be linearly unstable and f is sublinear with respect to u; a typical example being $a(x)u - b(x)u^2$, with $a(x+L) \equiv a(x) > 0$ and $b(x+L) \equiv b(x) > 0$.

1.1.2. *Main results.*

Before stating the main results, let us first introduce some notions which will play a fundamental role in this section. To make the thesis not lengthy, we refer the definitions of pulsating traveling waves and propagating terraces of Cauchy problem (E) to [18]. Next, we will define the corresponding ones in the free boundary problem (F).

Definition 1.1.4. (Pulsating semi-wave) *Given a positive periodic stationary state p, by a* **pulsating semi-wave solution** *of (F_h) connecting 0 to p, we mean any entire solution (u,h) satisfying, for some $T > 0$,*

$$\begin{cases} u(t, x-L) = u(t+T, x), & t \in \mathbb{R}, x \in (-\infty, h(t+T)), \\ h(t) + L = h(t+T), & t \in \mathbb{R}, \end{cases}$$

along with the asymptotics

$$u(+\infty, \cdot) = p(\cdot),$$

when the convergence is understood to hold locally uniformly in the space variable. Then ratio $c : \frac{L}{T} > 0$ is called the **average speed** *of this pulsating traveling wave with a right free boundary.*

(F_h) is a free boundary problem defined in the below, which has only a free boundary on the right side.

$$(F_h) \qquad \begin{cases} \partial_t u(t,x) - \partial_{xx}u(t,x) = f(x,u(t,x)), & t>0,\ -\infty<x<h(t), \\ u(t,h(t)) = 0, & t\geq 0, \\ h'(t) = -\mu\partial_x u(t,h(t)), & t>0, \\ h(0)=h_0,\ u(0,x)=u_0(x)>0, & -\infty<x<h_0. \end{cases}$$

The following definition on **steepness** plays a key role in the construction of the propagating terrace of (F_h).

Definition 1.1.5. *We will describe the steepness between different types of equations in the following:*

(i) *Let v_1, v_2 be two entire solutions of (E). we say that u_1 is* **steeper than** *u_2 if for any t_1, t_2 and x_1 in $\mathbb{R}$ such that $u_1(t_1,x_1) = u_2(t_2,x_1)$, we have either*

$$u_1(\cdot+t_1,\cdot) \equiv u_2(\cdot+t_2,\cdot) \text{ or } \partial_x u_1(t_1,x_1) < \partial_x u_2(t_2,x_1).$$

(ii) *Let $(u_1,h_1),(u_2,h_2)$ be two entire solutions of (F_h). We say that (u_1,h_1) is* **steeper than** *(u_2,h_2) if for any t_1,t_2 in $\mathbb{R}$ such that when $h_1(t_1)\geq h_2(t_2)$ we have*

$$u_1(t_1,x) \geq u_2(t_2,x) \text{ for } x\in(-\infty,h_2(t_2)),$$

and when $h_1(t_1) < h_2(t_2)$, we have

$$\partial_x u_1(t_1,x_1) < \partial_x u_2(t_2,x_1)$$

for any $x_1\in(-\infty,h_1(t_1))$ satisfying $u_1(t_1,x_1) = u_2(t_2,x_1)$.

(iii) *Let (u_1,h_1) be an entire solution of (F_h) and v_1 be that of (E). We say that (u_1,h_1) is* **steeper than** *v_1 if for any $t_1,t_2\in\mathbb{R}$ and $x_1\in(-\infty,h_1(t_1))$ such that $u_1(t_1,x_1)=v_2(t_2,x_1)$, we have*

$$\partial_x u_1(t_1,x_1) < \partial_x u_2(t_2,x_1).$$

Next, let us define the propagating terrace for the free boundary problem.

Definition 1.1.6. *A* **propagating terrace of** *(F_h) connecting 0 to p is a pair of finite sequence $(p_k)_{0\leq k\leq N}$, $(U_k)_{1\leq k\leq N}$ and a free boundary Hsuch that:*

- *Each p_k is an L-periodic stationary solution of (E_{per}) satisfying*

$$p=p_0>p_1>\cdots>p_N=0.$$

- *For each $1\leq k\leq N-1$, U_k is a pulsating traveling wave solution of (E) connecting p_k to p_{k-1}.*
- *(U_N,H) is a pulsating semi-wave solution 0 to p_{N-1}.*
- *The speed c_k of each U_k satisfies $0<c_1\leq c_2\leq\cdots\leq c_N$.*

Furthermore, a propagating terrace of (F_h) $T=((p_k)_{0\leq k\leq N},(U_k)_{1\leq k\leq N},H)$ connecting 0 to p is said to be minimal if it also satisfies the following:

- *For any propagating terrace $T'=((q_k)_{0\leq k\leq N},(\tilde{U}_k)_{1\leq k\leq N'},\tilde{H})$ connecting 0 to p, one has that*

$$\{p_k: 0\leq k\leq N\}\subset\{q_k:0\leq k\leq N'\}.$$

- *For each $1\leq k\leq N-1$, the pulsating traveling wave U_k is steeper than any other pulsating traveling wave connecting p_k to p_{k-1}.*
- *The pulsating semi-wave(U_N,H) is steeper than any other pulsating semi-wave connecting 0 to p_{N-1}.*

Next, let us introduce show the existence of a minimal propagating terrace of (F_h).

Theorem 1.1.7. *Let Assumption 1.1.1 hold. Then there exists a propagating terrace of* (F_h) $((p_k)_{0\le k\le N}$, $(U_k)_{1\le k\le N}, H)$ *that is minimal in the sense of Definition 1.1.6. Such a minimal propagating terrace of* (F_h) *is unique, in the sense that any minimal propagating terrace of* (F_h) *shares the same* $(p_k)_k$ *and that* $(U_k)_{1\le k\le N}$ *and* H *are unique up to time-shift. Moreover, it satisfies:*

(i) *For any* $0 \le k < N$*, the L-periodic stationary solution* p_k *is isolated and stable from below with respect to* (E_{per})*.*

(ii) $(p_k)_{0\le k\le N)}$ *and* $(U_k)_{1\le k\le N-1}$ *is steeper than any other entire solution of* (E) *between* 0 *and* p*. Moreover,* (U_N, H) *is steeper than any other entire solution of* (E) *or* (F_h) *between* 0 *and* p*.*

Remark 1.1.8. *According to the above theorem, we can get the existence of pulsating semi-wave connecting* 0 *to some steady state, which may be not* p*.*

From Theorem 1.10 in [18], we know that there is a unique minimal propagating terrace of (E) consisting of $((q_k)_{0\le k\le N'}, (V_k)_{1\le k\le N'})$ if Assumption 1.1.1 hold.

Theorem 1.1.9. *Let Assumption 1.1.1 hold and* $((q_k)_{0\le k\le N'}, (V_k)_{1\le k\le N'})$ *be the unique minimal propagating terrace of* (E)*. Then we have:*

(i) $N \le N'$, $(p_k)_{0\le k\le N-1} = (q_k)_{0\le k\le N-1}$ *and* $(U_k)_{1\le k\le N-1} = (V_k)_{1\le k\le N-1}$.

(ii) *If Assumption 1.1.1 holds for each* $\mu \in (0, +\infty)$*, there exists a sequence* $(\mu_i)_{0\le i\le N'}$*, which satisfies*

$$0 = \mu_0 < \mu_1 \le \cdots \le \mu_{N'-1} \le \mu_{N'} = +\infty$$

such that, for any $\mu \in (\mu_i, \mu_{i+1})$ *and* $0 \le i \le N'-1$*, the corresponding minimal propagating terrace of* (F_h) *consists of* $(p_k)_{0\le k\le i+1}$*,* $(U_k)_{1\le k\le i+1}$ *and* H*, where it holds that*

$$p_k = q_k \text{ for } 0 \le k \le i$$

and

$$U_k = V_k \text{ for } 1 \le k \le i$$

and (U_{i+1}, H) *is a pulsating semi-wave connecting* 0 *to* q_i*.*

Remark 1.1.10. *Assume* $f(x, u) = a(x)u(1-u)$*, where* $1 < a(x) < 2$ *for each* $x \in \mathbb{R}$*. By the results in* [10]*, we know that spreading must happen as long as* $2h_0$*, the length of initial datum* u_0*, is bigger than* π*. No matter how small* u_0 *and* μ *are. Thus, it is possible that Assumption 1.1.1 holds for each* $\mu \in (0, +\infty)$*. For more general* f*, if* ϕ_0 *is a compactly supported stationary solution of* $u'' + f(x, u) = 0$ *and Assumption 1.1.1 holds for some* $\mu_0 > 0$*, then we can easily check that Assumption 1.1.1 also holds for each* $\mu > 0$*.*

Following Theorem 1.1.7 and 1.1.9, we can give out some sufficient conditions on the existence of the pulsating semi-wave connecting 0 to p.

Corollary 1.1.11. *Let Assumption 1.1.1 hold for each* $\mu \in (0, +\infty)$*. There must exist a pulsating semi-wave connecting* 0 *to* p *if* μ *is sufficiently small.*

Corollary 1.1.12. *Let Assumption 1.1.1 and 1.1.2 hold. There must exist a pulsating semi-wave connecting* 0 *to* p*.*

The next theorem will tell us how solutions of (F) converge to the pulsating semi-wave without phase drift as $t \to +\infty$ if **spreading happens**, which means

$$u \to p$$

locally uniformly in $\mathbb{R}$ and

$$g \to -\infty \text{ and } h \to +\infty$$

as $t \to +\infty$.

Proposition 1.1.13. *Let Assumptions 1.1.1 and 1.1.3 hold and assume the existence of the pulsating semi-wave (U, H) connecting 0 to p. Then the pulsating semi-wave is unique up to time shift.*

Remark 1.1.14. *Notice that most of the above definitions and theorems in this subsection focus on the free boundary of the right side. However, we note that the corresponding results also hold for the left side in the same argument.*

Theorem 1.1.15. *Let Assumptions 1.1.1 and 1.1.3 hold. We further ssume that the pulsating semi-wave exists for the two sides. (U, H) is the pulsating semi-wave with a free boundary on the right side in the sense of Definition 1.1.4, while (U_*, H_*) is the corresponding one for the left side. Let (u, g, h) be a solution of (F) for which spreading happens, and given initial datum $u_0(x) < p(x) + \epsilon_0\psi(x)$ for $x \in [-h_0, h_0]$, where ϵ_0 is a sufficiently small positive constant (see Lemma 1.2.10). Then there exists constants T_r and T_l such that as $t \to +\infty$, we have*

$$\|u(t,\cdot) - U(t + T_r, \cdot)\|_{L^\infty([0,\min\{h(t),H(t+T_r)\}])} \to 0, \tag{1.4}$$

$$|h(t) - H(t + T_r)| \to 0 \text{ and } |h'(t) - H'(t + T_r)| \to 0, \tag{1.5}$$

$$\|u(t,\cdot) - U_*(t + T_l, \cdot)\|_{L^\infty([\max\{g(t),H_*(t+T_l)\},0])} \to 0, \tag{1.6}$$

$$|g(t) - H_*(t + T_l)| \to 0 \text{ and } |g'(t) - H_*'(t + T_l)| \to 0. \tag{1.7}$$

Remark 1.1.16. *Because the coefficient is periodic with respect to x, (U, H) and (U_*, H_*) may have different periodicities on time. Thus they may move at different average speeds.*

This section is organized as follows. In Subsection 1.2, we introduce some preliminary knowledges. In Subsection 1.3, we show the existence of the propagating terrace of (F_h) and its comparison with that of (E). The asymptotic behavior of problem (F) is proved in Subsection 1.4.

1.2. **Preliminaries.**

1.2.1. *Zero number.*

As our proof relies strongly on a zero number argument, let us first begin, for the sake of clarity, with some preliminary definitions and lemmas from [1].

Definition 1.2.1. *For any $u \in C^0(I)$, where I is an open interval, we denote by:*

- *$Z[u(\cdot); I]$ is the number of sign changes of u, that is the supremum over all $k \in \mathbb{N}$ such that there exist $x_1 < x_2 < ... < x_k$ real numbers with*
$$u(x_i) \cdot u(x_{i+1}) < 0 \text{ for all } i = 1, 2, ..., k - 1.$$
- *$SGN[u(\cdot); I]$ is the word consisting of + and -, describing the signs that appear alternately when looking at $u(\cdot)$ from the left to the right. We denote $SGN[0; I] = [\ ]$ the empty word.*

Let us recall some properties of Z and SGN:

Lemma 1.2.2. *Let $u(t, x) \not\equiv 0$ be a bounded solution of a parabolic equation of the form*

$$\partial_t u = a(t, x)\partial_{xx}u + b(t, x)\partial_x u + c(t, x)u,\ \ x \in (s(t), r(t)), t_1 < t < t_2,$$

where a, b, c are bounded continuous functions, and s, r either always equal to ∞, or are bounded continuous functions (we do not need s, r attain ∞ at the same time). We denote $I(t) := (s(t), r(t))$. The boundary condition is one of the followings:

$$u > 0;\ \ u < 0;\ \ u = 0;\ \ \partial_x u = 0\ \ t \in (t_1, t_2).$$

Then, for each $t \in (t_1, t_2)$, the zeros of $u(t, \cdot)$ do not accumulate in $I(t)$. Furthermore, *(i)* $Z[u(t,\cdot);I(t)]$ *and* $SGN[u(t,\cdot);I(t)]$ *are nonincreasing in* t*, that is, for any* $t' > t$*,*

$$Z[u(t',\cdot);I(t)] \leq Z[u(t,\cdot);I(t)],$$

$$SGN[u(t',\cdot);I(t)] \lhd SGN[u(t,\cdot);I(t)] \ (\textit{ where } \lhd \textit{ means being a subword});$$

(ii) if $u(t',x_0) = \partial_x u(t,x_0) = 0$ *for some* $t' \in (t_1,t_2)$ *and* $x_0 \in I(t)$*, then*

$$Z[u(t,\cdot);I(t)] > Z[u(s,\cdot);I(t)] \ \textit{for all } t \in (t_1,t') \textit{ and } s \in (t',t_2)$$

whenever $Z[u(s,\cdot);I(t)] < +\infty$*.*

One can also check that Z is semi-continuous with respect to the pointwise convergence, that is:

Lemma 1.2.3. *Let a sequence* $(u_n)_{n\in N}$ *and* u *in* $C^0(I)$ *such that for all* $x \in I$*,* $u_n(x) \to u(x)$*. Then, one has that*

$$u \equiv 0 \ \textit{or}$$

$$Z[u;I] \leq \liminf_{n\to\infty} Z[u_n;I] \ \textit{and } SGN[u;I] \lhd \liminf_{n\to\infty} SGN[u_n;I]. \tag{1.8}$$

Those results follow from [1]: though the author only considered bounded intervals for the space variable, most of the conclusions above can be shown similarly (see also [14] for more details).

1.2.2. *Hopf's lemma for parabolic equations.*
Let Q_T be a bounded open set of $\mathbb{R}^{N+1}$, contained in $(0,T)\times\mathbb{R}^N$, where $T > 0$. We denote by Q_T^* the parabolic interior of Q_T, that is the set of all points $(\bar{t},\bar{x})$ with the property: there exist a $\epsilon > 0$ such that $B_\epsilon(\bar{t},\bar{x}) \cap \{t < \bar{t}\} \subset Q_T$. Here $B_\epsilon(\bar{t},\bar{x})$ denotes the $(N+1)$-dimensional ball with radius ϵ and center $(\bar{t},\bar{x})$. Define also the parabolic boundary $\partial_p Q_T$ of Q_T, as $\partial_p Q_T = \overline{Q_T} - Q_T^*$.

Definition 1.2.4. *We say that a point* $(\bar{t},\bar{x}) \in \partial_p Q_T$ *has the property of the spherical cap if there exists an open ball* $B_r(t_0,x_0)$ *such that*

$$(\bar{t},\bar{x}) \in \partial B_r(t_0,x_0), \quad B_r(t_0,x_0) \cap \{t < \bar{t}\} \subset Q_T,$$

with $x_0 \neq \bar{x}$*.*

In the following,we denote by $C_r(\bar{t},\bar{x})$ a cap $B_r(t_0,x_0) \cap \{t < \bar{t}\}$ as the one appearing in the above definition. A version of Hopf's Lemma for parabolic equations was proven in [22].

Lemma 1.2.5. *Let* $\partial_t u - \Delta u \leq 0$ *in* Q_T^**. Let* $(\bar{t},\bar{x}) \in \partial_p Q_T$ *have the property of the spherical cap. If*

$$u(t,x) < u(\bar{t},\bar{x}), \quad \textit{for all } (t,x) \in C_r(\bar{t},\bar{x}),$$

then

$$\frac{\partial u}{\partial e}(\bar{t},\bar{x}) < 0, \tag{1.9}$$

where $e \in \mathbb{R}^{N+1}$ *is any direction such that*

$$(\bar{t},\bar{x}) + se \in \overline{C_r(\bar{t},\bar{x})}, \quad \textit{for } 0 < s < \Sigma(e),$$

and we also assume that the derivative in (1.9) *exists.*

1.2.3. *Some basic results.*
In this subsection we give some basic results about problem (F). Because $f(x,u) \in C^1(\mathbb{R}^2;\mathbb{R})$, most of the results can be obtained by a slight modification from [10, 11]. Thus we omit the proof if they are not necessary.

Lemma 1.2.6. (**Comparison principle**)*Suppoe* $T \in (0,\infty)$, $\bar{g}, \bar{h} \in C^1([0,T])$,$\bar{u} \in C(\bar{D}_T) \cap C^{1,2}(\bar{D}_T)$ *with* $D_T = \{(t,x)|\ 0 < t \le T, \bar{g}(t) < x < \bar{h}(t)\}$, *and*

$$\begin{cases} \bar{u}_t \ge \bar{u}_{xx} + f(x,\bar{u}), & 0 < t \le T, \bar{g}(t) < x < \bar{h}(t), \\ \bar{u}(t,\bar{h}(t)) = 0, \bar{h}' \ge -\mu \bar{u}_x(t,\bar{h}(t), & 0 < t \le T, \\ \bar{u}(t,\bar{g}(t)) = 0, \bar{g}' \le -\mu \bar{u}_x(t,\bar{g}(t), & 0 < t \le T. \end{cases}$$

If

$$\bar{g}(0) \le -h_0,\ \bar{h}_0 \ge h_0,\ \bar{u}(0,x) \ge u_0(x)\ in\ [\bar{g}(0), \bar{h}(0)],$$

where (u,g,h) *is a solution of* (F), *then for any* $t \in (0,T]$

$$\bar{g}(t) \le g(t),\ \bar{h}(t) \ge h(t),\ \bar{u}(t,x) \ge u(t,x)\ in\ [\bar{g}(t), \bar{h}(t)].$$

The triple $(\bar{u}, \bar{g}, \bar{h})$ in the above lemma is usually called an upper solution of (F). We can define a lower solution by reversing the inequalities in the obvious places.

Theorem 1.2.7. *For any given* $u_0 \in \mathscr{X}(h_0)$ *and any* $\alpha \in (0,1)$, *there a* $T > 0$ *such that problem* (F) *admits a unique solution*

$$(u,g,h) \in C^{(1+\alpha)/2, 1+\alpha}(\bar{G}_T) \times C^{1+\alpha/2}([0,T]) \times C^{1+\alpha/2}([0,T]);$$

moreover,

$$\|u\|_{C^{(1+\alpha)/2,1+\alpha}(\bar{G}_T)} + \|g\|_{C^{1+\alpha/2}([0,T])} + \|h\|_{C^{1+\alpha/2}([0,T])} \le C,$$

where $G_T = \|(t,x) \in \mathbb{R}^2 : x \in [g(t),h(t)], t \in [0,T]\}$, C *and* T *depend on* h_0, α, $\|a\|_{C^1([0,L])}$, $\|b\|_{C^1([0,L])}$, $\|u_0\|_{C^2([-h_0,h_0])}$ *and* $\|f\|_{C^1([0,L]\times[0,\|u_0\|_{C([-h_0,h_0])}+1])}$.

Lemma 1.2.8. (u,g,h) *is a solution to* (F) *defined for* $t \in [0,T_0)$ *for some* $T_0 \in (0,\infty)$, *and there exists* $C_1 > 0$ *such that*

$$u(t,x) < C_1\ for\ t \in [0,T_0)\ and\ x \in (g(t),h(t)).$$

Then, there exists C_2 *depending on* C^1 *but independent of* T_0 *such that*

$$-g'(t), h'(t) \in (0, C_2]\ for\ t \in (0,T_0).$$

Moreover, the solution can be extended to some interval $(0,T)$ *with* $T > T_0$.

Theorem 1.2.9. *Problem* (F) *has a unique solution defined on some maximal interval* $(0,T^*)$ *with* $T^* \in (0,\infty]$. *Moreover, when* $T^* < \infty$, *we have*

$$\lim_{t\to T^*} \|u(t,\cdot)\|_{C([g(t),h(t)])} = \infty.$$

If we further assume that $u_0(x) < p(x)$ *for* $x \in (-h_0,h_0)$, *then*

$$T^* = \infty.$$

The next two lemmas is about the uniform boundedness and exponential convergences of u.

Lemma 1.2.10. *Suppose that Assumption 1.1.3 holds. There exists positive constants* ϵ_0,δ_0 *and* T_0 *such that if* (u,g,h) *is a solution of* (F) *with initial datum* $u_0(x) \le p(x) + \epsilon_0 \psi(x)$ *for* $x \in (-h_0,h_0)$, *then we have*

$$u(t,x) < p(x) + e^{-\delta_0 t}\ for\ t \in (T_0,+\infty)\ and\ x \in (g(t),h(t)), \tag{1.10}$$

where ψ *is the eigenfunction defined in Assumption 1.1.3.*

Proof. Let us make a transformation first

$$\tilde{v}(t,x) := \frac{\tilde{u}(t,x) - p(x)}{\psi(x)},$$

where $\tilde{u}$ is the solution of (E). By calculation, we get

$$\partial_t \tilde{v} = \frac{\partial_t \tilde{u}}{\psi},$$

$$\partial_x \tilde{v} = \frac{(\partial_x \tilde{u} - p')\psi - (\tilde{u} - p)\psi'}{\psi^2}$$

and

$$\partial_{xx} \tilde{v} = \frac{[(\partial_{xx}\tilde{u} - p'')\psi - (\tilde{u} - p)\psi'']\psi^2 - 2\psi\psi'[(\partial_x \tilde{u} - p')\psi - (\tilde{u} - p)\psi']}{\psi^4}.$$

Replacing the above equalities into equation (E), then we get the equation for $\tilde{v}$:

$$\begin{cases} \partial_t \tilde{v} - \partial_{xx}\tilde{v} - 2\frac{\psi'}{\psi}\partial_x \tilde{v} = F(x,\tilde{v}), & t > 0,\ x \in \mathbb{R}, \\ \tilde{v}(0,x) = \frac{\tilde{u}_0(x) - p(x)}{\psi(x)}, & x \in \mathbb{R}, \end{cases} \tag{1.11}$$

where $F(x,v) = \frac{f(x, v\psi(x) + p(x)) - f(x,p(x))}{\psi(x)} - [\lambda + \partial_u f(x,p(x))]v$. It is easy to check that

$$\partial_v F(x,0) = -\lambda.$$

Then there exists a constant $\epsilon_0 > 0$ such that

$$\partial_v F(x,v) < -\frac{\lambda}{2}$$

holds for $x \in \mathbb{R}$ and $0 \le v \le \epsilon_0$. Next, we construct a new $V(t)$ as the supersplution of $\tilde{v}$:

$$\begin{cases} V'(t) = -\frac{\lambda}{2}V(t),\ t > 0, \\ V(0) = \epsilon_0. \end{cases} \tag{1.12}$$

Through comparison principle, it is easy to infer that $V(t)$ is a supersplution of $\tilde{v}(t,x)$ if given initial datum $0 \le \tilde{v}_0(x) < \epsilon_0$ for $x \in \mathbb{R}$. Because V strictly decreases with respect to t, it follows that

$$\tilde{v}(t,x) < \epsilon_0 \text{ for } t > 0,\ x \in \mathbb{R}.$$

We note that $\tilde{u}$ is a supersolution of (u,g,h), the solution of equation (F). Then by Theorem 1.2.9, (u,g,h) exists for all the time $t > 0$. (1.10) easily follows from the fact that V exponentially converges to 0 as $t \to +\infty$.

□

Lemma 1.2.11. *Let Assumptions 1.1.1 hold and (u,g,h) be a solution of (F) for which spreading happens. There exists a $c > 0$ such that*

$$\lim_{t \to +\infty} \inf\{u(t,x) - p(x) : |x| \le ct\} \ge 0. \tag{1.13}$$

Proof. Let $\phi_0 \in \mathscr{X}(l_0)$ be the compactly supported initial datum given in Assumption 1.1.1. This means that the solution $(\underline{u}, \underline{g}, \underline{h})$ of (F) with initial datum ϕ_0 converges locally uniformly to p as $t \to +\infty$. According to it, there exists a time $T > 0$ such that

$$\underline{u}(T,x) \ge \max\{\phi_0(x), \phi_0(x-L), \phi_0(x+L)\} \text{ for any } x \in \mathbb{R}.$$

For convenience, we mean $\underline{u} \equiv 0$ outside its free boundaries $\underline{g}$ and $\underline{h}$, so is for u. By the comparison principle, it follows that

$$\begin{aligned} \underline{u}(2T,x) &\ge \max\{\underline{u}(T,x), \underline{u}(T,x-L), \underline{u}(T,x+L)\} \\ &\ge \max\{\phi_0(x), \phi_0(x-L), \phi_0(x-2L), \phi_0(x+L), \phi_0(x+2L)\}. \end{aligned}$$

By induction, we obtain that for all $j \in \mathbb{N}$,

$$\underline{u}(jT, x) \geq \max\{\phi_0(x - iL) : i \in \mathbb{Z}, |i| \leq j\}.$$

Because (u, g, h) is a solution with initial datum u_0 for which spreading happens, we can assume that $u_0 \geq \phi_0$ without loss of generality. Applying the comparison principle one gets that

$$u(jT, x) \geq \max\{\phi_0(x - iL) : i \in \mathbb{Z}, |i| \leq j\}$$

for any $j \in \mathbb{N}$. Therefore, from Assumption (1.1.1), we have that for any $j \in \mathbb{N}$,

$$\begin{aligned} u(\tau + jT, x) &\geq \max\{\underline{u}(\tau, x - iL) : i \in \mathbb{Z}, |i| \leq j\}, \\ &\to p(x), \end{aligned}$$

where the convergence holds as $\tau \to +\infty$, uniformly with respect to $j \in \mathbb{N}$ and $x \in [-jL, jL]$.

Let us now define

$$c_* : L/T > 0 \tag{1.14}$$

and choose any c with $0 < c < c_*$. Denote by $\lceil y \rceil$ the ceiling function of y, that is, the least integer not smaller than y. Then for any $t \geq 0$, let

$$\tau(t) := t - \lceil \frac{ct}{L} \rceil T.$$

As $c < c_*$, one can easily check that $\tau \to +\infty$ as $t \to +\infty$. Thus,

$$\lim_{t \to +\infty} \inf\{u(\tau(t) + \lceil \frac{ct}{L} \rceil T, x) - p(x) : |x| \leq \lceil \frac{ct}{L} \rceil L\} \geq 0,$$

and, since $ct \leq \lceil \frac{ct}{L} \rceil L$ and $t = \tau(t) + \lceil \frac{ct}{L} \rceil T$ for all $t \geq 0$, we have (1.13).

□

1.3. **Propagating terrace with a free boundary.**

In this section, we will prove the existence and uniqueness of the propagating terrace of (F_h) and its relationship with that of (E). We call ω-limit set of a solution (u, h) of (F_h) the set of functions to which u may converge for large positive time. More rigorously, we define:

Definition 1.3.1. *Let $(u(t, x), h(t))$ be any solution of (F_h). We call $(v(t, x), l(t))$ an ω-limit orbit of (u, h) if there exist two sequences $t_j \to +\infty$ and $k_j \in \mathbb{Z}$ such that*

$$h(t + t_j) - k_j L \to l(t), \text{ as } j \to +\infty$$

locally uniformly on $\mathbb{R}$ and

$$u(t + t_j, x + k_j L) \to v(t, x), \text{ as } j \to +\infty$$

locally uniformly on $\mathbb{R} \times (-\infty, l(t))$.

Remark 1.3.2. *One can easily check that (v, l), an ω-limit orbit of (u, h), satisfies the Stefan boundary condition. Thus (v, l) is an entire solution of (F_h). Moreover, $(v(t + \tau, x + kL), j(t + \tau) - kL)$ is also an ω-limit orbit of (u, h) for any $\tau \in \mathbb{R}$ and $k \in \mathbb{Z}$.*

Definition 1.3.3. *We denote $(\hat{u}(t, x; a), \hat{h}(t; a))$ is the solution of (F_h) whose initial datum is given in the form*

$$u_0(x) = p(x)H(a - x) \text{ and } h_0 = a,$$

where a is a constant, H is the Heaviside function, which is defined by

$$\begin{cases} H(x) = 0, & x < 0, \\ H(x) = 1, & x \geq 0. \end{cases}$$

It is obvious that the initial datum of equation (F_h) should be at least continuous. However, $p(x)H(a-x)$ has a jump point at $x=0$. We solve this problem in the following way: first, we choose a sequence of continuous functions $H_n(x)$ satisfying

$$\begin{cases} H_n(x)=0,\ x\le \frac{1}{n} \text{ and } x\ge n, \\ H_n(x)>0,\ \frac{1}{n}<x<n, \\ H_n(x) \text{ increases with respect to } n \text{ for any } x\in\mathbb{R}, \\ H_n(x)\to 1 \text{ locally uniformly in } \mathbb{R}^+ \text{ as } n\to+\infty. \end{cases} \tag{1.15}$$

Moreover, we assume that $H_n(x)$ belongs to $C^2([\frac{1}{n},n])$. We say that (u_n,g_n,h_n) is the solution of (F) initialized by $p(x)H_n(a-x)$. By comparison principle, it is easy to see that $\{u_n\}_n$ and $\{h_n\}_n$ are increasing sequences. We define that

$$\begin{cases} \hat{u}(t,x;a)=\lim_{n\to\infty} u_n(t,x), \\ \hat{h}(t;a)=\lim_{n\to\infty} h_n(t). \end{cases}$$

It is not difficult to check that for any $t>0$, $(\hat{u}(t,x;a),\hat{h}(t;a))$ is a pair of solution of equation (F_h) satisfying the Stefan condition $\hat{h}'(t;a))=-\mu\hat{u}_x(t,h(t);a)$.

The following three lemmas are about some basic properties of $(\hat{u},\hat{h})$.

Lemma 1.3.4. *Let (u_n,g_n,h_n) be the sequence converging to $(\hat{u}(t,x;a),\hat{h}(t;a))$ in the sense of Definition 1.3.3. For any $T>\delta>0$, we have*

$$(\hat{u}(\cdot\,,\cdot\,;a),\hat{h}(t;a))\in C^{(1+\alpha)/2,1+\alpha}(\bar{\Omega}(\delta,T))\times C^{1+\alpha/2}([\delta,T]),$$

where $\bar{\Omega}(\delta,T)=\{(t,x)\in\mathbb{R}^2 : x\in(-\infty,\hat{h}(t;a)), t\in[\delta,T)\}$ and $0<\alpha<1$. Moreover, $\hat{h}'(t;a)$ is uniformly bounded for any $t>\delta$ and $a\in\mathbb{R}$.

Proof. Without loss of generality, we may assume $a=0$. Next, let us define a family of auxiliary functions $(Y_b)_{b\in\mathbb{R}}$, which is defined as follows:

$$\begin{cases} \partial_{xx}Y_b(x)+f(x,Y_b(x))=0,\ x\in(b-1,b), \\ Y_b(b-1)=p(b-1)+1 \text{ and } Y_b(b)=0. \end{cases} \tag{1.16}$$

Because $u_n(0,x)$ converges to $p(x)$ locally uniformly in $\mathbb{R}^-$, there must exist a $n_0\in\mathbb{N}$ such that $u_{n_0}(0,\cdot)$ is bigger than $\phi_0(\cdot+mL)$ for some integer m. By Assumption 1.1.1, we know that (u_n,g_n,h_n) will spread successfully for $n\ge n_0$. Then it follows that $\lim_{t\to+\infty} h_{n_0}(t;0)=+\infty$. For the monotonicity of h_{n_0}, there is a unique time t_0 such that $h_{n_0}(t_0;0)=1$. For any $n\ge n_0$ and $t>t_0$, by the boundary condition of Y_b we can easily check that $Y_{h_n(t)}(\cdot)$ does not have any tangent point or intersection point with $u_n(s,\cdot)$ for $0<s<t$. Then it is not difficult to infer that $Y_{h_n(t)}(x)\ge u_n(t,x)$ for $n\ge n_0$,$t>t_0$ and $x\in(h_n(t)-1,h_n(t)]$. It follows that $\partial_x u_n(t,h_n(t))\ge \partial_x Y_{h_n(t)}(h_n(t))$ for $n\ge n_0$ and $t>t_0$. For the periodicity of Y_b with respect to b, we know that $\partial_x u_n(t,h_n(t))$ is uniformly bounded for $n\ge n_0$ and $t>t_0$.. Furthermore, we can infer that $h_n'(t)$ is uniformly bounded for $n\ge n_0$ and $t>t_0$.. Because $h_n(t)$ converges to $\hat{h}(t;0)$ as $n\to+\infty$, we know that $\hat{h}$ is uniformly Lipschitz-continuous for $t>t_0$.

Because $\hat{u}$ is the convergence of u_n, $\hat{u}$ satisfies the equation away from the free boundary $\hat{h}$. It is also easy to check that $(\hat{u},\hat{h})$ satisfies the Stefan boundary condition almost every time.

Then by a similar argument of Theorem 2.1 in [10], we can make a transformation for changing the free boundary problem to the Dirichlet problem. Following the same argument of Theorem 2.1 in [10], we obtain the regularity of $(\hat{u},\hat{h})$ the same as that of problem (F) if $t>t_0$. We note that t_0 is only dependent on the length of the definition domain of Y_b. Thus if we shorten the domain of Y_b, t_0 can be chosen as small as we want.

□

Remark 1.3.5. *By standard L^p estimate, the sobolev embedding theorem and the H'older estimates for parabolic equations, we can further infer that $\hat{u}(t,\cdot;a) \in C^2((-\infty, \hat{h}(t;a)])$ for any $t > 0$.*

Lemma 1.3.6. *Let Assumption 1.1.1 be satisfied. For each $c \in (0, c_*)$, where c_* is defined in 1.14, one has*

$$\lim_{t\to+\infty} \sup_{x\le ct} |\hat{u}(t,x;a) - p(x)| = 0. \tag{1.17}$$

Proof. Following the method in Lemma 1.2.11, we can infer that

$$\hat{u}(jT, x; a) \ge \max\{\phi_0(x - iL : i \in \mathbb{Z}, i \le j\}$$

for all $j \in \mathbb{N}$. Then by the same argument, the lemma follows. □

Lemma 1.3.7. *Let (v, l) be any entire solution of (F_h). For any $t > 0$,*

(i) *If $l(t) \le \hat{h}(t;a)$, then $v(t,x) < \hat{u}(t,x;a)$ for $x \in (-\infty, l(t))$.*
(ii) *If $l(t) > \hat{h}(t;a)$, then $SGN[\hat{u}(t,\cdot\,;a) - v(t,\cdot); (-\infty, \hat{h}(t;a)]] = [+-]$.*

Proof. Without loss of generality, we may assume that $a = 0$. By our definition on $(\hat{u}, \hat{h})$, it is easy for us to observe that if both of $(u_n(0,x))_{n\in\mathbb{N}}$ and $(\tilde{u}_n(0,x))_{n\in\mathbb{N}}$ satisfies (1.15), then $(u_n(t,x), h_n(t))$ and $(\tilde{u}_n(t,x), \tilde{h}_n(t))$ converge to the same $(\hat{u}(t,x;0), \hat{h}(t;0)$ as $n \to +\infty$. That means $(\hat{u}, \hat{h})$ does not depend on the choice of $(u_n(0,x))_{n\in\mathbb{N}}$.

Next, let us consider the case $l(0) \ge 0 = a$. Because of $h_n(0) = -\frac{1}{n} \le l(0)$ and $g_n(0) = -n$, we can choose initial datums $(u_n(0,x))_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $u_n(0,x)$ and $v(0,x)$ have exactly two intersection points in $(-n, -\frac{1}{n})$. By Lemma 1.2.2, the number of intersection points between $u_n(t,x)$ and $v(t,x)$ does not increase before the time h_n and l meet. We denote the left and right intersection points of $u_n(t,x)$ and $v(t,x)$ as $y_n(t)$ and $z_n(t)$ for each n and $t > 0$ if they exist. It is easy to see that both of $y_n(t)$ and $z_n(t)$ are continuous functions.

For any $n \in \mathbb{R}$, we denote $t_n := \inf\{t : t > 0 \text{ and } h_n(t) = l(t)\}$. If $t_n = +\infty$, by Lemma 1.2.2, we know that it holds either that y_n and z_n meet at some time $t = s_0$ and disappear together for $t > s_0$, or that both of y_n and z_n exist for all the time $t > 0$.

If t_n is finite, we claim that either $y_n(t_n) < z_n(t_n) = h_n(t_n)$ or $y_n(t_n) = z_n(t_n) = h_n(t_n)$ happens. If y_n and z_n meets some time $s_0 < t_n$, by Lemma 1.2.2 it follows that (u_n, g_n, h_n) is a subsolution of (v, l) for any $t > s_0$. Then it is impossible that h_n and l meet at $t = t_n$. Therefore, if t_n is finite, both of y_n and z_n exist differently for $t < t_n$. If y_n and z_n meet at $t = t_n$, there are two possibilities: $y_n(t_n) = z_n(t_n) < h_n(t_n)$ or $y_n(t_n) = z_n(t_n) = h_n(t_n)$. Because the second one belongs to our claim, we only need to treat the first case. By the virtue of equation (1.16), we know that $h_n'(t)$ is uniformly bounded for any $t > \epsilon > 0$. For the case $y_n(t_n) = z_n(t_n) < h_n(t_n)$, we denote Q_{t_n} by $\{(x,t) : t \in (0, t_n) \text{ and } x \in (z_n(t), h_n(t))\}$. For the uniform boundedness of $h_n'(t)$ for $t > \epsilon$, we can easily check that $(t_n, h_n(t_n)) \in \partial_p Q_{t_n}$ has the property of the spherical cap. Thus by Hopf's Lemma, we can infer that $l'(t_n) > h_n'(t_n)$, which means h_n and l can not meet at $t = t_n$. By this contradiction, it follows that if t_n is finite and $y_n(t_n) = z_n(t_n)$, we deduce that $y_n(t_n) = z_n(t_n) = h_n(t_n) = l(t_n)$. By the same argument, we also have that if t_n is finite and $y_n(t_n) < z_n(t_n)$, then $y_n(t_n) < z_n(t_n) = h_n(t_n) = l(t_n)$. Thus, our claim has been claimed.

If t_n is finite and $y_n(t_n) < z_n(t_n) = h_n(t_n)$, it is easy to see that z_n disappears since $t = t_n$. In other words, there exists a small $\delta > 0$ such that $h_n(t) > l(t)$ for $t \in (t_n, t_n + \delta)$. Moreover, y_n is the unique intersection point between u_n and v for $t \in (t_n, t_n + \delta)$. Then We denote $\tilde{t}_n := \inf\{t : t > t_n \text{ and } h_n(t) = l(t)\}$. If $\tilde{t}_n = +\infty$, by Lemma 1.2.2 we know that y_n, the unique intersection point, always exists for $t > t_n$. If $\tilde{t}_n$ is finite, by Hopf's lemma, we can infer that $y_n(\tilde{t}_n) = h_n(\tilde{t}_n) = l(\tilde{t}_n)$. In this case, we have $h_n(t) < l(t)$ and $u_n(t,x) < l(t,x)$ for $t > \tilde{t}_n$ and $x \in (g_n(t), h_n(t))$.

For some $n_0 \in \mathbb{N}$, by Lemma 1.2.2, there is a time t_0 such that $h_{n_0}(t_0) < l(t_0)$ and both of $y_{n_0}(t_0)$ and $z_{n_0}(t_0)$ exist differently. Because of (1.15) and comparison principle, we know that $y_n(t_0)$ exists for any $n \geq n_0$ and decrease with respect to n. Moreover, we have

$$\lim_{n\to+\infty} y_n(t_0) = -\infty \tag{1.18}$$

for $u_n(0,x)$ converges to $p(x)$ locally uniformly in $(-\infty, 0)$ as $n \to +\infty$. If $\hat{h}(t_0; 0) > l(t_0)$, it follows that $h_n(t_0) > l(t_0)$ for large n. By the existence of $y_n(t_0)$, we know that $u_n(t_0, x) > v(t_0, x)$ for $x \in (y_n(t_0), l(t_0))$. Because of (1.18) and the monotonicity of $u_n(t,x)$ with respect to n, we get that $\hat{u}(t_0, x; 0) > v(t_0, x)$ for $x \in (-\infty, l(t_0))$. By comparison principle, we have $\hat{h}(t; 0) > l(t)$ and $\hat{u}(t, x; 0) > v(t, x)$ for $t > t_0$ and $x \in (-\infty, l(t))$.

If $\hat{h}(t_0; 0) = l(t_0)$, we have $h_n(t_0) \to l(t_0)$ as $n \to +\infty$. By the above argument, we know that $\hat{h}(t; 0) > l(t)$ for any $t > s > 0$, where s satisfies $\hat{h}(s; 0) = l(s)$. Then we can infer that $\hat{h}(t; 0) < l(t)$ for any $0 < t < t_0$. By the existence of $y_n(t_0)$, we know that $z_n(t_0)$ exists for any $n \geq n_0$ and increases with respect to n. Then we claim that

$$\lim_{n\to+\infty} z_n(t_0) = l(t_0). \tag{1.19}$$

Otherwise, we can get a contradiction on that $\hat{h}$ and l meet at $t = t_0$, by repeating the argument on how z_n disappears at $t = t_n$. Then (1.19) follows. And we know that $u_n(t_0, x) > v(t_0, x)$ for $x \in (y_n(t_0), z_n(t_0))$. For (1.18),(1.19) and the monotonicity of $u_n(t, x)$ with respect to n, we infer that $\hat{u}(t_0, x; 0) > v(t_0, x)$ for $x \in (-\infty, l(t_0))$.

If $\hat{h}(t_0; 0) < l(t_0)$, it easily follows $SGN[\hat{u}(t_0, \cdot\,; 0) - v(t_0, \cdot); (-\infty, \hat{h}(t_0; 0))] = [+-]$ from (1.18) and Lemma 1.2.3. And by Lemma 1.2.2, such a relationship will be kept for any time $t > t_0$ satisfying $\hat{h}(t; 0) < l(t)$.

Because t_0 can chosen arbitrarily small, this lemma has finished for the case $l(0) \geq 0$. As to the case $l(0) < 0$, it is much easier and can follow from the same argument, so we omit the proof.

□

Remark 1.3.8. *Assume (u, h) and (v, l) are two solutions of (F_h). By the argument of Lemma 1.3.7, it is easy to notice that the number of the intersection points between u and v will decrease at least 1 when h and l meets. It means that h and l can meet at most finite times. The fact also holds for solutions of (F).*

According to Lemma 1.3.7, we find that $(\hat{u}, \hat{h})$ is 'steeper than' any entire solution of (F_h) for $t > 0$ in the sense of Definition 1.1.5. By this property, we have the following lemma.

Lemma 1.3.9. *Let $a \in \mathbb{R}$ and let (v_1, l_1) be any $\omega - limit$ orbit of $(\hat{u}(t, x; a), \hat{h}(t; a))$. Then we have:*

(i) *If $l_1(0) = +\infty$, v_1 is steeper than any other entire solution of (E) between 0 and p.*

(ii) *If $l_1(0) < +\infty$, (v_1, l_1) is steeper than any other entire solution of (E) or (F_h) between 0 and p.*

Proof. (i) Fix $a \in \mathbb{R}$. And let the sequences $t_j \to +\infty$ and $k_j \in \mathbb{Z}$ be such that $\hat{h}(0 + t_j; a) - k_j L \to l_1(0) = +\infty$ and $u(t + t_j, x + k_j L) \to v_1(t, x)$ locally uniformly in $\mathbb{R}$ as $j \to +\infty$. According to Lemma 1.3.4, we know that $\hat{h}'(t; a)$ is uniformly bounded for any $t > 1$. Thus, for any $t \in \mathbb{R}$, $l_1(t) = +\infty$. By standard parabolic estimates, the convergence of u in fact holds in $C^1_{loc}(\mathbb{R}^2)$.

Then by the same argument of Lemma 2.8 in [18], v_1 is steeper than any other entire solution of (E) between 0 and p.

(ii) Fix $a \in \mathbb{R}$. And let the sequences $t_j \to +\infty$ and $k_j \in \mathbb{Z}$ be such that $\hat{h}(0 + t_j; a) - k_j L \to l_1(0) = +\infty$ and $u(t_j, x + k_j L) \to v_1(0, x)$ locally uniformly in $(-\infty, l(0))$ as $j \to +\infty$. For $\hat{h}'(t; a)$ is uniformly bounded for any $t > 1$, we can infer that $l_1(t)$ is bounded and Lipschitz continuous for any

$t \in \mathbb{R}$. Through the argument of Lemma 1.3.4, we can transform it into a Dirichlet problem around the free boundary and obtain the same regularity that $(v_1, l_1) \in C_{loc}^{(1+\alpha)/2, 1+\alpha} \times C_{loc}^{1+\alpha/2}$ for $0 < \alpha < 1$.

Let (v, l) be an entire solution of (F_h). Fix $T_0 \in \mathbb{R}$. If $l(T_0) < l_1(T_0)$, then we can find a j_0 such that $l(T_0) < \hat{h}(T_0 + t_j; a) - k_j L$ for any $j > j_0$. Through Lemma 1.3.7, we know that $v(T_0, x) < \hat{u}(T_0 + t_j, x + k_j L; a)$ holds for $x \in (-\infty, l(T_0))$ and $j > j_0$. Then it follows that $v(T_0, x) \le v_1(T_0, x)$ for $x \in (-\infty, l(T_0))$.

By Hopf's Lemma, we have that $l_1'(t) > 0$ for any $t \in \mathbb{R}$. If $l(T_0) = l_1(T_0)$, it holds that $l(T_0) < l_1(T_0 + \epsilon)$ for any $\epsilon > 0$. By the above argument, we know that $v(T_0, x) \le v_1(T_0 + \epsilon, x)$ for $x \in (-\infty, l(T_0))$. Because v_1 is continuous, it follows $v(T_0, x) \le v_1(T_0, x)$ for $x \in (-\infty, l(T_0))$.

If $l(T_0) > l_1(T_0)$, we can find a j_1 such that $l(T_0) > \hat{h}(T_0 + t_j; a) - k_j L$ for any $j > j_1$. Following the virtue of equation (1.16), for any $\epsilon_1 > 0$, we can find a sufficiently small $\delta_0 > 0$, such that $\hat{u}(t, x; a) < \epsilon_1$ for $t > 1$ and $x \in [\hat{h}(t; a) - \delta_0, \hat{h}(t; a)]$. Then we can find a δ_1 such that $l(T_0) - \delta_1 < l_1(T_0)$ and $l(T_0) - \delta_1 < \hat{h}(T_0 + t_j; a) - k_j L$ and $\hat{u}(T_0 + t_j, x + k_j L; a) < \frac{1}{2} v(T_0, x)$ hold for $j > j_1$ and $x \in [l(T_0) - \delta_1, \hat{h}(T_0 + t_j; a) - k_j L]$. It follows that

$$v_1(T_0, x) \le \frac{1}{2} v(T_0, x) < v(T_0, x) \text{ for } x \in [l(T_0) - \delta_1, l_1(T_0)]. \tag{1.20}$$

By Lemma 1.3.7, we know $SGN[\hat{u}(T_0 + t_j, \cdot + k_j L; a) - v(T_0, \cdot); (-\infty, l(T_0) - \delta_1]] = [+-]$. Then following (1.20) and Lemma 1.2.3, we have

$$SGN[v_1(T_0, \cdot) - v(T_0, \cdot); (-\infty, l_1(T_0)]] \lhd [+-]. \tag{1.21}$$

Because of $l(T_0) > l_1(T_0)$ and the uniform boundedness of l', l_1', there exists a small $\epsilon_2 > 0$ such that

$$l(s) > l_1(s) \text{ for } s \in [T_0 - \epsilon_2, T_0]. \tag{1.22}$$

By the same argument, (1.21) also holds at $t = T_0 - \epsilon_2$. That means

$$SGN[v_1(T_0 - \epsilon_2, \cdot) - v(T_0 - \epsilon_2, \cdot); (-\infty, l_1(T_0 - \epsilon_2)]] \lhd [+-].$$

Then it follows that either of

$$SGN[v_1(T_0 - \epsilon_2, \cdot) - v(T_0 - \epsilon_2, \cdot); (-\infty, l_1(T_0 - \epsilon_2)]] = [+-]$$

or

$$v_1(T_0 - \epsilon_2, x) \le v(T_0 - \epsilon_2, x) \text{ for } x \in (-\infty, l_1(T_0 - \epsilon_2))$$

holds. By (1.22) and Lemma 1.2.2, for the time $t = T_0$, we have either of

$$SGN[v_1(T_0, \cdot) - v(T_0, \cdot); (-\infty, l_1(T_0)]] = [+-]$$

or

$$v_1(T_0, x) < v(T_0, x) \text{ for } x \in (-\infty, l_1(T_0)).$$

Combining the above arguments together, (v_1, l_1) is steeper than (v, l) at $t = T_0$. Because T_0 is arbitrary, we know that (v_1, l_1) is steeper than any other entire solution of (F_h).

In a similar but easier argument, we also can infer (v_1, l_1) is steeper than any other entire solution of (E). □

Next, let us show how the propagating terrace of (F_h) is constructed.

Definition 1.3.10. *Let $x_0 \in \mathbb{R}$ be given. For any $0 < \beta < p(x_0)$ and $a < x_0$, let us define*

$$\tau(x_0, \beta; a) := \min\{t > 0 : \hat{u}(t, x_0; a) = \beta\}. \tag{1.23}$$

In the same sense, we also define

$$\tau(x_0, 0; a) := \{t > 0 : \hat{h}(t; a) = x_0\}. \tag{1.24}$$

Thanks to Lemma 1.3.6, $\tau(x_0, \beta; a)$ always exists and is finite for $\beta \in [0, p(x_0))$.

Lemma 1.3.11. *For any constants $a_1, a_2, x_0 \in \mathbb{R}$ satisfying $a_1 < a_2 < x_0$, we have*

$$\hat{h}'(\tau(x_0, 0; a_1); a_1) \leq \hat{h}'(\tau(x_0, 0; a_2); a_2) \tag{1.25}$$

and

$$\hat{u}(\tau(x_0, 0; a_1), x; a_1) < \hat{u}(\tau(x_0, 0; a_2), x; a_2) \tag{1.26}$$

for $x \in (-\infty, x_0)$.

Proof. For $a_1 < a_2$, it is obvious that $\tau(x_0, 0; a_1) > \tau(x_0, 0; a_2)$. Then we claim that

$$\hat{h}(\tau(x_0, 0; a_1) - \tau(x_0, 0; a_2); a_1) > a_2. \tag{1.27}$$

Otherwise, if $\hat{h}(\tau(x_0, 0; a_1) - \tau(x_0, 0; a_2); a_1) \leq a_2$, we know that $\hat{u}(\tau(x_0, 0; a_1) - \tau(x_0, 0; a_2), x; a_1) < p(x) = \hat{u}(0, x; a_2)$ for $x \in (-\infty, \hat{h}(\tau(x_0, 0; a_1) - \tau(x_0, 0; a_2); a_1)) \subseteq (-\infty, a_2)$. By the comparison principle, $(\hat{u}(\cdot\ ,\cdot\ ; a_2), \hat{h}(\cdot\ ; a_2))$ is a supersolution of $(\hat{u}(\cdot + \tau(x_0, 0; a_1) - \tau(x_0, 0; a_2), \cdot\ ; a_1), \hat{h}(\cdot + \tau(x_0, 0; a_1) - \tau(x_0, 0; a_2); a_1))$. Thus, it follows $\hat{h}(t + \tau(x_0, 0; a_1) - \tau(x_0, 0; a_2); a_1) \leq \hat{h}(t; a_2)$ for any $t > 0$. We know that at the time $t = \tau(x_0, 0; a_2)$ the two free boundaries will meet in $x = x_0$. Then we can infer that $\hat{h}'(t + \tau(x_0, 0; a_1) - \tau(x_0, 0; a_2); a_1) \geq \hat{h}'(t; a_2)$ when $t = \tau(x_0, 0; a_2)$. However, following Hopf's lemma, we have $\partial_x \hat{u}(\tau(x_0, 0; a_2), x_0; a_2) < \partial_x \hat{u}(\tau(x_0, 0; a_1), x_0; a_1)$ at the boundary, which contradicts with the former inducement. Thus, our claim (1.27) has been proved.

According to (1.27), we can find a small $\epsilon > 0$ such that

$$\hat{h}(\epsilon + \tau(x_0, 0; a_1) - \tau(x_0, 0; a_2); a_1) > \hat{h}(\epsilon; a_2)$$

and

$$SGN[\hat{u}(\epsilon, \cdot\ ; a_2) - \hat{u}(\epsilon + \tau(x_0, 0; a_1) - \tau(x_0, 0; a_2), \cdot\ ; a_1); (-\infty, \hat{h}(\epsilon; a_2)]] = [+-]$$

in a similar argument of Lemma 1.3.7. We claim $\hat{h}(t + \tau(x_0, 0; a_1) - \tau(x_0, 0; a_2); a_1) > \hat{h}(t; a_2)$ holds for any $t \in (\epsilon, \tau(x_0, 0; a_2))$. Otherwise, we can define $t_0 := \inf\{t : t \in (\epsilon, \tau(x_0, 0; a_2))$ and $\hat{h}(t_0 + \tau(x_0, 0; a_1) - \tau(x_0, 0; a_2); a_1) = \hat{h}(t_0; a_2)\}$. By following the argument for z_n in Lemma 1.3.7, we can infer that $\hat{h}(t + \tau(x_0, 0; a_1) - \tau(x_0, 0; a_2); a_1) < \hat{h}(t; a_2)$ for any $t > t_0$, which is a contradiction with the case $t = \tau(x_0, 0; a_2)$. Then repeating the argument for z_n at $t = \tau(x_0, 0; a_2)$, we obtain (1.25) and (1.26).

□

In a similar argument, we can extend Lemma 1.3.11 to $\tau(x_0, \beta; a)$ with $\beta > 0$.

Lemma 1.3.12. *For any constants $a_1, a_2, x_0 \in \mathbb{R}$ satisfying $a_1 < a_2 < x_0$ and $\beta \in (0, p(x_0))$, we have*

$$\hat{h}(\tau(x_0, \beta; a_1); a_1) > \hat{h}(\tau(x_0, \beta; a_2); a_2) \tag{1.28}$$

and

$$SGN[\hat{u}(\tau(x_0, \beta; a_2), \cdot\ ; a_2) - \hat{u}(\tau(x_0, \beta; a_1), \cdot\ ; a_1); (-\infty, \hat{h}(\tau(x_0, \beta; a_2); a_2)]] = [+-], \tag{1.29}$$

where the unique intersection point is $x = x_0$.

Lemma 1.3.13. *Let Assumption 1.1.1 be satisfied. Let $x_0 \in \mathbb{R}$ be given. For any $0 < \beta < p(x_0)$ and $a < x_0$, let us define*

$$\tau(x_0, \beta; a) := \min\{t > 0 : \hat{u}(t, x_0; a) = \beta\}.$$

Then the following limits exist:

$$h_\infty(t; \beta) = \lim_{a \to -\infty} \hat{h}(t + \tau(x_0, \beta; a); a)$$

and

$$w_\infty(t, x; \beta) = \lim_{a \to -\infty} \hat{u}(t + \tau(x_0, \beta; a), x; a).$$

If $h_\infty(\cdot;\beta) \equiv +\infty$, then $w_\infty(t,x;\beta)$ is an entire solution of (E) that is steeper than any other entire solution of (E). Otherwise, (w_∞, h_∞) is an entire solution of (F_h) that is steeper than any other entire solution of (E) or (F_h). Furthermore, the following alternative holds true: either $w_\infty(\cdot,\cdot;\beta)$ is a stationary solution of (E), or $\partial_t w_\infty(t,x;\beta) > 0$ for all $t \in \mathbb{R}$ and $x \in (-\infty, h_\infty(t;\beta))$. The former assettion is impossible for each β close enough to p_{x_0}. And $h_\infty(t;\beta)$ is a finite function for each β close enough to 0*.*

Proof. Fix some $x_0 \in \mathbb{R}$ and $\beta \in (0, p(x_0))$, we aim to prove the following limits exists for

$$h_\infty(t;\beta) = \lim_{a\to-\infty} \hat{h}(t + \tau(x_0,\beta;a);a)$$

and

$$w_\infty(t,x;\beta) = \lim_{a\to-\infty} \hat{u}(t + \tau(x_0,\beta;a),x;a)$$

for $t \in \mathbb{R}$ and $x \in (-\infty, h_\infty(t;\beta))$. It is easy to check that $\lim_{a\to-\infty} \tau(x_0,\beta;a) = +\infty$. By Lemma 1.3.12, $h_\infty(0;\beta)$ is well defined for $\hat{h}(\tau(x_0,\beta;a);a)$ strictly decreases with respect to $a \in (-\infty, x_0)$. If $h_\infty(0;\beta) = +\infty$, we can further infer that $h_\infty(t;\beta) = +\infty$ for any $t \in \mathbb{R}$, because $\hat{h}'(t;a)$ is uniformly bounded for $a \in \mathbb{R}$ and $t > \delta > 0$. Thus, by the argument of Section 3 in [18], it follows that the existence of

$$w_\infty(t,x;\beta) = \lim_{a\to-\infty} \hat{u}(t + \tau(x_0,\beta;a),x;a),$$

together with the convergence for the topology of $C^1_{loc}(\mathbb{R}^2)$ and the monotonicity in time of w_∞, that is either $\partial_t w_\infty > 0$ or $\partial_t w_\infty \equiv 0$ for $(t,x) \in \mathbb{R}^2$.

If f $h_\infty(0;\beta) < +\infty$, we know that $h_\infty(t;\beta)$ is finite for any $t \in \mathbb{R}$, because $\hat{h}'(t;a)$ is uniformly bounded for $a \in \mathbb{R}$ and $t > \delta > 0$. Let $(a_k)_{k\in\mathbb{N}}$ be a given sequence such that $a_k \to -\infty$ as $k \to +\infty$, and such that the following limits hold true:

$$\hat{h}(t + \tau(x_0,\beta;a_k);a_k) \to h_\infty(t;\beta)$$

and

$$\hat{u}(t + \tau(x_0,\beta;a_k),x;a_k) \to w_\infty(t,x;\beta)$$

as $k \to +\infty$, where the second convergence holds in $C^1_{loc}(\{(t,x) : t \in \mathbb{R}, x \in (-\infty, h_\infty(t;\beta))\})$. Because of the boundedness of $\hat{h}'$ for large time, h_∞ is Lipschitz continuous. Then by the argument of Lemma 1.3.4, (w_∞, h_∞) has the standard regularity for free boundary problem, $C^{(1+\alpha)/2,1+\alpha}_{loc} \times C^{1+\alpha/2}_{loc}$ with $0 < \alpha < 1$. Up to a subsequence, one may assume that $a_k - \lceil \frac{a_k}{L} \rceil L \to a_\infty$ in $[-L, 0]$. By observing that

$$\begin{aligned}
h_\infty(t;\beta) &= \lim_{k\to+\infty} \hat{h}(t + \tau(x_0,\beta;a_k);a_k) \\
&= \lim_{k\to+\infty} \hat{h}(t + \tau(x_0,\beta;a_k);a_k - \lceil \frac{a_k}{L} \rceil L) + \lceil \frac{a_k}{L} \rceil L \\
&= \lim_{k\to+\infty} \hat{h}(t + \tau(x_0,\beta;a_\infty + \lceil \frac{a_k}{L} \rceil L);a_\infty) + \lceil \frac{a_k}{L} \rceil L
\end{aligned} \tag{1.30}$$

and

$$\begin{aligned}
w_\infty(t,x;\beta) &= \lim_{k\to+\infty} \hat{u}(t + \tau(x_0,\beta;a_k),x;a_k) \\
&= \lim_{k\to+\infty} \hat{u}(t + \tau(x_0,\beta;a_k),x - \lceil \frac{a_k}{L} \rceil L;a_k - \lceil \frac{a_k}{L} \rceil L) \\
&= \lim_{k\to+\infty} \hat{u}(t + \tau(x_0,\beta;a_\infty + \lceil \frac{a_k}{L} \rceil L),x - \lceil \frac{a_k}{L} \rceil L;a_\infty).
\end{aligned} \tag{1.31}$$

According to Definition 1.3.1,(w_∞, h_∞) is an ω-limit orbit of $(\hat{u}(t,x;a_\infty), \hat{h}(t;a_\infty))$. By (ii) of Lemma 1.3.9, (w_∞, h_∞) is steeper than any other entire solution of (F_h) in the sense of Definition 1.1.5.

Recalling Definition 1.1.5, there is a unique entire solution (w_∞, h_∞) of (F_h) that is steeper than any other entire solution of (F_h) and such that $w_\infty(0,x_0) = \beta$. It follows that (w_∞, h_∞) does not depend on the choice of the sequence $\{a_k\}$.

Next, we want to show that the existence of (w_∞, h_∞) with a finite h_∞. Fix $a \in \mathbb{R}$ and we claim that $\limsup_{t\to+\infty} \hat{u}(t, \hat{h}(t;a) - M^{-1}; a) > 0$, where M is a positive constant such that

$$f(x,u) \le M^2 u \text{ for } x \in [0,L], u \in [0, \|p(\cdot)\|_{L^\infty}].$$

Otherwise, we can infer that $\lim_{t\to+\infty} \hat{u}(t, \hat{h}(t;a) - M^{-1}; a) = 0$. For some $\epsilon > 0$, there exists a time $T > 0$ such that $\hat{u}(t, \hat{h}(t;a) - M^{-1}; a) < \frac{\epsilon}{2}$ for $t > T$. Then we constructs a supersolution as follows:

$$\begin{cases} \bar{u} = \epsilon[2M(\bar{h}(t) - x) - M^2(\bar{h}(t) - x)^2], & t > 0, x \in (\bar{h}(t) - M^{-1}, \bar{h}(t)), \\ \bar{h}(t) = 2\mu\epsilon M t, & t > 0. \end{cases} \tag{1.32}$$

It is easy to check that for $t > 0$ and $x \in (\bar{h}(t) - M^{-1}, \bar{h}(t))$

$$\partial_t \bar{u} - \partial_{xx}\bar{u} \ge 2\epsilon M^2 \ge M^2 \bar{u} \ge f(x, \bar{u})$$

and

$$\bar{h}'(t) = 2\epsilon\mu M = -\mu \partial x \bar{u}(t, \bar{h}(t)).$$

Since $\hat{u}(t, \hat{h}(t;a) - M^{-1}; a) < \frac{\epsilon}{2}$ for $t > T$, it is easy to check that $(\bar{u}(\cdot\ , \cdot - \hat{h}(T;a) - M^{-1}), \bar{h}(\cdot) + \hat{h}(T;a) + M^{-1})$ is a supersolution of $(\hat{u}(\cdot + T, \cdot; a), \hat{h}(\cdot + T; a))$ for $t > 0$. It follows $\hat{h}(t+T;a) \le 2\mu\epsilon M t + \hat{h}(T;a) + M^{-1}$ for $t > 0$. Then we can infer that $\limsup_{t\to+\infty} \frac{\hat{h}(t)}{t} \le 2\mu\epsilon M$. Because ϵ is arbitrarily small, we can further induce that $\lim_{t\to+\infty} \frac{\hat{h}(t)}{t} = 0$, which is an apparent contradiction with Lemma 1.3.6. Therefore, our claim has been proved that is $\limsup_{t\to+\infty} \hat{u}(t, \hat{h}(t;a) - M^{-1}; a) > 0$.

We denote $\limsup_{t\to+\infty} \hat{u}(t, \hat{h}(t;a) - M^{-1}; a) = \beta_0 > 0$. Then we can find a sequence $(t_j)_{j\in\mathbb{N}}$ such that $\lim_{j\to+\infty} \hat{u}(t_j, \hat{h}(t_j;a) - M^{-1}; a) = \beta_0$ and $t_j \to +\infty$ as $j \to +\infty$. Up to a subsequence, we can assume that $\hat{h}(t_j;a) - M^{-1} \to x^*$ in $\mathbb{R}_{/L\mathbb{Z}}$, where $x^* \in [0,L]$. Without loss of generality, we can assume x^* is the same as x_0 in the above. By the argument for (1.30) and (1.31), we also can find an entire solution (w_0, h_0) of (F_h), which is an ω-limit orbit and satisfies $w_\infty(0,x_0) = \beta_0$ and $h_\infty(0) = x_0 + M^{-1}$. By the virtue of Lemma 1.3.7, we can infer that $\hat{h}(\tau(x_0,\beta_0;a);a) < x_0 + M^{-1}$ for any $a < x_0$. Then by repeating the argument for (1.30) and (1.31), the existence of $(w_\infty(\cdot,\cdot;\beta_0), h_\infty(\cdot;\beta_0))$ with a finite $h_\infty(\cdot;\beta_0)$ is obtained.

Moreover, by noticing that $\hat{h}(\tau(x_0,\beta;a);a) < \hat{h}(\tau(x_0,\beta_0;a);a) < x_0 + M^{-1}$ holds for $\beta < \beta_0$ and $a < x_0$, we can prove the existence of $(w_\infty(\cdot,\cdot;\beta), h_\infty(\cdot;\beta))$ with a finite $h_\infty(\cdot;\beta)$ for any $\beta \in (0,\beta_0)$.

Because (w_∞, h_∞) is steeper than any other entire solution of (F_h) (including the translation of itself on time), we infer that $w_\infty(t,x;\beta)$ increase with respect to t in the domain $\{(t,x) : t \in \mathbb{R}, x \in (-\infty, h_\infty(t;\beta)\}$. It follows that $\partial_t w_\infty \ge 0$ in the same domain. By the strong maximum principle, we infer that either $\partial_t w_\infty > 0$ or $\partial_t w_\infty \equiv 0$. However, the latter one contradicts with the fact $h'_\infty > 0$. Then we obtain the monotonicity in time of w_∞.

To conclude the proof of Lemma 1.3.13, let us show that when β is chosen close enough to $p(x_0)$ then w_∞ cannot be a stationary solution of (E). To show it let us notice that due to Assumption 1.1.1, the stationary solution p is isolated from below with respect to other stationary solutions of (E). Therefore, one can choose β close enough to $p(x_0)$ so that there is no stationary solution q of (E) with $q(x_0) = \beta$. Then due to Assumption 1.1.1, w_∞ is not a stationary solution of (E) and it converges to p as $t \to +\infty$. This completes the proof of this lemma. $\square$

Lemma 1.3.14. *Fix $\beta \in (0, p(x_0))$ and let (w_∞, h_∞) be the entire solution provided by Lemma 1.3.13. We have*

(i) *If $h_\infty(\cdot;\beta) \equiv +\infty$, then w_∞ is either a positive periodic stationary solution or a pulsating traveling wave of (E).*

(ii) *If $h_\infty(t;\beta)$ is finite for any $t \in \mathbb{R}$, then (w_∞, h_∞) is a pulsating semi-wave of (F_h).*

Proof. (i) This part easily follows the same argument of Lemma 4.1 in [18], because it does not matter with the free boundary.

(ii) We define the sequence the same as [18]

$$\tau_k := \tau(x_0, \beta; a - kL) - \tau(x_0, \beta; a - (k-1)L)$$

for $k \in \mathbb{N}^+$, so that for all $k \in \mathbb{N}^+$,

$$\tau(x_0, \beta; a - kL) = \Sigma_{i=0}^{k} \tau_i.$$

We claim that there exists some subsequence $(\tau_{k_j})_{j\in\mathbb{N}}$ converging to some $T > 0$. Otherwise there is no subsequence of $(\tau_k)_k$ converges to some positive constant. Following the argument on this case in Lemma 4.1 in [18], we get that w_∞ is L-periodic with respect to the space variable for all time, which is an apparent contradiction with the existence of the free boundary h_∞. Thus we can find a subsequence $(\tau_{k_j})_{j\in\mathbb{N}}$ converging to some $T > 0$. Following the argument on it in Lemma 4.1 in [18], we can show that (w_∞, h_∞) is a pulsating semi-wave of (F_h). □

<u>Proof of Theorem 1.1.7</u>:

We can conclude the proof of Theorem 1.1.7 by following the argument in Subsection 4.2 in [18],. Because most of the arguments there also hold for our free boundary problem without any modification, we only need to clarify three places, which relates with the free boundary.

The first one is Claim 4.6 in [18]. Generally speaking, solutions of (F_h) are natural subsolutions of positive solutions of (E) for the existence of the free boundary where u has to be 0. Thus, v, which is the supersolution of (E) defined in Claim 4.6 in [18], is also a supersolution of (F_h). As long as the chosen speed c is smaller than c_* given in Lemma 1.3.6, the result of Claim 4.6 still holds for our problem.

The second one is the statement after Step 3 in [18], which proves the sequence $(p_k)_k$ is finite. Because each p_k is isolated from below, we can infer that $\{p_k\}_k$ is monotonically decreasing. If $(p_k)_k$ is not finite, it converges uniformly to some $p_\infty \geq 0$. By Lemma 1.3.13, we know $h_\infty(t;\beta)$ is finite for small $\beta > 0$. Thus there exists a pulsating semi-wave connecting 0 to p_*, where p_* is a positive stationary solution of (E_{per}). For any $k \in \mathbb{N}$, it is apparently impossible that $0 < p_k < p_*$. Therefore it has to be $0 < p_* \leq p_\infty$. Then by the same argument in [18], it follows $(p_k)_k$ is finite.

The third one is the statement after Remark 4.7 in [18], which shows that the propagating terrace T^* of (E) is minimal. By Lemma 1.3.13 and 1.3.14, for small $\beta > 0$, we know that (w_∞, h_∞) is a pulsating semi-waveof (F_h), which is steeper than any other entire solution of (F_h). Combing it with the argument in [18], we can show the propagating terrace of (F_h), which is obtained from Lemma 1.3.14, is minimal in the sense of Definition 1.1.6. Moreover, up to some time shift, it is identically equal to any other minimal propagating terrace of (F_h).

Because other arguments of Subsection 4.2 in [18] do not relate with the free boundary, they can also be applied to our problem. Then it follows Theorem 1.1.7.

□

<u>Proof of Theorem 1.1.9</u>:

(i) If $N = 1$, it is trivial.

Next, let us assume $N > 1$. Because each of $(p_k)_{0\leq k\leq N-1}$ or $(q_k)_{0\leq k\leq N'-1}$ is a positive stationary solution of (E) and steeper than any other entire solution of (E), one of the following three possibilities must hold

$$p_i > q_j, \;\; p_i < q_j \text{ or } p_i = q_j$$

for any $0 \leq i \leq N-1$ and $0 \leq j \leq N'-1$. If $N' = 1$, p_1 must intersect with V_1 which connects 0 to p. However, it contradicts with the fact that p_1 is steeper than any other entire solution of (E). Thus we have $N' > 1$. Next, if $p_1 < q_1$, q_1 must intersect with U_1 which connects p_1 to p. This contradicts with the fact that q_1 is steeper than any other entire solution of (E). By the same reason, it is impossible that $p_1 > q_1$. Therefore, it follows $p_1 = q_1$. Because U_1 and V_1 are steeper than each other, we get $U_1 = V_1$ up to a time shift. Then this part follows from iterating the same argument.

(ii) If Assumption 1.1.1 holds for each $\mu > 0$, then there exists a propagating terrace of (F_h) $(p_k)_{0\leq k\leq N_\mu}, (U_k)_{1\leq k\leq N_\mu}, H)$ for any positive μ. We claim that $N_{\mu^*} \geq N_{\mu_*}$ if $\mu^* > \mu_*$. Otherwise, we have $N_{\mu^*} < N_{\mu_*}$ for some $\mu^* > \mu_*$. To make the explanation clearly, we denote the two propagating terraces by different symbols: $((p_k)_{0\leq k\leq N_{\mu^*}}, (U_k)_{1\leq k\leq N_{\mu^*}}, H)$ for $\mu = \mu^*$ while $((y_k)_{0\leq k\leq N_{\mu_*}}, (W_k)_{1\leq k\leq N_{\mu_*}}, G)$ for $\mu = \mu^*$. Because of $N_{\mu^*} < N_{\mu_*}$ and (i) of Theorem 1.1.9, we know that $p_{N_{\mu^*}-1} > y_{N_{\mu_*}-1}$. We note that $U_{N_{\mu^*}}$ connects with 0 to $p_{N_{\mu^*}-1}$ while $W_{N_{\mu_*}}$ connects with 0 to $y_{N_{\mu_*}-1}$. Then there exists a $M > 0$, such that $H(0) > G(-M)$ and $U_{N_{\mu^*}}(0,x) > W_{N_{\mu_*}}(-M,x)$ for $x \in (-\infty, G(-M))$. We define a time $t_0 := \sup\{t :$ for any $s \in (-M,t)$, it holds $H(0) > G(s)$ and $U_{N_{\mu^*}}(0,x) > W_{N_{\mu_*}}(s,x)$ for $x \in (-\infty, G(s))\}$. It easily follows that $H(0) \geq G(t_0)$ and $U_{N_{\mu^*}}(0,x) > W_{N_{\mu_*}}(t_0,x)$ for $x \in (-\infty, G(t_0))$. Moreover at $t = t_0$, it follows either

$$H(0) = G(t_0) \text{ or}$$

$$U_{N_{\mu^*}}(0,b_0) = W_{N_{\mu_*}}(t_0,b_0) \text{ and } \partial_x U_{N_{\mu^*}}(0,b_0) = \partial_x W_{N_{\mu_*}}(t_0,b_0) \text{ for some } b_0 \in (-\infty, G(t_0)).$$

Because of $\mu^* > \mu_*$, it is easy to check $(U_{N_{\mu^*}}(t,x), H(t))$ is a supersplution of $(W_{N_{\mu_*}}(t+t_0,x), G(t+t_0))$ for $t > 0$. Moreover, neither

$$H(t) = G(t+t_0) \text{ nor}$$

$$U_{N_{\mu^*}}(t,b_1) = W_{N_{\mu_*}}(t+t_0,b_1) \text{ and } \partial_x U_{N_{\mu^*}}(t,b_1) = \partial_x W_{N_{\mu_*}}(t+t_0,b_1) \text{ for some } b_1 \in (-\infty, G(t+t_0))$$

can happen for any $t > 0$. Thus $(U_{N_{\mu^*}}, H)$ and $(W_{N_{\mu_*}}, G)$ have different periodicities on time. Then we can infer that the average speed of $(U_{N_{\mu^*}}, H)$ is strictly faster than that of $(W_{N_{\mu_*}}, G)$. We denote $\hat{v}(t,x;H(0))$ is the solution of (E) with initial datum $p(x)X(H(0)-x)$. Because $\hat{v}(t,x;H(0))$ is a supersolution of $(U_{N_{\mu^*}}(t,x), H(t))$ for $t > 0$. Then by (i) of Theorem 1.1.9, we know that the average speed of $V_{N_{\mu^*}}$ is not less than that of $(U_{N_{\mu^*}}, H)$. By a similar argument, we also infer the average speed of $(U_{N_{\mu^*}}, H)$ is not less than that of $W_{N_{\mu^*}}$. However, by (i) of Theorem 1.1.9, we know that $V_{N_{\mu^*}}$ equals to $W_{N_{\mu^*}}$ up to a time shift. Thus, the average speed of $(U_{N_{\mu^*}}, H)$ equals that of $W_{N_{\mu^*}}$. By Theorem 1.1.7, we know that the average speed of $W_{N_{\mu^*}}$ is less than $(W_{N_{\mu_*}}, G)$. Until here, we get a contradiction on the speeds of $(U_{N_{\mu^*}}, H)$ and $(W_{N_{\mu_*}}, G)$. Thus, we have proved $N_{\mu^*} \geq N_{\mu_*}$ if $\mu^* > \mu_*$.

We can define $\mu_k := \sup\{\mu : N_\mu = k\}$. Then what left is to show $\mu_1 > 0$. By the virtue of equation (1.16), we know that for any $\mu > 0$ it holds $0 > \partial_x \hat{u}(t, \hat{h}(t;0); 0) > -m$ if $\hat{h}(t;0) \geq 1$, where m is a positive constant and independent on μ. By the Stefan condition, we know that the average speed of the free boundary $\hat{h}(t;0)$ uniformly converges to 0 as $\mu \to 0$. If the average speed of $\hat{h}(t;0)$ is smaller than that of V_1, it has to be $N_\mu = 1$. Then it follows $\mu_1 > 0$.

$\square$

As to Corollary 1.1.11 and 1.1.12, they are direct consequences of Theorem 1.1.7 and 1.1.9.

1.4. **Asymptotic Behavior.**

Proof of Proposition 1.1.13:

Through Theorem 1.1.7, we know that there exists $((p_k)_{0\leq k\leq N}, (U_k)_{1\leq k\leq N}, H)$, which is a minimal propagating terrace of (F_h). If $N > 1$, then (U_N, H) is a pulsating semi-wave connecting 0 to p_{N-1}. From (ii) of Theorem 1.1.7, we know that (U_N, H) is steeper than any other entire solution of (F_h) including (W, G). However, if $N > 1$, we can get a contradiction on the steepness by the fact that

$p_{N-1} < p$, similar to the argument in Theorem 1.1.9. Then we claim that $N = 1$ and (U_1, H) is a pulsating semi-wave connecting 0 to p. In the following, we call (U, H) instead of (U_1, H).

Assume that (U, H) and (W, G) are not the same up to any time shift. We claim that

$$\lim_{t\to+\infty} \frac{H(t)}{t} > \lim_{t\to+\infty} \frac{G(t)}{t}. \tag{1.33}$$

For (U, H) is steeper, if $H(t_1) = G(t_2)$, we have that $U(t_1, \cdot) > W(t_2, \cdot)$ holds for $x \in (-\infty, H(t_1))$. By the comparison principle and Hopf's lemma, we can infer that $H(t+t_1) > G(t+t_2)$ for any $t > 0$. Thus it follows that (U, H) and (W, G) have different periodicities on time. Then (1.33) is proved.

Next, we will make a contradiction on (1.33) through construction a subsolution of (W, G). To make the readers understand the method more clearly, we will treat the case $p \equiv 1$ first and consider the general case later. Here we assume that

$$p(\cdot) \equiv 1 \text{ and } \partial_u f(\cdot\ , 1) \equiv -\lambda, \tag{1.34}$$

where $\lambda > 0$. Then there exists a small positive constant ϵ such that

$$\partial_u f(x, u) \le \frac{\lambda}{2} \text{ for } x \in \mathbb{R} \text{ and } u \in [1-\epsilon, 1+\epsilon]. \tag{1.35}$$

We define

$$(\underline{U}, \underline{H}) := (U(t-\xi(t), x) - q(t)k(x - H(t-\xi(t))), H(t-\xi(t))),$$

where k is defined as

$$k(x) = 1 - e^x \text{ for } x \in (-\infty, 0]$$

and ξ, q are to be decided later. It is easy to see that k satisfies

$$1 > k(x) \ge 0 > k'(x) = k''(x) \ge -1 \text{ for } x \in (-\infty, 0].$$

Because (U, H) is a pulsating semi-wave connecting 0 to 1, we know that there exists a positive constant M, such that for any $t \in \mathbb{R}$

$$1 > U(t, x) > 1 - \frac{\epsilon}{2} \text{ for } x \in (-\infty, H(t) - M).$$

For the part $x \in (-\infty, H(t-\xi(t)) - M)$, we have

$$\begin{aligned} \partial_t \underline{U} - \partial_{xx} \underline{U} - f(x, \underline{U}) &= \partial_t U(1-\xi') - q'k + qk'H'(1-\xi') - \partial_{xx} U + qk'' - f(x, U - qk) \\ &= -\partial_t U \xi' - q'k + qk'H'(1-\xi') + qk'' + f(x, U) - f(x, U - qk). \end{aligned}$$

We note that $\partial_t U(t, x) > 0$ and $H'(t) > 0$ hold for any $t \in \mathbb{R}$ and $x \in (-\infty, H(t)]$. If assume

$$1 > \xi'(t) > 0, \frac{\epsilon}{2} > q(t) > 0 \tag{1.36}$$

for $t > 0$, then we get

$$\partial_t \underline{U} - \partial_{xx} \underline{U} - f(x, \underline{U}) \le -q'k - \frac{\lambda}{2} qk.$$

Define

$$q(t) := \frac{\epsilon}{2} e^{-\frac{\lambda}{2} t} \tag{1.37}$$

for $t > 0$, it follows that

$$\partial_t \underline{U} - \partial_{xx} \underline{U} - f(x, \underline{U}) \le 0$$

for $t > 0$ and $x \in (-\infty, \underline{H}(t) - M)$.

According to Lemma 1.3.13, we note that there exists a positive constant δ such that

$$\partial_t U(t, x) > \delta \text{ for } t \in \mathbb{R}, x \in [H(t) - M, H(t)].$$

For the part $x \in [H(t-\xi(t))-M, H(t-\xi(t)))$, we have

$$\begin{aligned}\partial_t \underline{U} - \partial_{xx}\underline{U} - f(x,\underline{U}) &= -\partial_t U\xi' - q'k + qk'H'(1-\xi') + qk'' + f(x,U) - f(x,U-qk)\\ &\leq -\delta\xi' + \frac{\lambda}{2}qk + mqk\end{aligned}$$

where $m := \|\partial_u f\|_{C([0,L]\times[0,1])}$. Then if

$$\delta\xi'(t) \geq (\frac{\lambda}{2}+m)q(t) \tag{1.38}$$

for $t > 0$, it follows that

$$\partial_t\underline{U} - \partial_{xx}\underline{U} - f(x,\underline{U}) \leq 0$$

for $t > 0$ and $x \in [\underline{H}(t)-M, \underline{H}(t)]$.

Next, let us check the free boundary condition. There exists a positive constant θ, such that $H'(t) > \theta$ holds for any $t \in \mathbb{R}$. Then if

$$\xi'(t) > \frac{\mu}{\theta}q(t) \tag{1.39}$$

for $t > 0$, we have

$$\underline{H}'(t) = (1-\xi')H' \leq -\mu\partial_x U - \xi'\theta \leq -\mu\partial_x U - \mu q = -\mu\partial_x\underline{U}(t,\underline{H}(t))$$

Then we can define

$$\xi(t) := M_0 - M_0 e^{-\frac{\lambda}{2}t}, \tag{1.40}$$

where $M_0 := 2\max\{\frac{\theta\epsilon}{\mu\lambda}, \frac{\lambda+2m}{2\delta\lambda}\epsilon\}$. It is easy to check that (1.36),(1.38) and (1.39) are satisfied simultaneously for $t > t_0$, where

$$t_0 := \frac{2}{\lambda}\ln\frac{M_0\lambda}{2}. \tag{1.41}$$

Finally, we need to check the initial datum. Because it holds that

$$\lim_{x\to-\infty} U(t_0,x) - q(t_0)k(x-H(t_0)) = 1 - q(t_0),$$

then we can find a sufficiently large time T_0 such that

$$\begin{cases} G(T_0+t_0) > \underline{H}(t_0),\\ W(T_0+t_0,x) > \underline{U}(t_0,x)\ x \in (-\infty, \underline{H}(t_0)).\end{cases} \tag{1.42}$$

Given $\xi(t) := M_0 - M_0 e^{-\frac{\lambda}{2}t}$ and $q(t) = \frac{\epsilon}{2}e^{-\frac{\lambda}{2}t}$, we know that $(\underline{U}(\cdot,\cdot), \underline{H}(\cdot))$ is a subsolution of $(W(T_0+\cdot,\cdot), G(T_0+\cdot))$ for $t > t_0$. Because ξ and q are exponential functions, we have

$$\lim_{t\to+\infty}\frac{H(t)}{t} = \lim_{t\to+\infty}\frac{\underline{H}(t)}{t} \leq \lim_{t\to+\infty}\frac{G(T_0+t)}{t},$$

which contradicts with (1.33). Thus we conclude (U,H) (or (W,G)) is the unique pulsating semiwaveup to time shift.

For general p, we rewrite the subsolution in the form

$$(\underline{U},\underline{H}) := (U(t-\xi(t),x) - q(t)k(x-H(t-\xi(t)))\psi(x), H(t-\xi(t))).$$

Because of loss of $\partial_u f(\cdot\ ,p(\cdot)) \equiv -\lambda$, we can not treat the tail of the pulsating traveling wave like above. Thus we need to make the following transformation like Lemma 1.2.10

$$W(t,x) := \frac{U(t,x)-p(x)}{\psi(x)},$$

and $\underline{W}$ is defined in the same way. Under this transformation, p is converted to 0 and W satisfies (1.11), with $\partial_w F(\cdot\ ,0) \equiv -\lambda$. At first, we only focus on the tail of the subsolution $(-\infty, \underline{H}(t) - M_2)$, where M_2 is sufficiently large. Then we have

$$\begin{aligned}&\partial_t \underline{W} - \partial_{xx}\underline{W} - 2\frac{\psi'}{\psi}\partial_x \underline{W} - F(x, \underline{W})\\ =&\ F(x,W) - F(x, W - qk) - \frac{\partial_t U \xi'}{\psi} - q'k + qk'H'(1-\xi') + qk'' + 2\frac{\psi'}{\psi}qk'.\end{aligned}$$

There exists a positive constant a such that

$$a > -2\frac{\psi'(x)}{\psi(x)} \text{ for any } x \in [0, L].$$

Then we define $k(x) := 1 - e^{ax}$ and still assume (1.36), it follows that

$$\begin{aligned}&\partial_t \underline{W} - \partial_{xx}\underline{W} - 2\frac{\psi'}{\psi}\partial_x \underline{W} - F(x, \underline{W})\\ \leq&\ F(x,W) - F(x, W - qk) - q'k + qk'(a + 2\frac{\psi'}{\psi})\\ \leq&\ F(x,W) - F(x, W - qk) - q'k\end{aligned}$$

Because of $\partial_w F(\cdot\ ,0) \equiv -\lambda$, we can choose appropriate q like the above to make $\underline{W}$ satisfy the subsolution condition in $(-\infty, \underline{H}(t) - M_2)$. As to the estimates on the front of the subsolution $[\underline{H}(t) - M_2, \underline{H}(t))$, the free boundary $\underline{H}(t)$ and the initial datum, we can calculate them in the same way as the case for $p \equiv 1$ without using any transformation. Then we can choose appropriate ξ and q to make that $(\underline{U}, \underline{H})$ is a subsolution of (W, G). Through the same argument on the average speed, this proposition is finished.

□

Lemma 1.4.1. *Let the assumptions of Theorem 1.1.15 hold and (u, g, h) be a solution of (F) for which spreading happens. Then there exist $\delta_0 > 0$ and $t_0 > 0$ such that*

$$u(t, 0) > p(0) - e^{-\delta_0 t} \textit{ for } t > t_0.$$

Proof. Let (U, H) and (U_*, H_*) be the pulsating semi-waves for the right side and that for the left side. By the same argument in Remark 1.3.5, we know that $\partial_{xx}U(t,\cdot) \in C((-\infty, H(t)])$ for any $t \in \mathbb{R}$. By differentiating $U(t, H(t)) \equiv 0$ on t, we obtain that $\partial_{xx}U(t, H(t)) = (H'(t))^2/\mu$. Thus $\partial_{xx}U$ is always bounded on the free boundary. Because of the periodicity of U, we can infer that $\partial_{xx}U(t,x)$ is uniformly bounded for $t \in \mathbb{R}$ and $x \in (-\infty, H(t))$. So is for $\partial_{xx}U_*(t,x)$. By Hopf's lemma, we know that $H'(t) > 0$ and $H_*'(t) < 0$ for any $t \in \mathbb{R}$. Then we can find a positive constant $\gamma < 1$ such that it holds $\partial_{xx}U(t, H(t)) > 2\gamma \max\{|\partial_{xx}U(t,x)| : x \in (-\infty, H(t))\}$ and $\partial_{xx}U_*(t, H_*(t)) > 2\gamma \max\{|\partial_{xx}U_*(t,x)| : x \in (H_*(t), +\infty)\}$ for any $t \in \mathbb{R}$.

We construct the subsolution $(\underline{u}, \underline{g}, \underline{h})$ as below:

$$\begin{cases}\underline{u}(t,x) = U(\gamma t - \xi(t), x) + U_*(\gamma t - \xi(t), x) - q(t)\psi(x) - p(x), & t > 0, x \in (\underline{g}(t), \underline{h}(t)),\\ \underline{u}(t, \underline{g}(t)) = \underline{u}(t, \underline{h}(t)) = 0, & t > 0,\end{cases} \tag{1.43}$$

where ξ and q are some functions to be determined later. Assume that

$$\xi(t), q(t) \in [0, 1] \tag{1.44}$$

for any $t > 0$. Note that if $\gamma t - \xi(t)$ and $\frac{1}{q(t)}$ is sufficiently large, $\underline{g}(t)$ and $\underline{h}(t)$ are the only two zero points of $\underline{u}(t,\cdot)$.

We divide the estimates into four domains:

$$\Omega_1 := \{(t,x) : t > 0, x \in (-1, \underline{h}(t)) \text{ and } U(t,x) \in (0, p(x) - \frac{\epsilon}{3}\psi(x))\},$$
$$\Omega_2 := \{(t,x) : t > 0, x \in (-1, \underline{h}(t)) \text{ and } U(t,x) \in (p(x) - \frac{\epsilon}{3}\psi(x), p(x))\},$$
$$\Omega_3 := \{(t,x) : t > 0, x \in (\underline{g}(t), 1) \text{ and } U_*(t,x) \in (p(x) - \frac{\epsilon}{3}\psi(x), p(x))\},$$
$$\Omega_4 := \{(t,x) : t > 0, x \in (\underline{g}(t), 1) \text{ and } U_*(t,x) \in (0, p(x) - \frac{\epsilon}{3}\psi(x))\},$$

where ϵ satisfies

$$-2\lambda < \partial_u F(x,u) < -\frac{\lambda}{2} \text{ for } x \in \mathbb{R} \text{ and } u \in [-\epsilon, \epsilon].$$

Because (U, H) is the pulsating semi-wave moving at a positive average speed, it is easy to check that there exist two positive contents α_0 and t_0 such that $p(x) - U(t,x) \leq e^{-\alpha_0 t}\psi(x)$ for $t > t_0$ and $x \in (-\infty, 1)$. Without loss generality, we may assume that $p(x) - U_*(t,x) \leq e^{-\alpha_0 t}\psi(x)$ also holds for $t > t_0$ and $x \in (-1, +\infty)$. We can also choose t_0 so large that $e^{-\alpha_0(\gamma t_0 - 1)} < \frac{\epsilon}{3}$.

For the domain Ω_2, let us make the transformation first:

$$\tilde{u}(t,x) := \frac{\underline{u}(t,x) - p(x)}{\psi(x)}.$$

Assume that for any $t > 0$, it holds

$$\xi'(t) \geq 0 \text{ and } \frac{\epsilon}{3} \geq q(t) \geq 0. \tag{1.45}$$

We note that $\partial_t U$,$\partial_t U_*$ are positive and $0 < \gamma < 1$. Then for $t > t_0$, we have

$$\begin{aligned}
&\partial_t \tilde{u} - \partial_{xx}\tilde{u} - 2\frac{\psi'}{\psi}\partial_x \tilde{u} - F(x, \tilde{u}) \\
<& - q' - \xi' \frac{\partial_t U + \partial_t U_*}{\psi} + F(x, \frac{U-p}{\psi}) + F(x, \frac{U_* - p}{\psi}) - F(x, \frac{U-p}{\psi} + \frac{U_* - p}{\psi} - q) \\
<& - q' + 2\lambda e^{-\alpha_0 t} + F(x, \frac{U-p}{\psi}) - F(x, \frac{U-p}{\psi} + \frac{U_* - p}{\psi} - q) \\
<& - q' + 2\lambda e^{-\alpha_0 t} - \frac{\lambda}{2} q.
\end{aligned}$$

If we assume

$$q(t) = \frac{\epsilon}{3} e^{-\beta t} \text{ for } t > 0, \tag{1.46}$$

where $\beta := \min\{\frac{\gamma\alpha_0}{2}, \frac{\lambda}{4}\}$, then there exist a time $t_1 > t_0$ such that $\partial_t \tilde{u} - \partial_{xx}\tilde{u} - 2\frac{\psi'}{\psi}\partial_x \tilde{u} - F(x, \tilde{u}) < 0$ in the domain Ω_2. In the same way, we also can show $\underline{u}$ is a subsolution in the domain Ω_3.

For the domain Ω_1, we directly calculate

$$\begin{aligned}
&\partial_t \underline{u} - \partial_{xx}\underline{u} - f(x, \underline{u}) \\
<& - \xi'(\partial_t U + \partial_t U_*) - q'\psi + f(x, U) + f(x, U_*) - f(x, p) + q\psi'' - f(x, U + U_* - p - q\psi) \\
\leq& - \xi' \partial_t U + O(e^{-\beta t}) + O(e^{-\alpha_0 t}) + f(x, U) - f(x, U + U_* - p - q\psi) \\
\leq& - \xi' \partial_t U + O(e^{-\beta t}).
\end{aligned}$$

In the domain Ω_1, there exists a $\delta > 0$ such that $\partial_t U > \delta$ for $(t,x) \in \Omega_1$.Then if given

$$\xi(t) = 1 - e^{-\frac{\beta}{2}t} \text{ for } t > 0, \tag{1.47}$$

there exists a time $t_2 > t_1$ such that $\underline{u}$ is a subsolution in Ω_1 for $t > t_2$. Ω_4 can be treated similarly.

We know that $\underline{h}$ is the unique zero point of $\underline{u}$ in $\mathbb{R}^+$ for large time t. Then let us check the subsolution condition for the boundary. Because of $U_*(\gamma t - \xi(t), \underline{h}(t)) - p(x) - q(t)\psi(x) = O(e^{-\beta t})$, it is easy to find that $\underline{h}(t) - H(\gamma t - \xi(t)) = O(e^{-\beta t})$. By noticing $\underline{u}(t, \underline{h}(t)) \equiv 0$ and $\partial_{xx}U$ is bounded, we have

$$\begin{aligned}\underline{h}'(t) &= \frac{q'(t)\psi(\underline{h}(t)) - (\gamma - \xi'(t))[\partial_t U(\gamma t - \xi(t), \underline{h}(t)) + \partial_t U_*(\gamma t - \xi(t), \underline{h}(t))]}{\partial_x U(\gamma t - \xi(t), \underline{h}(t)) + \partial_x U_*(\gamma t - \xi(t), \underline{h}(t)) - q(t)\psi'(\underline{h}(t)) - p'(\underline{h}(t))} \\ &= \frac{O(e^{-\beta t}) - (\gamma - \xi'(t))\partial_{xx}U(\gamma t - \xi(t), \underline{h}(t))}{O(e^{-\beta t}) + \partial_x U(\gamma t - \xi(t), \underline{h}(t))} \\ &= \frac{O(e^{-\beta t}) - (\frac{1}{2} - \frac{\xi'(t)}{2\gamma})\partial_{xx}U(\gamma t - \xi(t), H(\gamma t - \xi(t)))}{O(e^{-\beta t}) + \partial_x U(\gamma t - \xi(t), H(\gamma t - \xi(t)))}\end{aligned}$$

and

$$\begin{aligned}\partial_x \underline{u}(t, \underline{h}(t)) &= \partial_x U(\gamma t - \xi(t), \underline{h}(t)) + \partial_x U_*(\gamma t - \xi(t), \underline{h}(t)) - p'(\underline{h}(t)) - q(t)\psi'(\underline{h}(t)) \\ &= O(e^{-\beta t}) + \partial_x U(\gamma t - \xi(t), H(\gamma t - \xi(t))).\end{aligned}$$

Because of $\partial_{xx}U(t, H(t)) \equiv \mu \partial_x U^2(t, H(t))$, there exists a $t_3 > t_2$ such that for $t > t_3$, $\underline{h}'(t) \leq -\mu \partial_x \underline{u}(t, \underline{h}(t))$ and $\underline{h}$ is the unique zero point of $\underline{u}$ in $\mathbb{R}^+$. And $\underline{g}$ can be treated in the same way.

Then we get $(\underline{u}, \underline{g}, \underline{h})$ is a subsolution of (F) for $t > t_3$. Because (u, g, h) spreads, there must exists a time $\tilde{t}$ such that $u(\tilde{t}, x) > \underline{u}(t_3, x)$ for $x \in [\underline{g}(t_3), \underline{h}(t_3)]$. We know that $\underline{u}(t, 0)$ converges to $p(0)$ exponentially as $t \to +\infty$. Then this lemma follows.

□

Lemma 1.4.2. *Let the assumptions of Theorem 1.1.15 hold and (u, g, h) be a solution of (F) for which spreading happens. Then there exist constants T_0, T_1, T_2, q_0 and β_0 (the last two positive), such that for $t > T_0$ we have*

$$H(t + T_1) \leq h(t) \leq H(t + T_2) \tag{1.48}$$

and

$$U(t + T_1, x) - q_0 e^{-\beta_0 t} \leq u(t, x) \text{ for } x \in [0, H(t + T_1)) \tag{1.49}$$

and

$$u(t, x) \leq U(t + T_2, x) + q_0 e^{-\beta_0 t} \text{ for } x \in [0, h(t)). \tag{1.50}$$

Proof. We want to construct the subsolution in the following form

$$(\underline{U}, \underline{H}) := (U(t - \xi(t), x) - q(t)k(x - H(t - \xi(t)))\psi(x), H(t - \xi(t))).$$

Here define $\lambda_1 := \min\{\frac{\lambda}{2}, \frac{\delta_0}{2}\}$ and $q(t) := q^* e^{-\lambda_1 t}$, where $q^* > 0$ and δ_0 is defined in Lemma 1.4.1. Then by the same method in the proof of Theorem 1.1.13, we still can find appropriate k,ξ and t_0 so that $(\underline{U}, \underline{H})$ is a subsolution of (F_h) for $t > t_0$. Because $\xi(+\infty)$ is bounded, then there exists a time $t_1 (> t_0)$ such that

$$k(-\underline{H}(t)) > \frac{1}{2} \tag{1.51}$$

when $t > t_1$. Because (u, g, h) is a solution of (F) for which spreading happens, then we can find a large time T_0 such that

$$\underline{H}(t_1 + 1) < h(T_0) \tag{1.52}$$

and

$$\underline{U}(t_1 + 1, x) < u(T_0, x) \text{ for } x \in [0, \underline{H}(t_1 + 1)] \tag{1.53}$$

and

$$2M^*e^{-\delta_0 T_0}e^{-\delta_0 t} < q(t_1+1)\psi(0)e^{-\lambda_1 t} \text{ for } t > 0. \tag{1.54}$$

Through (1.51) and (1.54), it is not difficult for us to infer that

$$\underline{U}(t_1+1+t,0) < u(T_0+t,0) \text{ for } t > 0. \tag{1.55}$$

Combining (1.52), (1.53), (1.55) and the fact $(\underline{U},\underline{H})$ is a subsolution of (F_h) for $t > t_1+1$, we know that $(\underline{U}(t+t_1+1-T_0,\cdot),\underline{H}(t+t_1+1-T_0))$ is a subsolution of (u,g,h) on $\mathbb{R}^+$ when $t > T_0$. Note that $k \leq 1$ and $\xi(t)$,$U(t,x)$,$H(t)$ are increasing function with respect to t, then we have

$$H(t+t_1+1-T_0-\xi(+\infty)) \leq \underline{H}(t+t_1+1-T_0))$$

and

$$U(t+t_1+1-T_0-\xi(+\infty),x) - q^*\|\psi\|_{L^\infty([0,L])}e^{-\lambda_1(t_1+1-T_0)}e^{-\lambda_1 t} \leq \underline{U}(t+t_1+1-T_0,x).$$

Define

$$T_1 := t_1+1-T_0-\xi(+\infty),$$
$$q_0 := q^*\|\psi\|_{L^\infty([0,L])}e^{-\lambda_1(t_1+1-T_0)}$$

and

$$\beta_0 := \lambda_1,$$

it follows that the left part of (1.48) and (1.49).

By noticing Lemma 1.2.10, we can construct a supersolution

$$(\bar{U},\bar{H}) := (U(t+\xi(t),x)+q(t)k(x-H(t+\xi(t)))\psi(x), H(t+\xi(t)))$$

in a similar way. Then it follows the right part of (1.48) and (1.50).

□

By (1.37),(1.40) and (1.42), we notice that $|\xi(+\infty)-\xi(t_0)| = O(q(t_0))$ and $T_0 = O(q_{t_0})$, where M_0,T_0 and t_0 here are the notations in (1.40),(1.41) and (1.42). Then by following the virtue of Proposition 1.1.13 and Lemma 1.4.2, we have the following lemma.

Lemma 1.4.3. *Let the assumptions of Theorem 1.1.15 hold and (u,g,h) be a solution of (F) for which spreading happens. Then there exist a large constant $M > 0$ and a function $\gamma(\epsilon)$ satisfying $\lim_{\epsilon\to 0}\gamma(\epsilon) = 0$, such that if $M < h(t_1)-\epsilon < H(t_2) < h(t_1)+\epsilon$ and $\|u(t_1,\cdot)-U(t_2,\cdot)\|_{L^\infty([0,\min\{h(t_1),H(t_2)\}])} < \epsilon$ for some $t_1,t_2 > 0$, then we have for any $t > 0$*

$$|h(t+t_1)-H(t+t_2)| < \gamma(\epsilon)$$

and

$$\|u(t+t_1,\cdot)-U(t+t_2,\cdot)\|_{L^\infty([0,\min\{h(t+t_1),H(t+t_2)\}])} < \gamma(\epsilon).$$

Definition 1.4.4. *(1) X is a metric space. $\{\varphi_t\}_{t\geq 0}$ is a semi-flow on X, if it satisfies*

- *$\varphi_t : X \to X$ is continuous with respect to $t \in \mathbb{R}^+$ and $x \in X$,*
- *$\varphi_0(x) = x$ for all $x \in X$,*
- *$\varphi_t \circ \varphi_s = \varphi_{t+s}$ for all $t,s \geq 0$.*

(2) And the positive semi-orbit $O^+(x)$ is defined as

$$\{\varphi_t(x)|t \geq 0\}.$$

(3) $x \in X$ is called an equilibrium point, if it satisfies

$$\varphi_t(x) = x \ for\ all\ t \geq 0.$$

(4) $S \subset X$ is called an invariant set, if it satisfies

$$\varphi_t(S) = S \ for\ all\ t \geq 0.$$

(5) We call $z \in X$ is a $\omega-$ limit point of x, if there exists an increasing sequence $(t_k)_{k\in\mathbb{N}}$ such that

$$t_k \to +\infty \text{ and } \varphi_{t_k}(x) \to z$$

as $k \to +\infty$. $\omega(x)$ is the set of $\omega-$ limit points of x.

The following lemma can be easily found (for example, see [34]).

Lemma 1.4.5. *If $O^+(x)$ is relatively compact, then $\omega(x)$ is a nonempty compact invariant set.*

Definition 1.4.6. *For any continuous function $u \in C(\mathbb{R})$, we define $\sigma(u) := \min\{y : u(x) = 0 \text{ for } x \geq y\}$ and $\varsigma(u) := \max\{y : u(x) = 0 \text{ for } x \leq y\}$. X is the union of two metric spaces of continuous functions and defined in the below:*

$$A := \{u(x) : \sigma(u) < +\infty,\ \varsigma(u) > -\infty \text{ and } u(x) > 0 \text{ for } x \in (\varsigma(u), \sigma(u))\}$$

and

$$B := \{u(x) : \sigma(u) < +\infty,\ \varsigma(u) = -\infty \text{ and } u(x) > 0 \text{ for } x \in (-\infty, \sigma(u))\},$$

where the metric of $X := A \cup B$ is defined as follows:

$$d(u,v) := \|e^x(u-v)\|_{L^\infty} + |\sigma(u) - \sigma(v)| + |e^{\varsigma(u)} - e^{\varsigma(v)}|.$$

Proof of Theorem 1.1.15:

By Lemma 1.4.2, there exist two sufficiently large positive constants M_ϵ and T_ϵ for any $\epsilon > 0$ such that $|u(t,x) - p(x)| < \frac{\epsilon}{2}$ for any $t > T_\epsilon$ and $x \in [0, h(t) - M_\epsilon]$. Moreover, for appropriate M_ϵ, we also have $U(t,x) > p(x) - \epsilon$ for $t \in \mathbb{R}$ and $x \in (-\infty, H(t) - M_\epsilon + 1]$.

We denote the semiflow $\varphi_t[w_0] := w(t, x + t\frac{L}{T})$, where $w_0 \in X$ and T is the periodicity of the pulsating semi-wave (U, H). If $\varsigma(w_0) > -\infty$, w is the solution of (F) with initial datum w_0; if $\varsigma(w_0) = -\infty$, w is the solution of (F_h).

By Lemma 1.4.2, $|h(t) - t\frac{L}{T}|$ is uniformly bounded for any $t > 0$. Then we have $\sigma(\varphi_t[u_0])$ is uniformly bounded for $t > 0$ while $\varsigma(\varphi_t[u_0]) \to -\infty$ as $t \to +\infty$. It follows that $O^+(u_0)$ is relatively compact. Then by Lemma 1.3.1, $\omega(u_0) = \bigcap_{s\geq 0} \overline{\{\varphi_t(u_0) | t \geq s\}}$ is a nonempty compact invariant set.

We can find a sequence of positive integers $(n_j)_{j\in\mathbb{N}}$ such that $n_j \to +\infty$ and $\lim_{j\to+\infty} \varphi_{n_jT}[u_0] = \xi \in \omega(u_0)$ as $j \to +\infty$. It is easy to check that $\varphi_t(\xi)$ is well defined and belongs to $\omega(u_0)$ for all $t \in \mathbb{R}$ because $O^+(u_0)$ is relatively compact. Then we find a complete semi-orbit for ξ. For $\varsigma(\varphi_t[u_0]) \to -\infty$ as $t \to +\infty$, it is apparent that $(v(t,x), l(t)) := (\varphi_t(\xi), \sigma(\varphi_t(\xi)))$ is an entire solution of (F_h). By Lemma 1.4.2, we know that $\lim_{x\to-\infty} |v(t,x) - p(x)| = 0$ for any $t \in \mathbb{R}$.

We note that for any $t \in \mathbb{R}$, $(U(t,\cdot), H(t))$ is a fixed point of Q. By Lemma 1.3.9, we have $H'(t) \geq l'(s)$ if $H(t) = l(s)$. Then $\sigma(\varphi_{nT}[\xi])$ always decreases with respect to n. By Lemma 1.4.2, It is easy to see that $\sigma(\varphi_{nT}[\xi])$ converges to $H(t_0)$ for some t_0. We denote $\xi_* := \lim_{j\to} \varphi_{k_jT}[\xi]$, where $k_j \in \mathbb{N}$ for every $j \in \mathbb{N}$ and $\lim_{j\to+\infty} k_j = +\infty$. It is easy to check that $\sigma(\xi_*) = \sigma(\varphi_T(\xi_*) = H(t_0)$. Then we claim that $\xi_* = U(t_0, \cdot)$. Otherwise, by Lemma 1.3.9, we have $U(t_0, x) > \xi_*(x)$ for $x \in (-\infty, H(t_0))$. Then by comparison principle, it is easy to infer $\sigma(\varphi_T(\xi_*)) < H(t_0)$, which is a contradiction.

Because $\xi_* \in \omega(u_0)$, there exists a sequence of integers $(m_j)_{j\in\mathbb{N}}$ such that $m_j \to +\infty$ and $u(m_jT, x + m_jL)$ converges to $U(t_0, x)$ locally uniformly in $(H(t_0) - 2M_\epsilon, H(t_0))$ as $j \to +\infty$. Then we can find a m_{j_0} such that $m_{j_0}T > T_\epsilon$ and $|h(m_{j_0}T) - H(t_0 + m_{j_0}T)| < \epsilon$ and $|u(m_{j_0}T, x) - U(t_0 + m_{j_0}T, x)| < \epsilon$ for $x \in [h(m_{j_0}T) - M_\epsilon, \min\{h(m_{j_0}T), H(t_0 + m_{j_0}T)]$. We note that $|u(m_{j_0}T, x) - U(t_0 + m_{j_0}T, x)| < 2\epsilon$ for $x \in [0, h(m_{j_0}T) - M_\epsilon]$. Then by Lemma 1.4.3, we have $|h(t) - H(t + t_0)| < \gamma(2\epsilon)$ and $\|u(t,\cdot) - U(t + t_0, \cdot)\|_{L^\infty([0,\min\{h(t), H(t+t_0)\}])} < \gamma(2\epsilon)$ for any $t > m_{j_0}T$. Because ϵ can be arbitrarily small, (2.4.3) and (1.5) follow from repeating Lemma 1.4.3.

We can obtain (1.6) and (1.7) in a similar argument.

$\square$

2. **Spreading speed and profile for nonlinear Stefan problems in high space dimensions**

2.1. **Introduction.**

We are interested in the long-time limit of the spreading speed and profile determined by the following free boundary problem:

$$
\begin{cases}
u_t - \Delta u = f(u) & \text{for } x \in \Omega(t), t > 0, \\
u = 0 \text{ and } u_t = \mu |\nabla_x u|^2 & \text{for } x \in \Gamma(t), t > 0, \\
u(0,x) = u_0(x) & \text{for } x \in \Omega_0,
\end{cases} \tag{2.1}
$$

where $\Omega(t) \subset \mathbb{R}^N$ ($N \geq 2$) is bounded by the free boundary $\Gamma(t)$ (i.e., $\Gamma(t) = \partial\Omega(t)$), with $\Omega(0) = \Omega_0$, which is a bounded domain with smooth boundary $\partial\Omega_0$, and $u_0 \in C(\overline{\Omega}_0) \cap H^1(\Omega_0)$ is positive in Ω_0 and vanishes on $\partial\Omega_0$. μ is a given positive constant, and the nonlinearity $f(u)$ is assumed to be of monostable, bistable or combustion type, whose meanings will be made precise below.

When $f(u) \equiv 0$, (2.1) reduces to the classical one-phase Stefan problem, which arises in the study of the melting of ice in contact with water. Our motivation to study the nonlinear Stefan problem (2.1) mainly comes from the wish to better understand the spreading of a new species, where u is viewed as the density of such a species, and the free boundary represents the spreading front, beyond which the species cannot be observed (i.e., the species has density 0).

Starting from the pioneering works of Fisher [20] and Kolmogrov-Petrovski-Piskunov [28], such a spreading process is usually modeled by the Cauchy problem:

$$
\begin{cases}
U_t - \Delta U = f(U) & \text{for } x \in \mathbb{R}^N, \ t > 0, \\
U(0,x) = U_0(x) & \text{for } x \in \mathbb{R}^N,
\end{cases} \tag{2.2}
$$

where $U_0(x)$ is nonnegative and has nonempty compact support. In such a case, $U(t,x) > 0$ for all $x \in \mathbb{R}^N$ once $t > 0$, but one may specify a certain level set $\Gamma_\delta(t) := \{x : U(t,x) = \delta\}$ as the spreading front, where $\delta > 0$ is small, and $\Omega_\delta(t) := \{x : U(t,x) > \delta\}$ is regarded as the range where the species can be observed. A striking feature of the long time behavior of the front $\Gamma_\delta(t)$ is revealed by Aronson and Weinberger in their classical work [3], namely, when spreading happens (i.e., $U(t,x) \to 1$ as $t \to \infty$), $\Gamma_\delta(t)$ goes to infinity at a constant asymptotic speed in all directions, i.e., for any small $\epsilon > 0$, there exists $T > 0$ so that

$$
\Gamma_\delta(t) \subset A_\epsilon(t) := \{x \in \mathbb{R}^N : (c_0 - \epsilon)t \leq |x| \leq (c_0 + \epsilon)t\} \ \text{ for } t \geq T. \tag{2.3}
$$

The number c_0 is usually called the spreading speed of (2.2), and is determined by the well-known traveling wave problem

$$
Q'' - cQ' + f(Q) = 0, \ Q > 0 \text{ in } \mathbb{R}^1, \ Q(-\infty) = 0, \ Q(+\infty) = 1, \ Q(0) = 1/2. \tag{2.4}
$$

More precisely, in the monostable case, $c_0 > 0$ is the minimal value of c such that (2.4) has a solution Q_c (more accurately Q_c exists if and only if $c \geq c_0$); in the bistable and combustion cases, c_0 is the unique value of c such that (2.4) has a solution Q_c. Moreover, Q_c is unique when it exists for a given c.

When $U_0(x)$ is radially symmetric, then $U(t,x)$ is radially symmetric in x for any $t > 0$, and better estimates of the spreading speed and the profile of U near the front are available, which will be recalled briefly below.

We now look at the nonlinear Stefan problem (2.1), which is understood in the weak sense as described in [8], where it is shown that (2.1) has a unique weak solution defined for all $t > 0$. Further properties of (2.1) are obtained in [15], which include the following result:

Theorem A. *$\Omega(t)$ is expanding in the sense that $\overline{\Omega}_0 \subset \Omega(t) \subset \Omega(s)$ if $0 < t < s$. Moreover, $\Omega_\infty := \cup_{t>0}\Omega(t)$ is either the entire space $\mathbb{R}^N$, or it is a bounded set. Furthermore, when $\Omega_\infty = \mathbb{R}^N$,* *for all large t, $\Gamma(t)$ is a smooth closed hypersurface in $\mathbb{R}^N$, and there exists a continuous function $M(t)$ such that*

$$\Gamma(t) \subset \{x : M(t) - \frac{d_0}{2}\pi \le |x| \le M(t)\}; \tag{2.5}$$

and when Ω_∞ is bounded, $\lim_{t\to\infty} \|u(t,\cdot)\|_{L^\infty(\Omega(t))} = 0$. Here d_0 is the diameter of Ω_0.

It can be shown (see [12]) that when spreading happens (i.e., $u(t,x) \to 1$ as $t \to \infty$), there exists $c^* > 0$ such that

$$\lim_{t\to\infty} \frac{M(t)}{t} = c^*. \tag{2.6}$$

The number c^* is therefore called the asymptotic spreading speed of (2.1), which is determined by the following problem,

$$q'' - cq' + f(q) = 0,\ q > 0 \ \text{ in } (0,\infty),\ \ q(0) = 0,\ q(\infty) = 1. \tag{2.7}$$

The above discussion shows that when spreading happens, (2.2) and (2.1) exhibit similar asymptotic behavior: Their fronts can be approximated by spheres, which go to infinity at some constant asymptotic speed. Moreover, by [8], if u and $\Omega(t)$ in (2.1) are denoted by u_μ and $\Omega_\mu(t)$, respectively, then as $\mu \to \infty$,

$$\Omega_\mu(t) \to \mathbb{R}^N (\forall t > 0),\ \ u_\mu \to U \text{ in } C^{(1+\nu)/2,1+\nu}_{loc}((0,\infty)\times\mathbb{R}^N)\ (\forall \nu \in (0,1)),$$

where U is the unique solution of (2.2) with $U_0 = u_0$. Thus the Cauchy problem (2.2) may be regarded as the limiting problem of (2.1) as $\mu \to \infty$.

Fundamentally different behavior exists between (2.1) and (2.2). When $f(u)$ is of Fisher-KPP type (a special monotable case first considered by Fisher [20] and Kolmogrov-Petrovski-Piskunov [28]), it is known from [10] and [15] that the free boundary problem (2.1) exhibits a spreading-vanishing dichotomy: Either $\Omega(t)$ stays bounded for all t and $u \to 0$ as $t \to \infty$ (vanishing), or $\Omega(t)$ expands to $\mathbb{R}^N$ as described in (2.5), and $u \to 1$ as $t \to \infty$. Criteria for each to happen are also known. This is in sharp contrast to the "hair-trigger" phenomenon of the corresponding Cauchy problem (2.2) revealed in [3]: $U \to 1$ as $t \to \infty$ whenever U_0 is nonnegative and not identically zero (in other words, spreading always happens for (2.2) with such $f(u)$).

In this section, we consider the case that spreading happens to (2.1) with $f(u)$ of monostable, bistable or combustion type. By a simple comparison consideration, spreading also happens to the corresponding Cauchy problem (2.2) (with $U_0 = u_0$). We will show that, in this situation, underneath the similarities described before the last paragraph, there also exist fundamental differences between (2.1) and (2.2), once their spreading profiles are more closely examined.

The results of Theorem A allow us to reduce the problem to a much simpler situation, namely to the case with radial symmetry, without loss of much generality. More precisely, from (2.5) it can be easily shown (see [12]) that when spreading happens, the long time asymptotic behavior of the free boundary of (2.1) is largely determined by that of a radially symmetric problem, in the sense that,

$$|M(t) - h(t)| \le C \text{ for all large } t \text{ and some constant } C,$$

where $M(t)$ is given in (2.5), and $r = h(t)$ is the free boundary of a radially symmetric free boundary problem of the following form:

$$\begin{cases} u_t - u_{rr} - \frac{N-1}{r}u_r = f(u), & 0 < r < h(t),\ t > 0, \\ u_r(t,0) = 0,\ u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_r(t,h(t)), & t > 0, \\ h(0) = h_0,\ \ u(0,r) = u_0(r), & 0 \le r \le h_0, \end{cases} \tag{2.8}$$

where f is the same as in (2.1), μ and h_0 are given positive constants. The initial function u_0 is chosen from

$$\mathscr{K}(h_0) := \Big\{ \psi \in C^2([0,h_0]) : \psi'(0) = \psi(h_0) = 0,\ \psi(r) > 0 \text{ in } [0,h_0) \Big\}. \tag{2.9}$$

For any given $h_0 > 0$ and $u_0 \in \mathscr{K}(h_0)$, (2.8) has a classical solution defined for all $t > 0$ ([12]).

If $u_0(x)$ in (2.1) is radially symmetric, then (2.1) reduces to (2.8). Similarly, if we take

$$U_0(x) = \begin{cases} u_0(|x|), & |x| < h_0, \\ 0, & |x| \geq h_0, \end{cases} \tag{2.10}$$

with u_0 given in (2.8), then the unique solution of (2.2) is radially symmetric: $U = U(t, |x|)$. We will closely examine the spreading behavior determined by (2.8) and compare it with that of (2.2) with U_0 given in (2.10).

While the Cauchy problem (2.2) has been extensively studied in the past several decades and relatively well understood (some relevant results for (2.2) will be recalled below), the study of the nonlinear free boundary problem (2.8) is rather recent. Problem (2.8) with the Fisher (also called logistic) nonlinearity $f(u) = au - bu^2$ was investigated in [7], continuing a study initiated in [10] for the one space dimension case. A deduction of the free boundary condition based on ecological assumptions can be found in [6], but generally speaking, the role of this free boundary condition in the mechanism of spreading is still poorly understood.

In [11], problem (2.8) with a rather general $f(u)$ but in one space dimension was considered. In particular, if $f(u)$ is of monostable, or bistable, or combustion type, it was shown in [11] that (2.8) has a unique solution which is defined for all $t > 0$, and as $t \to \infty$, $h(t)$ either increases to a finite number h_∞, or it increases to $+\infty$. Moreover, in the former case, $u(t,r) \to 0$ uniformly in r, while in the latter case, $u(t,r) \to 1$ locally uniformly in $r \in [0, +\infty)$ (except for a transition case when f is of bistable or combustion type). The situation that

$$u \to 0 \text{ and } h \to h_\infty < +\infty$$

is called the **vanishing** case, and

$$u \to 1 \text{ and } h \to +\infty$$

is called the **spreading** case.

When spreading happens, it was shown in [11] that there exists $c^* > 0$ such that

$$\lim_{t\to\infty} \frac{h(t)}{t} = c^*.$$

The number c^* is the same as in (2.6). These conclusions remain valid in higher space dimensions ([12]).

Next we will describe the results more accurately. Firstly, let us recall in detail the three types of nonlinearities of f mentioned above:[1]

$$(\mathrm{f_M})\ \text{ monostable case, }\ (\mathrm{f_B})\ \text{ bistable case, }\ (\mathrm{f_C})\ \text{ combustion case.}$$

In the monostable case $(\mathrm{f_M})$, we assume that f is C^1 and it satisfies

$$f(0) = f(1) = 0,\ f'(0) > 0,\ f'(1) < 0,\ (1-u)f(u) > 0 \text{ for } u > 0, u \neq 1.$$

A typical example is $f(u) = u(1-u)$.

In the bistable case $(\mathrm{f_B})$, we assume that f is C^1 and it satisfies

$$\begin{cases} f(0) = f(\theta) = f(1) = 0, \\ f(u) < 0 \text{ in } (0,\theta),\ f(u) > 0 \text{ in } (\theta, 1),\ f(u) < 0 \text{ in } (1,\infty), \end{cases}$$

[1] While f being C^1 is enough for the results in this section on the radially symmetric problem (2.8), for Theorem A to hold, [15] requires additionally that $f \in C^{1,\alpha}([0,\delta])$ for some $\alpha \in (0,1)$ and small $\delta > 0$.

for some $\theta \in (0,1)$, $f'(0) < 0$, $f'(1) < 0$ and

$$\int_0^1 f(s)ds > 0.$$

A typical example is $f(u) = u(u-\theta)(1-u)$ with $\theta \in \big(0, \frac{1}{2}\big)$.

In the combustion case ($\mathrm{f_C}$), we assume that f is C^1 and it satisfies

$$f(u) = 0 \text{ in } [0,\theta], \quad f(u) > 0 \text{ in } (\theta,1), \; f'(1) < 0, \; f(u) < 0 \text{ in } [1,\infty)$$

for some $\theta \in (0,1)$, and there exists a small $\delta_0 > 0$ such that

$$f(u) \text{ is nondecreasing in } (\theta, \theta+\delta_0).$$

The asymptotic spreading speed c^* is determined in the following way.

Proposition 2.1.1 (Proposition 1.8 and Theorem 6.2 of [11]). *Suppose that f is of* ($\mathrm{f_M}$), *or* ($\mathrm{f_B}$), *or* ($\mathrm{f_C}$) *type. Then for any $\mu > 0$ there exists a unique $c^* = c^*(\mu) > 0$ and a unique solution q_{c^*} to* (2.7) *with $c = c^*$ such that $q'_{c^*}(0) = \frac{c^*}{\mu}$.*

We remark that this function q_{c^*} is shown in [11] to satisfy $q'_{c^*}(z) > 0$ for $z \geq 0$. We call q_{c^*} a *semi-wave with speed c^**, since the function $v(t,x) := q_{c^*}(c^*t - x)$ satisfies

$$\begin{cases} v_t = v_{xx} + f(v) \;\; \text{for} \;\; t \in \mathbb{R}^1, x < c^*t, \\ v(t, c^*t) = 0, \; -\mu v_x(t, c^*t) = c^*, \; v(t,-\infty) = 1. \end{cases}$$

In [16], sharper estimate of the spreading speed in one space dimension was obtained. More precisely it was shown in [16] that when spreading happens for (2.8) (with $N = 1$), there exists $\hat{H} \in \mathbb{R}$ such that

$$\lim_{t\to\infty} (h(t) - c^*t - \hat{H}) = 0, \; \lim_{t\to\infty} h'(t) = c^*, \tag{2.11}$$

$$\lim_{t\to\infty} \sup_{r\in[0,\,h(t)]} \Big| u(t,r) - q_{c^*}(h(t)-r) \Big| = 0. \tag{2.12}$$

In this section, we consider the case that the space dimension $N \geq 2$, and spreading happens for (2.8), namely

$$\lim_{t\to\infty} h(t) = \infty \text{ and } \lim_{t\to\infty} u(t,r) = 1 \text{ locally uniformly for } r \in [0,\infty).$$

We will show that in such a case, we still have (2.12) and $\lim_{t\to\infty} h'(t) = c^*$, but there exists $c_* > 0$ independent of N such that

$$\lim_{t\to\infty} \big[h(t) - c^*t + (N-1)c_* \log t\big] = \hat{h} \in \mathbb{R}^1. \tag{2.13}$$

Moreover, the constant c_* is given by

$$c_* = \frac{1}{\zeta\, c^*}, \quad \zeta = 1 + \frac{c^*}{\mu^2 \int_0^\infty q'_{c^*}(z)^2 e^{-c^* z} dz}.$$

The term $(N-1)c_* \log t$ in (2.13) will be called a logarithmic shifting term. For simplicity of notation, we will write $c_N = (N-1)c_*$. Thus from (2.13) and (2.12) we obtain

$$\lim_{t\to\infty} \sup_{r\in[0,\,h(t)]} \Big| u(t,r) - q_{c^*}(c^*t - c_N \log t + \hat{h} - r) \Big| = 0.$$

This indicates that as $t \to \infty$, u converges to the moving semi-wave profile q_{c^*} traveling with speed $c^* - c_N t^{-1}$.

For convenience of comparison, we now recall some well-known relevant results for the corresponding Cauchy problem (2.2). The classical paper of Aronson and Weinberger [3] contains a systematic investigation of this problem (see [2] for the case of one space dimension). Various sufficient conditions for $\lim_{t\to\infty} U(t,x) = 1$ ("spreading" or "propagation") and for $\lim_{t\to\infty} U(t,x) = 0$ ("vanishing" or "extinction") are given, and the way $U(t,x)$ approaches 1 as $t \to \infty$ is used to describe the spreading of a (biological or chemical) species. In particular, when spreading happens, it is shown in [3] that, in any space dimension $N \geq 1$, there exists $c_0 > 0$ independent of N, such that, for any small $\epsilon > 0$,

$$\begin{cases} \lim_{t\to\infty} \max_{|x|\geq(c_0+\epsilon)t} U(t,x) = 0, \\ \lim_{t\to\infty} \max_{|x|\leq(c_0-\epsilon)t} |U(t,x) - 1| = 0. \end{cases} \tag{2.14}$$

Clearly (2.3) is a consequence of (2.14) (with the same c_0). The relationship between the spreading speed determined by (2.1) and that determined by (2.2) is given by (see Theorem 6.2 of [11])

$$c_0 = \lim_{\mu\to\infty} c^*(\mu).$$

More details on the spreading behavior of the Cauchy problem can be found, for example, in [2, 3, 19, 20, 26, 27, 28, 37].

As we will explain below, fundamental differences arise between the free boundary problem and the Cauchy problem when we compare their spreading profiles closely. While the spreading profiles of all three basic cases ($\mathrm{f_M}$), ($\mathrm{f_B}$) and ($\mathrm{f_C}$) can be described in a unified fashion for the free boundary model (2.8) (see (2.11), (2.12) and (2.13)), where no logarithmic shifting occurs in space dimension $N = 1$, and a synchronized logarithmic shifting happens in dimensions $N \geq 2$, this is not the case for the Cauchy problem, where the monostable case may behave very differently from the other two cases: The (pulled) monostable case gives rise to logarithmic shifting in all dimensions $N \geq 1$, and the shifting coefficient is different from the other two cases when $N \geq 2$.

More precisely, in one space dimension, a classical result of Fife and McLeod [19] states that for f of type ($\mathrm{f_B}$), if spreading happens, i.e., $U(t,x) \to 1$ as $t \to \infty$, where U is the solution to (2.2), the spreading profile of U is described by

$$|U(t,x) - Q_{c_0}(c_0 t + x + C_-)| < Ke^{-\omega t} \text{ for } x < 0,$$
$$|U(t,x) - Q_{c_0}(c_0 t - x + C_+)| < Ke^{-\omega t} \text{ for } x > 0.$$

Here (c_0, Q_{c_0}) is the unique solution of (2.4), $C_\pm \in \mathbb{R}$, and K, ω are suitable positive constants. So as $t \to \infty$, U converges to the traveling wave profile Q_{c_0} moving at the exact speed c_0, and hence no logarithmic shifting occurs in this case.

The monostable case of (2.2) has very different behavior. Firstly we recall that (2.4) already behaves differently in the monostable case. Secondly, a logarithmic shifting occurs: When ($\mathrm{f_M}$) holds and furthermore $f(u) \leq f'(0)u$ for $u \in (0,1)$ (so f falls to the so called "pulled" case), there exist constants $C_\pm$ such that

$$\lim_{t\to\infty} \max_{x\geq 0} \left| U(t,x) - Q_{c_0}\Big(c_0 t - \frac{3}{c_0} \log t - x + C_+\Big) \right| = 0,$$

and

$$\lim_{t\to\infty} \max_{x\leq 0} \left| U(t,x) - Q_{c_0}\Big(c_0 t - \frac{3}{c_0} \log t + x + C_-\Big) \right| = 0.$$

This implies that as $t \to \infty$, U converges to the traveling wave profile Q_{c_0} moving with speed $c_0 - \frac{3}{c_0} t^{-1}$ instead of exactly c_0. The associated logarithmic shifting term $\frac{3}{c_0} \log t$ is known as the logarithmic Bramson correction term; see [5, 25, 32, 37] for more details.

For space dimension $N \geq 2$, if $U_0(x)$ is given by (2.10) and hence the unique solution U of (2.2) is spherically symmetric ($U = U(t,|x|)$), results in [23, 38] indicate that the Bramson correction term for the monostable case becomes

$$\frac{N+2}{c_0} \log t \text{ (for the pulled case of } f\text{)}, \quad \text{or} \quad \frac{N-1}{c_0} \log t \text{ (for the pushed case of } f\text{)},$$

that is, there exists some constant C such that for the pulled case of f,

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}^N}\left|U(t,|x|)-Q_{c_0}\Big(c_0t-\frac{N+2}{c_0}\log t+C-|x|\Big)\right|=0,$$

and for the pushed case of f,

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}^N}\left|U(t,|x|)-Q_{c_0}\Big(c_0t-\frac{N-1}{c_0}\log t+C-|x|\Big)\right|=0.$$

In the bistable case (as well as the combustion case), the Fife-McLeod result should be changed to (see [38])

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}^N}\left|U(t,|x|)-Q_{c_0}\Big(c_0t-\frac{N-1}{c_0}\log t+L-|x|\Big)\right|=0,$$

where L is some constant.

The above comparisons indicate that the singular behavior of the monostable case observed in the Cauchy problem does not exist anymore in the free boundary model, where all three cases behave in a rather synchronized manner.

The rest of the section is organized as follows. In Subsection 2.2, we describe how the constant c_N in the logarithmic shifting term is defined. In Subsection 2.3, we estimate $h(t)$ in several steps until the sharp term $c_N\log t$ appears in the upper and lower bounds of $h(t)$. The main convergence results of this section are proved in Subsection 2.4, where our arguments are based on the estimates obtained in Subsection 2.3, and on a new device very different from the energy methods used in [16] and [19].

A key step in this research is to find the exact form of the logarithmic shifting term $c_N\log t$. This relies on the discovery that sharp upper and lower solutions to (2.8) can be obtained by suitable perturbations of

$$h(t)=c^*t-c_N\log t,\ u(t,r)=\phi(\mu(c^*-c_Nt^{-1}),r-h(t)),$$

with the functions $\phi(\mu,z)$ and $\mu(\xi)$ defined in (2.15) and (2.20), respectively. This approach is completely different from that used for treating the corresponding Cauchy problem, and from that used to handle the one space dimension case in [16].

Our method to prove the convergence result in Subsection 2.4 also relies on innovative ideas. The method is very powerful and should have applications elsewhere. The spirit of the method is close to those in [39] and [18].

2.2. **Formula for c_N.**

In this subsection, we describe how c_N in the logarithmic shifting term is defined, and also give a key identity (see (2.21) below) to be used in the next subsection.

Let q_{c^*} be given by Proposition 2.1.1 and we define $\phi(z)$ to be the unique solution of the following initial value problem

$$\phi''+c^*\phi'+f(\phi)=0,\ \phi(0)=0,\ \phi'(0)=-c^*/\mu. \tag{2.15}$$

Clearly

$$\phi(z)=q_{c^*}(-z)\text{ for }z\le 0.$$

To stress its dependence on μ, we write $\phi(z)=\phi(\mu,z)$. Similarly we write $c^*=c^*(\mu)$. It is easily seen that for each given $\mu_0>0$, we can find $\epsilon_0>0$ such that $\phi(\mu,z)$ is defined for $z\in(-\infty,\epsilon_0]$ with

$$\phi_z(\mu,z)<0,\ \phi(\mu,\epsilon_0)<-\eta_0<0\text{ for }\mu\in[\mu_0/2,2\mu_0]\text{ and }z\le\epsilon_0.$$

From [11] we see that $\mu\to c^*(\mu)$ is strictly increasing. We will show below that it is a C^2 function. To this end, we need to recall some details contained in [11]. Under the assumptions of Proposition

2.1.1, it was shown in [11] that there exists a unique $c_0 > 0$ such that for each $c \in [0, c_0]$, the problem

$$P' = c - \frac{f(q)}{P} \text{ in } [0,1),\ P(1) = 0,\ P'(1) < 0 \tag{2.16}$$

has a unique solution $P_c(q)$, which necessarily satisfies

$$P_c'(1) = \frac{c - \sqrt{c^2 - 4f'(1)}}{2},\ P_c(q) > 0 \text{ in } (0,1).$$

Furthermore, the following monotonicity and continuity result holds.

Lemma 2.2.1 (Lemma 6.1 of [11]). *For any $0 \le c_1 < c_2 \le c_0$ and $\bar{c} \in [0, c_0]$,*

$$P_{c_1}(q) > P_{c_2}(q) \text{ in } [0,1),\ \lim_{c \to \bar{c}} P_c(q) = P_{\bar{c}}(q) \text{ uniformly in } [0,1].$$

Moreover, $P_{c_0}(0) = 0$ and $P_{c_0}(q) > 0$ in $(0,1)$.

From the proof of Theorem 6.2 in [11], we see that, for $\mu > 0$, $c^*(\mu)$ is the unique solution of

$$P_c(0) - \frac{c}{\mu} = 0,\ c \in [0, c_0].$$

We show below that $c \to P_c(0)$ is a C^2 function for $c \in (0, c_0)$.

Fix $c \in (0, c_0)$ and let $h \neq 0$ be sufficiently small so that $c + h \in (0, c_0)$. We then consider

$$\hat{P}_h(q) := \frac{P_{c+h}(q) - P_c(q)}{h},\ q \in [0,1].$$

Clearly

$$\hat{P}_h'(q) = 1 + \frac{f(q)}{P_c(q)P_{c+h}(q)} \hat{P}_h(q) \text{ in } [0,1),\ \hat{P}_h(1) = 0. \tag{2.17}$$

The unique solution of (2.17) is given by

$$\hat{P}_h(q) = -\int_q^1 e^{\int_q^\xi \frac{-f(s)}{P_c(s)P_{c+h}(s)} ds} d\xi,\ q \in [0,1).$$

Let us note that for q close to 1, $f(q)$ is close to $f'(1)(q-1)$ and $P_c(q)$ is close to $P_c'(1)(q-1)$. Hence, for fixed $q \in [0,1)$,

$$e^{\int_q^\xi \frac{-f(s)}{P_c(s)P_{c+h}(s)} ds} \to 0 \text{ as } \xi \to 1 \text{ uniformly in } c, h.$$

It follows that the integrand function

$$e^{\int_q^\xi \frac{-f(s)}{P_c(s)P_{c+h}(s)} ds}$$

is uniformly bounded in the set $\{(q,\xi) : 0 \le q \le \xi \le 1\}$. Letting $h \to 0$ in the expression for $\hat{P}_h(q)$ we obtain

$$\lim_{h \to 0} \hat{P}_h(q) = -\int_q^1 e^{\int_q^\xi \frac{-f(s)}{P_c(s)^2} ds} d\xi,\ q \in [0,1).$$

Therefore

$$\frac{d}{dc} P_c(q) = -\int_q^1 e^{\int_q^\xi \frac{-f(s)}{P_c(s)^2} ds} d\xi < 0 \text{ for } q \in [0,1). \tag{2.18}$$

By Lemma 2.2.1, we easily see from the above identity that $\frac{d}{dc} P_c(q)$ is continuous in c for $c \in (0, c_0)$. Moreover, $\frac{d}{dc} P_c(1) = 0$ and the continuity of $\frac{d}{dc} P_c(q)$ in c is uniform in $q \in [0,1]$.

From (2.18) we further obtain

$$\frac{d^2}{dc^2} P_c(0) = -2 \int_0^1 \left[e^{\int_0^\xi \frac{-f(s)}{P_c(s)^2} ds} \int_0^\xi \frac{f(s)}{P_c(s)^3} \frac{d}{dc} P_c(s) ds \right] d\xi, \tag{2.19}$$

provided that we can show the integral above is convergent. By (2.18) we can find $C_1 > 0$ such that

$$|\frac{d}{dc}P_c(s)| \le C_1 \text{ for } s \in [0,1].$$

For $\epsilon \in (0,1)$ sufficiently small, there exist $C_2, C_3 > 0$ such that

$$\frac{-f(s)}{P_c(s)^2} \le -C_2(1-s)^{-1},\ |\frac{f(s)}{P_c(s)^3}| \le C_3(1-s)^{-2} \text{ for } s \in [1-\epsilon, 1].$$

Hence, for $\xi \in [1-\epsilon, 1]$,

$$\begin{aligned}
&\left| e^{\int_0^\xi \frac{-f(s)}{P_c(s)^2}ds} \int_0^\xi \frac{f(s)}{P_c(s)^3}\frac{d}{dc}P_c(s)ds \right| \\
&\le C_1 e^{\int_0^\xi \frac{-f(s)}{P_c(s)^2}ds}\left[\int_0^{1-\epsilon} + \int_{1-\epsilon}^{\xi}\right] |\frac{f(s)}{P_c(s)^3}|ds \\
&\le C_1 C_\epsilon e^{-C_2\int_{1-\epsilon}^{\xi}(1-s)^{-1}ds}\left[C_\epsilon + C_3\int_{1-\epsilon}^{\xi}(1-s)^{-2}ds\right] \\
&\le C_\epsilon[(1-\xi)^{C_2} + (1-\xi)^{C_2-1}],
\end{aligned}$$

where we have used C_ϵ to denote various positive constants that depend on ϵ. Clearly this implies the convergence of the integral in the formula for $\frac{d^2}{dc^2}P_c(0)$ in (2.19). Moreover, by the continuous dependence of $P_c(q)$ and $\frac{d}{dc}P_c(q)$ on c, we find from (2.19) that $\frac{d^2}{dc^2}P_c(0)$ is continuous in c for $c \in (0, c_0)$. We have thus proved the following result.

Lemma 2.2.2. *The function $c \to P_c(0)$ is C^2 for $c \in (0, c_0)$.*

Define $\zeta(c,\mu) := P_c(0) - \frac{c}{\mu}$. Then

$$\partial_c\zeta(c,\mu) = \frac{d}{dc}P_c(0) - \frac{1}{\mu} < -\frac{1}{\mu} < 0.$$

Hence by the implicit function theorem we find that the unique solution $c = c^*(\mu)$ of $\zeta(c,\mu) = 0$, as a function of μ, is as smooth as ζ, and hence is C^2. Moreover

$$c^{*\prime}(\mu) = -\frac{\partial_\mu\zeta(c^*(\mu),\mu)}{\partial_c\zeta(c^*(\mu),\mu)} = -\frac{\mu^{-2}c^*(\mu)}{\partial_c\zeta(c^*(\mu),\mu)} > 0,$$

and

$$\left(\frac{c^*(\mu)}{\mu}\right)' = \frac{c^{*\prime}(\mu)}{\mu} - \frac{c^*(\mu)}{\mu^2} = \mu^{-2}c^*(\mu)\left[\frac{1}{-\mu\partial_c\zeta} - 1\right] < 0$$

since $-\partial_c\zeta > \mu^{-1}$.

From [11] we further have

$$\lim_{\mu\to\infty}\frac{c^*(\mu)}{\mu} = 0,\ \lim_{\mu\to 0}\frac{c^*(\mu)}{\mu} = P_0(0) > 0.$$

We now fix $\mu_0 > 0$ and denote $c_0^* = c^*(\mu_0)$. Therefore for each $\xi \in (0, \mu_0 P_0(0))$ there exists a unique $\mu = \mu(\xi)$ such that

$$\frac{c^*(\mu(\xi))}{\mu(\xi)} = \frac{\xi}{\mu_0},\ \xi \to \mu(\xi) \text{ is } C^2,\ \ \mu'(\xi) < 0,\ \ \mu(c_0^*) = \mu_0. \tag{2.20}$$

Here we have used the implicit function theorem and $\mu \to \frac{c^*(\mu)}{\mu}$ is C^2 to conclude that $\xi \to \mu(\xi)$ is C^2.

Let $g(\xi) := c^*(\mu(\xi))$. Then g is C^2 and $g'(\xi) = c^{*\prime}(\mu(\xi))\mu'(\xi) < 0$. The following identity will play a crucial role in the estimates of the next subsection.

$$g(c_0^* - c_N t^{-1}) - g(c_0^*) = -g'(c_0^*)(c_N t^{-1}) + \frac{1}{2} g''(\theta_t)(c_N^2 t^{-2}) \tag{2.21}$$

with $\theta_t \in (c_0^* - c_N t^{-1}, c_0^*)$, where c_N is given by

$$c_N = \Big[1 - g'(c_0^*)\Big]^{-1} \frac{N-1}{c_0^*}, \tag{2.22}$$

and $g'(c_0^*)$ can be calculated by the following formula:

Lemma 2.2.3.

$$g'(c_0^*) = -\frac{c_0^*}{\mu_0^2 \int_0^\infty q'_{c_0^*}(z)^2 e^{-c_0^* z} dz}. \tag{2.23}$$

Proof. By definition, $g'(c_0^*) = c^{*\prime}(\mu_0)\mu'(c_0^*)$. Using $c^*(\mu(\xi)) = \mu_0^{-1}\xi\mu(\xi)$, we obtain

$$c^{*\prime}(\mu(\xi))\mu'(\xi) = \mu_0^{-1}[\mu(\xi) + \xi\mu'(\xi)], \; \mu'(\xi) = \frac{\mu_0^{-1}\mu(\xi)}{c^{*\prime}(\mu(\xi)) - \mu_0^{-1}\xi}.$$

Hence

$$\mu'(c_0^*) = \frac{1}{c^{*\prime}(\mu_0) - \mu_0^{-1} c_0^*}.$$

By our earlier calculation, we have

$$c^{*\prime}(\mu_0) = -\left.\frac{\mu_0^{-2} c_0^*}{\frac{d}{dc} P_c(0) - \mu_0^{-1}}\right|_{c=c_0^*}.$$

Hence

$$g'(c_0^*) = \frac{c^{*\prime}(\mu_0)}{c^{*\prime}(\mu_0) - \mu_0^{-1} c_0^*} = \frac{1}{1 - \mu_0^{-1} c_0^* c^{*\prime}(\mu_0)^{-1}} = \left.\frac{1}{\mu_0 \frac{d}{dc} P_c(0)}\right|_{c=c_0^*}.$$

From (2.18) we obtain

$$\frac{d}{dc} P_c(0) = -\int_0^1 e^{\int_0^\xi \frac{-f(s)}{P_c(s)^2} ds} d\xi.$$

From [11] we know that

$$P_c(s) = P_c(q_c(z)) = q'_c(z) \text{ with } s = q_c(z), \text{ or equivalently } z = q_c^{-1}(s).$$

Therefore, making use of the change of variable $s = q_c(z)$, and the identity $f(q_c(z)) = -q''_c(z) + c q'_c(z)$, we obtain

$$\begin{aligned}
\int_0^\xi \frac{-f(s)}{P_c(s)^2} ds &= \int_0^{q_c^{-1}(\xi)} \frac{-f(q_c(z))}{q'_c(z)^2} q'_c(z) dz \\
&= \int_0^{q_c^{-1}(\xi)} \frac{q''_c(z) - c q'_c(z)}{q'_c(z)} dz \\
&= \log\left[\frac{q'_c(q_c^{-1}(\xi))}{q'_c(0)}\right] - c q_c^{-1}(\xi).
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{d}{dc}P_c(0) &= -\int_0^1 e^{\int_0^\xi \frac{-f(s)}{P_c(s)^2}ds} d\xi \\
&= -\int_0^1 \frac{q_c'(q_c^{-1}(\xi))}{q_c'(0)} e^{-cq_c^{-1}(\xi)} d\xi \\
&= -\int_0^\infty \frac{q_c'(z)}{q_c'(0)} e^{-cz} q_c'(z)dz \\
&= -\frac{\mu}{c}\int_0^\infty q_c'(z)^2 e^{-cz}dz.
\end{aligned}$$

Hence

$$g'(c_0^*) = \frac{-c_0^*}{\mu_0^2 \int_0^\infty q_{c_0^*}'(z)^2 e^{-c_0^* z}dz}.$$

□

2.3. **Sharp bounds.**
In this subsection we give some sharp estimates for $h(t)$. We always assume that f satisfies the conditions of Proposition 2.1.1. We fix $\mu_0 > 0$ and suppose that $(u(t,r), h(t))$ is the unique solution of (2.8) with $\mu = \mu_0$. Let c_0^* and c_N be defined as in the previous subsection (see (2.22)), and suppose that spreading happens:

$$\lim_{t\to\infty} h(t) = \infty,\ \lim_{t\to\infty} u(t,r) = 1 \text{ uniformly for } r \text{ in compact subsets of } [0,\infty). \tag{2.24}$$

We make these assumptions throughout this section. Our aim is to show the following result.

Theorem 2.3.1. *There exist positive constants C and T such that*

$$|h(t) - (c_0^* t - c_N \log t)| \le C \text{ for } t \ge T. \tag{2.25}$$

Moreover, for any $c \in (0, c_0^*)$, there exist positive constants M and σ such that

$$|u(t,r) - 1| \le Me^{-\sigma t} \text{ for } t > 0,\ r \in [0, ct]. \tag{2.26}$$

These conclusions will be proved by a sequence of lemmas.

2.3.1. *Rough bounds.* We start with some rough bounds for u and h.

Lemma 2.3.2. *The following conclusions hold:*

(i) *For any $c \in (0, c_0^*)$ and $\delta \in (0, -f'(1))$, there exist a positive constants $T_* > 0$ and $M > 0$ such that*

$$u(t,r) \le 1 + Me^{-\delta t},\ h(t) \ge ct \text{ for } t \ge T_* \text{ and } r \in [0, h(t)].$$

(ii) *There exists $\tilde{c} \in (0, c_0^*)$, $\tilde{\delta} \in (0, -f'(1))$, and $\tilde{T}_* > 0$, $\tilde{M} > 0$ such that*

$$u(t,r) \ge 1 - \tilde{M}e^{-\tilde{\delta} t} \text{ for } r \in [0, \tilde{c}t] \text{ and } t \ge \tilde{T}_*.$$

Proof. (i) Consider the equation $\eta'(t) = f(\eta)$ with initial value $\eta(0) = \|u_0\|_{L^\infty} + 1$. Then η is an upper solution of (1.1). So $u(t,x) \le \eta(t)$ for all $t \ge 0$. Since $f(u) < 0$ for $u > 1$, $\eta(t)$ is a decreasing function converging to 1 as $t \to \infty$. Hence there exists $T_* > 0$ such that $\eta(t) < 1+\rho$ and $\eta'(t) = f(\eta) \le \delta(1-\eta)$ for $t \ge T_*$. It follows that

$$u(t,x) \le \eta(t) \le 1 + \rho e^{-\delta(t-T_*)} \text{ for } 0 \le |x| \le h(t), t \ge T_*.$$

Next we take any $c \in (0, c_0^*)$ and show that for all large t, $h(t) \geq ct$. We construct a lower solution similar to the proof of Lemma 6.5 in [11]. Let us recall that for each $c \in (0, c_0^*)$, there exists a function $q^c(z)$ defined for $z \in [0, z^c]$ such that

$$q'' - cq' + f(q) = 0 \text{ in } [0, z^c];\ q(0) = q'(z^c) = 0;\ q'(z) > 0 \text{ in } [0, z^c).$$

Moreover, $Q^c := q^c(z^c) < 1$ and as $c \nearrow c_0^*$,

$$Q^c \nearrow 1,\ z^c \nearrow +\infty,\ \|q^c - q_{c_0^*}\|_{L^\infty([0,z^c])} \to 0.$$

See page 38 of [11] for details.

We now choose $c_1, c_2 \in (c, c_0^*)$ satisfying $c_1 < c_2$, $f(Q^{c_2}) > 0$, and define

$$k(t) := z^{c_2} + c_2 t - \frac{N-1}{c_1} \log t.$$

We can find $T_1 > 0$ such that

$$c_1 t \leq c_2 t - \frac{N-1}{c_1} \log t$$

for $t \geq T_1$. Set

$$w(t,r) := \begin{cases} q^{c_2}(k(t) - r), & c_2 t - \frac{N-1}{c_1} \log t \leq r \leq k(t), \\ q^{c_2}(z^{c_2}), & 0 \leq r \leq c_2 t - \frac{N-1}{c_1} \log t. \end{cases}$$

Since spreading happens we can find $T_2 > T_1$ such that

$$\begin{aligned} &k(T_1) \leq h(T_2) \\ &w(T_1, r) \leq u(T_2, r) \quad \text{for } r \in [0, k(T_1)] \end{aligned}$$

We note that

$$w_r(t,r) = 0 \quad \text{when} \quad 0 \leq r \leq c_2 t - \frac{N-1}{c_1} \log t.$$

Moreover, by (6.7) in [11],

$$k'(t) = c_2 - \frac{N-1}{c_1 t} \leq c_2 < \mu (q^{c_2})'(0) = -\mu w_r(t, k(t))$$

and

$$\begin{aligned} &w_t - \Delta w \\ &= k'(t)(q^{c_2})'(k(t) - r) - (q^{c_2})''(k(t) - r) + \frac{N-1}{r}(q^{c_2})'(k(t) - r) \\ &= f(q^{c_2}(k(t) - r)) + \left(\frac{N-1}{r} - \frac{N-1}{c_1 t} \right) (q^{c_2})'(k(t) - r) \\ &\leq f(w) \end{aligned}$$

for $r \in \left[c_2 t - \frac{N-1}{c_1} \log t, k(t) \right] \subset [c_1 t, k(t)]$ and

$$w_t - \Delta w = 0 < f(Q^{c_2}) = f(w)$$

for $r \in \left[0, c_2 t - \frac{N-1}{c_1} \log t \right]$.

Since w is C^1 in r, the above discussions show that $(w(t-T_2+T_1,r),k(t-T_2+T_1))$ is a lower solution of (1.1) for $t\geq T_2$. Hence there exists some $T_3\geq T_2$ such that for $t\geq T_3$,

$$\begin{aligned}h(t)&\geq k(t-T_2+T_1)=z^{c_2}+c_2(t-T_2+T_1)-\frac{N-1}{c_1}\log(t-T_2+T_1)\\&\geq z^{c_2}+c_1(t-T_2+T_1)\geq ct\end{aligned}$$

and

$$u(t,r)\geq w(t-T_2+T_1,r)\ \text{ for }\ r\in[0,k(t-T_1+T_2)]\supset[0,ct].$$

(ii) Since $w(t-T_2+T_1,r)\equiv q^{c_2}(z^{c_2})=Q^{c_2}>Q^c$ for $r\leq ct$ for all $t\geq T_3$, we find from the above estimates for u and h that

$$h(t)\geq ct,\ u(t,r)\geq Q^c \text{ for } 0\leq r\leq ct,\ t\geq T_3 \tag{2.27}$$

Since $f'(1)<0$, for any $\delta\in(0,-f'(1))$ we can find $\rho=\rho(\delta)\in(0,1)$ such that

$$f(u)\geq\delta(1-u)\ (u\in[1-\rho,1]),\quad f(u)\leq\delta(1-u)\ (u\in[1,1+\rho]).$$

Since $Q^c\to1$ as $c\nearrow c_0^*$, we may assume that $Q^c>1-\rho$.

Now for a given domain D we consider a solution $\psi=\psi_D$ to the following auxiliary problem:

$$\begin{cases}\psi_t-\Delta\psi=-\delta(\psi-1), & t>0,x\in D,\\ \psi\equiv Q^c, & t>0,x\in\partial D,\\ \psi\equiv Q^c, & t=0,x\in D.\end{cases} \tag{2.28}$$

The function $\Psi=\Psi_D=e^{\delta t}(\psi_D-Q^c)$ satisfies

$$\begin{cases}\Psi_t-\Delta\Psi=\delta e^{\delta t}(1-Q^c), & t>0,x\in D,\\ \Psi\equiv0, & t>0,x\in\partial D,\\ \Psi\equiv0, & t=0,x\in D.\end{cases} \tag{2.29}$$

Take

$$D=Q_{\tilde cT}:=\{x\in\mathbb{R}^N|-\tilde cT\leq x_i\leq\tilde cT,\ i=1,2,\cdots,N\}$$

with $\tilde c=c/\sqrt{N}$. Let $G(x,t;\xi,\tau)$ be the Green function for the problem (2.29). From page 84 of [21] one sees that this Green function can be expressed as follows:

$$G(x,t;\xi,\tau)=\prod_{i=1}^N\tilde G(x_i,t;\xi_i,\tau)$$

where $\tilde G$ is the Green function of the one space dimension problem:

$$\begin{cases}\Psi_t-\Psi_{xx}=g(t,x), & t>0,-\tilde cT\leq x\leq\tilde cT,\\ \Psi\equiv0, & t>0,x=\pm\tilde cT,\\ \Psi\equiv0, & t=0,-\tilde cT\leq x\leq\tilde cT.\end{cases}$$

Thus

$$\Psi_{Q_{\tilde cT}}(t,x)=\int_0^t\delta e^{\delta\tau}(1-Q^c)\int_{Q_{\tilde cT}}G(x,t;\xi,\tau)d\xi d\tau$$

For $\varepsilon\in(0,1)$, consider $(t,x)\in\mathbb{R}^{N+1}$ satisfying

$$|x_i|\leq(1-\varepsilon)\tilde cT,\ i=1,2,\cdots,N,\ 0\leq t\leq\frac{\varepsilon^2\tilde c^2T}{4}.$$

From the proof of Lemma 6.5 in [11] we find that for such (t,x), there exists $T_4 \geq T_3$ such that for $T \geq T_4$,
$$\int_{-\tilde{c}T}^{\tilde{c}T} \tilde{G}(x_i,t;\xi_i,\tau)d\xi_i \geq 1 - \frac{4}{\sqrt{\pi}}e^{-T/2} \geq 0.$$
Hence, for sufficiently large $T>0$ there exists $M_0>0$ such that
$$\int_{Q_{\tilde{c}T}} G(x,t;\xi,\tau)d\xi \geq 1 - M_0 e^{-T/2}.$$
From the above estimate we obtain
$$\begin{aligned}\Psi_{Q_{\tilde{c}T}}(t,x) &\geq \delta(1-Q^c)\int_0^t e^{\delta\tau}(1-M_0e^{-T/2})d\tau \\ &= (1-Q^c)(1-M_0e^{-T/2})(e^{\delta t}-1)\end{aligned}$$
for $T \geq T_4$, $|x_i| \leq (1-\varepsilon)\tilde{c}T$, $i=1,2,\cdots,N$, $0 \leq t \leq \frac{\varepsilon^2\tilde{c}^2}{4}T$.

Since $B_{\tilde{c}T} \subset Q_{\tilde{c}T} \subset B_{\sqrt{N}\tilde{c}T} \subset B_{cT}$, using (2.27) and a simple comparison argument we obtain
$$\psi_{Q_{\tilde{c}T}}(t,x) \leq \psi_{B_{cT}}(t,x) \leq u(t+T,|x|) \text{ for } t \geq 0,\ x \in Q_{\tilde{c}T}.$$
Hence
$$u(t+T,|x|) \geq \psi_{Q_{\tilde{c}T}}(t,x) \text{ for } t>0, x \in Q_{\tilde{c}T}. \tag{2.30}$$
Fix $T \geq T_4$. We have
$$\psi_{Q_{\tilde{c}T}}(t,x) = \Psi_{Q_{\tilde{c}T}}(t,x)e^{-\delta t} + Q^c \geq 1 - M_0e^{-T/2} - e^{-\delta t}$$
for $|x_i| \leq \tilde{c}T(1-\varepsilon)$, $i=1,2,\cdots,N$, $0 \leq t \leq \frac{\varepsilon^2\tilde{c}^2}{4}T$. Taking $t = \frac{\varepsilon^2\tilde{c}^2}{4}T$ we obtain
$$\psi_{Q_{\tilde{c}T}}\left(\frac{\varepsilon^2\tilde{c}^2}{4}T, x\right) \geq 1 - M_0e^{-T/2} - e^{-\varepsilon^2\tilde{c}^2\delta T/4}.$$
We only focus on small $\varepsilon>0$ such that $\varepsilon^2\tilde{c}^2\delta < 2$ so
$$\begin{aligned}\psi_{Q_{\tilde{c}T}}\left(\frac{\varepsilon^2\tilde{c}^2}{4}T, x\right) &\geq 1 - M_0e^{-\varepsilon^2\tilde{c}^2\delta T/4} - e^{-\varepsilon^2\tilde{c}^2\delta T/4} \\ &= 1 - (M_0+1)e^{-\varepsilon^2\tilde{c}^2\delta T/4}.\end{aligned}$$
This holds for $|x_i| \leq (1-\varepsilon)\tilde{c}T$, $i=1,2,\cdots,N$, $T \geq T_4$. Thus, by (2.30), for such T and x, we have
$$u\left(\frac{\varepsilon^2\tilde{c}^2}{4}T + T, |x|\right) \geq 1 - (M_0+1)e^{-\varepsilon^2\tilde{c}^2\delta T/4}.$$
Finally, if we rewrite
$$t = \frac{\varepsilon^2\tilde{c}^2}{4}T + T$$
then
$$T = \left(1+\frac{\varepsilon^2\tilde{c}^2}{4}\right)^{-1} t,$$
and
$$u(t,|x|) \geq 1 - (M_0+1)e^{-\tilde{\delta}t}$$
for $|x_i| \leq (1-\varepsilon)\left(1+\frac{\varepsilon^2\tilde{c}^2}{4}\right)^{-1}\tilde{c}t$, $i=1,2,\cdots,N$ and $t \geq T_5$ where $\tilde{\delta} := \frac{\varepsilon^2\tilde{c}^2}{4}\left(1+\frac{\varepsilon^2\tilde{c}^2}{4}\right)^{-1}\delta$ and $T_5 = \frac{\varepsilon^2\tilde{c}^2}{4}T_4 + T_4$. This is also true for $|x| \leq (1-\varepsilon)\left(1+\frac{\varepsilon^2\tilde{c}^2}{4}\right)^{-1}\tilde{c}t$. Since this is true for any

$\tilde{c} \in (0, c_0^*/\sqrt{N})$ and for any small $\varepsilon > 0$, the above estimate implies the conclusion in (ii). This completes the proof. $\square$

Lemma 2.3.3. *For any $c \in (0, c_0^*)$ there exist $M' > 0$, $T' > 0$ and $\delta' \in (0, -f'(1))$ such that*

$$u(t,r) \geq 1 - M'e^{-\delta' t},\ h(t) \geq c_0^* t - M' \log t \text{ for } r \in [0, ct] \text{ and } t \geq T'.$$

Proof. We first construct a lower solution. Define

$$\begin{aligned}
&\underline{u}(t,r) = (1 - \tilde{M}e^{-\tilde{\delta}t}) q_{c_0^*}(\underline{h}(t) - r),\\
&\underline{h}(t) = c_0^*(t - T_{**}) + \tilde{c}T_{**} - \frac{N-1}{\tilde{c}} \log \frac{t}{T_{**}} - \sigma \tilde{M}(e^{-\tilde{\delta}T_{**}} - e^{-\tilde{\delta}t}),\\
&\underline{g}(t) = \tilde{c}t,
\end{aligned}$$

where $\tilde{M}, \tilde{\delta}$ and $\tilde{c}$ are given in Lemma 2.3.2, $\sigma > 0$ and $T_{**} > T_*$ (T_* as in Lemma 2.3.2) will be chosen later. We will check that $(\underline{u}, \underline{g}, \underline{h})$ is a lower solution, that is,

$$\underline{u}_t - \left(\underline{u}_{rr} + \frac{N-1}{r} \underline{u}_r \right) \leq f(\underline{u}) \ \text{ for } \ t > T_{**}, \underline{g}(t) < r < \underline{h}(t), \tag{2.31}$$

$$\underline{u} \leq u \text{ for } \ t \geq T_{**},\ r = \underline{g}(t), \tag{2.32}$$

$$\underline{u} = 0,\ \underline{h}'(t) \leq -\mu \underline{u}_r \ \text{ for } \ t \geq T_{**},\ r = \underline{h}(t), \tag{2.33}$$

$$\underline{h}(T_{**}) \leq h(T_{**}),\ \underline{u}(T_{**}, r) \leq u(T_{**}, r) \ \text{ for } \ r \in [\underline{g}(T_{**}), \underline{h}(T_{**})]. \tag{2.34}$$

We first see that $\underline{h}(T_{**}) = \tilde{c}T_{**} \leq h(T_{**})$ from Lemma 2.3.2. Thus we have

$$\underline{u}(T_{**}, r) \leq 1 - \tilde{M}e^{-\tilde{\delta}T_{**}} \leq u(T_{**}, r) \ \text{ for } r \in [\underline{g}(T_{**}), \underline{h}(T_{**})]$$

from Lemma 2.3.2. Similarly we have

$$\underline{u}(t, \underline{g}(t)) = \underline{u}(t, \tilde{c}t) \leq 1 - \tilde{M}e^{\tilde{\delta}t} \leq u(t, \tilde{c}t) = u(t, \underline{g}(t))$$

for $t \geq T_{**}$ by Lemma 2.3.2.

Clearly $\underline{u}(t, \underline{h}(t)) = 0$. Next we calculate

$$\begin{aligned}
&\underline{h}'(t) = c_0^* - \frac{N-1}{\tilde{c}t} - \sigma \tilde{M}\tilde{\delta}e^{-\tilde{\delta}t} \leq c_0^* - \sigma\tilde{\delta}\tilde{M}e^{-\tilde{\delta}t},\\
&\underline{u}_r(t, \underline{h}(t)) = -(1 - \tilde{M}e^{-\tilde{\delta}t}) q'_{c_0^*}(0) = -\frac{c_0^* - c_0^*\tilde{M}e^{-\tilde{\delta}t}}{\mu},\\
&-\mu \underline{u}_r(t, \underline{h}(t)) = c_0^* - c_0^*\tilde{M}e^{-\tilde{\delta}t}.
\end{aligned}$$

Hence if we take $\sigma > 0$ so that $c_0^* \leq \sigma\tilde{\delta}$, then

$$\underline{h}'(t) \leq -\mu \underline{u}_r(t, \underline{h}(t)) \text{ for } t \geq T_{**}.$$

It remains to prove $\underline{u}_t - \left(\underline{u}_{rr} + \frac{N-1}{r}\underline{u}_r\right) - f(\underline{u}) \leq 0$. Put $\zeta = \underline{h}(t) - r$. Since

$$\begin{aligned}
&\underline{u}_t = \tilde{\delta}\tilde{M}e^{-\tilde{\delta}t} q_{c_0^*}(\zeta) + (1 - \tilde{M}e^{-\tilde{\delta}t}) \underline{h}'(t) q'_{c_0^*}(\zeta),\\
&\underline{u}_r = -(1 - \tilde{M}e^{-\tilde{\delta}t}) q'_{c_0^*}(\zeta),\\
&\underline{u}_{rr} = (1 - \tilde{M}'e^{-\tilde{\delta}t}) q''_{c_0^*}(\zeta),
\end{aligned}$$

we have for $t \geq T_{**}$ and $r \in (\tilde{c}t, \underline{h}(t))$,

$$
\begin{aligned}
&\underline{u}_t - \left(\underline{u}_{rr} + \frac{N-1}{r}\underline{u}_r\right) - f(\underline{u}) \\
&= \tilde{\delta}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta) + (1-\tilde{M}e^{-\tilde{\delta}t})\underline{h}'(t)q'_{c_0^*}(\zeta) \\
&\quad - (1-\tilde{M}e^{-\tilde{\delta}t})q''_{c_0^*}(\zeta) + \frac{N-1}{r}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta) - f((1-\tilde{M}e^{-\tilde{\delta}t})q_{c_0^*}(\zeta)) \\
&= \tilde{\delta}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta) + (1-\tilde{M}e^{-\tilde{\delta}t})\left(c_0^* - \frac{N-1}{\tilde{c}t} - \sigma\tilde{M}\tilde{\delta}e^{-\tilde{\delta}t}\right)q'_{c_0^*}(\zeta) \\
&\quad - (1-\tilde{M}e^{-\tilde{\delta}t})q''_{c_0^*}(\zeta) + \frac{N-1}{r}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta) - f((1-\tilde{M}e^{-\tilde{\delta}t})q_{c_0^*}(\zeta)) \\
&= \tilde{\delta}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta) + (1-\tilde{M}e^{-\tilde{\delta}t})(c_0^*q'_{c_0^*}(\zeta) - q''_{c_0^*}(\zeta)) \\
&\quad - \sigma\tilde{M}\tilde{\delta}e^{-\tilde{\delta}t}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta) + (1-\tilde{M}e^{-\tilde{\delta}t})\left(\frac{N-1}{r} - \frac{N-1}{\tilde{c}t}\right)q'_{c_0^*}(\zeta) \\
&\qquad\quad - f((1-\tilde{M}e^{-\tilde{\delta}t})q_{c_0^*}(\zeta)) \\
&\leq \tilde{\delta}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta) - \sigma\tilde{M}\tilde{\delta}e^{-\tilde{\delta}t}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta) \\
&\quad + (1-\tilde{M}e^{-\tilde{\delta}t})f(q_{c_0^*}(\zeta)) - f((1-\tilde{M}e^{-\tilde{\delta}t})q_{c_0^*}(\zeta)).
\end{aligned}
$$

Let us consider the term $(1-\tilde{M}e^{-\tilde{\delta}t})f(q_{c_0^*}(\zeta)) - f((1-\tilde{M}e^{-\tilde{\delta}t})q_{c_0^*}(\zeta))$, which is of the form

$$(1+\xi)f(u) - f((1+\xi)u).$$

The mean value theorem implies that

$$\xi f(u) + f(u) - f((1+\xi)u) = \xi f(u) - \xi f'(u + \theta_{\xi,u}\xi u)u$$

for some $\theta_{\xi,u} \in (0,1)$. Since $0 < \tilde{\delta} < -f'(1)$, we can find an $\eta > 0$ such that

$$
\begin{cases} \tilde{\delta} \leq -f'(u) \quad \text{for} \quad 1-\eta \leq u \leq 1+\eta, \\ f(u) \geq 0 \quad \text{for} \quad 1-\eta \leq u \leq 1. \end{cases} \tag{2.35}
$$

Since $q_{c_0^*}(\zeta) \to 1$ as $\zeta \to \infty$, there exists $\zeta_\eta > 0$ such that $q_{c_0^*}(\zeta) \geq 1-\eta/2$ for $\zeta \geq \zeta_\eta$. We may assume that $\tilde{M}e^{-\tilde{\delta}t} \leq \eta/2$ for $t \geq T_{**}$.

For $\zeta \geq \zeta_\eta$, we have

$$
\begin{aligned}
&\underline{u}_t - \left(\underline{u}_{rr} + \frac{N-1}{r}\underline{u}_r\right) - f(\underline{u}) \\
&\leq \tilde{\delta}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta) - \sigma\tilde{M}\tilde{\delta}e^{-\tilde{\delta}t}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta) \\
&\quad - \tilde{M}e^{-\tilde{\delta}t}\Big\{f(q_{c_0^*}(\zeta)) - f'\big(q_{c_0^*}(\zeta) - \theta'_{\zeta,t}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta)\big)q_{c_0^*}(\zeta)\Big\} \\
&= -\tilde{M}e^{-\tilde{\delta}t}f(q_{c_0^*}(\zeta)) - \sigma\tilde{M}\tilde{\delta}e^{-\tilde{\delta}t}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta) \\
&\quad + \tilde{M}e^{-\tilde{\delta}t}\Big\{f'\big(q_{c_0^*}(\zeta) - \theta'_{\zeta,t}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta)\big) + \tilde{\delta}\Big\}q_{c_0^*}(\zeta) \leq 0,
\end{aligned}
$$

for some $\theta'_{\zeta,t} \in (0,1)$. Here we have use the fact that

$$q_{c_0^*}(\zeta) - \theta'_{\zeta,t}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta) \geq q_{c_0^*}(\zeta) - \tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta) \geq 1-\eta$$

and hence $f'\big(q_{c_0^*}(\zeta) - \theta'_{\zeta,t}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta)\big) + \tilde{\delta} \leq 0$.

For $0 \le \zeta \le \zeta_\eta$, we have

$$\begin{aligned}
&\underline{u}_t - \left(\underline{u}_{rr} + \frac{N-1}{r}\underline{u}_r\right) - f(\underline{u})\\
\le&\tilde{\delta}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta) - \sigma\tilde{M}\tilde{\delta}e^{-\tilde{\delta}t}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta)\\
&\quad - \tilde{M}e^{-\tilde{\delta}t}\Big\{f(q_{c_0^*}(\zeta)) - f'\big(q_{c_0^*}(\zeta) - \theta'_{\zeta,t}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta)\big)q_{c_0^*}(\zeta)\Big\}\\
=& \ -\tilde{M}e^{-\tilde{\delta}t}f(q_{c_0^*}(\zeta)) - \sigma\tilde{M}\tilde{\delta}e^{-\tilde{\delta}t}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta)\\
&\quad + \tilde{M}e^{-\tilde{\delta}t}\Big\{f'\big(q_{c_0^*}(\zeta) - \theta'_{\zeta,t}\tilde{M}e^{-\tilde{\delta}t}q_{c_0^*}(\zeta)\big) + \tilde{\delta}\Big\}q_{c_0^*}(\zeta)\\
\le& \ -\tilde{M}e^{-\tilde{\delta}t}\min_{0\le s\le 1} f(s) - \sigma\tilde{M}\tilde{\delta}e^{-\tilde{\delta}t}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta)\\
&\qquad + \tilde{M}e^{-\tilde{\delta}t}\Big\{\max_{0\le s\le 1} f'(s) + \tilde{\delta}\Big\}\\
=& \ \tilde{M}e^{-\tilde{\delta}t}\Big\{-\min_{0\le s\le 1} f(s) + \max_{0\le s\le 1} f'(s) + \tilde{\delta} - \sigma\tilde{\delta}(1-\tilde{M}e^{-\tilde{\delta}t})q'_{c_0^*}(\zeta)\Big\}\\
\le& \ 0,
\end{aligned}$$

for sufficiently large $\sigma > 0$ and all large t. Finally we note that we can take $T_{**} > T_*$ so large that the above holds and $\tilde{c}t \le \underline{h}(t)$ for $t \ge T_{**}$.

Thus we have shown that (2.31)-(2.34) hold and $(\underline{u}, \underline{g}, \underline{h})$ is a lower solution of (2.8). It follows that

$$u(t,r) \ge \underline{u}(t,r),\ h(t) \ge \underline{h}(t) \text{ for } t \ge T_{**} \text{ and } r \in [\underline{g}(t), \underline{h}(t)].$$

Hence

$$\begin{aligned}
u(t,r) &\ge (1-\tilde{M}e^{-\tilde{\delta}t})q_{c_0^*}(\underline{h}(t)-r)\\
&\ge q_{c_0^*}(\underline{h}(t)-r) - \tilde{M}e^{-\tilde{\delta}t}
\end{aligned}$$

for $t \ge T_{**}$ and $\tilde{c}t \le r \le \underline{h}(t)$.

For any $c \in (0, c_0^*)$ and any $\kappa \in (0, c_0^* - c)$, there exists $T_{***} > 0$ such that for $t \ge T_{***}$ and $r \in [0, ct]$, we have

$$\underline{h}(t) - r \ge (c_0^* - c)t - \frac{N-1}{\tilde{c}}\log\frac{t}{T_{**}} + \tilde{c}T_{**} - \sigma\tilde{M} \ge \kappa t.$$

Since there exist $C > 0$ and $\beta > 0$ such that $q_{c_0^*}$ satisfies $q_{c_0^*}(z) \ge 1 - Ce^{-\beta z}$ for $z \ge 0$, we thus obtain

$$u(t,r) \ge 1 - Ce^{-\beta\kappa t} - \tilde{M}e^{-\tilde{\delta}t} = 1 - \tilde{M}'e^{-\delta' t} \tag{2.36}$$

for $t \ge T_{***}$ and $r \in [\tilde{c}t, ct]$, where $\delta' = \min\{\beta\kappa, \tilde{\delta}\}$.

Moreover, if $M_0 > (N-1)/\tilde{c}$, then

$$h(t) \ge \underline{h}(t) = c_0^*t - \frac{N-1}{\tilde{c}}\log t - \tilde{C} \ge c_0^*t - M_0\log t \text{ for all large } t.$$

Thus combined with (2.37) and Lemma 2.3.2, we find that

$$u(t,r) \ge 1 - M'e^{-\delta' t},\ h(t) \ge c_0^*t - M'\log t$$

for $t \ge T'$ and $r \in [0, ct]$ provided that M' and T' are chosen large enough. This completes the proof of Lemma 2.3.3. □

Clearly (2.26) follows directly from Lemmas 2.3.2 and 2.3.3. Let us note that from the proof of Lemma 2.3.3, we have, for $t \ge T'$ and $r \in [\tilde{c}t, c_0^*t - M'\log t]$,

$$u(t,r) \ge (1-\tilde{M}e^{-\tilde{\delta}t})q_{c_0^*}(c_0^*t - M'\log t - r).$$

Since

$$q_{c_0^*}(z) \geq 1 - M_1 e^{-\delta_1 z} \text{ for } z \in [0,\infty) \text{ and some } M_1, \delta_1 > 0,$$

we immediately obtain

$$u(t,r) \geq (1 - \tilde{M} e^{-\tilde{\delta} t})(1 - M_1 e^{-\delta_1 (c_0^* t - M' \log t - r)}) \tag{2.37}$$

for $t \geq T'$ and $r \in [\tilde{c}t, c_0^* t - M' \log t]$.

2.3.2. *Sharp bounds.* We now make use of the rough bounds for u and h to obtain sharp bounds for h. We first improve the estimate for $h(t)$ in Lemma 2.3.3.

Lemma 2.3.4. *There exist $C > 0$ and $T > 0$ such that*

$$h(t) \geq c_0^* t - c_N \log t - C \text{ for } t \geq T,$$

where c_N is given by (2.22).

Proof. With $B > 0$ a constant to be determined, and $\phi(z) = \phi(\mu, z)$ given in (2.15), we set

$$\tilde{k}(t) = c_0^* t - c_N \log t + B t^{-1} \log t,$$

$$\underline{v}(t,r) = \phi\left(\mu(c_0^* - c_N t^{-1}), r - \tilde{k}(t)\right) - t^{-2} \log t.$$

We have $\underline{v}(t, \tilde{k}(t)) = -t^{-2} \log t < 0$ for $t > 1$, and

$$\underline{v}(t, \tilde{k}(t) - t^{-1}) = \phi\left(\mu(c_0^* - c_N t^{-1}), -t^{-1}\right) - t^{-2} \log t = -\phi_r(\mu_0, 0) t^{-1} + o(t^{-1}) > 0$$

for all large t. Moreover,

$$\underline{v}_r(t,r) = \phi_r\left(\mu(c_0^* - c_N t^{-1}), r - \tilde{k}(t)\right) < 0 \text{ for all } t > 0 \text{ and } r \in (0, \tilde{k}(t)].$$

Therefore, there exists a unique $\underline{k}(t) \in (\tilde{k}(t) - t^{-1}, \tilde{k}(t))$ such that

$$\underline{v}(t, \underline{k}(t)) = 0 \text{ for all large } t.$$

By the implicit function theorem we know that $t \to \underline{k}(t)$ is smooth, and by the mean value theorem we obtain

$$\left[\phi_r(\mu_0, 0) + o(1)\right]\left[\underline{k}(t) - \tilde{k}(t)\right] = t^{-2} \log t.$$

Since $\phi_r(\mu_0, 0) = -c_0^*/\mu_0$, we thus obtain

$$\underline{k}(t) - \tilde{k}(t) = \left[-\frac{\mu_0}{c_0^*} + o(1)\right] t^{-2} \log t \quad \text{for all large } t. \tag{2.38}$$

Using $\underline{v}_t(t, \underline{k}(t)) + \underline{v}_r(t, \underline{k}(t)) \underline{k}'(t) = 0$ we obtain

$$\phi_\mu \cdot \mu' \cdot c_N t^{-2} + \phi_r \cdot \left[\underline{k}'(t) - \tilde{k}'(t)\right] + \left[1 + o(1)\right] 2t^{-3} \log t = 0.$$

It follows that

$$\underline{k}'(t) = \tilde{k}'(t) + O(t^{-2}) = c_0^* - c_N t^{-1} - B t^{-2} \log t + O(t^{-2})$$

for all large t.

We want to show that there exist positive constants M and T such that $(\underline{v}(t,r), \underline{k}(t))$ satisfies, for $t \geq T$ and $\underline{k}(t) - M \log t \leq r \leq \underline{k}(t)$,

$$\underline{v}(t, \underline{k}(t)) = 0, \ \ \underline{k}'(t) \leq -\mu_0 \underline{v}_r(t, \underline{k}(t)), \tag{2.39}$$

$$\underline{v}(t, \underline{k}(t) - M \log t) \leq u(t+s, k(t+s) - M \log(t+s)), \ \forall s > 0, \tag{2.40}$$

$$\underline{v}_t - \underline{v}_{rr} - \frac{N-1}{r} \underline{v}_r - f(\underline{v}) \leq 0. \tag{2.41}$$

Moreover, we will show that the above inequalities imply

(2.42) $\qquad \underline{k}(t) \le h(t+T_1),\ \underline{v}(t,r) \le u(t+T_1, r)$ for $r \in (\underline{k}(t) - M\log t, \underline{k}(t))$ and $t \ge T$.

Clearly the required estimate for $h(t)$ follows directly from (2.42) and (2.38).

By the definition of $\underline{k}(t)$, we have $\underline{v}(t,\underline{k}(t)) = 0$. We now calculate

$$\begin{aligned}\underline{v}_r(t,\underline{k}(t)) &= \phi_r(\mu(c_0^* - c_N t^{-1}), \underline{k}(t) - \tilde{k}(t))\\ &= \phi_r(\mu(c_0^* - c_N t^{-1}), 0) + \left[\phi_{rr}(\mu_0, 0) + o(1)\right]\left[\underline{k}(t) - \tilde{k}(t)\right]\\ &= -\frac{1}{\mu_0}(c_0^* - c_N t^{-1}) + \left[\phi_{rr}(\mu_0,0) + o(1)\right]\left[-\frac{\mu_0}{c_0^*} + o(1)\right] t^{-2}\log t.\end{aligned}$$

Using

$$\phi_{rr}(\mu_0, r) + c_0^*\phi_r(\mu_0, r) + f(\phi(\mu_0, r)) = 0$$

and $f(\phi(\mu_0, 0)) = f(0) = 0$, we obtain

$$\phi_{rr}(\mu_0, 0) = -c_0^*\phi_r(\mu_0, 0) = \frac{c_0^{*2}}{\mu_0}.$$

It follows that

$$\begin{aligned}-\mu_0\underline{v}_r(t,\underline{k}(t)) &= c_0^* - c_N t^{-1} + \mu_0 c_0^* t^{-2}\log t + o(t^{-2}\log t)\\ &> c_0^* - c_N t^{-1} - Bt^{-2}\log t + O(t^{-2})\\ &= \underline{k}'(t) \text{ for all large } t.\end{aligned}$$

Hence (2.39) holds.

Since

$$c_0^* t - M'\log t - \left[\underline{k}(t) - M\log t\right] = (c_N + M - M')\log t + o(1) > (M/2)\log t$$

for all large t, provided that $M > 2M'$, we obtain from (2.37) that

$$u(t, \underline{k}(t) - M\log t) \ge (1 - \tilde{M}e^{-\tilde{\delta}t})\left(1 - M_1 t^{-\delta_1 M/2}\right) > 1 - t^{-2}$$

for all large t, provided that $M > 4/\delta_1$. We now fix M such that $M > \max\{2M', 4/\delta_1\}$. Thus

$$u(t+s, \underline{k}(t+s) - M\log(t+s)) > 1 - (t+s)^{-2} > 1 - t^{-2}\log t > \underline{v}(t, \underline{k}(t) - M\log t)$$

for all large t and every $s > 0$. This proves (2.40).

Next we show (2.41). We have, with $\xi = c_0^* - c_N t^{-1}$,

$$\begin{aligned}\underline{v}_t =& \phi_\mu(\mu(\xi), r - \tilde{k}(t))\mu'(\xi)c_N t^{-2} - \phi_r(\mu(\xi), r - \tilde{k}(t))\tilde{k}'(t) + 2t^{-3}\log t - t^{-3}\\ =& O(t^{-2}) + \phi_r \cdot \big(-c_0^* + c_N t^{-1} + Bt^{-2}\log t - Bt^{-2}\big),\end{aligned}$$

and

$$\underline{v}_r(t,r) = \phi_r(\mu(\xi), r - \tilde{k}(t)),\ \underline{v}_{rr}(t,r) = \phi_{rr}(\mu(\xi), r - \tilde{k}(t)).$$

Hence,

$$\begin{aligned}&\underline{v}_t - \underline{v}_{rr} - \frac{N-1}{r}\underline{v}_r - f(\underline{v})\\ &= O(t^{-2}) + \phi_r\left[-c_0^* + c_N t^{-1} + Bt^{-2}\log t - Bt^{-2} - \frac{N-1}{r}\right] - \phi_{rr} - f\left(\phi - t^{-2}\log t\right)\\ &= O(t^{-2}) + \phi_r\left[g(\xi) - g(c_0^*) + c_N t^{-1} + Bt^{-2}\log t - Bt^{-2} - \frac{N-1}{r}\right]\\ &\quad - g(\xi)\phi_r - \phi_{rr} - f\left(\phi - t^{-2}\log t\right)\\ &= O(t^{-2}) + \phi_r J + f(\phi) - f\left(\phi - t^{-2}\log t\right),\end{aligned}$$

where

$$J := g(\xi) - g(c_0^*) + c_N t^{-1} + Bt^{-2}\log t - Bt^{-2} - \frac{N-1}{r}.$$

For $r \in [\underline{k}(t) - M\log t, \underline{k}(t)]$, we have

$$\begin{aligned} r &\geq \underline{k}(t) - M\log t \\ &= \tilde{k}(t) - M\log t + O(t^{-2}\log t) \\ &= c_0^* t - (c_N + M)\log t + Bt^{-1}\log t + O(t^{-2}\log t) \\ &\geq c_0^* t - M_2 \log t \quad \text{for all large } t, \end{aligned}$$

where $M_2 = c_N + M$. It follows that, for such r,

$$\begin{aligned} \frac{N-1}{r} &\leq \frac{N-1}{c_0^* t - M_2\log t} \\ &= \frac{N-1}{c_0^* t} + \frac{(N-1)M_2\log t}{c_0^{*2} t^2}\big[1 + o(1)\big]. \end{aligned}$$

Therefore

$$\begin{aligned} J &\geq -g'(c_0^*)c_N t^{-1} + c_N t^{-1} - \frac{N-1}{c_0^*}t^{-1} + \left[B - \frac{(N-1)M_2}{c_0^{*2}}\right] t^{-2}\log t + o(t^{-2}\log t) \\ &= \left[B - \frac{(N-1)M_2}{c_0^{*2}} + o(1)\right] t^{-2}\log t > 0 \end{aligned}$$

for all large t, provided that B is large enough.

We now fix $\epsilon_0 > 0$ small so that $f'(u) \leq -\sigma_0 < 0$ for $u \in [1 - 2\epsilon_0, 1 + 2\epsilon_0]$. Then when $\phi(\mu(\xi), r - \tilde{k}(t)) \in [1 - \epsilon_0, 1]$ we have

$$f(\phi) - f(\phi - t^{-2}\log t) \leq -\sigma_0 t^{-2}\log t$$

for all large t. Hence in such a case,

$$O(t^{-2}) + \phi_r J + f(\phi) - f(\phi - t^{-2}\log t) \leq O(t^{-2}) - \sigma_0 t^{-2}\log t < 0$$

for all large t.

If $\phi(\mu(\xi), r - \tilde{k}(t)) \in [0, 1 - \epsilon_0]$, then we can find $\sigma_1 > 0$ such that $\phi_r \leq -\sigma_1$, and hence

$$\phi_r J \leq -\sigma_1 \left[B - \frac{(N-1)M_2}{c_0^{*2}} + o(1)\right] t^{-2}\log t.$$

On the other hand, there exists $\sigma_2 > 0$ such that

$$f(\phi) - f(\phi - t^{-2}\log t) \leq \sigma_2 t^{-2}\log t.$$

Thus in this case we have

$$\begin{aligned} &O(t^{-2}) + \phi_r J + f(\phi) - f(\phi - t^{-2}\log t) \\ &\leq -\sigma_1 \left[B - \frac{(N-1)M_2}{c_0^{*2}} + o(1)\right] t^{-2}\log t + \sigma_2 t^{-2}\log t + O(t^{-2}) \\ &< 0 \end{aligned}$$

for all large t, provided that B is large enough. This proves (2.41).

We are now ready to show (2.42). Since as $t \to \infty$, $h(t) \to \infty$ and $u(t, r) \to 1$ locally uniformly in $r \in [0, \infty)$, we can find $T' > T$ such that

$$h(T') > \underline{k}(T),\ u(T', r) > \underline{v}(T, r) \text{ for } r \in [\underline{k}(T) - M\log T, \underline{k}(T)],$$

where $T > 0$ is a constant such that (2.39), (2.40) and (2.41) hold for $t \geq T$. We may now use a comparison argument to conclude that

$$h(T' + t) \geq \underline{k}(T + t),\ u(T' + t, r) \geq \underline{v}(T + t, r)$$

for $t > 0$, $r \in [\underline{k}(T + t) - M \log(T + t), \underline{k}(T + t)]$, which is equivalent to (2.42) with $T_1 = T' - T$. $\square$

Lemma 2.3.5. *There exist $C > 0$ and $T > 0$ such that*

$$h(t) \leq c_0^* t - c_N \log t + C \text{ for } t \geq T,$$

where c_N is given by (2.22).

Proof. With $B > 0$ and $C > 0$ constants to be determined, and $\phi(z) = \phi(\mu, z)$ given in (2.15), we set

$$\hat{k}(t) = c_0^* t - c_N \log t - Bt^{-1} \log t + C,$$
$$\overline{v}(t, r) = \phi\left(\mu(c_0^* - c_N t^{-1}), r - \hat{k}(t)\right) + t^{-2} \log t.$$

We have $\overline{v}(t, \hat{k}(t)) = t^{-2} \log t > 0$ for $t > 1$, and

$$\overline{v}(t, \hat{k}(t) + t^{-1}) = \phi\left(\mu(c_0^* - c_N t^{-1}), t^{-1}\right) + t^{-2} \log t = \left[\phi_r(\mu_0, 0) + o(1)\right] t^{-1} < 0$$

for all large t. Moreover,

$$\overline{v}_r(t, r) = \phi_r\left(\mu(c_0^* - c_N t^{-1}), r - \hat{k}(t)\right) < 0 \text{ for all } t > 0 \text{ and } r \in (0, \hat{k}(t)].$$

Therefore, there exists a unique $\overline{k}(t) \in (\hat{k}(t), \hat{k}(t) + t^{-1})$ such that

$$\overline{v}(t, \overline{k}(t)) = 0 \text{ for all large } t.$$

By the implicit function theorem we know that $t \to \overline{k}(t)$ is smooth, and by the mean value theorem we obtain

$$\left[\phi_r(\mu_0, 0) + o(1)\right]\left[\overline{k}(t) - \hat{k}(t)\right] = -t^{-2} \log t.$$

Since $\phi_r(\mu_0, 0) = -c_0^*/\mu_0$, we thus obtain

$$\overline{k}(t) - \hat{k}(t) = \left[\frac{\mu_0}{c_0^*} + o(1)\right] t^{-2} \log t \quad \text{for all large } t. \tag{2.43}$$

Using $\overline{v}_t(t, \overline{k}(t)) + \overline{v}_r(t, \overline{k}(t))\overline{k}'(t) = 0$ we obtain

$$\phi_\mu \cdot \mu' \cdot c_N t^{-2} + \phi_r \cdot \left[\overline{k}'(t) - \hat{k}'(t)\right] - \left[1 + o(1)\right] 2t^{-3} \log t = 0.$$

It follows that

$$\overline{k}'(t) = \hat{k}'(t) + O(t^{-2}) = c_0^* - c_N t^{-1} + Bt^{-2} \log t + O(t^{-2})$$

for all large t.

We want to show that, by choosing B and C properly, there exists a positive constant T such that $(\overline{v}(t, r), \overline{k}(t))$ satisfies, for $t \geq T$ and $1 \leq r \leq \overline{k}(t)$,

$$\overline{v}(t, \overline{k}(t)) = 0,\ \ \overline{k}'(t) \geq -\mu_0 \overline{v}_r(t, k(t)), \tag{2.44}$$

$$\overline{v}(t, 1) \geq u(t, 1), \tag{2.45}$$

$$\overline{v}_t - \overline{v}_{rr} - \frac{N - 1}{r} \overline{v}_r - f(\overline{v}) \geq 0, \tag{2.46}$$

and

$$\overline{k}(T) \geq h(T),\ \overline{v}(T, r) \geq u(T, r) \text{ for } r \in [1, h(T)]. \tag{2.47}$$

If these inequalities are proved, then we can apply a comparison argument to conclude that

$$\overline{k}(t) \geq h(t),\ \overline{v}(t, r) \geq u(t, r) \text{ for } r \in [1, h(t)] \text{ and } t \geq T. \tag{2.48}$$

Clearly the required estimate for $h(t)$ follows directly from (2.48) and (2.43).

By the definition of $\overline{k}(t)$, we have $\overline{v}(t,\overline{k}(t))=0$. We now calculate

$$\begin{aligned}\overline{v}_r(t,\overline{k}(t)) &= \phi_r(\mu(c_0^*-c_N t^{-1}),\overline{k}(t)-\hat{k}(t))\\ &= \phi_r(\mu(c_0^*-c_N t^{-1}),0)+\big[\phi_{rr}(\mu_0,0)+o(1)\big]\big[\overline{k}(t)-\hat{k}(t)\big]\\ &= -\frac{1}{\mu_0}(c_0^*-c_N t^{-1})+\big[\phi_{rr}(\mu_0,0)+o(1)\big]\left[\frac{\mu_0}{c_0^*}+o(1)\right]t^{-2}\log t\\ &= -\frac{1}{\mu_0}(c_0^*-c_N t^{-1})+c_0^* t^{-2}\log t+o(t^{-2}\log t).\end{aligned}$$

It follows that

$$\begin{aligned}-\mu_0\overline{v}_r(t,\overline{k}(t)) &= c_0^*-c_N t^{-1}-\mu_0 c_0^* t^{-2}\log t+o(t^{-2}\log t)\\ &< c_0^*-c_N t^{-1}+Bt^{-2}\log t+O(t^{-2})\\ &= \overline{k}\,'(t)\text{ for all large }t.\end{aligned}$$

Hence (2.44) holds.

Since

$$\overline{v}(t,1)=\phi\big(\mu(c_0^*-c_N t^{-1}),1-\hat{k}(t)\big)+t^{-2}\log t\geq 1-M_1 e^{\delta_1[1-\hat{k}(t)]}+t^{-2}\log t\geq 1+t^{-2}$$

for all large t, and by Lemma 2.3.2, $u(t,1)\leq 1+Me^{-\delta t}$ for all $t>0$, we find that

$$u(t,1)<\overline{v}(t,1)\text{ for all large }t.$$

This proves (2.45).

Next we show (2.46). We have, with $\xi=c_0^*-c_N t^{-1}$,

$$\begin{aligned}\overline{v}_t =&\phi_\mu(\mu(\xi),r-\hat{k}(t))\mu'(\xi)c_N t^{-2}-\phi_r(\mu(\xi),r-\hat{k}(t))\hat{k}'(t)-2t^{-3}\log t+t^{-3}\\ =&O(t^{-2})+\phi_r\cdot\big(-c_0^*+c_N t^{-1}-Bt^{-2}\log t+Bt^{-2}\big),\end{aligned}$$

and

$$\overline{v}_r(t,r)=\phi_r(\mu(\xi),r-\hat{k}(t)),\ \overline{v}_{rr}(t,r)=\phi_{rr}(\mu(\xi),r-\hat{k}(t)).$$

Hence,

$$\begin{aligned}&\overline{v}_t-\overline{v}_{rr}-\frac{N-1}{r}\overline{v}_r-f(\overline{v})\\ &=O(t^{-2})+\phi_r\left[-c_0^*+c_N t^{-1}-Bt^{-2}\log t+Bt^{-2}-\frac{N-1}{r}\right]-\phi_{rr}-f\left(\phi+t^{-2}\log t\right)\\ &=O(t^{-2})+\phi_r\left[g(\xi)-g(c_0^*)+c_N t^{-1}-Bt^{-2}\log t+Bt^{-2}-\frac{N-1}{r}\right]\\ &\quad -g(\xi)\phi_r-\phi_{rr}-f\left(\phi+t^{-2}\log t\right)\\ &=O(t^{-2})+\phi_r\hat{J}+f(\phi)-f\left(\phi+t^{-2}\log t\right),\end{aligned}$$

where

$$\hat{J}:=g(\xi)-g(c_0^*)+c_N t^{-1}-Bt^{-2}\log t+Bt^{-2}-\frac{N-1}{r}.$$

For $r\in[1,\overline{k}(t)]$, we have

$$\begin{aligned}\frac{N-1}{r}\geq\frac{N-1}{\overline{k}(t)}&=\frac{N-1}{\hat{k}(t)+o(t^{-1})}\\ &=\frac{N-1}{c_0^* t}+\frac{(N-1)c_N\log t}{{c_0^*}^2 t^2}\big[1+o(1)\big].\end{aligned}$$

Therefore, for such r,

$$\begin{aligned}\hat{J} &\le -g'(c_0^*)c_N t^{-1} + c_N t^{-1} - \frac{N-1}{c_0^*}t^{-1} - \left[B + \frac{(N-1)c_N}{c_0^{*2}}\right] t^{-2}\log t + o(t^{-2}\log t)\\ &= -\left[B + \frac{(N-1)c_N}{c_0^{*2}} + o(1)\right] t^{-2}\log t < 0\end{aligned}$$

for all large t.

We now fix $\epsilon_0 > 0$ small so that $f'(u) \le -\sigma_0 < 0$ for $u \in [1-2\epsilon_0, 1+2\epsilon_0]$. Then for $\phi(\mu(\xi), r-\hat{k}(t)) \in [1-\epsilon_0, 1]$ we have

$$f(\phi) - f(\phi + t^{-2}\log t) \ge \sigma_0 t^{-2}\log t$$

for all large t. Hence in such a case,

$$O(t^{-2}) + \phi_r \hat{J} + f(\phi) - f(\phi + t^{-2}\log t) \ge O(t^{-2}) + \sigma_0 t^{-2}\log t > 0$$

for all large t.

If $\phi(\mu(\xi), r - \hat{k}(t)) \in [0, 1-\epsilon_0]$, then we can find $\sigma_1 > 0$ such that $\phi_r \le -\sigma_1$, and hence

$$\phi_r \hat{J} \ge \sigma_1 \left[B + \frac{(N-1)c_N}{c_0^{*2}} + o(1)\right] t^{-2}\log t.$$

On the other hand, there exists $\sigma_2 > 0$ such that

$$f(\phi) - f(\phi + t^{-2}\log t) \ge -\sigma_2 t^{-2}\log t.$$

Thus in this case we have

$$\begin{aligned}&O(t^{-2}) + \phi_r \hat{J} + f(\phi) - f(\phi + t^{-2}\log t)\\ &\ge \sigma_1 \left[B + \frac{(N-1)c_N}{c_0^{*2}} + o(1)\right] t^{-2}\log t - \sigma_2 t^{-2}\log t + O(t^{-2})\\ &> 0\end{aligned}$$

for all large t, provided that B is large enough. This proves (2.46).

Finally we show that (2.47) holds if C is chosen suitably. Indeed, we set

$$C = h(T) - c_0^* T + c_N \log T + 2T.$$

Then

$$\overline{k}(T) = \hat{k}(T) + o(T^{-1}) = h(T) - BT^{-1}\log T + 2T + o(T^{-1}) > h(T) + T$$

for T large enough.

By enlarging T if necessary we have, for $r \in [1, h(T)]$,

$$\begin{aligned}\overline{v}(T,r) \ge \overline{v}(T,h(T)) &= \phi(\mu(c_0^* - c_N T^{-1}), h(T) - \hat{k}(T)) + T^{-2}\log T\\ &\ge \phi(\mu(c_0^* - c_N T^{-1}), -T) + T^{-2}\log T\\ &\ge 1 - M_1 e^{-\delta_1 T} + T^{-2}\log T\\ &> 1 + T^{-2},\end{aligned}$$

while

$$u(T,r) \le 1 + Me^{-\delta T}.$$

Therefore

$$\overline{v}(T,r) \ge u(T,r) \text{ for } r \in [1, h(T)]$$

provided that T is large enough. This proves (2.47). The proof of the lemma is now complete. $\square$

2.4. **Convergence.**
Throughout this subsection we assume that (u,h) is the unique solution of (2.8) with $\mu=\mu_0>0$, and spreading happens: As $t\to\infty$, $h(t)\to\infty$ and $u(t,r)\to 1$ for r in compact subsets of $[0,\infty)$. We will prove the following convergence result.

Theorem 2.4.1. *There exists a constant $\hat h\in\mathbb{R}^1$ such that*
$$\lim_{t\to\infty}\left\{h(t)-\left[c_0^*t-c_N\log t\right]\right\}=\hat h,\ \lim_{t\to\infty}h'(t)=c_0^*$$
and
$$\lim_{t\to\infty}\|u(t,\cdot)-q_{c_0^*}(h(t)-\cdot)\|_{L^\infty([0,h(t)])}=0.$$

Again we will prove this Theorem by a series of lemmas. By Lemmas 2.3.4 and 2.3.5 we know that there exist $C,T>0$ such that
$$-C\le h(t)-[c_0^*t-c_N\log t]\le C \text{ for } t\ge T.$$
We now denote
$$k(t)=c_0^*t-c_N\log t-2C$$
and define
$$v(t,r)=u(t,r+k(t)),\ g(t)=h(t)-k(t),\ t\ge T.$$
Clearly
$$C\le g(t)\le 3C \text{ for } t\ge T.$$
Moreover,
$$u_r=v_r,\ u_{rr}=v_{rr},\ u_t=v_t-(c_0^*-c_Nt^{-1})v_r,$$
and (v,g) satisfies
$$\begin{cases} v_t-v_{rr}-\left[c_0^*-c_Nt^{-1}+\frac{N-1}{r+k(t)}\right]v_r=f(v), & -k(t)\le r<g(t), t>T,\\ v(t,g(t))=0,\ g'(t)=-\mu_0v_r(t,g(t))-c_0^*+c_Nt^{-1}, & t>T.\end{cases}$$

2.4.1. *Limit along a subsequence of $t_n\to\infty$.* Let $t_n\to\infty$ be an arbitrary sequence satisfying $t_n>T$ for every $n\ge1$. Define
$$k_n(t)=k(t+t_n),\ v_n(t,r)=v(t+t_n,r),\ g_n(t)=g(t+t_n).$$

Lemma 2.4.2. *Subject to a subsequence,*
$$g_n\to G \text{ in } C^{1+\frac{\alpha}{2}}_{loc}(\mathbb{R}^1) \text{ and } \|v_n-V\|_{C^{\frac{1+\alpha}{2},1+\alpha}_{loc}(D_n)}\to0,$$
where $\alpha\in(0,1)$, $D_n=\{(t,r)\in D: r\le g_n(t)\}$, $D=\{(t,r): -\infty<r\le G(t),\ t\in\mathbb{R}^1\}$, and $(V(t,r),G(t))$ satisfies
$$\begin{cases} V_t-V_{rr}-c_0^*V_r=f(V), & (t,r)\in D,\\ V(t,G(t))=0,\ G'(t)=-\mu_0V_r(t,G(t))-c_0^*, & t\in\mathbb{R}^1.\end{cases}\tag{2.49}$$

Proof. By [12] there exists $C_0>0$ such that $0<h'(t)\le C_0$ for all $t>0$. It follows that
$$-c_0^*<g_n'(t)\le C_0 \text{ for } t+t_n \text{ large and every } n\ge1.$$
Define
$$s=\frac{r}{g_n(t)},\ w_n(t,s)=v_n(t,r).$$
Then $(w_n(t,s),g_n(t))$ satisfies
$$(w_n)_t-\frac{(w_n)_{ss}}{g_n(t)^2}-\left[sg_n'(t)+c_0^*-c_N(t+t_n)^{-1}+\frac{N-1}{g_n(t)s+k_n(t)}\right]\frac{(w_n)_s}{g_n(t)}=f(w_n)\tag{2.50}$$

for $-\frac{k_n(t)}{g_n(t)} \le s < 1, t > T - t_n$, and

$$w_n(t,1) = 0 \text{ for } t > T - t_n, \tag{2.51}$$

$$g_n'(t) = -\mu_0 \frac{(w_n)_s(t,1)}{g_n(t)} - c_0^* + c_N(t+t_n)^{-1} \text{ for } t > T - t_n. \tag{2.52}$$

For any given $R > 0$ and $T_0 \in \mathbb{R}^1$, using the partial interior-boundary L^p estimates (see Theorem 7.15 in [33]) to (2.50) and (2.51) over $[T_0 - 1, T_0 + 1] \times [-R-1, 1]$, we obtain, for any $p > 1$,

$$\|w_n\|_{W_p^{1,2}([T_0, T_0+1]\times[-R,1])} \le C_R \text{ for all large } n,$$

where C_R is a constant depending on R and p but independent of n and T_0. Therefore, for any $\alpha' \in (0,1)$, we can choose $p > 1$ large enough and use the Sobolev embedding theorem (see [31]) to obtain

$$\|w_n\|_{C^{\frac{1+\alpha'}{2}, 1+\alpha'}([T_0,\infty)\times[-R,1])} \le \tilde{C}_R \text{ for all large } n, \tag{2.53}$$

where $\tilde{C}_R$ is a constant depending on R and α' but independent of n and T_0.

From (2.52) and (2.53) we deduce

$$\|g_n\|_{C^{1+\frac{\alpha'}{2}}([T_0,\infty))} \le C_1 \text{ for all large } n,$$

with C_1 a constant independent of T_0 and n. Hence by passing to a subsequence we may assume that, as $n \to \infty$,

$$w_n \to W \text{ in } C_{loc}^{\frac{\alpha+1}{2}, 1+\alpha}(\mathbb{R}^1 \times (-\infty, 1]),\ g_n \to G \text{ in } C_{loc}^{1+\frac{\alpha}{2}}(\mathbb{R}^1),$$

where $\alpha \in (0, \alpha')$. Moreover, using (2.50),(2.51) and (2.52), we find that (W, G) satisfies in the $W_p^{1,2}$ sense (and hence classical sense by standard regularity theory),

$$\begin{cases} W_t - \frac{W_{ss}}{G(t)^2} - (sG'(t) + c_0^*)\frac{W_s}{G(t)} = f(W), & s \in (-\infty, 1],\ t \in \mathbb{R}^1, \\ W(t,1) = 0,\ G'(t) = -\mu_0 \frac{W_s(t,1)}{G(t)} - c_0^*, & t \in \mathbb{R}^1. \end{cases}$$

Define $V(t,r) = W(t, \frac{r}{G(t)})$. We easily see that (V, G) satisfies (2.49) and

$$\lim_{n\to\infty} \|v_n - V\|_{C_{loc}^{\frac{1+\alpha}{2}, 1+\alpha}(D_n)} = 0.$$

□

2.4.2. *Determine the limit pair* (V, G). We show by a sequence of lemmas that $G(t) \equiv G_0$ is a constant, and hence $V(t,r) = \phi(r - G_0)$.

Since $C \le g(t) \le 3C$ for $t \ge T$, we have

$$C \le G(t) \le 3C \text{ for } t \in \mathbb{R}^1.$$

By the proof of Lemma 2.3.5, we have, for $r \in [1 - k(t+t_n), g(t+t_n)]$ and $t + t_n$ large,

$$v_n(t,r) \le \phi\big(\mu(c_0^* - c_N(t+t_n)^{-1}), r - 3C\big) + (t+t_n)^{-2}\log(t+t_n).$$

Letting $n \to \infty$ we obtain

$$V(t,r) \le \phi(\mu_0, r - 3C) \text{ for all } t \in \mathbb{R}^1,\ r < G(t).$$

Define

$$R^* = \inf\big\{R : V(t,r) \le \phi(\mu_0, r - R) \text{ for all } (t,r) \in D\big\}.$$

Then

$$V(t,r) \le \phi(\mu_0, r - R^*) \text{ for all } (t,r) \in D$$

and

$$C \le \inf_{t\in\mathbb{R}^1} G(t) \le \sup_{t\in\mathbb{R}^1} G(t) \le R^* \le 3C.$$

Lemma 2.4.3. $R^* = \sup_{t\in\mathbb{R}^1} G(t)$.

Proof. Otherwise we have $R^* > \sup_{t\in\mathbb{R}^1} G(t)$. We are going to derive a contradiction.

Choose $\delta > 0$ such that

$$G(t) \le R^* - \delta \text{ for all } t \in \mathbb{R}^1.$$

We derive a contradiction in three steps. To simplify notations we will write $\phi(r)$ instead of $\phi(\mu_0, r)$.

Step 1. $V(t,r) < \phi(r - R^*)$ for all $t \in \mathbb{R}^1$ and $r \le G(t)$.

Otherwise there exists $(t_0, r_0) \in D$ such that

$$V(t_0, r_0) = \phi(r_0 - R^*) \ge \phi(-\delta) > 0.$$

Hence necessarily $r_0 < G(t_0)$. Since $V(t,r) \le \phi(r - R^*)$ in D, and $\phi(r - R^*)$ satisfies the first equation in (2.49), we can apply the strong maximum principle to conclude that $V(t,r) \equiv \phi(r - R^*)$ in $D_0 := \{(t,r) : r < G(t), t \le t_0\}$, which clearly contradicts with the assumption that $G(t) \le R^* - \delta$.

Step 2. $M_r := \inf_{t\in\mathbb{R}^1} \left[\phi(r - R^*) - V(t,r)\right] > 0$ for $r \in (-\infty, R^* - \delta]$. Here we assume that $V(t,r) = 0$ for $r > G(t)$.

Otherwise there exists $r_0 \in (-\infty, R^* - \delta]$ such that $M_{r_0} = 0$, since the definition of R^* implies $M_r \ge 0$ for all $r \le R^* - \delta$. By Step 1 we know that M_{r_0} is not achieved at any finite t. Therefore there exists $s_n \in \mathbb{R}^1$ with $|s_n| \to \infty$ such that

$$\phi(r_0 - R^*) = \lim_{n\to\infty} V(s_n, r_0).$$

Define

$$(V_n(t,r), G_n(t)) = (V(t + s_n, r), G(t + s_n)).$$

Then the same argument used in the proof of Lemma 2.4.2 shows that, by passing to a subsequence, $(V_n, G_n) \to (\tilde{V}, \tilde{G})$ with $(\tilde{V}, \tilde{G})$ satisfying

$$\tag{2.54} \begin{cases} \tilde{V}_t - \tilde{V}_{rr} - c_0^*\tilde{V}_r = f(\tilde{V}), & -\infty < r < \tilde{G}(t), t \in \mathbb{R}^1, \\ \tilde{V}(t, \tilde{G}(t)) = 0, & t \in \mathbb{R}^1. \end{cases}$$

Moreover,

$$\tag{2.55} \tilde{V}(t,r) \le \phi(r - R^*),\ \tilde{G}(t) \le R^* - \delta,\ \tilde{V}(0, r_0) = \phi(r_0 - R^*) > 0.$$

Since $\phi(r - R^*)$ satisfies (2.54) with $\tilde{G}(t)$ replaced by R^*, we can apply the strong maximum principle to conclude, from (2.55), that $\tilde{V}(t,r) \equiv \phi(r - R^*)$ for $t \le 0, r \le \tilde{G}(t)$, which is clearly impossible.

Step 3. Reaching a contradiction.

Choose $\epsilon_0 > 0$ small and $R_0 < 0$ large negative such that

$$\phi(r - R^*) \ge 1 - \epsilon_0 \text{ for } r \le R_0,\ f'(u) < 0 \text{ for } u \in [1 - 2\epsilon_0, 1 + 2\epsilon_0].$$

Then choose $\epsilon \in (0, \epsilon_0)$ such that

$$\phi(R_0 - R^* + \epsilon) \ge \phi(R_0 - R^*) - M_{R_0},\ \phi(r - R^* + \epsilon) \ge 1 - 2\epsilon_0 \text{ for } r \le R_0.$$

We consider the auxiliary problem

$$\tag{2.56} \begin{cases} \overline{V}_t - \overline{V}_{rr} - c_0^*\overline{V}_r = f(\overline{V}), & t > 0,\ r < R_0, \\ \overline{V}(t, R_0) = \phi(R_0 - R^* + \epsilon), & t > 0, \\ \overline{V}(0, r) = 1, & r < R_0. \end{cases}$$

Since the initial function is an upper solution of the corresponding stationary problem of (2.56), its unique solution $\overline{V}(t,r)$ is decreasing in t. Clearly $\underline{V}(t,r) := \phi(r-R^*+\epsilon)$ is a lower solution of (2.56). It follows from the comparison principle that

$$1 \geq \overline{V}(t,r) \geq \phi(r-R^*+\epsilon) \text{ for all } t>0,\ r<R_0.$$

Hence

$$V^*(r) := \lim_{t\to\infty} \overline{V}(t,r) \geq \phi(r-R^*+\epsilon),\ \forall r<R_0.$$

Moreover, V^* satisfies

$$-V^*_{rr} - c_0^* V^*_r = f(V^*) \text{ in } (-\infty,R_0),\ V^*(-\infty)=1,\ V^*(R_0)=\phi(R_0-R^*+\epsilon). \tag{2.57}$$

Write $\psi(r)=\phi(r-R^*+\epsilon)$. We notice that $\psi(r)$ also satisfies (2.57). Moreover

$$1-2\epsilon_0 \leq \psi(r) \leq V^*(r) \leq 1 \text{ for } r\in(-\infty,R_0].$$

Hence $W(r) := V^*(r)-\psi(r) \geq 0$ and there exists $c(r)<0$ such that

$$f(V^*(r))-f(\psi(r)) = c(r)W(r) \text{ in } (-\infty,R_0].$$

Therefore

$$-W''-c_0^*W' = c(r)W \text{ in } (-\infty,R_0),\ W(R_0)=0,$$

and by the maximum principle we deduce, for any $R<R_0$,

$$W(r) \leq W(R) \text{ for } r\in[R,R_0].$$

Letting $R\to-\infty$ we deduce $W(r)\leq 0$ in $(-\infty,R_0]$. It follows that $W\equiv 0$. Hence

$$V^*(r) \equiv \psi(r) = \phi(r-R^*+\epsilon).$$

We now look at $V(t,r)$, which satisfies the first equation in (2.56), and for any $t\in\mathbb{R}^1$,

$$V(t,r)\leq 1,\ V(t,R_0) \leq \phi(R_0-R^*) - M_{R_0} \leq \phi(R_0-R^*+\epsilon).$$

Therefore we can use the comparison principle to deduce that

$$V(s+t,r) \leq \overline{V}(t,r) \text{ for all } t>0, r<R_0, s\in\mathbb{R}^1.$$

Or equivalently

$$V(t,r) \leq \overline{V}(t-s,r) \text{ for all } t>s, r<R_0, s\in\mathbb{R}^1.$$

Letting $s\to-\infty$ we obtain

$$V(t,r) \leq V^*(r) = \phi(r-R^*+\epsilon) \text{ for all } r<R_0, t\in\mathbb{R}^1. \tag{2.58}$$

By Step 2 and the continuity of M_r in r, we have

$$M_r \geq \sigma > 0 \text{ for } r\in[R_0,R^*-\delta].$$

If $\epsilon_1\in(0,\epsilon]$ is small enough we have

$$\phi(r-R^*+\epsilon_1) \geq \phi(r-R^*)-\sigma \text{ for } r\in[R_0,R^*-\delta],$$

and hence

$$V(t,r)-\phi(r-R^*+\epsilon_1) \leq \sigma - M_r \leq 0 \text{ for } r\in[R_0,R^*-\delta],\ t\in\mathbb{R}^1.$$

Therefore we can combine with (2.58) to obtain

$$V(t,r)-\phi(r-R^*+\epsilon_1) \leq 0 \text{ for } r\in(-\infty,R^*-\delta],\ t\in\mathbb{R}^1,$$

for all small $\epsilon_1\in(0,\epsilon)$, which contradicts the definition of R^*. The proof is now complete. $\square$

Lemma 2.4.4. *There exists a sequence $\{s_n\}\subset\mathbb{R}^1$ such that*

$$G(t+s_n)\to R^*,\ V(t+s_n,r)\to\phi(r-R^*) \text{ as } n\to\infty$$

uniformly for (t,r) in compact subsets of $\mathbb{R}^1\times(-\infty,R^*]$.

Proof. There are two possibilities:

(i) $R^* = \sup_{t\in\mathbb{R}^1} G(t)$ is achieved at some finite $t = s_0$,

(ii) $R^* > G(t)$ for all $t \in \mathbb{R}^1$ and $G(s_n) \to R^*$ along some unbounded sequence s_n.

In case (i), necessarily $G'(s_0) = 0$. Since $V(t,r) \le \phi(r - R^*)$ for $r \le G(t)$ and $t \in \mathbb{R}^1$, with $V(s_0, G(s_0)) = \phi(G(s_0) - R^*) = \phi(0) = 0$, we can apply the strong maximum principle and the Hopf boundary lemma to conclude that

$$V_r(s_0, G(s_0)) > \phi'(0) \text{ unless } V(t,r) \equiv \phi(r - R^*) \text{ in } D_0 = \{(t,r) : r \le G(t), t \le s_0\}.$$

On the other hand, we have

$$V_r(s_0, G(s_0)) = -\mu_0^{-1}[G'(s_0) + c_0^*] = -\mu_0^{-1} c_0^* = \phi'(0).$$

Hence we must have $V(t,r) \equiv \phi(r - R^*)$ and $G(t) \equiv R^*$ in D_0. Using the uniqueness of (2.49) with a given initial value, we conclude that $V(t,r) \equiv \phi(r - R^*)$ for all $r \le G(t)$ and $t \in \mathbb{R}^1$. Thus the conclusion of the lemma holds by taking $s_n \equiv s_0$.

In case (ii), we consider the sequence

$$V_n(t,r) = V(t + s_n, r),\ G_n(t) = G(t + s_n).$$

By the same reasoning as in the proof of Lemma 2.4.2, we can show that, by passing to a subsequence,

$$V_n \to \tilde{V} \text{ in } C_{loc}^{\frac{1+\alpha}{2}, 1+\alpha}(D),\ G_n \to \tilde{G} \text{ in } C_{loc}^1(\mathbb{R}^1) \text{ and } (\tilde{V}, \tilde{G}) \text{ satisfies (2.49)},$$

where $D := \{(t,r) : -\infty < r \le \tilde{G}(t),\ t \in \mathbb{R}^1\}$. Moreover,

$$\tilde{G}(t) \le R^*,\ \tilde{G}(0) = R^*.$$

Hence we are back to case (i) and thus $\tilde{V}(t,r) \equiv \phi(r - R^*)$ in D, and $\tilde{G} \equiv R^*$. The conclusion of the lemma now follows easily. ☐

By the proof of Lemma 2.3.4, we have

$$v_n(t,r) \ge \phi\big(\mu(c_0^* - c_N(t + t_n)^{-1}), r - C\big) - (t + t_n)^{-2} \log(t + t_n)$$

for $r \in [\underline{k}(t + t_n) - k(t + t_n) - M \log(t + t_n), \underline{k}(t + t_n) - k(t + t_n)]$ and $t + t_n$ large. Letting $n \to \infty$ we obtain

$$V(t,r) \ge \phi(\mu_0, r - C) \text{ for all } t \in \mathbb{R}^1,\ r < G(t).$$

Define

$$R_* = \sup\big\{R : V(t,r) \ge \phi(\mu_0, r - R) \text{ for all } (t,r) \in D\big\}.$$

Then

$$V(t,r) \ge \phi(\mu_0, r - R_*) \text{ for all } (t,r) \in D$$

and

$$C \le R_* \le \inf_{t\in\mathbb{R}^1} G(t) \le \sup_{t\in\mathbb{R}^1} G(t) \le R^* \le 3C.$$

Lemma 2.4.5. *$R_* = \inf_{t\in\mathbb{R}^1} G(t)$, and there exists a sequence $\{\tilde{s}_n\} \subset \mathbb{R}^1$ such that*

$$G(t + \tilde{s}_n) \to R_*,\ V(t + \tilde{s}_n, r) \to \phi(r - R_*) \text{ as } n \to \infty$$

uniformly for (t,r) in compact subsets of $\mathbb{R}^1 \times (-\infty, R_*]$.

Proof. The proof uses similar arguments to those used to prove Lemmas 2.4.3 and 2.4.4, and we omit the details. ☐

Lemma 2.4.6. *$R_* = R^*$ and hence $G(t) \equiv G_0$ is a constant, which implies $V(t,r) = \phi(r - G_0)$.*

Proof. Argue indirectly we assume that $R_* < R^*$. Set $\epsilon = (R^* - R_*)/4$. We show next that there exists $T_\epsilon > 0$ such that

$$G(t) - R_* \le \epsilon \text{ and } G(t) - R^* \ge -\epsilon \text{ for } t \ge T_\epsilon, \tag{2.59}$$

which implies $R^* - R_* \le 2\epsilon$. This contradiction would complete the proof.

To prove (2.59), we use Lemmas 2.4.4 and 2.4.5, and a modification of the argument in section 3.3 of [16]. Indeed, by using Lemma 2.4.4 and constructing a suitable lower solution we can show that there exists $n_1 = n_1(\epsilon)$ large such that $G(t) - R^* \ge -\epsilon$ for all $t \ge s_{n_1}$. Similarly we can use Lemma 2.4.5 and construct a suitable upper solution to show that $G(t) - R_* \le \epsilon$ for all $t \ge \tilde{s}_{n_2}$ with $n_2 = n_2(\epsilon)$ large enough. Hence (2.59) holds for $t \ge T := \max\{s_{n_1}, \tilde{s}_{n_2}\}$. For completeness, the detailed constructions of the above mentioned upper and lower solutions are given in the Appendix at the end of the section. □

2.4.3. *Convergence of h and u.*

Lemma 2.4.7. *There exist a constant $C > 0$ and a function $\xi \in C^1(\mathbb{R}^1_+)$ such that $|\xi(t)| \le C$ for all $t > 0$,*

$$\lim_{t\to\infty} \left\{h(t) - \left[c_0^* t - c_N \log t + \xi(t)\right]\right\} = 0, \ \lim_{t\to\infty} \xi'(t) = 0,$$

and

$$\lim_{t\to\infty} \|u(t,\cdot) - q_{c_0^*}(h(t) - \cdot)\|_{L^\infty([0,h(t)])} = 0.$$

Proof. By Lemmas 2.4.2 and 2.4.6, we find that for any sequence $t_n \to \infty$, by passing to a subsequence, $h(t+t_n) - k(t+t_n) \to G_0$ in $C^{1+\frac{\alpha}{2}}_{loc}(\mathbb{R}^1)$. Hence $h'(t+t_n) \to c_0^*$ in $C^{\alpha/2}_{loc}(\mathbb{R}^1)$.

We now define

$$U(t,r) = u(t, r + h(t)) \ \text{ for } t > 0, \ r \in [-h(t), 0],$$

and

$$U_n(t,r) = U(t+t_n, r), \ h_n(t) = h(t+t_n).$$

It is easily checked that

$$\begin{cases} (U_n)_t - \left[h_n'(t) + \frac{N-1}{r+h_n(t)}\right](U_n)_r - (U_n)_{rr} = f(U_n), & t > -t_n, \ r \in (-h_n(t), 0], \\ U_n(t,0) = 0, \ (U_n)_r(t,0) = -h_n'(t)/\mu_0, & t > -t_n. \end{cases} \tag{2.60}$$

By the same reasoning as in the proof of Lemma 2.4.2, we can use the parabolic regularity to (2.60) plus Sobolev embedding to conclude that, by passing to a further subsequence, as $n \to \infty$,

$$U_n \to U \text{ in } C^{\frac{1+\alpha}{2}, 1+\alpha}_{loc}(\mathbb{R}^1 \times (-\infty, 0]),$$

and U satisfies, in view of $h_n'(t) \to c_0^*$,

$$\begin{cases} U_t - c_0^* U_r - U_{rr} = f(U), & t \in \mathbb{R}^1, \ r \in (-\infty, 0], \\ U(t,0) = 0, \ U_r(t,0) = -c_0^*/\mu_0, & t \in \mathbb{R}^1. \end{cases}$$

This is equivalent to (2.49) with $V = U$ and $G = 0$. Hence we may repeat the argument in Lemmas 2.4.2-2.4.5 to conclude that

$$U(t,r) \equiv \phi(\mu_0, r) \text{ for } (t,r) \in \mathbb{R}^1 \times (-\infty, 0].$$

Thus we have proved that, as $n \to \infty$,

$$u(t+t_n, r + h(t+t_n)) - q_{c_0^*}(-r) \to 0 \text{ in } C^{\frac{1+\alpha}{2}, 1+\alpha}_{loc}(\mathbb{R}^1 \times (-\infty, 0]).$$

Since $\{t_n\}$ is an arbitrary sequence converging to ∞, this implies that

$$\lim_{t\to\infty} \left[u(t, r + h(t)) - q_{c_0^*}(-r)\right] = 0 \text{ uniformly for } r \text{ in compact subsets of } (-\infty, 0].$$

Therefore, for every $L>0$,

$$\lim_{t\to\infty}\|u(t,\cdot)-q_{c_0^*}(h(t)-\cdot)\|_{L^\infty([h(t)-L,h(t)])}=0. \tag{2.61}$$

Similarly, the arbitrariness of $\{t_n\}$ implies that $h'(t)\to c_0^*$ as $t\to\infty$. Hence

$$\xi(t):=h(t)-[c_0^*t-c_N\log t]$$

satisfies

$$\xi'(t)\to 0 \text{ as } t\to\infty.$$

The boundedness of $\xi(t)$ is a direct consequence of (2.25).

It remains to strengthen (2.61) to

$$\lim_{t\to\infty}\|u(t,\cdot)-q_{c_0^*}(h(t)-\cdot)\|_{L^\infty([0,h(t)])}=0.$$

Let $(\underline{v}(t,r),\underline{k}(t))$ be as in the proof of Lemma 2.3.4, so that (2.39), (2.40) and (2.41) hold. Since as $t\to\infty$, $h(t)\to\infty$ and $u(t,r)\to 1$ locally uniformly in $r\in[0,\infty)$, we can find $T_2>0$ such that

$$h(T_2)>\underline{k}(T),\ u(T_2,r)>\underline{v}(T,r) \text{ for } r\in[0,\underline{k}(T)].$$

We note that $\underline{v}(T,r)$ is a strictly decreasing function of r. We now choose a smooth function $\tilde u_0(r)$ such that

$$\tilde u_0'(0)=\tilde u_0(\tilde h_0)=0,\ \tilde u_0'(r)<0,\ u(T_2,r)>\tilde u_0(r) \text{ in } (0,\tilde h_0], \text{ and } \tilde u_0(r)>\underline{v}(T,r) \text{ in } (0,\underline{k}(T)),$$

where $\tilde h_0\in(\underline{k}(T),h(T_2))$. We next consider the auxiliary problem

$$\begin{cases} u_t=u_{rr}+\frac{N-1}{r}u_r+f(u), & 0<r<h(t),\ t>0,\\ u(t,h(t))=0,\ h'(t)=-\mu_0u_r(t,h(t)), & t>0,\\ h(0)=\tilde h_0,\quad u(0,r)=\tilde u_0(r), & 0\le r\le\tilde h_0.\end{cases} \tag{2.62}$$

Let $(\tilde u,\tilde h)$ denote the unique solution of (2.62). By the comparison principle we have

$$h(t+T_2)\ge\tilde h(t),\ u(t+T_2,r)\ge\tilde u(t,r) \text{ for } t>0,\ r\in[0,\tilde h(t)].$$

Moreover, since $\tilde u_0'(r)<0$ we can use a reflection argument to show that $\tilde u_r(t,r)<0$ for $t>0$ and $r\in(0,\tilde h(t)]$. This reflection argument is similar in spirit to the well known moving plane argument used for elliptic problems. The idea is to treat (2.62) as an initial boundary value problem for $\tilde u=\tilde u(t,x)$ over the region $\Omega:=\{(t,x):t>0,|x|<\tilde h(t)\}$ in $\mathbb{R}^1\times\mathbb{R}^N$. For each point x_0 in the ball $\{|x|<\tilde h(t)\}$ but away from the origin, we consider a hyperplane H passing through x_0, which divides $\mathbb{R}^N$ into two half spaces H^- and H^+, where H^- denotes the half space that contains the origin. Denote $\Omega^+=\{(t,x)\in\Omega:x\in H^+\}$, and for each point $x\in H^+$, we denote by $x^*\in H^-$ its reflection in H, and define $\tilde u^*(t,x)=\tilde u(t,x^*)$ for $(t,x)\in\Omega^+$. Then on the parabolic boundary of Ω^+, $\tilde u-\tilde u^*\le 0$ but is not identically 0. We thus obtain by the maximum principle that $\tilde u-\tilde u^*\le 0$ in Ω^+ and strict inequality holds in the interior of Ω^+. Since $\tilde u(t,x_0)-\tilde u^*(t,x_0)=0$, we can apply the Hopf boundary lemma to conclude that

$$\partial_\nu\tilde u(t,x_0)=\frac{1}{2}\partial_\nu[\tilde u(t,x_0)-\tilde u^*(t,x_0)]<0,$$

where ν is a normal vector of H pointing away from the origin. The conclusion $\tilde u_r(t,r)<0$ is a simple consequence of this fact.

On the other hand, if T is large enough, our assumptions on $\tilde u(0,r)$ and $\tilde h(0)$ imply that spreading happens for $(\tilde u,\tilde h)$ (see [12]). Hence we can apply Lemma 2.3.4 to $(\tilde u,\tilde h)$ to conclude that there exist $\tilde T>0,\tilde T_1>0$ such that (2.42) holds when (u,h,T,T_1) there is replaced by $(\tilde u,\tilde h,\tilde T,\tilde T_1)$. We thus obtain

$$u(t+T_1+T_2,r)\ge\tilde u(t+T_1,r)\ge\underline{v}(t,r) \text{ for } r\in[\underline{k}(t)-M\log t,\underline{k}(t)] \text{ and } t\ge\tilde T.$$

It follows that

$$\liminf_{t\to\infty} \min_{r\in[0,h(t)-L]} u(t,r) \geq \liminf_{t\to\infty} \tilde{u}(t,h(t)-L) \geq \liminf_{t\to\infty} \underline{v}(t,h(t)-L) \geq \phi(\mu_0,-L+C).$$

Therefore, for any $\epsilon>0$ there exists $L_\epsilon>0$ large such that

$$u(t,r)\geq q_{c_0^*}(L_\epsilon - C)\geq 1-\epsilon \text{ for all } r\in[0,h(t)-L_\epsilon] \text{ and all large } t.$$

Since $q_{c_0^*}(r)<1$ is increasing in r, and by Lemma 2.3.2, $u(t,r)\leq 1+\epsilon$ for all large t, we deduce

$$|u(t,r)-q_{c_0^*}(h(t)-r)|\leq 2\epsilon \text{ for } r\in[0,h(t)-L_\epsilon] \text{ and all large } t.$$

We may now make use of (2.61) to obtain

$$\limsup_{t\to\infty} \|u(t,\cdot)-q_{c_0^*}(h(t)-\cdot)\|_{L^\infty([0,h(t)])} \leq \limsup_{t\to\infty} \|u(t,\cdot)-q_{c_0^*}(h(t)-\cdot)\|_{L^\infty([0,h(t)-L_\epsilon])} \leq 2\epsilon.$$

Since $\epsilon>0$ can be arbitrarily small, we obtain

$$\lim_{t\to\infty} \|u(t,\cdot)-q_{c_0^*}(h(t)-\cdot)\|_{L^\infty([0,h(t)])} = 0,$$

as we wanted. The proof is complete. □

2.4.4. *Improved convergence result for h.*

Lemma 2.4.8. *There exists $\hat{h}\in\mathbb{R}^1$ such that*

$$\lim_{t\to\infty}\left[h(t)-c_0^*t+c_N\log t\right]=\hat{h}.$$

Proof. By Lemma 2.4.7,

$$\xi(t)=h(t)-c_0^*t+c_N\log t\in[-C,C] \text{ for } t>0.$$

Set

$$\hat{h}=\liminf_{t\to\infty}\xi(t).$$

We will show that for any given small $\epsilon>0$,

$$\limsup_{t\to\infty}\xi(t)\leq \hat{h}+\epsilon. \tag{2.63}$$

The required conclusion clearly follows from (2.63).

We use a comparison argument to prove (2.63). Let $t_k\to\infty$ be chosen such that $\xi(t_k)\to\hat{h}$ as $k\to\infty$. For given small $\epsilon>0$, we define

$$\tilde{h}_k(t)=c_0^*(t+t_k)-c_N\log(t+t_k)+B\epsilon(1-e^{-\alpha t})+\hat{h}+\epsilon,\ t\geq 0,$$
$$\overline{u}_k(t,r)=\phi\big(\mu(c_0^*-c_N(t+t_k)^{-1}),r-\tilde{h}_k(t)\big)+\epsilon e^{-\alpha t},\ r\in[0,\tilde{h}_k(t)+\epsilon_0],$$

where α and B are positive constants to be determined later, and ϕ is given by (2.15), which is defined over $(-\infty,\epsilon_0]$. To simplify notations, we will write

$$\tilde{h}_k(t)=\tilde{h}(t),\ \overline{u}_k(t,r)=\overline{u}(t,r) \text{ unless their dependence on } k \text{ need to be stressed.}$$

We will choose α and B such that for all large k and small ϵ,

$$\limsup_{t\to\infty}\xi(t+t_k)\leq\hat{h}+C_0\epsilon,$$

where $C_0>0$ is a constant independent of ϵ and k. This clearly implies (2.63).

By definition, with the notation $\zeta=c_0^*-c_N(t+t_k)^{-1}$,

$$\overline{u}_r(t,r)=\phi_r(\mu(\zeta),r-\tilde{h}(t))<0 \text{ for } r\in[0,\tilde{h}(t)+\epsilon_0].$$

Moreover,

$$\overline{u}(t,\tilde{h}(t))=\phi(\mu(\zeta),0)+\epsilon e^{-\alpha t}>0\ \ (\forall t>0)$$

and

$$\overline{u}(t,\tilde{h}(t)+\epsilon_0)=\phi(\mu(\zeta),\epsilon_0)+\epsilon e^{-\alpha t}<0\ \ (\forall t>0)$$

provided that $\epsilon>0$ is small enough. Hence for such ϵ, there exists a unique $\overline{h}(t)=\overline{h}_k(t)\in(\tilde{h}(t),\tilde{h}(t)+\epsilon_0)$ such that

$$\overline{u}(t,\overline{h}(t))=0\ \ (\forall t>0).$$

Moreover, we could replace ϵ_0 by $C\epsilon$ with $C>0$ sufficiently large to conclude that $\overline{h}(t)<\tilde{h}(t)+C\epsilon$, and we can apply the implicit function theorem to conclude that $t\to\overline{h}(t)$ is a smooth function.

By the mean value theorem we have

$$\overline{u}(t,\overline{h}(t))-\overline{u}(t,\tilde{h}(t))=\big[\phi_r(\mu_0,0)+o_{\epsilon,k}(1)\big]\big[\overline{h}(t)-\tilde{h}(t)\big]=-\epsilon e^{-\alpha t}\ \ (\forall t>0),$$

where $o_{\epsilon,k}(1)\to 0$ as $\epsilon\to 0$ and $k\to\infty$, uniformly in $t>0$. It follows that

$$\overline{h}(t)-\tilde{h}(t)=\left[\frac{\mu_0}{c_0^*}+o_{\epsilon,k}(1)\right]\epsilon e^{-\alpha t}\ \ (\forall t>0). \tag{2.64}$$

Using $\frac{d}{dt}\overline{u}(t,\overline{h}(t))=0$ we deduce

$$\phi_\mu\cdot\mu'\cdot c_N(t+t_k)^{-2}+\phi_r\cdot\big[\overline{h}'(t)-\tilde{h}'(t)\big]-\alpha\epsilon e^{-\alpha t}=0.$$

Since $\phi_\mu\cdot\mu'>0$, it follows that

$$\begin{aligned}\overline{h}'(t)&>\tilde{h}'(t)+[\phi_r]^{-1}\alpha\epsilon e^{-\alpha t}\\ &=c_0^*-c_N(t+t_k)^{-1}+\alpha B\epsilon e^{-\alpha t}-\left[\frac{\mu_0}{c_0^*}+o_{\epsilon,k}(1)\right]\alpha\epsilon e^{-\alpha t}\\ &=c_0^*-c_N(t+t_k)^{-1}+\left[B-\frac{\mu_0}{c_0^*}+o_{\epsilon,k}(1)\right]\alpha\epsilon e^{-\alpha t}\ \ (\forall t>0).\end{aligned}$$

On the other hand, for all large k and small ϵ, we have

$$\begin{aligned}\overline{u}_r(t,\overline{h}(t))&=\phi_r(\mu(\zeta),\overline{h}(t)-\tilde{h}(t))\\ &=\phi_r(\mu(\zeta),0)+\big[\phi_{rr}(\mu_0,0)+o_{\epsilon,k}(1)\big]\big[\overline{h}(t)-\tilde{h}(t)\big]\\ &>-\frac{1}{\mu_0}\big[c_0^*-c_N(t+t_k)^{-1}\big]\ \ (\forall t>0)\end{aligned}$$

since $\phi_{rr}(\mu_0,0)=-c_0^*\phi_r(\mu_0,0)=(c_0^*)^2/\mu_0>0$. Therefore if we choose $B>\frac{\mu_0}{c_0^*}$, then for all large k and small ϵ,

$$\overline{h}'(t)>-\mu_0\overline{u}_r(t,\overline{h}(t))\ \ (\forall t>0). \tag{2.65}$$

Next we prove that by choosing α suitably small and enlarging B accordingly, we have

$$\overline{u}_t-\overline{u}_{rr}-\frac{N-1}{r}\overline{u}_r-f(\overline{u})>0 \text{ for } t>0,\ r\in(0,\overline{h}(t)] \tag{2.66}$$

and all large k and small ϵ.

We calculate

$$\begin{aligned}\overline{u}_t&=\phi_\mu\cdot\mu'\cdot c_N(t+t_k)^{-2}-\phi_r\cdot\tilde{h}'(t)-\epsilon\alpha e^{-\alpha t}\\ &>-\phi_r\big[c_0^*-c_N(t+t_k)^{-1}+B\epsilon\alpha e^{-\alpha t}\big]-\epsilon\alpha e^{-\alpha t}.\end{aligned}$$

Hence

$$\begin{aligned}
&\overline{u}_t - \overline{u}_{rr} - \frac{N-1}{r}\overline{u}_r - f(\overline{u})\\
&> -\phi_r\big[c_0^* - c_N(t+t_k)^{-1} + B\epsilon\alpha e^{-\alpha t} + \frac{N-1}{r}\big] - \phi_{rr} - f(\phi+\epsilon e^{-\alpha t}) - \epsilon\alpha e^{-\alpha t}\\
&= -\phi_r\tilde{J} + f(\phi) - f(\phi+\epsilon e^{-\alpha t}) - \epsilon\alpha e^{-\alpha t},
\end{aligned}$$

where

$$\tilde{J} := c_0^* - g(c_0^* - c_N(t+t_k)^{-1}) - c_N(t+t_k)^{-1} + B\epsilon\alpha e^{-\alpha t} + \frac{N-1}{r}.$$

For $r \in (0, \overline{h}(t)]$, we have

$$\begin{aligned}
\frac{N-1}{r} \geq \frac{N-1}{\overline{h}(t)} &= \frac{N-1}{\tilde{h}(t) + o_{\epsilon,k}(1)}\\
&= \frac{N-1}{c_0^*(t+t_k) - c_N\log(t+t_k) + \hat{h} + o_{\epsilon,k}(1)}\\
&= \frac{N-1}{c_0^*(t+t_k)} + \frac{(N-1)c_N\log(t+t_k)}{c_0^{*2}(t+t_k)^2}\big[1+o_{\epsilon,k}(1)\big].
\end{aligned}$$

Moreover,

$$c_0^* - g(c_0^* - c_N(t+t_k)^{-1}) = g'(c_0^*)c_N(t+t_k)^{-1} + O_k\big[(t+t_k)^{-2}\big].$$

Therefore,

$$\begin{aligned}
\tilde{J} &\geq \left\{c_N[g'(c_0^*)-1] + \frac{N-1}{c_0^*}\right\}(t+t_k)^{-1} + \frac{(N-1)c_N\log(t+t_k)}{c_0^{*2}(t+t_k)^2}\big[1+o_{\epsilon,k}(1)\big] + B\epsilon\alpha e^{-\alpha t}\\
&= \frac{(N-1)c_N\log(t+t_k)}{c_0^{*2}(t+t_k)^2}\big[1+o_{\epsilon,k}(1)\big] + B\epsilon\alpha e^{-\alpha t}\\
&> B\epsilon\alpha e^{-\alpha t} \quad (\forall t>0)
\end{aligned}$$

for all large k and small ϵ.

Choose $\delta_0 > 0$ small so that $f'(u) \leq -\sigma_0 < 0$ for $u \in [1-\delta_0, 1+\delta_0]$. Then for $\phi \in [1-\delta_0, 1)$ we have

$$f(\phi) - f(\phi+\epsilon e^{-\alpha t}) \geq \sigma_0\epsilon e^{-\alpha t}.$$

Thus for all large k and small ϵ and

$$(t,r) \in \Omega^1_{\epsilon,k} := \big\{(t,r) : \phi(\mu(c_0^* - c_N(t+t_k)^{-1}), r - \tilde{h}(t)) \in [1-\delta_0, 1)\big\},$$

we have

$$-\phi_r\tilde{J} + f(\phi) - f(\phi+\epsilon e^{-\alpha t}) - \epsilon\alpha e^{-\alpha t} \geq (\sigma_0 - \alpha)\epsilon e^{-\alpha t} > 0$$

provided that we take $\alpha = \sigma_0/2$.

For $\phi \in (0, 1-\delta_0)$, there exists $\sigma_1 > 0$ such that $\phi_r \leq -\sigma_1$; moreover, for all small ϵ,

$$f(\phi) - f(\phi+\epsilon e^{-\alpha t}) \geq -\sigma_2\epsilon e^{-\alpha t},$$

where $\sigma_2 = \max_{u\in[0,1]}|f'(u)|$. Therefore for all large k, small ϵ, and

$$(t,r) \in \Omega^2_{\epsilon,k} := \big\{(t,r) : \phi(\mu(c_0^* - c_N(t+t_k)^{-1}), r - \tilde{h}(t)) \in (0, 1-\delta_0)\big\},$$

we have

$$\begin{aligned}
&-\phi_r\tilde{J} + f(\phi) - f(\phi+\epsilon e^{-\alpha t}) - \epsilon\alpha e^{-\alpha t}\\
&\geq \sigma_1 B\epsilon\alpha e^{-\alpha t} - (\sigma_2+\alpha)\epsilon e^{-\alpha t}\\
&= (\sigma_1 B\alpha - \sigma_2 - \alpha)\epsilon e^{-\alpha t} > 0
\end{aligned}$$

provided that $\sigma_1 B\alpha > \sigma_2 + \alpha$. With $\alpha = \sigma_0/2$, this is achieved by taking $B \geq \frac{4\sigma_2+2\sigma_0}{\sigma_1\sigma_0}$. This proves that (2.66) holds for all large k and small ϵ.

We show below that for all large k and small ϵ,

$$h(t_k) < \overline{h}_k(0),\ u(t_k,r) \leq \overline{u}_k(0,r) \text{ for } r \in [0, h(t_k)]. \tag{2.67}$$

Since

$$h(t_k) - \tilde{h}_k(0) = \xi(t_k) - \hat{h} - \epsilon \to -\epsilon \text{ as } k \to \infty,$$

we have, in view of (2.64),

$$h(t_k) < \tilde{h}_k(0) < \overline{h}_k(0)$$

for all large k, say $k \geq k_1(\epsilon)$, and all small ϵ.

By Lemma 2.4.7,

$$\lim_{k\to\infty} \|u(t_k,\cdot) - \phi(\mu_0, \cdot - h(t_k))\|_{L^\infty([0,h(t_k)])} = 0.$$

Since

$$\mu(c_0^* - c_N t_k^{-1}) \to \mu_0,\ h(t_k) - \tilde{h}_k(0) + \epsilon \to 0 \text{ as } k \to \infty,$$

we deduce

$$\|u(t_k,\cdot) - \phi(\mu(c_0^* - c_N t_k^{-1}), \cdot - \tilde{h}_k(0) + \epsilon)\|_{L^\infty([0,h(t_k)])} \to 0 \text{ as } k \to \infty.$$

Therefore there exists $k_2(\epsilon) \geq k_1(\epsilon)$ such that for $k \geq k_2(\epsilon)$,

$$\begin{aligned} u(t_k,r) &\leq \phi(\mu(c_0^* - c_N t_k^{-1}), r - \tilde{h}_k(0) + \epsilon) + \epsilon \\ &< \phi(\mu(c_0^* - c_N t_k^{-1}), r - \tilde{h}_k(0)) + \epsilon = \overline{u}_k(0,r) \quad (\forall r \in [0, h(t_k)]). \end{aligned}$$

Thus (2.67) holds for all small ϵ and $k \geq k_2(\epsilon)$. By enlarging $k_2(\epsilon)$ if necessary we may assume that (2.65) and (2.66) both hold for $k \geq k_2(\epsilon)$ and all small $\epsilon > 0$.

In view of (2.65), (2.66), (2.67) and the fact that $\overline{u}_r(t,0) < 0,\ u_r(t,0) = 0$, we can use a standard comparison argument to conclude that

$$h(t+t_k) \leq \overline{h}(t),\ u(t_k+t,r) \leq \overline{u}(t,r) \quad (\forall t > 0,\ \forall r \in [0, h(t_k+t)])$$

for all small $\epsilon > 0$ and $k \geq k_2(\epsilon)$. It follows that

$$\begin{aligned} \xi(t+t_k) &= h(t+t_k) - \tilde{h}(t) + B\epsilon(1 - e^{-\alpha t}) + \hat{h} + \epsilon \\ &= h(t+t_k) - \overline{h}(t) - \left[\frac{\mu_0}{c_0^*} + o_{\epsilon,k}(1)\right] \epsilon e^{-\alpha t} + B\epsilon(1 - e^{-\alpha t}) + \hat{h} + \epsilon \\ &\leq -\left[\frac{\mu_0}{c_0^*} + o_{\epsilon,k}(1)\right] \epsilon e^{-\alpha t} + B\epsilon(1 - e^{-\alpha t}) + \hat{h} + \epsilon \\ &\to \hat{h} + (B+1)\epsilon \ \text{ as } t \to \infty. \end{aligned}$$

Therefore

$$\limsup_{t\to\infty} \xi(t) \leq \hat{h} + (B+1)\epsilon,$$

as we wanted. This completes the proof. □

2.5. **Appendix: Further details for the proof of Lemma 2.4.6.**
For completeness, we give the detailed proof of the facts that for any given $\epsilon > 0$, there exists $n_1 = n_1(\epsilon)$ and $n_2 = n_2(\epsilon)$ such that

$$G(t) - R^* \geq -\epsilon \ (\forall t \geq s_{n_1}),\ G(t) - R_* \leq \epsilon \ (\forall t \geq \tilde{s}_{n_2}).$$

From the inequalities

$$\phi(r - R_*) \leq V(t,r) \leq \phi(r - R^*)$$

we have

$$|1 - V(t,r)| \leq Ce^{\beta r}$$

for some $C > 0$ and β >0. Therefore, for any $\varepsilon > 0$, there exists $K > 0$ and $T > 0$ such that

$$\sup_{r\in(-\infty,-K]} |V(\tilde{s}_n, r) - \phi(r - R_*)| < \varepsilon. \tag{2.68}$$

for $\tilde{s}_n > T$. Let $H(t) = G(t) + c_0^* t$, $W(t,r) = V(t, r - c_0^* t)$. (W, H) satisfies

$$\begin{cases} W_t - W_{rr} = f(W), t \in \mathbb{R}^1, r \leq H(t) \\ W(t, H(t)) = 0, H'(t) = -\mu_0 W_r(t, H(t)) \end{cases} \tag{2.69}$$

By Lemma 2.4.5 and (2.68), there exists $n_1 = n_1(\varepsilon)$ such that, for $n \geq n_1$,

$$G(\tilde{s}_n) \leq R_* + \varepsilon \tag{2.70}$$

$$V(\tilde{s}_n, r) \leq \phi(r - R_* - \varepsilon) + \varepsilon \ \text{ for } \ r \leq R_*. \tag{2.71}$$

We note that we can find $N > 1$ independent of $\varepsilon > 0$ such that

$$\phi(r - R_* - \varepsilon) + \varepsilon \leq (1 + N\varepsilon)\phi(r - R_* - N\varepsilon) \ \text{ for } \ r \leq R_* + \varepsilon. \tag{2.72}$$

Next we remark that for any $\delta \in (0, -f'(1))$ there exists $\eta > 0$ such that

$$\begin{cases} \delta \leq -f'(u) & \text{for} \quad 1 - \eta \leq u \leq 1 + \eta, \\ f(u) \geq 0 & \text{for} \quad 1 - \eta \leq u \leq 1. \end{cases}$$

Let us define an upper solution for problem (2.69) as follows:

$$\overline{H}(t) := R_* + N\varepsilon + c_0^* t + N\varepsilon\sigma(1 - e^{-\delta(t - \tilde{s}_n)})$$
$$\overline{W}(t,r) := (1 + N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi(r - \overline{H}(t))$$

Since $\lim_{r\to-\infty} \overline{W}(t,r) > 1$, there exists a smooth function $\overline{K}(t)$ of $t \geq \tilde{s}_n$ such that $\overline{K}(t) \to -\infty$ as $t \to \infty$ and $\overline{W}(t, \overline{K}(t)) > 1$. We will check that the triple $(\overline{W}, \overline{H}, \overline{K})$ is an upper solution for $t \geq \tilde{s}_n$, that is,

$$\overline{W}_t - \overline{W}_{rr} \geq f(\overline{W}) \text{ for } t > \tilde{s}_n, r \in [\overline{K}(t), \overline{H}(t)] \tag{2.73}$$

$$\overline{W}(t, \overline{K}(t)) \geq W(t, \overline{K}(t)) \text{ for } t \geq \tilde{s}_n, \tag{2.74}$$

$$\overline{W}(t, \overline{H}(t)) = 0, \overline{H}'(t) \geq -\mu_0 \overline{W}_r(t, \overline{H}(t)) \ \text{ for } t \geq \tilde{s}_n, \tag{2.75}$$

$$H(\tilde{s}_n) \leq \overline{H}(\tilde{s}_n),\ W(\tilde{s}_n, r) \leq \overline{W}(\tilde{s}_n, r) \ \text{ for } \ r \in [\overline{K}(\tilde{s}_n), H(\tilde{s}_n)]. \tag{2.76}$$

From (2.70) we have

$$H(\tilde{s}_n) = G(\tilde{s}_n) + c_0^* \tilde{s}_n \leq R_* + N\varepsilon + c_0^* \tilde{s}_n = \overline{H}(\tilde{s}_n).$$

We also have, in view of (2.71),

$$\begin{aligned}\overline{W}(\tilde{s}_n, r) &= (1+N\varepsilon)\phi(r-\overline{H}(\tilde{s}_n))\\&= (1+N\varepsilon)\phi(r-R_*-N\varepsilon-c_0^*\tilde{s}_n)\\&\geq \phi(r-R_*-\varepsilon-c_0^*\tilde{s}_n)+\varepsilon\\&\geq V(\tilde{s}_n, r-c_0^*\tilde{s}_n) = W(\tilde{s}_n, r)\end{aligned}$$

for $r \leq H(\tilde{s}_n)$. Thus (2.76) holds.

We next show (2.75). By definition $\overline{W}(t,\overline{H}(t)) = 0$ and direct calculation gives

$$\begin{aligned}&\overline{H}'(t) = c_0^* + N\varepsilon\sigma\delta e^{-\delta(t-\tilde{s}_n)},\\&-\mu_0\overline{W}_r(t,\overline{H}(t)) = c_0^* + N\varepsilon c_0^* e^{-\delta(t-\tilde{s}_n)}.\end{aligned}$$

Hence if we take $\sigma > 0$ so that $c_0^* \leq \sigma\delta$ then

$$\overline{H}'(t) \geq -\mu_0\overline{W}_r(t,\overline{H}(t)).$$

This proves (2.75).

Since $W \leq 1$, by the definition of $\overline{K}(t)$, (2.74) clearly holds. Finally we show (2.73). Put $z = r - \overline{H}(t)$. Since

$$\begin{aligned}\overline{W}_t &= -\delta N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z) - (1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\overline{H}'(t)\phi'(z)\\&= -\delta N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z) - (1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})(c_0^*+\sigma N\varepsilon\delta e^{-\delta(t-\tilde{s}_n)})\phi'(z),\end{aligned}$$

and

$$\overline{W}_{rr} = (1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi''(z),$$

we have

$$\begin{aligned}&\overline{W}_t - \overline{W}_{rr} - f(\overline{W})\\&= -\delta N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z) - (1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})(c_0^*+\sigma N\varepsilon\delta e^{-\delta(t-\tilde{s}_n)})\phi'(z)\\&\qquad - (1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi''(z) - f((1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi(z))\\&= -\delta N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z) + (1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\{-\phi''(z) - c_0^*\phi'(z)\}\\&\qquad - \sigma N\varepsilon\delta(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})e^{-\delta(t-\tilde{s}_n)}\phi'(z)\\&= -\delta N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z) - \sigma N\varepsilon\delta e^{-\delta(t-\tilde{s}_n)}(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi'(z)\\&\qquad + (1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})f(\phi(z)) - f((1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi(z)).\end{aligned}$$

Now we consider the term $(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})f(\phi(z)) - f((1+N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z))$. Denote

$$F(\xi,u) := (1+\xi)f(u) - f((1+\xi)u).$$

The mean value theorem yields

$$F(\xi,u) = \xi f(u) + f(u) - f((1+\xi)u) = \xi f(u) - \xi f'(u+\theta_{\xi,u}\xi u)u$$

for some $\theta_{\xi,u} \in (0,1)$. Since $\phi(z) \to 1$ as $z \to -\infty$, there exists $z_\eta < 0$ such that $\phi(z) \geq 1-\eta$ for $z \leq z_\eta$.

For $r - \overline{H}(t) \le z_\eta$, we have

$$
\begin{aligned}
&\overline{W}_t - \overline{W}_{rr} - f(\overline{W}) \\
= &- \delta N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z) - \sigma N\varepsilon\delta e^{-\delta(t-\tilde{s}_n)}(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi'(z) + F(N\varepsilon e^{-\delta(t-\tilde{s}_n)}, \phi(z)) \\
= &- \sigma N\varepsilon\delta e^{-\delta(t-\tilde{s}_n)}(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi'(z) + N\varepsilon e^{-\delta(t-\tilde{s}_n)} f(\phi(z)) \\
&+ N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z)\Big\{ - f'\big(\phi(z) + \theta' N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z)\big) - \delta\Big\} \\
\ge &\; 0,
\end{aligned}
$$

where $\theta' = \theta'(t,z) \in (0,1)$. We note that by shrinking ε we can guarantee that $N\varepsilon < \eta$ and so $1 + N\varepsilon e^{-\delta(t-\tilde{s}_n)} \le 1+\eta$ for $t \ge \tilde{s}_n$.

On the other hand for $z_\eta \le r - \overline{H}(t) \le 0$, we obtain

$$
\begin{aligned}
&\overline{W}_t - \overline{W}_{rr} - f(\overline{W}) \\
&= N\varepsilon e^{-\delta(t-\tilde{s}_n)} f(\phi(z)) - \sigma N\varepsilon\delta e^{-\delta(t-\tilde{s}_n)}(1+N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi'(z) \\
&\quad + N\varepsilon e^{-\delta(t-\tilde{s}_n)} \Big\{ -f'\big(\phi(z) + \theta' N\varepsilon e^{-\delta(t-\tilde{s}_n)}\phi(z)\big) - \delta\Big\}\phi(z) \\
&\ge N\varepsilon e^{-\delta(t-\tilde{s}_n)} \min_{0\le s\le 1} f(s) + \sigma\delta N\varepsilon e^{-\delta(t-\tilde{s}_n)} Q_\eta - N\varepsilon e^{-\delta(t-\tilde{s}_n)} \left(\max_{0\le s\le 1+\eta} f'(s) + \delta \right) \\
&= N\varepsilon e^{-\delta(t-\tilde{s}_n)} \left\{ \min_{0\le s\le 1} f(s) - \max_{0\le s\le 1+\eta} f'(s) - \delta + \sigma\delta Q_\eta \right\} \\
&\ge 0,
\end{aligned}
$$

where $Q_\eta := \min_{z_\eta \le z \le 0} |\phi'(z)| > 0$ provided that σ is large positive. Thus $\overline{W}_t - \overline{W}_{rr} - f(\overline{W}) \ge 0$ for sufficiently large $\sigma > 0$.

We may now apply the comparison principle to conclude that

$$W(t,r) \le \overline{W}(t,r), \quad H(t) \le \overline{H}(t) \;\; \text{for } t \ge \tilde{s}_n \text{ and } r \in (\overline{K}(t), H(t)],$$

in particular

$$G(t) \le R_* + N\varepsilon(\sigma+1)$$

for $t \ge \tilde{s}_n$. By shrinking ε we obtain

$$G(t) \le R_* + \epsilon$$

for $t \ge \tilde{s}_n$ and $n \ge n_1$.

Next we show $G(t) \ge R^* - \epsilon$ for all large $t > 0$. As in the construction of upper solution, for any $\varepsilon > 0$, there exists $n_2 = n_2(\varepsilon)$ such that, for $n \ge n_2$,

$$R^* - \varepsilon \le G(s_n), \tag{2.77}$$

$$\phi(r - R^* + \varepsilon) - \varepsilon \le V(s_n, r) \;\; \text{for} \;\; r \le R^* - \varepsilon. \tag{2.78}$$

We note that we can find $N > 1$ which does not depend on $\varepsilon > 0$ such that

$$(1 - N\varepsilon)\phi(r - R^* + N\varepsilon) \le \phi(r - R^* + \varepsilon) - \varepsilon \;\; \text{for} \;\; r \le R^* - \varepsilon.$$

Now we define a lower solution as follows:

$$
\begin{aligned}
\underline{H}(t) &:= R^* - N\varepsilon + c_0^* t - N\varepsilon\sigma(1 - e^{-\delta(t-s_n)}), \\
\underline{W}(t,r) &:= (1 - N\varepsilon e^{-\delta(t-s_n)})\phi(r - \underline{H}(t)).
\end{aligned}
$$

Since $V(t,r) \geq \phi(r - R_*)$, there exists $C > 0$ and $\beta > 0$ such that V satisfies $V(t,r) \geq 1 - Ce^{\beta r}$ for all $r \leq 0$, that is, W satisfies

$$W(t,r) \geq 1 - Ce^{\beta(r - c_0^* t)}.$$

We fix $c > 0$ so that $\delta \leq \beta(c + c_0^*)$. By enlarging n we may assume that $C \leq N\varepsilon e^{\delta s_n}$. Let $\underline{K}(t) \equiv -ct$. We will check that the triple $(\underline{W}, \underline{H}, \underline{K})$ is a lower solution for $t \geq s_n$, that is,

$$\underline{W}_t - \underline{W}_{rr} \leq f(\underline{W}) \text{ for } t > s_n, r \in [\underline{K}(t), \underline{H}(t)] \tag{2.79}$$

$$\underline{W}(t, \underline{K}(t)) \leq W(t, \underline{K}(t)) \text{ for } t \geq s_n, \tag{2.80}$$

$$\underline{W}(t, \underline{H}(t)) = 0, \underline{H}'(t) \geq -\mu_0 \underline{W}_r(t, \underline{H}(t)) \quad \text{for } t \geq s_n, \tag{2.81}$$

$$\underline{H}(s_n) \leq H(s_n),\ W(s_n, r) \leq \underline{W}(s_n, r) \quad \text{for} \quad r \in [\underline{K}(s_n), H(s_n)]. \tag{2.82}$$

From (2.77) we have

$$\underline{H}(s_n) = R^* - N\varepsilon + c_0^* s_n \leq R^* - \varepsilon + c_0^* s_n \leq G(s_n) + c_0^* s_n = H(s_n)$$

We also have

$$\begin{aligned}
\underline{W}(s_n, r) &= (1 - N\varepsilon)\phi(r - \underline{H}(s_n)) \\
&= (1 - N\varepsilon)\phi(r - R^* + N\varepsilon - c_0^* s_n) \\
&\leq \phi(r - R^* + \varepsilon - c_0^* \tilde{s}_n) - \varepsilon \\
&\leq V(s_n, r - c_0^* s_n) = W(s_n, r)
\end{aligned}$$

for $r \leq \underline{H}(s_n)$. Hence (2.82) holds.

We next show (2.81). By definition $\underline{W}(t, \underline{H}(t)) = 0$, and direct calculation gives

$$\begin{aligned}
&\underline{H}'(t) = c_0^* - N\varepsilon\sigma\delta e^{-\delta(t - s_n)}, \\
&- \mu_0 \underline{W}_r(t, \underline{H}(t)) = c_0^* - N\varepsilon c_0^* e^{-\delta(t - s_n)}.
\end{aligned}$$

Hence if we take $\sigma > 0$ so that $c_0^* \leq \sigma\delta$ then

$$\underline{H}'(t) \leq -\mu_0 \underline{W}_r(t, \underline{H}(t)).$$

This proves (2.81).

For $t \geq s_n$, we have

$$\begin{aligned}
\underline{W}(t, \underline{K}(t)) = \underline{W}(t, -ct) &\leq (1 - N\varepsilon e^{-\delta(t - s_n)}) \\
&= 1 - N\varepsilon e^{\delta s_n} e^{-\delta t} \leq 1 - Ce^{-\delta t} \\
&\leq 1 - Ce^{-\beta(c + c_0^*)t} \leq W(t, -ct) = W(t, \underline{K}(t)).
\end{aligned}$$

Hence (2.80) holds.

Finally we show (2.79). Put $\zeta = r - \underline{H}(t)$. Since

$$\begin{aligned}
\underline{W}_t &= \delta N\varepsilon e^{-\delta(t - s_n)}\phi(z) - (1 - N\varepsilon e^{-\delta(t - s_n)})\underline{H}'(t)\phi'(\zeta) \\
&= \delta N\varepsilon e^{-\delta(t - s_n)}\phi(z) - (1 - N\varepsilon e^{-\delta(t - s_n)})(c_0^* - \sigma N\varepsilon\delta e^{-\delta(t - s_n)})\phi'(\zeta),
\end{aligned}$$

and

$$\underline{W}_{rr} = (1 - N\varepsilon e^{-\delta(t - s_n)})\phi''(\zeta),$$

we have

$$
\begin{aligned}
&\underline{W}_t - \underline{W}_{rr} - f(\underline{W})\\
&= \delta N\varepsilon e^{-\delta(t-s_n)}\phi(\zeta) - (1 - N\varepsilon e^{-\delta(t-s_n)})(c_0^* - \sigma N\varepsilon\delta e^{-\delta(t-s_n)})\phi'(\zeta)\\
&\qquad - (1 - N\varepsilon e^{-\delta(t-s_n)})\phi''(\zeta) - f((1 - N\varepsilon e^{-\delta(t-s_n)}\phi(\zeta))\\
&= \delta N\varepsilon e^{-\delta(t-s_n)}\phi(\zeta) + (1 - N\varepsilon e^{-\delta(t-s_n)})\{-\phi''(z) - c_0^*\phi'(\zeta)\}\\
&\quad + \sigma N\varepsilon\delta e^{-\delta(t-s_n)}(1 - N\varepsilon e^{-\delta(t-s_n)})\phi'(\zeta)\\
&= \delta N\varepsilon e^{-\delta(t-s_n)}\phi(\zeta) + \sigma N\varepsilon\delta e^{-\delta(t-s_n)}(1 - N\varepsilon e^{-\delta(t-s_n)})\phi'(\zeta)\\
&\quad + (1 - N\varepsilon e^{-\delta(t-s_n)})f(\phi(\zeta)) - f((1 - N\varepsilon e^{-\delta(t-s_n)})\phi(\zeta))\\
&= \delta N\varepsilon e^{-\delta(t-s_n)}\phi(\zeta) + \sigma N\varepsilon\delta e^{-\delta(t-s_n)}(1 - N\varepsilon e^{-\delta(t-s_n)})\phi'(z) + F(-N\varepsilon e^{-\delta(t-s_n)}, \phi(\zeta)).
\end{aligned}
$$

Since $\phi(\zeta) \to 1$ as $\zeta \to -\infty$, there exists $\zeta_\eta < 0$ such that $\phi(\zeta) \geq 1-\eta/2$ for $\zeta \leq \zeta_\eta$. For $r-\underline{H}(t) \leq \zeta_\eta$, we have

$$
\begin{aligned}
&\underline{W}_t - \underline{W}_{rr} - f(\underline{W})\\
&= \delta N\varepsilon e^{-\delta(t-s_n)}\phi(z) + \sigma N\varepsilon\delta e^{-\delta(t-s_n)}(1 - N\varepsilon e^{-\delta(t-s_n)}))\phi'(\zeta)\\
&\qquad - N\varepsilon e^{-\delta(t-s_n)}\Big\{f(\phi(\zeta)) - f'(\phi(\zeta) - \theta'' N\varepsilon e^{-\delta(t-s_n)}\phi(\zeta))\phi(\zeta)\Big\}\\
&= \ - N\varepsilon e^{-\delta(t-s_n)}f(\phi(\zeta)) + \sigma N\varepsilon\delta e^{-\delta(t-s_n)}(1 - N\varepsilon e^{-\delta(t-s_n)})\phi'(\zeta)\\
&\qquad + N\varepsilon e^{-\delta(t-s_n)}\Big\{f'(\phi(\zeta) - \theta'' N\varepsilon e^{-\delta(t-s_n)}\phi(\zeta)) + \delta\Big\}\phi(\zeta)\\
&\leq 0,
\end{aligned}
$$

where $\theta'' = \theta''(t, z) \in (0, 1)$. We note that by shrinking ε we can guarantee that $N\varepsilon < \eta/2$ and so

$$1 \geq \phi(\zeta) - \theta'' N\varepsilon^{-\delta(t-s_n)}\phi(\zeta) \geq \phi(\zeta) - N\varepsilon e^{-\delta(t-s_n)}\phi(\zeta) \geq 1 - \eta.$$

On the other hand for $z_\eta \leq r - \overline{H}(t) \leq 0$ and $t \geq s_n$, we obtain

$$
\begin{aligned}
&\underline{W}_t - \underline{W}_{rr} - f(\underline{W})\\
&= \ - N\varepsilon e^{-\delta(t-s_n)}f(\phi(\zeta)) + \sigma N\varepsilon\delta e^{-\delta(t-\tilde{s}_n)}(1 - N\varepsilon e^{-\delta(t-\tilde{s}_n)})\phi'(\zeta)\\
&\qquad + N\varepsilon e^{-\delta(t-s_n)}\left\{f'\big(\phi(\zeta) - \theta'' N\varepsilon e^{-\delta(t-s_n)}\phi(\zeta)\big) + \delta\right\}\phi(\zeta)\\
&\leq \ - N\varepsilon e^{-\delta(t-s_n)}\min_{0\leq s\leq 1} f(s) + \sigma\delta N\varepsilon e^{-\delta(t-s_n)}(1 - N\varepsilon e^{-\delta(t-s_n)})\phi'(\zeta)\\
&\qquad + N\varepsilon e^{-\delta(t-\tilde{s}_n)}\left(\max_{0\leq s\leq 1+\eta} f'(s) + \delta\right)\\
&\leq N\varepsilon e^{-\delta(t-s_n)}\left\{-\min_{0\leq s\leq 1} f(s) + \max_{0\leq s\leq 1+\eta} f'(s) + \delta - \sigma\delta\left(1 - \frac{\eta}{2}\right)Q'_\eta\right\}\\
&\leq 0,
\end{aligned}
$$

by taking $\sigma > 0$ sufficiently large, where $Q'_\eta := \min_{\zeta_\eta\leq\zeta\leq 0}|\phi'(\zeta)| > 0$.

We may now apply the comparison principle to conclude that

$$\underline{W}(t, r) \leq W(t, r), \quad \underline{H}(t) \leq H(t) \quad \text{for } t \geq s_n \text{ and } r \in (-ct, \underline{H}(t)],$$

and in particular,

$$R^* - N\varepsilon(\sigma + 1) \leq G(t)$$

for $t \geq s_n$. By shrinking ε we obtain

$$R^* - \epsilon \leq G(t)$$

for $t \geq s_n$ and $n \geq n_2$. $\square$

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