The Euler Characteristic Formula for Logarithmic Minimal Degenerations of Surfaces with Kodaira Dimension Zero

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Abstract

In this paper, the Euler characteristic formula for projective logarithmic minimal degenerations of surfaces with Kodaira dimension zero over a 1-dimensional complex disk is proved under a reasonable assumption and as its application, we show that any degenerations of abelian or hyperelliptic surfaces have relatively projective log minimal models whose singular fibre has only V-normal crossing singularity and the possible local fundamental groups of the singular points of the total spaces of type II and III (in the generalized sense) degenerations are determined.

Contents

1 Introduction

Based on the 2-dimensional minimal model theory, Kodaira classified the singular fibres of degenerations of elliptic curves ([25], Theorem 6.2). It is quite natural that many people have been interested in the degenerations of surfaces with Kodaira dimension zero as a next problem. The first effort began by his student Iitaka and Ueno who studied the first kind degeneration (i.e., degeneration with the finite monodromy) of abelian surfaces with a principal polarization ([64] and [65]) while in that time, 3-dimensional minimal model theory had not been known. After Kulikov succeeded to construct the minimal models of degenerations of algebraic K3 surfaces in the analytic category from semistable degenerations and to classify their singular fibres, extension to the case of the other surfaces with Kodaira dimension zero has been done (see for example, [45], [36]). As for the non-semistable case, there are works due to Crauder and Morrison who classified triple point free degeneration ([8], [9]). 3-dimensional minimal model theory in the projective category was established by Mori ([37]) but we can not start studying the degenerations from minimal models because of their complexity while it has been known that log minimal models of degenerations of elliptic curves behaves nicely (see [49], (8.9) Added in Proof.). After the establishment of 3-dimensional log minimal model theory, we introduced the notion of a *logarithmic minimal degeneration* in [44] as a good intermediate model to a minimal model which acts like a "quotient" of minimal semistable degeneration by the transformation group induced from a semistable reduction. Of course, because of the non-uniqueness of minimal models, the transformation group does not act holomorphically on the total space in general.

Definition 1.1 Let X be a normal $\mathbf{Q}$-Gorenstein 3-fold with a relatively projective connected morphism $f : X \to \mathcal{D}$ to a unit disk $\mathcal{D} := \{z \in \boldsymbol{C}; |z| < 1\}$ which is smooth over $\mathcal{D}^* := \mathcal{D} \setminus \{0\}$ and put $\Theta := f^*(0)_{\text{red}}$. $f : X \to \mathcal{D}$ is called a *projective logarithmic (or abbreviated, log) minimal degeneration* of surfaces with Kodaira dimension zero if the following two conditions hold.

(1) $K_X + \Theta \sim_{\boldsymbol{Q}} 0$.

(2) A log 3-fold (X, Θ) has only divisorially log terminal singularity.

(3) Each component of Θ is $\mathbf{Q}$-Cartier.

We note that any degenerations of algebraic surfaces with Kodaira dimension zero over a 1-dimensional unit disk $\mathcal{D}$ are bimeromorphically equivalent to a projective log minimal degeneration. In fact, we can take a bimeromorphic model $g : Y \to \mathcal{D}$, where g is a relatively projective connected morphism from a smooth 3-fold Y such that each component of the singular fibre $g^*(0)$ is smooth and Supp $g^*(0)$ has only simple normal crossing singularity by the Hironaka's theorem ([19]). By the existence theorem of log minimal models established in [27], Theorem 1.4, [58], we can run the log minimal program with respect to $K_Y + g^*(0)_{\text{red}}$ starting from Y to get a log minimal model $f : X \to \mathcal{D}$ satisfying the conditions (2) and (3) as above. By the base point free theorem in [41], we infer that $K_X + \Theta \sim_{\boldsymbol{Q}} 0$.

Definition 1.2 Let G be a finite group and $\rho : G \to \mathrm{GL}(3, \boldsymbol{C}^3)$ be a faithful representation. Let $\boldsymbol{C}^3/(G, \rho)$ denote the quotient of $\boldsymbol{C}^3$ by the action of G defined by ρ. We assume that the quotient map $\boldsymbol{C}^3 \to \boldsymbol{C}^3/(G, \rho)$ is étale in codimension one. A pair (X, D) which consists of a normal complex analytic space X and a reduced divisor D on X is said to *have singularity of type* $V_1(G, \rho)$ (*resp.* $V_2(G, \rho)$) *at* $p \in X$ if there exists an analytic isomorphism $\varphi : (X, p) \to (\boldsymbol{C}^3/(G, \rho), 0)$ between germs and a hypersurface H in $\boldsymbol{C}^3$ defined by the equation $z = 0$ (resp. $xy = 0$), where x, y and z are semi-invariant coordinates of $\boldsymbol{C}^3$ at 0 such that $D = \varphi^*(H/(G, \rho))$. In particular, if G is cyclic with a generator $\sigma \in G$ and $(\rho(\sigma)^*x, \rho(\sigma)^*y, \rho(\sigma)^*z) = (\zeta^a x, \zeta^b y, \zeta^c z)$, where a, b, $c \in \boldsymbol{Z}$ and ζ is a primitive r-th root of unity for some coordinate x, y and z of $\boldsymbol{C}^3$ at 0, we shall use the notation $V_1(r; a, b, c)$ (resp. $V_2(r; a, b, c)$) instead of $V_1(G, \rho)$ (resp. $V_2(G, \rho)$).

Remark 1.1 We note that if (X, D) has singularity of type $V_i(G, \rho)$ at p, the local fundamental group at p of the singularity of X is isomorphic to G by its definition.

Let $\Theta = \sum_i \Theta_i$ be the irreducible decomposition and put $\Delta_i := \mathrm{Diff}_{\Theta_i}(\Theta - \Theta_i)$ for any i. For $p \in X$, let $d(p)$ be the number of irreducible components of Θ passing through $p \in X$. Then the following holds.

(a) For any i, Θ_i is normal, Δ_i is a standard boundary and (Θ_i, Δ_i) is log terminal (see [58], Lemma 3.6, (3.2.3) and Corollary 3.10).

(b) $d(p) \leq 3$.

(c) If $d(p) = 2$, (X, Θ) has singularity of type $V_2(r; a, b, 1)$ at p, where $r \in \boldsymbol{N}$, $a, b \in \boldsymbol{Z}$ and $(r, a, b) = 1$ (see [7], Theorem 16.15.2).

(d) If $d(p) = 3$, $p \in \Theta \subset X$ is analytically isomorphic to the germ of the origin $0 \in \{(x, y, z); xyz = 0\} \subset \boldsymbol{C}^3$ (see [7], Theorem 16.15.1).

(e) For any i and $p \in \Theta_i \setminus \mathrm{Supp}\, \Delta_i$, if Θ_i is smooth at p, then X is smooth at p (see [58], Corollary 3.7).

One of the aims of this paper is to give the following Euler characteristic formula for log minimal degenerations with $K_X + \Theta$ being Cartier. We note here that the study of log minimal degenerations of surfaces with Kodaira dimension zero reduces to this case by taking the log canonical cover with respect to $K_X + \Theta$ globally (see §6).

Theorem 1.1 *Let $f : X \to \mathcal{D}$ be a projective log minimal degeneration of surfaces with Kodaira dimension zero such that $K_X + \Theta$ is Cartier. Let $f^*(0) = \sum_i m_i \Theta_i$ be the irreducible decomposition. Then for $t \in \mathcal{D}^*$, the following formula holds.*

$$e_{\mathrm{top}}(X_t) = \sum m_i (e_{\mathrm{orb}}(\Theta_i \setminus \Delta_i) + \sum_{p \in \Theta_i \setminus \Delta_i} \delta_p(X, \Theta_i)),$$

where $X_t := f^*(t)$, $e_{\mathrm{orb}}(\Theta_i \setminus \Delta_i)$ is the orbifold Euler number of $\Theta_i \setminus \Delta_i$ and $\delta_p(X, \Theta_i)$ is the invariant of the singularity of the pair (X, Θ_i) at $p \in \Theta_i \setminus \Delta_i$ which is well defined and can be calculated explicitly as explained in the next section.

The above formula turns out to be quite useful for further study of degenerations. In fact, we apply the following corollary to the study on non-semistable degenerations of abelian or hyperelliptic surfaces.

Corollary 1.1 *Let notation and assumptions be as in Theorem 1.1. Assume that $e_{\mathrm{top}}(X_t) = 0$ for $t \in \mathcal{D}^*$. Then, for any i, we have $e_{\mathrm{orb}}(\Theta_i \setminus \Delta_i) = 0$ and for any $p \in \Theta_i \setminus \Delta_i$, (X, Θ) has only singularity of type $V_1(r; a, -a, 1)$ at p, where $(r, a) = 1$.*

Based on the result of Corollary 1.1, we shall prove the following theorem.

Theorem 1.2 *Let $f : X \to \mathcal{D}$ be a projective log minimal degeneration of abelian or hyperelliptic surfaces. Then the possible singularities of (X, Θ) at $p \in X$ are the following three types :*

(0) *X is smooth at $p \in X$ and Θ has only normal crossing singularity at p,*

(1) *(X,Θ) has singularity of type $V_2(r;a,b,1)$ at p, where $r \in \boldsymbol{N}$, $a,b \in \boldsymbol{Z}$ and $(r,a,b)=1$.*

(2) *(X,Θ) has singularity of type $V_1(G,\rho)$ at p.*

More precisely, if f is of type II, *we have $r = 2,3,4$ or 6 in (1), and $G \simeq \boldsymbol{Z}/n\boldsymbol{Z}$ or $\boldsymbol{Z}/2\boldsymbol{Z} \oplus \boldsymbol{Z}/n\boldsymbol{Z}$, where $n = 2,3,4$ or 6 in (2). The dual graph of Θ is a linear chain or a cycle. Moreover, there exists a projective bimeromorphic morphism $\psi : X \to X^\mu$ over $\mathcal{D}$ such that for the induced projective degeneration $f^\mu : X^\mu \to \mathcal{D}$, we have $K_{X^\mu} \sim_{\boldsymbol{Q}} 0$ and $f^{\mu *}(0) = m\Theta^\mu$ for some $m \in \boldsymbol{N}$, where $\Theta^\mu := \psi_*\Theta$. The possible types of singularity of (X^μ, Θ^μ) and the dual graph of the support of the singular fibre are the same as ones of (X,Θ) (but the components of the singular fiber may become non-normal). If f is of type* III, *we have $r = 2$ in (1), and (2) is reduced to the following three types.*

(III-2.1) *(X,Θ) has singularity of type $V_1(r;a,-a,1)$ at p, where $r = 2,3,4$ or 6, $a \in \boldsymbol{Z}$ and $(r,a) = 1$,*

(III-2.2) *(X,Θ) has singularity of type $V_1(2;1,0,1)$ at p,*

(III-2.3) *(X,Θ) has singularity of type $V_1(G,\rho)$ at p, where $G \simeq \boldsymbol{Z}/2\boldsymbol{Z} \oplus \boldsymbol{Z}/2\boldsymbol{Z}$ and letting $\{\sigma,\tau\}$ denote a set of generators,*

$$\rho(\sigma) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \rho(\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In particular, if f is of type III, *then X has only canonical quotient singularity.*

For the definition of types I, II and III, see Definition 5.2.

Problem 1.1 Let $f : X \to \mathcal{D}$ a projective log minimal degeneration of abelian or hyperelliptic surfaces of type III. Applying the log minimal program to f with respect to K_X, we see that there exists a projective bimeromorphic map $\psi : X - \to X^\mu$ over $\mathcal{D}$ such that for the induced projective degeneration $f^\mu : X^\mu \to \mathcal{D}$, we have $K_{X^\mu} \sim_{\boldsymbol{Q}} 0$ and $f^{\mu *}(0) = m\Theta^\mu$ for some $m \in \boldsymbol{N}$, where $\Theta^\mu := \psi_*\Theta$ and that X^μ has only canonical singularity but the possible types of singularity of (X^μ, Θ^μ) may differ from the ones of (X,Θ). So determination of the types of singularity of (X^μ,Θ^μ) remains to be done.

In §2, we define the invariant δ_p, and prove Riemann-Roch formula for divisors on singular 3-fold under some assumptions (Proposition 2.3) to prove Theorem 1.1 and Corollary 1.1. In §3, we classify type II and III log surfaces (for the definition, see Definition 3.4) under certain typical assumption which are supposed to appear canonically as the components of the singular fibres of the degeneration of surfaces with Kodaira dimension zero. In §4, we give a theory to calculate local fundamental groups seeing differents, which will be used to determine the singularity of the total spaces from the information of log surfaces obtained in the previous section. In §5, we prove Theorem 1.2 by using the results in the previous sections.

Notation and Conventions

In this paper, we shall use terminologies defined in [27] or [58], but let us fix again the notion of *log terminal* to avoid confusions. Let X be a normal log variety defined over an algebraically closed field or a normal Stein space or a germ of a normal complex analytic space with a point $p \in X$ with a Q-boundary Δ such that $K_X + D$ is $\boldsymbol{Q}$-Cartier. Take a projective resolution $\mu : Y \to X$ such that each components of the support of $\mu_*^{-1}\Delta + \sum_{i\in I} E_i$ are smooth and cross normally, where $\{E_i\}_{i\in I}$ is a set of all the exceptional divisors of μ and put $d_i := \mathrm{mult}_{E_i}(K_Y + \mu_*^{-1}\Delta + \sum_{i\in I} E_i - \mu^*(K_X+\Delta)) \in \boldsymbol{Q}$ for $i \in I$. The pair (X, Δ) is said to be *log terminal*, if all d_i are positive for some μ. log terminal pair (X, Δ) is said to be *divisorially log terminal*, if the exceptional loci of μ is purely one codimensional for some μ. (X, Δ) is said to be *purely log terminal*, if all d_i are positive for any μ.

In this paper, we shall use the following notation:

$\nu : X^\nu \to X$: The normalization of a scheme X.

$\mathrm{Diff}_{\Gamma^\nu}(\Delta)$: $\boldsymbol{Q}$-divisor which is called Shokurov's different satisfying

$$\nu^*(K_X + \Gamma + \Delta) = K_{\Gamma^\nu} + \mathrm{Diff}_{\Gamma^\nu}(\Delta),$$

where Γ is a reduced divisor on a normal variety X and $\Gamma + \Delta$ is a $\boldsymbol{Q}$-boundary on X such that $K_X + \Gamma + \Delta$ is $\boldsymbol{Q}$-Cartier. (see [58], §3, [7], §16).

Δ^Y : $\boldsymbol{Q}$-divisor on Y satisfying $K_Y + \Delta^Y = f^*(K_X + \Delta)$, where $f : Y \to X$ is a birational morphism between normal varieties and Δ is a $\boldsymbol{Q}$-boundary on X such that $K_X + \Delta$ is $\boldsymbol{Q}$-Cartier.

$\mathrm{ind}_p(D)$: The smallest positive integer r such that rD is Cartier on the germ of X at p, where D is a $\boldsymbol{Q}$-Cartier $\boldsymbol{Q}$-divisor on a normal variety or a normal complex analytic space X.

$\mathrm{Ind}(D)$: The smallest positive integer r such that $rD \sim 0$, where D is a $\boldsymbol{Q}$-Cartier $\boldsymbol{Q}$-divisor on a normal variety X such that $D \sim_{\boldsymbol{Q}} 0$.

$\mathrm{mult}_\Gamma D$: The coefficient of the irreducible decomposition of D at a prime divisor Γ, where D is a $\boldsymbol{Q}$-divisor on a normal variety X.

$\mathrm{Exc} f$: Exceptional loci of a birational morphism f.

$\sim$: Linear equivalence.

$\sim_{\boldsymbol{Q}}$: $\boldsymbol{Q}$-linear equivalence.

$\lceil \Delta \rceil$: Round up of a $\boldsymbol{Q}$-divisor Δ.

$\lfloor \Delta \rfloor$: Round down of a $\boldsymbol{Q}$-divisor Δ.

$\{\Delta\}$: Fractional part of the boundary Δ.

e_{top} : Topological Euler characteristic.

e_{orb} : Orbifold Euler number.

$\rho(X/Y)$: Relative Picard number of a normal $\boldsymbol{Q}$-factorial variety X over a variety Y.

Σ_d : Hirzebruch surface of degree d.

$\mathrm{Card}\, \mathcal{S}$: Cardinality of a set $\mathcal{S}$.

For a normal complete surface S with at worst Du Val singularities, we shall write

$$\mathrm{Sing\ S} = \sum_{\mathcal{T}} \nu(\mathcal{T})\mathcal{T},$$

where $\nu(\mathcal{T})$ denotes the number of singular points on S of type $\mathcal{T}$.

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2 The Euler characteristic formula

Firstly, let us recall the following result due to Crauder and Morrison.

Proposition 2.1 ([8], Proposition (A.1)) *Let X be a smooth 3-fold and let D be a complete effective divisor on X whose support has only simple normal crossing singularities. Then the following holds.*

$$\chi(\mathcal{O}_D) = \sum_i m_i \chi(\mathcal{O}_{D_i}) + \frac{1}{6}(D^3 - \sum_i m_i D_i^3) + \frac{1}{4}(D^2 - \sum_i m_i D_i^2)K_X,$$

where $D = \sum_i m_i D_i$ is the irreducible decomposition.

Let (X, p) be a germ of 3-dimensional terminal singularity at p whose index r is equal to or greater than 2. Take a Du Val element $S \in |-K_X|$ passing through p, where we say that $S \in |-K_X|$ is a Du Val element, if S is a reduced normal $\mathbf{Q}$-Cartier divisor on X passing through p such that S has a Du Val singularity at p. The canonical cover $\pi : \tilde{X} \to X$ with respect to K_X induces a covering of Du Val singularities $\pi : \tilde{S} := \pi^{-1}(S) \to S$. There is a coordinate system x, y and z of $\boldsymbol{C}^3$ which are semi-invariant under the action of the Galois group Gal $(\tilde{S}/S)$ such that $\tilde{p} := \pi^{-1}(p) \in \tilde{S}$ is analytically isomorphic to the germ of the origin of the hypersurface defined by a equation $f(x, y, z) = 0$. Let σ be a generator of Gal $(\tilde{S}/S)$ and let ζ be a primitive r-th root of unity. The actions of σ are completely classified into the following 6 types (see [50]).

(1) $\tilde{p} \in \tilde{S}$ is of type A_{n-1} and $p \in S$ is of type A_{rn-1} $(n \geq 1)$. $f = xy + z^n$, $\sigma^* x = \zeta^a x$, $\sigma^* y = \zeta^{-a} y$ and $\sigma^* z = z$, where $(r, a) = 1$.

(2) $\tilde{p} \in \tilde{S}$ is of type A_{2n-2} and $p \in S$ is of type D_{2n+1} $(n \geq 2)$. $r = 4$, $f = x^2 + y^2 + z^{2n-1}$, $\sigma^* x = \zeta x$, $\sigma^* y = \zeta^3 y$ and $\sigma^* z = \zeta^2 z$.

(3) $\tilde{p} \in \tilde{S}$ is of type A_{2n-1} and $p \in S$ is of type D_{n+2} $(n \geq 2)$. $r = 2$, $f = x^2 + y^2 + z^{2n}$, $\sigma^* x = x$, $\sigma^* y = -y$ and $\sigma^* z = -z$.

(4) $\tilde{p} \in \tilde{S}$ is of type D_4 and $p \in S$ is of type E_6. $r = 3$, $f = x^2 + y^3 + z^3$, $\sigma^* x = x$, $\sigma^* y = \zeta y$ and $\sigma^* z = \zeta^2 z$.

(5) $\tilde{p} \in \tilde{S}$ is of type D_{n+1} and $p \in S$ is of type D_{2n}. $r = 2$, $f = x^2 + y^2 z + z^n$, $\sigma^* x = -x$, $\sigma^* y = -y$ and $\sigma^* z = z$.

(6) $\tilde{p} \in \tilde{S}$ is of type E_6 and $p \in S$ is of type E_7. $r = 2$, $f = x^2 + y^3 + z^4$, $\sigma^* x = -x$, $\sigma^* y = y$ and $\sigma^* z = -z$.

Definition 2.1 For $p \in S \subset X$ as above, we define the rational number $c_p(X,S) \in \boldsymbol{Q}$ as follows:

$$c_p(X,S) := \begin{cases} 0 & \text{Case } p \in X \text{ Gorenstein}, \\ n\{r-(1/r)\} & \text{Case (1)}, \\ 3(2n+3)/4 & \text{Case (2)}, \\ 3 & \text{Case (3)}, \\ 16/3 & \text{Case (4)}, \\ 3n/2 & \text{Case (5)}, \\ 9/2 & \text{Case (6)}. \end{cases}$$

Definition 2.2 Let $p \in S \subset X$ be as above. we define the rational number $\delta_p(X,S) \in \boldsymbol{Q}$ as follows:

$$\delta_p(X,S) := e_p(S) - \frac{1}{o_p(S)} - c_p(X,S) \in \boldsymbol{Q},$$

where $e_p(S)$ is the Euler number of the inverse image of p by the morphism induced by the minimal resolution and $o_p(S)$ is the order of the local fundamental group of S at p.

If the index of X at p is equal to or greater than 2, we obtain the following table.

Table I

	$e_p(S)$	$o_p(S)$	$c_p(X,S)$	$\delta_p(X,S)$
(1)	rn	rn	$n\{r-(1/r)\}$	$(n^2-1)/rn$
(2)	$2n+2$	$8n-4$	$3(2n+3)/4$	$n(n-1)/(2n-1)$
(3)	$n+3$	$4n$	3	$(4n^2-1)/4n$
(4)	7	24	$16/3$	$13/8$
(5)	$2n+1$	$8(n-1)$	$3n/2$	$(4n^2+4n-9)/8(n-1)$
(6)	8	48	$9/2$	$167/48$

Proposition 2.2 *$\delta_p(X,S) \geq 0$. $\delta_p(X,S) = 0$ if and only if (X,S) has only singularity of type $V_1(r;a,-a,1)$ at p, where $(r,a)=1$.*

Proof. If $p \in X$ is Gorenstein, it is easy to see that $\delta_p(X,S) = e_p(S) - 1/o_p(S) \geq 0$ and that $\delta_p(X,S) = 0$ if and only if X and S is smooth at p. Assume that the index of $p \in X$ is equal to or greater than 2. If we have $\delta_p(X,S) = 0$, we infer that $\tilde{p} \in \tilde{S}$ is smooth (hence $\tilde{p} \in \tilde{X}$) from Table I. Thus we get the assertion. ∎

We give a proof of the following Reid's Riemann-Roch formula, which seems to be more clear than the one in [50], (9.2) to see the last satement in the theorem which is a crucial point for our subsequent argument.

Theorem 2.1 ([50], Theorem 9.1 (I)) *Let X be a projective surface with at worst Du Val singularities and let D be a Weil divisor on X. Then*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2}D(D-K_X) + \sum_{p\in X} c_p(D),$$

where $c_p(D)$ is the rational number which depends only on the local analytic type of $p \in X$ and $\mathcal{O}_X(D)$.

Proof. Let $\mu : Y \to X$ be the minimal resolution of X. Put $\Gamma := \lceil \mu^* D \rceil - \lfloor \mu^* D \rfloor$. Then there exists the following exact sequence:

$$0 \to \mathcal{O}_Y(\lfloor \mu^* D \rfloor) \to \mathcal{O}_Y(\lceil \mu^* D \rceil) \to \mathcal{O}_\Gamma(\lceil \mu^* D \rceil) \to 0.$$

Since we have $\mu_* \mathcal{O}_Y(\lfloor \mu^* D \rfloor) \simeq \mathcal{O}_Y(D)$ by [52], Theorem 2.1 and $R^i \mu_* \mathcal{O}_Y(\lceil \mu^* D \rceil) = 0$ for $i > 0$ by [52], Theorem 2.2, we obtain the following exact sequence:

$$0 \to \mathcal{O}_X(D) \to \mu_* \mathcal{O}_Y(\lceil \mu^* D \rceil) \to \mu_* \mathcal{O}_\Gamma(\lceil \mu^* D \rceil) \to R^1 \mu_* \mathcal{O}_Y(\lfloor \mu^* D \rfloor) \to 0$$

to get

$$\begin{aligned} \chi(\mathcal{O}_X(D)) &= \chi(\mu_* \mathcal{O}_Y(\lceil \mu^* D \rceil)) \\ &\quad - \text{length Ker}\{\mu_* \mathcal{O}_\Gamma(\lceil \mu^* D \rceil) \to R^1 \mu_* \mathcal{O}_Y(\lfloor \mu^* D \rfloor)\}. \end{aligned}$$

Put $\Delta := \lceil \mu^* D \rceil - \mu^* D$. Then $\chi(\mu_* \mathcal{O}_Y(\lceil \mu^* D \rceil))$ can be written as follows:

$$\begin{aligned} \chi(\mu_* \mathcal{O}_Y(\lceil \mu^* D \rceil)) &= \chi(\mathcal{O}_Y(\lceil \mu^* D \rceil)) \\ &= \chi(\mathcal{O}_Y) + \frac{1}{2}\lceil \mu^* D \rceil(\lceil \mu^* D \rceil - K_Y) \\ &= \chi(\mathcal{O}_X) + \frac{1}{2}(\mu^* D + \Delta)(\mu^* D + \Delta - \mu^* K_X) \\ &= \chi(\mathcal{O}_X) + \frac{1}{2}(D^2 - DK_X) + \frac{1}{2}\Delta^2. \end{aligned}$$

Putting $c_p(D) := (1/2)(\Delta|_{\mu^{-1}(p)})^2 - \text{length } \mathcal{T}(D)_p$, where

$$\mathcal{T}(D) := \text{Ker}\{\mu_* \mathcal{O}_\Gamma(\lceil \mu^* D \rceil) \to R^1 \mu_* \mathcal{O}_Y(\lfloor \mu^* D \rfloor)\},$$

we get

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2}D(D - K_X) + \sum_{p \in X} c_p(D).$$

As for the last assertion, for any two Weil divisors D_1 and D_2 such that $D_1 - D_2$ is Cartier at $p \in X$, since Δ for $D = D_1$ and $D = D_2$ is the same and we have $\mathcal{T}(D_1)_p = \mathcal{T}(D_2)_p \otimes_{\mathcal{O}_{X,p}} \mathcal{O}_X(D_1 - D_2)_p$, we infer that $c_p(D_1) = c_p(D_2)$. Thus we get the assertion. ∎

Lemma 2.1 *Let X be a germ of reduced irreducible normal $\boldsymbol{Q}$-Gorenstein analytic spaces and D be a non zero $\boldsymbol{Q}$-boundary on X. Assume that (X, D) is canonical and that the center on X of any divisor E_j with discrepancy zero is contained in the support of D, then X has only terminal singularities.*

Proof. Let $\mu : Y \to X$ be a Hironaka resolution of X and write $K_Y = \mu^* K_X + \sum_{j \in J} a_j E_j$ and $\mu^* D = \mu_*^{-1} D + \sum_{j \in J} \nu_j E_j$, where $\{E_j | j \in J\}$ is the set of all the μ-exceptional divisors and a_j, ν_j are non-negative rational numbers for any $j \in J$. By the assumption, we have $a_j \geq \nu_j$ for any $j \in J$. By the choice of our resolution, $\nu_j = 0$ implies that E_j is obtained by blowing up the center which is not contained in the support of the weak transform of some multiple of D. Thus we get the assertion. ∎

Proposition 2.3 *Let X be a normal $\mathbf{Q}$-Gorenstein 3-fold and D be an effective complete Cartier divisor on X such that the log 3-fold (X, D_{red}) is divisorially log terminal and that X is smooth outside the support of D. Assume that $K_X + D_{\text{red}}$ is Cartier and each irreducible component of D is algebraic and $\mathbf{Q}$-Cartier. Then the following formula holds :*

$$\begin{aligned} \chi(\mathcal{O}_D) &= \sum_i m_i \chi(\mathcal{O}_{D_i}) + \frac{1}{6}(D^3 - \sum_i m_i D_i^3) + \frac{1}{4}(D^2 - \sum_i m_i D_i^2) K_X \\ &\quad - \frac{1}{12}\sum_i m_i \sum_{p \in D_i^o} c_p(X, D_i), \end{aligned}$$

where $D = \sum_i m_i D_i$ is the irreducible decomposition and $D_i^o := D_i \setminus \cup_{j \neq i} D_j$.

Proof. We calculate the contribution of singularities to the formula in Proposition 2.1 using Reid's Riemann-Roch for surfaces with Du Val singularities and examining the original proof of Proposition 2.1. By the assumptions, we infer that if X is not smooth or X is smooth but D_{red} is not a normal crossing divisor at $p \in X$, then $p \in D_i^o$ for some i, X has only terminal singularity by Lemma 2.1 and D_i has at worst Du Val singularity at $p \in D_i \subset X$ since (X, D_i) is canonical for any i. We note that if a singularity $p \in X$ is Gorenstein then this singularity does not contribute to the Riemann-Roch. Assume that $p \in D_i^o$ for some i and that $r := \text{ind}\,_p K_X \geq 2$. Take the canonical cover $\pi : \tilde{X} \to X$ locally at p (here we used the same notation X for an open neighbourhood of $p \in X$). Putting $\tilde{p} := \pi^{-1}(p)$ and $\tilde{D}_i := \pi^{-1}(D_i)$, let $\vartheta_i \in \mathcal{O}_{\tilde{X}}$ be a defining equation of $\tilde{D}_i$ and let σ be the generator of the covering action. For any $l \in \mathrm{N}$, there is the following exact sequence locally at $\tilde{p}$ which is compatible with the action of σ.

$$0 \to \vartheta_i^l \mathcal{O}_{\tilde{D}_i} \to \mathcal{O}_{(l+1)\tilde{D}_i} \to \mathcal{O}_{l\tilde{D}_i} \to 0.$$

Since $H^1(<\sigma>, \vartheta_i^l \mathcal{O}_{\tilde{D}_i}) = 0$, taking the invariant part of the above exact sequence, we obtain the following exact sequence locally at p:

$$0 \to (\vartheta_i^l \mathcal{O}_{\tilde{D}_i})^{<\sigma>} \to \mathcal{O}_{(l+1)D_i} \to \mathcal{O}_{lD_i} \to 0,$$

where $(\vartheta_i^l \mathcal{O}_{\tilde{D}_i})^{<\sigma>}$ is the σ-invariant part of $\vartheta_i^l \mathcal{O}_{\tilde{D}_i}$. We note that in fact, $(\vartheta_i^l \mathcal{O}_{\tilde{D}_i})^{<\sigma>}$ is a restriction of the divisorial sheaf $\mathcal{F}_l := \text{Ker}\,\{\mathcal{O}_{(l+1)D_i} \to \mathcal{O}_{lD_i}\}$ on D_i. Define $a \in \boldsymbol{N}$ so that $\sigma^* \vartheta_i = \zeta^{-a} \vartheta_i$ where ζ is a primitive r-th root of unity and note that $(a, r) = 1$ since $K_X + D_{\text{red}}$ is Cartier. we note that for any $\varphi \in \mathcal{O}_{\tilde{D}_i}$, $\vartheta_i^l \varphi \in \vartheta_i^l \mathcal{O}_{\tilde{D}_i}$ is σ-invariant if and only if $\sigma^* \varphi = \zeta^{al} \varphi$, so we have $\mathcal{F}_l \simeq \{\varphi \in \pi_* \mathcal{O}_{\tilde{D}_i}; \sigma^* \varphi = \zeta^{al} \varphi\}$ locally at $p \in D_i^o$. Assuming that $\tilde{p} \in \tilde{D}_i$ is smooth (the case (1), $n = 1$) for simplicity, we calculate the summation of the contribution to the Riemann-Roch for $\mathcal{F}_l$ ($1 \leq l \leq m_i - 1$), that is, $\sum_{l=1}^{m_i-1} c_p(\mathcal{F}_l)$ as follows. Note that there exists a natural number $d_i \in \boldsymbol{N}$ such that $m_i = r d_i$ since $m_i D_i$ is Cartier at p.

$$\begin{aligned}
\sum_{l=1}^{m_i-1} c_p(\mathcal{F}_l) &= \sum_{l=1}^{m_i-1} -\frac{\overline{al}(r - \overline{al})}{2r} = -d_i \sum_{l=1}^{r-1} \frac{\overline{al}(r - \overline{al})}{2r} = -d_i \sum_{l=1}^{r-1} \frac{l(r-l)}{2r} \\
&= -m_i \frac{r^2 - 1}{12r},
\end{aligned}$$

where $\overline{al} \in \boldsymbol{Z}$ is the unique integer such that $\overline{al} \equiv al \pmod{r}$ and $0 \leq \overline{al} \leq r - 1$. The other cases can be treated similarly using the $\mathbf{Q}$-smoothing method as explained in [50]. ∎

Proof of Theorem 1.1. Since f is flat, we have $\chi(\mathcal{O}_{X_t}) = \chi(\mathcal{O}_{X_0})$ for $t \in \mathcal{D}^*$, where $X_0 := f^*(0)$. Using Proposition 2.3 and the assumption that $K_X + \Theta \sim_{\boldsymbol{Q}} 0$, we obtain

$$\begin{aligned}
e_{\text{top}}(X_t) &= 12\chi(\mathcal{O}_{X_t}) \\
&= \sum_i m_i (12\chi(\mathcal{O}_{\Theta_i}) - 2\Theta_i^3 + 3\sum_j \Theta_i^2 \Theta_j - \sum_{p \in \Theta_i \setminus \Delta_i} c_p(X, \Theta_i)). \qquad (2.1)
\end{aligned}$$

On the other hand, since for any i, Δ_i is either 0, disjoint union of smooth elliptic curves or a cycle of rational curves (see Lemma 3.4), we have

$$\begin{aligned}
K_{\Theta_i}^2 + e_{\text{top}}(\Delta_i) = \Delta_i^2 + e_{\text{top}}(\Delta_i) &= \sum_{j \neq i} \Theta_i \Theta_j^2 + 3 \sum_{j,k \neq i \text{and} j<k} \Theta_i \Theta_j \Theta_k \\
&= \sum_{j \neq i} \Theta_i \Theta_j^2 + \frac{3}{2} \sum_{i \neq j, k \text{and} j \neq k} \Theta_i \Theta_j \Theta_k \\
&= 2\Theta_i^3 - \frac{1}{2}\sum_j \Theta_i \Theta_j^2 - 3\sum_j \Theta_i^2 \Theta_j + \frac{3}{2}\sum_{j,k} \Theta_i \Theta_j \Theta_k.
\end{aligned}$$

Therefore, we have

$$\sum_i m_i(K^2_{\Theta_i} + e_{\text{top}}(\Delta_i)) = \sum_i m_i(2\Theta_i^3 - 3\sum_j \Theta_i^2\Theta_j), \tag{2.2}$$

since $\sum_i m_i\Theta_i \sim 0$. From (2.1) and (2.2), we obtain

$$e_{\text{top}}(X_t) = \sum_i m_i(12\chi(\mathcal{O}_{\Theta_i}) - K^2_{\Theta_i} - e_{\text{top}}(\Delta_i) - \sum_{p\in\Theta_i\setminus\Delta_i} c_p(X,\Theta_i)).$$

For any i, let $\Theta_i' \to \Theta_i$ be the minimal resolution of the singularities of Θ_i. Since we have

$$e_{\text{top}}(\Theta_i') = e_{\text{top}}(\Theta_i) + \sum_{p\in\Theta_i}(e_p(\Theta_i) - 1)$$

and Θ_i has only Du Val singularities, we have

$$e_{\text{top}}(\Theta_i) = 12\chi(\mathcal{O}_{\Theta_i}) - K^2_{\Theta_i} - \sum_{p\in\Theta_i}(e_p(\Theta_i) - 1)$$

by Noether's equality. Thus we obtain

$$\begin{aligned} e_{\text{top}}(X_t) &= \sum_i m_i\{e_{\text{top}}(\Theta_i\setminus\Delta_i) + \sum_{p\in\Theta_i\setminus\Delta_i}(e_p(\Theta_i) - 1 - c_p(X,\Theta_i))\} \\ &= \sum m_i(e_{\text{orb}}(\Theta_i\setminus\Delta_i) + \sum_{p\in\Theta_i\setminus\Delta_i}\delta_p(X,\Theta_i)). \end{aligned}$$

■

Proof of Corollary 1.1. Since we have $e_{\text{orb}}(\Theta_i\setminus\Delta_i) \geq 0$ for any i by Miyaoka's inequality ([35], Theorem 1.1), we obtain the assertion from Proposition 2.2. ■

3 Structures of log surfaces with a standard boundary whose log canonical divisors are numerically trivial

3.1 $\mathcal{S}$-extractions and $\mathcal{S}$-elementary transformations

Definition 3.1 A normal log variety (X,Δ) defined over an algebraically closed field is said to be *a log variety with a standard boundary*, if $\text{mult}_\Gamma\Delta \in \mathcal{S} := \{(b-1)/b | b \in \boldsymbol{N}\cup\{\infty\}\}$ for any prime divisor Γ.

Definition 3.2 Let $(S,\Gamma+\Delta)$ be a normal log surface with a $\boldsymbol{Q}$-boundary $\Gamma+\Delta$, where Γ is a prime divisor on S. For a point $p\in\Gamma$, we define a rational number $m_p(\Gamma^\nu;\Delta)\in\boldsymbol{Q}$ as follows.

$$m_p(\Gamma^\nu;\Delta) := \text{mult}_p\text{Diff}_{\Gamma^\nu}(\Delta).$$

Let (S,Δ) be a normal log surface with a standard boundary defined over an algebraically closed field k. Assume that (S,Δ) is log canonical and $\lfloor\Delta\rfloor\neq 0$. Take $p\in\lfloor\Delta\rfloor$ and let Γ be an irreducible component of the germ of $\lfloor\Delta\rfloor$ passing through p. Then $m_p(\Gamma;\Delta-\Gamma)$ can be written as follows:

$$m_p(\Gamma;\Delta-\Gamma) = \frac{n-1}{n} + \sum_{b=2}^{\infty}\frac{b-1}{b}\frac{k_b}{n},$$

where $n\in\boldsymbol{N}$ and k_b is a non-negative integer for all b and $k_b = 0$ except for finite number of b. We note that this is well known in the case of characteristic 0 but in fact this is just a conclusion of the intersection theory, so we don't have to worry about the characteristic of k about this matter. Since (S, Δ) is log canonical, we have

$$\frac{n-1}{n} + \sum_{b=2}^{\infty} \frac{b-1}{b} \frac{k_b}{n} \leq 1.$$

Noting that $(b-1)/b \geq 1/2$, we get $\sum_{b=2}^{\infty} k_b \leq 2$. Therefore, one of the following three cases (1), (2) and (3) occurs.

(1) $k_b = 0$ for all b,

(2) There exists b' such that $k_{b'} = 1$ and $k_b = 0$ if $b \neq b'$,

(3) $k_2 = 2$ and $k_b = 0$ if $b \neq 2$.

So we have

$$m_p(\Gamma; \Delta - \Gamma) = \begin{cases} (n-1)/n & \text{Case (1)}, \\ (bn-1)/bn & \text{Case (2)}, \\ 1 & \text{Case (3)}. \end{cases}$$

Let $\rho : T \to S$ be a birational morphism from a normal surface T which is the blow up of p if S is smooth at p or is obtained from the minimal resolution of $p \in S$ by contracting all the exceptional divisors that do not intersect the strict transform of Γ if S is singular at p. We call this ρ an *$\mathcal{S}$-extraction.*

Lemma 3.1 *(T, Δ^T) is a log surface with a standard boundary.*

Proof. Let E be the exceptional prime divisor for ρ. We only have to show $m := \mathrm{mult}_E \Delta^T \in \mathcal{S}$. By definitions, we have

$$K_{\Gamma'} + (\Delta' - \Gamma')|_{\Gamma'} + mE|_{\Gamma'} = \rho^*(K_\Gamma + \mathrm{Diff}_\Gamma(\Delta - \Gamma)),$$

where Δ' and Γ' are the strict transform of Δ and Γ respectively, hence

$$(\Delta' - \Gamma', \Gamma')_{p'} + m = m_p(\Gamma; \Delta - \Gamma),$$

where $p' := \rho^{-1}(p) \cap \Gamma'$ and $(\Delta' - \Gamma', \Gamma')_{p'}$ denotes the local intersection number of $\Delta' - \Gamma'$ and Γ' at p'. If we are in the cases (1) or (2), we see that $(\Delta' - \Gamma', \Gamma')_{p'} = 0$, hence $m = m_p(\Gamma; \Delta - \Gamma) \in \mathcal{S}$. Assume that we are in the case (3). Since $(\Delta' - \Gamma', \Gamma')_{p'} \in (1/2)\boldsymbol{Z}$, $m \geq 0$ and $m = 1 - (\Delta' - \Gamma', \Gamma')_{p'}$, we have $m = 0$, $1/2$ or 1. Thus we get the assertion. ∎

Assume that there exists a proper surjective connected k-morphism $\varphi : S \to B$ to a smooth curve B defined over k such that $\rho(S/B) = 1$ and that the support of $\lfloor \Delta \rfloor$ is not entirely contained in a fibre of φ and that $K_S + \Delta$ is relatively numerically trivial over B. Let $\hat{\rho} : T \to \hat{S}$ be a birational morphism to a normal surface $\hat{S}$ obtained by contracting the other component of the fibre of $\varphi \circ \rho$ which contains E and put $\hat{\Delta} := \hat{\rho}_* \Delta^T$. Then $(\hat{S}, \hat{\Delta})$ with the induced morphism $\hat{\varphi} : \hat{S} \to B$ satisfies the same conditions as (S, Δ) and $\varphi : S \to B$. We call this transformation a *$\mathcal{S}$-elementary transformation.*

Definition 3.3 For $t \in B$, we define the $\boldsymbol{Q}$-boundaries $\Delta_\varphi^+(t)$ and $\Delta_\varphi^-(t)$ on S as follows.

$$\begin{aligned} \Delta_\varphi^+(t) &:= \Delta + (1 - \mathrm{mult}_{C_\varphi(t)} \Delta) C_\varphi(t), \\ \Delta_\varphi^-(t) &:= \Delta - (\mathrm{mult}_{C_\varphi(t)} \Delta) C_\varphi(t), \end{aligned}$$

where $C_\varphi(t) := \varphi^*(t)_{\mathrm{red}}$.

Lemma 3.2 *Assume that $(S, \Delta^+_\varphi(t))$ is not log canonical over $t \in B$. Then one of the followings (a) or (b) holds.*

(a) *φ is smooth over $t \in B$ and $\Delta^-_\varphi(t) = \lfloor \Delta^-_\varphi(t) \rfloor$. $\Delta^-_\varphi(t)$ is smooth in the neighbourhood of $C_\varphi(t)$ and $\Delta^-_\varphi(t) \cdot C_\varphi(t) = 2p$, where $\Delta^-_\varphi(t) \cdot C_\varphi(t)$ denotes the intersection cycle.*

(b) *There exists a sequence of $\mathcal{S}$-elementary transformations over $t \in B$ such that for the resulting log surface $(\hat{S}, \hat{\Delta})$, $(\hat{S}, \Delta^+(\hat{\varphi}; t))$ is log canonical in the neighbourhood of the fibre over $t \in B$.*

Proof. Assume that $(S, \Delta^+_\varphi(t))$ is not log canonical at $p_0 \in C_\varphi(t)$. From

$$\sum_{p \in C_\varphi(t)} m_p(C_\varphi(t); \Delta^-_\varphi(t)) = 2 \text{ and } m_{p_0}(C_\varphi(t); \Delta^-_\varphi(t)) > 1,$$

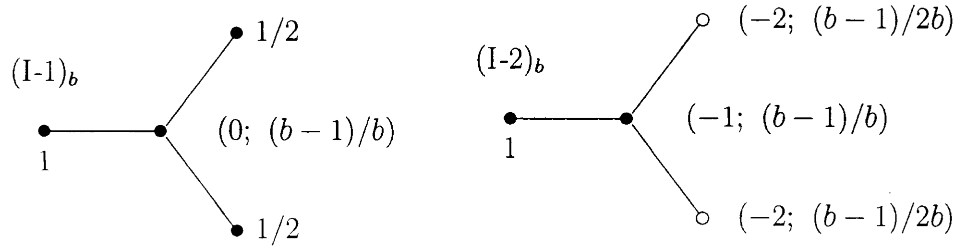
we deduce that $m_p(C_\varphi(t); \Delta^-_\varphi(t)) < 1$ for any $p \neq p_0$, that is, $(S, \Delta^+_\varphi(t))$ is purely log terminal at $p \in C_\varphi(t)$ if $p \neq p_0$. In particular, we see that $\lfloor \Delta^-_\varphi(t) \rfloor \cap C_\varphi(t) = \{p_0\}$. Let $\rho : T \to S$ be the $\mathcal{S}$-extraction at p_0 and let E be the exceptional divisor for ρ. Put $\psi := \varphi \circ \rho$. Then we have

$$2 = (\Delta^-_\varphi(t), \psi^*(t)) \geq (\Delta^-_\varphi(t)', C_\varphi(t)') + (\Delta^-_\varphi(t)', E) \geq (\Delta'_-, C_\varphi(t)') + 1,$$

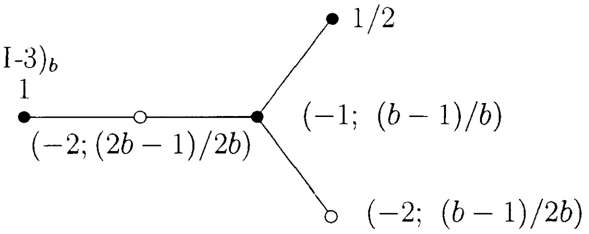
where Δ'_- and $C_\varphi(t)'$ are the strict transforms of $\Delta^-_\varphi(t)$ and $C_\varphi(t)$ respectively. Let $p'_0 \in T$ be as in the proof of Lemma 3.1. Suppose that $p'_0 \in C_\varphi(t)'$. Since $(\Delta^-_\varphi(t)', C_\varphi(t)')_{p'_0} \geq 1$, we have $(\Delta^-_\varphi(t)', C_\varphi(t)') = (\Delta^-_\varphi(t)', E) = 1$ and $\psi^*(t) = C_\varphi(t)' + E$. Noting that $C_\varphi(t)' \cap E = \{p'_0\}$ and T is smooth at p'_0, we see that T is smooth in the neighbourhood of the fibre over $t \in B$ and that $C_\varphi(t)'$ and E are (-1)-curves with $(C_\varphi(t)', E) = 1$, hence we are in the case (a). In the case that $p'_0 \notin C_\varphi(t)'$, by a $\mathcal{S}$-elementary transformation, we may assume that S is smooth at p_0 and $(C_\varphi(t), \lfloor \Delta^-_\varphi(t) \rfloor)_{p_0} = 1$ at first. Moreover we may assume that $(\lfloor \Delta^-_\varphi(t) \rfloor, \{\Delta^-_\varphi(t)\})_{p_0} = 0$ after operating $\mathcal{S}$-elementary transformations. Thus we are in the case (b). ∎

Lemma 3.3 *Assume that $(S, \Delta^+_\varphi(t))$ is log canonical over $t \in B$. Then one of the followings* (I) *or* (II) *holds.*

(I) $\deg \lfloor \mathrm{Diff}_{C_\varphi(t)}(\Delta^-_\varphi(t)) \rfloor = 1$, *then the dual graph of* $\mathrm{Supp}\, \mu^*\varphi^*(t) \cup \mathrm{Supp}\, \mu^{-1}_* \Delta^-_\varphi(t)$ *is one of the three types* $(\text{I-1})_b$, $(\text{I-2})_b$ *or* $(\text{I-3})_b$ *as below, where $\mu : M \to S$ is the minimal resolution of the singularities of S over $t \in B$.*

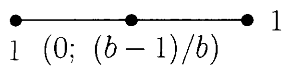


(I-3)$_b$

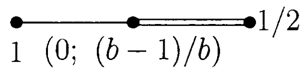


(II) *If* $\deg\lfloor \mathrm{Diff}_{C_\varphi(t)}(\Delta_\varphi^-(t))\rfloor = 2$, *then there exists a* $\mathcal{S}$*-elementary transformation such that for the resulting log surface* $(\hat{S}, \hat{\Delta})$, *the dual graph of* $\mathrm{Supp}\, \hat{\mu}^*\hat{\varphi}^*(t) \cup \mathrm{Supp}\, \hat{\mu}_*^{-1}\hat{\Delta}_\varphi^-(t)$ *is one of the three types* (II-1)$_b$, (II-2)$_b$ *or* (II-3)$_{b,k}$ ($k \geq 1$) *as below, where* $\hat{\mu} : \hat{M} \to \hat{S}$ *is the minimal resolution of the singularities of* $\hat{S}$ *over* $t \in B$.

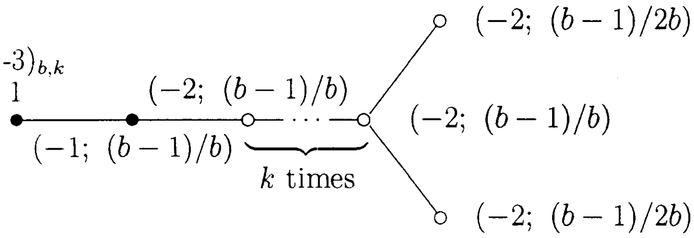
(II-1)$_b$



(II-2)$_b$



(II-3)$_{b,k}$



In the above dual graphs, $\bullet$ *denotes a germ of curves or a smooth rational curve which is not* μ-, *(resp.,* $\hat{\mu}$ *)-exceptional and* $\circ$ *denotes a* μ-, *(resp.,* $\hat{\mu}$ *)-exceptional curve. The numbers attached on* $\bullet$ *are the multiplicities in* Δ^M, *(resp.,* $\hat{\Delta}^{\hat{M}}$ *).* $(*;*)$ *denotes (the self-intersection number; the multiplicities in* Δ^M, *(resp.,* $\hat{\Delta}^{\hat{M}}$ *)).* b *is a natural number or* ∞ *such that* $\mathrm{mult}_{C_\varphi(t)}\Delta$, *(resp.,* $\mathrm{mult}_{C_{\hat{\varphi}}(t)}\hat{\Delta}$ *)* $= (b-1)/b$.

Proof. Since we have $\sum_{p \in C_\varphi(t)} m_p(C_\varphi(t); \Delta_\varphi^-(t)) = 2$, we have $\deg\lfloor \mathrm{Diff}_{C_\varphi(t)}(\Delta_\varphi^-(t))\rfloor = 1$ or 2. Firstly assume that $\deg\lfloor \mathrm{Diff}_{C_\varphi(t)}(\Delta_\varphi^-(t))\rfloor = 1$. Since we have $m_p(C_\varphi(t); \Delta_\varphi^-(t)) \in \mathcal{S}$ for any $p \in C_\varphi(t)$, there exists three points p_1, p_2 and $p_3 \in C_\varphi(t)$ such that

$$m_p(C_\varphi(t); \Delta_\varphi^-(t)) = \begin{cases} 1 & \text{if } p = p_0, \\ 1/2 & \text{if } p = p_1 \text{ or } p_2, \\ 0 & \text{otherwise} \end{cases}$$

and one of the following three case occurs. (1-1) S is smooth at p_1 and p_2, (1-2) S has Du val singularities of type A_1 at p_1 and p_2 or (1-3) S is smooth at p_1 and has a Du val singularity of type A_1 at p_2. Assume that S is not smooth at p_0 and let $\mu : M \to S$ be the minimal resolution of

$\{p_i|i=0,1,2\}$. Write $\mu^*C_\varphi(t) = \mu_*^{-1}C_\varphi(t) + \sum_{i=1}^s (1/2)E_i + \sum_{j=1}^n l_jF_j$, where $l_j > 0$ for any j and $\{E_i|1 \le i \le s\}$, $\{F_j|1 \le j \le n\}$ are μ-exceptional divisors over p_i ($i = 1, 2$) and p_0 respectively. Say $(F_1, \mu_*^{-1}C_\varphi(t)) = 1$, then we have

$$0 = (\mu^*C_\varphi(t), \mu_*^{-1}C_\varphi(t)) = (\mu_*^{-1}C_\varphi(t))^2 + (1/2)s + l_1$$

and $l_1 + (1/2)s = 1$, since $(\mu_*^{-1}C_\varphi(t))^2 = -1$. We know that $0 < l_1 < 1$, so we see that S must be smooth at p_0 in the cases (1-1) or (1-2) and that we are in the cases (I-1) or (I-2) respectively. Assume that we are in the case (1-3). Let $\rho : T \to S$ be the $\mathcal{S}$-extraction at p_0 and $\bar{F}_1$ be the ρ-exceptional divisor. Since we have $\rho^*C_\varphi(t) := \rho_*^{-1}C_\varphi(t) + (1/2)\bar{F}_1$, we see that $\bar{F}_1^2 = -2$ and that T is smooth in the neighbourhood of $\bar{F}_1$, hence we are in the case (I-3). Secondarily, assume that $\deg\lfloor \mathrm{Diff}_{C_\varphi(t)}(\Delta_\varphi^-(t))\rfloor = 2$. Take two points p_0 and $p_1 \in C_\varphi(t)$ such that

$$m_p(C_\varphi(t); \Delta_\varphi^-(t)) = \begin{cases} 1 & \text{if } p = p_0 \text{ or } p_1, \\ 0 & \text{otherwise.} \end{cases}$$

By operating a $\mathcal{S}$-elementary transformation at $p_0 \in \lfloor \Delta_\varphi^-(t) \rfloor$, we may assume that S is smooth at p_0. If S is smooth at p_1, then we see that we are in the cases (II-1) or (II-2). Assume S is not smooth at p_1 and let $\mu : M \to S$ be the minimal resolution of p_1. Write $\mu^*C_\varphi(t) = \mu_*^{-1}C_\varphi(t) + \sum_{j=1}^n l_jF_j$ and $\mu^*K_S = K_M + \sum_{j=1}^n a_jF_j$ where $a_j \ge 0$, $l_j > 0$ for any j and $\{F_j|1 \le j \le n\}$ are μ-exceptional divisors over p_1. Say $(F_1, \mu_*^{-1}C_\varphi(t)) = 1$, then we have $0 = (\mu^*C_\varphi(t), \mu_*^{-1}C_\varphi(t)) = (\mu_*^{-1}C_\varphi(t))^2 + l_1$, hence $l_1 = 1$ and $a_1 = 0$. So we deduce that $(S, C_\varphi(t))$ is not purely log terminal but S has a Du Val singularity at p_1 and $(C_\varphi(t), \Delta_\varphi^-(t))_{p_1} = 0$. So we are in the case $\text{(II-3)}_{b,k}$. ∎

Using the technique of $\mathcal{S}$-elementary transformations as above, we can obtain partial generalization of Proposition 3.2 to the positive characteristic case. We note that Proposition 3.2 is proved by the covering trick and the Hodge theory both of which does not work well in the positive characteristic case.

Proposition 3.1 *Let (S, Δ) be a normal projective log surface with a standard boundary defined over an algebraically closed field k such that (S, Δ) is log canonical and $K_S + \Delta$ is numerically trivial. Assume that there exists a proper surjective connected k-morphism $\varphi : S \to B$ onto a smooth projective curve B defined over k and that there exists an irreducible component of $\lfloor \Delta \rfloor$ which is not contained in a fibre of φ. Then we have $8(K_S + \Delta) \sim 0$ or $12(K_S + \Delta) \sim 0$ and in particular if* char $k \neq 2$, *we have $4(K_S + \Delta) \sim 0$ or $6(K_S + \Delta) \sim 0$, where $D \sim 0$ for $\boldsymbol{Q}$-divisor D means that D is integral and is linearly equivalent to 0.*

Proof. By contracting all the components of singular fibres of φ except one in each fibre and operating $\mathcal{S}$-elementary transformations, we may assume that $\rho(S/B) = 1$ and all of the fibres of φ are of type (a) as in Lemma 3.2 or of types $\text{(I-}i)_b$ or $\text{(II-}i)_b$ ($i = 1$, 2 or 3) as in Lemma 3.3. Let Γ be an irreducible component of $\lfloor \Delta \rfloor$. Since $m_p(\Gamma; \Delta - \Gamma) \in \mathcal{S}$ and $\sum_{p\in\Gamma} m_p(\Gamma; \Delta - \Gamma) = 2$, the possible values of $m_p(\Gamma; \Delta - \Gamma)$ are 0, 1, 1/2, 2/3, 3/4 or 5/6. So we deduce that $4(K_M + \Delta^M)$ or $6(K_M + \Delta^M)$is not integral if and only if φ has a fibre of types (I-2)_b or (II-3)_b ($b = 4$, 6). Assume that char $k \neq 2$ and that φ has a fibre of type (I-2)_4 or (II-3)_4. Then there exists distinct three points p_0, p_1 and p_2 in Γ such that

$$m_p(\Gamma; \Delta - \Gamma) = \begin{cases} 1/2 & \text{if } p = p_0, \\ 3/4 & \text{if } p = p_1 \text{ or } p_2, \\ 0 & \text{otherwise} \end{cases}$$

and the induced morphism $\varphi : \Gamma \simeq \boldsymbol{P}^1 \to B$ has degree 2 which is separable by the assumption and branches at $\{p_i|0 \le i \le 2\}$ but which is absurd by the Hurwitz's formula. Assume that φ has a fibre of type (I-2)$_6$ or (II-3)$_6$. Then there exists distinct three points p_0, p_1 and p_2 in Γ such that

$$m_p(\Gamma; \Delta - \Gamma) = \begin{cases} 1/2 & \text{if } p = p_0, \\ 2/3 & \text{if } p = p_1, \\ 5/6 & \text{if } p = p_2, \\ 0 & \text{otherwise} \end{cases}$$

and we can derive the absurdity as in the same way as above. Therefore, we only have to check that if $r(K_M + \Delta^M)$ is an integral divisor on M for some $r \in \boldsymbol{N}$, then we have $r(K_M + \Delta^M) \sim 0$ or $(r-1)(K_M + \Delta^M) \sim 0$. Put $D := r(K_M + \Delta^M)$. If the genus of B is zero, then M is a rational surface, hence $D \sim 0$. Assume that the genus of B is positive. Then $\lfloor \Delta^M \rfloor$ contains an smooth elliptic curve Γ and M is birationally elliptic ruled. From $\chi(\mathcal{O}_M(D)) = \chi(\mathcal{O}_M) = 0$, we have $h^0(\mathcal{O}_M(D)) + h^2(\mathcal{O}_M(D)) = h^1(\mathcal{O}_M(D))$. If $h^2(\mathcal{O}_M(D)) \neq 0$, then $h^0(\mathcal{O}_M(K_M - D)) \neq 0$ by the Serre duality and for an ample divisor H, we have $0 \leq (K_M - D, H) = (K_M, H) = -(\Delta^M, H) < 0$ which is absurd. Thus we get $h^2(\mathcal{O}_M(D)) = 0$ and $h^0(\mathcal{O}_M(D)) = h^1(\mathcal{O}_M(D))$. Assume that $h^2(\mathcal{O}_M(D - \Gamma)) \neq 0$. Then $h^0(\mathcal{O}_M(K_M - D + \Gamma)) \neq 0$ by the Serre duality again and $(K_M + \Gamma, H) \geq 0$, hence $\Delta^M = \Gamma$ and $(r-1)(K_M + \Delta^M) \sim 0$. So we may assume that $h^2(\mathcal{O}_M(D - \Gamma)) = 0$. By the assumption, there exists a surjection: $H^1(\mathcal{O}_M(D)) \to H^1(\mathcal{O}_\Gamma(D)) \simeq H^1(\mathcal{O}_\Gamma)$, hence $h^0(\mathcal{O}_M(D)) = h^1(\mathcal{O}_M(D)) > 0$. Thus we get the assertion. ∎

3.2 Case $e_{\text{orb}}(\tilde{S} \setminus \tilde{\Delta}) = 0$ and $\lfloor \Delta \rfloor \neq 0$

Let (S, Δ) be a projective log surface with a standard boundary defined over an algebraically closed field k. Assume that (S, Δ) is log terminal and $K_S + \Delta$ is numerically trivial. (S, Δ) can be roughly classified into the following three types.

I : $\lfloor \Delta \rfloor = 0$,

II : $\lfloor \Delta \rfloor \neq 0$ and $\lfloor \mathrm{Diff}_{\lfloor \Delta \rfloor^\nu}(\Delta - \lfloor \Delta \rfloor) \rfloor = 0$,

III : $\lfloor \Delta \rfloor \neq 0$ and $\lfloor \mathrm{Diff}_{\lfloor \Delta \rfloor^\nu}(\Delta - \lfloor \Delta \rfloor) \rfloor \neq 0$,

where ν denotes the normalization map $\nu : \lfloor \Delta \rfloor^\nu \to \lfloor \Delta \rfloor$.

Definition 3.4 Log surfaces (S, Δ) with the above conditions are said to be *of type* I, II *and* III respectively.

The classification of the log surfaces as above in the case that $K_S + \Delta$ is Cartier is well known.

Lemma 3.4 *Assume that $K_S + \Delta$ is Cartier, then one of the following hold.*

(1) *S is either an abelian surface, a hyperelliptic surface or a normal surface with at worst Du Val singularities whose minimal resolution is a $K3$ surface and $\Delta = 0$.*

(2) *S is a rational or birationally elliptic ruled surface with Du Val singularities and Δ is either a smooth elliptic curve or a disjoint union of two smooth elliptic curves and the support of Δ is disjoint from the singular loci of S.*

(3) *S is a rational surface with Du Val singularities and Δ is a connected cycle of smooth rational curves which is disjoint from the singular loci of S.*

Proof. By the definition and assumptions, the proof reduces to the well known results (see [28]). ∎

In what follows we assume that char $k = 0$.

Lemma 3.5 *Let (S, Δ) be a germ of log surfaces with a standard boundary and let $(\tilde{S}, \tilde{\Delta})$ be the log canonical cover of (S, Δ). If (S, Δ) is log terminal, then $(\tilde{S}, \tilde{\Delta})$ is also log terminal.*

Proof. If the number of irreducible components of $\lfloor \Delta \rfloor$ is less than 2, then (S, Δ) is purely log terminal, hence so is $(\tilde{S}, \tilde{\Delta})$ (see [58], Corollary 2.2). If not, we get the assertion since the index of $K_S + \Delta$ is 1 in this case. ■

By the log abundance theorem for surfaces, there exists a global log canonical cover $(\tilde{S}, \tilde{\Delta})$ of (S, Δ).

Lemma 3.6 *If (S, Δ) is of type* I (*resp.,* II, *resp.,* III), *then $(\tilde{S}, \tilde{\Delta})$ is also of type* I (*resp.,* II, *resp.,* III).

Proof. Firstly, we prove that $(\tilde{S}, \tilde{\Delta})$ is log terminal. By Lemma 3.5, we only have to check that any irreducible components of $\lfloor \Delta \rfloor$ does not have self-intersections. Assume that there exists an irreducible component $\tilde{\Gamma}$ which has self-intersections. Let Γ be the image of $\tilde{\Gamma}$ on S. Since we have

$$K_{\tilde{S}} + \pi^{-1}(\Gamma) = \pi^*(K_S + \Gamma + \{\Delta\}),$$

where π is the covering morphism $\pi : \tilde{S} \to S$, $(\tilde{S}, \pi^{-1}(\Gamma))$ is purely log terminal, hence so is $(\tilde{S}, \tilde{\Gamma})$, but which is absurd. Noting that $\tilde{\Delta} = \pi^{-1}(\Delta)$ and that $\lfloor \mathrm{Diff}_{\lfloor \tilde{\Delta} \rfloor^{\nu}}(0) \rfloor = \pi^{-1} \lfloor \mathrm{Diff}_{\lfloor \Delta \rfloor^{\nu}}(\Delta - \lfloor \Delta \rfloor) \rfloor$, we get the assertion. ■

The following result is essentially due to S. Tsunoda.

Proposition 3.2 (c.f. [63], Theorem 2.1)

(1) *If (S, Δ) is of type* II, *then $4(K_S + \Delta) \sim 0$ or $6(K_S + \Delta) \sim 0$.*

(2) *If (S, Δ) is of type* III, *then $2(K_S + \Delta) \sim 0$.*

Proof. The argument in the proof of [63], Theorem 2.1 can also be applied in the case in which S is rational. If S is not rational, we only have to apply the argument in the last part of the proof of Proposition3.1 ■

Let (S, Δ) be a projective log surface with a standard boundary defined over the complex number field $\boldsymbol{C}$. Assume that (S, Δ) is log terminal and that $K_S + \Delta$ is numerically trivial. Let $(\tilde{S}, \tilde{\Delta})$ be the log canonical cover. To classify log surfaces as above, we need some conditions for $(\tilde{S}, \tilde{\Delta})$ especially in the case $\lfloor \Delta \rfloor \neq 0$. In this section, we shall classify (S, Δ) with $\lfloor \Delta \rfloor \neq 0$ under the condition that $e_{\text{orb}}(\tilde{S} \setminus \tilde{\Delta}) = 0$ which seems to be the most fundamental case. In general, applying the log minimal program for S with respect $K_S + \{\Delta\}$, we get a birational morphism $\tau : S \to S^{\flat}$ to a projective normal surface $S^{\flat}$ such that (1) $K_{S^{\flat}} + \{\Delta^{\flat}\}$ is nef, where $\Delta^{\flat} := \tau_* \Delta$, (2) $-(K_{S^{\flat}} + \{\Delta^{\flat}\})$ is ample and $\rho(S^{\flat}) = 1$ or (3) there exists a projective surjective morphism $\varphi^{\flat} : S^{\flat} \to B$ onto a smooth projective curve B with $\rho(S^{\flat}/B) = 1$ and $-(K_{S^{\flat}} + \{\Delta^{\flat}\})$ is relatively ample with respect to $\varphi^{\flat}$. Assume that we are in the case (1). Then we have $\lfloor \Delta^{\flat} \rfloor = 0$ and $K_S + \Delta = \tau^*(K_{S^{\flat}} + \{\Delta^{\flat}\})$. On the other hand, we have $K_S + \{\Delta\} - \tau^*(K_{S^{\flat}} + \{\Delta^{\flat}\}) \geq 0$ by our construction. Thus we see that the condition (1) is equivalent to $\lfloor \Delta \rfloor = 0$. The following is the key lemma for our purpose.

Lemma 3.7 *Assume that $e_{\text{orb}}(\tilde{S} \setminus \tilde{\Delta}) = 0$. Then* $\mathrm{Exc}\, \tau \subset \lfloor \Delta \rfloor$.

Proof. We may assume that $\lfloor\Delta\rfloor \neq 0$. Let $\tau^\natural : S \to S^\natural$ be a birational extremal contraction with respect to $K_S + \{\Delta\}$. Let E denote the exceptional divisor for $\tau^\natural$ and put $\Delta^\natural := \tau^\natural_*\Delta$. Assume that $K_S + \Delta$ is Cartier firstly. By [29], Theorem 0.1, we see that E is a (-1)-curve and that $(E, \Delta) = 1$. Moreover, there exists a point $p \in E$ such that $E \cap \operatorname{Sing} S = \emptyset$ or $\{p\}$ and E contracts to a smooth point of $S^\natural$. Therefore, if E is not contained in Supp Δ, we have

$$e_{\mathrm{orb}}(S \setminus \Delta) - e_{\mathrm{orb}}(S^\natural \setminus \Delta^\natural) = \rho(S) - \rho(S^\natural) - (1 - \frac{1}{o_p(S)}) = \frac{1}{o_p(S)}.$$

On the other hand, we have $e_{\mathrm{orb}}(S \setminus \Delta) = 0$ and $e_{\mathrm{orb}}(S^\natural \setminus \Delta^\natural) \geq 0$, which is absurd. If $K_S + \Delta$ is not Cartier, we take log canonical covers $(\tilde{S}, \tilde{\Delta})$ and $(\tilde{S}^\natural, \tilde{\Delta}^\natural)$ of (S, Δ) and $(S^\natural, \Delta^\natural)$ respectively such that there exists a birational morphism $\tilde{\tau}^\natural : \tilde{S} \to \tilde{S}^\natural$ induced by $\tau^\natural$. Let π denote the covering morphism $\tilde{S} \to S$. Since we have $(K_{\tilde{S}}, \pi^*E) = \deg \pi (K_S + \{\Delta\}, E) < 0$, we see that $\pi^{-1}E \subset \operatorname{Supp} \tilde{\Delta} = \operatorname{Supp} \pi^{-1}\lfloor\Delta\rfloor$ by the previous argument, hence we conclude that $E \subset \lfloor\Delta\rfloor$. ■

3.2.1 Case Type II

Let (S, Δ) be a projective log surface with a standard boundary and assume that there exists a structure of a conic fibration $\varphi : S \to B$ with $\rho(S/B) = 1$, where B is a smooth projective connected curve such that φ has only fibres of types listed in Lemma 3.3. We denote the number of fibres of type $\mathcal{T}$ by $\nu(\mathcal{T})$ if it is finite.

Lemma 3.8 *Assume that (S, Δ) is of type* II *and that $e_{\mathrm{orb}}(\tilde{S} \setminus \tilde{\Delta}) = 0$. Then the followings holds.*

(1) *There exists a projective surjective morphism $\varphi : S \to B$ onto a smooth projective curve B with $\rho(S/B) = 1$,*

(2) *$(S, \Delta^+_\varphi(t))$ is log canonical for any $t \in B$ and*

(3) Supp $\lfloor \mathrm{Diff}_{C_\varphi(t)}(\Delta^-_\varphi(t))\rfloor \subset \lfloor\Delta^-_\varphi(t)\rfloor$ *for any $t \in B$.*

Proof. Firstly, assume that $K_S + \Delta$ is Cartier. If (1) does not hold, S is a rank one Gorenstein log del Pezzo surface by Lemma 3.7. Therefore we have $0 = e_{\mathrm{orb}}(S \setminus \Delta) = e_{\mathrm{orb}}(S) = 3 - \sum_{p \in S}\{1 - (1/o_p(S))\}$, which contradicts the Table IV. Hence (1) holds under the assumption that $K_S + \Delta$ is Cartier. Assume that $h^1(\mathcal{O}_S) = 1$. Then we have $0 = e_{\mathrm{orb}}(S \setminus \Delta) = -\sum_{p \in S}\{1 - (1/o_p(S))\}$, hence φ gives a structure of relatively minimal elliptic ruled surface on S. Noting that the induced morphism $\varphi : \Delta \to B$ is étale, we see that (2) and (3) hold. If $h^1(\mathcal{O}_S) = 0$, we have

$$4 = \sum_{t \in B} \sum_{p \in C(t)} (1 - \frac{1}{o_p(S)}) = \nu((\text{I-2})_1) + \sum_{k \geq 1} \frac{4k-1}{4k} \nu((\text{II-3})_{1,k}).$$

We note here that we do not have to operate any $\mathcal{S}$-elementary transformation by examining the proof of Lemma 3.3. By the Hurwitz's formula we have $\nu((\text{I-2})_1) + \sum_{k \geq 1} \nu((\text{II-3})_{1,k}) \leq 4$, hence we deduce that $\nu((\text{I-2})_1) = 4$ and $\nu((\text{II-3})_{1,k}) = 0$ $(k \geq 1)$. Thus we get the assertion (2) and (3) under the assumption that $K_S + \Delta$ is Cartier. We note that $\Delta^2 = K_S^2 = 0$ in each of the above cases. Assume that $K_S + \Delta$ is not necessarily Cartier and that there exists a birational extremal contraction $\tau^\natural : S \to S^\natural$ with respect to $K_S + \{\Delta\}$ and let E be as in Lemma 3.7. Then by the assumption and Lemma 3.7, we have $(K_S + \Delta, E) = (K_S + \{\Delta\}, E) + E^2 + (\lfloor\Delta\rfloor - E, E) < 0$, since $(K_S + \{\Delta\}, E) < 0$, $E^2 < 0$ and $(\lfloor\Delta\rfloor - E, E) = 0$, but which is absurd. Thus if (1) does not hold, $-(K_S + \{\Delta\})$ is ample and $\rho(S) = 1$, which implies that $\tilde{\Delta} = \pi^*\lfloor\Delta\rfloor$ is ample. Thus we get the absurdity. By taking the Stein factorization, there exists a finite morphism $\varpi : \tilde{B} \to B$ from a smooth projective curve $\tilde{B}$ and a proper surjective connected morphism $\tilde{\varphi} : \tilde{S} \to \tilde{B}$ such that $\varphi \circ \pi = \varpi \circ \tilde{\varphi}$. Since we have

$K_{\tilde{S}} + \sum_{\varpi(\tilde{t})=t} \Delta^+(\tilde{\varphi}; \tilde{t}) = \pi^*(K_S + \Delta_\varphi^+(t))$ and we know that $(\tilde{S}, \sum_{\varpi(\tilde{t})=t} \Delta^+(\tilde{\varphi}; \tilde{t}))$ is log canonical, we conclude that (2) holds. As for (3), we get the assertion by the following diagram:

$$\begin{array}{ccc} \operatorname{Supp} \sum_{\varpi(\tilde{t})=t} \operatorname{Diff}_{C(\tilde{\varphi};\tilde{t})}(\Delta^-(\tilde{\varphi}; \tilde{t})) & = & \operatorname{Supp} \pi^{-1}\lfloor \operatorname{Diff}_{C_\varphi(t)}(\Delta_\varphi^-(t))\rfloor \\ \cap & & \cap \\ \sum_{\varpi(\tilde{t})=t} \Delta^-(\tilde{\varphi}; \tilde{t}) & = & \pi^{-1}\lfloor \Delta_\varphi^-(t)\rfloor . \end{array}$$

■

Let (S, Δ) be a projective log surface with a standard boundary and assume that there exists a structure of a conic fibration $\varphi : S \to B$ with $\rho(S/B) = 1$, where B is a smooth projective connected curve such that φ has only fibres of types listed in Lemma 3.3. If φ has fibres of type $\mathcal{T}_1, \ldots, \mathcal{T}_r$ and generically of type $\mathcal{T}$, then we shall write $\operatorname{Typ}(S, \Delta; \varphi) = (\mathcal{T}_1 + \cdots + \mathcal{T}_r; \mathcal{T})$.

Proposition 3.3 *Type* II *log surfaces* (S, Δ) *with* $e_{\text{orb}}(\tilde{S} \setminus \tilde{\Delta}) = 0$ *are classified into the following 22 types modulo* $\mathcal{S}$*-elementary transformations.*

(*a*) $S \simeq \boldsymbol{P}_E(\mathcal{E})$, *where* $\mathcal{E}$ *is a rank two vector bundle on an elliptic curve* E *and* $\Delta = \Gamma$ (*a*-1), $\Gamma_1 + \Gamma_2$ (*a*-2), $\Gamma + (1/2)\Xi$ (*a*-3) *or* $\Gamma + (1/2)\Xi_1 + (1/2)\Xi_2$ (*a*-4), *where* Γ, Γ_i, Ξ *and* Ξ_j *are smooth elliptic curves which are disjoint from each other.*

(*b*) $S \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ *and*

$$\Delta = \Gamma + \sum_{i=1}^{2}(1/2)\Xi_{1,i} + \begin{cases} \sum_{j=1}^4 (1/2)\Xi_{2,j}, \\ \sum_{j=1}^3 (2/3)\Xi_{2,j}, \\ (1/2)\Xi_{2,1} + \sum_{j=2}^3 (3/4)\Xi_{2,j} \text{ or} \\ (1/2)\Xi_{2,1} + (2/3)\Xi_{2,2} + (5/6)\Xi_{2,3}, \end{cases}$$

where Γ *and* $\Xi_{1,i}$ *are fibres of the first projection for any* i *and* $\Xi_{2,j}$ *is a fibre of the second projection for any* j.

(*c*) *There exists a sequence of* $\mathcal{S}$*-elementary transformations such that for the resulting log surface* $(\hat{S}, \hat{\Delta})$, $\hat{S} \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ *and*

$$\hat{\Delta} = \hat{\Gamma}_1 + \hat{\Gamma}_2 + \begin{cases} \sum_{j=1}^4 (1/2)\hat{\Xi}_j, \\ \sum_{j=1}^3 (2/3)\hat{\Xi}_j, \\ (1/2)\hat{\Xi}_1 + \sum_{j=2}^3 (3/4)\hat{\Xi}_j \text{ or} \\ (1/2)\hat{\Xi}_1 + (2/3)\hat{\Xi}_2 + (5/6)\hat{\Xi}_3, \end{cases}$$

where $\hat{\Gamma}_i$ *is a fibre of the first projection for any* i *and* $\hat{\Xi}_j$ *is a fibre of the second projection for any* j.

(*d*) *There exists a structure of* $\boldsymbol{P}^1$*-fibration* $\varphi : S \to \boldsymbol{P}^1$ *with* $\rho(S/\boldsymbol{P}^1) = 1$ *such that one of the followings in* Table II *holds.*

Table II

	Δ	$\operatorname{Typ}(S, \Delta; \varphi)$
(*d*-1)	Γ	$(4(\text{I-2})_1; (\text{II-1})_1)$
(*d*-2)	$\Gamma + (1/2)\Xi_h$	$(4(\text{I-3})_1; (\text{I-1})_1)$
(*d*-3)	$\Gamma + (1/2)\Xi_h + (1/2)\Xi_{v,1} + (1/2)\Xi_{v,2}$	$(2(\text{I-1})_2 + 2(\text{I-3})_1; (\text{I-1})_1)$
(*d*-4)	$\Gamma + (1/2)\Xi_h + (1/2)\Xi_{v,1} + (3/4)\Xi_{v,2}$	$((\text{I-1})_4 + (\text{I-3})_2 + (\text{I-3})_1; (\text{I-1})_1)$
(*d*-5)	$\Gamma + (1/2)\Xi_h + \sum_{j=1}^3 (1/2)\Xi_{v,j}$	$(2(\text{I-3})_2 + (\text{I-1})_2; (\text{I-1})_1)$
(*d*-6)	$\Gamma + (1/2)\Xi_h + (2/3)\Xi_{v,1} + (2/3)\Xi_{v,2}$	$((\text{I-1})_3 + (\text{I-3})_1 + (\text{I-3})_3; (\text{I-1})_1)$

Γ in (d-1) and Ξ_h in (d-2) denote a smooth elliptic curve with $\Gamma^2 = \Xi_h^2 = 0$. Γ in (d-2), (d-3), (d-4), (d-5) and (d-6), Ξ_h in (d-3), (d-4), (d-5) and (d-6) denote smooth rational curves with $\Gamma^2 = \Xi_h^2 = 0$ which are horizontal with respect to φ. $\Xi_{v,j}$ in (d-3), (d-4), (d-5) and (d-6) denote smooth rational curves which are vertical with respect to φ for any j.

(e) *There exists a structure of $\boldsymbol{P}^1$-fibration $\varphi : S \to \boldsymbol{P}^1$ with $\rho(S/\boldsymbol{P}^1) = 1$ and there exists a sequence of $\mathcal{S}$-elementary transformations such that for the resulting log surface $(\hat{S}, \hat{\Delta})$, one of the followings in* Table III *holds.*

Table III

	$\hat{\Delta}$	$\mathrm{Typ}(\hat{S}, \hat{\Delta}; \hat{\varphi})$
(e-1)	$\hat{\Gamma} + \sum_{j=1}^{3}(1/2)\hat{\Xi}_j$	$(2(\text{I-2})_2 + (\text{II-1})_2;\ (\text{II-1})_1)$
(e-2)	$\hat{\Gamma} + \sum_{j=1}^{2}(1/2)\hat{\Xi}_j$	$(2(\text{I-2})_1 + 2(\text{II-1})_2;\ (\text{II-1})_1)$
(e-3)	$\hat{\Gamma} + \sum_{j=1}^{2}(2/3)\hat{\Xi}_j$	$((\text{I-2})_1 + (\text{I-2})_3 + (\text{II-1})_3;\ (\text{II-1})_1)$
(e-4)	$\hat{\Gamma} + (1/2)\hat{\Xi}_1 + (3/4)\hat{\Xi}_2$	$((\text{I-2})_1 + (\text{I-2})_2 + (\text{II-1})_4;\ (\text{II-1})_1)$

$\hat{\Gamma}$ denote a smooth rational curve with $(\hat{\Gamma}, \hat{\varphi}^*(t)) = 2$ for $t \in \boldsymbol{P}^1$ and $\hat{\Xi}_i$ denote a smooth rational curve which are vertical with respect to $\hat{\varphi}$ for any i.

Proof. Let Γ_1 be an irreducible component of $\lfloor\Delta\rfloor$. Then we have $2g(\Gamma_1) - 2 + \deg \mathrm{Diff}_{\Gamma_1}(\Delta - \Gamma_1) = 0$, where $g(\Gamma_1)$ denote the genus of Γ_1. (1) Assume that $g(\Gamma_1) = 1$. Then $\deg \mathrm{Diff}_{\Gamma_1}(\Delta - \Gamma_1) = 0$ and φ has only fibres of types $(\text{I-1})_1$, $(\text{I-2})_1$ or $(\text{II-1})_1$ by Lemma 3.3 and Lemma 3.8. (1-1) Assume that $g(B) = 1$. Since the induced morphism $\Gamma_1 \to B$ is étale, type $(\text{I-2})_1$ fibre does not exist, hence we are in the case (a). (1-2) Assume that $g(B) = 0$. Applying the Hurwitz's formula to the induced double covering $\Gamma_1 \to B$, we obtain $\nu((\text{I-2})_1) = 4$, hence we are in the case (d-1). (2) Assume that $g(\Gamma_1) = 0$. Then we have $\sum_{p \in \Gamma_1} m_p(\Gamma_1; \Delta - \Gamma_1) = 2$. (2-1) Assume that Γ_1 is a section of φ. Then φ has only fibres of types $(\text{I-1})_b$, $(\text{I-3})_b$ or $(\text{II-1})_b$. (2-1-1) Assume that $\Gamma_1 = \lfloor\Delta\rfloor$. Then φ has only fibres of types $(\text{I-1})_b$ or $(\text{I-3})_b$ and we have $\sum_{b\geq 1}\{(b-1)/b\}\nu((\text{I-1})_b) + \sum_{b'\geq 1}\{(2b'-1)/2b'\}\nu((\text{I-3})_{b'})$. Let Ξ_h be the sum of all the horizontal components of $\mathrm{Supp}\{\Delta\}$. If Ξ_h is reducible, then φ has only fibres of type $(\text{I-1})_b$, hence we are in one of the cases (b). Assume that Ξ_h is irreducible. Then we have $e_{\mathrm{top}}(\Xi_h) = 4 - \sum_{b'\geq 1}\nu((\text{I-3})_{b'})$ by the Hurwitz's formula. On the other hand, since we have $K_S + \Gamma_1 + \Xi_h + \{\Delta\}_v \sim_{\boldsymbol{Q}} (1/2)\Xi_h$, where $\{\Delta\}_v$ denotes the vertical part of $\{\Delta\}$, we have $e_{\mathrm{top}}(\Xi_h) = \deg \mathrm{Diff}_{\Xi_h}(\{\Delta\}_v) - (1/2)\Xi_h^2 = 2\sum_{b\geq 1}\{(b-1)/b\}\nu((\text{I-1})_b) + \sum_{b'\geq 1}\{(b'-1)/b'\}\nu((\text{I-3})_{b'}) - (1/2)\Xi_h^2$, hence $\Xi_h^2 = 0$. Since $e_{\mathrm{top}}(\Xi_h) \leq 2$ and $e_{\mathrm{top}}(\Xi_h) \in 2\boldsymbol{Z}$, we have $\sum_{b'\geq 1}\nu((\text{I-3})_{b'}) = 2$ or 4. If $\sum_{b'\geq 1}\nu((\text{I-3})_{b'}) = 4$, then we have $e_{\mathrm{top}}(\Xi_h) = 0$ and $\nu((\text{I-3})_1) = 4$, hence we are in the case (d-2). Assume that $\sum_{b'\geq 1}\nu((\text{I-3})_{b'}) = 2$. Then we have $e_{\mathrm{top}}(\Xi_h) = 2$ and we see that we are in the cases (d-3), (d-4), (d-5) and (d-6). (2-1-2) Assume that $\lfloor\Delta\rfloor$ is decomposed into two sections Γ_1 and Γ_2. Then φ has only fibres of type $(\text{II-1})_b$ after $\mathcal{S}$-elementary transformations, hence we are in the cases (c). (2-2) Assume that $(\Gamma_1, \varphi^*(t)) = 2$ for $t \in \boldsymbol{P}^1$. Then φ has only fibres of types $(\text{I-2})_b$ or $(\text{II-1})_b$ after $\mathcal{S}$-elementary transformations. We note that $\sum_{b\geq 1}\{(b-1)/b\}\nu((\text{I-2})_b) + 2\sum_{b'\geq 1}\{(b'-1)/b'\}\nu((\text{II-1})_{b'}) = 2$. By the Hurwitz's formula, we have $\sum_{b\geq 1}\nu((\text{I-2})_b) = 2$. Thus we see that we are in the cases (e-1), (e-2), (e-3) and (e-4). ■

We prove the lemma needed later which was taught to the author by Prof. A. Fujiki.

Lemma 3.9 *Let X be a complex manifold and let G be a finite subgroup of the holomorphic automorphism group* Aut X. *Let $\Pi : \mathcal{U} \to X$ be the universal cover of X. Then there exists a discrete subgroup $\tilde{G}$ of* Aut $\mathcal{U}$ *which acts on $\mathcal{U}$ properly discontinuously such that $\mathcal{U}/\tilde{G} \simeq X/G$*

Proof. Let $\mathrm{Aut}_\Pi\mathcal{U}$ be the subgroup of $\mathrm{Aut}\,\mathcal{U}$ which consists of all the elements $\tilde{\gamma} \in \mathrm{Aut}\,\mathcal{U}$ such that there exists an element $\gamma \in \mathrm{Aut}\,X$ which satisfies $\gamma \circ \Pi = \Pi \circ \tilde{\gamma}$. Then there exists the following exact sequence of groups: $1 \longrightarrow \pi_1(X) \longrightarrow \mathrm{Aut}_\Pi\mathcal{U} \longrightarrow \mathrm{Aut}\,X \longrightarrow 1$. Let $\tilde{G}$ be the inverse image of G by the third map in the above exact sequence. We note that $\tilde{G}$ is a discrete subgroup of $\mathrm{Aut}_\Pi\mathcal{U}$ and that we have the following exact sequence: $1 \longrightarrow \pi_1(X) \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$. Therefore, we see that $\tilde{G}$ acts on $\mathcal{U}$ properly discontinuously and that $\mathcal{U}/\tilde{G} \simeq (\mathcal{U}/\pi_1(X))/G \simeq X/G$. ∎

Let (X, D) be a normal log variety with reduced boundary D. Assume that a finite group G acts on X faithfully preserving D. Let $f : X \to X/G$ be the quotient morphism and assume that any component of D is not contained in the ramification divisor of f. For a prime divisor Γ on X/G, take a prime divisor $\tilde{\Gamma}$ on X contained in $f^{-1}(\Gamma)$ and let $G_{\tilde{\Gamma}}$ denote the subgroup of G consisting of all the element of G which acts on $\tilde{\Gamma}$ trivially. The order $|G_{\tilde{\Gamma}}|$ of $G_{\tilde{\Gamma}}$, which is nothing but the ramification index of f at $\tilde{\Gamma}$, does not depend on the choice of $\tilde{\Gamma}$. Put $e_\Gamma := |G_{\tilde{\Gamma}}|$. We define the $\boldsymbol{Q}$-boundary D_G on X/G as follows.

$$D_G := D/G + \sum_{\Gamma;prime} \frac{e_\Gamma - 1}{e_\Gamma}\Gamma,$$

where the summation is taken over all the prime divisors Γ on X/G. We note that by the definition, we have $K_X + D = f^*(K_{X/G} + D_G)$ if $K_{X/G} + D_G$ is $\boldsymbol{Q}$-Cartier.

Here is a corollary of Proposition 3.3.

Corollary 3.1 *Let (S, Δ) be a type* II *log surface with $e_{\mathrm{orb}}(\tilde{S}\setminus\tilde{\Delta}) = 0$. Then there exists a relatively minimal elliptic ruled surface $\pi : P \to E$ over an elliptic curve E which admits a π-equivariant action of a finite abelian group G with $|G| = l\mathrm{Ind}(K_S+\Delta)$, where $l = 1$, 2 or 4 and there exists a G-invariant reduced divisor D on P which is a smooth elliptic curve or a disjoint union of two smooth elliptic curves such that $(S, \Delta) \simeq (P/G, D_G)$. In particular, (S, Δ) is uniformizable to $\boldsymbol{C} \times \boldsymbol{P}^1$ in the sense of R. Kobayashi, S. Nakamura and F. Sakai (see* [23], [24]*).*

Proof. We only have to consider the case where $B \simeq \boldsymbol{P}^1$. We define a $\boldsymbol{Q}$-divisor δ on B as $\delta := \sum_{t\in B}\{(m_t b_t - 1)/m_t b_t\}t$, where $m_t := \mathrm{mult}_{C_\varphi(t)}\varphi^*(t)$ and $b_t := (1 - \mathrm{mult}_{C_\varphi(t)}\Delta)^{-1}$ for $t \in B$. Then by checking the list in Proposition3.3, we see that $\deg\delta = 2$. Put $r := \mathrm{Ind}(K_B + \delta)$. We can also check that $r = \mathrm{Ind}(K_S + \Delta)$. Let $\boldsymbol{C}(x) \subset K := \boldsymbol{C}(x, y)$ be the field extension induced by φ. We define a finite Galois field extension $K \subset L$ as follows. Take a rational function $\alpha(x)$ on B such that $\mathrm{div}\,\alpha(x) = r(K_B + \delta)$. If $\Xi_h = 0$, we put $L := K(\sqrt[r]{\alpha(x)})$. If $\Xi_h \neq 0$. Let $\Xi_{h,\eta}$ be the restriction of Ξ_h to the generic fibre $S_\eta \simeq \boldsymbol{P}^1(\boldsymbol{C}(x))$ of φ. Then the defining equation of $\Xi_{h,\eta}$ can be written as $y^2 = \beta(x)$. Let $\Xi_{h,\bar{\eta}}$ be the pull back of $\Xi_{h,\eta}$ on $\boldsymbol{P}^1(\boldsymbol{C}(x, \sqrt{\beta(x)}))$. By substituting y for $y' = (y - \sqrt{\beta(x)})/(y + \sqrt{\beta(x)})$, we may assume that $\Xi_{h,\bar{\eta}}$ corresponds to $y = 0, \infty$. We put $L := K(\sqrt[r]{\alpha(x)}, \sqrt{\beta(x)}, \sqrt{y})$. We note that L is a Kummer extension of K by the construction. Let $f : P \to S$ be the normalization of S in L. We show that P is a desired relatively minimal elliptic ruled surface. Let $f_1 : P_1 \to S$ be the normalization of S in $K(\sqrt[r]{\alpha(x)})$ and $\varphi_1 : P_1 \to B_1$ be the induced morphism from φ by taking the Stein factorization. By our construction, B_1 is nothing but the log canonical cover of (B, δ), hence B_1 is an elliptic curve. Since locally on S, we can write $\mathrm{div}\,\varphi^*\alpha(x) = \{r(b_t - 1)/b_t\}C_\varphi(t)$, $f_1^*C_\varphi(t)$ is Cartier, hence φ_1 is smooth and the ramification index at the generic point of prime divisors in $f_1^{-1}(C_\varphi(t))$ is b_t. Thus in the case where $\Xi_h = 0$, we get the assertion. Assume that $\Xi_h \neq 0$. Then we see that $f_1^{-1}(\Xi_h)$ is smooth and $e_{\mathrm{top}}(f_1^{-1}(\Xi_h)) = 0$ from $K_{P_1} + f_1^{-1}(\Gamma) + (1/2)f_1^{-1}(\Xi_h) = f_1^*(K_S + \Delta) \sim_{\boldsymbol{Q}} 0$ and $(f_1^{-1}(\Xi_h))^2 = 0$. Let $f_2 : P_2 \to P_1$

be the normalization of P_1 in $K(\sqrt[r]{\alpha(x)}, \sqrt{\beta(x)})$ and $\varphi_2 : P_2 \to B_2$ be the induced morphism from φ_1. If $K(\sqrt[r]{\alpha(x)}, \sqrt{\beta(x)}) \neq K(\sqrt[r]{\alpha(x)})$, then $f_1^{-1}(\Xi_h)$ is an elliptic curve and $B_2 \simeq f_1^{-1}(\Xi_h)$ by the construction. Therefore, P_2 is also a relatively minimal elliptic ruled surface and $f_2^{-1} f_1^{-1}(\Xi_h)$ is a disjoint union of two sections over which $f_3 : P \to P_2$ ramifies with the index two, where f_3 is the normalization map of P_2 in L. Thus we conclude that P is a relatively minimal elliptic ruled surface with a section $f^{-1}(\lfloor \Delta \rfloor)$. The last assertion follows from Lemma 3.9. ■

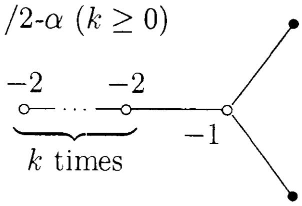
3.2.2 Case Type III

Lemma 3.10 (c.f. [3], [6] and [24], §5. Appendix) *Let $(S, \Delta; p)$ be a germ of normal log surface with $\boldsymbol{Q}$-boundary Δ at p such that $\lfloor \Delta \rfloor = 0$. Assume that (S, Δ) is purely log terminal and* $\mathrm{ind}_p(K_S + \Delta) = 2$. *Then there exists a resolution $f : T \to S$ such that $f^{-1}(p) \cup \operatorname{Supp} f_*^{-1}\Delta$ has only normal crossing singularities whose dual graph is one of the following types.*

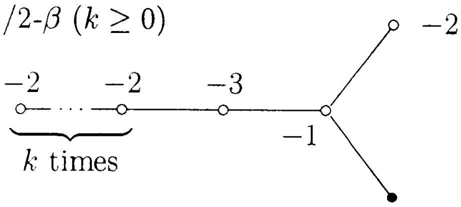
(1) *Case $p \in S$ smooth.*

$A_0/2$

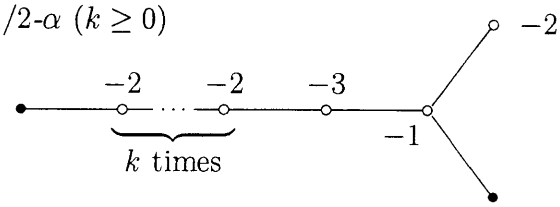
$A_{2k+1}/2$-α $(k \geq 0)$



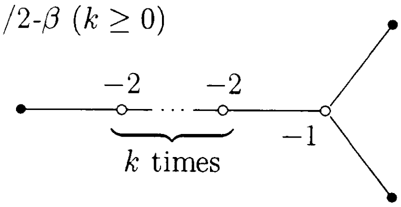
$A_{2k+2}/2$-β $(k \geq 0)$



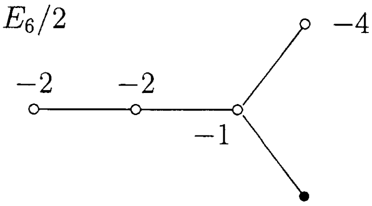
$D_{2k+5}/2$-α $(k \geq 0)$



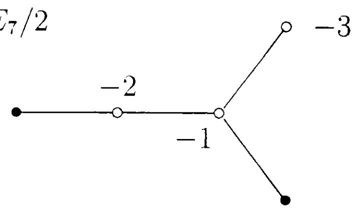
$D_{2k+4}/2$-β $(k \geq 0)$



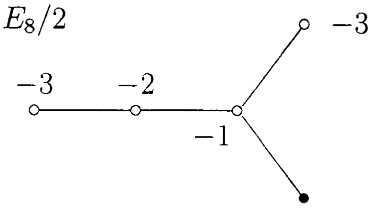
$E_6/2$



$E_7/2$

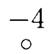


$E_8/2$

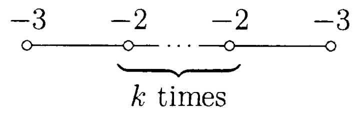


(2) *Case $p \in S$ singular.*

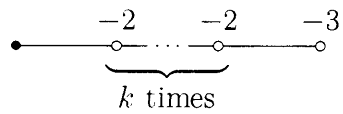
$A_1/2$-γ



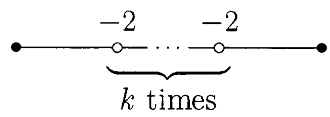
$A_{2k+3}/2$-δ $(k \geq 0)$



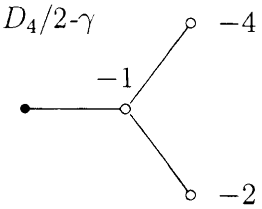
$A_{2k+2}/2$-ϵ $(k \geq 0)$



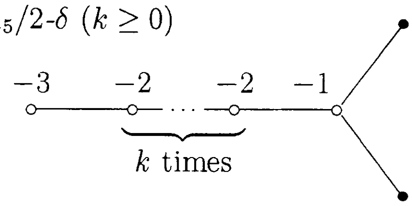
$A_{2k+1}/2$-ζ $(k \geq 1)$



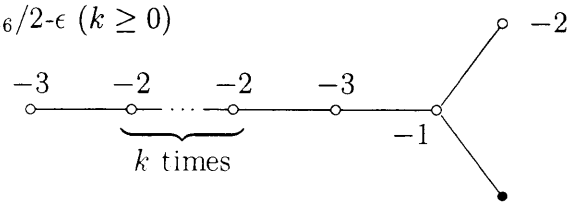
$D_4/2$-γ



$D_{2k+5}/2$-δ $(k \geq 0)$



$D_{2k+6}/2$-ϵ $(k \geq 0)$



In the above dual graphs $\bullet$ *denotes the strict transform of* Supp $\{\Delta\}$ *by* f *and* $\circ$ *denotes an* f*-exceptional rational curve. The number attached on* $\circ$ *is its self-intersection number.*

Proof. We give a proof for the convenience of the readers. In the case where S is smooth at p, the proof reduces to the well know fact about the resolution of plane curves, so we omit the proof in this case. Assume that S is singular at p and let $\mu : M \to S$ be the minimal resolution of $p \in S$. Put $\Xi :=$ Supp Δ and $\Xi' := \mu_*^{-1}\Xi$. Let $\mu^{-1}(p) = \cup_{i=1}^{l} E_i$ be the irreducible decomposition. Then we can write $K_M + (1/2)\Xi' + \sum_{i=1}^{l} a_i E_i = \mu^*(K_S + \Delta)$, where $a_i = 0$ or $1/2$ for any i by the assumption. Since we have $(K_M + (1/2)\Xi' + \sum_{i=1}^{l} a_i E_i, E_j) = 0$, we have $(\Xi' + \sum_{i \neq j} 2a_i E_i, E_j) = 4 + 2(1 - a_j)E_j^2$

for any j. Assume that $a_{i_0} = 0$ for some i_0, say for $i_0 = 1$. Then we see that $(\Xi' + \sum_{i \neq 1} 2a_i E_i, E_1) = 0$ since $E_1^2 \leq -2$, which implies that $\mu^{-1}(p) = \cup_{a_i=0} E \amalg \cup_{a_i=1/2} E_i$. Since $\mu^{-1}(p)$ is connected, we infer that $a_i = 0$ and $(\Xi', E_i) = 0$ for any i, hence $\mathrm{ind}_p(K_S + \Delta) = 1$, which is absurd. Thus we obtain $a_i = 1/2$ for any i and $(\Xi' + \sum_{i \neq j} E_i, E_j) = 4 + E_j^2$ for any j, which implies that $E_i^2 = -2, -3$ or -4 for any i. Assume that there exists E_{i_0}, say E_1, such that $E_1^2 = -4$. Then since we have $(\Xi' + \sum_{i \neq j} E_i, E_1) = 0$, we see that $\mu^{-1}(p) = E_1$ and $(\Xi', E_1) = 0$, which implies that we are in the case $A_1/2$-γ. Thus we may assume that $E_i^2 = -2, -3$. We note that we have $(\Xi' + \sum_{i \neq j} E_i, E_j) = 1$ if $E_j^2 = -3$ and 2 if $E_j^2 = -2$. Assume that there exists E_{i_0}, say E_1, such that $E_1^2 = -3$. Then it is easily seen that we are in the cases $A_{2k+3}/2$-δ $(k \geq 0)$ or $A_{2k+2}/2$-ϵ $(k \geq 0)$. Thus we may assume that $E_i^2 = -2$ for any i. Assume that $(\Xi', E_{i_0}) = 1$ for some i_0. Then by the inductive argument, we see that we are in the cases $A_{2k+1}/2$-ζ $(k \geq 2)$. Assume that $(\Xi', E_{i_0}) = 2$ for some i_0. Then we have $\mu^{-1}(p) = E_{i_0}$ since $(\sum_{i \neq i_0} E_i, E_{i_0}) = 0$, from which we deduce that we are in the cases $A_3/2$-ζ, $D_4/2$-γ, $D_{2k+5}/2$-δ $(k \geq 0)$ and $D_{2k+6}/2$-ϵ $(k \geq 0)$. ■

Definition 3.5 A normal complex projective surface S is called a *rank one log del Pezzo surface* if S has at worst quotient singularities, $-K_S$ is ample and the Picard number one and moreover, if S is Gorenstein, S is called a *rank one Gorenstein log del Pezzo surface.*

Singular rank one Gorenstein log del Pezzo surfaces are classified into the following 27 types listed as follows (see [14], [34]). We need the list below to study type III log surfaces.

Table IV

	$(-K_S)^2$	Sing S	$e_{\mathrm{orb}}(S)$		$(-K_S)^2$	Sing S	$e_{\mathrm{orb}}(S)$
(1)	8	A_1	5/2	(15)	1	E_8	241/120
(2)	6	$A_2 + A_1$	11/6	(16)	1	$E_7 + A_1$	73/48
(3)	5	A_4	11/5	(17)	1	$E_7 + A_2$	65/48
(4)	4	D_5	25/12	(18)	1	A_8	19/9
(5)	4	$A_3 + 2A_1$	5/4	(19)	1	$A_7 + A_1$	13/8
(6)	3	E_6	49/24	(20)	1	$A_5 + A_2 + A_1$	1
(7)	3	$A_5 + A_1$	5/3	(21)	1	D_8	49/24
(8)	3	$3A_2$	1	(22)	1	$D_6 + 2A_1$	17/16
(9)	2	E_7	97/48	(23)	1	$D_5 + A_3$	4/3
(10)	2	$D_6 + A_1$	25/16	(24)	1	$2D_4$	5/4
(11)	2	A_7	17/8	(25)	1	$4A_2$	1/3
(12)	2	$D_4 + A_3$	11/8	(26)	1	$2A_3 + 2A_1$	1/2
(13)	2	$A_5 + A_2$	3/2	(27)	1	$2A_4$	7/5
(14)	2	$2A_3 + A_1$	1				

Proposition 3.4 *Let (S, Δ) be of type* III. *Assume that $K_S + \Delta$ is Cartier and that $e_{\mathrm{orb}}(S \setminus \Delta) = 0$. Then there exists a birational morphism $\tau : S \to S^\flat$ which is composed of contractions of (-1)-curves with* Exc $\tau \subset \Delta$ *to a normal projective surface $S^\flat$ and $(S^\flat, \Delta^\flat)$ satisfies one of the followings.*

(*a*) *$S^\flat \simeq \Sigma_d$, where Σ_d is a Hirzebruch surface and $\Delta^\flat = \sum_{i=1}^4 \Gamma_i^\flat$ where $\Gamma_1^\flat$ and $\Gamma_2^\flat$ are two disjoint sections, $\Gamma_3^\flat$ and $\Gamma_4^\flat$ are two fibres.*

(*b*) *$S^\flat$ is a rank one Gorenstein log del Pezzo surface with* Sing $S^\flat = 3A_2$, $A_1 + 2A_3$, *or* $A_1 + A_2 + A_5$ *and $\Delta^\flat$ is a rational curve with only one node as its singularities.*

(*c*) *$S^\flat$ has a structure of a conic fibration $\varphi^\flat : S^\flat \to \boldsymbol{P}^1$ with $\rho(S^\flat/\boldsymbol{P}^1) = 1$ and* $\mathrm{Typ}(S^\flat, \Delta^\flat; \varphi^\flat) = (2(\text{I-2})_1 + (\text{II-1})_\infty; (\text{II-1})_1)$. *$\Delta^\flat = \Gamma_1^\flat + \Gamma_2^\flat$, where $\Gamma_1^\flat$ is a smooth rational curve such that $(\Gamma_1^\flat, \varphi^{\flat *}(t)) = 2$ for $t \in \boldsymbol{P}^1$ and that $(\Gamma_1^\flat)^2 = 0$ and $\Gamma_2^\flat$ is a fibre of $\varphi^\flat$.*

Proof. As we have already seen in Lemma 3.7, by applying the log minimal program on S with respect to K_S, we get a birational morphism $\tau : S \to S^\flat$ with Exc $\tau \subset \Delta$ to a normal projective surface $S^\flat$ (1) which is a rank one Gorenstein log del Pezzo surface or (2) which admits a structure of conic fibration $\varphi^\flat : S^\flat \to \boldsymbol{P}^1$ with $\rho(S^\flat/\boldsymbol{P}^1) = 1$. Assume that we are in the case (1). Since we have $e_{\text{orb}}(S^\flat) = e_{\text{top}}(\Delta^\flat) \in \boldsymbol{N}$, we see that $S^\flat \simeq \boldsymbol{P}^2$ and $e_{\text{top}}(\Delta^\flat) = 3$ or Sing $S^\flat = 3A_2$; $A_1 + 2A_3$, or $A_1 + A_2 + A_5$ and $e_{\text{top}}(\Delta^\flat) = 1$ by Table IV. Thus we are in the cases (a) and (b). Assume that we are in the case (2). By the assumption, we have $4 = e_{\text{top}}(\Delta^\flat) + \nu((\text{I-2})_1) + \sum_{k\geq 1}\{(4k-1)/(4k)\}\nu((\text{II-3})_{1,k})$. where $\nu((\text{I-2})_1)$ and $\nu((\text{II-3})_{1,k})$ denote the number of fibres of type $(\text{I-2})_1$ and $(\text{II-3})_{1,k}$ respectively. Assume that $\Delta^\flat$ contains an irreducible component $\Gamma_0^\flat$ with $(\Gamma_0^\flat, \varphi^{\flat *}(t)) = 2$ for $t \in \boldsymbol{P}^1$. Since $\Delta^\flat - \Gamma_0^\flat$ is composed of fibres of $\varphi^\flat$, we can write $\Delta^\flat = \Gamma_0^\flat + \Gamma_1^\flat$ where $\Gamma_1^\flat$ is a fibre of $\varphi^\flat$ and we see that $e_{\text{top}}(\Delta^\flat) = 2$ and that $\nu((\text{I-2})_1) + \sum_{k\geq 1}\{(4k-1)/(4k)\}\nu((\text{II-3})_{1,k}) = 2$. On the other hand, since $\varphi^\flat$ induces a double cover $\varphi^\flat : \Gamma_0^\flat \simeq \boldsymbol{P}^1 \to \boldsymbol{P}^1$, we have $\nu((\text{I-2})_1) + \sum_{k\geq 1}\nu((\text{II-3})_{1,k}) \leq 2$ by the Hurwitz's formula. So we have $\nu((\text{I-2})_1) = 2$ and $\nu((\text{II-3})_{1,k}) = 0$ for any k, hence we are in the case (c). Assume that there exist two sections of $\varphi^\flat$, $\Gamma_1^\flat$ and $\Gamma_2^\flat$. Then we see that $\nu((\text{I-2})_1) = \nu((\text{II-3})_{1,k}) = 0$ for any k and that $\varphi^\flat : S^\flat \to \boldsymbol{P}^1$ is smooth and $e_{\text{top}}(\Delta^\flat) = 4$. Thus we are in the case (a). ∎

Let (S, Δ) be of type III and assume that $K_S + \Delta$ is not Cartier and let $\lfloor\Delta\rfloor = \sum_{i=0}^{b}\Gamma_i$ be the irreducible decomposition. Then $b \geq 1$ and Γ_i's form a linear chain of smooth rational curves, that is, $(\Gamma_i, \Gamma_j) = 0$, if $|i-j| > 1$, 1, if $|i-j| = 1$ and $\{p \in \lfloor\Delta\rfloor \mid \text{ind}_p(K_S + \Delta) = 2\}$, namely the set of points $p \in \lfloor\Delta\rfloor$ at which $K_S + \Delta$ is not Cartier consists of four points p_i $(1 \leq i \leq 4)$, where $p_i \in \Gamma_0$ for $i = 1, 2$ and $p_i \in \Gamma_b$ for $i = 3, 4$ by [44], Lemma 3.2 (1). Comparing the Euler numbers, we have

$$e_{\text{top}}(\tilde{\Delta}) = 2e_{\text{top}}(\lfloor\Delta\rfloor) - 4. \tag{3.3}$$

Let $(S^\flat, \Delta^\flat)$ be a log surface obtained from (S, Δ) by applying log minimal program with respect to $K_S + \{\Delta\}$. Write $\Delta^\flat = \Gamma^\flat + (1/2)\Xi^\flat$, where $\Gamma^\flat := \lfloor\Delta^\flat\rfloor$ and $\Xi^\flat := 2\{\Delta^\flat\}$. Put $d := (-K_{\tilde{S}^\flat})^2$. We note that we have

$$e_{\text{top}}(\Xi^{\flat\nu}) = \deg \text{Diff}_{\Xi^{\flat\nu}}(\Gamma^\flat) - \frac{1}{2}(\Xi^\flat)^2 \tag{3.4}$$

from $K_{S^\flat} + \Gamma^\flat + (1/2)\Xi^\flat \sim_{\boldsymbol{Q}} 0$. Let $\mu^\flat : M^\flat \to S^\flat$ be the minimal resolution of $S^\flat$ and put $\delta K^2 := K_{S^\flat}^2 - K_{M^\flat}^2$, $\delta\rho := \rho(M^\flat) - \rho(S^\flat)$. Then we have

$$K_{S^\flat}^2 = K_{M^\flat}^2 + \delta K^2 = 10 - \rho(M^\flat) + \delta K^2 = 10 - \rho(S^\flat) - \delta\rho + \delta K^2.$$

On the other hand, we have

$$K_{S^\flat}^2 = (\Gamma^\flat + \frac{1}{2}\Xi^\flat)^2 = \frac{d}{2} + (\Gamma^\flat, \Xi^\flat) + \frac{1}{4}(\Xi^\flat)^2,$$

hence

$$\delta K^2 - \delta\rho = \frac{d}{2} + (\Gamma^\flat, \Xi^\flat) + \frac{1}{4}(\Xi^\flat)^2 + \rho(S^\flat) - 10. \tag{3.5}$$

If we have $\rho(S^\flat) = 1$, for any divisor D on $S^\flat$, there exists $a \in \boldsymbol{Q}$ such that D is numerically equivalent to $a\Gamma^\flat$. Noting that $D^2 = a^2(\Gamma^\flat)^2 = (1/2)da^2$ by $(\Gamma^\flat)^2 = (1/2)(\tilde{\Delta}^\flat)^2 = (1/2)(-K_{\tilde{S}^\flat})^2 = d/2$ and that $(\Gamma^\flat, D) = a(\Gamma^\flat)^2 = (1/2)da$, we deduce that

$$D^2 = (2/d)(\Gamma^\flat, D)^2. \tag{3.6}$$

Assume that $e_{\text{orb}}(\tilde{S} \setminus \tilde{\Delta}) = 0$. Then from Lemma 3.7, we have

$$\rho(S) - \rho(S^\flat) = e_{\text{top}}(\lfloor\Delta\rfloor) - e_{\text{top}}(\lfloor\Delta^\flat\rfloor). \tag{3.7}$$

Let LC $(S^\flat, \Delta^\flat)$ be the set which consists of all point $p \in S^\flat$ such that $(S^\flat, \Delta^\flat)$ is not log terminal at p. We see that $\mathrm{LC}(S^\flat, \Delta^\flat)$ consists of at most two points and for $p \in \mathrm{LC}(S^\flat, \Delta^\flat)$, $\Gamma^\flat$ and $\Xi^\flat$ are smooth at p and $(\Gamma^\flat, \Xi^\flat)_p = 2$. Let $\pi^\flat : \tilde{S}^\flat \to S^\flat$ be the log canonical cover of $(S^\flat, \Delta^\flat)$. Put $\tilde{\Delta}^\flat := \pi^{\flat -1}\Delta^\flat$ and $s := \mathrm{Card}\ \mathrm{LC}(S^\flat, \Delta^\flat)$. Comparing the Euler numbers again, we have

$$e_{\mathrm{top}}(\tilde{\Delta}^\flat) = 2e_{\mathrm{top}}(\lfloor \Delta^\flat \rfloor) - 4 + s. \tag{3.8}$$

If $\mathrm{ind}(K_{S^\flat} + \Delta^\flat) = 1$ for $p \in S^\flat \setminus \lfloor \Delta^\flat \rfloor$, then $p \notin \Xi^\flat$ and $S^\flat$ has at worst Du Val singularity at p. In this situation, we shall say that $(S^\flat, \Delta^\flat)$ has singularity of type A_n at p, if $S^\flat$ has Du Val singularity of type A_n at p for instance. Let $\psi : U \to B$ be a complex elliptic surface over a smooth complete curve B with a section such that U is minimal over B. We shall write

$$\begin{aligned}\mathrm{Typ}(U;\psi) &= \sum_{k\geq 1}\nu(\mathrm{I}_k)\mathrm{I}_k + \sum_{l\geq 0}\nu(\mathrm{I}_l^*)\mathrm{I}_l^* + \nu(\mathrm{II})\mathrm{II} + \nu(\mathrm{II}^*)\mathrm{II}^* + \nu(\mathrm{III})\mathrm{III} + \nu(\mathrm{III}^*)\mathrm{III}^* \\ &\quad + \nu(\mathrm{IV})\mathrm{IV} + \nu(\mathrm{IV}^*)\mathrm{IV}^*,\end{aligned}$$

where $\nu(\mathcal{T})$ denotes the number of singular fibres of type $\mathcal{T}$ in the Kodaira's notation ([25]). It turns out that the following lemma works better than the fixed point formula in our cases.

Lemma 3.11 *Let G denote a finite subgroup of k-automorphism group of normal variety X defined over an algebraically closed field k and let $f : X \to X/G$ be the quotient morphism. Assume that X/G is $\boldsymbol{Q}$-factorial. Then f induces the isomorphism* $\mathrm{Pic}\,(X/G) \otimes \boldsymbol{Q} \simeq (\mathrm{Pic}\, X \otimes \boldsymbol{Q})^G$.

Proof. The injectivity is trivial. As for the surjectivity, we only have to note that $f^*f_*D = \sum_{g\in G} g^*D$ for any $\boldsymbol{Q}$-divisor D on X, where f_*D denotes the pushforward of D by f as a cycle. ■

Assume that $\Xi^\flat \neq 0$ and that there exists a birational morphism $\eta : W \to S^\flat$ from a normal surface W with $\Delta^W \geq 0$ such that $\eta_*^{-1}\Xi^\flat$ is nef and $(\eta_*^{-1}\Xi^\flat)^2 = 0$. By applying the log abundance theorem (see [13] or [11], Theorem 11.1.3) to $K_W + \Delta^W + \varepsilon\Xi^\flat \sim_{\boldsymbol{Q}} \varepsilon\Xi^\flat$ for sufficiently small positive rational number ε, we obtain a proper surjective connected morphism $\varphi : W \to \boldsymbol{P}^1$. In what follows, we shall frequently use this technique. In Propositions 3.5, 3.6, 3.7, 3.8 and 3.9, a log surface (S, Δ) is assumed to be of type III such that $K_S + \Delta$ is not Cartier and $e_{\mathrm{orb}}(\tilde{S} \setminus \tilde{\Delta}) = 0$.

Proposition 3.5 *Assume that $\tilde{S}$ is smooth. Then there exists a birational morphism $\tau : S \to S^\flat$ with* $\mathrm{Exc}\,\tau \subset \lfloor \Delta \rfloor$ *such that there exists a conic fibration $\varphi^\flat : S^\flat \to \boldsymbol{P}^1$ with $\rho(S^\flat/\boldsymbol{P}^1) = 1$ and that $(S^\flat, \Delta^\flat)$ is log terminal, where $\Delta^\flat := \tau_*\Delta$. Moreover, there exists a sequence of $\mathcal{S}$-elementary transformation such that for the resulting log surface $(\hat{S}^\flat, \hat{\Delta}^\flat)$, $\hat{S}^\flat \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ with $\hat{\varphi}^\flat$ being the second projection and $\hat{\Delta}^\flat = \sum_{i=0}^2 \hat{\Gamma}_i^\flat + (1/2)\sum_{j=1}^2 \hat{\Xi}_j^\flat$, where $\hat{\Gamma}_i^\flat$ is a fibre of the first projection for $i = 1, 2$ and $\hat{\Gamma}_0^\flat$, $\hat{\Xi}_j^\flat$ are fibres of the second projection for $j = 1, 2$.*

Proof. From the exact sequence: $0 \to \Omega^1_{\tilde{S}} \to \Omega^1_{\tilde{S}}(\log \tilde{\Delta}) \to \mathcal{O}_{\tilde{\Delta}^{\flat\nu}} \to 0$, we have the following exact sequence:

$$0 \to H^0(\Omega^1_{\tilde{S}}(\log \tilde{\Delta})) \to H^0(\mathcal{O}_{\tilde{\Delta}^{\flat\nu}}) \to \mathrm{Pic}\ \tilde{S} \otimes \boldsymbol{C} \to 0,$$

since $\mathrm{Pic}\ \tilde{S} \otimes \boldsymbol{Q}$ is generated by all the irreducible components of $\tilde{\Delta}$. By taking the invariant subspaces under the induced action of $G := \mathrm{Gal}(\tilde{S}/S)$, we get the the following exact sequence:

$$0 \to H^0(\Omega^1_{\tilde{S}}(\log \tilde{\Delta}))^G \to H^0(\mathcal{O}_{\lfloor \Delta \rfloor^\nu}) \to \mathrm{Pic}\ S \otimes \boldsymbol{C} \to 0,$$

where $H^0(\Omega^1_{\tilde{S}}(\log \tilde{\Delta}))^G$ is the invariant subspace under the G-action. Since $\tilde{S} \setminus \tilde{\Delta}$ is a two dimensional algebraic torus, we see that $h^0(\Omega^1_{\tilde{S}}(\log \tilde{\Delta})) = 2$ and that the natural morphism $\bigwedge^2 H^0(\Omega^1_{\tilde{S}}(\log \tilde{\Delta})) \to H^0(\mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + \tilde{\Delta}))$ is an isomorphism. Since the eigenvalue of the induced action of a generator of G

on $H^0(\mathcal{O}_{\tilde{S}}(K_{\tilde{S}}+\tilde{\Delta}))$ is -1, $H^0(\Omega^1_{\tilde{S}}(\log\tilde{\Delta}))^G$ is one dimensional. Thus we get $\rho(S)=e_{\mathrm{top}}(\lfloor\Delta\rfloor)-2$. Firstly, we show that there exists a $\boldsymbol{P}^1$-fibration $\varphi: S\to \boldsymbol{P}^1$. Assume that $\rho(S^\flat)=1$. From (3.7), we get $e_{\mathrm{top}}(\lfloor\Delta^\flat\rfloor)=3$ which implies $e_{\mathrm{top}}(\tilde{\Delta}^\flat)=2+s$ by (3.8), hence $\rho(\tilde{S}^\flat)=e_{\mathrm{top}}(\tilde{\Delta}^\flat)-2=s$. We note that $\Xi^\flat\neq 0$ since $s>0$. By Noether's equality, we have $K^2_{\tilde{S}^\flat}=10-s$. Since $S^\flat$ has only Du Val singularities of type A_1, we have $\delta K^2=0$. Combining this with $\delta\rho=4-(\Gamma^\flat,\Xi^\flat)$, we obtain $(\Xi^\flat)^2-2s=0$ from (3.5). Let $\iota: S^{\flat\prime}\to S^\flat$ be the extraction of all the divisors over $\mathrm{LC}(S^\flat,\Delta^\flat)$ whose log discrepancy with respect to $(S^\flat,\Delta^\flat)$ are 0. Noting that $(\iota_*^{-1}\Xi^\flat)^2=(\Xi^\flat)^2-2s=0$ by our construction and that $\Xi^\flat$ is irreducible by our assumption that $\rho(S^\flat)=1$ and $\tilde{S}^\flat$ is smooth, some multiple of $\iota_*^{-1}\Xi^\flat$ defines a $\boldsymbol{P}^1$-fibration $\varphi^{\flat\prime}: S^{\flat\prime}\to\boldsymbol{P}^1$. Since τ factors into $\iota\circ\tau'$, where $\tau': S\to S^{\flat\prime}$ is a birational morphism, we conclude that there exists a $\boldsymbol{P}^1$-fibration $\varphi: S\to\boldsymbol{P}^1$ and we may assume that $S^\flat$ has a structure of conic fibration $\varphi^\flat: S^\flat\to\boldsymbol{P}^1$ with $\rho(S^\flat/\boldsymbol{P}^1)=1$. Under this assumption, we have $e_{\mathrm{top}}(\lfloor\Delta^\flat\rfloor)=4$, $e_{\mathrm{top}}(\tilde{\Delta}^\flat)=4+s$, $\rho(\tilde{S}^\flat)=2+s$, $K^2_{\tilde{S}^\flat}=8-s$, $(\Gamma^\flat)^2=4-(1/2)s$ and $(\Xi^\flat)^2=2s$ in the same way as above. Let $\Gamma^\flat=\sum_{i=0}^2\Gamma_i^\flat$ be the irreducible decomposition and assume that $\Gamma_0^\flat$ is horizontal with respect to $\varphi^\flat$. We may assume that $\Gamma_1^\flat$ is contained in a fibre of $\varphi^\flat$ since $(\Gamma^\flat,\varphi^*(t))\leq 2$ for $t\in\boldsymbol{P}^1$. Firstly, assume that $(\Gamma_0^\flat,\varphi^*(t))=2$ for $t\in\boldsymbol{P}^1$. Then $\Gamma_1^\flat$, $\Gamma_2^\flat$ and $\Xi^\flat$ are contained in fibres of $\varphi^\flat$. Noting that $(\Gamma_0^\flat,\Gamma_i^\flat)=1$ for $i=1,2$, we have $(\Gamma_0^\flat,\Xi^\flat)=0$. Thus we conclude that $\Xi^\flat=0$ and that $s=0$, hence $(\Gamma_0^\flat)^2=0$. Since $(-K_{S^\flat},C)=(\Gamma^\flat,C)>0$ for any irreducible curve C on $S^\flat$, we see that $\overline{\mathrm{NE}}_{K_{S^\flat}}(S^\flat)=0$ and that $\overline{\mathrm{NE}}(S^\flat)$ is spanned by exactly two extremal rays with respect to $K_{S^\flat}$ one of which corresponds to $\varphi^\flat$. Let $\check{\varphi}^\flat$ be the extremal contraction of the other extremal ray. If $\check{\varphi}^\flat$ is birational, then, by Lemma 3.7, Exc $\check{\varphi}^\flat\subset\Gamma^\flat$, which is absurd. Thus $\check{\varphi}^\flat$ defines another conic fibration $\check{\varphi}^\flat\to\boldsymbol{P}^1$ with $\rho(S^\flat/\boldsymbol{P}^1)=1$ such that $\Gamma_1^\flat$ and $\Gamma_2^\flat$ is horizontal with respect to $\check{\varphi}^\flat$. Thus we may assume that Γ_0 is a section of φ at first. Assume that $\Gamma_2^\flat$ is contained in a fibre of $\varphi^\flat$. Then we have $(\Gamma_0^\flat)^2=-(1/2)s$. Since $(\Gamma_0^\flat)^2\in\boldsymbol{Z}$, we have $s=0$ or 2. We may assume that $s=0$ for if we assume that $s=2$, then we have $(\Gamma_0^\flat)^2=-1$ and we can use the argument in the case $\rho(S^\flat)=1$. Under the assumption that $s=0$, there exists another another conic fibration $\check{\varphi}^\flat\to\boldsymbol{P}^1$ with $\rho(S^\flat/\boldsymbol{P}^1)=1$ such that $\Gamma_1^\flat$ and $\Gamma_2^\flat$ is horizontal with respect to $\check{\varphi}^\flat$, by the same argument as above. Thus we may assume that Γ_0 and Γ_2 are sections of $\varphi^\flat$ at first. Then we have $(\Xi^\flat,\varphi^{\flat *}(t))=0$ for $t\in\boldsymbol{P}^1$ and $s=0$. Since we have $\rho(\tilde{S}^\flat)=2$, $(S^\flat,\Delta^{\flat+}_{\varphi^\flat}(t))$ is log canonical for any $t\in\boldsymbol{P}^1$ and Supp $\lfloor\mathrm{Diff}_{C_{\varphi^\flat}(t)}(\Delta^{\flat-}_{\varphi^\flat}(t))\rfloor\subset\lfloor\Delta^{\flat-}_{\varphi^\flat}(t)\rfloor$ for any $t\in\boldsymbol{P}^1$ by the argument in the proof of Lemma 3.8. Therefore, possibly after $\mathcal{S}$-elementary transformations, $\varphi^\flat$ has only fibres of type $(\mathrm{II}\text{-}1)_b$ as in Lemma 3.3. Thus we get the assertion. ■

Assume that $\tilde{S}$ is not smooth. By Proposition 3.4, we see that $\rho(\tilde{S})=e_{\mathrm{top}}(\tilde{\Delta})$ and all the irreducible components of $\tilde{\Delta}$ give a complete basis of Pic $\tilde{S}\otimes\boldsymbol{Q}$, hence we get

$$\rho(S)=e_{\mathrm{top}}(\tilde{\Delta})/2+1=e_{\mathrm{top}}(\lfloor\Delta\rfloor)-1.$$

by Lemma 3.11. Combining this with (3.7), we obtain $e_{\mathrm{top}}(\lfloor\Delta^\flat\rfloor)=1+\rho(S^\flat)$. From (3.8), we have $\rho(\tilde{S}^\flat)=e_{\mathrm{top}}(\tilde{\Delta}^\flat)=2\rho(S^\flat)-2+s$. Let $\tilde{\mu}^\flat:\tilde{M}^\flat\to\tilde{S}^\flat$ be the minimal resolution of $\tilde{S}^\flat$ and put $\delta\tilde{\rho}:=\rho(\tilde{M}^\flat)-\rho(\tilde{S}^\flat)$. Then by Noether's equality, we have

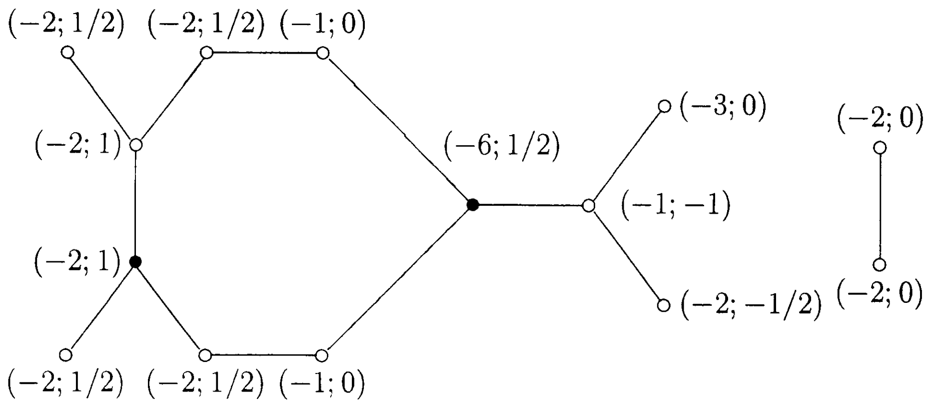
$$\begin{aligned} d &= K^2_{\tilde{M}^\flat}=10-\rho(\tilde{M}^\flat)\\ &= 10-2\rho(S^\flat)-s-\delta\tilde{\rho}\\ &= -2\rho(S^\flat)-s+\begin{cases} 8, & \text{if Sing } \tilde{S}=4A_1,\\ 6, & \text{if Sing } \tilde{S}=3A_2,\\ 5, & \text{if Sing } \tilde{S}=A_1+2A_3,\\ 4 & \text{if Sing } \tilde{S}=A_1+A_2+A_5. \end{cases} \end{aligned} \tag{3.9}$$

Proposition 3.6 *Assume that* $\mathrm{Sing}\,\tilde{S} = 3A_2$. *Then one of the followings holds.*

(1) $S^\flat \simeq \boldsymbol{P}^2$ *and* $\Delta^\flat = \Gamma^\flat + (1/2)\Xi^\flat$, *where* $\Xi^\flat$ *is an irreducible quartic curve with three cusps and* $\Gamma^\flat$ *is a double tangent.*

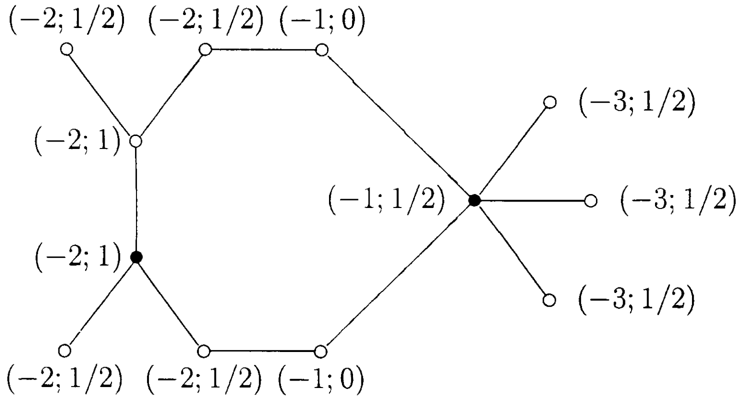
(2) $S^\flat$ *is a rank one Gorenstein log del Pezzo surface with* $\mathrm{Sing}\,S^\flat = A_1 + A_2$ *and there exists a birational morphism* $\lambda : U \to S^\flat$ *from a smooth projective surface* U *such that* U *admits a structure of elliptic surface with a section* $\psi : U \to \boldsymbol{P}^1$ *which is minimal over* $\boldsymbol{P}^1$ *with* $\mathrm{Typ}(U;\psi) = \mathrm{I}_1^* + \mathrm{II} + \mathrm{I}_3$ *and that* $\Delta^{\flat\, U} = (1/2)\psi^*(t+u)$ *where* $\psi^*(t)$ *and* $\psi^*(u)$ *are the singular fibres of type* I_1^* *and* II *respectively.*

Proof. The possible singular types of $(S^\flat, \Delta^\flat)$ on $S^\flat \setminus \lfloor \Delta^\flat \rfloor$ are types A_2, $A_0/2$, $A_2/2\text{-}\beta$ and $A_2/2\text{-}\epsilon$ and we see that $\Xi^\flat$ is irreducible if $\rho(S^\flat) = 1$. In what follows, $\nu(\mathcal{T})$ denotes the number of points of type $\mathcal{T}$. We note that we have $2\nu(A_2) + \nu(A_2/2\text{-}\beta) + \nu(A_2/2\text{-}\epsilon) = 3$. Firstly, we consider the case in which $S^\flat$ is a rank one log del Pezzo surface. Assume that $s = 2$. Then we have $\rho(\tilde{S}^\flat) = 2$, $d = 2$, $(\Gamma^\flat)^2 = 1$, $(\Gamma^\flat, \Xi^\flat) = 4$ and $(\Xi^\flat)^2 = 16$. Thus we have $\delta K^2 - \delta\rho = 0$. On the other hand, since we have $\delta K^2 = (1/3)\nu(A_2/2\text{-}\epsilon)$ and $\delta\rho = 2\nu(A_2) + \nu(A_2/2\text{-}\epsilon)$, we have $\delta K^2 - \delta\rho = -2\nu(A_2) - (2/3)\nu(A_2/2\text{-}\epsilon)$, hence $\nu(A_2) = \nu(A_2/2\text{-}\epsilon) = 0$ and $\nu(A_2/2\text{-}\beta) = 3$, which implies that we are in the case (1). Assume that $s = 1$. Then we have $d = 3$ and $\rho(\tilde{S}^\flat) = 1$. We note that we have $(\Gamma^\flat, \Xi^\flat) \equiv 1 \pmod 2$ from $(\Gamma^\flat)^2 = 3/2$, so we have $(\Gamma^\flat, \Xi^\flat) = 3$, $(\Xi^\flat)^2 = 6$ and $\delta K^2 - \delta\rho = -3$. On the other hand, since we have $\delta K^2 = (1/3)\nu(A_2/2\text{-}\epsilon)$ and $\delta\rho = 1 + 2\nu(A_2) + \nu(A_2/2\text{-}\epsilon)$, we get $\delta K^2 - \delta\rho = -1 - 2\nu(A_2) - (2/3)\nu(A_2/2\text{-}\epsilon)$. Therefore, we obtain $3\nu(A_2) + 2\nu(A_2/2\text{-}\epsilon) = 3$. Hence $(\nu(A_2), \nu(A_2/2\text{-}\epsilon)) = (1,0)$ or $(0,3)$. Since we have $\deg \mathrm{Diff}_{\Xi^{\flat\nu}}(\Gamma^\flat) = 3 + 2\nu(A_2/2\text{-}\beta) + (2/3)\nu(A_2/2\text{-}\epsilon)$, we have $e_{\mathrm{top}}(\Xi^{\flat\nu}) = 2\nu(A_2/2\text{-}\beta) + (2/3)\nu(A_2/2\text{-}\epsilon)$. Assume that $(\nu(A_2), \nu(A_2/2\text{-}\epsilon)) = (1,0)$. Then we have $\nu(A_2/2\text{-}\beta) = 1$ and $e_{\mathrm{top}}(\Xi^{\flat\nu}) = 2$, i.e., $\Xi^{\flat\nu} \simeq \boldsymbol{P}^1$. We can see that there exists a birational morphism $\eta : W \to S^\flat$ from a smooth rational surface with $\rho(W) = 13$ such that $\mathrm{Supp}\,\eta_*^{-1}\Delta^\flat \cup \mathrm{Exc}\,\eta$ has only simple normal crossing singularities whose dual graph is as follows.



In the above dual graph, • denotes a curves which is not η-exceptional and ∘ denotes a η-exceptional curve. $(*;*)$ denotes (the self-intersection number; the multiplicities in $\Delta^{\flat W}$. We shall use the same notations in the subsequent dual graphs.

We see that there exists a birational morphism $v : W \to U$ to a smooth rational surface U with $\rho(U) = 10$ such that $\eta = \lambda \circ v$, where $\lambda : U \to S^\flat$ is a birational morphism and that $\lambda_*^{-1}\Xi^\flat$ is nef with $(\lambda_*^{-1}\Xi^\flat)^2 = 0$ and $(\Delta^{\flat U} - (1/2)\lambda_*^{-1}\Xi^\flat, \lambda_*^{-1}\Xi^\flat) = 0$. We note that $\Delta^{\flat U} \geq 0$. Applying the log abundance theorem to $K_U + \Delta^{\flat U} + \varepsilon\lambda_*^{-1}\Xi^\flat$ for sufficiently small positive rational number ε, we obtain a connected surjective morphism $\psi : U \to \boldsymbol{P}^1$, which turns out to be a minimal elliptic fibration since $K_U + \Delta^{\flat U} \sim_{\boldsymbol{Q}} 0$ and $\rho(U) = 10$. Put $t := \psi(\lambda_*^{-1}\Gamma^\flat)$ and $u := \psi(\lambda_*^{-1}\Xi^\flat)$. Then we see that $\psi^*(t)$ and $\psi^*(u)$ are singular fibre of type I_1^* and II respectively and that $\Delta^{\flat\ U} = (1/2)\psi^*(t+u)$. Since we have $\sum_{v \neq t,u} e_{\mathrm{top}}(\psi^{-1}(v)) = 3$ and there exists $v \in \boldsymbol{P}^1$ such that the singular fibre $\psi^*(v)$ has the dual graph containing A_2 as its dual subgraph, we deduce that $\psi^*(v)$ is of type I_3 and that ψ is smooth over $\boldsymbol{P}^1 \setminus \{t, u, v\}$. Thus we are in the case (2). We shall show that the case $(\nu(A_2), \nu(A_2/2\text{-}\epsilon)) = (0,3)$ does not occurs. Assume that $(\nu(A_2), \nu(A_2/2\text{-}\epsilon)) = (0,3)$. Then we have $\nu(A_2/2\text{-}\beta) = 0$ and $\Xi^{\flat\nu} \simeq \boldsymbol{P}^1$. We see that that there exists a birational morphism $\eta : W \to S^\flat$, where W is a rational surface with $\rho(W) = 11$, such that Supp $\eta_*^{-1}\Delta^\flat \cup \mathrm{Exc}\ \eta$ has only simple normal crossing singularities whose dual graph is as follows.



Let $v : W \to U$ be a contraction of $\eta_*^{-1}\Xi^\flat$. Then we see that U is smooth rational surface which admits a structure of an elliptic fibration $\psi : U \to \boldsymbol{P}^1$ by the same argument as above, such that there exists two points t, $u \in \boldsymbol{P}^1$ with $\psi^*(t)$ and $\psi^*(u)$ being singular fibres of type I_1^* and IV respectively. From the above dual graph, we can see that there exists a smooth rational curve E on U such that $(\psi^*(t), E) = 1$ and $(\psi^*(u), E) = 3$ but which is absurd. Secondarily, we consider the case in which there exists a conic fibration $\varphi^\flat : S^\flat \to \boldsymbol{P}^1$ with $\rho(S^\flat/\boldsymbol{P}^1) = 1$. In this case, we note that we have $e_{\mathrm{top}}(\lfloor\Delta^\flat\rfloor) = 3$. Let $\Gamma^\flat = \Gamma_0^\flat + \Gamma_1^\flat$ be the irreducible decomposition. Assume that $s = 2$. Then we have $d = 0$ and $\rho(\tilde{S}^\flat) = 4$, hence the self-intersection numbers of all the irreducible components of $\tilde{\Delta}^\flat$ are (a) -1, -2, -2, -3 or (b) -1, -1, -3, -3. The case (a) is excluded for if we are in the case (a), the -1-curve and the (-3)-curve is invariant under the action of the covering transformation group, but which is absurd. Thus we may assume that $(\Gamma_0^\flat)^2 = 0$, $(\Gamma_1^\flat)^2 = -2$. Since Pic $S^\flat \otimes \boldsymbol{Q}$ is generated by $\Gamma_0^\flat$ and $\Gamma_1^\flat$, we have $\Xi^\flat \sim_{\boldsymbol{Q}} 6\Gamma_0^\flat + 2\Gamma_1^\flat$ from $(\Gamma_i^\flat, \Xi^\flat) = 2$ for $i = 0$, 1, which implies that $(\Xi^\flat)^2 = 16$. Thus we have $0 = \delta K^2 - \delta\rho = -2\nu(A_1) - (2/3)\nu(A_2/2\text{-}\epsilon)$, hence $\nu(A_2) = \nu(A_2/2\text{-}\epsilon) = 0$ and $\nu(A_2/2\text{-}\beta) = 3$. Moreover, since we have $\deg \mathrm{Diff}_{\Xi^{\flat\nu}}(\Gamma^\flat) = 4 + 2\nu(A_2/2\text{-}\beta) = 10$, we have $e_{\mathrm{top}}(\Xi^{\flat\nu}) = 2$. Since $\varphi^\flat$ is smooth, It is easily seen that $\Gamma_0^\flat$ is a fibre, $\Gamma_0^\flat$ is a section of $\varphi^\flat$ and $\Xi^\flat$ is irreducible, hence $\Xi^{\flat\nu} \simeq \boldsymbol{P}^1$. $\varphi^\flat$ induces a double cover $\varphi^\flat : \Xi^{\flat\nu} \to \boldsymbol{P}^1$ with at least four branching point, which contradicts the Hurwitz's formula. Assume that $s = 1$. Then we have $d = 1$ and $\rho(\tilde{S}^\flat) = 3$, hence the self-intersection numbers of all the irreducible components of $\tilde{\Delta}^\flat$ are -1, -1, -2. Assume that $\mathrm{LC}(S^\flat, \Delta^\flat) \cap \Gamma_0^\flat \neq \emptyset$. Then we have $(\Gamma_0^\flat)^2 = -1$ and $(\Gamma_1^\flat)^2 = -1/2$ and we see that $\Gamma_1^\flat$ supports an extremal ray with respect to $K_{S^\flat} + \Delta^\flat$. Contracting $\Gamma_1^\flat$, this case reduces to the rank one log del Pezzo case. Assume that $s = 0$. Then we have $d = 2$ and $\rho(\tilde{S}^\flat) = 2$, hence the self-intersection numbers of all the irreducible components of $\tilde{\Delta}^\flat$ are -1, -1, hence $(\Gamma_i^\flat)^2 = -1/2$ for

$i = 0, 1$. By the same way as in the previous argument, we conclude that this case also reduces to the rank one log del Pezzo case. Thus we get the assertion. ∎

Example 3.1 The example of the case (1) is well known and goes back to [66]. To show the existence, we only have to take a dual curve of a nodal cubic as $\Xi^\flat$. There exists exactly one double tangent by the Plücker's formulae. It is also well known that for a suitable choice of homogeneous coordinates, a defining equation $f(X,Y,Z)$ of a nodal cubic curve can be written as $f(X,Y,Z) = Y^2Z - X^2(X+Z)$ and the defining equation $\hat{f}(X,Y,Z)$ of its dual curve is calculated to be $\hat{f}(X,Y,Z) = 4X^3Z + 4X^4 - 36XY^2Z - 8X^2Y^2 - 27Y^2Z^2 + 4Y^4$. (see, for example, [5], p.585, [10], p.131, Exercise (4.7)[1] or [32], Table 6.8). In particular, we see that such a log surface $(S^\flat, \Delta^\flat)$ is unique up to isomorphisms. As for the case (2), the existence of a minimal rational elliptic surface with a section $\psi : U \to \boldsymbol{P}^1$ with $\mathrm{Typ}(U;\psi) = \mathrm{I}_1^* + \mathrm{II} + \mathrm{I}_3$ is known (see [46]). By the list in [43], we see that $\mathrm{MW}(U_\eta) = \boldsymbol{Z}P$ for some $P \in \mathrm{MW}(U_\eta)$ with $< P, P >= 1/12$, where $\mathrm{MW}(U_\eta)$ denotes the Mordell-Weil group of the generic fibre U_η and $< *, * >$ denotes the height paring in the Shioda's sense. Put $Q := 3P$. Then from the formula (8.12) in [57], we have

$$\frac{3}{4} =< Q, Q >= 2 + 2(QO) - \sum_{v\in R} \mathrm{contr}_v(Q),$$

where we followed the notations in [57]. Let $\psi^*(t) = \Theta_{t,0} + \Theta_{t,1} + \Theta_{t,2} + \Theta_{t,3} + 2\Theta_{t,4} + 2\Theta_{t,5}$ be the type I_1^* singular fibre and $\psi^*(v) = \Theta_{v,0} + \Theta_{v,1} + \Theta_{v,2}$ be the type I_3 fibre, where $\Theta_{t,0}$ and $\Theta_{v,0}$ are the components which intersect the section (O). From (8.16) in [57], we have

$$\mathrm{contr}_t(Q) = \begin{cases} 0 & \text{if } (Q\Theta_{t,0}) = 1, \\ 1 & \text{if } (Q\Theta_{t,1}) = 1, \\ 5/4 & \text{if } (Q\Theta_{t,2}) = 1 \text{ or } (Q\Theta_{t,3}) = 1 \end{cases}$$

and

$$\mathrm{contr}_v(Q) = \begin{cases} 0 & \text{if } (Q\Theta_{v,0}) = 1, \\ 2/3 & \text{if } (Q\Theta_{v,1}) = 1 \text{ or } (Q\Theta_{v,2}) = 1. \end{cases}$$

Noting that $(QO) \in \boldsymbol{Z}$, we see that $(Q\Theta_{t,2}) = 1$ or $(Q\Theta_{t,3}) = 1$ and that $(Q\Theta_{v,0}) = 1$ and $(QO) = 0$. Since $\omega_U = \psi^*\mathcal{O}_{\boldsymbol{P}^1}(-1)$, we have $2K_U + \psi^*(t+u) \sim 0$, where $\psi^*(u)$ is the type II singular fiber. Let $\lambda : U \to S^\flat$ be the contraction of all the curves (O), (Q), $\Theta_{t,i}$ $(0 \le i \le 4)$ and $\Theta_{v,j}$ $(1 \le j \le 2)$ and put $\Delta^\flat := (1/2)\lambda_*\psi^*(t+u)$. Then $(S^\flat, \Delta^\flat)$ gives an example of the log surfaces of type (2) as in Proposition 3.6.

Proposition 3.7 *Assume that* $\mathrm{Sing}\,\tilde{S} = A_1 + 2A_3$. *Then there exists a birational morphism* $\lambda : U \to S^\flat$ *from a smooth projective surface* U *which admits a structure of an elliptic surface with a section* $\psi : U \to \boldsymbol{P}^1$ *which is minimal over* $\boldsymbol{P}^1$. *And one of the followings holds.*

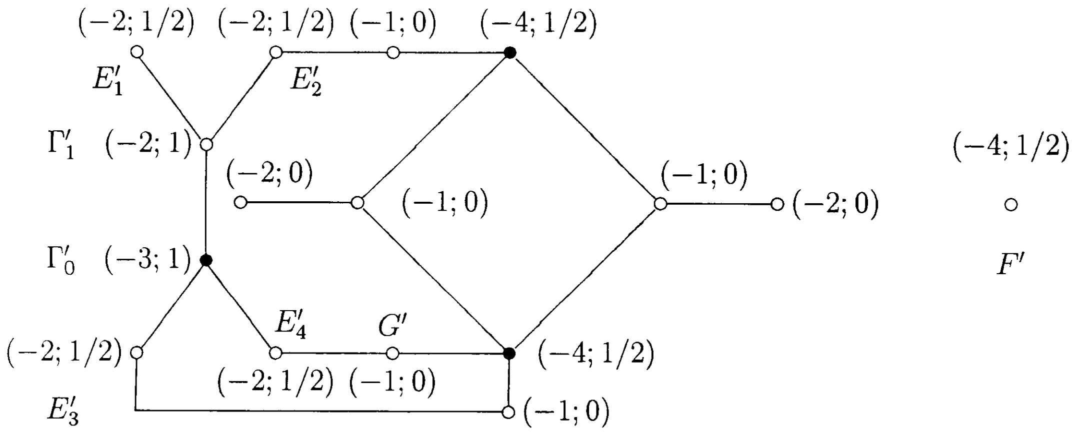
(1) $S^\flat \simeq \boldsymbol{P}^2$ *and* $\mathrm{Typ}(U;\psi) = \mathrm{I}_1^* + \mathrm{III} + \mathrm{I}_2$,

(2) $S^\flat$ *is a rank one Gorenstein log del Pezzo surface with* $\mathrm{Sing}\,S^\flat = 2A_1 + A_3$ *and* $\mathrm{Typ}(U;\psi) = \mathrm{I}_1^* + \mathrm{I}_1 + \mathrm{I}_4$.

Moreover, $\Delta^{\flat\, U} = (1/2)\psi^*(t+u)$ *where* $\psi^*(t)$ *is the singular fibres of type* I_1^* *and* $\psi^*(u)$ *is the singular fibres of type* III *in the case* (1), I_1 *in the case* (2).

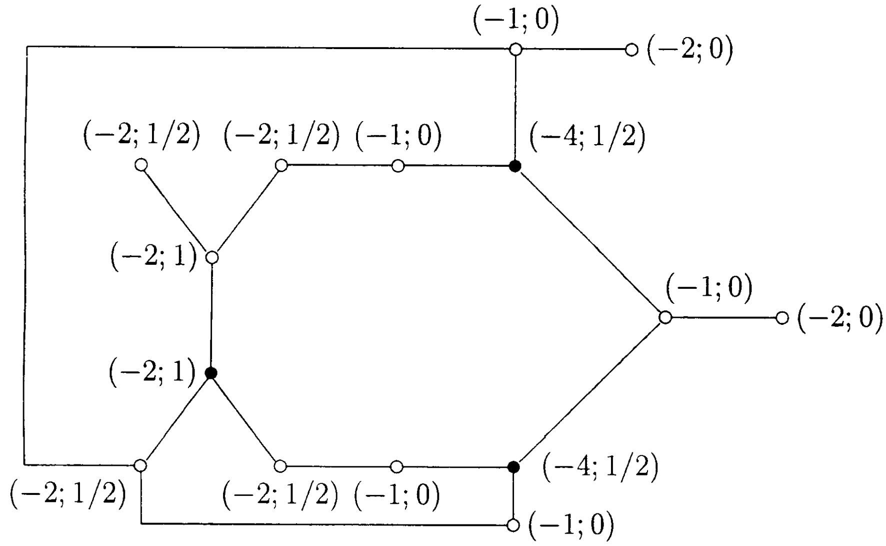
[1] Unfortunately, there is a minor misprint in [10], p131 which says "... cuspidal cubic ...". "cuspidal " should be "nodal ".

Proof. The possible singular types of $(S^\flat, \Delta^\flat)$ on $S^\flat \setminus \lfloor \Delta^\flat \rfloor$ are types A_3, $A_0/2$, $A_1/2\text{-}\alpha$, $A_1/2\text{-}\gamma$, $A_3/2\text{-}\alpha$, $A_3/2\text{-}\delta$ and $A_3/2\text{-}\zeta$. We note that we have $\nu(A_1/2\text{-}\alpha) + \nu(A_1/2\text{-}\gamma) = 1$ and $2\nu(A_3) + \nu(A_3/2\text{-}\alpha) + \nu(A_3/2\text{-}\delta) + \nu(A_3/2\text{-}\zeta) = 2$. From $\delta K^2 = \nu(A_1/2\text{-}\gamma) + \nu(A_3/2\text{-}\delta)$ and $\delta\rho = 4 - (\Gamma^\flat, \Xi^\flat) + 3\nu(A_3) + \nu(A_1/2\text{-}\gamma) + 2\nu(A_3/2\text{-}\delta) + \nu(A_3/2\text{-}\zeta)$, we obtain $\delta K^2 - \delta\rho = (\Gamma^\flat, \Xi^\flat) - 4 - 3\nu(A_3) - \nu(A_3/2\text{-}\delta) - \nu(A_3/2\text{-}\zeta)$, hence we have $3\nu(A_3) + \nu(A_3/2\text{-}\delta) + \nu(A_3/2\text{-}\zeta) = 4 - (1/4)(\Xi^\flat)^2$. Moreover, since we have $\deg \mathrm{Diff}_{\Xi^{\flat\nu}}(\Gamma^\flat) = (\Gamma^\flat, \Xi^\flat) + 2\nu(A_1/2\text{-}\alpha) + 4\nu(A_3/2\text{-}\alpha) + 2\nu(A_3/2\text{-}\zeta)$, we have $e_{\mathrm{top}}(\Xi^{\flat\nu}) = (\Gamma^\flat, \Xi^\flat) + 2\nu(A_1/2\text{-}\alpha) + 4\nu(A_3/2\text{-}\alpha) + 2\nu(A_3/2\text{-}\zeta) - (1/2)(\Xi^\flat)^2$. Firstly, we consider the case in which $S^\flat$ is a rank one log del Pezzo surface. In this case, we have $d = 3 - s$, hence $(\Gamma^\flat)^2 = (3-s)/2$. We note that we have $s = 1$ or $s = 2$ since $\rho(\tilde{S}^\flat) = s$. Assume that $s = 2$. Then we have $(\Gamma^\flat)^2 = 1/2$, which is absurd since $\Gamma^\flat \cap \mathrm{Sing}\, S^\flat = \emptyset$ by our assumption. Thus we have $d = 2$, hence $(\Gamma^\flat)^2 = 1$ and $(\Xi^\flat)^2 = (\Gamma^\flat, \Xi^\flat)^2$. We note that $(\Gamma^\flat, \Xi^\flat) = 2$ or 4 since $(\Gamma^\flat)^2 \in \boldsymbol{Z}$. Assume that $(\Gamma^\flat, \Xi^\flat) = 4$. Then we have $3\nu(A_3) + \nu(A_3/2\text{-}\delta) + \nu(A_3/2\text{-}\zeta) = 0$, hence we have $\nu(A_3) = \nu(A_3/2\text{-}\delta) = \nu(A_3/2\text{-}\zeta) = 0$ and $\nu(A_3/2\text{-}\alpha) = 2$. We note that we have $e_{\mathrm{top}}(\Xi^{\flat\nu}) = 4 + 2\nu(A_1/2\text{-}\alpha) \geq 4$, so we see that $\Xi^\flat$ is reducible and consists of two or three irreducible components since $\rho(S^\flat) = 1$. In fact, we show that $\Xi^\flat$ consists of exactly three irreducible components. Assume that $\Xi^\flat$ consists of two irreducible components $\Xi_1^\flat$ and $\Xi_2^\flat$. Since we have $e_{\mathrm{top}}(\Xi^{\flat\nu}) \leq 4$, we have $\nu(A_1/2\text{-}\alpha) = 0$, $\nu(A_1/2\text{-}\gamma) = 1$ and $\Xi_1^{\flat\nu}$, $\Xi_1^{\flat\nu} \simeq \boldsymbol{P}^1$. We note that we have $(\Xi_1^\flat, \Xi_2^\flat) = 4$ since $(\Xi_i^\flat)^2 = (\Gamma^\flat, \Xi_i^\flat)^2 = 4$ for $i = 1, 2$. Thus we see that there exists a birational morphism $\eta' : W' \to S^\flat$, where W' is a rational surface with $\rho(W') = 14$ such that $\mathrm{Supp}\, \eta'^{-1}_* \Delta^\flat \cup \mathrm{Exc}\, \eta'$ has only simple normal crossing singularities whose dual graph is as follows.

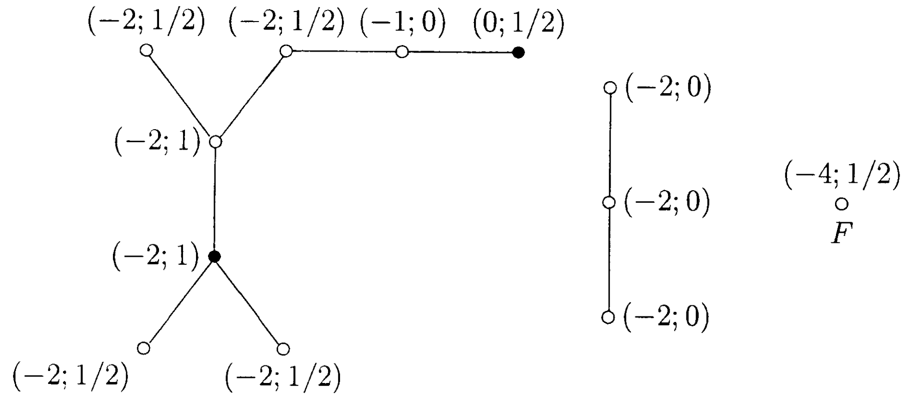


From the above dual graph, we see that there exists a birational morphism $v : W' \to W$ to a rational surface with $\rho(W) = 12$ such that η' factors into $v \circ \eta$ where $\eta : W \to S^\flat$ is a birational morphism and that $\eta_*^{-1}\Xi^\flat$ is nef and $(\eta_*^{-1}\Xi^\flat)^2 = 0$. By the log abundance theorem, some multiple of $\eta_*^{-1}\Xi^\flat$ determines the structure of elliptic fibration with a section $\psi_W : W \to \boldsymbol{P}^1$. We see that $\psi_W^*(u) = \eta_*^{-1}\Xi^\flat$ is a singular fibre of type III where $u := \psi_W(\eta_*^{-1}\Xi^\flat)$. Write $\Delta^{\flat W} = \Gamma_0 + \Gamma_1 + (1/2)(\sum_{i=1}^4 E_i + \eta_*^{-1}\Xi^\flat + F)$, where $\Gamma_0 := v_*\Gamma_0'$, $\Gamma_1' := v_*\Gamma_1'$, $E_i := v_*E_i'$ $(1 \leq i \leq 4)$ and $F := v_*F'$. Let e_0 be a (-1)-curve on W which is contained in a fibre of ψ_W and let $v_0 : W := W_0 \to W_1$ be the contraction of e_0. Assume that $(e_0, F) = 0$. We can see that $(e_0, \sum_{i=1}^4 E_i) < 2$ by the semi-negativity of fiber components. Since we have $(e_0, \Delta^{\flat W}) = (e_0, -K_W) = 1$, we have $(e_0, \sum_{i=1}^4 E_i) \in 2\boldsymbol{Z}$, hence we obtain $(e_0, \sum_{i=1}^4 E_i) = 0$ and $(e_0, \Gamma_0 + \Gamma_1) = 1$. Noting that $\rho(W) = 12$, we have $(e_0, \Gamma_0) = 1$ and $(e_0, \Gamma_1) = 0$. Let $\psi_{W_1} : W_1 \to \boldsymbol{P}^1$ be the induced morphism from ψ and put $t := \psi_W(\Gamma_0)$. Then we see that $\psi_{W_1}^*(t)$ is a singular fibre of type I_1^*. Let e_1 be a (-1)-curve on W_1 which is contained in a fibre of ψ_{W_1}, $v_1 : W_1 \to W_2$ be the contraction of e_1 and put $F^{(1)} := v_{0*}F$. We note that $(e_1, F^{(1)}) = 2$.

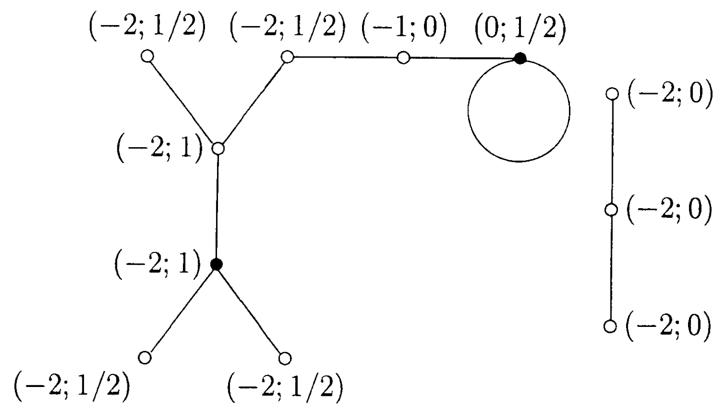
Let $\psi_{W_2} : W_2 \longrightarrow \boldsymbol{P}^1$ be the induced morphism from ψ_{W_1} and put $v := \psi_{W_1}(F^{(1)})$. Then we see that $\psi^*_{W_2}(v)$ is a singular fibre of type II or of type I_1 and that $2K_{W_2} + \psi^*_{W_2}(t+u+v) \sim 0$. On the other hand, since $\psi_{W_2} : W_2 \longrightarrow \boldsymbol{P}^1$ is a rational elliptic surface with a section which is minimal over $\boldsymbol{P}^1$, we have $\omega_{W_2} = \psi^*_{W_2}\mathcal{O}_{\boldsymbol{P}^1}(-1)$, which is a contradiction. Thus we may assume that $(e_0, F) > 0$. In the same way as the above argument, we see that $(e_0, F) = 1$, hence $(e_0, \sum_{i=1}^4 E_i) = 1$. Since $\rho(W) = 12$, we see that $(e_0, E_3) = 1$ or $(e_0, E_4) = 1$. Say $(e_0, E_4) = 1$. By contracting e_0 and E_4, we obtain a minimal elliptic fibration such that the singular fibre over t is of type I_1^*. Thus we see that $\psi^*_W(t) = 2\Gamma_0 + 2\Gamma_1 + \sum_{i=1}^3 E_i + 3E_4 + 4e_0 + F$. Put $G := v_*G'$. The above decomposition implies that $(\psi^*_W(t), G) \geq 3(E_4, G) = 3$. On the other hand, we have $(\psi^*_W(u), G) = 1$, but which is absurd. Thus we conclude that $\Xi^\flat$ has three irreducible components assuming $(\Gamma^\flat, \Xi^\flat) = 4$. Let $\Xi^\flat = \sum_{i=1}^3 \Xi_i^\flat$ be the irreducible decomposition such that $(\Gamma^\flat, \Xi_1^\flat) = 2$ and $(\Gamma^\flat, \Xi_i^\flat) = 1$ for $i = 2$, 3. For $i = 2$, 3, we have $(\Xi_1^\flat + \Xi_i^\flat)^2 = (\Gamma^\flat, \Xi_1^\flat + \Xi_i^\flat)^2 = 9$, hence $(\Xi_1^\flat, \Xi_i^\flat) = 2$. In the same way, we have $(\Xi_2^\flat + \Xi_3^\flat)^2 = (\Gamma^\flat, \Xi_2^\flat + \Xi_3^\flat)^2 = 4$, hence $(\Xi_2^\flat, \Xi_3^\flat) = 1$. Thus we infer that $\nu(A_1/2\text{-}\alpha) = 1$ and $\nu(A_1/2\text{-}\gamma) = 0$, that is, $S^\flat \simeq \boldsymbol{P}^2$, $\Xi_1^\flat$ is a conic and $\Xi_i^\flat$ is a line for $i = 2$, 3. Moreover there exists a birational morphism $\eta : W \longrightarrow S^\flat$ from a smooth rational surface W with $\rho(W) = 12$ such that $\mathrm{Supp}\ \eta_*^{-1}\Delta^\flat \cup \mathrm{Exc}\ \eta$ has only simple normal crossing singularities whose dual graph is as follows.



We see that there exists a birational morphism $\lambda : W \longrightarrow U$ to a smooth rational surface U with $\rho(U) = 10$ such that $\lambda_*^{-1}\Xi^\flat$ is nef and $(\lambda_*^{-1}\Xi^\flat)^2 = 0$. Thus we get a rational elliptic surface with a section $\psi : U \longrightarrow \boldsymbol{P}^1$ which is minimal over $\boldsymbol{P}^1$ with singular fibres $\psi^*(t)$ of type I_1^* and $\psi^*(u)$ of type III. Noting that we have $\sum_{v\neq t,u} e_{\mathrm{top}}(\psi^{-1}(v)) = 2$ and that there exists a singular fibre $\psi^*(v)$ whose dual graph contains a subgraph of type A_1, we see that $\psi^*(v)$ is of type I_2 and ψ is smooth over $\boldsymbol{P}^1 \setminus \{t, u, v\}$. Thus we see that we are in the case (1). Assume that $(\Gamma^\flat, \Xi^\flat) = 2$. Then we have $(\Xi^\flat)^2 = (\Gamma^\flat, \Xi^\flat)^2 = 4$, which implies that $\Xi^\flat$ is irreducible, and $3\nu(A_3) + \nu(A_3/2\text{-}\delta) + \nu(A_3/2\text{-}\zeta) = 3$. Since we have $\nu(A_3/2\text{-}\delta) + \nu(A_3/2\text{-}\zeta) \leq 2$, we see that $\nu(A_3) = 1$, $\nu(A_3/2\text{-}\alpha) = \nu(A_3/2\text{-}\delta) = \nu(A_3/2\text{-}\zeta) = 0$ and we get $e_{\mathrm{top}}(\Xi^{\flat\nu}) = 2\nu(A_1/2\text{-}\alpha)$. We note that $\nu(A_1/2\text{-}\alpha) = 0$ or 1 but in fact we can show that $\nu(A_1/2\text{-}\alpha) = 1$ as follows. Assume that $\nu(A_1/2\text{-}\alpha) = 0$. Then we have $\nu(A_1/2\text{-}\gamma) = 1$ and $e_{\mathrm{top}}(\Xi^{\flat\nu}) = 0$, hence $\Xi^\flat$ is a smooth elliptic curve. We see that there exists a birational morphism $\eta : W \longrightarrow S^\flat$ from a smooth rational surface W with $\rho(W) = 11$ such that $\mathrm{Supp}\ \eta_*^{-1}\Delta^\flat \cup \mathrm{Exc}\ \eta$ has only simple normal crossing singularities whose dual graph is as follows.



Some multiple of $\eta_*^{-1}\Xi^\flat$ determines an elliptic fibration $\psi_W : W \to \boldsymbol{P}^1$ with a section. Let e_0 be a (-1)-curve on W which is contained in a fibre of ψ_W. Then we have $(e_0, F) = 2$ from $2(K_W + \Delta^{\flat W}) \sim 0$. Let $v : W \to U$ be the contraction e_0 and $\psi : U \to \boldsymbol{P}^1$ be the induced morphism ψ_W. Then ψ is minimal since $\rho(U) = 10$ and v_*F supports a singular fibre of type I_1 or II. We see that $v_*\Delta^{\flat W} = (1/2)\psi^*(t + u + v)$, where $\psi^*(t)$ is a singular fibre of type I_1^*, $\psi^*(u) = v_*\eta_*^{-1}\Xi^\flat$ and $\psi^*(v) = v_*F$, but which is absurd since $\omega_U = \psi^*\mathcal{O}_{\boldsymbol{P}^1}(-1)$. Thus we conclude that $\nu(A_1/2\text{-}\alpha) = 1$, $\nu(A_1/2\text{-}\gamma) = 0$, hence $e_{\mathrm{top}}(\Xi^\flat) = 2$, that is, $\Xi^\flat \simeq \boldsymbol{P}^1$. We see that there exists a birational morphism $\lambda : U \to S^\flat$ from a smooth rational surface U with $\rho(U) = 10$ such that Supp $\lambda_*^{-1}\Delta^\flat \cup \operatorname{Exc}\lambda$ has only normal crossing singularities whose dual graph is as follows.



Some multiple of $\lambda_*^{-1}\Xi^\flat$ determines a minimal elliptic fibration with a section $\psi : U \to \boldsymbol{P}^1$ with singular fibres $\psi^*(t)$ of type I_1^* and $\psi^*(u)$ of type I_1. Since we have $\sum_{v \neq t,u} e_{\mathrm{top}}(\psi^{-1}(v)) = 4$ and there exists a singular fibre $\psi^*(v)$ whose dual graph has a subgraph of type A_3, we see that $\psi^*(v)$ is of type I_4 and ψ is smooth except over t, u and v. Thus we are in the case (2). Secondarily, we consider the case in which $S^\flat$ has a structure of a conic fibration $\varphi^\flat : S^\flat \to \boldsymbol{P}^1$. In this case, we have $d = 1 - s$, hence $(\Gamma^\flat)^2 = (1 - s)/2$. Assume that $s = 2$. Then we have $(\Gamma^\flat)^2 = -1/2$, which is absurd since $\Gamma^\flat \cap \operatorname{Sing} S^\flat$ by our assumption. Assume that $s = 1$. Then we have $\rho(\tilde{S}^\flat) = 3$, hence the self-intersection numbers of all the irreducible components of $\tilde{\Delta}^\flat$ are -1, -2 and -3, which implies that all the irreducible components of $\tilde{\Delta}^\flat$ are invariant under the action of the covering transformation group. This contradicts the assumption $s = 1$. Thus we conclude that $s = 0$. Since $\rho(\tilde{S}^\flat) = 2$, the self-intersection numbers of all the irreducible components of $\tilde{\Delta}^\flat$ are -1 and -2. Thus we infer that there exists an irreducible component $\Gamma_0^\flat \subset \Gamma^\flat$ such that $(\Gamma_0^\flat)^2 = -1/2$. Contracting $\Gamma_0^\flat$, our case reduces to the rank one log del Pezzo case. ∎

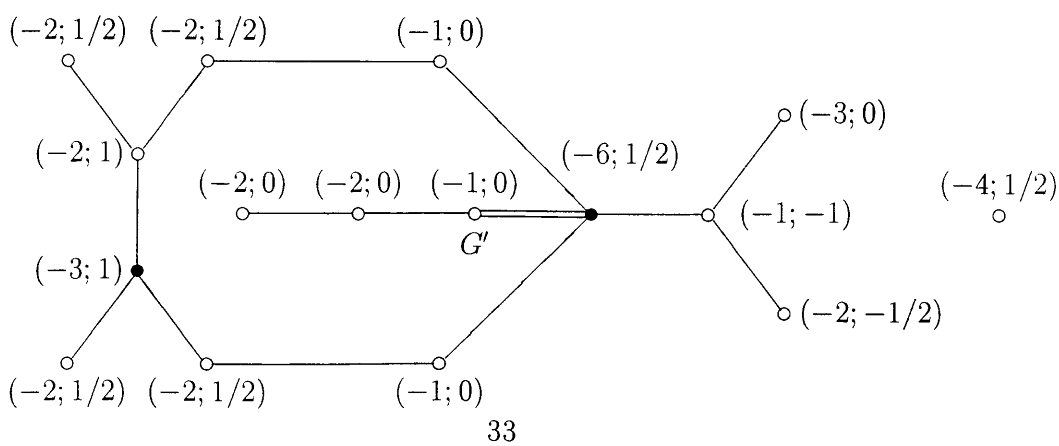
Example 3.2 By [46], there exists minimal rational elliptic surface with a section $\psi : U \to \boldsymbol{P}^1$ with $\mathrm{Typ}(U;\psi) = \mathrm{I}_1^* + \mathrm{I}_1 + \mathrm{I}_4$ from which we can easily construct an example of $(S^\flat, \Delta^\flat)$ in (2). We can also construct an example of $(S^\flat, \Delta^\flat)$ in (1) from a pair $(\boldsymbol{P}^1, \text{line} + (1/2)\text{line} + (1/2)\text{line} + (1/2)\text{conic})$ with properly chosen alignment.

Proposition 3.8 *Assume that* $\mathrm{Sing}\, \tilde{S} = A_5 + A_2 + A_1$. *Then* $S^\flat$ *is a rank one Gorenstein log del Pezzo surface with* $\mathrm{Sing}\, S^\flat = A_1$ *and there exists a birational morphism* $\lambda : U \to S^\flat$ *from a smooth projective surface* U *such that* U *admits a structure of elliptic surface with a section* $\psi : U \to \boldsymbol{P}^1$ *which is minimal over* $\boldsymbol{P}^1$ *with* $\mathrm{Typ}(U;\psi) = \mathrm{I}_1^* + \mathrm{II} + \mathrm{I}_3$ *and that* $\Delta^{\flat\, U} = (1/2)\psi^*(t+u)$ *where* $\psi^*(t)$ *and* $\psi^*(u)$ *are the singular fibres of type* I_1^* *and* II *respectively.*

Proof. The possible singular types of $(S^\flat, \Delta^\flat)$ on $S^\flat \setminus \lfloor \Delta \rfloor$ are types $A_0/2$, $A_1/2\text{-}\alpha$, $A_1/2\text{-}\gamma$, $A_2/2\text{-}\beta$, $A_2/2\text{-}\epsilon$, $A_5/2\text{-}\alpha$, $A_5/2\text{-}\delta$ and $A_5/2\text{-}\zeta$. We note that $\nu(A_1/2\text{-}\alpha) + \nu(A_1/2\text{-}\gamma) = 1$, $\nu(A_2/2\text{-}\beta) + \nu(A_2/2\text{-}\epsilon) = 1$ and $\nu(A_5/2\text{-}\alpha) + \nu(A_5/2\text{-}\delta) + \nu(A_5/2\text{-}\zeta) = 1$. Firstly we consider the case in which $S^\flat$ is a rank one log del Pezzo surface. Assume that $s = 2$. Then we have $\rho(\tilde{S}^\flat) = 2$ and the self-intersection numbers of all the irreducible components of $\tilde{\Delta}^\flat$ are -1 and -3, which is absurd. Thus we have $s = 1$, hence $\rho(\tilde{S}^\flat) = 1$. Since $d = 1$, we have $(\Gamma^\flat)^2 = 1/2$, hence $(\Gamma^\flat, \Xi^\flat) = 3$, $(\Xi^\flat)^2 = 18$ and $\delta K^2 - \delta\rho = -1$. From $\delta K^2 = \nu(A_1/2\text{-}\gamma) + (1/3)\nu(A_2/2\text{-}\epsilon) + \nu(A_5/2\text{-}\delta)$ and $\delta\rho = 1 + \nu(A_1/2\text{-}\gamma) + \nu(A_2/2\text{-}\epsilon) + 3\nu(A_5/2\text{-}\delta) + 3\nu(A_5/2\text{-}\zeta)$ we obtain $\delta K^2 - \delta\rho = -1 - (2/3)\nu(A_2/2\text{-}\epsilon) - 2\nu(A_5/2\text{-}\delta) - 3\nu(A_5/2\text{-}\zeta)$. Thus we get $(2/3)\nu(A_2/2\text{-}\epsilon) + 2\nu(A_5/2\text{-}\delta) + 3\nu(A_5/2\text{-}\zeta) = 0$, hence $\nu(A_2/2\text{-}\epsilon) = \nu(A_5/2\text{-}\delta) = \nu(A_5/2\text{-}\zeta) = 0$ and $\nu(A_2/2\text{-}\beta) = \nu(A_5/2\text{-}\alpha) = 1$. Moreover, since we have

$$
\begin{aligned}
\deg \mathrm{Diff}_{\Xi^{\flat\nu}}(\Gamma^\flat) &= 3 + 2\nu(A_1/2\text{-}\alpha) + 2\nu(A_2/2\text{-}\beta) + 6\nu(A_5/2\text{-}\alpha) \\
&= 11 + 2\nu(A_1/2\text{-}\alpha),
\end{aligned}
$$

we have $e_{\mathrm{top}}(\Xi^{\flat\nu}) = \deg \mathrm{Diff}_{\Xi^{\flat\nu}}(\Gamma^\flat) - 9 = 2 + 2\nu(A_1/2\text{-}\alpha)$. The number of irreducible components of $\Xi^\flat$ is one or two. Firstly, we shall show that $\Xi^\flat$ consists of two irreducible components. Assume that $\Xi^\flat$ is irreducible. Then since we have $e_{\mathrm{top}}(\Xi^{\flat\nu}) \le 2$, we have $\nu(A_1/2\text{-}\alpha) = 0$ and $\nu(A_1/2\text{-}\gamma) = 1$. Thus we see that there exists a birational morphism $\eta' : W' \to S^\flat$ from a smooth rational surface W' with $\rho(W') = 15$ such that $\mathrm{Supp}\, \eta'^{-1}_* \Delta^\flat \cup \mathrm{Exc}\, \eta'$ has only simple normal crossing singularities whose dual graph is as follows.



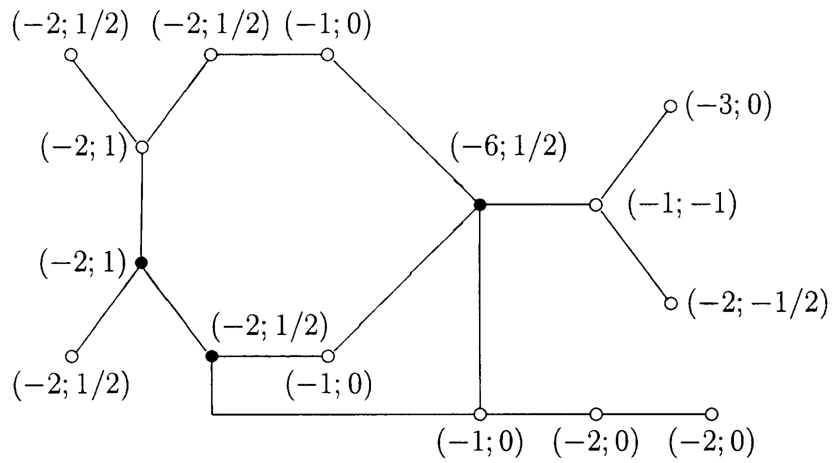
From the above dual graph, we see that there exists a birational morphism $v : W' \to W$ to a rational surface W with $\rho(W) = 12$ such that η' factors into $\eta \circ v$ where $\eta : W \to S^\flat$ is a birational morphism and that $(\eta_*^{-1}\Xi^\flat)^2 = 0$. We note that $\Delta^{\flat W} \geq 0$. Some multiple of $\eta_*^{-1}\Xi^\flat$ determines an elliptic fibration $\psi_W : W \to \boldsymbol{P}^1$ with a section such that $\psi_W^*(u) = \eta_*^{-1}\Xi^\flat$ is a singular fibre of type II. Put $G := v_*G'$. Since $(G, \psi_W^*(u)) = (G, \eta_*^{-1}\Xi^\flat) = 2$ we have $(G, \psi^*(t)) = 2$, where $t := \psi_W(\eta_*^{-1}\Gamma^\flat)$. On the other hand, by the previous argument in the proof of Proposition 3.7, we have $(G, \psi_W^*(t)) \geq 4$, which is a contradiction. Thus we conclude that $\Xi^\flat$ consists of two irreducible components. Let $\Xi^\flat = \Xi_1^\flat + \Xi_2^\flat$ be the irreducible decomposition such that $(\Gamma^\flat, \Xi_1^\flat) = 2$ and $(\Gamma^\flat, \Xi_2^\flat) = 1$. We note that since we have $(\Xi_1^\flat)^2 = 2(\Gamma^\flat, \Xi_1^\flat)^2 = 8$ and $(\Xi_2^\flat)^2 = 2(\Gamma^\flat, \Xi_2^\flat)^2 = 2$, we have $18 = (\Xi^\flat)^2 = 10 + 2(\Xi_1^\flat, \Xi_2^\flat)$, hence $(\Xi_1^\flat, \Xi_2^\flat) = 4$, which implies that $\nu(A_1/2\text{-}\alpha) = 1$ and $\nu(A_1/2\text{-}\gamma) = 0$, hence $e_{\mathrm{top}}(\Xi^{\flat\nu}) = 4$, that is, $\Xi_i^{\flat\nu} \simeq \boldsymbol{P}^1$ $(i = 1, 2)$. From $K_{S^\flat} + \Xi_2^\flat + \Gamma^\flat + (1/2)\Xi_1^\flat \sim_{\boldsymbol{Q}} (1/2)\Xi_2^\flat$, we obtain

$$\deg \mathrm{Diff}_{\Xi_2^{\flat\nu}}(\Gamma^\flat + \frac{1}{2}\Xi_1^\flat) = e_{\mathrm{top}}(\Xi_2^{\flat\nu}) + \frac{1}{2}(\Xi_2^\flat)^2 = 3.$$

On the other hand, we have

$$\begin{aligned} \deg \mathrm{Diff}_{\Xi_2^{\flat\nu}}(\Gamma^\flat + \frac{1}{2}\Xi_1^\flat) &= (\Gamma^\flat, \Xi_2^\flat) + \frac{1}{2}(\Xi_1^\flat, \Xi_2^\flat) + \deg \mathrm{Diff}_{\Xi_2^{\flat\nu}}(0) \\ &= 3 + \deg \mathrm{Diff}_{\Xi_2^{\flat\nu}}(0). \end{aligned}$$

Thus we get $\deg \mathrm{Diff}_{\Xi_2^{\flat\nu}}(0) = 0$ and consequently, we infer that the type $A_2/2\text{-}\beta$ point lies on $\Xi_1^\flat$. Thus we see that there exists a birational morphism $\eta : W \to S^\flat$ from a smooth rational surface W with $\rho(W) = 13$ such that $\mathrm{Supp}\, \eta_*^{-1}\Delta^\flat \cup \mathrm{Exc}\, \eta$ has only simple normal crossing singularities whose dual graph is as follows.



We see that there exists a birational morphism $v : W \to U$ to a rational surface with $\rho(U) = 10$ and $\Delta^{\flat U} \geq 0$ such that η factors into $\lambda \circ v$, where $\lambda : U \to S^\flat$ is a birational morphism and that $\lambda_*^{-1}\Xi^\flat$ is nef with $(\lambda_*^{-1}\Xi^\flat)^2 = 0$. Some multiple $\lambda_*^{-1}\Xi^\flat$ determines an elliptic fibration $\psi : U \to \boldsymbol{P}^1$ and we see that $\mathrm{Typ}(U; \psi) = \mathrm{I}_1^* + \mathrm{II} + \mathrm{I}_3$ as in the same way in the proof of Proposition 3.6. Secondarily, we consider the case in which there exists a structure of a conic fibration $\varphi^\flat : S^\flat \to \boldsymbol{P}^1$. Assume that $s = 2$. Then we have $\rho(\tilde{S}^\flat) = 4$ and the self-intersection numbers of all the irreducible components are (*a*) -1, -2, -2 and -5, (*b*) -1, -2, -3 and -4 or (*c*) -1, -1, -3 and -5, which is absurd with the assumption $s = 2$. Assume that $s = 1$. Then we have $\rho(\tilde{S}^\flat) = 3$ and the self-intersection numbers of all the irreducible components are -1, -2 and -4, which is absurd with the assumption

$s = 1$. Assume that $s = 0$. Then we have $\rho(\tilde{S}^\flat) = 2$ and the self-intersection numbers of all the irreducible components are -1, -3. Thus we see that there there exists an irreducible component $\Gamma_0^\flat \subset \Gamma^\flat$ such that $(\Gamma_0^\flat)^2 = -1/2$. Contracting $\Gamma_0^\flat$, our case reduces to the rank one log del Pezzo case. Thus we get the assertion. ∎

Example 3.3 We shall follow the notations in Example 3.1. By [57], Theorem 8.6, (8.12), we have

$$\frac{1}{12} = < -P, -P > = 2 + 2((-P), (O)) - \mathrm{contr}_t(-P) - \mathrm{contr}_v(-P)$$

and $((-P), (O)) \in \boldsymbol{Z}$, we see that $((-P), \Theta_{t,i}) = 1$ for $i = 2$ or 3 and that $((-P), \Theta_{v,j}) = 1$ for $j = 1$ or 2. hence $((-P), (O)) = 0$. From [57], Theorem 8.6, (8.11), we have

$$-\frac{1}{4} = < -P, Q > = 1 - ((-P), (Q)) - \mathrm{contr}_t(-P, Q).$$

Noting that by [57], (8.16),

$$\mathrm{contr}_t(-P, Q) = \begin{cases} 5/4 & \text{if } (-P) \text{ and } (Q) \text{ intersects the same components of } \psi^*(t) \\ 3/4 & \text{otherwise} \end{cases}$$

and $((-P), (Q)) \in \boldsymbol{Z}$, we infer that $((-P), (Q)) = 0$ and $((-P), \Theta_{t,i}) = ((Q), \Theta_{t,i}) = 1$ for $i = 2$ or 3. Let $\lambda : U \to S^\flat$ be the contraction of all the curves (O), $(-P)$, (Q), $\{\Theta_{t,j} | j \neq i,\ 5\}$ and $\{\Theta_{v,k} | k = 1,\ 2\}$ and put $\Delta^\flat := (1/2)\lambda_*\psi^*(t + u)$. Then $(S^\flat, \Delta^\flat)$ gives an example of the log surfaces as in Proposition 3.8.

Proposition 3.9 *Assume that* $\mathrm{Sing}\, \tilde{S} = 4A_1$. *Then one of the following holds.*

(1) $S^\flat \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ *and* $\Delta^\flat = \sum_{i=0}^{1} \Gamma_i^\flat + (1/2)\sum_{j=1}^{4} \Xi_j^\flat$, *where* $\Gamma_0^\flat$, $\Xi_j^\flat$ *are fibres of the first projection for* $j = 1,\ 2$ *and* $\Gamma_1^\flat$, $\Xi_j^\flat$ *are fibres of the second projection for* $j = 3,\ 4$.

(2) $(S^\flat, \Delta^\flat)$ *is log terminal and* $S^\flat$ *has a structure of conic fibration* $\varphi^\flat : S^\flat \to \boldsymbol{P}^1$ *with* $\rho(S^\flat/\boldsymbol{P}^1) = 1$ *and* $\mathrm{Typ}(S^\flat, \Delta^\flat; \varphi^\flat) = ((\text{I-2})_\infty + (\text{I-2})_1 + (\text{II-1})_2; (\text{II-1})_1)$ *possibly after operating* $\boldsymbol{S}$*-elementary transformations.* $\Delta^\flat = \Gamma_0^\flat + \Gamma_1^\flat + (1/2)\Xi^\flat$, *where* $\Gamma_0^\flat$ *is a smooth rational curve with* $(\Gamma_0^\flat, \varphi^{\flat *}(t)) = 2$ *for* $t \in \boldsymbol{P}^1$ *and* $(\Gamma_0^\flat)^2 = 0$, $\Gamma_1^\flat$ *and* $\Xi^\flat$ *are fibres of* $\varphi^\flat$ *with reduced structure.*

(3) $S^\flat$ *is a rank one Gorenstein log del Pezzo surface with* $\mathrm{Sing}\, S^\flat = A_1$ *and there exists a birational morphism* $\lambda : U \to S^\flat$ *from a smooth projective surface* U *such that* U *admits a structure of elliptic surface with a section* $\psi : U \to \boldsymbol{P}^1$ *which is minimal over* $\boldsymbol{P}^1$ *with* $\mathrm{Typ}(U; \psi) = \mathrm{I}_2^* + 2\mathrm{I}_1$ *and that* $\Delta^{\flat\, U} = (1/2)\psi^*(t + u)$ *where* $\psi^*(t)$ *and* $\psi^*(u)$ *are the singular fibres of type* I_2^* *and* I_1 *respectively.*

Proof. Firstly, we consider the case in which $S^\flat$ is a rank one log del Pezzo surface. In this case, since we have $s = \rho(\tilde{S}^\flat) \geq 2$, we have $s = 2$ and $(\Gamma^\flat, \Xi^\flat) = 4$, hence $d = 4$ by (3.9), $(\Gamma^\flat)^2 = 2$ and $(\Xi^\flat)^2 = 8$ by (3.6). The possible of singular types on $S^\flat \setminus \lfloor \Delta^\flat \rfloor$ are types A_1, $A_0/2$, $A_1/2$-α, $A_1/2$-γ. We note that we have $2\nu(A_1) + \nu(A_1/2\text{-}\alpha) + \nu(A_1/2\text{-}\gamma) = 4$. Since we have $\delta K^2 = \nu(A_1/2\text{-}\gamma)$ and $\delta\rho = \nu(A_1) + \nu(A_1/2\text{-}\gamma)$, we have $\delta K^2 - \delta\rho = -\nu(A_1)$. On the other hand, we have $\delta K^2 - \delta\rho = -1$ by (3.5). Thus we obtain $\nu(A_1) = 1$ and $\nu(A_1/2\text{-}\alpha) + \nu(A_1/2\text{-}\gamma) = 2$. Noting that we have $\deg \mathrm{Diff}_{\Xi^{\flat\nu}}(\Gamma^\flat) = 4 + 2\nu(A_1/2\text{-}\alpha)$, we obtain $e_{\mathrm{top}}(\Xi^{\flat\nu}) = 2\nu(A_1/2\text{-}\alpha)$. We shall show that $\Xi^\flat$ is reducible. Assume the contrary. Since we have $e_{\mathrm{top}}(\Xi^{\flat\nu}) = 2\nu(A_1/2\text{-}\alpha) \leq 2$, we have $\nu(A_1/2\text{-}\alpha) \geq 1$, hence $\nu(A_1/2\text{-}\gamma) \geq 1$. Therefore, we see that there exists a birational morphism $\eta : W \to S^\flat$ from a smooth rational surface with $\rho(W) = 10 + \nu(A_1/2\text{-}\gamma)$ such that $\Delta^{\flat W} \geq 0$, $(\Delta^{\flat W} - (1/2)\eta_*^{-1}\Xi^\flat, \eta_*^{-1}\Xi^\flat) = 0$ and

$(\eta_*^{-1}\Xi^\flat)^2 = 0$. Some multiple of $\eta_*^{-1}\Xi^\flat$ defines an elliptic fibration with a section $\psi_W : W \to \boldsymbol{P}^1$. Put $t := \psi_W(\eta_*^{-1}\Gamma^\flat)$ and $u := \psi_W(\eta_*^{-1}\Xi^\flat)$. We see that $\psi_W^*(t)$ is a singular fibre of type I_2^* and $\psi_W^*(u)$ is a singular fibre of type I_0, if $\nu(A_1/2\text{-}\gamma) = 2$, of type I_1, if $\nu(A_1/2\text{-}\gamma) = 1$, in particular, ψ_W is minimal over t and u. Consequently, we can write $K_W + (1/2)\psi_W^*(t+u) + (1/2)F \sim_{\boldsymbol{Q}} 0$, where F is a sum of disjoint (-4)-curves which comes from the minimal resolution of singularities of type $A_1/2\text{-}\gamma$. Let $\psi : U \to \boldsymbol{P}^1$ be the minimal model of ψ_W and $v : W \to U$ be the induced morphism. Then we have $K_U + (1/2)\psi^*(t+u) + (1/2)v_*F \sim_{\boldsymbol{Q}} 0$ which implies that $v_*F = 0$ since $\omega_U \simeq \psi^*\mathcal{O}_{\boldsymbol{P}^1}(-1)$. Thus we have $K_W + (1/2)\psi_W^*(t+u) + (1/2)F = v^*(K_U + (1/2)\psi^*(t+u))$, hence $K_W + (1/2)F = v^*K_U$, but which is absurd. Thus we conclude that $\Xi^\flat$ is reducible. We note that we can write $\Xi^\flat = \Xi_1^\flat + \Xi_2^\flat$, where $\Xi_i^\flat$ are irreducible curves with $(\Gamma^\flat, \Xi_i^\flat) = 2$ and hence $(\Xi_i^\flat)^2 = 2$ by (3.6) for $i = 1, 2$. From $8 = (\Xi^\flat)^2 = 4 + 2(\Xi_1^\flat, \Xi_2^\flat)$, we have $(\Xi_1^\flat, \Xi_2^\flat) = 2$, hence $\nu(A_1/2\text{-}\alpha) = 2$ and $\nu(A_1/2\text{-}\gamma) = 0$. Thus we see that there exists a birational morphism $\lambda : U \to S^\flat$ from a smooth rational surface with $\rho(U) = 10$ such that $\Delta^{\flat U} \geq 0$ and $(\Delta^{\flat U} - \lambda_*^{-1}\Xi^\flat, \lambda_*^{-1}\Xi^\flat) = 0$, $\lambda_*^{-1}\Xi^\flat$ is nef and $(\lambda_*^{-1}\Xi^\flat)^2 = 0$. Some multiple defines an elliptic fibration with a section $\psi : U \to \boldsymbol{P}^1$ which is necessarily minimal over $\boldsymbol{P}^1$ such that $\psi_W^*(t)$ is a singular fibre of type I_2^*, $\psi_W^*(u)$ is a singular fibre of type I_2, where $t := \psi_W(\lambda_*^{-1}\Gamma^\flat)$ and $u := \psi_W(\lambda_*^{-1}\Xi^\flat)$. Noting that there exists a singular fibre $\psi^*(v)$ whose dual graph contains a Dynkin diagram of type A_1 as a subgraph and that we have $\sum_{v \neq t,u} e_{\mathrm{top}}(\psi^{-1}(v)) = 2$, we infer that $\psi^*(v)$ is of type I_2 and ψ is smooth over $\boldsymbol{P}^1 \setminus \{t, u, v\}$. Thus we conclude that we are in the case (2). Secondarily, we consider the case in which there exists a structure of a conic fibration $\varphi^\flat : S^\flat \to \boldsymbol{P}^1$ with $\rho(S^\flat/\boldsymbol{P}^1) = 1$. Let $\Gamma^\flat = \Gamma_0^\flat + \Gamma_1^\flat$ be the irreducible decomposition. We note that we have $d = 4 - s$ by (3.9), hence we have $(\Gamma^\flat)^2 = 2 - (1/2)s$ and $(\Gamma_0^\flat)^2 + (\Gamma_1^\flat)^2 = -(1/2)s$. Since we have $\delta K^2 = \nu(A_1/2\text{-}\gamma)$ and $\delta\rho = \nu(A_1) + \nu(A_1/2\text{-}\gamma) + 4 - (\Gamma^\flat, \Xi^\flat)$, we have $\delta K^2 - \delta\rho = -\nu(A_1) - 4 + (\Gamma^\flat, \Xi^\flat)$. On the other hand, we have $\delta K^2 - \delta\rho = (\Gamma^\flat, \Xi^\flat) + (1/4)(\Xi^\flat)^2 - 6 - (1/2)s$ by (3.5), hence we obtain $(\Xi^\flat)^2 = 8 + 2s - 4\nu(A_1)$. Combining this with (3.4), we obtain $e_{\mathrm{top}}(\Xi^{\flat\nu}) = (\Gamma^\flat, \Xi^\flat) - 4 - s + 2\nu(A_1) + 2\nu(A_1/2\text{-}\alpha)$. We note that if $\rho(\tilde{S}^\flat) = 2$, $(S^\flat, \Delta^+(\varphi^\flat; t)) \subset \lfloor \Delta_{\varphi^\flat}^{\flat -}(t) \rfloor$ is log canonical and $\mathrm{Supp}\lfloor \mathrm{Diff}_{C_{\varphi^\flat}(t)}(\Delta_{\varphi^\flat}^{\flat -}(t)) \rfloor \subset \lfloor \Delta_{\varphi^\flat}^{\flat -}(t) \rfloor$ for any $t \in \boldsymbol{P}^1$ by the same argument as in the proof of Lemma 3.8. Assume that $(\Gamma_0^\flat, \varphi^{\flat *}(t)) = 2$ for $t \in \boldsymbol{P}^1$. Then $\Gamma_1^\flat$ and $\Xi^\flat$ are contained in fibres of $\varphi^\flat$, hence $s = 0$ and $\rho(\tilde{S}^\flat) = 2$. Therefore, φ has only fibres of type $(\text{I-}2)_b$ and $(\text{II-}1)_b$ by Lemma 3.3. We note that $\varphi^{\flat *}(\varphi^\flat(\Gamma_1^\flat))$ is a fibre of type $(\text{I-}2)_\infty$. Since $\varphi^\flat$ induces a double cover $\varphi^\flat : \Gamma_0^\flat \to \boldsymbol{P}^1$, $\varphi^\flat$ has exactly one fibre of type $(\text{I-}2)_1$ by Hurwitz's formula. Thus we conclude that $\varphi^\flat$ has exactly one fibre of type $(\text{II-}1)_2$ since $\sum_{p \in \Gamma_0^\flat} m_p(\Gamma_0^\flat, \Delta^\flat - \Gamma_0^\flat) = 2$ and the other fibres are of type $(\text{II-}1)_1$, hence we are in the case (2). Assume that $(\Gamma_0^\flat, \varphi^{\flat *}(t)) = 1$ for $t \in \boldsymbol{P}^1$. We shall show that $\Gamma_1^\flat$ is contained in a fibre. Assume the contrary. Then $(\Xi^\flat)^2 = 0$, hence $s = 0$, which implies $\nu(A_1) = 2$. Take type A_1 point $p \in S^\flat \setminus \lfloor \Delta^\flat \rfloor$ and take $C_{\varphi^\flat}(t)$ passing through p. Since $S^\flat$ has only Du Val singularities of type A_1, we can write $\varphi^\flat(t) = 2C_{\varphi^\flat}(t)$, hence $(\Gamma_i^\flat, C_{\varphi^\flat}(t)) = 1/2$ for $i = 0, 1$. From $(K_{S^\flat} + \Gamma^\flat, C_{\varphi^\flat}(t)) = 0$, we have $\deg \mathrm{Diff}_{C_{\varphi^\flat}(t)}(\Gamma^\flat) = 2$, which implies $m_p(C_{\varphi^\flat}(t); \Gamma^\flat) = 0$. Thus we get absurdity and we conclude that $\Gamma_1^\flat$ is contained in a fibre. We note that $(\Gamma_0^\flat)^2 = -(1/2)s$, hence the case $s = 1$ reduces to the case in which $S^\flat$ is rank one log del Pezzo surface by contracting $\Gamma_0^\flat$. Let $\Xi_h^\flat$ be the horizontal part of $\Xi^\flat$ with respect to $\varphi^\flat$. Assume that $s = 0$. In this case, we may assume that $\Xi_h^\flat$ is reducible for if $\Xi_h^\flat$ is irreducible, since $\varphi^\flat$ has only fibres of type $(\text{I-}1)_1$, $(\text{I-}1)_2$ and $(\text{I-}3)_1$, we have $\nu((\text{I-}1)_2) = 0$ and $\nu((\text{I-}3)_1) = 2$, hence $(\Xi_h^\flat)^2 = 0$ by applying Hurwitz's formula to the double cover $\varphi^\flat : \Xi_h^\flat \to \boldsymbol{P}^1$, thus some multiple defines another conic fibration $\tilde{\varphi}^\flat : S^\flat \to \boldsymbol{P}^1$ with $(\Gamma_1^\flat, \tilde{\varphi}^{\flat *}(t)) = 2$, which case was already considered. Assuming that $\Xi_h^\flat$ is reducible, we see that $\varphi^\flat$ has only fibres of type $(\text{I-}2)_b$ $(b = 1, 2, \infty)$. Thus we see that we are in the case (1). Assume that $s = 2$. Since $\Gamma_0^\flat$ is a section of $\varphi^\flat$ which does not pass through any singular point of $S^\flat$, $\varphi^\flat$ is smooth, hence $\nu(A_1) = \nu(A_1/2\text{-}\gamma) = 0$ and $\nu(A_1/2\text{-}\alpha) = 4$, which implies that $(\Xi^\flat)^2 = 12$ and $e_{\mathrm{top}}(\Xi^{\flat\nu}) = 6$. We see that $\Xi^\flat$ is reducible.

Let $\Xi^\flat = \Xi_1^\flat + \Xi_2^\flat$ be the irreducible decomposition of $(\Gamma_0^\flat, \Xi_1^\flat) = 2$ and $(\Gamma_1^\flat, \Xi_2^\flat) = 2$. Since $\Xi_1^\flat$ is not a fibre of $\varphi^\flat$, $(\Gamma_1^\flat, \Xi_1^\flat) > 0$, which contradicts $(K_{S^\flat} + \Delta^\flat, \Gamma_1^\flat) = 0$. ∎

Example 3.4 The existence of minimal rational elliptic surface with a section $\psi : U \to \boldsymbol{P}^1$ with $\mathrm{Typ}(U;\psi) = \mathrm{I}_2^* + 2\mathrm{I}_2$ is known (see [46]). From the list in [43], we see that $\mathrm{MW}(U_\eta) = (\boldsymbol{Z}/2\boldsymbol{Z})^{\oplus 2}$. Take $P \in \mathrm{MW}(U_\eta) \setminus \{0\}$. From [57], Theorem 8.6, (8.12), we have

$$0 =< P, P >= 2 + 2(PO) - \mathrm{contr}_t(P) - \mathrm{contr}_u(P) - \mathrm{contr}_v(P),$$

hence

$$(PO) = \frac{1}{2}(\mathrm{contr}_t(P) + \mathrm{contr}_u(P) + \mathrm{contr}_v(P)) - 1.$$

Let $\psi^*(t) = \sum_{i=0}^{3} \Theta_{t,i} + 2\sum_{i=4}^{6} \Theta_{t,i}$ be the type I_2^* singular fibre with $(\Theta_{t,5}, \Theta_{t,i}) = 0$ for any i such that $0 \le i \le 3$, $\psi^*(u) = \Theta_{u,1} + \Theta_{u,i}$, $\psi^*(t) = \Theta_{u,1} + \Theta_{u,i}$ be the type I_2 singular fibre and assume that the 0-section intersects $\Theta_{t,0}$, $\Theta_{u,0}$ and $\Theta_{v,0}$. Then from [57], (8.16), we have

$$\mathrm{contr}_t(P) = \begin{cases} 0 & \text{if } (P\Theta_{t,0}) = 1, \\ 1 & \text{if } (P\Theta_{t,1}) = 1, \\ 3/2 & \text{otherwise} \end{cases} \quad \text{and } \mathrm{contr}_{v'}(P) = \begin{cases} 0 & \text{if } (P\Theta_{v',0}) = 1, \\ 1/2 & \text{if } (P\Theta_{v',1}) = 1 \end{cases}$$

for $v' = u$, v. Since $\mathrm{contr}_t(P) + \mathrm{contr}_u(P) + \mathrm{contr}_v(P) \in 2\boldsymbol{N}$, there exists exactly two types of P, (1) $(P\Theta_{t,1}) = 1$, $(P\Theta_{u,1}) = 1$ and $(P\Theta_{v,1}) = 1$ or (2) $(P\Theta_{t,i}) = 1$ for some $i > 1$, $(P\Theta_{u,1}) = 1$ and $(P\Theta_{v,0}) = 1$. Take two elements P, $Q \in \mathrm{MW}(U_\eta) \setminus \{0\}$ which is distinct from each other. Assume that both of P and Q are of type (1). Then from [57], Theorem 8.6, (8.11), we have

$$0 =< P, Q >= 1 - (PQ) - \mathrm{contr}_t(P, Q) - \mathrm{contr}_u(P, Q) - \mathrm{contr}_v(P, Q),$$

hence $(PQ) = -1$ from [57], (8.16), which is absurd. Consequently, there exists $P \in \mathrm{MW}(U_\eta)$ of type (2). We may assume that $(P\Theta_{v,0}) = 1$. Let $\lambda : U \to S^\flat$ be the contraction of all the curves (O), (P), $\{\Theta_{t,i} | i \neq 5\}$ and $\{\Theta_{v,j} | j = 1,\ 2\}$ and put $\Delta^\flat := (1/2)\lambda_*\psi^*(t + u)$. Then $(S^\flat, \Delta^\flat)$ gives an example of the log surfaces as in Proposition 3.9.

4 Generalized local fundamental groups for analytic singularities with Weil divisors

In this section, we give a theory to calculate local fundamental groups from differents. Let us review here the theory due to Prill. For a germ of normal complex analytic spaces (X, p), put $\mathrm{Reg}\ X := \mathrm{projlim}_{p \in \mathcal{U};\mathrm{open}} \mathrm{Reg}\ \mathcal{U}$, where $\mathrm{Reg}\ \mathcal{U}$ is the smooth loci of $\mathcal{U}$. $\pi_1^{\mathrm{loc}}(\mathrm{Reg}\ X) := \mathrm{projlim}_{p \in \mathcal{U};\mathrm{open}} \pi_1(\mathrm{Reg}\ \mathcal{U})$ called the *local fundamental group* for (X, p). We denote by $\hat{\pi}_1^{\mathrm{loc}}(\mathrm{Reg}\ X)$ its profinite completion which is called the *local algebraic fundamental group* of (X, p). Let Σ be an analytically closed proper subset of X. According to Prill ([47], §IIB), there exists a contractible open neighbourhood U of p such that there exists a neighbourhood basis $\{U_\lambda\}_{\lambda \in \Lambda}$ of p satisfying the condition that $U_\lambda \setminus \Sigma$ is a deformation retract of $U \setminus \Sigma$ for any $\lambda \in \Lambda$. By the definition, we have $\pi_1(U \setminus \Sigma) = \mathrm{projlim}_{p \in \mathcal{U};\mathrm{open}} \pi_1(\mathcal{U} \setminus \Sigma)$. We call such U as above a *Prill's good neighbourhood with regard to* Σ and we say that $\{U_\lambda\}_{\lambda \in \Lambda}$ is a *neighbourhood basis associated with* U. Recall that U_λ is also a Prill's good neighbourhood with regard to Σ and for any two Prill's good neighbourhood U and U', $U \setminus \Sigma$ and $U' \setminus \Sigma$ have the same homotopy type. In particular, we have $\pi_1^{\mathrm{loc}}(\mathrm{Reg}\ X) \simeq \pi_1(\mathrm{Reg}\ U)$ for a Prill's good neighbourhood U with regard to $\mathrm{Sing}\ X$.

4.1 B(D)-local fundamental groups

In this section, we introduce a generalized local fundamental group for a pair consisting of a germ of a normal complex analytic space and a Weil divisor on it, which turns out to appear canonically when we calculate local fundamental groups from differents. To introduce the generalized notion of local fundamental groups, let us briefly review here the theory of universal ramified coverings due to M. Kato ([20]), M. Namba ([42]) and J.P. Serre ([55], Appendix 6.4) according to M. Namba. Let B be an integral effective divisor on a connected complex manifold M and let $B := \sum_{i \in I} b_i B_i$ be the irreducible decomposition of B. Fix a base point $x \in M \setminus \mathrm{Supp}\ B$ and let γ_i be a loop which starts from x and goes around B_i once in a counterclockwise direction with the center being a smooth point of $\mathrm{Supp}\ B$ on B_i. Let $\mathcal{N}(M, B, x) \subset \pi_1(M \setminus \mathrm{Supp}\ B, x)$ denote the normal subgroup generated by all the conjugates of the loops $\{\gamma_i^{b_i}\}_{i \in I}$. Recall that $\mathcal{N}(M, B, x)$ is known to be independent from the choice of such loops. We define a *B-fundamental group* of M by putting

$$\overline{\pi}_1^B(M, x) := \pi_1(M \setminus \mathrm{Supp}\ B, x)/\mathcal{N}(M, B, x).$$

Here, let us fix our terminology from the category theory. By a *projective system*, we mean a category $\mathcal{I}$ such that $\mathrm{Hom}_{\mathcal{I}}(\lambda, \mu)$ is empty or consists of exactly one element $f_{\lambda,\mu}$ satisfying $f_{\lambda,\mu} \circ f_{\mu,\nu} = f_{\lambda,\nu}$ for any λ, μ, $\nu \in \mathrm{Ob}\ \mathcal{I}$. An object $\alpha \in \mathrm{Ob}\ \mathcal{I}$ (resp. $\omega \in \mathrm{Ob}\ \mathcal{I}$) is called an *initial object* (resp. a *final object*) if $\mathrm{Card\ Hom}_{\mathcal{I}}(\alpha, \lambda) = 1$ (resp. $\mathrm{Card\ Hom}_{\mathcal{I}}(\lambda, \omega) = 1$) for any $\lambda \in \mathrm{Ob}\ \mathcal{I}$. A projective system $\mathcal{I}$ is said to be *cofilterd* if, for any given two objects λ, $\mu \in \mathrm{Ob}\ \mathcal{I}$, there exists $\nu \in \mathrm{Ob}\ \mathcal{I}$ with $\mathrm{Card\ Hom}_{\mathcal{I}}(\nu, \lambda) = \mathrm{Card\ Hom}_{\mathcal{I}}(\nu, \mu) = 1$. A covariant functor $\Phi : \mathcal{I}^\circ \to \mathcal{I}'^\circ$ between injective systems $\mathcal{I}^\circ$ and $\mathcal{I}'^\circ$ is said to be *cofinal*, if, for any given $\lambda' \in \mathrm{Ob}\ \mathcal{I}'^\circ$, there exists $\lambda \in \mathrm{Ob}\ \mathcal{I}^\circ$ such that $\mathrm{Card\ Hom}_{\mathcal{I}'^\circ}(\lambda', \Phi(\lambda)) = 1$. We shall also say that a projective subsystem $\mathcal{I}'$ in a projective system $\mathcal{I}$ is cofinal in $\mathcal{I}$ if the dual embedding functor from $\mathcal{I}'^\circ$ to $\mathcal{I}^\circ$ is cofinal. (see [2], Appendix (1.5), [18], Exposé I, Definition 2.7 and Definition 8.1.1). A finite covering $f : N \to M$ from a connected normal complex analytic space N which is étale over $M \setminus \mathrm{Supp}\ B$ is said to be branching at most (resp. branching) at B, if the ramification index $e_{\tilde{B}_{i,j}}(f)$ of f at any prime divisor $\tilde{B}_{i,j}$ such that $f(\tilde{B}_{i,j}) = B_i$ divides (resp. is equals to) b_i for any $i \in I$. Let $FC^{\leq B}(M)$ (resp. $FC^B(M)$) denote the category of finite coverings over M branching at most (resp. branching) at B. Let $FGC^{\leq B}(M)$ (resp. $FGC^B(M)$ denote the full subcategory of $FC^{\leq B}(M)$ whose objects consists of Galois covers over M. Triplet (N, f, y), where $(N, f) \in \mathrm{Ob}\ FC^{\leq B}(M)$ and $y \in f^{-1}(x)$ are called *pointed finite coverings branching at most at B*. Pointed finite coverings branching at most at B and morphisms $f_{\lambda,\mu} \in \mathrm{Hom}_{FC^{\leq B}(M)}((N_\mu, f_\mu), (N_\lambda, f_\lambda))$ such that $f_{\lambda,\mu}(y_\mu) = y_\lambda$, where (N_μ, f_μ, y_μ) and $(N_\lambda, f_\lambda, y_\lambda)$ are two pointed finite coverings branching at most at B form a projective system denoted by $FC^{\leq B}(M)^p$. We also define the projective subsystems $F(G)C^{(\leq)B}(M)^p$ in the same way. From [42], Lemma 1.3.1, Theorem 1.3.8 and Theorem 1.3.9, we see that there exists a canonical functor Ψ from $FC^{\leq B}(M)^p$ to the projective system of subgroups of finite indices in $\overline{\pi}_1^B(M, x)$ such that

$$\Psi((N, f, y)) = f_* \pi_1(N \setminus \mathrm{Supp}\ f^{-1}B, y)/\mathcal{N}(M, B, x) \subset \overline{\pi}_1^B(M, x)$$

for $(N, f, y) \in \mathrm{Ob} FC^{\leq B}(M)^p$ and that the functor Ψ defines an equivalence between the above two projective systems. Thus by using the basic group theory, we obtain the following lemma.

Lemma 4.1 *$FGC^{\leq B}(M)^p$ (resp. $FGC^B(M)^p$) is cofilterd and cofinal in $FC^{\leq B}(M)^p$ (resp. $FC^B(M)^p$) and hence in particular, $FGC^B(M)^p$ is cofinal in $FGC^{\leq B}(M)^p$ if $FGC^B(M)^p$ is not empty.*

Remark 4.1 Let $\overline{\pi}_1^B(M, x)^\wedge$ denote the profinite completion of $\overline{\pi}_1^B(M, x)$ called the *B-algebraic fundamental group* of M. Assume that $FGC^B(M)^p$ is not empty. Then by Lemma 4.1, we have

$$\overline{\pi}_1^B(M, x)^\wedge \simeq \mathrm{projlim}_{(N,f,y) \in \mathrm{Ob} FGC^B(M)^p} \mathrm{Gal}\ (N/M),$$

where Gal $(N/M) := \overline{\pi}_1^B(M,x)/f_*\overline{\pi}_1^B(N,y)$.

In what follows, we shall use the following notation. Let X be a normal Stein space or a germ of normal complex analytic spaces with a point $p \in X$. Weil X is the free abelian group generated by prime divisors on X and Div X is the subgroup of Weil X generated by Cartier divisors. $\mathrm{Div}_{\boldsymbol{Q}} X$ is the Q-submodule of Weil $X \otimes \boldsymbol{Q}$ generated by Div X. Let $f : Y \to X$ be a finite morphism between normal Stein spaces or germs of normal complex analytic spaces. The pull-back homomorphism $f^* :$ Weil $X \to$ Weil Y canonically extends to a homomorphism $f^* :$ Weil $X \otimes \boldsymbol{Q} \to$ Weil $Y \otimes \boldsymbol{Q}$.

Definition 4.1 For a germ of normal complex analytic spaces (X,p) and $B \in$ Weil X, we define a *B-local fundamental group* of X with respect to B as follows:

$$\overline{\pi}^B_{1,\mathrm{loc}}(\mathrm{Reg}\ X) := \mathrm{projlim}_{p\in\mathcal{U};\mathrm{open}}\overline{\pi}_1^B(\mathrm{Reg}\ \mathcal{U}).$$

Moreover, by $\overline{\pi}^B_{1,\mathrm{loc}}(\mathrm{Reg}\ X)^\wedge$, we mean the profinite completion of $\overline{\pi}^B_{1,\mathrm{loc}}(\mathrm{Reg}\ X)$.

Remark 4.2 We note that apparently, we have $\overline{\pi}^B_{1,\mathrm{loc}}(\mathrm{Reg}\ X) = \pi_1^{\mathrm{loc}}(\mathrm{Reg}\ X)$ and if B is reduced.

Definition 4.2 For $D \in$ Weil $X \otimes \boldsymbol{Q}$, a finite surjective morphism $f : Y \to X$, where Y is a germ of irreducible normal complex analytic spaces such that $f^*D \in$ Weil Y is called a *integral cover with respect to D*. A integral cover $f : Y \to X$ with respect to D is called a *strict integral cover*, if $e_{\tilde{\Gamma}}(f) = e_\Gamma(D)$ for any prime divisors $\tilde{\Gamma}$ on Y and Γ on X such that $f(\tilde{\Gamma}) = \Gamma$, where $e_{\tilde{\Gamma}}(f)$ denotes the ramification index of f at $\tilde{\Gamma}$ and $e_\Gamma(D) := [\boldsymbol{Z}(\mathrm{mult}_\Gamma D) : \boldsymbol{Z}(\mathrm{mult}_\Gamma D) \cap \boldsymbol{Z}] \in \boldsymbol{N}$. By $\mathrm{Int}^m(X;D)$ (resp. $\mathrm{Int}^\dagger(X;D)$), we mean a category of integral covers (resp. a category of strict integral covers) with respect to D. We shall also define categories $\mathrm{Int}^{m(\dagger)}(G)(X;D)^{(p)}$ similarly as before.

Let $\mathcal{X}$ be an arcwise connected, locally arcwise connected, Hausdorff topological space, A continuous map $f : \mathcal{Y} \to \mathcal{X}$ from a Hausdorff topological space $\mathcal{Y}$ with discrete finite fibres is called a *finite topological covering* if for any $x \in \mathcal{X}$, there exists an arcwise connected open neighbourhood $\mathcal{U}$ of $x \in \mathcal{X}$ such that the restriction of π to each arcwise connected component of of $\pi^{-1}(\mathcal{U})$ gives a homeomorphism onto $\mathcal{U}$. The following lemma is nothing but a consequence from the first covering homotopy theorem (see [60], 11.3).

Lemma 4.2 *Let $f : \mathcal{Y} \to \mathcal{X}$ be a connected finite topological covering. Assume that $\mathcal{X}$ is paracompact and let $\mathcal{Z} \subset \mathcal{X}$ be a topological subspace which is a deformation retract of $\mathcal{X}$. Then $\tilde{\mathcal{Z}} := \pi^{-1}(\mathcal{Z}) \subset \mathcal{Y}$ is a deformation retract of $\mathcal{Y}$. In particular, $\tilde{\mathcal{Z}}$ is also arcwise connected and $\pi_1(\tilde{\mathcal{Z}}) = \pi_1(\mathcal{Y})$.*

For a germ of normal complex analytic space (X,p), let U be a Prill's good neighbourhood with regard to a proper analytically closed subset $\Sigma \subset X$ and let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a neighbourhood basis associated with U. We put $U^- := U \setminus \Sigma$ and $U_\lambda^- := U_\lambda \setminus \Sigma$. Take any connected finite topological covering $f^- : V^- \to U^-$. Then V^- has the unique analytic structure such that $f^- : V^- \to U^-$ is étale. By the Grauert-Remmert's theorem, f^- extends uniquely to a finite cover $f : V \to U$, where V is a normal complex analytic space such that $f^{-1}(U^-) = V^-$ (see [15], §2, Satz 8 and [17], XII, Theorem 5.4). Recall here that $f^{-1}(p)$ consists of exactly one point, for if $f^{-1}(p) = \{q_1, \ldots, q_n\}$ and $n \geq 2$, where q_i are distinct from each other, then by [12], 1.10, Lemma 2, there exists $\lambda \in \Lambda$ such that $f^{-1}(U_\lambda) = \coprod_{i=1}^n W_{\lambda,i}$, where $W_{\lambda,i}$ is an open neighbourhood of q_i for $i = 1, \ldots, n$. Since $f^{-1}(U_\lambda^-) \subset f^{-1}(U_\lambda)$ is connected by Lemma 4.2, there exists i, say i_0, such that $f^{-1}(U_\lambda^-) \subset W_{\lambda,i_0}$ and $f^{-1}(U_\lambda^-) \cap W_{\lambda,i}$ is empty if $i \neq i_0$, but which is absurd for $f^{-1}(U_\lambda^-) \cap W_{\lambda,i} = W_{i,\lambda} \setminus f^{-1}(\Sigma)$ is non-empty for any i. Thus we see that any connected finite topological covering $f^- : V^- \to U^-$ determines a finite surjective morphism $f : Y := (V,q) \to (X,p)$ from a germ of normal complex analytic spaces Y uniquely up to isomorphisms, where $f^{-1}(p) = \{q\}$. For two connected finite topological coverings $f_1^- : V_1^- \to U^-$ and $f_2^- : V_2^- \to U^-$, let $f_i : V_i \to U^-$ be the extended finite covers and $f : Y_i \to X$ be the corresponding finite surjective morphisms as above for $i = 1$, 2. By [17], Exposé XII, Proposition 5.3, the restriction map $r : \mathrm{Hom}_U(V_1, V_2) \to \mathrm{Hom}_{U^-}(V_1^-, V_2^-)$ is bijective and composed with the canonical injection $\mathrm{Hom}_U(V_1, V_2) \to \mathrm{Hom}_X(Y_1, Y_2)$, r^{-1} gives a canonical injection $\mathrm{Hom}_{U^-}(V_1^-, V_2^-) \to \mathrm{Hom}_X(Y_1, Y_2)$. From the above argument, we see that we have a canonical faithful functor $\mathcal{P}$ called a *Prill functor* from the category of connected topological finite coverings over U^- denoted by $FT(U^-)$ to the category of germs of normal complex analytic spaces which is finite over X and étale outside over Σ denoted by $FC(X, \Sigma)$.

Lemma 4.3 *A Prill functor defines an equivalence between the categories $FT(U^-)$ and $FC(X, \Sigma)$.*

Proof. We note that the restriction functor $\mathcal{R}_\lambda : FT(U^-) \to FT(U_\lambda^-)$ defines an equivalence of categories between $FT(U^-)$ and $FT(U_\lambda^-)$ since these are known to be determined up to equivalences by the corresponding fundamental groups. Put $(V_{i,\lambda}^-, f_{i,\lambda}^-) := \mathcal{R}_\lambda((V_i^-, f_i^-)) \in \mathrm{Ob}\ FT(U_\lambda)$ and $V_{i,\lambda} := f_i^{-1}(U_\lambda)$ for $i = 1, 2$. Note also that the canonical map

$$\mathrm{injlim}_{\lambda\in\Lambda}\mathrm{Hom}\ _{U_\lambda}(V_{1,\lambda}, V_{2,\lambda}) \to \mathrm{Hom}\ _{FC(X,\Sigma)}((Y_1, f_1), (Y_2, f_2))$$

is bijective and that we have

$$\mathrm{injlim}_{\lambda\in\Lambda}\mathrm{Hom}\ _{FT(U_\lambda^-)}((V_{1,\lambda}^-, f_{1,\lambda}^-), (V_{2,\lambda}^-, f_{2,\lambda}^-)) = \mathrm{Hom}\ _{FT(U^-)}((V_1^-, f_1^-), (V_2^-, f_2^-)).$$

Therefore, we conclude that the canonical map

$$\mathrm{Hom}\ _{FT(U^-)}((V_1^-, f_1^-), (V_2^-, f_2^-)) \to \mathrm{Hom}\ _{FC(X,\Sigma)}((Y_1, f_1), (Y_2, f_2))$$

is bijective, which implies that the functor $\mathcal{P}$ is faithfully full. Take any $(Y, f) \in \mathrm{Ob}\ FC(X, \Sigma)$. Then f is represented by a finite cover $f : V_\lambda \to U_\lambda$ for some $\lambda \in \Lambda$, where V_λ is connected. Since f is étale over U_λ^- and $V_\lambda^- := f^{-1}(U_\lambda^-)$ is also connected, we obtain an object $(V_\lambda^-, f|_{V_\lambda^-}) \in \mathrm{Ob}\ FT(U_\lambda^-)$ which goes to $(Y, f) \in \mathrm{Ob}\ FC(X, \Sigma)$ via $\mathcal{P} \circ \mathcal{R}_\lambda^{-1}$. Thus we conclude that $\mathcal{P}$ is essentially surjective, and hence $\mathcal{P}$ defines an equivalence. ■

Remark 4.3 It is obvious that a Prill functor also defines an equivalence between the full subcategory of Galois objects of $FT(U^-)$ and $FC(X, \Sigma)$. Note that giving a pointing to an object of $FT(U^-)$ and $FC(X, \Sigma)$ has essentially the same meaning since the number of pointings for $(V^-, f^-) \in \mathrm{Ob}\ FT(U^-)$ and the number of pointings for $(Y, f) \in \mathrm{Ob}\ FC(X, \Sigma)$ are both $\deg f = \deg f^-$. Thus we see that $\mathcal{P}$ induces an equivalence between $FT(U^-)^p$ and $FC(X, \Sigma)^p$.

For $\mathbf{Q}$-divisor D on X, let $\mathcal{B}_X(D)$ be the set of all the prime divisors on X such that $e_\Gamma(D) > 1$ and put $D^\vee := \sum_{\Gamma\in\mathcal{B}_X(D)} e_\Gamma(D)\Gamma \in \mathrm{Weil}\ X$. Combined with Lemma 4.1, Lemma 4.3 yields the following proposition.

Proposition 4.1 *There exists a canonical functor $\mathcal{P} : FC^{D^\vee}(\mathrm{Reg}\ U) \to \mathrm{Int}^\dagger(X; D)$ which defines an equivalence between the categories $FC^{D^\vee}(\mathrm{Reg}\ U)$ and $\mathrm{Int}^\dagger(X; D)$. In particular, $\mathrm{Int}^\dagger G(X; D)^p$ is cofilterd and cofinal in $\mathrm{Int}^\dagger(X; D)^p$.*

Remark 4.4 From Proposition 4.1, we deduce that $\mathrm{projlim}_{(Y,f,i_Y)\in \mathrm{Ob\ Int}^\dagger G(X;D)^p}\mathrm{Gal}\ (Y/X)$ is isomorphic to $\overline{\pi}_{1,\mathrm{loc}}^{D^\vee}(\mathrm{Reg}\ X)^\wedge$ in the category of profinite groups.

Definition 4.3 For $\mathbf{Q}$-divisor D on X, we define a group $\pi_{1,X,p}^{\mathrm{loc}}[D]$ by $\pi_{1,X,p}^{\mathrm{loc}}[D] := \overline{\pi}_{1,\mathrm{loc}}^{D^\vee}(\mathrm{Reg}\ X)$, which is called *the D-local fundamental group* for $((X, p), D)$ and we denote by $\hat{\pi}_{1,X,p}^{\mathrm{loc}}[D]$ its profinite completion.

Remark 4.5 We note that $\pi_{1,X,p}^{\mathrm{loc}}[D]$ depends only on the class $[D] \in \mathrm{Weil}\ X \otimes \boldsymbol{Q}/\mathrm{Weil}\ X$ and that if $D \in \mathrm{Weil}\ X$, $\pi_{1,X,p}^{\mathrm{loc}}[D] \simeq \pi_1^{\mathrm{loc}}(\mathrm{Reg}\ X)$.

4.2 The category of Cartier covers and Comparison Theorem

In this section, we introduce a category which is easier to handle with than the category of integral covers. Let (X, p) be a germ of irreducible normal complex analytic spaces and let $\mathcal{M}_X$ denote the field of germs of meromorphic functions on X. In what follows, we fix an algebraic closure $\overline{\mathcal{M}}_X$ of $\mathcal{M}_X$ and the inclusion $i_X : \mathcal{M}_X \to \overline{\mathcal{M}}_X$. Take any $D \in \mathrm{Div}_{\boldsymbol{Q}} X$ and fix it. Recall that a holomorphic map between complex analytic spaces is said to be *finite*, if it is proper with discrete finite fibres.

Definition 4.4 A finite surjective morphism $f : Y \to X$, where Y is a germ of irreducible normal complex analytic spaces such that f^*D is integral and Cartier, is called an *Cartier cover with respect to D*. A Cartier cover $f : Y \to X$ with respect to D is called *Cartier Galois cover with respect to D* if f is Galois.

Definition 4.5 Cartier covers (resp. Cartier Galois covers) of X with respect to D form a full subcategory of complex analytic germs denoted by $\mathrm{Cart}^m(X; D)$ (resp. $\mathrm{Cart}^m G(X; D)$). For $(Y, f) \in \mathrm{Ob}\,\mathrm{Cart}^m(X; D)$, an injective homomorphism $i_Y : \mathcal{M}_Y \to \overline{\mathcal{M}}_X$, where $\mathcal{M}_Y$ is the meromorphic function field of Y, such that $i_Y \circ f^* = i_X$ is called a *pointing.* Triplet (Y, f, i_Y) composed of $(Y, f) \in \mathrm{Ob}\,\mathrm{Cart}^m(X; D)$ and a pointing i_Y are called *pointed Cartier covers with respect to D.* Pointed Cartier covers (resp. pointed Cartier Galois covers) with respect to D and morphisms $f_{\lambda,\mu} \in \mathrm{Hom}_{\,\mathrm{Cart}^m(X;D)}((Y_\mu, f_\mu), (Y_\lambda, f_\lambda))$ satisfying $i_{Y_\mu} \circ f^*_{\lambda,\mu} = i_{Y_\lambda}$ form a projective system denoted by $\mathrm{Cart}^m(X; D)^p$ (resp. $\mathrm{Cart}^m G(X; D)^p$).

Non-zero $\boldsymbol{C}$-algebra $\mathcal{A}$ is called *a complex analytic ring* if there exists a surjective $\boldsymbol{C}$-algebra homomorphism $\mathcal{O}^{\mathrm{an}}_{E,0} \to \mathcal{A}$, where $E \simeq \boldsymbol{C}^n$ for some n. Recall that the category of complex analytic rings $\mathcal{A}$ which are finite $\mathcal{O}_X$-modules and that category of germs of complex analytic spaces which are finite over X are dual to each other via the contravariant functor Specan_X (see [12] and [16], VI). The structure morphism $f^* : \mathcal{O}_X \to \mathcal{A}$ is injective if and only if $f : \mathrm{Specan}_X \mathcal{A} \to X$ is surjective by the Remmert's proper mapping theorem (see, for example, [12], 1.18). Let φ be a meromorphic function on X such that $\mathcal{O}_X(-rD) = \varphi \mathcal{O}_X$ and let $\pi : \tilde{X} \to X$ be the index one cover with respect to D obtained by taking a r-th root of φ, where $r := \mathrm{ind}_p D$ and fix a pointing $i_{\tilde{X}}$. Take any $(Y, f, i_Y) \in \mathrm{Ob}\,\mathrm{Cart}^m(X; D)^p$. For simplicity, assume that $\mathcal{M}_X \subset \mathcal{M}_{\tilde{X}} \subset \overline{\mathcal{M}}_X$ and $\mathcal{M}_X \subset \mathcal{M}_Y \subset \overline{\mathcal{M}}_X$. The assumption on Y implies that there exists a meromorphic function ψ on Y such that $\varphi \mathcal{O}_Y = \psi^r \mathcal{O}_Y$, which implies that there exists a unit $u \in \mathcal{O}_Y^\times$ such that $\psi^r = u\varphi$. Since $\mathcal{O}_Y$ is a henselian local ring whose residue field is the complex number field (see, for example, [1], Ch. III, §20, Proposition 20.6), we see that $\sqrt[r]{u} \in \mathcal{O}_Y^\times$, hence $\mathcal{M}_X(\sqrt[r]{\varphi}) \subset \mathcal{M}_Y$. Consequently, there exists a $\mathcal{O}_X$-homomorphism $\pi_* \mathcal{O}_{\tilde{X}} \to f_* \mathcal{O}_Y$ which induces a finite surjective morphism $\varpi_Y : Y = \mathrm{Specan}_X f_* \mathcal{O}_Y \to \tilde{X} = \mathrm{Specan}_X \pi_* \mathcal{O}_{\tilde{X}}$ satisfying $f = \pi \circ \varpi_Y$. The above argument implies that $\mathrm{Card}\,\mathrm{Hom}_{\,\mathrm{Cart}^m(X;D)^p}((Y, f, i_Y), (\tilde{X}, \pi, i_{\tilde{X}})) = 1$ for any $(Y, f, i_Y) \in \mathrm{Ob}\,\mathrm{Cart}^m(X; D)^p$, that is, $(\tilde{X}, \pi, i_{\tilde{X}})$ is a final object, or equivalently, a colimit of $\mathrm{Cart}^m(X; D)^p$. Let $\varpi_Y(i_Y, i_{\tilde{X}})$ denote the element of $\mathrm{Hom}_{\,\mathrm{Cart}^m(X;D)^p}((Y, f, i_Y), (\tilde{X}, \pi, i_{\tilde{X}}))$.

Definition 4.6 A Cartier cover $f : Y \to X$ with respect to D is called a *strict Cartier cover with respect to D*, if $\varpi_Y(i_Y, i_{\tilde{X}})$ is étale in codimension one for any pointings $i_Y, i_{\tilde{X}}$.

Remark 4.6 $\varpi_Y(i_Y, i_{\tilde{X}})$ is étale in codimension one if and only if $\varpi_Y(i_Y, i_{\tilde{X}})$ is étale over $\mathrm{Reg}\, X$ by the purity of branch loci (see [1], V, §39, (39.8) or [12], 4.2).

Definition 4.7 A strict Cartier cover $f : Y \to X$ with respect to D is called a *strict Cartier Galois cover* with respect to D, if f is Galois.

Remark 4.7 Take another pointings i'_Y and $i'_{\tilde{X}}$ of (Y, f), $(\tilde{X}, \pi) \in \mathrm{Ob}\ \mathrm{Cart}^m(X; D)^p$ respectively and assume that $f : Y \to X$ is Galois. Then, by the Galois theory, there exist two isomorphisms $\alpha(i_Y, i'_Y) \in \mathrm{Hom}_{\ \mathrm{Cart}^m(X;D)^p}((Y, f, i_Y), (Y, f, i'_Y))$ and $\beta(i_{\tilde{X}}, i'_{\tilde{X}}) \in \mathrm{Hom}_{\ \mathrm{Cart}^m(X;D)^p}((\tilde{X}, \pi, i_{\tilde{X}}), (\tilde{X}, \pi, i'_{\tilde{X}}))$ such that the following diagram in $\mathrm{Cart}^m(X; D)^p$ commutes.

$$\begin{array}{ccc} (Y, f, i_Y) & \xrightarrow{\alpha(i_Y, i'_Y)} & (Y, f, i'_Y) \\ {\scriptstyle \varpi_Y(i_Y, i_{\tilde{X}})}\downarrow & & \downarrow{\scriptstyle \varpi_Y(i'_Y, i'_{\tilde{X}})} \\ (\tilde{X}, \pi, i_{\tilde{X}}) & \xrightarrow{\beta(i_{\tilde{X}}, i'_{\tilde{X}})} & (\tilde{X}, \pi, i'_{\tilde{X}}) \end{array}$$

Therefore, $(Y, f) \in \mathrm{Ob}\ \mathrm{Cart}^m(X; D)^p$ is a strict Cartier Galois cover if one of $\varpi_Y(i_Y, i_{\tilde{X}}) \in \mathrm{Hom}_{\ \mathrm{Cart}^m(X;D)}((Y, f), (\tilde{X}, \pi))$ is étale in codimension one. One can also check easily that the same holds even if f is not Galois.

Let $\mathrm{Cart}^\dagger(X; D)$ (resp, $\mathrm{Cart}^\dagger(X; D)^p$) denote the full subcategory of $\mathrm{Cart}^m(X; D)$ (resp. projective subsystem $\mathrm{Cart}^m(X; D)^p$) whose objects are strict Cartier covers with respect to D (resp. pointed strict Cartier covers with respect to D) and let $\mathrm{Cart}^\dagger G(X; D)$ (resp, $\mathrm{Cart}^\dagger G(X; D)^p$) denote the full subcategory of $\mathrm{Cart}^\dagger(X; D)$ (resp. projective subsystem $\mathrm{Cart}^\dagger(X; D)^p$) whose objects are strict Cartier Galois covers with respect to D (resp. pointed strict Cartier Galois covers with respect to D). Let $f_{\lambda,\mu} : (Y_\mu, f_\mu, i_{Y_\mu}) \to (Y_\lambda, f_\lambda, i_{Y_\lambda})$ be a morphism in $\mathrm{Cart}^\dagger G(X; D)^p$ and assume that $\mathcal{M}_X \subset \mathcal{M}_{Y_\lambda} \subset \mathcal{M}_{Y_\mu} \subset \bar{\mathcal{M}}_X$ for simplicity. Then by the Galois theory, there exists a canonical surjective homomorphism $g_{\lambda,\mu} : \mathrm{Gal}\ (\mathcal{M}_{Y_\mu}/\mathcal{M}_X) \to \mathrm{Gal}\ (\mathcal{M}_{Y_\lambda}/\mathcal{M}_X)$ which is nothing but the restriction map. Thus Galois groups $\mathrm{Gal}\ (Y/X) := \mathrm{Gal}\ (\mathcal{M}_Y/\mathcal{M}_X)$ for $(Y, f, i_Y) \in \mathrm{Cart}^\dagger G(X; D)^p$ form a projective system with the induced morphisms from $\mathrm{Cart}^\dagger G(X; D)^p$. Here is our comparison theorem.

Theorem 4.1 *There exists a canonical isomorphism* :

$$\hat{\pi}^{\mathrm{loc}}_{1,X,p}[D] \simeq \mathrm{projlim}_{(Y,f,i_Y) \in \mathrm{Ob}\ \mathrm{Cart}^\dagger G(X;D)^p} \mathrm{Gal}\ (Y/X)$$

in the category of profinite groups for any $D \in \mathrm{Div}_{\boldsymbol{Q}} X$.

To prove the above theorem, we need some lemmas and propositions as follows. Let $\mathcal{C}$ be a category and let $X \in \mathrm{Ob}\ \mathcal{C}$ be an object of $\mathcal{C}$ and $G \subset \mathrm{Aut}\ X$ be a subgroup of the automorphism group of X.

Definition 4.8 An epimorphism $f : X \to Y$ in $\mathcal{C}$ is said to be *Galois with the Galois group* G, if $G = \mathrm{Aut}_{\ Y} X := \{\sigma \in \mathrm{Aut}\ X | f \circ \sigma = f\}$ and for any morphism $f' : X \to Y'$ such that $G \subset \mathrm{Aut}_{\ Y'} X$, there exists a unique morphism $\varphi : Y \to Y'$ satisfying $f' = \varphi \circ f$.

Remark 4.8 Assume that two Galois morphisms $f : X \to Y$ and $f' : X \to Y'$ with the Galois group G are given. Then by the universal mapping property, there exists an isomorphism $\varphi : Y \to Y'$ such that $f' = \varphi \circ f$, that is, Galois morphisms with the Galois group G is unique up to this equivalence.

Example 4.1 Let $\mathcal{F} :=$ (Fields) be a category of fields such that $\mathrm{Hom}_{\mathcal{F}}(K_1, K_2)$ is empty or consists of inclusions for any $K_1, K_2 \in \mathrm{Ob}\ \mathcal{F}$. For any finite extension $i : K_1 \to K_2$, i is a Galois extension if and only if its dual $i^\circ : K_2^\circ \to K_1^\circ$ in the dual category $\mathcal{F}^\circ$ is Galois by the Galois theory.

Definition 4.9 For any two morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, we define a subgroup $\mathrm{Aut}^f_Z X \subset \mathrm{Aut}\ X \times \mathrm{Aut}_Z Y$ as $\mathrm{Aut}^f_Z X := \{(\tilde{\sigma}, \sigma) | f \circ \tilde{\sigma} = \sigma \circ f\}$.

Lemma 4.4 *Let* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ *and* $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ *be two Galois morphisms and assume that the second projection* $p_2 : \mathrm{Aut}^f_Z X \to \mathrm{Aut}_Z Y$ *is surjective. Then* $h := g \circ f$ *is also Galois.*

Proof. Take any $h' \in \mathrm{Hom}_{\mathcal{C}}(X, Z')$ with $\mathrm{Aut}_Z X \subset \mathrm{Aut}_{Z'} X$. Since $\mathrm{Aut}_Y X \subset \mathrm{Aut}_Z X$ and f is Galois, there exists a morphism $\psi : Y \to Z'$ such that $\psi \circ f = h'$. Take any $\sigma \in \mathrm{Aut}_Z Y$. Then there exists $\tilde{\sigma} \in \mathrm{Aut}\, X$ such that $f \circ \tilde{\sigma} = \sigma \circ f$ by the assumption. Since $\tilde{\sigma} \in \mathrm{Aut}_Z X \subset \mathrm{Aut}_{Z'} X$, we have $h' = h' \circ \tilde{\sigma} = \psi \circ f \circ \tilde{\sigma} = \psi \circ \sigma \circ f$, hence $\psi \circ f = \psi \circ \sigma \circ f$. Since f is an epimorphism, we deduce that $\psi = \psi \circ \sigma$, that is, $\mathrm{Aut}_Z Y \subset \mathrm{Aut}_{Z'} Y$. Thus we conclude that there exists a morphism $\varphi : Z \to Z'$ such that $\varphi \circ g = \psi$. Obviously φ satisfies $\varphi \circ h = h'$. As for the uniqueness of φ, Let $\varphi' : Z \to Z'$ be another morphism satisfying $\varphi' \circ h = h'$. Then $\psi \circ f = h' = \varphi' \circ h = \varphi' \circ g \circ f$, hence $\varphi' \circ g = \psi = \varphi \circ g$. Since g is also an epimorphism, we obtain $\varphi' = \varphi$. ■

Remark 4.9 The assumption in Lemma 4.4 is satisfied in the following two theoretically important cases.

(1) Let f and g are finite Galois covers between normal algebraic varieties over an algebraically closed field or normal connected complex analytic spaces. Assume that there exits a Zariski closed subset or an analytic subset Σ on Y with $\mathrm{codim}_Y \Sigma \leq 2$ such that the restriction f^- of f to $X^- := X \setminus f^{-1}(\Sigma)$ gives the algebraic universal cover of $Y^- := Y \setminus \Sigma$, that is, $\hat{\pi}_1(X^-) = \{1\}$. Moreover assume that Y^- is invariant under the action of Gal (Y/Z). Take any $\sigma \in$ Gal (Y/Z). Since σ acts on Y^-, there exists an automorphism $\tilde{\sigma}^-$ on X^- such that $f^- \circ \tilde{\sigma}^- = \sigma \circ f^-$ by the property of algebraic universal cover. $\tilde{\sigma}^-$ extends uniquely to an automorphism $\tilde{\sigma}$ on X satisfying $f \circ \tilde{\sigma} = \sigma \circ f$ by the normality (see also [6], §1 and [62], Lemma 2.1).

(2) Let f and g are finite Galois covers between germs of normal complex analytic spaces. Assume that X is obtained from Y by taking a r-th root of a primitive principal divisor $P = \mathrm{div}\, \varphi$ on Y such that $\mathcal{O}_Y(P) \subset \mathcal{M}_Y$ is invariant under the action of Gal (Y/Z) (for the definition of primitive principal divisors, see [58], 2.3). Take any $\sigma \in$ Gal (Y/Z). Then by the assumption, we have $\sigma^* \varphi = u\varphi$ for some unit $u \in \mathcal{O}_Y^\times$. As in the previous argument, there exists a unit $v \in \mathcal{O}_Y^\times$ such that $v^r = u$. Since we can write $\mathcal{M}_X = \mathcal{M}_Y[T]/(T^r - \varphi)$, it is obvious that σ^* lifts to an automorphism $\tilde{\sigma}^*$ on $\mathcal{M}_X$ by putting $\tilde{\sigma}^* T = vT$. Thus we see that any elements of Gal (Y/Z) lift to elements of Gal (X/Z).

Lemma 4.5 (c.f., [54]) *Let $\mathcal{A}$ be an integral complex analytic ring and $\mathcal{M}$ be its quotient field. Let $\mathcal{A}_\mathcal{L}$ be the normalization of $\mathcal{A}$ in a finite extension field $\mathcal{L}$ of $\mathcal{M}$. Then $\mathcal{A}_\mathcal{L}$ is also an integral complex analytic ring which is a finite $\mathcal{A}$-module.*

Proof. Recall that $\mathcal{A}$ is noetherian ([16], II, Proposition 2.3). [54], Theorem 4 says that $\mathcal{A}$ is N-1, hence N-2 by [30], Ch. 12, Corollary 1, that is, $\mathcal{A}_\mathcal{L}$ is a finite $\mathcal{A}$-module. By [54], Theorem 1, $\mathcal{A}$ is a finite $\mathcal{O}^{\mathrm{an}}_{\boldsymbol{C}^n,0}$-module for some $\boldsymbol{C}^n$, hence so is $\mathcal{A}_\mathcal{L}$. Thus by [54], Theorem 3, we conclude that $\mathcal{A}_\mathcal{L}$ is a complex analytic ring. ■

Remark 4.10 Lemma 4.5 implies that if we are given a finite extension field $\mathcal{L}$ of the meromorphic function field $\mathcal{M}_X$ of an irreducible germ of complex analytic spaces X, there exists a germ of normal complex analytic spaces Y with a finite surjective morphism $f : Y \to X$ such that $\mathcal{M}_Y$ is isomorphic to $\mathcal{L}$ over $\mathcal{M}_X$ and such Y as above is uniquely determined up to isomorphisms over X.

The following proposition can be also derived from Proposition 4.1, but we shall give an algebraic proof for further research such as extending our theory to the positive characteristic case.

Proposition 4.2 $\mathrm{Cart}^\dagger G(X; D)^p$ *is cofilterd and is cofinal in* $\mathrm{Cart}^\dagger(X; D)^p$ *for any* $D \in \mathrm{Div}_{\boldsymbol{Q}} X$.

Proof. Firstly, we prove the first statement. Take any two objects $(Y_\lambda, f_\lambda, i_{Y_\lambda})$, $(Y_\mu, f_\mu, i_{Y_\mu}) \in \mathrm{Cart}^\dagger G(X; D)^p$. Let $\mathcal{L} := i_{Y_\lambda}(\mathcal{M}_{Y_\lambda}) \vee i_{Y_\mu}(\mathcal{M}_{Y_\mu})$ be the minimal subfield of $\overline{\mathcal{M}}_X$ containing $i_{Y_\lambda}(\mathcal{M}_{Y_\lambda})$ and $i_{Y_\mu}(\mathcal{M}_{Y_\mu})$. We note that $\mathcal{L}$ is a finite Galois extension of $\mathcal{M}_X$ by its definition. Let $g : Z \to X$ be the normalization of X in $\mathcal{L}$ as explained in Remark 4.10. By the construction, we get an object

$(Z, g, i_Z) \in \mathrm{Ob}\ \mathrm{Cart}^m(X; D)^p$ dominating both of $(Y_\lambda, f_\lambda, i_{Y_\lambda})$ and $(Y_\mu, f_\mu, i_{Y_\mu})$ in $\mathrm{Cart}^m(X; D)^p$. Let $(\tilde{X}, \pi, i_{\tilde{X}}) \in \mathrm{Ob}\ \mathrm{Cart}^\dagger G(X; D)^p$ be a final object of $\mathrm{Cart}^\dagger G(X; D)^p$. From the equality $i_{\tilde{X}} = i_Y \circ \varpi_Y(i_Y, i_{\tilde{X}})^*$, we see that there exists a canonical embedding:

$$\Phi_{i_{\tilde{X}}} : \mathrm{Cart}^\dagger G(X; D)^p \to \mathrm{Cart}^\dagger G(\tilde{X}; 0)^p, \tag{4.10}$$

depending on the choice of pointings $i_{\tilde{X}}$ for $(\tilde{X}, \pi) \in \mathrm{Ob}\ \mathrm{Cart}^\dagger G(X; D)$, such that $\Phi_{i_{\tilde{X}}}((Y, f, i_Y)) = (Y, \varpi_Y(i_Y, i_{\tilde{X}}), i_Y) \in \mathrm{Ob}\ \mathrm{Cart}^\dagger G(\tilde{X}; 0)^p$ for $(Y, f, i_Y) \in \mathrm{Ob}\ \mathrm{Cart}^\dagger G(X; D)^p$. Since $\mathrm{Cart}^\dagger G(\tilde{X}; 0)^p$ is cofilterd as explained in Remark 4.3, there exists $(W, h, i_W) \in \mathrm{Ob}\ \mathrm{Cart}^\dagger G(\tilde{X}; 0)^p$ which dominates both of $(Y_\lambda, \varpi_{Y_\lambda}(i_{Y_\lambda}, i_{\tilde{X}}), i_{Y_\lambda})$ and $(Y_\mu, \varpi_{Y_\mu}(i_{Y_\mu}, i_{\tilde{X}}), i_{Y_\mu})$ in $\mathrm{Cart}^\dagger G(\tilde{X}; 0)^p$. By the construction of $(Z, g, i_Z) \in \mathrm{Ob}\ \mathrm{Cart}^m(X; D)^p$, i_Z factors into $i_W \circ \tau^*$, where $\tau^* : \mathcal{M}_Z \to \mathcal{M}_W$ is an injective homomorphism. Let τ be the induced morphism $\tau : W \to Z$. Then we see that h factors into $\varpi_Z(i_Z, i_{\tilde{X}}) \circ \tau$. Since h is étale in codimension one, hence so is $\varpi_Z(i_Z, i_{\tilde{X}})$. Thus we conclude that $(Z, g, i_Z) \in \mathrm{Ob}\ \mathrm{Cart}^\dagger G(X; D)^p$ and consequently, $\mathrm{Cart}^\dagger G(X; D)^p$ is cofilterd. As for second statement, take any $(Y, f, i_Y) \in \mathrm{Ob}\ \mathrm{Cart}^\dagger (X; D)^p$ and let $\{i_Y^{(k)} | k = 1, 2, \ldots n\}$ be all the pointings for $(Y, f) \in \mathrm{Ob}\ \mathrm{Cart}^\dagger (X; D)$. Let $\mathcal{L}$ be the minimal subfield of $\overline{\mathcal{M}}_X$ containing all the subfields $i_Y^{(1)}(\mathcal{M}_Y)$, $\ldots$, $i_Y^{(1)}(\mathcal{M}_Y)$. Since $\mathcal{L}$ is a finite Galois extension of $\mathcal{M}_X$ by its construction, we have an object $(Z, g, i_Z) \in \mathrm{Ob}\ \mathrm{Cart}^m G(X; D)^p$ dominating all the $(Y, f, i_Y^{(1)})$, $\ldots$, $(Y, f, i_Y^{(n)}) \in \mathrm{Ob}\ \mathrm{Cart}^\dagger (X; D)^p$, where $g : Z \to X$ is the normalization of X in $\mathcal{L}$. In the same way as in the previous argument, we conclude that $(Z, g, i_Z) \in \mathrm{Ob}\ \mathrm{Cart}^\dagger G(X; D)^p$. ■

Proof of Theorem 4.1. By Proposition 4.1 and Remark 4.4, we only have to show that the full subcategory $\mathrm{Cart}^\dagger G(X; D)^p$ is cofinal in $\mathrm{Int}^\dagger G(X; D)^p$ (see [18], Exposé I, Proposition 8.1.3 or [2], Appendix, Corollary (2.5)). Choose any object $(Y, f, i_Y) \in \mathrm{Ob}\ \mathrm{Int}^\dagger G(X; D)^p$ and let $\pi_Y : \tilde{Y} \to Y$ be the index one cover with respect to f^*D. We can choose a pointing $i_{\tilde{Y}}$ so that a triple $(\tilde{Y}, \tilde{f}, i_{\tilde{Y}})$ becomes an object in $\mathrm{Int}^\dagger (X; D)^p$ dominating (Y, f, i_Y). From Remark 4.9, we deduce that $(\tilde{Y}, \tilde{f}, i_{\tilde{Y}}) \in \mathrm{Ob}\ \mathrm{Cart}^\dagger G(X; D)^p$ by its construction. ■

4.3 Universal Cartier covers

Let U be a Prill's good neighbourhood with regard to $\mathrm{Sing}\ X$ and $\{U_\lambda\}_{\lambda \in \Lambda}$ its associated neighbourhood basis. Take any $(Y, f, i_Y) \in \mathrm{Ob}\ \mathrm{Cart}^\dagger (X; 0)^p$ and put $(V^-, f^-, y) := \mathcal{P}^{-1}(Y, f, i_Y) \in FT(U^-)$, where $\mathcal{P}$ is a Prill functor. Let $f : V \to U$ be the extended finite cover of f^-. By Lemma 4.2, V is a Prill's good neighbourhood with regard to $f^{-1}(\mathrm{Sing}\ X)$ with $\{V_\lambda\}_{\lambda \in \Lambda}$ being its associated neighbourhood basis. Thus we have $\pi_1(V^-) = \mathrm{projlim}_{q \in \mathcal{V};\mathrm{open}} \pi_1(\mathcal{V} \setminus f^{-1}(\mathrm{Sing}\ X)) = \mathrm{projlim}_{q \in \mathcal{V};\mathrm{open}} \pi_1(\mathrm{Reg}\ \mathcal{V}) = \pi_1^{\mathrm{loc}}(\mathrm{Reg}\ Y)$ since $\mathrm{Reg}\ \mathcal{V} \cap f^{-1}(\mathrm{Sing}\ X)$ is a closed analytic subspace of codimension at least two in $\mathrm{Reg}\ \mathcal{V}$. (see, for example, [47], III, Corollary 2.). In particular, we see that $(Y, f, i_Y) \in \mathrm{Ob}\ \mathrm{Cart}^\dagger (X; 0)^p$ is an initial object of $\mathrm{Cart}^\dagger (X; 0)^p$ if and only if $\hat{\pi}_1^{\mathrm{loc}}(\mathrm{Reg}\ Y) = \{1\}$.

Definition 4.10 For $D \in \mathrm{Div}_{\boldsymbol{Q}} X$, a strict Cartier Galois cover $\pi^\dagger : X^\dagger \to X$ with respect to D is called an *algebraic universal Cartier cover*, or abbreviated, a *universal Cartier cover* with respect to D if $\hat{\pi}_1^{\mathrm{loc}}(\mathrm{Reg}\ X^\dagger) = \{1\}$.

Remark 4.11 Singularity with trivial local algebraic fundamental group is quite restrictive one. For example, $\hat{\pi}_1^{\mathrm{loc}}(\mathrm{Reg} X) = \{1\}$ implies $\mathrm{Div}_{\boldsymbol{Q}} X \cap \mathrm{Weil}\ X = \mathrm{Div}\ X$. Moreover, if we assume, in addition, that (X, p) is analytically $\boldsymbol{Q}$-factorial, then $\mathcal{O}_X$ is factorial (see, for example, [5], Satz 1.4).

Proposition 4.3 *For $D \in \mathrm{Div}_{\boldsymbol{Q}} X$, take the index one cover $\pi : \tilde{X} \to X$ with respect to D, the there exists the universal Cartier cover of X with respect to D if and only if $\hat{\pi}_1^{\mathrm{loc}}(\mathrm{Reg}\ \tilde{X})$ is finite.*

Proof. Assume that $\hat{\pi}_1^{\mathrm{loc}}(\mathrm{Reg}\,\tilde{X})$ is finite and take a final object $(\tilde{X}, \pi, i_{\tilde{X}}) \in \mathrm{Ob\ Cart}^{\dagger}G(X;D)^p$. By the assumption, $\mathrm{Cart}^{\dagger}(\tilde{X};0)^p$ has an initial object $(Y, f, i_Y) \in \mathrm{Ob\ Cart}^{\dagger}G(\tilde{X};0)^p$ such that $\hat{\pi}_1^{\mathrm{loc}}(\mathrm{Reg}\,Y) = \{1\}$. We note that i_Y is also a pointing for $(Y, \pi \circ f) \in \mathrm{Ob\ Cart}^m(X;D)$ since we have $i_Y \circ f^* \circ \pi^* = i_{\tilde{X}} \circ \pi^* = i_X$. Consider an object $(Y, \pi \circ f, i_Y) \in \mathrm{Ob\ Cart}^m(X;D)^p$. By the argument in Remark 4.9, (1), we see that $\pi \circ f$ is Galois. Since $\varpi_Y(i_Y, i_{\tilde{X}}) = f$ is étale in codimension one, we conclude that $(Y, \pi \circ f, i_Y) \in \mathrm{Ob\ Cart}^{\dagger}G(X;D)^p$. Conversely, assume that there exists a pointed universal Cartier cover $(X^{\dagger}, \pi^{\dagger}, i_{X^{\dagger}}) \in \mathrm{Ob\ Cart}^{\dagger}G(X;D)^p$ with respect to D. Then $\Phi_{i_{\tilde{X}}}((X^{\dagger}, \pi^{\dagger}, i_{X^{\dagger}})) \in \mathrm{Ob\ Cart}^{\dagger}G(\tilde{X};0)^p$ is an initial object of $\mathrm{Ob\ Cart}^{\dagger}(\tilde{X};0)^p$, hence $\hat{\pi}_1^{\mathrm{loc}}(\mathrm{Reg}\,\tilde{X})$ is finite. ■

Remark 4.12 Assume that (X, Δ) is purely log terminal, where Δ is a standard $\mathbf{Q}$-boundary. Then $(\tilde{X}, \Delta_{\tilde{X}})$ is known to be canonical, hence $\tilde{X}$ has only canonical singularity if we assume that $\lfloor \Delta \rfloor = 0$ or $\tilde{X}$ is $\mathbf{Q}$-Gorenstein. Thus if $\dim X \le 3$, then $\hat{\pi}_1^{\mathrm{loc}}(\mathrm{Reg}\,\tilde{X})$ is finite by [56], Theorem 3.6.

Proposition 4.4 (c.f., [17], Exposé IX, Remark 5.8) *For $D \in \mathrm{Div}_{\boldsymbol{Q}}X$, Let $\pi : \tilde{X} \to X$ be the index one cover with respect to D. Then there exists the following exact sequence in the category of profinite groups :*

$$\{1\} \longrightarrow \hat{\pi}_1^{\mathrm{loc}}(\mathrm{Reg}\,\tilde{X}) \longrightarrow \hat{\pi}_{1,X,p}^{\mathrm{loc}}[D] \longrightarrow \mathrm{Gal}\,(\tilde{X}/X) \simeq \boldsymbol{Z}/r\boldsymbol{Z} \longrightarrow \{1\}, \tag{4.11}$$

where $r := \mathrm{ind}_p D$.

Proof. Recall that we have a canonical embedding $\Phi_{i_{\tilde{X}}} : \mathrm{Cart}^{\dagger}(X;D)^p \to \mathrm{Cart}^{\dagger}(\tilde{X};0)^p$ as in (4.10). Since we have the exact sequence:

$$\{1\} \longrightarrow \mathrm{projlim}_{(Y,f,i_Y) \in \mathrm{ObCart}^{\dagger}G(X;D)^p} \mathrm{Gal}\,(Y/\tilde{X}) \longrightarrow \hat{\pi}_{1,X,p}^{\mathrm{loc}}[D] \longrightarrow \mathrm{Gal}\,(\tilde{X}/X) \longrightarrow \{1\},$$

we only have to show that $\mathrm{Cart}^{\dagger}G(X;D)^p$ is cofinal in $\mathrm{Cart}^{\dagger}G(\tilde{X};0)^p$ via the functor $\Phi_{i_{\tilde{X}}}$. Choose any object $(Y, f, i_Y) \in \mathrm{Ob\ Cart}^{\dagger}G(\tilde{X};0)^p$. Then we see that $(Y, \pi \circ f, i_Y) \in \mathrm{Ob\ Int}^{\dagger}(X;D)^p$ since $\pi^{-1}(\mathrm{Reg}\,X \setminus \mathrm{Supp}\,B) \subset \mathrm{Reg}\,\tilde{X}$ and $\pi \circ f$ is étale over $\mathrm{Reg}\,X \setminus \mathrm{Supp}\,B$. By Proposition 4.1, There exists an object $(Z, g, i_Z) \in \mathrm{Ob\ Int}^{\dagger}G(X;D)^p$ dominating the object $(Y, \pi \circ f, i_Y)$. Since $\mathrm{Cart}^{\dagger}G(X;D)^p$ is cofinal in $\mathrm{Int}^{\dagger}G(X;D)^p$ (see the proof of Theorem 4.1), (Z, g, i_Z) is dominated by some object in $\mathrm{Cart}^{\dagger}G(X;D)^p$. Thus we get the assertion. ■

Corollary 4.1 *A pointed universal Cartier cover $(X^{\dagger}, \pi^{\dagger}, i_{X^{\dagger}}) \in \mathrm{Ob\ Cart}^{\dagger}G(X;D)^p$ is an initial object, or equivalently, a limit of $\mathrm{Cart}^{\dagger}(X;D)^p$ and vice versa. In particular, a universal Cartier cover with respect to D is unique up to isomorphisms over X if it exists.*

Proof. Let $(\tilde{X}, \pi, i_{\tilde{X}}) \in \mathrm{Ob\ Cart}^{\dagger}G(X;D)^p$ be a final object of $\mathrm{Cart}^{\dagger}(X;D)^p$. As we noted firstly in this section, $(X^{\dagger}, \varpi_{X^{\dagger}}(i_{X^{\dagger}}, i_{\tilde{X}}), i_{X^{\dagger}}) \in \mathrm{Ob\ Cart}^{\dagger}(\tilde{X};0)^p$ is an initial object of $\mathrm{Cart}^{\dagger}(\tilde{X};0)^p$, hence $(X^{\dagger}, \pi^{\dagger}, i_{X^{\dagger}}) \in \mathrm{Ob\ Cart}^{\dagger}G(X;D)^p$ is also an initial object of $\mathrm{Cart}^{\dagger}(X;D)^p$. On the contrary, assume that there exists an initial object $(X^{\dagger\prime}, \pi^{\dagger\prime}, i_{X^{\dagger\prime}})$ of $\mathrm{Cart}^{\dagger}(X;D)^p$. By Proposition4.2, we see that $(X^{\dagger\prime}, \pi^{\dagger\prime}, i_{X^{\dagger\prime}}) \in \mathrm{Ob\ Cart}^{\dagger}G(X;D)^p$. Since $\hat{\pi}_{1,X,p}^{\mathrm{loc}}[D]$ is finite, $\hat{\pi}_1^{\mathrm{loc}}(\mathrm{Reg}\,\tilde{X})$ is also finite by Proposition 4.4. Therefore there exists a pointed universal Cartier cover $(X^{\dagger}, \pi^{\dagger}, i_{X^{\dagger}}) \in \mathrm{Ob\ Cart}^{\dagger}G(X;D)^p$ by Proposition 4.3 which is also an initial object of $\mathrm{Cart}^{\dagger}(X;D)^p$ and hence isomorphic to $(X^{\dagger\prime}, \pi^{\dagger\prime}, i_{X^{\dagger\prime}})$. Thus we conclude that $\hat{\pi}_1^{\mathrm{loc}}(\mathrm{Reg}\,X^{\dagger\prime}) = \{1\}$. ■

The following Lemma is an algebraic generalization of Brieskorn's fundamental lemma.

Lemma 4.6 (c.f., [5], Lemma 2.6) *Let $f : (X, p) \to (Y, q)$ be a finite morphism between germs of normal complex analytic spaces (X, p) and (Y, q). Then for any $D \in \mathrm{Div}_{\boldsymbol{Q}}Y$, there exists a canonical homomorphism $f_* : \hat{\pi}_{1,X,p}^{\mathrm{loc}}[f^*D] \to \hat{\pi}_{1,Y,q}^{\mathrm{loc}}[D]$ which satisfies $|\hat{\pi}_{1,Y,q}^{\mathrm{loc}}[D] : \mathrm{Im}\,f_*| \le \deg f$. In particular, if $\hat{\pi}_{1,X,p}^{\mathrm{loc}}[f^*D]$ is finite, so is $\hat{\pi}_{1,Y,q}^{\mathrm{loc}}[D]$.*

Proof. For a given pointing $i_Y : \mathcal{M}_Y \to \overline{\mathcal{M}}_Y$, we choose a pointing $i_X : \mathcal{M}_X \to \overline{\mathcal{M}}_X = \overline{\mathcal{M}}_Y$ such that $i_X \circ f^* = i_Y$. Take any $(Z, \alpha, i_Z) \in \text{Ob Cart}^\dagger G(Y; D)^p$. Let $\nu : W \to Y$ be the normalization of Y in $i_X(\mathcal{M}_X) \vee i_Z(\mathcal{M}_Z)$. We note that there exist morphisms $\beta : W \to X$ and $\gamma : W \to Z$ such that $\alpha \circ \gamma = f \circ \beta = \nu$. Since $\beta^* f^* D = \gamma^* \alpha^* D \in \text{Div } W$ and $\beta : W \to X$ is Galois (see, for example, [39], Theorem 3.6.3), we have $(W, \beta, i_W) \in \text{Ob Cart}^m G(X; f^*D)^p$ for a suitable pointing i_W. Let $(\tilde{X}, \pi_X, i_{\tilde{X}})$ (resp. $(\tilde{Y}, \pi_Y, i_{\tilde{Y}})$) be a final object of $\text{Cart}^m G(X; f^*D)^p$ (resp. $\text{Cart}^m G(Y; D)^p$). Since $(\tilde{X}, f \circ \pi_X, i_{\tilde{X}}) \in \text{Ob Cart}^m(Y; D)^p$, there exists a morphism $\omega_{\tilde{X}}(i_{\tilde{X}}, i_{\tilde{Y}}) : (\tilde{X}, f \circ \pi_X, i_{\tilde{X}}) \to (\tilde{Y}, \pi_Y, i_{\tilde{Y}})$ in $\text{Cart}^m(Y; D)^p$. Let $\nu^\natural : Y^\natural \to Y$ be the normalization of Y in $i_Z(\mathcal{M}_Z) \cap i_{\tilde{X}}(\mathcal{M}_{\tilde{X}})$. Since the induced finite morphism $\delta : Z \to Y^\natural$ is Galois, $i_Z(\mathcal{M}_Z)$ and $i_{\tilde{X}}(\mathcal{M}_{\tilde{X}})$ are linearly disjoint over $i_{Y^\natural}(\mathcal{M}_{Y^\natural})$, that is, $i_Z(\mathcal{M}_Z) \otimes_{i_{Y^\natural}(\mathcal{M}_{Y^\natural})} i_{\tilde{X}}(\mathcal{M}_{\tilde{X}}) \simeq i_W(\mathcal{M}_W)$ (see, for example, [39], Exercise 4.2.3). Let $\eta \in W$ be the generic point of a prime divisor on W and $\xi \in Z$ (resp. $\tilde{\xi} \in \tilde{X}$, resp. $\xi^\natural \in Y^\natural$) be its image on Z (resp. $\tilde{X}$, resp. $Y^\natural$). Consider the canonical morphism $\kappa : i_Z(\mathcal{O}_{Z,\xi}) \otimes_{i_{Y^\natural}(\mathcal{O}_{Y^\natural,\xi^\natural})} i_{\tilde{X}}(\mathcal{O}_{\tilde{X},\tilde{\xi}}) \to i_W(\mathcal{O}_{W,\eta})$ and put $S := i_{Y^\natural}(\mathcal{O}_{Y^\natural}) \setminus \{0\}$. We note that since $i_Z(\mathcal{O}_{Z,\xi})$ is flat over $i_{Y^\natural}(\mathcal{O}_{Y^\natural,\xi^\natural})$ by our construction, $i_Z(\mathcal{O}_{Z,\xi}) \otimes_{i_{Y^\natural}(\mathcal{O}_{Y^\natural,\xi^\natural})} i_{\tilde{X}}(\mathcal{O}_{\tilde{X},\tilde{\xi}})$ is a free $i_{\tilde{X}}(\mathcal{O}_{\tilde{X},\tilde{\xi}})$-module, in particular, a torsion free $i_{Y^\natural}(\mathcal{O}_{Y^\natural,\xi^\natural})$-module. Since $S^{-1}\kappa : S^{-1}(i_Z(\mathcal{O}_{Z,\xi}) \otimes_{i_{Y^\natural}(\mathcal{O}_{Y^\natural,\xi^\natural})} i_{\tilde{X}}(\mathcal{O}_{\tilde{X},\tilde{\xi}})) \simeq S^{-1}(i_Z(\mathcal{O}_{Z,\xi})) \otimes_{i_{Y^\natural}(\mathcal{M}_{Y^\natural})} S^{-1}(i_{\tilde{X}}(\mathcal{O}_{\tilde{X},\tilde{\xi}})) \to i_W(\mathcal{M}_W)$ is injective by the previous argument, so is κ, hence, in particular, Im κ is a normal subring of $i_W(\mathcal{O}_{W,\eta})$ whose total quotient ring coincides $i_W(\mathcal{M}_W)$ which implies that Im $\kappa = i_W(\mathcal{M}_W)$. Thus we conclude that κ is an isomorphism and that $i_W(\mathcal{O}_{W,\eta})$ is flat over $i_{\tilde{X}}(\mathcal{O}_{\tilde{X},\tilde{\xi}})$, which implies that $(W, \beta, i_W) \in \text{Ob Cart}^\dagger G(X; f^*D)^p$. The canonical inclusion Gal $(W/X) \simeq$ Gal $(i_Z(\mathcal{M}_Z)/i_Z(\mathcal{M}_Z) \cap i_X(\mathcal{M}_X)) \to$ Gal (Z/Y) induces a homomorphism $f_* : \hat{\pi}^{\text{loc}}_{1,X,p}[f^*D] \to \hat{\pi}^{\text{loc}}_{1,Y,q}[D]$. Since we have [Gal (Z/Y) : Gal (W/X)] $= [i_Z(\mathcal{M}_Z) \cap i_X(\mathcal{M}_X) : i_Y(\mathcal{M}_Y)] \leq \deg f$, we get the assertion (see also [4], §7.1, Corollaire 3). ■

4.4 Lefshetz type theorem for D-local algebraic fundamental groups

The aim of this section is to state and prove the Lefshetz type theorem for D-local algebraic fundamental groups.

Lemma 4.7 (c.f., [21], Corollary 10.8) *Take any $D \in \text{Div}_{\boldsymbol{Q}} X \cap \text{Weil } X$ and let $\pi : \tilde{X} \to X$ be the index one cover with respect to D. Assume that there exists a normal prime divisor Γ passing through $p \in X$ such that the following three conditions hold.*

(1) $\tilde{\Gamma} := \pi^{-1}\Gamma$ *is normal,*

(2) Γ *does not contained in* Supp D,

(3) *there exists an analytic closed subset $\Sigma \subset X$ with $\text{codim}_X \Sigma \geq 2$ and $\text{codim}_\Gamma(\Sigma \cap \Gamma) \geq 2$ such that $D|_U$ is Cartier and $D_\Gamma := j_\Gamma^* D \in \text{Div } \Gamma$, where $U := X \setminus \Sigma$ and $j_\Gamma : \Gamma_0 := \Gamma \setminus \Sigma \to \Gamma$ is the natural embedding.*

Then $D \in \text{Div } X$.

Proof. Consider the exact sequence:

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}(\pi^*D - \tilde{\Gamma}) \longrightarrow \mathcal{O}_{\tilde{X}}(\pi^*D) \longrightarrow \mathcal{O}_{\tilde{\Gamma}}(\pi^*D) \longrightarrow 0.$$

Note that $\mathcal{O}_{\tilde{\Gamma}}(\pi^*D)$ and $\pi^*\mathcal{O}_\Gamma(D_\Gamma)$ are both invertible and coincide on $\tilde{\Gamma} \setminus \pi^{-1}(\Sigma)$, hence we have $\mathcal{O}_{\tilde{\Gamma}}(\pi^*D) = \pi^*\mathcal{O}_\Gamma(D_\Gamma)$ by the normality of $\tilde{\Gamma}$. Since $\pi_*^G := \Gamma_X^G \circ \pi_*$ is an exact functor, where $G :=$ Gal $(\tilde{X}/X)$, the above exact sequence induces a surjective map

$$\alpha : \mathcal{O}_X(D) = \pi_*^G \mathcal{O}_{\tilde{X}}(\pi^*D) \to \pi_*^G \mathcal{O}_{\tilde{\Gamma}}(\pi^*D) = \mathcal{O}_\Gamma(D_\Gamma) \otimes (\pi|_{\tilde{\Gamma}})_*^G \mathcal{O}_{\tilde{\Gamma}} = \mathcal{O}_\Gamma(D_\Gamma).$$

Take $\varphi_\Gamma \in \mathcal{M}_\Gamma$ such that div $\varphi_\Gamma = -D|_\Gamma$. By the above argument, we have $\varphi \in \mathcal{M}_X$ such that $\alpha(\varphi) = \varphi_\Gamma$. Since $(D + \text{div } \varphi)|_\Gamma = D_\Gamma + \text{div } \varphi_\Gamma = 0$ and $D + \text{div } \varphi$ is $\mathbf{Q}$-Cartier, we deduce that $\Gamma \cap \text{Supp } (D + \text{div } \varphi) = \emptyset$, that is, $D + \text{div } \varphi = 0$, hence $D \in \text{Div } X$. ∎

Lemma 4.8 (c.f., [48], Lemma 1.12, [53]) *Let X be a normal complex analytic space embedded in some domain in $\boldsymbol{C}^n$. Consider the hypersurfaces H_τ on $\boldsymbol{C}^n$ parametrized by $\tau \in \boldsymbol{P}^n$ which is defined by a linear equation $\tau_0 + \sum_{i=1}^n \tau_i z_i = 0$, where $z_1, \ldots, z_n$ is a complete coordinate system of $\boldsymbol{C}^n$. Then there exist a non-empty open subset $U \subset \boldsymbol{P}^n$ and a countable union Z of closed analytic subsets of U such that for any $\tau \in U \setminus Z$, $H_\tau \cap X$ is a normal hypersurface on X.*

Proof. Take an analytic open subset $U \subset \boldsymbol{P}^n$ such that for any $\tau \in U$, $\bar{H}_\tau := H_\tau \cap X$ is non empty and H_τ does not contain X. Since the base point free linear system $\{\bar{H}_\tau\}_{\tau \in U}$ on X induces a base point free linear system on Reg X, we have Sing $\bar{H}_\tau \subset$ Sing X and codim $_{\bar{H}_\tau}$Sing $\bar{H}_\tau \geq 2$ for any $\tau \in U \setminus Z$, where Z is a countable union of closed analytic subsets of U by Bertini's theorem. Moreover, we may assume that for $k = 1, \ldots, d-2$, $\bar{H}_\tau$ does not contain any maximal dimensional components of (Sing $X) \cap S_{k+1}(\mathcal{O}_X)$, where $d := \dim X$ and $S_k(\mathcal{O}_X)$ is a closed analytic set consisting of points at which the profoundity of $\mathcal{O}_X$ does not exceed k. Since we have (Sing $\bar{H}_\tau) \cap S_k(\mathcal{O}_{\bar{H}_\tau}) \subset$ (Sing $X) \cap S_{k+1}(\mathcal{O}_X)$, we see that dim(Sing $\bar{H}_\tau) \cap S_k(\mathcal{O}_{\bar{H}_\tau}) \leq k - 2$ for any k, hence $\bar{H}_\tau$ is normal for any $\tau \in U \setminus Z$ by [12], 2.27, Theorem. ∎

Remark 4.13 Let X be a normal Stein space. For $\mathbf{Q}$-divisor Δ on X, let $\text{Mult}_X(\Delta) \subset \boldsymbol{Q}$ denote the subset consisting of all the multiplicities of Δ at prime divisors on X. We note that for general normal hyperplanes $\bar{H}_\tau$, we have $\text{Mult}_{\bar{H}_\tau}(\text{Diff}_{\bar{H}_\tau}(\Delta)) \subset \text{Mult}_X(\Delta)$.

To state the Lefshetz type theorem, we need to fix some sort of general conditions. We shall consider the following conditions assuming $\dim X \geq 2$.

$(M1)$ Δ is a standard $\mathbf{Q}$-boundary.

$(M2)$ (X, Δ) is divisorially log terminal.

$(M2)^*$ $(M2)^\alpha$ (X, Δ) is divisorially log terminal and $\{\Delta\} = 0$ or $(M2)^\beta$ (X, Δ) is purely log terminal.

$(M3)$ There exists an irreducible component Γ of $\lfloor \Delta \rfloor$ passing through $p \in X$ such that $K_X + \Gamma$ is $\mathbf{Q}$-Cartier.

Remark 4.14 $(M2)^*$ is a slightly stronger condition than $(M2)$.

Proposition 4.5 *Assume the conditions $(M1)$, $(M2)$ and $(M3)$. Then*

$$\text{ind}_p(K_X + \Delta) = \text{ind}_p(K_\Gamma + \text{Diff}_\Gamma(\Delta - \Gamma)).$$

Proof. Put $r_\Gamma := \text{ind}_p(K_\Gamma + \text{Diff}_\Gamma(\Delta - \Gamma))$. Firstly, we note that (X, Γ) is purely log terminal and that $\Gamma \cap \text{Supp } (\Delta - \Gamma)$ is purely one codimensional in Γ since $\Delta - \Gamma$ is $\mathbf{Q}$-Cartier by the conditions $(M2)$ and $(M3)$. We show that $r_\Gamma(K_X + \Delta)$ is an integral divisor on X and is Cartier at general points of any prime divisors on Γ. By taking general hyperplane sections, we only have to check that if $\dim X = 2$, then $r_\Gamma(K_X + \Delta)$ is Cartier. This can be checked by the classification of log canonical singularities with a standard $\mathbf{Q}$-boundary due to S. Nakamura (see, §3.1 or [24], Theorem 3.1), but we can also argue in this way as follows. We note that $p \in X$ is a cyclic quotient singular point with the order, say, n by the condition $(M2)$. If $\Gamma \cap (\lfloor \Delta \rfloor - \Gamma) \neq \emptyset$, then X is smooth, hence this case is trivial. Assume that $\Gamma \cap (\lfloor \Delta \rfloor - \Gamma) = \emptyset$. Since we can see that (X, Δ) is purely log terminal in this case from the condition $(M2)$, we can write $\Delta = \Gamma + d\Xi$ for a prime divisor Ξ such that $(\Gamma, \Xi)_p = 1$ and for some $d = (l-1)/l$, where l is a natural number and we have $\mathrm{mult}_p\mathrm{Diff}_\Gamma(\Delta-\Gamma) = (nl-1)/(nl)$, as in [59], Lemma 2.25, which implies $r_\Gamma(K_X + \Delta) \in \mathrm{Div}\ X$. Going back to the general case, we see that $D := K_X + \Delta \in \mathrm{Div}_{\boldsymbol{Q}} X$ and Γ satisfies the conditions in Lemma 4.7 using [58], Corollary 2.2 and Lemma 3.6, hence we conclude that $r_\Gamma(K_X + \Delta) \in \mathrm{Div}\ X$. ■

Remark 4.15 We note that $\mathrm{Diff}_\Gamma(\Delta - \Gamma)$ is also a standard **Q**-boundary, since $\mathrm{Diff}_{\tilde{\Gamma}}((\Delta - \Gamma)_{\tilde{X}})$ is a **Q**-boundary (see [58], (2.4.1)).

Example 4.2 Let X be the germ of $\boldsymbol{C}^2$ at the origin and put $\Gamma := \mathrm{div}\ z$ and $\Delta := \mathrm{div}\ z + (1/n)\mathrm{div}\ w + (1/n)\mathrm{div}\ (z+w)$, where (z, w) is a system of coordinates and $n \in \boldsymbol{N}$. Then we have $\mathrm{ind}_0(K_X + \Delta) = n$ while $\mathrm{ind}_0(K_\Gamma + \mathrm{Diff}_\Gamma(\Delta - \Gamma)) = n/2$ (resp. n) if n is even (resp. if n is odd), which explains why we need the assumptions in Proposition 4.5.

A directed set $(\Lambda, \geq)$ naturally forms a cofilterd projective system assuming that for λ, $\mu \in \Lambda$, Card $\mathrm{Hom}_\Lambda(\lambda, \mu) = 1$ if and only if $\lambda \geq \mu$. We call this projective system Λ a *cofilterd index projective system.* Let us recall the following basic result (see for example, [51]).

Lemma 4.9 *Let $\phi : \Lambda' \to \Lambda$ be a covariant functor between cofilterd index projective systems and $G : \Lambda \to$ (Top. groups), $H : \Lambda' \to$ (Top. groups) be two covariant functors to the category of topological groups. Assume that the following three conditions (a), (b) and (c) hold.*

(a) *$G_\lambda := G(\lambda)$ and $H_{\lambda'} := H(\lambda')$ are compact for any $\lambda \in \mathrm{Ob}\ \Lambda$ and $\lambda' \in \mathrm{Ob}\ \Lambda'$.*

(b) *$G(\lambda \to \mu)$ and $H(\lambda' \to \mu')$ are all surjective.*

(c) *There exists a natural transformation $\Psi : G \circ \phi \to H$ such that $\Psi(\lambda') : G_{\phi(\lambda')} \to H_{\lambda'}$ are surjective for any $\lambda' \in \mathrm{Ob}\ \Lambda'$.*

Then there exists a canonical surjective morphism in (Top. groups) :

$$\psi : \mathrm{projlim}\ _{\lambda \in \mathrm{Ob}\Lambda} G_\lambda \to \mathrm{projlim}\ _{\lambda' \in \mathrm{Ob}\Lambda'} H_{\lambda'}.$$

Let Δ be a **Q**-divisor on X such that $K_X + \Delta$ is **Q**-Cartier. In what follows, we put

$$\mathcal{I}_1^{\dagger(m)}(G)(X, \Delta)^{(p)} := \mathcal{I}_1^{\dagger(m)}(G)(X; K_X + \Delta)^{(p)}.$$

Theorem 4.2 *Assume the conditions $(M1)$, $(M2)^*$ and $(M3)$. Then there exists a canonical continuous surjective homomorphism :*

$$\psi_\Gamma : \hat{\pi}^{\mathrm{loc}}_{1,\Gamma,p}[\mathrm{Diff}_\Gamma(\Delta - \Gamma)] \to \hat{\pi}^{\mathrm{loc}}_{1,X,p}[\Delta].$$

Proof. For any $(Y, f) \in \mathrm{Ob}\ \mathrm{Cart}^m(X, \Delta)$, Γ and $\Gamma_Y := f^{-1}\Gamma$ are normal by [58], Lemma 3.6 and Corollary 2.2, hence they are irreducible since $f^{-1}(p)$ consists of just one point. A canonical inclusion $\mathcal{O}_\Gamma \to \mathrm{injlim}_{(Y,f,i_Y) \in \mathrm{ObCart}^m(X,\Delta)^p} \mathcal{O}_{\Gamma_Y}$ extends to an inclusion $i_\Gamma : \mathcal{O}_\Gamma \to \overline{\mathcal{M}}_\Gamma$ and we fix this i_Γ. Then we have a canonical functor $\phi_\Gamma^{(p)} : \mathrm{Cart}^m(X, \Delta)^{(p)} \to \mathrm{Cart}^m(\Gamma, \mathrm{Diff}_\Gamma(\Delta - \Gamma))^{(p)}$ such that $\phi((Y, f)) = (\Gamma_Y, f_\Gamma)$, where $f_\Gamma := f|_{\Gamma_Y}$. Take any $(Y, f) \in \mathrm{Ob}\ \mathrm{Cart}^\dagger(X, \Delta)$. We shall show that $(\Gamma_Y, f_\Gamma) \in \mathrm{Ob}\ \mathrm{Cart}^\dagger(\Gamma, \mathrm{Diff}_\Gamma(\Delta - \Gamma))$. Let $\tilde{X} \to X$ be the log canonical cover with respect to $K_X + \Delta$. Note that $\Gamma_{\tilde{X}}$ is also normal and $\pi_\Gamma := \pi|\Gamma_Y : \tilde{\Gamma} := \Gamma_{\tilde{X}} \to \Gamma$ is the log canonical with respect to $K_\Gamma + \mathrm{Diff}_\Gamma(\Delta - \Gamma)$ by Proposition 4.5. Take any pointings $i_{\tilde{\Gamma}}$ and i_{Γ_Y} for $(\tilde{\Gamma}, \pi_\Gamma)$ and $(\Gamma_Y, f_\Gamma) \in \mathrm{Ob}\ \mathrm{Cart}^m(\Gamma, \mathrm{Diff}_\Gamma(\Delta - \Gamma))$. We note also that $\varpi_{\Gamma_Y}(i_{\Gamma_Y}, i_{\tilde{\Gamma}}) = \varpi_Y(i_Y, i_{\tilde{X}})|_{\Gamma_Y}$ for some pointings i_Y and $i_{\tilde{X}}$ of (Y, f) and

$(\tilde{X},\pi) \in \text{Ob Cart}^\dagger(X,\Delta)$. $(\Gamma_Y, f_\Gamma) \in \text{Ob Cart}^\dagger(\Gamma, \text{Diff}_\Gamma(\Delta-\Gamma))$. By the covering theorem in [61], $(\tilde{X},\tilde{\Delta})$ is divisorially log terminal of index one, which implies that $\tilde{X}$ is smooth in codimension two, hence, in particular, we have $\text{codim}_{\tilde{\Gamma}}(\text{Sing }\tilde{X}\cap\tilde{\Gamma}) \geq 2$. Since $\varpi_Y(i_Y, i_{\tilde{X}})$ is étale over Reg $\tilde{X}$, we conclude that $\varpi_{\Gamma_Y}(i_{\Gamma_Y}, i_{\tilde{\Gamma}})$ is étale in codimension one and $(\Gamma_Y, f_\Gamma) \in \text{Ob Cart}^\dagger(\Gamma, \text{Diff}_\Gamma(\Delta-\Gamma))$. In other words, $\phi^{(p)}$ induces a functor $\phi_\Gamma^{(p)} : \text{Cart}^\dagger(G)(X,\Delta)^{(p)} \to \text{Cart}^\dagger(G)(\Gamma, \text{Diff}_\Gamma(\Delta-\Gamma))^{(p)}$, where we used the same notation $\phi^{(p)}$. Consider the two functors

$$G_X : \text{Cart}^\dagger G(X,\Delta)^p \to (\text{Top. groups}) \text{ and } G_\Gamma : \text{Cart}^\dagger G(\Gamma, \text{Diff}_\Gamma(\Delta-\Gamma))^p \to (\text{Top. groups})$$

such that $G_X((Y,f,i_Y)) = \text{Gal }(Y/X)$ and $G_\Gamma((\Gamma',g,i_{\Gamma'})) = \text{Gal }(\Gamma'/\Gamma)$. Since f is étale over a general points of Γ for any $(Y,f) \in \text{Ob Cart}^\dagger G(X,\Delta)$, there exists a natural equivalence $\Psi_\Gamma : G_\Gamma \circ \phi^p \to G_X$, which induces the desired surjection $\psi_\Gamma : \hat{\pi}^{\text{loc}}_{1,\Gamma,p}[\text{Diff}_\Gamma(\Delta-\Gamma)] \to \hat{\pi}^{\text{loc}}_{1,X,p}[\Delta]$ by Lemma 4.9.
■

Remark 4.16 Assume the conditions $(M1)$, $(M2)^\alpha$ and $(M3)$. Then, combined with Remark4.5, Theorem 4.2 says that there exists a surjection $\psi_\Gamma : \hat{\pi}^{\text{loc}}_{1,\Gamma,p}[\text{Diff}_\Gamma(\Delta-\Gamma)] \to \hat{\pi}^{\text{loc}}_1(\text{Reg }X)$. For example, if $\dim X = 4$, $\hat{\pi}^{\text{loc}}_{1,\Gamma,p}[\text{Diff}_\Gamma(\Delta-\Gamma)]$ is finite under the assumptions as explained in Remark 4.12, hence so is $\hat{\pi}^{\text{loc}}_1(\text{Reg }X)$.

Letting notation and assumptions be as in Theorem 4.2, we obtain the following corollary.

Corollary 4.2 *Assume that the universal Cartier cover of* Γ *with respect to* $K_\Gamma+\text{Diff}_\Gamma(\Delta-\Gamma)$ *exists. Then there exists the universal Cartier cover of* X *with respect to* $K_X+\Delta$. *Moreover, there exists the following exact sequence :*

$$\{1\} \longrightarrow \hat{\pi}^{\text{loc}}_1(\text{Reg }\Gamma_{X^\dagger}) \longrightarrow \hat{\pi}^{\text{loc}}_{1,\Gamma,p}[\text{Diff}_\Gamma(\Delta-\Gamma)] \longrightarrow \hat{\pi}^{\text{loc}}_{1,X,p}[\Delta] \longrightarrow \{1\}, \tag{4.12}$$

where $\pi^\dagger : X^\dagger \to X$ *is the universal Cartier cover of* X *with respect to* $K_X+\Delta$.

Proof. The first assertion follows from Proposition 4.3 and Proposition 4.4. As for the last statement, let $\pi^\dagger_\Gamma : \Gamma^\dagger \to \Gamma$ be the universal Cartier cover of Γ with respect to $K_\Gamma + \text{Diff}_\Gamma(\Delta-\Gamma)$. Then the induced morphism $\tau^\dagger_\Gamma : \Gamma^\dagger \to \Gamma_{X^\dagger}$ is the universal Cartier cover of $\Gamma_{X^\dagger}$ since $\tau^\dagger_\Gamma$ is étale in codimension one, which implies that $\text{Gal }(\Gamma^\dagger/\Gamma_{X^\dagger}) \simeq \hat{\pi}^{\text{loc}}_1(\text{Reg }\Gamma_{X^\dagger})$, hence we obtain the desired exact sequence. ■

Remark 4.17 Let notation be as above. Assume that (X,p) is a three dimensional $\mathbf{Q}$-Gorenstein singularity and that (X,Γ) is purely log terminal with Sing X $\subset \Gamma$. Then we see that $(\tilde{X},\tilde{p})$ has only terminal singularities and that $(X^\dagger,p^\dagger)$ is an isolated compound Du Val singularity (see, [31], Theorem 5.2). We also note that $\Gamma^\dagger$ is smooth and that $\Gamma_{X^\dagger} \in |-K_{X^\dagger}|$ is a Du Val element. Moreover, the above exact sequence (4.12) reduces to the following exact sequence:

$$\{1\} \longrightarrow \pi^{\text{loc}}_1(\text{Reg }\Gamma_{X^\dagger}) \longrightarrow \pi^{\text{loc}}_{1,\Gamma,p}[\text{Diff}_\Gamma(0)] \longrightarrow \pi^{\text{loc}}_1(\text{Reg }X) \longrightarrow \{1\}, \tag{4.13}$$

which enables us to calculate the local fundamental group of the germ (X,p), since $\pi^{\text{loc}}_1(\text{Reg }\Gamma_{X^\dagger})$ and $\pi^{\text{loc}}_{1,\Gamma,p}[\text{Diff}_\Gamma(0)]$ have faithful representations to the special unitary group $SU(2,\boldsymbol{C})$ and the unitary group $U(2,\boldsymbol{C})$ respectively, both of which are classified. It is important to determine the pair $(X^\dagger, \pi^{\text{loc}}_1(\text{Reg }X))$ which will lead us to the classification 3-dimensional purely log terminal singularities.

5 Types of degenerations of algebraic surfaces with Kodaira dimension zero

Definition 5.1 (Minimal Semistable Degeneration) A minimal model $X \to \mathcal{D}$ obtained from a projective semistable degeneration of surfaces with non-negative Kodaira dimension $g : Y \to \mathcal{D}$ by applying the Minimal Model Program is called a projective minimal semistable degeneration of surfaces.

A projective log minimal degeneration of Kodaira dimension zero is related to a minimal semistable degeneration as in the following way.

Lemma 5.1 *Let $f : X \to \mathcal{D}$ be a projective log minimal degeneration of surfaces with non-negative Kodaira dimension. Then there exists a finite covering $\tau : \mathcal{D}^\sigma \to \mathcal{D}$, a projective minimal semistable degeneration $f^\sigma : X^\sigma \to \mathcal{D}^\sigma$ which is bimeromorphically equivalent to $X \times_{\mathcal{D}} \mathcal{D}^\sigma$ over $\mathcal{D}^\sigma$ and a generically finite morphism $\pi : X^\sigma \to X$ such that $f \circ \pi = \tau \circ f^\sigma$ and $K_{X^\sigma} + \Theta^\sigma = \pi^*(K_X + \Theta)$, where $\Theta^\sigma := f^{\sigma *}(0)$.*

$$\begin{array}{ccc} X^\sigma & \xrightarrow{\pi} & X \\ {\scriptstyle f^\sigma}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{D}^\sigma & \xrightarrow{\tau} & \mathcal{D} \end{array}$$

Proof. We use the idea explained in [58], §2. Let $\mu : Y \to X$ be a projective resolution of X such that the support of the singular fiber of the induced morphism $g : Y \to \mathcal{D}$ has only simple normal crossings as its singularities. By the semistable reduction theorem ([22]), there exists a finite covering $\tau : \mathcal{D}^\sigma \to \mathcal{D}$ and a projective resolution $Y^\sigma \to Y \times_{\mathcal{D}} \mathcal{D}^\sigma$ such that the induced degeneration $g^\sigma : Y^\sigma \to \mathcal{D}^\sigma$ is semistable. Let $\pi' : X' \to X$ be the normalization of X in the meromorphic function field of Y^σ and let $\varphi : X^\sigma \to X'$ be a minimal model over X' obtained by applying the Minimal Model Program to the induced morphism $\tilde{Y} \to X'$. Then, by the ramification formula, we have $K_{X'} + \Theta' = \pi'^*(K_X + \Theta)$, where $\Theta' := \pi'^{-1}\Theta$. Since (X', Θ') is log canonical, we infer that $K_{X^\sigma} + \Theta^\sigma = \varphi^*(K_{X'} + \Theta')$. The induced morphism $f^\sigma : X^\sigma \to \mathcal{D}^\sigma$ as the Stein factorization of the morphism $X^\sigma \to \mathcal{D}$ gives the desired minimal semistable degeneration. ∎

Definition 5.2 (cf. Definition 3.4) A log minimal degeneration $f : X \to \mathcal{D}$ of surfaces of Kodaira dimension zero is said to be *of type* I (resp. *of type* II, resp. *of type* III), if there exists an irreducible component Θ_i of Θ such that $(\Theta_i, \mathrm{Diff}_{\Theta_i}(\Theta - \Theta_i))$ is of type I (resp. of type II, resp. of type III).

For a projective log minimal degenerations of surfaces of Kodaira dimension zero $f : X \to \mathcal{D}$, take a projective minimal semistable degeneration $f^\sigma : X^\sigma \to \mathcal{D}^\sigma$ obtained from f as in Lemma 5.1. Then the following holds.

Proposition 5.1 *f is of type* I (*resp. of type* II, *resp. of type* III) *if and only if f^σ is of type* I (*resp. of type* II, *resp. of type* III). *Moreover two projective log minimal degenerations $f_j : X_j \to \mathcal{D}$ ($j = 1, 2$) which are bimeromorphically equivalent to each other over $\mathcal{D}$ have exactly the same types as each other, i.e., types* I, II *and* III *are bimeromorphic notion which are independent from the choice of log minimal models.*

Proof. Let Θ_i be an irreducible component of Θ and Θ_i^σ be an irreducible component of $\pi^{-1}(\Theta_i)$ dominating Θ_i. Since we have $K_{X^\sigma} + \Theta^\sigma = \pi^*(K_X + \Theta)$, we have $K_{\Theta_i^\sigma} + \Delta_i^\sigma = \pi^*(K_{\Theta_i} + \Delta_i)$, where $\Delta_i^\sigma := \mathrm{Diff}_{\Theta_i^\sigma}(\Theta^\sigma - \Theta_i^\sigma)$ and $\Delta_i := \mathrm{Diff}_{\Theta_i}(\Theta - \Theta_i)$, hence $\lfloor \Delta_i^\sigma \rfloor = \pi^{-1}(\lfloor \Delta_i \rfloor)$ and

$$\lfloor \mathrm{Diff}_{\lfloor \Delta_i^\sigma \rfloor^\nu}(\Delta_i^\sigma - \lfloor \Delta_i^\sigma \rfloor) \rfloor = \pi^{\nu -1} \lfloor \mathrm{Diff}_{\lfloor \Delta_i \rfloor^\nu}(\Delta_i - \lfloor \Delta_i \rfloor) \rfloor,$$

where $\pi^\nu : \lfloor\Delta_i^\sigma\rfloor^\nu \to \lfloor\Delta_i\rfloor^\nu$ is the induced morphism by π between the normalization of $\lfloor\Delta_i^\sigma\rfloor$ and $\lfloor\Delta_i\rfloor$. Let $\pi^\dagger : X^\dagger \to X^\sigma$ be the index one cover of X^σ with respect to K_{X^σ} and $f^\dagger : X^\dagger \to \mathcal{D}^\dagger$ be the degeneration obtained by Stein factorization. Then putting $\Theta^\dagger := \pi^{\dagger-1}(\Theta^\sigma)$, $f^\dagger$ is also a projective minimal semistable degeneration with $K_{X^\dagger} + \Theta^\dagger$ being Cartier. As in the same way as above, letting $\Theta_i^\dagger$ be an irreducible component of $\pi^{\dagger-1}(\Theta_i^\sigma)$ dominating Θ_i^σ, we have $\lfloor\Delta_i^\dagger\rfloor = \pi^{\dagger-1}(\lfloor\Delta_i^\sigma\rfloor)$ and $\lfloor\mathrm{Diff}_{\lfloor\Delta_i^\dagger\rfloor^\nu}(\Delta_i^\dagger - \lfloor\Delta_i^\dagger\rfloor)\rfloor = \pi^{\dagger\nu-1}\lfloor\mathrm{Diff}_{\lfloor\Delta_i^\sigma\rfloor^\nu}(\Delta_i^\sigma - \lfloor\Delta_i^\sigma\rfloor)\rfloor$, where $\Delta_i^\dagger := \mathrm{Diff}_{\Theta_i^\dagger}(\Theta^\dagger - \Theta_i^\dagger)$ and $\pi^{\dagger\nu} : \lfloor\Delta_i^\dagger\rfloor^\nu \to \lfloor\Delta_i^\sigma\rfloor^\nu$ is the induced morphism by $\pi^\dagger$ between the normalization of $\lfloor\Delta_i^\dagger\rfloor$ and $\lfloor\Delta_i^\sigma\rfloor$. By Lemma 3.4, if $f^\dagger$ is of type I (resp. of type II, resp. of type III), then for any irreducible component $\Theta_i^\dagger$ of $\Theta^\dagger$, $(\Theta_i^\dagger, \mathrm{Diff}_{\Theta_i^\dagger}(\Theta^\dagger - \Theta_i^\dagger))$ is of type I (resp. of type II, resp. of type III). Thus we infer the first assertion. As for the last assertion, construct two projective minimal semistable degeneration, $f_j^\sigma : X_j^\sigma \to \mathcal{D}^\sigma$ $(j = 1, 2)$ as in Lemma 5.1 from $f_j : X_j \to \mathcal{D}$ $(j = 1, 2)$ such that f_1^σ and f_2^σ are bimeromorphically equivalent over $\mathcal{D}^\sigma$. Since there exists a sequence of flops between f_1^σ and f_2^σ (see [26], Theorem 4.9), it is easily seen that f_1^σ and f_2^σ have the same type, so we get the last assertion. ■

Remark 5.1 From Proposition 5.1, for a projective log minimal degenerations of surfaces of Kodaira dimension zero $f : X \to \mathcal{D}$, we can see that if f is of type I (resp. of type II, resp. of type III), then for any irreducible component Θ_i of Θ, $(\Theta_i, \mathrm{Diff}_{\Theta_i}(\Theta - \Theta_i))$ is of type I (resp. of type II, resp. of type III).

6 Non-semistable degenerations of abelian or hyperelliptic surfaces

In this section we prove Theorem 1.2. Let $f : X \to \mathcal{D}$ be a projective log minimal degeneration of surfaces with Kodaira dimension zero. Take the log canonical cover $\pi : \tilde{X} \to X$ with respect to $K_X + \Theta$, where $\Theta := f^*(0)_{\mathrm{red}}$ and let $\tilde{f} : \tilde{X} \to \tilde{\mathcal{D}}$ be the induced degeneration via Stein factorization. For any irreducible component Θ_i of Θ, (X, Θ_i) is purely log terminal, hence so is $(\tilde{X}, \pi^{-1}\Theta_i)$. Thus the irreducible decomposition $\pi^{-1}\Theta_i = \sum_j \tilde{\Theta}_j^i$ is disjoint, which implies that each component of $\tilde{\Theta} := \tilde{f}^*(0)_{\mathrm{red}}$ is $\mathbf{Q}$-Cartier hence $\tilde{X}$ is $\mathbf{Q}$-Gorenstein. By [61], Covering Theorem, we see that $(\tilde{X}, \tilde{\Theta})$ is divisorially log terminal. Thus we conclude that $\tilde{f} : \tilde{X} \to \tilde{\mathcal{D}}$ is a projective log minimal degeneration of surfaces with Kodaira dimension zero with $K_{\tilde{X}} + \tilde{\Theta}$ being Cartier. Let $\pi_i : \tilde{\Theta}_i \to \Theta_i$ be the log canonical cover with respect to $K_{\Theta_i} + \Delta_i$, where $\Delta_i := \mathrm{Diff}_{\Theta_i}(\Theta - \Theta_i)$.

Lemma 6.1 *There exists an étale morphism* $\tilde{\Theta}_j^i \to \tilde{\Theta}_i$.

Proof. Put $r_i := \mathrm{Min}\{n \in \mathbf{N} | n(K_{\Theta_i} + \Delta_i) \sim 0\}$. By Proposition 4.5, we infer that there exists an open neighbourhood U of X containing Θ_i such that $\mathcal{O}_U(r_i(K_U + \Theta|_U)) \in \mathrm{Tor\ Pic}^\circ U$. Let m_i be the order of $\mathcal{O}_U(r_i(K_U + \Theta|_U))$. For any connected component V of $\pi^{-1}U$, $\pi|_V$ factors into $\pi|_V = \beta \circ \alpha$, where $\alpha : V \to W$ is étale of degree m_i and $\beta : W \to U$ is finite of degree r_i. We note $\pi|_V$ also factors into $\pi_i \circ \omega$, where $\omega : \pi^{-1}\Theta_i \to \tilde{\Theta}_i$ is finite. Since $\pi|_V$ is cyclic, we see that $\beta^{-1}\Theta_i \simeq \tilde{\Theta}_i$ and that ω is étale. Thus we get the assertion. ■

Assume that $e_{\mathrm{top}}(X_t) = 0$ for $t \in \mathcal{D}^*$. From the first part of Corollary 1.1 and Lemma 6.1, we obtain $e_{\mathrm{orb}}(\tilde{\Theta}_i \setminus \tilde{\Delta}_i) = 0$, where $\tilde{\Delta}_i := \pi_i^{-1}\lfloor\Delta_i\rfloor$. Let $\pi_p : (\tilde{X}, \tilde{p}) \to (X, p)$ be the log canonical cover of the germ of X at $p \in \Theta_i \setminus \lfloor\Delta_i\rfloor$ with respect to $K_X + \Theta_i$. The last part of Corollary 1.1 says that $(\tilde{X}, \pi_p^{-1}\Theta_i)$ has only singularity of type $V_1(r; a, -a, 1)$ at $\tilde{p} \in \tilde{X}$, where $(r, a) = 1$. Thus by the exact sequence (4.13) in the previous section, we have $\pi_1^{\mathrm{loc}}(\mathrm{Reg}\ (X, p)) \simeq \pi_{1,\Theta_i,p}^{\mathrm{loc}}[\mathrm{Diff}_{\Theta_i}(0)]$. So when we want to calculate the local fundamental group $\pi_1^{\mathrm{loc}}(\mathrm{Reg}\ (X, p))$, we only have to calculate

$\pi^{\rm loc}_{1,\Theta_i,p}[\mathrm{Diff}_{\Theta_i}(0)]$. We note that the proof of the first assertion of Theorem 1.2 is straightforward since (X,p) has the universal Cartier cover $\pi^\dagger : (X^\dagger, p^\dagger) \to (X,p)$ with respect to $K_X + \Theta_i$ such that $(X^\dagger, p^\dagger)$ is smooth (see Proposition 4.3).

6.1 Case Type II

In this section, we prove Theorem 1.2 in the case of type II. From proposition 3.3, For $p \in \Theta_i \setminus \Delta_i$, possible $\pi^{\rm loc}_{1,\Theta_i,p}[\mathrm{Diff}_{\Theta_i}(0)]$ is calculated to be $\boldsymbol{Z}/n\boldsymbol{Z}$ or $\boldsymbol{Z}/2\boldsymbol{Z} \oplus \boldsymbol{Z}/n\boldsymbol{Z}$, where $n = 2, 3, 4$ or 6. In applying the log minimal model program on $f : X \to \mathcal{D}$ with respect to K_X, we see that each extremal contraction contracts a prime divisor to a curve and reducing to the surface case, that contracting generic curve does not intersect Supp $\{\mathrm{Diff}_{\Theta_i}(\Theta - \Theta_i)\}$. So the last assertion in the case of type II follows from the following Lemma.

Lemma 6.2 *Let X be a normal $\boldsymbol{Q}$-Gorenstein 3-fold and S_1, S_2, E be mutually distinct $\boldsymbol{Q}$-Cartier prime divisors on X such that $S_1 \cap S_2 = \emptyset$. Putting $D := S_1 + S_2 + E$, assume that (X, D) is divisorially log terminal and that there exists an extremal contraction $\varphi : X \to X^\natural$ to a normal 3-fold $X^\natural$ such that S_i is φ-ample for $i = 1$, 2 and the following (1), (2) and (3) hold.*

(1) $-K_X$ *is φ-ample,*

(2) $K_X + D$ *is numerically trivial over $X^\natural$,*

(3) *E is contracted to a curve on $X^\natural$ and general fibres of the induced morphism $\varphi : E \to \varphi(E)$ does not intersects* Supp $\{\mathrm{Diff}_E(D - E)\}$.

Then $(X^\natural, D^\natural)$ has only divisorially log terminal singularities, where $D^\natural := \varphi_* D$.

Proof. Let F be any exceptional divisor in the function field of $X^\natural$ centered on $X^\natural$ whose log discrepancy with respect to $K_{X^\natural} + D^\natural$ is non-positive. If $F = E$, the conditions (1) and (3) imply that $X^\natural$ is smooth and $D^\natural$ has simple normal crossings at the generic point of the center of F. If $F \neq E$, log discrepancy at F with respect to $K_X + D$ is non-positive from the condition (2), hence the support of F at X is a curve C contained in $S_1 \cap E$. If C is contracted to a point by φ, then $(S_2 . C) > 0$ which contradicts the assumption $S_1 \cap S_2 = \emptyset$. So $X^\natural$ is smooth and $D^\natural$ has simple normal crossings at the generic point of the center of F also in this case. Thus we get the assertion by [61], Divisorial Log Terminal Theorem. ■

6.2 Case Type III

The the results of Theorem 1.2 in the case of type III follows from the Propositions 3.4, 3.5, 3.6, 3.7, 3.8, 3.9 and the following lemma.

Lemma 6.3 *Let (X, p) be a three dimensional $\boldsymbol{Q}$-Gorenstein singularity and Γ be a prime divisor on X passing through $p \in X$ such that (X, Γ) is purely log terminal. Let $\pi : (\tilde{X}, \tilde{p}) \to (X, p)$ be the log canonical cover with respect to $K_X + \Gamma$ and put $\tilde{\Gamma} := \pi^{-1}\Gamma$. Assume that $(\Gamma, p) \simeq (\boldsymbol{C}^2, 0)$ and* $\mathrm{Diff}_\Gamma(0) = (1/2)\mathrm{div}(z^2 + w^n)$ *($n \geq 2$), where (z, w) is a system of coordinate of Γ at $p \in \Gamma$ and that $(\tilde{X}, \tilde{\Gamma})$ has singularity of type $V_1(r; a, -a, 1)$ at $\tilde{p} \in \tilde{X}$, where $(r, a) = 1$. Then we have $n = 2$.*

Proof. It can be easily checked that $(X^\dagger, p^\dagger)$ and $(\Gamma^\dagger, p^\dagger)$ are both smooth and that $\hat{\pi}^{\rm loc}_{1,\Gamma,p}[\mathrm{Diff}_\Gamma(0)] \simeq \hat{\pi}^{\rm loc}_1(\mathrm{Reg}\ X) \simeq G$, where G is the dihedral group of the order $2n$ (see also [40]). Let $G = < a, b; a^n = 1, b^2 = 1, b^{-1}ab = a^{-1} >$ be a presentation of G and $\rho_\Gamma : G \to U(2, \boldsymbol{C})$ be a corresponding representation with respect to $(\Gamma, \mathrm{Diff}_\Gamma(0))$ defined as follows.

$$\rho_\Gamma(a) = \begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{pmatrix}, \quad \rho_\Gamma(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\rho_X : G \to U(3, \boldsymbol{C})$ be a corresponding faithful representation with respect to (X, p). Since $\Gamma^\dagger \subset X^\dagger$ is invariant under the action of G through ρ_X, ρ_X is equivalent to $\rho_\Gamma \oplus \chi$ for some character $\chi : G \to \boldsymbol{C}^\times$. Let K be the kernel of the character $G \to \mathrm{Aut}\ \mathcal{O}_{X^\dagger}(K_{X^\dagger} + \Gamma^\dagger)/m_{p^\dagger}\mathcal{O}_{X^\dagger}(K_{X^\dagger} + \Gamma^\dagger)$ induced by ρ_X. Then we see that $K = \mathrm{Ker}\ \det\ \rho_\Gamma$. Since we have $\det\ \rho_\Gamma(a) = 1$ and $\det\ \rho_\Gamma(b) = -1$, we have $< a > \subset K$ and $b \notin K$. Noting that we have $2 = [G :< a >] = [G : K][K :< a >]$, we obtain $K =< a >$. We also note that we have ord $\chi(a) = n$ since $\tilde{p} \in \tilde{X} \simeq \boldsymbol{C}^3/K$ is isolated. On the other hand, since we have $\chi(b^{-1}ab) = \chi(a^{-1})$, we get $\chi(a)^2 = 1$. Thus we conclude that $n = 2$. ■

The last assertion of the Theorem 1.2 in the case of type III follows from [48], Theorem 3.1.

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